# ALGEBRAIC CURVES

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## Contents



## 1. Introduction

0BRW In this chapter we develop some of the theory of algebraic curves. A reference covering algebraic curves over the complex numbers is the book [ACGH85].

What we already know. Besides general algebraic geometry, we have already proved some specific results on algebraic curves. Here is a list.

(1) We have discussed affine opens of and ample invertible sheaves on 1 dimensional Noetherian schemes in Varieties, Section 36.
(2) We have seen a curve is either affine or projective in Varieties, Section 41.
(3) We have discussed degrees of locally free modules on proper curves in Varieties, Section 42.
(4) We have discussed the Picard scheme of a nonsingular projective curve over an algebraically closed field in Picard Schemes of Curves, Section 1.

This is a chapter of the Stacks Project, version 40a39686, compiled on May 27, 2017.

## 2. Curves and function fields

0BXX In this section we elaborate on the results of Varieties, Section 4 in the case of curves.

0BXY **Lemma 2.1.** *Let $k$ be a field. Let $X$ be a curve and $Y$ a proper variety. Let $U \subset X$ be a nonempty open and let $f : U \to Y$ be a morphism. If $x \in X$ is a closed point such that $\mathcal{O}_{X,x}$ is a discrete valuation ring, then there exists an open $U \subset U' \subset X$ containing $x$ and a morphism of varieties $f' : U' \to Y$ extending $f$.*

**Proof.** This is a special case of Morphisms, Lemma 40.5. $\square$

0BXZ **Lemma 2.2.** *Let $k$ be a field. Let $X$ be a normal curve and $Y$ a proper variety. The set of rational maps from $X$ to $Y$ is the same as the set of morphisms $X \to Y$.*

**Proof.** This is clear from Lemma 2.1 as every local ring is a discrete valuation ring (for example by Varieties, Lemma 42.15). $\square$

0CCK **Lemma 2.3.** *Let $k$ be a field. Let $f : X \to Y$ be a nonconstant morphism of curves over $k$. If $Y$ is normal, then $f$ is flat.*

**Proof.** Pick $x \in X$ mapping to $y \in Y$. Then $\mathcal{O}_{Y,y}$ is either a field or a discrete valuation ring (Varieties, Lemma 42.15). Since $f$ is nonconstant it is dominant (as it must map the generic point of $X$ to the generic point of $Y$). This implies that $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ is injective (Morphisms, Lemma 8.6). Hence $\mathcal{O}_{X,x}$ is torsion free as a $\mathcal{O}_{Y,y}$-module and therefore $\mathcal{O}_{X,x}$ is flat as a $\mathcal{O}_{Y,y}$-module by More on Algebra, Lemma 20.10. $\square$

0CCL **Lemma 2.4.** *Let $k$ be a field. Let $f : X \to Y$ be a morphism of schemes over $k$. Assume*

1. *$Y$ is separated over $k$,*
2. *$X$ is proper of dimension $\leq 1$ over $k$,*
3. *$f(Z)$ has at least two points for every irreducible component $Z \subset X$ of dimension 1.*

*Then $f$ is finite.*

**Proof.** The morphism $f$ is proper by Morphisms, Lemma 39.7. Thus $f(X)$ is closed and images of closed points are closed. Let $y \in Y$ be the image of a closed point in $X$. Then $f^{-1}(\{y\})$ is a closed subset of $X$ not containing any of the generic points of irreducible components of dimension 1 by condition (3). It follows that $f^{-1}(\{y\})$ is finite. Hence $f$ is finite over an open neighbourhood of $y$ by More on Morphisms, Lemma 38.5 (if $Y$ is Noetherian, then you can use the easier Cohomology of Schemes, Lemma 21.2). Since we've seen above that there are enough of these points $y$, the proof is complete. $\square$

0BY0 **Lemma 2.5.** *Let $k$ be a field. Let $X \to Y$ be a morphism of varieties with $Y$ proper and $X$ a curve. There exists a factorization $X \to \overline{X} \to Y$ where $X \to \overline{X}$ is an open immersion and $\overline{X}$ is a projective curve.*

**Proof.** This is clear from Lemma 2.1 and Varieties, Lemma 41.6. $\square$

Here is the main theorem of this section. We will say a morphism $f : X \to Y$ of varieties is *constant* if the image $f(X)$ consists of a single point $y$ of $Y$. If this happens then $y$ is a closed point of $Y$ (since the image of a closed point of $X$ will be a closed point of $Y$).

0BY1 **Theorem 2.6.** *Let $k$ be a field. The following categories are canonically equivalent*

(1) *The category of finitely generated field extensions $K/k$ of transcendence degree* 1.
(2) *The category of curves and dominant rational maps.*
(3) *The category of normal projective curves and nonconstant morphisms.*
(4) *The category of nonsingular projective curves and nonconstant morphisms.*
(5) *The category of regular projective curves and nonconstant morphisms.*
(6) *The category of normal proper curves and nonconstant morphisms.*

**Proof.** The equivalence between categories (1) and (2) is the restriction of the equivalence of Varieties, Theorem 4.1. Namely, a variety is a curve if and only if its function field has transcendence degree 1, see for example Varieties, Lemma 20.3.

The categories in (3), (4), (5), and (6) are the same. First of all, the terms "regular" and "nonsingular" are synonyms, see Properties, Definition 9.1. Being normal and regular are the same thing for Noetherian 1-dimensional schemes (Properties, Lemmas 9.4 and 12.6). See Varieties, Lemma 42.15 for the case of curves. Thus (3) is the same as (5). Finally, (6) is the same as (3) by Varieties, Lemma 41.4.

If $f : X \to Y$ is a nonconstant morphism of nonsingular projective curves, then $f$ sends the generic point $\eta$ of $X$ to the generic point $\xi$ of $Y$. Hence we obtain a morphism $k(Y) = \mathcal{O}_{Y,\xi} \to \mathcal{O}_{X,\eta} = k(X)$ in the category (1). Conversely, suppose that we have a map $k(Y) \to k(X)$ in the category (1). Then we obtain a morphism $U \to Y$ for some nonempty open $U \subset X$. By Lemma 2.1 this extends to all of $X$ and we obtain a morphism in the category (5). Thus we see that there is a fully faithful functor (5)$\to$(1).

To finish the proof we have to show that every $K/k$ in (1) is the function field of a normal projective curve. We already know that $K = k(X)$ for some curve $X$. After replacing $X$ by its normalization (which is a variety birational to $X$) we may assume $X$ is normal (Varieties, Lemma 26.1). Then we choose $X \to \overline{X}$ with $\overline{X} \setminus X = \{x_1, \ldots, x_n\}$ as in Varieties, Lemma 41.6. Since $X$ is normal and since each of the local rings $\mathcal{O}_{\overline{X},x_i}$ is normal we conclude that $\overline{X}$ is a normal projective curve as desired. (Remark: We can also first compactify using Varieties, Lemma 41.5 and then normalize using Varieties, Lemma 26.1. Doing it this way we avoid using the somewhat tricky Morphisms, Lemma 50.16.) $\square$

0BY2 **Definition 2.7.** Let $k$ be a field. Let $X$ be a curve. A *nonsingular projective model of* $X$ is a pair $(Y, \varphi)$ where $Y$ is a nonsingular projective curve and $\varphi : k(X) \to k(Y)$ is an isomorphism of function fields.

A nonsingular projective model is determined up to unique isomorphism by Theorem 2.6. Thus we often say "the nonsingular projective model". We usually drop $\varphi$ from the notation. Warning: it needn't be the case that $Y$ is smooth over $k$ but Lemma 2.8 shows this can only happen in positive characteristic.

0BY3 **Lemma 2.8.** *Let $k$ be a field. Let $X$ be a curve and let $Y$ be the nonsingular projective model of $X$. If $k$ is perfect, then $Y$ is a smooth projective curve.*

**Proof.** See Varieties, Lemma 42.15 for example. $\square$

0BY4 **Lemma 2.9.** *Let $k$ be a field. Let $X$ be a geometrically irreducible curve over $k$. For a field extension $K/k$ denote $Y_K$ a nonsingular projective model of $(X_K)_{red}$.*

(1) *If $X$ is proper, then $Y_K$ is the normalization of $X_K$.*
(2) *There exists $K/k$ finite purely inseparable such that $Y_K$ is smooth.*
(3) *Whenever $Y_K$ is smooth*[1] *we have $H^0(Y_K, \mathcal{O}_{Y_K}) = K$.*
(4) *Given a commutative diagram*

$$\begin{tikzcd} \Omega & K' \arrow[l] \\ K \arrow[u] & k \arrow[l] \arrow[u] \end{tikzcd}$$

*of fields such that $Y_K$ and $Y_{K'}$ are smooth, then $Y_\Omega = (Y_K)_\Omega = (Y_{K'})_\Omega$.*

**Proof.** Let $X'$ be a nonsingular projective model of $X$. Then $X'$ and $X$ have isomorphic nonempty open subschemes. In particular $X'$ is geometrically irreducible as $X$ is (some details omitted). Thus we may assume that $X$ is projective.

Assume $X$ is proper. Then $X_K$ is proper and hence the normalization $(X_K)^\nu$ is proper as a scheme finite over a proper scheme (Varieties, Lemma 26.1 and Morphisms, Lemmas 42.10 and 39.4). On the other hand, $X_K$ is irreducible as $X$ is geometrically irreducible. Hence $X_K^\nu$ is proper, normal, irreducible, and birational to $(X_K)_{red}$. This proves (1) because a proper curve is projective (Varieties, Lemma 41.4).

Proof of (2). As $X$ is proper and we have (1), we can apply Varieties, Lemma 26.3 to find $K/k$ finite purely inseparable such that $Y_K$ is geometrically normal. Then $Y_K$ is geometrically regular as normal and regular are the same for curves (Properties, Lemma 12.6). Then $Y$ is a smooth variety by Varieties, Lemma 12.6.

If $Y_K$ is geometrically reduced, then $Y_K$ is geometrically integral (Varieties, Lemma 9.2) and we see that $H^0(Y_K, \mathcal{O}_{Y_K}) = K$ by Varieties, Lemma 25.2. This proves (3) because a smooth variety is geometrically reduced (even geometrically regular, see Varieties, Lemma 12.6).

If $Y_K$ is smooth, then for every extension $\Omega/K$ the base change $(Y_K)_\Omega$ is smooth over $\Omega$ (Morphisms, Lemma 32.5). Hence it is clear that $Y_\Omega = (Y_K)_\Omega$. This proves (4). □

## 3. Linear series

0CCM We deviate from the classical story (see Remark 3.6) by defining linear series in the following manner.

0CCN **Definition 3.1.** Let $k$ be a field. Let $X$ be a proper scheme of dimension $\leq 1$ over $k$. Let $d \geq 0$ and $r \geq 0$. A *linear series of degree $d$ and dimension $r$* is a pair $(\mathcal{L}, V)$ where $\mathcal{L}$ in an invertible $\mathcal{O}_X$-module of degree $d$ (Varieties, Definition 42.1) and $V \subset H^0(X, \mathcal{L})$ is a $k$-subvector space of dimension $r+1$. We will abbreviate this by saying $(\mathcal{L}, V)$ is a $\mathfrak{g}^r_d$ on $X$.

We will mostly use this when $X$ is a nonsingular proper curve. In fact, the definition above is just one way to generalize the classical definition of a $\mathfrak{g}^r_d$. For example, if $X$ is a proper curve, then one can generalize linear series by allowing $\mathcal{L}$ to be a torsion free coherent $\mathcal{O}_X$-module of rank 1. On a nonsingular curve every torsion

[1]Or even geometrically reduced.

free coherent module is locally free, so this agrees with our notion for nonsingular proper curves.

The following lemma explains the geometric meaning of linear series for proper nonsingular curves.

0CCP **Lemma 3.2.** *Let $k$ be a field. Let $X$ be a nonsingular proper curve over $k$. Let $(\mathcal{L}, V)$ be a $\mathfrak{g}^r_d$ on $X$. Then there exists a morphism*

$$\varphi : X \longrightarrow \mathbf{P}^r_k = Proj(k[T_0, \ldots, T_r])$$

*of varieties over $k$ and a map $\alpha : \varphi^*\mathcal{O}_{\mathbf{P}^r_k}(1) \to \mathcal{L}$ such that $\varphi^*T_0, \ldots, \varphi^*T_r$ are sent to a basis of $V$ by $\alpha$.*

**Proof.** Let $s_0, \ldots, s_r \in V$ be a $k$-basis. Since $X$ is nonsingular the image $\mathcal{L}' \subset \mathcal{L}$ of the map $s_0, \ldots, s_r : \mathcal{O}_X^{\oplus r} \to \mathcal{L}$ is an invertible $\mathcal{O}_X$-module for example by Divisors, Lemma 11.11. Then we use Constructions, Lemma 13.1 to get a morphism

$$\varphi = \varphi_{(\mathcal{L}', (s_0, \ldots, s_r))} : X \longrightarrow \mathbf{P}^r_k$$

as in the statement of the lemma. □

0CCQ **Lemma 3.3.** *Let $k$ be a field. Let $X$ be a proper scheme of dimension $\leq 1$ over $k$. If $X$ has a $\mathfrak{g}^r_d$, then $X$ has a $\mathfrak{g}^s_d$ for all $0 \leq s \leq r$.*

**Proof.** This is true because a vector space $V$ of dimension $r + 1$ over $k$ has a linear subspace of dimension $s + 1$ for all $0 \leq s \leq r$. □

0CCR **Lemma 3.4.** *Let $k$ be a field. Let $X$ be a nonsingular proper curve over $k$. Let $(\mathcal{L}, V)$ be a $\mathfrak{g}^1_d$ on $X$. Then the morphism $\varphi : X \to \mathbf{P}^1_k$ of Lemma 3.2 has degree $\leq d$.*

**Proof.** By Lemma 3.2 we see that $\mathcal{L}' = \varphi^*\mathcal{O}_{\mathbf{P}^1_k}(1)$ has a nonzero map $\mathcal{L}' \to \mathcal{L}$. Hence by Varieties, Lemma 42.12 we see that $\deg(\mathcal{L}') \leq d$. On the other hand, we have

$$\deg(\mathcal{L}') = \deg(X/\mathbf{P}^1_k) \deg(\mathcal{O}_{\mathbf{P}^1_k}(1))$$

by Varieties, Lemma 42.11. This finishes the proof as the degree of $\mathcal{O}_{\mathbf{P}^1_k}(1)$ is 1. □

0CCS **Lemma 3.5.** *Let $k$ be a field. Let $X$ be a proper curve over $k$ with $H^0(X, \mathcal{O}_X) = k$. If $X$ has a $\mathfrak{g}^r_d$, then $r \leq d$. If equality holds, then $H^1(X, \mathcal{O}_X) = 0$, i.e., the genus of $X$ (Definition 6.1) is 0.*

**Proof.** Let $(\mathcal{L}, V)$ be a $\mathfrak{g}^r_d$. Since this will only increase $r$, we may assume $V = H^0(X, \mathcal{L})$. Choose a nonzero element $s \in V$. Then the zero scheme of $s$ is an effective Cartier divisor $D \subset X$, we have $\mathcal{L} = \mathcal{O}_X(D)$, and we have a short exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{L} \to \mathcal{L}|_D \to 0$$

see Divisors, Lemma 14.10 and Remark 14.11. By Varieties, Lemma 42.9 we have $\deg(D) = \deg(\mathcal{L}) = d$. Since $D$ is an Artinian scheme we have $\mathcal{L}|_D \cong \mathcal{O}_D$[2]. Thus

$$\dim_k H^0(D, \mathcal{L}|_D) = \dim_k H^0(D, \mathcal{O}_D) = \deg(D) = d$$

On the other hand, by assumption $\dim_k H^0(X, \mathcal{O}_X) = 1$ and $\dim H^0(X, \mathcal{L}) = r + 1$. We conclude that $r + 1 \leq 1 + d$, i.e., $r \leq d$ as in the lemma.

[2]In our case this follows from Divisors, Lemma 17.1 as $D \to \operatorname{Spec}(k)$ is finite.

Assume equality holds. Then $H^0(X, \mathcal{L}) \to H^0(X, \mathcal{L}|_D)$ is surjective. If we knew that $H^1(X, \mathcal{L})$ was zero, then we would conclude that $H^1(X, \mathcal{O}_X)$ is zero by the long exact cohomology sequence and the proof would be complete. Our strategy will be to replace $\mathcal{L}$ by a large power which has vanishing. As $\mathcal{L}|_D$ is the trivial invertible module (see above), we can find a section $t$ of $\mathcal{L}$ whose restriction of $D$ generates $\mathcal{L}|_D$. Consider the multiplication map

$$\mu : H^0(X, \mathcal{L}) \otimes_k H^0(X, \mathcal{L}) \longrightarrow H^0(X, \mathcal{L}^{\otimes 2})$$

and consider the short exact sequence

$$0 \to \mathcal{L} \xrightarrow{s} \mathcal{L}^{\otimes 2} \to \mathcal{L}^{\otimes 2}|_D \to 0$$

Since $H^0(\mathcal{L}) \to H^0(\mathcal{L}|_D)$ is surjective and since $t$ maps to a trivialization of $\mathcal{L}|_D$ we see that $\mu(H^0(X, \mathcal{L}) \otimes t)$ gives a subspace of $H^0(X, \mathcal{L}^{\otimes 2})$ surjecting onto the global sections of $\mathcal{L}^{\otimes 2}|_D$. Thus we see that

$$\dim H^0(X, \mathcal{L}^{\otimes 2}) = r + 1 + d = 2r + 1 = \deg(\mathcal{L}^{\otimes 2}) + 1$$

Ok, so $\mathcal{L}^{\otimes 2}$ has the same property as $\mathcal{L}$, i.e., that the dimension of the space of global sections is equal to the degree plus one. Since $\mathcal{L}$ is ample (Varieties, Lemma 42.13) there exists some $n_0$ such that $\mathcal{L}^{\otimes n}$ has vanishing $H^1$ for all $n \geq n_0$ (Cohomology of Schemes, Lemma 16.1). Thus applying the argument above to $\mathcal{L}^{\otimes n}$ with $n = 2^m$ for some sufficiently large $m$ we conclude the lemma is true. $\square$

0CCT **Remark 3.6** (Classical definition)**.** Let $X$ be a smooth projective curve over an algebraically closed field $k$. We say two effective Cartier divisors $D, D' \subset X$ are *linearly equivalent* if and only if $\mathcal{O}_X(D) \cong \mathcal{O}_X(D')$ as $\mathcal{O}_X$-modules. Since $\text{Pic}(X) = \text{Cl}(X)$ (Divisors, Lemma 25.7) we see that $D$ and $D'$ are linearly equivalent if and only if the Weil divisors associated to $D$ and $D'$ define the same element of $\text{Cl}(X)$. Given an effective Cartier divisor $D \subset X$ of degree $d$ the *complete linear system* or *complete linear series* $|D|$ of $D$ is the set of effective Cartier divisors $E \subset X$ which are linearly equivalent to $D$. Another way to say it is that $|D|$ is the set of closed points of the fibre of the morphism

$$\gamma_d : \underline{\text{Hilb}}^d_{X/k} \longrightarrow \underline{\text{Pic}}^d_{X/k}$$

(Picard Schemes of Curves, Lemma 6.7) over the closed point corresponding to $\mathcal{O}_X(D)$. This gives $|D|$ a natural scheme structure and it turns out that $|D| \cong \mathbf{P}^m_k$ with $m + 1 = h^0(\mathcal{O}_X(D))$. In fact, more canonically we have

$$|D| = \mathbf{P}(H^0(X, \mathcal{O}_X(D))^\vee)$$

where $(-)^\vee$ indicates $k$-linear dual and $\mathbf{P}$ is as in Intersection Theory, Section 23. In this language a *linear system* or a *linear series* on $X$ is a closed subvariety $L \subset |D|$ which can be cut out by linear equations. If $L$ has dimension $r$, then $L = \mathbf{P}(V^\vee)$ where $V \subset H^0(X, \mathcal{O}_X(D))$ is a linear subspace of dimension $r + 1$. Thus the classical linear series $L \subset |D|$ corresponds to the linear series $(\mathcal{O}_X(D), V)$ as defined above.

## 4. Riemann-Roch and duality

0B5B Let $k$ be a field. Let $X$ be a proper scheme of dimension $\leq 1$ over $k$. In Varieties, Section 42 we have defined the degree of a locally free $\mathcal{O}_X$-module $\mathcal{E}$ of constant rank by the formula

0BRX (4.0.1) $$\deg(\mathcal{E}) = \chi(X, \mathcal{E}) - \text{rank}(\mathcal{E})\chi(X, \mathcal{O}_X)$$

see Varieties, Definition 42.1. In the chapter on Chow Homology we defined the first chern class of $\mathcal{E}$ as an operation on cycles (Chow Homology, Section 35) and we proved that

0BRY (4.0.2)
$$\deg(\mathcal{E}) = \deg(c_1(\mathcal{E}) \cap [X]_1)$$

see Chow Homology, Lemma 41.3. Combining (4.0.1) and (4.0.2) we obtain our first version of the Riemann-Roch formula

0BRZ (4.0.3)
$$\chi(X, \mathcal{E}) = \deg(c_1(\mathcal{E}) \cap [X]_1) + \text{rank}(\mathcal{E})\chi(X, \mathcal{O}_X)$$

If $\mathcal{L}$ is an invertible $\mathcal{O}_X$-module, then we can also consider the numerical intersection $(\mathcal{L} \cdot X)$ as defined in Varieties, Definition 43.3. However, this does not give anything new as

0BS0 (4.0.4)
$$(\mathcal{L} \cdot X) = \deg(\mathcal{L})$$

by Varieties, Lemma 43.12. If $\mathcal{L}$ is ample, then this integer is positive and is called the degree

0BS1 (4.0.5)
$$\deg_{\mathcal{L}}(X) = (\mathcal{L} \cdot X) = \deg(\mathcal{L})$$

of $X$ with respect to $\mathcal{L}$, see Varieties, Definition 43.10.

To obtain a true Riemann-Roch theorem we would like to write $\chi(X, \mathcal{O}_X)$ as the degree of a canonical zero cycle on $X$. We refer to [Ful98] for a fully general version of this. We will use duality to get a formula in the case where $X$ is Gorenstein; however, in some sense this is a cheat (for example because this method cannot work in higher dimension).

