1. Introduction

In this chapter we use the material from the preceding sections to give criteria under which a presheaf of sets on the category of schemes is an algebraic space. Some of this material comes from the work of Artin, see \[\text{Art69b}, \text{Art70}, \text{Art73}, \text{Art71b}, \text{Art71a}, \text{Art69a}, \text{Art69c}, \text{and Art74}\]. However, our method will be to use as much as possible arguments similar to those of the paper by Keel and Mori, see \[\text{KM97}\].

2. Conventions

The standing assumption is that all schemes are contained in a big fppf site \(\text{Sch}_{fppf}\). And all rings \(A\) considered have the property that \(\text{Spec}(A)\) is (isomorphic) to an object of this big site.

Let \(S\) be a scheme and let \(X\) be an algebraic space over \(S\). In this chapter and the following we will write \(X \times_S X\) for the product of \(X\) with itself (in the category of algebraic spaces over \(S\)), instead of \(X \times X\).

3. Morphisms representable by algebraic spaces

Here we define the notion of one presheaf being relatively representable by algebraic spaces over another, and we prove some properties of this notion.
**Definition 3.1.** Let $S$ be a scheme contained in $\text{Sch}_{\text{fppf}}$. Let $F, G$ be presheaves on $\text{Sch}_{\text{fppf}}/S$. We say a morphism $a : F \to G$ is representable by algebraic spaces if for every $U \in \text{Ob}((\text{Sch}/S)_{\text{fppf}})$ and any $\xi : U \to G$ the fiber product $U \times_{\xi,G} F$ is an algebraic space.

Here is a sanity check.

**Lemma 3.2.** Let $S$ be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over $S$. Then $f$ is representable by algebraic spaces.

**Proof.** This is formal. It relies on the fact that the category of algebraic spaces over $S$ has fibre products, see Spaces, Lemma 7.3.

**Lemma 3.3.** Let $S$ be a scheme. Let

$$
\begin{array}{ccc}
G' \times_G F & \longrightarrow & F \\
\downarrow^{a'} & & \downarrow^{a} \\
G' & \longrightarrow & G
\end{array}
$$

be a fibre square of presheaves on $(\text{Sch}/S)_{\text{fppf}}$. If $a$ is representable by algebraic spaces so is $a'$.

**Proof.** Omitted. Hint: This is formal.

**Lemma 3.4.** Let $S$ be a scheme contained in $\text{Sch}_{\text{fppf}}$. Let $F, G : (\text{Sch}/S)_{\text{fppf}} \to \text{Sets}$. Let $a : F \to G$ be representable by algebraic spaces. If $G$ is a sheaf, then so is $F$.

**Proof.** (Same as the proof of Spaces, Lemma 3.5) Let $\{\varphi_i : T_i \to T\}$ be a covering of the site $(\text{Sch}/S)_{\text{fppf}}$. Let $s_i \in F(T_i)$ which satisfy the sheaf condition. Then $\sigma_i = a(s_i) \in G(T_i)$ satisfy the sheaf condition also. Hence there exists a unique $\sigma \in G(T)$ such that $\sigma_i = \sigma|_{T_i}$. By assumption $F' = h_T \times_{\sigma,G,a} F$ is a sheaf. Note that $(\varphi_i, s_i) \in F'(T_i)$ satisfy the sheaf condition also, and hence come from some unique $(\text{id}_T, s) \in F'(T)$. Clearly $s$ is the section of $F$ we are looking for.

**Lemma 3.5.** Let $S$ be a scheme contained in $\text{Sch}_{\text{fppf}}$. Let $F, G : (\text{Sch}/S)_{\text{fppf}} \to \text{Sets}$. Let $a : F \to G$ be representable by algebraic spaces. Then $\Delta_{F/G} : F \to F \times_G F$ is representable by algebraic spaces.

**Proof.** (Same as the proof of Spaces, Lemma 3.6) Let $U$ be a scheme. Let $\xi = (\xi_1, \xi_2) \in (F \times_G F)(U)$. Set $\xi' = a(\xi_1) = a(\xi_2) \in G(U)$. By assumption there exist an algebraic space $V$ and a morphism $V \to U$ representing the fibre product $U \times_{\xi,G} F$. In particular, the elements $\xi_1, \xi_2$ give morphisms $f_1, f_2 : U \to V$ over $U$. Because $V$ represents the fibre product $U \times_{\xi', G} F$ and because $\xi' = a \circ \xi_1 = a \circ \xi_2$ we see that if $g : U' \to U$ is a morphism then

$$
g^*\xi_1 = g^*\xi_2 \iff f_1 \circ g = f_2 \circ g.
$$

In other words, we see that $U \times_{\xi', F \times_G F} F$ is represented by $V \times_{\Delta, V \times V, (f_1, f_2)} U$ which is an algebraic space.

The proof of Lemma 3.6 below is actually slightly tricky. Namely, we cannot use the argument of the proof of Spaces, Lemma 11.1 because we do not yet know that a composition of transformations representable by algebraic spaces is representable by algebraic spaces. In fact, we will use this lemma to prove that statement.
Lemma 3.6. Let $S$ be a scheme contained in $\text{Sch}_{\text{fppf}}$. Let $F,G : (\text{Sch}/S)^{\text{op}}_{\text{fppf}} \to \text{Sets}$. Let $a : F \to G$ be representable by algebraic spaces. If $G$ is an algebraic space, then so is $F$.

Proof. We have seen in Lemma 3.4 that $F$ is a sheaf.

Let $U$ be a scheme and let $U \to G$ be a surjective étale morphism. In this case $U \times_G F$ is an algebraic space. Let $W$ be a scheme and let $W \to U \times_G F$ be a surjective étale morphism.

First we claim that $W \to F$ is representable. To see this let $X$ be a scheme and let $X \to F$ be a morphism. Then
\[ W \times_F X = W \times_{U \times_G F} U \times_G F \times_F X = W \times_{U \times_G F} (U \times_G X) \]
Since both $U \times_G F$ and $G$ are algebraic spaces we see that this is a scheme.

Next, we claim that $W \to F$ is surjective and étale (this makes sense now that we know it is representable). This follows from the formula above since both $W \to U \times_G F$ and $U \to G$ are étale and surjective, hence $W \times_{U \times_G F} (U \times_G X) \to U \times_G X$ and $U \times_G X \to X$ are surjective and étale, and the composition of surjective étale morphisms is surjective and étale.

Set $R = W \times_F W$. By the above $R$ is a scheme and the projections $t,s : R \to W$ are étale. It is clear that $R$ is an equivalence relation, and $W \to F$ is a surjection of sheaves. Hence $R$ is an étale equivalence relation and $F = W/R$. Hence $F$ is an algebraic space by Spaces, Theorem 10.5.

Lemma 3.7. Let $S$ be a scheme. Let $a : F \to G$ be a map of presheaves on $(\text{Sch}/S)^{\text{op}}_{\text{fppf}}$. Suppose $a : F \to G$ is representable by algebraic spaces. If $X$ is an algebraic space over $S$, and $X \to G$ is a map of presheaves then $X \times_G F$ is an algebraic space.

Proof. By Lemma 3.3 the transformation $X \times_G F \to X$ is representable by algebraic spaces. Hence it is an algebraic space by Lemma 3.6.

Lemma 3.8. Let $S$ be a scheme. Let
\[ F \xrightarrow{a} G \xrightarrow{b} H \]
be maps of presheaves on $(\text{Sch}/S)^{\text{op}}_{\text{fppf}}$. If $a$ and $b$ are representable by algebraic spaces, so is $b \circ a$.

Proof. Let $T$ be a scheme over $S$, and let $T \to H$ be a morphism. By assumption $T \times_H G$ is an algebraic space. Hence by Lemma 3.7 we see that $T \times_H F = (T \times_H G) \times_G F$ is an algebraic space as well.

Lemma 3.9. Let $S$ be a scheme. Let $F_i, G_i : (\text{Sch}/S)^{\text{op}}_{\text{fppf}} \to \text{Sets}, i = 1,2$. Let $a_i : F_i \to G_i, i = 1,2$ be representable by algebraic spaces. Then
\[ a_1 \times a_2 : F_1 \times F_2 \longrightarrow G_1 \times G_2 \]
is a representable by algebraic spaces.

Proof. Write $a_1 \times a_2$ as the composition $F_1 \times F_2 \to G_1 \times F_2 \to G_1 \times G_2$. The first arrow is the base change of $a_1$ by the map $G_1 \times F_2 \to G_1$, and the second arrow is the base change of $a_2$ by the map $G_1 \times G_2 \to G_2$. Hence this lemma is a formal consequence of Lemmas 3.8 and 3.3.
Lemma 3.10. Let $S$ be a scheme. Let $a : F \to G$ and $b : G \to H$ be transformations of functors $(\text{Sch}/S)^{\text{op}}_{fppf} \to \text{Sets}$. Assume

1. $\Delta : G \to G \times_H G$ is representable by algebraic spaces, and
2. $b \circ a : F \to H$ is representable by algebraic spaces.

Then $a$ is representable by algebraic spaces.

Proof. Let $U$ be a scheme over $S$ and let $\xi \in G(U)$. Then

$$U \times_{\xi,G,a} F = (U \times_{b(\xi),H,b \circ a} F) \times_{(\xi,a),(G \times_H G),\Delta} G$$

Hence the result using Lemma 3.7.

Lemma 3.11. Let $S \in \text{Ob}(\text{Sch}_{fppf})$. Let $F$ be a presheaf of sets on $(\text{Sch}/S)_{fppf}$.

Assume

1. $F$ is a sheaf for the Zariski topology on $(\text{Sch}/S)_{fppf}$,
2. there exists an index set $I$ and subfunctors $F_i \subset F$ such that
   a. each $F_i$ is an fppf sheaf,
   b. each $F_i \to F$ is representable by algebraic spaces,
   c. $\prod F_i \to F$ becomes surjective after fppf sheafification.

Then $F$ is an fppf sheaf.

