1. Introduction

A reference is the book [Har66].

The goals of this chapter are the following:

1. Define what it means to have a dualizing complex $\omega^\bullet_A$ over a Noetherian ring $A$, namely
   (a) we have $\omega^\bullet_A \in D^+(A)$,
   (b) the cohomology modules $H^i(\omega^\bullet_A)$ are all finite $A$-modules,
   (c) $\omega^\bullet_A$ has finite injective dimension, and
   (d) we have $A \to R\text{Hom}_A(\omega^\bullet_A, \omega^\bullet_A)$ is a quasi-isomorphism.

2. List elementary properties of dualizing complexes.

3. Show a dualizing complex gives rise to a dimension function.

4. Show a dualizing complex gives rise to a good notion of a reflexive hull.

5. Prove the finiteness theorem when a dualizing complex exists.

2. Essential surjections and injections

We will mostly work in categories of modules, but we may as well make the definition in general.

**Definition 2.1.** Let $\mathcal{A}$ be an abelian category.

1. An injection $A \subset B$ of $\mathcal{A}$ is essential, or we say that $B$ is an essential extension of $A$, if every nonzero subobject $B' \subset B$ has nonzero intersection with $A$.

2. A surjection $f : A \to B$ of $\mathcal{A}$ is essential if for every proper subobject $A' \subset A$ we have $f(A') \neq B$.

Some lemmas about this notion.

**Lemma 2.2.** Let $A$ be an abelian category.

1. If $A \subset B$ and $B \subset C$ are essential extensions, then $A \subset C$ is an essential extension.

2. If $A \subset B$ is an essential extension and $C \subset B$ is a subobject, then $A \cap C \subset C$ is an essential extension.

3. If $A \to B$ and $B \to C$ are essential surjections, then $A \to C$ is an essential surjection.

4. Given an essential surjection $f : A \to B$ and a surjection $A \to C$ with kernel $K$, the morphism $C \to B/f(K)$ is an essential surjection.

**Proof.** Omitted. \(\square\)

**Lemma 2.3.** Let $R$ be a ring. Let $M$ be an $R$-module. Let $E = \text{colim} E_i$ be a filtered colimit of $R$-modules. Suppose given a compatible system of essential injections $M \to E_i$ of $R$-modules. Then $M \to E$ is an essential extension of $M$.

**Proof.** Immediate from the definitions and the fact that filtered colimits are exact (Algebra, Lemma 8.9). \(\square\)

**Lemma 2.4.** Let $R$ be a ring. Let $M \subset N$ be $R$-modules. The following are equivalent

1. $M \subset N$ is an essential extension,

2. for all $x \in N$ there exists an $f \in R$ such that $fx \in M$ and $fx \neq 0$.
Proof. Assume (1) and let \( x \in N \) be a nonzero element. By (1) we have \( Rx \cap M \neq 0 \). This implies (2).

Assume (2). Let \( N' \subset N \) be a nonzero submodule. Pick \( x \in N' \) nonzero. By (2) we can find \( f \in M \) with \( fx \in N \) and \( fx \neq 0 \). Thus \( N' \cap M \neq 0 \).

3. Injective modules

Some results about injective modules over rings.

**Lemma 3.1.** Let \( R \) be a ring. Any product of injective \( R \)-modules is injective.

**Proof.** Special case of Homology, Lemma 23.3. \( \square \)

**Lemma 3.2.** Let \( R \to S \) be a flat ring map. If \( E \) is an injective \( S \)-module, then \( E \) is injective as an \( R \)-module.

**Proof.** This is true because \( \text{Hom}_R(M, E) = \text{Hom}_S(M \otimes_R S, E) \) by Algebra, Lemma 13.3 and the fact that tensoring with \( S \) is exact. \( \square \)

**Lemma 3.3.** Let \( R \to S \) be an epimorphism of rings. Let \( E \) be an \( S \)-module. If \( E \) is injective as an \( R \)-module, then \( E \) is an injective \( S \)-module.

**Proof.** This is true because \( \text{Hom}_R(N, E) = \text{Hom}_S(N, E) \) for any \( S \)-module \( N \), see Algebra, Lemma 104.14. \( \square \)

**Lemma 3.4.** Let \( R \to S \) be a ring map. If \( E \) is an injective \( R \)-module, then \( \text{Hom}_R(S, E) \) is an injective \( S \)-module.

**Proof.** This is true because \( \text{Hom}_S(N, \text{Hom}_R(S, E)) = \text{Hom}_R(N, E) \) by Algebra, Lemma 13.3. \( \square \)

**Lemma 3.5.** Let \( R \) be a ring. Let \( I \) be an injective \( R \)-module. Let \( E \subset I \) be a submodule. The following are equivalent

1. \( E \) is injective, and
2. for all \( E \subset E' \subset I \) with \( E \subset E' \) essential we have \( E = E' \).

In particular, an \( R \)-module is injective if and only if every essential extension is trivial.

**Proof.** The final assertion follows from the first and the fact that the category of \( R \)-modules has enough injectives (More on Algebra, Section 44).

Assume (1). Let \( E \subset E' \subset I \) as in (2). Then the map \( \text{id}_E : E \to E \) can be extended to a map \( \alpha : E' \to E \). The kernel of \( \alpha \) has to be zero because it intersects \( E \) trivially and \( E' \) is an essential extension. Hence \( E = E' \).

Assume (2). Let \( M \subset N \) be \( R \)-modules and let \( \varphi : M \to E \) be an \( R \)-module map. In order to prove (1) we have to show that \( \varphi \) extends to a morphism \( N \to E \). Consider the set \( \mathcal{S} \) of pairs \( (M', \varphi') \) where \( M \subset M' \subset N \) and \( \varphi' : M' \to E \) is an \( R \)-module map agreeing with \( \varphi \) on \( M \). We define an ordering on \( \mathcal{S} \) by the rule \( (M', \varphi') \leq (M'', \varphi'') \) if and only if \( M' \subset M'' \) and \( \varphi''|_{M'} = \varphi' \). It is clear that we can take the maximum of a totally ordered subset of \( \mathcal{S} \). Hence by Zorn’s lemma we may assume \( (M, \varphi) \) is a maximal element.

Choose an extension \( \psi : N \to I \) of \( \varphi \) composed with the inclusion \( E \to I \). This is possible as \( I \) is injective. If \( \psi(N) \subset E \), then \( \psi \) is the desired extension. If \( \psi(N) \) is not contained in \( E \), then by (2) the inclusion \( E \subset E + \psi(N) \) is not essential. hence
we can find a nonzero submodule \( K \subset E + \psi(N) \) meeting \( E \) in 0. This means that \( M' = \psi^{-1}(E + K) \) strictly contains \( M \). Thus we can extend \( \varphi \) to \( M' \) using

\[
M' \xrightarrow{\psi|_{M'}} E + K \to (E + K)/K = E
\]

This contradicts the maximality of \( (M, \varphi) \).

**Example 3.6.** Let \( R \) be a reduced ring. Let \( p \subset R \) be a minimal prime so that \( K = R_p \) is a field (Algebra, Lemma 24.1). Then \( K \) is an injective \( R \)-module. Namely, we have \( \text{Hom}_R(M, K) = \text{Hom}_K(M_p, K) \) for any \( R \)-module \( M \). Since localization is an exact functor and taking duals is an exact functor on \( K \)-vector spaces we conclude \( \text{Hom}_R(-, K) \) is an exact functor, i.e., \( K \) is an injective \( R \)-module.

**Lemma 3.7.** Let \( R \) be a ring. Let \( E \) be an \( R \)-module. The following are equivalent

1. \( E \) is an injective \( R \)-module, and
2. given an ideal \( I \subset R \) and a module map \( \varphi : I \to E \) there exists an extension of \( \varphi \) to an \( R \)-module map \( R \to E \).

**Proof.** The implication (1) \( \Rightarrow \) (2) follows from the definitions. Thus we assume (2) holds and we prove (1). First proof: The lemma follows from More on Algebra, Lemma 44.4. Second proof: Since \( R \) is a generator for the category of \( R \)-modules, the lemma follows from Injectives, Lemma 11.5.

Third proof: We have to show that every essential extension \( E \subset E' \) is trivial, see Lemma 3.5. Pick \( x \in E' \) and set \( I = \{ f \in R \mid fx \in E \} \). The map \( I \to E \), \( f \mapsto fx \) extends to \( \psi : R \to E \) by (2). Then \( x' = x - \psi(1) \) is an element of \( E' \) whose annihilator in \( E'/E \) is \( I \) and which is annihilated by \( I \) as an element of \( E' \). Thus \( Rx' = (R/I)x' \) does not intersect \( E \). Since \( E \subset E' \) is an essential extension it follows that \( x' \in E \) as desired.

**Lemma 3.8.** Let \( R \) be a Noetherian ring. A direct sum of injective modules is injective.

**Proof.** Let \( E_i \) be a family of injective modules parametrized by a set \( I \). Set \( E = \bigcup E_i \). To show that \( E \) is injective we use Lemma 3.7. Thus let \( \varphi : I \to E \) be a module map from an ideal of \( R \) into \( E \). As \( I \) is a finite \( R \)-module (because \( R \) is Noetherian) we can find finitely many elements \( i_1, \ldots, i_r \in I \) such that \( \varphi \) maps into \( \bigcup_{j=1}^r E_{i_j} \). Then we can extend \( \varphi \) into \( \bigcup_{j=1}^r E_{i_j} \) using the injectivity of the modules \( E_{i_j} \).

**Lemma 3.9.** Let \( R \) be a Noetherian ring. Let \( S \subset R \) be a multiplicative subset. If \( E \) is an injective \( R \)-module, then \( S^{-1}E \) is an injective \( S^{-1}R \)-module.

**Proof.** Since \( R \to S^{-1}R \) is an epimorphism of rings, it suffices to show that \( S^{-1}E \) is injective as an \( R \)-module, see Lemma 3.3. To show this we use Lemma 3.7. Thus let \( I \subset R \) be an ideal and let \( \varphi : I \to S^{-1}E \) be an \( R \)-module map. As \( I \) is a finitely presented \( R \)-module (because \( R \) is Noetherian) we can find finitely many elements \( f \in S \) and \( e \in E \) such that \( f \varphi \) is the composition \( I \to E \to S^{-1}E \) (Algebra, Lemma 10.2). Then we can extend \( I \to E \) to a homomorphism \( R \to E \). Then the composition

\[
R \to E \to S^{-1}E \xrightarrow{f^{-1}} S^{-1}E
\]

is the desired extension of \( \varphi \) to \( R \).
Lemma 3.10. Let $R$ be a Noetherian ring. Let $I$ be an injective $R$-module.

1. Let $f \in R$. Then $E = \bigcup I[f^n] = I[f^{\infty}]$ is an injective submodule of $I$.
2. Let $J \subset R$ be an ideal. Then the $J$-power torsion submodule $I[J^{\infty}]$ is an injective submodule of $I$.

Proof. We will use Lemma 3.5 to prove (1). Suppose that $E \subset E' \subset I$ and that $E'$ is an essential extension of $E$. We will show that $E' = E$. If not, then we can find $x \in E'$ and $x \notin E$. Let $J = \{a \in R \mid ax \in E'\}$. Since $R$ is Noetherian we can choose $x$ with $J$ maximal. Since $R$ is Noetherian we can write $J = (g_1, \ldots, g_t)$ for some $g_i \in R$. Say $f^{n_i}$ annihilates $g_ix$. Set $n = \max\{n_i\}$. Then $x' = f^n x$ is an element of $E'$ not in $E$ and is annihilated by $J$. By maximality of $J$ we see that $Rx' = (R/J)x' \cap E = (0)$. Hence $E'$ is not an essential extension of $E$ a contradiction.

To prove (2) write $J = (f_1, \ldots, f_t)$. Then $I[J^{\infty}]$ is equal to

$$(\cdots((I[f_1^{\infty}])[f_2^{\infty}])\cdots)[f_t^{\infty}]$$

and the result follows from (1) and induction. \qed


$$0 \to E[x] \to \text{Hom}_A(A[x], E[x]) \to \text{Hom}_A(A[x], E[x]) \to 0$$

where the first map sends $p \in E[x]$ to $f \mapsto fp$ and the second map sends $\varphi$ to $f \mapsto \varphi(xf) - x\varphi(f)$. The second map is surjective because $\text{Hom}_A(A[x], E[x]) = \prod_{n \geq 0} E[x]$ as an abelian group and the map sends $(e_n)$ to $(e_{n+1} - xe_n)$ which is surjective. As an $A$-module we have $E[x] \cong \bigoplus_{n \geq 0} E$ which is injective by Lemma 3.8. Hence the $A[x]$-module $\text{Hom}_A(A[x], I[x])$ is injective by Lemma 3.4 and the proof is complete. \qed

4. Projective covers

In this section we briefly discuss projective covers.

Definition 4.1. Let $R$ be a ring. A surjection $P \to M$ of $R$-modules is said to be a **projective cover**, or sometimes a **projective envelope**, if $P$ is a projective $R$-module and $P \to M$ is an essential surjection.

Projective covers do not always exist. For example, if $k$ is a field and $R = k[x]$ is the polynomial ring over $k$, then the module $M = R/(x)$ does not have a projective cover. Namely, for any surjection $f : P \to M$ with $P$ projective over $R$, the proper submodule $(x - 1)P$ surjects onto $M$. Hence $f$ is not essential.

Lemma 4.2. Let $R$ be a ring and let $M$ be an $R$-module. If a projective cover of $M$ exists, then it is unique up to isomorphism.

Proof. Let $P \to M$ and $P' \to M$ be projective covers. Because $P$ is a projective $R$-module and $P' \to M$ is surjective, we can find an $R$-module map $\alpha : P \to P'$ compatible with the maps to $M$. Since $P' \to M$ is essential, we see that $\alpha$ is surjective. As $P'$ is a projective $R$-module we can choose a direct sum decomposition
\[ P = \text{Ker}(\alpha) \oplus P'. \] Since \( P' \to M \) is surjective and since \( P \to M \) is essential we conclude that \( \text{Ker}(\alpha) \) is zero as desired. \( \square \)

Here is an example where projective covers exist.

**Lemma 4.3.** Let \((R, m, \kappa)\) be a local ring. Any finite \( R \)-module has a projective cover.

**Proof.** Let \( M \) be a finite \( R \)-module. Let \( r = \dim_{\kappa}(M/mM) \). Choose \( x_1, \ldots, x_r \in M \) mapping to a basis of \( M/mM \). Consider the map \( f: R^\oplus r \to M \). By Nakayama's lemma this is a surjection (Algebra, Lemma 19.1). If \( N \subset R^\oplus r \) is a proper submodule, then \( N/mN \to \kappa^\oplus r \) is not surjective (by Nakayama's lemma again) hence \( N/mN \to M/mM \) is not surjective. Thus \( f \) is an essential surjection. \( \square \)

5. Injective hulls

In this section we briefly discuss injective hulls.

**Definition 5.1.** Let \( R \) be a ring. A injection \( M \to I \) of \( R \)-modules is said to be an injective hull if \( I \) is a injective \( R \)-module and \( M \to I \) is an essential injection.

Injective hulls always exist.

**Lemma 5.2.** Let \( R \) be a ring. Any \( R \)-module has an injective hull.

**Proof.** Let \( M \) be an \( R \)-module. By More on Algebra, Section 44 the category of \( R \)-modules has enough injectives. Choose an injection \( M \to I \) with \( I \) an injective \( R \)-module. Consider the set \( S \) of submodules \( M \subset E \subset I \) such that \( E \) is an essential extension of \( M \). We order \( S \) by inclusion. If \( \{E_\alpha\} \) is a totally ordered subset of \( S \), then \( \bigcup E_\alpha \) is an essential extension of \( M \) too (Lemma 2.3). Thus we can apply Zorn’s lemma and find a maximal element \( E \in S \). We claim \( M \subset E \) is an injective hull, i.e., \( E \) is an injective \( R \)-module. This follows from Lemma 3.5. \( \square \)

**Lemma 5.3.** Let \( R \) be a ring. Let \( M, N \) be \( R \)-modules and let \( M \to E \) and \( N \to E' \) be injective hulls. Then

1. for any \( R \)-module map \( \varphi: M \to N \) there exists an \( R \)-module map \( \psi: E \to E' \) such that

\[
\begin{array}{ccc}
M & \longrightarrow & E \\
\varphi \downarrow & & \downarrow \psi \\
N & \longrightarrow & E'
\end{array}
\]

commutes,

2. if \( \varphi \) is injective, then \( \psi \) is injective,

3. if \( \varphi \) is an essential injection, then \( \psi \) is an isomorphism,

4. if \( \varphi \) is an isomorphism, then \( \psi \) is an isomorphism,

5. if \( M \to I \) is an embedding of \( M \) into an injective \( R \)-module, then there is an isomorphism \( I \cong E \oplus I' \) compatible with the embeddings of \( M \).

In particular, the injective hull \( E \) of \( M \) is unique up to isomorphism.

**Proof.** Part (1) follows from the fact that \( E' \) is an injective \( R \)-module. Part (2) follows as \( \text{Ker}(\psi) \cap M = 0 \) and \( E \) is an essential extension of \( M \). Assume \( \varphi \) is an essential injection. Then \( E \cong \psi(E) \subset E' \) by (2) which implies \( E' = \psi(E) \oplus E'' \) because \( E \) is injective. Since \( E' \) is an essential extension of \( M \) (Lemma 2.2) we get
\(E'' = 0\). Part (4) is a special case of (3). Assume \(M \to I\) as in (5). Choose a map \(\alpha : E \to I\) extending the map \(M \to I\). Arguing as before we see that \(\alpha\) is injective. Thus as before \(\alpha(E)\) splits off from \(I\). This proves (5).

**Example 5.4.** Let \(R\) be a domain with fraction field \(K\). Then \(R \subset K\) is an injective hull of \(R\). Namely, by Example 3.6 we see that \(K\) is an injective \(R\)-module and by Lemma 2.4 we see that \(R \subset K\) is an essential extension.

**Definition 5.5.** An object \(X\) of an additive category is called **indecomposable** if it is nonzero and if \(X = Y \oplus Z\), then either \(Y = 0\) or \(Z = 0\).

**Lemma 5.6.** Let \(R\) be a ring. Let \(E\) be an indecomposable injective \(R\)-module. Then

1. \(E\) is the injective hull of any nonzero submodule of \(E\),
2. the intersection of any two nonzero submodules of \(E\) is nonzero,
3. \(\text{End}_R(E,E)\) is a noncommutative local ring with maximal ideal those \(\varphi : E \to E\) whose kernel is nonzero, and
4. the set of zerodivisors on \(E\) is a prime ideal \(p\) of \(R\) and \(E\) is an injective \(R_p\)-module.

**Proof.** Part (1) follows from Lemma 5.3 Part (2) follows from part (1) and the definition of injective hulls.

Proof of (3). Set \(A = \text{End}_R(E,E)\) and \(I = \{\varphi \in A \mid \text{Ker}(f) \neq 0\}\). The statement means that \(I\) is a two sided ideal and that any \(\varphi \in A\), \(\varphi \notin I\) is invertible. Suppose \(\varphi\) and \(\psi\) are not injective. Then \(\text{Ker}(\varphi) \cap \text{Ker}(\psi)\) is nonzero by (2). Hence \(\varphi + \psi \in I\). It follows that \(I\) is a two sided ideal. If \(\varphi \in A\), \(\varphi \notin I\), then \(E \cong \varphi(E) \subset E\) is an injective submodule, hence \(E = \varphi(E)\) because \(E\) is indecomposable.

Proof of (4). Consider the ring map \(R \to A\) and let \(p \subset R\) be the inverse image of the maximal ideal \(I\). Then it is clear that \(p\) is a prime ideal and that \(R \to A\) extends to \(R_p \to A\). Thus \(E\) is an \(R_p\)-module. It follows from Lemma 3.3 that \(E\) is injective as an \(R_p\)-module.

**Lemma 5.7.** Let \(p \subset R\) be a prime of a ring \(R\). Let \(E\) be the injective hull of \(R/p\). Then

1. \(E\) is indecomposable,
2. \(E\) is the injective hull of \(\kappa(p)\),
3. \(E\) is the injective hull of \(\kappa(p)\) over the ring \(R_p\).

**Proof.** As \(R/p \subset \kappa(p)\) we can extend the embedding to a map \(\kappa(p) \to E\). Hence (2) holds. For \(f \in R\), \(f \notin p\) the map \(f : \kappa(p) \to \kappa(p)\) is an isomorphism hence the map \(f : E \to E\) is an isomorphism, see Lemma 5.3 Thus \(E\) is an \(R_p\)-module. It is injective as an \(R_p\)-module by Lemma 3.3. Finally, let \(E' \subset E\) be a nonzero injective \(R\)-submodule. Then \(J = (R/p) \cap E'\) is nonzero. After shrinking \(E'\) we may assume that \(E'\) is the injective hull of \(J\) (see Lemma 5.3 for example). Observe that \(R/p\) is an essential extension of \(J\) for example by Lemma 2.4. Hence \(E' \to E\) is an isomorphism by Lemma 5.3 part (3). Hence \(E\) is indecomposable.

**Lemma 5.8.** Let \(R\) be a Noetherian ring. Let \(E\) be an indecomposable injective \(R\)-module. Then there exists a prime ideal \(p\) of \(R\) such that \(E\) is the injective hull of \(\kappa(p)\).
Proof. Let \( \mathfrak{p} \) be the prime ideal found in Lemma 5.6. Say \( \mathfrak{p} = (f_1, \ldots, f_r) \). Pick a nonzero element \( x \in \bigcap \text{Ker}(f_i : E \to E) \), see Lemma 5.6. Then \((R_\mathfrak{p})x\) is a module isomorphic to \( \kappa(\mathfrak{p}) \) inside \( E \). We conclude by Lemma 5.6. \( \square \)

Proposition 5.9 (Structure of injective modules over Noetherian rings). Let \( R \) be a Noetherian ring. Every injective module is a direct sum of indecomposable injective modules. Every indecomposable injective module is the injective hull of the residue field at a prime.

Proof. The second statement is Lemma 5.8. For the first statement, let \( I \) be an injective \( R \)-module. We will use transfinite induction to construct \( I_\alpha \subset I \) for ordinals \( \alpha \) which are direct sums of indecomposable injective \( R \)-modules \( E_{\beta+1} \) for \( \beta < \alpha \). For \( \alpha = 0 \) we let \( I_0 = 0 \). Suppose given an ordinal \( \alpha \) such that \( I_\alpha \) has been constructed. Then \( I_\alpha \) is an injective \( R \)-module by Lemma 3.8. Hence \( I \cong I_\alpha \oplus I' \). If \( I' = 0 \) we are done. If not, then \( I' \) has an associated prime by Algebra, Lemma 62.7. Thus \( I' \) contains a copy of \( R/\mathfrak{p} \) for some prime \( \mathfrak{p} \). Hence \( I' \) contains an indecomposable submodule \( E \) by Lemmas 5.3 and 5.7. Set \( I_{\alpha+1} = I_\alpha \oplus E_\alpha \). If \( \alpha \) is a limit ordinal and \( I_\alpha \) has been constructed for \( \beta < \alpha \), then we set \( I_\alpha = \bigcup_{\beta < \alpha} I_\beta \). Observe that \( I_\alpha = \bigoplus_{\beta < \alpha} E_{\beta+1} \). This concludes the proof. \( \square \)

6. Duality over Artinian local rings

Let \((R, \mathfrak{m}, \kappa)\) be an artinian local ring. Recall that this implies \( R \) is Noetherian and that \( R \) has finite length as an \( R \)-module. Moreover an \( R \)-module is finite if and only if it has finite length. We will use these facts without further mention in this section. Please see Algebra, Sections 50 and 51 and Algebra, Proposition 59.6 for more details.