0BS2 **Lemma 4.1.** *Let $X$ be a proper scheme of dimension $\leq 1$ over a field $k$. There exists a dualizing complex $\omega_X^\bullet$ with the following properties*

(1) $H^i(\omega_X^\bullet)$ *is nonzero only for* $i = -1, 0$,
(2) $\omega_X = H^{-1}(\omega_X^\bullet)$ *is a coherent Cohen-Macaulay module whose support is the irreducible components of dimension* 1,
(3) *for $x \in X$ closed, the module $H^0(\omega_{X,x}^\bullet)$ is nonzero if and only if either*
    (a) $\dim(\mathcal{O}_{X,x}) = 0$ *or*
    (b) $\dim(\mathcal{O}_{X,x}) = 1$ *and $\mathcal{O}_{X,x}$ is not Cohen-Macaulay,*
(4) *there are functorial isomorphisms* $\text{Ext}^i_X(K, \omega_X^\bullet) = \text{Hom}_k(H^{-i}(X, K), k)$ *compatible with shifts for* $K \in D_{QCoh}(X)$,
(5) *there are functorial isomorphisms* $\text{Hom}(\mathcal{F}, \omega_X) = \text{Hom}_k(H^1(X, \mathcal{F}), k)$ *for $\mathcal{F}$ quasi-coherent on $X$,*
(6) *if $X \to \text{Spec}(k)$ is smooth of relative dimension 1, then $\omega_X \cong \Omega_{X/k}$.*

**Proof.** Denote $f : X \to \text{Spec}(k)$ the structure morphism. We start with the relative dualizing complex

$$\omega_X^\bullet = \omega_{X/k}^\bullet = a(\mathcal{O}_{\text{Spec}(k)}) = f^!\mathcal{O}_{\text{Spec}(k)}$$

as described in Dualizing Complexes, Remark 28.6. Then property (4) holds by construction. Observe that $\omega_X^\bullet$ is also the dualizing complex normalized relative to $\omega_{\text{Spec}(k)}^\bullet = \mathcal{O}_{\text{Spec}(k)}$, i.e., it is the dualizing complex $\omega_X^\bullet$ as in Dualizing Complexes, Example 39.1 with $A = k$ and $\omega_A = k[0]$. Parts (1) and (2) follow from Dualizing Complexes, Lemma 39.4. For a closed point $x \in X$ we see that $\omega_{X,x}^\bullet$ is a normalized dualizing complex over $\mathcal{O}_{X,x}$, see Dualizing Complexes, Lemma 38.1. Assertion (3) then follows from Dualizing Complexes, Lemma 40.2. Assertion (5) follows from

Dualizing Complexes, Lemma 39.5 for coherent $\mathcal{F}$ and in general by unwinding (4) for $K = \mathcal{F}[0]$ and $i = -1$. Assertion (6) follows from Dualizing Complexes, Lemma 30.7. $\square$

0BS3 **Lemma 4.2.** *Let $X$ be a proper scheme over a field $k$ which is Cohen-Macaulay and equidimensional of dimension 1. There exists a dualizing module $\omega_X$ with the following properties*

1. *$\omega_X$ is a coherent Cohen-Macaulay module whose support is $X$,*
2. *there are functorial isomorphisms $\mathrm{Ext}^i_X(K, \omega_X[1]) = \mathrm{Hom}_k(H^{-i}(X, K), k)$ compatible with shifts for $K \in D_{QCoh}(X)$,*
3. *there are functorial isomorphisms $\mathrm{Ext}^{1+i}(\mathcal{F}, \omega_X) = \mathrm{Hom}_k(H^{-i}(X, \mathcal{F}), k)$ for $\mathcal{F}$ quasi-coherent on $X$.*

**Proof.** Let us take $\omega_X$ normalized as in as in Dualizing Complexes, Example 39.2. Then the statements follow from Lemma 4.1 and the fact that $\omega_X^\bullet = \omega_X[1]$ as $X$ is Cohen-Macualay (Dualizing Complexes, Lemma 40.3). $\square$

0BS4 **Remark 4.3.** Let $X$ be a proper scheme of dimension $\leq 1$ over a field $k$. Let $\omega_X^\bullet$ be as in Lemma 4.1. If $\mathcal{E}$ is a finite locally free $\mathcal{O}_X$-module with dual $\mathcal{E}^\vee$ then we have canonical isomorphisms

$$\mathrm{Hom}_k(H^{-i}(X, \mathcal{E}), k) = H^i(X, \mathcal{E}^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_X} \omega_X^\bullet)$$

This follows from the lemma and Cohomology, Lemma 43.11. If $X$ is Cohen-Macaulay and equidimensional of dimension 1, then we have canonical isomorphisms

$$\mathrm{Hom}_k(H^{-i}(X, \mathcal{E}), k) = H^{1-i}(X, \mathcal{E}^\vee \otimes_{\mathcal{O}_X} \omega_X)$$

where $\omega_X$ is as in Lemma 4.2. In particular if $\mathcal{L}$ is an invertible $\mathcal{O}_X$-module, then we have

$$\dim_k H^0(X, \mathcal{L}) = \dim_k H^1(X, \mathcal{L}^{\otimes -1} \otimes_{\mathcal{O}_X} \omega_X)$$

and

$$\dim_k H^1(X, \mathcal{L}) = \dim_k H^0(X, \mathcal{L}^{\otimes -1} \otimes_{\mathcal{O}_X} \omega_X)$$

We can use Lemmas 4.1 and 4.2 to get a relation between the euler characteristic of $\mathcal{O}_X$ and the euler characteristic of the dualizing complex or the dualizing module.

0BS5 **Lemma 4.4.** *Let $X$ be a proper scheme of dimension $\leq 1$ over a field $k$. With $\omega_X^\bullet$ as in Lemma 4.1 we have*

$$\chi(X, \mathcal{O}_X) = \chi(X, \omega_X^\bullet)$$

*If $X$ is Cohen-Macaulay and equidimensional of dimension 1, then*

$$\chi(X, \mathcal{O}_X) = -\chi(X, \omega_X)$$

*with $\omega_X$ as in Lemma 4.2.*

**Proof.** We define the right hand side of the first formula as follows:

$$\chi(X, \omega_X^\bullet) = \sum\nolimits_{i \in \mathbf{Z}} (-1)^i \dim_k H^i(X, \omega_X^\bullet)$$

This is well defined because $\omega_X^\bullet$ is in $D^b_{Coh}(\mathcal{O}_X)$, but also because

$$H^i(X, \omega_X^\bullet) = \mathrm{Ext}^i(\mathcal{O}_X, \omega_X^\bullet) = H^{-i}(X, \mathcal{O}_X)$$

which is always finite dimensional and nonzero only if $i = 0, -1$. This of course also proves the first formula. The second is a consequence of the first because $\omega_X^\bullet = \omega_X[1]$ in the CM case. □

We will use Lemma 4.4 to get the desired formula for $\chi(X, \mathcal{O}_X)$ in the case that $\omega_X$ is invertible, i.e., that $X$ is Gorenstein. The statement is that $-1/2$ of the first chern class of $\omega_X$ capped with the cycle $[X]_1$ associated to $X$ is a natural zero cycle on $X$ with half-integer coefficients whose degree is $\chi(X, \mathcal{O}_X)$. The occurence of fractions in the statement of Riemann-Roch cannot be avoided.

0BS6 **Lemma 4.5** (Riemann-Roch)**.** *Let $X$ be a proper scheme over a field $k$ which is Gorenstein and equidimensional of dimension 1. Let $\omega_X$ be as in Lemma 4.2. Then*

(1) *$\omega_X$ is an invertible $\mathcal{O}_X$-module,*
(2) $\deg(\omega_X) = -2\chi(X, \mathcal{O}_X)$,
(3) *for a locally free $\mathcal{O}_X$-module $\mathcal{E}$ of constant rank we have*

$$\chi(X, \mathcal{E}) = \deg(\mathcal{E}) - \tfrac{1}{2} rank(\mathcal{E}) \deg(\omega_X)$$

*and* $\dim_k(H^i(X, \mathcal{E})) = \dim_k(H^{1-i}(X, \mathcal{E}^\vee \otimes_{\mathcal{O}_X} \omega_X))$ *for all* $i \in \mathbf{Z}$.

Nonsingular (normal) curves are Gorenstein, see Dualizing Complexes, Lemma 41.4.

**Proof.** It follows more or less from the definition of the Gorenstein property that the dualizing sheaf is invertible, see Dualizing Complexes, Section 41. By (4.0.3) applied to $\omega_X$ we have

$$\chi(X, \omega_X) = \deg(c_1(\omega_X) \cap [X]_1) + \chi(X, \mathcal{O}_X)$$

Combined with Lemma 4.4 this gives

$$2\chi(X, \mathcal{O}_X) = -\deg(c_1(\omega_X) \cap [X]_1) = -\deg(\omega_X)$$

the second equality by (4.0.2). Putting this back into (4.0.3) for $\mathcal{E}$ gives the displayed formula of the lemma. The symmetry in dimensions is a consequence of duality for $X$, see Remark 4.3. □

## 5. Some vanishing results

0B5C

0BY5 **Lemma 5.1.** *Let $k$ be a field. Let $X$ be a proper scheme over $k$ having dimension 1 and $H^0(X, \mathcal{O}_X) = k$. Then $X$ is Cohen-Macaulay and equidimensional of dimension 1.*

**Proof.** Since $\Gamma(X, \mathcal{O}_X) = k$ has no nontrivial idempotents, we see that $X$ is connected. This already shows that $X$ is equidimensional of dimension 1 (any irreducible component of dimension 0 would be a connected component). Let $\mathcal{I} \subset \mathcal{O}_X$ be the maximal coherent submodule supported in closed points. Then $\mathcal{I}$ exists (Divisors, Lemma 4.6) and is globally generated (Varieties, Lemma 31.3). Since $1 \in \Gamma(X, \mathcal{O}_X)$ is not a section of $\mathcal{I}$ we conclude that $\mathcal{I} = 0$. Thus $X$ does not have embedded points (Divisors, Lemma 4.6). Thus $X$ has $(S_1)$ by Divisors, Lemma 4.3. Hence $X$ is Cohen-Macaulay. □

In this section we work in the following situation.

0B5D **Situation 5.2.** Here $k$ is a field, $X$ is a proper scheme over $k$ having dimension 1 and $H^0(X, \mathcal{O}_X) = k$.

By Lemma 5.1 the scheme $X$ is Cohen-Macaulay and equidimensional of dimension 1. We denote $\omega_X$ the dualizing module of $X$ as in Dualizing Complexes, Example 39.2. Then Lemmas 4.1 and 4.2 show that $\omega_X$ has nonvanishing $H^1$ and in fact $\dim_k H^1(X, \omega_X) = 1$. It turns out that anything slightly more "positive" than $\omega_X$ has vanishing $H^1$.

0B5E **Lemma 5.3.** *In Situation 5.2. Given an exact sequence*

$$\omega_X \to \mathcal{F} \to \mathcal{Q} \to 0$$

*of coherent $\mathcal{O}_X$-modules with $H^1(X, \mathcal{Q}) = 0$ (for example if $\dim(Supp(\mathcal{Q})) = 0$), then either $H^1(X, \mathcal{F}) = 0$ or $\mathcal{F} = \omega_X \oplus \mathcal{Q}$.*

**Proof.** (The parenthetical statement follows from Cohomology of Schemes, Lemma 9.10.) Since $H^0(X, \mathcal{O}_X) = k$ is dual to $H^1(X, \omega_X)$ (see Section 4) we see that $\dim H^1(X, \omega_X) = 1$. The sheaf $\omega_X$ represents the functor $\mathcal{F} \mapsto \mathrm{Hom}_k(H^1(X, \mathcal{F}), k)$ on the category of coherent $\mathcal{O}_X$-modules (Dualizing Complexes, Lemma 39.5). Consider an exact sequence as in the statement of the lemma and assume that $H^1(X, \mathcal{F}) \neq 0$. Since $H^1(X, \mathcal{Q}) = 0$ we see that $H^1(X, \omega_X) \to H^1(X, \mathcal{F})$ is an isomorphism. By the universal property of $\omega_X$ stated above, we conclude there is a map $\mathcal{F} \to \omega_X$ whose action on $H^1$ is the inverse of this isomorphism. The composition $\omega_X \to \mathcal{F} \to \omega_X$ is the identity (by the universal property) and the lemma is proved. $\square$

0B62 **Lemma 5.4.** *In Situation 5.2. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module which is globally generated and not isomorphic to $\mathcal{O}_X$. Then $H^1(X, \omega_X \otimes \mathcal{L}) = 0$.*

**Proof.** By duality as discussed in Section 4 we have to show that $H^0(X, \mathcal{L}^{\otimes -1}) = 0$. If not, then we can choose a global section $t$ of $\mathcal{L}^{\otimes -1}$ and a global section $s$ of $\mathcal{L}$ such that $st \neq 0$. However, then $st$ is a constant multiple of 1, by our assumption that $H^0(X, \mathcal{O}_X) = k$. It follows that $\mathcal{L} \cong \mathcal{O}_X$, which is a contradiction. $\square$

0B5F **Lemma 5.5.** *In Situation 5.2. Given an exact sequence*

$$\omega_X \to \mathcal{F} \to \mathcal{Q} \to 0$$

*of coherent $\mathcal{O}_X$-modules with $\dim(Supp(\mathcal{Q})) = 0$ and $\dim_k H^0(X, \mathcal{Q}) \geq 2$ and such that there is no nonzero submodule $\mathcal{Q}' \subset \mathcal{F}$ such that $\mathcal{Q}' \to \mathcal{Q}$ is injective. Then the submodule of $\mathcal{F}$ generated by global sections surjects onto $\mathcal{Q}$.*

**Proof.** Let $\mathcal{F}' \subset \mathcal{F}$ be the submodule generated by global sections and the image of $\omega_X \to \mathcal{F}$. Since $\dim_k H^0(X, \mathcal{Q}) \geq 2$ and $\dim_k H^1(X, \omega_X) = \dim_k H^0(X, \mathcal{O}_X) = 1$, we see that $\mathcal{F}' \to \mathcal{Q}$ is not zero and $\omega_X \to \mathcal{F}'$ is not an isomorphism. Hence $H^1(X, \mathcal{F}') = 0$ by Lemma 5.3 and our assumption on $\mathcal{F}$. Consider the short exact sequence

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{Q}/\mathrm{Im}(\mathcal{F}' \to \mathcal{Q}) \to 0$$

If the quotient on the right is nonzero, then we obtain a contradiction because then $H^0(X, \mathcal{F})$ is bigger than $H^0(X, \mathcal{F}')$. $\square$

Here is an example global generation statement.

0B5G **Lemma 5.6.** *In Situation 5.2 assume that $X$ is integral. Let $0 \to \omega_X \to \mathcal{F} \to \mathcal{Q} \to 0$ be a short exact sequence of coherent $\mathcal{O}_X$-modules with $\mathcal{F}$ torsion free, $\dim(Supp(\mathcal{Q})) = 0$, and $\dim_k H^0(X, \mathcal{Q}) \geq 2$. Then $\mathcal{F}$ is globally generated.*

**Proof.** Consider the submodule $\mathcal{F}'$ generated by the global sections. By Lemma 5.5 we see that $\mathcal{F}' \to \mathcal{Q}$ is surjective, in particular $\mathcal{F}' \neq 0$. Since $X$ is a curve, we see that $\mathcal{F}' \subset \mathcal{F}$ is an inclusion of rank 1 sheaves, hence $\mathcal{Q}' = \mathcal{F}/\mathcal{F}'$ is supported in finitely many points. To get a contradiction, assume that $\mathcal{Q}'$ is nonzero. Then we see that $H^1(X, \mathcal{F}') \neq 0$. Then we get a nonzero map $\mathcal{F}' \to \omega_X$ by the universal property (Dualizing Complexes, Lemma 39.5). The image of the composition $\mathcal{F}' \to \omega_X \to \mathcal{F}$ is generated by global sections, hence is inside of $\mathcal{F}'$. Thus we get a nonzero self map $\mathcal{F}' \to \mathcal{F}'$. Since $\mathcal{F}'$ is torsion free of rank 1 on a proper curve this has to be an automorphism (details omitted). But then this implies that $\mathcal{F}'$ is contained in $\omega_X \subset \mathcal{F}$ contradicting the surjectivity of $\mathcal{F}' \to \mathcal{Q}$. ☐

0B5H **Lemma 5.7.** *In Situation 5.2. Let $\mathcal{L}$ be a very ample invertible $\mathcal{O}_X$-module with $\deg(\mathcal{L}) \geq 2$. Then $\omega_X \otimes_{\mathcal{O}_X} \mathcal{L}$ is globally generated.*

**Proof.** Assume $k$ is algebraically closed. Let $x \in X$ be a closed point. Let $C_i \subset X$ be the irreducible components and for each $i$ let $x_i \in C_i$ be the generic point. By Varieties, Lemma 22.2 we can choose a section $s \in H^0(X, \mathcal{L})$ such that $s$ vanishes at $x$ but not at $x_i$ for all $i$. The corresponding module map $s : \mathcal{O}_X \to \mathcal{L}$ is injective with cokernel $\mathcal{Q}$ supported in finitely many points and with $H^0(X, \mathcal{Q}) \geq 2$. Consider the corresponding exact sequence

$$0 \to \omega_X \to \omega_X \otimes \mathcal{L} \to \omega_X \otimes \mathcal{Q} \to 0$$

By Lemma 5.5 we see that the module generated by global sections surjects onto $\omega_X \otimes \mathcal{Q}$. Since $x$ was arbitrary this proves the lemma. Some details omitted.

We will reduce the case where $k$ is not algebraically closed, to the algebraically closed field case. We suggest the reader skip the rest of the proof. Choose an algebraic closure $\overline{k}$ of $k$ and consider the base change $X_{\overline{k}}$. Let us check that $X_{\overline{k}} \to \mathrm{Spec}(\overline{k})$ is an example of Situation 5.2. By flat base change (Cohomology of Schemes, Lemma 5.2) we see that $H^0(X_{\overline{k}}, \mathcal{O}) = \overline{k}$. By Varieties, Lemma 13.1 we see that $X_{\overline{k}}$ is Cohen-Macaulay. The scheme $X_{\overline{k}}$ is proper over $\overline{k}$ (Morphisms, Lemma 39.5) and equidimensional of dimension 1 (Morphisms, Lemma 27.3). The pullback of $\omega_X$ to $X_{\overline{k}}$ is the dualizing module of $X_{\overline{k}}$ by Dualizing Complexes, Lemma 23.1. The pullback of $\mathcal{L}$ to $X_{\overline{k}}$ is very ample (Morphisms, Lemma 36.8). The degree of the pullback of $\mathcal{L}$ to $X_{\overline{k}}$ is equal to the degree of $\mathcal{L}$ on $X$ (Varieties, Lemma 42.2). Finally, we see that $\omega_X \otimes \mathcal{L}$ is globally generated if and only if its base change is so (Varieties, Lemma 22.1). In this way we see that the result follows from the result in the case of an algebraically closed ground field. ☐

## 6. The genus of a curve

0BY6 If $X$ is a smooth projective curve over an algebraically closed field, then we've previously defined the genus of $X$ as the dimension of $H^1(X, \mathcal{O}_X)$, see Picard Schemes of Curves, Definition 6.3. Let us generalize this as follows.

0BY7 **Definition 6.1.** Let $k$ be a field. Let $X$ be a proper scheme over $k$ having dimension 1 and $H^0(X, \mathcal{O}_X) = k$. Then the *genus* of $X$ is $g = \dim_k H^1(X, \mathcal{O}_X)$.

This is sometimes called the *arithmetic genus* of $X$. In the literature the arithmetic genus of a proper curve $X$ over $k$ is sometimes defined as

$$p_a(X) = 1 - \chi(X, \mathcal{O}_X) = 1 - \dim_k H^0(X, \mathcal{O}_X) + \dim_k H^1(X, \mathcal{O}_X)$$

This agrees with our definition when it applies because we assume $H^0(X, \mathcal{O}_X) = k$. But note that

(1) $p_a(X)$ can be negative, and
(2) $p_a(X)$ depends on the base field $k$ and should be written $p_a(X/k)$.

For example if $k = \mathbf{Q}$ and $X = \mathbf{P}^1_{\mathbf{Q}(i)}$ then $p_a(X/\mathbf{Q}) = -1$ and $p_a(X/\mathbf{Q}(i)) = 0$.

The assumption that $H^0(X, \mathcal{O}_X) = k$ in our definition has two consequences. On the one hand, it means there is no confusion about the base field. On the other hand, it implies the scheme $X$ is Cohen-Macaulay and equidimensional of dimension 1 (Lemma 5.1). Letting $\omega_X$ be the dualizing module as in Dualizing Complexes, Example 39.2 we see that

0BY8 (6.1.1) $$g = \dim_k H^1(X, \mathcal{O}_X) = \dim_k H^0(X, \omega_X)$$

by duality (see Remark 4.3).

If $X$ is proper over $k$ of dimension $\leq 1$ and $H^0(X, \mathcal{O}_X)$ is not equal to the ground field $k$, instead of using the arithmetic genus $p_a(X)$ given by the displayed formula above we shall use the invariant $\chi(X, \mathcal{O}_X)$. In fact, it is advocated in [Ser55, page 276] and [Hir95, Introduction] that we should call $\chi(X, \mathcal{O}_X)$ the arithmetic genus.

0BY9 **Lemma 6.2.** *Let $k'/k$ be a field extension. Let $X$ be a proper scheme over $k$ having dimension 1 and $H^0(X, \mathcal{O}_X) = k$. Then $X_{k'}$ is a proper scheme over $k'$ having dimension 1 and $H^0(X_{k'}, \mathcal{O}_{X_{k'}}) = k'$. Moreover the genus of $X_{k'}$ is equal to the genus of $X$.*

**Proof.** The dimension of $X_{k'}$ is 1 for example by Morphisms, Lemma 27.3. The morphism $X_{k'} \to \mathrm{Spec}(k')$ is proper by Morphisms, Lemma 39.5. The equality $H^0(X_{k'}, \mathcal{O}_{X_{k'}}) = k'$ follows from Cohomology of Schemes, Lemma 5.2. The equality of the genus follows from the same lemma. □

0C19 **Lemma 6.3.** *Let $k$ be a field. Let $X$ be a proper scheme over $k$ having dimension 1 and $H^0(X, \mathcal{O}_X) = k$. If $X$ is Gorenstein, then*

$$\deg(\omega_X) = 2g - 2$$

*where $g$ is the genus of $X$ and $\omega_X$ is as in Lemma 4.2.*

**Proof.** Immediate from Lemma 4.5. □

0C1A **Lemma 6.4.** *Let $X$ be a smooth proper curve over a field $k$ with $H^0(X, \mathcal{O}_X) = k$. Then*

$$\dim_k H^0(X, \Omega_{X/k}) = g \quad \text{and} \quad \deg(\Omega_{X/k}) = 2g - 2$$

*where $g$ is the genus of $X$.*

**Proof.** By Lemma 4.1 we have $\Omega_{X/k} = \omega_X$. Hence the formulas hold by (6.1.1) and Lemma 6.3. □

## 7. Plane curves

0BYA Let $k$ be a field. A *plane curve* will be a curve $X$ which is isomorphic to a closed subscheme of $\mathbf{P}^2_k$. Often the embedding $X \to \mathbf{P}^2_k$ will be considered given. By Divisors, Example 28.2 a curve is determined by the corresponding homogeneous ideal

$$I(X) = \mathrm{Ker}\left(k[T_0, T_2, T_2] \longrightarrow \bigoplus \Gamma(X, \mathcal{O}_X(n))\right)$$

Recall that in this situation we have

$$X = \mathrm{Proj}(k[T_0, T_2, T_2]/I)$$

as closed subschemes of $\mathbf{P}^2_k$. For more general information on these constructions we refer the reader to Divisors, Example 28.2 and the references therein. It turns out that $I(X) = (F)$ for some homogeneous polynomial $F \in k[T_0, T_1, T_2]$, see Lemma 7.1. Since $X$ is irreducible, it follows that $F$ is irreducible, see Lemma 7.2. Moreover, looking at the short exact sequence

$$0 \to \mathcal{O}_{\mathbf{P}^2_k}(-d) \xrightarrow{F} \mathcal{O}_{\mathbf{P}^2_k} \to \mathcal{O}_X \to 0$$

where $d = \deg(F)$ we find that $H^0(X, \mathcal{O}_X) = k$ and that $X$ has genus $(d-1)(d-2)/2$, see proof of Lemma 7.3.