Proof. Let $T \in \text{Ob}(\text{Sch}/S)_{fppf}$ and let $s \in F(T)$. By (2)(c) there exists an fppf covering $\{T_j \to T\}$ such that $s|_{T_j}$ is a section of $F_{\alpha(j)}$ for some $\alpha(j) \in I$. Let $W_j \subset T$ be the image of $T_j \to T$ which is an open subscheme Morphisms, Lemma 26.9. By (2)(b) we see $F_{\alpha(j)} \times_{F,s|_{W_j}} W_j \to W_j$ is a monomorphism of algebraic spaces through which $T_j$ factors. Since $\{T_j \to W_j\}$ is an fppf covering, we conclude that $F_{\alpha(j)} \times_{F,s|_{W_j}} W_j = W_j$, in other words $s|_{W_j} \in F_{\alpha(j)}(W_j)$. Hence we conclude that $\prod F_i \to F$ is surjective for the Zariski topology.

Let $\{T_j \to T\}$ be an fppf covering in $(\text{Sch}/S)_{fppf}$. Let $s,s' \in F(T)$ with $s|_{T_j} = s'|_{T_j}$ for all $j$. We want to show that $s,s'$ are equal. As $F$ is a Zariski sheaf by (1) we may work Zariski locally on $T$. By the result of the previous paragraph we may assume there exist $i$ such that $s \in F_i(T)$. Then we see that $s'|_{T_i}$ is a section of $F_i$. By (2)(b) we see $F_i \times_{F,s'} T \to T$ is a monomorphism of algebraic spaces through which all of the $T_j$ factor. Hence we conclude that $s' \in F_i(T)$. Since $F_i$ is a sheaf for the fppf topology we conclude that $s = s'$.

Let $\{T_j \to T\}$ be an fppf covering in $(\text{Sch}/S)_{fppf}$ and let $s_j \in F(T_j)$ such that $s_j|_{T_j \times T_j'} = s_j'|_{T_j \times T_j'}$. By assumption (2)(b) we may refine the covering and assume that $s_j \in F_{\alpha(j)}(T_j)$ for some $\alpha(j) \in I$. Let $W_j \subset T$ be the image of $T_j \to T$ which is an open subscheme Morphisms, Lemma 26.9. Then $\{T_j \to W_j\}$ is an fppf covering. Since $F_{\alpha(j)}$ is a sub presheaf of $F$ we see that the two restrictions of $s_j$ to $T_j \times_{W_j} T_j$ agree as elements of $F_{\alpha(j)}(T_j \times_{W_j} T_j)$. Hence, the sheaf condition for $F_{\alpha(j)}$ implies there exists $s_j' \in F_{\alpha(j)}(W_j)$ whose restriction to $T_j$ is $s_j$. For a pair of indices $j$ and $j'$ the sections $s_j'|_{W_j \cap W_j'}$, and $s_j'|_{W_j \cap W_j'}$ of $F$ agree by the result of the previous paragraph. This finishes the proof by the fact that $F$ is a Zariski sheaf.

4. Properties of maps of presheaves representable by algebraic spaces

Here is the definition that makes this work.
Definition 4.1. Let $S$ be a scheme. Let $a : F \to G$ be a map of presheaves on $(\text{Sch}/S)_{\text{fppf}}$ which is representable by algebraic spaces. Let $\mathcal{P}$ be a property of morphisms of algebraic spaces which

1. is preserved under any base change, and
2. is fppf local on the base, see Descent on Spaces, Definition 9.1.

In this case we say that $a$ has property $\mathcal{P}$ if for every scheme $U$ and $\xi : U \to G$ the resulting morphism of algebraic spaces $U \times_G F \to U$ has property $\mathcal{P}$.

It is important to note that we will only use this definition for properties of morphisms that are stable under base change, and local in the fppf topology on the base. This is not because the definition doesn’t make sense otherwise; rather it is because we may want to give a different definition which is better suited to the property we have in mind.

The definition above applies, for example to the properties of being “surjective”, “quasi-compact”, “étale”, “flat”, “separated”, “(locally) of finite type”, “(locally) quasi-finite”, “(locally) of finite presentation”, “proper”, and “a closed immersion”. In other words, $a$ is surjective (resp. quasi-compact, étale, flat, separated, (locally) of finite type, (locally) quasi-finite, (locally) of finite presentation, proper, a closed immersion) if for every scheme $T$ and map $\xi : T \to G$ the morphism of algebraic spaces $T \times_\xi G \to T$ is surjective (resp. quasi-compact, étale, flat, separated, (locally) of finite type, (locally) quasi-finite, (locally) of finite presentation, proper, a closed immersion).

Next, we check consistency with the already existing notions. By Lemma 3.2 any morphism between algebraic spaces over $S$ is representable by algebraic spaces. And by Morphisms of Spaces, Lemmas 5.3, 8.7, 36.2, 28.5, 4.12, 23.4, 26.6, 27.4, 37.2, 12.1 the definition of surjective (resp. quasi-compact, étale, flat, separated, (locally) of finite type, (locally) quasi-finite, (locally) of finite presentation, proper, a closed immersion) above agrees with the already existing definition of morphisms of algebraic spaces.

Some formal lemas follow.

Lemma 4.2. Let $S$ be a scheme. Let $\mathcal{P}$ be a property as in Definition 4.1. Let $G' \times_G F \to F \xrightarrow{a} G$ be a fibre square of presheaves on $(\text{Sch}/S)_{\text{fppf}}$. If $a$ is representable by algebraic spaces and has $\mathcal{P}$ so does $a'$.

Proof. Omitted. Hint: This is formal. \qed

Lemma 4.3. Let $S$ be a scheme. Let $\mathcal{P}$ be a property as in Definition 4.1, and assume $\mathcal{P}$ is stable under composition. Let

\[
\begin{array}{ccc}
F & \xrightarrow{a} & G \\
\downarrow{a'} & & \downarrow{a}
\end{array}
\]

be a fibre square of presheaves on $(\text{Sch}/S)_{\text{fppf}}$. If $a$ is representable by algebraic spaces and has $\mathcal{P}$ so does $a'$.

Proof. Omitted. Hint: This is formal. \qed
be maps of presheaves on \((\text{Sch}/S)_{\text{fppf}}\). If \(a, b\) are representable by algebraic spaces and has \(\mathcal{P}\) so does \(b \circ a\).

**Proof.** Omitted. Hint: See Lemma 3.8 and use stability under composition. \(\square\)

**Lemma 4.4.** Let \(S\) be a scheme. Let \(F_i, G_i : (\text{Sch}/S)_{\text{fppf}} \to \text{Sets}, i = 1, 2\). Let \(a_i : F_i \to G_i, i = 1, 2\) be representable by algebraic spaces. Let \(\mathcal{P}\) be a property as in Definition 4.1 which is stable under composition. If \(a_1\) and \(a_2\) have property \(\mathcal{P}\) so does \(a_1 \times a_2 : F_1 \times F_2 \to G_1 \times G_2\).

**Proof.** Note that the lemma makes sense by Lemma 3.9. Proof omitted. \(\square\)

**Lemma 4.5.** Let \(S\) be a scheme. Let \(F, G : (\text{Sch}/S)_{\text{opp fppf}} \to \text{Sets}\). Let \(a : F \to G\) be representable by algebraic spaces. Let \(\mathcal{P}, \mathcal{P}'\) be properties as in Definition 4.1. Suppose that for any morphism \(f : X \to Y\) of algebraic spaces over \(S\) we have \(\mathcal{P}(f) \Rightarrow \mathcal{P}'(f)\). If \(a\) has property \(\mathcal{P}\), then \(a\) has property \(\mathcal{P}'\).

**Proof.** Formal. \(\square\)

**Lemma 4.6.** Let \(S\) be a scheme. Let \(F, G : (\text{Sch}/S)_{\text{opp fppf}} \to \text{Sets}\) be sheaves. Let \(a : F \to G\) be representable by algebraic spaces, flat, locally of finite presentation, and surjective. Then \(a : F \to G\) is surjective as a map of sheaves.

**Proof.** Let \(T\) be a scheme over \(S\) and let \(g : T \to G\) be a \(T\)-valued point of \(G\). By assumption \(T' = F \times_G T\) is an algebraic space and the morphism \(T' \to T\) is a flat, locally of finite presentation, and surjective morphism of algebraic spaces. Let \(U \to T'\) be a surjective étale morphism, where \(U\) is a scheme. Then by the definition of flat morphisms of algebraic spaces the morphism of schemes \(U \to T\) is flat. Similarly for “locally of finite presentation”. The morphism \(U \to T\) is surjective also, see Morphisms of Spaces, Lemma 5.3 Hence we see that \(\{U \to T\}\) is an fppf covering such that \(g|_U \in G(U)\) comes from an element of \(F(U)\), namely the map \(U \to T' \to F\). This proves the map is surjective as a map of sheaves, see Sites, Definition 12.1. \(\square\)

### 5. Bootstraping the diagonal

**Lemma 5.1.** Let \(S\) be a scheme. If \(F\) is a presheaf on \((\text{Sch}/S)_{\text{fppf}}\). The following are equivalent:

1. \(\Delta_F : F \to F \times F\) is representable by algebraic spaces,
2. for every scheme \(T\) any map \(T \to F\) is representable by algebraic spaces, and
3. for every algebraic space \(X\) any map \(X \to F\) is representable by algebraic spaces.

**Proof.** Assume (1). Let \(X \to F\) be as in (3). Let \(T\) be a scheme, and let \(T \to F\) be a morphism. Then we have

\[
T \times_F X = (T \times_S X) \times_{F \times F, \Delta} F
\]

which is an algebraic space by Lemma 3.7 and (1). Hence \(X \to F\) is representable, i.e., (3) holds. The implication (3) \(\Rightarrow\) (2) is trivial. Assume (2). Let \(T\) be a scheme, and let \((a, b) : T \to F \times F\) be a morphism. Then

\[
F \times_{\Delta_F, F \times F} T = T \times_{a, F, b} T
\]
which is an algebraic space by assumption. Hence $\Delta_F$ is representable by algebraic spaces, i.e., (1) holds.

