DESCENT

Contents

1. Introduction

0239   In the chapter on topologies on schemes (see Topologies, Section 1) we introduced Zariski, étale, fppf, smooth, syntomic and fpqc coverings of schemes. In this chapter we discuss what kind of structures over schemes can be descended through such coverings. See for example [Gro95a], [Gro95b], [Gro95e], [Gro95f], [Gro95c], and [Gro95d]. This is also meant to introduce the notions of descent, descent data, effective descent data, in the less formal setting of descent questions for quasi-coherent sheaves, schemes, etc. The formal notion, that of a stack over a site, is discussed in the chapter on stacks (see Stacks, Section 1).

2. Descent data for quasi-coherent sheaves

023A   In this chapter we will use the convention where the projection maps $\mathrm{pr}_i : X \times \ldots \times X \to X$ are labeled starting with $i = 0$. Hence we have $\mathrm{pr}_0, \mathrm{pr}_1 : X \times X \to X$, $\mathrm{pr}_0, \mathrm{pr}_1, \mathrm{pr}_2 : X \times X \times X \to X$, etc.

023B   **Definition 2.1.** Let S be a scheme. Let $\{f_i : S_i \to S\}_{i \in I}$ be a family of morphisms with target S.

(1) A *descent datum* $(\mathcal{F}_i, \varphi_{ij})$ *for quasi-coherent sheaves* with respect to the given family is given by a quasi-coherent sheaf $\mathcal{F}_i$ on S_i for each $i \in I$, an isomorphism of quasi-coherent $\mathcal{O}_{S_i \times_S S_j}$-modules $\varphi_{ij} : \mathrm{pr}_0^* \mathcal{F}_i \to \mathrm{pr}_1^* \mathcal{F}_j$ for each pair $(i, j) \in I^2$ such that for every triple of indices $(i, j, k) \in I^3$ the diagram

$$
\begin{array}{ccc}
\mathrm{pr}_0^* \mathcal{F}_i & \xrightarrow{\ \mathrm{pr}_{02}^* \varphi_{ik}\ } & \mathrm{pr}_2^* \mathcal{F}_k \\
& & \\
{\scriptstyle \mathrm{pr}_{01}^* \varphi_{ij}} \searrow & & \nearrow {\scriptstyle \mathrm{pr}_{12}^* \varphi_{jk}} \\
& \mathrm{pr}_1^* \mathcal{F}_j &
\end{array}
$$

of $\mathcal{O}_{S_i \times_S S_j \times_S S_k}$-modules commutes. This is called the *cocycle condition*.

(2) A *morphism* $\psi : (\mathcal{F}_i, \varphi_{ij}) \to (\mathcal{F}_i', \varphi_{ij}')$ *of descent data* is given by a family $\psi = (\psi_i)_{i \in I}$ of morphisms of $\mathcal{O}_{S_i}$-modules $\psi_i : \mathcal{F}_i \to \mathcal{F}_i'$ such that all the diagrams

$$
\begin{array}{ccc}
\mathrm{pr}_0^* \mathcal{F}_i & \xrightarrow{\ \varphi_{ij}\ } & \mathrm{pr}_1^* \mathcal{F}_j \\
{\scriptstyle \mathrm{pr}_0^* \psi_i} \downarrow & & \downarrow {\scriptstyle \mathrm{pr}_1^* \psi_j} \\
\mathrm{pr}_0^* \mathcal{F}_i' & \xrightarrow{\ \varphi_{ij}'\ } & \mathrm{pr}_1^* \mathcal{F}_j'
\end{array}
$$

commute.

A good example to keep in mind is the following. Suppose that $S = \bigcup S_i$ is an open covering. In that case we have seen descent data for sheaves of sets in Sheaves, Section 33 where we called them "glueing data for sheaves of sets with respect to the given covering". Moreover, we proved that the category of glueing data is equivalent to the category of sheaves on S. We will show the analogue in the setting above when $\{S_i \to S\}_{i \in I}$ is an fpqc covering.

In the extreme case where the covering $\{S \to S\}$ is given by id_S a descent datum is necessarily of the form $(\mathcal{F}, \mathrm{id}_{\mathcal{F}})$. The cocycle condition guarantees that the identity on $\mathcal{F}$ is the only permitted map in this case. The following lemma shows in particular that to every quasi-coherent sheaf of $\mathcal{O}_S$-modules there is associated a unique descent datum with respect to any given family.

023C **Lemma 2.2.** *Let $\mathcal{U} = \{U_i \to U\}_{i \in I}$ and $\mathcal{V} = \{V_j \to V\}_{j \in J}$ be families of morphisms of schemes with fixed target. Let $(g, \alpha : I \to J, (g_i)) : \mathcal{U} \to \mathcal{V}$ be a morphism of families of maps with fixed target, see Sites, Definition 8.1. Let $(\mathcal{F}_j, \varphi_{jj'})$ be a descent datum for quasi-coherent sheaves with respect to the family $\{V_j \to V\}_{j \in J}$. Then*

(1) *The system*
$$\left(g_i^* \mathcal{F}_{\alpha(i)}, (g_i \times g_{i'})^* \varphi_{\alpha(i)\alpha(i')}\right)$$
is a descent datum with respect to the family $\{U_i \to U\}_{i \in I}$.
(2) *This construction is functorial in the descent datum $(\mathcal{F}_j, \varphi_{jj'})$.*
(3) *Given a second morphism $(g', \alpha' : I \to J, (g_i'))$ of families of maps with fixed target with $g = g'$ there exists a functorial isomorphism of descent data*
$$(g_i^* \mathcal{F}_{\alpha(i)}, (g_i \times g_{i'})^* \varphi_{\alpha(i)\alpha(i')}) \cong ((g_i')^* \mathcal{F}_{\alpha'(i)}, (g_i' \times g_{i'}')^* \varphi_{\alpha'(i)\alpha'(i')}).$$

Proof. Omitted. Hint: The maps $g_i^* \mathcal{F}_{\alpha(i)} \to (g_i')^* \mathcal{F}_{\alpha'(i)}$ which give the isomorphism of descent data in part (3) are the pullbacks of the maps $\varphi_{\alpha(i)\alpha'(i)}$ by the morphisms $(g_i, g_i') : U_i \to V_{\alpha(i)} \times_V V_{\alpha'(i)}$. $\qquad\square$

Any family $\mathcal{U} = \{S_i \to S\}_{i \in I}$ is a refinement of the trivial covering $\{S \to S\}$ in a unique way. For a quasi-coherent sheaf $\mathcal{F}$ on S we denote simply $(\mathcal{F}|_{S_i}, can)$ the descent datum with respect to $\mathcal{U}$ obtained by the procedure above.

023D **Definition 2.3.** Let S be a scheme. Let $\{S_i \to S\}_{i \in I}$ be a family of morphisms with target S.

(1) Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_S$-module. We call the unique descent on $\mathcal{F}$ datum with respect to the covering $\{S \to S\}$ the *trivial descent datum*.
(2) The pullback of the trivial descent datum to $\{S_i \to S\}$ is called the *canonical descent datum*. Notation: $(\mathcal{F}|_{S_i}, can)$.
(3) A descent datum $(\mathcal{F}_i, \varphi_{ij})$ for quasi-coherent sheaves with respect to the given covering is said to be *effective* if there exists a quasi-coherent sheaf $\mathcal{F}$ on S such that $(\mathcal{F}_i, \varphi_{ij})$ is isomorphic to $(\mathcal{F}|_{S_i}, can)$.

023E **Lemma 2.4.** *Let S be a scheme. Let $S = \bigcup U_i$ be an open covering. Any descent datum on quasi-coherent sheaves for the family $\mathcal{U} = \{U_i \to S\}$ is effective. Moreover, the functor from the category of quasi-coherent $\mathcal{O}_S$-modules to the category of descent data with respect to $\mathcal{U}$ is fully faithful.*

Proof. This follows immediately from Sheaves, Section 33 and the fact that being quasi-coherent is a local property, see Modules, Definition 10.1. $\qquad\square$

To prove more we first need to study the case of modules over rings.

3. Descent for modules

023F   Let $R \to A$ be a ring map. By Simplicial, Example 5.5 this gives rise to a cosimplicial R-algebra

$$A \rightrightarrows A \otimes_R A \Rrightarrow A \otimes_R A \otimes_R A$$

Let us denote this $(A/R)_\bullet$ so that $(A/R)_n$ is the $(n+1)$-fold tensor product of A over R. Given a map $\varphi : [n] \to [m]$ the R-algebra map $(A/R)_\bullet(\varphi)$ is the map

$$a_0 \otimes \ldots \otimes a_n \longmapsto \prod_{\varphi(i)=0} a_i \otimes \prod_{\varphi(i)=1} a_i \otimes \ldots \otimes \prod_{\varphi(i)=m} a_i$$

where we use the convention that the empty product is 1. Thus the first few maps, notation as in Simplicial, Section 5, are

$$
\begin{array}{ccccc}
\delta_0^1 & : & a_0 & \mapsto & 1 \otimes a_0 \\
\delta_1^1 & : & a_0 & \mapsto & a_0 \otimes 1 \\
\sigma_0^0 & : & a_0 \otimes a_1 & \mapsto & a_0 a_1 \\
\delta_0^2 & : & a_0 \otimes a_1 & \mapsto & 1 \otimes a_0 \otimes a_1 \\
\delta_1^2 & : & a_0 \otimes a_1 & \mapsto & a_0 \otimes 1 \otimes a_1 \\
\delta_2^2 & : & a_0 \otimes a_1 & \mapsto & a_0 \otimes a_1 \otimes 1 \\
\sigma_0^1 & : & a_0 \otimes a_1 \otimes a_2 & \mapsto & a_0 a_1 \otimes a_2 \\
\sigma_1^1 & : & a_0 \otimes a_1 \otimes a_2 & \mapsto & a_0 \otimes a_1 a_2 \\
\end{array}
$$

and so on.

An R-module M gives rise to a cosimplicial $(A/R)_\bullet$-module $(A/R)_\bullet \otimes_R M$. In other words $M_n = (A/R)_n \otimes_R M$ and using the R-algebra maps $(A/R)_n \to (A/R)_m$ to define the corresponding maps on $M \otimes_R (A/R)_\bullet$.

The analogue to a descent datum for quasi-coherent sheaves in the setting of modules is the following.

023G   **Definition 3.1.** Let $R \to A$ be a ring map.

(1) A *descent datum (N, φ) for modules with respect to $R \to A$* is given by an A-module N and an isomorphism of $A \otimes_R A$-modules

$$\varphi : N \otimes_R A \to A \otimes_R N$$

such that the *cocycle condition* holds: the diagram of $A \otimes_R A \otimes_R A$-module maps

$$
\begin{array}{ccc}
N \otimes_R A \otimes_R A & \xrightarrow{\varphi_{02}} & A \otimes_R A \otimes_R N \\
& {\scriptstyle \varphi_{01}} \searrow \quad \nearrow {\scriptstyle \varphi_{12}} & \\
& A \otimes_R N \otimes_R A &
\end{array}
$$

commutes (see below for notation).
(2) A *morphism $(N, \varphi) \to (N', \varphi')$ of descent data* is a morphism of A-modules $\psi : N \to N'$ such that the diagram

$$
\begin{array}{ccc}
N \otimes_R A & \xrightarrow{\varphi} & A \otimes_R N \\
{\scriptstyle \psi \otimes \mathrm{id}_A} \downarrow & & \downarrow {\scriptstyle \mathrm{id}_A \otimes \psi} \\
N' \otimes_R A & \xrightarrow{\varphi'} & A \otimes_R N'
\end{array}
$$

is commutative.

In the definition we use the notation that $\varphi_{01} = \varphi \otimes \mathrm{id}_A$, $\varphi_{12} = \mathrm{id}_A \otimes \varphi$, and $\varphi_{02}(n \otimes 1 \otimes 1) = \sum a_i \otimes 1 \otimes n_i$ if $\varphi(n \otimes 1) = \sum a_i \otimes n_i$. All three are $A \otimes_R A \otimes_R A$-module homomorphisms. Equivalently we have

$$\varphi_{ij} = \varphi \otimes_{(A/R)_1, \, (A/R)_\bullet(\tau_{ij}^2)} (A/R)_2$$

where $\tau_{ij}^2 : [1] \to [2]$ is the map $0 \mapsto i$, $1 \mapsto j$. Namely, $(A/R)_\bullet(\tau_{02}^2)(a_0 \otimes a_1) = a_0 \otimes 1 \otimes a_1$, and similarly for the others[1].

We need some more notation to be able to state the next lemma. Let (N, φ) be a descent datum with respect to a ring map $R \to A$. For $n \geq 0$ and $i \in [n]$ we set

$$N_{n,i} = A \otimes_R \ldots \otimes_R A \otimes_R N \otimes_R A \otimes_R \ldots \otimes_R A$$

with the factor N in the ith spot. It is an $(A/R)_n$-module. If we introduce the maps $\tau_i^n : [0] \to [n]$, $0 \mapsto i$ then we see that

$$N_{n,i} = N \otimes_{(A/R)_0, \, (A/R)_\bullet(\tau_i^n)} (A/R)_n$$

For $0 \leq i \leq j \leq n$ we let $\tau_{ij}^n : [1] \to [n]$ be the map such that 0 maps to i and 1 to j. Similarly to the above the homomorphism φ induces isomorphisms

$$\varphi_{ij}^n = \varphi \otimes_{(A/R)_1, \, (A/R)_\bullet(\tau_{ij}^n)} (A/R)_n : N_{n,i} \longrightarrow N_{n,j}$$

of $(A/R)_n$-modules when $i < j$. If $i = j$ we set $\varphi_{ij}^n = \mathrm{id}$. Since these are all isomorphisms they allow us to move the factor N to any spot we like. And the cocycle condition exactly means that it does not matter how we do this (e.g., as a composition of two of these or at once). Finally, for any $\beta : [n] \to [m]$ we define the morphism

$$N_{\beta,i} : N_{n,i} \to N_{m,\beta(i)}$$

as the unique $(A/R)_\bullet(\beta)$-semi linear map such that

$$N_{\beta,i}(1 \otimes \ldots \otimes n \otimes \ldots \otimes 1) = 1 \otimes \ldots \otimes n \otimes \ldots \otimes 1$$

for all $n \in N$. This hints at the following lemma.

023H **Lemma 3.2.** *Let $R \to A$ be a ring map. Given a descent datum (N, φ) we can associate to it a cosimplicial $(A/R)_\bullet$-module $N_\bullet$[2] by the rules $N_n = N_{n,n}$ and given $\beta : [n] \to [m]$ setting we define*

$$N_\bullet(\beta) = (\varphi_{\beta(n)m}^m) \circ N_{\beta,n} : N_{n,n} \longrightarrow N_{m,m}.$$

This procedure is functorial in the descent datum.

Proof. Here are the first few maps where $\varphi(n \otimes 1) = \sum \alpha_i \otimes x_i$

δ_0^1	:	N	$\to$	$A \otimes N$	n	$\mapsto$	$1 \otimes n$
δ_1^1	:	N	$\to$	$A \otimes N$	n	$\mapsto$	$\sum \alpha_i \otimes x_i$
σ_0^0	:	$A \otimes N$	$\to$	N	$a_0 \otimes n$	$\mapsto$	$a_0 n$
δ_0^2	:	$A \otimes N$	$\to$	$A \otimes A \otimes N$	$a_0 \otimes n$	$\mapsto$	$1 \otimes a_0 \otimes n$
δ_1^2	:	$A \otimes N$	$\to$	$A \otimes A \otimes N$	$a_0 \otimes n$	$\mapsto$	$a_0 \otimes 1 \otimes n$
δ_2^2	:	$A \otimes N$	$\to$	$A \otimes A \otimes N$	$a_0 \otimes n$	$\mapsto$	$\sum a_0 \otimes \alpha_i \otimes x_i$
σ_0^1	:	$A \otimes A \otimes N$	$\to$	$A \otimes N$	$a_0 \otimes a_1 \otimes n$	$\mapsto$	$a_0 a_1 \otimes n$
σ_1^1	:	$A \otimes A \otimes N$	$\to$	$A \otimes N$	$a_0 \otimes a_1 \otimes n$	$\mapsto$	$a_0 \otimes a_1 n$

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[1]Note that $\tau_{ij}^2 = \delta_k^2$, if $\{i, j, k\} = [2] = \{0, 1, 2\}$, see Simplicial, Definition 2.1.

[2]We should really write $(N, \varphi)_\bullet$.

with notation as in Simplicial, Section 5. We first verify the two properties $\sigma_0^0 \circ \delta_0^1 = \mathrm{id}$ and $\sigma_0^0 \circ \delta_0^1 = \mathrm{id}$. The first one, $\sigma_0^0 \circ \delta_0^1 = \mathrm{id}$, is clear from the explicit description of the morphisms above. To prove the second relation we have to use the cocycle condition (because it does not hold for an arbitrary isomorphism $\varphi : N \otimes_R A \to A \otimes_R N$). Write $p = \sigma_0^0 \circ \delta_1^1 : N \to N$. By the description of the maps above we deduce that p is also equal to

$$p = \varphi \otimes \mathrm{id} : N = (N \otimes_R A) \otimes_{(A \otimes_R A)} A \longrightarrow (A \otimes_R N) \otimes_{(A \otimes_R A)} A = N$$

Since φ is an isomorphism we see that p is an isomorphism. Write $\varphi(n \otimes 1) = \sum \alpha_i \otimes x_i$ for certain $\alpha_i \in A$ and $x_i \in N$. Then $p(n) = \sum \alpha_i x_i$. Next, write $\varphi(x_i \otimes 1) = \sum \alpha_{ij} \otimes y_j$ for certain $\alpha_{ij} \in A$ and $y_j \in N$. Then the cocycle condition says that

$$\sum \alpha_i \otimes \alpha_{ij} \otimes y_j = \sum \alpha_i \otimes 1 \otimes x_i.$$

This means that $p(n) = \sum \alpha_i x_i = \sum \alpha_i \alpha_{ij} y_j = \sum \alpha_i p(x_i) = p(p(n))$. Thus p is a projector, and since it is an isomorphism it is the identity.

To prove fully that $N_\bullet$ is a cosimplicial module we have to check all 5 types of relations of Simplicial, Remark 5.3. The relations on composing σ's are obvious. The relations on composing δ's come down to the cocycle condition for φ. In exactly the same way as above one checks the relations $\sigma_j \circ \delta_j = \sigma_j \circ \delta_{j+1} = \mathrm{id}$. Finally, the other relations on compositions of δ's and σ's hold for any φ whatsoever. $\square$

Note that to an R-module M we can associate a canonical descent datum, namely $(M \otimes_R A, can)$ where $can : (M \otimes_R A) \otimes_R A \to A \otimes_R (M \otimes_R A)$ is the obvious map: $(m \otimes a) \otimes a' \mapsto a \otimes (m \otimes a')$.

023I   **Lemma 3.3.** *Let $R \to A$ be a ring map. Let M be an R-module. The cosimplicial $(A/R)_\bullet$-module associated to the canonical descent datum is isomorphic to the cosimplicial module $(A/R)_\bullet \otimes_R M$.*

Proof. Omitted. $\square$

023J   **Definition 3.4.** Let $R \to A$ be a ring map. We say a descent datum (N, φ) is *effective* if there exists an R-module M and an isomorphism of descent data from $(M \otimes_R A, can)$ to (N, φ).

Let $R \to A$ be a ring map. Let (N, φ) be a descent datum. We may take the cochain complex $s(N_\bullet)$ associated with $N_\bullet$ (see Simplicial, Section 25). It has the following shape:

$$N \to A \otimes_R N \to A \otimes_R A \otimes_R N \to \ldots$$

We can describe the maps. The first map is the map

$$n \longmapsto 1 \otimes n - \varphi(n \otimes 1).$$

The second map on pure tensors has the values

$$a \otimes n \longmapsto 1 \otimes a \otimes n - a \otimes 1 \otimes n + a \otimes \varphi(n \otimes 1).$$

It is clear how the pattern continues.

In the special case where $N = A \otimes_R M$ we see that for any $m \in M$ the element $1 \otimes m$ is in the kernel of the first map of the cochain complex associated to the cosimplicial module $(A/R)_\bullet \otimes_R M$. Hence we get an extended cochain complex

023K   (3.4.1) $0 \to M \to A \otimes_R M \to A \otimes_R A \otimes_R M \to \ldots$

Here we think of the 0 as being in degree -2, the module M in degree -1, the module $A \otimes_R M$ in degree 0, etc. Note that this complex has the shape

$$0 \to R \to A \to A \otimes_R A \to A \otimes_R A \otimes_R A \to \ldots$$

when $M = R$.

023L **Lemma 3.5.** *Suppose that $R \to A$ has a section. Then for any R-module M the extended cochain complex (3.4.1) is exact.*

Proof. By Simplicial, Lemma 28.5 the map $R \to (A/R)_\bullet$ is a homotopy equivalence of cosimplicial R-algebras (here R denotes the constant cosimplicial R-algebra). Hence $M \to (A/R)_\bullet \otimes_R M$ is a homotopy equivalence in the category of cosimplicial R-modules, because $\otimes_R M$ is a functor from the category of R-algebras to the category of R-modules, see Simplicial, Lemma 28.4. This implies that the induced map of associated complexes is a homotopy equivalence, see Simplicial, Lemma 28.6. Since the complex associated to the constant cosimplicial R-module M is the complex

$$M \xrightarrow{\ 0\ } M \xrightarrow{\ 1\ } M \xrightarrow{\ 0\ } M \xrightarrow{\ 1\ } M \ldots$$

we win (since the extended version simply puts an extra M at the beginning). $\square$

023M **Lemma 3.6.** *Suppose that $R \to A$ is faithfully flat, see Algebra, Definition 39.1. Then for any R-module M the extended cochain complex (3.4.1) is exact.*

Proof. Suppose we can show there exists a faithfully flat ring map $R \to R'$ such that the result holds for the ring map $R' \to A' = R' \otimes_R A$. Then the result follows for $R \to A$. Namely, for any R-module M the cosimplicial module $(M \otimes_R R') \otimes_{R'} (A'/R')_\bullet$ is just the cosimplicial module $R' \otimes_R (M \otimes_R (A/R)_\bullet)$. Hence the vanishing of cohomology of the complex associated to $(M \otimes_R R') \otimes_{R'} (A'/R')_\bullet$ implies the vanishing of the cohomology of the complex associated to $M \otimes_R (A/R)_\bullet$ by faithful flatness of $R \to R'$. Similarly for the vanishing of cohomology groups in degrees -1 and 0 of the extended complex (proof omitted).

But we have such a faithful flat extension. Namely $R' = A$ works because the ring map $R' = A \to A' = A \otimes_R A$ has a section $a \otimes a' \mapsto aa'$ and Lemma 3.5 applies. $\square$

Here is how the complex relates to the question of effectivity.

039W **Lemma 3.7.** *Let $R \to A$ be a faithfully flat ring map. Let (N, φ) be a descent datum. Then (N, φ) is effective if and only if the canonical map*

$$A \otimes_R H^0(s(N_\bullet)) \longrightarrow N$$

is an isomorphism.

Proof. If (N, φ) is effective, then we may write $N = A \otimes_R M$ with $\varphi = can$. It follows that $H^0(s(N_\bullet)) = M$ by Lemmas 3.3 and 3.6. Conversely, suppose the map of the lemma is an isomorphism. In this case set $M = H^0(s(N_\bullet))$. This is an R-submodule of N, namely $M = \{n \in N \mid 1 \otimes n = \varphi(n \otimes 1)\}$. The only thing to check is that via the isomorphism $A \otimes_R M \to N$ the canonical descent data agrees with φ. We omit the verification. $\square$

039X **Lemma 3.8.** *Let $R \to A$ be a faithfully flat ring map, and let $R \to R'$ be faithfully flat. Set $A' = R' \otimes_R A$. If all descent data for $R' \to A'$ are effective, then so are all descent data for $R \to A$.*

Proof. Let (N, φ) be a descent datum for $R \to A$. Set $N' = R' \otimes_R N = A' \otimes_A N$, and denote $\varphi' = \mathrm{id}_{R'} \otimes \varphi$ the base change of the descent datum φ. Then (N', φ') is a descent datum for $R' \to A'$ and $H^0(s(N'_\bullet)) = R' \otimes_R H^0(s(N_\bullet))$. Moreover, the map $A' \otimes_{R'} H^0(s(N'_\bullet)) \to N'$ is identified with the base change of the A-module map $A \otimes_R H^0(s(N)) \to N$ via the faithfully flat map $A \to A'$. Hence we conclude by Lemma 3.7. $\square$

Here is the main result of this section. Its proof may seem a little clumsy; for a more highbrow approach see Remark 3.11 below.

023N **Proposition 3.9.** *Let $R \to A$ be a faithfully flat ring map. Then*

(1) *any descent datum on modules with respect to $R \to A$ is effective,*
(2) *the functor $M \mapsto (A \otimes_R M, can)$ from R-modules to the category of descent data is an equivalence, and*
(3) *the inverse functor is given by $(N, \varphi) \mapsto H^0(s(N_\bullet))$.*

Proof. We only prove (1) and omit the proofs of (2) and (3). As $R \to A$ is faithfully flat, there exists a faithfully flat base change $R \to R'$ such that $R' \to A' = R' \otimes_R A$ has a section (namely take $R' = A$ as in the proof of Lemma 3.6). Hence, using Lemma 3.8 we may assume that $R \to A$ has a section, say $\sigma : A \to R$. Let (N, φ) be a descent datum relative to $R \to A$. Set

$$M = H^0(s(N_\bullet)) = \{n \in N \mid 1 \otimes n = \varphi(n \otimes 1)\} \subset N$$

By Lemma 3.7 it suffices to show that $A \otimes_R M \to N$ is an isomorphism.

Take an element $n \in N$. Write $\varphi(n \otimes 1) = \sum a_i \otimes x_i$ for certain $a_i \in A$ and $x_i \in N$. By Lemma 3.2 we have $n = \sum a_i x_i$ in N (because $\sigma_0^0 \circ \delta_1^1 = \mathrm{id}$ in any cosimplicial object). Next, write $\varphi(x_i \otimes 1) = \sum a_{ij} \otimes y_j$ for certain $a_{ij} \in A$ and $y_j \in N$. The cocycle condition means that

$$\sum a_i \otimes a_{ij} \otimes y_j = \sum a_i \otimes 1 \otimes x_i$$

in $A \otimes_R A \otimes_R N$. We conclude two things from this:

(1) applying σ to the first A we get $\sum \sigma(a_i)\varphi(x_i \otimes 1) = \sum \sigma(a_i) \otimes x_i$,
(2) applying σ to the middle A we get $\sum_i a_i \otimes \sum_j \sigma(a_{ij})y_j = \sum a_i \otimes x_i$.

Part (1) shows that $\sum \sigma(a_i)x_i \in M$. Applying this to x_i we see that $\sum \sigma(a_{ij})y_i \in M$ for all i. Multiplying out the equation in (2) we conclude that $\sum_i a_i(\sum_j \sigma(a_{ij})y_j) = \sum a_i x_i = n$. Hence $A \otimes_R M \to N$ is surjective. Finally, suppose that $m_i \in M$ and $\sum a_i m_i = 0$. Then we see by applying φ to $\sum a_i m_i \otimes 1$ that $\sum a_i \otimes m_i = 0$. In other words $A \otimes_R M \to N$ is injective and we win. $\square$

023O **Remark 3.10.** Let R be a ring. Let $f_1, \ldots, f_n \in R$ generate the unit ideal. The ring $A = \prod_i R_{f_i}$ is a faithfully flat R-algebra. We remark that the cosimplicial ring $(A/R)_\bullet$ has the following ring in degree n:

$$\prod_{i_0, \ldots, i_n} R_{f_{i_0} \cdots f_{i_n}}$$

Hence the results above recover Algebra, Lemmas 24.2, 24.1 and 24.5. But the results above actually say more because of exactness in higher degrees. Namely, it implies that Čech cohomology of quasi-coherent sheaves on affines is trivial. Thus we get a second proof of Cohomology of Schemes, Lemma 2.1.

039Y **Remark 3.11.** Let R be a ring. Let $A_\bullet$ be a cosimplicial R-algebra. In this setting a descent datum corresponds to an cosimplicial $A_\bullet$-module $M_\bullet$ with the property that for every $n, m \geq 0$ and every $\varphi : [n] \to [m]$ the map $M(\varphi) : M_n \to M_m$ induces an isomorphism

$$M_n \otimes_{A_n, A(\varphi)} A_m \longrightarrow M_m.$$

Let us call such a cosimplicial module a *cartesian module.* In this setting, the proof of Proposition 3.9 can be split in the following steps

(1) If $R \to R'$ and $R \to A$ are faithfully flat, then descent data for A/R are effective if descent data for $(R' \otimes_R A)/R'$ are effective.
(2) Let A be an R-algebra. Descent data for A/R correspond to cartesian $(A/R)_\bullet$-modules.
(3) If $R \to A$ has a section then $(A/R)_\bullet$ is homotopy equivalent to R, the constant cosimplicial R-algebra with value R.
(4) If $A_\bullet \to B_\bullet$ is a homotopy equivalence of cosimplicial R-algebras then the functor $M_\bullet \mapsto M_\bullet \otimes_{A_\bullet} B_\bullet$ induces an equivalence of categories between cartesian $A_\bullet$-modules and cartesian $B_\bullet$-modules.

For (1) see Lemma 3.8. Part (2) uses Lemma 3.2. Part (3) we have seen in the proof of Lemma 3.5 (it relies on Simplicial, Lemma 28.5). Moreover, part (4) is a triviality if you think about it right!

4. Descent for universally injective morphisms

08WE   Numerous constructions in algebraic geometry are made using techniques of *descent*, such as constructing objects over a given space by first working over a somewhat larger space which projects down to the given space, or verifying a property of a space or a morphism by pulling back along a covering map. The utility of such techniques is of course dependent on identification of a wide class of *effective descent morphisms.* Early in the Grothendieckian development of modern algebraic geometry, the class of morphisms which are *quasi-compact* and *faithfully flat* was shown to be effective for descending objects, morphisms, and many properties thereof.

As usual, this statement comes down to a property of rings and modules. For a homomorphism $f : R \to S$ to be an effective descent morphism for modules, Grothendieck showed that it is sufficient for f to be faithfully flat. However, this excludes many natural examples: for instance, any split ring homomorphism is an effective descent morphism. One natural example of this even arises in the proof of faithfully flat descent: for $f : R \to S$ any ring homomorphism, $1_S \otimes f : S \to S \otimes_R S$ is split by the multiplication map whether or not it is flat.
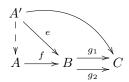
One may then ask whether there is a natural ring-theoretic condition implying effective descent for modules which includes both the case of a faithfully flat morphism and that of a split ring homomorphism. It may surprise the reader (at least it surprised this author) to learn that a complete answer to this question has been known since around 1970! Namely, it is not hard to check that a necessary condition for $f : R \to S$ to be an effective descent morphism for modules is that f must be *universally injective* in the category of R-modules, that is, for any R-module M, the map $1_M \otimes f : M \to M \otimes_R S$ must be injective. This then turns out to be a sufficient condition as well. For example, if f is split in the category of R-modules (but not necessarily in the category of rings), then f is an effective descent morphism for modules.

The history of this result is a bit involved: it was originally asserted by Olivier [Oli70], who called universally injective morphisms *pure*, but without a clear indication of proof. One can extract the result from the work of Joyal and Tierney [JT84], but to the best of our knowledge, the first free-standing proof to appear in the literature is that of Mesablishvili [Mes00]. The first purpose of this section is to expose Mesablishvili's proof; this requires little modification of his original presentation aside from correcting typos, with the one exception that we make explicit the relationship between the customary definition of a descent datum in algebraic geometry and the one used in [Mes00]. The proof turns out to be entirely category-theoretic, and consequently can be put in the language of monads (and thus applied in other contexts); see [JT04].

The second purpose of this section is to collect some information about which properties of modules, algebras, and morphisms can be descended along universally injective ring homomorphisms. The cases of finite modules and flat modules were treated by Mesablishvili [Mes02].

08WF **4.1. Category-theoretic preliminaries.** We start by recalling a few basic notions from category theory which will simplify the exposition. In this subsection, fix an ambient category.

For two morphisms $g_1, g_2 : B \to C$, recall that an *equalizer* of g_1 and g_2 is a morphism $f : A \to B$ which satisfies $g_1 \circ f = g_2 \circ f$ and is universal for this property. This second statement means that any commutative diagram

$$
\begin{array}{ccc}
A' & & \\
\vdots & \searrow^{e} & \\
\downarrow & & \\
A \xrightarrow{f} & B \underset{g_2}{\overset{g_1}{\rightrightarrows}} & C
\end{array}
$$

without the dashed arrow can be uniquely completed. We also say in this situation that the diagram

08WG (4.1.1)
$$
A \xrightarrow{f} B \underset{g_2}{\overset{g_1}{\rightrightarrows}} C
$$

is an equalizer. Reversing arrows gives the definition of a *coequalizer*. See Categories, Sections 10 and 11.

Since it involves a universal property, the property of being an equalizer is typically not stable under applying a covariant functor. Just as for monomorphisms and epimorphisms, one can get around this in some cases by exhibiting splittings.

08WH **Definition 4.2.** A *split equalizer* is a diagram (4.1.1) with $g_1 \circ f = g_2 \circ f$ for which there exist auxiliary morphisms $h : B \to A$ and $i : C \to B$ such that

08WI (4.2.1) $$ h \circ f = 1_A, \quad f \circ h = i \circ g_1, \quad i \circ g_2 = 1_B. $$

The point is that the equalities among arrows force (4.1.1) to be an equalizer: the map e factors uniquely through f by writing $e = f \circ (h \circ e)$. Consequently, applying a covariant functor to a split equalizer gives a split equalizer; applying a contravariant functor gives a *split coequalizer*, whose definition is apparent.

08WJ   **4.3. Universally injective morphisms.** Recall that *Rings* denotes the category
of commutative rings with 1. For an object R of *Rings* we denote Mod_R the category
of R-modules.

08WK   **Remark 4.4.** Any functor $F : \mathcal{A} \to \mathcal{B}$ of abelian categories which is exact and
takes nonzero objects to nonzero objects reflects injections and surjections. Namely,
exactness implies that F preserves kernels and cokernels (compare with Homology,
Section 7). For example, if $f : R \to S$ is a faithfully flat ring homomorphism, then
$\bullet \otimes_R S : \mathrm{Mod}_R \to \mathrm{Mod}_S$ has these properties.

Let R be a ring. Recall that a morphism $f : M \to N$ in Mod_R is *universally
injective* if for all $P \in \mathrm{Mod}_R$, the morphism $f \otimes 1_P : M \otimes_R P \to N \otimes_R P$ is
injective. See Algebra, Definition 82.1.

08WL   **Definition 4.5.** A ring map $f : R \to S$ is *universally injective* if it is universally
injective as a morphism in Mod_R.

08WM   **Example 4.6.** Any split injection in Mod_R is universally injective. In particular,
any split injection in *Rings* is universally injective.

08WN   **Example 4.7.** For a ring R and $f_1, \ldots, f_n \in R$ generating the unit ideal, the
morphism $R \to R_{f_1} \oplus \ldots \oplus R_{f_n}$ is universally injective. Although this is immediate
from Lemma 4.8, it is instructive to check it directly: we immediately reduce to
the case where R is local, in which case some f_i must be a unit and so the map
$R \to R_{f_i}$ is an isomorphism.

08WP   **Lemma 4.8.** *Any faithfully flat ring map is universally injective.*

Proof. This is a reformulation of Algebra, Lemma 82.11. $\square$

The key observation from [Mes00] is that universal injectivity can be usefully re-
formulated in terms of a splitting, using the usual construction of an injective
cogenerator in Mod_R.

08WQ   **Definition 4.9.** Let R be a ring. Define the contravariant functor $C : \mathrm{Mod}_R \to$
Mod_R by setting
$$C(M) = \mathrm{Hom}_{Ab}(M, \mathbf{Q}/\mathbf{Z}),$$
with the R-action on $C(M)$ given by $rf(s) = f(rs)$.

This functor was denoted $M \mapsto M^\vee$ in More on Algebra, Section 55.

08WR   **Lemma 4.10.** *For a ring R, the functor $C : Mod_R \to Mod_R$ is exact and reflects
injections and surjections.*

Proof. Exactness is More on Algebra, Lemma 55.6 and the other properties follow
from this, see Remark 4.4. $\square$

08WS   **Remark 4.11.** We will use frequently the standard adjunction between Hom and
tensor product, in the form of the natural isomorphism of contravariant functors

08WT   (4.11.1) $C(\bullet_1 \otimes_R \bullet_2) \cong \mathrm{Hom}_R(\bullet_1, C(\bullet_2)) : \mathrm{Mod}_R \times \mathrm{Mod}_R \to \mathrm{Mod}_R$

taking $f : M_1 \otimes_R M_2 \to \mathbf{Q}/\mathbf{Z}$ to the map $m_1 \mapsto (m_2 \mapsto f(m_1 \otimes m_2))$. See Algebra,
Lemma 14.5. A corollary of this observation is that if

$$C(M) \rightrightarrows C(N) \longrightarrow C(P)$$

is a split coequalizer diagram in Mod_R, then so is

$$C(M \otimes_R Q) \rightrightarrows C(N \otimes_R Q) \longrightarrow C(P \otimes_R Q)$$

for any $Q \in \mathrm{Mod}_R$.

08WU **Lemma 4.12.** *Let R be a ring. A morphism $f : M \to N$ in Mod_R is universally injective if and only if $C(f) : C(N) \to C(M)$ is a split surjection.*

Proof. By (4.11.1), for any $P \in \mathrm{Mod}_R$ we have a commutative diagram

$$
\begin{array}{ccc}
\mathrm{Hom}_R(P, C(N)) & \xrightarrow{\ \mathrm{Hom}_R(P, C(f))\ } & \mathrm{Hom}_R(P, C(M)) \\
\downarrow{\scriptstyle\cong} & & \downarrow{\scriptstyle\cong} \\
C(P \otimes_R N) & \xrightarrow{\ C(1_P \otimes f)\ } & C(P \otimes_R M).
\end{array}
$$

If f is universally injective, then $1_{C(M)} \otimes f : C(M) \otimes_R M \to C(M) \otimes_R N$ is injective, so both rows in the above diagram are surjective for $P = C(M)$. We may thus lift $1_{C(M)} \in \mathrm{Hom}_R(C(M), C(M))$ to some $g \in \mathrm{Hom}_R(C(N), C(M))$ splitting $C(f)$. Conversely, if $C(f)$ is a split surjection, then both rows in the above diagram are surjective, so by Lemma 4.10, $1_P \otimes f$ is injective. $\square$

08WV **Remark 4.13.** Let $f : M \to N$ be a universally injective morphism in Mod_R. By choosing a splitting g of $C(f)$, we may construct a functorial splitting of $C(1_P \otimes f)$ for each $P \in \mathrm{Mod}_R$. Namely, by (4.11.1) this amounts to splitting $\mathrm{Hom}_R(P, C(f))$ functorially in P, and this is achieved by the map $g \circ \bullet$.

08WW **4.14. Descent for modules and their morphisms.** Throughout this subsection, fix a ring map $f : R \to S$. As seen in Section 3 we can use the language of cosimplicial algebras to talk about descent data for modules, but in this subsection we prefer a more down to earth terminology.

For $i = 1, 2, 3$, let S_i be the i-fold tensor product of S over R. Define the ring homomorphisms $\delta_0^1, \delta_1^1 : S_1 \to S_2$, $\delta_{01}^1, \delta_{02}^1, \delta_{12}^1 : S_1 \to S_3$, and $\delta_0^2, \delta_1^2, \delta_2^2 : S_2 \to S_3$ by the formulas

$$
\begin{aligned}
\delta_0^1(a_0) &= 1 \otimes a_0 \\
\delta_1^1(a_0) &= a_0 \otimes 1 \\
\delta_0^2(a_0 \otimes a_1) &= 1 \otimes a_0 \otimes a_1 \\
\delta_1^2(a_0 \otimes a_1) &= a_0 \otimes 1 \otimes a_1 \\
\delta_2^2(a_0 \otimes a_1) &= a_0 \otimes a_1 \otimes 1 \\
\delta_{01}^1(a_0) &= 1 \otimes 1 \otimes a_0 \\
\delta_{02}^1(a_0) &= 1 \otimes a_0 \otimes 1 \\
\delta_{12}^1(a_0) &= a_0 \otimes 1 \otimes 1.
\end{aligned}
$$

In other words, the upper index indicates the source ring, while the lower index indicates where to insert factors of 1. (This notation is compatible with the notation introduced in Section 3.)

Recall[3] from Definition 3.1 that for $M \in \mathrm{Mod}_S$, a *descent datum* on M relative to f is an isomorphism

$$\theta : M \otimes_{S,\delta_0^1} S_2 \longrightarrow M \otimes_{S,\delta_1^1} S_2$$

of S_2-modules satisfying the *cocycle condition*

08WX   (4.14.1) $(\theta \otimes \delta_2^2) \circ (\theta \otimes \delta_2^0) = (\theta \otimes \delta_2^1) : M \otimes_{S,\delta_{01}^1} S_3 \to M \otimes_{S,\delta_{12}^1} S_3.$

Let $DD_{S/R}$ be the category of S-modules equipped with descent data relative to f.

For example, for $M_0 \in \mathrm{Mod}_R$ and a choice of isomorphism $M \cong M_0 \otimes_R S$ gives rise to a descent datum by identifying $M \otimes_{S,\delta_0^1} S_2$ and $M \otimes_{S,\delta_1^1} S_2$ naturally with $M_0 \otimes_R S_2$. This construction in particular defines a functor $f^* : \mathrm{Mod}_R \to DD_{S/R}$.

08WY   **Definition 4.15.** The functor $f^* : \mathrm{Mod}_R \to DD_{S/R}$ is called *base extension along* f. We say that f is a *descent morphism for modules* if f^* is fully faithful. We say that f is an *effective descent morphism for modules* if f^* is an equivalence of categories.

Our goal is to show that for f universally injective, we can use θ to locate M_0 within M. This process makes crucial use of some equalizer diagrams.

08WZ   **Lemma 4.16.** *For $(M,\theta) \in DD_{S/R}$, the diagram*
(4.16.1)

08X0 $M \xrightarrow{\;\theta \circ (1_M \otimes \delta_0^1)\;} M \otimes_{S,\delta_1^1} S_2 \overset{(\theta \otimes \delta_2^2) \circ (1_M \otimes \delta_0^2)}{\underset{1_{M \otimes S_2} \otimes \delta_1^2}{\rightrightarrows}} M \otimes_{S,\delta_{12}^1} S_3$

is a split equalizer.

Proof. Define the ring homomorphisms $\sigma_0^0 : S_2 \to S_1$ and $\sigma_0^1, \sigma_1^1 : S_3 \to S_2$ by the formulas

$$\sigma_0^0(a_0 \otimes a_1) = a_0 a_1$$
$$\sigma_0^1(a_0 \otimes a_1 \otimes a_2) = a_0 a_1 \otimes a_2$$
$$\sigma_1^1(a_0 \otimes a_1 \otimes a_2) = a_0 \otimes a_1 a_2.$$

We then take the auxiliary morphisms to be $1_M \otimes \sigma_0^0 : M \otimes_{S,\delta_1^1} S_2 \to M$ and $1_M \otimes \sigma_0^1 : M \otimes_{S,\delta_{12}^1} S_3 \to M \otimes_{S,\delta_1^1} S_2$. Of the compatibilities required in (4.2.1), the first follows from tensoring the cocycle condition (4.14.1) with σ_1^1 and the others are immediate. $\qquad\square$

08X1   **Lemma 4.17.** *For $(M,\theta) \in DD_{S/R}$, the diagram*
(4.17.1)

08X2 $C(M \otimes_{S,\delta_{12}^1} S_3) \overset{C((\theta \otimes \delta_2^2) \circ (1_M \otimes \delta_0^2))}{\underset{C(1_{M \otimes S_2} \otimes \delta_1^2)}{\rightrightarrows}} C(M \otimes_{S,\delta_1^1} S_2) \xrightarrow{\;C(\theta \circ (1_M \otimes \delta_0^1))\;} C(M).$

obtained by applying C to (4.16.1) is a split coequalizer.