To find smooth plane curves it is easiest to write explicit equations. Let $p$ denote the characteristic of $k$. If $p$ does not divide $d$, then we can take

$$F = T_0^d + T_1^d + T_2^d$$

The corresponding curve $X = V_+(F)$ is called the *Fermat curve* of degree $d$. It is smooth because on each standard affine piece $D_+(T_i)$ we obtain a curve isomorphic to the affine curve

$$\mathrm{Spec}(k[x, y]/(x^d + y^d + 1))$$

The ring map $k \to k[x, y]/(x^d + y^d + 1)$ is smooth by Algebra, Lemma 135.15 as $dx^{d-1}$ and $dy^{d-1}$ generate the unit ideal in $k[x, y]/(x^d + y^d + 1)$. If $p|d$ but $p \neq 3$ then you can use the equation

$$F = T_0^{d-1}T_1 + T_1^{d-1}T_2 + T_2^{d-1}T_0$$

Namely, on the affine pieces you get $x + x^{d-1}y + y^{d-1}$ with derivatives $1 - x^{d-2}y$ and $x^{d-1} - y^{d-2}$ whose common zero set (of all three) is empty[3]. We leave it to the reader to make examples in characteristic 3.

More generally for any field $k$ and any $n$ and $d$ there exists a smooth hypersurface of degree $d$ in $\mathbf{P}^n_k$, see for example [Poo05].

Of course, in this way we only find smooth curves whose genus is a triangular number. To get smooth curves of an arbitrary genus one can look for smooth curves lying on $\mathbf{P}^1 \times \mathbf{P}^1$ (insert future reference here).

0BYB **Lemma 7.1.** *Let $Z \subset \mathbf{P}^2_k$ be a closed subscheme which is equidimensional of dimension 1 and has no embedded points (equivalently $Z$ is Cohen-Macaulay). Then the ideal $I(Z) \subset k[T_0, T_1, T_2]$ corresponding to $Z$ is principal.*

**Proof.** This is a special case of Divisors, Lemma 28.3 (see also Varieties, Lemma 32.4). The parenthetical statement follows from the fact that a 1 dimensional Noetherian scheme is Cohen-Macaulay if and only if it has no embedded points, see Divisors, Lemma 4.4. □

0BYC **Lemma 7.2.** *Let $Z \subset \mathbf{P}^2_k$ be as in Lemma 7.1 and let $I(Z) = (F)$ for some $F \in k[T_0, T_1, T_2]$. Then $Z$ is a curve if and only if $F$ is irreducible.*

[3] Namely, as $x^{d-1} = y^{d-2}$, then $0 = x + x^{d-1}y + y^{d-1} = x + 2x^{d-1}y$. Since $x \neq 0$ because $1 = x^{d-2}y$ we get $0 = 1 + 2x^{d-2}y = 3$ which is absurd unless $3 = 0$.

**Proof.** If $F$ is reducible, say $F = F'F''$ then let $Z'$ be the closed subscheme of $\mathbf{P}^2_k$ defined by $F'$. It is clear that $Z' \subset Z$ and that $Z' \neq Z$. Since $Z'$ has dimension 1 as well, we conclude that either $Z$ is not reduced, or that $Z$ is not irreducible. Conversely, write $Z = \sum a_i D_i$ where $D_i$ are the irreducible components of $Z$, see Divisors, Lemmas 15.8 and 15.9. Let $F_i \in k[T_0, T_1, T_2]$ be the homogeneous polynomial generating the ideal of $D_i$. Then it is clear that $F$ and $\prod F_i^{a_i}$ cut out the same closed subscheme of $\mathbf{P}^2_k$. Hence $F = \lambda \prod F_i^{a_i}$ for some $\lambda \in k^*$ because both generate the ideal of $Z$. Thus we see that if $F$ is irreducible, then $Z$ is a prime divisor, i.e., a curve. $\square$

0BYD **Lemma 7.3.** *Let $Z \subset \mathbf{P}^2_k$ be as in Lemma 7.1 and let $I(Z) = (F)$ for some $F \in k[T_0, T_1, T_2]$. Then $H^0(Z, \mathcal{O}_Z) = k$ and the genus of $Z$ is $(d-1)(d-2)/2$ where $d = \deg(F)$.*

**Proof.** Let $S = k[T_0, T_1, T_2]$. There is an exact sequence of graded modules

$$0 \to S(-d) \xrightarrow{F} S \to S/(F) \to 0$$

Denote $i : Z \to \mathbf{P}^2_k$ the given closed immersion. Applying the exact functor $\widetilde{\ }$ (Constructions, Lemma 8.4) we obtain

$$0 \to \mathcal{O}_{\mathbf{P}^2_k}(-d) \to \mathcal{O}_{\mathbf{P}^2_k} \to i_*\mathcal{O}_Z \to 0$$

because $F$ generates the ideal of $Z$. Note that the cohomology groups of $\mathcal{O}_{\mathbf{P}^2_k}(-d)$ and $\mathcal{O}_{\mathbf{P}^2_k}$ are given in Cohomology of Schemes, Lemma 8.1. On the other hand, we have $H^q(Z, \mathcal{O}_Z) = H^q(\mathbf{P}^2_k, i_*\mathcal{O}_Z)$ by Cohomology of Schemes, Lemma 2.4. Applying the long exact cohomology sequence we first obtain that

$$k = H^0(\mathbf{P}^2_k, \mathcal{O}_{\mathbf{P}^2_k}) \longrightarrow H^0(Z, \mathcal{O}_Z)$$

is an isomorphism and next that the boundary map

$$H^1(Z, \mathcal{O}_Z) \longrightarrow H^2(\mathbf{P}^2_k, \mathcal{O}_{\mathbf{P}^2_k}(-d)) \cong k[T_0, T_1, T_2]_{d-3}$$

is an isomorphism. Since it is easy to see that the dimension of this is $(d-1)(d-2)/2$ the proof is finished. $\square$

0CCU **Lemma 7.4.** *Let $Z \subset \mathbf{P}^2_k$ be as in Lemma 7.1 and let $I(Z) = (F)$ for some $F \in k[T_0, T_1, T_2]$. If $Z \to \mathrm{Spec}(k)$ is smooth in at least one point and $k$ is infinite, then there exists a closed point $z \in Z$ contained in the smooth locus such that $\kappa(z)/k$ is finite separable of degree at most $d$.*

**Proof.** Suppose that $z' \in Z$ is a point where $Z \to \mathrm{Spec}(k)$ is smooth. After renumbering the coordinates if necessary we may assume $z'$ is contained in $D_+(T_0)$. Set $f = F(1, x, y) \in k[x, y]$. Then $Z \cap D_+(X_0)$ is isomorphic to the spectrum of $k[x, y]/(f)$. Let $f_x, f_y$ be the partial derivatives of $f$ with respect to $x, y$. Since $z'$ is a smooth point of $Z/k$ we see that either $f_x$ or $f_y$ is nonzero in $z'$ (see discussion in Algebra, Section 135). After renumbering the coordinates we may assume $f_y$ is not zero at $z'$. Hence there is a nonempty open subscheme $V \subset Z \cap D_+(X_0)$ such that the projection

$$p : V \longrightarrow \mathrm{Spec}(k[x])$$

is étale. Because the degree of $f$ as a polynomial in $y$ is at most $d$, we see that the degrees of the fibres of the projection $p$ are at most $d$ (see discussion in Morphisms, Section 53). Moreover, as $p$ is étale the image of $p$ is an open $U \subset \mathrm{Spec}(k[x])$. Finally, since $k$ is infinite, the set of $k$-rational points $U(k)$ of $U$ is infinite, in particular not empty. Pick any $t \in U(k)$ and let $z \in V$ be a point mapping to $t$. Then $z$ works. $\square$

## 8. Curves of genus zero

0C6L Later we will need to know what a proper genus zero curve looks like. It turns out that a Gorenstein proper genus zero curve is a plane curve of degree 2, i.e., a conic, see Lemma 8.3. A general proper genus zero curve is obtain from a nonsingular one (over a bigger field) by a pushout procedure, see Lemma 8.5. Since a nonsingular curve is Gorenstein, these two results cover all possible cases.

0C6M **Lemma 8.1.** *Let $X$ be a proper curve over a field $k$ with $H^0(X, \mathcal{O}_X) = k$. If $X$ has genus 0, then every invertible $\mathcal{O}_X$-module $\mathcal{L}$ of degree 0 is trivial.*

**Proof.** Namely, we have $\dim_k H^0(X, \mathcal{L}) \geq 0+1-0 = 1$ by Riemann-Roch (Lemma 4.5), hence $\mathcal{L}$ has a nonzero section, hence $\mathcal{L} \cong \mathcal{O}_X$ by Varieties, Lemma 42.12. $\square$

0C6T **Lemma 8.2.** *Let $X$ be a proper curve over a field $k$ with $H^0(X, \mathcal{O}_X) = k$. Assume $X$ has genus 0. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module of degree $d > 0$. Then we have*

(1) $\dim_k H^0(X, \mathcal{L}) = d + 1$ *and* $\dim_k H^1(X, \mathcal{L}) = 0$,
(2) *$\mathcal{L}$ is very ample and defines a closed immersion into $\mathbf{P}^d_k$.*

**Proof.** By definition of degree and genus we have

$$\dim_k H^0(X, \mathcal{L}) - \dim_k H^1(X, \mathcal{L}) = d + 1$$

Let $s$ be a nonzero section of $\mathcal{L}$. Then the zero scheme of $s$ is an effective Cartier divisor $D \subset X$, we have $\mathcal{L} = \mathcal{O}_X(D)$ and we have a short exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{L} \to \mathcal{L}|_D \to 0$$

see Divisors, Lemma 14.10 and Remark 14.11. Since $H^1(X, \mathcal{O}_X) = 0$ by assumption, we see that $H^0(X, \mathcal{L}) \to H^0(X, \mathcal{L}|_D)$ is surjective. As $\mathcal{L}|_D$ is generated by global sections (because $\dim(D) = 0$, see Varieties, Lemma 31.3) we conclude that the invertible module $\mathcal{L}$ is generated by global sections. In fact, since $D$ is an Artinian scheme we have $\mathcal{L}|_D \cong \mathcal{O}_D$[4] and hence we can find a section $t$ of $\mathcal{L}$ whose restriction of $D$ generates $\mathcal{L}|_D$. The short exact sequence also shows that $H^1(X, \mathcal{L}) = 0$.

For $n \geq 1$ consider the multiplication map

$$\mu_n : H^0(X, \mathcal{L}) \otimes_k H^0(X, \mathcal{L}^{\otimes n}) \longrightarrow H^0(X, \mathcal{L}^{\otimes n+1})$$

We claim this is surjective. To see this we consider the short exact sequence

$$0 \to \mathcal{L}^{\otimes n} \xrightarrow{s} \mathcal{L}^{\otimes n+1} \to \mathcal{L}^{\otimes n+1}|_D \to 0$$

The sections of $\mathcal{L}^{\otimes n+1}$ coming from the left in this sequence are in the image of $\mu_n$. On the other hand, since $H^0(\mathcal{L}) \to H^0(\mathcal{L}|_D)$ is surjective and since $t^n$ maps to a trivialization of $\mathcal{L}^{\otimes n}|_D$ we see that $\mu_n(H^0(X, \mathcal{L}) \otimes t^n)$ gives a subspace of $H^0(X, \mathcal{L}^{\otimes n+1})$ surjecting onto the global sections of $\mathcal{L}^{\otimes n+1}|_D$. This proves the claim.

[4]In our case this follows from Divisors, Lemma 17.1 as $D \to \mathrm{Spec}(k)$ is finite.

Observe that $\mathcal{L}$ is ample by Varieties, Lemma 42.13. Hence Morphisms, Lemma 41.17 gives an isomorphism

$$X \longrightarrow \text{Proj}\left(\bigoplus\nolimits_{n \geq 0} H^0(X, \mathcal{L}^{\otimes n})\right)$$

Since the maps $\mu_n$ are surjective for all $n \geq 1$ we see that the graded algebra on the right hand side is a quotient of the symmetric algebra on $H^0(X, \mathcal{L})$. Choosing a $k$-basis $s_0, \ldots, s_d$ of $H^0(X, \mathcal{L})$ we see that it is a quotient of a polynomial algebra in $d+1$ variables. Since quotients of graded rings correspond to closed immersions of Proj (Constructions, Lemma 11.5) we find a closed immersion $X \to \mathbf{P}^d_k$. We omit the verification that this morphism is the morphism of Constructions, Lemma 13.1 associated to the sections $s_0, \ldots, s_d$ of $\mathcal{L}$. □

0C6N **Lemma 8.3.** *Let $X$ be a proper curve over a field $k$ with $H^0(X, \mathcal{O}_X) = k$. If $X$ is Gorenstein and has genus $0$, then $X$ is isomorphic to a plane curve of degree $2$.*

**Proof.** Consider the invertible sheaf $\mathcal{L} = \omega_X^{\otimes -1}$ where $\omega_X$ is as in Lemma 4.2. Then $\deg(\omega_X) = -2$ by Lemma 6.3 and hence $\deg(\mathcal{L}) = 2$. By Lemma 8.2 we conclude that choosing a basis $s_0, s_1, s_2$ of the $k$-vector space of global sections of $\mathcal{L}$ we obtain a closed immersion

$$\varphi_{(\mathcal{L}, (s_0, s_1, s_2))} : X \longrightarrow \mathbf{P}^2_k$$

Thus $X$ is a plane curve of some degree $d$. Let $F \in k[T_0, T_1, T_2]_d$ be its equation (Lemma 7.1). Because the genus of $X$ is 0 we see that $d$ is 1 or 2 (Lemma 7.3). Observe that $F$ restricts to the zero section on $\varphi(X)$ and hence $F(s_0, s_1, s_2)$ is the zero section of $\mathcal{L}^{\otimes 2}$. Because $s_0, s_1, s_2$ are linearly independent we see that $F$ cannot be linear, i.e., $d = \deg(F) \geq 2$. Thus $d = 2$ and the proof is complete. □

0C6U **Proposition 8.4** (Characterization of the projective line)**.** *Let $k$ be a field. Let $X$ be a proper curve over $k$. The following are equivalent*

1. $X \cong \mathbf{P}^1_k$,
2. *$X$ is smooth and geometrically irreducible over $k$, $X$ has genus $0$, and $X$ has an invertible module of odd degree,*
3. *$X$ is geometrically integral over $k$, $X$ has genus $0$, $X$ is Gorenstein, and $X$ has an invertible sheaf of odd degree,*
4. *$H^0(X, \mathcal{O}_X) = k$, $X$ has genus $0$, $X$ is Gorenstein, and $X$ has an invertible sheaf of odd degree,*
5. *$X$ is geometrically integral over $k$, $X$ has genus $0$, and $X$ has an invertible $\mathcal{O}_X$-module of degree $1$,*
6. *$H^0(X, \mathcal{O}_X) = k$, $X$ has genus $0$, and $X$ has an invertible $\mathcal{O}_X$-module of degree $1$,*
7. *$H^1(X, \mathcal{O}_X) = 0$ and $X$ has an invertible $\mathcal{O}_X$-module of degree $1$,*
8. *$H^1(X, \mathcal{O}_X) = 0$ and $X$ has closed points $x_1, \ldots, x_n$ such that $\mathcal{O}_{X,x_i}$ is normal and $\gcd([\kappa(x_i) : k]) = 1$, and*
9. *add more here.*

**Proof.** We will prove that each condition (2) – (8) implies (1) and we omit the verification that (1) implies (2) – (8).

Assume (2). A smooth scheme over $k$ is geometrically reduced (Varieties, Lemma 24.4) and regular (Varieties, Lemma 24.3). Hence $X$ is Gorenstein (Dualizing Complexes, Lemma 41.4). Thus we reduce to (3).

Assume (3). Since $X$ is geometrically integral over $k$ we have $H^0(X, \mathcal{O}_X) = k$ by Varieties, Lemma 25.2. and we reduce to (4).

Assume (4). Since $X$ is Gorenstein the dualizing module $\omega_X$ as in Lemma 4.2 has degree $\deg(\omega_X) = -2$ by Lemma 6.3. Combined with the assumed existence of an odd degree invertible module, we conclude there exists an invertible module of degree 1. In this way we reduce to (6).

Assume (5). Since $X$ is geometrically integral over $k$ we have $H^0(X, \mathcal{O}_X) = k$ by Varieties, Lemma 25.2. and we reduce to (6).

Assume (6). Then $X \cong \mathbf{P}^1_k$ by Lemma 8.2.

Assume (7). Observe that $\kappa = H^0(X, \mathcal{O}_X)$ is a field finite over $k$ by Varieties, Lemma 25.2. If $d = [\kappa : k] > 1$, then every invertible sheaf has degree divisible by $d$ and there cannot be an invertible sheaf of degree 1. Hence $d = 1$ and we reduce to case (6).

Assume (8). Observe that $\kappa = H^0(X, \mathcal{O}_X)$ is a field finite over $k$ by Varieties, Lemma 25.2. Since $\kappa \subset \kappa(x_i)$ we see that $k = \kappa$ by the assumption on the gcd of the degrees. The same condition allows us to find integers $a_i$ such that $1 = \sum a_i[\kappa(x_i) : k]$. Because $x_i$ defines an effective Cartier divisor on $X$ by Varieties, Lemma 42.15 we can consider the invertible module $\mathcal{O}_X(\sum a_i x_i)$. By our choice of $a_i$ the degree of $\mathcal{L}$ is 1. Thus $X \cong \mathbf{P}^1_k$ by Lemma 8.2. □

0DJB **Lemma 8.5.** *Let $X$ be a proper curve over a field $k$ with $H^0(X, \mathcal{O}_X) = k$. Assume $X$ is singular and has genus 0. Then there exists a diagram*

$$\begin{tikzcd} x' \arrow[r] \arrow[d] & X' \arrow[r] \arrow[d, "\nu"] & \mathrm{Spec}(k') \arrow[d] \\ x \arrow[r] & X \arrow[r] & \mathrm{Spec}(k) \end{tikzcd}$$

*where*

1. *$k'/k$ is a nontrivial finite extension,*
2. *$X' \cong \mathbf{P}^1_{k'}$,*
3. *$x'$ is a $k'$-rational point of $X'$,*
4. *$x$ is a $k$-rational point of $X$,*
5. *$X' \setminus \{x'\} \to X \setminus \{x\}$ is an isomorphism,*
6. *$0 \to \mathcal{O}_X \to \nu_*\mathcal{O}_{X'} \to k'/k \to 0$ is a short exact sequence where $k'/k = \kappa(x')/\kappa(x)$ indicates the skyscraper sheaf on the point $x$.*

**Proof.** Let $\nu : X' \to X$ be the normalization of $X$, see Varieties, Sections 26 and 39. Since $X$ is singular $\nu$ is not an isomorphism. Then $k' = H^0(X', \mathcal{O}_{X'})$ is a finite extension of $k$ (Varieties, Lemma 25.2). The short exact sequence

$$0 \to \mathcal{O}_X \to \nu_*\mathcal{O}_{X'} \to \mathcal{Q} \to 0$$

and the fact that $\mathcal{Q}$ is supported in finitely many closed points give us that

1. $H^1(X', \mathcal{O}_{X'}) = 0$, i.e., $X'$ has genus 0 as a curve over $k'$,
2. there is a short exact sequence $0 \to k \to k' \to H^0(X, \mathcal{Q}) \to 0$.

In particular $k'/k$ is a nontrivial extension.

Next, we consider what is often called the *conductor ideal*

$$\mathcal{I} = \mathcal{H}om_{\mathcal{O}_X}(\nu_*\mathcal{O}_{X'}, \mathcal{O}_X)$$

This is a quasi-coherent $\mathcal{O}_X$-module. We view $\mathcal{I}$ as an ideal in $\mathcal{O}_X$ via the map $\varphi \mapsto \varphi(1)$. Thus $\mathcal{I}(U)$ is the set of $f \in \mathcal{O}_X(U)$ such that $f(\nu_*\mathcal{O}_{X'}(U)) \subset \mathcal{O}_X(U)$. In other words, the condition is that $f$ annihilates $\mathcal{Q}$. In other words, there is a defining exact sequence

$$0 \to \mathcal{I} \to \mathcal{O}_X \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{Q}, \mathcal{Q})$$

Let $U \subset X$ be an affine open containing the support of $\mathcal{Q}$. Then $V = \mathcal{Q}(U) = H^0(X, \mathcal{Q})$ is a $k$-vector space of dimension $n-1$. The image of $\mathcal{O}_X(U) \to \mathrm{Hom}_k(V, V)$ is a commutative subalgebra, hence has dimension $\leq n-1$ over $k$ (this is a property of commutative subalgebras of matrix algebras; details omitted). We conclude that we have a short exact sequence

$$0 \to \mathcal{I} \to \mathcal{O}_X \to \mathcal{A} \to 0$$

where $\mathrm{Supp}(\mathcal{A}) = \mathrm{Supp}(\mathcal{Q})$ and $\dim_k H^0(X, \mathcal{A}) \leq n-1$. On the other hand, the description $\mathcal{I} = \mathcal{H}om_{\mathcal{O}_X}(\nu_*\mathcal{O}_{X'}, \mathcal{O}_X)$ provides $\mathcal{I}$ with a $\nu_*\mathcal{O}_{X'}$-module structure such that the inclusion map $\mathcal{I} \to \nu_*\mathcal{O}_{X'}$ is a $\nu_*\mathcal{O}_{X'}$-module map. We conclude that $\mathcal{I} = \nu_*\mathcal{I}'$ for some quasi-coherent sheaf of ideals $\mathcal{I}' \subset \mathcal{O}_{X'}$, see Morphisms, Lemma 11.6. Define $\mathcal{A}'$ as the cokernel:

$$0 \to \mathcal{I}' \to \mathcal{O}_{X'} \to \mathcal{A}' \to 0$$

Combining the exact sequences so far we obtain a short exact sequence $0 \to \mathcal{A} \to \nu_*\mathcal{A}' \to \mathcal{Q} \to 0$. Using the estimate above, combined with $\dim_k H^0(X, \mathcal{Q}) = n-1$, gives

$$\dim_k H^0(X', \mathcal{A}') = \dim_k H^0(X, \mathcal{A}) + \dim_k H^0(X, \mathcal{Q}) \leq 2n - 2$$

However, since $X'$ is a curve over $k'$ we see that the left hand side is divisible by $n$ (Varieties, Lemma 42.10). As $\mathcal{A}$ and $\mathcal{A}'$ cannot be zero, we conclude that $\dim_k H^0(X', \mathcal{A}') = n$ which means that $\mathcal{I}'$ is the ideal sheaf of a $k'$-rational point $x'$. By Proposition 8.4 we find $X' \cong \mathbf{P}^1_{k'}$. Going back to the equalities above, we conclude that $\dim_k H^0(X, \mathcal{A}) = 1$. This means that $\mathcal{I}$ is the ideal sheaf of a $k$-rational point $x$. Then $\mathcal{A} = \kappa(x) = k$ and $\mathcal{A}' = \kappa(x') = k'$ as skyscraper sheaves. Comparing the exact sequences given above, this immediately implies the result on structure sheaves as stated in the lemma. □

0DJC **Example 8.6.** In fact, the situation described in Lemma 8.5 occurs for any nontrivial finite extension $k'/k$. Namely, we can consider

$$A = \{f \in k'[x] \mid f(0) \in k\}$$

The spectrum of $A$ is an affine curve, which we can glue to the spectrum of $B = k'[y]$ using the isomorphism $A_x \cong B_y$ sending $x^{-1}$ to $y$. The result is a proper curve $X$ with $H^0(X, \mathcal{O}_X) = k$ and singular point $x$ corresponding to the maximal ideal $A \cap (x)$. The normalization of $X$ is $\mathbf{P}^1_{k'}$ exactly as in the lemma.

## 9. Geometric genus

0BYE If $X$ is a proper and **smooth** curve over $k$ with $H^0(X, \mathcal{O}_X) = k$, then

$$p_g(X) = \dim_k H^0(X, \Omega_{X/k})$$

is called the *geometric genus* of $X$. By Lemma 6.4 the geometric genus of $X$ agrees with the (arithmetic) genus. However, in higher dimensions there is a difference between the geometric genus and the arithmetic genus, see Remark 9.2.