Lemma 6.1. Let \((R, \mathfrak{m}, \kappa)\) be an artinian local ring. Let \( E \) be an injective hull of \( \kappa \). For every finite \( R \)-module \( M \) we have

\[
\text{length}_R(M) = \text{length}_R(\text{Hom}_R(M, E))
\]

In particular, the injective hull \( E \) of \( \kappa \) is a finite \( R \)-module.

Proof. Because \( E \) is an essential extension of \( \kappa \) we have \( \kappa = E[\mathfrak{m}] \) where \( E[\mathfrak{m}] \) is the \( \mathfrak{m} \)-torsion in \( E \) (notation as in More on Algebra, Section 66). Hence \( \text{Hom}_R(\kappa, E) \cong \kappa \) and the equality of lengths holds for \( M = \kappa \). We prove the displayed equality of the lemma by induction on the length of \( M \). If \( M \) is nonzero there exists a surjection \( M \to \kappa \) with kernel \( M' \). Since the functor \( M \to \text{Hom}_R(M, E) \) is exact we obtain a short exact sequence

\[
0 \to \text{Hom}_R(\kappa, E) \to \text{Hom}_R(M, E) \to \text{Hom}_R(M', E) \to 0.
\]

Additivity of length for this sequence and the sequence \( 0 \to M' \to M \to \kappa \to 0 \) and the equality for \( M' \) (induction hypothesis) and \( \kappa \) implies the equality for \( M \). The final statement of the lemma follows as \( E = \text{Hom}_R(R, E) \). \( \square \)

Lemma 6.2. Let \((R, \mathfrak{m}, \kappa)\) be an artinian local ring. Let \( E \) be an injective hull of \( \kappa \). For any finite \( R \)-module \( M \) the evaluation map

\[
M \longrightarrow \text{Hom}_R(\text{Hom}_R(M, E), E)
\]

is an isomorphism. In particular \( R = \text{Hom}_R(E, E) \).
**Proof.** Observe that the displayed arrow is injective. Namely, if \( x \in M \) is a nonzero element, then there is a nonzero map \( Rx \to \kappa \) which we can extend to a map \( \varphi : M \to E \) that doesn’t vanish on \( x \). Since the source and target of the arrow have the same length by Lemma 6.1, we conclude it is an isomorphism. The final statement follows on taking \( M = R \). □

To state the next lemma, denote \( \text{Mod}^{fg}_R \) the category of finite \( R \)-modules over a ring \( R \).

**Lemma 6.3.** Let \((R, m, \kappa)\) be an artinian local ring. Let \( E \) be an injective hull of \( \kappa \). The functor \( D(-) = \text{Hom}_R(-, E) \) induces an exact anti-equivalence \( \text{Mod}^{fg}_R \to \text{Mod}^{fg}_R \) and \( D \circ D \cong \text{id} \).

**Proof.** We have seen that \( D \circ D = \text{id} \) on \( \text{Mod}^{fg}_R \) in Lemma 6.2. It follows immediately that \( D \) is an anti-equivalence. □

**Lemma 6.4.** Assumptions and notation as in Lemma 6.3. Let \( I \subset R \) be an ideal and \( M \) a finite \( R \)-module. Then \( D(M[I]) = D(M)/ID(M) \) and \( D(M/IM) = D(M)[I] \).

**Proof.** Say \( I = (f_1, \ldots, f_t) \). Consider the map
\[
M^{\oplus t} \to M
\]
with cokernel \( M/IM \). Applying the exact functor \( D \) we conclude that \( D(M/IM) \) is \( D(M)[I] \). The other case is proved in the same way. □

### 7. Injective hull of the residue field

Most of our results will be for Noetherian local rings in this section.

**Lemma 7.1.** Let \( R \to S \) be a surjective map of local rings with kernel \( I \). Let \( E \) be the injective hull of the residue field of \( R \) over \( S \). Then \( E[I] \) is the injective hull of the residue field of \( S \) over \( S \).

**Proof.** Observe that \( E[I] = \text{Hom}_R(S, E) \) as \( S = R/I \). Hence \( E[I] \) is an injective \( S \)-module by Lemma 3.4. Since \( E \) is an essential extension of \( \kappa = R/m_R \) it follows that \( E[I] \) is an essential extension of \( \kappa \) as well. The result follows. □

**Lemma 7.2.** Let \( (R, m, \kappa) \) be a local ring. Let \( E \) be the injective hull of \( \kappa \). Let \( M \) be a \( m \)-power torsion \( R \)-module with \( n = \dim_\kappa(M[m]) < \infty \). Then \( M \) is isomorphic to a submodule of \( E^{\oplus n} \).

**Proof.** Observe that \( E^{\oplus n} \) is the injective hull of \( \kappa^{\oplus n} = M[m] \). Thus there is an \( R \)-module map \( M \to E^{\oplus n} \) which is injective on \( M[m] \). Since \( M \) is \( m \)-power torsion the inclusion \( M[m] \subset M \) is an essential extension (for example by Lemma 2.4) we conclude that the kernel of \( M \to E^{\oplus n} \) is zero. □

**Lemma 7.3.** Let \( (R, m, \kappa) \) be a Noetherian local ring. Let \( E \) be an injective hull of \( \kappa \) over \( R \). Let \( E_n \) be an injective hull of \( \kappa \) over \( R/m^n \). Then \( E = \bigcup E_n \) and \( E_n = E[m^n] \).

**Proof.** We have \( E_n = E[m^n] \) by Lemma 7.1. We have \( E = \bigcup E_n \) because \( \bigcup E_n = E[m^\infty] \) is an injective \( R \)-submodule which contains \( \kappa \), see Lemma 3.10. □
The following lemma tells us the injective hull of the residue field of a Noetherian local ring only depends on the completion.

**Lemma 7.4.** Let $R \to S$ be a flat local homomorphism of local Noetherian rings such that $R/m_R \cong S/m_RS$. Then the injective hull of the residue field of $R$ is the injective hull of the residue field of $S$.

**Proof.** Set $\kappa = R/m_R = S/m_S$. Let $E_R$ be the injective hull of $\kappa$ over $R$. Let $E_S$ be the injective hull of $\kappa$ over $S$. Observe that $E_S$ is an injective $R$-module by Lemma 3.2. Choose an extension $E_R \to E_S$ of the identification of residue fields. This map is an isomorphism by Lemma 7.3 because $R \to S$ induces an isomorphism $R/m^n_R \to S/m^n_S$ for all $n$. □

**Lemma 7.5.** Let $(R, m, \kappa)$ be a Noetherian local ring. Let $E$ be an injective hull of $\kappa$ over $R$. Then $\text{Hom}_R(E, E)$ is canonically isomorphic to the completion of $R$.

**Proof.** Write $E = \bigcup E_n$ with $E_n = E[m^n]$ as in Lemma 7.3. Any endomorphism of $E$ preserves this filtration. Hence

$$\text{Hom}_R(E, E) = \lim_{\leftarrow} \text{Hom}_R(E_n, E_n).$$

The lemma follows as $\text{Hom}_R(E_n, E_n) = \text{Hom}_{R/m^n}(E_n, E_n) = R/m^n$ by Lemma 6.2. □

**Lemma 7.6.** Let $(R, m, \kappa)$ be a Noetherian local ring. Let $E$ be an injective hull of $\kappa$ over $R$. Then $E$ satisfies the descending chain condition.

**Proof.** If $E \subset M_1 \subset M_2 \ldots$ is a sequence of submodules, then

$$\text{Hom}_R(E, E) \to \text{Hom}_R(M_1, E) \to \text{Hom}_R(M_2, E) \to \ldots$$

is sequence of surjections. By Lemma 7.5 each of these is a module over the completion $R^\wedge = \text{Hom}_R(E, E)$. Since $R^\wedge$ is Noetherian (Algebra, Lemma 94.10) the sequence stabilizes: $\text{Hom}_R(M_n, E) = \text{Hom}_R(M_{n+1}, E) = \ldots$. Since $E$ is injective, this can only happen if $\text{Hom}_R(M_n/M_{n+1}, E)$ is zero. However, if $M_n/M_{n+1}$ is nonzero, then it contains a nonzero element annihilated by $m$, because $E$ is $m$-power torsion by Lemma 7.3. In this case $M_n/M_{n+1}$ has a nonzero map into $E$, contradicting the assumed vanishing. This finishes the proof. □

**Lemma 7.7.** Let $(R, m, \kappa)$ be a Noetherian local ring. Let $E$ be an injective hull of $\kappa$.

1. For an $R$-module $M$ the following are equivalent:
   (a) $M$ satisfies the ascending chain condition,
   (b) $M$ is a finite $R$-module, and
   (c) there exist $n, m$ and an exact sequence $R^{\oplus m} \to R^{\oplus n} \to M \to 0$.

2. For an $R$-module $M$ the following are equivalent:
   (a) $M$ satisfies the descending chain condition,
   (b) $M$ is $m$-power torsion and $\dim_k(M[m]) < \infty$, and
   (c) there exist $n, m$ and an exact sequence $0 \to M \to E^{\oplus n} \to E^{\oplus m}$.

**Proof.** We omit the proof of (1). Let $M$ be an $R$-module with the descending chain condition. Let $x \in M$. Then $m^n x$ is a descending chain of submodules, hence stabilizes. Thus $m^n x = m^{n+1} x$ for some $n$. By Nakayama’s lemma (Algebra, Lemma 19.1) this implies $m^n x = 0$, for some $n$. Then $m^n E$ is a sequence of submodules, then

$$m^n E \subset \bigcap_{i=1}^\infty E_i = 0.$$
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i.e., $x$ is $m$-power torsion. Since $M[m]$ is a vector space over $\kappa$ it has to be finite dimensional in order to have the descending chain condition.

Assume that $M$ is $m$-power torsion and has a finite dimensional $m$-torsion submodule $M[m]$. By Lemma 7.2 we see that $M$ is a submodule of $E^{\oplus n}$ for some $n$. Consider the quotient $N = E^{\oplus n}/M$. By Lemma 7.6 the module $E$ has the descending chain condition hence so do $E^{\oplus n}$ and $N$. Therefore $N$ satisfies (2)(a) which implies $N$ satisfies (2)(b) by the second paragraph of the proof. Thus by Lemma 7.2 again we see that $N$ is a submodule of $E^{\oplus m}$ for some $m$. Thus we have a short exact sequence $0 \to M \to E^{\oplus n} \to E^{\oplus m}$.

Assume we have a short exact sequence $0 \to M \to E^{\oplus n} \to E^{\oplus m}$. Since $E$ satisfies the descending chain condition by Lemma 7.6 so does $M$. □

Proposition 7.8 (Matlis duality). Let $(R, m, \kappa)$ be a complete local Noetherian ring. Let $E$ be an injective hull of $\kappa$ over $R$. The functor $D(\cdot) = \text{Hom}_R(\cdot, E)$ induces an anti-equivalence

\[ \{ \text{R-modules with the descending chain condition} \} \leftrightarrow \{ \text{R-modules with the ascending chain condition} \} \]

and we have $D \circ D = \text{id}$ on either side of the equivalence.

Proof. By Lemma 7.5 we have $R = \text{Hom}_R(E, E) = D(E)$. Of course we have $E = \text{Hom}_R(R, E) = D(R)$. Since $E$ is injective the functor $D$ is exact. The result now follows immediately from the description of the categories in Lemma 7.7. □

8. Local cohomology

Let $A$ be a ring and let $I \subset A$ be a finitely generated ideal (if $I$ is not finitely generated perhaps a different definition should be used). Let $Z = V(I) \subset \text{Spec}(A)$. Recall that the category $I^\infty$-torsion of $I$-power torsion modules only depends on the closed subset $Z$ and not on the choice of the finitely generated ideal $I$ such that $Z = V(I)$, see More on Algebra, Lemma 65.6. In this section we will consider the functor

$H^0_I : \text{Mod}_A \to I^\infty$-torsion, \quad $M \mapsto M[I^\infty] = \bigcup M[I^n]$

which sends $M$ to the submodule of $I$-power torsion as well as its relationship with the functors

$\mathcal{H}_Z : \text{Ab}(X) \to \text{Ab}(Z)$

and $\Gamma_Z(-) = \Gamma(Z, \mathcal{H}_Z(-))$ of Cohomology, Section 22.

Let $A$ be a ring and let $I$ be a finitely generated ideal. Note that $I^\infty$-torsion is a Grothendieck abelian category (direct sums exist, filtered colimits are exact, and $\bigoplus A/I^n$ is a generator by More on Algebra, Lemma 65.2). Hence the derived category $D(I^\infty$-torsion) exists, see Injectives, Remark 13.3. Our functor $H^0_I$ is left exact and has a derived extension which we will denote

$R\Gamma_I : D(A) \to D(I^\infty$-torsion).

Warning: this functor does not deserve the name local cohomology unless the ring $A$ is Noetherian. The functors $H^0_I$, $R\Gamma_I$, and the satellites $H^p_I$ only depend on the closed subset $Z \subset \text{Spec}(A)$ and not on the choice of the finitely generated ideal $I$ such that $V(I) = Z$. However, we insist on using the subscript $I$ for the functors above as the notation $R\Gamma_Z$ is going to be used for a different functor, see (8.4.1).
which agrees with the functor $R\Gamma_I$ only (as far as we know) in case $A$ is Noetherian (see Lemma 8.9).

**Lemma 8.1.** Let $A$ be a ring and let $I \subset A$ be a finitely generated ideal. The functor $R\Gamma_I$ is right adjoint to the functor $D(I^{\infty}-\text{torsion}) \to D(A)$.

**Proof.** This follows from the fact that taking $I$-power torsion submodules is the right adjoint to the inclusion functor $I^{\infty}-\text{torsion} \to \text{Mod}_A$. See Derived Categories, Lemma 28.4.

**Lemma 8.2.** Let $A$ be a ring and let $I \subset A$ be a finitely generated ideal. For any object $K$ of $D(A)$ we have

$$R\Gamma_I(K) = \text{hocolim} R\text{Hom}(A/I^n, K)$$

in $D(A)$ and

$$R^q\Gamma_I(K) = \text{colim}_n \text{Ext}^q_A(A/I^n, K)$$

as modules for all $q \in \mathbb{Z}$.

**Proof.** Let $J^\bullet$ be a $K$-injective complex representing $K$. Then

$$R\Gamma_I(K) = J^\bullet[I^{\infty}] = \text{colim} J^\bullet[I^n] = \text{colim}_A \text{Hom}_A(A/I^n, J^\bullet)$$

By Derived Categories, Lemma 31.4 we obtain the first equality. The second equality is clear because $H^q(\text{Hom}_A(A/I^n, J^\bullet)) = \text{Ext}^q_A(A/I^n, K)$ and because filtered colimits are exact in the category of abelian groups.

**Lemma 8.3.** Let $A$ be a ring and let $I \subset A$ be a finitely generated ideal. Let $K^\bullet$ be a complex of $A$-modules such that $f: K^\bullet \to K^\bullet$ is an isomorphism for some $f \in I$, i.e., $K^\bullet$ is a complex of $A_f$-modules. Then $R\Gamma_I(K^\bullet) = 0$.

**Proof.** Namely, in this case the cohomology modules of $R\Gamma_I(K^\bullet)$ are both $f$-power torsion and $f$ acts by automorphisms. Hence the cohomology modules are zero and hence the object is zero.

Let $A$ be a ring and $I \subset A$ a finitely generated ideal. By More on Algebra, Lemma 65.5 the category of $I$-power torsion modules is a Serre subcategory of the category of all $A$-modules, hence there is a functor

$$D(I^{\infty}-\text{torsion}) \to D(I^{\infty}-\text{torsion})(A)$$

see Derived Categories, Section 13.

**Lemma 8.4.** Let $A$ be a ring and let $I \subset A$ be a finitely generated ideal. Let $M$ and $N$ be $I$-power torsion modules.

1. $\text{Hom}_{D(A)}(M, N) = \text{Hom}_{D(I^{\infty}-\text{torsion})}(M, N)$,
2. $\text{Ext}^1_{D(A)}(M, N) = \text{Ext}^1_{D(I^{\infty}-\text{torsion})}(M, N)$,
3. $\text{Ext}^2_{D(I^{\infty}-\text{torsion})}(M, N) \to \text{Ext}^2_{D(A)}(M, N)$ is not surjective in general,
4. (8.3.1) is not an equivalence in general.

**Proof.** Parts (1) and (2) follow immediately from the fact that $I$-power torsion forms a Serre subcategory of $\text{Mod}_A$. Part (4) follows from part (3).

For part (3) let $A$ be a ring with an element $f \in A$ such that $A[f]$ contains a nonzero element $x$ and $A$ contains elements $x_n$ with $f^n x_n = x$. Such a ring $A$ exists because we can take

$$A = \mathbb{Z}[f, x, x_n]/(fx, f^n x_n - x)$$
Given $A$ set $I = (f)$. Then the exact sequence

$$0 \to A[f] \to A \xrightarrow{f} A \to A/ fA \to 0$$

defines an element in $\text{Ext}^2_A(A/fA, A[f])$. We claim this element does not come from an element of $\text{Ext}^2_B(f, \text{-torsion})(A/fA, A[f])$. Namely, if it did, then there would be an exact sequence

$$0 \to A[f] \to M \to N \to A/fA \to 0$$

where $M$ and $N$ are $f$-power torsion modules defining the same 2 extension class. Since $A \to A$ is a complex of free modules and since the 2 extension classes are the same we would be able to find a map

$$0 \to A[f] \xrightarrow{\varphi} A \xrightarrow{\psi} A/fA \to 0$$

(some details omitted). Then we could replace $M$ by the image of $\varphi$ and $N$ by the image of $\psi$. Then $M$ would be a cyclic module, hence $f^n M = 0$ for some $n$. Considering $\varphi(x_{n+1})$ we get a contradiction with the fact that $f^{n+1} x_n = x$ is nonzero in $A[f]$. □

Let $A$ be a ring and let $I \subset A$ be a finitely generated ideal. Set $Z = V(I) \subset \text{Spec}(A)$. We will construct a functor

$$(8.4.1) \quad R\Gamma_Z : D(A) \to D_{I, \text{-torsion}}(A).$$

which is right adjoint to the inclusion functor. The cohomology modules of $R\Gamma_Z(K)$ are the local cohomology groups of $K$ with respect to $Z$. In fact, we will show $R\Gamma_Z$ computes cohomology with support in $Z$ for the associated complex of quasi-coherent sheaves on $\text{Spec}(A)$. By Lemma 8.4 this functor will in general not be equal to $R\Gamma_I(-)$ even viewed as functors into $D(A)$.

**Lemma 8.5.** Let $A$ be a ring and let $I \subset A$ be a finitely generated ideal. There exists a right adjoint $R\Gamma_Z$ (8.4.1) to the inclusion functor $D_{I, \text{-torsion}}(A) \to D(A)$. In fact, if $I$ is generated by $f_1, \ldots, f_r \in A$, then we have

$$R\Gamma_Z(K) = (A \to \prod_{i_0} A_{f_{i_0}} \to \prod_{i_0 < i_1} A_{f_{i_0}f_{i_1}} \to \cdots \to A_{f_{i_1} \cdots f_r}) \otimes_A K$$

functorially in $K$.

**Proof.** Say $I = (f_1, \ldots, f_r)$ be an ideal. Let $K^\bullet$ be a complex of $A$-modules. There is a canonical map of complexes

$$(A \to \prod_{i_0} A_{f_{i_0}} \to \prod_{i_0 < i_1} A_{f_{i_0}f_{i_1}} \to \cdots \to A_{f_{i_1} \cdots f_r}) \to A.$$

from the extended Čech complex to $A$. Tensoring with $K^\bullet$, taking associated total complex, we get a map

$$\text{Tot} \left( K^\bullet \otimes_A (A \to \prod_{i_0} A_{f_{i_0}} \to \prod_{i_0 < i_1} A_{f_{i_0}f_{i_1}} \to \cdots \to A_{f_{i_1} \cdots f_r}) \right) \to K^\bullet$$

in $D(A)$. We claim the cohomology modules of the complex on the left are $I$-power torsion, i.e., the LHS is an object of $D_{I, \text{-torsion}}(A)$. Namely, we have

$$(A \to \prod_{i_0} A_{f_{i_0}} \to \prod_{i_0 < i_1} A_{f_{i_0}f_{i_1}} \to \cdots \to A_{f_{i_1} \cdots f_r}) = \text{colim} K(A, f_1^{n_1}, \ldots, f_r^{n_r})$$

...
by More on Algebra, Lemma 21.13. Moreover, multiplication by \( f^n \) on the complex \( K(A, f_1^n, \ldots, f_r^n) \) is homotopic to zero by More on Algebra, Lemma 21.6. Since

\[
H^q(LHS) = \text{colim} H^q(\text{Tot}(K^\bullet \otimes_A K(A, f_1^n, \ldots, f_r^n)))
\]

we obtain our claim. On the other hand, if \( K^\bullet \) is an object of \( D_{I^{\infty}, \text{torsion}}(A) \), then the complexes \( K^\bullet \otimes_A A_{f_{i_0} \cdots f_{i_p}} \) have vanishing cohomology. Hence in this case the map \( LHS \to K^\bullet \) is an isomorphism in \( D(A) \). The construction

\[
R\Gamma_Z(K^\bullet) = \text{Tot} \left( K^\bullet \otimes_A \left( A \to \prod_{i_0} A_{f_{i_0}} \to \prod_{i_0 < i_1} A_{f_{i_0}f_{i_1}} \to \cdots \to A_{f_1 \cdots f_r} \right) \right)
\]

is functorial in \( K^\bullet \) and defines an exact functor \( D(A) \to D_{I^{\infty}, \text{torsion}}(A) \) between triangulated categories. It follows formally from the existence of the natural transformation \( R\Gamma_Z \to \text{id} \) given above and the fact that this evaluates to an isomorphism on \( K^\bullet \) in the subcategory, that \( R\Gamma_Z \) is the desired right adjoint.

**Lemma 8.6.** Let \( A \) be a ring and let \( I \subset A \) be a finitely generated ideal. Let \( K^\bullet \) be a complex of \( A \)-modules such that \( f : K^\bullet \to K^\bullet \) is an isomorphism for some \( f \in I \), i.e., \( K^\bullet \) is a complex of \( A_f \)-modules. Then \( R\Gamma_Z(K^\bullet) = 0 \).

**Proof.** Namely, in this case the cohomology modules of \( R\Gamma_Z(K^\bullet) \) are both \( f \)-power torsion and \( f \) acts by automorphisms. Hence the cohomology modules are zero and hence the object is zero.

**Lemma 8.7.** Let \( A \) be a ring and let \( I \subset A \) be a finitely generated ideal. For \( K, L \in D(A) \) general we have

\[
R\Gamma_Z(K \otimes_A L) = K \otimes_A R\Gamma_Z(L) = R\Gamma_Z(K) \otimes_A R\Gamma_Z(L)
\]

If \( K \) or \( L \) is in \( D_{I^{\infty}, \text{torsion}}(A) \) then so is \( K \otimes_A L \).

**Proof.** By Lemma 8.5 we know that \( R\Gamma_Z \) is given by \( C \otimes^L - \) for some \( C \in D(A) \). Hence, for \( K, L \in D(A) \) general we have

\[
R\Gamma_Z(K \otimes_A L) = K \otimes^L L \otimes_A C = K \otimes_A R\Gamma_Z(L)
\]

The other equalities follow formally from this one. This also implies the last statement of the lemma.

The following lemma tells us that the functor \( R\Gamma_Z \) is related to local cohomology.

**Lemma 8.8.** Let \( A \) be a ring and let \( I \) be a finitely generated ideal. With \( Z = V(I) \subset X = \text{Spec}(A) \) there is a functorial isomorphism

\[
R\Gamma_Z(K^\bullet) = R\Gamma_Z(\tilde{K}^\bullet)
\]

where on the left we have (8.4.1) and on the right we have the functor of Cohomology, Section 22.

