

DESIRABLES

02B4

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1. Introduction

02B5 This is basically just a list of things that we want to put in the stacks project. As we add material to the Stacks project continuously this is always somewhat behind the current state of the Stacks project. In fact, it may have been a mistake to try and list things we should add, because it seems impossible to keep it up to date.

Last updated: Wednesday, September 21, 2016.

2. Conventions

02B7 We should have a chapter with a short list of conventions used in the document. This chapter already exists, see Conventions, Section 1, but a lot more could be added there. Especially useful would be to find “hidden” conventions and tacit assumptions and put those there.

This is a chapter of the Stacks Project, version f2cc092b, compiled on Jun 21, 2017.

3. Sites and Topoi

- 02BA We have a chapter on sites and sheaves, see Sites, Section 1. We have a chapter on ringed sites (and topoi) and modules on them, see Modules on Sites, Section 1. We have a chapter on cohomology in this setting, see Cohomology on Sites, Section 1. But a lot more could be added, especially in the chapter on cohomology.

4. Stacks

- 02BB We have a chapter on (abstract) stacks, see Stacks, Section 1. It would be nice if
- (1) improve the discussion on “stackyfication”,
 - (2) give examples of stackyfication,
 - (3) more examples in general,
 - (4) improve the discussion of gerbes.

Example result which has not been added yet: Given a sheaf of abelian groups \mathcal{F} over \mathcal{C} the set of equivalence classes of gerbes banded by \mathcal{F} is bijective to $H^2(\mathcal{C}, \mathcal{F})$.

5. Simplicial methods

- 03MZ We have a chapter on simplicial methods, see Simplicial, Section 1. This has to be reviewed and improved. The discussion of the relationship between simplicial homotopy (also known as combinatorial homotopy) and Kan complexes should be improved upon. There is a chapter on simplicial spaces, see Simplicial Spaces, Section 1. This chapter briefly discusses simplicial topological spaces, simplicial sites, and simplicial topoi. We can further develop “simplicial algebraic geometry” to discuss simplicial schemes (or simplicial algebraic spaces, or simplicial algebraic stacks) and treat geometric questions, their cohomology, etc.

6. Cohomology of schemes

- 02BE There is already a chapter on cohomology of quasi-coherent sheaves, see Cohomology of Schemes, Section 1. We have a chapter discussing the derived category of quasi-coherent sheaves on a scheme, see Derived Categories of Schemes, Section 1. We have a chapter discussing coherent duality for proper morphisms of Noetherian schemes, see Dualizing Complexes, Section 1. We also have chapters on étale cohomology of schemes and on crystalline cohomology of schemes. But most of the material in these chapters is very basic and a lot more could/should be added there.

7. Deformation theory à la Schlessinger

- 02BF We have a chapter on this material, see Formal Deformation Theory, Section 1. What is needed is worked out examples of the general theory, for example the case of representations of a fixed abstract group. We have a chapter, see Deformation Theory, Section 1 which discusses deformations of rings (and modules), deformations of ringed spaces (and sheaves of modules), deformations of ringed topoi (and sheaves of modules). In this chapter we use the naive cotangent complex to describe obstructions, first order deformations, and infinitesimal automorphisms. What is needed is more direct applications of this material. There is also a chapter discussing the full cotangent complex, see Cotangent, Section 1.

8. Definition of algebraic stacks

02BK An algebraic stack is a stack in groupoids over the category of schemes with the fppf topology that has a diagonal representable by algebraic spaces and is the target of a surjective smooth morphism from a scheme. The notion “Deligne-Mumford stack” will be reserved for a stack as in [DM69]. We will reserve the term “Artin stack” for a stack such as in the papers by Artin [Art69], and [Art74]. (See also [CdJ02].) In other words, an Artin stack will be an algebraic stack with some reasonable finiteness and separatedness conditions.

9. Examples of schemes, algebraic spaces, algebraic stacks

02BL It really is not that hard to show that \mathcal{M}_g is an algebraic stack for $g \geq 2$. We should really have a long list of moduli problems here and prove they are all algebraic stacks. Some of them we can prove are algebraic using Artin approximation. For example the Kontsevich moduli space in characteristic $p > 0$.