For singular curves, we will define the geometric genus as follows.

0BYF **Definition 9.1.** Let $k$ be a field. Let $X$ be a geometrically irreducible curve over $k$. The *geometric genus* of $X$ is the genus of a smooth projective model of $X$ possibly defined over an extension field of $k$ as in Lemma 2.9.

If $k$ is perfect, then the nonsingular projective model $Y$ of $X$ is smooth (Lemma 2.8) and the geometric genus of $X$ is just the genus of $Y$. But if $k$ is not perfect, this may not be true. In this case we choose an extension $K/k$ such that the nonsingular projective model $Y_K$ of $(X_K)_{red}$ is a smooth projective curve and we define the geometric genus of $X$ to be the genus of $Y_K$. This is well defined by Lemmas 2.9 and 6.2.

0BYG **Remark 9.2.** Suppose that $X$ is a $d$-dimensional proper smooth variety over an algebraically closed field $k$. Then the *arithmetic genus* is often defined as $p_a(X) = (-1)^d(\chi(X, \mathcal{O}_X) - 1)$ and the *geometric genus* as $p_g(X) = \dim_k H^0(X, \Omega^d_{X/k})$. In this situation the arithmetic genus and the geometric genus no longer agree even though it is still true that $\omega_X \cong \Omega^d_{X/k}$. For example, if $d = 2$, then we have

$$\begin{aligned} p_a(X) - p_g(X) &= h^0(X, \mathcal{O}_X) - h^1(X, \mathcal{O}_X) + h^2(X, \mathcal{O}_X) - 1 - h^0(X, \Omega^2_{X/k}) \\ &= -h^1(X, \mathcal{O}_X) + h^2(X, \mathcal{O}_X) - h^0(X, \omega_X) \\ &= -h^1(X, \mathcal{O}_X) \end{aligned}$$

where $h^i(X, \mathcal{F}) = \dim_k H^i(X, \mathcal{F})$ and where the last equality follows from duality. Hence for a surface the difference $p_g(X) - p_a(X)$ is always nonnegative; it is sometimes called the irregularity of the surface. If $X = C_1 \times C_2$ is a product of smooth projective curves of genus $g_1$ and $g_2$, then the irregularity is $g_1 + g_2$.

## 10. Riemann-Hurwitz

0C1B Let $k$ be a field. Let $f : X \to Y$ be a morphism of smooth curves over $k$. Then we obtain a canonical exact sequence

$$f^*\Omega_{Y/k} \xrightarrow{\text{d}f} \Omega_{X/k} \longrightarrow \Omega_{X/Y} \longrightarrow 0$$

by Morphisms, Lemma 31.9. Since $X$ and $Y$ are smooth, the sheaves $\Omega_{X/k}$ and $\Omega_{Y/k}$ are invertible modules, see Morphisms, Lemma 32.12. Assume the first map is nonzero, i.e., assume $f$ is generically étale, see Lemma 10.1. Let $R \subset X$ be the closed subscheme cut out by the different $\mathfrak{D}_f$ of $f$. By Dualizing Complexes, Lemma 54.6 this is the same as the vanishing locus of $\text{d}f$, it is an effective Cartier divisor, and we get

$$f^*\Omega_{Y/k} \otimes_{\mathcal{O}_X} \mathcal{O}_X(R) = \Omega_{X/k}$$

In particular, if $X$, $Y$ are projective with $k = H^0(Y, \mathcal{O}_Y) = H^0(X, \mathcal{O}_X)$ and $X$, $Y$ have genus $g_X$, $g_Y$, then we get the Riemann-Hurwitz formula

$$\begin{aligned} 2g_X - 2 &= \deg(\Omega_{X/k}) \\ &= \deg(f^*\Omega_{Y/k} \otimes_{\mathcal{O}_X} \mathcal{O}_X(R)) \\ &= \deg(f)\deg(\Omega_{Y/k}) + \deg(R) \\ &= \deg(f)(2g_Y - 2) + \deg(R) \end{aligned}$$

The first and last equality by Lemma 6.4. The second equality by the isomorphism of invertible sheaves given above. The third equality by additivity of degrees (Varieties, Lemma 42.7), the formula for the degree of a pullback (Varieties, Lemma 42.11), and finally the formula for the degree of $\mathcal{O}_X(R)$ (Varieties, Lemma 42.9).

To use the Riemann-Hurwitz formula we need to compute $\deg(R) = \dim_k \Gamma(R, \mathcal{O}_R)$. By the structure of zero dimensional schemes over $k$ (see for example Varieties, Lemma 20.2), we see that $R$ is a finite disjoint union of spectra of Artinian local rings $R = \coprod_{x \in R} \mathrm{Spec}(\mathcal{O}_{R,x})$ with each $\mathcal{O}_{R,x}$ of finite dimension over $k$. Thus

$$\deg(R) = \sum\nolimits_{x \in R} \dim_k \mathcal{O}_{R,x} = \sum\nolimits_{x \in R} d_x[\kappa(x) : k]$$

with

$$d_x = \text{length}_{\mathcal{O}_{R,x}} \mathcal{O}_{R,x} = \text{length}_{\mathcal{O}_{X,x}} \mathcal{O}_{R,x}$$

the multiplicity of $x$ in $R$ (see Algebra, Lemma 51.12). Let $x \in X$ be a closed point with image $y \in Y$. Looking at stalks we obtain an exact sequence

$$\Omega_{Y/k,y} \to \Omega_{X/k,x} \to \Omega_{X/Y,x} \to 0$$

Choosing local generators $\eta_x$ and $\eta_y$ of the (free rank 1) modules $\Omega_{X/k,x}$ and $\Omega_{Y/k,y}$ we see that $\eta_y \mapsto h\eta_x$ for some nonzero $h \in \mathcal{O}_{X,x}$. By definition $R$ is cut out by $h$. By the exact sequence we see that $\Omega_{X/Y,x} \cong \mathcal{O}_{X,x}/h\mathcal{O}_{X,x}$ as $\mathcal{O}_{X,x}$-modules. Thus we find the following equalities

$$\begin{aligned} d_x &= \text{length}_{\mathcal{O}_{X,x}}(\mathcal{O}_{X,x}/h\mathcal{O}_{X,x}) \\ &= \text{length}_{\mathcal{O}_{X,x}}(\Omega_{X/Y,x}) \\ &= \text{ord}_{\mathcal{O}_{X,x}}(h) \\ &= \text{ord}_{\mathcal{O}_{X,x}}(\text{“}\eta_y/\eta_x\text{”}) \end{aligned}$$

The first equality by our definition of $d_x$. The second by the above. The third equality is the definition of ord, see Algebra, Definition 120.2. The fourth equality is a mnemonic. Since $\mathcal{O}_{X,x}$ is a discrete valuation ring, the integer $\text{ord}_{\mathcal{O}_{X,x}}(h)$ just the valuation of $h$.

Here is a case where one can "calculate" the multiplicity $d_x$ in terms of other invariants. Namely, if $\kappa(x)$ is separable over $k$, then we may choose $\eta_x = \mathrm{d}s$ and $\eta_y = \mathrm{d}t$ where $s$ and $t$ are uniformizers in $\mathcal{O}_{X,x}$ and $\mathcal{O}_{Y,y}$ (Lemma 10.3). Then $t \mapsto us^{e_x}$ for some unit $u \in \mathcal{O}_{X,x}$ where $e_x$ is the ramification index of the extension $\mathcal{O}_{Y,y} \subset \mathcal{O}_{X,x}$. Hence we get

$$\eta_y = \mathrm{d}t = \mathrm{d}(us^{e_x}) = es^{e_x - 1}u\mathrm{d}s + s^{e_x}\mathrm{d}u$$

Writing $\mathrm{d}u = w\mathrm{d}s$ for some $w \in \mathcal{O}_{X,x}$ we see that

$$\text{“}\eta_y/\eta_x\text{”} = es^{e_x - 1}u + s^{e_x}w = (e_x u + sw)s^{e_x - 1}$$

We conclude that the order of vanishing of this is $e_x - 1$ unless the characteristic of $\kappa(x)$ is $p > 0$ and $p$ divides $e_x$ in which case the order of vanishing is $> e_x - 1$.

Combining all of the above we find that if $k$ has characteristic zero, then

$$2g_x - 2 = (2g_Y - 2)\deg(f) + \sum\nolimits_{x \in X} (e_x - 1)[\kappa(x) : k]$$

where $e_x$ is the ramification index of $\mathcal{O}_{X,x}$ over $\mathcal{O}_{Y,f(x)}$. This precise formula will hold if and only if all the ramification is tame, i.e., when the residue field extensions $\kappa(x)/\kappa(y)$ are separable and $e_x$ is prime to the characteristic of $k$, although the arguments above are insufficient to prove this. We refer the reader to Lemma 10.4 and its proof.

0C1C **Lemma 10.1.** *Let $k$ be a field. Let $f : X \to Y$ be a morphism of smooth curves over $k$. The following are equivalent*

1. *$df : f^*\Omega_{Y/k} \to \Omega_{X/k}$ is nonzero,*
2. *$\Omega_{X/Y}$ is supported on a proper closed subset of $X$,*
3. *there exists a nonempty open $U \subset X$ such that $f|_U : U \to Y$ is unramified,*
4. *there exists a nonempty open $U \subset X$ such that $f|_U : U \to Y$ is étale,*
5. *the extension $k(Y) \subset k(X)$ of function fields is finite separable.*

**Proof.** Since $X$ and $Y$ are smooth, the sheaves $\Omega_{X/k}$ and $\Omega_{Y/k}$ are invertible modules, see Morphisms, Lemma 32.12. Using the exact sequence

$$f^*\Omega_{Y/k} \longrightarrow \Omega_{X/k} \longrightarrow \Omega_{X/Y} \longrightarrow 0$$

of Morphisms, Lemma 31.9 we see that (1) and (2) are equivalent and equivalent to the condition that $f^*\Omega_{Y/k} \to \Omega_{X/k}$ is nonzero in the generic point. The equivalence of (2) and (3) follows from Morphisms, Lemma 33.2. The equivalence between (3) and (4) follows from Morphisms, Lemma 34.16 and the fact that flatness is automatic (Lemma 2.3). To see the equivalence of (5) and (4) use Algebra, Lemma 138.9. Some details omitted. □

0C1D **Lemma 10.2.** *Let $f : X \to Y$ be a morphism of smooth proper curves over a field $k$ which satisfies the equivalent conditions of Lemma 10.1. If $k = H^0(Y, \mathcal{O}_Y) = H^0(X, \mathcal{O}_X)$ and $X$ and $Y$ have genus $g_X$ and $g_y$, then*

$$2g_X - 2 = (2g_Y - 2)\deg(f) + \deg(R)$$

*where $R \subset X$ is the effective Cartier divisor cut out by the different of $f$.*

**Proof.** See discussion above; we used Dualizing Complexes, Lemma 54.6, Lemma 6.4, and Varieties, Lemmas 42.7 and 42.11. □

0C1E **Lemma 10.3.** *Let $X \to \mathrm{Spec}(k)$ be smooth of relative dimension 1 at a closed point $x \in X$. If $\kappa(x)$ is separable over $k$, then for any uniformizer $s$ in the discrete valuation ring $\mathcal{O}_{X,x}$ the element $ds$ freely generates $\Omega_{X/k,x}$ over $\mathcal{O}_{X,x}$.*

**Proof.** The ring $\mathcal{O}_{X,x}$ is a discrete valuation ring by Algebra, Lemma 138.3. Since $x$ is closed $\kappa(x)$ is finite over $k$. Hence if $\kappa(x)/k$ is separable, then any uniformizer $s$ maps to a nonzero element of $\Omega_{X/k,x} \otimes_{\mathcal{O}_{X,x}} \kappa(x)$ by Algebra, Lemma 138.4. Since $\Omega_{X/k,x}$ is free of rank 1 over $\mathcal{O}_{X,x}$ the result follows. □

0C1F **Lemma 10.4.** *Notation and assumptions as in Lemma 10.2. For a closed point $x \in X$ let $d_x$ be the multiplicity of $x$ in $R$. Then*

$$2g_X - 2 = (2g_Y - 2)\deg(f) + \sum d_x[\kappa(x) : k]$$

*Moreover, we have the following results*

(1) $d_x = length_{\mathcal{O}_{X,x}}(\Omega_{X/Y,x})$,
(2) $d_x \geq e_x - 1$ *where* $e_x$ *is the ramification index of* $\mathcal{O}_{X,x}$ *over* $\mathcal{O}_{Y,y}$,
(3) $d_x = e_x - 1$ *if and only if* $\mathcal{O}_{X,x}$ *is tamely ramified over* $\mathcal{O}_{Y,y}$.

**Proof.** By Lemma 10.2 and the discussion above (which used Varieties, Lemma 20.2 and Algebra, Lemma 51.12) it suffices to prove the results on the multiplicity $d_x$ of $x$ in $R$. Part (1) was proved in the discussion above. In the discussion above we proved (2) and (3) only in the case where $\kappa(x)$ is separable over $k$. In the rest of the proof we give a uniform treatment of (2) and (3) using material on differents of quasi-finite Gorenstein morphisms.

First, observe that $f$ is a quasi-finite Gorenstein morphism. This is true for example because $f$ is a flat quasi-finite morphism and $X$ is Gorenstein (see Dualizing Complexes, Lemma 43.6) or because it was shown in the proof of Dualizing Complexes, Lemma 54.6 (which we used above). Thus $\omega_{X/Y}$ is invertible by Dualizing Complexes, Lemma 53.2 and the same remains true after replacing $X$ by opens and after performing a base change by some $Y' \to Y$. We will use this below without further mention.

Choose affine opens $U \subset X$ and $V \subset Y$ such that $x \in U$, $y \in V$, $f(U) \subset V$, and $x$ is the only point of $U$ lying over $y$. Write $U = \text{Spec}(A)$ and $V = \text{Spec}(B)$. Then $R \cap U$ is the different of $f|_U : U \to V$. By Dualizing Complexes, Lemma 52.4 formation of the different commutes with arbitrary base change in our case. By our choice of $U$ and $V$ we have

$$A \otimes_B \kappa(y) = \mathcal{O}_{X,x} \otimes_{\mathcal{O}_{Y,y}} \kappa(y) = \mathcal{O}_{X,x}/(s^{e_x})$$

where $e_x$ is the ramification index as in the statement of the lemma. Let $C = \mathcal{O}_{X,x}/(s^{e_x})$ viewed as a finite algebra over $\kappa(y)$. Let $\mathfrak{D}_{C/\kappa(y)}$ be the different of $C$ over $\kappa(y)$ in the sense of Dualizing Complexes, Definition 52.1. It suffices to show: $\mathfrak{D}_{C/\kappa(y)}$ is nonzero if and only if the the extension $\mathcal{O}_{Y,y} \subset \mathcal{O}_{X,x}$ is tamely ramified and in the tamely ramified case $\mathfrak{D}_{C/\kappa(y)}$ is equal to the ideal generated by $s^{e_x - 1}$ in $C$. Recall that tame ramification means exactly that $\kappa(x)/\kappa(y)$ is separable and that the characteristic of $\kappa(y)$ does not divide $e_x$. On the other hand, the different of $C/\kappa(y)$ is nonzero if and only if $\tau_{C/\kappa(y)} \in \omega_{C/\kappa(y)}$ is nonzero. Namely, since $\omega_{C/\kappa(y)}$ is an invertible $C$-module (as the base change of $\omega_{A/B}$) it is free of rank 1, say with generator $\lambda$. Write $\tau_{C/\kappa(y)} = h\lambda$ for some $h \in C$. Then $\mathfrak{D}_{C/\kappa(y)} = (h) \subset C$ whence the claim. By Dualizing Complexes, Lemma 48.9 we have $\tau_{C/\kappa(y)} \neq 0$ if and only if $\kappa(x)/\kappa(y)$ is separable and $e_x$ is prime to the characteristic. Finally, even if $\tau_{C/\kappa(y)}$ is nonzero, then it is still the case that $s\tau_{C/\kappa(y)} = 0$ because $s\tau_{C/\kappa(y)} : C \to \kappa(y)$ sends $c$ to the trace of the nilpotent operator $sc$ which is zero. Hence $sh = 0$, hence $h \in (s^{e_x - 1})$ which proves that $\mathfrak{D}_{C/\kappa(y)} \subset (s^{e_x - 1})$ always. Since $(s^{e_x - 1}) \subset C$ is the smallest nonzero ideal, we have proved the final assertion. $\square$

## 11. Inseparable maps

0CCV Some remarks on the behaviour of the genus under inseparable maps.

0CCW **Lemma 11.1.** *Let $k$ be a field. Let $f : X \to Y$ be a surjective morphism of curves over $k$. If $X$ is smooth over $k$ and $Y$ is normal, then $Y$ is smooth over $k$.*

**Proof.** Let $y \in Y$. Pick $x \in X$ mapping to $y$. By Varieties, Lemma 24.9 it suffices to show that $f$ is flat at $x$. This follows from Lemma 2.3. $\square$

0CCX **Lemma 11.2.** *Let $k$ be a field of characteristic $p > 0$. Let $f : X \to Y$ be a nonconstant morphism of proper nonsingular curves over $k$. If the extension $k(Y) \subset k(X)$ of function fields is purely inseparable, then there exists a factorization*

$$X = X_0 \to X_1 \to \ldots \to X_n = Y$$

*such that each $X_i$ is a proper nonsingular curve and $X_i \to X_{i+1}$ is a degree $p$ morphism with $k(X_{i+1}) \subset k(X_i)$ inseparable.*

**Proof.** This follows from Theorem 2.6 and the fact that a finite purely inseparable extension of fields can always be gotten as a sequence of (inseparable) extensions of degree $p$, see Fields, Lemma 14.5. $\square$

0CCY **Lemma 11.3.** *Let $k$ be a field of characteristic $p > 0$. Let $f : X \to Y$ be a nonconstant morphism of proper nonsingular curves over $k$. If $X$ is smooth and $k(Y) \subset k(X)$ is inseparable of degree $p$, then there is a unique isomorphism $Y = X^{(p)}$ such that $f$ is $F_{X/k}$.*

**Proof.** The relative frobenius morphism $F_{X/k} : X \to X^{(p)}$ is constructed in Varieties, Section 34. Observe that $X^{(p)}$ is a smooth proper curve over $k$ as a base change of $X$. The morphism $F_{X/k}$ has degree $p$ by Varieties, Lemma 34.10. Thus $k(X^{(p)})$ and $k(Y)$ are both subfields of $k(X)$ with $[k(X) : k(Y)] = [k(X) : k(X^{(p)})] = p$. To prove the lemma it suffices to show that $k(Y) = k(X^{(p)})$ inside $k(X)$. See Theorem 2.6.

Write $K = k(X)$. Consider the map $\text{d} : K \to \Omega_{K/k}$. It follows from Lemma 10.1 that both $k(Y)$ is contained in the kernel of d. By Varieties, Lemma 34.7 we see that $k(X^{(p)})$ is in the kernel of d. Since $X$ is a smooth curve we know that $\Omega_{K/k}$ is a vector space of dimension 1 over $K$. Then More on Algebra, Lemma 42.2. implies that $\text{Ker}(\text{d}) = kK^p$ and that $[K : kK^p] = p$. Thus $k(Y) = kK^p = k(X^{(p)})$ for reasons of degree. $\square$

0CCZ **Lemma 11.4.** *Let $k$ be a field of characteristic $p > 0$. Let $f : X \to Y$ be a nonconstant morphism of proper nonsingular curves over $k$. If $X$ is smooth and $k(Y) \subset k(X)$ is purely inseparable, then there is a unique $n \geq 0$ and a unique isomorphism $Y = X^{(p^n)}$ such that $f$ is the $n$-fold relative Frobenius of $X/k$.*

**Proof.** The $n$-fold relative Frobenius of $X/k$ is defined in Varieties, Remark 34.11. The lemma follows by combining Lemmas 11.3 and 11.2. $\square$

0CD0 **Lemma 11.5.** *Let $k$ be a field of characteristic $p > 0$. Let $f : X \to Y$ be a nonconstant morphism of proper nonsingular curves over $k$. Assume*

1. *$X$ is smooth,*
2. *$H^0(X, \mathcal{O}_X) = k$,*
3. *$k(X)/k(Y)$ is purely inseparable.*

*Then $Y$ is smooth, $H^0(Y, \mathcal{O}_Y) = k$, and the genus of $Y$ is equal to the genus of $X$.*

**Proof.** By Lemma 11.4 we see that $Y = X^{(p^n)}$ is the base change of $X$ by $F^n_{\text{Spec}(k)}$. Thus $Y$ is smooth and the result on the cohomology and genus follows from Lemma 6.2. $\square$

0CD1 **Example 11.6.** This example will show that the genus can change under a purely inseparable morphism of nonsingular projective curves. Let $k$ be a field of characteristic 3. Assume there exists an element $a \in k$ which is not a 3rd power. For example $k = \mathbf{F}_3(a)$ would work. Let $X$ be the plane curve with homogeneous equation

$$F = T_1^2 T_0 - T_2^3 + aT_0^3$$

as in Section 7. On the affine piece $D_+(T_0)$ using coordinates $x = T_1/T_0$ and $y = T_2/T_0$ we obtain $x^2 - y^3 + a = 0$ which defines a nonsingular affine curve. Moreover, the point at infinity $(0 : 1 : 0)$ is a smooth point. Hence $X$ is a nonsingular projective curve of genus 1 (Lemma 7.3). On the other hand, consider the morphism $f : X \to \mathbf{P}^1_k$ which on $D_+(T_0)$ sends $(x, y)$ to $y \in \mathbf{A}^1_k \subset \mathbf{P}^1_k$. Then $f$ is a morphism of proper nonsingular curves over $k$ inducing an inseparable function field extension of degree $p$ but the genus of $X$ is 1 and the genus of $\mathbf{P}^1_k$ is 0.