**Proof.** Denote \( \mathcal{F}^\bullet = \tilde{K}^\bullet \) be the complex of quasi-coherent \( \mathcal{O}_X \)-modules on \( X \) associated to \( K^\bullet \). By Cohomology, Section 22 there exists a distinguished triangle

\[
R\Gamma_Z(X, \mathcal{F}^\bullet) \to R\Gamma(X, \mathcal{F}^\bullet) \to R\Gamma(U, \mathcal{F}^\bullet) \to R\Gamma_Z(X, \mathcal{F}^\bullet)[1]
\]

where \( U = X \setminus Z \). We know that \( R\Gamma(X, \mathcal{F}^\bullet) = K^\bullet \) for example by Derived Categories of Schemes, Lemma 3.7. Say \( I = (f_1, \ldots, f_r) \). Then we obtain a finite affine open covering \( \mathcal{U} : U = D(f_1) \cup \ldots \cup D(f_r) \). By Derived Categories of Schemes, Lemma 9.4 the alternating Čech complex

\[
\text{Tot}(\mathcal{C}_{alt}(\mathcal{U}, \mathcal{F}^\bullet))
\]
computes $R\Gamma(U, F^\bullet)$. Working through the definitions we find

$$R\Gamma(U, F^\bullet) = \text{Tot} \left( K^\bullet \otimes_A \left( \prod_{i_0} A f_{i_0} \to \prod_{i_0<i_1} A f_{i_0} f_{i_1} \to \ldots \to A f_{i_1} \ldots f_r \right) \right)$$

It is clear that $R\Gamma(X, F^\bullet) \to R\Gamma(U, F^\bullet)$ is given by the map from $A$ into $\prod A f_i$. Hence we conclude that

$$R\Gamma_Z(X, F^\bullet) = \text{Tot} \left( K^\bullet \otimes_A (A \to \prod_{i_0} A f_{i_0} \to \prod_{i_0<i_1} A f_{i_0} f_{i_1} \to \ldots \to A f_{i_1} \ldots f_r) \right)$$

By Lemma 8.5 this complex computes $R\Gamma_Z(K^\bullet)$ and we see the lemma holds. \(\square\)

Let $A$ be a ring and let $I \subset A$ be a finitely generated ideal. Set $Z = V(I) \subset \text{Spec}(A)$. There is a natural transformation of functors

$$(8.8.1) \quad \rho : R\Gamma_I(-) \to R\Gamma_Z(-)$$

Namely, given a complex of $A$-modules $K^\bullet$ the canonical map $R\Gamma_I(K^\bullet) \to K^\bullet$ in $D(A)$ factors (uniquely) through $R\Gamma_Z(K^\bullet)$ as $R\Gamma_I(K^\bullet)$ has $I$-power torsion cohomology modules (see Lemma 8.1). In general this map is not an isomorphism (we’ve seen this above).

**Lemma 8.9.** Let $A$ be a Noetherian ring and let $I \subset A$ be an ideal. Denote $j : D(I^{\infty}\text{-torsion}) \to D_{I^{\infty}\text{-torsion}}(A)$ the functor (8.3.1).

1. (1) the adjunction $j(R\Gamma_I(K)) \to K$ is an isomorphism for $K \in D_{I^{\infty}\text{-torsion}}(A)$,
2. the functor $j$ is an equivalence,
3. the transformation of functors $\rho$ is an isomorphism,

**Proof.** A formal argument, which we omit, shows that it suffices to prove (1).

Let $M$ be an $I$-power torsion $A$-module. Choose an embedding $M \to J$ into an injective $A$-module. Then $J[I^{\infty}]$ is an injective $A$-module, see Lemma 3.10 and we obtain an embedding $M \to J[I^{\infty}]$. Thus every $I$-power torsion module has an injective resolution $M \to J^\bullet$ with $J^n$ also $I$-power torsion. It follows that $R\Gamma_I(M) = M$ (this is not a triviality and this is not true in general if $A$ is not Noetherian). Next, suppose that $K \in D_{I^{\infty}\text{-torsion}}(A)$. Then the spectral sequence

$$R^q\Gamma_I(H^p(K)) \Rightarrow R^{q+p}\Gamma_I(K)$$

(Derived Categories, Lemma 21.3) converges and above we have seen that only the terms with $q = 0$ are nonzero. Thus we see that $R\Gamma_I(K) \to K$ is an isomorphism.

Suppose $K$ is an arbitrary object of $D_{I^{\infty}\text{-torsion}}(A)$. We have

$$R^q\Gamma_I(K) = \text{colim} \text{Ext}_A^q(A/I^n, K)$$

by Lemma 8.2. Choose $f_1, \ldots, f_r \in A$ generating $I$. Let $K^\bullet_n = K(A, f_1^n, \ldots, f_r^n)$ be the Koszul complex with terms in degrees $-r, \ldots, 0$. Since the pro-objects $\{A/I^n\}$ and $\{K^\bullet_n\}$ in $D(A)$ are the same by More on Algebra, Lemma 67.18 we see that

$$R^q\Gamma_I(K) = \text{colim} \text{Ext}_A^q(K^\bullet_n, K)$$

Pick any complex $K^\bullet$ of $A$-modules representing $K$. Since $K^\bullet_n$ is a finite complex of finite free modules we see that

$$\text{Ext}_A^q(K^\bullet_n, K) = H^q(\text{Tot}((K^\bullet_n)^\vee \otimes_A K^\bullet))$$

where $(K^\bullet_n)^\vee$ is the dual of the complex $K^\bullet_n$. See More on Algebra, Lemma 58.2. As $(K^\bullet_n)^\vee$ is a complex of finite free $A$-modules sitting in degrees $0, \ldots, r$ we see that the terms of the complex $\text{Tot}((K^\bullet_n)^\vee \otimes_A K^\bullet)$ are the same as the terms of
the complex $\text{Tot}((K_n^\bullet)^V \otimes_A \tau_{\geq q-r-2}K^\bullet)$ in degrees $q-1$ and higher. Hence we see that
\[
\text{Ext}^q_A(K_n, K) = \text{Ext}^q_A(K_n, \tau_{\geq q-r-2}K)
\]
for all $n$. It follows that
\[
R^q\Gamma_I(K) = R^q\Gamma_I(\tau_{\geq q-r-2}K) = H^q(\tau_{\geq q-r-2}K) = H^q(K)
\]
Thus we see that the map $R\Gamma_I(K) \to K$ is an isomorphism.

**Lemma 8.10.** If $A$ is a Noetherian ring and $I = (f_1, \ldots, f_r)$ an ideal. There are canonical isomorphisms
\[
R\Gamma_I(A) \to (A \to \prod_{i_0} A_{f_{i_0}} \to \prod_{i_0 < i_1} A_{f_{i_0}f_{i_1}} \to \ldots \to A_{f_{1}\ldots f_r}) \to R\Gamma_Z(A)
\]
in $D(A)$.

**Proof.** This follows from Lemma 8.9 and the computation of the functor $R\Gamma_Z$ in Lemma 8.5.

**Lemma 8.11.** Let $A \to B$ be a ring map. Let $I \subset A$ be a finitely generated ideal. Let $Z = V(I) \subset \text{Spec}(A)$ and $Y = V(IB) \subset \text{Spec}(B)$. For $K$ in $D(A)$ we have $R\Gamma_Z(K) \otimes^L_A B = R\Gamma_Y(K \otimes^L_A B)$.

**Proof.** This follows from uniqueness of adjoint functors as both $R\Gamma_Z(-) \otimes^L_A B$ and $R\Gamma_Y(- \otimes^L_A B)$ are right adjoint to the functor $D(I_B)^{\text{torsion}}(B) \to D(A)$. Alternatively, one can use the description of $R\Gamma_Z$ and $R\Gamma_Y$ in terms of alternating Čech complexes (Lemma 8.5) and use that formation of the extended Čech complex commutes with base change.

**Lemma 8.12.** If $A \to B$ is a homomorphism of Noetherian rings and $I \subset A$ is an ideal, then in $D(B)$ we have
\[
R\Gamma_I(A) \otimes^L_A B = R\Gamma_Z(A) \otimes^L_A B = R\Gamma_Y(B) = R\Gamma_{IB}(B)
\]
where $Y = V(IB) \subset \text{Spec}(B)$.

**Proof.** Combine Lemmas 8.10 and 8.11.

The following lemma is the analogue of More on Algebra, Lemma 67.26 for complexes with torsion cohomologies.

**Lemma 8.13.** Let $A \to B$ be a flat ring map and let $I \subset A$ be a finitely generated ideal such that $A/I = B/IB$. Then base change and restriction induce quasi-inverse equivalences $D_{I^{\text{torsion}}}(A) \to D_{(IB)^{\text{torsion}}}(B)$.

**Proof.** More precisely the functors are $K \to K \otimes^L_A B$ for $K$ in $D_{I^{\text{torsion}}}(A)$ and $M \to M_A$ for $M$ in $D_{(IB)^{\text{torsion}}}(B)$. The reason this works is that $H^i(K \otimes^L_A B) = H^i(K) \otimes_A B = H^i(K)$. The first equality holds as $A \to B$ is flat and the second by More on Algebra, Lemma 66.2.

The following lemma was shown for $\text{Hom}$ and $\text{Ext}^1$ of modules in More on Algebra, Lemmas 66.3 and 66.8.
Lemma 8.14. Let $A \to B$ be a flat ring map and let $I \subset A$ be a finitely generated ideal such that $A/I \to B/IB$ is an isomorphism. For $K \in D_{I\text{-}\text{torsion}}(A)$ and $L \in D(A)$ the map
\[ R\text{Hom}_A(K, L) \to R\text{Hom}_B(K \otimes_A B, L \otimes_A B) \]
is a quasi-isomorphism. In particular, if $M$, $N$ are $A$-modules and $M$ is $I$-power torsion, then the canonical map
\[ \text{Ext}^i_A(M, N) \to \text{Ext}^i_B(M \otimes_A B, N \otimes_A B) \]
is an isomorphism for all $i$.

Proof. Let $Z = V(I) \subset \text{Spec}(A)$ and $Y = V(IB) \subset \text{Spec}(B)$. Since the cohomology modules of $K$ are $I$-power torsion, the canonical map $R\Gamma_Z(L) \in D(A)$. Similarly, the cohomology modules of $K \otimes_A B$ are $IB$-power torsion and we have an isomorphism
\[ R\text{Hom}_B(K \otimes_A B, R\Gamma_Y(L \otimes_A B)) \to R\text{Hom}_B(K \otimes_A B, L \otimes_A B) \]
in $D(B)$. By Lemma 8.11 we have $R\Gamma_Z(L) \otimes_A B = R\Gamma_Y(L \otimes_A B)$. Hence it suffices to show that the map
\[ R\text{Hom}_A(K, R\Gamma_Z(L)) \to R\text{Hom}_B(K \otimes_A B, R\Gamma_Z(L) \otimes_A B) \]
is a quasi-isomorphism. This follows from Lemma 8.13.

□

9. Depth

In this section we revisit the notion of depth introduced in Algebra, Section 70.

Lemma 9.1. Let $A$ be a Noetherian ring, let $I \subset A$ be an ideal, and let $M$ be a finite $A$-module such that $IM \neq M$. Then the following integers are equal:

1. $\text{depth}_I(M)$,
2. the smallest integer $i$ such that $\text{Ext}^i_A(A/I, M)$ is nonzero, and
3. the smallest integer $i$ such that $H^i_I(M)$ is nonzero.

Moreover, we have $\text{Ext}^i_A(N, M) = 0$ for $i < \text{depth}_I(M)$ for any finite $A$-module $N$ annihilated by a power of $I$.

Proof. We prove the equality of (1) and (2) by induction on $\text{depth}_I(M)$ which is allowed by Algebra, Lemma 70.4.

Base case. If $\text{depth}_I(M) = 0$, then $I$ is contained in the union of the associated primes of $M$ (Algebra, Lemma 62.9). By prime avoidance (Algebra, Lemma 14.2) we see that $I \subset \mathfrak{p}$ for some associated prime $\mathfrak{p}$. Hence $\text{Hom}_A(A/I, M)$ is nonzero. Thus equality holds in this case.

Assume that $\text{depth}_I(M) > 0$. Let $f \in I$ be $M$-regular. Consider the short exact sequence
\[ 0 \to M \to M \to M/fM \to 0 \]
and the associated long exact sequence for $\text{Ext}^i_A(A/I, -)$. Note that $\text{Ext}^i_A(A/I, M)$ is a finite $A/I$-module (Algebra, Lemmas 69.9 and 69.8). Hence we obtain
\[ \text{Hom}_A(A/I, M/fM) = \text{Ext}^1_A(A/I, M) \]
and short exact sequences

\[ 0 \to \text{Ext}^i_A(A/I, M) \to \text{Ext}^i_A(A/I, M/IM) \to \text{Ext}^{i+1}_A(A/I, M) \to 0 \]

Thus the equality of (1) and (2) by induction.

Observe that \( \text{depth}_I(M) = \text{depth}_{I^n}(M) \) for all \( n \geq 1 \) for example by Algebra, Lemma 67.8. Hence by the equality of (1) and (2) we see that \( \text{Ext}^i_A(A/I^n, M) = 0 \) for all \( n \) and \( i \) \( < \) \( \text{depth}_I(M) \). Let \( N \) be a finite \( A \)-module annihilated by a power of \( I \). Then we can choose a short exact sequence

\[ 0 \to N' \to (A/I^n)^{\oplus m} \to N \to 0 \]

for some \( n, m \geq 0 \). Then \( \text{Hom}_A(N, M) \subset \text{Hom}_A((A/I^n)^{\oplus m}, M) \) and \( \text{Ext}^i_A(N, M) \subset \text{Ext}^{i-1}_A(N', M) \) for \( i < \text{depth}_I(M) \). Thus a simply induction argument shows that the final statement of the lemma holds.

Finally, we prove that (3) is equal to (1) and (2). We have \( H^p_I(M) = \text{colim} \text{Ext}^p_A(A/I^n, M) \) by Lemma 8.2. Thus we see that \( H^1_I(M) = 0 \) for \( i < \text{depth}_I(M) \). For \( i = \text{depth}_I(M) \), using the vanishing of \( \text{Ext}^{i-1}_A(I/I^n, M) \) we see that the map \( \text{Ext}^i_A(A/I, M) \to H^i_I(M) \) is injective which proves nonvanishing in the correct degree.

**Lemma 9.2.** Let \( A \) be a ring and let \( I \subset A \) be a finitely generated ideal. Let \( M \) be an \( A \)-module. Let \( Z = V(I) \). Then \( H^0_I(M) = H^0_Z(M) \). Let \( N \) be the common value and set \( M' = M/N \). Then

\begin{enumerate}
\item \( H^p_I(M') = 0 \text{ and } H^p_Z(M') = 0 \) for all \( p > 0 \),
\item \( H^p_I(M') = 0 \text{ and } H^p_Z(M') = 0 \) for all \( p > 0 \).
\end{enumerate}

**Proof.** By definition \( H^0_I(M) = M[I^\infty] \) is \( I \)-power torsion. By Lemma 8.5 we see that \( H^p_Z(M) = \text{Ker}(M \to M_{f_1} \times \ldots \times M_{f_r}) \) if \( I = (f_1, \ldots, f_r) \). Thus \( H^p_I(M) \subset H^p_Z(M) \) and conversely, if \( x \in H^p_Z(M) \), then it is annihilated by \( f_i^{e_i} \) for some \( e_i \geq 1 \) hence annihilated by some power of \( I \). This proves the first equality and moreover \( N \) is \( I \)-power torsion. By Lemma 8.4 we see that \( \text{Ref}_I(N) = N \). By Lemma 8.5 we see that \( \text{Ref}_Z(N) = N \). This proves the higher vanishing of \( H^p_I(N) \) and \( H^p_Z(N) \) in (1) and (2). The vanishing of \( H^0_I(M') \) and \( H^0_Z(M') \) follow from the preceding remarks and the fact that \( M' \) is \( I \)-power torsion free by More on Algebra, Lemma 65.4. The equality of higher cohomologies for \( M \) and \( M' \) follow immediately from the long exact cohomology sequence. \( \square \)

### 10. Formally catenary rings

In this section we prove a theorem of Ratliff [Rat71] that a Noetherian local ring is universally catenary if and only if it is formally catenary.

**Definition 10.1.** A Noetherian local ring \( A \) is formally catenary if for every minimal prime \( p \subset A \) the ring \( A^\wedge/pA^\wedge \) is equidimensional.

The following lemma can be used to construct finite type extensions from given finite type extensions of the formal completion.

**Lemma 10.2.** Let \( A \) be a Noetherian ring and \( I \) an ideal. Let \( B \) be a finite type \( A \)-algebra. Let \( B^\wedge \to C \) be a surjective ring map with kernel \( J \) where \( B^\wedge \) is the \( I \)-adic completion. If \( J/J^2 \) is annihilated by \( I^c \) for some \( c \geq 0 \), then \( C \) is isomorphic to the completion of a finite type \( A \)-algebra.
Proof. Since $B^\wedge$ is Noetherian (Algebra, Lemma \textbf{94.10}), we see that $J$ is a finitely generated ideal. Hence we conclude from Algebra, Lemma \textbf{20.5} that

$$\text{Spec}(C) \setminus V(IC) \longrightarrow \text{Spec}(B^\wedge) \setminus V(IB^\wedge)$$

is an open and closed immersion. Let $V \subset \text{Spec}(B^\wedge) \setminus V(IB^\wedge)$ be the complement of the image viewed as an open and closed subscheme. Let $Z \subset \text{Spec}(B^\wedge)$ be the scheme theoretic closure of $V$. Write $Z = \text{Spec}(C')$. Then

$$\text{Spec}(C \times C') = \text{Spec}(C) \amalg Z \longrightarrow \text{Spec}(B^\wedge)$$

is a finite morphism of schemes which is an isomorphism away from $V(IB^\wedge)$. Hence the corresponding ring map $B^\wedge \to C \times C'$ is finite and becomes an isomorphism on inverting any element of $I$. Since $B \to B^\wedge$ is a flat map (Algebra, Lemma \textbf{94.3}) inducing an isomorphism $B/IB \to B^\wedge/IB^\wedge$ we may apply More on Algebra, Proposition \textbf{66.15} and Remark \textbf{66.19} to it. We conclude that $C \times C'$ is isomorphic to $D \otimes_B B^\wedge$ for some finite $B$-algebra $D$. Then $D/IB \cong C/IC \times C'/IC'$. Let $\pi \in D/ID$ be the idempotent corresponding to the factor $C/IC$. By More on Algebra, Lemma \textbf{6.9} there exists an étale ring map $B \to B'$ which induces an isomorphism $B/IB \to B'/IB'$ such that $D' = D \otimes_B B'$ contains an idempotent $e$ lifting $\pi$. Since $C \times C'$ is $I$-adically complete the pair $(C \times C', IC \times IC')$ is henselian (More on Algebra, Lemma \textbf{7.3}). Thus we can factor the map $B \to C \times C'$ through $B'$. Doing so we may replace $B$ by $B'$ and $D$ by $D'$. Then we find that $D = D_e \times D_{1-e} = D/(1-e) \times D/(e)$ is a product of finite type $A$-algebras and the completion of the first part is $C$ and the completion of the second part is $C'$. $\square$

Lemma \textbf{10.3}. Let $(A, \mathfrak{m})$ be a Noetherian local ring which is not formally catenary. Then $A$ is not universally catenary.

Proof. By assumption there exists a minimal prime $\mathfrak{p} \subset A$ such that $A^\wedge/\mathfrak{p}A^\wedge$ is not equidimensional. After replacing $A$ by $A/\mathfrak{p}$ we may assume that $A$ is a domain and that $A^\wedge$ is not equidimensional. Let $\mathfrak{q}$ be a minimal prime of $A^\wedge$ such that $d = \dim(A^\wedge/\mathfrak{q})$ is minimal and hence $0 < d < \dim(A)$. We prove the lemma by induction on $d$.

The case $d = 1$. In this case $\dim(A^\wedge_{\mathfrak{q}}) = 0$. Hence $A^\wedge_{\mathfrak{q}}$ is Artinian local and we see that for some $n > 0$ the ideal $J = \mathfrak{q}^n$ maps to zero in $A^\wedge_{\mathfrak{q}}$. It follows that $\mathfrak{m}$ is the only associated prime of $J/J^2$, whence $\mathfrak{m}^n$ annihilates $J/J^2$ for some $m > 0$. Thus we can use Lemma \textbf{10.2} to find $A \to B$ of finite type such that $B^\wedge \cong A^\wedge/J$. It follows that $\mathfrak{m}_B = \sqrt{\mathfrak{m}B}$ is a maximal ideal with the same residue field as $\mathfrak{m}$ and $B^\wedge$ is the $\mathfrak{m}_B$-adic completion (Algebra, Lemma \textbf{94.18}). Then

$$\dim(B_{\mathfrak{m}_B}) = \dim(B^\wedge) = 1 = d.$$ 

Since we have the factorization $A \to B \to A^\wedge/J$ the inverse image of $\mathfrak{q}/J$ is a prime $\mathfrak{q}' \subset \mathfrak{m}_B$ lying over $(0)$ in $A$. Thus, if $A$ were universally catenary, the dimension formula (Algebra, Lemma \textbf{110.1}) would give

$$\dim(B_{\mathfrak{m}_B}) \geq \dim((B/\mathfrak{q})_{\mathfrak{m}_B})$$

$$= \dim(A) + \text{trdeg}_{f.f.(A)}(f.f.(B/\mathfrak{q}')) - \text{trdeg}_{\kappa(\mathfrak{m})}(\kappa(\mathfrak{m}_B))$$

$$= \dim(A) + \text{trdeg}_{f.f.(A)}(f.f.(B/\mathfrak{q}'))$$

This contradictions finishes the argument in case $d = 1$. 

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Assume $d > 1$. Let $Z \subseteq \text{Spec}(A^\wedge)$ be the union of the irreducible components distinct from $V(q)$. Let $r_1, \ldots, r_m \subseteq A^\wedge$ be the prime ideals corresponding to irreducible components of $V(q) \cap Z$ of dimension $> 0$. Choose $f \in m, f \not\in A \cap r_j$ using prime avoidance (Algebra, Lemma 14.1). Then $\dim(A/fA) = \dim(A) - 1$ and there is some irreducible component of $V(q, f)$ of dimension $d - 1$. Thus $A/fA$ is not formally catenary and the invariant $d$ has decreased. By induction $A/fA$ is not universally catenary, hence $A$ is not universally catenary.

$\square$

**Lemma 10.4.** Let $A \to B$ be a flat local ring map of local Noetherian rings. Assume $B$ is catenary and equidimensional. Then

1. $B/\mathfrak{p}B$ is equidimensional for all $\mathfrak{p} \subset A$,
2. $A$ is catenary and equidimensional.