Proof. Omitted. $\qquad\square$

_____

[3]To be precise, our θ here is the inverse of φ from Definition 3.1.

08X3 **Lemma 4.18.** *The diagram*

08X4 (4.18.1)
$$S_1 \xrightarrow{\ \delta_1^1\ } S_2 \underset{\delta_1^2}{\overset{\delta_2^2}{\rightrightarrows}} S_3$$

is a split equalizer.

Proof. In Lemma 4.16, take $(M, \theta) = f^*(S)$. $\qquad\qquad\qquad\square$

This suggests a definition of a potential quasi-inverse functor for f^*.

08X5 **Definition 4.19.** Define the functor $f_* : DD_{S/R} \to \mathrm{Mod}_R$ by taking $f_*(M, \theta)$ to be the R-submodule of M for which the diagram

08X6 (4.19.1)
$$f_*(M, \theta) \longrightarrow M \underset{1_M \otimes \delta_1^1}{\overset{\theta \circ (1_M \otimes \delta_0^1)}{\rightrightarrows}} M \otimes_{S, \delta_1^1} S_2$$

is an equalizer.

Using Lemma 4.16 and the fact that the restriction functor $\mathrm{Mod}_S \to \mathrm{Mod}_R$ is right adjoint to the base extension functor $\bullet \otimes_R S : \mathrm{Mod}_R \to \mathrm{Mod}_S$, we deduce that f_* is right adjoint to f^*.

We are ready for the key lemma. In the faithfully flat case this is a triviality (see Remark 4.21), but in the general case some argument is needed.

08X7 **Lemma 4.20.** *If f is universally injective, then the diagram*
(4.20.1)

08X8
$$f_*(M, \theta) \otimes_R S \xrightarrow{\theta \circ (1_M \otimes \delta_0^1)} M \otimes_{S, \delta_1^1} S_2 \underset{1_M \otimes_{S_2} \otimes \delta_1^2}{\overset{(\theta \otimes \delta_2^2) \circ (1_M \otimes \delta_0^2)}{\rightrightarrows}} M \otimes_{S, \delta_{12}^1} S_3$$

obtained by tensoring (4.19.1) over R with S is an equalizer.

Proof. By Lemma 4.12 and Remark 4.13, the map $C(1_N \otimes f) : C(N \otimes_R S) \to C(N)$ can be split functorially in N. This gives the upper vertical arrows in the commutative diagram

in which the compositions along the columns are identity morphisms. The second row is the coequalizer diagram (4.17.1); this produces the dashed arrow. From the top right square, we obtain auxiliary morphisms $C(f_*(M, \theta)) \to C(M)$ and $C(M) \to C(M \otimes_{S, \delta_1^1} S_2)$ which imply that the first row is a split coequalizer diagram.

By Remark 4.11, we may tensor with S inside C to obtain the split coequalizer diagram

$$C(M \otimes_{S,\delta_2^2 \circ \delta_1^1} S_3) \xrightarrow[C(1_{M \otimes S_2} \otimes \delta_1^2)]{C((\theta \otimes \delta_2^2) \circ (1_M \otimes \delta_0^2))} C(M \otimes_{S,\delta_1^1} S_2) \xrightarrow{C(\theta \circ (1_M \otimes \delta_0^1))} C(f_*(M,\theta) \otimes_R S).$$

By Lemma 4.10, we conclude (4.20.1) must also be an equalizer. $\square$

08X9 **Remark 4.21.** If f is a split injection in Mod_R, one can simplify the argument by splitting f directly, without using C. Things are even simpler if f is faithfully flat; in this case, the conclusion of Lemma 4.20 is immediate because tensoring over R with S preserves all equalizers.

08XA **Theorem 4.22.** *The following conditions are equivalent.*
  (a) *The morphism f is a descent morphism for modules.*
  (b) *The morphism f is an effective descent morphism for modules.*
  (c) *The morphism f is universally injective.*

Proof. It is clear that (b) implies (a). We now check that (a) implies (c). If f is not universally injective, we can find $M \in \mathrm{Mod}_R$ such that the map $1_M \otimes f : M \to M \otimes_R S$ has nontrivial kernel N. The natural projection $M \to M/N$ is not an isomorphism, but its image in $DD_{S/R}$ is an isomorphism. Hence f^* is not fully faithful.

We finally check that (c) implies (b). By Lemmas 4.16 and 4.20, for $(M,\theta) \in DD_{S/R}$, the natural map $f^* f_*(M,\theta) \to M$ is an isomorphism of S-modules. On the other hand, for $M_0 \in \mathrm{Mod}_R$, we may tensor (4.18.1) with M_0 over R to obtain an equalizer sequence, so $M_0 \to f_* f^* M_0$ is an isomorphism. Consequently, f_* and f^* are quasi-inverse functors, proving the claim. $\square$

08XB **4.23. Descent for properties of modules.** Throughout this subsection, fix a universally injective ring map $f : R \to S$, an object $M \in \mathrm{Mod}_R$, and a ring map $R \to A$. We now investigate the question of which properties of M or A can be checked after base extension along f. We start with some results from [Mes02].

08XC **Lemma 4.24.** *If $M \in \mathrm{Mod}_R$ is flat, then $C(M)$ is an injective R-module.*

Proof. Let $0 \to N \to P \to Q \to 0$ be an exact sequence in Mod_R. Since M is flat,

$$0 \to N \otimes_R M \to P \otimes_R M \to Q \otimes_R M \to 0$$

is exact. By Lemma 4.10,

$$0 \to C(Q \otimes_R M) \to C(P \otimes_R M) \to C(N \otimes_R M) \to 0$$

is exact. By (4.11.1), this last sequence can be rewritten as

$$0 \to \mathrm{Hom}_R(Q, C(M)) \to \mathrm{Hom}_R(P, C(M)) \to \mathrm{Hom}_R(N, C(M)) \to 0.$$

Hence $C(M)$ is an injective object of Mod_R. $\square$

08XD **Theorem 4.25.** *If $M \otimes_R S$ has one of the following properties as an S-module*
  (a) *finitely generated;*
  (b) *finitely presented;*
  (c) *flat;*
  (d) *faithfully flat;*

(e) *finite projective;*

then so does M as an R-module (and conversely).

Proof. To prove (a), choose a finite set $\{n_i\}$ of generators of $M \otimes_R S$ in Mod_S. Write each n_i as $\sum_j m_{ij} \otimes s_{ij}$ with $m_{ij} \in M$ and $s_{ij} \in S$. Let F be the finite free R-module with basis e_{ij} and let $F \to M$ be the R-module map sending e_{ij} to m_{ij}. Then $F \otimes_R S \to M \otimes_R S$ is surjective, so $\mathrm{Coker}(F \to M) \otimes_R S$ is zero and hence $\mathrm{Coker}(F \to M)$ is zero. This proves (a).

To see (b) assume $M \otimes_R S$ is finitely presented. Then M is finitely generated by (a). Choose a surjection $R^{\oplus n} \to M$ with kernel K. Then $K \otimes_R S \to S^{\oplus r} \to M \otimes_R S \to 0$ is exact. By Algebra, Lemma 5.3 the kernel of $S^{\oplus r} \to M \otimes_R S$ is a finite S-module. Thus we can find finitely many elements $k_1, \ldots, k_t \in K$ such that the images of $k_i \otimes 1$ in $S^{\oplus r}$ generate the kernel of $S^{\oplus r} \to M \otimes_R S$. Let $K' \subset K$ be the submodule generated by $k_1, \ldots, k_t$. Then $M' = R^{\oplus r}/K'$ is a finitely presented R-module with a morphism $M' \to M$ such that $M' \otimes_R S \to M \otimes_R S$ is an isomorphism. Thus $M' \cong M$ as desired.

To prove (c), let $0 \to M' \to M'' \to M \to 0$ be a short exact sequence in Mod_R. Since $\bullet \otimes_R S$ is a right exact functor, $M'' \otimes_R S \to M \otimes_R S$ is surjective. So by Lemma 4.10 the map $C(M \otimes_R S) \to C(M'' \otimes_R S)$ is injective. If $M \otimes_R S$ is flat, then Lemma 4.24 shows $C(M \otimes_R S)$ is an injective object of Mod_S, so the injection $C(M \otimes_R S) \to C(M'' \otimes_R S)$ is split in Mod_S and hence also in Mod_R. Since $C(M \otimes_R S) \to C(M)$ is a split surjection by Lemma 4.12, it follows that $C(M) \to C(M'')$ is a split injection in Mod_R. That is, the sequence

$$0 \to C(M) \to C(M'') \to C(M') \to 0$$

is split exact. For $N \in \mathrm{Mod}_R$, by (4.11.1) we see that

$$0 \to C(M \otimes_R N) \to C(M'' \otimes_R N) \to C(M' \otimes_R N) \to 0$$

is split exact. By Lemma 4.10,

$$0 \to M' \otimes_R N \to M'' \otimes_R N \to M \otimes_R N \to 0$$

is exact. This implies M is flat over R. Namely, taking M' a free module surjecting onto M we conclude that $\mathrm{Tor}_1^R(M, N) = 0$ for all modules N and we can use Algebra, Lemma 75.8. This proves (c).

To deduce (d) from (c), note that if $N \in \mathrm{Mod}_R$ and $M \otimes_R N$ is zero, then $M \otimes_R S \otimes_S (N \otimes_R S) \cong (M \otimes_R N) \otimes_R S$ is zero, so $N \otimes_R S$ is zero and hence N is zero.

To deduce (e) at this point, it suffices to recall that M is finitely generated and projective if and only if it is finitely presented and flat. See Algebra, Lemma 78.2. $\square$

There is a variant for R-algebras.

08XE **Theorem 4.26.** *If $A \otimes_R S$ has one of the following properties as an S-algebra*

(a) *of finite type;*
(b) *of finite presentation;*
(c) *formally unramified;*
(d) *unramified;*
(e) *étale;*

then so does A as an R-algebra (and of course conversely).

Proof. To prove (a), choose a finite set $\{x_i\}$ of generators of $A \otimes_R S$ over S. Write each x_i as $\sum_j y_{ij} \otimes s_{ij}$ with $y_{ij} \in A$ and $s_{ij} \in S$. Let F be the polynomial R-algebra on variables e_{ij} and let $F \to M$ be the R-algebra map sending e_{ij} to y_{ij}. Then $F \otimes_R S \to A \otimes_R S$ is surjective, so $\mathrm{Coker}(F \to A) \otimes_R S$ is zero and hence $\mathrm{Coker}(F \to A)$ is zero. This proves (a).

To see (b) assume $A \otimes_R S$ is a finitely presented S-algebra. Then A is finite type over R by (a). Choose a surjection $R[x_1, \ldots, x_n] \to A$ with kernel I. Then $I \otimes_R S \to S[x_1, \ldots, x_n] \to A \otimes_R S \to 0$ is exact. By Algebra, Lemma 6.3 the kernel of $S[x_1, \ldots, x_n] \to A \otimes_R S$ is a finitely generated ideal. Thus we can find finitely many elements $y_1, \ldots, y_t \in I$ such that the images of $y_i \otimes 1$ in $S[x_1, \ldots, x_n]$ generate the kernel of $S[x_1, \ldots, x_n] \to A \otimes_R S$. Let $I' \subset I$ be the ideal generated by $y_1, \ldots, y_t$. Then $A' = R[x_1, \ldots, x_n]/I'$ is a finitely presented R-algebra with a morphism $A' \to A$ such that $A' \otimes_R S \to A \otimes_R S$ is an isomorphism. Thus $A' \cong A$ as desired.

To prove (c), recall that A is formally unramified over R if and only if the module of relative differentials $\Omega_{A/R}$ vanishes, see Algebra, Lemma 148.3 or [GD67, Proposition 17.2.1]. Since $\Omega_{(A \otimes_R S)/S} = \Omega_{A/R} \otimes_R S$, the vanishing descends by Theorem 4.22.

To deduce (d) from the previous cases, recall that A is unramified over R if and only if A is formally unramified and of finite type over R, see Algebra, Lemma 151.2.

To prove (e), recall that by Algebra, Lemma 151.8 or [GD67, Théorème 17.6.1] the algebra A is étale over R if and only if A is flat, unramified, and of finite presentation over R. $\square$

08XF **Remark 4.27.** It would make things easier to have a faithfully flat ring homomorphism $g : R \to T$ for which $T \to S \otimes_R T$ has some extra structure. For instance, if one could ensure that $T \to S \otimes_R T$ is split in *Rings*, then it would follow that every property of a module or algebra which is stable under base extension and which descends along faithfully flat morphisms also descends along universally injective morphisms. An obvious guess would be to find g for which T is not only faithfully flat but also injective in Mod_R, but even for $R = \mathbf{Z}$ no such homomorphism can exist.

5. Fpqc descent of quasi-coherent sheaves

023R The main application of flat descent for modules is the corresponding descent statement for quasi-coherent sheaves with respect to fpqc-coverings.

023S **Lemma 5.1.** *Let S be an affine scheme. Let $\mathcal{U} = \{f_i : U_i \to S\}_{i=1,\ldots,n}$ be a standard fpqc covering of S, see Topologies, Definition 9.10. Any descent datum on quasi-coherent sheaves for $\mathcal{U} = \{U_i \to S\}$ is effective. Moreover, the functor from the category of quasi-coherent $\mathcal{O}_S$-modules to the category of descent data with respect to $\mathcal{U}$ is fully faithful.*

Proof. This is a restatement of Proposition 3.9 in terms of schemes. First, note that a descent datum ξ for quasi-coherent sheaves with respect to $\mathcal{U}$ is exactly the same as a descent datum ξ' for quasi-coherent sheaves with respect to the covering $\mathcal{U}' = \{\coprod_{i=1,\ldots,n} U_i \to S\}$. Moreover, effectivity for ξ is the same as effectivity for

ξ'. Hence we may assume $n = 1$, i.e., $\mathcal{U} = \{U \to S\}$ where U and S are affine. In this case descent data correspond to descent data on modules with respect to the ring map

$$\Gamma(S, \mathcal{O}) \longrightarrow \Gamma(U, \mathcal{O}).$$

Since $U \to S$ is surjective and flat, we see that this ring map is faithfully flat. In other words, Proposition 3.9 applies and we win. $\square$

023T   **Proposition 5.2.** *Let S be a scheme. Let $\mathcal{U} = \{\varphi_i : U_i \to S\}$ be an fpqc covering, see Topologies, Definition 9.1. Any descent datum on quasi-coherent sheaves for $\mathcal{U} = \{U_i \to S\}$ is effective. Moreover, the functor from the category of quasi-coherent $\mathcal{O}_S$-modules to the category of descent data with respect to $\mathcal{U}$ is fully faithful.*

Proof. Let $S = \bigcup_{j \in J} V_j$ be an affine open covering. For $j, j' \in J$ we denote $V_{jj'} = V_j \cap V_{j'}$ the intersection (which need not be affine). For $V \subset S$ open we denote $\mathcal{U}_V = \{V \times_S U_i \to V\}_{i \in I}$ which is a fpqc-covering (Topologies, Lemma 9.8). By definition of an fpqc covering, we can find for each $j \in J$ a finite set K_j, a map $\underline{i} : K_j \to I$, affine opens $U_{\underline{i}(k),k} \subset U_{\underline{i}(k)}$, $k \in K_j$ such that $\mathcal{V}_j = \{U_{\underline{i}(k),k} \to V_j\}_{k \in K_j}$ is a standard fpqc covering of V_j. And of course, $\mathcal{V}_j$ is a refinement of $\mathcal{U}_{V_j}$. Picture

$$
\begin{array}{ccccc}
\mathcal{V}_j & \longrightarrow & \mathcal{U}_{V_j} & \longrightarrow & \mathcal{U} \\
\downarrow & & \downarrow & & \downarrow \\
V_j & = & V_j & \longrightarrow & S
\end{array}
$$

where the top horizontal arrows are morphisms of families of morphisms with fixed target (see Sites, Definition 8.1).

To prove the proposition you show successively the faithfulness, fullness, and essential surjectivity of the functor from quasi-coherent sheaves to descent data.

Faithfulness. Let $\mathcal{F}$, $\mathcal{G}$ be quasi-coherent sheaves on S and let $a, b : \mathcal{F} \to \mathcal{G}$ be homomorphisms of $\mathcal{O}_S$-modules. Suppose $\varphi_i^*(a) = \varphi_i^*(b)$ for all i. Pick $s \in S$. Then $s = \varphi_i(u)$ for some $i \in I$ and $u \in U_i$. Since $\mathcal{O}_{S,s} \to \mathcal{O}_{U_i,u}$ is flat, hence faithfully flat (Algebra, Lemma 39.17) we see that $a_s = b_s : \mathcal{F}_s \to \mathcal{G}_s$. Hence $a = b$.

Fully faithfulness. Let $\mathcal{F}$, $\mathcal{G}$ be quasi-coherent sheaves on S and let $a_i : \varphi_i^* \mathcal{F} \to \varphi_i^* \mathcal{G}$ be homomorphisms of $\mathcal{O}_{U_i}$-modules such that $\mathrm{pr}_0^* a_i = \mathrm{pr}_1^* a_j$ on $U_i \times_U U_j$. We can pull back these morphisms to get morphisms

$$a_k : \varphi_{\underline{i}(k)}^* \mathcal{F}|_{U_{\underline{i}(k),k}} \longrightarrow \varphi_{\underline{i}(k)}^* \mathcal{G}|_{U_{\underline{i}(k),k}}$$

$k \in K_j$ with notation as above. Moreover, Lemma 2.2 assures us that these define a morphism between (canonical) descent data on $\mathcal{V}_j$. Hence, by Lemma 5.1, we get correspondingly unique morphisms $a_j : \mathcal{F}|_{V_j} \to \mathcal{G}|_{V_j}$. To see that $a_j|_{V_{jj'}} = a_{j'}|_{V_{jj'}}$ we use that both a_j and $a_{j'}$ agree with the pullback of the morphism $(a_i)_{i \in I}$ of (canonical) descent data to any covering refining both $\mathcal{V}_{j,V_{jj'}}$ and $\mathcal{V}_{j',V_{jj'}}$, and using the faithfulness already shown. For example the covering $\mathcal{V}_{jj'} = \{V_k \times_S V_{k'} \to V_{jj'}\}_{k \in K_j, k' \in K_{j'}}$ will do.

Essential surjectivity. Let $\xi = (\mathcal{F}_i, \varphi_{ii'})$ be a descent datum for quasi-coherent sheaves relative to the covering $\mathcal{U}$. Pull back this descent datum to get descent data ξ_j for quasi-coherent sheaves relative to the coverings $\mathcal{V}_j$ of V_j. By Lemma 5.1

once again there exist quasi-coherent sheaves $\mathcal{F}_j$ on V_j whose associated canonical descent datum is isomorphic to ξ_j. By fully faithfulness (proved above) we see there are isomorphisms

$$\phi_{jj'} : \mathcal{F}_j|_{V_{jj'}} \longrightarrow \mathcal{F}_{j'}|_{V_{jj'}}$$

corresponding to the isomorphism of descent data between the pullback of ξ_j and $\xi_{j'}$ to $\mathcal{V}_{jj'}$. To see that these maps $\phi_{jj'}$ satisfy the cocycle condition we use faithfulness (proved above) over the triple intersections $V_{jj'j''}$. Hence, by Lemma 2.4 we see that the sheaves $\mathcal{F}_j$ glue to a quasi-coherent sheaf $\mathcal{F}$ as desired. We still have to verify that the canonical descent datum relative to $\mathcal{U}$ associated to $\mathcal{F}$ is isomorphic to the descent datum we started out with. This verification is omitted. $\square$

6. Galois descent for quasi-coherent sheaves

0CDQ   Galois descent for quasi-coherent sheaves is just a special case of fpqc descent for quasi-coherent sheaves. In this section we will explain how to translate from a Galois descent to an fpqc descent and then apply earlier results to conclude.

Let k'/k be a field extension. Then $\{\mathrm{Spec}(k') \to \mathrm{Spec}(k)\}$ is an fpqc covering. Let X be a scheme over k. For a k-algebra A we set $X_A = X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(A)$. By Topologies, Lemma 9.8 we see that $\{X_{k'} \to X\}$ is an fpqc covering. Observe that

$$X_{k'} \times_X X_{k'} = X_{k' \otimes_k k'} \quad \text{and} \quad X_{k'} \times_X X_{k'} \times_X X_{k'} = X_{k' \otimes_k k' \otimes_k k'}$$

Thus a descent datum for quasi-coherent sheaves with respect to $\{X_{k'} \to X\}$ is given by a quasi-coherent sheaf $\mathcal{F}$ on $X_{k'}$, an isomorphism $\varphi : \mathrm{pr}_0^*\mathcal{F} \to \mathrm{pr}_1^*\mathcal{F}$ on $X_{k' \otimes_k k'}$ which satisfies an obvious cocycle condition on $X_{k' \otimes_k k' \otimes_k k'}$. We will work out what this means in the case of a Galois extension below.

Let k'/k be a finite Galois extension with Galois group $G = \mathrm{Gal}(k'/k)$. Then there are k-algebra isomorphisms

$$k' \otimes_k k' \longrightarrow \prod_{\sigma \in G} k', \quad a \otimes b \longrightarrow \prod a\sigma(b)$$

and

$$k' \otimes_k k' \otimes_k k' \longrightarrow \prod_{(\sigma,\tau) \in G \times G} k', \quad a \otimes b \otimes c \longrightarrow \prod a\sigma(b)\sigma(\tau(c))$$

The reason for choosing here $a\sigma(b)\sigma(\tau(c))$ and not $a\sigma(b)\tau(c)$ is that the formulas below simplify but it isn't strictly necessary. Given $\sigma \in G$ we denote

$$f_\sigma = \mathrm{id}_X \times \mathrm{Spec}(\sigma) : X_{k'} \longrightarrow X_{k'}$$

Please keep in mind that because $\mathrm{Spec}(-)$ is a contravariant functor we have $f_{\sigma\tau} = f_\tau \circ f_\sigma$ and not the other way around. Using the first isomorphism above we obtain an identification

$$X_{k' \otimes_k k'} = \coprod_{\sigma \in G} X_{k'}$$

such that pr_0 corresponds to the map

$$\coprod_{\sigma \in G} X_{k'} \xrightarrow{\coprod \mathrm{id}} X_{k'}$$

and such that pr_1 corresponds to the map

$$\coprod_{\sigma \in G} X_{k'} \xrightarrow{\coprod f_\sigma} X_{k'}$$

Thus we see that a descent datum φ on $\mathcal{F}$ over $X_{k'}$ corresponds to a family of isomorphisms $\varphi_\sigma : \mathcal{F} \to f_\sigma^* \mathcal{F}$. To work out the cocycle condition we use the identification

$$X_{k' \otimes_k k' \otimes_k k'} = \coprod_{(\sigma,\tau) \in G \times G} X_{k'}.$$

we get from our isomorphism of algebras above. Via this identification the map pr_{01} corresponds to the map

$$\coprod_{(\sigma,\tau) \in G \times G} X_{k'} \longrightarrow \coprod_{\sigma \in G} X_{k'}$$

which maps the summand with index (σ, τ) to the summand with index σ via the identity morphism. The map pr_{12} corresponds to the map

$$\coprod_{(\sigma,\tau) \in G \times G} X_{k'} \longrightarrow \coprod_{\sigma \in G} X_{k'}$$

which maps the summand with index (σ, τ) to the summand with index τ via the morphism f_σ. Finally, the map pr_{02} corresponds to the map

$$\coprod_{(\sigma,\tau) \in G \times G} X_{k'} \longrightarrow \coprod_{\sigma \in G} X_{k'}$$

which maps the summand with index (σ, τ) to the summand with index $\sigma\tau$ via the identity morphism. Thus the cocycle condition

$$\mathrm{pr}_{02}^* \varphi = \mathrm{pr}_{12}^* \varphi \circ \mathrm{pr}_{01}^* \varphi$$

translates into one condition for each pair (σ, τ), namely

$$\varphi_{\sigma\tau} = f_\sigma^* \varphi_\tau \circ \varphi_\sigma$$

as maps $\mathcal{F} \to f_{\sigma\tau}^* \mathcal{F}$. (Everything works out beautifully; for example the target of φ_σ is $f_\sigma^* \mathcal{F}$ and the source of $f_\sigma^* \varphi_\tau$ is $f_\sigma^* \mathcal{F}$ as well.)

0CDR **Lemma 6.1.** *Let k'/k be a (finite) Galois extension with Galois group G. Let X be a scheme over k. The category of quasi-coherent $\mathcal{O}_X$-modules is equivalent to the category of systems $(\mathcal{F}, (\varphi_\sigma)_{\sigma \in G})$ where*

    (1) *$\mathcal{F}$ is a quasi-coherent module on $X_{k'}$,*
    (2) *$\varphi_\sigma : \mathcal{F} \to f_\sigma^* \mathcal{F}$ is an isomorphism of modules,*
    (3) *$\varphi_{\sigma\tau} = f_\sigma^* \varphi_\tau \circ \varphi_\sigma$ for all $\sigma, \tau \in G$.*

Here $f_\sigma = id_X \times \mathrm{Spec}(\sigma) : X_{k'} \to X_{k'}$.

Proof. As seen above a datum $(\mathcal{F}, (\varphi_\sigma)_{\sigma \in G})$ as in the lemma is the same thing as a descent datum for the fpqc covering $\{X_{k'} \to X\}$. Thus the lemma follows from Proposition 5.2. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

A slightly more general case of the above is the following. Suppose we have a surjective finite étale morphism $X \to Y$ and a finite group G together with a group homomorphism $G^{opp} \to \mathrm{Aut}_Y(X), \sigma \mapsto f_\sigma$ such that the map

$$G \times X \longrightarrow X \times_Y X, \quad (\sigma, x) \longmapsto (x, f_\sigma(x))$$

is an isomorphism. Then the same result as above holds.

0D1V **Lemma 6.2.** *Let $X \to Y$, G, and $f_\sigma : X \to X$ be as above. The category of quasi-coherent $\mathcal{O}_Y$-modules is equivalent to the category of systems $(\mathcal{F}, (\varphi_\sigma)_{\sigma \in G})$ where*

    (1) *$\mathcal{F}$ is a quasi-coherent $\mathcal{O}_X$-module,*
    (2) *$\varphi_\sigma : \mathcal{F} \to f_\sigma^* \mathcal{F}$ is an isomorphism of modules,*

(3) $\varphi_{\sigma\tau} = f_\sigma^* \varphi_\tau \circ \varphi_\sigma$ *for all* $\sigma, \tau \in G$.

Proof. Since $X \to Y$ is surjective finite étale $\{X \to Y\}$ is an fpqc covering. Since $G \times X \to X \times_Y X$, $(\sigma, x) \mapsto (x, f_\sigma(x))$ is an isomorphism, we see that $G \times G \times X \to X \times_Y X \times_Y X$, $(\sigma, \tau, x) \mapsto (x, f_\sigma(x), f_{\sigma\tau}(x))$ is an isomorphism too. Using these identifications, the category of data as in the lemma is the same as the category of descent data for quasi-coherent sheaves for the covering $\{x \to Y\}$. Thus the lemma follows from Proposition 5.2. $\square$

7. Descent of finiteness properties of modules

05AY   In this section we prove that one can check quasi-coherent module has a certain finiteness conditions by checking on the members of a covering.

05AZ   **Lemma 7.1.** *Let X be a scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $\{f_i : X_i \to X\}_{i \in I}$ be an fpqc covering such that each $f_i^* \mathcal{F}$ is a finite type $\mathcal{O}_{X_i}$-module. Then $\mathcal{F}$ is a finite type $\mathcal{O}_X$-module.*

Proof. Omitted. For the affine case, see Algebra, Lemma 83.2. $\square$

09UB   **Lemma 7.2.** *Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of locally ringed spaces. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_Y$-modules. If*

(1) *f is open as a map of topological spaces,*
(2) *f is surjective and flat, and*
(3) *$f^* \mathcal{F}$ is of finite type,*

then $\mathcal{F}$ is of finite type.

Proof. Let $y \in Y$ be a point. Choose a point $x \in X$ mapping to y. Choose an open $x \in U \subset X$ and elements $s_1, \ldots, s_n$ of $f^* \mathcal{F}(U)$ which generate $f^* \mathcal{F}$ over U. Since $f^* \mathcal{F} = f^{-1} \mathcal{F} \otimes_{f^{-1} \mathcal{O}_Y} \mathcal{O}_X$ we can after shrinking U assume $s_i = \sum t_{ij} \otimes a_{ij}$ with $t_{ij} \in f^{-1} \mathcal{F}(U)$ and $a_{ij} \in \mathcal{O}_X(U)$. After shrinking U further we may assume that t_{ij} comes from a section $s_{ij} \in \mathcal{F}(V)$ for some $V \subset Y$ open with $f(U) \subset V$. Let N be the number of sections s_{ij} and consider the map

$$\sigma = (s_{ij}) : \mathcal{O}_V^{\oplus N} \to \mathcal{F}|_V$$

By our choice of the sections we see that $f^* \sigma|_U$ is surjective. Hence for every $u \in U$ the map

$$\sigma_{f(u)} \otimes_{\mathcal{O}_{Y, f(u)}} \mathcal{O}_{X, u} : \mathcal{O}_{X, u}^{\oplus N} \longrightarrow \mathcal{F}_{f(u)} \otimes_{\mathcal{O}_{Y, f(u)}} \mathcal{O}_{X, u}$$

is surjective. As f is flat, the local ring map $\mathcal{O}_{Y, f(u)} \to \mathcal{O}_{X, u}$ is flat, hence faithfully flat (Algebra, Lemma 39.17). Hence $\sigma_{f(u)}$ is surjective. Since f is open, $f(U)$ is an open neighbourhood of y and the proof is done. $\square$

05B0   **Lemma 7.3.** *Let X be a scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $\{f_i : X_i \to X\}_{i \in I}$ be an fpqc covering such that each $f_i^* \mathcal{F}$ is an $\mathcal{O}_{X_i}$-module of finite presentation. Then $\mathcal{F}$ is an $\mathcal{O}_X$-module of finite presentation.*

Proof. Omitted. For the affine case, see Algebra, Lemma 83.2. $\square$

082U   **Lemma 7.4.** *Let X be a scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $\{f_i : X_i \to X\}_{i \in I}$ be an fpqc covering such that each $f_i^* \mathcal{F}$ is locally generated by r sections as an $\mathcal{O}_{X_i}$-module. Then $\mathcal{F}$ is locally generated by r sections as an $\mathcal{O}_X$-module.*

Proof. By Lemma 7.1 we see that $\mathcal{F}$ is of finite type. Hence Nakayama's lemma (Algebra, Lemma 20.1) implies that $\mathcal{F}$ is generated by r sections in the neighbourhood of a point $x \in X$ if and only if $\dim_{\kappa(x)} \mathcal{F}_x \otimes \kappa(x) \leq r$. Choose an i and a point $x_i \in X_i$ mapping to x. Then $\dim_{\kappa(x)} \mathcal{F}_x \otimes \kappa(x) = \dim_{\kappa(x_i)}(f_i^* \mathcal{F})_{x_i} \otimes \kappa(x_i)$ which is $\leq r$ as $f_i^* \mathcal{F}$ is locally generated by r sections. $\square$

05B1 **Lemma 7.5.** *Let X be a scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $\{f_i : X_i \to X\}_{i \in I}$ be an fpqc covering such that each $f_i^* \mathcal{F}$ is a flat $\mathcal{O}_{X_i}$-module. Then $\mathcal{F}$ is a flat $\mathcal{O}_X$-module.*

Proof. Omitted. For the affine case, see Algebra, Lemma 83.2. $\square$

05B2 **Lemma 7.6.** *Let X be a scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $\{f_i : X_i \to X\}_{i \in I}$ be an fpqc covering such that each $f_i^* \mathcal{F}$ is a finite locally free $\mathcal{O}_{X_i}$-module. Then $\mathcal{F}$ is a finite locally free $\mathcal{O}_X$-module.*

Proof. This follows from the fact that a quasi-coherent sheaf is finite locally free if and only if it is of finite presentation and flat, see Algebra, Lemma 78.2. Namely, if each $f_i^* \mathcal{F}$ is flat and of finite presentation, then so is $\mathcal{F}$ by Lemmas 7.5 and 7.3. $\square$

The definition of a locally projective quasi-coherent sheaf can be found in Properties, Section 21.

05JZ **Lemma 7.7.** *Let X be a scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $\{f_i : X_i \to X\}_{i \in I}$ be an fpqc covering such that each $f_i^* \mathcal{F}$ is a locally projective $\mathcal{O}_{X_i}$-module. Then $\mathcal{F}$ is a locally projective $\mathcal{O}_X$-module.*

Proof. Omitted. For Zariski coverings this is Properties, Lemma 21.2. For the affine case this is Algebra, Theorem 95.6. $\square$

05VF **Remark 7.8.** Being locally free is a property of quasi-coherent modules which does not descend in the fpqc topology. Namely, suppose that R is a ring and that M is a projective R-module which is a countable direct sum $M = \bigoplus L_n$ of rank 1 locally free modules, but not locally free, see Examples, Lemma 34.5. Then M becomes free on making the faithfully flat base change

$$R \longrightarrow \bigoplus_{m \geq 1} \bigoplus_{(i_1, \ldots, i_m) \in \mathbf{Z}^{\oplus m}} L_1^{\otimes i_1} \otimes_R \ldots \otimes_R L_m^{\otimes i_m}$$

But we don't know what happens for fppf coverings. In other words, we don't know the answer to the following question: Suppose $A \to B$ is a faithfully flat ring map of finite presentation. Let M be an A-module such that $M \otimes_A B$ is free. Is M a locally free A-module? It turns out that if A is Noetherian, then the answer is yes. This follows from the results of [Bas63]. But in general we don't know the answer. If you know the answer, or have a reference, please email stacks.project@gmail.com.

We also add here two results which are related to the results above, but are of a slightly different nature.

05B3 **Lemma 7.9.** *Let $f : X \to Y$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Assume f is a finite morphism. Then $\mathcal{F}$ is an $\mathcal{O}_X$-module of finite type if and only if $f_* \mathcal{F}$ is an $\mathcal{O}_Y$-module of finite type.*

Proof. As f is finite it is affine. This reduces us to the case where f is the morphism $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ given by a finite ring map $A \to B$. Moreover, then $\mathcal{F} = \widetilde{M}$ is the sheaf of modules associated to the B-module M. Note that M is finite as

a B-module if and only if M is finite as an A-module, see Algebra, Lemma 7.2. Combined with Properties, Lemma 16.1 this proves the lemma. $\square$

05B4 **Lemma 7.10.** *Let $f : X \to Y$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Assume f is finite and of finite presentation. Then $\mathcal{F}$ is an $\mathcal{O}_X$-module of finite presentation if and only if $f_*\mathcal{F}$ is an $\mathcal{O}_Y$-module of finite presentation.*

Proof. As f is finite it is affine. This reduces us to the case where f is the morphism $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ given by a finite and finitely presented ring map $A \to B$. Moreover, then $\mathcal{F} = \widetilde{M}$ is the sheaf of modules associated to the B-module M. Note that M is finitely presented as a B-module if and only if M is finitely presented as an A-module, see Algebra, Lemma 36.23. Combined with Properties, Lemma 16.2 this proves the lemma. $\square$

8. Quasi-coherent sheaves and topologies, I

03DR The results in this section say there is a natural equivalence between the category quasi-coherent modules on a scheme S and the category of quasi-coherent modules on many of the sites associated to S in the chapter on topologies.

Let S be a scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_S$-module. Consider the functor

03DS (8.0.1) $$(Sch/S)^{opp} \longrightarrow Ab, \quad (f : T \to S) \longmapsto \Gamma(T, f^*\mathcal{F}).$$

03DT **Lemma 8.1.** *Let S be a scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_S$-module. Let $\tau \in \{Zariski, \acute{e}tale, smooth, syntomic, fppf, fpqc\}$. The functor defined in (8.0.1) satisfies the sheaf condition with respect to any τ-covering $\{T_i \to T\}_{i \in I}$ of any scheme T over S.*

Proof. For $\tau \in \{Zariski, \acute{e}tale, smooth, syntomic, fppf\}$ a τ-covering is also a fpqc-covering, see the results in Topologies, Lemmas 4.2, 5.2, 6.2, 7.2, and 9.7. Hence it suffices to prove the theorem for a fpqc covering. Assume that $\{f_i : T_i \to T\}_{i \in I}$ is an fpqc covering where $f : T \to S$ is given. Suppose that we have a family of sections $s_i \in \Gamma(T_i, f_i^* f^*\mathcal{F})$ such that $s_i|_{T_i \times_T T_j} = s_j|_{T_i \times_T T_j}$. We have to find the correspond section $s \in \Gamma(T, f^*\mathcal{F})$. We can reinterpret the s_i as a family of maps $\varphi_i : f_i^* \mathcal{O}_T = \mathcal{O}_{T_i} \to f_i^* f^*\mathcal{F}$ compatible with the canonical descent data associated to the quasi-coherent sheaves $\mathcal{O}_T$ and $f^*\mathcal{F}$ on T. Hence by Proposition 5.2 we see that we may (uniquely) descend these to a map $\mathcal{O}_T \to f^*\mathcal{F}$ which gives us our section s. $\square$

We may in particular make the following definition.

03DU **Definition 8.2.** Let $\tau \in \{Zariski, \acute{e}tale, smooth, syntomic, fppf\}$. Let S be a scheme. Let Sch_τ be a big site containing S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_S$-module.

(1) The *structure sheaf of the big site* $(Sch/S)_\tau$ is the sheaf of rings $T/S \mapsto \Gamma(T, \mathcal{O}_T)$ which is denoted $\mathcal{O}$ or $\mathcal{O}_S$.
(2) If $\tau = Zariski$ or $\tau = \acute{e}tale$ the *structure sheaf of the small site* S_{Zar} or $S_{\acute{e}tale}$ is the sheaf of rings $T/S \mapsto \Gamma(T, \mathcal{O}_T)$ which is denoted $\mathcal{O}$ or $\mathcal{O}_S$.
(3) The *sheaf of $\mathcal{O}$-modules associated to $\mathcal{F}$* on the big site $(Sch/S)_\tau$ is the sheaf of $\mathcal{O}$-modules $(f : T \to S) \mapsto \Gamma(T, f^*\mathcal{F})$ which is denoted $\mathcal{F}^a$ (and often simply $\mathcal{F}$).

(4) If $\tau = Zariski$ or $\tau = \acute{e}tale$ the *sheaf of $\mathcal{O}$-modules associated to $\mathcal{F}$* on the small site S_{Zar} or $S_{\acute{e}tale}$ is the sheaf of $\mathcal{O}$-modules $(f : T \to S) \mapsto \Gamma(T, f^* \mathcal{F})$ which is denoted $\mathcal{F}^a$ (and often simply $\mathcal{F}$).

Note how we use the same notation $\mathcal{F}^a$ in each case. No confusion can really arise from this as by definition the rule that defines the sheaf $\mathcal{F}^a$ is independent of the site we choose to look at.

03FG   **Remark 8.3.** In Topologies, Lemma 3.12 we have seen that the small Zariski site of a scheme S is equivalent to S as a topological space in the sense that the categories of sheaves are naturally equivalent. Now that S_{Zar} is also endowed with a structure sheaf $\mathcal{O}$ we see that sheaves of modules on the ringed site $(S_{Zar}, \mathcal{O})$ agree with sheaves of modules on the ringed space $(S, \mathcal{O}_S)$.

070R   **Remark 8.4.** Let $f : T \to S$ be a morphism of schemes. Each of the morphisms of sites f_{sites} listed in Topologies, Section 11 becomes a morphism of ringed sites. Namely, each of these morphisms of sites $f_{sites} : (Sch/T)_\tau \to (Sch/S)_{\tau'}$, or $f_{sites} : (Sch/S)_\tau \to S_{\tau'}$ is given by the continuous functor $S'/S \mapsto T \times_S S'/S$. Hence, given S'/S we let

$$f^\sharp_{sites} : \mathcal{O}(S'/S) \longrightarrow f_{sites,*}\mathcal{O}(S'/S) = \mathcal{O}(T \times_S S'/T)$$

be the usual map $\mathrm{pr}^\sharp_{S'} : \mathcal{O}(S') \to \mathcal{O}(T \times_S S')$. Similarly, the morphism $i_f : Sh(T_\tau) \to Sh((Sch/S)_\tau)$ for $\tau \in \{Zar, \acute{e}tale\}$, see Topologies, Lemmas 3.13 and 4.13, becomes a morphism of ringed topoi because $i_f^{-1}\mathcal{O} = \mathcal{O}$. Here are some special cases:

(1) The morphism of big sites $f_{big} : (Sch/X)_{fppf} \to (Sch/Y)_{fppf}$, becomes a morphism of ringed sites

$$(f_{big}, f^\sharp_{big}) : ((Sch/X)_{fppf}, \mathcal{O}_X) \longrightarrow ((Sch/Y)_{fppf}, \mathcal{O}_Y)$$

as in Modules on Sites, Definition 6.1. Similarly for the big syntomic, smooth, étale and Zariski sites.

(2) The morphism of small sites $f_{small} : X_{\acute{e}tale} \to Y_{\acute{e}tale}$ becomes a morphism of ringed sites

$$(f_{small}, f^\sharp_{small}) : (X_{\acute{e}tale}, \mathcal{O}_X) \longrightarrow (Y_{\acute{e}tale}, \mathcal{O}_Y)$$

as in Modules on Sites, Definition 6.1. Similarly for the small Zariski site.

Let S be a scheme. It is clear that given an $\mathcal{O}$-module on (say) $(Sch/S)_{Zar}$ the pullback to (say) $(Sch/S)_{fppf}$ is just the fppf-sheafification. To see what happens when comparing big and small sites we have the following.

070S   **Lemma 8.5.** *Let S be a scheme. Denote*

$$
\begin{array}{lll}
id_{\tau,Zar} & : & (Sch/S)_\tau \to S_{Zar}, \quad \tau \in \{Zar, \acute{e}tale, smooth, syntomic, fppf\} \\
id_{\tau,\acute{e}tale} & : & (Sch/S)_\tau \to S_{\acute{e}tale}, \quad \tau \in \{\acute{e}tale, smooth, syntomic, fppf\} \\
id_{small,\acute{e}tale,Zar} & : & S_{\acute{e}tale} \to S_{Zar},
\end{array}
$$

the morphisms of ringed sites of Remark 8.4. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_S$-modules which we view a sheaf of $\mathcal{O}$-modules on S_{Zar}. Then

(1) *$(id_{\tau,Zar})^* \mathcal{F}$ is the τ-sheafification of the Zariski sheaf*

$$(f : T \to S) \longmapsto \Gamma(T, f^* \mathcal{F})$$

on $(Sch/S)_\tau$, and

(2) $(id_{small,\acute{e}tale,Zar})^*\mathcal{F}$ is the étale sheafification of the Zariski sheaf

$$(f : T \to S) \longmapsto \Gamma(T, f^*\mathcal{F})$$

on $S_{\acute{e}tale}$.