0CD2 **Proposition 11.7.** *Let $k$ be a field of characteristic $p > 0$. Let $f : X \to Y$ be a nonconstant morphism of proper smooth curves over $k$. Then we can factor $f$ as*

$$X \longrightarrow X^{(p^n)} \longrightarrow Y$$

*where $X^{(p^n)} \to Y$ is a nonconstant morphism of proper smooth curves inducing a separable field extension $k(X^{(p^n)})/k(Y)$, we have*

$$X^{(p^n)} = X \times_{\mathrm{Spec}(k), F^n_{\mathrm{Spec}(k)}} \mathrm{Spec}(k),$$

*and $X \to X^{(p^n)}$ is the $n$-fold relative frobenius of $X$.*

**Proof.** By Fields, Lemma 14.6 there is a subextension $k(X)/E/k(Y)$ such that $k(X)/E$ is purely inseparable and $E/k(Y)$ is separable. By Theorem 2.6 this corresponds to a factorization $X \to Z \to Y$ of $f$ with $Z$ a nonsingular proper curve. Apply Lemma 11.4 to the morphism $X \to Z$ to conclude. □

0CD3 **Lemma 11.8.** *Let $k$ be a field of characteristic $p > 0$. Let $X$ be a smooth proper curve over $k$. Let $(\mathcal{L}, V)$ be a $\mathfrak{g}^r_d$ with $r \geq 1$. Then one of the following two is true*

1. *there exists a $\mathfrak{g}^1_d$ whose corresponding morphism $X \to \mathbf{P}^1_k$ (Lemma 3.2) is generically étale (i.e., is as in Lemma 10.1), or*
2. *there exists a $\mathfrak{g}^r_{d'}$ on $X^{(p)}$ where $d' \leq d/p$.*

**Proof.** Pick two $k$-linearly independent elements $s, t \in V$. Then $f = s/t$ is the rational function defining the morphism $X \to \mathbf{P}^1_k$ corresponding to the linear series $(\mathcal{L}, ks + kt)$. If this morphism is not generically étale, then $f \in k(X^{(p)})$ by Proposition 11.7. Now choose a basis $s_0, \ldots, s_r$ of $V$ and let $\mathcal{L}' \subset \mathcal{L}$ be the invertible sheaf generated by $s_0, \ldots, s_r$. Set $f_i = s_i/s_0$ in $k(X)$. If for each pair $(s_0, s_i)$ we have $f_i \in k(X^{(p)})$, then the morphism

$$\varphi = \varphi_{(\mathcal{L}', (s_0, \ldots, s_r))} : X \longrightarrow \mathbf{P}^1_k = \mathrm{Proj}(k[T_0, \ldots, T_r])$$

factors through $X^{(p)}$ as this is true over the affine open $D_+(T_0)$ and we can extend the morphism over the affine part to the whole of the smooth curve $X^{(p)}$ by Lemma 2.2. Introducing notation, say we have the factorization

$$X \xrightarrow{F_{X/k}} X^{(p)} \xrightarrow{\psi} \mathbf{P}^r_k$$

of $\varphi$. Then $\mathcal{N} = \psi^*\mathcal{O}_{\mathbf{P}^1_k}(1)$ is an invertible $\mathcal{O}_{X^{(p)}}$-module with $\mathcal{L}' = F_{X/k}^*\mathcal{N}$ and with $\psi^*T_0, \ldots, \psi^*T_r$ $k$-linearly independent (as they pullback to $s_0, \ldots, s_r$ on $X$). Finally, we have

$$d = \deg(\mathcal{L}) \geq \deg(\mathcal{L}') = \deg(F_{X/k})\deg(\mathcal{N}) = p\deg(\mathcal{N})$$

as desired. Here we used Varieties, Lemmas 42.12, 42.11, and 34.10. $\square$

0CD4 **Lemma 11.9.** *Let $k$ be a field. Let $X$ be a smooth proper curve over $k$ with $H^0(X, \mathcal{O}_X) = k$ and genus $g \geq 2$. Then there exists a closed point $x \in X$ with $\kappa(x)/k$ separable of degree $\leq 2g - 2$.*

**Proof.** Set $\omega = \Omega_{X/k}$. By Lemma 6.4 this has degree $2g - 2$ and has $g$ global sections. Thus we have a $\mathfrak{g}^{g-1}_{2g-2}$. By the trivial Lemma 3.3 there exists a $\mathfrak{g}^1_{2g-2}$ and by Lemma 3.4 we obtain a morphism

$$\varphi : X \longrightarrow \mathbf{P}^1_k$$

of some degree $d \leq 2g - 2$. Since $\varphi$ is flat (Lemma 2.3) and finite (Lemma 2.4) it is finite locally free of degree $d$ (Morphisms, Lemma 45.2). Pick any rational point $t \in \mathbf{P}^1_k$ and any point $x \in X$ with $\varphi(x) = t$. Then

$$d \geq [\kappa(x) : \kappa(t)] = [\kappa(x) : k]$$

for example by Morphisms, Lemmas 53.3 and 53.2. Thus if $k$ is perfect (for example has characteristic zero or is finite) then the lemma is proved. Thus we reduce to the case discussed in the next paragraph.

Assume that $k$ is an infinite field of characteristic $p > 0$. As above we will use that $X$ has a $\mathfrak{g}^{g-1}_{2g-2}$. The smooth proper curve $X^{(p)}$ has the same genus as $X$. Hence its genus is $> 0$. We conclude that $X^{(p)}$ does not have a $\mathfrak{g}^{g-1}_d$ for any $d \leq g - 1$ by Lemma 3.5. Applying Lemma 11.8 to our $\mathfrak{g}^{g-1}_{2g-2}$ (and noting that $2g - 2/p \leq g - 1$) we conclude that possibility (2) does not occur. Hence we obtain a morphism

$$\varphi : X \longrightarrow \mathbf{P}^1_k$$

which is generically étale (in the sense of the lemma) and has degree $\leq 2g - 2$. Let $U \subset X$ be the nonempty open subscheme where $\varphi$ is étale. Then $\varphi(U) \subset \mathbf{P}^1_k$ is a nonempty Zariski open and we can pick a $k$-rational point $t \in \varphi(U)$ as $k$ is infinite. Let $u \in U$ be a point with $\varphi(u) = t$. Then $\kappa(u)/\kappa(t)$ is separable (Morphisms, Lemma 34.7), $\kappa(t) = k$, and $[\kappa(u) : k] \leq 2g - 2$ as before. $\square$

The following lemma does not really belong in this section but we don't know a good place for it elsewhere.

0C1G **Lemma 11.10.** *Let $X$ be a smooth curve over a field $k$. Let $\overline{x} \in X_{\overline{k}}$ be a closed point with image $x \in X$. The ramification index of $\mathcal{O}_{X,x} \subset \mathcal{O}_{X_{\overline{k}},\overline{x}}$ is the inseparable degree of $\kappa(x)/k$.*

**Proof.** After shrinking $X$ we may assume there is an étale morphism $\pi : X \to \mathbf{A}^1_k$, see Morphisms, Lemma 34.20. Then we can consider the diagram of local rings

$$\begin{tikzcd}
\mathcal{O}_{X_{\overline{k}},\overline{x}} & \mathcal{O}_{\mathbf{A}^1_{\overline{k}},\pi(\overline{x})} \arrow[l] \\
\mathcal{O}_{X,x} \arrow[u] & \mathcal{O}_{\mathbf{A}^1_k,\pi(x)} \arrow[l] \arrow[u]
\end{tikzcd}$$

The horizontal arrows have ramification index 1 as they correspond to étale morphisms. Moreover, the extension $\kappa(x)/\kappa(\pi(x))$ is separable hence $\kappa(x)$ and $\kappa(\pi(x))$ have the same inseparable degree over $k$. By multiplicativity of ramification indices it suffices to prove the result when $x$ is a point of the affine line.

Assume $X = \mathbf{A}^1_k$. In this case, the local ring of $X$ at $x$ looks like

$$\mathcal{O}_{X,x} = k[t]_{(P)}$$

where $P$ is an irreducible monic polynomial over $k$. Then $P(t) = Q(t^q)$ for some separable polynomial $Q \in k[t]$, see Fields, Lemma 12.1. Observe that $\kappa(x) = k[t]/(P)$ has inseparable degree $q$ over $k$. On the other hand, over $\overline{k}$ we can factor $Q(t) = \prod(t - \alpha_i)$ with $\alpha_i$ pairwise distinct. Write $\alpha_i = \beta_i^q$ for some unique $\beta_i \in \overline{k}$. Then our point $\overline{x}$ corresponds to one of the $\beta_i$ and we conclude because the ramification index of

$$k[t]_{(P)} \longrightarrow \overline{k}[t]_{(t-\beta_i)}$$

is indeed equal to $q$ as the uniformizer $P$ maps to $(t - \beta_i)^q$ times a unit. $\square$

## 12. Glueing and squishing

0C1H Below we will indicate $k[\epsilon]$ the algebra of dual numbers over $k$ as defined in Varieties, Definition 16.1.

0C1I **Lemma 12.1.** *Let $k$ be an algebraically closed field. Let $k \subset A$ be a ring extension such that $A$ has exactly two $k$-sub algebras, then either $A = k \times k$ or $A = k[\epsilon]$.*

**Proof.** The assumption means $k \neq A$ and any subring $k \subset C \subset A$ is equal to either $k$ or $A$. Let $t \in A$, $t \notin k$. Then $A$ is generated by $t$ over $k$. Hence $A = k[x]/I$ for some ideal $I$. If $I = (0)$, then we have the subalgebra $k[x^2]$ which is not allowed. Otherwise $I$ is generated by a monic polynomial $P$. Write $P = \prod_{i=1}^d (t - a_i)$. If $d > 2$, then the subalgebra generated by $(t-a_1)(t-a_2)$ gives a contradiction. Thus $d = 2$. If $a_1 \neq a_2$, then $A = k \times k$, if $a_1 = a_2$, then $A = k[\epsilon]$. $\square$

0C1J **Example 12.2** (Glueing points)**.** Let $k$ be an algebraically closed field. Let $f : X' \to X$ be a morphism of algebraic $k$-schemes. We say $X$ is obtained by glueing $a$ and $b$ in $X'$ if the following are true:

(1) $a, b \in X'(k)$ are distinct points which map to the same point $x \in X(k)$,
(2) $f$ is finite and $f^{-1}(X \setminus \{x\}) \to X \setminus \{x\}$ is an isomorphism,
(3) there is a short exact sequence

$$0 \to \mathcal{O}_X \to f_*\mathcal{O}_{X'} \xrightarrow{a-b} x_*k \to 0$$

where arrow on the right sends a local section $h$ of $f_*\mathcal{O}_{X'}$ to the difference $h(a) - h(b) \in k$.

If this is the case, then there also is a short exact sequence

$$0 \to \mathcal{O}_X^* \to f_*\mathcal{O}_{X'}^* \xrightarrow{ab^{-1}} x_*k^* \to 0$$

where arrow on the right sends a local section $h$ of $f_*\mathcal{O}_{X'}^*$ to the multiplicative difference $h(a)h(b)^{-1} \in k^*$.

0C1K **Example 12.3** (Squishing a tangent vector)**.** Let $k$ be an algebraically closed field. Let $f : X' \to X$ be a morphism of algebraic $k$-schemes. We say $X$ is obtained by squishing the tangent vector $\vartheta$ in $X'$ if the following are true:

(1) $\vartheta : \mathrm{Spec}(k[\epsilon]) \to X'$ is a closed immersion over $k$ such that $f \circ \vartheta$ factors through a point $x \in X(k)$,
(2) $f$ is finite and $f^{-1}(X \setminus \{x\}) \to X \setminus \{x\}$ is an isomorphism,
(3) there is a short exact sequence

$$0 \to \mathcal{O}_X \to f_*\mathcal{O}_{X'} \xrightarrow{\vartheta} x_*k \to 0$$

where arrow on the right sends a local section $h$ of $f_*\mathcal{O}_{X'}$ to the coefficient of $\epsilon$ in $\vartheta^\sharp(h) \in k[\epsilon]$.

If this is the case, then there also is a short exact sequence

$$0 \to \mathcal{O}_X^* \to f_*\mathcal{O}_{X'}^* \xrightarrow{\vartheta} x_*k \to 0$$

where arrow on the right sends a local section $h$ of $f_*\mathcal{O}_{X'}^*$ to $\mathrm{d}\log(\vartheta^\sharp(h))$ where $\mathrm{d}\log : k[\epsilon]^* \to k$ is the homomorphism of abelian groups sending $a + b\epsilon$ to $b/a \in k$.

0C1L **Lemma 12.4.** *Let $k$ be an algebraically closed field. Let $f : X' \to X$ be a finite morphism algebraic $k$-schemes such that $\mathcal{O}_X \subset f_*\mathcal{O}_{X'}$ and such that $f$ is an isomorphism away from a finite set of points. Then there is a factorization*

$$X' = X_n \to X_{n-1} \to \ldots \to X_1 \to X_0 = X$$

*such that each $X_i \to X_{i-1}$ is either the glueing of two points or the squishing of a tangent vector (see Examples 12.2 and 12.3).*

**Proof.** Let $U \subset X$ be the maximal open set over which $f$ is an isomorphism. Then $X \setminus U = \{x_1, \ldots, x_n\}$ with $x_i \in X(k)$. We will consider factorizations $X' \to Y \to X$ of $f$ such that both morphisms are finite and

$$\mathcal{O}_X \subset g_*\mathcal{O}_Y \subset f_*\mathcal{O}_{X'}$$

where $g : Y \to X$ is the given morphism. By assumption $\mathcal{O}_{X,x} \to (f_*\mathcal{O}_{X'})_x$ is an isomorphism onless $x = x_i$ for some $i$. Hence the cokernel

$$f_*\mathcal{O}_{X'}/\mathcal{O}_X = \bigoplus \mathcal{Q}_i$$

is a direct sum of skyscraper sheaves $\mathcal{Q}_i$ supported at $x_1, \ldots, x_n$. Because the displayed quotient is a coherent $\mathcal{O}_X$-module, we conclude that $\mathcal{Q}_i$ has finite length over $\mathcal{O}_{X,x_i}$. Hence we can argue by induction on the sum of these lengths, i.e., the length of the whole cokernel.

If $n > 1$, then we can define an $\mathcal{O}_X$-subalgebra $\mathcal{A} \subset f_*\mathcal{O}_{X'}$ by taking the inverse image of $\mathcal{Q}_1$. This will give a nontrivial factorization and we win by induction.

Assume $n = 1$. We abbreviate $x = x_1$. Consider the finite $k$-algebra extension

$$A = \mathcal{O}_{X,x} \subset (f_*\mathcal{O}_{X'})_x = B$$

Note that $\mathcal{Q} = \mathcal{Q}_1$ is the skyscraper sheaf with value $B/A$. We have a $k$-subalgebra $A \subset A + \mathfrak{m}_A B \subset B$. If both inclusions are strict, then we obtain a nontrivial factorization and we win by induction as above. If $A + \mathfrak{m}_A B = B$, then $A = B$ by Nakayama, then $f$ is an isomorphism and there is nothing to prove. We conclude that we may assume $B = A + \mathfrak{m}_A B$. Set $C = B/\mathfrak{m}_A B$. If $C$ has more than 2 $k$-subalgebras, then we obtain a subalgebra between $A$ and $B$ by taking the inverse image in $B$. Thus we may assume $C$ has exactly 2 $k$-subalgebras. Thus $C = k \times k$ or $C = k[\epsilon]$ by Lemma 12.1. In this case $f$ is correspondingly the glueing two points or the squishing of a tangent vector. $\square$

0C1M **Lemma 12.5.** *Let $k$ be an algebraically closed field. If $f : X \to X'$ is the glueing of two points $a, b$ as in Example 12.2, then there is an exact sequence*

$$k^* \to \mathrm{Pic}(X) \to \mathrm{Pic}(X') \to 0$$

*The first map is zero if $a$ and $b$ are on different connected components of $X'$ and injective if $X'$ is proper and $a$ and $b$ are on the same connected component of $X'$.*

**Proof.** The map $\mathrm{Pic}(X) \to \mathrm{Pic}(X')$ is surjective by Varieties, Lemma 36.7. Using the short exact sequence

$$0 \to \mathcal{O}_X^* \to f_*\mathcal{O}_{X'}^* \xrightarrow{ab^{-1}} x_*k^* \to 0$$

we obtain

$$H^0(X', \mathcal{O}_{X'}^*) \xrightarrow{ab^{-1}} k^* \to H^1(X, \mathcal{O}_X^*) \to H^1(X, f_*\mathcal{O}_{X'}^*)$$

We have $H^1(X, f_*\mathcal{O}_{X'}^*) \subset H^1(X', \mathcal{O}_{X'}^*)$ (for example by the Leray spectral sequence, see Cohomology, Lemma 14.4). Hence the kernel of $\mathrm{Pic}(X) \to \mathrm{Pic}(X')$ is the cokernel of $ab^{-1} : H^0(X', \mathcal{O}_{X'}^*) \to k^*$. If $a$ and $b$ are on different connected components of $X'$, then $ab^{-1}$ is surjective (also for example if $X'$ is affine). Because $k$ is algebraically closed any regular function on a reduced connected proper scheme over $k$ comes from an element of $k$, see Varieties, Lemma 9.3. Thus $ab^{-1}$ is zero if $X'$ is proper and $a$ and $b$ are on the same connected component. $\square$

0C1N **Lemma 12.6.** *Let $k$ be an algebraically closed field. If $f : X \to X'$ is the squishing of a tangent vector $\vartheta$ as in Example 12.3, then there is an exact sequence*

$$(k, +) \to \mathrm{Pic}(X) \to \mathrm{Pic}(X') \to 0$$

*and the first map is injective if $X'$ is proper and reduced.*

**Proof.** The map $\mathrm{Pic}(X) \to \mathrm{Pic}(X')$ is surjective by Varieties, Lemma 36.7. Using the short exact sequence

$$0 \to \mathcal{O}_X^* \to f_*\mathcal{O}_{X'}^* \xrightarrow{\vartheta} x_*k \to 0$$

of Example 12.3 we obtain

$$H^0(X', \mathcal{O}_{X'}^*) \xrightarrow{\vartheta} k \to H^1(X, \mathcal{O}_X^*) \to H^1(X, f_*\mathcal{O}_{X'}^*)$$

We have $H^1(X, f_*\mathcal{O}_{X'}^*) \subset H^1(X', \mathcal{O}_{X'}^*)$ (for example by the Leray spectral sequence, see Cohomology, Lemma 14.4). Hence the kernel of $\mathrm{Pic}(X) \to \mathrm{Pic}(X')$ is the cokernel of the map $\vartheta : H^0(X', \mathcal{O}_{X'}^*) \to k$. Because $k$ is algebraically closed any regular function on a reduced connected proper scheme over $k$ comes from an element of $k$, see Varieties, Lemma 9.3. Thus the final statement of the lemma. $\square$

## 13. Multicross and nodal singularities

0C1P In this section we discuss the simplest possible curve singularities.

Let $k$ be a field. Consider the complete local $k$-algebra

0C1U (13.0.1) $$A = \{(f_1, \ldots, f_n) \in k[[t]] \times \ldots \times k[[t]] \mid f_1(0) = \ldots = f_n(0)\}$$

In the language introduced in Varieties, Definition 38.4 we see that $A$ is a wedge of $n$ copies of the power series ring in 1 variable over $k$. Observe that $k[[t]] \times \ldots \times k[[t]]$

is the integral closure of $A$ in its total ring of fractions. Hence the $\delta$-invariant of $A$ is $n-1$. There is an isomorphism

$$k[[x_1, \ldots, x_n]]/(\{x_i x_j\}_{i \neq j}) \longrightarrow A$$

obtained by sending $x_i$ to $(0, \ldots, 0, t, 0, \ldots, 0)$ in $A$. It follows that $\dim(A) = 1$ and $\dim_k \mathfrak{m}/\mathfrak{m}^2 = n$. In particular, $A$ is regular if and only if $n = 1$.

0C1V **Lemma 13.1.** *Let $k$ be a separably closed field. Let $A$ be a 1-dimensional reduced Nagata local $k$-algebra with residue field $k$. Then*

$$\delta\text{-invariant } A \geq \text{number of branches of } A - 1$$

*If equality holds, then $A^\wedge$ is as in (13.0.1).*

**Proof.** Since the residue field of $A$ is separably closed, the number of branches of $A$ is equal to the number of geometric branches of $A$, see More on Algebra, Definition 87.6. The inequality holds by Varieties, Lemma 38.6. Assume equality holds. We may replace $A$ by the completion of $A$; this does not change the number of branches or the $\delta$-invariant, see More on Algebra, Lemma 88.7 and Varieties, Lemma 37.6. Then $A$ is strictly henselian, see Algebra, Lemma 148.9. By Varieties, Lemma 38.5 we see that $A$ is a wedge of complete discrete valuation rings. Each of these is isomorphic to $k[[t]]$ by Algebra, Lemma 154.10. Hence $A$ is as in (13.0.1). $\square$

0C1W **Definition 13.2.** Let $k$ be an algebraically closed field. Let $X$ be an algebraic 1-dimensional $k$-scheme. Let $x \in X$ be a closed point. We say $x$ defines a *multicross singularity* if the completion $\mathcal{O}_{X,x}^\wedge$ is isomorphic to (13.0.1) for some $n \geq 2$. We say $x$ is a *node*, or an *ordinary double point*, or *defines a nodal singularity* if $n = 2$.

These singularities are in some sense the simplest kind of singularities one can have on a curve over an algebraically closed field.

0C1X **Lemma 13.3.** *Let $k$ be an algebraically closed field. Let $X$ be a reduced algebraic 1-dimensional $k$-scheme. Let $x \in X$. The following are equivalent*

1. *$x$ defines a multicross singularity,*
2. *the $\delta$-invariant of $X$ at $x$ is the number of branches of $X$ at $x$ minus 1,*
3. *there is a sequence of morphisms $U_n \to U_{n-1} \to \ldots \to U_0 = U \subset X$ where $U$ is an open neighbourhood of $x$, where $U_n$ is nonsingular, and where each $U_i \to U_{i-1}$ is the glueing of two points as in Example 12.2.*

**Proof.** The equivalence of (1) and (2) is Lemma 13.1.

Assume (3). We will argue by descending induction on $i$ that all singularities of $U_i$ are multicross. This is true for $U_n$ as $U_n$ has no singular points. If $U_i$ is gotten from $U_{i+1}$ by glueing $a, b \in U_{i+1}$ to a point $c \in U_i$, then we see that

$$\mathcal{O}_{U_i,c}^\wedge \subset \mathcal{O}_{U_{i+1},a}^\wedge \times \mathcal{O}_{U_{i+1},b}^\wedge$$

is the set of elements having the same residue classes in $k$. Thus the number of branches at $c$ is the sum of the number of branches at $a$ and $b$, and the $\delta$-invariant at $c$ is the sum of the $\delta$-invariants at $a$ and $b$ plus 1 (because the displayed inclusion has codimension 1). This proves that (2) holds as desired.

Assume the equivalent conditions (1) and (2). We may choose an open $U \subset X$ such that $x$ is the only singular point of $U$. Then we apply Lemma 12.4 to the normalization morphism

$$U^\nu = U_n \to U_{n-1} \to \ldots \to U_1 \to U_0 = U$$

All we have to do is show that in none of the steps we are squishing a tangent vector. Suppose $U_{i+1} \to U_i$ is the smallest $i$ such that this is the squishing of a tangent vector $\theta$ at $u' \in U_{i+1}$ lying over $u \in U_i$. Arguing as above, we see that $u_i$ is a multicross singularity (because the maps $U_i \to \ldots \to U_0$ are glueing of pairs of points). But now the number of branches at $u'$ and $u$ is the same and the $\delta$-invariant of $U_i$ at $u$ is 1 bigger than the $\delta$-invariant of $U_{i+1}$ at $u'$. By Lemma 13.1 this implies that $u$ cannot be a multicross singularity which is a contradiction. □

0CDZ **Lemma 13.4.** *Let $k$ be an algebraically closed field. Let $X$ be a reduced algebraic 1-dimensional $k$-scheme. Let $x \in X$ be a multicross singularity (Definition 13.2). If $X$ is Gorenstein, then $x$ is a node.*

**Proof.** The map $\mathcal{O}_{X,x} \to \mathcal{O}_{X,x}^\wedge$ is flat and unramified in the sense that $\kappa(x) = \mathcal{O}_{X,x}^\wedge/\mathfrak{m}_x\mathcal{O}_{X,x}^\wedge$. (See More on Algebra, Section 39.) Thus $X$ is Gorenstein implies $\mathcal{O}_{X,x}$ is Gorenstein, implies $\mathcal{O}_{X,x}^\wedge$ is Gorenstein by Dualizing Complexes, Lemma 41.8. Thus it suffices to show that the ring $A$ in (13.0.1) with $n \geq 2$ is Gorenstein if and only if $n = 2$.

If $n = 2$, then $A = k[[x, y]]/(xy)$ is a complete intersection and hence Gorenstein. For example this follows from Dualizing Complexes, Lemma 41.7 applied to $k[[x, y]] \to A$ and the fact that the regular local ring $k[[x, y]]$ is Gorenstein by Dualizing Complexes, Lemma 41.4.