**Proof.** Let $\mathfrak{p} \subset A$ be a prime ideal. Let $q \subset B$ be a prime minimal over $\mathfrak{p}B$. Then $q \cap A = \mathfrak{p}$ by going down for $A \to B$ (Algebra, Lemma 38.17). Hence $A_{\mathfrak{p}} \to B_q$ is a flat local ring map with special fibre of dimension 0 and hence

$$\dim(A_{\mathfrak{p}}) = \dim(B_q) = \dim(B) - \dim(B/q)$$

(Algebra, Lemma 109.7). The second equality because $B$ is equidimensional and catenary. Thus $\dim(B/q)$ is independent of the choice of $q$ and we conclude that $B/\mathfrak{p}B$ is equidimensional of dimension $\dim(B) - \dim(A_{\mathfrak{p}})$. On the other hand, we have $\dim(B/\mathfrak{p}B) = \dim(A/\mathfrak{p}) + \dim(B/m_AB)$ and $\dim(B) = \dim(A) + \dim(B/m_AB)$ by flatness (see lemma cited above) and we get

$$\dim(A_{\mathfrak{p}}) = \dim(A) - \dim(\mathfrak{p}/\mathfrak{p}^\prime)$$

for all $\mathfrak{p} \in A$. Applying this to all minimal primes in $A$ we see that $A$ is equidimensional. If $\mathfrak{p} \subset \mathfrak{p}^\prime$ is a strict inclusion with no primes in between, then we may apply the above to the prime $\mathfrak{p}^\prime/\mathfrak{p}$ in $A/\mathfrak{p}$ because $A/\mathfrak{p} \to B/\mathfrak{p}B$ is flat and $B/\mathfrak{p}B$ is equidimensional, to get

$$1 = \dim((A/\mathfrak{p})_{\mathfrak{p}^\prime}) = \dim(A/\mathfrak{p}) - \dim(A/\mathfrak{p}^\prime)$$

Thus $\mathfrak{p} \mapsto \dim(A/\mathfrak{p})$ is a dimension function and we conclude that $A$ is catenary. $\square$

**Lemma 10.5.** Let $A$ be a formally catenary Noetherian local ring. Then $A$ is universally catenary.

**Proof.** We may replace $A$ by $A/\mathfrak{p}$ where $\mathfrak{p}$ is a minimal prime of $A$, see Algebra, Lemma 102.7. Thus we may assume that $A^\wedge$ is equidimensional. It suffices to show that every local ring essentially of finite type over $A$ is catenary (see for example Algebra, Lemma 102.5). Hence it suffices to show that $A[x_1, \ldots, x_n]_m$ is catenary where $m \subset A[x_1, \ldots, x_n]$ is a maximal ideal lying over $m_A$, see Algebra, Lemma 52.5 (and Algebra, Lemmas 102.6 and 102.4). Let $m' \subset A^\wedge[x_1, \ldots, x_n]$ be the unique maximal ideal lying over $m$. Then

$$A[x_1, \ldots, x_n]_m \to A^\wedge[x_1, \ldots, x_n]_{m'}$$

is local and flat (Algebra, Lemma 94.3). Hence it suffices to show that the ring on the right hand side is equidimensional and catenary, see Lemma 10.4. It is catenary because complete local rings are universally catenary (Algebra, Remark 150.9). Pick any minimal prime $q$ of $A^\wedge[x_1, \ldots, x_n]_{m'}$. Then $q = \mathfrak{p}A^\wedge[x_1, \ldots, x_n]_{m'}$ for some minimal prime $\mathfrak{p}$ of $A^\wedge$ (small detail omitted). Hence

$$\dim(A^\wedge[x_1, \ldots, x_n]_{m'}/q) = \dim(A^\wedge/\mathfrak{p}) + n = \dim(A^\wedge) + n$$
the first equality by Algebra, Lemma \[109.7\] and the second because \( A^\wedge \) is equidimensional. This finishes the proof.

\[\square\]

**Proposition 10.6** (Ratliff). A Noetherian local ring is universally catenary if and only if it is formally catenary.

**Proof.** Combine Lemmas \([10.3]\) and \([10.5]\). \(\square\)

11. Finiteness of local cohomology, I

In this section we discuss a baby case of the finiteness theorem. Here is a lemma of Faltings which shows that it suffices to prove local cohomology modules have an annihilator in order to prove that they are finite modules.

**Lemma 11.1.** Let \( A \) be a Noetherian ring, \( I \subset A \) an ideal, \( M \) a finite \( A \)-module, and \( n \geq 0 \) an integer. Let \( Z = V(I) \). The following are equivalent

1. \( H^i_Z(M) \) is finite for \( i \leq n \),
2. there exists an \( e \geq 0 \) such that \( I^e \) annihilates \( H^i_Z(M) \) for \( i \leq n \), and
3. there exists an ideal \( J \subset A \) with \( V(J) \subset Z \) such that \( J \) annihilates \( H^i_Z(M) \) for \( i \leq n \).

**Proof.** We prove the lemma by induction on \( n \). For \( n = 0 \) we have \( H^0_Z(M) \subset M \) is finite, hence (1), (2), and (3) are true. Assume that \( n > 0 \).

If (1) is true, then, since \( H^i_Z(M) = H^i_I(M) \) (Lemma \[8.9\]) is \( I \)-power torsion, we see that (2) holds. It is clear that (2) implies (3).

Assume (3) is true. Let \( N = H^0_Z(M) \) and \( M' = M/N \). By Lemma \[9.2\] we may replace \( M \) by \( M' \). Thus we may assume that \( H^0_Z(M) = 0 \). This means that depth\(_I(M) > 0 \) (Lemma \[9.1\]). Pick \( f \in I \) a nonzerodivisor on \( M \). After raising \( f \) to a suitable power, we may assume \( f \in J \) as \( V(J) \subset V(I) \). Then the long exact local cohomology sequence associated to the short exact sequence

\[
0 \to M \to M \to M/fM \to 0
\]

turns into short exact sequences

\[
0 \to H^i_Z(M) \to H^i_Z(M/fM) \to H^{i+1}_Z(M) \to 0
\]

for \( i < n \). We conclude that \( J^2 \) annihilates \( H^i_Z(M/fM) \) for \( i < n \). By induction hypothesis we see that \( H^i_Z(M/fM) \) is finite for \( i < n \). Using the short exact sequence once more we see that \( H^{i+1}_Z(M) \) is finite for \( i < n \) as desired. \(\square\)

The following result of Faltings allows us to prove finiteness of local cohomology at the level of local rings.

**Lemma 11.2.** Let \( A \) be a Noetherian ring, \( I \subset A \) an ideal, \( M \) a finite \( A \)-module, and \( n \geq 0 \) an integer. Let \( Z = V(I) \). The following are equivalent

1. the modules \( H^i_Z(M) \) are finite for \( i \leq n \), and
2. for all \( \mathfrak{p} \in \text{Spec}(A) \) the modules \( H^i_Z(M)_{\mathfrak{p}} \), \( i \leq n \) are finite \( A_{\mathfrak{p}} \)-modules.

**Proof.** The implication (1) \( \Rightarrow \) (2) is immediate. We prove the converse by induction on \( n \). The case \( n = 0 \) is clear because both (1) and (2) are always true in that case.

Assume \( n > 0 \) and that (2) is true. Let \( N = H^0_Z(M) \) and \( M' = M/N \). By Lemma \[9.2\] we may replace \( M \) by \( M' \). Thus we may assume that \( H^0_Z(M) = 0 \). This means
that \( \text{depth}_x(M) > 0 \) (Lemma 9.1). Pick \( f \in I \) a nonzerodivisor on \( M \) and consider the short exact sequence

\[
0 \to M \to M \to M/IM \to 0
\]

which produces a long exact sequence

\[
0 \to H^0_Z(M/IM) \to H^1_Z(M) \to H^1_Z(M/IM) \to H^2_Z(M) \to \ldots
\]

and similarly after localization. Thus assumption (2) implies that the modules \( H^i_Z(M/IM)_p \) are finite for \( i < n \). Hence by induction assumption \( H^i_Z(M/IM) \) are finite for \( i < n \).

Let \( p \) be a prime of \( A \) which is associated to \( H^i_Z(M) \) for some \( i \leq n \). Say \( p \) is the annihilator of the element \( x \in H^i_Z(M) \). Then \( p \in Z \), hence \( f \in p \). Thus \( fx = 0 \) and hence \( x \) comes from an element of \( H^{-1}_Z(M/IM) \) by the boundary map \( \delta \) in the long exact sequence above. It follows that \( p \) is an associated prime of the finite module \( \text{Im}(\delta) \). We conclude that \( \text{Ass}(H^i_Z(M)) \) is finite for \( i \leq n \), see Algebra, Lemma 62.5.

Recall that

\[
H^i_Z(M) \subseteq \prod_{p \in \text{Ass}(H^i_Z(M))} H^i_Z(M)_p
\]

by Algebra, Lemma 62.19. Since by assumption the modules on the right hand side are finite and \( I \)-power torsion, we can find integers \( e_p, i \geq 0 \), \( i \leq n \), \( p \in \text{Ass}(H^i_Z(M)) \) such that \( f^{e_p} \) annihilates \( H^i_Z(M)_p \). We conclude that \( f^e \) with \( e = \max\{e_p, i\} \) annihilates \( H^i_Z(M) \) for \( i \leq n \). By Lemma 11.1 we see that \( H^i_Z(M) \) is finite for \( i \leq n \).

**Lemma 11.3.** Let \( X \) be a locally Noetherian scheme. Let \( j : U \to X \) be the inclusion of an open subscheme with complement \( Z \). Let \( F \) be a coherent \( O_U \)-module. Assume

1. \( X \) is Nagata,
2. \( X \) is universally catenary, and
3. for \( x \in \text{Ass}(F) \) and \( z \in Z \cap \{x\} \) we have \( \dim(O_{X,x,z}) \geq 2 \).

Then \( j_* F \) is coherent.

**Proof.** The statement is local on \( X \), hence we may assume \( X \) is affine. Then \( U \) is quasi-compact, hence \( \text{Ass}(F) \) is finite (Divisors, Lemma 2.5). Thus we may argue by induction on the number of associated points. Let \( x \in U \) be a generic point of an irreducible component of the support of \( F \). By Divisors, Lemma 2.5 we have \( x \in \text{Ass}(F) \). By our choice of \( x \) we have \( \dim(F_x) = 0 \) as \( O_{X,x} \)-module. Hence \( F_x \) has finite length as an \( O_{X,x} \)-module (Algebra, Lemma 61.3). Thus we may use induction on this length.

Let \( \pi : Y \to X \) be the normalization of \( X \) in \( \text{Spec}(\kappa(x)) \), see Morphisms, Section 48. By Morphisms, Lemma 48.13 the morphism \( \pi \) is finite. Hence \( \mathcal{G} = \pi_* \mathcal{O}_Y \) is a coherent \( O_X \)-module by Cohomology of Schemes, Lemma 9.9. We have \( \mathcal{G}_x = \kappa(x) \), see Morphisms, Lemma 48.1. Choose a nonzero map \( \varphi_x : F_x \to \kappa(x) = G_x \). By Cohomology of Schemes, Lemma 9.6 there is an open \( x \in V \subseteq U \) and a map \( \varphi_V : F|_V \to G|_V \) whose stalk at \( x \) is \( \varphi_x \). Choose \( f \in \Gamma(X, \mathcal{O}_X) \), which does not vanish at \( x \) such that \( D(f) \subseteq V \). By Cohomology of Schemes, Lemma 10.4 (for example) we see that \( \varphi_V \) extends to \( f^n \mathcal{F} \to \mathcal{G}|_U \) for some \( n \). Precomposing with multiplication by \( f^n \) we obtain a map \( \mathcal{F} \to \mathcal{G}|_U \) whose stalk at \( x \) is nonzero. Let \( \mathcal{F}' \subseteq \mathcal{F} \) be the kernel. Note that \( \text{Ass}(\mathcal{F}') \subseteq \text{Ass}(\mathcal{F}) \), see Divisors, Lemma 2.4.
Since \( \text{length}_{\mathcal{O}_{X,z}}(F') = \text{length}_{\mathcal{O}_{X,z}}(F) - 1 \) we may apply the induction hypothesis to conclude \( j_*F' \) is coherent. If we can show that \( j_*(\mathcal{G}|_U) \) is coherent, then we consider the exact sequence

\[
0 \to j_*F' \to j_*F \to j_*(\mathcal{G}|_U)
\]

By Schemes, Lemma 24.1 the sheaf \( j_*F \) is quasi-coherent. Hence the image of \( j_*F \) in \( j_*(\mathcal{G}|_U) \) is coherent by Cohomology of Schemes, Lemma 9.3. Finally, \( j_*F \) is coherent by Cohomology of Schemes, Lemma 9.2.

It remains to prove that \( j_*(\mathcal{G}|_U) \) is coherent. We claim Divisors, Lemma 2.11 applies to

\[
\mathcal{G} \longrightarrow j_*(\mathcal{G}|_U)
\]

which finishes the proof. Let \( z \in X \). If \( z \in U \), then the map is an isomorphism on stalks as \( j_*((\mathcal{G}|_U)|_U) = \mathcal{G}|_U \). If \( z \in Z \), then \( z \not\in \text{Ass}(j_*((\mathcal{G}|_U)|_U)) \) (Divisors, Lemmas 5.9 and 5.3). Thus it suffices to show that \( \text{depth}(\mathcal{G}_z) \geq 2 \). Let \( y_1, \ldots, y_n \in Y \) be the points mapping to \( z \). By Algebra, Lemma 70.8 it suffices to show that \( \text{depth}(\mathcal{O}_{Y,y_i}) \geq 2 \) for \( i = 1, \ldots, n \). If not, then by Properties, Lemma 12.5 we see that \( \dim(\mathcal{O}_{Y,y_i}) = 1 \) for some \( i \). This is impossible by the dimension formula (Morphisms, Lemma 31.1) for \( \pi : Y \to \{x\} \) and assumption (3).

**Lemma 11.4.** Let \( X \) be a locally Noetherian scheme. Let \( j : U \to X \) be the inclusion of an open subscheme with complement \( Z \). Let \( F \) be a coherent \( \mathcal{O}_U \)-module. Assume

1. \( X \) is universally catenary,
2. for every \( z \in Z \) the formal fibres of \( \mathcal{O}_{X,z} \) are \((S_1)\), and
3. for \( x \in \text{Ass}(F) \) and \( z \in Z \cap \{x\} \) we have \( \dim(\mathcal{O}_{X,z}) \geq 2 \).

Then \( j_*F \) is coherent.

**Proof.** The statement is local on \( X \), hence we may assume \( X \) is affine. Say \( X = \text{Spec}(A) \) and \( Z = V(I) \). By Properties, Lemma 20.4 there exists a coherent \( \mathcal{O}_X \)-module \( F' \) whose restriction to \( U \) is isomorphic to \( F \). Say \( F' \) corresponds to the finite \( A \)-module \( M \). Note that \( j_*F \) is quasi-coherent (Schemes, Lemma 24.1) and corresponds to the \( A \)-module \( H^0(U,F) \). By Lemma 8.8 we obtain an exact sequence

\[
0 \to H^0_Z(M) \to M \to H^0(U,F) \to H^1_Z(M) \to 0
\]

The final zero because \( H^i(X,F') = 0 \) as \( X \) is affine and \( F' \) is quasi-coherent (Cohomology of Schemes, Lemma 2.2). Thus \( j_*F \) is coherent if and only if \( H^1_Z(M) \) is finite.

By Lemma 11.2 we may assume that \( A \) is a local Noetherian ring. Let \( m \) be the maximal ideal. We may assume \( I \subset m \), otherwise the lemma is trivial. Let \( A^\wedge \) be the completion of \( A \), let \( Z^\wedge = V(I A^\wedge) \), and let \( M^\wedge = M \otimes_A A^\wedge \) be the completion of \( M \) (Algebra, Lemma 94.2). Then \( H^1_Z(M) \otimes_A A^\wedge = H^1_Z(M^\wedge) \) by Lemma 8.11 and flatness of \( A \to A^\wedge \) (Algebra, Lemma 94.3). Hence it suffices to show that \( H^1_Z(M^\wedge) \) is finite, see Algebra, Lemma 81.2.

The ring \( A^\wedge \) is universally catenary and Nagata (Algebra, Remark 150.9 and Lemma 151.29). Thus, reading the arguments in the first paragraph of the proof backwards, we have the desired finiteness of \( H^1_Z(M^\wedge) \) by Lemma 11.3 provided assumption
(3) of that lemma are satisfied. Let \( p^\wedge \subset q^\wedge \subset A^\wedge \) be primes with \( p^\wedge \in \text{Ass}(M^\wedge) \) and \( q^\wedge \in Z^\wedge \). We have to show that \( \dim((A^\wedge/p^\wedge)_{q^\wedge}) \geq 2 \). Observe that

\[
\text{Ass}_{A^\wedge}(M^\wedge) = \bigcup_{p \in \text{Ass}_{A}(M)} \text{Ass}_{A^\wedge}(A^\wedge \otimes_A \kappa(p))
\]

by Algebra, Lemma \[64.5\] By assumption (2) we see that \( p^\wedge \) is minimal over \( pA^\wedge \) with \( p = A \cap p^\wedge \in \text{Ass}(M) \). On the other hand, \( q = A \cap q^\wedge \) is a point of \( Z \).

By assumption (1) \( A/p \) is universally catenary, hence \( A^\wedge/pA^\wedge \) is equidimensional, see Proposition \[10.6\]. Since \( A^\wedge/pA^\wedge \) is catenary, we conclude that \( (A^\wedge/pA^\wedge)_{q^\wedge} \) is equidimensional too. Hence

\[
\dim((A^\wedge/p^\wedge)_{q^\wedge}) = \dim((A^\wedge/pA^\wedge)_{q^\wedge}) \geq \dim((A/p)_{q}) \geq 2
\]

The first inequality by Algebra, Lemma \[109.7\] and the last by assumption (3).

\[
\mathbf{Remark\ 11.5.}\ \text{Let} \ X \ \text{be a locally Noetherian scheme. Let} \ j : U \to X \ \text{be the inclusion of an open subscheme with complement} \ Z. \ \text{Let} \ F \ \text{be a coherent} \ \mathcal{O}_U\text{-module. If there exists an} \ x \in \text{Ass}(F) \ \text{and} \ z \in Z \cap \{x\} \ \text{such that} \ \text{dim}(\mathcal{O}_{X,x,z}) \leq 1, \ \text{then} \ j_*F \ \text{is not coherent. To prove this we can do a flat base change to the spectrum of} \ \mathcal{O}_{X,z}. \ \text{Let} \ X' = \{x\}. \ \text{The assumption implies} \ \mathcal{O}_{X'\cap U} \subseteq F. \ \text{Thus it suffices to see that} \ j_*\mathcal{O}_{X'\cap U} \ \text{is not coherent. This is clear because} \ X' = \{x, z\}, \ \text{hence} \ j_*\mathcal{O}_{X'\cap U} \ \text{corresponds to} \ \kappa(x) \ \text{as an} \ \mathcal{O}_{X,z}\text{-module which cannot be finite as} \ x \ \text{is not a closed point.}
\]

\section{12. Torsion versus complete modules}

Let \( A \) be a ring and let \( I \) be a finitely generated ideal. In this case we can consider the derived category \( D_{I^\infty,\text{torsion}}(A) \) of complexes with \( I \)-power torsion cohomology modules (Section \[8\]) and the derived category \( D_{\text{comp}}(A,I) \) of derived complete complexes (More on Algebra, Section \[67\]). In this section we show these categories are equivalent. A more general statement can be found in \cite{DG02}.

\[
\text{Lemma\ 12.1.}\ \text{Let} \ A \ \text{be a ring and let} \ I \ \text{be a finitely generated ideal. Let} \ \Gamma_Z^A \ \text{be as in Lemma \[8.3\]. Let} \ ^\wedge \ \text{denote derived completion as in More on Algebra, Lemma \[67.9\]. For an object} \ K \ \text{in} \ D(A) \ \text{we have}
\]

\[
\Gamma_Z^A(K^\wedge) = \Gamma_Z^A(K) \ \text{and} \ (\Gamma_Z^A(K))^\wedge = K^\wedge
\]

in \( D(A) \).

\[
\text{Proof.}\ \text{Choose} \ f_1, \ldots, f_r \in A \ \text{generating} \ I.\ \text{Recall that}
\]

\[
K^\wedge = R \text{Hom}(\prod \to A_{f_{i_0}} \to \prod A_{f_{i_{01}}}, \to \cdots \to A_{f_1 \cdots f_r}, K)
\]

by More on Algebra, Lemma \[67.9\]. Hence the cone \( C = \text{Cone}(K \to K^\wedge) \) is given by

\[
R \text{Hom}(\prod A_{f_{i_0}} \to \prod A_{f_{i_{01}}}, \to \cdots \to A_{f_1 \cdots f_r}, K)
\]

which can be represented by a complex endowed with a finite filtration whose successive quotients are isomorphic to

\[
R \text{Hom}(A_{f_{i_0} \cdots f_p}, K), \ p > 0
\]

These complexes vanish on applying \( \Gamma_Z^A \), see Lemma \[8.6\]. Applying \( \Gamma_Z^A \) to the distinguished triangle \( K \to K^\wedge \to C \to K[1] \) we see that the first formula of the lemma is correct.
Recall that
\[ R\Gamma_Z(K) = K \otimes L (A \to \prod A_{f_0} \to \prod A_{f_{i_0} f_i} \to \ldots \to A_{f_1 \ldots f_r}) \]
by Lemma \[8.5\]. Hence the cone \( C = \text{Cone}(R\Gamma_Z(K) \to K) \) can be represented by a complex endowed with a finite filtration whose successive quotients are isomorphic to
\[ K \otimes_A A_{f_{0 \ldots f_p}}, \quad p > 0 \]
These complexes vanish on applying \( \wedge \), see More on Algebra, Lemma \[67.10\]. Applying derived completion to the distinguished triangle \( R\Gamma_Z(K) \to K \to C \to R\Gamma_Z(K)[1] \) we see that the second formula of the lemma is correct. \( \square \)

The following result is a special case of a very general phenomenon concerning admissible subcategories of a triangulated category.

**Proposition 12.2.** Let \( A \) be a ring and let \( I \subset A \) be a finitely generated ideal. The functors \( R\Gamma_Z \) and \( \wedge \) define quasi-inverse equivalences of categories
\[ D_{I^{\infty}\text{-torsion}}(A) \leftrightarrow D_{\text{comp}}(A, I) \]

**Proof.** Follows immediately from Lemma \[12.1\]. \( \square \)

The following addendum of the proposition above makes the correspondence on morphisms more precise.

**Lemma 12.3.** With notation as in Lemma \[12.1\]. For objects \( K, L \) in \( D(A) \) there is a canonical isomorphism
\[ R\text{Hom}(K^\wedge, L^\wedge) \to R\text{Hom}(R\Gamma_Z(K), R\Gamma_Z(L)) \]
in \( D(A) \).

**Proof.** Say \( I = (f_1, \ldots, f_r) \). Denote \( C = (A \to \prod A_{f_i} \to \ldots \to A_{f_1 \ldots f_r}) \) the alternating Čech complex. Then derived completion is given by \( R\text{Hom}(C, -) \) and local cohomology by \( C \otimes L \). Combining the isomorphism
\[ R\text{Hom}(K \otimes L C, L \otimes L C) = R\text{Hom}(K, R\text{Hom}(C, L \otimes L C)) \]
(More on Algebra, Lemma \[58.1\]) and the map
\[ L \to R\text{Hom}(C, L \otimes L C) \]
(More on Algebra, Lemma \[58.5\]) we obtain a map
\[ \gamma: R\text{Hom}(K, L) \to R\text{Hom}(K \otimes L C, L \otimes L C) \]
On the other hand, the right hand side is derived complete as it is equal to
\[ R\text{Hom}(C, R\text{Hom}(K, L \otimes L C)) \]
Thus \( \gamma \) factors through the derived completion of \( R\text{Hom}(K, L) \) by the universal property of derived completion. However, the derived completion goes inside the \( R\text{Hom} \) by More on Algebra, Lemma \[67.11\] and we obtain the desired map.