Let $\mathcal{G}$ be a sheaf of $\mathcal{O}$-modules on $S_{\acute{e}tale}$. Then

(3) $(id_{\tau,\acute{e}tale})^*\mathcal{G}$ is the τ-sheafification of the étale sheaf

$$(f : T \to S) \longmapsto \Gamma(T, f^*_{small}\mathcal{G})$$

where $f_{small} : T_{\acute{e}tale} \to S_{\acute{e}tale}$ is the morphism of ringed small étale sites of Remark 8.4.

Proof. Proof of (1). We first note that the result is true when $\tau = Zar$ because in that case we have the morphism of topoi $i_f : Sh(T_{Zar}) \to Sh((Sch/S)_{Zar})$ such that $id_{\tau,Zar} \circ i_f = f_{small}$ as morphisms $T_{Zar} \to S_{Zar}$, see Topologies, Lemmas 3.13 and 3.17. Since pullback is transitive (see Modules on Sites, Lemma 13.3) we see that $i_f^*(id_{\tau,Zar})^*\mathcal{F} = f^*_{small}\mathcal{F}$ as desired. Hence, by the remark preceding this lemma we see that $(id_{\tau,Zar})^*\mathcal{F}$ is the τ-sheafification of the presheaf $T \mapsto \Gamma(T, f^*\mathcal{F})$.

The proof of (3) is exactly the same as the proof of (1), except that it uses Topologies, Lemmas 4.13 and 4.17. We omit the proof of (2). $\square$

03FH **Remark 8.6.** Remark 8.4 and Lemma 8.5 have the following applications:

(1) Let S be a scheme. The construction $\mathcal{F} \mapsto \mathcal{F}^a$ is the pullback under the morphism of ringed sites $id_{\tau,Zar} : ((Sch/S)_\tau, \mathcal{O}) \to (S_{Zar}, \mathcal{O})$ or the morphism $id_{small,\acute{e}tale,Zar} : (S_{\acute{e}tale}, \mathcal{O}) \to (S_{Zar}, \mathcal{O})$.
(2) Let $f : X \to Y$ be a morphism of schemes. For any of the morphisms f_{sites} of ringed sites of Remark 8.4 we have

$$(f^*\mathcal{F})^a = f^*_{sites}\mathcal{F}^a.$$

This follows from (1) and the fact that pullbacks are compatible with compositions of morphisms of ringed sites, see Modules on Sites, Lemma 13.3.

03DV **Lemma 8.7.** *Let S be a scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_S$-module. Let $\tau \in \{Zariski, \acute{e}tale, smooth, syntomic, fppf\}$.*

(1) *The sheaf $\mathcal{F}^a$ is a quasi-coherent $\mathcal{O}$-module on $(Sch/S)_\tau$, as defined in Modules on Sites, Definition 23.1.*
(2) *If $\tau = Zariski$ or $\tau = \acute{e}tale$, then the sheaf $\mathcal{F}^a$ is a quasi-coherent $\mathcal{O}$-module on S_{Zar} or $S_{\acute{e}tale}$ as defined in Modules on Sites, Definition 23.1.*

Proof. Let $\{S_i \to S\}$ be a Zariski covering such that we have exact sequences

$$\bigoplus\nolimits_{k \in K_i} \mathcal{O}_{S_i} \longrightarrow \bigoplus\nolimits_{j \in J_i} \mathcal{O}_{S_i} \longrightarrow \mathcal{F}|_{S_i} \longrightarrow 0$$

for some index sets K_i and J_i. This is possible by the definition of a quasi-coherent sheaf on a ringed space (See Modules, Definition 10.1).

Proof of (1). Let $\tau \in \{Zariski, fppf, \acute{e}tale, smooth, syntomic\}$. It is clear that $\mathcal{F}^a|_{(Sch/S_i)_\tau}$ also sits in an exact sequence

$$\bigoplus\nolimits_{k \in K_i} \mathcal{O}|_{(Sch/S_i)_\tau} \longrightarrow \bigoplus\nolimits_{j \in J_i} \mathcal{O}|_{(Sch/S_i)_\tau} \longrightarrow \mathcal{F}^a|_{(Sch/S_i)_\tau} \longrightarrow 0$$

Hence $\mathcal{F}^a$ is quasi-coherent by Modules on Sites, Lemma 23.3.

Proof of (2). Let $\tau = \acute{e}tale$. It is clear that $\mathcal{F}^a|_{(S_i)_{\acute{e}tale}}$ also sits in an exact sequence

$$\bigoplus\nolimits_{k \in K_i} \mathcal{O}|_{(S_i)_{\acute{e}tale}} \longrightarrow \bigoplus\nolimits_{j \in J_i} \mathcal{O}|_{(S_i)_{\acute{e}tale}} \longrightarrow \mathcal{F}^a|_{(S_i)_{\acute{e}tale}} \longrightarrow 0$$

Hence $\mathcal{F}^a$ is quasi-coherent by Modules on Sites, Lemma 23.3. The case $\tau = Zariski$ is similar (actually, it is really tautological since the corresponding ringed topoi agree). $\square$

0GN7 **Lemma 8.8.** *Let S be a scheme. Let $\tau \in \{Zariski, \acute{e}tale, smooth, syntomic, fppf\}$. Each of the functors $\mathcal{F} \mapsto \mathcal{F}^a$ of Definition 8.2*

$$QCoh(\mathcal{O}_S) \to QCoh((Sch/S)_\tau, \mathcal{O}) \quad or \quad QCoh(\mathcal{O}_S) \to QCoh(S_\tau, \mathcal{O})$$

is fully faithful.

Proof. (By Lemma 8.7 we do indeed get functors as indicated.) We may and do identify $\mathcal{O}_S$-modules on S with modules on $(S_{Zar}, \mathcal{O}_S)$. The functor $\mathcal{F} \mapsto \mathcal{F}^a$ on quasi-coherent modules $\mathcal{F}$ is given by pullback by a morphism f of ringed sites, see Remark 8.6. In each case the functor f_* is given by restriction along the inclusion functor $S_{Zar} \to S_\tau$ or $S_{Zar} \to (Sch/S)_\tau$ (see discussion of how these morphisms of sites are defined in Topologies, Section 11). Combining this with the description of $f^*\mathcal{F} = \mathcal{F}^a$ we see that $f_* f^* \mathcal{F} = \mathcal{F}$ provided that $\mathcal{F}$ is quasi-coherent. Then we see that

$$\mathrm{Hom}_{\mathcal{O}}(\mathcal{F}^a, \mathcal{G}^a) = \mathrm{Hom}_{\mathcal{O}}(f^*\mathcal{F}, f^*\mathcal{G}) = \mathrm{Hom}_{\mathcal{O}_S}(\mathcal{F}, f_* f^* \mathcal{G}) = \mathrm{Hom}_{\mathcal{O}_S}(\mathcal{F}, \mathcal{G})$$

as desired. $\square$

03DX **Proposition 8.9.** *Let S be a scheme. Let $\tau \in \{Zariski, \acute{e}tale, smooth, syntomic, fppf\}$.*

(1) *The functor $\mathcal{F} \mapsto \mathcal{F}^a$ defines an equivalence of categories*

$$QCoh(\mathcal{O}_S) \longrightarrow QCoh((Sch/S)_\tau, \mathcal{O})$$

between the category of quasi-coherent sheaves on S and the category of quasi-coherent $\mathcal{O}$-modules on the big τ site of S.

(2) *Let $\tau = Zariski$ or $\tau = \acute{e}tale$. The functor $\mathcal{F} \mapsto \mathcal{F}^a$ defines an equivalence of categories*

$$QCoh(\mathcal{O}_S) \longrightarrow QCoh(S_\tau, \mathcal{O})$$

between the category of quasi-coherent sheaves on S and the category of quasi-coherent $\mathcal{O}$-modules on the small τ site of S.

Proof. We have seen in Lemma 8.7 that the functor is well defined. By Lemma 8.8 the functor is fully faithful. To finish the proof we will show that a quasi-coherent $\mathcal{O}$-module on $(Sch/S)_\tau$ gives rise to a descent datum for quasi-coherent sheaves relative to a τ-covering of S. Having produced this descent datum we will appeal to Proposition 5.2 to get the corresponding quasi-coherent sheaf on S.

Let $\mathcal{G}$ be a quasi-coherent $\mathcal{O}$-modules on the big τ site of S. By Modules on Sites, Definition 23.1 there exists a τ-covering $\{S_i \to S\}_{i \in I}$ of S such that each of the restrictions $\mathcal{G}|_{(Sch/S_i)_\tau}$ has a global presentation

$$\bigoplus\nolimits_{k \in K_i} \mathcal{O}|_{(Sch/S_i)_\tau} \longrightarrow \bigoplus\nolimits_{j \in J_i} \mathcal{O}|_{(Sch/S_i)_\tau} \longrightarrow \mathcal{G}|_{(Sch/S_i)_\tau} \longrightarrow 0$$

for some index sets J_i and K_i. We claim that this implies that $\mathcal{G}|_{(Sch/S_i)_\tau}$ is $\mathcal{F}_i^a$ for some quasi-coherent sheaf $\mathcal{F}_i$ on S_i. Namely, this is clear for the direct

sums $\bigoplus_{k \in K_i} \mathcal{O}|_{(Sch/S_i)_\tau}$ and $\bigoplus_{j \in J_i} \mathcal{O}|_{(Sch/S_i)_\tau}$. Hence we see that $\mathcal{G}|_{(Sch/S_i)_\tau}$ is a cokernel of a map $\varphi : \mathcal{K}_i^a \to \mathcal{L}_i^a$ for some quasi-coherent sheaves $\mathcal{K}_i, \mathcal{L}_i$ on S_i. By the fully faithfulness of $(\ )^a$ we see that $\varphi = \phi^a$ for some map of quasi-coherent sheaves $\phi : \mathcal{K}_i \to \mathcal{L}_i$ on S_i. Then it is clear that $\mathcal{G}|_{(Sch/S_i)_\tau} \cong \mathrm{Coker}(\phi)^a$ as claimed.

Since $\mathcal{G}$ lives on all of the category $(Sch/S)_\tau$ we see that

$$(\mathrm{pr}_0^* \mathcal{F}_i)^a \cong \mathcal{G}|_{(Sch/(S_i \times_S S_j))_\tau} \cong (\mathrm{pr}_1^* \mathcal{F})^a$$

as $\mathcal{O}$-modules on $(Sch/(S_i \times_S S_j))_\tau$. Hence, using fully faithfulness again we get canonical isomorphisms

$$\phi_{ij} : \mathrm{pr}_0^* \mathcal{F}_i \longrightarrow \mathrm{pr}_1^* \mathcal{F}_j$$

of quasi-coherent modules over $S_i \times_S S_j$. We omit the verification that these satisfy the cocycle condition. Since they do we see by effectivity of descent for quasi-coherent sheaves and the covering $\{S_i \to S\}$ (Proposition 5.2) that there exists a quasi-coherent sheaf $\mathcal{F}$ on S with $\mathcal{F}|_{S_i} \cong \mathcal{F}_i$ compatible with the given descent data. In other words we are given $\mathcal{O}$-module isomorphisms

$$\phi_i : \mathcal{F}^a|_{(Sch/S_i)_\tau} \longrightarrow \mathcal{G}|_{(Sch/S_i)_\tau}$$

which agree over $S_i \times_S S_j$. Hence, since $\mathcal{H}om_{\mathcal{O}}(\mathcal{F}^a, \mathcal{G})$ is a sheaf (Modules on Sites, Lemma 27.1), we conclude that there is a morphism of $\mathcal{O}$-modules $\mathcal{F}^a \to \mathcal{G}$ recovering the isomorphisms ϕ_i above. Hence this is an isomorphism and we win.

The case of the sites $S_{\acute{e}tale}$ and S_{Zar} is proved in the exact same manner. $\square$

05VG **Lemma 8.10.** *Let S be a scheme. Let $\tau \in \{Zariski, \acute{e}tale, smooth, syntomic, fppf\}$. Let $\mathcal{P}$ be one of the properties of modules[4] defined in Modules on Sites, Definitions 17.1, 23.1, and 28.1. The equivalences of categories*

$$QCoh(\mathcal{O}_S) \longrightarrow QCoh((Sch/S)_\tau, \mathcal{O}) \quad and \quad QCoh(\mathcal{O}_S) \longrightarrow QCoh(S_\tau, \mathcal{O})$$

defined by the rule $\mathcal{F} \mapsto \mathcal{F}^a$ seen in Proposition 8.9 have the property

$$\mathcal{F} \text{ has } \mathcal{P} \Leftrightarrow \mathcal{F}^a \text{ has } \mathcal{P} \text{ as an } \mathcal{O}\text{-module}$$

except (possibly) when $\mathcal{P}$ is "locally free" or "coherent". If $\mathcal{P} =$ "coherent" the equivalence holds for $QCoh(\mathcal{O}_S) \to QCoh(S_\tau, \mathcal{O})$ when S is locally Noetherian and τ is Zariski or étale.

Proof. This is immediate for the global properties, i.e., those defined in Modules on Sites, Definition 17.1. For the local properties we can use Modules on Sites, Lemma 23.3 to translate "$\mathcal{F}^a$ has $\mathcal{P}$" into a property on the members of a covering of X. Hence the result follows from Lemmas 7.1, 7.3, 7.4, 7.5, and 7.6. Being coherent for a quasi-coherent module is the same as being of finite type over a locally Noetherian scheme (see Cohomology of Schemes, Lemma 9.1) hence this reduces to the case of finite type modules (details omitted). $\square$

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[4] The list is: free, finite free, generated by global sections, generated by r global sections, generated by finitely many global sections, having a global presentation, having a global finite presentation, locally free, finite locally free, locally generated by sections, locally generated by r sections, finite type, of finite presentation, coherent, or flat.

9. Cohomology of quasi-coherent modules and topologies

0GN8 In this section we prove that cohomology of quasi-coherent modules is independent of the choice of topology.

03FI **Lemma 9.1.** *Let S be a scheme. Let*

(a) *$\tau \in \{Zariski, fppf, \acute{e}tale, smooth, syntomic\}$ and $\mathcal{C} = (Sch/S)_\tau$, or*
(b) *let $\tau = \acute{e}tale$ and $\mathcal{C} = S_{\acute{e}tale}$, or*
(c) *let $\tau = Zariski$ and $\mathcal{C} = S_{Zar}$.*

Let $\mathcal{F}$ be an abelian sheaf on $\mathcal{C}$. Let $U \in \mathrm{Ob}(\mathcal{C})$ be affine. Let $\mathcal{U} = \{U_i \to U\}_{i=1,\ldots,n}$ be a standard affine τ-covering in $\mathcal{C}$. Then

(1) *$\mathcal{V} = \{\coprod_{i=1,\ldots,n} U_i \to U\}$ is a τ-covering of U,*
(2) *$\mathcal{U}$ is a refinement of $\mathcal{V}$, and*
(3) *the induced map on Čech complexes (Cohomology on Sites, Equation (8.2.1))*

$$\check{\mathcal{C}}^\bullet(\mathcal{V}, \mathcal{F}) \longrightarrow \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})$$

is an isomorphism of complexes.

Proof. This follows because

$$\left(\coprod_{i_0=1,\ldots,n} U_{i_0}\right) \times_U \ldots \times_U \left(\coprod_{i_p=1,\ldots,n} U_{i_p}\right) = \coprod_{i_0,\ldots,i_p \in \{1,\ldots,n\}} U_{i_0} \times_U \ldots \times_U U_{i_p}$$

and the fact that $\mathcal{F}(\coprod_a V_a) = \prod_a \mathcal{F}(V_a)$ since disjoint unions are τ-coverings. $\square$

03FJ **Lemma 9.2.** *Let S be a scheme. Let $\mathcal{F}$ be a quasi-coherent sheaf on S. Let τ, $\mathcal{C}$, U, $\mathcal{U}$ be as in Lemma 9.1. Then there is an isomorphism of complexes*

$$\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}^a) \cong s((A/R)_\bullet \otimes_R M)$$

(see Section 3) where $R = \Gamma(U, \mathcal{O}_U)$, $M = \Gamma(U, \mathcal{F}^a)$ and $R \to A$ is a faithfully flat ring map. In particular

$$\check{H}^p(\mathcal{U}, \mathcal{F}^a) = 0$$

for all $p \geq 1$.

Proof. By Lemma 9.1 we see that $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}^a)$ is isomorphic to $\check{\mathcal{C}}^\bullet(\mathcal{V}, \mathcal{F}^a)$ where $\mathcal{V} = \{V \to U\}$ with $V = \coprod_{i=1,\ldots n} U_i$ affine also. Set $A = \Gamma(V, \mathcal{O}_V)$. Since $\{V \to U\}$ is a τ-covering we see that $R \to A$ is faithfully flat. On the other hand, by definition of $\mathcal{F}^a$ we have that the degree p term $\check{\mathcal{C}}^p(\mathcal{V}, \mathcal{F}^a)$ is

$$\Gamma(V \times_U \ldots \times_U V, \mathcal{F}^a) = \Gamma(\mathrm{Spec}(A \otimes_R \ldots \otimes_R A), \mathcal{F}^a) = A \otimes_R \ldots \otimes_R A \otimes_R M$$

We omit the verification that the maps of the Čech complex agree with the maps in the complex $s((A/R)_\bullet \otimes_R M)$. The vanishing of cohomology is Lemma 3.6. $\square$

03DW **Proposition 9.3.** *Let S be a scheme. Let $\mathcal{F}$ be a quasi-coherent sheaf on S. Let $\tau \in \{Zariski, \acute{e}tale, smooth, syntomic, fppf\}$.*

(1) *There is a canonical isomorphism*

$$H^q(S, \mathcal{F}) = H^q((Sch/S)_\tau, \mathcal{F}^a).$$

(2) *There are canonical isomorphisms*

$$H^q(S, \mathcal{F}) = H^q(S_{Zar}, \mathcal{F}^a) = H^q(S_{\acute{e}tale}, \mathcal{F}^a).$$

Proof. The result for $q = 0$ is clear from the definition of $\mathcal{F}^a$. Let $\mathcal{C} = (Sch/S)_\tau$, or $\mathcal{C} = S_{\text{étale}}$, or $\mathcal{C} = S_{Zar}$.

We are going to apply Cohomology on Sites, Lemma 10.9 with $\mathcal{F} = \mathcal{F}^a$, $\mathcal{B} \subset \text{Ob}(\mathcal{C})$ the set of affine schemes in $\mathcal{C}$, and $\text{Cov} \subset \text{Cov}_{\mathcal{C}}$ the set of standard affine τ-coverings. Assumption (3) of the lemma is satisfied by Lemma 9.2. Hence we conclude that $H^p(U, \mathcal{F}^a) = 0$ for every affine object U of $\mathcal{C}$.

Next, let $U \in \text{Ob}(\mathcal{C})$ be any separated object. Denote $f : U \to S$ the structure morphism. Let $U = \bigcup U_i$ be an affine open covering. We may also think of this as a τ-covering $\mathcal{U} = \{U_i \to U\}$ of U in $\mathcal{C}$. Note that $U_{i_0} \times_U \ldots \times_U U_{i_p} = U_{i_0} \cap \ldots \cap U_{i_p}$ is affine as we assumed U separated. By Cohomology on Sites, Lemma 10.7 and the result above we see that

$$H^p(U, \mathcal{F}^a) = \check{H}^p(\mathcal{U}, \mathcal{F}^a) = H^p(U, f^*\mathcal{F})$$

the last equality by Cohomology of Schemes, Lemma 2.6. In particular, if S is separated we can take $U = S$ and $f = \text{id}_S$ and the proposition is proved. We suggest the reader skip the rest of the proof (or rewrite it to give a clearer exposition).

Choose an injective resolution $\mathcal{F} \to \mathcal{I}^\bullet$ on S. Choose an injective resolution $\mathcal{F}^a \to \mathcal{J}^\bullet$ on $\mathcal{C}$. Denote $\mathcal{J}^n|_S$ the restriction of $\mathcal{J}^n$ to opens of S; this is a sheaf on the topological space S as open coverings are τ-coverings. We get a complex

$$0 \to \mathcal{F} \to \mathcal{J}^0|_S \to \mathcal{J}^1|_S \to \ldots$$

which is exact since its sections over any affine open $U \subset S$ is exact (by the vanishing of $H^p(U, \mathcal{F}^a)$, $p > 0$ seen above). Hence by Derived Categories, Lemma 18.6 there exists map of complexes $\mathcal{J}^\bullet|_S \to \mathcal{I}^\bullet$ which in particular induces a map

$$R\Gamma(\mathcal{C}, \mathcal{F}^a) = \Gamma(S, \mathcal{J}^\bullet) \longrightarrow \Gamma(S, \mathcal{I}^\bullet) = R\Gamma(S, \mathcal{F}).$$

Taking cohomology gives the map $H^n(\mathcal{C}, \mathcal{F}^a) \to H^n(S, \mathcal{F})$ which we have to prove is an isomorphism. Let $\mathcal{U} : S = \bigcup U_i$ be an affine open covering which we may think of as a τ-covering also. By the above we get a map of double complexes

$$\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{J}) = \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{J}|_S) \longrightarrow \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}).$$

This map induces a map of spectral sequences

$$^\tau E_2^{p,q} = \check{H}^p(\mathcal{U}, \underline{H}^q(\mathcal{F}^a)) \longrightarrow E_2^{p,q} = \check{H}^p(\mathcal{U}, \underline{H}^q(\mathcal{F}))$$

The first spectral sequence converges to $H^{p+q}(\mathcal{C}, \mathcal{F})$ and the second to $H^{p+q}(S, \mathcal{F})$. On the other hand, we have seen that the induced maps $^\tau E_2^{p,q} \to E_2^{p,q}$ are bijections (as all the intersections are separated being opens in affines). Whence also the maps $H^n(\mathcal{C}, \mathcal{F}^a) \to H^n(S, \mathcal{F})$ are isomorphisms, and we win. $\square$

03LC   **Proposition 9.4.** *Let $f : T \to S$ be a morphism of schemes.*

     (1) *The equivalences of categories of Proposition 8.9 are compatible with pullback. More precisely, we have $f^*(\mathcal{G}^a) = (f^*\mathcal{G})^a$ for any quasi-coherent sheaf $\mathcal{G}$ on S.*

     (2) *The equivalences of categories of Proposition 8.9 part (1) are* **not** *compatible with pushforward in general.*

(3) *If f is quasi-compact and quasi-separated, and $\tau \in \{Zariski, \acute{e}tale\}$ then f_* and $f_{small,*}$ preserve quasi-coherent sheaves and the diagram*

$$
\begin{array}{ccc}
QCoh(\mathcal{O}_T) & \xrightarrow{\quad f_* \quad} & QCoh(\mathcal{O}_S) \\
{\scriptstyle \mathcal{F} \mapsto \mathcal{F}^a} \downarrow & & \downarrow {\scriptstyle \mathcal{G} \mapsto \mathcal{G}^a} \\
QCoh(T_\tau, \mathcal{O}) & \xrightarrow{\quad f_{small,*} \quad} & QCoh(S_\tau, \mathcal{O})
\end{array}
$$

is commutative, i.e., $f_{small,*}(\mathcal{F}^a) = (f_*\mathcal{F})^a$.

Proof. Part (1) follows from the discussion in Remark 8.6. Part (2) is just a warning, and can be explained in the following way: First the statement cannot be made precise since f_* does not transform quasi-coherent sheaves into quasi-coherent sheaves in general. Even if this is the case for f (and any base change of f), then the compatibility over the big sites would mean that formation of $f_*\mathcal{F}$ commutes with any base change, which does not hold in general. An explicit example is the quasi-compact open immersion $j : X = \mathbf{A}_k^2 \setminus \{0\} \to \mathbf{A}_k^2 = Y$ where k is a field. We have $j_*\mathcal{O}_X = \mathcal{O}_Y$ but after base change to $\mathrm{Spec}(k)$ by the 0 map we see that the pushforward is zero.

Let us prove (3) in case $\tau = \acute{e}tale$. Note that f, and any base change of f, transforms quasi-coherent sheaves into quasi-coherent sheaves, see Schemes, Lemma 24.1. The equality $f_{small,*}(\mathcal{F}^a) = (f_*\mathcal{F})^a$ means that for any étale morphism $g : U \to S$ we have $\Gamma(U, g^*f_*\mathcal{F}) = \Gamma(U \times_S T, (g')^*\mathcal{F})$ where $g' : U \times_S T \to T$ is the projection. This is true by Cohomology of Schemes, Lemma 5.2. $\square$

071N **Lemma 9.5.** *Let $f : T \to S$ be a quasi-compact and quasi-separated morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent sheaf on T. For either the étale or Zariski topology, there are canonical isomorphisms $R^i f_{small,*}(\mathcal{F}^a) = (R^i f_*\mathcal{F})^a$.*

Proof. We prove this for the étale topology; we omit the proof in the case of the Zariski topology. By Cohomology of Schemes, Lemma 4.5 the sheaves $R^i f_*\mathcal{F}$ are quasi-coherent so that the assertion makes sense. The sheaf $R^i f_{small,*}\mathcal{F}^a$ is the sheaf associated to the presheaf

$$
U \longmapsto H^i(U \times_S T, \mathcal{F}^a)
$$

where $g : U \to S$ is an object of $S_{\acute{e}tale}$, see Cohomology on Sites, Lemma 7.4. By our conventions the right hand side is the étale cohomology of the restriction of $\mathcal{F}^a$ to the localization $T_{\acute{e}tale}/U \times_S T$ which equals $(U \times_S T)_{\acute{e}tale}$. By Proposition 9.3 this is presheaf the same as the presheaf

$$
U \longmapsto H^i(U \times_S T, (g')^*\mathcal{F}),
$$

where $g' : U \times_S T \to T$ is the projection. If U is affine then this is the same as $H^0(U, R^i f_*'(g')^*\mathcal{F})$, see Cohomology of Schemes, Lemma 4.6. By Cohomology of Schemes, Lemma 5.2 this is equal to $H^0(U, g^*R^i f_*\mathcal{F})$ which is the value of $(R^i f_*\mathcal{F})^a$ on U. Thus the values of the sheaves of modules $R^i f_{small,*}(\mathcal{F}^a)$ and $(R^i f_*\mathcal{F})^a$ on every affine object of $S_{\acute{e}tale}$ are canonically isomorphic which implies they are canonically isomorphic. $\square$

10. Quasi-coherent sheaves and topologies, II

0GN9   We continue the discussion comparing quasi-coherent modules on a scheme S with quasi-coherent modules on any of the sites associated to S in the chapter on topologies.

0GNA   **Lemma 10.1.** *In Lemma 8.5 the morphism of ringed sites* $id_{small,\acute{e}tale,Zar}$: $S_{\acute{e}tale} \to S_{Zar}$ *is flat.*

Proof. Let us denote $\epsilon = id_{small,\acute{e}tale,Zar}$ and $\mathcal{O}_{\acute{e}tale}$ and $\mathcal{O}_{Zar}$ the structure sheaves on $S_{\acute{e}tale}$ and S_{Zar}. We have to show that $\mathcal{O}_{\acute{e}tale}$ is a flat $\epsilon^{-1}\mathcal{O}_{Zar}$-module. Recall that étale morphisms are open, see Morphisms, Lemma 36.13. It follows (from the construction of pullback on sheaves) that $\epsilon^{-1}\mathcal{O}_{Zar}$ is the sheafification of the presheaf $\mathcal{O}'$ on $S_{\acute{e}tale}$ which sends an étale morphism $f : V \to S$ to $\mathcal{O}_S(f(V))$. If both V and $U = f(V) \subset S$ are affine, then $V \to U$ is an étale morphism of affines, hence corresponds to an étale ring map. Since étale ring maps are flat, we see that $\mathcal{O}_S(U) = \mathcal{O}'(V) \to \mathcal{O}_{\acute{e}tale}(V) = \mathcal{O}_V(V)$ is flat. Finally, for every étale morphism $f : V \to S$, i.e., object of $S_{\acute{e}tale}$, there is an affine open covering $V = \bigcup V_i$ such that $f(V_i)$ is an affine open in S for all i[5]. Thus the result by Modules on Sites, Lemma 28.4. $\square$

06VE   **Lemma 10.2.** *Let S be a scheme. Let $\tau \in \{Zariski, \acute{e}tale, smooth, syntomic, fppf\}$. The functors*

$$QCoh(\mathcal{O}_S) \longrightarrow Mod((Sch/S)_\tau, \mathcal{O}) \quad and \quad QCoh(\mathcal{O}_S) \longrightarrow Mod(S_\tau, \mathcal{O})$$

defined by the rule $\mathcal{F} \mapsto \mathcal{F}^a$ seen in Proposition 8.9 are

  (1) *fully faithful,*
  (2) *commmute with direct sums,*
  (3) *commmute with colimits,*
  (4) *right exact,*
  (5) *exact as a functor $QCoh(\mathcal{O}_S) \to Mod(S_{\acute{e}tale}, \mathcal{O})$,*
  (6) **not** *exact as a functor $QCoh(\mathcal{O}_S) \to Mod((Sch/S)_\tau, \mathcal{O})$ in general,*
  (7) *given two quasi-coherent $\mathcal{O}_S$-modules $\mathcal{F}, \mathcal{G}$ we have $(\mathcal{F}\otimes_{\mathcal{O}_S}\mathcal{G})^a = \mathcal{F}^a\otimes_{\mathcal{O}}\mathcal{G}^a$,*
  (8) *if $\tau = \acute{e}tale$ or $\tau = Zariski$, given two quasi-coherent $\mathcal{O}_S$-modules $\mathcal{F}, \mathcal{G}$ such that $\mathcal{F}$ is of finite presentation we have $(\mathcal{H}om_{\mathcal{O}_S}(\mathcal{F}, \mathcal{G}))^a = \mathcal{H}om_{\mathcal{O}}(\mathcal{F}^a, \mathcal{G}^a)$ in $Mod(S_\tau, \mathcal{O})$,*
  (9) *given two quasi-coherent $\mathcal{O}_S$-modules $\mathcal{F}, \mathcal{G}$ we do* **not** *have $(\mathcal{H}om_{\mathcal{O}_S}(\mathcal{F}, \mathcal{G}))^a = \mathcal{H}om_{\mathcal{O}}(\mathcal{F}^a, \mathcal{G}^a)$ in $Mod((Sch/S)_\tau, \mathcal{O})$ in general even if $\mathcal{F}$ is of finite presentation, and*
 (10) *given a short exact sequence $0 \to \mathcal{F}_1^a \to \mathcal{E} \to \mathcal{F}_2^a \to 0$ of $\mathcal{O}$-modules then $\mathcal{E}$ is quasi-coherent[6], i.e., $\mathcal{E}$ is in the essential image of the functor.*

Proof. Part (1) we saw in Proposition 8.9.

We have seen in Schemes, Section 24 that a colimit of quasi-coherent sheaves on a scheme is a quasi-coherent sheaf. Moreover, in Remark 8.6 we saw that $\mathcal{F} \mapsto \mathcal{F}^a$ is the pullback functor for a morphism of ringed sites, hence commutes with all colimits, see Modules on Sites, Lemma 14.3. Thus (3) and its special case (2) hold.

---

[5]Namely, for $y \in V$, we pick an affine open $y \in V' \subset V$ with $f(V')$ contained in an affine open $U \subset S$. Then we pick an affine open $f(y) \in U' \subset f(V')$. Then $V'' = f^{-1}(U') \subset V'$ is affine as it is equal to $U' \times_U V'$ and $f(V'') = U'$ is affine too.

[6]Warning: This is misleading. See part (6).

This also shows that the functor is right exact (i.e., commutes with finite colimits), hence (4).

The functor $QCoh(\mathcal{O}_S) \to Mod(S_{\text{étale}}, \mathcal{O})$, $\mathcal{F} \mapsto \mathcal{F}^a$ is left exact because an étale morphism is flat, see Morphisms, Lemma 36.12. This proves (5).

To see (6), suppose that $S = \operatorname{Spec}(\mathbf{Z})$. Then $2 : \mathcal{O}_S \to \mathcal{O}_S$ is injective but the associated map of $\mathcal{O}$-modules on $(Sch/S)_\tau$ isn't injective because $2 : \mathbf{F}_2 \to \mathbf{F}_2$ isn't injective and $\operatorname{Spec}(\mathbf{F}_2)$ is an object of $(Sch/S)_\tau$.

Part (7) holds because, as mentioned above, the functor $\mathcal{F} \mapsto \mathcal{F}^a$ is the pullback functor for a morphism of ringed sites and such commute with tensor products by Modules on Sites, Lemma 26.2.

Part (8) is obvious if $\tau = Zariski$ because the category of $\mathcal{O}$-modules on S_{Zar} is the same as the category of $\mathcal{O}_S$-modules on the topological space S. If $\tau = étale$ then (8) holds because, as mentioned above, the functor $\mathcal{F} \mapsto \mathcal{F}^a$ is the pullback functor for the flat morphism of ringed sites $(S_{\text{étale}}, \mathcal{O}) \to (S_{Zar}, \mathcal{O}_S)$, see Lemma 10.1. Pullback by flat morphisms of ringed sites commutes with taking internal hom out of a finitely presented module by Modules on Sites, Lemma 31.4.

To see (9), suppose that $S = \operatorname{Spec}(\mathbf{Z})$. Let $\mathcal{F} = \operatorname{Coker}(2 : \mathcal{O}_S \to \mathcal{O}_S)$ and $\mathcal{G} = \mathcal{O}_S$. Then $\mathcal{F}^a = \operatorname{Coker}(2 : \mathcal{O} \to \mathcal{O})$ and $\mathcal{G}^a = \mathcal{O}$. Hence $\mathcal{H}om_{\mathcal{O}}(\mathcal{F}^a, \mathcal{G}^a) = \mathcal{O}[2]$ is equal to the 2-torsion in $\mathcal{O}$, which is not zero, see proof of (6). On the other hand, the module $\mathcal{H}om_{\mathcal{O}_S}(\mathcal{F}, \mathcal{G})$ is zero.

Proof of (10). Let $0 \to \mathcal{F}_1^a \to \mathcal{E} \to \mathcal{F}_2^a \to 0$ be a short exact sequence of $\mathcal{O}$-modules with $\mathcal{F}_1$ and $\mathcal{F}_2$ quasi-coherent on S. Consider the restriction

$$0 \to \mathcal{F}_1 \to \mathcal{E}|_{S_{Zar}} \to \mathcal{F}_2$$

to S_{Zar}. By Proposition 9.3 we see that on any affine $U \subset S$ we have $H^1(U, \mathcal{F}_1^a) = H^1(U, \mathcal{F}_1) = 0$. Hence the sequence above is also exact on the right. By Schemes, Section 24 we conclude that $\mathcal{F} = \mathcal{E}|_{S_{Zar}}$ is quasi-coherent. Thus we obtain a commutative diagram

$$
\begin{array}{ccccccc}
\mathcal{F}_1^a & \longrightarrow & \mathcal{F}^a & \longrightarrow & \mathcal{F}_2^a & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 \longrightarrow & \mathcal{F}_1^a & \longrightarrow & \mathcal{E} & \longrightarrow & \mathcal{F}_2^a & \longrightarrow 0
\end{array}
$$

To finish the proof it suffices to show that the top row is also right exact. To do this, denote once more $U = \operatorname{Spec}(A) \subset S$ an affine open of S. We have seen above that $0 \to \mathcal{F}_1(U) \to \mathcal{E}(U) \to \mathcal{F}_2(U) \to 0$ is exact. For any affine scheme V/U, $V = \operatorname{Spec}(B)$ the map $\mathcal{F}_1^a(V) \to \mathcal{E}(V)$ is injective. We have $\mathcal{F}_1^a(V) = \mathcal{F}_1(U) \otimes_A B$ by definition. The injection $\mathcal{F}_1^a(V) \to \mathcal{E}(V)$ factors as

$$\mathcal{F}_1(U) \otimes_A B \to \mathcal{E}(U) \otimes_A B \to \mathcal{E}(V)$$

Considering A-algebras B of the form $B = A \oplus M$ we see that $\mathcal{F}_1(U) \to \mathcal{E}(U)$ is universally injective (see Algebra, Definition 82.1). Since $\mathcal{E}(U) = \mathcal{F}(U)$ we conclude that $\mathcal{F}_1 \to \mathcal{F}$ remains injective after any base change, or equivalently that $\mathcal{F}_1^a \to \mathcal{F}^a$ is injective. $\qquad\square$

0GNB   **Lemma 10.3.** *Let S be a scheme. The category $QCoh(S_{\text{étale}}, \mathcal{O})$ of quasi-coherent modules on $S_{\text{étale}}$ has the following properties:*

(1) *Any direct sum of quasi-coherent sheaves is quasi-coherent.*
(2) *Any colimit of quasi-coherent sheaves is quasi-coherent.*
(3) *The kernel and cokernel of a morphism of quasi-coherent sheaves is quasi-coherent.*
(4) *Given a short exact sequence of $\mathcal{O}$-modules $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ if two out of three are quasi-coherent so is the third.*
(5) *Given two quasi-coherent $\mathcal{O}$-modules the tensor product is quasi-coherent.*
(6) *Given two quasi-coherent $\mathcal{O}$-modules $\mathcal{F}$, $\mathcal{G}$ such that $\mathcal{F}$ is of finite presentation. then the internal hom $\mathcal{H}om_{\mathcal{O}}(\mathcal{F}, \mathcal{G})$ is quasi-coherent.*

Proof. The corresponding facts hold for quasi-coherent modules on the scheme S, see Schemes, Section 24. The proof will be to use Lemma 10.2 to transfer these truths to $S_{\acute{e}tale}$.

Proof of (1). Let $\mathcal{F}_i$, $i \in I$ be a family of objects of $QCoh(S_{\acute{e}tale}, \mathcal{O})$. Write $\mathcal{F}_i = \mathcal{G}_i^a$ for some quasi-coherent modules $\mathcal{G}_i$ on S. Then $\bigoplus \mathcal{F}_i = (\bigoplus \mathcal{G}_i)^a$ by the lemma cited and we conclude.

Proof of (2). Let $\mathcal{I} \to QCoh(S_{\acute{e}tale}, \mathcal{O})$, $i \mapsto \mathcal{F}_i$ be a diagram. Write $\mathcal{F}_i = \mathcal{G}_i^a$ so we get a diagram $\mathcal{I} \to QCoh(\mathcal{O}_S)$. Then $\operatorname{colim} \mathcal{F}_i = (\operatorname{colim} \mathcal{G}_i)^a$ by the lemma cited and we conclude.

Proof of (3). Let $a : \mathcal{F} \to \mathcal{F}'$ be an arrow of $QCoh(S_{\acute{e}tale}, \mathcal{O})$. Write $a = b^a$ for some map $b : \mathcal{G} \to \mathcal{G}'$ of quasi-coherent modules on S. By the lemma cited we have $\operatorname{Ker}(a) = \operatorname{Ker}(b)^a$ and $\operatorname{Coker}(a) = \operatorname{Coker}(b)^a$ and we conclude.

Proof of (4). This follows from (3) except in the case when we know $\mathcal{F}_1$ and $\mathcal{F}_3$ are quasi-coherent. In this case write $\mathcal{F}_1 = \mathcal{G}_1^a$ and $\mathcal{F}_3 = \mathcal{G}_3^a$ with $\mathcal{G}_i$ quasi-coherent on S. By Lemma 10.2 part (10) we conclude.

Proof of (5). Let $\mathcal{F}$ and $\mathcal{F}'$ be in $QCoh(S_{\acute{e}tale}, \mathcal{O})$. Write $\mathcal{F} = \mathcal{G}^a$ and $\mathcal{F}' = (\mathcal{G}')^a$ with $\mathcal{G}$ and $\mathcal{G}'$ quasi-coherent on S. By the lemma cited we have $\mathcal{F} \otimes_{\mathcal{O}} \mathcal{F}' = (\mathcal{G} \otimes_{\mathcal{O}_S} \mathcal{G}')^a$ and we conclude.

Proof of (6). Let $\mathcal{F}$ and $\mathcal{G}$ be in $QCoh(S_{\acute{e}tale}, \mathcal{O})$ with $\mathcal{F}$ of finite presentation. Write $\mathcal{F} = \mathcal{H}^a$ and $\mathcal{G} = (\mathcal{I})^a$ with $\mathcal{H}$ and $\mathcal{I}$ quasi-coherent on S. By Lemma 8.10 we see that $\mathcal{H}$ is of finite presentation. By Lemma 10.2 part (8) we have $\mathcal{H}om_{\mathcal{O}}(\mathcal{F}, \mathcal{G}) = (\mathcal{H}om_{\mathcal{O}_S}(\mathcal{H}, \mathcal{I}))^a$ and we conclude. $\square$

0GNC **Lemma 10.4.** *Let S be a scheme. Let $\tau \in \{Zariski, \acute{e}tale, smooth, syntomic, fppf\}$. The category $QCoh((Sch/S)_\tau, \mathcal{O})$ of quasi-coherent modules on $(Sch/S)_\tau$ has the following properties:*

(1) *Any direct sum of quasi-coherent sheaves is quasi-coherent.*
(2) *Any colimit of quasi-coherent sheaves is quasi-coherent.*
(3) *The cokernel of a morphism of quasi-coherent sheaves is quasi-coherent.*
(4) *Given a short exact sequence of $\mathcal{O}$-modules $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ if $\mathcal{F}_1$ and $\mathcal{F}_3$ are quasi-coherent so is $\mathcal{F}_2$.*
(5) *Given two quasi-coherent $\mathcal{O}$-modules the tensor product is quasi-coherent.*
(6) *Given two quasi-coherent $\mathcal{O}$-modules $\mathcal{F}$, $\mathcal{G}$ such that $\mathcal{F}$ is finite locally free, the internal hom $\mathcal{H}om_{\mathcal{O}}(\mathcal{F}, \mathcal{G})$ is quasi-coherent.*

Proof. The corresponding facts hold for quasi-coherent modules on the scheme S, see Schemes, Section 24. The proof will be to use Lemma 10.2 to transfer these truths to $(Sch/S)_\tau$.

Proof of (1). Let $\mathcal{F}_i$, $i \in I$ be a family of objects of $QCoh((Sch/S)_\tau, \mathcal{O})$. Write $\mathcal{F}_i = \mathcal{G}_i^a$ for some quasi-coherent modules $\mathcal{G}_i$ on S. Then $\bigoplus \mathcal{F}_i = (\bigoplus \mathcal{G}_i)^a$ by the lemma cited and we conclude.

Proof of (2). Let $\mathcal{I} \to QCoh((Sch/S)_\tau, \mathcal{O})$, $i \mapsto \mathcal{F}_i$ be a diagram. Write $\mathcal{F}_i = \mathcal{G}_i^a$ so we get a diagram $\mathcal{I} \to QCoh(\mathcal{O}_S)$. Then $\operatorname{colim} \mathcal{F}_i = (\operatorname{colim} \mathcal{G}_i)^a$ by the lemma cited and we conclude.

Proof of (3). Let $a : \mathcal{F} \to \mathcal{F}'$ be an arrow of $QCoh((Sch/S)_\tau, \mathcal{O})$. Write $a = b^a$ for some map $b : \mathcal{G} \to \mathcal{G}'$ of quasi-coherent modules on S. By the lemma cited we have $\operatorname{Coker}(a) = \operatorname{Coker}(b)^a$ (because a cokernel is a colimit) and we conclude.

Proof of (4). Write $\mathcal{F}_1 = \mathcal{G}_1^a$ and $\mathcal{F}_3 = \mathcal{G}_3^a$ with $\mathcal{G}_i$ quasi-coherent on S. By Lemma 10.2 part (10) we conclude.

Proof of (5). Let $\mathcal{F}$ and $\mathcal{F}'$ be in $QCoh((Sch/S)_\tau, \mathcal{O})$. Write $\mathcal{F} = \mathcal{G}^a$ and $\mathcal{F}' = (\mathcal{G}')^a$ with $\mathcal{G}$ and $\mathcal{G}'$ quasi-coherent on S. By the lemma cited we have $\mathcal{F} \otimes_{\mathcal{O}} \mathcal{F}' = (\mathcal{G} \otimes_{\mathcal{O}_S} \mathcal{G}')^a$ and we conclude.