Assume $n > 2$. If $A$ where Gorenstein, then $A$ would be a dualizing complex over $A$ (Dualizing Complexes, Definition 41.1). Then $R\operatorname{Hom}(k, A)$ would be equal to $k[n]$ for some $n \in \mathbf{Z}$, see Dualizing Complexes, Lemma 16.12. It would follow that $\operatorname{Ext}^1_A(k, A) \cong k$ or $\operatorname{Ext}^1_A(k, A) = 0$ (depending on the value of $n$; in fact $n$ has to be $-1$ but it doesn't matter to us here). Using the exact sequence

$$0 \to \mathfrak{m}_A \to A \to k \to 0$$

we find that

$$\operatorname{Ext}^1_A(k, A) = \operatorname{Hom}_A(\mathfrak{m}_A, A)/A$$

where $A \to \operatorname{Hom}_A(\mathfrak{m}_A, A)$ is given by $a \mapsto (a' \mapsto aa')$. Let $e_i \in \operatorname{Hom}_A(\mathfrak{m}_A, A)$ be the element that sends $(f_1, \ldots, f_n) \in \mathfrak{m}_A$ to $(0, \ldots, 0, f_i, 0, \ldots, 0)$. The reader verifies easily that $e_1, \ldots, e_{n-1}$ are $k$-linearly independent in $\operatorname{Hom}_A(\mathfrak{m}_A, A)/A$. Thus $\dim_k \operatorname{Ext}^1_A(k, A) \geq n - 1 \geq 2$ which finishes the proof. (Observe that $e_1 + \ldots + e_n$ is the image of 1 under the map $A \to \operatorname{Hom}_A(\mathfrak{m}_A, A)$.) □

## 14. Torsion in the Picard group

0C1Y In this section we bound the torsion in the Picard group of a 1-dimensional proper scheme over a field. We will use this in our study of semistable reduction for curves.

There does not seem to be an elementary way to obtain the result of Lemma 14.1. Analyzing the proof there are two key ingredients: (1) there is an abelian variety classifying degree zero invertible sheaves on a smooth projective curve and (2) the structure of torsion points on an abelian variety can be determined.

0C1Z **Lemma 14.1.** *Let $k$ be an algebraically closed field. Let $X$ be a smooth projective curve of genus $g$ over $k$.*

(1) *If $n \geq 1$ is invertible in $k$, then* $\operatorname{Pic}(X)[n] \cong (\mathbf{Z}/n\mathbf{Z})^{\oplus 2g}$.

(2) *If the characteristic of $k$ is $p > 0$, then there exists an integer $0 \leq f \leq g$ such that* $\text{Pic}(X)[p^m] \cong (\mathbf{Z}/p^m\mathbf{Z})^{\oplus f}$ *for all $m \geq 1$.*

**Proof.** Let $\text{Pic}^0(X) \subset \text{Pic}(X)$ denote the subgroup of invertible sheaves of degree 0. In other words, there is a short exact sequence

$$0 \to \text{Pic}^0(X) \to \text{Pic}(X) \xrightarrow{\deg} \mathbf{Z} \to 0.$$

The group $\text{Pic}^0(X)$ is the $k$-points of the group scheme $\underline{\text{Pic}}^0_{X/k}$, see Picard Schemes of Curves, Lemma 6.7. The same lemma tells us that $\underline{\text{Pic}}^0_{X/k}$ is a $g$-dimensional abelian variety over $k$ as defined in Groupoids, Definition 9.1. Thus we conclude by the results of Groupoids, Proposition 9.11. $\square$

0CDU **Lemma 14.2.** *Let $k$ be a field. Let $n$ be prime to the characteristic of $k$. Let $X$ be a smooth proper curve over $k$ with $H^0(X, \mathcal{O}_X) = k$ and of genus $g$.*

(1) *If $g = 1$ then there exists a finite separable extension $k'/k$ such that $X_{k'}$ has a $k'$-rational point and* $\text{Pic}(X_{k'})[n] \cong (\mathbf{Z}/n\mathbf{Z})^{\oplus 2}$.
(2) *If $g \geq 2$ then there exists a finite separable extension $k'/k$ with $[k' : k] \leq (2g-2)(n^{2g})!$ such that $X_{k'}$ has a $k'$-rational point and* $\text{Pic}(X_{k'})[n] \cong (\mathbf{Z}/n\mathbf{Z})^{\oplus 2g}$.

**Proof.** Assume $g \geq 2$. First we may choose a finite separable extension of degree at most $2g-2$ such that $X$ acquires a rational point. Thus we may assume $X$ has a $k$-rational point $x \in X(k)$ but now we have to prove the lemma with $[k' : k] \leq\leq (n^{2g})!$. Let $k \subset k^{sep} \subset \overline{k}$ be a separable algebraic closure inside an algebraic closure. By Lemma 14.1 we have

$$\text{Pic}(X_{\overline{k}})[n] \cong (\mathbf{Z}/n\mathbf{Z})^{\oplus 2g}$$

By Picard Schemes of Curves, Lemma 7.2 we conclude that

$$\text{Pic}(X_{k^{sep}})[n] \cong (\mathbf{Z}/n\mathbf{Z})^{\oplus 2g}$$

By Picard Schemes of Curves, Lemma 7.2 there is a continuous action

$$\text{Gal}(k^{sep}/k) \longrightarrow \text{Aut}(\text{Pic}(X_{k^{sep}})[n]$$

and the lemma is true for the fixed field $k'$ of the kernel of this map. The kernel is open because the action is continuous which implies that $k'/k$ is finite. By Galois theory $\text{Gal}(k'/k)$ is the image of the displayed arrow. Since the permutation group of a set of cardinality $n^{2g}$ has cardinality $(n^{2g})!$ we conclude by Galois theory that $[k' : k] \leq (n^{2g})!$. (Of course this proves the lemma with the bound $|\text{GL}_{2g}(\mathbf{Z}/n\mathbf{Z})|$, but all we want here is that there is some bound.)

If the genus is 1, then there is no upper bound on the degree of a finite separable field extension over which $X$ acquires a rational point (details omitted). Still, there is such an extension for example by Varieties, Lemma 24.6. The rest of the proof is the same as in the case of $g \geq 2$. $\square$

0C20 **Proposition 14.3.** *Let $k$ be an algebraically closed field. Let $X$ be a proper scheme over $k$ which is reduced, connected, and has dimension 1. Let $g$ be the genus of $X$ and let $g_{geom}$ be the sum of the geometric genera of the irreducible components of $X$. For any prime $\ell$ different from the characteristic of $k$ we have*

$$\dim_{\mathbf{F}_\ell} \text{Pic}(X)[\ell] \leq g + g_{geom}$$

*and equality holds if and only if all the singularities of $X$ are multicross.*

**Proof.** Let $\nu : X^\nu \to X$ be the normalization (Varieties, Lemma 39.2). Choose a factorization

$$X^\nu = X_n \to X_{n-1} \to \ldots \to X_1 \to X_0 = X$$

as in Lemma 12.4. Let us denote $h_i^0 = \dim_k H^0(X_i, \mathcal{O}_{X_i})$ and $h_i^1 = \dim_k H^1(X_i, \mathcal{O}_{X_i})$. By Lemmas 12.5 and 12.6 for each $n > i \geq 0$ we have one of the following there possibilities

(1) $X_i$ is obtained by glueing $a, b \in X_{i+1}$ which are on different connected components: in this case $\text{Pic}(X_i) = \text{Pic}(X_{i+1})$, $h_{i+1}^0 = h_i^0 + 1$, $h_{i+1}^1 = h_i^1$,
(2) $X_i$ is obtained by glueing $a, b \in X_{i+1}$ which are on the same connected component: in this case there is a short exact sequence
$$0 \to k^* \to \text{Pic}(X_i) \to \text{Pic}(X_{i+1}) \to 0,$$
and $h_{i+1}^0 = h_i^0$, $h_{i+1}^1 = h_i^1 - 1$,
(3) $X_i$ is obtained by squishing a tangent vector in $X_{i+1}$: in this case there is a short exact sequence
$$0 \to (k, +) \to \text{Pic}(X_i) \to \text{Pic}(X_{i+1}) \to 0,$$
and $h_{i+1}^0 = h_i^0$, $h_{i+1}^1 = h_i^1 - 1$.

To prove the statements on dimensions of cohomology groups of the structure sheaf, use the exact sequences in Examples 12.2 and 12.3. Since $k$ is algebraically closed of characteristic prime to $\ell$ we see that $(k, +)$ and $k^*$ are $\ell$-divisible and with $\ell$-torsion $(k, +)[\ell] = 0$ and $k^*[\ell] \cong \mathbf{F}_\ell$. Hence

$$\dim_{\mathbf{F}_\ell} \text{Pic}(X_{i+1})[\ell] - \dim_{\mathbf{F}_\ell} \text{Pic}(X_i)[\ell]$$

is zero, except in case (2) where it is equal to $-1$. At the end of this process we get the normalization $X^\nu = X_n$ which is a disjoint union of smooth projective curves over $k$. Hence we have

(1) $h_n^1 = g_{geom}$ and
(2) $\dim_{\mathbf{F}_\ell} \text{Pic}(X_n)[\ell] = 2g_{geom}$.

The last equality by Lemma 14.1. Since $g = h_0^1$ we see that the number of steps of type (2) and (3) is at most $h_0^1 - h_n^1 = g - g_{geom}$. By our comptation of the differences in ranks we conclude that

$$\dim_{\mathbf{F}_\ell} \text{Pic}(X)[\ell] \leq g - g_{geom} + 2g_{geom} = g + g_{geom}$$

and equality holds if and only if no steps of type (3) occur. This indeed means that all singularities of $X$ are multicross by Lemma 13.3. Conversely, if all the singularities are multicross, then Lemma 13.3 guarantees that we can find a sequence $X^\nu = X_n \to \ldots \to X_0 = X$ as above such that no steps of type (3) occur in the sequence and we find equality holds in the lemma (just glue the local sequences for each point to find one that works for all singular points of $x$; some details omitted). □

## 15. Genus versus geometric genus

0CE0 Let $k$ be a field with algebraic closure $\overline{k}$. Let $X$ be a proper scheme of dimension $\leq 1$ over $k$. We define $g_{geom}(X/k)$ to be the sum of the geometric genera of the irreducible components of $X_{\overline{k}}$ which have dimension 1.

0CE1 **Lemma 15.1.** *Let $k$ be a field. Let $X$ be a proper scheme of dimension $\leq 1$ over $k$. Then*

$$g_{geom}(X/k) = \sum\nolimits_{C \subset X} g_{geom}(C/k)$$

*where the sum is over irreducible components $C \subset X$ of dimension 1.*

**Proof.** This is immediate from the definition and the fact that an irreducible component $\overline{Z}$ of $X_{\overline{k}}$ maps onto an irreducible component $Z$ of $X$ (Varieties, Lemma 8.10) of the same dimension (Morphisms, Lemma 27.3 applied to the generic point of $\overline{Z}$). $\square$

0CE2 **Lemma 15.2.** *Let $k$ be a field. Let $X$ be a proper scheme of dimension $\leq 1$ over $k$. Then*

(1) *We have $g_{geom}(X/k) = g_{geom}(X_{red}/k)$.*
(2) *If $X' \to X$ is a birational proper morphism, then $g_{geom}(X'/k) = g_{geom}(X/k)$.*
(3) *If $X^\nu \to X$ is the normalization morphism, then $g_{geom}(X^\nu/k) = g_{geom}(X/k)$.*

**Proof.** Part (1) is immediate from Lemma 15.1. If $X' \to X$ is proper birational, then it is finite and an isomorphism over a dense open (see Varieties, Lemmas 17.2 and 17.3). Hence $X'_{\overline{k}} \to X_{\overline{k}}$ is an isomorphism over a dense open. Thus the irreducible components of $X'_{\overline{k}}$ and $X_{\overline{k}}$ are in bijective correspondence and the corresponding components have isomorphic function fields. In particular these components have isomorphic nonsingular projective models and hence have the same geometric genera. This proves (2). Part (3) follows from (1) and (2) and the fact that $X^\nu \to X_{red}$ is birational (Morphisms, Lemma 51.7). $\square$

0CE3 **Lemma 15.3.** *Let $k$ be a field. Let $X$ be a proper scheme of dimension $\leq 1$ over $k$. Let $f : Y \to X$ be a finite morphism such that there exists a dense open $U \subset X$ over which $f$ is a closed immersion. Then*

$$\dim_k H^1(X, \mathcal{O}_X) \geq \dim_k H^1(Y, \mathcal{O}_Y)$$

**Proof.** Consider the exact sequence

$$0 \to \mathcal{G} \to \mathcal{O}_X \to f_*\mathcal{O}_Y \to \mathcal{F} \to 0$$

of coherent sheaves on $X$. By assumption $\mathcal{F}$ is supported in finitely many closed points and hence has vanishing higher cohomology (Varieties, Lemma 31.3). On the other hand, we have $H^2(X, \mathcal{G}) = 0$ by Cohomology, Proposition 21.7. It follows formally that the induced map $H^1(X, \mathcal{O}_X) \to H^1(X, f_*\mathcal{O}_Y)$ is surjective. Since $H^1(X, f_*\mathcal{O}_Y) = H^1(Y, \mathcal{O}_Y)$ (Cohomology of Schemes, Lemma 2.4) we conclude the lemma holds. $\square$

0CE4 **Lemma 15.4.** *Let $k$ be a field. Let $X$ be a proper scheme of dimension $\leq 1$ over $k$. If $X' \to X$ is a birational proper morphism, then*

$$\dim_k H^1(X, \mathcal{O}_X) \geq \dim_k H^1(X', \mathcal{O}_{X'})$$

*If $X$ is reduced, $H^0(X, \mathcal{O}_X) \to H^0(X', \mathcal{O}_{X'})$ is surjective, and equality holds, then $X' = X$.*

**Proof.** If $f : X' \to X$ is proper birational, then it is finite and an isomorphism over a dense open (see Varieties, Lemmas 17.2 and 17.3). Thus the inequality by

Lemma 15.3. Assume $X$ is reduced. Then $\mathcal{O}_X \to f_*\mathcal{O}_{X'}$ is injective and we obtain a short exact sequence

$$0 \to \mathcal{O}_X \to f_*\mathcal{O}_{X'} \to \mathcal{F} \to 0$$

Under the assumptions given in the second statement, we conclude from the long exact cohomology sequence that $H^0(X, \mathcal{F}) = 0$. Then $\mathcal{F} = 0$ because $\mathcal{F}$ is generated by global sections (Varieties, Lemma 31.3). and $\mathcal{O}_X = f_*\mathcal{O}_{X'}$. Since $f$ is affine this implies $X = X'$. $\square$

0CE5 **Lemma 15.5.** *Let $k$ be a field. Let $C$ be a proper curve over $k$. Set $\kappa = H^0(C, \mathcal{O}_C)$. Then*

$$[\kappa : k]_s \dim_\kappa H^1(C, \mathcal{O}_C) \geq g_{geom}(C/k)$$

**Proof.** Varieties, Lemma 25.2 implies $\kappa$ is a field and a finite extension of $k$. By Fields, Lemma 14.8 we have $[\kappa : k]_s = |\operatorname{Mor}_k(\kappa, \overline{k})|$ and hence $\operatorname{Spec}(\kappa \otimes_k \overline{k})$ has $[\kappa : k]_s$ points each with residue field $\overline{k}$. Thus

$$C_{\overline{k}} = \bigcup\nolimits_{t \in \operatorname{Spec}(\kappa \otimes_k \overline{k})} C_t$$

(set theoretic union). Here $C_t = C \times_{\operatorname{Spec}(\kappa), t} \operatorname{Spec}(\overline{k})$ where we view $t$ as a $k$-algebra map $t : \kappa \to \overline{k}$. The conclusion is that $g_{geom}(C/k) = \sum_t g_{geom}(C_t/\overline{k})$ and the sum is over an index set of size $[\kappa : k]_s$. We have

$$H^0(C_t, \mathcal{O}_{C_t}) = \overline{k} \quad \text{and} \quad \dim_{\overline{k}} H^1(C_t, \mathcal{O}_{C_t}) = \dim_\kappa H^1(C, \mathcal{O}_C)$$

by cohomology and base change (Cohomology of Schemes, Lemma 5.2). Observe that the normalization $C_t^\nu$ is the disjoint union of the nonsingular projective models of the irreducible components of $C_t$ (Morphisms, Lemma 51.6). Hence $\dim_{\overline{k}} H^1(C_t^\nu, \mathcal{O}_{C_t^\nu})$ is equal to $g_{geom}(C_t/\overline{k})$. By Lemma 15.3 we have

$$\dim_{\overline{k}} H^1(C_t, \mathcal{O}_{C_t}) \geq \dim_{\overline{k}} H^1(C_t^\nu, \mathcal{O}_{C_t^\nu})$$

and this finishes the proof. $\square$

0CE6 **Lemma 15.6.** *Let $k$ be a field. Let $X$ be a proper scheme of dimension $\leq 1$ over $k$. Let $\ell$ be a prime number invertible in $k$. Then*

$$\dim_{\mathbf{F}_\ell} \operatorname{Pic}(X)[\ell] \leq \dim_k H^1(X, \mathcal{O}_X) + g_{geom}(X/k)$$

*where $g_{geom}(X/k)$ is as defined above.*

**Proof.** The map $\operatorname{Pic}(X) \to \operatorname{Pic}(X_{\overline{k}})$ is injective by Varieties, Lemma 29.3. By Cohomology of Schemes, Lemma 5.2 $\dim_k H^1(X, \mathcal{O}_X)$ equals $\dim_{\overline{k}} H^1(X_{\overline{k}}, \mathcal{O}_{X_{\overline{k}}})$. Hence we may assume $k$ is algebraically closed.

Let $X_{red}$ be the reduction of $X$. Then the surjection $\mathcal{O}_X \to \mathcal{O}_{X_{red}}$ induces a surjection $H^1(X, \mathcal{O}_X) \to H^1(X, \mathcal{O}_{X_{red}})$ because cohomology of quasi-coherent sheaves vanishes in degrees $\geq 2$ by Cohomology, Proposition 21.7. Since $X_{red} \to X$ induces an isomorphism on irreducible components over $\overline{k}$ and an isomorphism on $\ell$-torsion in Picard groups (Picard Schemes of Curves, Lemma 7.2) we may replace $X$ by $X_{red}$. In this way we reduce to Proposition 14.3. $\square$

## 16. Nodal curves

0C46 We have already defined ordinary double points over algebraically closed fields, see Definition 13.2. Namely, if $x \in X$ is a closed point of a 1-dimensional algebraic scheme over an algebraically closed field $k$, then $x$ is an ordinary double point if and only if

$$\mathcal{O}^\wedge_{X,x} \cong k[[x, y]]/(xy)$$

See discussion following (13.0.1) in Section 13.

0C47 **Definition 16.1.** Let $k$ be a field. Let $X$ be a 1-dimensional locally algebraic $k$-scheme.

(1) We say a closed point $x \in X$ is a *node*, or an *ordinary double point*, or *defines a nodal singularity* if there exists an ordinary double point $\overline{x} \in X_{\overline{k}}$ mapping to $x$.
(2) We say the *singularities of $X$ are at-worst-nodal* if all closed points of $X$ are either in the smooth locus of the structure morphism $X \to \mathrm{Spec}(k)$ or are ordinary double points.

Often a 1-dimensional algebraic scheme $X$ is called a *nodal curve* if the singularities of $X$ are at worst nodal. Sometimes a nodal curve is required to be proper. Since a nodal curve so defined need not be irreducible, this conflicts with our earlier definition of a curve as a variety of dimension 1.

0C48 **Lemma 16.2.** *Let $(A, \mathfrak{m})$ be a regular local ring of dimension 2. Let $I \subset \mathfrak{m}$ be an ideal.*

(1) *If $A/I$ is reduced, then $I = (0)$, $I = \mathfrak{m}$, or $I = (f)$ for some nonzero $f \in \mathfrak{m}$.*
(2) *If $A/I$ has depth 1, then $I = (f)$ for some nonzero $f \in \mathfrak{m}$.*

**Proof.** Assume $I \neq 0$. Write $I = (f_1, \ldots, f_r)$. As $A$ is a UFD (More on Algebra, Lemma 94.7) we can write $f_i = fg_i$ where $f$ is the gcd of $f_1, \ldots, f_r$. Thus the gcd of $g_1, \ldots, g_r$ is 1 which means that there is no height 1 prime ideal over $g_1, \ldots, g_r$. Since $\dim(A) = 2$ this implies that $V(g_1, \ldots, g_r) = \{\mathfrak{m}\}$, i.e., $\mathfrak{m} = \sqrt{(g_1, \ldots, g_r)}$.

Assume $A/I$ reduced, i.e., $I$ radical. If $f$ is a unit, then since $I$ is radical we see that $I = \mathfrak{m}$. If $f \in \mathfrak{m}$, then we see that $f^n$ maps to zero in $A/I$. Hence $f \in I$ by reducedness and we conclude $I = (f)$.

Assume $A/I$ has depth 1. Then $\mathfrak{m}$ is not an associated prime of $A/I$. Since the class of $f$ modulo $I$ is annihilated by $g_1, \ldots, g_r$, this implies that the class of $f$ is zero in $A/I$. Thus $I = (f)$ as desired. $\square$

Let $\kappa$ be a field and let $V$ be a vector space over $\kappa$. We will say $q \in \mathrm{Sym}^2_\kappa(V)$ is *nondegenerate* if the induced $\kappa$-linear map $V^\vee \to V$ is an isomorphism. If $q = \sum_{i \leq j} a_{ij} x_i x_j$ for some $\kappa$-basis $x_1, \ldots, x_n$ of $V$, then this means that the determinant of the matrix

$$\begin{pmatrix} 2a_{11} & a_{12} & \ldots \\ a_{12} & 2a_{22} & \ldots \\ \ldots & \ldots & \ldots \end{pmatrix}$$

is nonzero. This is equivalent to the condition that the partial derivatives of $q$ with respect to the $x_i$ cut out 0 scheme theoretically.

0C49 **Lemma 16.3.** *Let $k$ be a field. Let $(A, \mathfrak{m}, \kappa)$ be a Noetherian local $k$-algebra. The following are equivalent*

(1) *$\kappa/k$ is separable, $A$ is reduced,* $\dim_\kappa(\mathfrak m/\mathfrak m^2) = 2$*, and there exists a nondegenerate $q \in Sym^2_\kappa(\mathfrak m/\mathfrak m^2)$ which maps to zero in $\mathfrak m^2/\mathfrak m^3$,*
(2) *$\kappa/k$ is separable, depth$(A) = 1$,* $\dim_\kappa(\mathfrak m/\mathfrak m^2) = 2$*, and there exists a nondegenerate $q \in Sym^2_\kappa(\mathfrak m/\mathfrak m^2)$ which maps to zero in $\mathfrak m^2/\mathfrak m^3$,*
(3) *$\kappa/k$ is separable, $A^\wedge \cong \kappa[[x, y]]/(ax^2 + bxy + cy^2)$ as a $k$-algebra where $ax^2 + bxy + cy^2$ is a nondegenerate quadratic form over $\kappa$.*

**Proof.** Assume (3). Then $A^\wedge$ is reduced because $ax^2+bxy+cy^2$ is either irreducible or a product of two nonassociated prime elements. Hence $A \subset A^\wedge$ is reduced. It follows that (1) is true.

Assume (1). Then $A$ cannot be Artinian, since it would not be reduced because $\mathfrak m \neq (0)$. Hence $\dim(A) \geq 1$, hence $\text{depth}(A) \geq 1$ by Algebra, Lemma 151.3. On the other hand $\dim(A) = 2$ implies $A$ is regular which contradicts the existence of $q$ by Algebra, Lemma 105.1. Thus $\dim(A) \leq 1$ and we conclude $\text{depth}(A) = 1$ by Algebra, Lemma 71.3. It follows that (2) is true.