To show that the map of the lemma is an isomorphism we may assume that \( K \) and \( L \) are derived complete, i.e., \( K = K^\wedge \) and \( L = L^\wedge \). In this case we are looking at the map
\[ \gamma: R\text{Hom}(K, L) \to R\text{Hom}(R\Gamma_Z(K), R\Gamma_Z(L)) \]
By Proposition \[12.2\] we know that the cohomology groups of the left and the right hand side coincide. In other words, we have to check that the map \( \gamma \) sends a
morphism $\alpha : K \to L$ in $D(A)$ to the morphism $R\Gamma_Z(\alpha) : R\Gamma_Z(K) \to R\Gamma_Z(L)$. We omit the verification (hint: note that $R\Gamma_Z(\alpha)$ is just the map $\alpha \otimes \text{id}_C : K \otimes^L C \to L \otimes^L C$ which is almost the same as the construction of the map in More on Algebra, Lemma \[58.5\]. \hfill \Box

13. Trivial duality for a ring map

Let $A \to B$ be a ring homomorphism. Consider the functor

$$\text{Hom}(B, -) : \text{Mod}_A \longrightarrow \text{Mod}_B, \quad M \mapsto \text{Hom}_A(B, M)$$

This functor is left exact and has a derived extension $R\text{Hom}(B, -) : D(A) \to D(B)$. Note that for every $K \in D(A)$ there is a canonical map $i_* R\text{Hom}(B, K) \to K$ where $i_* : D(B) \to D(A)$ is the obvious functor.

**Lemma 13.1.** With notation as above. The functor $R\text{Hom}(B, -)$ is the right adjoint to the functor $i_* : D(B) \to D(A)$.

**Proof.** This is a consequence of the fact that $i_*$ and $\text{Hom}_A(B, -)$ are adjoint functors by Algebra, Lemma \[13.3\] See Derived Categories, Lemma \[28.4\]. \hfill \Box

**Lemma 13.2.** With notation as above. For $K$ in $D(A)$ we have $R^q \text{Hom}(B, K) = \text{Ext}_A^q(B, K)$ as $A$-modules (the left hand side starts out as a $B$-module).

**Proof.** Omitted. \hfill \Box

Let $A$ be a Noetherian ring. We will denote

$$D_{\text{Coh}}(A) \subset D(A)$$

the full subcategory consisting of those objects $K$ of $D(A)$ whose cohomology modules are all finite $A$-modules. This makes sense by Derived Categories, Section \[13\] because as $A$ is Noetherian, the subcategory of finite $A$-modules is a Serre subcategory of $\text{Mod}_A$.

**Lemma 13.3.** With notation as above, assume $A \to B$ is a finite ring map of Noetherian rings. Then $R\text{Hom}(B, -)$ maps $D_{\text{Coh}}(A)$ into $D_{\text{Coh}}(B)$.

**Proof.** We have to show: if $K \in D^+(A)$ has finite cohomology modules, then the complex $R\text{Hom}(B, K)$ has finite cohomology modules too. This follows for example from Lemma \[13.2\] if we can show the ext modules $\text{Ext}_A^q(B, K)$ are finite $A$-modules. Since $K$ is bounded below there is a convergent spectral sequence

$$\text{Ext}_A^p(B, H^q(K)) \Rightarrow \text{Ext}_B^{p+q}(B, K)$$

This finishes the proof as the modules $\text{Ext}_A^p(B, H^q(K))$ are finite by Algebra, Lemma \[69.9\]. \hfill \Box

**Remark 13.4.** Let $A$ be a ring and let $I \subset A$ be an ideal. Set $B = A/I$. In this case the functor $\text{Hom}_A(B, -)$ is equal to the functor

$$\text{Mod}_A \longrightarrow \text{Mod}_B, \quad M \mapsto M[I]$$

which sends $M$ to the submodule of $I$-torsion.
14. Sections with support in a closed subscheme

Let \( i : (Z, \mathcal{O}_Z) \to (X, \mathcal{O}_X) \) be a morphism of ringed spaces such that \( i \) is a homomorphism onto a closed subset and such that \( i^\sharp : \mathcal{O}_X \to i_* \mathcal{O}_Z \) is surjective. (For example a closed immersion of schemes.) Let \( \mathcal{I} = \text{Ker}(i^\sharp) \). For a sheaf of \( \mathcal{O}_X \)-modules \( \mathcal{F} \) the sheaf

\[
\mathcal{H}om_{\mathcal{O}_X}(i_* \mathcal{O}_Z, \mathcal{F})
\]

a sheaf of \( \mathcal{O}_X \)-modules annihilated by \( \mathcal{I} \). Hence by Modules, Lemma 13.4 there is a sheaf of \( \mathcal{O}_Z \)-modules, which we will denote \( \mathcal{H}om(\mathcal{O}_Z, \mathcal{F}) \), such that

\[
i_* \mathcal{H}om(\mathcal{O}_Z, \mathcal{F}) = \mathcal{H}om_{\mathcal{O}_X}(i_* \mathcal{O}_Z, \mathcal{F})
\]
as \( \mathcal{O}_X \)-modules. We spell out what this means.

**Lemma 14.1.** With notation as above. The functor \( \mathcal{H}om(\mathcal{O}_Z, -) \) is a right adjoint to the functor \( i_* : \text{Mod}(\mathcal{O}_Z) \to \text{Mod}(\mathcal{O}_X) \). For \( V \subset Z \) open we have

\[
\Gamma(V, \mathcal{H}om(\mathcal{O}_Z, \mathcal{F})) = \{ s \in \Gamma(U, \mathcal{F}) \mid \mathcal{I}s = 0 \}
\]
where \( U \subset X \) is an open whose intersection with \( Z \) is \( V \).

**Proof.** Let \( \mathcal{G} \) be a sheaf of \( \mathcal{O}_Z \)-modules. Then

\[
\mathcal{H}om_{\mathcal{O}_X}(i_* \mathcal{G}, \mathcal{F}) = \mathcal{H}om_{\mathcal{O}_X}(i_* \mathcal{G}, \mathcal{H}om_{\mathcal{O}_X}(i_* \mathcal{O}_Z, \mathcal{F})) = \mathcal{H}om_{\mathcal{O}_Z}(\mathcal{G}, \mathcal{H}om(\mathcal{O}_Z, \mathcal{F}))
\]
The first equality by Modules, Lemma 19.5 and the second by the fully faithfulness of \( i_* \), see Modules, Lemma 13.4 The description of sections is left to the reader. □

The functor

\[
\text{Mod}(\mathcal{O}_X) \to \text{Mod}(\mathcal{O}_Z), \quad \mathcal{F} \mapsto \mathcal{H}om(\mathcal{O}_Z, \mathcal{F})
\]
is left exact and has a derived extension

\[
R \mathcal{H}om(\mathcal{O}_Z, -) : D(\mathcal{O}_X) \to D(\mathcal{O}_Z).
\]

**Lemma 14.2.** With notation as above. The functor \( R \mathcal{H}om(\mathcal{O}_Z, -) \) is the right adjoint of the functor \( i_* : D(\mathcal{O}_Z) \to D(\mathcal{O}_X) \).

**Proof.** This is a consequence of the fact that \( i_* \) and \( \mathcal{H}om(\mathcal{O}_Z, -) \) are adjoint functors by Lemma 14.1 See Derived Categories, Lemma 28.4 □

**Lemma 14.3.** With notation as above. We have

\[
i_* R \mathcal{H}om(\mathcal{O}_Z, K) = R \mathcal{H}om(i_* \mathcal{O}_Z, K)
\]
in \( D(\mathcal{O}_X) \) for all \( K \) in \( D(\mathcal{O}_X) \).

**Proof.** This is immediate from the construction of the functor \( R \mathcal{H}om(\mathcal{O}_Z, -) \). □

**Lemma 14.4.** In the situation above, assume \( i : Z \to X \) is a pseudo-coherent morphism of schemes (for example if \( X \) is locally Noetherian). Then

1. \( R \mathcal{H}om(\mathcal{O}_Z, -) \) maps \( D^+_{QCoh}(\mathcal{O}_X) \) into \( D^+_{QCoh}(\mathcal{O}_Z) \), and
2. if \( X = \text{Spec}(A) \) and \( Z = \text{Spec}(B) \), then the diagram

\[
\begin{array}{ccc}
D^+(B) & \longrightarrow & D^+_{QCoh}(\mathcal{O}_Z) \\
R \mathcal{H}om(B, -) & \longrightarrow & \downarrow \\
D^+(A) & \longrightarrow & D^+_{QCoh}(\mathcal{O}_X) \\
\end{array}
\]

is commutative.
**Proof.** To explain the parenthetical remark, if $X$ is locally Noetherian, then $i$ is pseudo-coherent by More on Morphisms, Lemma 42.9.

Let $K$ be an object of $D_{QCoh}^+(O_X)$. To prove (1), by Morphisms, Lemma 42.11 it suffices to show that $i_*$ applied to $H^n(R\mathcal{H}om(O_Z,K))$ produces a quasi-coherent module on $X$. By Lemma 14.3 this means we have to show that $R\mathcal{H}om(i_*O_Z,K)$ is in $D_{QCoh}(O_X)$. Since $i$ is pseudo-coherent the sheaf $O_Z$ is a pseudo-coherent $O_X$-module. Hence the result follows from Derived Categories of Schemes, Lemma 10.8.

Assume $X = \text{Spec}(A)$ and $Z = \text{Spec}(B)$ as in (2). Let $I^\bullet$ be a bounded below complex of injective $A$-modules representing an object $K$ of $D^+(A)$. Then we know that $R\mathcal{H}om(B,K) = \mathcal{H}om_A(B,I^\bullet)$ viewed as a complex of $B$-modules. Choose a quasi-isomorphism

$$\tilde{I}^\bullet \longrightarrow I^\bullet$$

where $I^\bullet$ is a bounded below complex of injective $O_X$-modules. It follows from the description of the functor $\mathcal{H}om(O_Z,-)$ in Lemma 14.1 that there is a map

$$\mathcal{H}om_A(B,I^\bullet) \longrightarrow \Gamma(Z,\mathcal{H}om(O_Z,I^\bullet))$$

Observe that $\mathcal{H}om(O_Z,I^\bullet)$ represents $R\mathcal{H}om(O_Z,\tilde{K})$. Applying the universal property of the $\tilde{}$ functor we obtain a map

$$\mathcal{H}om_A(B,I^\bullet) \longrightarrow R\mathcal{H}om(O_Z,\tilde{K})$$

in $D(O_Z)$. We may check that this map is an isomorphism in $D(O_Z)$ after applying $i_*$. However, once we apply $i_*$ we obtain the isomorphism of Derived Categories of Schemes, Lemma 10.8 via the identification of Lemma 14.3. □

**Lemma 14.5.** In this situation above. Assume $X$ is a locally Noetherian scheme. Then $R\mathcal{H}om(O_Z,-)$ maps $D^+_{Coh}(O_X)$ into $D^+_{Coh}(O_Z)$.

**Proof.** The question is local on $X$, hence we may assume that $X$ is affine. Say $X = \text{Spec}(A)$ and $Z = \text{Spec}(B)$ with $A$ Noetherian and $A \to B$ surjective. In this case, we can apply Lemma 14.3 to translate the question into algebra. The corresponding algebra result is a consequence of Lemma 13.3. □

### 15. Dualizing complexes

In this section we define dualizing complexes for Noetherian rings.

**Definition 15.1.** Let $A$ be a Noetherian ring. A dualizing complex is a complex of $A$-modules $\omega_A^\bullet$ such that

1. $\omega_A^\bullet$ has finite injective dimension,
2. $H^i(\omega_A^\bullet)$ is a finite $A$-module for all $i$, and
3. $A \to R\mathcal{H}om(\omega_A^\bullet,\omega_A^\bullet)$ is a quasi-isomorphism.

This definition takes some time getting used to. It is perhaps a good idea to prove some of the following lemmas yourself without reading the proofs.

**Lemma 15.2.** Let $A$ be a Noetherian ring. If $\omega_A^\bullet$ is a dualizing complex, then the functor

$$D : K \longmapsto R\mathcal{H}om(K,\omega_A^\bullet)$$
is an anti-equivalence $D_{Coh}(A) \to D_{Coh}(A)$ which exchanges $D^+_{Coh}(A)$ and $D^-_{Coh}(A)$ and induces an equivalence $D^b_{Coh}(A) \to D^b_{Coh}(A)$. Moreover $D \circ D$ is isomorphic to the identity functor.

**Proof.** Let $K$ be an object of $D_{Coh}(A)$. Pick an integer $n$ and consider the distinguished triangle

$$\tau_{\leq n} K \to K \to \tau_{\geq n+1} K \to \tau_{\leq n} K[1]$$

see Derived Categories, Remark [12.4.13]. Since $\omega_A^\bullet$ has finite injective dimension we see that $R\text{Hom}(\tau_{\geq n+1} K, \omega_A^\bullet)$ has vanishing cohomology in degrees $\geq n - c$ for some constant $c$. On the other hand, we obtain a spectral sequence

$$\text{Ext}^q_A(H^{-q}(\tau_{\leq n} K, \omega_A^\bullet)) \Rightarrow \text{Ext}^{p+q}_A(\tau_{\leq n} K, \omega_A^\bullet) = H^{p+q}(R\text{Hom}(\tau_{\leq n} K, \omega_A^\bullet))$$

which shows that these cohomology modules are finite. Since for $n > p + q + c$ this is equal to $H^{p+q}(R\text{Hom}(K, \omega_A^\bullet))$ we see that $R\text{Hom}(K, \omega_A^\bullet)$ is indeed an object of $D_{Coh}(A)$. By More on Algebra, Lemma [69.2.12] and the assumptions on the dualizing isomorphism

$$K = R\text{Hom}(\omega_A^\bullet, \omega_A^\bullet) \otimes^L_A K \to R\text{Hom}(R\text{Hom}(K, \omega_A^\bullet), \omega_A^\bullet)$$

Thus our functor has a quasi-inverse and the proof is complete. □

**Lemma 15.3.** Let $A$ be a Noetherian ring. Let $K \in D^b_{Coh}(A)$. Let $m$ be a maximal ideal of $A$. If $H^i(K)/mH^i(K) \neq 0$, then there exists a finite $A$-module $E$ annihilated by a power of $m$ and a map $K \to E[-i]$ which is nonzero on $H^i(K)$.

**Proof.** Let $I$ be the injective hull of the residue field of $m$. If $H^i(K)/mH^i(K) \neq 0$, then there exists a nonzero map $H^i(K) \to I$. Since $I$ is injective, we can lift this to a nonzero map $K \to I[-i]$. Recall that $I = \bigcup I[m^n]$, see Lemma [7.2.1] and that each of the modules $E = I[m^n]$ is of the desired type. Thus it suffices to prove that

$$\text{Hom}_{D(A)}(K, I) = \text{colim} \text{Hom}_{D(A)}(K, I[m^n])$$

This would be immediate if $K$ where a compact object (or a perfect object) of $D(A)$. This is not the case, but $K$ is a pseudo-coherent object which is enough here. Namely, we can represent $K$ by a bounded above complex of finite free $R$-modules $K^\bullet$. In this case the Hom groups above are computed by using $\text{Hom}_{K(A)}(K^\bullet, -)$. As each $K^n$ is finite free the limit statement holds and the proof is complete. □

Let $R$ be a ring. We will say that an object $L$ of $D(R)$ is **invertible** if there is an open covering $\text{Spec}(R) = \bigcup D(f_i)$ such that $L \otimes_R R_{f_i} \cong R_{f_i}[-n_i]$ for some integers $n_i$. In this case, the function

$$p \mapsto n_p,$$

where $n_p$ is the unique integer such that $H^{nr}(L \otimes \kappa(p)) \neq 0$

is locally constant on $\text{Spec}(R)$. In particular, it follows that $L = \bigoplus H^n(L)[-n]$ which gives a well defined complex of $R$-modules (with zero differentials) representing $L$. Since each $H^n(L)$ is finite projective and nonzero for only a finite number of $n$ we also see that $L$ is a perfect object of $D(R)$.

**Lemma 15.4.** Let $A$ be a Noetherian ring. Let $F : D^b_{Coh}(A) \to D^b_{Coh}(A)$ be an $A$-linear equivalence of categories. Then $F(A)$ is an invertible object of $D(A)$.
Proof. Let $m \subset A$ be a maximal ideal with residue field $\kappa$. Consider the object $F(\kappa)$. Since $\kappa = \text{Hom}_{D(A)}(\kappa, \kappa)$ we find that all cohomology groups of $F(\kappa)$ are annihilated by $m$. We also see that

$$\text{Ext}^i_A(\kappa, \kappa) = \text{Ext}^i_A(F(\kappa), F(\kappa)) = \text{Hom}_{D(A)}(F(\kappa), F(\kappa)[-i])$$

is zero for $i < 0$. Say $H^a(F(\kappa)) \neq 0$ and $H^b(F(\kappa)) \neq 0$ with $a$ minimal and $b$ maximal (so in particular $a \leq b$). Then there is a nonzero map

$$F(\kappa) \rightarrow H^b(F(\kappa))[-b] \rightarrow H^a(F(\kappa))[-b] \rightarrow F(\kappa)[a - b]$$

in $D(A)$ (nonzero because it induces a nonzero map on cohomology). This proves that $b = a$. We conclude that $F(\kappa) = \kappa[-a]$.

Let $G$ be a quasi-inverse to our functor $F$. Arguing as above we find an integer $b$ such that $G(\kappa) = \kappa[-b]$. On composing we find $a + b = 0$. Let $E$ be a finite $A$-module which is annihilated by a power of $m$. Arguing by induction on the length of $E$ we find that $G(E) = E'[-b]$ for some finite $A$-module $E'$ annihilated by a power of $m$. Then $E[-a] = F(E')$. Next, we consider the groups

$$\text{Ext}^i_A(E', E') = \text{Ext}^i_A(F(A), F(E')) = \text{Hom}_{D(A)}(F(A), E'[-a + i])$$

The left hand side is nonzero if and only if $i = 0$ and then we get $E'$. Applying this with $E = E' = \kappa$ and using Nakayama’s lemma this implies that $H^j(F(A))$ is zero for $j > a$ and generated by 1 element for $j = a$. On the other hand, if $H^j(F(A))_m$ is not zero for some $j < a$, then there is a map $F(A) \rightarrow E[-a + i]$ for some $i < 0$ and some $E$ (Lemma 15.3). Thus we see that $F(A)_m = M[-a]$ for some $A_m$-module $M$ generated by 1 element. However, since

$$A_m = \text{Hom}_{D(A)}(A, A)_m = \text{Hom}_{D(A)}(F(A), F(A))_m = \text{Hom}_{A_m}(M, M)$$

we see that $M \cong A_m$. We conclude that there exists an element $f \in A$, $f \notin m$ such that $F(A)_f$ is isomorphic to $A_f[-a]$. This finishes the proof. \qed

Lemma 15.5. Let $A$ be a Noetherian ring. If $\omega_A$ and $(\omega_A')^\bullet$ are dualizing complexes, then $(\omega_A')^\bullet$ is quasi-isomorphic to $\omega_A^\bullet \otimes_A^L L$ for some invertible object $L$ of $D(A)$.

Proof. By Lemmas 15.2 and 15.4 the functor $K \mapsto R\text{Hom}(R\text{Hom}(K, \omega_A^\bullet), (\omega_A')^\bullet)$ maps $A$ to an invertible object $L$. In other words, there is an isomorphism

$$L \rightarrow R\text{Hom}(\omega_A^\bullet, (\omega_A')^\bullet)$$

Since $L$ has finite tor dimension, this means that we can apply More on Algebra, Lemma 69.2 to see that

$$R\text{Hom}(\omega_A^\bullet, (\omega_A')^\bullet) \otimes_A^L K \rightarrow R\text{Hom}(R\text{Hom}(K, \omega_A^\bullet), (\omega_A')^\bullet)$$

is an isomorphism for $K$ in $D_{\text{coh}}^b(A)$. In particular, setting $K = \omega_A^\bullet$ finishes the proof. \qed

Lemma 15.6. Let $A$ be a Noetherian ring. Let $B = S^{-1}A$ be a localization. If $\omega_A^\bullet$ is a dualizing complex, then $\omega_A^\bullet \otimes_A B$ is a dualizing complex for $B$.

Proof. Let $\omega_A^\bullet \rightarrow I^\bullet$ be a quasi-isomorphism with $I^\bullet$ a bounded complex of injectives. Then $S^{-1}I^\bullet$ is a bounded complex of injective $B = S^{-1}A$-modules (Lemma 3.9) representing $\omega_A^\bullet \otimes_A B$. Thus $\omega_A^\bullet \otimes_A B$ has finite injective dimension. Since
Lemma 15.7. Let $A$ be a Noetherian ring. Let $f_1, \ldots, f_n \in A$ generate the unit ideal. If $\omega_A^\bullet$ is a complex of $A$-modules such that $(\omega_A^\bullet_{f_i})$ is a dualizing complex for $A_{f_i}$ for all $i$, then $\omega_A^\bullet$ is a dualizing complex for $A$.

Proof. Consider the double complex $A \otimes B \to R\text{Hom}(\omega_A^\bullet \otimes_A B, \omega_A^\bullet \otimes_A B)$ is a quasi-isomorphism as formation of internal hom commutes with flat base change in this case, see More on Algebra, Lemma 69.3.

Lemma 15.8. Let $A \to B$ be a surjective homomorphism of Noetherian rings. Let $\omega_A^\bullet$ be a dualizing complex. Then $R\text{Hom}(B, \omega_A^\bullet)$ is a dualizing complex for $B$.

Proof. Let $\omega_A^\bullet \to I^\bullet$ be a quasi-isomorphism with $I^\bullet$ a bounded complex of injectives. Then $\text{Hom}_A(B, I^\bullet)$ is a bounded complex of injective $A$-modules (Lemma 3.4) representing $R\text{Hom}(B, \omega_A^\bullet)$. Thus $R\text{Hom}(B, \omega_A^\bullet)$ has finite injective dimension. By Lemma 13.3 it is an object of $D_{\text{Cont}}(B)$. Finally, we compute

\[
\text{Hom}_{D_{\text{Cont}}(B)}(R\text{Hom}(B, \omega_A^\bullet), R\text{Hom}(B, \omega_A^\bullet)) = \text{Hom}_{D_{\text{Cont}}(A)}(R\text{Hom}(B, \omega_A^\bullet), \omega_A^\bullet) = B
\]

and for $n \neq 0$ we compute

\[
\text{Hom}_{D_{\text{Cont}}(B)}(R\text{Hom}(B, \omega_A^\bullet), R\text{Hom}(B, \omega_A^\bullet)[n]) = \text{Hom}_{D_{\text{Cont}}(A)}(R\text{Hom}(B, \omega_A^\bullet), \omega_A^\bullet[n]) = 0
\]

which proves the last property of a dualizing complex. In the displayed equations, the first equality holds by Lemma 13.1 and the second equality holds by Lemma 15.2.

Lemma 15.9. Let $A$ be a Noetherian ring. If $\omega_A^\bullet$ is a dualizing complex, then $\omega_A^\bullet \otimes_A A[x]$ is a dualizing complex for $A[x]$.

Proof. Set $B = A[x]$ and $\omega_B^\bullet = \omega_A^\bullet \otimes_A B$. It follows from Lemma 56.4 that $\omega_B^\bullet$ has finite injective dimension. Since $H^i(\omega_B^\bullet) = H^i(\omega_A^\bullet) \otimes_A B$ by flatness of $A \to B$ we see that $\omega_A^\bullet \otimes_A B$ has finite cohomology modules. Finally, the map

\[
B \to R\text{Hom}(\omega_B^\bullet, \omega_B^\bullet)
\]
is a quasi-isomorphism as formation of internal hom commutes with flat base change in this case, see More on Algebra, Lemma 69.3.

**Proposition 15.10.** Let $A$ be a Noetherian ring which has a dualizing complex. Then any $A$-algebra essentially of finite type over $A$ has a dualizing complex.