Proof of (6). Write $\mathcal{F} = \mathcal{H}^a$ for some quasi-coherent $\mathcal{O}_S$-module. By Lemma 8.10 we see that $\mathcal{H}$ is finite locally free. The problem is Zariski local on S (details omitted) hence we may assume $\mathcal{H} = \mathcal{O}_S^{\oplus n}$ is finite free. Then $\mathcal{F} = \mathcal{O}^{\oplus n}$ and $\mathcal{H}om_{\mathcal{O}}(\mathcal{F}, \mathcal{G}) = \mathcal{G}^{\oplus n}$ is quasi-coherent. $\qquad\square$

0GND **Example 10.5.** Let S be a scheme. Let $\mathcal{F}$ and $\mathcal{G}$ be quasi-coherent modules on $(Sch/S)_\tau$ for one of the topologies τ considered in Lemma 10.4. In general it is not the case that $\mathcal{H}om_{\mathcal{O}}(\mathcal{F}, \mathcal{G})$ is quasi-coherent even if $\mathcal{F}$ is of finite presentation. Namely, say $S = \operatorname{Spec}(\mathbf{Z})$, $\mathcal{F} = \operatorname{Coker}(2 : \mathcal{O} \to \mathcal{O})$, and $\mathcal{G} = \mathcal{O}$. Then $\mathcal{H}om_{\mathcal{O}}(\mathcal{F}, \mathcal{G}) = \mathcal{O}[2]$ is equal to the 2-torsion in $\mathcal{O}$, which is not quasi-coherent.

0GNE **Lemma 10.6.** *Let S be a scheme.*

(1) *The category $QCoh((Sch/S)_{fppf}, \mathcal{O})$ has colimits and they agree with colimits in the categories $Mod((Sch/S)_{Zar}, \mathcal{O})$, $Mod((Sch/S)_{étale}, \mathcal{O})$, and $Mod((Sch/S)_{fppf}, \mathcal{O})$.*

(2) *Given $\mathcal{F}, \mathcal{G}$ in $QCoh((Sch/S)_{fppf}, \mathcal{O})$ the tensor products $\mathcal{F} \otimes_{\mathcal{O}} \mathcal{G}$ computed in $Mod((Sch/S)_{Zar}, \mathcal{O})$, $Mod((Sch/S)_{étale}, \mathcal{O})$, or $Mod((Sch/S)_{fppf}, \mathcal{O})$ agree and the common value is an object of $QCoh((Sch/S)_{fppf}, \mathcal{O})$.*

(3) *Given $\mathcal{F}, \mathcal{G}$ in $QCoh((Sch/S)_{fppf}, \mathcal{O})$ with $\mathcal{F}$ finite locally free (in fppf, or equivalently étale, or equivalently Zariski topology) the internal homs $\mathcal{H}om_{\mathcal{O}}(\mathcal{F}, \mathcal{G})$ computed in $Mod((Sch/S)_{Zar}, \mathcal{O})$, $Mod((Sch/S)_{étale}, \mathcal{O})$, or $Mod((Sch/S)_{fppf}, \mathcal{O})$ agree and the common value is an object of $QCoh((Sch/S)_{fppf}, \mathcal{O})$.*

Proof. This lemma collects the results shown above in a slightly different manner. First of all, by Lemma 10.4 we already know the output of the construction in (1), (2), or (3) ends up in $QCoh((Sch/S)_\tau, \mathcal{O})$. It remains to show in each case that the result is independent of the topology used. The key to this is that the equivalence $QCoh(\mathcal{O}_S) \to QCoh((Sch/S)_\tau, \mathcal{O})$, $\mathcal{F} \mapsto \mathcal{F}^a$ of Proposition 8.9 is given by the same formula independent of the choice of the topology $\tau \in \{Zariski, étale, fppf\}$.

Proof of (1). Let $\mathcal{I} \to QCoh((Sch/S)_{fppf}, \mathcal{O})$, $i \mapsto \mathcal{F}_i$ be a diagram. Write $\mathcal{F}_i = \mathcal{G}_i^a$ so we get a diagram $\mathcal{I} \to QCoh(\mathcal{O}_S)$. Then $\operatorname{colim} \mathcal{F}_i = (\operatorname{colim} \mathcal{G}_i)^a$ in $Mod((Sch/S)_\tau, \mathcal{O})$ for $\tau \in \{Zariski, étale, fppf\}$ by Lemma 10.2. This proves (1).

Proof of (2). Write $\mathcal{F} = \mathcal{H}^a$ and $\mathcal{G} = (\mathcal{I})^a$ with $\mathcal{H}$ and $\mathcal{I}$ quasi-coherent on S. Then $\mathcal{F} \otimes_{\mathcal{O}} \mathcal{G} = (\mathcal{H} \otimes_{\mathcal{O}} \mathcal{I})^a$ in $Mod((Sch/S)_\tau, \mathcal{O})$ for $\tau \in \{Zariski, étale, fppf\}$ by Lemma 10.2. This proves (2).

Proof of (3). Let $\mathcal{F}$ and $\mathcal{G}$ be in $QCoh((Sch/S)_{fppf}, \mathcal{O})$. Write $\mathcal{F} = \mathcal{H}^a$ with $\mathcal{H}$ quasi-coherent on S. By Lemma 8.10 we have

$\mathcal{F}$ finite locally free in fppf topology $\Leftrightarrow \mathcal{H}$ finite locally free on S

$\Leftrightarrow \mathcal{F}$ finite locally free in étale topology

$\Leftrightarrow \mathcal{H}$ finite locally free on S

$\Leftrightarrow \mathcal{F}$ finite locally free in Zariski topology

This explains the parenthetical statement of part (3). Now, if these equivalent conditions hold, then $\mathcal{H}$ is finite locally free. The construction of $\mathcal{H}om_{\mathcal{O}}(\mathcal{F}, \mathcal{G})$ in Modules on Sites, Section 27 depends only on $\mathcal{F}$ and $\mathcal{G}$ as presheaves of modules (only whether the output $\mathcal{H}om$ is a sheaf depends on whether $\mathcal{F}$ and $\mathcal{G}$ are sheaves). $\square$

11. Quasi-coherent modules and affines

0GZT   Let S be a scheme[7]. Let $\tau \in \{Zariski, étale, smooth, syntomic, fppf\}$. Recall that $(Aff/S)_\tau$ is the full subcategory of $(Sch/S)_\tau$ whose objects are affine turned into a site by declaring the coverings to be the standard τ-coverings. By Topologies, Lemmas 3.10, 4.11, 5.9, 6.9, and 7.11 we have an equivalence of topoi $g : Sh((Aff/S)_\tau) \to Sh((Sch/S)_\tau)$ whose pullback functor is given by restriction. Recalling that $\mathcal{O}$ denotes the structure sheaf on $(Sch/S)_\tau$, let us temporarily and pedantically denote $\mathcal{O}_{Aff}$ the restriction of $\mathcal{O}$ to $(Aff/S)_\tau$. Then we obtain an equivalence

0GZU   (11.0.1) $(Sh((Aff/S)_\tau), \mathcal{O}_{Aff}) \longrightarrow (Sh((Sch/S)_\tau), \mathcal{O})$

of ringed topoi. Having said this we can compare quasi-coherent modules as well.

0GZV   **Lemma 11.1.** *Let S be a scheme. Let $\mathcal{F}$ be a presheaf of $\mathcal{O}_{Aff}$-modules on $(Aff/S)_{fppf}$. The following are equivalent*

(1) *for every morphism $U \to U'$ of $(Aff/S)_{fppf}$ the map $\mathcal{F}(U') \otimes_{\mathcal{O}(U')} \mathcal{O}(U) \to \mathcal{F}(U)$ is an isomorphism,*

(2) *$\mathcal{F}$ is a sheaf on $(Aff/S)_{Zar}$ and a quasi-coherent module on the ringed site $((Aff/S)_{Zar}, \mathcal{O}_{Aff})$ in the sense of Modules on Sites, Definition 23.1,*

(3) *same as in (2) for the étale topology,*

(4) *same as in (2) for the smooth topology,*

(5) *same as in (2) for the syntomic topology,*

(6) *same as in (2) for the fppf topology,*

(7) *$\mathcal{F}$ corresponds to a quasi-coherent module on $(Sch/S)_{Zar}$, $(Sch/S)_{étale}$, $(Sch/S)_{smooth}$, $(Sch/S)_{syntomic}$, or $(Sch/S)_{fppf}$ via the equivalence (11.0.1),*

(8) *$\mathcal{F}$ comes from a unique quasi-coherent $\mathcal{O}_S$-module $\mathcal{G}$ by the procedure described in Section 8.*

Proof. Since the notion of a quasi-coherent module is intrinsic (Modules on Sites, Lemma 23.2) we see that the equivalence (11.0.1) induces an equivalence between categories of quasi-coherent modules. Proposition 8.9 says the topology we use to study quasi-coherent modules on Sch/S does not matter and it also tells us that (8) is the same as (7). Hence we see that (2) – (8) are all equivalent.

---

[7]In this section, as in Topologies, Section 11, we choose our sites $(Sch/S)_\tau$ to have the same underlying category for $\tau \in \{Zariski, étale, smooth, syntomic, fppf\}$. Then also the sites $(Aff/S)_\tau$ have the same underlying category.

Assume the equivalent conditions (2) – (8) hold and let $\mathcal{G}$ be as in (8). Let $h :$ $U \to U' \to S$ be a morphism of Aff/S. Denote $f : U \to S$ and $f' : U' \to S$ the structure morphisms, so that $f = f' \circ h$. We have $\mathcal{F}(U') = \Gamma(U', (f')^*\mathcal{G})$ and $\mathcal{F}(U) = \Gamma(U, f^*\mathcal{G}) = \Gamma(U, h^*(f')^*\mathcal{G})$. Hence (1) holds by Schemes, Lemma 7.3.

Assume (1) holds. To finish the proof it suffices to prove (2). Let U be an object of $(\mathit{Aff}/S)_{Zar}$. Say $U = \mathrm{Spec}(R)$. A standard open covering $U = U_1 \cup \ldots \cup U_n$ is given by $U_i = D(f_i)$ for some elements $f_1, \ldots, f_n \in R$ generating the unit ideal of R. By property (1) we see that

$$\mathcal{F}(U_i) = \mathcal{F}(U) \otimes_R R_{f_i} = \mathcal{F}(U)_{f_i}$$

and

$$\mathcal{F}(U_i \cap U_j) = \mathcal{F}(U) \otimes_R R_{f_i f_j} = \mathcal{F}(U)_{f_i f_j}$$

Thus we conclude from Algebra, Lemma 24.1 that $\mathcal{F}$ is a sheaf on $(\mathit{Aff}/S)_{Zar}$. Choose a presentation

$$\bigoplus\nolimits_{k \in K} R \longrightarrow \bigoplus\nolimits_{l \in L} R \longrightarrow \mathcal{F}(U) \longrightarrow 0$$

by free R-modules. By property (1) and the right exactness of tensor product we see that for every morphism $U' \to U$ in $(\mathit{Aff}/S)_{Zar}$ we obtain a presentation

$$\bigoplus\nolimits_{k \in K} \mathcal{O}_{Aff}(U') \longrightarrow \bigoplus\nolimits_{l \in L} \mathcal{O}_{Aff}(U') \longrightarrow \mathcal{F}(U') \longrightarrow 0$$

In other words, we see that the restriction of $\mathcal{F}$ to the localized category $(\mathit{Aff}/S)_{Zar}/U$ has a presentation

$$\bigoplus\nolimits_{k \in K} \mathcal{O}_{Aff}|_{(\mathit{Aff}/S)_{Zar}/U} \longrightarrow \bigoplus\nolimits_{l \in L} \mathcal{O}_{Aff}|_{(\mathit{Aff}/S)_{Zar}/U} \longrightarrow \mathcal{F}|_{(\mathit{Aff}/S)_{Zar}/U} \longrightarrow 0$$

With apologies for the horrible notation, this finishes the proof. $\qquad\square$

We continue the discussion started in the introduction to this section. Let $\tau \in \{Zariski, \text{é}tale\}$. Recall that $S_{affine,\tau}$ is the full subcategory of S_τ whose objects are affine turned into a site by declaring the coverings to be the standard τ coverings. See Topologies, Definitions 3.7 and 4.8. By Topologies, Lemmas 3.11, resp. 4.12 we have an equivalence of topoi $g : Sh(S_{affine,\tau}) \to Sh(S_\tau)$, whose pullback functor is given by restriction. Recalling that $\mathcal{O}$ denotes the structure sheaf on S_τ let us temporarily and pedantically denote $\mathcal{O}_{affine}$ the restriction of $\mathcal{O}$ to $S_{affine,\tau}$. Then we obtain an equivalence

0GZW   (11.1.1) $(Sh(S_{affine,\tau}), \mathcal{O}_{affine}) \longrightarrow (Sh(S_\tau), \mathcal{O})$

of ringed topoi. Having said this we can compare quasi-coherent modules as well.

0GZX   **Lemma 11.2.** *Let S be a scheme. Let $\tau \in \{Zariski, \text{é}tale\}$. Let $\mathcal{F}$ be a presheaf of $\mathcal{O}_{affine}$-modules on $S_{affine,\tau}$. The following are equivalent*
   (1) *for every morphism $U \to U'$ of $S_{affine,\tau}$ the map $\mathcal{F}(U') \otimes_{\mathcal{O}(U')} \mathcal{O}(U) \to \mathcal{F}(U)$ is an isomorphism,*
   (2) *$\mathcal{F}$ is a sheaf on $S_{affine,\tau}$ and a quasi-coherent module on the ringed site $(S_{affine,\tau}, \mathcal{O}_{affine})$ in the sense of Modules on Sites, Definition 23.1,*
   (3) *$\mathcal{F}$ corresponds to a quasi-coherent module on S_τ via the equivalence (11.1.1),*
   (4) *$\mathcal{F}$ comes from a unique quasi-coherent $\mathcal{O}_S$-module $\mathcal{G}$ by the procedure described in Section 8.*

Proof. Let us prove this in the case of the étale topology.

Assume (1) holds. To show that $\mathcal{F}$ is a sheaf, let $\mathcal{U} = \{U_i \to U\}_{i=1,\ldots,n}$ be a covering of $S_{affine,\acute{e}tale}$. The sheaf condition for $\mathcal{F}$ and $\mathcal{U}$, by our assumption on $\mathcal{F}$. reduces to showing that

$$0 \to \mathcal{F}(U) \to \prod \mathcal{F}(U) \otimes_{\mathcal{O}(U)} \mathcal{O}(U_i) \to \prod \mathcal{F}(U) \otimes_{\mathcal{O}(U)} \mathcal{O}(U_i \times_U U_j)$$

is exact. This is true because $\mathcal{O}(U) \to \prod \mathcal{O}(U_i)$ is faithfully flat (by Lemma 9.1 and the fact that coverings in $S_{affine,\acute{e}tale}$ are standard étale coverings) and we may apply Lemma 3.6. Next, we show that $\mathcal{F}$ is quasi-coherent on $S_{affine,\acute{e}tale}$. Namely, for U in $S_{affine,\acute{e}tale}$, set $R = \mathcal{O}(U)$ and choose a presentation

$$\bigoplus_{k \in K} R \longrightarrow \bigoplus_{l \in L} R \longrightarrow \mathcal{F}(U) \longrightarrow 0$$

by free R-modules. By property (1) and the right exactness of tensor product we see that for every morphism $U' \to U$ in $S_{affine,\acute{e}tale}$ we obtain a presentation

$$\bigoplus_{k \in K} \mathcal{O}(U') \longrightarrow \bigoplus_{l \in L} \mathcal{O}(U') \longrightarrow \mathcal{F}(U') \longrightarrow 0$$

In other words, we see that the restriction of $\mathcal{F}$ to the localized category $S_{affine,etale}/U$ has a presentation

$$\bigoplus_{k \in K} \mathcal{O}_{affine}|_{S_{affine,\acute{e}tale}/U} \longrightarrow \bigoplus_{l \in L} \mathcal{O}_{affine}|_{S_{affine,\acute{e}tale}/U} \longrightarrow \mathcal{F}|_{S_{affine,\acute{e}tale}/U} \longrightarrow 0$$

as required to show that $\mathcal{F}$ is quasi-coherent. With apologies for the horrible notation, this finishes the proof that (1) implies (2).

Since the notion of a quasi-coherent module is intrinsic (Modules on Sites, Lemma 23.2) we see that the equivalence (11.1.1) induces an equivalence between categories of quasi-coherent modules. Thus we have the equivalence of (2) and (3).

The equivalence of (3) and (4) follows from Proposition 8.9.

Let us assume (4) and prove (1). Namely, let $\mathcal{G}$ be as in (4). Let $h : U \to U' \to S$ be a morphism of $S_{affine,\acute{e}tale}$. Denote $f : U \to S$ and $f' : U' \to S$ the structure morphisms, so that $f = f' \circ h$. We have $\mathcal{F}(U') = \Gamma(U', (f')^*\mathcal{G})$ and $\mathcal{F}(U) = \Gamma(U, f^*\mathcal{G}) = \Gamma(U, h^*(f')^*\mathcal{G})$. Hence (1) holds by Schemes, Lemma 7.3.

We omit the proof in the case of the Zariski topology. $\qquad\square$

12. Parasitic modules

07AF Parasitic modules are those which are zero when restricted to schemes flat over the base scheme. Here is the formal definition.

06ZL **Definition 12.1.** Let S be a scheme. Let $\tau \in \{Zar, \acute{e}tale, smooth, syntomic, fppf\}$. Let $\mathcal{F}$ be a presheaf of $\mathcal{O}$-modules on $(Sch/S)_\tau$.
  (1) $\mathcal{F}$ is called *parasitic*[8] if for every flat morphism $U \to S$ we have $\mathcal{F}(U) = 0$.
  (2) $\mathcal{F}$ is called *parasitic for the τ-topology* if for every τ-covering $\{U_i \to S\}_{i \in I}$ we have $\mathcal{F}(U_i) = 0$ for all i.

If $\tau = fppf$ this means that $\mathcal{F}|_{U_{Zar}} = 0$ whenever $U \to S$ is flat and locally of finite presentation; similar for the other cases.

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[8]This may be nonstandard notation.

0755 **Lemma 12.2.** *Let S be a scheme. Let $\tau \in \{Zar, \acute{e}tale, smooth, syntomic, fppf\}$. Let $\mathcal{G}$ be a presheaf of $\mathcal{O}$-modules on $(Sch/S)_\tau$.*

    (1) *If $\mathcal{G}$ is parasitic for the τ-topology, then $H^p_\tau(U, \mathcal{G}) = 0$ for every U open in S, resp. étale over S, resp. smooth over S, resp. syntomic over S, resp. flat and locally of finite presentation over S.*

    (2) *If $\mathcal{G}$ is parasitic then $H^p_\tau(U, \mathcal{G}) = 0$ for every U flat over S.*

Proof. Proof in case $\tau = fppf$; the other cases are proved in the exact same way. The assumption means that $\mathcal{G}(U) = 0$ for any $U \to S$ flat and locally of finite presentation. Apply Cohomology on Sites, Lemma 10.9 to the subset $\mathcal{B} \subset \mathrm{Ob}((Sch/S)_{fppf})$ consisting of $U \to S$ flat and locally of finite presentation and the collection Cov of all fppf coverings of elements of $\mathcal{B}$. $\square$

07AG **Lemma 12.3.** *Let $f : T \to S$ be a morphism of schemes. For any parasitic $\mathcal{O}$-module on $(Sch/T)_\tau$ the pushforward $f_* \mathcal{F}$ and the higher direct images $R^i f_* \mathcal{F}$ are parasitic $\mathcal{O}$-modules on $(Sch/S)_\tau$.*

Proof. Recall that $R^i f_* \mathcal{F}$ is the sheaf associated to the presheaf

$$U \mapsto H^i((Sch/U \times_S T)_\tau, \mathcal{F})$$

see Cohomology on Sites, Lemma 7.4. If $U \to S$ is flat, then $U \times_S T \to T$ is flat as a base change. Hence the displayed group is zero by Lemma 12.2. If $\{U_i \to U\}$ is a τ-covering then $U_i \times_S T \to T$ is also flat. Hence it is clear that the sheafification of the displayed presheaf is zero on schemes U flat over S. $\square$

0756 **Lemma 12.4.** *Let S be a scheme. Let $\tau \in \{Zar, \acute{e}tale\}$. Let $\mathcal{G}$ be a sheaf of $\mathcal{O}$-modules on $(Sch/S)_{fppf}$ such that*

    (1) *$\mathcal{G}|_{S_\tau}$ is quasi-coherent, and*

    (2) *for every flat, locally finitely presented morphism $g : U \to S$ the canonical map $g^*_{\tau, small}(\mathcal{G}|_{S_\tau}) \to \mathcal{G}|_{U_\tau}$ is an isomorphism.*

Then $H^p(U, \mathcal{G}) = H^p(U, \mathcal{G}|_{U_\tau})$ for every U flat and locally of finite presentation over S.

Proof. Let $\mathcal{F}$ be the pullback of $\mathcal{G}|_{S_\tau}$ to the big fppf site $(Sch/S)_{fppf}$. Note that $\mathcal{F}$ is quasi-coherent. There is a canonical comparison map $\varphi : \mathcal{F} \to \mathcal{G}$ which by assumptions (1) and (2) induces an isomorphism $\mathcal{F}|_{U_\tau} \to \mathcal{G}|_{U_\tau}$ for all $g : U \to S$ flat and locally of finite presentation. Hence in the short exact sequences

$$0 \to \mathrm{Ker}(\varphi) \to \mathcal{F} \to \mathrm{Im}(\varphi) \to 0$$

and

$$0 \to \mathrm{Im}(\varphi) \to \mathcal{G} \to \mathrm{Coker}(\varphi) \to 0$$

the sheaves $\mathrm{Ker}(\varphi)$ and $\mathrm{Coker}(\varphi)$ are parasitic for the fppf topology. By Lemma 12.2 we conclude that $H^p(U, \mathcal{F}) \to H^p(U, \mathcal{G})$ is an isomorphism for $g : U \to S$ flat and locally of finite presentation. Since the result holds for $\mathcal{F}$ by Proposition 9.3 we win. $\square$

13. Fpqc coverings are universal effective epimorphisms

023P We apply the material above to prove an interesting result, namely Lemma 13.7. By Sites, Section 12 this lemma implies that the representable presheaves on any of the sites $(Sch/S)_\tau$ are sheaves for $\tau \in \{Zariski, fppf, \acute{e}tale, smooth, syntomic\}$. First we prove a helper lemma.

02KI **Lemma 13.1.** *For a scheme X denote $|X|$ the underlying set. Let $f : X \to S$ be a morphism of schemes. Then*

$$|X \times_S X| \to |X| \times_{|S|} |X|$$

is surjective.

Proof. Follows immediately from the description of points on the fibre product in Schemes, Lemma 17.5. $\square$

0EUA **Lemma 13.2.** *Let $\{f_i : X_i \to X\}_{i \in I}$ be a family of morphisms of affine schemes. The following are equivalent*

(1) *for any quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ we have*

$$\Gamma(X, \mathcal{F}) = Equalizer\left(\ \textstyle\prod_{i \in I} \Gamma(X_i, f_i^* \mathcal{F}) \rightrightarrows \prod_{i,j \in I} \Gamma(X_i \times_X X_j, (f_i \times f_j)^* \mathcal{F}) \ \right)$$

(2) *$\{f_i : X_i \to X\}_{i \in I}$ is a universal effective epimorphism (Sites, Definition 12.1) in the category of affine schemes.*

Proof. Assume (2) holds and let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Consider the scheme (Constructions, Section 4)

$$X' = \underline{\mathrm{Spec}}_X(\mathcal{O}_X \oplus \mathcal{F})$$

where $\mathcal{O}_X \oplus \mathcal{F}$ is an $\mathcal{O}_X$-algebra with multiplication $(f, s)(f', s') = (ff', fs' + f's)$. If $s_i \in \Gamma(X_i, f_i^* \mathcal{F})$ is a section, then s_i determines a unique element of

$$\Gamma(X' \times_X X_i, \mathcal{O}_{X' \times_X X_i}) = \Gamma(X_i, \mathcal{O}_{X_i}) \oplus \Gamma(X_i, f_i^* \mathcal{F})$$

Proof of equality omitted. If $(s_i)_{i \in I}$ is in the equalizer of (1), then, using the equality

$$\mathrm{Mor}(T, \mathbf{A}_{\mathbf{Z}}^1) = \Gamma(T, \mathcal{O}_T)$$

which holds for any scheme T, we see that these sections define a family of morphisms $h_i : X' \times_X X_i \to \mathbf{A}_{\mathbf{Z}}^1$ with $h_i \circ \mathrm{pr}_1 = h_j \circ \mathrm{pr}_2$ as morphisms $(X' \times_X X_i) \times_{X'} (X' \times_X X_j) \to \mathbf{A}_{\mathbf{Z}}^1$. Since we've assume (2) we obtain a morphism $h : X' \to \mathbf{A}_{\mathbf{Z}}^1$ compatible with the morphisms h_i which in turn determines an element $s \in \Gamma(X, \mathcal{F})$. We omit the verification that s maps to s_i in $\Gamma(X_i, f_i^* \mathcal{F})$.

Assume (1). Let T be an affine scheme and let $h_i : X_i \to T$ be a family of morphisms such that $h_i \circ \mathrm{pr}_1 = h_j \circ \mathrm{pr}_2$ on $X_i \times_X X_j$ for all $i, j \in I$. Then

$$\prod h_i^\sharp : \Gamma(T, \mathcal{O}_T) \to \prod \Gamma(X_i, \mathcal{O}_{X_i})$$

maps into the equalizer and we find that we get a ring map $\Gamma(T, \mathcal{O}_T) \to \Gamma(X, \mathcal{O}_X)$ by the assumption of the lemma for $\mathcal{F} = \mathcal{O}_X$. This ring map corresponds to a morphism $h : X \to T$ such that $h_i = h \circ f_i$. Hence our family is an effective epimorphism.

Let $p : Y \to X$ be a morphism of affines. We will show the base changes $g_i : Y_i \to Y$ of f_i form an effective epimorphism by applying the result of the previous paragraph. Namely, if $\mathcal{G}$ is a quasi-coherent $\mathcal{O}_Y$-module, then

$$\Gamma(Y, \mathcal{G}) = \Gamma(X, p_* \mathcal{G}), \quad \Gamma(Y_i, g_i^* \mathcal{G}) = \Gamma(X, f_i^* p_* \mathcal{G}),$$

and

$$\Gamma(Y_i \times_Y Y_j, (g_i \times g_j)^* \mathcal{G}) = \Gamma(X, (f_i \times f_j)^* p_* \mathcal{G})$$

by the trivial base change formula (Cohomology of Schemes, Lemma 5.1). Thus we see property (1) lemma holds for the family g_i. $\square$

0EUB **Lemma 13.3.** *Let $\{f_i : X_i \to X\}_{i \in I}$ be a family of morphisms of schemes.*

(1) *If the family is universal effective epimorphism in the category of schemes, then $\coprod f_i$ is surjective.*
(2) *If X and X_i are affine and the family is a universal effective epimorphism in the category of affine schemes, then $\coprod f_i$ is surjective.*

Proof. Omitted. Hint: perform base change by $\mathrm{Spec}(\kappa(x)) \to X$ to see that any $x \in X$ has to be in the image. $\square$

0EUC **Lemma 13.4.** *Let $\{f_i : X_i \to X\}_{i \in I}$ be a family of morphisms of schemes. If for every morphism $Y \to X$ with Y affine the family of base changes $g_i : Y_i \to Y$ forms an effective epimorphism, then the family of f_i forms a universal effective epimorphism in the category of schemes.*

Proof. Let $Y \to X$ be a morphism of schemes. We have to show that the base changes $g_i : Y_i \to Y$ form an effective epimorphism. To do this, assume given a scheme T and morphisms $h_i : Y_i \to T$ with $h_i \circ \mathrm{pr}_1 = h_j \circ \mathrm{pr}_2$ on $Y_i \times_Y Y_j$. Choose an affine open covering $Y = \bigcup V_\alpha$. Set $V_{\alpha,i}$ equal to the inverse image of V_α in Y_i. Then we see that $V_{\alpha,i} \to V_\alpha$ is the base change of f_i by $V_\alpha \to X$. Thus by assumption the family of restrictions $h_i|_{V_{\alpha,i}}$ come from a morphism of schemes $h_\alpha : V_\alpha \to T$. We leave it to the reader to show that these agree on overlaps and define the desired morphism $Y \to T$. See discussion in Schemes, Section 14. $\square$

0EUD **Lemma 13.5.** *Let $\{f_i : X_i \to X\}_{i \in I}$ be a family of morphisms of affine schemes. Assume the equivalent assumption of Lemma 13.2 hold and that moreover for any morphism of affines $Y \to X$ the map*

$$\coprod X_i \times_X Y \longrightarrow Y$$

is a submersive map of topological spaces (Topology, Definition 6.3). Then our family of morphisms is a universal effective epimorphism in the category of schemes.

Proof. By Lemma 13.4 it suffices to base change our family of morphisms by $Y \to X$ with Y affine. Set $Y_i = X_i \times_X Y$. Let T be a scheme and let $h_i : Y_i \to T$ be a family of morphisms such that $h_i \circ \mathrm{pr}_1 = h_j \circ \mathrm{pr}_2$ on $Y_i \times_Y Y_j$. Note that Y as a set is the coequalizer of the two maps from $\coprod Y_i \times_Y Y_j$ to $\coprod Y_i$. Namely, surjectivity by the affine case of Lemma 13.3 and injectivity by Lemma 13.1. Hence there is a set map of underlying sets $h : Y \to T$ compatible with the maps h_i. By the second condition of the lemma we see that h is continuous! Thus if $y \in Y$ and $U \subset T$ is an affine open neighbourhood of $h(y)$, then we can find an affine open $V \subset Y$ such that $h(V) \subset U$. Setting $V_i = Y_i \times_Y V = X_i \times_X V$ we can use the result proved in Lemma 13.2 to see that $h|_V : V \to U \subset T$ comes from a unique morphism of affine schemes $h_V : V \to U$ agreeing with $h_i|_{V_i}$ as morphisms of schemes for all i. Glueing these h_V (see Schemes, Section 14) gives a morphism $Y \to T$ as desired. $\square$

03N0 **Lemma 13.6.** *Let $\{f_i : T_i \to T\}_{i \in I}$ be a fpqc covering. Suppose that for each i we have an open subset $W_i \subset T_i$ such that for all $i, j \in I$ we have $pr_0^{-1}(W_i) = pr_1^{-1}(W_j)$ as open subsets of $T_i \times_T T_j$. Then there exists a unique open subset $W \subset T$ such that $W_i = f_i^{-1}(W)$ for each i.*

Proof. Apply Lemma 13.1 to the map $\coprod_{i \in I} T_i \to T$. It implies there exists a subset $W \subset T$ such that $W_i = f_i^{-1}(W)$ for each i, namely $W = \bigcup f_i(W_i)$. To see that W is open we may work Zariski locally on T. Hence we may assume that T

is affine. Using Topologies, Definition 9.1 we may choose a standard fpqc covering $\{g_j : V_j \to T\}_{j \in J}$ which refines $\{T_i \to T\}_{i \in I}$. Let $\alpha : J \to I$ and $h_j : V_j \to T_{\alpha(j)}$ be as in Sites, Definition 8.1. Then $g_j^{-1}(W) = h_j^{-1}(W_{\alpha(j)})$. Thus we may assume $\{f_i : T_i \to T\}$ is a standard fpqc covering. In this case we may apply Morphisms, Lemma 25.12 to the morphism $\coprod T_i \to T$ to conclude that W is open. $\square$

023Q   **Lemma 13.7.** *Let $\{T_i \to T\}$ be an fpqc covering, see Topologies, Definition 9.1.
         Then $\{T_i \to T\}$ is a universal effective epimorphism in the category of schemes, see
         Sites, Definition 12.1. In other words, every representable functor on the category of
         schemes satisfies the sheaf condition for the fpqc topology, see Topologies, Definition
         9.13.*

Proof. Let S be a scheme. We have to show the following: Given morphisms $\varphi_i : T_i \to S$ such that $\varphi_i|_{T_i \times_T T_j} = \varphi_j|_{T_i \times_T T_j}$ there exists a unique morphism $T \to S$ which restricts to φ_i on each T_i. In other words, we have to show that the functor $h_S = \mathrm{Mor}_{Sch}(-, S)$ satisfies the sheaf property for the fpqc topology.

If $\{T_i \to T\}$ is a Zariski covering, then this follows from Schemes, Lemma 14.1. Thus Topologies, Lemma 9.14 reduces us to the case of a covering $\{X \to Y\}$ given by a single surjective flat morphism of affines.

First proof. By Lemma 8.1 we have the sheaf condition for quasi-coherent modules for $\{X \to Y\}$. By Lemma 13.6 the morphism $X \to Y$ is universally submersive. Hence we may apply Lemma 13.5 to see that $\{X \to Y\}$ is a universal effective epimorphism.

Second proof. Let $R \to A$ be the faithfully flat ring map corresponding to our surjective flat morphism $\pi : X \to Y$. Let $f : X \to S$ be a morphism such that $f \circ \mathrm{pr}_1 = f \circ \mathrm{pr}_2$ as morphisms $X \times_Y X = \mathrm{Spec}(A \otimes_R A) \to S$. By Lemma 13.1 we see that as a map on the underlying sets f is of the form $f = g \circ \pi$ for some (set theoretic) map $g : \mathrm{Spec}(R) \to S$. By Morphisms, Lemma 25.12 and the fact that f is continuous we see that g is continuous.

Pick $y \in Y = \mathrm{Spec}(R)$. Choose $U \subset S$ affine open containing $g(y)$. Say $U = \mathrm{Spec}(B)$. By the above we may choose an $r \in R$ such that $y \in D(r) \subset g^{-1}(U)$. The restriction of f to $\pi^{-1}(D(r))$ into U corresponds to a ring map $B \to A_r$. The two induced ring maps $B \to A_r \otimes_{R_r} A_r = (A \otimes_R A)_r$ are equal by assumption on f. Note that $R_r \to A_r$ is faithfully flat. By Lemma 3.6 the equalizer of the two arrows $A_r \to A_r \otimes_{R_r} A_r$ is R_r. We conclude that $B \to A_r$ factors uniquely through a map $B \to R_r$. This map in turn gives a morphism of schemes $D(r) \to U \to S$, see Schemes, Lemma 6.4.

What have we proved so far? We have shown that for any prime $\mathfrak{p} \subset R$, there exists a standard affine open $D(r) \subset \mathrm{Spec}(R)$ such that the morphism $f|_{\pi^{-1}(D(r))} : \pi^{-1}(D(r)) \to S$ factors uniquely through some morphism of schemes $D(r) \to S$. We omit the verification that these morphisms glue to the desired morphism $\mathrm{Spec}(R) \to S$. $\square$

0BMN   **Lemma 13.8.** *Consider schemes X, Y, Z and morphisms $a, b : X \to Y$ and a
         morphism $c : Y \to Z$ with $c \circ a = c \circ b$. Set $d = c \circ a = c \circ b$. If there exists an fpqc
         covering $\{Z_i \to Z\}$ such that*

         (1) *for all i the morphism $Y \times_{c,Z} Z_i \to Z_i$ is the coequalizer of $(a, 1) : X \times_{d,Z}
              Z_i \to Y \times_{c,Z} Z_i$ and $(b, 1) : X \times_{d,Z} Z_i \to Y \times_{c,Z} Z_i$, and*

(2) *for all i and i' the morphism $Y \times_{c,Z} (Z_i \times_Z Z_{i'}) \to (Z_i \times_Z Z_{i'})$ is the coequalizer of $(a,1) : X \times_{d,Z} (Z_i \times_Z Z_{i'}) \to Y \times_{c,Z} (Z_i \times_Z Z_{i'})$ and $(b,1) : X \times_{d,Z} (Z_i \times_Z Z_{i'}) \to Y \times_{c,Z} (Z_i \times_Z Z_{i'})$*

then c is the coequalizer of a and b.

Proof. Namely, for a scheme T a morphism $Z \to T$ is the same thing as a collection of morphism $Z_i \to T$ which agree on overlaps by Lemma 13.7. $\square$

14. Descent of finiteness and smoothness properties of morphisms

02KJ In this section we show that several properties of morphisms (being smooth, locally of finite presentation, and so on) descend under faithfully flat morphisms. We start with an algebraic version. (The "Noetherian" reader should consult Lemma 14.2 instead of the next lemma.)

02KK **Lemma 14.1.** *Let $R \to A \to B$ be ring maps. Assume $R \to B$ is of finite presentation and $A \to B$ faithfully flat and of finite presentation. Then $R \to A$ is of finite presentation.*

Proof. Consider the algebra $C = B \otimes_A B$ together with the pair of maps $p, q : B \to C$ given by $p(b) = b \otimes 1$ and $q(b) = 1 \otimes b$. Of course the two compositions $A \to B \to C$ are the same. Note that as $p : B \to C$ is flat and of finite presentation (base change of $A \to B$), the ring map $R \to C$ is of finite presentation (as the composite of $R \to B \to C$).

We are going to use the criterion Algebra, Lemma 127.3 to show that $R \to A$ is of finite presentation. Let S be any R-algebra, and suppose that $S = \mathrm{colim}_{\lambda \in \Lambda} S_\lambda$ is written as a directed colimit of R-algebras. Let $A \to S$ be an R-algebra homomorphism. We have to show that $A \to S$ factors through one of the S_λ. Consider the rings $B' = S \otimes_A B$ and $C' = S \otimes_A C = B' \otimes_S B'$. As B is faithfully flat of finite presentation over A, also B' is faithfully flat of finite presentation over S. By Algebra, Lemma 168.1 part (2) applied to the pair $(S \to B', B')$ and the system (S_λ) there exists a $\lambda_0 \in \Lambda$ and a flat, finitely presented S_{λ_0}-algebra B_{λ_0} such that $B' = S \otimes_{S_{\lambda_0}} B_{\lambda_0}$. For $\lambda \geq \lambda_0$ set $B_\lambda = S_\lambda \otimes_{S_{\lambda_0}} B_{\lambda_0}$ and $C_\lambda = B_\lambda \otimes_{S_\lambda} B_\lambda$.

We interrupt the flow of the argument to show that $S_\lambda \to B_\lambda$ is faithfully flat for λ large enough. (This should really be a separate lemma somewhere else, maybe in the chapter on limits.) Since $\mathrm{Spec}(B_{\lambda_0}) \to \mathrm{Spec}(S_{\lambda_0})$ is flat and of finite presentation it is open (see Morphisms, Lemma 25.10). Let $I \subset S_{\lambda_0}$ be an ideal such that $V(I) \subset \mathrm{Spec}(S_{\lambda_0})$ is the complement of the image. Note that formation of the image commutes with base change. Hence, since $\mathrm{Spec}(B') \to \mathrm{Spec}(S)$ is surjective, and $B' = B_{\lambda_0} \otimes_{S_{\lambda_0}} S$ we see that $IS = S$. Thus for some $\lambda \geq \lambda_0$ we have $IS_\lambda = S_\lambda$. For this and all greater λ the morphism $\mathrm{Spec}(B_\lambda) \to \mathrm{Spec}(S_\lambda)$ is surjective.

By analogy with the notation in the first paragraph of the proof denote $p_\lambda, q_\lambda : B_\lambda \to C_\lambda$ the two canonical maps. Then $B' = \mathrm{colim}_{\lambda \geq \lambda_0} B_\lambda$ and $C' = \mathrm{colim}_{\lambda \geq \lambda_0} C_\lambda$. Since B and C are finitely presented over R there exist (by Algebra, Lemma 127.3 applied several times) a $\lambda \geq \lambda_0$ and an R-algebra maps $B \to B_\lambda$, $C \to C_\lambda$ such that the diagram

$$
\begin{array}{ccc}
C & \longrightarrow & C_\lambda \\
{\scriptstyle p}\big\uparrow\big\uparrow{\scriptstyle q} & & {\scriptstyle p_\lambda}\big\uparrow\big\uparrow{\scriptstyle q_\lambda} \\
B & \longrightarrow & B_\lambda
\end{array}
$$

is commutative. OK, and this means that $A \to B \to B_\lambda$ maps into the equalizer of p_λ and q_λ. By Lemma 3.6 we see that S_λ is the equalizer of p_λ and q_λ. Thus we get the desired ring map $A \to S_\lambda$ and we win. $\square$

Here is an easier version of this dealing with the property of being of finite type.

0367 **Lemma 14.2.** *Let $R \to A \to B$ be ring maps. Assume $R \to B$ is of finite type and $A \to B$ faithfully flat and of finite presentation. Then $R \to A$ is of finite type.*

Proof. By Algebra, Lemma 168.2 there exists a commutative diagram

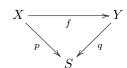$$\begin{array}{ccc} R \longrightarrow & A_0 \longrightarrow & B_0 \\ \| & \downarrow & \downarrow \\ R \longrightarrow & A \longrightarrow & B \end{array}$$

with $R \to A_0$ of finite presentation, $A_0 \to B_0$ faithfully flat of finite presentation and $B = A \otimes_{A_0} B_0$. Since $R \to B$ is of finite type by assumption, we may add some elements to A_0 and assume that the map $B_0 \to B$ is surjective! In this case, since $A_0 \to B_0$ is faithfully flat, we see that as

$$(A_0 \to A) \otimes_{A_0} B_0 \cong (B_0 \to B)$$

is surjective, also $A_0 \to A$ is surjective. Hence we win. $\square$

02KL **Lemma 14.3.** *Let*

[DG67, IV, 17.7.5 (i) and (ii)].

$$\begin{array}{ccc} X & \xrightarrow{\ f\ } & Y \\ & {\scriptstyle p}\searrow \quad \swarrow {\scriptstyle q} & \\ & S & \end{array}$$

be a commutative diagram of morphisms of schemes. Assume that f is surjective, flat and locally of finite presentation and assume that p is locally of finite presentation (resp. locally of finite type). Then q is locally of finite presentation (resp. locally of finite type).

Proof. The problem is local on S and Y. Hence we may assume that S and Y are affine. Since f is flat and locally of finite presentation, we see that f is open (Morphisms, Lemma 25.10). Hence, since Y is quasi-compact, there exist finitely many affine opens $X_i \subset X$ such that $Y = \bigcup f(X_i)$. Clearly we may replace X by $\coprod X_i$, and hence we may assume X is affine as well. In this case the lemma is equivalent to Lemma 14.1 (resp. Lemma 14.2) above. $\square$

We use this to improve some of the results on morphisms obtained earlier.

02KM **Lemma 14.4.** *Let*

$$\begin{array}{ccc} X & \xrightarrow{\ f\ } & Y \\ & {\scriptstyle p}\searrow \quad \swarrow {\scriptstyle q} & \\ & S & \end{array}$$

be a commutative diagram of morphisms of schemes. Assume that

(1) *f is surjective, and syntomic (resp. smooth, resp. étale),*
(2) *p is syntomic (resp. smooth, resp. étale).*

Then q is syntomic (resp. smooth, resp. étale).

Proof. Combine Morphisms, Lemmas 30.16, 34.19, and 36.19 with Lemma 14.3 above. $\square$

Actually we can strengthen this result as follows.