Assume (2). Since the depth of $A$ is the same as the depth of $A^\wedge$ (More on Algebra, Lemma 39.2) and since the other conditions are insensitive to completion, we may assume that $A$ is complete. Choose $\kappa \to A$ as in More on Algebra, Lemma 34.11. Since $\dim_\kappa(\mathfrak m/\mathfrak m^2) = 2$ we can choose $x_0, y_0 \in \mathfrak m$ which map to a basis. We obtain a continuous $\kappa$-algebra map

$$\kappa[[x, y]] \longrightarrow A$$

by the rules $x \mapsto x_0$ and $y \mapsto y_0$. Let $q$ be the class of $ax_0^2 + bx_0y_0 + cy_0^2$ in $\text{Sym}^2_\kappa(\mathfrak m/\mathfrak m^2)$. Write $Q(x, y) = ax^2 + bxy + cy^2$ viewed as a polynomial in two variables. Then we see that

$$Q(x_0, y_0) = ax_0^2 + bx_0y_0 + cy_0^2 = \sum\nolimits_{i+j=3} a_{ij}x_0^iy_0^j$$

for some $a_{ij}$ in $A$. We want to prove that we can increase the order of vanishing by changing our choice of $x_0$, $y_0$. Suppose that $x_1, y_1 \in \mathfrak m^2$. Then

$$Q(x_0 + x_1, y_0 + y_1) = Q(x_0, y_0) + (2ax_0 + by_0)x_1 + (bx_0 + 2cy_0)y_1 \bmod \mathfrak m^4$$

Nondegeneracy of $Q$ means exactly that $2ax_0 + by_0$ and $bx_0 + 2cy_0$ are a $\kappa$-basis for $\mathfrak m/\mathfrak m^2$, see discussion preceding the lemma. Hence we can certainly choose $x_1, y_1 \in \mathfrak m^2$ such that $Q(x_0 + x_1, y_0 + y_1) \in \mathfrak m^4$. Continuing in this fashion by induction we can find $x_i, y_i \in \mathfrak m^{i+1}$ such that

$$Q(x_0 + x_1 + \ldots + x_n, y_0 + y_1 + \ldots + y_n) \in \mathfrak m^{n+3}$$

Since $A$ is complete we can set $x_\infty = \sum x_i$ and $y_\infty = \sum y_i$ and we can consider the map $\kappa[[x, y]] \longrightarrow A$ sending $x$ to $x_\infty$ and $y$ to $y_\infty$. This map induces a surjection $\kappa[[x, y]]/(Q) \longrightarrow A$ by Algebra, Lemma 95.1. By Lemma 16.2 the kernel of $k[[x, y]] \to A$ is principal. But the kernel cannot contain a proper divisor of $Q$ as such a divisor would have degree 1 in $x, y$ and this would contradict $\dim(\mathfrak m/\mathfrak m^2) = 2$. Hence $Q$ generates the kernel as desired. $\square$

0C4A **Lemma 16.4.** *Let $k$ be a field. Let $(A, \mathfrak m, \kappa)$ be a Nagata local $k$-algebra. The following are equivalent*

(1) *$k \to A$ is as in Lemma 16.3,*
(2) *$\kappa/k$ is separable, $A$ is reduced of dimension 1, the $\delta$-invariant of $A$ is 1, and $A$ has 2 geometric branches.*

*If this holds, then the integral closure $A'$ of $A$ in its total ring of fractions has either* 1 *or* 2 *maximal ideals $\mathfrak{m}'$ and the extensions $\kappa(\mathfrak{m}')/k$ are separable.*

**Proof.** In both cases $A$ and $A^\wedge$ are reduced. In case (2) because the completion of a reduced local Nagata ring is reduced (More on Algebra, Lemma 39.6). In both cases $A$ and $A^\wedge$ have dimension 1 (More on Algebra, Lemma 39.1). The $\delta$-invariant and the number of geometric branches of $A$ and $A^\wedge$ agree by Varieties, Lemma 37.6 and More on Algebra, Lemma 88.7. Let $A'$ be the integral closure of $A$ in its total ring of fractions as in Varieties, Lemma 37.2. By Varieties, Lemma 37.5 we see that $A' \otimes_A A^\wedge$ plays the same role for $A^\wedge$. Thus we may replace $A$ by $A^\wedge$ and assume $A$ is complete.

Assume (1) holds. It suffices to show that $A$ has two geometric branches and $\delta$-invariant 1. We may assume $A = \kappa[[x, y]]/(ax^2 + bxy + cy^2)$ with $q = ax^2 + bxy + cy^2$ nondegenerate. There are two cases.

Case I: $q$ splits over $\kappa$. In this case we may after changing coordinates assume that $q = xy$. Then we see that

$$A' = \kappa[[x, y]]/(x) \times \kappa[[x, y]]/(y)$$

Case II: $q$ does not split. In this case $c \neq 0$ and nondegenerate means $b^2 - 4ac \neq 0$. Hence $\kappa' = \kappa[t]/(a + bt + ct^2)$ is a degree 2 separable extension of $\kappa$. Then $t = y/x$ is integral over $A$ and we conclude that

$$A' = \kappa'[[x]]$$

with $y$ mapping to $tx$ on the right hand side.

In both cases one verifies by hand that the $\delta$-invariant is 1 and the number of geometric branches is 2. In this way we see that (1) implies (2). Moreover we conclude that the final statement of the lemma holds.

Assume (2) holds. More on Algebra, Lemma 87.7 implies $A'$ either has two maximal ideals or $A'$ has one maximal ideal and $[\kappa(\mathfrak{m}') : \kappa]_s = 2$.

Case I: $A'$ has two maximal ideals $\mathfrak{m}'_1$, $\mathfrak{m}'_2$ with residue fields $\kappa_1$, $\kappa_2$. Since the $\delta$-invariant is the length of $A'/A$ and since there is a surjection $A'/A \to (\kappa_1 \times \kappa_2)/\kappa$ we see that $\kappa = \kappa_1 = \kappa_2$. Since $A$ is complete (and henselian by Algebra, Lemma 148.9) and $A'$ is finite over $A$ we see that $A' = A_1 \times A_2$ (by Algebra, Lemma 148.4). Since $A'$ is a normal ring it follows that $A_1$ and $A_2$ are discrete valuation rings. Hence $A_1$ and $A_2$ are isomorphic to $\kappa[[t]]$ (as $k$-algebras) by More on Algebra, Lemma 34.12. Since the $\delta$-invariant is 1 we conclude that $A$ is the wedge of $A_1$ and $A_2$ (Varieties, Definition 38.4). It follows easily that $A \cong \kappa[[x, y]]/(xy)$.

Case II: $A'$ has a single maximal ideal $\mathfrak{m}'$ with residue field $\kappa'$ and $[\kappa' : \kappa]_s = 2$. Arguing exactly as in Case I we see that $[\kappa' : \kappa] = 2$ and $\kappa'$ is separable over $\kappa$. Since $A'$ is normal we see that $A'$ is isomorphic to $\kappa'[[t]]$ (see reference above). Since $A'/A$ has length 1 we conclude that

$$A = \{f \in \kappa'[[t]] \mid f(0) \in \kappa\}$$

Then a simple computation shows that $A$ as in case (1). $\square$

0C4B **Lemma 16.5.** *Let $k$ be a field. Let $A = k[[x_1, \ldots, x_n]]$. Let $I = (f_1, \ldots, f_m) \subset A$ be an ideal. For any $r \geq 0$ the ideal in $A/I$ generated by the $r \times r$-minors of the matrix $(\partial f_j/\partial x_i)$ is independent of the choice of the generators of $I$ or the regular system of parameters $x_1, \ldots, x_n$ of $A$.*

**Proof.** The "correct" proof of this lemma is to prove that this ideal is the $(n-r)$th Fitting ideal of a module of continuous differentials of $A/I$ over $k$. Here is a direct proof. If $g_1, \ldots g_l$ is a second set of generators of $I$, then we can write $g_s = \sum a_{sj} f_j$ and we have the equality of matrices

$$(\partial g_s/\partial x_i) = (a_{sj})(\partial f_j/\partial x_i) + (\partial a_{sj}/\partial x_i f_j)$$

The final term is zero in $A/I$. By the Cauchy-Binet formula we see that the ideal of minors for the $g_s$ is contained in the ideal for the $f_j$. By symmetry these ideals are the same. If $y_1, \ldots, y_n \in \mathfrak{m}_A$ is a second regular system of parameters, then the matrix $(\partial y_j/\partial x_i)$ is invertible and we can use the chain rule for differentiation. Some details omitted. $\square$

0C4C **Lemma 16.6.** *Let $k$ be a field. Let $A = k[[x_1, \ldots, x_n]]$. Let $I = (f_1, \ldots, f_m) \subset \mathfrak{m}_A$ be an ideal. The following are equivalent*

1. *$k \to A/I$ is as in Lemma 16.3,*
2. *$A/I$ is reduced and the $(n-1) \times (n-1)$ minors of the matrix $(\partial f_j/\partial x_i)$ generate $I + \mathfrak{m}_A$,*
3. *depth$(A/I) = 1$ and the $(n-1) \times (n-1)$ minors of the matrix $(\partial f_j/\partial x_i)$ generate $I + \mathfrak{m}_A$.*

**Proof.** By Lemma 16.5 we may change our system of coordinates and the choice of generators during the proof.

If (1) holds, then we may change coordinates such that $x_1, \ldots, x_{n-2}$ map to zero in $A/I$ and $A/I = k[[x_{n-1}, x_n]]/(ax_{n-1}^2 + bx_{n-1}x_n + cx_n^2)$ for some nondegenerate quadric $ax_{n-1}^2 + bx_{n-1}x_n + cx_n^2$. Then we can explicitly compute to show that both (2) and (3) are true.

Assume the $(n-1) \times (n-1)$ minors of the matrix $(\partial f_j/\partial x_i)$ generate $I + \mathfrak{m}_A$. Suppose that for some $i$ and $j$ the partial derivative $\partial f_j/\partial x_i$ is a unit in $A$. Then we may use the system of parameters $f_j, x_1, \ldots, x_{i-1}, \hat{x}_i, x_{i+1}, \ldots, x_n$ and the generators $f_j, f_1, \ldots, f_{j-1}, \hat{f}_j, f_{j+1}, \ldots, f_m$ of $I$. Then we get a regular system of parameters $x_1, \ldots, x_n$ and generators $x_1, f_2, \ldots, f_m$ of $I$. Next, we look for an $i \geq 2$ and $j \geq 2$ such that $\partial f_j/\partial x_i$ is a unit in $A$. If such a pair exists, then we can make a replacement as above and assume that we have a regular system of parameters $x_1, \ldots, x_n$ and generators $x_1, x_2, f_3, \ldots, f_m$ of $I$. Continuing, in finitely many steps we reach the situation where we have a regular system of parameters $x_1, \ldots, x_n$ and generators $x_1, \ldots, x_t, f_{t+1}, \ldots, f_m$ of $I$ such that $\partial f_j/\partial x_i \in \mathfrak{m}_A$ for all $i, j \geq t + 1$.

In this case the matrix of partial derivatives has the following block shape

$$\begin{pmatrix} I_{t \times t} & * \\ 0 & \mathfrak{m}_A \end{pmatrix}$$

Hence every $(n-1) \times (n-1)$-minor is in $\mathfrak{m}_A^{n-1-t}$. Note that $I \neq \mathfrak{m}_A$ otherwise the ideal of minors would contain 1. It follows that $n - 1 - t \leq 1$ because there is an element of $\mathfrak{m}_A \setminus \mathfrak{m}_A^2 + I$ (otherwise $I = \mathfrak{m}_A$ by Nakayama). Thus $t \geq n - 2$. We have seen that $t \neq n$ above and similarly if $t = n - 1$, then there is an invertible

$(n-1) \times (n-1)$-minor which is disallowed as well. Hence $t = n-2$. Then $A/I$ is a quotient of $k[[x_{n-1}, x_n]]$ and Lemma 16.2 implies in both cases (2) and (3) that $I$ is generated by $x_1, \ldots, x_{n-2}, f$ for some $f = f(x_{n-1}, x_n)$. In this case the condition on the minors exactly says that the quadratic term in $f$ is nondegenerate, i.e., $A/I$ is as in Lemma 16.3. □

0C4D **Lemma 16.7.** *Let $k$ be a field. Let $X$ be a 1-dimensional algebraic $k$-scheme. Let $x \in X$ be a closed point. The following are equivalent*

1. *$x$ is a node,*
2. *$k \to \mathcal{O}_{X,x}$ is as in Lemma 16.3,*
3. *any $\overline{x} \in X_{\overline{k}}$ mapping to $x$ defines a nodal singularity,*
4. *$\kappa(x)/k$ is separable, $\mathcal{O}_{X,x}$ is reduced, and the first Fitting ideal of $\Omega_{X/k}$ generates $\mathfrak{m}_x$ in $\mathcal{O}_{X,x}$,*
5. *$\kappa(x)/k$ is separable, depth$(\mathcal{O}_{X,x}) = 1$, and the first Fitting ideal of $\Omega_{X/k}$ generates $\mathfrak{m}_x$ in $\mathcal{O}_{X,x}$,*
6. *$\kappa(x)/k$ is separable and $\mathcal{O}_{X,x}$ is reduced, has $\delta$-invariant 1, and has 2 geometric branches.*

**Proof.** First assume that $k$ is algebraically closed. In this case the equivalence of (1) and (3) is trivial. The equivalence of (1) and (3) with (2) holds because the only nondegenerate quadric in two variables is $xy$ up to change in coordinates. The equivalence of (1) and (6) is Lemma 13.1. After replacing $X$ by an affine neighbourhood of $x$, we may assume there is a closed immersion $X \to \mathbf{A}^n_k$ mapping $x$ to 0. Let $f_1, \ldots, f_m \in k[x_1, \ldots, x_n]$ be generators for the ideal $I$ of $X$ in $\mathbf{A}^n_k$. Then $\Omega_{X/k}$ corresponds to the $R = k[x_1, \ldots, x_n]/I$-module $\Omega_{R/k}$ which has a presentation

$$R^{\oplus m} \xrightarrow{(\partial f_j/\partial x_i)} R^{\oplus n} \to \Omega_{R/k} \to 0$$

(See Algebra, Sections 130 and 132.) The first Fitting ideal of $\Omega_{R/k}$ is thus the ideal generated by the $(n-1) \times (n-1)$-minors of the matrix $(\partial f_j/\partial x_i)$. Hence (2), (4), (5) are equivalent by Lemma 16.6 applied to the completion of $k[x_1, \ldots, x_n] \to R$ at the maximal ideal $(x_1, \ldots, x_n)$.

Now assume $k$ is an arbitrary field. In cases (2), (4), (5), (6) the residue field $\kappa(x)$ is separable over $k$. Let us show this holds as well in cases (1) and (3). Namely, let $Z \subset X$ be the closed subscheme of $X$ defined by the first Fitting ideal of $\Omega_{X/k}$. The formation of $Z$ commutes with field extension (Divisors, Lemma 10.1). If (1) or (3) is true, then there exists a point $\overline{x}$ of $X_{\overline{k}}$ such that $\overline{x}$ is an isolated point of multiplicity 1 of $Z_{\overline{k}}$ (as we have the equivalence of the conditions of the lemma over $\overline{k}$). In particular $Z_{\overline{x}}$ is geometrically reduced at $\overline{x}$ (because $\overline{k}$ is algebraically closed). Hence $Z$ is geometrically reduced at $x$ (Varieties, Lemma 6.6). In particular, $Z$ is reduced at $x$, hence $Z = \text{Spec}(\kappa(x))$ in a neighbourhood of $x$ and $\kappa(x)$ is geometrically reduced over $k$. This means that $\kappa(x)/k$ is separable (Algebra, Lemma 43.1).

The argument of the previous paragraph shows that if (1) or (3) holds, then the first Fitting ideal of $\Omega_{X/k}$ generates $\mathfrak{m}_x$. Since $\mathcal{O}_{X,x} \to \mathcal{O}_{X_{\overline{k}},\overline{x}}$ is flat and since $\mathcal{O}_{X_{\overline{k}},\overline{x}}$ is reduced and has depth 1, we see that (4) and (5) hold (use Algebra, Lemmas 158.2 and 157.2). Conversely, (4) implies (5) by Algebra, Lemma 151.3. If (5) holds, then $Z$ is geometrically reduced at $x$ (because $\kappa(x)/k$ separable and $Z$ is $x$ in a neighbourhood). Hence $Z_{\overline{k}}$ is reduced at any point $\overline{x}$ of $X_{\overline{k}}$ lying over $x$. In other words, the first fitting ideal of $\Omega_{X_{\overline{k}}/\overline{k}}$ generates $\mathfrak m_{\overline{x}}$ in $\mathcal{O}_{X_{\overline{k}},\overline{x}}$. Moreover, since $\mathcal{O}_{X,x} \to \mathcal{O}_{X_{\overline{k}},\overline{x}}$ is flat we see that $\text{depth}(\mathcal{O}_{X_{\overline{k}},\overline{x}}) = 1$ (see reference above). Hence (5) holds for $\overline{x} \in X_{\overline{k}}$ and we conclude that (3) holds (because of the equivalence over algebraically closed fields). In this way we see that (1), (3), (4), (5) are equivalent.

The equivalence of (2) and (6) follows from Lemma 16.4.

Finally, we prove the equivalence of (2) = (6) with (1) = (3) = (4) = (5). First we note that the geometric number of branches of $X$ at $x$ and the geometric number of branches of $X_{\overline{k}}$ at $\overline{x}$ are equal by Varieties, Lemma 38.2. We conclude from the information available to us at this point that in all cases this number is equal to 2. On the other hand, in case (1) it is clear that $X$ is geometrically reduced at $x$, and hence

$$\delta\text{-invariant of } X \text{ at } x \leq \delta\text{-invariant of } X_{\overline{k}} \text{ at } \overline{x}$$

by Varieties, Lemma 37.8. Since in case (1) the right hand side is 1, this forces the $\delta$-invariant of $X$ at $x$ to be 1 (because if it were zero, then $\mathcal{O}_{X,x}$ would be a discrete valuation ring by Varieties, Lemma 37.4 which is unibranch, a contradiction). Thus (5) holds. Conversely, if (2) = (5) is true, then assumptions (a), (b), (c) of Varieties, Lemma 26.5 hold for $x \in X$ by Lemma 16.4. Thus Varieties, Lemma 37.9 applies and shows that we have equality in the above displayed inequality. We conclude that (5) holds for $\overline{x} \in X_{\overline{k}}$ and we are back in case (1) by the equivalence of the conditions over an algebraically closed field. ☐

0CBT **Remark 16.8** (The quadratic extension associated to a node)**.** Let $k$ be a field. Let $(A, \mathfrak m, \kappa)$ be a Noetherian local $k$-algebra. Assume that either $(A, \mathfrak m, \kappa)$ is as in Lemma 16.3, or $A$ is Nagata as in Lemma 16.4, or $A$ is complete and as in Lemma 16.6. Then $A$ defines canonically a degree 2 separable $\kappa$-algebra $\kappa'$ as follows

(1) let $q = ax^2 + bxy + cy^2$ be a nondegenerate quadric as in Lemma 16.3 with coordinates $x, y$ chosen such that $a \neq 0$ and set $\kappa' = \kappa[x]/(ax^2 + bx + c)$,
(2) let $A' \supset A$ be the integral closure of $A$ in its total ring of fractions and set $\kappa' = A'/\mathfrak m A'$, or
(3) let $\kappa'$ be the $\kappa$-algebra such that $\text{Proj}(\bigoplus_{n \geq 0} \mathfrak m^n/\mathfrak m^{n+1}) = \text{Spec}(\kappa')$.

The equivalence of (1) and (2) was shown in the proof of Lemma 16.4. We omit the equivalence of this with (3). If $X$ is a locally Noetherian $k$-scheme and $x \in X$ is a point such that $\mathcal{O}_{X,x} = A$, then (3) shows that $\text{Spec}(\kappa') = X^\nu \times_X \text{Spec}(\kappa)$ where $\nu : X^\nu \to X$ is the normalization morphism.

0CBU **Remark 16.9** (Trivial quadratic extension)**.** Let $k$ be a field. Let $(A, \mathfrak m, \kappa)$ be as in Remark 16.8 and let $\kappa'/\kappa$ be the associated separable algebra of degree 2. Then the following are equivalent

(1) $\kappa' \cong \kappa \times \kappa$ as $\kappa$-algebra,
(2) the form $q$ of Lemma 16.3 can be chosen to be $xy$,
(3) $A$ has two branches,
(4) the extension $A'/A$ of Lemma 16.4 has two maximal ideals, and
(5) $A^\wedge \cong \kappa[[x, y]]/(xy)$ as a $k$-algebra.

The equivalence between these conditions has been shown in the proof of Lemma 16.4. If $X$ is a locally Noetherian $k$-scheme and $x \in X$ is a point such that $\mathcal{O}_{X,x} = A$, then this means exactly that there are two points $x_1, x_2$ of the normalization $X^\nu$ lying over $x$ and that $\kappa(x) = \kappa(x_1) = \kappa(x_2)$.

0CBV **Definition 16.10.** Let $k$ be a field. Let $X$ be a 1-dimensional algebraic $k$-scheme. Let $x \in X$ be a closed point. We say $x$ is a *split node* if $x$ is a node, $\kappa(x) = k$, and the equivalent assertions of Remark 16.9 hold for $A = \mathcal{O}_{X,x}$.

We formulate the obligatory lemma stating what we already know about this concept.

0CBW **Lemma 16.11.** *Let $k$ be a field. Let $X$ be a 1-dimensional algebraic $k$-scheme. Let $x \in X$ be a closed point. The following are equivalent*

(1) *$x$ is a split node,*
(2) *$x$ is a node and there are exactly two points $x_1, x_2$ of the normalization $X^\nu$ lying over $x$ with $k = \kappa(x_1) = \kappa(x_2)$,*
(3) *$\mathcal{O}^\wedge_{X,x} \cong k[[x, y]]/(xy)$ as a $k$-algebra, and*
(4) *add more here.*

**Proof.** This follows from the discussion in Remark 16.9 and Lemma 16.7. □

0C56 **Lemma 16.12.** *Let $K/k$ be an extension of fields. Let $X$ be a locally algebraic $k$-scheme of dimension 1. Let $y \in X_K$ be a point with image $x \in X$. The following are equivalent*

(1) *$x$ is a closed point of $X$ and a node, and*
(2) *$y$ is a closed point of $Y$ and a node.*

**Proof.** If $x$ is a closed point of $X$, then $y$ is too (look at residue fields). But conversely, this need not be the case, i.e., it can happen that a closed point of $Y$ maps to a nonclosed point of $X$. However, in this case $y$ cannot be a node. Namely, then $X$ would be geometrically unibranch at $x$ (because $x$ would be a generic point of $X$ and $\mathcal{O}_{X,x}$ would be Artinian and any Artinian local ring is geometrically unibranch), hence $Y$ is geometrically unibranch at $y$ (Varieties, Lemma 38.3), which means that $y$ cannot be a node by Lemma 16.7. Thus we may and do assume that both $x$ and $y$ are closed points.