In particular if $F$ is a presheaf satisfying the equivalent conditions of the lemma, then for any morphism $X \to F$ where $X$ is an algebraic space it makes sense to say that $X \to F$ is surjective (resp. étale, flat, locally of finite presentation) by using Definition 4.1.

Before we actually do the bootstrap we prove a fun lemma.

**Lemma 5.2.** Let $S$ be a scheme. Let

$$
\begin{array}{ccc}
E & \rightarrow & F \\
\downarrow f & & \downarrow g \\
H & \rightarrow & G \\
\end{array}
$$

be a cartesian diagram of sheaves on $(\text{Sch}/S)_{\text{fppf}}$, so $E = H \times_G F$. If

1. $g$ is representable by algebraic spaces, surjective, flat, and locally of finite presentation, and
2. $a$ is representable by algebraic spaces, separated, and locally quasi-finite

then $b$ is representable (by schemes) as well as separated and locally quasi-finite.

**Proof.** Let $T$ be a scheme, and let $T \to G$ be a morphism. We have to show that $T \times_G H$ is an algebraic space, and that the morphism $T \times_G H \to T$ is separated and locally quasi-finite. Thus we may base change the whole diagram to $T$ and assume that $G$ is a scheme. In this case $F$ is an algebraic space. Let $U$ be a scheme, and let $U \to F$ be a surjective étale morphism. Then $U \to F$ is representable, surjective, flat and locally of finite presentation by Morphisms of Spaces, Lemmas 36.7 and 36.8. By Lemma 3.8 $U \to G$ is surjective, flat and locally of finite presentation also. Note that the base change $E \times_F U \to U$ of $a$ is still separated and locally quasi-finite (by Lemma 4.2). Hence we may replace the upper part of the diagram of the lemma by $E \times_F U \to U$. In other words, we may assume that $F \to G$ is a surjective, flat morphism of schemes which is locally of finite presentation. In particular, $\{F \to G\}$ is an fppf covering of schemes. By Morphisms of Spaces, Proposition 44.2 we conclude that $E$ is a scheme also. By Descent, Lemma 35.1 the fact that $E = H \times_G F$ means that we get a descent datum on $E$ relative to the fppf covering $\{F \to G\}$. By More on Morphisms, Lemma 39.1 this descent datum is effective. By Descent, Lemma 35.1 again this implies that $H$ is a scheme. By Descent, Lemmas 19.5 and 19.22 it now follows that $b$ is separated and locally quasi-finite.

Here is the result that the section title refers to.

**Lemma 5.3.** Let $S$ be a scheme. Let $F : (\text{Sch}/S)^{\text{opp}}_{\text{fppf}} \to \text{Sets}$ be a functor. Assume that

1. the presheaf $F$ is a sheaf,
2. there exists an algebraic space $X$ and a map $X \to F$ which is representable by algebraic spaces, surjective, flat and locally of finite presentation.

Then $\Delta_F$ is representable (by schemes).
**Proof.** Let \( U \to X \) be a surjective étale morphism from a scheme towards \( X \). Then \( U \to X \) is representable, surjective, flat and locally of finite presentation by Morphisms of Spaces, Lemmas 36.7 and 36.8 By Lemma 4.3 the composition \( U \to F \) is representable by algebraic spaces, surjective, flat and locally of finite presentation also. Thus we see that \( R = U \times F \) is an algebraic space, see Lemma 3.7. The morphism of algebraic spaces \( R \to U \times_S U \) is a monomorphism, hence separated (as the diagonal of a monomorphism is an isomorphism, see Morphisms of Spaces, Lemma 10.2). Since \( U \to F \) is locally of finite presentation, both morphisms \( R \to U \) are locally of finite presentation, see Lemma 4.2. Hence \( R \to U \times_S U \) is locally of finite type (use Morphisms of Spaces, Lemmas 27.5 and 23.6). Altogether this means that \( R \to U \times_S U \) is a monomorphism which is locally of finite type, hence a separated and locally quasi-finite morphism, see Morphisms of Spaces, Lemma 26.10.

Now we are ready to prove that \( \Delta_F \) is representable. Let \( T \) be a scheme, and let \((a, b) : T \to F \times F\) be a morphism. Set

\[
T' = (U \times_S U) \times_{F \times F} T.
\]

Note that \( U \times_S U \to F \times F \) is representable by algebraic spaces, surjective, flat and locally of finite presentation by Lemma 4.4. Hence \( T' \) is an algebraic space, and the projection morphism \( T' \to T \) is surjective, flat, and locally of finite presentation. Consider \( Z = T \times_{F \times F} F \) (this is a sheaf) and

\[
Z' = T' \times_{U \times_S U} R = T' \times_T Z.
\]

We see that \( Z' \) is an algebraic space, and \( Z' \to T' \) is separated and locally quasi-finite by the discussion in the first paragraph of the proof which showed that \( R \) is an algebraic space and that the morphism \( R \to U \times_S U \) has those properties. Hence we may apply Lemma 5.2 to the diagram

\[
\begin{array}{ccc}
Z' & \to & T' \\
\downarrow & & \downarrow \\
Z & \to & T
\end{array}
\]

and we conclude. \( \square \)

Here is a variant of the result above.

**Lemma 5.4.** Let \( S \) be a scheme. Let \( F : (\text{Sch}/S)^{\text{opp}}_{fppf} \to \text{Sets} \) be a functor. Let \( X \) be a scheme and let \( X \to F \) be representable by algebraic spaces and locally quasi-finite. Then \( X \to F \) is representable (by schemes).

**Proof.** Let \( T \) be a scheme and let \( T \to F \) be a morphism. We have to show that the algebraic space \( X \times_F T \) is representable by a scheme. Consider the morphism

\[
X \times_F T \to X \times_{\text{Spec}(\mathbb{Z})} \text{Spec}(A)
\]

Since \( X \times_F T \to T \) is locally quasi-finite, so is the displayed arrow (Morphisms of Spaces, Lemma 26.8). On the other hand, the displayed arrow is a monomorphism and hence separated (Morphisms of Spaces, Lemma 10.3). Thus \( X \times_F T \) is a scheme by Morphisms of Spaces, Proposition 24.2 \( \square \)
6. Bootstrap

We warn the reader right away that the result of this section will be superseded by the stronger Theorem [10.1]. On the other hand, the theorem in this section is quite a bit easier to prove and still provides quite a bit of insight into how things work, especially for those readers mainly interested in Deligne-Mumford stacks.

In Spaces, Section 6, we defined an algebraic space as a sheaf in the fppf topology whose diagonal is representable, and such that there exist a surjective étale morphism from a scheme towards it. In this section we show that a sheaf in the fppf topology whose diagonal is representable by algebraic spaces and which has an étale surjective covering by an algebraic space is also an algebraic space. In other words, the category of algebraic spaces is an enlargement of the category of schemes by those fppf sheaves \( F \) which have a representable diagonal and an étale covering by a scheme. The result of this section says that doing the same process again starting with the category of algebraic spaces, does not lead to yet another category.

Another motivation for the material in this section is that it will guarantee later that a Deligne-Mumford stack whose inertia stack is trivial is equivalent to an algebraic space, see Algebraic Stacks, Lemma [13.2].

Here is the main result of this section (as we mentioned above this will be superseded by the stronger Theorem [10.1]).

**Theorem 6.1.** Let \( S \) be a scheme. Let \( F : (\text{Sch}/S)_{\text{fppf}}^{\text{op}} \to \text{Sets} \) be a functor. Assume that

1. the presheaf \( F \) is a sheaf,
2. the diagonal morphism \( F \to F \times F \) is representable by algebraic spaces, and
3. there exists an algebraic space \( X \) and a map \( X \to F \) which is surjective, and étale.

Then \( F \) is an algebraic space.

**Proof.** We will use the remarks directly below Definition 4.4 without further mention. In the situation of the theorem, let \( U \to X \) be a surjective étale morphism from a scheme towards \( X \). By Lemma 3.8 \( U \to F \) is surjective and étale also. Hence the theorem boils down to proving that \( \Delta_F \) is representable. This follows immediately from Lemma 5.3. On the other hand we can circumvent this lemma and show directly \( F \) is an algebraic space as in the next paragraph.

Let \( U \) be a scheme, and let \( U \to F \) be surjective and étale. Set \( R = U \times_F U \), which is an algebraic space (see Lemma 5.1). The morphism of algebraic spaces \( R \to U \times_S U \) is a monomorphism, hence separated (as the diagonal of a monomorphism is an isomorphism). Moreover, since \( U \to F \) is étale, we see that \( U \to U \) is étale, by Lemma 4.2. In particular, we see that \( R \to U \) is locally quasi-finite, see Morphisms of Spaces, Lemma 36.5. We conclude that also \( R \to U \times_S U \) is locally quasi-finite by Morphisms of Spaces, Lemma 26.8. Hence Morphisms of Spaces, Proposition 44.2 applies and \( R \) is a scheme. Hence \( F = U/R \) is an algebraic space according to Spaces, Theorem 10.5.

7. Finding opens

First we prove a lemma which is a slight improvement and generalization of Spaces, Lemma 10.2 to quotient sheaves associated to groupoids.
Lemma 7.1. Let $S$ be a scheme. Let $(U, R, s, t, c)$ be a groupoid scheme over $S$. Let $g : U' \to U$ be a morphism. Assume

1. the composition

$$U' \times_{g, U, t} R \xrightarrow{h} R \xrightarrow{pr} U$$

has an open image $W \subset U$, and
2. the resulting map $h : U' \times_{g, U, t} R \to W$ defines a surjection of sheaves in the fppf topology.

Let $R' = R |_{U'}$ be the restriction of $R$ to $U'$. Then the map of quotient sheaves

$$U'/R' \to U/R$$

in the fppf topology is representable, and is an open immersion.