05B5   **Lemma 14.5.** *Let*
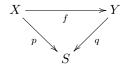
$$X \xrightarrow{\quad f \quad} Y$$
$$p \searrow \quad \swarrow q$$
$$S$$

be a commutative diagram of morphisms of schemes. Assume that
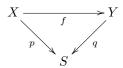
(1) *f is surjective, flat, and locally of finite presentation,*
(2) *p is smooth (resp. étale).*

Then q is smooth (resp. étale).

Proof. Assume (1) and that p is smooth. By Lemma 14.3 we see that q is locally of finite presentation. By Morphisms, Lemma 25.13 we see that q is flat. Hence now it suffices to show that the fibres of q are smooth, see Morphisms, Lemma 34.3. Apply Varieties, Lemma 25.9 to the flat surjective morphisms $X_s \to Y_s$ for $s \in S$ to conclude. We omit the proof of the étale case. $\square$

05B6   **Remark 14.6.** With the assumptions (1) and p smooth in Lemma 14.5 it is not automatically the case that $X \to Y$ is smooth. A counter example is $S = \mathrm{Spec}(k)$, $X = \mathrm{Spec}(k[s])$, $Y = \mathrm{Spec}(k[t])$ and f given by $t \mapsto s^2$. But see also Lemma 14.7 for some information on the structure of f.

05B7   **Lemma 14.7.** *Let*

$$X \xrightarrow{\quad f \quad} Y$$
$$p \searrow \quad \swarrow q$$
$$S$$

be a commutative diagram of morphisms of schemes. Assume that

(1) *f is surjective, flat, and locally of finite presentation,*
(2) *p is syntomic.*

Then both q and f are syntomic.

Proof. By Lemma 14.3 we see that q is of finite presentation. By Morphisms, Lemma 25.13 we see that q is flat. By Morphisms, Lemma 30.10 it now suffices to show that the local rings of the fibres of $Y \to S$ and the fibres of $X \to Y$ are local complete intersection rings. To do this we may take the fibre of $X \to Y \to S$ at a point $s \in S$, i.e., we may assume S is the spectrum of a field. Pick a point $x \in X$ with image $y \in Y$ and consider the ring map

$$\mathcal{O}_{Y,y} \longrightarrow \mathcal{O}_{X,x}$$

This is a flat local homomorphism of local Noetherian rings. The local ring $\mathcal{O}_{X,x}$ is a complete intersection. Thus may use Avramov's result, see Divided Power Algebra, Lemma 8.9, to conclude that both $\mathcal{O}_{Y,y}$ and $\mathcal{O}_{X,x}/\mathfrak{m}_y\mathcal{O}_{X,x}$ are complete intersection rings. $\square$

The following type of lemma is occasionally useful.

06NB **Lemma 14.8.** *Let $X \to Y \to Z$ be morphism of schemes. Let P be one of the following properties of morphisms of schemes: flat, locally finite type, locally finite presentation. Assume that $X \to Z$ has P and that $\{X \to Y\}$ can be refined by an fppf covering of Y. Then $Y \to Z$ is P.*

Proof. Let $\operatorname{Spec}(C) \subset Z$ be an affine open and let $\operatorname{Spec}(B) \subset Y$ be an affine open which maps into $\operatorname{Spec}(C)$. The assumption on $X \to Y$ implies we can find a standard affine fppf covering $\{\operatorname{Spec}(B_j) \to \operatorname{Spec}(B)\}$ and lifts $x_j : \operatorname{Spec}(B_j) \to X$. Since $\operatorname{Spec}(B_j)$ is quasi-compact we can find finitely many affine opens $\operatorname{Spec}(A_i) \subset X$ lying over $\operatorname{Spec}(B)$ such that the image of each x_j is contained in the union $\bigcup \operatorname{Spec}(A_i)$. Hence after replacing each $\operatorname{Spec}(B_j)$ by a standard affine Zariski coverings of itself we may assume we have a standard affine fppf covering $\{\operatorname{Spec}(B_i) \to \operatorname{Spec}(B)\}$ such that each $\operatorname{Spec}(B_i) \to Y$ factors through an affine open $\operatorname{Spec}(A_i) \subset X$ lying over $\operatorname{Spec}(B)$. In other words, we have ring maps $C \to B \to A_i \to B_i$ for each i. Note that we can also consider

$$C \to B \to A = \prod A_i \to B' = \prod B_i$$

and that the ring map $B \to \prod B_i$ is faithfully flat and of finite presentation.

The case $P = flat$. In this case we know that $C \to A$ is flat and we have to prove that $C \to B$ is flat. Suppose that $N \to N' \to N''$ is an exact sequence of C-modules. We want to show that $N \otimes_C B \to N' \otimes_C B \to N'' \otimes_C B$ is exact. Let H be its cohomology and let H' be the cohomology of $N \otimes_C B' \to N' \otimes_C B' \to N'' \otimes_C B'$. As $B \to B'$ is flat we know that $H' = H \otimes_B B'$. On the other hand $N \otimes_C A \to N' \otimes_C A \to N'' \otimes_C A$ is exact hence has zero cohomology. Hence the map $H \to H'$ is zero (as it factors through the zero module). Thus $H' = 0$. As $B \to B'$ is faithfully flat we conclude that $H = 0$ as desired.

The case $P = locally\ finite\ type$. In this case we know that $C \to A$ is of finite type and we have to prove that $C \to B$ is of finite type. Because $B \to B'$ is of finite presentation (hence of finite type) we see that $A \to B'$ is of finite type, see Algebra, Lemma 6.2. Therefore $C \to B'$ is of finite type and we conclude by Lemma 14.2.

The case $P = locally\ finite\ presentation$. In this case we know that $C \to A$ is of finite presentation and we have to prove that $C \to B$ is of finite presentation. Because $B \to B'$ is of finite presentation and $B \to A$ of finite type we see that $A \to B'$ is of finite presentation, see Algebra, Lemma 6.2. Therefore $C \to B'$ is of finite presentation and we conclude by Lemma 14.1. $\square$

15. Local properties of schemes

0347 It often happens one can prove the members of a covering of a scheme have a certain property. In many cases this implies the scheme has the property too. For example, if S is a scheme, and $f : S' \to S$ is a surjective flat morphism such that S' is a reduced scheme, then S is reduced. You can prove this by looking at local rings and using Algebra, Lemma 164.2. We say that the property of being reduced *descends through flat surjective morphisms*. Some results of this type are collected in Algebra, Section 164 and for schemes in Section 19. Some analogous results on descending properties of morphisms are in Section 14.

On the other hand, there are examples of surjective flat morphisms $f : S' \to S$ with S reduced and S' not, for example the morphism $\operatorname{Spec}(k[x]/(x^2)) \to \operatorname{Spec}(k)$.

Hence the property of being reduced does not *ascend along flat morphisms*. Having infinite residue fields is a property which does ascend along flat morphisms (but does not descend along surjective flat morphisms of course). Some results of this type are collected in Algebra, Section 163.

Finally, we say that a property is *local for the flat topology* if it ascends along flat morphisms and descends along flat surjective morphisms. A somewhat silly example is the property of having residue fields of a given characteristic. To be more precise, and to tie this in with the various topologies on schemes, we make the following formal definition.

0348 **Definition 15.1.** Let $\mathcal{P}$ be a property of schemes. Let $\tau \in \{fpqc, fppf, syntomic, smooth, \acute{e}tale, Zariski\}$. We say $\mathcal{P}$ is *local in the τ-topology* if for any τ-covering $\{S_i \to S\}_{i \in I}$ (see Topologies, Section 2) we have

$$S \text{ has } \mathcal{P} \Leftrightarrow \text{each } S_i \text{ has } \mathcal{P}.$$

To be sure, since isomorphisms are always coverings we see (or require) that property $\mathcal{P}$ holds for S if and only if it holds for any scheme S' isomorphic to S. In fact, if $\tau = fpqc, fppf, syntomic, smooth, \acute{e}tale$, or $Zariski$, then if S has $\mathcal{P}$ and $S' \to S$ is flat, flat and locally of finite presentation, syntomic, smooth, étale, or an open immersion, then S' has $\mathcal{P}$. This is true because we can always extend $\{S' \to S\}$ to a τ-covering.

We have the following implications: $\mathcal{P}$ is local in the fpqc topology $\Rightarrow$ $\mathcal{P}$ is local in the fppf topology $\Rightarrow$ $\mathcal{P}$ is local in the syntomic topology $\Rightarrow$ $\mathcal{P}$ is local in the smooth topology $\Rightarrow$ $\mathcal{P}$ is local in the étale topology $\Rightarrow$ $\mathcal{P}$ is local in the Zariski topology. This follows from Topologies, Lemmas 4.2, 5.2, 6.2, 7.2, and 9.7.

0349 **Lemma 15.2.** *Let $\mathcal{P}$ be a property of schemes. Let $\tau \in \{fpqc, fppf, \acute{e}tale, smooth, syntomic\}$. Assume that*
  (1) *the property is local in the Zariski topology,*
  (2) *for any morphism of affine schemes $S' \to S$ which is flat, flat of finite presentation, étale, smooth or syntomic depending on whether τ is fpqc, fppf, étale, smooth, or syntomic, property $\mathcal{P}$ holds for S' if property $\mathcal{P}$ holds for S, and*
  (3) *for any surjective morphism of affine schemes $S' \to S$ which is flat, flat of finite presentation, étale, smooth or syntomic depending on whether τ is fpqc, fppf, étale, smooth, or syntomic, property $\mathcal{P}$ holds for S if property $\mathcal{P}$ holds for S'.*

Then $\mathcal{P}$ is τ local on the base.

Proof. This follows almost immediately from the definition of a τ-covering, see Topologies, Definition 9.1 7.1 4.1 5.1, or 6.1 and Topologies, Lemma 9.9, 7.4, 4.4, 5.4, or 6.4. Details omitted. $\square$

034A **Remark 15.3.** In Lemma 15.2 above if $\tau = smooth$ then in condition (3) we may assume that the morphism is a (surjective) standard smooth morphism. Similarly, when $\tau = syntomic$ or $\tau = \acute{e}tale$.

16. Properties of schemes local in the fppf topology

034B In this section we find some properties of schemes which are local on the base in the fppf topology.
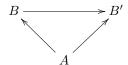
034C **Lemma 16.1.** *The property $\mathcal{P}(S) = $ "S is locally Noetherian" is local in the fppf topology.*

Proof. We will use Lemma 15.2. First we note that "being locally Noetherian" is local in the Zariski topology. This is clear from the definition, see Properties, Definition 5.1. Next, we show that if $S' \to S$ is a flat, finitely presented morphism of affines and S is locally Noetherian, then S' is locally Noetherian. This is Morphisms, Lemma 15.6. Finally, we have to show that if $S' \to S$ is a surjective flat, finitely presented morphism of affines and S' is locally Noetherian, then S is locally Noetherian. This follows from Algebra, Lemma 164.1. Thus (1), (2) and (3) of Lemma 15.2 hold and we win. $\square$

0368 **Lemma 16.2.** *The property $\mathcal{P}(S) = $ "S is Jacobson" is local in the fppf topology.*

Proof. We will use Lemma 15.2. First we note that "being Jacobson" is local in the Zariski topology. This is Properties, Lemma 6.3. Next, we show that if $S' \to S$ is a flat, finitely presented morphism of affines and S is Jacobson, then S' is Jacobson. This is Morphisms, Lemma 16.9. Finally, we have to show that if $f : S' \to S$ is a surjective flat, finitely presented morphism of affines and S' is Jacobson, then S is Jacobson. Say $S = \mathrm{Spec}(A)$ and $S' = \mathrm{Spec}(B)$ and $S' \to S$ given by $A \to B$. Then $A \to B$ is finitely presented and faithfully flat. Moreover, the ring B is Jacobson, see Properties, Lemma 6.3.

By Algebra, Lemma 168.10 there exists a diagram

$$
\begin{array}{ccc}
B & \longrightarrow & B' \\
& \nwarrow \quad \nearrow & \\
& A &
\end{array}
$$

with $A \to B'$ finitely presented, faithfully flat and quasi-finite. In particular, $B \to B'$ is finite type, and we see from Algebra, Proposition 35.19 that B' is Jacobson. Hence we may assume that $A \to B$ is quasi-finite as well as faithfully flat and of finite presentation.

Assume A is not Jacobson to get a contradiction. According to Algebra, Lemma 35.5 there exists a nonmaximal prime $\mathfrak{p} \subset A$ and an element $f \in A$, $f \notin \mathfrak{p}$ such that $V(\mathfrak{p}) \cap D(f) = \{\mathfrak{p}\}$.

This leads to a contradiction as follows. First let $\mathfrak{p} \subset \mathfrak{m}$ be a maximal ideal of A. Pick a prime $\mathfrak{m}' \subset B$ lying over $\mathfrak{m}$ (exists because $A \to B$ is faithfully flat, see Algebra, Lemma 39.16). As $A \to B$ is flat, by going down see Algebra, Lemma 39.19, we can find a prime $\mathfrak{q} \subset \mathfrak{m}'$ lying over $\mathfrak{p}$. In particular we see that $\mathfrak{q}$ is not maximal. Hence according to Algebra, Lemma 35.5 again the set $V(\mathfrak{q}) \cap D(f)$ is infinite (here we finally use that B is Jacobson). All points of $V(\mathfrak{q}) \cap D(f)$ map to $V(\mathfrak{p}) \cap D(f) = \{\mathfrak{p}\}$. Hence the fibre over $\mathfrak{p}$ is infinite. This contradicts the fact that $A \to B$ is quasi-finite (see Algebra, Lemma 122.5 or more explicitly Morphisms, Lemma 20.10). Thus the lemma is proved. $\square$

0BAL **Lemma 16.3.** *The property $\mathcal{P}(S) = $ "every quasi-compact open of S has a finite number of irreducible components" is local in the fppf topology.*

Proof. We will use Lemma 15.2. First we note that $\mathcal{P}$ is local in the Zariski topology. Next, we show that if $T \to S$ is a flat, finitely presented morphism

of affines and S has a finite number of irreducible components, then so does T. Namely, since $T \to S$ is flat, the generic points of T map to the generic points of S, see Morphisms, Lemma 25.9. Hence it suffices to show that for $s \in S$ the fibre T_s has a finite number of generic points. Note that T_s is an affine scheme of finite type over $\kappa(s)$, see Morphisms, Lemma 15.4. Hence T_s is Noetherian and has a finite number of irreducible components (Morphisms, Lemma 15.6 and Properties, Lemma 5.7). Finally, we have to show that if $T \to S$ is a surjective flat, finitely presented morphism of affines and T has a finite number of irreducible components, then so does S. This follows from Topology, Lemma 8.5. Thus (1), (2) and (3) of Lemma 15.2 hold and we win. $\square$

17. Properties of schemes local in the syntomic topology

0369   In this section we find some properties of schemes which are local on the base in the syntomic topology.

036A   **Lemma 17.1.** *The property $\mathcal{P}(S) =$ "S is locally Noetherian and (S_k)" is local in the syntomic topology.*

Proof. We will check (1), (2) and (3) of Lemma 15.2. As a syntomic morphism is flat of finite presentation (Morphisms, Lemmas 30.7 and 30.6) we have already checked this for "being locally Noetherian" in the proof of Lemma 16.1. We will use this without further mention in the proof. First we note that $\mathcal{P}$ is local in the Zariski topology. This is clear from the definition, see Cohomology of Schemes, Definition 11.1. Next, we show that if $S' \to S$ is a syntomic morphism of affines and S has $\mathcal{P}$, then S' has $\mathcal{P}$. This is Algebra, Lemma 163.4 (use Morphisms, Lemma 30.2 and Algebra, Definition 136.1 and Lemma 135.3). Finally, we show that if $S' \to S$ is a surjective syntomic morphism of affines and S' has $\mathcal{P}$, then S has $\mathcal{P}$. This is Algebra, Lemma 164.5. Thus (1), (2) and (3) of Lemma 15.2 hold and we win. $\square$

036B   **Lemma 17.2.** *The property $\mathcal{P}(S) =$ "S is Cohen-Macaulay" is local in the syntomic topology.*

Proof. This is clear from Lemma 17.1 above since a scheme is Cohen-Macaulay if and only if it is locally Noetherian and (S_k) for all $k \geq 0$, see Properties, Lemma 12.3. $\square$

18. Properties of schemes local in the smooth topology

034D   In this section we find some properties of schemes which are local on the base in the smooth topology.

034E   **Lemma 18.1.** *The property $\mathcal{P}(S) =$ "S is reduced" is local in the smooth topology.*

Proof. We will use Lemma 15.2. First we note that "being reduced" is local in the Zariski topology. This is clear from the definition, see Schemes, Definition 12.1. Next, we show that if $S' \to S$ is a smooth morphism of affines and S is reduced, then S' is reduced. This is Algebra, Lemma 163.7. Finally, we show that if $S' \to S$ is a surjective smooth morphism of affines and S' is reduced, then S is reduced. This is Algebra, Lemma 164.2. Thus (1), (2) and (3) of Lemma 15.2 hold and we win. $\square$

034F   **Lemma 18.2.** *The property $\mathcal{P}(S) =$ "S is normal" is local in the smooth topology.*

Proof. We will use Lemma 15.2. First we show "being normal" is local in the Zariski topology. This is clear from the definition, see Properties, Definition 7.1. Next, we show that if $S' \to S$ is a smooth morphism of affines and S is normal, then S' is normal. This is Algebra, Lemma 163.9. Finally, we show that if $S' \to S$ is a surjective smooth morphism of affines and S' is normal, then S is normal. This is Algebra, Lemma 164.3. Thus (1), (2) and (3) of Lemma 15.2 hold and we win. $\square$

036C **Lemma 18.3.** *The property $\mathcal{P}(S) =$ "S is locally Noetherian and (R_k)" is local in the smooth topology.*

Proof. We will check (1), (2) and (3) of Lemma 15.2. As a smooth morphism is flat of finite presentation (Morphisms, Lemmas 34.9 and 34.8) we have already checked this for "being locally Noetherian" in the proof of Lemma 16.1. We will use this without further mention in the proof. First we note that $\mathcal{P}$ is local in the Zariski topology. This is clear from the definition, see Properties, Definition 12.1. Next, we show that if $S' \to S$ is a smooth morphism of affines and S has $\mathcal{P}$, then S' has $\mathcal{P}$. This is Algebra, Lemmas 163.5 (use Morphisms, Lemma 34.2, Algebra, Lemmas 137.3 and 140.3). Finally, we show that if $S' \to S$ is a surjective smooth morphism of affines and S' has $\mathcal{P}$, then S has $\mathcal{P}$. This is Algebra, Lemma 164.6. Thus (1), (2) and (3) of Lemma 15.2 hold and we win. $\square$

036D **Lemma 18.4.** *The property $\mathcal{P}(S) =$ "S is regular" is local in the smooth topology.*

Proof. This is clear from Lemma 18.3 above since a locally Noetherian scheme is regular if and only if it is locally Noetherian and (R_k) for all $k \geq 0$. $\square$

036E **Lemma 18.5.** *The property $\mathcal{P}(S) =$ "S is Nagata" is local in the smooth topology.*

Proof. We will check (1), (2) and (3) of Lemma 15.2. First we note that being Nagata is local in the Zariski topology. This is Properties, Lemma 13.6. Next, we show that if $S' \to S$ is a smooth morphism of affines and S is Nagata, then S' is Nagata. This is Morphisms, Lemma 18.1. Finally, we show that if $S' \to S$ is a surjective smooth morphism of affines and S' is Nagata, then S is Nagata. This is Algebra, Lemma 164.7. Thus (1), (2) and (3) of Lemma 15.2 hold and we win. $\square$

19. Variants on descending properties

06QL Sometimes one can descend properties, which are not local. We put results of this kind in this section. See also Section 14 on descending properties of morphisms, such as smoothness.

06QM **Lemma 19.1.** *If $f : X \to Y$ is a flat and surjective morphism of schemes and X is reduced, then Y is reduced.*

Proof. The result follows by looking at local rings (Schemes, Definition 12.1) and Algebra, Lemma 164.2. $\square$

06QN **Lemma 19.2.** *Let $f : X \to Y$ be a morphism of algebraic spaces. If f is locally of finite presentation, flat, and surjective and X is regular, then Y is regular.*

Proof. This lemma reduces to the following algebra statement: If $A \to B$ is a faithfully flat, finitely presented ring homomorphism with B Noetherian and regular, then A is Noetherian and regular. We see that A is Noetherian by Algebra, Lemma 164.1 and regular by Algebra, Lemma 110.9. $\square$

20. Germs of schemes

04QQ
04QR **Definition 20.1.** Germs of schemes.

(1) A pair (X, x) consisting of a scheme X and a point $x \in X$ is called the *germ of X at x*.
(2) A *morphism of germs $f : (X, x) \to (S, s)$* is an equivalence class of morphisms of schemes $f : U \to S$ with $f(x) = s$ where $U \subset X$ is an open neighbourhood of x. Two such f, f' are said to be equivalent if and only if f and f' agree in some open neighbourhood of x.
(3) We define the *composition of morphisms of germs* by composing representatives (this is well defined).

Before we continue we need one more definition.

04QS **Definition 20.2.** Let $f : (X, x) \to (S, s)$ be a morphism of germs. We say f is *étale* (resp. *smooth*) if there exists a representative $f : U \to S$ of f which is an étale morphism (resp. a smooth morphism) of schemes.

21. Local properties of germs

04QT
04N1 **Definition 21.1.** Let $\mathcal{P}$ be a property of germs of schemes. We say that $\mathcal{P}$ is *étale local* (resp. *smooth local*) if for any étale (resp. smooth) morphism of germs $(U', u') \to (U, u)$ we have $\mathcal{P}(U, u) \Leftrightarrow \mathcal{P}(U', u')$.

Let (X, x) be a germ of a scheme. The dimension of X at x is the minimum of the dimensions of open neighbourhoods of x in X, and any small enough open neighbourhood has this dimension. Hence this is an invariant of the isomorphism class of the germ. We denote this simply $\dim_x(X)$. The following lemma tells us that the assertion $\dim_x(X) = d$ is an étale local property of germs.

04N4 **Lemma 21.2.** *Let $f : U \to V$ be an étale morphism of schemes. Let $u \in U$ and $v = f(u)$. Then $\dim_u(U) = \dim_v(V)$.*

Proof. In the statement $\dim_u(U)$ is the dimension of U at u as defined in Topology, Definition 10.1 as the minimum of the Krull dimensions of open neighbourhoods of u in U. Similarly for $\dim_v(V)$.

Let us show that $\dim_v(V) \geq \dim_u(U)$. Let V' be an open neighbourhood of v in V. Then there exists an open neighbourhood U' of u in U contained in $f^{-1}(V')$ such that $\dim_u(U) = \dim(U')$. Suppose that $Z_0 \subset Z_1 \subset \ldots \subset Z_n$ is a chain of irreducible closed subschemes of U'. If $\xi_i \in Z_i$ is the generic point then we have specializations $\xi_n \rightsquigarrow \xi_{n-1} \rightsquigarrow \ldots \rightsquigarrow \xi_0$. This gives specializations $f(\xi_n) \rightsquigarrow f(\xi_{n-1}) \rightsquigarrow \ldots \rightsquigarrow f(\xi_0)$ in V'. Note that $f(\xi_j) \neq f(\xi_i)$ if $i \neq j$ as the fibres of f are discrete (see Morphisms, Lemma 36.7). Hence we see that $\dim(V') \geq n$. The inequality $\dim_v(V) \geq \dim_u(U)$ follows formally.

Let us show that $\dim_u(U) \geq \dim_v(V)$. Let U' be an open neighbourhood of u in U. Note that $V' = f(U')$ is an open neighbourhood of v by Morphisms, Lemma 25.10. Hence $\dim(V') \geq \dim_v(V)$. Pick a chain $Z_0 \subset Z_1 \subset \ldots \subset Z_n$ of irreducible closed subschemes of V'. Let $\xi_i \in Z_i$ be the generic point, so we have specializations

$\xi_n \rightsquigarrow \xi_{n-1} \rightsquigarrow \ldots \rightsquigarrow \xi_0$. Since $\xi_0 \in f(U')$ we can find a point $\eta_0 \in U'$ with $f(\eta_0) = \xi_0$. Consider the map of local rings

$$\mathcal{O}_{V',\xi_0} \longrightarrow \mathcal{O}_{U',\eta_0}$$

which is a flat local ring map by Morphisms, Lemma 36.12. Note that the points ξ_i correspond to primes of the ring on the left by Schemes, Lemma 13.2. Hence by going down (see Algebra, Section 41) for the displayed ring map we can find a sequence of specializations $\eta_n \rightsquigarrow \eta_{n-1} \rightsquigarrow \ldots \rightsquigarrow \eta_0$ in U' mapping to the sequence $\xi_n \rightsquigarrow \xi_{n-1} \rightsquigarrow \ldots \rightsquigarrow \xi_0$ under f. This implies that $\dim_u(U) \geq \dim_v(V)$. $\qquad \square$

Let (X, x) be a germ of a scheme. The isomorphism class of the local ring $\mathcal{O}_{X,x}$ is an invariant of the germ. The following lemma says that the property $\dim(\mathcal{O}_{X,x}) = d$ is an étale local property of germs.

04N8   **Lemma 21.3.** *Let $f : U \to V$ be an étale morphism of schemes. Let $u \in U$ and $v = f(u)$. Then $\dim(\mathcal{O}_{U,u}) = \dim(\mathcal{O}_{V,v})$.*

Proof. The algebraic statement we are asked to prove is the following: If $A \to B$ is an étale ring map and $\mathfrak{q}$ is a prime of B lying over $\mathfrak{p} \subset A$, then $\dim(A_{\mathfrak{p}}) = \dim(B_{\mathfrak{q}})$. This is More on Algebra, Lemma 44.2. $\qquad \square$

Let (X, x) be a germ of a scheme. The isomorphism class of the local ring $\mathcal{O}_{X,x}$ is an invariant of the germ. The following lemma says that the property "$\mathcal{O}_{X,x}$ is regular" is an étale local property of germs.

0AH7   **Lemma 21.4.** *Let $f : U \to V$ be an étale morphism of schemes. Let $u \in U$ and $v = f(u)$. Then $\mathcal{O}_{U,u}$ is a regular local ring if and only if $\mathcal{O}_{V,v}$ is a regular local ring.*

Proof. The algebraic statement we are asked to prove is the following: If $A \to B$ is an étale ring map and $\mathfrak{q}$ is a prime of B lying over $\mathfrak{p} \subset A$, then $A_{\mathfrak{p}}$ is regular if and only if $B_{\mathfrak{q}}$ is regular. This is More on Algebra, Lemma 44.3. $\qquad \square$

22. Properties of morphisms local on the target

02KN   Suppose that $f : X \to Y$ is a morphism of schemes. Let $g : Y' \to Y$ be a morphism of schemes. Let $f' : X' \to Y'$ be the base change of f by g:

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

Let $\mathcal{P}$ be a property of morphisms of schemes. Then we can wonder if (a) $\mathcal{P}(f) \Rightarrow \mathcal{P}(f')$, and also whether the converse (b) $\mathcal{P}(f') \Rightarrow \mathcal{P}(f)$ is true. If (a) holds whenever g is flat, then we say $\mathcal{P}$ is preserved under flat base change. If (b) holds whenever g is surjective and flat, then we say $\mathcal{P}$ descends through flat surjective base changes. If $\mathcal{P}$ is preserved under flat base changes and descends through flat surjective base changes, then we say $\mathcal{P}$ is flat local on the target. Compare with the discussion in Section 15. This turns out to be a very important notion which we formalize in the following definition.

02KO   **Definition 22.1.** Let $\mathcal{P}$ be a property of morphisms of schemes over a base. Let $\tau \in \{fpqc, fppf, syntomic, smooth, \acute{e}tale, Zariski\}$. We say $\mathcal{P}$ is τ *local on the base*, or τ *local on the target*, or *local on the base for the τ-topology* if for any τ-covering $\{Y_i \to Y\}_{i \in I}$ (see Topologies, Section 2) and any morphism of schemes $f : X \to Y$ over S we have

$$f \text{ has } \mathcal{P} \Leftrightarrow \text{each } Y_i \times_Y X \to Y_i \text{ has } \mathcal{P}.$$

To be sure, since isomorphisms are always coverings we see (or require) that property $\mathcal{P}$ holds for $X \to Y$ if and only if it holds for any arrow $X' \to Y'$ isomorphic to $X \to Y$. If a property is τ-local on the target then it is preserved by base changes by morphisms which occur in τ-coverings. Here is a formal statement.

04QU   **Lemma 22.2.** *Let $\tau \in \{fpqc, fppf, syntomic, smooth, \acute{e}tale, Zariski\}$. Let $\mathcal{P}$ be a property of morphisms which is τ local on the target. Let $f : X \to Y$ have property $\mathcal{P}$. For any morphism $Y' \to Y$ which is flat, resp. flat and locally of finite presentation, resp. syntomic, resp. étale, resp. an open immersion, the base change $f' : Y' \times_Y X \to Y'$ of f has property $\mathcal{P}$.*

   Proof. This is true because we can fit $Y' \to Y$ into a family of morphisms which forms a τ-covering. $\qquad\square$

   A simple often used consequence of the above is that if $f : X \to Y$ has property $\mathcal{P}$ which is τ-local on the target and $f(X) \subset V$ for some open subscheme $V \subset Y$, then also the induced morphism $X \to V$ has $\mathcal{P}$. Proof: The base change f by $V \to Y$ gives $X \to V$.

06QP   **Lemma 22.3.** *Let $\tau \in \{fppf, syntomic, smooth, \acute{e}tale\}$. Let $\mathcal{P}$ be a property of morphisms which is τ local on the target. For any morphism of schemes $f : X \to Y$ there exists a largest open $W(f) \subset Y$ such that the restriction $X_{W(f)} \to W(f)$ has $\mathcal{P}$. Moreover,*

   (1) *if $g : Y' \to Y$ is flat and locally of finite presentation, syntomic, smooth, or étale and the base change $f' : X_{Y'} \to Y'$ has $\mathcal{P}$, then $g(Y') \subset W(f)$,*
   (2) *if $g : Y' \to Y$ is flat and locally of finite presentation, syntomic, smooth, or étale, then $W(f') = g^{-1}(W(f))$, and*
   (3) *if $\{g_i : Y_i \to Y\}$ is a τ-covering, then $g_i^{-1}(W(f)) = W(f_i)$, where f_i is the base change of f by $Y_i \to Y$.*

   Proof. Consider the union W of the images $g(Y') \subset Y$ of morphisms $g : Y' \to Y$ with the properties:

   (1) g is flat and locally of finite presentation, syntomic, smooth, or étale, and
   (2) the base change $Y' \times_{g,Y} X \to Y'$ has property $\mathcal{P}$.

   Since such a morphism g is open (see Morphisms, Lemma 25.10) we see that $W \subset Y$ is an open subset of Y. Since $\mathcal{P}$ is local in the τ topology the restriction $X_W \to W$ has property $\mathcal{P}$ because we are given a covering $\{Y' \to W\}$ of W such that the pullbacks have $\mathcal{P}$. This proves the existence and proves that $W(f)$ has property (1). To see property (2) note that $W(f') \supset g^{-1}(W(f))$ because $\mathcal{P}$ is stable under base change by flat and locally of finite presentation, syntomic, smooth, or étale morphisms, see Lemma 22.2. On the other hand, if $Y'' \subset Y'$ is an open such that $X_{Y''} \to Y''$ has property $\mathcal{P}$, then $Y'' \to Y$ factors through W by construction, i.e., $Y'' \subset g^{-1}(W(f))$. This proves (2). Assertion (3) follows from (2) because each morphism $Y_i \to Y$ is flat and locally of finite presentation, syntomic, smooth, or étale by our definition of a τ-covering. $\qquad\square$

02KP **Lemma 22.4.** *Let $\mathcal{P}$ be a property of morphisms of schemes over a base. Let $\tau \in \{fpqc, fppf, \acute{e}tale, smooth, syntomic\}$. Assume that*

(1) *the property is preserved under flat, flat and locally of finite presentation, étale, smooth, or syntomic base change depending on whether τ is fpqc, fppf, étale, smooth, or syntomic (compare with Schemes, Definition 18.3),*

(2) *the property is Zariski local on the base.*

(3) *for any surjective morphism of affine schemes $S' \to S$ which is flat, flat of finite presentation, étale, smooth or syntomic depending on whether τ is fpqc, fppf, étale, smooth, or syntomic, and any morphism of schemes $f : X \to S$ property $\mathcal{P}$ holds for f if property $\mathcal{P}$ holds for the base change $f' : X' = S' \times_S X \to S'$.*

Then $\mathcal{P}$ is τ local on the base.

Proof. This follows almost immediately from the definition of a τ-covering, see Topologies, Definition 9.1 7.1 4.1 5.1, or 6.1 and Topologies, Lemma 9.9, 7.4, 4.4, 5.4, or 6.4. Details omitted. $\square$

034G **Remark 22.5.** (This is a repeat of Remark 15.3 above.) In Lemma 22.4 above if $\tau = smooth$ then in condition (3) we may assume that the morphism is a (surjective) standard smooth morphism. Similarly, when $\tau = syntomic$ or $\tau = \acute{e}tale$.

23. Properties of morphisms local in the fpqc topology on the target

02YJ In this section we find a large number of properties of morphisms of schemes which are local on the base in the fpqc topology. By contrast, in Examples, Section 65 we will show that the properties "projective" and "quasi-projective" are not local on the base even in the Zariski topology.

02KQ **Lemma 23.1.** *The property $\mathcal{P}(f) = $ "f is quasi-compact" is fpqc local on the base.*

Proof. A base change of a quasi-compact morphism is quasi-compact, see Schemes, Lemma 19.3. Being quasi-compact is Zariski local on the base, see Schemes, Lemma 19.2. Finally, let $S' \to S$ be a flat surjective morphism of affine schemes, and let $f : X \to S$ be a morphism. Assume that the base change $f' : X' \to S'$ is quasi-compact. Then X' is quasi-compact, and $X' \to X$ is surjective. Hence X is quasi-compact. This implies that f is quasi-compact. Therefore Lemma 22.4 applies and we win. $\square$

02KR **Lemma 23.2.** *The property $\mathcal{P}(f) = $ "f is quasi-separated" is fpqc local on the base.*

Proof. Any base change of a quasi-separated morphism is quasi-separated, see Schemes, Lemma 21.12. Being quasi-separated is Zariski local on the base (from the definition or by Schemes, Lemma 21.6). Finally, let $S' \to S$ be a flat surjective morphism of affine schemes, and let $f : X \to S$ be a morphism. Assume that the base change $f' : X' \to S'$ is quasi-separated. This means that $\Delta' : X' \to X' \times_{S'} X'$ is quasi-compact. Note that Δ' is the base change of $\Delta : X \to X \times_S X$ via $S' \to S$. By Lemma 23.1 this implies Δ is quasi-compact, and hence f is quasi-separated. Therefore Lemma 22.4 applies and we win. $\square$

02KS **Lemma 23.3.** *The property $\mathcal{P}(f) = $ "f is universally closed" is fpqc local on the base.*

Proof. A base change of a universally closed morphism is universally closed by definition. Being universally closed is Zariski local on the base (from the definition or by Morphisms, Lemma 41.2). Finally, let $S' \to S$ be a flat surjective morphism of affine schemes, and let $f : X \to S$ be a morphism. Assume that the base change $f' : X' \to S'$ is universally closed. Let $T \to S$ be any morphism. Consider the diagram

$$
\begin{array}{ccccc}
X' & \longleftarrow & S' \times_S T \times_S X & \longrightarrow & T \times_S X \\
\downarrow & & \downarrow & & \downarrow \\
S' & \longleftarrow & S' \times_S T & \longrightarrow & T
\end{array}
$$

in which both squares are cartesian. Thus the assumption implies that the middle vertical arrow is closed. The right horizontal arrows are flat, quasi-compact and surjective (as base changes of $S' \to S$). Hence a subset of T is closed if and only if its inverse image in $S' \times_S T$ is closed, see Morphisms, Lemma 25.12. An easy diagram chase shows that the right vertical arrow is closed too, and we conclude $X \to S$ is universally closed. Therefore Lemma 22.4 applies and we win. $\square$

02KT   **Lemma 23.4.** *The property $\mathcal{P}(f) =$ "f is universally open" is fpqc local on the base.*

Proof. The proof is the same as the proof of Lemma 23.3. $\square$

0CEW   **Lemma 23.5.** *The property $\mathcal{P}(f) =$ "f is universally submersive" is fpqc local on the base.*

Proof. The proof is the same as the proof of Lemma 23.3 using that a quasi-compact flat surjective morphism is universally submersive by Morphisms, Lemma 25.12. $\square$

02KU   **Lemma 23.6.** *The property $\mathcal{P}(f) =$ "f is separated" is fpqc local on the base.*

Proof. A base change of a separated morphism is separated, see Schemes, Lemma 21.12. Being separated is Zariski local on the base (from the definition or by Schemes, Lemma 21.7). Finally, let $S' \to S$ be a flat surjective morphism of affine schemes, and let $f : X \to S$ be a morphism. Assume that the base change $f' : X' \to S'$ is separated. This means that $\Delta' : X' \to X' \times_{S'} X'$ is a closed immersion, hence universally closed. Note that Δ' is the base change of $\Delta : X \to X \times_S X$ via $S' \to S$. By Lemma 23.3 this implies Δ is universally closed. Since it is an immersion (Schemes, Lemma 21.2) we conclude Δ is a closed immersion. Hence f is separated. Therefore Lemma 22.4 applies and we win. $\square$

02KV   **Lemma 23.7.** *The property $\mathcal{P}(f) =$ "f is surjective" is fpqc local on the base.*

Proof. This is clear. $\square$

0H8H   **Lemma 23.8.** *The property $\mathcal{P}(f) =$ "f is quasi-compact and dominant" is fpqc local on the base.*

Proof. By Morphisms, Lemma 8.4, quasi-compact dominant morphisms are preserved by flat pullback. The other direction is easier. Indeed, quasi-compactness is fpqc local on the base by Lemma 23.1. Being dominant is clearly Zariski local on the base. Finally, let $S' \to S$ be a surjective morphism of schemes (not necessarily flat), and let $f : X \to S$ be a morphism. Assume that the base change $f' : X' \to S'$

is dominant. Then $X' \to S' \to S$ is a composite of two dominant morphisms, hence dominant. Since this is also the composite $X' \to X \to S$, it follows that $X \to S$ is dominant. $\square$
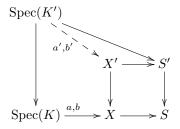
Dominant morphisms are not preserved by flat pullback in full generality, but the following special case is worth mentioning. See also Lemma 24.2.

0H8I **Lemma 23.9.** *Let E/k be a field extension. Then a morphism $X \to Y$ over k is dominant if and only if the pullback $X_E \to Y_E$ is dominant.*

Proof. By Morphisms, Lemma 23.4, the morphism $\mathrm{Spec}(E) \to \mathrm{Spec}(k)$ is universally open. So $Y_E \to Y$ is open. Therefore, if $X \to Y$ is dominant, Morphisms, Lemma 8.5 gives that $X_E \to Y_E$ is dominant. Conversely, suppose that $X_E \to Y_E$ is dominant. The morphism $Y_E \to Y$ is surjective by Morphisms, Lemma 9.4, and so the composition $X_E \to Y_E \to Y$ is dominant. This is also the composition $X_E \to X \to Y$, and so $X \to Y$ is dominant. $\square$

02KW **Lemma 23.10.** *The property $\mathcal{P}(f) =$"f is universally injective" is fpqc local on the base.*

Proof. A base change of a universally injective morphism is universally injective (this is formal). Being universally injective is Zariski local on the base; this is clear from the definition. Finally, let $S' \to S$ be a flat surjective morphism of affine schemes, and let $f : X \to S$ be a morphism. Assume that the base change $f' : X' \to S'$ is universally injective. Let K be a field, and let $a, b : \mathrm{Spec}(K) \to X$ be two morphisms such that $f \circ a = f \circ b$. As $S' \to S$ is surjective and by the discussion in Schemes, Section 13 there exists a field extension K'/K and a morphism $\mathrm{Spec}(K') \to S'$ such that the following solid diagram commutes

$$
\begin{array}{ccc}
\mathrm{Spec}(K') & & \\
\downarrow \quad \searrow^{a',b'} & & \\
& X' \longrightarrow S' & \\
\downarrow & \downarrow \quad \downarrow & \\
\mathrm{Spec}(K) \xrightarrow{a,b} X \longrightarrow S &
\end{array}
$$

As the square is cartesian we get the two dotted arrows a', b' making the diagram commute. Since $X' \to S'$ is universally injective we get $a' = b'$, by Morphisms, Lemma 10.2. Clearly this forces $a = b$ (by the discussion in Schemes, Section 13). Therefore Lemma 22.4 applies and we win.

An alternative proof would be to use the characterization of a universally injective morphism as one whose diagonal is surjective, see Morphisms, Lemma 10.2. The lemma then follows from the fact that the property of being surjective is fpqc local on the base, see Lemma 23.7. (Hint: use that the base change of the diagonal is the diagonal of the base change.) $\square$

0CEX **Lemma 23.11.** *The property $\mathcal{P}(f) =$"f is a universal homeomorphism" is fpqc local on the base.*

Proof. This can be proved in exactly the same manner as Lemma 23.3. Alternatively, one can use that a map of topological spaces is a homeomorphism if and

only if it is injective, surjective, and open. Thus a universal homeomorphism is the same thing as a surjective, universally injective, and universally open morphism. Thus the lemma follows from Lemmas 23.7, 23.10, and 23.4. $\square$

02KX   **Lemma 23.12.** *The property $\mathcal{P}(f) =$ "f is locally of finite type" is fpqc local on the base.*

Proof. Being locally of finite type is preserved under base change, see Morphisms, Lemma 15.4. Being locally of finite type is Zariski local on the base, see Morphisms, Lemma 15.2. Finally, let $S' \to S$ be a flat surjective morphism of affine schemes, and let $f : X \to S$ be a morphism. Assume that the base change $f' : X' \to S'$ is locally of finite type. Let $U \subset X$ be an affine open. Then $U' = S' \times_S U$ is affine and of finite type over S'. Write $S = \mathrm{Spec}(R)$, $S' = \mathrm{Spec}(R')$, $U = \mathrm{Spec}(A)$, and $U' = \mathrm{Spec}(A')$. We know that $R \to R'$ is faithfully flat, $A' = R' \otimes_R A$ and $R' \to A'$ is of finite type. We have to show that $R \to A$ is of finite type. This is the result of Algebra, Lemma 126.1. It follows that f is locally of finite type. Therefore Lemma 22.4 applies and we win. $\square$

02KY   **Lemma 23.13.** *The property $\mathcal{P}(f) =$ "f is locally of finite presentation" is fpqc local on the base.*

Proof. Being locally of finite presentation is preserved under base change, see Morphisms, Lemma 21.4. Being locally of finite type is Zariski local on the base, see Morphisms, Lemma 21.2. Finally, let $S' \to S$ be a flat surjective morphism of affine schemes, and let $f : X \to S$ be a morphism. Assume that the base change $f' : X' \to S'$ is locally of finite presentation. Let $U \subset X$ be an affine open. Then $U' = S' \times_S U$ is affine and of finite type over S'. Write $S = \mathrm{Spec}(R)$, $S' = \mathrm{Spec}(R')$, $U = \mathrm{Spec}(A)$, and $U' = \mathrm{Spec}(A')$. We know that $R \to R'$ is faithfully flat, $A' = R' \otimes_R A$ and $R' \to A'$ is of finite presentation. We have to show that $R \to A$ is of finite presentation. This is the result of Algebra, Lemma 126.2. It follows that f is locally of finite presentation. Therefore Lemma 22.4 applies and we win. $\square$

02KZ   **Lemma 23.14.** *The property $\mathcal{P}(f) =$ "f is of finite type" is fpqc local on the base.*

Proof. Combine Lemmas 23.1 and 23.12. $\square$

02L0   **Lemma 23.15.** *The property $\mathcal{P}(f) =$ "f is of finite presentation" is fpqc local on the base.*

Proof. Combine Lemmas 23.1, 23.2 and 23.13. $\square$

02L1   **Lemma 23.16.** *The property $\mathcal{P}(f) =$ "f is proper" is fpqc local on the base.*

Proof. The lemma follows by combining Lemmas 23.3, 23.6 and 23.14. $\square$

02L2   **Lemma 23.17.** *The property $\mathcal{P}(f) =$ "f is flat" is fpqc local on the base.*

Proof. Being flat is preserved under arbitrary base change, see Morphisms, Lemma 25.8. Being flat is Zariski local on the base by definition. Finally, let $S' \to S$ be a flat surjective morphism of affine schemes, and let $f : X \to S$ be a morphism. Assume that the base change $f' : X' \to S'$ is flat. Let $U \subset X$ be an affine open. Then $U' = S' \times_S U$ is affine. Write $S = \mathrm{Spec}(R)$, $S' = \mathrm{Spec}(R')$, $U = \mathrm{Spec}(A)$, and $U' = \mathrm{Spec}(A')$. We know that $R \to R'$ is faithfully flat, $A' = R' \otimes_R A$ and $R' \to A'$

is flat. Goal: Show that $R \to A$ is flat. This follows immediately from Algebra, Lemma 39.8. Hence f is flat. Therefore Lemma 22.4 applies and we win. $\qquad\square$

02L3   **Lemma 23.18.** *The property $\mathcal{P}(f) =$ "f is an open immersion" is fpqc local on the base.*

Proof. The property of being an open immersion is stable under base change, see Schemes, Lemma 18.2. The property of being an open immersion is Zariski local on the base (this is obvious).