**Proof.** This follows from a combination of Lemmas 15.6, 15.8, and 15.9.

**Lemma 15.11.** Let $A$ be a Noetherian ring. Let $\omega_A^\bullet$ be a dualizing complex. Let $m \subset A$ be a maximal ideal and set $\kappa = A/m$. Then $R\text{Hom}_A(\kappa, \omega_A^\bullet) \cong \kappa[n]$ for some $n \in \mathbb{Z}$.

**Proof.** This is true because $R\text{Hom}_A(\kappa, \omega_A^\bullet)$ is a dualizing complex over $\kappa$ (Lemma 15.8), because dualizing complexes over $\kappa$ are unique up to shifts (Lemma 15.5), and because $\kappa$ is a dualizing complex over $\kappa$.

### 16. Dualizing complexes over local rings

In this section $(A, m, \kappa)$ will be a Noetherian local ring endowed with a dualizing complex $\omega_A^\bullet$ such that the integer $n$ of Lemma 15.11 is zero. More precisely, we assume that $R\text{Hom}_A(\kappa, \omega_A^\bullet) = \kappa[0]$. In this case we will say that the dualizing complex is normalized. Observe that a normalized dualizing complex is unique up to isomorphism and that any other dualizing complex for $A$ is isomorphic to a shift of a normalized one (Lemma 15.5).

**Lemma 16.1.** Let $(A, m, \kappa)$ be a Noetherian local ring with normalized dualizing complex $\omega_A^\bullet$. Let $A \to B$ be surjective. Then $\omega_B^\bullet = R\text{Hom}_A(B, \omega_A^\bullet)$ is a normalized dualizing complex for $B$.

**Proof.** By Lemma 15.8 the complex $\omega_B^\bullet$ is dualizing for $B$. We compute

$$R\text{Hom}_B(\kappa, R\text{Hom}_A(B, \omega_A^\bullet)) = R\text{Hom}_A(\kappa, \omega_A^\bullet) \cong \kappa[0]$$

The first equality by Lemma 13.1.

**Lemma 16.2.** Let $(A, m, \kappa)$ be a Noetherian local ring. Let $F$ be an $A$-linear self-equivalence of the category of finite length $A$-modules. Then $F$ is isomorphic to the identity functor.

**Proof.** Since $\kappa$ is the unique simple object of the category we have $F(\kappa) \cong \kappa$. Since our category is abelian, we find that $F$ is exact. Hence $F(E)$ has the same length as $E$ for all finite length modules $E$. Since $\text{Hom}(E, \kappa) = \text{Hom}(F(E), F(\kappa)) \cong \text{Hom}(F(E), \kappa)$ we conclude from Nakayama’s lemma that $E$ and $F(E)$ have the same number of generators. Hence $F(A/m^n)$ is a cyclic $A$-module. Pick a generator $e \in F(A/m^n)$. Since $F$ is $A$-linear we conclude that $m^n e = 0$. The map $A/m^n \to F(A/m^n)$ has to be an isomorphism as the lengths are equal. Pick an element $e \in \text{lim} F(A/m^n)$ which maps to a generator for all $n$ (small argument omitted). Then we obtain a system of isomorphisms $A/m^n \to F(A/m^n)$ compatible with all $A$-module maps $A/m^n \to A/m^n$ (by $A$-linearity of $F$ again). Since any finite length module is a cokernel of a map between direct sums of cyclic modules, we obtain the isomorphism of the lemma.
Lemma 16.3. Let \((A, \mathfrak{m}, \kappa)\) be a Noetherian local ring with normalized dualizing complex \(\omega_A^\bullet\). Let \(E\) be an injective hull of \(\kappa\). Then there exists a functorial isomorphism

\[
R\Hom(N, \omega_A^\bullet) = \Hom_A(N, E)[0]
\]
for \(N\) running through the finite length \(A\)-modules.

Proof. By induction on the length of \(N\) we see that \(R\Hom(N, \omega_A^\bullet)\) is a module of finite length sitting in degree 0. Thus \(R\Hom_A(-, \omega_A^\bullet)\) induces an anti-equivalence on the category of finite length modules. Since the same is true for \(\Hom_A(-, E)\) by Proposition 7.8 we see that

\[
N \mapsto \Hom_A(R\Hom(N, \omega_A^\bullet), E)
\]
is an equivalence as in Lemma 16.2 Hence it is isomorphic to the identity functor. Since \(\Hom_A(-, E)\) applied twice is the identity (Proposition 7.8) we obtain the statement of the lemma. \qed

Lemma 16.4. Let \((A, \mathfrak{m}, \kappa)\) be a Noetherian local ring with normalized dualizing complex \(\omega_A^\bullet\). Let \(M\) be a finite \(A\)-module and let \(d = \dim(\text{Supp}(M))\). Then

1. if \(\Ext_A^i(M, \omega_A^\bullet)\) is nonzero, then \(i \in \{-d, \ldots, 0\}\),
2. the dimension of the support of \(\Ext_A^i(M, \omega_A^\bullet)\) is at most \(-i\),
3. \(\text{depth}(M)\) is the smallest integer \(\delta \geq 0\) such that \(\Ext_A^\delta(M, \omega_A^\bullet) \neq 0\).

Proof. We prove this by induction on \(d\). If \(d = 0\), this follows from Lemma 16.3 and Matlis duality (Proposition 7.8) which guarantees that \(\Hom_A(M, E)\) is nonzero if \(M\) is nonzero.

Assume the result holds for modules with support of dimension < \(d\) and that \(M\) has depth > 0. Choose an \(f \in \mathfrak{m}\) which is a nonzerodivisor on \(M\) and consider the short exact sequence

\[
0 \to M \to M \to M/fM \to 0
\]
Since \(\dim(\text{Supp}(M/fM)) = d - 1\) (Algebra, Lemma 62.10) we may apply the induction hypothesis. Writing \(E^i = \Ext_A^i(M, \omega_A^\bullet)\) and \(F^i = \Ext_A^i(M/fM, \omega_A^\bullet)\) we obtain a long exact sequence

\[
\cdots \to F^i \to E^i \xrightarrow{f} E^i \to F^{i+1} \to \cdots
\]
By induction \(E^i/fE^i = 0\) for \(i+1 \notin \{-\dim(\text{Supp}(M/fM)), \ldots, -\text{depth}(M/fM)\}\).

By Nakayama’s lemma (Algebra, Lemma 19.1) and Algebra, Lemma 70.7 we conclude \(E^i = 0\) for \(i \notin \{-\dim(\text{Supp}(M)), \ldots, -\text{depth}(M)\}\). Moreover, in the boundary case \(i = -\text{depth}(M)\) we deduce that \(E^i\) is nonzero as \(F^{i+1}\) is nonzero by induction. Since \(E^i/fE^i \subset F^{i+1}\) we get

\[
\dim(\text{Supp}(F^{i+1})) \geq \dim(\text{Supp}(E^i/fE^i)) \geq \dim(\text{Supp}(E^i)) - 1
\]
(see lemma used above) we also obtain the dimension estimate (2).

If \(M\) has depth 0 and \(d > 0\) we let \(N = M[\mathfrak{m}^\infty]\) and set \(M' = M/N\) (compare with Lemma 9.2). Then \(M'\) has depth > 0 and \(\dim(\text{Supp}(M')) = d\). Thus we know the result for \(M'\) and since \(R\Hom(N, \omega_A^\bullet) = \Hom_A(N, E)\) (Lemma 16.3) the long exact cohomology sequence of \(\text{Ext}\)'s implies the result for \(M\). \qed

Lemma 16.5. Let \((A, \mathfrak{m}, \kappa)\) be a Noetherian local ring with normalized dualizing complex \(\omega_A^\bullet\). Let \(M\) be a finite \(A\)-module. The following are equivalent
(1) $M$ is Cohen-Macaulay,
(2) $\text{Ext}^i_A(M, \omega_A^*)$ is nonzero for a single $i$,
(3) $\text{Ext}^i_A(M, \omega_A^*)$ is zero for $i \neq \dim(\text{Supp}(M))$.

Denote $CM_d$ the category of finite Cohen-Macaulay $A$-modules of depth $d$. Then $M \mapsto \text{Ext}^{-\delta}_A(M, \omega_A^*)$ defines an anti-auto-equivalence of $CM_d$.

**Proof.** We will use the results of Lemma 16.4 without further mention. Fix a finite module $M$. If $M$ is Cohen-Macaulay, then only $\text{Ext}^{-d}_A(M, \omega_A^*)$ can be nonzero, hence (1) $\Rightarrow$ (3). The implication (3) $\Rightarrow$ (2) is immediate. Assume (2) and let $N = \text{Ext}^{-\delta}_A(M, \omega_A^*)$ be the nonzero Ext where $\delta = \text{depth}(M)$. Then, since $M[0] = R\text{Hom}_A(R\text{Hom}_A(M, \omega_A^*), \omega_A^*) = R\text{Hom}_A(N[\delta], \omega_A^*)$ (Lemma 15.2) we conclude that $M = \text{Ext}^{-\delta}_A(N, \omega_A^*)$. Thus $\delta \geq \dim(\text{Supp}(M))$. However, since we also know that $\delta \leq \dim(\text{Supp}(M))$ (Algebra, Lemma 70.3) we conclude that $M$ is Cohen-Macaulay.

To prove the final statement, it suffices to show that $N = \text{Ext}^{-d}_A(M, \omega_A^*)$ is in $CM_d$ for $M$ in $CM_d$. Above we have seen that $M[0] = R\text{Hom}_A(N[d], \omega_A^*)$ and this proves the desired result by the equivalence of (1) and (3).

**Lemma 16.6.** Let $(A, m, \kappa)$ be a Noetherian local ring with normalized dualizing complex $\omega_A^*$. If $\dim(A) = 0$, then $\omega_A^* \cong E[0]$ where $E$ is an injective hull of the residue field.

**Proof.** Immediate from Lemma 16.3.

**Lemma 16.7.** Let $(A, m, \kappa)$ be a Noetherian local ring with normalized dualizing complex. Let $I \subseteq m$ be an ideal of finite length. Set $B = A/I$. Then there is a distinguished triangle

$$\omega_B^* \to \omega_A^* \to \text{Hom}_A(I, E)[0] \to \omega_B^*[1]$$

in $D(A)$ where $E$ is an injective hull of $\kappa$ and $\omega_B^*$ is a normalized dualizing complex for $B$.

**Proof.** Use the short exact sequence $0 \to I \to A \to B \to 0$ and Lemmas 16.3 and 16.1.

**Lemma 16.8.** Let $(A, m, \kappa)$ be a Noetherian local ring with normalized dualizing complex $\omega_A^*$. Let $f \in m$ be a nonzerodivisor. Set $B = A/(f)$. Then there is a distinguished triangle

$$\omega_B^* \to \omega_A^* \to \omega_A^* \to \omega_B^*[1]$$

in $D(A)$ where $\omega_B^*$ is a normalized dualizing complex for $B$.

**Proof.** Use the short exact sequence $0 \to A \to A \to B \to 0$ and Lemma 16.1.

**Lemma 16.9.** Let $A \to B$ be a local homomorphism of Noetherian local rings. Let $\omega_A^*$ be a normalized dualizing complex. If $A \to B$ is flat and $m_A B = m_B$, then $\omega_A^* \otimes_A B$ is a normalized dualizing complex for $B$.

**Proof.** It is clear that $\omega_A^* \otimes_A B$ is in $D^b_{\text{Coh}}(B)$. Let $\kappa_A$ and $\kappa_B$ be the residue fields of $A$ and $B$. By More on Algebra, Lemma 69.3 we see that

$$R\text{Hom}_B(\kappa_B, \omega_B^* \otimes_A B) = R\text{Hom}_A(\kappa_A, \omega_A^* \otimes_A B = \kappa_A[0] \otimes_A B = \kappa_B[0]$$
Thus $\omega_A \otimes_A B$ has finite injective dimension by More on Algebra, Lemma 56.5.

Finally, we can use the same arguments to see that

$$R\text{Hom}_B(\omega_A \otimes_A B, \omega_A \otimes_A B) = R\text{Hom}_A(\omega_A, \omega_A) \otimes_A B = A \otimes_A B = B$$

as desired.

\[\square\]

**Lemma 16.10.** Let $(A, m, \kappa)$ be a Noetherian local ring with normalized dualizing complex $\omega_A$. Let $p$ be a minimal prime of $A$ with $\dim(A/p) = e$. Then $H^i(\omega_A)_p$ is nonzero if and only if $i = -e$.

**Proof.** Since $A_p$ has dimension zero, there exists an integer $n > 0$ such that $p^n A_p$ is zero. Set $B = A/p^n$ and $\omega_B = R\text{Hom}_A(B, \omega_A)$. Since $B_p = A_p$ we see that $(\omega_B)_p \cong (\omega_A)_p$ by using More on Algebra, Lemma 69.3. By Lemma 16.1 we may replace $A$ by $B$. After doing so, we see that $\dim(A) = e$. Then we see that $H^i(\omega_A)_p$ can only be nonzero if $i = -e$ by Lemma 16.4. On the other hand, since $(\omega_A)_p$ is a dualizing complex for the nonzero ring $A_p$ (Lemma 15.6) we see that the remaining module has to be nonzero. 

\[\square\]

### 17. The dimension function of a dualizing complex

Our results in the local setting have the following consequence: a Noetherian ring with a dualizing complex is a universally catenary ring of finite dimension.

**Lemma 17.1.** Let $A$ be a Noetherian ring. Let $p$ be a minimal prime of $A$. Then $H^i(\omega_A)_p$ is nonzero for exactly one $i$.

**Proof.** The complex $\omega_A \otimes_A A_p$ is a dualizing complex for $A_p$ (Lemma 15.6). The dimension of $A_p$ is zero as $p$ is minimal. Hence the result follows from Lemma 16.6.

Let $A$ be a Noetherian ring and let $\omega_A$ be a dualizing complex. Lemma 15.11 allows us to define a function

$$\delta = \delta_{\omega_A} : \text{Spec}(A) \to \mathbb{Z}$$

by mapping $p$ to the integer of Lemma 15.11 for the dualizing complex $(\omega_A)_p$ over $A_p$ (Lemma 15.6) and the residue field $\kappa(p)$. To be precise, we define $\delta(p)$ to be the unique integer such that

$$(\omega_A)_p[-\delta(p)]$$

is a normalized dualizing complex over the Noetherian local ring $A_p$.

**Lemma 17.2.** Let $A$ be a Noetherian ring and let $\omega_A$ be a dualizing complex. Let $A \to B$ be a surjective ring map and let $\omega_B = R\text{Hom}(B, \omega_A)$ be the dualizing complex for $B$ of Lemma 15.8. Then we have

$$\delta_{\omega_B} = \delta_{\omega_A}|_{\text{Spec}(B)}$$

**Proof.** This follows from the definition of the functions and Lemma 16.1.

**Lemma 17.3.** Let $A$ be a Noetherian ring and let $\omega_A$ be a dualizing complex. The function $\delta = \delta_{\omega_A}$ defined above is a dimension function (Topology, Definition 19.1).

**Proof.** Let $p \subset q$ be an immediate specialization. We have to show that $\delta(q) = \delta(p) + 1$. We may replace $A$ by $A/p$, the complex $\omega_A$ by $\omega_A/p$, $R\text{Hom}(A/p, \omega_A)$, the prime $p$ by $(0)$, and the prime $q$ by $q/p$, see Lemma 17.2. Thus we may assume that $A$ is a domain, $p = (0)$, and $q$ is a prime ideal of height 1.
Then $H^i(\omega^*_A)_{(0)}$ is nonzero for exactly one $i$, say $i_0$, by Lemma 17.1. In fact $i_0 = -\delta((0))$ because $(\omega^*_A)_{(0)}[-\delta((0))]$ is a normalized dualizing complex over the field $A_{(0)}$.

On the other hand $(\omega^*_A)_q[-\delta(q)]$ is a normalized dualizing complex for $A_q$. By Lemma 16.10 we see that $H^e((\omega^*_A)_q[-\delta(q)])_{(0)} = H^{e-\delta(q)}(\omega^*_A)_{(0)}$

is nonzero only for $e = -\dim(A_q) = -1$. We conclude $-\delta((0)) = -1 - \delta(p)$ as desired.

**Lemma 17.4.** Let $A$ be a Noetherian ring which has a dualizing complex. Then $A$ is universally catenary of finite dimension.

**Proof.** Because Spec($A$) has a dimension function by Lemma 17.3 it is catenary, see Topology, Lemma 19.2. Hence $A$ is catenary, see Algebra, Lemma 102.2. It follows from Proposition 15.10 that $A$ is universally catenary.

Because any dualizing complex $\omega^*_A$ is in $D^b_{\text{coh}}(A)$ the values of the function $\delta^*_A$ in minimal primes are bounded by Lemma 16.1. On the other hand, for a maximal ideal $m$ with residue field $\kappa$ the integer $i = -\delta(m)$ is the unique integer such that $\text{Ext}^i_A(\kappa, \omega^*_A)$ is nonzero (Lemma 15.11). Since $\omega^*_A$ has finite injective dimension these values are bounded too. Since the dimension of $A$ is the maximal value of $\delta(p) - \delta(m)$ where $p \subseteq m$ are a pair consisting of a minimal prime and a maximal prime we find that the dimension of Spec($A$) is bounded.

**Lemma 17.5.** Let $(A, m, \kappa)$ be a Noetherian local ring with normalized dualizing complex $\omega^*_A$. Let $d = \dim(A)$ and $\omega_A = H^{-d}(\omega^*_A)$. Then

1. the support of $\omega_A$ is the union of the irreducible components of Spec($A$) of dimension $d$,
2. $\omega_A$ satisfies $(S_2)$, see Algebra, Definition 147.1.

**Proof.** We will use Lemma 16.4 without further mention. By Lemma 16.10 the support of $\omega_A$ contains the irreducible components of dimension $d$. Let $p \subseteq A$ be a prime. By Lemma 17.3 the complex $(\omega^*_A)_p[\dim(A/p)]$ is a normalized dualizing complex for $A_p$. Hence if $\dim(A/p) + \dim(A_p) < d$, then $(\omega_A)_p = 0$. This proves the support of $\omega_A$ is the union of the irreducible components of dimension $d$, because the complement of this union is exactly the primes $p$ of $A$ for which $\dim(A/p) + \dim(A_p) < d$ as $A$ is catenary (Lemma 17.4). On the other hand, if $\dim(A/p) + \dim(A_p) = d$, then

$$(\omega_A)_p = H^{-\dim(A_p)}((\omega^*_A)_p[\dim(A/p)])$$

Hence in order to prove $\omega_A$ has $(S_2)$ it suffices to show that the depth of $\omega_A$ is at least $\min(\dim(A), 2)$. We prove this by induction on $\dim(A)$. The case $\dim(A) = 0$ is trivial.

Assume $\dim(A) > 0$. Choose a nonzerodivisor $f \in m$ and set $B = A/fA$. Then $\dim(B) = \dim(A) - 1$ and we may apply the induction hypothesis to $B$. By Lemma 16.8 we see that multiplication by $f$ is injective on $\omega_A$ and we get $\omega_A/f\omega_A \subseteq \omega_B$. This proves the depth of $\omega_A$ is at least 1. If $\dim(A) > 1$, then $\dim(B) > 0$ and $\omega_B$ has depth $> 0$. Hence $\omega_A$ has depth $> 1$ and we conclude in this case.
Assume \( \dim(A) > 0 \) and \( \depth(A) = 0 \). Let \( I = A[m^\infty] \) and set \( B = A/I \). Then \( B \) has depth \( \geq 1 \) and \( \omega_A = \omega_B \) by Lemma \[16.6\]. Since we proved the result for \( \omega_B \) above the proof is done.

\[ \square \]

18. The local duality theorem

The main result in this section is due to Grothendieck.

**Lemma 18.1.** Let \((A, m, \kappa)\) be a Noetherian local ring. Let \( \omega_A^* \) be a normalized dualizing complex. Let \( Z = V(m) \subset \text{Spec}(A) \). Then \( E = R^0\Gamma_Z(\omega_A^*) \) is an injective hull of \( \kappa \) and \( R\Gamma_Z (\omega_A^*) = E[0] \).

**Proof.** By Lemma \[8.9\] we have \( R\Gamma_m = R\Gamma_Z \). Thus

\[ R\Gamma_Z (\omega_A^*) = R\Gamma_m (\omega_A^*) = \hocolim R\text{Hom}(A/m^n, \omega_A^*) \]

by Lemma \[8.2\]. Let \( E' \) be an injective hull of the residue field. By Lemma \[16.3\] we can find isomorphisms

\[ R\text{Hom}(A/m^n, \omega_A^*) \cong \text{Hom}_A(A/I^n, E')[0] \]

compatible with transition maps. Since \( E' = \bigcup E'[m^n] = \colim \text{Hom}_A(A/I^n, E') \) by Lemma \[7.3\] we conclude that \( E \cong E' \) and that all other cohomology groups of the complex \( R\Gamma_Z (\omega_A^*) \) are zero.

\[ \square \]

**Remark 18.2.** Let \((A, m, \kappa)\) be a Noetherian local ring with a normalized dualizing complex \( \omega_A^* \). By Lemma \[18.1\] above we see that \( R\Gamma_Z (\omega_A^*) \) is an injective hull of the residue field placed in degree 0. In fact, this gives a “construction” or “realization” of the injective hull which is slightly more canonical than just picking any old injective hull. Namely, a normalized dualizing complex is unique up to isomorphism, with group of automorphisms the group of units of \( \kappa \), whereas an injective hull of \( \kappa \) is unique up to isomorphism, with group of automorphisms the group of units of the completion \( A^\wedge \) of \( A \) with respect to \( m \).

Here is the main result of this section.

**Theorem 18.3.** Let \((A, m, \kappa)\) be a Noetherian local ring. Let \( \omega_A^* \) be a normalized dualizing complex. Let \( E \) be an injective hull of the residue field. Let \( Z = V(m) \subset \text{Spec}(A) \). Denote \( \wedge \) derived completion with respect to \( m \). Then

\[ R\text{Hom}(K, \omega_A^*)^\wedge \cong R\text{Hom}(R\Gamma_Z(K), E[0]) \]

for \( K \) in \( D(A) \).

**Proof.** Observe that \( E[0] \cong R\Gamma_Z(\omega_A^*) \) by Lemma \[18.1\]. By More on Algebra, Lemma \[67.11\] completion on the left hand side goes inside. Thus we have to prove

\[ R\text{Hom}(K^\wedge, (\omega_A^*)^\wedge) = R\text{Hom}(R\Gamma_Z(K), R\Gamma_Z(\omega_A^*)) \]

This follows from the equivalence between \( D_{\text{comp}}(A, m) \) and \( D_{m\text{-torsion}}(A) \) given in Proposition \[12.2\]. More precisely, it is a special case of Lemma \[12.3\].

\[ \square \]

Here is a special case of the theorem above.

**Lemma 18.4.** Let \((A, m, \kappa)\) be a Noetherian local ring. Let \( \omega_A^* \) be a normalized dualizing complex. Let \( E \) be an injective hull of the residue field. Let \( K \in D_{\text{Coh}}(A) \). Then

\[ \text{Ext}_A^1(K, \omega_A^*)^\wedge = \text{Hom}_A(H_m^0(K), E) \]

where \( ^\wedge \) denotes \( m \)-adic completion.
Proof. By Lemma 15.2 we see that $R\text{Hom}(K, \omega_A^\bullet)$ is an object of $D_{\text{Coh}}(A)$. It follows that the cohomology modules of the derived completion of $R\text{Hom}(K, \omega_A^\bullet)$ are equal to the usual completions $\text{Ext}^i_A(K, \omega_A^\bullet)^\wedge$ by More on Algebra, Lemma 67.20. On the other hand, we have $R\Gamma_m = R\Gamma_Z$ for $Z = V(m)$ by Lemma 8.9. Moreover, the functor $\text{Hom}_A(-, E)$ is exact hence factors through cohomology. Hence the lemma is consequence of Theorem 18.3. □

19. Dualizing complexes on schemes

We define a dualizing complex on a locally Noetherian scheme to be a complex which affine locally comes from a dualizing complex on the corresponding ring. This is not completely standard but agrees with all definitions in the literature on Noetherian schemes of finite dimension.