Proof. Note that $W$ is an $R$-invariant open subscheme of $U$. This is true because the set of points of $W$ is the set of points of $U$ which are equivalent in the sense of Groupoids, Lemma 3.4 to a point of $g(U') \subset U$ (the lemma applies as $j : R \to U \times_S U$ is a pre-equivalence relation by Groupoids, Lemma 11.2). Also $g : U' \to U$ factors through $W$. Let $R_W$ be the restriction of $R$ to $W$. Then it follows that $R'$ is also the restriction of $R_W$ to $U'$. Hence we can factor the map of sheaves of the lemma as

$$U'/R' \to W/R_W \to U/R$$

By Groupoids, Lemma 18.6 we see that the first arrow is an isomorphism of sheaves. Hence it suffices to show the lemma in case $g$ is the immersion of an $R$-invariant open into $U$.

Assume $U' \subset U$ is an $R$-invariant open and $g$ is the inclusion morphism. Set $F = U/R$ and $F' = U'/R'$. By Groupoids, Lemma 18.5 or 18.6 the map $F' \to F$ is injective. Let $\xi \in F(T)$. We have to show that $T \times_{\xi, F} F'$ is representable by an open subscheme of $T$. There exists an fppf covering $\{f_i : T_i \to T\}$ such that $\xi|_{T_i}$ is the image via $U \to U/R$ of a morphism $a_i : T_i \to U$. Set $V_i = s^{-1}(U')$. We claim that $V_i \times_T T_j = T_i \times_T V_j$ as open subschemes of $T_i \times_T T_j$.

As $a_i \circ pr_0$ and $a_j \circ pr_1$ are morphisms $T_i \times_T T_j \to U$ which both map to the section $\xi|_{T_i \times_T T_j} \in F(T_i \times_T T_j)$ we can find an fppf covering $\{f_{ijk} : T_{ijk} \to T_i \times_T T_j\}$ and morphisms $r_{ijk} : T_{ijk} \to R$ such that

$$a_i \circ pr_0 \circ f_{ijk} = s \circ r_{ijk}, \quad a_j \circ pr_1 \circ f_{ijk} = t \circ r_{ijk},$$

see Groupoids, Lemma 18.4. Since $U'$ is $R$-invariant we have $s^{-1}(U') = t^{-1}(U')$ and hence $f_{ijk}(V_i \times_T T_j) = f_{ijk}(T_i \times_T V_j)$. As $\{f_{ijk}\}$ is surjective this implies the claim above. Hence by Descent, Lemma 9.2 there exists an open subscheme $V \subset T$ such that $f^{-1}_i(V) = V_i$. We claim that $V$ represents $T \times_{\xi, F} F'$.

As a first step, we will show that $\xi|_V$ lies in $F'(V) \subset F(V)$. Namely, the family of morphisms $\{V_i \to V\}$ is an fppf covering, and by construction we have $\xi|_{V_i} \in F'(V_i)$. Hence by the sheaf property of $F'$ we get $\xi|_V \in F'(V)$. Finally, let $T' \to T$ be a morphism of schemes and that $\xi|_{T'} \in F'(T')$. To finish the proof we have to show that $T' \to T$ factors through $V$. We can find a fppf covering $\{T'_j \to T'\}_{j \in J}$ and morphisms $b_j : T'_j \to U'$ such that $\xi|_{T'_j}$ is the image via $U' \to U/R$ of $b_j$. Clearly, it is enough to show that the compositions $T'_j \to T$ factor through $V$. Hence we
may assume that $\xi|_{T'}$ is the image of a morphism $b : T' \to U'$. Now, it is enough to show that $T' \times_T T_i \to T_i$ factors through $V_i$. Over the scheme $T' \times_T T_i$, the restriction of $\xi$ is the image of two elements of $(U/R)(T' \times_T T_i)$, namely $a_i \circ pr_1$, and $b \circ pr_0$, the second of which factors through the $R$-invariant open $U'$. Hence by Groupoids, Lemma 18.6 we see that $\xi$ is an isomorphism. By the construction of $U$ we see that $\xi$ is $\text{pr}_1$-surjective. Since $U$ is an affine scheme $U$ may assume that $s,t \in u$.

Step 2. Assume $s,t : U \to T$ are Cohen-Macaulay and locally of finite presentation. Let $u \in U$ be a point of finite type. By More on Groupoids, Lemma 11.4 there exists an affine scheme $U'$ and a morphism $g : U' \to U$ such that

1. $g$ is an immersion,
2. $u \in U'$,
3. $g$ is locally of finite presentation,
4. $h$ is flat, locally of finite presentation and locally quasi-finite, and
5. the morphisms $s', t' : R' \to U'$ are flat, locally of finite presentation and locally quasi-finite.

\section{Slicing equivalence relations}

In this section we explain how to “improve” a given equivalence relation by slicing. This is not a kind of “étale slicing” that you may be used to but a much coarser kind of slicing.

\textbf{Lemma 8.1.} Let $S$ be a scheme. Let $j : R \to U \times_S U$ be an equivalence relation on schemes over $S$. Assume $s, t : R \to U$ are flat and locally of finite presentation. Then there exists an equivalence relation $j' : R' \to U' \times_S U'$ on schemes over $S$, and an isomorphism $U'/R' \to U/R$ induced by a morphism $U' \to U$ which maps $R'$ into $R$ such that $s', t' : R' \to U$ are flat, locally of finite presentation and locally quasi-finite.

\textbf{Proof.} We will prove this lemma in several steps. We will use without further mention that an equivalence relation gives rise to a groupoid scheme and that the restriction of an equivalence relation is an equivalence relation, see Groupoids, Lemmas 3.2, 11.3, and 16.3.

Step 1: We may assume that $s, t : R \to U$ are locally of finite presentation and Cohen-Macaulay morphisms. Namely, as in More on Groupoids, Lemma 7.1 let $g : U' \to U$ be the open subscheme such that $t^{-1}(U') \subset R$ is the maximal open over which $s : R \to U$ is Cohen-Macaulay, and denote $R'$ the restriction of $R$ to $U'$.

By the lemma cited above we see that

$$t^{-1}(U') \xrightarrow{\text{h}} U' \times_{g,U,t} R \xrightarrow{\text{pr}_1} R \xrightarrow{s} U$$

is surjective. Since $h$ is flat and locally of finite presentation, we see that $\{h\}$ is a fpqc covering. Hence by Groupoids, Lemma 18.6 we see that $U'/R' \to U/R$ is an isomorphism. By the construction of $U'$ we see that $s', t'$ are Cohen-Macaulay and locally of finite presentation.

Step 2. Assume $s, t$ are Cohen-Macaulay and locally of finite presentation. Let $u \in U$ be a point of finite type. By More on Groupoids, Lemma 11.4 there exists an affine scheme $U'$ and a morphism $g : U' \to U$ such that

1. $g$ is an immersion,
2. $u \in U'$,
3. $g$ is locally of finite presentation,
4. $h$ is flat, locally of finite presentation and locally quasi-finite, and
5. the morphisms $s', t' : R' \to U'$ are flat, locally of finite presentation and locally quasi-finite.
Here we have used the notation introduced in More on Groupoids, Situation 11.1.

Step 3. For each point \( u \in U \) which is of finite type choose a \( g_u : U'_u \to U \) as in Step 2 and denote \( R'_u \) the restriction of \( R \) to \( U'_u \). Denote \( h_u = s \circ p_1 : U'_u \times_{g_u, U, t} R \to U \). Set \( U' = \coprod_{u \in U} U'_u \), and \( g = \coprod g_u \). Let \( R' \) be the restriction of \( R \) to \( U' \) as above.

We claim that the pair \((U', g)\) works\(^2\) Note that

\[
R' = \coprod_{u_1, u_2 \in U} (U'_{u_1} \times_{g_{u_1}, U, t} R) \times_R (R \times_{s, U, g_{u_2}} U'_{u_2})
\]

Hence the projection \( s' : R' \to U' = \coprod U'_{u_2} \) is flat, locally of finite presentation and locally quasi-finite as a base change of \( \coprod h_u \). Since each \( h_u \) is flat and locally of finite presentation we conclude that \( h \) is flat and locally of finite presentation. In particular, the image of \( h \) is open (see Morphisms, Lemma 26.9) and since the set of points of finite type is dense (see Morphisms, Lemma 17.7) we conclude that the image of \( h \) is \( U \). This implies that \( \{ h \} \) is an fpqc covering. By Groupoids, Lemma 18.6 this means that \( U'/R' \to U/R \) is an isomorphism. This finishes the proof of the lemma. \( \square \)

9. Quotient by a subgroupoid

We need one more lemma before we can do our final bootstrap. Let us discuss what is going on in terms of “plain” groupoids before embarking on the scheme theoretic version.

Let \( C \) be a groupoid, see Categories, Definition 2.5. As discussed in Groupoids, Section 11 this corresponds to a quintuple \((\text{Ob}, \text{Arrows}, s, t, c)\). Suppose we are given a subset \( P \subseteq \text{Arrows} \) such that \((\text{Ob}, P, s|_P, t|_P, c|_P)\) is also a groupoid and such that there are no nontrivial automorphisms in \( P \). Then we can construct the quotient groupoid \((\text{Ob}, \text{Arrows}, \pi, \ell, \tau)\) as follows:

1. \( \text{Ob} = \text{Ob}/P \) is the set of \( P \)-isomorphism classes,

2. \( \text{Arrows} = P \setminus \text{Arrows}/P \) is the set of arrows in \( C \) up to pre-composing and post-composing by arrows of \( P \),

3. the source and target maps \( \pi, \ell : P \setminus \text{Arrows}/P \to \text{Ob}/P \) are induced by \( s, t \),

4. composition is defined by the rule \( \tau(\pi, \ell) = c(a, b) \) which is well defined.

In fact, it turns out that the original groupoid \((\text{Ob}, \text{Arrows}, s, t, c)\) is canonically isomorphic to the restriction (see discussion in Groupoids, Section 10) of the groupoid \((\text{Ob}, \text{Arrows}, \pi, \ell, \tau)\) via the quotient map \( g : \text{Ob} \to \text{Ob} \). Recall that this means that

\[
\text{Arrows} = \text{Ob} \times_{g, \text{Ob}, \ell} \text{Arrows} \times_{\pi, \text{Ob}, g} \text{Ob}
\]

which holds as \( P \) has no nontrivial automorphisms. We omit the details.