Let $S' \to S$ be a flat surjective morphism of affine schemes, and let $f : X \to S$ be a morphism. Assume that the base change $f' : X' \to S'$ is an open immersion. We claim that f is an open immersion. Then f' is universally open, and universally injective. Hence we conclude that f is universally open by Lemma 23.4, and universally injective by Lemma 23.10. In particular $f(X) \subset S$ is open. If for every affine open $U \subset f(X)$ we can prove that $f^{-1}(U) \to U$ is an isomorphism, then f is an open immersion and we're done. If $U' \subset S'$ denotes the inverse image of U, then $U' \to U$ is a faithfully flat morphism of affines and $(f')^{-1}(U') \to U'$ is an isomorphism (as $f'(X')$ contains U' by our choice of U). Thus we reduce to the case discussed in the next paragraph.

Let $S' \to S$ be a flat surjective morphism of affine schemes, let $f : X \to S$ be a morphism, and assume that the base change $f' : X' \to S'$ is an isomorphism. We have to show that f is an isomorphism also. It is clear that f is surjective, universally injective, and universally open (see arguments above for the last two). Hence f is bijective, i.e., f is a homeomorphism. Thus f is affine by Morphisms, Lemma 45.4. Since

$$\mathcal{O}(S') \to \mathcal{O}(X') = \mathcal{O}(S') \otimes_{\mathcal{O}(S)} \mathcal{O}(X)$$

is an isomorphism and since $\mathcal{O}(S) \to \mathcal{O}(S')$ is faithfully flat this implies that $\mathcal{O}(S) \to \mathcal{O}(X)$ is an isomorphism. Thus f is an isomorphism. This finishes the proof of the claim above. Therefore Lemma 22.4 applies and we win. $\qquad\square$

02L4   **Lemma 23.19.** *The property $\mathcal{P}(f) =$ "f is an isomorphism" is fpqc local on the base.*

Proof. Combine Lemmas 23.7 and 23.18. $\qquad\square$

02L5   **Lemma 23.20.** *The property $\mathcal{P}(f) =$ "f is affine" is fpqc local on the base.*

Proof. A base change of an affine morphism is affine, see Morphisms, Lemma 11.9. Being affine is Zariski local on the base, see Morphisms, Lemma 11.3. Finally, let $g : S' \to S$ be a flat surjective morphism of affine schemes, and let $f : X \to S$ be a morphism. Assume that the base change $f' : X' \to S'$ is affine. In other words, X' is affine, say $X' = \operatorname{Spec}(A')$. Also write $S = \operatorname{Spec}(R)$ and $S' = \operatorname{Spec}(R')$. We have to show that X is affine.

By Lemmas 23.1 and 23.6 we see that $X \to S$ is separated and quasi-compact. Thus $f_* \mathcal{O}_X$ is a quasi-coherent sheaf of $\mathcal{O}_S$-algebras, see Schemes, Lemma 24.1. Hence $f_* \mathcal{O}_X = \widetilde{A}$ for some R-algebra A. In fact $A = \Gamma(X, \mathcal{O}_X)$ of course. Also, by flat base change (see for example Cohomology of Schemes, Lemma 5.2) we have $g^* f_* \mathcal{O}_X = f'_* \mathcal{O}_{X'}$. In other words, we have $A' = R' \otimes_R A$. Consider the canonical morphism

$$X \longrightarrow \operatorname{Spec}(A)$$

over S from Schemes, Lemma 6.4. By the above the base change of this morphism to S' is an isomorphism. Hence it is an isomorphism by Lemma 23.19. Therefore Lemma 22.4 applies and we win. $\square$

02L6 **Lemma 23.21.** *The property $\mathcal{P}(f) =$ "f is a closed immersion" is fpqc local on the base.*

Proof. Let $f : X \to Y$ be a morphism of schemes. Let $\{Y_i \to Y\}$ be an fpqc covering. Assume that each $f_i : Y_i \times_Y X \to Y_i$ is a closed immersion. This implies that each f_i is affine, see Morphisms, Lemma 11.10. By Lemma 23.20 we conclude that f is affine. It remains to show that $\mathcal{O}_Y \to f_* \mathcal{O}_X$ is surjective. For every $y \in Y$ there exists an i and a point $y_i \in Y_i$ mapping to y. By Cohomology of Schemes, Lemma 5.2 the sheaf $f_{i,*}(\mathcal{O}_{Y_i \times_Y X})$ is the pullback of $f_* \mathcal{O}_X$. By assumption it is a quotient of $\mathcal{O}_{Y_i}$. Hence we see that

$$\left(\mathcal{O}_{Y,y} \longrightarrow (f_* \mathcal{O}_X)_y \right) \otimes_{\mathcal{O}_{Y,y}} \mathcal{O}_{Y_i, y_i}$$

is surjective. Since $\mathcal{O}_{Y_i, y_i}$ is faithfully flat over $\mathcal{O}_{Y,y}$ this implies the surjectivity of $\mathcal{O}_{Y,y} \longrightarrow (f_* \mathcal{O}_X)_y$ as desired. $\square$

02L7 **Lemma 23.22.** *The property $\mathcal{P}(f) =$ "f is quasi-affine" is fpqc local on the base.*
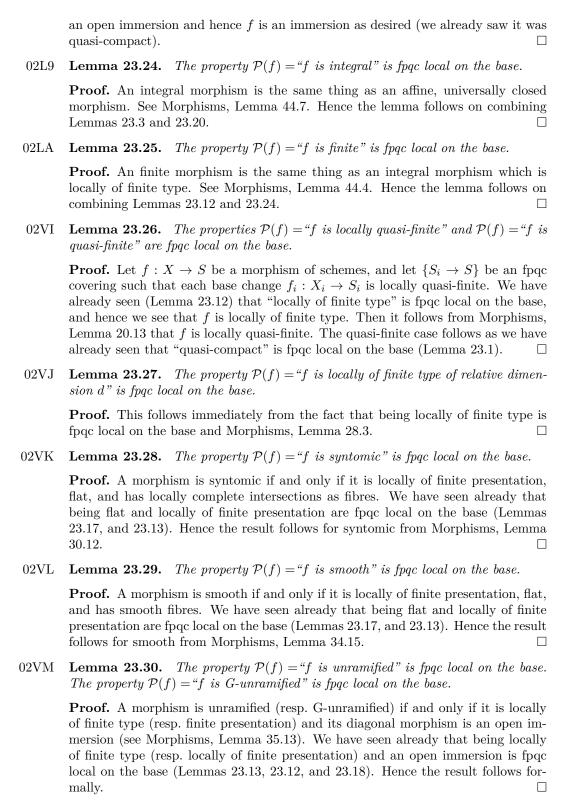
Proof. Let $f : X \to Y$ be a morphism of schemes. Let $\{g_i : Y_i \to Y\}$ be an fpqc covering. Assume that each $f_i : Y_i \times_Y X \to Y_i$ is quasi-affine. This implies that each f_i is quasi-compact and separated. By Lemmas 23.1 and 23.6 this implies that f is quasi-compact and separated. Consider the sheaf of $\mathcal{O}_Y$-algebras $\mathcal{A} = f_* \mathcal{O}_X$. By Schemes, Lemma 24.1 it is a quasi-coherent $\mathcal{O}_Y$-algebra. Consider the canonical morphism

$$j : X \longrightarrow \underline{\mathrm{Spec}}_Y(\mathcal{A})$$

see Constructions, Lemma 4.7. By flat base change (see for example Cohomology of Schemes, Lemma 5.2) we have $g_i^* f_* \mathcal{O}_X = f_{i,*} \mathcal{O}_{X'}$ where $g_i : Y_i \to Y$ are the given flat maps. Hence the base change j_i of j by g_i is the canonical morphism of Constructions, Lemma 4.7 for the morphism f_i. By assumption and Morphisms, Lemma 13.3 all of these morphisms j_i are quasi-compact open immersions. Hence, by Lemmas 23.1 and 23.18 we see that j is a quasi-compact open immersion. Hence by Morphisms, Lemma 13.3 again we conclude that f is quasi-affine. $\square$

02L8 **Lemma 23.23.** *The property $\mathcal{P}(f) =$ "f is a quasi-compact immersion" is fpqc local on the base.*

Proof. Let $f : X \to Y$ be a morphism of schemes. Let $\{Y_i \to Y\}$ be an fpqc covering. Write $X_i = Y_i \times_Y X$ and $f_i : X_i \to Y_i$ the base change of f. Also denote $q_i : Y_i \to Y$ the given flat morphisms. Assume each f_i is a quasi-compact immersion. By Schemes, Lemma 23.8 each f_i is separated. By Lemmas 23.1 and 23.6 this implies that f is quasi-compact and separated. Let $X \to Z \to Y$ be the factorization of f through its scheme theoretic image. By Morphisms, Lemma 6.3 the closed subscheme $Z \subset Y$ is cut out by the quasi-coherent sheaf of ideals $\mathcal{I} = \mathrm{Ker}(\mathcal{O}_Y \to f_* \mathcal{O}_X)$ as f is quasi-compact. By flat base change (see for example Cohomology of Schemes, Lemma 5.2; here we use f is separated) we see $f_{i,*} \mathcal{O}_{X_i}$ is the pullback $q_i^* f_* \mathcal{O}_X$. Hence $Y_i \times_Y Z$ is cut out by the quasi-coherent sheaf of ideals $q_i^* \mathcal{I} = \mathrm{Ker}(\mathcal{O}_{Y_i} \to f_{i,*} \mathcal{O}_{X_i})$. By Morphisms, Lemma 7.7 the morphisms $X_i \to Y_i \times_Y Z$ are open immersions. Hence by Lemma 23.18 we see that $X \to Z$ is

an open immersion and hence f is an immersion as desired (we already saw it was quasi-compact). $\square$

02L9 **Lemma 23.24.** *The property $\mathcal{P}(f) =$ "f is integral" is fpqc local on the base.*

Proof. An integral morphism is the same thing as an affine, universally closed morphism. See Morphisms, Lemma 44.7. Hence the lemma follows on combining Lemmas 23.3 and 23.20. $\square$

02LA **Lemma 23.25.** *The property $\mathcal{P}(f) =$ "f is finite" is fpqc local on the base.*

Proof. An finite morphism is the same thing as an integral morphism which is locally of finite type. See Morphisms, Lemma 44.4. Hence the lemma follows on combining Lemmas 23.12 and 23.24. $\square$

02VI **Lemma 23.26.** *The properties $\mathcal{P}(f) =$ "f is locally quasi-finite" and $\mathcal{P}(f) =$ "f is quasi-finite" are fpqc local on the base.*

Proof. Let $f : X \to S$ be a morphism of schemes, and let $\{S_i \to S\}$ be an fpqc covering such that each base change $f_i : X_i \to S_i$ is locally quasi-finite. We have already seen (Lemma 23.12) that "locally of finite type" is fpqc local on the base, and hence we see that f is locally of finite type. Then it follows from Morphisms, Lemma 20.13 that f is locally quasi-finite. The quasi-finite case follows as we have already seen that "quasi-compact" is fpqc local on the base (Lemma 23.1). $\square$

02VJ **Lemma 23.27.** *The property $\mathcal{P}(f) =$ "f is locally of finite type of relative dimension d" is fpqc local on the base.*

Proof. This follows immediately from the fact that being locally of finite type is fpqc local on the base and Morphisms, Lemma 28.3. $\square$

02VK **Lemma 23.28.** *The property $\mathcal{P}(f) =$ "f is syntomic" is fpqc local on the base.*

Proof. A morphism is syntomic if and only if it is locally of finite presentation, flat, and has locally complete intersections as fibres. We have seen already that being flat and locally of finite presentation are fpqc local on the base (Lemmas 23.17, and 23.13). Hence the result follows for syntomic from Morphisms, Lemma 30.12. $\square$

02VL **Lemma 23.29.** *The property $\mathcal{P}(f) =$ "f is smooth" is fpqc local on the base.*

Proof. A morphism is smooth if and only if it is locally of finite presentation, flat, and has smooth fibres. We have seen already that being flat and locally of finite presentation are fpqc local on the base (Lemmas 23.17, and 23.13). Hence the result follows for smooth from Morphisms, Lemma 34.15. $\square$

02VM **Lemma 23.30.** *The property $\mathcal{P}(f) =$ "f is unramified" is fpqc local on the base. The property $\mathcal{P}(f) =$ "f is G-unramified" is fpqc local on the base.*

Proof. A morphism is unramified (resp. G-unramified) if and only if it is locally of finite type (resp. finite presentation) and its diagonal morphism is an open immersion (see Morphisms, Lemma 35.13). We have seen already that being locally of finite type (resp. locally of finite presentation) and an open immersion is fpqc local on the base (Lemmas 23.13, 23.12, and 23.18). Hence the result follows formally. $\square$

02VN **Lemma 23.31.** *The property $\mathcal{P}(f) = $"$f$ is étale" is fpqc local on the base.*

Proof. A morphism is étale if and only if it flat and G-unramified. See Morphisms, Lemma 36.16. We have seen already that being flat and G-unramified are fpqc local on the base (Lemmas 23.17, and 23.30). Hence the result follows. $\square$

02VO **Lemma 23.32.** *The property $\mathcal{P}(f) = $"$f$ is finite locally free" is fpqc local on the base. Let $d \geq 0$. The property $\mathcal{P}(f) = $"$f$ is finite locally free of degree d" is fpqc local on the base.*

Proof. Being finite locally free is equivalent to being finite, flat and locally of finite presentation (Morphisms, Lemma 48.2). Hence this follows from Lemmas 23.25, 23.17, and 23.13. If $f : Z \to U$ is finite locally free, and $\{U_i \to U\}$ is a surjective family of morphisms such that each pullback $Z \times_U U_i \to U_i$ has degree d, then $Z \to U$ has degree d, for example because we can read off the degree in a point $u \in U$ from the fibre $(f_* \mathcal{O}_Z)_u \otimes_{\mathcal{O}_{U,u}} \kappa(u)$. $\square$

02YK **Lemma 23.33.** *The property $\mathcal{P}(f) = $"$f$ is a monomorphism" is fpqc local on the base.*

Proof. Let $f : X \to S$ be a morphism of schemes. Let $\{S_i \to S\}$ be an fpqc covering, and assume each of the base changes $f_i : X_i \to S_i$ of f is a monomorphism. Let $a, b : T \to X$ be two morphisms such that $f \circ a = f \circ b$. We have to show that $a = b$. Since f_i is a monomorphism we see that $a_i = b_i$, where $a_i, b_i : S_i \times_S T \to X_i$ are the base changes. In particular the compositions $S_i \times_S T \to T \to X$ are equal. Since $\coprod S_i \times_S T \to T$ is an epimorphism (see e.g. Lemma 13.7) we conclude $a = b$. $\square$

0694 **Lemma 23.34.** *The properties*

$$\mathcal{P}(f) = \text{"}f \text{ is a Koszul-regular immersion"},$$
$$\mathcal{P}(f) = \text{"}f \text{ is an } H_1\text{-regular immersion", and}$$
$$\mathcal{P}(f) = \text{"}f \text{ is a quasi-regular immersion"}$$

are fpqc local on the base.

Proof. We will use the criterion of Lemma 22.4 to prove this. By Divisors, Definition 21.1 being a Koszul-regular (resp. H_1-regular, quasi-regular) immersion is Zariski local on the base. By Divisors, Lemma 21.4 being a Koszul-regular (resp. H_1-regular, quasi-regular) immersion is preserved under flat base change. The final hypothesis (3) of Lemma 22.4 translates into the following algebra statement: Let $A \to B$ be a faithfully flat ring map. Let $I \subset A$ be an ideal. If IB is locally on $\text{Spec}(B)$ generated by a Koszul-regular (resp. H_1-regular, quasi-regular) sequence in B, then $I \subset A$ is locally on $\text{Spec}(A)$ generated by a Koszul-regular (resp. H_1-regular, quasi-regular) sequence in A. This is More on Algebra, Lemma 32.4. $\square$

24. Properties of morphisms local in the fppf topology on the target

02YL In this section we find some properties of morphisms of schemes for which we could not (yet) show they are local on the base in the fpqc topology which, however, are local on the base in the fppf topology.

02YM **Lemma 24.1.** *The property $\mathcal{P}(f) = $"$f$ is an immersion" is fppf local on the base.*

Proof. The property of being an immersion is stable under base change, see Schemes, Lemma 18.2. The property of being an immersion is Zariski local on the base. Finally, let $\pi : S' \to S$ be a surjective morphism of affine schemes, which is flat and locally of finite presentation. Note that $\pi : S' \to S$ is open by Morphisms, Lemma 25.10. Let $f : X \to S$ be a morphism. Assume that the base change $f' : X' \to S'$ is an immersion. In particular we see that $f'(X') = \pi^{-1}(f(X))$ is locally closed. Hence by Topology, Lemma 6.4 we see that $f(X) \subset S$ is locally closed. Let $Z \subset S$ be the closed subset $Z = \overline{f(X)} \setminus f(X)$. By Topology, Lemma 6.4 again we see that $f'(X')$ is closed in $S' \setminus Z'$. Hence we may apply Lemma 23.21 to the fpqc covering $\{S' \setminus Z' \to S \setminus Z\}$ and conclude that $f : X \to S \setminus Z$ is a closed immersion. In other words, f is an immersion. Therefore Lemma 22.4 applies and we win. $\square$

0H8J **Lemma 24.2.** *The property $\mathcal{P}(f) =$ "f is dominant" is fppf local on the base.*

Proof. By Morphisms, Lemma 8.5, dominant morphisms are preserved under pullback by open morphisms, hence by flat morphisms locally of finite presentation (Morphisms, Lemma 25.10). The other direction is easier. Indeed, being dominant is clearly Zariski local on the base. Next, let $S' \to S$ be a surjective morphism of schemes (not necessarily flat or locally of finite presentation), and let $f : X \to S$ be a morphism. Assume that the base change $f' : X' \to S'$ is dominant. Then $X' \to S' \to S$ is a composition of two dominant morphisms, hence dominant. Since this is also the composition $X' \to X \to S$, it follows that $X \to S$ is dominant. $\square$

25. Application of fpqc descent of properties of morphisms

02LB The following lemma may seem a bit frivolous but turns out is a useful tool in studying étale and unramified morphisms.

06NC **Lemma 25.1.** *Let $f : X \to Y$ be a flat, quasi-compact, surjective monomorphism. Then f is an isomorphism.*

Proof. As f is a flat, quasi-compact, surjective morphism we see $\{X \to Y\}$ is an fpqc covering of Y. The diagonal $\Delta : X \to X \times_Y X$ is an isomorphism (Schemes, Lemma 23.2). This implies that the base change of f by f is an isomorphism. Hence we see f is an isomorphism by Lemma 23.19. $\square$

We can use this lemma to show the following important result; we also give a proof avoiding fpqc descent. We will discuss this and related results in more detail in Étale Morphisms, Section 14.

02LC **Lemma 25.2.** *A universally injective étale morphism is an open immersion.*

First proof. Let $f : X \to Y$ be an étale morphism which is universally injective. Then f is open (Morphisms, Lemma 36.13) hence we can replace Y by $f(X)$ and we may assume that f is surjective. Then f is bijective and open hence a homeomorphism. Hence f is quasi-compact. Thus by Lemma 25.1 it suffices to show that f is a monomorphism. As $X \to Y$ is étale the morphism $\Delta_{X/Y} : X \to X \times_Y X$ is an open immersion by Morphisms, Lemma 35.13 (and Morphisms, Lemma 36.16). As f is universally injective $\Delta_{X/Y}$ is also surjective, see Morphisms, Lemma 10.2. Hence $\Delta_{X/Y}$ is an isomorphism, i.e., $X \to Y$ is a monomorphism. $\square$

Second proof. Let $f : X \to Y$ be an étale morphism which is universally injective. Then f is open (Morphisms, Lemma 36.13) hence we can replace Y by $f(X)$ and we may assume that f is surjective. Since the hypotheses remain satisfied after any base change, we conclude that f is a universal homeomorphism. Therefore f is integral, see Morphisms, Lemma 45.5. It follows that f is finite by Morphisms, Lemma 44.4. It follows that f is finite locally free by Morphisms, Lemma 48.2. To finish the proof, it suffices that f is finite locally free of degree 1 (a finite locally free morphism of degree 1 is an isomorphism). There is decomposition of Y into open and closed subschemes V_d such that $f^{-1}(V_d) \to V_d$ is finite locally free of degree d, see Morphisms, Lemma 48.5. If V_d is not empty, we can pick a morphism $\mathrm{Spec}(k) \to V_d \subset Y$ where k is an algebraically closed field (just take the algebraic closure of the residue field of some point of V_d). Then $\mathrm{Spec}(k) \times_Y X \to \mathrm{Spec}(k)$ is a disjoint union of copies of $\mathrm{Spec}(k)$, by Morphisms, Lemma 36.7 and the fact that k is algebraically closed. However, since f is universally injective, there can only be one copy and hence $d = 1$ as desired. $\square$

We can reformulate the hypotheses in the lemma above a bit by using the following characterization of flat universally injective morphisms.

09NP **Lemma 25.3.** *Let $f : X \to Y$ be a morphism of schemes. Let X^0 denote the set of generic points of irreducible components of X. If*

(1) *f is flat and separated,*
(2) *for $\xi \in X^0$ we have $\kappa(f(\xi)) = \kappa(\xi)$, and*
(3) *if $\xi, \xi' \in X^0$, $\xi \neq \xi'$, then $f(\xi) \neq f(\xi')$,*

then f is universally injective.

Proof. We have to show that $\Delta : X \to X \times_Y X$ is surjective, see Morphisms, Lemma 10.2. As $X \to Y$ is separated, the image of Δ is closed. Thus if Δ is not surjective, we can find a generic point $\eta \in X \times_S X$ of an irreducible component of $X \times_S X$ which is not in the image of Δ. The projection $\mathrm{pr}_1 : X \times_Y X \to X$ is flat as a base change of the flat morphism $X \to Y$, see Morphisms, Lemma 25.8. Hence generalizations lift along pr_1, see Morphisms, Lemma 25.9. We conclude that $\xi = \mathrm{pr}_1(\eta) \in X^0$. However, assumptions (2) and (3) guarantee that the scheme $(X \times_Y X)_{f(\xi)}$ has at most one point for every $\xi \in X^0$. In other words, we have $\Delta(\xi) = \eta$ a contradiction. $\square$

Thus we can reformulate Lemma 25.2 as follows.

09NQ **Lemma 25.4.** *Let $f : X \to Y$ be a morphism of schemes. Let X^0 denote the set of generic points of irreducible components of X. If*

(1) *f is étale and separated,*
(2) *for $\xi \in X^0$ we have $\kappa(f(\xi)) = \kappa(\xi)$, and*
(3) *if $\xi, \xi' \in X^0$, $\xi \neq \xi'$, then $f(\xi) \neq f(\xi')$,*

then f is an open immersion.

Proof. Immediate from Lemmas 25.3 and 25.2. $\square$

0F4J **Lemma 25.5.** *Let $f : X \to Y$ be a morphism of schemes which is locally of finite type. Let Z be a closed subset of X. If there exists an fpqc covering $\{Y_i \to Y\}$ such that the inverse image $Z_i \subset Y_i \times_Y X$ is proper over Y_i (Cohomology of Schemes, Definition 26.2) then Z is proper over Y.*

Proof. Endow Z with the reduced induced closed subscheme structure, see Schemes, Definition 12.5. For every i the base change $Y_i \times_Y Z$ is a closed subscheme of $Y_i \times_Y X$ whose underlying closed subset is Z_i. By definition (via Cohomology of Schemes, Lemma 26.1) we conclude that the projections $Y_i \times_Y Z \to Y_i$ are proper morphisms. Hence $Z \to Y$ is a proper morphism by Lemma 23.16. Thus Z is proper over Y by definition. $\square$

0D2P **Lemma 25.6.** *Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $\{g_i : S_i \to S\}_{i \in I}$ be an fpqc covering. Let $f_i : X_i \to S_i$ be the base change of f and let $\mathcal{L}_i$ be the pullback of $\mathcal{L}$ to X_i. The following are equivalent*

(1) *$\mathcal{L}$ is ample on X/S, and*
(2) *$\mathcal{L}_i$ is ample on X_i/S_i for every $i \in I$.*

Proof. The implication (1) $\Rightarrow$ (2) follows from Morphisms, Lemma 37.9. Assume $\mathcal{L}_i$ is ample on X_i/S_i for every $i \in I$. By Morphisms, Definition 37.1 this implies that $X_i \to S_i$ is quasi-compact and by Morphisms, Lemma 37.3 this implies $X_i \to S$ is separated. Hence f is quasi-compact and separated by Lemmas 23.1 and 23.6.

This means that $\mathcal{A} = \bigoplus_{d \geq 0} f_* \mathcal{L}^{\otimes d}$ is a quasi-coherent graded $\mathcal{O}_S$-algebra (Schemes, Lemma 24.1). Moreover, the formation of $\mathcal{A}$ commutes with flat base change by Cohomology of Schemes, Lemma 5.2. In particular, if we set $\mathcal{A}_i = \bigoplus_{d \geq 0} f_{i,*} \mathcal{L}_i^{\otimes d}$ then we have $\mathcal{A}_i = g_i^* \mathcal{A}$. It follows that the natural maps $\psi_d : f^* \mathcal{A}_d \to \mathcal{L}^{\otimes d}$ of $\mathcal{O}_X$ pullback to give the natural maps $\psi_{i,d} : f_i^* (\mathcal{A}_i)_d \to \mathcal{L}_i^{\otimes d}$ of $\mathcal{O}_{X_i}$-modules. Since $\mathcal{L}_i$ is ample on X_i/S_i we see that for any point $x_i \in X_i$, there exists a $d \geq 1$ such that $f_i^*(\mathcal{A}_i)_d \to \mathcal{L}_i^{\otimes d}$ is surjective on stalks at x_i. This follows either directly from the definition of a relatively ample module or from Morphisms, Lemma 37.4. If $x \in X$, then we can choose an i and an $x_i \in X_i$ mapping to x. Since $\mathcal{O}_{X,x} \to \mathcal{O}_{X_i,x_i}$ is flat hence faithfully flat, we conclude that for every $x \in X$ there exists a $d \geq 1$ such that $f^* \mathcal{A}_d \to \mathcal{L}^{\otimes d}$ is surjective on stalks at x. This implies that the open subset $U(\psi) \subset X$ of Constructions, Lemma 19.1 corresponding to the map $\psi : f^* \mathcal{A} \to \bigoplus_{d \geq 0} \mathcal{L}^{\otimes d}$ of graded $\mathcal{O}_X$-algebras is equal to X. Consider the corresponding morphism

$$r_{\mathcal{L},\psi} : X \longrightarrow \underline{\mathrm{Proj}}_S(\mathcal{A})$$

It is clear from the above that the base change of $r_{\mathcal{L},\psi}$ to S_i is the morphism $r_{\mathcal{L}_i,\psi_i}$ which is an open immersion by Morphisms, Lemma 37.4. Hence $r_{\mathcal{L},\psi}$ is an open immersion by Lemma 23.18 and we conclude $\mathcal{L}$ is ample on X/S by Morphisms, Lemma 37.4. $\square$

26. Properties of morphisms local on the source

036F It often happens one can prove a morphism has a certain property after precomposing with some other morphism. In many cases this implies the morphism has the property too. We formalize this in the following definition.

036G **Definition 26.1.** Let $\mathcal{P}$ be a property of morphisms of schemes. Let $\tau \in \{Zariski, fpqc, fppf, \acute{e}tale, smooth, syntomic\}$. We say $\mathcal{P}$ is τ *local on the source*, or *local on the source for the τ-topology* if for any morphism of schemes $f : X \to Y$ over S, and any τ-covering $\{X_i \to X\}_{i \in I}$ we have

$$f \text{ has } \mathcal{P} \Leftrightarrow \text{each } X_i \to Y \text{ has } \mathcal{P}.$$

To be sure, since isomorphisms are always coverings we see (or require) that property $\mathcal{P}$ holds for $X \to Y$ if and only if it holds for any arrow $X' \to Y'$ isomorphic to $X \to Y$. If a property is τ-local on the source then it is preserved by precomposing with morphisms which occur in τ-coverings. Here is a formal statement.

04QV **Lemma 26.2.** *Let $\tau \in \{fpqc, fppf, syntomic, smooth, \acute{e}tale, Zariski\}$. Let $\mathcal{P}$ be a property of morphisms which is τ local on the source. Let $f : X \to Y$ have property $\mathcal{P}$. For any morphism $a : X' \to X$ which is flat, resp. flat and locally of finite presentation, resp. syntomic, resp. étale, resp. an open immersion, the composition $f \circ a : X' \to Y$ has property $\mathcal{P}$.*

Proof. This is true because we can fit $X' \to X$ into a family of morphisms which forms a τ-covering. $\square$

0CEY **Lemma 26.3.** *Let $\tau \in \{fppf, syntomic, smooth, \acute{e}tale\}$. Let $\mathcal{P}$ be a property of morphisms which is τ local on the source. For any morphism of schemes $f : X \to Y$ there exists a largest open $W(f) \subset X$ such that the restriction $f|_{W(f)} : W(f) \to Y$ has $\mathcal{P}$. Moreover, if $g : X' \to X$ is flat and locally of finite presentation, syntomic, smooth, or étale and $f' = f \circ g : X' \to Y$, then $g^{-1}(W(f)) = W(f')$.*

Proof. Consider the union W of the images $g(X') \subset X$ of morphisms $g : X' \to X$ with the properties:

    (1) g is flat and locally of finite presentation, syntomic, smooth, or étale, and
    (2) the composition $X' \to X \to Y$ has property $\mathcal{P}$.

Since such a morphism g is open (see Morphisms, Lemma 25.10) we see that $W \subset X$ is an open subset of X. Since $\mathcal{P}$ is local in the τ topology the restriction $f|_W : W \to Y$ has property $\mathcal{P}$ because we are given a τ covering $\{X' \to W\}$ of W such that the pullbacks have $\mathcal{P}$. This proves the existence of $W(f)$. The compatibility stated in the last sentence follows immediately from the construction of $W(f)$. $\square$

036H **Lemma 26.4.** *Let $\mathcal{P}$ be a property of morphisms of schemes. Let $\tau \in \{fpqc, fppf, \acute{e}tale, smooth, syntomic\}$. Assume that*

    (1) *the property is preserved under precomposing with flat, flat locally of finite presentation, étale, smooth or syntomic morphisms depending on whether τ is fpqc, fppf, étale, smooth, or syntomic,*
    (2) *the property is Zariski local on the source,*
    (3) *the property is Zariski local on the target,*
    (4) *for any morphism of affine schemes $f : X \to Y$, and any surjective morphism of affine schemes $X' \to X$ which is flat, flat of finite presentation, étale, smooth or syntomic depending on whether τ is fpqc, fppf, étale, smooth, or syntomic, property $\mathcal{P}$ holds for f if property $\mathcal{P}$ holds for the composition $f' : X' \to Y$.*

Then $\mathcal{P}$ is τ local on the source.

Proof. This follows almost immediately from the definition of a τ-covering, see Topologies, Definition 9.1 7.1 4.1 5.1, or 6.1 and Topologies, Lemma 9.9, 7.4, 4.4, 5.4, or 6.4. Details omitted. (Hint: Use locality on the source and target to reduce the verification of property $\mathcal{P}$ to the case of a morphism between affines. Then apply (1) and (4).) $\square$

036I **Remark 26.5.** (This is a repeat of Remarks 15.3 and 22.5 above.) In Lemma 26.4 above if $\tau = smooth$ then in condition (4) we may assume that the morphism is a (surjective) standard smooth morphism. Similarly, when $\tau = syntomic$ or $\tau = \acute{e}tale$.

27. Properties of morphisms local in the fpqc topology on the source

036J Here are some properties of morphisms that are fpqc local on the source.

036K **Lemma 27.1.** *The property $\mathcal{P}(f) = $"f is flat" is fpqc local on the source.*

Proof. Since flatness is defined in terms of the maps of local rings (Morphisms, Definition 25.1) what has to be shown is the following algebraic fact: Suppose $A \to B \to C$ are local homomorphisms of local rings, and assume $B \to C$ is flat. Then $A \to B$ is flat if and only if $A \to C$ is flat. If $A \to B$ is flat, then $A \to C$ is flat by Algebra, Lemma 39.4. Conversely, assume $A \to C$ is flat. Note that $B \to C$ is faithfully flat, see Algebra, Lemma 39.17. Hence $A \to B$ is flat by Algebra, Lemma 39.10. (Also see Morphisms, Lemma 25.13 for a direct proof.) $\square$

036L **Lemma 27.2.** *Then property $\mathcal{P}(f : X \to Y) = $"for every $x \in X$ the map of local rings $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is injective" is fpqc local on the source.*

Proof. Omitted. This is just a (probably misguided) attempt to be playful. $\square$

28. Properties of morphisms local in the fppf topology on the source

036M Here are some properties of morphisms that are fppf local on the source.

036N **Lemma 28.1.** *The property $\mathcal{P}(f) = $"f is locally of finite presentation" is fppf local on the source.*

Proof. Being locally of finite presentation is Zariski local on the source and the target, see Morphisms, Lemma 21.2. It is a property which is preserved under composition, see Morphisms, Lemma 21.3. This proves (1), (2) and (3) of Lemma 26.4. The final condition (4) is Lemma 14.1. Hence we win. $\square$

036O **Lemma 28.2.** *The property $\mathcal{P}(f) = $"f is locally of finite type" is fppf local on the source.*

Proof. Being locally of finite type is Zariski local on the source and the target, see Morphisms, Lemma 15.2. It is a property which is preserved under composition, see Morphisms, Lemma 15.3, and a flat morphism locally of finite presentation is locally of finite type, see Morphisms, Lemma 21.8. This proves (1), (2) and (3) of Lemma 26.4. The final condition (4) is Lemma 14.2. Hence we win. $\square$

036P **Lemma 28.3.** *The property $\mathcal{P}(f) = $"f is open" is fppf local on the source.*

Proof. Being an open morphism is clearly Zariski local on the source and the target. It is a property which is preserved under composition, see Morphisms, Lemma 23.3, and a flat morphism of finite presentation is open, see Morphisms, Lemma 25.10 This proves (1), (2) and (3) of Lemma 26.4. The final condition (4) follows from Morphisms, Lemma 25.12. Hence we win. $\square$

036Q **Lemma 28.4.** *The property $\mathcal{P}(f) = $"f is universally open" is fppf local on the source.*

Proof. Let $f : X \to Y$ be a morphism of schemes. Let $\{X_i \to X\}_{i \in I}$ be an fppf covering. Denote $f_i : X_i \to X$ the compositions. We have to show that f is universally open if and only if each f_i is universally open. If f is universally open, then also each f_i is universally open since the maps $X_i \to X$ are universally open and compositions of universally open morphisms are universally open (Morphisms, Lemmas 25.10 and 23.3). Conversely, assume each f_i is universally open. Let $Y' \to Y$ be a morphism of schemes. Denote $X' = Y' \times_Y X$ and $X_i' = Y' \times_Y X_i$. Note that $\{X_i' \to X'\}_{i \in I}$ is an fppf covering also. The morphisms $f_i' : X_i' \to Y'$ are open by assumption. Hence by the Lemma 28.3 above we conclude that $f' : X' \to Y'$ is open as desired. $\square$

29. Properties of morphisms local in the syntomic topology on the source

036R Here are some properties of morphisms that are syntomic local on the source.

036S **Lemma 29.1.** *The property* $\mathcal{P}(f) =$ *"f is syntomic" is syntomic local on the source.*

Proof. Combine Lemma 26.4 with Morphisms, Lemma 30.2 (local for Zariski on source and target), Morphisms, Lemma 30.3 (pre-composing), and Lemma 14.4 (part (4)). $\square$

30. Properties of morphisms local in the smooth topology on the source

036T Here are some properties of morphisms that are smooth local on the source. Note also the (in some respects stronger) result on descending smoothness via flat morphisms, Lemma 14.5.

036U **Lemma 30.1.** *The property* $\mathcal{P}(f) =$ *"f is smooth" is smooth local on the source.*

Proof. Combine Lemma 26.4 with Morphisms, Lemma 34.2 (local for Zariski on source and target), Morphisms, Lemma 34.4 (pre-composing), and Lemma 14.4 (part (4)). $\square$

31. Properties of morphisms local in the étale topology on the source

036V Here are some properties of morphisms that are étale local on the source.

036W **Lemma 31.1.** *The property* $\mathcal{P}(f) =$ *"f is étale" is étale local on the source.*

Proof. Combine Lemma 26.4 with Morphisms, Lemma 36.2 (local for Zariski on source and target), Morphisms, Lemma 36.3 (pre-composing), and Lemma 14.4 (part (4)). $\square$

03X4 **Lemma 31.2.** *The property* $\mathcal{P}(f) =$ *"f is locally quasi-finite" is étale local on the source.*

Proof. We are going to use Lemma 26.4. By Morphisms, Lemma 20.11 the property of being locally quasi-finite is local for Zariski on source and target. By Morphisms, Lemmas 20.12 and 36.6 we see the precomposition of a locally quasi-finite morphism by an étale morphism is locally quasi-finite. Finally, suppose that $X \to Y$ is a morphism of affine schemes and that $X' \to X$ is a surjective étale morphism of affine schemes such that $X' \to Y$ is locally quasi-finite. Then $X' \to Y$ is of finite type, and by Lemma 14.2 we see that $X \to Y$ is of finite type also. Moreover, by

assumption $X' \to Y$ has finite fibres, and hence $X \to Y$ has finite fibres also. We conclude that $X \to Y$ is quasi-finite by Morphisms, Lemma 20.10. This proves the last assumption of Lemma 26.4 and finishes the proof. $\square$

03YV **Lemma 31.3.** *The property $\mathcal{P}(f) = $"$f$ is unramified" is étale local on the source. The property $\mathcal{P}(f) = $"$f$ is G-unramified" is étale local on the source.*

Proof. We are going to use Lemma 26.4. By Morphisms, Lemma 35.3 the property of being unramified (resp. G-unramified) is local for Zariski on source and target. By Morphisms, Lemmas 35.4 and 36.5 we see the precomposition of an unramified (resp. G-unramified) morphism by an étale morphism is unramified (resp. G-unramified). Finally, suppose that $X \to Y$ is a morphism of affine schemes and that $f : X' \to X$ is a surjective étale morphism of affine schemes such that $X' \to Y$ is unramified (resp. G-unramified). Then $X' \to Y$ is of finite type (resp. finite presentation), and by Lemma 14.2 (resp. Lemma 14.1) we see that $X \to Y$ is of finite type (resp. finite presentation) also. By Morphisms, Lemma 34.16 we have a short exact sequence

$$0 \to f^*\Omega_{X/Y} \to \Omega_{X'/Y} \to \Omega_{X'/X} \to 0.$$

As $X' \to Y$ is unramified we see that the middle term is zero. Hence, as f is faithfully flat we see that $\Omega_{X/Y} = 0$. Hence $X \to Y$ is unramified (resp. G-unramified), see Morphisms, Lemma 35.2. This proves the last assumption of Lemma 26.4 and finishes the proof. $\square$

32. Properties of morphisms étale local on source-and-target

04QW Let $\mathcal{P}$ be a property of morphisms of schemes. There is an intuitive meaning to the phrase "$\mathcal{P}$ is étale local on the source and target". However, it turns out that this notion is not the same as asking $\mathcal{P}$ to be both étale local on the source and étale local on the target. Before we discuss this further we give two silly examples.

04QX **Example 32.1.** Consider the property $\mathcal{P}$ of morphisms of schemes defined by the rule $\mathcal{P}(X \to Y) = $"$Y$ is locally Noetherian". The reader can verify that this is étale local on the source and étale local on the target (omitted, see Lemma 16.1). But it is **not** true that if $f : X \to Y$ has $\mathcal{P}$ and $g : Y \to Z$ is étale, then $g \circ f$ has $\mathcal{P}$. Namely, f could be the identity on Y and g could be an open immersion of a locally Noetherian scheme Y into a non locally Noetherian scheme Z.

The following example is in some sense worse.

04QY **Example 32.2.** Consider the property $\mathcal{P}$ of morphisms of schemes defined by the rule $\mathcal{P}(f : X \to Y) = $"for every $y \in Y$ which is a specialization of some $f(x)$, $x \in X$ the local ring $\mathcal{O}_{Y,y}$ is Noetherian". Let us verify that this is étale local on the source and étale local on the target. We will freely use Schemes, Lemma 13.2.

Local on the target: Let $\{g_i : Y_i \to Y\}$ be an étale covering. Let $f_i : X_i \to Y_i$ be the base change of f, and denote $h_i : X_i \to X$ the projection. Assume $\mathcal{P}(f)$. Let $f(x_i) \rightsquigarrow y_i$ be a specialization. Then $f(h_i(x_i)) \rightsquigarrow g_i(y_i)$ so $\mathcal{P}(f)$ implies $\mathcal{O}_{Y,g_i(y_i)}$ is Noetherian. Also $\mathcal{O}_{Y,g_i(y_i)} \to \mathcal{O}_{Y_i,y_i}$ is a localization of an étale ring map. Hence $\mathcal{O}_{Y_i,y_i}$ is Noetherian by Algebra, Lemma 31.1. Conversely, assume $\mathcal{P}(f_i)$ for all i. Let $f(x) \rightsquigarrow y$ be a specialization. Choose an i and $y_i \in Y_i$ mapping to y. Since x can be viewed as a point of $\mathrm{Spec}(\mathcal{O}_{Y,y}) \times_Y X$ and $\mathcal{O}_{Y,y} \to \mathcal{O}_{Y_i,y_i}$ is faithfully flat, there exists a point $x_i \in \mathrm{Spec}(\mathcal{O}_{Y_i,y_i}) \times_Y X$ mapping to x. Then $x_i \in X_i$,

and $f_i(x_i)$ specializes to y_i. Thus we see that $\mathcal{O}_{Y_i, y_i}$ is Noetherian by $\mathcal{P}(f_i)$ which implies that $\mathcal{O}_{Y, y}$ is Noetherian by Algebra, Lemma 164.1.