Lemma 19.1. Let $X$ be a locally Noetherian scheme. Let $K$ be an object of $D(\mathcal{O}_X)$. The following are equivalent

1. For every affine open $U = \text{Spec}(A) \subset X$ there exists a dualizing complex $\omega_A^\bullet$ for $A$ such that $K|_U$ is isomorphic to the image of $\omega_A^\bullet$ by the functor $\gamma^\leftarrow: D(A) \to D(\mathcal{O}_U)$.
2. There is an affine open covering $X = \bigcup U_i$, $U_i = \text{Spec}(A_i)$ such that for each $i$ there exists a dualizing complex $\omega_i^\bullet$ for $A_i$ such that $K|_U$ is isomorphic to the image of $\omega_i^\bullet$ by the functor $\gamma^\leftarrow: D(A_i) \to D(\mathcal{O}_{U_i})$.

Proof. Assume (2) and let $U = \text{Spec}(A)$ be an affine open of $X$. Since condition (2) implies that $K$ is in $D_{\text{QCoh}}(\mathcal{O}_X)$ we find an object $\omega_A^\bullet$ in $D(A)$ whose associated complex of quasi-coherent sheaves is isomorphic to $K|_U$, see Derived Categories of Schemes, Lemma 3.4. We will show that $\omega_A^\bullet$ is a dualizing complex for $A$ which will finish the proof.

Since $X = \bigcup U_i$ is an open covering, we can find a standard open covering $U = D(f_1) \cup \ldots \cup D(f_m)$ such that each $D(f_j)$ is a standard open in one of the affine opens $U_i$, see Schemes, Lemma 11.5. Say $D(f_j) = D(g_j)$ for $g_j \in A_{i_j}$. Then $A_{f_j} \cong (A_{i_j})_{g_j}$ and we have

$$(\omega_A^\bullet)_{f_j} \cong (\omega_i^\bullet)_{g_j}$$

in the derived category by Derived Categories of Schemes, Lemma 3.4. By Lemma 15.6 we find that the complex $(\omega_A^\bullet)_{f_j}$ is a dualizing complex over $A_{f_j}$ for $j = 1, \ldots, m$. This implies that $\omega_A^\bullet$ is dualizing by Lemma 15.7. □

Definition 19.2. Let $X$ be a locally Noetherian scheme. An object $K$ of $D(\mathcal{O}_X)$ is called a dualizing complex if $K$ satisfies the equivalent conditions of Lemma 19.1.

Please see remarks made at the beginning of this section.

Lemma 19.3. Let $A$ be a Noetherian ring and let $X = \text{Spec}(A)$. Let $K, L$ be objects of $D(A)$. If $K \in D_{\text{Coh}}(A)$ and $L$ has finite injective dimension, then

$$R\text{Hom}(\tilde{K}, \tilde{L}) = \widetilde{R\text{Hom}(K, L)}$$

in $D(\mathcal{O}_X)$.

Proof. We may assume that $L$ is given by a finite complex $I^\bullet$ of injective $A$-modules. By induction on the length of $I^\bullet$ and compatibility of the constructions with distinguished triangles, we reduce to the case that $L = I[0]$ where $I$ is an injective $A$-module. In this case, Derived Categories of Schemes, Lemma 10.8 tells
us that the $n$th cohomology sheaf of $R\Hom(K, L)$ is the sheaf associated to the presheaf

$$D(f) \mapsto \Ext^n_{A_f}(K \otimes_A A_f, I \otimes_A A_f)$$

Since $A$ is Noetherian, the $A_f$-module $I \otimes_A A_f$ is injective (Lemma 3.9). Hence we see that

$$\Ext^n_{A_f}(K \otimes_A A_f, I \otimes_A A_f) = \Hom_{A_f}(H^{-n}(K \otimes_A A_f), I \otimes_A A_f)$$

$$= \Hom_A(H^{-n}(K), I) \otimes_A A_f$$

The last equality because $H^{-n}(K)$ is a finite $A$-module. This proves that the canonical map

$$\widetilde{R\Hom}(K, L) \to R\Hom(K, L)$$

is a quasi-isomorphism in this case and the proof is done. \hfill \Box

**Lemma 19.4.** Let $K$ be a dualizing complex on a locally Noetherian scheme $X$. Then $K$ is an object of $D\Coh(\mathcal{O}_X)$ and $D = R\Hom(-, K)$ induces an anti-equivalence

$$D : D\Coh(\mathcal{O}_X) \to D\Coh(\mathcal{O}_X)$$

which comes equipped with a canonical isomorphism $id \to D \circ D$. If $X$ is quasi-compact, then $D$ exchanges $D_{\Coh}^b(\mathcal{O}_X)$ and $D_{\Coh}(\mathcal{O}_X)$ and induces an equivalence $D_{\Coh}^b(\mathcal{O}_X) \to D_{\Coh}^b(\mathcal{O}_X)$.

**Proof.** Let $U \subset X$ be an affine open. Say $U = \Spec(A)$ and let $\omega_A^\bullet$ be a dualizing complex for $A$ corresponding to $K|_U$ as in Lemma 19.1. By Lemma 19.3 the diagram

$$\begin{array}{ccc}
D_{\Coh}(A) & \longrightarrow & D_{\Coh}(\mathcal{O}_U) \\
\| & & \| \\
\bigcap R\Hom(-, \omega_A^\bullet) & \longrightarrow & \bigcap R\Hom(-, K|_U) \\
\| & & \| \\
D_{\Coh}(A) & \longrightarrow & D(\mathcal{O}_U)
\end{array}$$

commutes. We conclude that $D$ sends $D_{\Coh}(\mathcal{O}_X)$ into $D_{\Coh}(\mathcal{O}_X)$. Moreover, the canonical map

$$L \to R\Hom(R\Hom(L, K), K)$$

(Cohomology on Sites, Lemma 26.5) is an isomorphism for all $L$ because this is true on affines by Lemma 15.2. The statement on boundedness properties of the functor $D$ in the quasi-compact case also follow from the corresponding statements of Lemma 15.2. \hfill \Box

Let $X$ be a locally ringed space. We will say that an object $L$ of $D(\mathcal{O}_X)$ is invertible if there is an open covering $X = \bigcup U_i$ such that $L|_{U_i} \cong \mathcal{O}_{U_i}[-n_i]$ for some integers $n_i$. In this case, the function

$$x \mapsto n_x, \quad \text{where } n_x \text{ is the unique integer such that } H^{n_x}(L_x) \neq 0$$

is locally constant on $X$. In particular, it follows that $L = \bigoplus H^n(L)[-n]$ which gives a well defined complex of $\mathcal{O}_X$-modules (with zero differentials) representing $L$. In particular $L$ is a perfect object of $D(\mathcal{O}_X)$.

**Lemma 19.5.** Let $X$ be a locally Noetherian scheme. If $K$ and $K'$ are dualizing complexes on $X$, then $K'$ is isomorphic to $K \otimes_{\mathcal{O}_X} L$ for some invertible object $L$ of $D(\mathcal{O}_X)$.
Proof. Set

\[ L = R \text{Hom}_{\mathcal{O}_X}(K, K') \]

This is an invertible object of \( D(\mathcal{O}_X) \), because affine locally this is true, see Lemma 15.5 and its proof. The evaluation map \( L \otimes_{\mathcal{O}_X} K \to K' \) is an isomorphism for the same reason.

\[ \square \]

**Lemma 19.6.** Let \( X \) be a locally Noetherian scheme. Let \( \omega^*_X \) be a dualizing complex on \( X \). Then \( X \) is universally catenary and the function \( X \to \mathbb{Z} \) defined by \( x \mapsto \delta(x) \) such that \( \omega^*_{X,x}[-\delta(x)] \) is a normalized dualizing complex over \( \mathcal{O}_{X,x} \) is a dimension functor.

**Proof.** Immediate from the affine case Lemma 17.3 and the definitions. \( \square \)

**20. Twisted inverse image**

References for this section are [Nee96] and [LN07]. Let \( f : X \to Y \) be a morphism of schemes. In some papers, a **twisted inverse image** for \( f \) is defined to be a right adjoint to the functor \( Rf_* : D_{QCoh}(\mathcal{O}_X) \to D_{QCoh}(\mathcal{O}_X) \). However, this terminology is not universally accepted and we refrain from giving a formal definition. We also will not use the notation \( f^! \) for such a functor, as this would clash (for general morphisms \( f \)) with the notation in [Har66].

**Lemma 20.1.** Let \( f : X \to Y \) be a morphism between quasi-separated and quasi-compact schemes. The functor \( Rf_* : D_{QCoh}(X) \to D_{QCoh}(Y) \) has a right adjoint.

**Proof.** We will prove a right adjoint exists by verifying the hypotheses of Derived Categories, Proposition 35.2. First off, the category \( D_{QCoh}(\mathcal{O}_X) \) has direct sums, see Derived Categories of Schemes, Lemma 3.1. The category \( D_{QCoh}(\mathcal{O}_X) \) is compactly generated by Derived Categories of Schemes, Theorem 14.3. Since \( X \) and \( Y \) are quasi-compact and quasi-separated, so is \( f \), see Schemes, Lemmas 21.14 and 21.15. Hence the functor \( Rf_* \) commutes with direct sums, see Derived Categories of Schemes, Lemma 3.2. This finishes the proof. \( \square \)

**Example 20.2.** Let \( A \to B \) be a ring map. Let \( Y = \text{Spec}(A) \) and \( X = \text{Spec}(B) \) and \( f : X \to Y \) the morphism corresponding to \( A \to B \). Then \( Rf_* \) corresponds to restriction \( D(B) \to D(A) \) via the equivalences \( D(B) \to D_{QCoh}(\mathcal{O}_X) \) and \( D(A) \to D_{QCoh}(\mathcal{O}_Y) \). Hence the right adjoint corresponds to the functor \( K \mapsto R\text{Hom}(B, K) \) of Section 13.

**Example 20.3.** If \( f : X \to Y \) is a separated finite type morphism of Noetherian schemes, then twisted inverse image does not map \( D_{Coh}(\mathcal{O}_Y) \) into \( D_{Coh}(\mathcal{O}_X) \). Namely, let \( k \) be a field and consider the morphism \( f : \mathbb{A}_k^1 \to \text{Spec}(k) \). By Example 20.2 this corresponds to the question of whether \( R\text{Hom}(B, -) \) maps \( D_{Coh}(A) \) into \( D_{Coh}(B) \) where \( A = k \) and \( B = k[x] \). This is not true because

\[ R\text{Hom}(k[x], k) = \left( \prod_{n \geq 0} k \right) [0] \]

which is not a finite \( k[x] \)-module. Hence \( a(\mathcal{O}_Y) \) does not have coherent cohomology sheaves.
Example 20.4. If \( f : X \to Y \) is a proper or even finite morphism of Noetherian schemes, then twisted inverse image does not map \( D_{QCoh}(\mathcal{O}_Y) \) into \( D_{QCoh}(\mathcal{O}_X) \). Namely, let \( k \) be a field, let \( k[e] \) be the dual numbers over \( k \), let \( X = \text{Spec}(k) \), and let \( Y = \text{Spec}(k[e]) \). Then \( \text{Ext}^i_{\mathcal{O}_Y}(k,k) \) is nonzero for all \( i \geq 0 \). Hence \( a(\mathcal{O}_Y) \) is not bounded above by Example 20.2.

Lemma 20.5. Let \( f : X \to Y \) be a morphism of quasi-compact and quasi-separated schemes. Let \( a : D_{QCoh}(\mathcal{O}_Y) \to D_{QCoh}(\mathcal{O}_X) \) be the right adjoint to \( Rf_* \) of Lemma 20.7. Then a maps \( D_{QCoh}(\mathcal{O}_Y) \) into \( D_{QCoh}(\mathcal{O}_X) \).

Proof. By Derived Categories of Schemes, Lemma 4.1 the functor \( Rf_* \) has finite cohomological dimension. In other words, there exist an integer \( N \) such that \( H^i(\mathcal{Rf}_*L) = 0 \) for \( i \geq N + c \) if \( H^j(L) = 0 \) for \( j \geq c \). Say \( K \in D_{QCoh}(\mathcal{O}_Y) \) has \( H^k(K) = 0 \) for \( k \geq c \). Then

\[
\text{Hom}_{D(\mathcal{O}_X)}(\tau_{\leq -N}a(K), a(K)) = \text{Hom}_{D(\mathcal{O}_Y)}(Rf_*\tau_{\leq -N}a(K), K) = 0
\]

by what we said above. Clearly, this implies that \( a(K) \) is bounded below. \( \square \)

We often want to know whether the twisted inverse image commutes with base change. Thus we consider a cartesian square

\[
\begin{array}{ccc}
X' & \xrightarrow{g} & X \\
\downarrow{f'} & & \downarrow{f} \\
Y' & \xrightarrow{g} & Y
\end{array}
\]

(20.5.1)

of quasi-compact and quasi-separated schemes. Denote

\[
a : D_{QCoh}(\mathcal{O}_Y) \to D_{QCoh}(\mathcal{O}_X),
\]

\[
a' : D_{QCoh}(\mathcal{O}_Y') \to D_{QCoh}(\mathcal{O}_X'),
\]

\[
b : D_{QCoh}(\mathcal{O}_X) \to D_{QCoh}(\mathcal{O}_X'),
\]

\[
b' : D_{QCoh}(\mathcal{O}_Y) \to D_{QCoh}(\mathcal{O}_Y')
\]

the right adjoints to \( Rf_* \), \( Rf'_* \), \( Rg_* \), and \( Rg'_* \) (Lemma 20.1). Since \( Rf_* \circ Rg'_* = Rg_* \circ Rf'_* \) we get

\[
b' \circ a = a' \circ b.
\]

Another compatibility comes from the base change map of Cohomology, Remark 29.2. It induces a transformation of functors

\[
Lg^* \circ Rf_* \longrightarrow Rf'_* \circ L(g')^*
\]

on derived categories of sheaves with quasi-coherent cohomology. Hence a transformation between the right adjoints in the opposite direction

\[
a \circ Rg_* \leftarrow Rg'_* \circ a'
\]

Lemma 20.6. In diagram (20.5.1) assume that \( g \) is flat or more generally that \( f \) and \( g \) are Tor independent. Then \( a \circ Rg_* \leftarrow Rg'_* \circ a' \) is an isomorphism.

Proof. In this case the base change map \( Lg^* \circ Rf_* K \longrightarrow Rf'_* \circ L(g')^* K \) is an isomorphism for every \( K \) in \( D_{QCoh}(\mathcal{O}_X) \) by Derived Categories of Schemes, Lemma 17.3. Thus the corresponding transformation between adjoint functors is an isomorphism as well. \( \square \)
Let $f : X \to Y$ be a morphism of quasi-compact and quasi-separated schemes. Let $V \subset Y$ be a quasi-compact open subscheme and set $U = f^{-1}(V)$. This gives a cartesian square

$$
\begin{array}{ccc}
U & \rightarrow & X \\
\downarrow_{f|U} & & \downarrow f \\
V & \rightarrow & Y
\end{array}
$$

as in (20.5.1). By Lemma 20.6 the map $\xi : a \circ Rj_* \leftarrow Rj'_* \circ a'$ is an isomorphism where $a$ and $a'$ are the twisted inverse images corresponding to $f$ and $f|_U$. We obtain a transformation of functors $D_{\text{QCoh}}(\mathcal{O}_Y) \to D_{\text{QCoh}}(\mathcal{O}_U)$

(20.6.1) $$(j')^* \circ a \to (j')^* \circ a \circ Rj_* \circ j^* \xrightarrow{\xi^{-1}} (j')^* \circ Rj'_* \circ a' \circ j^* \to a' \circ j^*$$

where the first arrow comes from $\text{id} \to Rj_* \circ j^*$ and the final arrow from the isomorphism $((j')^* \circ Rj'_*) \to \text{id}$. In particular, we see that (20.6.1) is an isomorphism when evaluated on $K$ if and only if $a(K)|_U \to a(Rj_*(K|_V))|_U$ is an isomorphism.

**Example 20.7.** There is a finite morphism $f : X \to Y$ of Noetherian schemes such that (20.6.1) is not an isomorphism when evaluated on some $K \in D_{\text{coh}}(\mathcal{O}_Y)$. Namely, let $X = \text{Spec}(B) \to Y = \text{Spec}(A)$ with $A = k[x, \epsilon]$ where $k$ is a field and $\epsilon^2 = 0$ and $B = k[x] = A/(\epsilon)$. For $n \in \mathbb{N}$ set $M_n = A/(\epsilon, x^n)$. Observe that

$$\text{Ext}_A^n(B, M_n) = M_n, \quad i \geq 0$$

because $B$ has the free periodic resolution $\cdots \to A \to A \to A$ with maps given by multiplication by $\epsilon$. Consider the object $K = \bigoplus K_n[n] = \prod K_n[n]$ of $D_{\text{Coh}}(A)$ (equality in $D(A)$ by Derived Categories, Lemmas 31.2 and 32.3). Then we see that $a(K)$ corresponds to $R\text{Hom}(B, K)$ by Example 20.2 and

$$H^0(R\text{Hom}(B, K)) = \text{Ext}_A^0(B, K) = \prod_{n \geq 1} \text{Ext}_A^n(B, M_n) = \prod_{n \geq 1} M_n$$

by the above. But this module has elements which are not annihilated by any power of $x$, whereas the complex $K$ does have every element of its cohomology annihilated by a power of $x$. In other words, for the map (20.6.1) with $V = D(x)$ and $U = D(x)$ and the complex $K$ cannot be an isomorphism because $(j')^*(a(K))$ is nonzero and $a'(j^*K)$ is zero.

**Lemma 20.8.** Let $f : X \to Y$ be a morphism of quasi-compact and quasi-separated schemes. Let $V \subset Y$ be quasi-compact open with inverse image $U \subset X$. If for every $Q \in D_{\text{QCoh}}^+(\mathcal{O}_Y)$ supported on $Y \setminus V$ the twisted inverse image $a(Q)$ is supported on $X \setminus U$, then (20.6.1) is an isomorphism on all $K$ in $D_{\text{QCoh}}^+(\mathcal{O}_Y)$.

**Proof.** Choose a distinguished triangle

$$K \to Rj_*K|_V \to Q \to K[1]$$

Observe that $Q$ is supported on $Y \setminus V$ (Derived Categories of Schemes, Definition 7.4). Applying the twisted inverse image $a$ we obtain a distinguished triangle

$$a(K) \to a(Rj_*K|_V) \to a(Q) \to a(K)[1]$$

on $X$. If $a(Q)$ is supported on $X \setminus U$, then restricting to $U$ the map $a(K)|_U \to a(Rj_*K|_V)|_U$ is an isomorphism, i.e., (20.6.1) is an isomorphism. $\square$
**Lemma 20.9.** Let $f : X \to Y$ be a proper\footnote{This proof works for those morphisms of quasi-compact and quasi-separated schemes such that $Rf_*P$ is pseudo-coherent for all $P$ perfect on $X$. It follows easily from a theorem of Kiehl [Kie72] that this holds if $f$ is proper and pseudo-coherent. This is the correct generality for this lemma and some of the other results in this section.} morphism of Noetherian schemes. The assumption and hence the conclusion of Lemma 20.8 holds for all opens $V$ of $Y$.

**Proof.** Let $Q \in D^+_{QCoh}(\mathcal{O}_Y)$ be supported on $Y \setminus V$. To get a contradiction, assume that $a(Q)$ is not supported on $X \setminus U$. Then we can find a perfect complex $P_U$ on $U$ and a nonzero map $P_U \to a(Q)|_U$ (follows from Derived Categories of Schemes, Theorem 14.3). Then using Derived Categories of Schemes, Lemma 12.9 we may assume there is a perfect complex $P$ on $X$ and a map $P \to a(Q)$ whose restriction to $U$ is nonzero. By definition of the twisted inverse image this map is adjoint to a map $Rf_*P \to Q$.

Because $f$ is proper and $X$ and $Y$ Noetherian, the complex $Rf_*P$ is pseudo-coherent, see Derived Categories of Schemes, Lemmas 6.1 and 10.4. Thus we may apply Derived Categories of Schemes, Lemma 15.3 and get a map $I \to \mathcal{O}_V$ of perfect complexes whose restriction to $V$ is an isomorphism such that the composition $I \otimes_{\mathcal{O}_V} Rf_*P \to Rf_*P \to K$ is zero. By Derived Categories of Schemes, Lemma 17.1 we have $I \otimes_{\mathcal{O}_V} Rf_*P = Rf_* (Lf^*I \otimes_{\mathcal{O}_X} P)$. We conclude that the composition $Lf^*I \otimes_{\mathcal{O}_X} P \to P \to a(K)$ is zero. However, the restriction to $U$ is the map $P|_U \to a(K)|_U$ which we assumed to be nonzero. This contradiction finishes the proof.

**Lemma 20.10.** Let $f : X \to Y$ be a proper morphism of Noetherian schemes. Let $a$ be the twisted inverse image. Then the canonical map

$$Rf_* R\mathcal{H}om(L, a(K)) \to R\mathcal{H}om(Rf_*L, K)$$

is an isomorphism for all $L \in D_{QCoh}(\mathcal{O}_X)$ and all $K \in D^+_{QCoh}(\mathcal{O}_Y)$.

**Proof.** Since $a$ is the right adjoint to $Rf_*$ there is an adjunction map $Rf_*a(K) \to K$. On the other hand, there is a canonical map

$$Rf_* R\mathcal{H}om(L, a(K)) \to R\mathcal{H}om(Rf_*L, Rf_*a(K))$$

which works on the level of complexes. Combining these we obtain the map of the lemma. Taking $H^n(V, -)$ for an open $V$ of $Y$ with inverse image $U$ in $X$ we get

$$\text{Hom}_{D(\mathcal{O}_U)}(L|_U, a(K)|_U) \to \text{Hom}_{D(\mathcal{O}_V)}(Rf_*L|_V, K|_V)$$

see Cohomology, Lemma 34.1. Since we’ve shown above that $a(K)|_U$ is the twisted inverse image of $K|_V$ (Lemma 20.9) the two sides of this arrow are isomorphic. We omit the verification that the two maps agree. A similar argument works for $H^n(V, -)$. Thus the map defined above is an isomorphism on cohomology and hence an isomorphism in the derived category.

Let $f : X \to Y$ be a morphism of quasi-compact and quasi-separated schemes. Let $a$ be the twisted inverse image (Lemma 20.1). There is a canonical map

$$Lf^* K \otimes_{\mathcal{O}_X} a(\mathcal{O}_Y) \to a(K)$$

(20.10.1)
functorial in $K$ and compatible with distinguished triangles. Namely, this map is adjoint to a map
\[
Rf_*(Lf^*K \otimes_{O_X} a(O_Y)) = K \otimes_{O_X} Rf_*(a(O_Y)) \to K
\]
(equality by Derived Categories of Schemes, Lemma [17.1] for which we use the adjunction map $Rf_*a(O_Y) \to O_Y$ and the identity on $K$).

**Lemma 20.11.** Let $f : X \to Y$ be a morphism of quasi-compact and quasi-separated schemes. The map (20.10.1) is an isomorphism for every perfect object $K$ of $D(O_Y)$.