The following lemma holds in much greater generality, but this is the version we use in the proof of the final bootstrap (after which we can more easily prove the more general versions of this lemma).

\(^2\)Here we should check that \( U' \) is not too large, i.e., that it is isomorphic to an object of the category \( \text{Sch}_{fppf} \), see Section 3. This is a purely set theoretical matter; let us use the notion of size of a scheme introduced in Sets, Section 4. Note that each \( U'_u \) has size at most the size of \( U \) and that the cardinality of the index set is at most the cardinality of \(|U|\) which is bounded by the size of \( U \). Hence \( U' \) is isomorphic to an object of \( \text{Sch}_{fppf} \) by Sets, Lemma 9.9 part (6).
Lemma 9.1. Let $S$ be a scheme. Let $(U, R, s, t, c)$ be a groupoid scheme over $S$. Let $P \to R$ be monomorphism of schemes. Assume that

1. $(U, P, s|_P, t|_P, c|_{P \times_U t|_P})$ is a groupoid scheme,
2. $s|_P, t|_P : P \to U$ are finite locally free,
3. $f|_P : P \to U \times_S U$ is a monomorphism.
4. $U$ is affine, and
5. $j : R \to U \times_S U$ is separated and locally quasi-finite,

Then $U/P$ is representable by an affine scheme $\overline{U}$, the quotient morphism $U \to \overline{U}$ is finite locally free, and $P = U \times_{\overline{U}} U$. Moreover, $R$ is the restriction of a groupoid scheme $(\overline{U}, \overline{R}, \overline{s}, \overline{t}, \overline{c})$ on $\overline{U}$ via the quotient morphism $U \to \overline{U}$.

Proof. Conditions (1), (2), (3), and (4) and Groupoids, Proposition 21.8 imply the affine scheme $\overline{U}$ representing $U/P$ exists, the morphism $U \to \overline{U}$ is finite locally free, and $P = U \times_{\overline{U}} U$. The identification $P = U \times_{\overline{U}} U$ is such that $t|_P = pr_0$ and $s|_P = pr_1$, and such that composition is equal to $pr_{02} : U \times_{\overline{U}} U \times U \to U \times_{\overline{U}} U$. A product of finite locally free morphisms is finite locally free (see Spaces, Lemma 5.7 and Morphisms, Lemmas 46.4 and 46.3). To get $\overline{R}$ we are going to descend the scheme $R$ via the finite locally free morphism $U \times S U \to \overline{U} \times S \overline{U}$. Namely, note that

$$(U \times S U) \times (\overline{U} \times S \overline{U}) (U \times S U) = P \times S P$$

by the above. Thus giving a descent datum (see Descent, Definition 30.1) for $R/U \times S U/\overline{U} \times S \overline{U}$ consists of an isomorphism

$$\varphi : R \times (U \times S U) \times (\overline{U} \times S \overline{U}) (P \times S P) \to (P \times S P) \times s \times (U \times S U) R$$

over $P \times S P$ satisfying a cocycle condition. We define $\varphi$ on $T$-valued points by the rule

$$\varphi : (r, (p, p')) \mapsto ((p, p'), p^{-1} \circ r \circ p')$$

where the composition is taken in the groupoid category $(U(T), R(T), s, t, c)$. This makes sense because for $(r, (p, p'))$ to be a $T$-valued point of the source of $\varphi$ it needs to be the case that $t(r) = t(p)$ and $s(r) = t(p')$. Note that this map is an isomorphism with inverse given by $((p, p'), (r', p')) \mapsto (p \circ r' \circ (p')^{-1}, (p, p'))$. To check the cocycle condition we have to verify that $\varphi_{02} = \varphi_{12} \circ \varphi_{01}$ as maps over

$$(U \times S U) \times (\overline{U} \times S \overline{U}) (U \times S U) \times (\overline{U} \times S \overline{U}) (U \times S U) = (P \times S P) \times s \times (U \times S U) \times (\overline{U} \times S \overline{U}) (P \times S P)$$

By explicit calculation we see that

$$\varphi_{02} (r, (p_1, p'_1), (p_2, p'_2)) \mapsto ((p_1, p'_1), (p_2, p'_2), (p_1 \circ p_2)^{-1} \circ r \circ (p'_1 \circ p'_2))$$
$$\varphi_{01} (r, (p_1, p'_1), (p_2, p'_2)) \mapsto ((p_1, p'_1), (p_2, p'_2), (p_1 \circ p_2, p'_1 \circ p'_2))$$
$$\varphi_{12} (p_1, r, (p_2, p'_2)) \mapsto ((p_1, p'_1), (p_2, p'_2), (p_1 \circ p_2, p'_1 \circ p'_2))$$

(with obvious notation) which implies what we want. As $j$ is separated and locally quasi-finite by (5) we may apply More on Morphisms, Lemma 39.1 to get a scheme $\overline{R} \to \overline{U} \times S \overline{U}$ and an isomorphism

$$R \to \overline{R} \times (\overline{U} \times S \overline{U}) (U \times S U)$$

which identifies the descent datum $\varphi$ with the canonical descent datum on $\overline{R} \times (\overline{U} \times S \overline{U}) (U \times S U)$, see Descent, Definition 30.10.

Since $U \times S U \to \overline{U} \times S \overline{U}$ is finite locally free we conclude that $R \to \overline{R}$ is finite locally free as a base change. Hence $R \to \overline{R}$ is surjective as a map of sheaves on
Our choice of $\varphi$ implies that given $T$-valued points $r, r' \in R(T)$ these have the same image in $\overline{R}$ if and only if $p^{-1} \circ r \circ p'$ for some $p, p' \in P(T)$. Thus $\overline{R}$ represents the sheaf

$$T \mapsto \overline{R(T)} = P(T) \setminus R(T)/P(T)$$

with notation as in the discussion preceding the lemma. Hence we can define the groupoid structure on $(\overline{U} = U/P, \overline{R} = P \setminus R/P)$ exactly as in the discussion of the “plain” groupoid case. It follows from this that $(U, R, s, t, c)$ is the pullback of this groupoid structure via the morphism $U \to \overline{U}$. This concludes the proof. □

10. Final bootstrap

The following result goes quite a bit beyond the earlier results.

**Theorem 10.1.** Let $S$ be a scheme. Let $F : (\text{Sch}/S)^{\text{op}}_{\text{fppf}} \to \text{Sets}$ be a functor. Any one of the following conditions implies that $F$ is an algebraic space:

1. $F = U/R$ where $(U, R, s, t, c)$ is a groupoid in algebraic spaces over $S$ such that $s, t$ are flat and locally of finite presentation, and $j = (t, s) : R \to U \times_S U$ is an equivalence relation,
2. $F = U/R$ where $(U, R, s, t, c)$ is a groupoid scheme over $S$ such that $s, t$ are flat and locally of finite presentation, and $j = (t, s) : R \to U \times_S U$ is an equivalence relation,
3. $F$ is a sheaf and there exists an algebraic space $U$ and a morphism $U \to F$ which is representable by algebraic spaces, surjective, flat and locally of finite presentation,
4. $F$ is a sheaf and there exists a scheme $U$ and a morphism $U \to F$ which is representable by algebraic spaces or schemes, surjective, flat and locally of finite presentation,
5. $F$ is a sheaf, $\Delta_F$ is representable by algebraic spaces, and there exists an algebraic space $U$ and a morphism $U \to F$ which is surjective, flat, and locally of finite presentation, or
6. $F$ is a sheaf, $\Delta_F$ is representable, and there exists a scheme $U$ and a morphism $U \to F$ which is surjective, flat, and locally of finite presentation.

**Proof.** Trivial observations: (6) is a special case of (5) and (4) is a special case of (3). We first prove that cases (5) and (3) reduce to case (1). Namely, by bootstrapping the diagonal Lemma 5.3 we see that (3) implies (5). In case (5) we set $R = U \times_F U$ which is an algebraic space by assumption. Moreover, by assumption both projections $s, t : R \to U$ are surjective, flat and locally of finite presentation. The map $j : R \to U \times_S U$ is clearly an equivalence relation. By Lemma 4.6 the map $U \to F$ is a surjection of sheaves. Thus $F = U/R$ which reduces us to case (1).

Next, we show that (1) reduces to (2). Namely, let $(U, R, s, t, c)$ be a groupoid in algebraic spaces over $S$ such that $s, t$ are flat and locally of finite presentation, and $j = (t, s) : R \to U \times_S U$ is an equivalence relation. Choose a scheme $U'$ and a surjective étale morphism $U' \to U$. Let $R' = R|_{U'}$ be the restriction of $R$ to $U'$. By Groupoids in Spaces, Lemma 18.6 we see that $U/R = U'/R'$. Since $s', t' : R' \to U'$ are also flat and locally of finite presentation (see More on Groupoids in Spaces, Lemma 6.1) this reduces us to the case where $U$ is a scheme. As $j$ is an equivalence relation we see that $j$ is a monomorphism. As $s : R \to U$ is locally of finite
presentation we see that \( j : R \to U \times_S U \) is locally of finite type, see Morphisms of Spaces, Lemma \[23.6\] By Morphisms of Spaces, Lemma \[26.10\] we see that \( j \) is locally quasi-finite and separated. Hence if \( U \) is a scheme, then \( R \) is a scheme by Morphisms of Spaces, Proposition \[44.2\] Thus we reduce to proving the theorem in case (2).