Here are some items for the list of moduli problems mentioned above.

- (1) \mathcal{M}_g , i.e., moduli of smooth projective curves of genus g ,
- (2) $\overline{\mathcal{M}}_g$, i.e., moduli of stable genus g curves,
- (3) \mathcal{A}_g , i.e., principally polarized abelian schemes of genus g ,
- (4) $\mathcal{M}_{1,1}$, i.e., 1-pointed smooth projective genus 1 curves,
- (5) $\mathcal{M}_{g,n}$, i.e., smooth projective genus g -curves with n pairwise distinct labeled points,
- (6) $\overline{\mathcal{M}}_{g,n}$, i.e., stable n -pointed nodal projective genus g -curves,
- (7) $\mathcal{H}om_S(\mathcal{X}, \mathcal{Y})$, moduli of morphisms (with suitable conditions on the stacks \mathcal{X}, \mathcal{Y} and the base scheme S),
- (8) $Bun_G(X) = \mathcal{H}om_S(X, BG)$, the stack of G -bundles of the geometric Langlands programme (with suitable conditions on the scheme X , the group scheme G , and the base scheme S),
- (9) $\mathcal{P}ic_{\mathcal{X}/S}$, i.e., the Picard stack associated to an algebraic stack over a base scheme (or space).

How about the algebraic space you get from the deformation theory of a general surface in \mathbf{P}^3 with a node? (I mean where you deform it to a general smooth surface in \mathbf{P}^3 .) Perhaps we can talk about some small dimensional examples here too. For example the stack where you have \mathbf{A}^1 with a $B(\mathbf{Z}/2)$ sitting at 0. Bugeyed covers. And so on.

10. Properties of algebraic stacks

02BM This is perhaps one of the easier projects to work on, as most of the basic theory is there now. An interesting project is discussing the various ways of defining what a proper algebraic stack is. Of course these things are really properties of morphisms of stacks. We can define singularities (up to smooth factors) etc. Prove that a connected normal stack is irreducible, etc.

11. Lisse étale site of an algebraic stack

02BN This has been introduced in Cohomology of Stacks, Section 11. An example to show that it is not functorial with respect to 1-morphisms of algebraic stacks is discussed in Examples, Section 51. Of course a lot more could be said about this,

but it turns out to be very useful to prove things using the “big” étale site as much as possible.

12. Things you always wanted to know but were afraid to ask

02BO There are going to be lots of lemmas that you use over and over again that are useful but aren’t really mentioned specifically in the literature, or it isn’t easy to find references for. Bag of tricks.

Example: Given two groupoids in schemes $R \rightrightarrows U$ and $R' \rightrightarrows U'$ what does it mean to have a 1-morphism $[U/R] \rightarrow [U'/R']$ purely in terms of groupoids in schemes.

13. Quasi-coherent sheaves on stacks

02BP These are defined and discussed in the chapter Cohomology of Stacks, Section 1. Derived categories of modules are discussed in the chapter Derived Categories of Stacks, Section 1. A lot more could be added to these chapters.

14. Flat and smooth

02BR Artin’s theorem that having a flat surjection from a scheme is a replacement for the smooth surjective condition. This is now available as Criteria for Representability, Theorem 16.1.

15. Artin’s representability theorem

02BS This is discussed in the chapter Artin’s Axioms, Section 1. We also have an application, see Quot, Theorem 5.12. There should be a lot more applications and the chapter itself has to be cleaned up as well.

16. DM stacks are finitely covered by schemes

02BT We already have the corresponding result for algebraic spaces, see Limits of Spaces, Section 16. For DM stacks, there is Gabber’s lemma. Somewhere in Asterisque about Faltings proof of Mordell?

17. Martin Olsson’s paper on properness

02BU This proves two notions of proper are the same. The first part of this is now available in the form of Chow’s lemma for algebraic stacks, see More on Morphisms of Stacks, Theorem 10.3. As a consequence we show that it suffices to use DVR’s in checking the valuative criterion for properness for algebraic stacks in certain cases, see More on Morphisms of Stacks, Section 11.

18. Proper pushforward of coherent sheaves

02BV We can start working on this now that we have Chow’s lemma for algebraic stacks, see previous section.