Local on the source: Let $\{h_i : X_i \to X\}$ be an étale covering. Let $f_i : X_i \to Y$ be the composition $f \circ h_i$. Assume $\mathcal{P}(f)$. Let $f(x_i) \rightsquigarrow y$ be a specialization. Then $f(h_i(x_i)) \rightsquigarrow y$ so $\mathcal{P}(f)$ implies $\mathcal{O}_{Y, y}$ is Noetherian. Thus $\mathcal{P}(f_i)$ holds. Conversely, assume $\mathcal{P}(f_i)$ for all i. Let $f(x) \rightsquigarrow y$ be a specialization. Choose an i and $x_i \in X_i$ mapping to x. Then y is a specialization of $f_i(x_i) = f(x)$. Hence $\mathcal{P}(f_i)$ implies $\mathcal{O}_{Y, y}$ is Noetherian as desired.

We claim that there exists a commutative diagram

$$
\begin{array}{ccc}
U & \xrightarrow{\ h\ } & V \\
{\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\
X & \xrightarrow{\ f\ } & Y
\end{array}
$$

with surjective étale vertical arrows, such that h has $\mathcal{P}$ and f does not have $\mathcal{P}$. Namely, let

$$ Y = \operatorname{Spec}\left(\mathbf{C}[x_n; n \in \mathbf{Z}]/(x_n x_m; n \neq m) \right) $$

and let $X \subset Y$ be the open subscheme which is the complement of the point all of whose coordinates $x_n = 0$. Let $U = X$, let $V = X \amalg Y$, let a, b the obvious map, and let $h : U \to V$ be the inclusion of $U = X$ into the first summand of V. The claim above holds because U is locally Noetherian, but Y is not.

What should be the correct notion of a property which is étale local on the source-and-target? We think that, by analogy with Morphisms, Definition 14.1 it should be the following.

04QZ **Definition 32.3.** Let $\mathcal{P}$ be a property of morphisms of schemes. We say $\mathcal{P}$ is *étale local on source-and-target* if

    (1) (stable under precomposing with étale maps) if $f : X \to Y$ is étale and $g : Y \to Z$ has $\mathcal{P}$, then $g \circ f$ has $\mathcal{P}$,

    (2) (stable under étale base change) if $f : X \to Y$ has $\mathcal{P}$ and $Y' \to Y$ is étale, then the base change $f' : Y' \times_Y X \to Y'$ has $\mathcal{P}$, and

    (3) (locality) given a morphism $f : X \to Y$ the following are equivalent
        (a) f has $\mathcal{P}$,
        (b) for every $x \in X$ there exists a commutative diagram

$$
\begin{array}{ccc}
U & \xrightarrow{\ h\ } & V \\
{\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\
X & \xrightarrow{\ f\ } & Y
\end{array}
$$

        with étale vertical arrows and $u \in U$ with $a(u) = x$ such that h has $\mathcal{P}$.

It turns out this definition excludes the behavior seen in Examples 32.1 and 32.2. We will compare this to the definition in the paper [DM69] by Deligne and Mumford in Remark 32.8. Moreover, a property which is étale local on the source-and-target is étale local on the source and étale local on the target. Finally, the converse is almost true as we will see in Lemma 32.6.

04R0 **Lemma 32.4.** *Let $\mathcal{P}$ be a property of morphisms of schemes which is étale local on source-and-target. Then*

(1) $\mathcal{P}$ *is étale local on the source,*
(2) $\mathcal{P}$ *is étale local on the target,*
(3) $\mathcal{P}$ *is stable under postcomposing with étale morphisms: if $f : X \to Y$ has $\mathcal{P}$ and $g : Y \to Z$ is étale, then $g \circ f$ has $\mathcal{P}$, and*
(4) $\mathcal{P}$ *has a permanence property: given $f : X \to Y$ and $g : Y \to Z$ étale such that $g \circ f$ has $\mathcal{P}$, then f has $\mathcal{P}$.*

Proof. We write everything out completely.

Proof of (1). Let $f : X \to Y$ be a morphism of schemes. Let $\{X_i \to X\}_{i \in I}$ be an étale covering of X. If each composition $h_i : X_i \to Y$ has $\mathcal{P}$, then for each $x \in X$ we can find an $i \in I$ and a point $x_i \in X_i$ mapping to x. Then $(X_i, x_i) \to (X, x)$ is an étale morphism of germs, and $\mathrm{id}_Y : Y \to Y$ is an étale morphism, and h_i is as in part (3) of Definition 32.3. Thus we see that f has $\mathcal{P}$. Conversely, if f has $\mathcal{P}$ then each $X_i \to Y$ has $\mathcal{P}$ by Definition 32.3 part (1).

Proof of (2). Let $f : X \to Y$ be a morphism of schemes. Let $\{Y_i \to Y\}_{i \in I}$ be an étale covering of Y. Write $X_i = Y_i \times_Y X$ and $h_i : X_i \to Y_i$ for the base change of f. If each $h_i : X_i \to Y_i$ has $\mathcal{P}$, then for each $x \in X$ we pick an $i \in I$ and a point $x_i \in X_i$ mapping to x. Then $(X_i, x_i) \to (X, x)$ is an étale morphism of germs, $Y_i \to Y$ is étale, and h_i is as in part (3) of Definition 32.3. Thus we see that f has $\mathcal{P}$. Conversely, if f has $\mathcal{P}$, then each $X_i \to Y_i$ has $\mathcal{P}$ by Definition 32.3 part (2).

Proof of (3). Assume $f : X \to Y$ has $\mathcal{P}$ and $g : Y \to Z$ is étale. For every $x \in X$ we can think of $(X, x) \to (X, x)$ as an étale morphism of germs, $Y \to Z$ is an étale morphism, and $h = f$ is as in part (3) of Definition 32.3. Thus we see that $g \circ f$ has $\mathcal{P}$.

Proof of (4). Let $f : X \to Y$ be a morphism and $g : Y \to Z$ étale such that $g \circ f$ has $\mathcal{P}$. Then by Definition 32.3 part (2) we see that $\mathrm{pr}_Y : Y \times_Z X \to Y$ has $\mathcal{P}$. But the morphism $(f, 1) : X \to Y \times_Z X$ is étale as a section to the étale projection $\mathrm{pr}_X : Y \times_Z X \to X$, see Morphisms, Lemma 36.18. Hence $f = \mathrm{pr}_Y \circ (f, 1)$ has $\mathcal{P}$ by Definition 32.3 part (1). $\square$

The following lemma is the analogue of Morphisms, Lemma 14.4.

04R1 **Lemma 32.5.** *Let $\mathcal{P}$ be a property of morphisms of schemes which is étale local on source-and-target. Let $f : X \to Y$ be a morphism of schemes. The following are equivalent:*

(a) *f has property $\mathcal{P}$,*
(b) *for every $x \in X$ there exists an étale morphism of germs $a : (U, u) \to (X, x)$, an étale morphism $b : V \to Y$, and a morphism $h : U \to V$ such that $f \circ a = b \circ h$ and h has $\mathcal{P}$,*
(c) *for any commutative diagram*

$$
\begin{array}{ccc}
U & \xrightarrow{\ h\ } & V \\
{\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\
X & \xrightarrow{\ f\ } & Y
\end{array}
$$

with a, b étale the morphism h has $\mathcal{P}$,
(d) *for some diagram as in (c) with $a : U \to X$ surjective h has $\mathcal{P}$,*
(e) *there exists an étale covering $\{Y_i \to Y\}_{i \in I}$ such that each base change $Y_i \times_Y X \to Y_i$ has $\mathcal{P}$,*

(f) *there exists an étale covering $\{X_i \to X\}_{i \in I}$ such that each composition $X_i \to Y$ has $\mathcal{P}$,*

(g) *there exists an étale covering $\{Y_i \to Y\}_{i \in I}$ and for each $i \in I$ an étale covering $\{X_{ij} \to Y_i \times_Y X\}_{j \in J_i}$ such that each morphism $X_{ij} \to Y_i$ has $\mathcal{P}$.*

Proof. The equivalence of (a) and (b) is part of Definition 32.3. The equivalence of (a) and (e) is Lemma 32.4 part (2). The equivalence of (a) and (f) is Lemma 32.4 part (1). As (a) is now equivalent to (e) and (f) it follows that (a) equivalent to (g).

It is clear that (c) implies (a). If (a) holds, then for any diagram as in (c) the morphism $f \circ a$ has $\mathcal{P}$ by Definition 32.3 part (1), whereupon h has $\mathcal{P}$ by Lemma 32.4 part (4). Thus (a) and (c) are equivalent. It is clear that (c) implies (d). To see that (d) implies (a) assume we have a diagram as in (c) with $a : U \to X$ surjective and h having $\mathcal{P}$. Then $b \circ h$ has $\mathcal{P}$ by Lemma 32.4 part (3). Since $\{a : U \to X\}$ is an étale covering we conclude that f has $\mathcal{P}$ by Lemma 32.4 part (1).   □

It seems that the result of the following lemma is not a formality, i.e., it actually uses something about the geometry of étale morphisms.

04R2   **Lemma 32.6.** *Let $\mathcal{P}$ be a property of morphisms of schemes. Assume*

(1) *$\mathcal{P}$ is étale local on the source,*

(2) *$\mathcal{P}$ is étale local on the target, and*

(3) *$\mathcal{P}$ is stable under postcomposing with open immersions: if $f : X \to Y$ has $\mathcal{P}$ and $Y \subset Z$ is an open subscheme then $X \to Z$ has $\mathcal{P}$.*

Then $\mathcal{P}$ is étale local on the source-and-target.

Proof. Let $\mathcal{P}$ be a property of morphisms of schemes which satisfies conditions (1), (2) and (3) of the lemma. By Lemma 26.2 we see that $\mathcal{P}$ is stable under precomposing with étale morphisms. By Lemma 22.2 we see that $\mathcal{P}$ is stable under étale base change. Hence it suffices to prove part (3) of Definition 32.3 holds.

More precisely, suppose that $f : X \to Y$ is a morphism of schemes which satisfies Definition 32.3 part (3)(b). In other words, for every $x \in X$ there exists an étale morphism $a_x : U_x \to X$, a point $u_x \in U_x$ mapping to x, an étale morphism $b_x : V_x \to Y$, and a morphism $h_x : U_x \to V_x$ such that $f \circ a_x = b_x \circ h_x$ and h_x has $\mathcal{P}$. The proof of the lemma is complete once we show that f has $\mathcal{P}$. Set $U = \coprod U_x$, $a = \coprod a_x$, $V = \coprod V_x$, $b = \coprod b_x$, and $h = \coprod h_x$. We obtain a commutative diagram

$$
\begin{array}{ccc}
U & \xrightarrow{\ h\ } & V \\
{\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\
X & \xrightarrow{\ f\ } & Y
\end{array}
$$

with a, b étale, a surjective. Note that h has $\mathcal{P}$ as each h_x does and $\mathcal{P}$ is étale local on the target. Because a is surjective and $\mathcal{P}$ is étale local on the source, it suffices to prove that $b \circ h$ has $\mathcal{P}$. This reduces the lemma to proving that $\mathcal{P}$ is stable under postcomposing with an étale morphism.

During the rest of the proof we let $f : X \to Y$ be a morphism with property $\mathcal{P}$ and $g : Y \to Z$ is an étale morphism. Consider the following statements:

(-) With no additional assumptions $g \circ f$ has property $\mathcal{P}$.

(A) Whenever Z is affine $g \circ f$ has property $\mathcal{P}$.

(AA) Whenever X and Z are affine $g \circ f$ has property $\mathcal{P}$.

(AAA) Whenever X, Y, and Z are affine $g \circ f$ has property $\mathcal{P}$.

Once we have proved (-) the proof of the lemma will be complete.

Claim 1: (AAA) $\Rightarrow$ (AA). Namely, let $f : X \to Y$, $g : Y \to Z$ be as above with X, Z affine. As X is affine hence quasi-compact we can find finitely many affine open $Y_i \subset Y$, $i = 1, \ldots, n$ such that $X = \bigcup_{i=1,\ldots,n} f^{-1}(Y_i)$. Set $X_i = f^{-1}(Y_i)$. By Lemma 22.2 each of the morphisms $X_i \to Y_i$ has $\mathcal{P}$. Hence $\coprod_{i=1,\ldots,n} X_i \to \coprod_{i=1,\ldots,n} Y_i$ has $\mathcal{P}$ as $\mathcal{P}$ is étale local on the target. By (AAA) applied to $\coprod_{i=1,\ldots,n} X_i \to \coprod_{i=1,\ldots,n} Y_i$ and the étale morphism $\coprod_{i=1,\ldots,n} Y_i \to Z$ we see that $\coprod_{i=1,\ldots,n} X_i \to Z$ has $\mathcal{P}$. Now $\{\coprod_{i=1,\ldots,n} X_i \to X\}$ is an étale covering, hence as $\mathcal{P}$ is étale local on the source we conclude that $X \to Z$ has $\mathcal{P}$ as desired.

Claim 2: (AAA) $\Rightarrow$ (A). Namely, let $f : X \to Y$, $g : Y \to Z$ be as above with Z affine. Choose an affine open covering $X = \bigcup X_i$. As $\mathcal{P}$ is étale local on the source we see that each $f|_{X_i} : X_i \to Y$ has $\mathcal{P}$. By (AA), which follows from (AAA) according to Claim 1, we see that $X_i \to Z$ has $\mathcal{P}$ for each i. Since $\{X_i \to X\}$ is an étale covering and $\mathcal{P}$ is étale local on the source we conclude that $X \to Z$ has $\mathcal{P}$.

Claim 3: (AAA) $\Rightarrow$ (-). Namely, let $f : X \to Y$, $g : Y \to Z$ be as above. Choose an affine open covering $Z = \bigcup Z_i$. Set $Y_i = g^{-1}(Z_i)$ and $X_i = f^{-1}(Y_i)$. By Lemma 22.2 each of the morphisms $X_i \to Y_i$ has $\mathcal{P}$. By (A), which follows from (AAA) according to Claim 2, we see that $X_i \to Z_i$ has $\mathcal{P}$ for each i. Since $\mathcal{P}$ is local on the target and $X_i = (g \circ f)^{-1}(Z_i)$ we conclude that $X \to Z$ has $\mathcal{P}$.

Thus to prove the lemma it suffices to prove (AAA). Let $f : X \to Y$ and $g : Y \to Z$ be as above X, Y, Z affine. Note that an étale morphism of affines has universally bounded fibres, see Morphisms, Lemma 36.6 and Lemma 57.9. Hence we can do induction on the integer n bounding the degree of the fibres of $Y \to Z$. See Morphisms, Lemma 57.8 for a description of this integer in the case of an étale morphism. If $n = 1$, then $Y \to Z$ is an open immersion, see Lemma 25.2, and the result follows from assumption (3) of the lemma. Assume $n > 1$.

Consider the following commutative diagram

$$
\begin{array}{ccccc}
X \times_Z Y & \xrightarrow{f_Y} & Y \times_Z Y & \xrightarrow{\mathrm{pr}} & Y \\
\downarrow & & \downarrow & & \downarrow \\
X & \xrightarrow{f} & Y & \xrightarrow{g} & Z
\end{array}
$$

Note that we have a decomposition into open and closed subschemes $Y \times_Z Y = \Delta_{Y/Z}(Y) \amalg Y'$, see Morphisms, Lemma 35.13. As a base change the degrees of the fibres of the second projection $\mathrm{pr} : Y \times_Z Y \to Y$ are bounded by n, see Morphisms, Lemma 57.5. On the other hand, $\mathrm{pr}|_{\Delta(Y)} : \Delta(Y) \to Y$ is an isomorphism and every fibre has exactly one point. Thus, on applying Morphisms, Lemma 57.8 we conclude the degrees of the fibres of the restriction $\mathrm{pr}|_{Y'} : Y' \to Y$ are bounded by $n - 1$. Set $X' = f_Y^{-1}(Y')$. Picture

$$
\begin{array}{ccccc}
X \amalg X' & \xrightarrow{f \amalg f'} & \Delta(Y) \amalg Y' & \longrightarrow & Y \\
\| & & \| & & \| \\
X \times_Z Y & \xrightarrow{f_Y} & Y \times_Z Y & \xrightarrow{\mathrm{pr}} & Y
\end{array}
$$

As $\mathcal{P}$ is étale local on the target and hence stable under étale base change (see Lemma 22.2) we see that f_Y has $\mathcal{P}$. Hence, as $\mathcal{P}$ is étale local on the source, $f' = f_Y|_{X'}$ has $\mathcal{P}$. By induction hypothesis we see that $X' \to Y$ has $\mathcal{P}$. As $\mathcal{P}$ is local on the source, and $\{X \to X \times_Z Y, X' \to X \times_Y Z\}$ is an étale covering, we conclude that $\mathrm{pr} \circ f_Y$ has $\mathcal{P}$. Note that $g \circ f$ can be viewed as a morphism $g \circ f : X \to g(Y)$. As $\mathrm{pr} \circ f_Y$ is the pullback of $g \circ f : X \to g(Y)$ via the étale covering $\{Y \to g(Y)\}$, and as $\mathcal{P}$ is étale local on the target, we conclude that $g \circ f : X \to g(Y)$ has property $\mathcal{P}$. Finally, applying assumption (3) of the lemma once more we conclude that $g \circ f : X \to Z$ has property $\mathcal{P}$. $\qquad\square$

04R3   **Remark 32.7.** Using Lemma 32.6 and the work done in the earlier sections of this chapter it is easy to make a list of types of morphisms which are étale local on the source-and-target. In each case we list the lemma which implies the property is étale local on the source and the lemma which implies the property is étale local on the target. In each case the third assumption of Lemma 32.6 is trivial to check, and we omit it. Here is the list:

   (1) flat, see Lemmas 27.1 and 23.17,
   (2) locally of finite presentation, see Lemmas 28.1 and 23.13,
   (3) locally finite type, see Lemmas 28.2 and 23.12,
   (4) universally open, see Lemmas 28.4 and 23.4,
   (5) syntomic, see Lemmas 29.1 and 23.28,
   (6) smooth, see Lemmas 30.1 and 23.29,
   (7) étale, see Lemmas 31.1 and 23.31,
   (8) locally quasi-finite, see Lemmas 31.2 and 23.26,
   (9) unramified, see Lemmas 31.3 and 23.30,
   (10) G-unramified, see Lemmas 31.3 and 23.30, and
   (11) add more here as needed.

04R4   **Remark 32.8.** At this point we have three possible definitions of what it means for a property $\mathcal{P}$ of morphisms to be "étale local on the source and target":

   (ST) $\mathcal{P}$ is étale local on the source and $\mathcal{P}$ is étale local on the target,
   (DM) (the definition in the paper [DM69, Page 100] by Deligne and Mumford) for every diagram

$$
\begin{array}{ccc}
U & \xrightarrow{\ h\ } & V \\
{\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\
X & \xrightarrow{\ f\ } & Y
\end{array}
$$

   with surjective étale vertical arrows we have $\mathcal{P}(h) \Leftrightarrow \mathcal{P}(f)$, and
   (SP) $\mathcal{P}$ is étale local on the source-and-target.

In this section we have seen that (SP) $\Rightarrow$ (DM) $\Rightarrow$ (ST). The Examples 32.1 and 32.2 show that neither implication can be reversed. Finally, Lemma 32.6 shows that the difference disappears when looking at properties of morphisms which are stable under postcomposing with open immersions, which in practice will always be the case.

0CEZ **Lemma 32.9.** *Let $\mathcal{P}$ be a property of morphisms of schemes which is étale local on the source-and-target. Given a commutative diagram of schemes*

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & Y \end{array} \quad \text{with points} \quad \begin{array}{ccc} x' & \longrightarrow & y' \\ \downarrow & & \downarrow \\ x & \longrightarrow & y \end{array}$$

such that g' is étale at x' and g is étale at y', then $x \in W(f) \Leftrightarrow x' \in W(f')$ where $W(-)$ is as in Lemma 26.3.

Proof. Lemma 26.3 applies since $\mathcal{P}$ is étale local on the source by Lemma 32.4.

Assume $x \in W(f)$. Let $U' \subset X'$ and $V' \subset Y'$ be open neighbourhoods of x' and y' such that $f'(U') \subset V'$, $g'(U') \subset W(f)$ and $g'|_{U'}$ and $g|_{V'}$ are étale. Then $f \circ g'|_{U'} = g \circ f'|_{U'}$ has $\mathcal{P}$ by property (1) of Definition 32.3. Then $f'|_{U'} : U' \to V'$ has property $\mathcal{P}$ by (4) of Lemma 32.4. Then by (3) of Lemma 32.4 we conclude that $f'_{U'} : U' \to Y'$ has $\mathcal{P}$. Hence $U' \subset W(f')$ by definition. Hence $x' \in W(f')$.

Assume $x' \in W(f')$. Let $U' \subset X'$ and $V' \subset Y'$ be open neighbourhoods of x' and y' such that $f'(U') \subset V'$, $U' \subset W(f')$ and $g'|_{U'}$ and $g|_{V'}$ are étale. Then $U' \to Y'$ has $\mathcal{P}$ by definition of $W(f')$. Then $U' \to V'$ has $\mathcal{P}$ by (4) of Lemma 32.4. Then $U' \to Y$ has $\mathcal{P}$ by (3) of Lemma 32.4. Let $U \subset X$ be the image of the étale (hence open) morphism $g'|'_U : U' \to X$. Then $\{U' \to U\}$ is an étale covering and we conclude that $U \to Y$ has $\mathcal{P}$ by (1) of Lemma 32.4. Thus $U \subset W(f)$ by definition. Hence $x \in W(f)$. $\square$

0CF0 **Lemma 32.10.** *Let k be a field. Let $n \geq 2$. For $1 \leq i, j \leq n$ with $i \neq j$ and $d \geq 0$ denote $T_{i,j,d}$ the automorphism of $\mathbf{A}_k^n$ given in coordinates by*

$$(x_1, \ldots, x_n) \longmapsto (x_1, \ldots, x_{i-1}, x_i + x_j^d, x_{i+1}, \ldots, x_n)$$

Let $W \subset \mathbf{A}_k^n$ be a nonempty open subscheme such that $T_{i,j,d}(W) = W$ for all i, j, d as above. Then either $W = \mathbf{A}_k^n$ or the characteristic of k is $p > 0$ and $\mathbf{A}_k^n \setminus W$ is a finite set of closed points whose coordinates are algebraic over $\mathbf{F}_p$.

Proof. We may replace k by any extension field in order to prove this. Let Z be an irreducible component of $\mathbf{A}_k^n \setminus W$. Assume $\dim(Z) \geq 1$, to get a contradiction. Then there exists an extension field k'/k and a k'-valued point $\xi = (\xi_1, \ldots, \xi_n) \in (k')^n$ of $Z_{k'} \subset \mathbf{A}_{k'}^n$ such that at least one of $x_1, \ldots, x_n$ is transcendental over the prime field. Claim: the orbit of ξ under the group generated by the transformations $T_{i,j,d}$ is Zariski dense in $\mathbf{A}_{k'}^n$. The claim will give the desired contradiction.

If the characteristic of k' is zero, then already the operators $T_{i,j,0}$ will be enough since these transform ξ into the points

$$(\xi_1 + a_1, \ldots, \xi_n + a_n)$$

for arbitrary $(a_1, \ldots, a_n) \in \mathbf{Z}_{\geq 0}^n$. If the characteristic is $p > 0$, we may assume after renumbering that ξ_n is transcendental over $\mathbf{F}_p$. By successively applying the operators $T_{i,n,d}$ for $i < n$ we see the orbit of ξ contains the elements

$$(\xi_1 + P_1(\xi_n), \ldots, \xi_{n-1} + P_{n-1}(\xi_n), \xi_n)$$

for arbitrary $(P_1, \ldots, P_{n-1}) \in \mathbf{F}_p[t]$. Thus the Zariski closure of the orbit contains the coordinate hyperplane $x_n = \xi_n$. Repeating the argument with a different coordinate, we conclude that the Zariski closure contains $x_i = \xi_i + P(\xi_n)$ for any

$P \in \mathbf{F}_p[t]$ such that $\xi_i + P(\xi_n)$ is transcendental over $\mathbf{F}_p$. Since there are infinitely many such P the claim follows.

Of course the argument in the preceding paragraph also applies if $Z = \{z\}$ has dimension 0 and the coordinates of z in $\kappa(z)$ are not algebraic over $\mathbf{F}_p$. The lemma follows. $\square$

0CF1 **Lemma 32.11.** *Let $\mathcal{P}$ be a property of morphisms of schemes. Assume*

(1) *$\mathcal{P}$ is étale local on the source,*
(2) *$\mathcal{P}$ is smooth local on the target,*
(3) *$\mathcal{P}$ is stable under postcomposing with open immersions: if $f : X \to Y$ has $\mathcal{P}$ and $Y \subset Z$ is an open subscheme then $X \to Z$ has $\mathcal{P}$.*

Given a commutative diagram of schemes

$$
\begin{array}{ccc}
X' & \xrightarrow{\ f'\ } & Y' \\
{\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\
X & \xrightarrow{\ f\ } & Y
\end{array}
\qquad \text{with points} \qquad
\begin{array}{ccc}
x' & \longrightarrow & y' \\
\downarrow & & \downarrow \\
x & \longrightarrow & y
\end{array}
$$

such that g is smooth y' and $X' \to X \times_Y Y'$ is étale at x', then $x \in W(f) \Leftrightarrow x' \in W(f')$ where $W(-)$ is as in Lemma 26.3.

Proof. Since $\mathcal{P}$ is étale local on the source we see that $x \in W(f)$ if and only if the image of x in $X \times_Y Y'$ is in $W(X \times_Y Y' \to Y')$. Hence we may assume the diagram in the lemma is cartesian.

Assume $x \in W(f)$. Since $\mathcal{P}$ is smooth local on the target we see that $(g')^{-1} W(f) = W(f) \times_Y Y' \to Y'$ has $\mathcal{P}$. Hence $(g')^{-1} W(f) \subset W(f')$. We conclude $x' \in W(f')$.

Assume $x' \in W(f')$. For any open neighbourhood $V' \subset Y'$ of y' we may replace Y' by V' and X' by $U' = (f')^{-1} V'$ because $V' \to Y'$ is smooth and hence the base change $W(f') \cap U' \to V'$ of $W(f') \to Y'$ has property $\mathcal{P}$. Thus we may assume there exists an étale morphism $Y' \to \mathbf{A}_Y^n$ over Y, see Morphisms, Lemma 36.20. Picture

$$
\begin{array}{ccc}
X' & \longrightarrow & Y' \\
\downarrow & & \downarrow \\
\mathbf{A}_X^n & \xrightarrow{\ f_n\ } & \mathbf{A}_Y^n \\
\downarrow & & \downarrow \\
X & \xrightarrow{\ f\ } & Y
\end{array}
$$

By Lemma 32.6 (and because étale coverings are smooth coverings) we see that $\mathcal{P}$ is étale local on the source-and-target. By Lemma 32.9 we see that $W(f')$ is the inverse image of the open $W(f_n) \subset \mathbf{A}_X^n$. In particular $W(f_n)$ contains a point lying over x. After replacing X by the image of $W(f_n)$ (which is open) we may assume $W(f_n) \to X$ is surjective. Claim: $W(f_n) = \mathbf{A}_X^n$. The claim implies f has $\mathcal{P}$ as $\mathcal{P}$ is local in the smooth topology and $\{\mathbf{A}_Y^n \to Y\}$ is a smooth covering.

Essentially, the claim follows as $W(f_n) \subset \mathbf{A}_X^n$ is a "translation invariant" open which meets every fibre of $\mathbf{A}_X^n \to X$. However, to produce an argument along these lines one has to do étale localization on Y to produce enough translations and it becomes a bit annoying. Instead we use the automorphisms of Lemma 32.10

and étale morphisms of affine spaces. We may assume $n \geq 2$. Namely, if $n = 0$, then we are done. If $n = 1$, then we consider the diagram

$$
\begin{array}{ccc}
\mathbf{A}_X^2 & \xrightarrow{f_2} & \mathbf{A}_Y^2 \\
{\scriptstyle p}\downarrow & & \downarrow \\
\mathbf{A}_X^1 & \xrightarrow{f_1} & \mathbf{A}_Y^1
\end{array}
$$

We have $p^{-1}(W(f_1)) \subset W(f_2)$ (see first paragraph of the proof). Thus $W(f_2) \to X$ is still surjective and we may work with f_2. Assume $n \geq 2$.

For any $1 \leq i, j \leq n$ with $i \neq j$ and $d \geq 0$ denote $T_{i,j,d}$ the automorphism of $\mathbf{A}^n$ defined in Lemma 32.10. Then we get a commutative diagram

$$
\begin{array}{ccc}
\mathbf{A}_X^n & \xrightarrow{f_n} & \mathbf{A}_Y^n \\
{\scriptstyle T_{i,j,d}}\downarrow & & \downarrow{\scriptstyle T_{i,j,d}} \\
\mathbf{A}_X^n & \xrightarrow{f_n} & \mathbf{A}_Y^n
\end{array}
$$

whose vertical arrows are isomorphisms. We conclude that $T_{i,j,d}(W(f_n)) = W(f_n)$. Applying Lemma 32.10 we conclude for any $x \in X$ the fibre $W(f_n)_x \subset \mathbf{A}_x^n$ is either $\mathbf{A}_x^n$ (this is what we want) or $\kappa(x)$ has characteristic $p > 0$ and $W(f_n)_x$ is the complement of a finite set $Z_x \subset \mathbf{A}_x^n$ of closed points. The second possibility cannot occur. Namely, consider the morphism $T_p : \mathbf{A}^n \to \mathbf{A}^n$ given by

$$(x_1, \ldots, x_n) \mapsto (x_1 - x_1^p, \ldots, x_n - x_n^p)$$

As above we get a commutative diagram

$$
\begin{array}{ccc}
\mathbf{A}_X^n & \xrightarrow{f_n} & \mathbf{A}_Y^n \\
{\scriptstyle T_p}\downarrow & & \downarrow{\scriptstyle T_p} \\
\mathbf{A}_X^n & \xrightarrow{f_n} & \mathbf{A}_Y^n
\end{array}
$$

The morphism $T_p : \mathbf{A}_X^n \to \mathbf{A}_X^n$ is étale at every point lying over x and the morphism $T_p : \mathbf{A}_Y^n \to \mathbf{A}_Y^n$ is étale at every point lying over the image of x in Y. (Details omitted; hint: compute the derivatives.) We conclude that

$$T_p^{-1}(W) \cap \mathbf{A}_x^n = W \cap \mathbf{A}_x^n$$

by Lemma 32.9 (we've already seen $\mathcal{P}$ is étale local on the source-and-target). Since $T_p : \mathbf{A}_x^n \to \mathbf{A}_x^n$ is finite étale of degree $p^n > 1$ we see that if Z_x is not empty then it contains $T_p^{-1}(Z_x)$ which is bigger. This contradiction finishes the proof. $\qquad\square$

33. Properties of morphisms of germs local on source-and-target

04R5   In this section we discuss the analogue of the material in Section 32 for morphisms of germs of schemes.

04NB **Definition 33.1.** Let $\mathcal{Q}$ be a property of morphisms of germs of schemes. We say $\mathcal{Q}$ is *étale local on the source-and-target* if for any commutative diagram

$$
\begin{array}{ccc}
(U', u') & \xrightarrow{\ h'\ } & (V', v') \\
{\scriptstyle a}\Big\downarrow & & \Big\downarrow{\scriptstyle b} \\
(U, u) & \xrightarrow{\ h\ } & (V, v)
\end{array}
$$

of germs with étale vertical arrows we have $\mathcal{Q}(h) \Leftrightarrow \mathcal{Q}(h')$.

04R6 **Lemma 33.2.** *Let $\mathcal{P}$ be a property of morphisms of schemes which is étale local on the source-and-target. Consider the property $\mathcal{Q}$ of morphisms of germs defined by the rule*

$$\mathcal{Q}((X, x) \to (S, s)) \Leftrightarrow \text{there exists a representative } U \to S \text{ which has } \mathcal{P}$$

Then $\mathcal{Q}$ is étale local on the source-and-target as in Definition 33.1.

Proof. If a morphism of germs $(X, x) \to (S, s)$ has $\mathcal{Q}$, then there are arbitrarily small neighbourhoods $U \subset X$ of x and $V \subset S$ of s such that a representative $U \to V$ of $(X, x) \to (S, s)$ has $\mathcal{P}$. This follows from Lemma 32.4. Let

$$
\begin{array}{ccc}
(U', u') & \xrightarrow{\ h'\ } & (V', v') \\
{\scriptstyle a}\Big\downarrow & & \Big\downarrow{\scriptstyle b} \\
(U, u) & \xrightarrow{\ h\ } & (V, v)
\end{array}
$$

be as in Definition 33.1. Choose $U_1 \subset U$ and a representative $h_1 : U_1 \to V$ of h. Choose $V_1' \subset V'$ and an étale representative $b_1 : V_1' \to V$ of b (Definition 20.2). Choose $U_1' \subset U'$ and representatives $a_1 : U_1' \to U_1$ and $h_1' : U_1' \to V_1'$ of a and h' with a_1 étale. After shrinking U_1' we may assume $h_1 \circ a_1 = b_1 \circ h_1'$. By the initial remark of the proof, we are trying to show $u' \in W(h_1') \Leftrightarrow u \in W(h_1)$ where $W(-)$ is as in Lemma 26.3. Thus the lemma follows from Lemma 32.9. $\square$

04R7 **Lemma 33.3.** *Let $\mathcal{P}$ be a property of morphisms of schemes which is étale local on source-and-target. Let Q be the associated property of morphisms of germs, see Lemma 33.2. Let $f : X \to Y$ be a morphism of schemes. The following are equivalent:*

(1) *f has property $\mathcal{P}$, and*
(2) *for every $x \in X$ the morphism of germs $(X, x) \to (Y, f(x))$ has property $\mathcal{Q}$.*

Proof. The implication $(1) \Rightarrow (2)$ is direct from the definitions. The implication $(2) \Rightarrow (1)$ also follows from part (3) of Definition 32.3. $\square$

A morphism of germs $(X, x) \to (S, s)$ determines a well defined map of local rings. Hence the following lemma makes sense.

04ND **Lemma 33.4.** *The property of morphisms of germs*

$$\mathcal{P}((X, x) \to (S, s)) = \mathcal{O}_{S,s} \to \mathcal{O}_{X,x} \text{ is flat}$$

is étale local on the source-and-target.

Proof. Given a diagram as in Definition 33.1 we obtain the following diagram of local homomorphisms of local rings

$$
\begin{array}{ccc}
\mathcal{O}_{U',u'} & \longleftarrow & \mathcal{O}_{V',v'} \\
\uparrow & & \uparrow \\
\mathcal{O}_{U,u} & \longleftarrow & \mathcal{O}_{V,v}
\end{array}
$$

Note that the vertical arrows are localizations of étale ring maps, in particular they are essentially of finite presentation, flat, and unramified (see Algebra, Section 143). In particular the vertical maps are faithfully flat, see Algebra, Lemma 39.17. Now, if the upper horizontal arrow is flat, then the lower horizontal arrow is flat by an application of Algebra, Lemma 39.10 with $R = \mathcal{O}_{V,v}$, $S = \mathcal{O}_{U,u}$ and $M = \mathcal{O}_{U',u'}$. If the lower horizontal arrow is flat, then the ring map

$$
\mathcal{O}_{V',v'} \otimes_{\mathcal{O}_{V,v}} \mathcal{O}_{U,u} \longleftarrow \mathcal{O}_{V',v'}
$$

is flat by Algebra, Lemma 39.7. And the ring map

$$
\mathcal{O}_{U',u'} \longleftarrow \mathcal{O}_{V',v'} \otimes_{\mathcal{O}_{V,v}} \mathcal{O}_{U,u}
$$

is a localization of a map between étale ring extensions of $\mathcal{O}_{U,u}$, hence flat by Algebra, Lemma 143.8. $\square$

04NI **Lemma 33.5.** *Consider a commutative diagram of morphisms of schemes*

$$
\begin{array}{ccc}
U' & \longrightarrow & V' \\
\downarrow & & \downarrow \\
U & \longrightarrow & V
\end{array}
$$

with étale vertical arrows and a point $v' \in V'$ mapping to $v \in V$. Then the morphism of fibres $U'_{v'} \to U_v$ is étale.

Proof. Note that $U'_v \to U_v$ is étale as a base change of the étale morphism $U' \to U$. The scheme U'_v is a scheme over V'_v. By Morphisms, Lemma 36.7 the scheme V'_v is a disjoint union of spectra of finite separable field extensions of $\kappa(v)$. One of these is $v' = \mathrm{Spec}(\kappa(v'))$. Hence $U'_{v'}$ is an open and closed subscheme of U'_v and it follows that $U'_{v'} \to U'_v \to U_v$ is étale (as a composition of an open immersion and an étale morphism, see Morphisms, Section 36). $\square$

Given a morphism of germs of schemes $(X, x) \to (S, s)$ we can define the *fibre* as the isomorphism class of germs (U_s, x) where $U \to S$ is any representative. We will often abuse notation and just write (X_s, x).

04NJ **Lemma 33.6.** *Let $d \in \{0, 1, 2, \ldots, \infty\}$. The property of morphisms of germs*

$$
\mathcal{P}_d((X, x) \to (S, s)) = \text{the local ring } \mathcal{O}_{X_s,x} \text{ of the fibre has dimension } d
$$

is étale local on the source-and-target.

Proof. Given a diagram as in Definition 33.1 we obtain an étale morphism of fibres $U'_{v'} \to U_v$ mapping u' to u, see Lemma 33.5. Hence the result follows from Lemma 21.3. $\square$

04NK **Lemma 33.7.** *Let $r \in \{0, 1, 2, \ldots, \infty\}$. The property of morphisms of germs*

$$\mathcal{P}_r((X, x) \to (S, s)) \Leftrightarrow trdeg_{\kappa(s)} \kappa(x) = r$$

is étale local on the source-and-target.

Proof. Given a diagram as in Definition 33.1 we obtain the following diagram of local homomorphisms of local rings

$$
\begin{array}{ccc}
\mathcal{O}_{U', u'} & \longleftarrow & \mathcal{O}_{V', v'} \\
\uparrow & & \uparrow \\
\mathcal{O}_{U, u} & \longleftarrow & \mathcal{O}_{V, v}
\end{array}
$$

Note that the vertical arrows are localizations of étale ring maps, in particular they are unramified (see Algebra, Section 143). Hence $\kappa(u')/\kappa(u)$ and $\kappa(v')/\kappa(v)$ are finite separable field extensions. Thus we have $\mathrm{trdeg}_{\kappa(v)}\kappa(u) = \mathrm{trdeg}_{\kappa(v')}\kappa(u)$ which proves the lemma. $\qquad\square$

Let (X, x) be a germ of a scheme. The dimension of X at x is the minimum of the dimensions of open neighbourhoods of x in X, and any small enough open neighbourhood has this dimension. Hence this is an invariant of the isomorphism class of the germ. We denote this simply $\dim_x(X)$.

04NL **Lemma 33.8.** *Let $d \in \{0, 1, 2, \ldots, \infty\}$. The property of morphisms of germs*

$$\mathcal{P}_d((X, x) \to (S, s)) \Leftrightarrow \dim_x(X_s) = d$$

is étale local on the source-and-target.

Proof. Given a diagram as in Definition 33.1 we obtain an étale morphism of fibres $U'_{v'} \to U_v$ mapping u' to u, see Lemma 33.5. Hence now the equality $\dim_u(U_v) = \dim_{u'}(U'_{v'})$ follows from Lemma 21.2. $\qquad\square$

34. Descent data for schemes over schemes

023U Most of the arguments in this section are formal relying only on the definition of a descent datum. In Simplicial Spaces, Section 27 we will examine the relationship with simplicial schemes which will somewhat clarify the situation.

023V **Definition 34.1.** Let $f : X \to S$ be a morphism of schemes.

(1) Let $V \to X$ be a scheme over X. A *descent datum for $V/X/S$* is an isomorphism $\varphi : V \times_S X \to X \times_S V$ of schemes over $X \times_S X$ satisfying the *cocycle condition* that the diagram

$$
\begin{array}{ccc}
V \times_S X \times_S X & \xrightarrow{\varphi_{02}} & X \times_S X \times_S V \\
{\scriptstyle \varphi_{01}} \searrow & & \nearrow {\scriptstyle \varphi_{12}} \\
& X \times_S V \times_S X &
\end{array}
$$

commutes (with obvious notation).
(2) We also say that the pair $(V/X, \varphi)$ is a *descent datum relative to $X \to S$*.

(3) A *morphism* $g : (V/X, \varphi) \to (V'/X, \varphi')$ *of descent data relative to* $X \to S$ is a morphism $g : V \to V'$ of schemes over X such that the diagram

$$
\begin{array}{ccc}
V \times_S X & \xrightarrow{\ \varphi\ } & X \times_S V \\
{\scriptstyle g \times \mathrm{id}_X} \big\downarrow & & \big\downarrow {\scriptstyle \mathrm{id}_X \times g} \\
V' \times_S X & \xrightarrow{\ \varphi'\ } & X \times_S V'
\end{array}
$$

commutes.

There are all kinds of "miraculous" identities which arise out of the definition above. For example the pullback of φ via the diagonal morphism $\Delta : X \to X \times_S X$ can be seen as a morphism $\Delta^* \varphi : V \to V$. This because $X \times_{\Delta, X \times_S X} (V \times_S X) = V$ and also $X \times_{\Delta, X \times_S X} (X \times_S V) = V$. In fact, $\Delta^* \varphi$ is equal to the identity. This is a good exercise if you are unfamiliar with this material.

02VP **Remark 34.2.** Let $X \to S$ be a morphism of schemes. Let $(V/X, \varphi)$ be a descent datum relative to $X \to S$. We may think of the isomorphism φ as an isomorphism

$$(X \times_S X) \times_{\mathrm{pr}_0, X} V \longrightarrow (X \times_S X) \times_{\mathrm{pr}_1, X} V$$

of schemes over $X \times_S X$. So loosely speaking one may think of φ as a map $\varphi : \mathrm{pr}_0^* V \to \mathrm{pr}_1^* V$[9]. The cocycle condition then says that $\mathrm{pr}_{02}^* \varphi = \mathrm{pr}_{12}^* \varphi \circ \mathrm{pr}_{01}^* \varphi$. In this way it is very similar to the case of a descent datum on quasi-coherent sheaves.

Here is the definition in case you have a family of morphisms with fixed target.

023W **Definition 34.3.** Let S be a scheme. Let $\{X_i \to S\}_{i \in I}$ be a family of morphisms with target S.

(1) A *descent datum* (V_i, φ_{ij}) *relative to the family* $\{X_i \to S\}$ is given by a scheme V_i over X_i for each $i \in I$, an isomorphism $\varphi_{ij} : V_i \times_S X_j \to X_i \times_S V_j$ of schemes over $X_i \times_S X_j$ for each pair $(i, j) \in I^2$ such that for every triple of indices $(i, j, k) \in I^3$ the diagram

$$
\begin{array}{ccc}
V_i \times_S X_j \times_S X_k & \xrightarrow{\ \ \mathrm{pr}_{02}^* \varphi_{ik}\ \ } & X_i \times_S X_j \times_S V_k \\
{\scriptstyle \mathrm{pr}_{01}^* \varphi_{ij}} \searrow & & \nearrow {\scriptstyle \mathrm{pr}_{12}^* \varphi_{jk}} \\
& X_i \times_S V_j \times_S X_k &
\end{array}
$$

of schemes over $X_i \times_S X_j \times_S X_k$ commutes (with obvious notation).