Assume \( F = U/R \) where \((U, R, s, t, c)\) is a groupoid scheme over \( S \) such that \( s, t \) are flat and locally of finite presentation, and \( j = (t, s) : R \to U \times_S U \) is an equivalence relation. By Lemma \[8.1\] we reduce to that case where \( s, t \) are flat, locally of finite presentation, and locally quasi-finite. Let \( U = \bigcup_{i \in I} U_i \) be an affine open covering (with index set \( I \) of cardinality \( \leq \) than the size of \( U \) to avoid set theoretic problems later – most readers can safely ignore this remark). Let \( (U_i, R_i, s_i, t_i, c_i) \) be the restriction of \( R \) to \( U_i \). It is clear that \( s_i, t_i \) are still flat, locally of finite presentation, and locally quasi-finite as \( R_i \) is the open subscheme \( s^{-1}(U_i) \cap t^{-1}(U_i) \) of \( R \) and \( s_i, t_i \) are the restrictions of \( s, t \) to this open. By Lemma \[7.1\] (or the simpler Spaces, Lemma \[10.2\]) the map \( U_i/R_i \to U/R \) is representable by open immersions. Hence if we can show that \( F_i = U_i/R_i \) is an algebraic space, then \( \coprod_{i \in I} F_i \) is an algebraic space by Spaces, Lemma \[8.3\]. As \( U = \bigcup U_i \) is an open covering it is clear that \( \coprod F_i \to F \) is surjective. Thus it follows that \( U/R \) is an algebraic space, by Spaces, Lemma \[8.4\] In this way we reduce to the case where \( U \) is affine and \( s, t \) are flat, locally of finite presentation, and locally quasi-finite and \( j \) is an equivalence.

Assume \((U, R, s, t, c)\) is a groupoid scheme over \( S \), with \( U \) affine, such that \( s, t \) are flat, locally of finite presentation, and locally quasi-finite, and \( j \) is an equivalence relation. Choose \( u \in U \). We apply More on Groupoids in Spaces, Lemma \[12.9\] to \( u \in U, R, s, t, c \). We obtain an affine scheme \( U' \), an étale morphism \( g : U' \to U \), a point \( u' \in U' \) with \( \kappa(u) = \kappa(u') \) such that the restriction \( R' = R|_{U'} \) is quasi-split over \( u' \). Note that the image \( g(U') \) is open as \( g \) is étale and contains \( u' \). Hence, repeatedly applying the lemma, we can find finitely many points \( u_i \in U, i = 1, \ldots, n \) affine schemes \( U'_i \), étale morphisms \( g_i : U'_i \to U \), points \( u'_i \in U'_i \) with \( g(u'_i) = u_i \) such that (a) each restriction \( R'_i \) is quasi-split over some point in \( U'_i \) and (b) \( U = \bigcup_{i=1, \ldots, n} g_i(U'_i) \). Now we rerun the last part of the argument in the preceding paragraph: Using Lemma \[7.1\] (or the simpler Spaces, Lemma \[10.2\]) the map \( U'_i/R'_i \to U/R \) is representable by open immersions. If we can show that \( F_i = U'_i/R'_i \) is an algebraic space, then \( \coprod_{i \in I} F_i \) is an algebraic space by Spaces, Lemma \[8.3\] As \( \{g_i : U'_i \to U\} \) is an étale covering it is clear that \( \coprod F_i \to F \) is surjective. Thus it follows that \( U/R \) is an algebraic space, by Spaces, Lemma \[8.4\] In this way we reduce to the case where \( U \) is affine and \( s, t \) are flat, locally of finite presentation, and locally quasi-finite, \( j \) is an equivalence, and \( R \) is quasi-split over \( u \) for some \( u \in U \).

Assume \((U, R, s, t, c)\) is a groupoid scheme over \( S \), with \( U \) affine, \( u \in U \) such that \( s, t \) are flat, locally of finite presentation, and locally quasi-finite and \( j = (t, s) : R \to U \times_S U \) is an equivalence relation and \( R \) is quasi-split over \( u \). Let \( P \subset R \) be a quasi-splitting of \( R \) over \( u \). By Lemma \[9.1\] we see that \((U, R, s, t, c)\) is the restriction of a groupoid \((U, R, \pi, \tau, \sigma)\) by a surjective finite locally free morphism \( U \to \mathcal{U} \) such that \( P = U \times_{\mathcal{U}} \mathcal{U} \). Note that \( s, t \) are the base changes of the morphisms \( \pi, \tau \) by \( U \to \mathcal{U} \). As \( \{U \to \mathcal{U}\} \) is an fppf covering we conclude \( \pi, \tau \) are flat, locally of finite presentation, and locally quasi-finite, see Descent, Lemmas \[19.13\], \[19.9\] and \[19.22\]
Consider the commutative diagram
\[
\begin{array}{ccc}
U \times_U U & \rightarrow & P \\
\downarrow & & \downarrow \\
\Upsilon & \rightarrow & R
\end{array}
\]

It is a general fact about restrictions that the outer four corners form a cartesian diagram. By the equality we see the inner square is cartesian. Since \( P \) is open in \( R \) (by definition of a quasi-splitting) we conclude that \( \Upsilon \) is an open immersion by Descent, Lemma \([19.14]\). An application of Groupoids, Lemma \([18.5]\) shows that \( U/R = U/R \). Hence we have reduced to the case where \((U, R, s, t, c)\) is a groupoid scheme over \( S \), with \( U \) affine, \( u \in U \) such that \( s, t \) are flat, locally of finite presentation, and locally quasi-finite and \( j = (t, s) : R \rightarrow U \times S U \) is an equivalence relation and \( e : U \rightarrow R \) is an open immersion!

But of course, if \( e \) is an open immersion and \( s, t \) are flat and locally of finite presentation then the morphisms \( t, s \) are étale. For example you can see this by applying More on Groupoids, Lemma \([4.1]\) which shows that \( \Omega_{R/U} = 0 \) which in turn implies that \( s, t : R \rightarrow U \) is G-unramified (see Morphisms, Lemma \([36.2]\)), which in turn implies that \( s, t \) are étale (see Morphisms, Lemma \([37.16]\)). And if \( s, t \) are étale then finally \( U/R \) is an algebraic space by Spaces, Theorem \([10.5]\). □

11. Applications

As a first application we obtain the following fundamental fact:

A sheaf which is fppf locally an algebraic space is an algebraic space.

This is the content of the following lemma. Note that assumption (2) is equivalent to the condition that \( F|_{(Sch/S)_\text{fppf}} \) is an algebraic space, see Spaces, Lemma \([16.4]\)

Assumption (3) is a set theoretic condition which may be ignored by those not worried about set theoretic questions.

**Lemma 11.1.** Let \( S \) be a scheme. Let \( F : (Sch/S)^\text{opp}_{\text{fppf}} \rightarrow \text{Sets} \) be a functor. Let \( \{S_i \rightarrow S\}_{i \in I} \) be a covering of \( (Sch/S)^\text{fppf} \). Assume that

1. \( F \) is a sheaf,
2. each \( F_i = h_{S_i} \times F \) is an algebraic space, and
3. \( \coprod_{i \in I} F_i \) is an algebraic space (see Spaces, Lemma \([8.3]\)).

Then \( F \) is an algebraic space.

**Proof.** Consider the morphism \( \prod F_i \rightarrow F \). This is the base change of \( \prod S_i \rightarrow S \) via \( F \rightarrow S \). Hence it is representable, locally of finite presentation, flat and surjective by our definition of an fppf covering and Lemma \([4.2]\). Thus Theorem \([10.1]\) applies to show that \( F \) is an algebraic space. □

As a second application we obtain

Any fppf descent datum for algebraic spaces is effective.

This is the content of the following lemma.
Lemma 11.2. Let $S$ be a scheme. Let $\{X_i \to X\}_{i \in I}$ be an fppf covering of algebraic spaces over $S$. Assume $I$ is countable\footnote{We can allow larger index sets here if we can bound the size of the algebraic spaces which we are descending. If we ever need this we will add a more precise statement here.} Then any descent datum for algebraic spaces relative to $\{X_i \to X\}$ is effective.

Proof. By Descent on Spaces, Lemma\textsuperscript{20.1} this translates into the statement that an fppf sheaf $F$ endowed with a map $F \to X$ is an algebraic space provided that each $F \times_X X_i$ is an algebraic space. The restriction on the cardinality of $I$ implies that coproducts of algebraic spaces indexed by $I$ are algebraic spaces, see Spaces, Lemma\textsuperscript{8.3} and Sets, Lemma\textsuperscript{9.9}. The morphism $\coprod_i F \times_X X_i \to F$ is representable by algebraic spaces (as the base change of $\coprod_i X_i \to X$, see Lemma\textsuperscript{3.3}), and surjective, flat, and locally of finite presentation (as the base change of $\coprod_i X_i \to X$, see Lemma\textsuperscript{4.2}). Hence the lemma follows from Theorem\textsuperscript{10.1}. $\square$

Here is a different type of application.

Lemma 11.3. Let $S$ be a scheme. Let $a : F \to G$ and $b : G \to H$ be transformations of functors $(\text{Sch}/S)^{\text{opp}}_{\text{fppf}} \to \text{Sets}$. Assume
1. $F, G, H$ are sheaves,
2. $a : F \to G$ is representable by algebraic spaces, flat, locally of finite presentation, and surjective, and
3. $b \circ a : F \to H$ is representable by algebraic spaces.

Then $b$ is representable by algebraic spaces.

Proof. Let $U$ be a scheme over $S$ and let $\xi \in H(U)$. We have to show that $U \times_{\xi, H} G$ is an algebraic space. On the other hand, we know that $U \times_{\xi, H} F$ is an algebraic space and that $U \times_{\xi, H} F \to U \times_{\xi, H} G$ is representable by algebraic spaces, flat, locally of finite presentation, and surjective as a base change of the morphism $a$ (see Lemma\textsuperscript{4.2}). Thus the result follows from Theorem\textsuperscript{10.1}. $\square$

Here is a special case of Lemma\textsuperscript{11.1} where we do not need to worry about set theoretical issues.

Lemma 11.4. Let $S$ be a scheme. Let $F : (\text{Sch}/S)^{\text{opp}}_{\text{fppf}} \to \text{Sets}$ be a functor. Let $\{S_i \to S\}_{i \in I}$ be a covering of $(\text{Sch}/S)_{\text{fppf}}$. Assume that
1. $F$ is a sheaf,
2. each $F_i = h_{S_i} \times F$ is an algebraic space, and
3. the morphisms $F_i \to S_i$ are of finite type.

Then $F$ is an algebraic space.