19. Keel and Mori

02BW See [KM97]. This material has been incorporated throughout the Stacks project. See for example More on Groupoids, Section 13 and More on Groupoids in Spaces, Section 15.

20. Add more here

02BX Actually, no we should never have started this list as part of the Stacks project itself!

21. Other chapters

Preliminaries

- (1) Introduction
- (2) Conventions
- (3) Set Theory
- (4) Categories
- (5) Topology
- (6) Sheaves on Spaces
- (7) Sites and Sheaves
- (8) Stacks
- (9) Fields
- (10) Commutative Algebra
- (11) Brauer Groups
- (12) Homological Algebra
- (13) Derived Categories
- (14) Simplicial Methods
- (15) More on Algebra
- (16) Smoothing Ring Maps
- (17) Sheaves of Modules
- (18) Modules on Sites
- (19) Injectives
- (20) Cohomology of Sheaves
- (21) Cohomology on Sites
- (22) Differential Graded Algebra
- (23) Divided Power Algebra
- (24) Hypercoverings

Schemes

- (25) Schemes
- (26) Constructions of Schemes
- (27) Properties of Schemes
- (28) Morphisms of Schemes
- (29) Cohomology of Schemes
- (30) Divisors
- (31) Limits of Schemes
- (32) Varieties
- (33) Topologies on Schemes
- (34) Descent
- (35) Derived Categories of Schemes
- (36) More on Morphisms
- (37) More on Flatness
- (38) Groupoid Schemes
- (39) More on Groupoid Schemes
- (40) Étale Morphisms of Schemes

Topics in Scheme Theory

- (41) Chow Homology
- (42) Intersection Theory
- (43) Picard Schemes of Curves
- (44) Adequate Modules
- (45) Dualizing Complexes
- (46) Duality for Schemes
- (47) Discriminants and Differents
- (48) Local Cohomology
- (49) Algebraic Curves
- (50) Resolution of Surfaces
- (51) Semistable Reduction
- (52) Fundamental Groups of Schemes
- (53) Étale Cohomology
- (54) Crystalline Cohomology
- (55) Pro-étale Cohomology

Algebraic Spaces

- (56) Algebraic Spaces
- (57) Properties of Algebraic Spaces
- (58) Morphisms of Algebraic Spaces
- (59) Decent Algebraic Spaces
- (60) Cohomology of Algebraic Spaces
- (61) Limits of Algebraic Spaces
- (62) Divisors on Algebraic Spaces
- (63) Algebraic Spaces over Fields
- (64) Topologies on Algebraic Spaces
- (65) Descent and Algebraic Spaces
- (66) Derived Categories of Spaces
- (67) More on Morphisms of Spaces
- (68) Flatness on Algebraic Spaces
- (69) Groupoids in Algebraic Spaces
- (70) More on Groupoids in Spaces
- (71) Bootstrap
- (72) Pushouts of Algebraic Spaces

Topics in Geometry

- (73) Quotients of Groupoids
- (74) More on Cohomology of Spaces
- (75) Simplicial Spaces
- (76) Formal Algebraic Spaces
- (77) Restricted Power Series
- (78) Resolution of Surfaces Revisited

Deformation Theory	(94) Introducing Algebraic Stacks
(79) Formal Deformation Theory	(95) More on Morphisms of Stacks
(80) Deformation Theory	(96) The Geometry of Stacks
(81) The Cotangent Complex	Topics in Moduli Theory
(82) Deformation Problems	(97) Moduli Stacks
Algebraic Stacks	(98) Moduli of Curves
(83) Algebraic Stacks	Miscellany
(84) Examples of Stacks	(99) Examples
(85) Sheaves on Algebraic Stacks	(100) Exercises
(86) Criteria for Representability	(101) Guide to Literature
(87) Artin's Axioms	(102) Desirables
(88) Quot and Hilbert Spaces	(103) Coding Style
(89) Properties of Algebraic Stacks	(104) Obsolete
(90) Morphisms of Algebraic Stacks	(105) GNU Free Documentation Li-
(91) Limits of Algebraic Stacks	cense
(92) Cohomology of Algebraic Stacks	(106) Auto Generated Index
(93) Derived Categories of Stacks	

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