Choose algebraic closures $\overline{k}$, $\overline{K}$ and a map $\overline{k} \to \overline{K}$ extending the given map $k \to K$. Using the equivalence of (1) and (3) in Lemma 16.7 we reduce to the case where $k$ and $K$ are algebraically closed. In this case we can argue as in the proof of Lemma 16.7 that the geometric number of branches and $\delta$-invariants of $X$ at $x$ and $Y$ at $y$ are the same. Another argument can be given by choosing an isomorphism $k[[x_1, \ldots, x_n]]/(g_1, \ldots, g_m) \to \mathcal{O}^\wedge_{X,x}$ of $k$-algebras as in Varieties, Lemma 21.1. By Varieties, Lemma 21.2 this gives an isomorphism $K[[x_1, \ldots, x_n]]/(g_1, \ldots, g_m) \to \mathcal{O}^\wedge_{Y,y}$ of $K$-algebras. By definition we have to show that

$$k[[x_1, \ldots, x_n]]/(g_1, \ldots, g_m) \cong k[[s, t]]/(st)$$

if and only if

$$K[[x_1, \ldots, x_n]]/(g_1, \ldots, g_m) \cong K[[s, t]]/(st)$$

We encourage the reader to prove this for themselves. Since $k$ and $K$ are algebraically closed fields, this is the same as asking these rings to be as in Lemma 16.3. Via Lemma 16.6 this translates into a statement about the $(n-1) \times (n-1)$-minors of the matrix $(\partial g_j/\partial x_i)$ which is clearly independent of the field used. We omit the details. □

0C57 **Lemma 16.13.** *Let $k$ be a field. Let $X$ be a locally algebraic $k$-scheme of dimension $1$. Let $Y \to X$ be an étale morphism. Let $y \in Y$ be a point with image $x \in X$. The following are equivalent*

1. *$x$ is a closed point of $X$ and a node, and*
2. *$y$ is a closed point of $Y$ and a node.*

**Proof.** By Lemma 16.12 we may base change to the algebraic closure of $k$. Then the residue fields of $x$ and $y$ are $k$. Hence the map $\mathcal{O}_{X,x}^\wedge \to \mathcal{O}_{Y,y}$ is an isomorphism (for example by Étale Morphisms, Lemma 11.3 or More on Algebra, Lemma 39.8). Thus the lemma is clear. $\square$

0CD6 **Lemma 16.14.** *Let $k'/k$ be a finite separable field extension. Let $X$ be a locally algebraic $k'$-scheme of dimension $1$. Let $x \in X$ be a closed point. The following are equivalent*

1. *$x$ is a node, and*
2. *$x$ is a node when $X$ viewed as a locally algebraic $k$-scheme.*

**Proof.** Follows immediately from the characterization of nodes in Lemma 16.7. $\square$

0C4E **Lemma 16.15.** *Let $k$ be a field. Let $X$ be a locally algebraic $k$-scheme equidimensional of dimension $1$. The following are equivalent*

1. *the singularities of $X$ are at-worst-nodal, and*
2. *$X$ is a local complete intersection over $k$ and the closed subscheme $Z \subset X$ cut out by the first fitting ideal of $\Omega_{X/k}$ is unramified over $k$.*

**Proof.** We urge the reader to find their own proof of this lemma; what follows is just putting together earlier results and may hide what is really going on.

Assume (2). Since $Z \to \mathrm{Spec}(k)$ is quasi-finite (Morphisms, Lemma 33.10) we see that the residue fields of points $x \in Z$ are finite over $k$ (as well as separable) by Morphisms, Lemma 19.5. Hence each $x \in Z$ is a closed point of $X$ by Morphisms, Lemma 19.2. The local ring $\mathcal{O}_{X,x}$ is Cohen-Macaulay by Algebra, Lemma 133.3. Since $\dim(\mathcal{O}_{X,x}) = 1$ by dimension theory (Varieties, Section 20), we conclude that $\text{depth}(\mathcal{O}_{X,x})) = 1$. Thus $x$ is a node by Lemma 16.7. If $x \in X$, $x \not\in Z$, then $X \to \mathrm{Spec}(k)$ is smooth at $x$ by Divisors, Lemma 10.3.

Assume (1). Under this assumption $X$ is geometrically reduced at every closed point (see Varieties, Lemma 6.6). Hence $X \to \mathrm{Spec}(k)$ is smooth on a dense open by Varieties, Lemma 24.7. Thus $Z$ is closed and consists of closed points. By Divisors, Lemma 10.3 the morphism $X \setminus Z \to \mathrm{Spec}(k)$ is smooth. Hence $X \setminus Z$ is a local complete intersection by Morphisms, Lemma 32.7 and the definition of a local complete intersection in Morphisms, Definition 29.1. By Lemma 16.7 for every point $x \in Z$ the local ring $\mathcal{O}_{Z,x}$ is equal to $\kappa(x)$ and $\kappa(x)$ is separable over $k$. Thus $Z \to \mathrm{Spec}(k)$ is unramified (Morphisms, Lemma 33.11). Finally, Lemma 16.7 via part (3) of Lemma 16.3, shows that $\mathcal{O}_{X,x}$ is a complete intersection in the sense of Divided Power Algebra, Definition 8.5. However, Divided Power Algebra, Lemma 8.8 and Morphisms, Lemma 29.9 show that this agrees with the notion used to define a local complete intersection scheme over a field and the proof is complete. $\square$

## 17. Families of nodal curves

0C58 In the Stacks project curves are irreducible varieties of dimension 1, but in the literature a "semi-stable curve" or a "nodal curve" is usually not irreducible and often assumed to be proper, especially when used in a phrase such as "family of semistable curves" or "family of nodal curves", or "nodal family". Thus it is a bit difficult for us to choose a terminology which is consistent with the literature as well as internally consistent. Moreover, we really want to first study the notion introduced in the following lemma (which is local on the source).

0C59 **Lemma 17.1.** *Let $f : X \to S$ be a morphism of schemes. The following are equivalent*

1. *$f$ is flat, locally of finite presentation, every nonempty fibre $X_s$ is equidimensional of dimension 1, and $X_s$ has at-worst-nodal singularities, and*
2. *$f$ is syntomic of relative dimension 1 and the closed subscheme $Sing(f) \subset X$ defined by the first Fitting ideal of $\Omega_{X/S}$ is unramified over $S$.*

**Proof.** Recall that the formation of $\text{Sing}(f)$ commutes with base change, see Divisors, Lemma 10.1. Thus the lemma follows from Lemma 16.15, Morphisms, Lemma 29.11, and Morphisms, Lemma 33.12. (We also use the trivial Morphisms, Lemmas 29.6 and 29.7.) ☐

0C5A **Definition 17.2.** Let $f : X \to S$ be a morphism of schemes. We say $f$ is *at-worst-nodal of relative dimension* 1 if $f$ satisfies the equivalent conditions of Lemma 17.1.

Here are some reasons for the cumbersome terminology[5]. First, we want to make sure this notion is not confused with any of the other notions in the literature (see introduction to this section). Second, we can imagine several generalizations of this notion to morphisms of higher relative dimension (for example, one can ask for morphisms which are étale locally compositions of at-worst-nodal morphisms or one can ask for morphisms whose fibres are higher dimensional but have at worst ordinary double points).

0CD7 **Lemma 17.3.** *A smooth morphism of relative dimension* 1 *is at-worst-nodal of relative dimension* 1*.*

**Proof.** Omitted. ☐

0C5B **Lemma 17.4.** *Let $f : X \to S$ be at-worst-nodal of relative dimension* 1*. Then the same is true for any base change of $f$.*

**Proof.** This is true because the base change of a syntomic morphism is syntomic (Morphisms, Lemma 29.4), the base change of a morphism of relative dimension 1 has relative dimension 1 (Morphisms, Lemma 28.2), the formation of $\text{Sing}(f)$ commutes with base change (Divisors, Lemma 10.1), and the base change of an unramified morphism is unramified (Morphisms, Lemma 33.5). ☐

The following lemma tells us that we can check whether a morphism is at-worst-nodal of relative dimension 1 on the fibres.

0DSC **Lemma 17.5.** *Let $f : X \to S$ be a morphism of schemes which is flat and locally of finite presentation. Then there is a maximal open subscheme $U \subset X$ such that*

[5]But please email the maintainer of the Stacks project if you have a better suggestion.

*$f|_U : U \to S$ is at-worst-nodal of relative dimension 1. Moreover, formation of $U$ commutes with arbitrary base change.*

**Proof.** By Morphisms, Lemma 29.12 we find that there is such an open where $f$ is syntomic. Hence we may assume that $f$ is a syntomic morphism. In particular $f$ is a Cohen-Macaulay morphism (Dualizing Complexes, Lemmas 53.1 and 43.4). Thus $X$ is a disjoint union of open and closed subschemes on which $f$ has given relative dimension, see Morphisms, Lemma 28.4. This decomposition is preserved by arbitrary base change, see Morphisms, Lemma 28.2. Discarding all but one piece we may assume $f$ is syntomic of relative dimension 1. Let $\text{Sing}(f) \subset X$ be the closed subscheem defined by the first fitting ideal of $\Omega_{X/S}$. There is a maximal open subscheme $W \subset \text{Sing}(f)$ such that $W \to S$ is unramified and its formation commutes with base change (Morphisms, Lemma 33.15). Since also formation of $\text{Sing}(f)$ commutes with base change (Divisors, Lemma 10.1), we see that

$$U = (X \setminus \text{Sing}(f)) \cup W$$

is the maximal open subscheme of $X$ such that $f|_U : U \to S$ is at-worst-nodal of relative dimension 1 and that formation of $U$ commutes with base change. □

0C5C **Lemma 17.6.** *Let $f : X \to S$ be at-worst-nodal of relative dimension 1. If $Y \to X$ is an étale morphism, then the composition $g : Y \to S$ is at-worst-nodal of relative dimension 1.*

**Proof.** Observe that $g$ is flat and locally of finite presentation as a composition of morphisms which are flat and locally of finite presentation (use Morphisms, Lemmas 34.11, 34.12, 20.3, and 24.5). Thus it suffices to prove the fibres have at-worst-nodal singularities. This follows from Lemma 16.13 (and the fact that the composition of an étale morphism and a smooth morphism is smooth by Morphisms, Lemmas 34.5 and 32.4). □

0CD8 **Lemma 17.7.** *Let $S' \to S$ be an étale morphism of schemes. Let $f : X \to S'$ be at-worst-nodal of relative dimension 1. Then the composition $g : X \to S$ is at-worst-nodal of relative dimension 1.*

**Proof.** Observe that $g$ is flat and locally of finite presentation as a composition of morphisms which are flat and locally of finite presentation (use Morphisms, Lemmas 34.11, 34.12, 20.3, and 24.5). Thus it suffices to prove the fibres of $g$ have at-worst-nodal singularities. This follows from Lemma 16.14 and the analogous result for smooth points. □

0C5D **Lemma 17.8.** *Let $f : X \to S$ be a morphism of schemes. Let $\{U_i \to X\}$ be an étale covering. The following are equivalent*

1. *$f$ is at-worst-nodal of relative dimension 1,*
2. *each $U_i \to S$ is at-worst-nodal of relative dimension 1.*

*In other words, being at-worst-nodal of relative dimension 1 is étale local on the source.*

**Proof.** One direction we have seen in Lemma 17.6. For the other direction, observe that being locally of finite presentation, flat, or to have relative dimension 1 is étale local on the source (Descent, Lemmas 25.1, 24.1, and 30.8). Taking fibres we reduce to the case where $S$ is the spectrum of a field. In this case the result follows from Lemma 16.13 (and the fact that being smooth is étale local on the source by Descent, Lemma 27.1). $\square$

0C5E **Lemma 17.9.** *Let $f : X \to S$ be a morphism of schemes. Let $\{U_i \to S\}$ be an fpqc covering. The following are equivalent*

1. *$f$ is at-worst-nodal of relative dimension 1,*
2. *each $X \times_S U_i \to U_i$ is at-worst-nodal of relative dimension 1.*

*In other words, being at-worst-nodal of relative dimension 1 is fpqc local on the target.*

**Proof.** One direction we have seen in Lemma 17.4. For the other direction, observe that being locally of finite presentation, flat, or to have relative dimension 1 is fpqc local on the target (Descent, Lemmas 20.11, 20.15, and Morphisms, Lemma 27.3). Taking fibres we reduce to the case where $S$ is the spectrum of a field. In this case the result follows from Lemma 16.12 (and the fact that being smooth is fpqc local on the target by Descent, Lemma 20.27). $\square$

0C5F **Lemma 17.10.** *Let $S = \lim S_i$ be a limit of a directed system of schemes with affine transition morphisms. Let $0 \in I$ and let $f_0 : X_0 \to Y_0$ be a morphism of schemes over $S_0$. Assume $S_0$, $X_0$, $Y_0$ are quasi-compact and quasi-separated. Let $f_i : X_i \to Y_i$ be the base change of $f_0$ to $S_i$ and let $f : X \to Y$ be the base change of $f_0$ to $S$. If*

1. *$f$ is at-worst-nodal of relative dimension 1, and*
2. *$f_0$ is locally of finite presentation,*

*then there exists an $i \geq 0$ such that $f_i$ is at-worst-nodal of relative dimension 1.*

**Proof.** By Limits, Lemma 8.14 there exists an $i$ such that $f_i$ is syntomic. Then $X_i = \coprod_{d \geq 0} X_{i,d}$ is a disjoint union of open and closed subschemes such that $X_{i,d} \to Y_i$ has relative dimension $d$, see Morphisms, Lemma 29.14. Because of the behaviour of dimensions of fibres under base change given in Morphisms, Lemma 27.3 we see that $X \to X_i$ maps into $X_{i,1}$. Then there exists an $i' \geq i$ such that $X_{i'} \to X_i$ maps into $X_{i,1}$, see Limits, Lemma 4.10. Thus $f_{i'} : X_{i'} \to Y_{i'}$ is syntomic of relative dimension 1 (by Morphisms, Lemma 27.3 again). Consider the morphism $\text{Sing}(f_{i'}) \to Y_{i'}$. We know that the base change to $Y$ is an unramified morphism. Hence by Limits, Lemma 8.4 we see that after increasing $i'$ the morphism $\text{Sing}(f_{i'}) \to Y_{i'}$ becomes unramified. This finishes the proof. $\square$

0CBX **Lemma 17.11.** *Let $f : T \to S$ be a morphism of schemes. Let $t \in T$ with image $s \in S$. Assume*

1. *$f$ is flat at $t$,*
2. *$\mathcal{O}_{S,s}$ is Noetherian,*
3. *$f$ is locally of finite type,*
4. *$t$ is a split node of the fibre $T_s$.*

*Then there exists an $h \in \mathfrak{m}_s^\wedge$ and an isomorphism*

$$\mathcal{O}_{T,t}^\wedge \cong \mathcal{O}_{S,s}^\wedge[[x, y]]/(xy - h)$$

*of $\mathcal{O}_{S,s}^\wedge$-algebras.*

**Proof.** We replace $S$ by $\text{Spec}(\mathcal{O}_{S,s})$ and $T$ by the base change to $\text{Spec}(\mathcal{O}_{S,s})$. Then $T$ is locally Noetherian and hence $\mathcal{O}_{T,t}$ is Noetherian. Set $A = \mathcal{O}_{S,s}^\wedge$, $\mathfrak{m} = \mathfrak{m}_A$, and $B = \mathcal{O}_{T,t}^\wedge$. By More on Algebra, Lemma 39.7 we see that $A \to B$ is flat. Since

$\mathcal{O}_{T,t}/\mathfrak{m}_s\mathcal{O}_{T,t} = \mathcal{O}_{T_s,t}$ we see that $B/\mathfrak{m}B = \mathcal{O}^\wedge_{T_s,t}$. By assumption (4) and Lemma 16.11 we conclude there exist $\overline{u}, \overline{v} \in B/\mathfrak{m}B$ such that the map

$$(A/\mathfrak{m})[[x,y]] \longrightarrow B/\mathfrak{m}B, \quad x \longmapsto \overline{u}, x \longmapsto \overline{v}$$

is surjective with kernel $(xy)$.

Assume we have $n \geq 1$ and $u, v \in B$ mapping to $\overline{u}, \overline{v}$ such that

$$uv = h + \delta$$

for some $h \in A$ and $\delta \in \mathfrak{m}^n B$. We claim that there exist $u', v' \in B$ with $u - u', v - v' \in \mathfrak{m}^n B$ such that

$$u'v' = h' + \delta'$$

for some $h' \in A$ and $\delta' \in \mathfrak{m}^{n+1}B$. To see this, write $\delta = \sum f_i b_i$ with $f_i \in \mathfrak{m}^n$ and $b_i \in B$. Then write $b_i = a_i + ub_{i,1} + vb_{i,2} + \delta_i$ with $a_i \in A$, $b_{i,1}, b_{i,2} \in B$ and $\delta_i \in \mathfrak{m}B$. This is possible because the residue field of $B$ agrees with the residue field of $A$ and the images of $u$ and $v$ in $B/\mathfrak{m}B$ generate the maximal ideal. Then we set

$$u' = u - \sum b_{i,2} f_i, \quad v' = v - \sum b_{i,1} f_i$$

and we obtain

$$u'v' = h + \delta - \sum (b_{i,1}u + b_{i,2}v) f_i + \sum c_{ij} f_i f_j = h + \sum a_i f_i + \sum f_i \delta_i + \sum c_{ij} f_i f_j$$

for some $c_{i,j} \in B$. Thus we get a formula as above with $h' = h + \sum a_i f_i$ and $\delta' = \sum f_i \delta_i + \sum c_{ij} f_i f_j$.

Arguing by induction and starting with any lifts $u_1, v_1 \in B$ of $\overline{u}, \overline{v}$ the result of the previous paragraph shows that we find a sequence of elements $u_n, v_n \in B$ and $h_n \in A$ such that $u_n - u_{n+1} \in \mathfrak{m}^n B$, $v_n - v_{n+1} \in \mathfrak{m}^n B$, $h_n - h_{n+1} \in \mathfrak{m}^n$, and such that $u_n v_n - h_n \in \mathfrak{m}^n B$. Since $A$ and $B$ are complete we can set $u_\infty = \lim u_n$, $v_\infty = \lim v_n$, and $h_\infty = \lim h_n$, and then we obtain $u_\infty v_\infty = h_\infty$ in $B$. Thus we have an $A$-algebra map

$$A[[x,y]]/(xy - h_\infty) \longrightarrow B$$

sending $x$ to $u_\infty$ and $v$ to $v_\infty$. This is a map of flat $A$-algebras which is an isomorphism after dividing by $\mathfrak{m}$. It is surjective modulo $\mathfrak{m}$ and hence surjective by completeness and Algebra, Lemma 95.1. Then we can apply Algebra, Lemma 98.1 to conclude it is an isomorphism. $\square$

Consider the morphism of schemes

$$\mathrm{Spec}(\mathbf{Z}[u,v,a]/(uv-a)) \longrightarrow \mathrm{Spec}(\mathbf{Z}[a])$$

The next lemma shows that this morphism is a model for the étale local structure of a nodal family of curves. If you know a proof of this lemma avoiding the use of Artin approximation, then please email stacks.project@gmail.com.

0CBY **Lemma 17.12.** *Let $f : X \to S$ be a morphism of schemes. Assume that $f$ is at-worst-nodal of relative dimension 1. Let $x \in X$ be a point which is a singular point of the fibre $X_s$. Then there exists a commutative diagram of schemes*

$$\begin{tikzcd}
X \arrow[d] & U \arrow[l] \arrow[r] \arrow[dr] & W \arrow[r] \arrow[d] & \mathrm{Spec}(\mathbf{Z}[u,v,a]/(uv-a)) \arrow[d] \\
S & & V \arrow[ll] \arrow[r] & \mathrm{Spec}(\mathbf{Z}[a])
\end{tikzcd}$$

*with $X \leftarrow U$, $S \leftarrow V$, and $U \to W$ étale morphisms, and with the right hand square cartesian, such that there exists a point $u \in U$ mapping to $x$ in $X$.*

**Proof.** We first use absolute Noetherian approximation to reduce to the case of schemes of finite type over $\mathbf{Z}$. The question is local on $X$ and $S$. Hence we may assume that $X$ and $S$ are affine. Then we can write $S = \text{Spec}(R)$ and write $R$ as a filtered colimit $R = \text{colim}\, R_i$ of finite type $\mathbf{Z}$-algebras. Using Limits, Lemma 10.1 we can find an $i$ and a morphism $f_i : X_i \to \text{Spec}(R_i)$ whose base change to $S$ is $f$. After increasing $i$ we may assume that $f_i$ is at-worst-nodal of relative dimension 1, see Lemma 17.10. The image $x_i \in X_i$ of $x$ will be a singular point of its fibre, for example because the formation of $\text{Sing}(f)$ commutes with base change (Divisors, Lemma 10.1). If we can prove the lemma for $f_i : X_i \to S_i$ and $x_i$, then the lemma follows for $f : X \to S$ by base change. Thus we reduce to the case studied in the next paragraph.

Assume $S$ is of finite type over $\mathbf{Z}$. Let $s \in S$ be the image of $x$. Recall that $\kappa(x)$ is a finite separable extension of $\kappa(s)$, for example because $\text{Sing}(f) \to S$ is unramified or because $x$ is a node of the fibre $X_s$ and we can apply Lemma 16.7. Furthermore, let $\kappa'/\kappa(x)$ be the degree 2 separable algebra associated to $\mathcal{O}_{X_s,x}$ in Remark 16.8. By More on Morphisms, Lemma 31.2 we can choose an étale neighbourhood $(V, v) \to (S, s)$ such that the extension $\kappa(v)/\kappa(s)$ realizes either the extension $\kappa(x)/\kappa(s)$ in case $\kappa' \cong \kappa(x) \times \kappa(x)$ or the extension $\kappa'/\kappa(s)$ if $\kappa'$ is a field. After replacing $X$ by $X \times_S V$ and $S$ by $V$ we reduce to the situation described in the next paragraph.

Assume $S$ is of finite type over $\mathbf{Z}$ and $x \in X_s$ is a split node, see Definition 16.10. By Lemma 17.11 we see that there exists an $\mathcal{O}_{S,s}$-algebra isomorphism

$$\mathcal{O}^\wedge_{X,x} \cong \mathcal{O}^\wedge_{S,s}[[s,t]]/(st - h)$$

for some $h \in \mathfrak{m}^\wedge_s \subset \mathcal{O}^\wedge_{S,s}$. In other words, if we consider the homomorphism

$$\sigma : \mathbf{Z}[a] \longrightarrow \mathcal{O}^\wedge_{S,s}$$

sending $a$ to $h$, then there exists an $\mathcal{O}_{S,s}$-algebra isomorphism

$$\mathcal{O}^\wedge_{X,x} \longrightarrow \mathcal{O}^\wedge_{Y_\sigma, y_\sigma}$$

where

$$Y_\sigma = \text{Spec}(\mathbf{Z}[u, v, t]/(uv - a)) \times_{\text{Spec}(\mathbf{Z}[a]),\sigma} \text{Spec}(\mathcal{O}^\wedge_{S,s})$$

and $y_\sigma$ is the point of $Y_\sigma$ lying over the closed point of $\text{Spec}(\mathcal{O}^\wedge_{S,s})$ and having coordinates $u, v$ equal to zero. Since $\mathcal{O}_{S,s}$ is a G-ring by More on Algebra, Proposition 46.12 we may apply More on Morphisms, Lemmas 32.3 and 32.6 with $T = S \times \text{Spec}(\mathbf{Z}[a])$ and $Y = S \times \text{Spec}(\mathbf{Z}[u, v, a]/(uv - a))$ to get the conclusion of the lemma. $\square$

## 18. Other chapters

Preliminaries

(1) Introduction
(2) Conventions
(3) Set Theory
(4) Categories
(5) Topology
(6) Sheaves on Spaces
(7) Sites and Sheaves
(8) Stacks
(9) Fields
(10) Commutative Algebra
(11) Brauer Groups

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