Proof. We will use Lemma\textsuperscript{11.1} above. To do this we will show that the assumption that $F_i$ is of finite type over $S_i$ to prove that the set theoretic condition in the lemma is satisfied (after perhaps refining the given covering of $S$ a bit). We suggest the reader skip the rest of the proof.

If $S'_i \to S_i$ is a morphism of schemes then
\[ h_{S'_i} \times F = h_{S'_i} \times h_{S_i} \times F = h_{S'_i} \times h_{S_i} F_i \]
is an algebraic space of finite type over $S_i$, see Spaces, Lemma [23.3] and Morphisms of Spaces, Lemma [23.3]. Thus we may assume: (a) each $S_i$ is affine, and (b) the cardinality of $I$ is at most the cardinality of the set of points of $S$. (Since to cover all of $S$ it is enough that each point is in the image of $S_i \to S$ for some $i$.)

Since each $S_i$ is affine and each $F_i$ of finite type over $S_i$ we conclude that $F_i$ is quasi-compact. Hence by Properties of Spaces, Lemma [6.3] we can find an affine $U_i \in \text{Ob}((\text{Sch}/S)_{\text{fppf}})$ and a surjective étale morphism $U_i \to F_i$. The fact that $F_i \to S_i$ is locally of finite type then implies that $U_i \to S_i$ is locally of finite type, and in particular $U_i \to S$ is locally of finite type. By Sets, Lemma [9.7] we conclude that $\text{size}(U_i) \leq \text{size}(S)$. Since also $|I| \leq \text{size}(S)$ we conclude that $\prod_{i \in I} U_i$ is isomorphic to an object of $(\text{Sch}/S)_{\text{fppf}}$ by Sets, Lemma [9.5] and the construction of $\text{Sch}$. This implies that $\prod F_i$ is an algebraic space by Spaces, Lemma [8.3] and we win.

Lemma 11.5. Assume $B \to S$ and $(U,R,s,t,c)$ are as in Groupoids in Spaces, Definition [19.1] (1). For any scheme $T$ over $S$ and objects $x,y$ of $[U/R]$ over $T$ the sheaf $\text{Isom}(x,y)$ on $(\text{Sch}/T)_{\text{fppf}}$ is an algebraic space.

Proof. By Groupoids in Spaces, Lemma [21.3] there exists an fppf covering $\{T_i \to T\}_{i \in I}$ such that $\text{Isom}(x,y)|_{(\text{Sch}/T_i)_{\text{fppf}}}$ is an algebraic space for each $i$. By Spaces, Lemma [16.4] this means that each $F_i = h_{S_i} \times \text{Isom}(x,y)$ is an algebraic space. Thus to prove the lemma we only have to verify the set theoretic condition that $\prod F_i$ is an algebraic space of Lemma [11.1] above to conclude. To do this we use Spaces, Lemma [8.3] which requires showing that $I$ and the $F_i$ are not “too large”. We suggest the reader skip the rest of the proof.

Choose $U' \in \text{Ob}(\text{Sch}/S)_{\text{fppf}}$ and a surjective étale morphism $U' \to U$. Let $R'$ be the restriction of $R$ to $U'$. Since $[U/R] = [U'/R']$ we may, after replacing $U$ by $U'$, assume that $U$ is a scheme. (This step is here so that the fibre products below are over a scheme.)

Note that if we refine the covering $\{T_i \to T\}$ then it remains true that each $F_i$ is an algebraic space. Hence we may assume that each $T_i$ is affine. Since $T_i \to T$ is locally of finite presentation, this then implies that $\text{size}(T_i) \leq \text{size}(T)$, see Sets, Lemma [9.7]. We may also assume that the cardinality of the index set $I$ is at most the cardinality of the set of points of $T$ since to get a covering it suffices to check that each point of $T$ is in the image. Hence $|I| \leq \text{size}(T)$. Choose $W \in \text{Ob}((\text{Sch}/S)_{\text{fppf}})$ and a surjective étale morphism $W \to R$. Note that in the proof of Groupoids in Spaces, Lemma [21.3] we showed that $F_i$ is representable by $T_i \times_{(y_i,x_i),U \times_R U} R$ for some $x_i,y_i : T_i \to U$. Hence now we see that $V_i = T_i \times_{(y_i,x_i),U \times_R U} W$ is a scheme which comes with an étale surjection $V_i \to F_i$. By Sets, Lemma [9.6] we see that

$$\text{size}(V_i) \leq \max\{\text{size}(T_i), \text{size}(W)\} \leq \max\{\text{size}(T), \text{size}(W)\}$$

Hence, by Sets, Lemma [9.5] we conclude that

$$\text{size}(\prod_{i \in I} V_i) \leq \max\{|I|, \text{size}(T), \text{size}(W)\}.$$ 

Hence we conclude by our construction of $\text{Sch}$ that $\prod_{i \in I} V_i$ is isomorphic to an object $V$ of $(\text{Sch}/S)_{\text{fppf}}$. This verifies the hypothesis of Spaces, Lemma [8.3] and we win. □
Lemma 11.6. Let $S$ be a scheme. Consider an algebraic space $F$ of the form $F = U/R$ where $(U,R,s,t,c)$ is a groupoid in algebraic spaces over $S$ such that $s,t$ are flat and locally of finite presentation, and $j = (t,s) : R \to U \times_S U$ is an equivalence relation. Then $U \to F$ is surjective, flat, and locally of finite presentation.

Proof. This is almost but not quite a triviality. Namely, by Groupoids in Spaces, Lemma 18.5 and the fact that $U/R$ is a monomorphism we see that $R = U \times_F U$. Choose a scheme $W$ and a surjective étale morphism $W \to F$. As $U \to F$ is a surjection of sheaves we can find an fppf covering $\{W_i \to W\}$ and maps $W_i \to U$ lifting the morphisms $W_i \to F$. Then we see that

$$W_i \times_F U = W_i \times_U U \times_F U = W_i \times_U t R$$

and the projection $W_i \times_F U \to W_i$ is the base change of $t : R \to U$ hence flat and locally of finite presentation, see Morphisms of Spaces, Lemmas 28.4 and 27.3. Hence by Descent on Spaces, Lemmas 10.11 and 10.8 we see that $U \to F$ is flat and locally of finite presentation. It is surjective by Spaces, Remark 5.2. $\square$

Lemma 11.7. Let $S$ be a scheme. Let $X \to B$ be a morphism of algebraic spaces over $S$. Let $G$ be a group algebraic space over $B$ and let $a : G \times_B X \to X$ be an action of $G$ on $X$ over $B$. If

1. $a$ is a free action, and
2. $G \to B$ is flat and locally of finite presentation,

then $X/G$ (see Groupoids in Spaces, Definition 18.1) is an algebraic space and $X \to X/G$ is surjective, flat, and locally of finite presentation.

Proof. The fact that $X/G$ is an algebraic space is immediate from Theorem 10.1 and the definitions. Namely, $X/G = X/R$ where $R = G \times_B X$. The morphisms $s,t : G \times_B X \to X$ are flat and locally of finite presentation (clear for $s$ as a base change of $G \to B$ and by symmetry using the inverse it follows for $t$) and the morphism $j : G \times_B X \to X \times_B X$ is a monomorphism by Groupoids in Spaces, Lemma 8.3 as the action is free. The assertions about the morphism $X \to X/G$ follow from Lemma 11.6. $\square$

Lemma 11.8. Let $\{S_i \to S\}_{i \in I}$ be a covering of $(\text{Sch}/S)_{\text{fppf}}$. Let $G$ be a group algebraic space over $S$, and denote $G_i = G_{S_i}$ the base changes. Suppose given

1. for each $i \in I$ an fppf $G_i$-torsor $X_i$ over $S_i$,
2. for each $i,j \in I$ a $G_{S_i \times_S S_j}$-equivariant isomorphism $\phi_{ij} : X_i \times_{S_i} S_j \to S_i \times_{S} X_j$ satisfying the cocycle condition over every $S_i \times_{S} S_j \times_{S} S_j$.

Then there exists an fppf $G$-torsor $X$ over $S$ whose base change to $S_i$ is isomorphic to $X_i$ such that we recover the descent datum $\phi_{ij}$.

Proof. We may think of $X_i$ as a sheaf on $(\text{Sch}/S)_{\text{fppf}}$, see Spaces, Section 16. By Sites, Section 25 the descent datum $(X_i, \phi_{ij})$ is effective in the sense that there exists a unique sheaf $X$ on $(\text{Sch}/S)_{\text{fppf}}$ which recovers the algebraic spaces $X_i$ after restricting back to $(\text{Sch}/S_i)_{\text{fppf}}$. Hence we see that $X_i = h_{S_i} \times X$. By Lemma 11.1 we see that $X$ is an algebraic space, modulo verifying that $\coprod X_i$ is an algebraic space which we do at the end of the proof. By the equivalence of categories in Sites, Lemma 25.3 the action maps $G_i \times_{S_i} X_i \to X_i$ glue to give a map $a : G \times_S X \to X$. Now we have to show that $a$ is an action and that $X$ is a pseudo-torsor, and fppf locally trivial (see Groupoids in Spaces, Definition 9.3). These may be checked
etale covering, hence we suggest the reader skip the rest of the proof, which is purely set theoretical. Pick $G$ fppf locally, and hence follow from the corresponding properties of the actions $G_i \times_S X_i \to X_i$. Hence the lemma is true.

We have the same image in the quotient of $U$ for all $i$, then $(U_i) : \bigcup_{i} U_i \to F$ is an algebraic space in the sense of Algebraic Spaces, Definition 6.1. Thus we get a second notion of algebraic spaces by working in the etale topology.

This notion is (a priori) weaker than the notion introduced in Algebraic Spaces, Definition 6.1 since a sheaf in the fppf topology is certainly a sheaf in the etale topology. However, the notions are equivalent as is shown by the following lemma.

**Lemma 12.1.** Denote the common underlying category of $\text{Sch}_{fppf}$ and $\text{Sch}_{\text{etale}}$ by $\text{Sch}_\alpha$ (see Topologies, Remark 9.4). Let $S$ be an object of $\text{Sch}_\alpha$.

$$F : (\text{Sch}_\alpha \times S)^{opp} \to \text{Sets}$$

be a presheaf with the following properties:

(1) $F$ is a sheaf for the etale topology,

(2) the diagonal $\Delta : F \to F \times F$ is representable, and

(3) there exists $U \in \text{Ob}(\text{Sch}_\alpha \times S)$ and $U \to F$ which is surjective and etale.