(2) A *morphism* $\psi : (V_i, \varphi_{ij}) \to (V_i', \varphi_{ij}')$ *of descent data* is given by a family $\psi = (\psi_i)_{i \in I}$ of morphisms of X_i-schemes $\psi_i : V_i \to V_i'$ such that all the diagrams

$$
\begin{array}{ccc}
V_i \times_S X_j & \xrightarrow{\ \varphi_{ij}\ } & X_i \times_S V_j \\
{\scriptstyle \psi_i \times \mathrm{id}} \big\downarrow & & \big\downarrow {\scriptstyle \mathrm{id} \times \psi_j} \\
V_i' \times_S X_j & \xrightarrow{\ \varphi_{ij}'\ } & X_i \times_S V_j'
\end{array}
$$

commute.

---

[9]Unfortunately, we have chosen the "wrong" direction for our arrow here. In Definitions 34.1 and 34.3 we should have the opposite direction to what was done in Definition 2.1 by the general principle that "functions" and "spaces" are dual.

This is the notion that comes up naturally for example when the question arises whether the fibred category of relative curves is a stack in the fpqc topology (it isn't – at least not if you stick to schemes).

02VQ **Remark 34.4.** Let S be a scheme. Let $\{X_i \to S\}_{i \in I}$ be a family of morphisms with target S. Let (V_i, φ_{ij}) be a descent datum relative to $\{X_i \to S\}$. We may think of the isomorphisms φ_{ij} as isomorphisms

$$(X_i \times_S X_j) \times_{\mathrm{pr}_0, X_i} V_i \longrightarrow (X_i \times_S X_j) \times_{\mathrm{pr}_1, X_j} V_j$$

of schemes over $X_i \times_S X_j$. So loosely speaking one may think of φ_{ij} as an isomorphism $\mathrm{pr}_0^* V_i \to \mathrm{pr}_1^* V_j$ over $X_i \times_S X_j$. The cocycle condition then says that $\mathrm{pr}_{02}^* \varphi_{ik} = \mathrm{pr}_{12}^* \varphi_{jk} \circ \mathrm{pr}_{01}^* \varphi_{ij}$. In this way it is very similar to the case of a descent datum on quasi-coherent sheaves.

The reason we will usually work with the version of a family consisting of a single morphism is the following lemma.

023X **Lemma 34.5.** *Let S be a scheme. Let $\{X_i \to S\}_{i \in I}$ be a family of morphisms with target S. Set $X = \coprod_{i \in I} X_i$, and consider it as an S-scheme. There is a canonical equivalence of categories*

$$\begin{array}{c} \textit{category of descent data} \\ \textit{relative to the family } \{X_i \to S\}_{i \in I} \end{array} \longrightarrow \begin{array}{c} \textit{category of descent data} \\ \textit{relative to } X/S \end{array}$$

which maps (V_i, φ_{ij}) to (V, φ) with $V = \coprod_{i \in I} V_i$ and $\varphi = \coprod \varphi_{ij}$.

Proof. Observe that $X \times_S X = \coprod_{ij} X_i \times_S X_j$ and similarly for higher fibre products. Giving a morphism $V \to X$ is exactly the same as giving a family $V_i \to X_i$. And giving a descent datum φ is exactly the same as giving a family φ_{ij}. $\square$

023Y **Lemma 34.6.** *Pullback of descent data for schemes over schemes.*

(1) *Let*

$$\begin{array}{ccc} X' & \xrightarrow{f} & X \\ a' \downarrow & & \downarrow a \\ S' & \xrightarrow{h} & S \end{array}$$

be a commutative diagram of morphisms of schemes. The construction

$$(V \to X, \varphi) \longmapsto f^*(V \to X, \varphi) = (V' \to X', \varphi')$$

where $V' = X' \times_X V$ and where φ' is defined as the composition

$$V' \times_{S'} X' \mathbin{=\!=\!=} (X' \times_X V) \times_{S'} X' \mathbin{=\!=\!=} (X' \times_{S'} X') \times_{X \times_S X} (V \times_S X)$$

$$\downarrow {\scriptstyle id \times \varphi}$$

$$X' \times_{S'} V' \mathbin{=\!=\!=} X' \times_{S'} (X' \times_X V) \mathbin{=\!=\!=} (X' \times_{S'} X') \times_{X \times_S X} (X \times_S V)$$

defines a functor from the category of descent data relative to $X \to S$ to the category of descent data relative to $X' \to S'$.

(2) *Given two morphisms $f_i : X' \to X$, $i = 0, 1$ making the diagram commute the functors f_0^* and f_1^* are canonically isomorphic.*

Proof. We omit the proof of (1), but we remark that the morphism φ' is the morphism $(f \times f)^*\varphi$ in the notation introduced in Remark 34.2. For (2) we indicate which morphism $f_0^*V \to f_1^*V$ gives the functorial isomorphism. Namely, since f_0 and f_1 both fit into the commutative diagram we see there is a unique morphism $r : X' \to X \times_S X$ with $f_i = \mathrm{pr}_i \circ r$. Then we take

$$
\begin{aligned}
f_0^*V &= X' \times_{f_0, X} V \\
&= X' \times_{\mathrm{pr}_0 \circ r, X} V \\
&= X' \times_{r, X \times_S X} (X \times_S X) \times_{\mathrm{pr}_0, X} V \\
&\xrightarrow{\varphi} X' \times_{r, X \times_S X} (X \times_S X) \times_{\mathrm{pr}_1, X} V \\
&= X' \times_{\mathrm{pr}_1 \circ r, X} V \\
&= X' \times_{f_1, X} V \\
&= f_1^*V
\end{aligned}
$$

We omit the verification that this works. $\qquad\square$

02VR **Definition 34.7.** With $S, S', X, X', f, a, a', h$ as in Lemma 34.6 the functor

$$(V, \varphi) \longmapsto f^*(V, \varphi)$$

constructed in that lemma is called the *pullback functor* on descent data.

02VS **Lemma 34.8** (Pullback of descent data for schemes over families)**.** *Let* $\mathcal{U} = \{U_i \to S'\}_{i \in I}$ *and* $\mathcal{V} = \{V_j \to S\}_{j \in J}$ *be families of morphisms with fixed target. Let* $\alpha : I \to J$, $h : S' \to S$ *and* $g_i : U_i \to V_{\alpha(i)}$ *be a morphism of families of maps with fixed target, see Sites, Definition 8.1.*

(1) *Let* $(Y_j, \varphi_{jj'})$ *be a descent datum relative to the family* $\{V_j \to S'\}$*. The system*

$$\left(g_i^* Y_{\alpha(i)}, (g_i \times g_{i'})^* \varphi_{\alpha(i)\alpha(i')}\right)$$

(with notation as in Remark 34.4) is a descent datum relative to $\mathcal{V}$*.*

(2) *This construction defines a functor between descent data relative to* $\mathcal{U}$ *and descent data relative to* $\mathcal{V}$*.*

(3) *Given a second* $\alpha' : I \to J$, $h' : S' \to S$ *and* $g_i' : U_i \to V_{\alpha'(i)}$ *morphism of families of maps with fixed target, then if* $h = h'$ *the two resulting functors between descent data are canonically isomorphic.*

(4) *These functors agree, via Lemma 34.5, with the pullback functors constructed in Lemma 34.6.*

Proof. This follows from Lemma 34.6 via the correspondence of Lemma 34.5. $\quad\square$

02VT **Definition 34.9.** With $\mathcal{U} = \{U_i \to S'\}_{i \in I}$, $\mathcal{V} = \{V_j \to S\}_{j \in J}$, $\alpha : I \to J$, $h : S' \to S$, and $g_i : U_i \to V_{\alpha(i)}$ as in Lemma 34.8 the functor

$$(Y_j, \varphi_{jj'}) \longmapsto (g_i^* Y_{\alpha(i)}, (g_i \times g_{i'})^* \varphi_{\alpha(i)\alpha(i')})$$

constructed in that lemma is called the *pullback functor* on descent data.

If $\mathcal{U}$ and $\mathcal{V}$ have the same target S, and if $\mathcal{U}$ refines $\mathcal{V}$ (see Sites, Definition 8.1) but no explicit pair (α, g_i) is given, then we can still talk about the pullback functor since we have seen in Lemma 34.8 that the choice of the pair does not matter (up to a canonical isomorphism).

023Z **Definition 34.10.** Let S be a scheme. Let $f : X \to S$ be a morphism of schemes.

(1) Given a scheme U over S we have the *trivial descent datum* of U relative to id $: S \to S$, namely the identity morphism on U.

(2) By Lemma 34.6 we get a *canonical descent datum* on $X \times_S U$ relative to $X \to S$ by pulling back the trivial descent datum via f. We often denote $(X \times_S U, can)$ this descent datum.

(3) A descent datum (V, φ) relative to X/S is called *effective* if (V, φ) is isomorphic to the canonical descent datum $(X \times_S U, can)$ for some scheme U over S.

Thus being effective means there exists a scheme U over S and an isomorphism $\psi : V \to X \times_S U$ of X-schemes such that φ is equal to the composition

$$V \times_S X \xrightarrow{\psi \times \mathrm{id}_X} X \times_S U \times_S X = X \times_S X \times_S U \xrightarrow{\mathrm{id}_X \times \psi^{-1}} X \times_S V$$

02VU **Definition 34.11.** Let S be a scheme. Let $\{X_i \to S\}$ be a family of morphisms with target S.

(1) Given a scheme U over S we have a *canonical descent datum* on the family of schemes $X_i \times_S U$ by pulling back the trivial descent datum for U relative to $\{\mathrm{id} : S \to S\}$. We denote this descent datum $(X_i \times_S U, can)$.

(2) A descent datum (V_i, φ_{ij}) relative to $\{X_i \to S\}$ is called *effective* if there exists a scheme U over S such that (V_i, φ_{ij}) is isomorphic to $(X_i \times_S U, can)$.

35. Fully faithfulness of the pullback functors

02VV It turns out that the pullback functor between descent data for fpqc-coverings is fully faithful. In other words, morphisms of schemes satisfy fpqc descent. The goal of this section is to prove this. The reader is encouraged instead to prove this him/herself. The key is to use Lemma 13.7.

02VW **Lemma 35.1.** *A surjective and flat morphism is an epimorphism in the category of schemes.*

Proof. Suppose we have $h : X' \to X$ surjective and flat and $a, b : X \to Y$ morphisms such that $a \circ h = b \circ h$. As h is surjective we see that a and b agree on underlying topological spaces. Pick $x' \in X'$ and set $x = h(x')$ and $y = a(x) = b(x)$. Consider the local ring maps

$$a_x^\sharp, b_x^\sharp : \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$$

These become equal when composed with the flat local homomorphism $h_{x'}^\sharp : \mathcal{O}_{X,x} \to \mathcal{O}_{X',x'}$. Since a flat local homomorphism is faithfully flat (Algebra, Lemma 39.17) we conclude that $h_{x'}^\sharp$ is injective. Hence $a_x^\sharp = b_x^\sharp$ which implies $a = b$ as desired. $\square$

02VX **Lemma 35.2.** *Let $h : S' \to S$ be a surjective, flat morphism of schemes. The base change functor*

$$Sch/S \longrightarrow Sch/S', \quad X \longmapsto S' \times_S X$$

is faithful.

Proof. Let X_1, X_2 be schemes over S. Let $\alpha, \beta : X_2 \to X_1$ be morphisms over S. If α, β base change to the same morphism then we get a commutative diagram as

follows

$$X_2 \longleftarrow S' \times_S X_2 \longrightarrow X_2$$
$$\downarrow \alpha \qquad\qquad \downarrow \qquad\qquad \downarrow \beta$$
$$X_1 \longleftarrow S' \times_S X_1 \longrightarrow X_1$$

Hence it suffices to show that $S' \times_S X_2 \to X_2$ is an epimorphism. As the base change of a surjective and flat morphism it is surjective and flat (see Morphisms, Lemmas 9.4 and 25.8). Hence the lemma follows from Lemma 35.1. $\square$

0240   **Lemma 35.3.** *In the situation of Lemma 34.6 assume that $f : X' \to X$ is surjective and flat. Then the pullback functor is faithful.*

Proof. Let (V_i, φ_i), $i = 1, 2$ be descent data for $X \to S$. Let $\alpha, \beta : V_1 \to V_2$ be morphisms of descent data. Suppose that $f^*\alpha = f^*\beta$. Our task is to show that $\alpha = \beta$. Note that α, β are morphisms of schemes over X, and that $f^*\alpha$, $f^*\beta$ are simply the base changes of α, β to morphisms over X'. Hence the lemma follows from Lemma 35.2. $\square$

Here is the key lemma of this section.

0241   **Lemma 35.4.** *In the situation of Lemma 34.6 assume*
   (1) *$\{f : X' \to X\}$ is an fpqc covering (for example if f is surjective, flat, and quasi-compact), and*
   (2) *$S = S'$.*
Then the pullback functor is fully faithful.

Proof. Assumption (1) implies that f is surjective and flat. Hence the pullback functor is faithful by Lemma 35.3. Let (V, φ) and (W, ψ) be two descent data relative to $X \to S$. Set $(V', \varphi') = f^*(V, \varphi)$ and $(W', \psi') = f^*(W, \psi)$. Let $\alpha' : V' \to W'$ be a morphism of descent data for X' over S. We have to show there exists a morphism $\alpha : V \to W$ of descent data for X over S whose pullback is α'.

Recall that V' is the base change of V by f and that φ' is the base change of φ by $f \times f$ (see Remark 34.2). By assumption the diagram

$$V' \times_S X' \xrightarrow{\ \varphi'\ } X' \times_S V'$$
$$\alpha' \times \mathrm{id} \downarrow \qquad\qquad\qquad \downarrow \mathrm{id} \times \alpha'$$
$$W' \times_S X' \xrightarrow{\ \psi'\ } X' \times_S W'$$

commutes. We claim the two compositions

$$V' \times_V V' \xrightarrow{\ \mathrm{pr}_i\ } V' \xrightarrow{\ \alpha'\ } W' \longrightarrow W \ , \quad i = 0, 1$$

are the same. The reader is advised to prove this themselves rather than read the rest of this paragraph. (Please email if you find a nice clean argument.) Let v_0, v_1 be points of V' which map to the same point $v \in V$. Let $x_i \in X'$ be the image of v_i, and let x be the point of X which is the image of v in X. In other words, $v_i = (x_i, v)$ in $V' = X' \times_X V$. Write $\varphi(v, x) = (x, v')$ for some point v' of V. This is possible because φ is a morphism over $X \times_S X$. Denote $v_i' = (x_i, v')$ which is a point of V'. Then a calculation (using the definition of φ') shows that $\varphi'(v_i, x_j) = (x_i, v_j')$. Denote $w_i = \alpha'(v_i)$ and $w_i' = \alpha'(v_i')$. Now we may write $w_i = (x_i, u_i)$ for some

point u_i of W, and $w_i' = (x_i, u_i')$ for some point u_i' of W. The claim is equivalent to the assertion: $u_0 = u_1$. A formal calculation using the definition of ψ' (see Lemma 34.6) shows that the commutativity of the diagram displayed above says that

$$((x_i, x_j), \psi(u_i, x)) = ((x_i, x_j), (x, u_j'))$$

as points of $(X' \times_S X') \times_{X \times_S X} (X \times_S W)$ for all $i, j \in \{0, 1\}$. This shows that $\psi(u_0, x) = \psi(u_1, x)$ and hence $u_0 = u_1$ by taking ψ^{-1}. This proves the claim because the argument above was formal and we can take scheme points (in other words, we may take $(v_0, v_1) = \mathrm{id}_{V' \times_V V'}$).

At this point we can use Lemma 13.7. Namely, $\{V' \to V\}$ is a fpqc covering as the base change of the morphism $f : X' \to X$. Hence, by Lemma 13.7 the morphism $\alpha' : V' \to W' \to W$ factors through a unique morphism $\alpha : V \to W$ whose base change is necessarily α'. Finally, we see the diagram

$$
\begin{array}{ccc}
V \times_S X & \xrightarrow{\varphi} & X \times_S V \\
{\scriptstyle \alpha \times \mathrm{id}} \downarrow & & \downarrow {\scriptstyle \mathrm{id} \times \alpha} \\
W \times_S X & \xrightarrow{\psi} & X \times_S W
\end{array}
$$

commutes because its base change to $X' \times_S X'$ commutes and the morphism $X' \times_S X' \to X \times_S X$ is surjective and flat (use Lemma 35.2). Hence α is a morphism of descent data $(V, \varphi) \to (W, \psi)$ as desired. $\square$

The following two lemmas have been obsoleted by the improved exposition of the previous material. But they are still true!

0242 **Lemma 35.5.** *Let $X \to S$ be a morphism of schemes. Let $f : X \to X$ be a selfmap of X over S. In this case pullback by f is isomorphic to the identity functor on the category of descent data relative to $X \to S$.*

Proof. This is clear from Lemma 34.6 since it tells us that $f^* \cong \mathrm{id}^*$. $\square$

0243 **Lemma 35.6.** *Let $f : X' \to X$ be a morphism of schemes over a base scheme S. Assume there exists a morphism $g : X \to X'$ over S, for example if f has a section. Then the pullback functor of Lemma 34.6 defines an equivalence of categories between the category of descent data relative to X/S and X'/S.*

Proof. Let $g : X \to X'$ be a morphism over S. Lemma 35.5 above shows that the functors $f^* \circ g^* = (g \circ f)^*$ and $g^* \circ f^* = (f \circ g)^*$ are isomorphic to the respective identity functors as desired. $\square$

040J **Lemma 35.7.** *Let $f : X \to X'$ be a morphism of schemes over a base scheme S. Assume $X \to S$ is surjective and flat. Then the pullback functor of Lemma 34.6 is a faithful functor from the category of descent data relative to X'/S to the category of descent data relative to X/S.*

Proof. We may factor $X \to X'$ as $X \to X \times_S X' \to X'$. The first morphism has a section, hence induces an equivalence of categories of descent data by Lemma 35.6. The second morphism is surjective and flat, hence induces a faithful functor by Lemma 35.3. $\square$

040K   **Lemma 35.8.** *Let $f : X \to X'$ be a morphism of schemes over a base scheme S. Assume $\{X \to S\}$ is an fpqc covering (for example if f is surjective, flat and quasi-compact). Then the pullback functor of Lemma 34.6 is a fully faithful functor from the category of descent data relative to X'/S to the category of descent data relative to X/S.*

Proof. We may factor $X \to X'$ as $X \to X \times_S X' \to X'$. The first morphism has a section, hence induces an equivalence of categories of descent data by Lemma 35.6. The second morphism is an fpqc covering hence induces a fully faithful functor by Lemma 35.4. $\qquad\square$

02VZ   **Lemma 35.9.** *Let S be a scheme. Let $\mathcal{U} = \{U_i \to S\}_{i \in I}$, and $\mathcal{V} = \{V_j \to S\}_{j \in J}$, be families of morphisms with target S. Let $\alpha : I \to J$, $\mathrm{id} : S \to S$ and $g_i : U_i \to V_{\alpha(i)}$ be a morphism of families of maps with fixed target, see Sites, Definition 8.1. Assume that for each $j \in J$ the family $\{g_i : U_i \to V_j\}_{\alpha(i)=j}$ is an fpqc covering of V_j. Then the pullback functor*

$$\text{descent data relative to } \mathcal{V} \longrightarrow \text{descent data relative to } \mathcal{U}$$

of Lemma 34.8 is fully faithful.

Proof. Consider the morphism of schemes

$$g : X = \coprod\nolimits_{i \in I} U_i \longrightarrow Y = \coprod\nolimits_{j \in J} V_j$$

over S which on the ith component maps into the $\alpha(i)$th component via the morphism $g_{\alpha(i)}$. We claim that $\{g : X \to Y\}$ is an fpqc covering of schemes. Namely, by Topologies, Lemma 9.4 for each j the morphism $\{\coprod_{\alpha(i)=j} U_i \to V_j\}$ is an fpqc covering. Thus for every affine open $V \subset V_j$ (which we may think of as an affine open of Y) we can find finitely many affine opens $W_1, \ldots, W_n \subset \coprod_{\alpha(i)=j} U_i$ (which we may think of as affine opens of X) such that $V = \bigcup_{i=1,\ldots,n} g(W_i)$. This provides enough affine opens of Y which can be covered by finitely many affine opens of X so that Topologies, Lemma 9.2 part (3) applies, and the claim follows. Let us write $DD(X/S)$, resp. $DD(\mathcal{U})$ for the category of descent data with respect to X/S, resp. $\mathcal{U}$, and similarly for Y/S and $\mathcal{V}$. Consider the diagram

$$
\begin{array}{ccc}
DD(Y/S) & \longrightarrow & DD(X/S) \\
\uparrow{\scriptstyle\text{Lemma 34.5}} & & \uparrow{\scriptstyle\text{Lemma 34.5}} \\
DD(\mathcal{V}) & \longrightarrow & DD(\mathcal{U})
\end{array}
$$

This diagram is commutative, see the proof of Lemma 34.8. The vertical arrows are equivalences. Hence the lemma follows from Lemma 35.4 which shows the top horizontal arrow of the diagram is fully faithful. $\qquad\square$

The next lemma shows that, in order to check effectiveness, we may always Zariski refine the given family of morphisms with target S.

02VY   **Lemma 35.10.** *Let S be a scheme. Let $\mathcal{U} = \{U_i \to S\}_{i \in I}$, and $\mathcal{V} = \{V_j \to S\}_{j \in J}$, be families of morphisms with target S. Let $\alpha : I \to J$, $\mathrm{id} : S \to S$ and $g_i : U_i \to V_{\alpha(i)}$ be a morphism of families of maps with fixed target, see Sites, Definition 8.1. Assume that for each $j \in J$ the family $\{g_i : U_i \to V_j\}_{\alpha(i)=j}$ is a Zariski covering (see Topologies, Definition 3.1) of V_j. Then the pullback functor*

$$\text{descent data relative to } \mathcal{V} \longrightarrow \text{descent data relative to } \mathcal{U}$$

of Lemma 34.8 is an equivalence of categories. In particular, the category of schemes over S is equivalent to the category of descent data relative to any Zariski covering of S.

Proof. The functor is faithful and fully faithful by Lemma 35.9. Let us indicate how to prove that it is essentially surjective. Let $(X_i, \varphi_{ii'})$ be a descent datum relative to $\mathcal{U}$. Fix $j \in J$ and set $I_j = \{i \in I \mid \alpha(i) = j\}$. For $i, i' \in I_j$ note that there is a canonical morphism

$$c_{ii'} : U_i \times_{g_i, V_j, g_{i'}} U_{i'} \rightarrow U_i \times_S U_{i'}.$$

Hence we can pullback $\varphi_{ii'}$ by this morphism and set $\psi_{ii'} = c_{ii'}^* \varphi_{ii'}$ for $i, i' \in I_j$. In this way we obtain a descent datum $(X_i, \psi_{ii'})$ relative to the Zariski covering $\{g_i : U_i \rightarrow V_j\}_{i \in I_j}$. Note that $\psi_{ii'}$ is an isomorphism from the open $X_{i, U_i \times_{V_j} U_{i'}}$ of X_i to the corresponding open of $X_{i'}$. It follows from Schemes, Section 14 that we may glue $(X_i, \psi_{ii'})$ into a scheme Y_j over V_j. Moreover, the morphisms $\varphi_{ii'}$ for $i \in I_j$ and $i' \in I_{j'}$ glue to a morphism $\varphi_{jj'} : Y_j \times_S V_{j'} \rightarrow V_j \times_S Y_{j'}$ satisfying the cocycle condition (details omitted). Hence we obtain the desired descent datum $(Y_j, \varphi_{jj'})$ relative to $\mathcal{V}$. $\square$

02W0 **Lemma 35.11.** *Let S be a scheme. Let $\mathcal{U} = \{U_i \rightarrow S\}_{i \in I}$, and $\mathcal{V} = \{V_j \rightarrow S\}_{j \in J}$, be fpqc-coverings of S. If $\mathcal{U}$ is a refinement of $\mathcal{V}$, then the pullback functor*

descent data relative to $\mathcal{V} \longrightarrow$ descent data relative to $\mathcal{U}$

is fully faithful. In particular, the category of schemes over S is identified with a full subcategory of the category of descent data relative to any fpqc-covering of S.

Proof. Consider the fpqc-covering $\mathcal{W} = \{U_i \times_S V_j \rightarrow S\}_{(i,j) \in I \times J}$ of S. It is a refinement of both $\mathcal{U}$ and $\mathcal{V}$. Hence we have a 2-commutative diagram of functors and categories

$$\begin{array}{ccc} DD(\mathcal{V}) & \longrightarrow & DD(\mathcal{U}) \\ & \searrow \quad \swarrow & \\ & DD(\mathcal{W}) & \end{array}$$

Notation as in the proof of Lemma 35.9 and commutativity by Lemma 34.8 part (3). Hence clearly it suffices to prove the functors $DD(\mathcal{V}) \rightarrow DD(\mathcal{W})$ and $DD(\mathcal{U}) \rightarrow DD(\mathcal{W})$ are fully faithful. This follows from Lemma 35.9 as desired. $\square$

040L **Remark 35.12.** Lemma 35.11 says that morphisms of schemes satisfy fpqc descent. In other words, given a scheme S and schemes X, Y over S the functor

$$(Sch/S)^{opp} \longrightarrow Sets, \quad T \longmapsto \mathrm{Mor}_T(X_T, Y_T)$$

satisfies the sheaf condition for the fpqc topology. The simplest case of this is the following. Suppose that $T \rightarrow S$ is a surjective flat morphism of affines. Let $\psi_0 : X_T \rightarrow Y_T$ be a morphism of schemes over T which is compatible with the canonical descent data. Then there exists a unique morphism $\psi : X \rightarrow Y$ whose base change to T is ψ_0. In fact this special case follows in a straightforward manner from Lemma 35.4. And, in turn, that lemma is a formal consequence of the following two facts: (a) the base change functor by a faithfully flat morphism is faithful, see Lemma 35.2 and (b) a scheme satisfies the sheaf condition for the fpqc topology, see Lemma 13.7.

0AP4 **Lemma 35.13.** *Let $X \to S$ be a surjective, quasi-compact, flat morphism of schemes. Let (V, φ) be a descent datum relative to X/S. Suppose that for all $v \in V$ there exists an open subscheme $v \in W \subset V$ such that $\varphi(W \times_S X) \subset X \times_S W$ and such that the descent datum $(W, \varphi|_{W \times_S X})$ is effective. Then (V, φ) is effective.*

Proof. Let $V = \bigcup W_i$ be an open covering with $\varphi(W_i \times_S X) \subset X \times_S W_i$ and such that the descent datum $(W_i, \varphi|_{W_i \times_S X})$ is effective. Let $U_i \to S$ be a scheme and let $\alpha_i : (X \times_S U_i, can) \to (W_i, \varphi|_{W_i \times_S X})$ be an isomorphism of descent data. For each pair of indices (i, j) consider the open $\alpha_i^{-1}(W_i \cap W_j) \subset X \times_S U_i$. Because everything is compatible with descent data and since $\{X \to S\}$ is an fpqc covering, we may apply Lemma 13.6 to find an open $U_{ij} \subset U_i$ such that $\alpha_i^{-1}(W_i \cap W_j) = X \times_S U_{ij}$. Now the identity morphism on $W_i \cap W_j$ is compatible with descent data, hence comes from a unique morphism $\varphi_{ij} : U_{ij} \to U_{ji}$ over S (see Remark 35.12). Then $(U_i, U_{ij}, \varphi_{ij})$ is a glueing data as in Schemes, Section 14 (proof omitted). Thus we may assume there is a scheme U over S such that $U_i \subset U$ is open, $U_{ij} = U_i \cap U_j$ and $\varphi_{ij} = \mathrm{id}_{U_i \cap U_j}$, see Schemes, Lemma 14.1. Pulling back to X we can use the α_i to get the desired isomorphism $\alpha : X \times_S U \to V$. $\square$

36. Descending types of morphisms

02W1 In the following we study the question as to whether descent data for schemes relative to a fpqc-covering are effective. The first remark to make is that this is not always the case. We will see this in Algebraic Spaces, Example 14.2. Even projective morphisms do not always satisfy descent for fpqc-coverings, by Examples, Lemma 66.1.

On the other hand, if the schemes we are trying to descend are particularly simple, then it is sometime the case that for whole classes of schemes descent data are effective. We will introduce terminology here that describes this phenomenon abstractly, even though it may lead to confusion if not used correctly later on.

02W2 **Definition 36.1.** Let $\mathcal{P}$ be a property of morphisms of schemes over a base. Let $\tau \in \{Zariski, fpqc, fppf, \acute{e}tale, smooth, syntomic\}$. We say *morphisms of type $\mathcal{P}$ satisfy descent for τ-coverings* if for any τ-covering $\mathcal{U} : \{U_i \to S\}_{i \in I}$ (see Topologies, Section 2), any descent datum (X_i, φ_{ij}) relative to $\mathcal{U}$ such that each morphism $X_i \to U_i$ has property $\mathcal{P}$ is effective.

Note that in each of the cases we have already seen that the functor from schemes over S to descent data over $\mathcal{U}$ is fully faithful (Lemma 35.11 combined with the results in Topologies that any τ-covering is also a fpqc-covering). We have also seen that descent data are always effective with respect to Zariski coverings (Lemma 35.10). It may be prudent to only study the notion just introduced when $\mathcal{P}$ is either stable under any base change or at least local on the base in the τ-topology (see Definition 22.1) in order to avoid erroneous arguments (relying on $\mathcal{P}$ when descending halfway).

Here is the obligatory lemma reducing this question to the case of a covering given by a single morphism of affines.

02W3 **Lemma 36.2.** *Let $\mathcal{P}$ be a property of morphisms of schemes over a base. Let $\tau \in \{fpqc, fppf, \acute{e}tale, smooth, syntomic\}$. Suppose that*

    (1) *$\mathcal{P}$ is stable under any base change (see Schemes, Definition 18.3),*

(2) *if $Y_j \to V_j$, $j = 1, \ldots, m$ have $\mathcal{P}$, then so does $\coprod Y_j \to \coprod V_j$, and*
(3) *for any surjective morphism of affines $X \to S$ which is flat, flat of finite presentation, étale, smooth or syntomic depending on whether τ is fpqc, fppf, étale, smooth, or syntomic, any descent datum (V, φ) relative to X over S such that $\mathcal{P}$ holds for $V \to X$ is effective.*

Then morphisms of type $\mathcal{P}$ satisfy descent for τ-coverings.

Proof. Let S be a scheme. Let $\mathcal{U} = \{\varphi_i : U_i \to S\}_{i \in I}$ be a τ-covering of S. Let $(X_i, \varphi_{ii'})$ be a descent datum relative to $\mathcal{U}$ and assume that each morphism $X_i \to U_i$ has property $\mathcal{P}$. We have to show there exists a scheme $X \to S$ such that $(X_i, \varphi_{ii'}) \cong (U_i \times_S X, can)$.

Before we start the proof proper we remark that for any family of morphisms $\mathcal{V} : \{V_j \to S\}$ and any morphism of families $\mathcal{V} \to \mathcal{U}$, if we pullback the descent datum $(X_i, \varphi_{ii'})$ to a descent datum $(Y_j, \varphi_{jj'})$ over $\mathcal{V}$, then each of the morphisms $Y_j \to V_j$ has property $\mathcal{P}$ also. This is true because of assumption (1) that $\mathcal{P}$ is stable under any base change and the definition of pullback (see Definition 34.9). We will use this without further mention.

First, let us prove the lemma when S is affine. By Topologies, Lemma 9.9, 7.4, 4.4, 5.4, or 6.4 there exists a standard τ-covering $\mathcal{V} : \{V_j \to S\}_{j=1,\ldots,m}$ which refines $\mathcal{U}$. The pullback functor $DD(\mathcal{U}) \to DD(\mathcal{V})$ between categories of descent data is fully faithful by Lemma 35.11. Hence it suffices to prove that the descent datum over the standard τ-covering $\mathcal{V}$ is effective. By assumption (2) we see that $\coprod Y_j \to \coprod V_j$ has property $\mathcal{P}$. By Lemma 34.5 this reduces us to the covering $\{\coprod_{j=1,\ldots,m} V_j \to S\}$ for which we have assumed the result in assumption (3) of the lemma. Hence the lemma holds when S is affine.

Assume S is general. Let $V \subset S$ be an affine open. By the properties of site the family $\mathcal{U}_V = \{V \times_S U_i \to V\}_{i \in I}$ is a τ-covering of V. Denote $(X_i, \varphi_{ii'})_V$ the restriction (or pullback) of the given descent datum to $\mathcal{U}_V$. Hence by what we just saw we obtain a scheme X_V over V whose canonical descent datum with respect to $\mathcal{U}_V$ is isomorphic to $(X_i, \varphi_{ii'})_V$. Suppose that $V' \subset V$ is an affine open of V. Then both $X_{V'}$ and $V' \times_V X_V$ have canonical descent data isomorphic to $(X_i, \varphi_{ii'})_{V'}$. Hence, by Lemma 35.11 again we obtain a canonical morphism $\rho^V_{V'} : X_{V'} \to X_V$ over S which identifies $X_{V'}$ with the inverse image of V' in X_V. We omit the verification that given affine opens $V'' \subset V' \subset V$ of S we have $\rho^V_{V''} = \rho^V_{V'} \circ \rho^{V'}_{V''}$.

By Constructions, Lemma 2.1 the data $(X_V, \rho^V_{V'})$ glue to a scheme $X \to S$. Moreover, we are given isomorphisms $V \times_S X \to X_V$ which recover the maps $\rho^V_{V'}$. Unwinding the construction of the schemes X_V we obtain isomorphisms

$$V \times_S U_i \times_S X \longrightarrow V \times_S X_i$$

compatible with the maps $\varphi_{ii'}$ and compatible with restricting to smaller affine opens in X. This implies that the canonical descent datum on $U_i \times_S X$ is isomorphic to the given descent datum and we win. $\square$
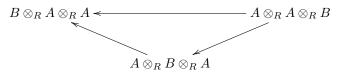
37. Descending affine morphisms

0244   In this section we show that "affine morphisms satisfy descent for fpqc-coverings". Here is the formal statement.

0245 **Lemma 37.1.** *Let S be a scheme. Let $\{X_i \to S\}_{i \in I}$ be an fpqc covering, see Topologies, Definition 9.1. Let $(V_i/X_i, \varphi_{ij})$ be a descent datum relative to $\{X_i \to S\}$. If each morphism $V_i \to X_i$ is affine, then the descent datum is effective.*

Proof. Being affine is a property of morphisms of schemes which is local on the base and preserved under any base change, see Morphisms, Lemmas 11.3 and 11.9. Hence Lemma 36.2 applies and it suffices to prove the statement of the lemma in case the fpqc-covering is given by a single $\{X \to S\}$ flat surjective morphism of affines. Say $X = \mathrm{Spec}(A)$ and $S = \mathrm{Spec}(R)$ so that $R \to A$ is a faithfully flat ring map. Let (V, φ) be a descent datum relative to X over S and assume that $V \to X$ is affine. Then $V \to X$ being affine implies that $V = \mathrm{Spec}(B)$ for some A-algebra B (see Morphisms, Definition 11.1). The isomorphism φ corresponds to an isomorphism of rings

$$\varphi^{\sharp} : B \otimes_R A \longleftarrow A \otimes_R B$$

as $A \otimes_R A$-algebras. The cocycle condition on φ says that

$$
\begin{array}{ccc}
B \otimes_R A \otimes_R A & \longleftarrow & A \otimes_R A \otimes_R B \\
& \searrow & \downarrow \\
& A \otimes_R B \otimes_R A &
\end{array}
$$

is commutative. Inverting these arrows we see that we have a descent datum for modules with respect to $R \to A$ as in Definition 3.1. Hence we may apply Proposition 3.9 to obtain an R-module $C = \mathrm{Ker}(B \to A \otimes_R B)$ and an isomorphism $A \otimes_R C \cong B$ respecting descent data. Given any pair $c, c' \in C$ the product cc' in B lies in C since the map φ is an algebra homomorphism. Hence C is an R-algebra whose base change to A is isomorphic to B compatibly with descent data. Applying Spec we obtain a scheme U over S such that $(V, \varphi) \cong (X \times_S U, can)$ as desired. $\square$

03I0 **Lemma 37.2.** *Let S be a scheme. Let $\{X_i \to S\}_{i \in I}$ be an fpqc covering, see Topologies, Definition 9.1. Let $(V_i/X_i, \varphi_{ij})$ be a descent datum relative to $\{X_i \to S\}$. If each morphism $V_i \to X_i$ is a closed immersion, then the descent datum is effective.*

Proof. This is true because a closed immersion is an affine morphism (Morphisms, Lemma 11.10), and hence Lemma 37.1 applies. $\square$

38. Descending quasi-affine morphisms

0246 In this section we show that "quasi-affine morphisms satisfy descent for fpqc-coverings". Here is the formal statement.

0247 **Lemma 38.1.** *Let S be a scheme. Let $\{X_i \to S\}_{i \in I}$ be an fpqc covering, see Topologies, Definition 9.1. Let $(V_i/X_i, \varphi_{ij})$ be a descent datum relative to $\{X_i \to S\}$. If each morphism $V_i \to X_i$ is quasi-affine, then the descent datum is effective.*

Proof. Being quasi-affine is a property of morphisms of schemes which is preserved under any base change, see Morphisms, Lemmas 13.3 and 13.5. Hence Lemma 36.2 applies and it suffices to prove the statement of the lemma in case the fpqc-covering is given by a single $\{X \to S\}$ flat surjective morphism of affines. Say $X = \mathrm{Spec}(A)$ and $S = \mathrm{Spec}(R)$ so that $R \to A$ is a faithfully flat ring map. Let (V, φ) be a descent datum relative to X over S and assume that $\pi : V \to X$ is quasi-affine.

According to Morphisms, Lemma 13.3 this means that

$$V \longrightarrow \underline{\mathrm{Spec}}_X(\pi_*\mathcal{O}_V) = W$$

is a quasi-compact open immersion of schemes over X. The projections $\mathrm{pr}_i : X \times_S X \to X$ are flat and hence we have

$$\mathrm{pr}_0^*\pi_*\mathcal{O}_V = (\pi \times \mathrm{id}_X)_*\mathcal{O}_{V \times_S X}, \quad \mathrm{pr}_1^*\pi_*\mathcal{O}_V = (\mathrm{id}_X \times \pi)_*\mathcal{O}_{X \times_S V}$$

by flat base change (Cohomology of Schemes, Lemma 5.2). Thus the isomorphism $\varphi : V \times_S X \to X \times_S V$ (which is an isomorphism over $X \times_S X$) induces an isomorphism of quasi-coherent sheaves of algebras

$$\varphi^\sharp : \mathrm{pr}_0^*\pi_*\mathcal{O}_V \longrightarrow \mathrm{pr}_1^*\pi_*\mathcal{O}_V$$

on $X \times_S X$. The cocycle condition for φ implies the cocycle condition for $\varphi^\sharp$. Another way to say this is that it produces a descent datum φ' on the affine scheme W relative to X over S, which moreover has the property that the morphism $V \to W$ is a morphism of descent data. Hence by Lemma 37.1 (or by effectivity of descent for quasi-coherent algebras) we obtain a scheme $U' \to S$ with an isomorphism $(W, \varphi') \cong (X \times_S U', can)$ of descent data. We note in passing that U' is affine by Lemma 23.20.

And now we can think of V as a (quasi-compact) open $V \subset X \times_S U'$ with the property that it is stable under the descent datum

$$can : X \times_S U' \times_S X \to X \times_S X \times_S U', (x_0, u', x_1) \mapsto (x_0, x_1, u').$$

In other words $(x_0, u') \in V \Rightarrow (x_1, u') \in V$ for any x_0, x_1, u' mapping to the same point of S. Because $X \to S$ is surjective we immediately find that V is the inverse image of a subset $U \subset U'$ under the morphism $X \times_S U' \to U'$. Because $X \to S$ is quasi-compact, flat and surjective also $X \times_S U' \to U'$ is quasi-compact flat and surjective. Hence by Morphisms, Lemma 25.12 this subset $U \subset U'$ is open and we win. $\square$

39. Descent data in terms of sheaves

02W4   Here is another way to think about descent data in case of a covering on a site.

02W5   **Lemma 39.1.** *Let* $\tau \in \{Zariski, fppf, \acute{e}tale, smooth, syntomic\}$[10]. *Let* Sch_τ *be a big* τ-*site. Let* $S \in \mathrm{Ob}(Sch_\tau)$. *Let* $\{S_i \to S\}_{i \in I}$ *be a covering in the site* $(Sch/S)_\tau$. *There is an equivalence of categories*

$$\left\{ \begin{array}{c} descent\ data\ (X_i, \varphi_{ii'})\ such\ that \\ each\ X_i \in \mathrm{Ob}((Sch/S)_\tau) \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} sheaves\ F\ on\ (Sch/S)_\tau\ such\ that \\ each\ h_{S_i} \times F\ is\ representable \end{array} \right\}.$$

Moreover,

(1) *the objects representing* $h_{S_i} \times F$ *on the right hand side correspond to the schemes* X_i *on the left hand side, and*

(2) *the sheaf* F *is representable if and only if the corresponding descent datum* $(X_i, \varphi_{ii'})$ *is effective.*

---

[10]The fact that fpqc is missing is not a typo. See discussion in Topologies, Section 9.

Proof. We have seen in Section 13 that representable presheaves are sheaves on the site $(Sch/S)_\tau$. Moreover, the Yoneda lemma (Categories, Lemma 3.5) guarantees that maps between representable sheaves correspond one to one with maps between the representing objects. We will use these remarks without further mention during the proof.

Let us construct the functor from right to left. Let F be a sheaf on $(Sch/S)_\tau$ such that each $h_{S_i} \times F$ is representable. In this case let X_i be a representing object in $(Sch/S)_\tau$. It comes equipped with a morphism $X_i \to S_i$. Then both $X_i \times_S S_{i'}$ and $S_i \times_S X_{i'}$ represent the sheaf $h_{S_i} \times F \times h_{S_{i'}}$ and hence we obtain an isomorphism

$$\varphi_{ii'} : X_i \times_S S_{i'} \to S_i \times_S X_{i'}$$

It is straightforward to see that the maps $\varphi_{ii'}$ are morphisms over $S_i \times_S S_{i'}$ and satisfy the cocycle condition. The functor from right to left is given by this construction $F \mapsto (X_i, \varphi_{ii'})$.

Let us construct a functor from left to right. For each i denote F_i the sheaf h_{X_i}. The isomorphisms $\varphi_{ii'}$ give isomorphisms

$$\varphi_{ii'} : F_i \times h_{S_{i'}} \longrightarrow h_{S_i} \times F_{i'}$$

over $h_{S_i} \times h_{S_{i'}}$. Set F equal to the coequalizer in the following diagram

$$\coprod_{i,i'} F_i \times h_{S_{i'}} \underset{\mathrm{pr}_1 \circ \varphi_{ii'}}{\overset{\mathrm{pr}_0}{\rightrightarrows}} \coprod_i F_i \longrightarrow F$$

The cocycle condition guarantees that $h_{S_i} \times F$ is isomorphic to F_i and hence representable. The functor from left to right is given by this construction $(X_i, \varphi_{ii'}) \mapsto F$.

We omit the verification that these constructions are mutually quasi-inverse functors. The final statements (1) and (2) follow from the constructions. $\square$

02W6   **Remark 39.2.** In the statement of Lemma 39.1 the condition that $h_{S_i} \times F$ is representable is equivalent to the condition that the restriction of F to $(Sch/S_i)_\tau$ is representable.

40. Other chapters

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    (3) Set Theory
    (4) Categories
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    (6) Sheaves on Spaces
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Schemes

    (26) Schemes
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