Then $F$ is an algebraic space in the sense of Algebraic Spaces, Definition 6.1.

**Proof.** Note that properties (2) and (3) of the lemma and the corresponding properties (2) and (3) of Algebraic Spaces, Definition 6.1, are independent of the topology. This is true because these properties involve only the notion of a fibre product of presheaves, maps of presheaves, the notion of a representable transformation of functors, and what it means for such a transformation to be surjective and etale. Thus all we have to prove is that an etale sheaf $F$ with properties (2) and (3) is also an fppf sheaf.

To do this, let $R = U \times_{F} U$. By (2) the presheaf $R$ is representable by a scheme and by (3) the projections $R \to U$ are etale. Thus $j : R \to U \times_S U$ is an etale equivalence relation. Moreover $U \to F$ identifies $F$ as the quotient of $U$ by $R$ for the etale topology: (a) if $T \to F$ is a morphism, then $\{T \times_{U} R \to T\}$ is an etale covering, hence $U \to F$ is a surjection of sheaves for the etale topology, (b) if $a,b : T \to U$ map to the same section of $F$, then $(a,b) : T \to R$ hence $a$ and $b$ have the same image in the quotient of $U$ by $R$ for the etale topology. Next, let...
U/R denote the quotient sheaf in the fppf topology which is an algebraic space by Spaces, Theorem 10.5. Thus we have morphisms (transformations of functors)

\[ U \to F \to U/R. \]

By the aforementioned Spaces, Theorem 10.5, the composition is representable, surjective, and étale. Hence for any scheme T and morphism \( T \to U/R \) the fibre product \( V = T \times_{U/R} U \) is a scheme surjective and étale over \( T \). In other words, \( \{ V \to U \} \) is an étale covering. This proves that \( U \to U/R \) is surjective as a map of sheaves in the étale topology. It follows that \( F \to U/R \) is surjective as a map of sheaves in the étale topology. On the other hand, the map \( F \to U/R \) is injective (as a map of presheaves) since \( R = U \times_{U/R} U \) again by Spaces, Theorem 10.5. It follows that \( F \to U/R \) is an isomorphism of étale sheaves, see Sites, Lemma 12.2 which concludes the proof.

\[ \square \]

In fact, it suffices to have a smooth cover by a scheme and it suffices to assume the diagonal is representable by algebraic spaces.

**Lemma 12.2.** Denote the common underlying category of \( \text{Sch}_{\text{fppf}} \) and \( \text{Sch}_{\text{étale}} \) by \( \text{Sch}_\alpha \) (see Topologies, Remark 9.1). Let \( S \) be an object of \( \text{Sch}_\alpha \).

\[ F : (\text{Sch}_\alpha/S)_{\text{op}} \to \text{Sets} \]

be a presheaf with the following properties:

1. \( F \) is a sheaf for the étale topology,
2. the diagonal \( \Delta : F \to F \times F \) is representable by algebraic spaces, and
3. there exists \( U \in \text{Ob}(\text{Sch}_\alpha/S) \) and \( U \to F \) which is surjective and smooth.

Then \( F \) is an algebraic space in the sense of Algebraic Spaces, Definition 6.1.

**Proof.** The proof mirrors the proof of Lemma 12.1. Let \( R = U \times_F U \). By (2) the presheaf \( R \) is an algebraic space and by (3) the projections \( R \to U \) are smooth and surjective. Denote \( (U, R, s, t, c) \) the groupoid associated to the equivalence relation \( j : R \to U \times_S U \) (see Groupoids in Spaces, Lemma 11.3). By Theorem 10.1 we see that \( X = U/R \) (quotient in the fppf-topology) is an algebraic space. Using that the smooth topology and the étale topology have the same sheaves (by More on Morphisms, Lemma 28.7) we see the map \( U \to F \) identifies \( F \) as the quotient of \( U \) by \( R \) for the smooth topology (details omitted). Thus we have morphisms (transformations of functors)

\[ U \to F \to X. \]

By Lemma 11.6, we see that \( U \to X \) is surjective, flat and locally of finite presentation. By Groupoids in Spaces, Lemma 18.5 (and the fact that \( j \) is a monomorphism) we have \( R = U \times_X U \). By Descent on Spaces, Lemma 10.24, we conclude that \( U \to X \) is smooth and surjective (as the projections \( R \to U \) are smooth and surjective and \( \{ U \to X \} \) is an fppf covering). Hence for any scheme \( T \) and morphism \( T \to X \) the fibre product \( T \times_X U \) is an algebraic space surjective and smooth over \( T \). Choose a scheme \( V \) and a surjective étale morphism \( V \to T \times_X U \). Then \( \{ V \to T \} \) is a smooth covering such that \( V \to T \to X \) lifts to a morphism \( V \to U \). This proves that \( U \to X \) is surjective as a map of sheaves in the smooth topology. It follows that \( F \to X \) is surjective as a map of sheaves in the smooth topology. On the other hand, the map \( F \to X \) is injective (as a map of presheaves) since \( R = U \times_X U \). It follows that \( F \to X \) is an isomorphism of smooth (= étale) sheaves, see Sites, Lemma 12.2 which concludes the proof.

\[ \square \]
# 13. Other chapters

<table>
<thead>
<tr>
<th>Preliminaries</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>(1) Introduction</td>
<td>(43) Adequate Modules</td>
</tr>
<tr>
<td>(2) Conventions</td>
<td>(44) Dualizing Complexes</td>
</tr>
<tr>
<td>(3) Set Theory</td>
<td>(45) Étale Cohomology</td>
</tr>
<tr>
<td>(4) Categories</td>
<td>(46) Crystalline Cohomology</td>
</tr>
<tr>
<td>(5) Topology</td>
<td>(47) Pro-étale Cohomology</td>
</tr>
<tr>
<td>(6) Sheaves on Spaces</td>
<td>(48) Algebraic Spaces</td>
</tr>
<tr>
<td>(7) Sites and Sheaves</td>
<td>(49) Properties of Algebraic Spaces</td>
</tr>
<tr>
<td>(8) Stacks</td>
<td>(50) Morphisms of Algebraic Spaces</td>
</tr>
<tr>
<td>(9) Fields</td>
<td>(51) Decent Algebraic Spaces</td>
</tr>
<tr>
<td>(10) Commutative Algebra</td>
<td>(52) Cohomology of Algebraic Spaces</td>
</tr>
<tr>
<td>(11) Brauer Groups</td>
<td>(53) Limits of Algebraic Spaces</td>
</tr>
<tr>
<td>(12) Homological Algebra</td>
<td>(54) Divisors on Algebraic Spaces</td>
</tr>
<tr>
<td>(13) Derived Categories</td>
<td>(55) Algebraic Spaces over Fields</td>
</tr>
<tr>
<td>(14) Simplicial Methods</td>
<td>(56) Topologies on Algebraic Spaces</td>
</tr>
<tr>
<td>(15) More on Algebra</td>
<td>(57) Descent and Algebraic Spaces</td>
</tr>
<tr>
<td>(16) Smoothing Ring Maps</td>
<td>(58) Derived Categories of Spaces</td>
</tr>
<tr>
<td>(17) Sheaves of Modules</td>
<td>(59) More on Morphisms of Spaces</td>
</tr>
<tr>
<td>(18) Modules on Sites</td>
<td>(60) Pushouts of Algebraic Spaces</td>
</tr>
<tr>
<td>(19) Injectives</td>
<td>(61) Groupoids in Algebraic Spaces</td>
</tr>
<tr>
<td>(20) Cohomology of Sheaves</td>
<td>(62) More on Groupoids in Spaces</td>
</tr>
<tr>
<td>(21) Cohomology on Sites</td>
<td>(63) Bootstrap</td>
</tr>
<tr>
<td>(22) Differential Graded Algebra</td>
<td></td>
</tr>
<tr>
<td>(23) Divided Power Algebra</td>
<td></td>
</tr>
<tr>
<td>(24) Hypercoverings</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Topics in Geometry</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>(64) Quotients of Groupoids</td>
<td></td>
</tr>
<tr>
<td>(65) Simplicial Spaces</td>
<td></td>
</tr>
<tr>
<td>(66) Formal Algebraic Spaces</td>
<td></td>
</tr>
<tr>
<td>(67) Restricted Power Series</td>
<td></td>
</tr>
<tr>
<td>(68) Resolution of Surfaces</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Deformation Theory</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>(69) Formal Deformation Theory</td>
<td></td>
</tr>
<tr>
<td>(70) Deformation Theory</td>
<td></td>
</tr>
<tr>
<td>(71) The Cotangent Complex</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Algebraic Stacks</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>(72) Algebraic Stacks</td>
<td></td>
</tr>
<tr>
<td>(73) Examples of Stacks</td>
<td></td>
</tr>
<tr>
<td>(74) Sheaves on Algebraic Stacks</td>
<td></td>
</tr>
<tr>
<td>(75) Criteria for Representability</td>
<td></td>
</tr>
<tr>
<td>(76) Artin's Axioms</td>
<td></td>
</tr>
<tr>
<td>(77) Quot and Hilbert Spaces</td>
<td></td>
</tr>
<tr>
<td>(78) Properties of Algebraic Stacks</td>
<td></td>
</tr>
<tr>
<td>(79) Morphisms of Algebraic Stacks</td>
<td></td>
</tr>
<tr>
<td>(80) Cohomology of Algebraic Stacks</td>
<td></td>
</tr>
<tr>
<td>(81) Derived Categories of Stacks</td>
<td></td>
</tr>
<tr>
<td>(82) Introducing Algebraic Stacks</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Miscellany</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>(11) Chow Homology</td>
<td></td>
</tr>
<tr>
<td>(12) Intersection Theory</td>
<td></td>
</tr>
</tbody>
</table>
References