**Proof.** For a perfect object $K$ on $Y$ and $L \in D_{QCoh}(O_X)$ we have
\[
\text{Hom}_{D(O_Y)}(Rf_!L, K) = \text{Hom}_{D(O_X)}(Rf_*L \otimes_{O_Y} K^\wedge, O_Y)
\]  
\[
= \text{Hom}_{D(O_X)}(L \otimes_{O_X} Lf^*K^\wedge, a(O_Y))
\]  
\[
= \text{Hom}_{D(O_X)}(L, a(O_Y) \otimes_{O_X} Lf^*K)
\]
Hence the result by the Yoneda lemma. \qed

**Lemma 20.12.** Let $X$ be a quasi-compact and quasi-separated scheme. Let $i : Z \to X$ be a pseudo-coherent closed immersion (if $X$ is Noetherian, then any closed immersion is pseudo-coherent). Let $a : D_{QCoh}(O_X) \to D_{QCoh}(O_Z)$ be the twisted inverse image, i.e., the right adjoint to $Ri_*$. Then there is a functorial isomorphism
\[
a(K) = R\text{Hom}(O_Z, K)
\]
for $K \in D^+(O_X)$.

**Proof.** (The parenthetical statement follows from More on Morphisms, Lemma 42.9.) By Lemma 14.2 the functor $R\text{Hom}(O_Z, \cdot)$ is a right adjoint to $Ri_* : D(O_Z) \to D(O_X)$. Moreover, by Lemma 14.4 and Lemma 20.5 both $R\text{Hom}(O_Z, \cdot)$ and a map $D^+_{QCoh}(O_X)$ into $D^+_{QCoh}(O_Z)$. Hence we obtain the isomorphism by uniqueness of adjoint functors. \qed

### 21. Base change for twisted inverse image

The map (20.6.1) is a special case of a base change map. Namely, suppose that we have a diagram (20.5.1)

\[
\begin{array}{ccc}
X' & \overset{g'}{\longrightarrow} & X \\
\downarrow{f'} & & \downarrow{f} \\
Y' & \overset{g}{\longrightarrow} & Y
\end{array}
\]

where $f$ and $g$ are Tor independent. Then we can consider the morphism of functors $D_{QCoh}(O_Y) \to D_{QCoh}(O_X')$ given by the composition

(21.0.1) $L(g')^* \circ a \to L(g')^* \circ a \circ Rg_* \circ Lg^* \leftarrow L(g')^* \circ Rg'_* \circ a' \circ Lg^* \to a' \circ Lg^*$

The first arrow comes from the adjunction map $id \to Rg_* Lg^*$ and the last arrow from the adjunction map $L(g')^* Rg'_* \to id$. We need the assumption on Tor independence to invert the arrow in the middle, see Lemma (20.6). Alternatively, we can think of (21.0.1) by adjointness of $L(g')^*$ and $R(g')_*$ as a natural transformation
\[
a \to a \circ Rg_* \circ Lg^* \leftarrow Rg'_* \circ a' \circ Lg^*
\]
were again the second arrow is invertible. If $M \in D_{QCoh}(\mathcal{O}_X)$ and $K \in D_{QCoh}(\mathcal{O}_Y)$ then on Yoneda functors this map is given by

$$
\text{Hom}_X(M, a(K)) = \text{Hom}_Y(Rf_*M, K) \\
\rightarrow \text{Hom}_Y(Rf_*M, Rg_*Lg^*K) \\
= \text{Hom}_Y(Lg^*Rf_*M, Lg^*K) \\
\leftarrow \text{Hom}_Y(Rf'_*L(g')^*M, Lg^*K) \\
= \text{Hom}_X(L(g')^*M, a'(Lg^*K)) \\
= \text{Hom}_X(M, Rg'_*a'(Lg^*K))
$$

(were the arrow pointing left is invertible by the base change theorem given in Derived Categories of Schemes, Lemma 17.3) which makes things a little bit more explicit.

In this section we first prove that the base change map is an isomorphism in some cases and then we prove that the base change map satisfies some natural compatibilities with regards to stacking squares as in Cohomology, Remarks 29.3 and 29.4 for the usual base change map. We suggest the reader skip the rest of this section on a first reading.

**Lemma 21.1.** In diagram (20.5.1) assume

1. $g : Y' \to Y$ is a morphism of affine schemes,
2. $f : X \to Y$ is proper,
3. $Y$ Noetherian, and
4. $f$ and $g$ are Tor independent.

Then the base change map (21.0.1) induces an isomorphism

$$L(g')^*a(K) \to a'(Lg^*K)$$

in the following cases

1. for all $K \in D_{QCoh}(\mathcal{O}_X)$ if $f$ is flat, or
2. for $K \in D^+_{QCoh}(\mathcal{O}_X)$ if $g$ has finite Tor dimension.

**Proof.** Write $Y = \text{Spec}(A)$ and $Y' = \text{Spec}(A')$. As a base change of an affine morphism, the morphism $g'$ is affine. Hence $Rg'_*$ reflects isomorphisms, see Derived Categories of Schemes, Lemma 5.1. Thus (21.0.1) is an isomorphism for $K \in D_{QCoh}(\mathcal{O}_X)$ if and only if the map $a(K) \to a(Rg_*Lg^*K) = Rg'_*a'(Lg^*K)$ induces an isomorphism

$$a(K) \otimes_{\mathcal{O}_X} g'_*\mathcal{O}_{X'} \to a(Rg_*Lg^*K)$$

(see Derived Categories of Schemes, Lemma 5.2). As $D_{QCoh}(\mathcal{O}_X)$ is generated by perfect objects (see Derived Categories of Schemes, Theorem 14.3), it suffices to check we obtain an isomorphism after applying the functor $\text{Hom}_X(M, -)$ where $M$ is perfect on $X$. On the left hand side we get

$$\text{Hom}_X(M, a(K) \otimes_{\mathcal{O}_X} g'_*\mathcal{O}_{X'}) = H^0(R\Gamma(X, R\text{Hom}(M, a(K))) \otimes^L_{A'} A')$$

$$= H^0(R\Gamma(Y, R\text{Hom}(Rf_*M, K)) \otimes^L_{A'} A')$$

The first equality by Derived Categories of Schemes, Lemma 17.6. The second equality by Lemma 20.10. In the case that $f$ is flat the complex $Rf_*M$ is perfect on $Y$ (Derived Categories of Schemes, Lemma 18.1) and in general the complex
\( Rf_*M \) is pseudo-coherent on \( Y \) (Derived Categories of Schemes, Lemmas 6.1 and 10.4). Thus we get on the right hand side

\[
\text{Hom}_X(M, a(Rg_*Lg^*K)) = \text{Hom}_Y(Rf_*M, Rg_*Lg^*K) = H^0(Y, R\text{Hom}(Rf_*M, K)) \otimes_{\mathcal{O}_Y} Lg_! K
\]

The first equality by definition of \( a \). The second equality by Derived Categories of Schemes, Lemma 5.2. The third equality by Derived Categories of Schemes, Lemma 17.6. Thus we get the same outcome as before. We omit the verification that our map induces the given identifications. \( \square \)

**Lemma 21.2.** Consider a commutative diagram

\[
\begin{array}{ccc}
X' & \longrightarrow & X \\
\downarrow f' & & \downarrow f \\
Y' & \longrightarrow & Y \\
\downarrow g' & & \downarrow g \\
Z' & \longrightarrow & Z
\end{array}
\]

of quasi-compact and quasi-separated schemes where both diagrams are cartesian and where \( f \) and \( l \) as well as \( g \) and \( m \) are Tor independent. Then the maps for the two squares compose to give the base change map for the outer rectangle (see proof for a precise statement).

**Proof.** It follows from the assumptions that \( g \circ f \) and \( m \) are Tor independent (details omitted), hence the statement makes sense. In this proof we write \( k^* \) in place of \( Lk^* \) and \( f_* \) instead of \( Rf_* \). Let \( a, b, \) and \( c \) be the twisted inverse image for \( f, g, \) and \( g \circ f \) and similarly for the primed versions. The arrow corresponding to the top square is the composition

\[
\gamma_{\text{top}} : k^* \circ a \rightarrow k^* \circ a \circ l_* \circ l^* \xrightarrow{\xi_{\text{top}}} k^* \circ k_* \circ a' \circ l^* \rightarrow a' \circ l^*
\]

where \( \xi_{\text{top}} : k_* \circ a' \rightarrow a \circ l_* \) is an isomorphism (hence can be inverted) and is the arrow “dual” to the base change map \( l^* \circ f_* \rightarrow f'_* \circ k^* \). The outer arrows come from the canonical maps \( 1 \rightarrow l_* \circ l^* \) and \( k^* \circ k_* \rightarrow 1 \). Similarly for the second square we have

\[
\gamma_{\text{bot}} : l^* \circ b \rightarrow l^* \circ b \circ m_* \circ m^* \xrightarrow{\xi_{\text{bot}}} l^* \circ l_* \circ b' \circ m^* \rightarrow b' \circ m^*
\]

For the outer rectangle we get

\[
\gamma_{\text{rect}} : k^* \circ c \rightarrow k^* \circ c \circ m_* \circ m^* \xrightarrow{\xi_{\text{rect}}} k^* \circ k_* \circ c' \circ m^* \rightarrow c' \circ m^*
\]

We have \((g \circ f)_* = g_* \circ f_*\) and hence \( c = a \circ b \) and similarly \( c' = a' \circ b' \). The statement of the lemma is that \( \gamma_{\text{rect}} \) is equal to the composition

\[
k^* \circ c = k^* \circ a \circ b \xrightarrow{\gamma_{\text{top}}} a' \circ l^* \circ b \xrightarrow{\gamma_{\text{bot}}} a' \circ b' \circ m^* = c' \circ m^*
\]
To see this we contemplate the following diagram:

![Diagram with arrows and labels]

Going down the right hand side we have the composition and going down the left hand side we have $\gamma_{\text{rect}}$. All the quadrilaterals on the right hand side of this diagram commute by Categories, Lemma 27.2 or more simply the discussion preceding Categories, Definition 27.1. Hence we see that it suffices to show the diagram

\[
\begin{align*}
 a \circ l_* \circ l^* \circ b \circ m_* & \leftarrow a \circ b \circ m_* \\
 k_* \circ a' \circ l^* \circ b \circ m_* & \\
 k_* \circ a' \circ l^* \circ l_* \circ b' & \leftarrow k_* \circ a' \circ b'
\end{align*}
\]

becomes commutative if we invert the arrows $\xi_{\text{top}}$, $\xi_{\text{bot}}$, and $\xi_{\text{rect}}$ (note that this is different from asking the diagram to be commutative). However, the diagram

![Diagram with arrows and labels]

Lemma 21.3. Consider a commutative diagram
\[
\begin{array}{ccc}
X'' & \xrightarrow{g'} & X' \xrightarrow{g} X \\
\downarrow{f''} & & \downarrow{f} \\
Y'' & \xrightarrow{h'} & Y' \xrightarrow{h} Y
\end{array}
\]
of quasi-compact and quasi-separated schemes where both diagrams are cartesian and where \(f\) and \(h\) as well as \(f'\) and \(h'\) are Tor independent. Then the maps of quasi-compact and quasi-separated schemes where both diagrams are cartesian and where \(f\) and \(h\) as well as \(f'\) and \(h'\) are Tor independent. Then the maps
\[
\delta_{\text{left}} : (g')^* \circ a' \to (g')^* \circ a' \circ (h')^* \circ (h')^* \xrightarrow{\xi_{\text{left}}} (g')^* \circ (g')^* \circ a'' \circ (h')^* \to a'' \circ (h')^* 
\]
where \(\xi_{\text{left}} : g_{a'}^* \circ a' \to a \circ h_{a'}^* \) is an isomorphism (hence can be inverted) and is the arrow “dual” to the base change map \(h^* \to f_{a'}^* \circ g^*\). The outer arrows come from the canonical maps \(1 \to h_{a'}^* \circ h^* \) and \(g^* \circ g_{a'}^* \to 1\). Similarly for the left square we have
\[
\gamma_{\text{left}} : (g')^* \circ a' \to (g')^* \circ a' \circ (h')^* \circ (h')^* \xrightarrow{\xi_{\text{right}}} (g')^* \circ (g')^* \circ a'' \circ (h')^* \to a'' \circ (h')^* 
\]
For the outer rectangle we get
\[
\gamma_{\text{rect}} : k^* \circ a \xrightarrow{\xi_{\text{rect}}} k^* \circ a \circ m_{a} \circ m^* \xrightarrow{\xi_{\text{rect}}} k^* \circ a'' \circ m^* 
\]
where \(k = g \circ g'\) and \(m = h \circ h'\). We have \(k^* = (g')^* \circ g^*\) and \(m^* = (h')^* \circ h^*\). The statement of the lemma is that \(\gamma_{\text{rect}}\) is equal to the composition
\[
k^* \circ a = (g')^* \circ g^* \circ a \xrightarrow{\gamma_{\text{right}}} (g')^* \circ a' \circ h^* \xrightarrow{\gamma_{\text{right}}} (h')^* \circ h^* = a'' \circ m^*
\]
To see this we contemplate the following diagram

\[
\begin{array}{c}
\xi_{\text{right}}
\end{array}
\]

Going down the right hand side we have the composition and going down the left hand side we have \( \gamma_{\text{rect}} \). All the quadrilaterals on the right hand side of this diagram commute by Categories, Lemma 27.2 or more simply the discussion preceding Categories, Definition 27.1. Hence we see that it suffices to show that

\[
g_* \circ (g')_* \circ a'' \xrightarrow{\xi_{\text{left}}} g_* \circ a' \circ (h')_* \xrightarrow{\xi_{\text{right}}} a \circ h_* \circ (h')_*
\]

is equal to \( \xi_{\text{rect}} \). This is the statement dual to Cohomology, Remark 29.4 and the proof is complete.

\[\square\]

**Remark 21.4.** Consider a commutative diagram

\[
\begin{array}{c}
X'' & \xrightarrow{k'} & X' & \xrightarrow{k} & X \\
\downarrow{f''} & & \downarrow{f'} & & \downarrow{f} \\
Y'' & \xrightarrow{l'} & Y' & \xrightarrow{l} & Y \\
\downarrow{g''} & & \downarrow{g'} & & \downarrow{g} \\
Z'' & \xrightarrow{m'} & Z' & \xrightarrow{m} & Z
\end{array}
\]

of quasi-compact and quasi-separated schemes where all squares are cartesian and where \((f, l), (g, m), (f', l'), (g', m')\) are Tor independent pairs of maps. Let \(a, a', a'', b, b', b''\) be the twisted inverse image for \(f, f', f'', g, g', g''\). Let us label the squares of the diagram \(A, B, C, D\) as follows

\[
\begin{array}{c}
A & B \\
C & D
\end{array}
\]
Then the maps (21.0.1) for the squares are (where we use $k^* = Lk^*$, etc)

\[
\begin{align*}
\gamma_A &: (k')^* \circ a' \to a'' \circ (l')^* \\
\gamma_B &: k^* \circ a \to a' \circ l^* \\
\gamma_C &: (l')^* \circ b' \to b'' \circ (m')^* \\
\gamma_D &: l^* \circ b \to b' \circ m^*
\end{align*}
\]

For the $2 \times 1$ and $1 \times 2$ rectangles we have four further base change maps

\[
\begin{align*}
\gamma_{A+B} &: (k \circ k')^* \circ a \to a'' \circ (l \circ l')^* \\
\gamma_{C+D} &: (l \circ l')^* \circ b \to b'' \circ (m \circ m')^* \\
\gamma_{A+C} &: (k')^* \circ (a' \circ b') \to (a'' \circ b'') \circ (m')^* \\
\gamma_{A+C} &: k^* \circ (a \circ b) \to (a' \circ b') \circ m^*
\end{align*}
\]

By Lemma 21.3 we have

\[
\gamma_{A+B} = \gamma_A \circ \gamma_B, \quad \gamma_{C+D} = \gamma_C \circ \gamma_D
\]

and by Lemma 21.2 we have

\[
\gamma_{A+C} = \gamma_C \circ \gamma_A, \quad \gamma_{B+D} = \gamma_D \circ \gamma_B
\]

Here it would be more correct to write $\gamma_{A+B} = (\gamma_A \ast \text{id}_{l'}) \circ (\text{id}_{k'} \ast \gamma_B)$ with notation as in Categories, Section 27 and similarly for the others. However, we continue the abuse of notation used in the proofs of Lemmas 21.2 and 21.3 of dropping $\ast$ products with identities as one can figure out which ones to add as long as the source and target of the transformation is known. Having said all of this we find (a priori) two transformations

\[
(k')^* \circ k^* \circ a \circ b \to a'' \circ b'' \circ (m')^* \circ m^*
\]

namely

\[
\gamma_C \circ \gamma_A \circ \gamma_D \circ \gamma_B = \gamma_{A+C} \circ \gamma_{B+D}
\]

and

\[
\gamma_C \circ \gamma_D \circ \gamma_A \circ \gamma_B = \gamma_{C+D} \circ \gamma_{A+B}
\]

The point of this remark is to point out that these transformations are equal. Namely, to see this it suffices to show that

\[
\begin{align*}
(k')^* \circ a' \circ l^* \circ b &\xrightarrow{\gamma_D} (k')^* \circ a' \circ b' \circ m^* \\
\gamma_A &\downarrow \quad \gamma_A \downarrow
\end{align*}
\]

\[
\begin{align*}
a'' \circ (l')^* \circ l^* \circ b &\xrightarrow{\gamma_D} a'' \circ (l')^* \circ b' \circ m^*
\end{align*}
\]

commutes. This is true by Categories, Lemma 27.2 or more simply the discussion preceding Categories, Definition 27.1.

### 22. Flat and proper morphisms

The correct generality for this section would be to consider proper perfect morphisms of quasi-compact and quasi-separated schemes, see [LN07].

**Lemma 22.1.** Let $f : X \to Y$ be a flat and proper morphism of Noetherian schemes. Let $a$ be the twisted inverse image. Then $a$ commutes with direct sums.
Lemma 22.2. Let $f : X \to Y$ be a flat and proper morphism of Noetherian schemes. Let $a$ be the twisted inverse image. Let $T \subset Y$ be closed. Then

1. if $Q \in D_{QCoh}(Y)$ is supported on $T$, then $a(Q)$ is supported on $f^{-1}(T)$,
2. the map \eqref{20.6.1} is an isomorphism for all $K \in D_{QCoh}(O_Y)$, and
3. the canonical map

$$Rf_*R\text{Hom}(L,a(K)) \to R\text{Hom}(Rf_*L,K)$$

is an isomorphism for all $L \in D_{QCoh}(O_X)$ and all $K \in D_{QCoh}(O_Y)$.

Proof. Arguing exactly as in the proof of Lemma \ref{20.10} we see that (2) implies (3). Arguing exactly as in the proof of Lemma \ref{20.8} we see that (1) implies (2).

Proof of (1). We will use the notation $D_{QCoh,T}(O_Y)$ and $D_{QCoh,f^{-1}(T)}(O_X)$ to denote complexes whose cohomology sheaves are supported on $T$ and $f^{-1}(T)$. By Lemma \ref{22.1} the functor $a$ commutes with direct sums. Hence the strictly full, saturated, triangulated subcategory $D$ with objects

$$\{Q \in D_{QCoh,T}(O_Y) \mid a(Q) \in D_{QCoh,f^{-1}(T)}(O_X)\}$$

is preserved by direct sums (and hence derived colimits). On the other hand, the category $D_{QCoh,T}(O_Y)$ is generated by a perfect object $E$ (see Derived Categories of Schemes, Lemma \ref{14.5}). By Lemma \ref{20.9} we see that $E \in D$. By Derived Categories, Lemma \ref{34.3} every object $Q$ of $D_{QCoh,T}(O_Y)$ is a derived colimit of a system $Q_1 \to Q_2 \to Q_3 \to \ldots$ such that the cones of the transition maps are direct sums of shifts of $E$. Arguing by induction we see that $E_n \in D$ for all $n$ and finally that $Q$ is in $D$. Thus (1) is true.

Lemma 22.3. Let $f : X \to Y$ be a proper flat morphism of Noetherian schemes. The map \eqref{20.10.1} is an isomorphism for every object $K$ of $D_{QCoh}(O_Y)$.

Proof. By Lemma \ref{22.1} we know that $a$ commutes with direct sums. Hence the collection of objects of $D_{QCoh}(O_Y)$ for which \eqref{20.10.1} is an isomorphism is a strictly full, saturated, triangulated subcategory of $D_{QCoh}(O_Y)$ which is moreover preserved under taking direct sums. Since $D_{QCoh}(O_Y)$ is a module category (Derived Categories of Schemes, Theorem \ref{16.3}) generated by a single perfect object (Derived Categories of Schemes, Theorem \ref{14.3}) we can argue as in More on Algebra, Remark \ref{47.11} to see that it suffices to prove \eqref{20.10.1} is an isomorphism for a single perfect object. However, the result holds for perfect objects, see Lemma \ref{20.11}.
Lemma 22.4. Let $f : X \to Y$ be a proper flat morphism of Noetherian schemes. Let $g : Y' \to Y$ be a morphism of finite type. Then the base change map \([21.0.1]\) is an isomorphism for all $K \in D_{Qcoh}(\mathcal{O}_X)$.

Proof. By Lemma 22.2 formation of the functors $a$ and $a'$ commutes with restriction to opens of $Y$ and $Y'$. Hence we may assume $Y' \to Y$ is a morphism of affine schemes. In this case the statement follows from Lemma 21.1. □

Lemma 22.5. Let $f : X = \mathbb{P}_Y^1 \to Y$ be the projection where $Y$ is a Noetherian scheme. Let $a$ be the twisted inverse image. Then $a(\mathcal{O}_Y)$ is isomorphic to $\mathcal{O}_X(-2)[1]$.

Proof. Recall that there is an identification $Rf_*(\mathcal{O}_X(-2)[1]) = \mathcal{O}_Y$, see Cohomology of Schemes, Lemma 8.3 or 8.4. This determines a map $\mathcal{O}_X(-2)[1] \to a(\mathcal{O}_Y)$. By Lemma 20.9 construction of the twisted inverse image is local on the base. In particular, to check that $\mathcal{O}_X(-2)[1] \to a(\mathcal{O}_Y)$ is an isomorphism, we may work locally on $Y$. In other words, we may assume $Y$ is affine. In the affine case the sheaves $\mathcal{O}_X$ and $\mathcal{O}_X(-1)$ generate $D_{Qcoh}(X)$, see Derived Categories of Schemes, Lemma 14.4. Hence it suffices to show that the maps

$$H^{-n+1}(X, \mathcal{O}(-2)) = \text{Hom}_{D(\mathcal{O}_X)}(\mathcal{O}_X[n], \mathcal{O}_X(-2)[1])$$

$$\to \text{Hom}_{D(\mathcal{O}_X)}(\mathcal{O}_X[n], a(\mathcal{O}_Y))$$

$$= \text{Hom}_{D(\mathcal{O}_Y)}(Rf_*(\mathcal{O}_X)[n], \mathcal{O}_Y)$$

$$= H^{-n}(Y, \mathcal{O}_Y)$$

and

$$H^{-n+1}(X, \mathcal{O}_X(-1)) = \text{Hom}_{D(\mathcal{O}_X)}(\mathcal{O}_X(-1)[n], \mathcal{O}_X(-2)[1])$$

$$\to \text{Hom}_{D(\mathcal{O}_X)}(\mathcal{O}_X(-1)[n], a(\mathcal{O}_Y))$$

$$= \text{Hom}_{D(\mathcal{O}_Y)}(Rf_*(\mathcal{O}_X(-1)[n], \mathcal{O}_Y)$$

$$= 0$$

(where we used Cohomology of Schemes, Lemma 8.1) are isomorphisms for all $n \in \mathbb{Z}$. This is clear from the explicit computation of cohomology in Cohomology of Schemes, Lemma 8.1. □

Example 22.6. The base change map \([21.0.1]\) is not an isomorphism if $f$ is proper and perfect and $g$ is perfect. Let $k$ be a field. Let $Y = \mathbb{A}_k^2$ and let $f : X \to Y$ be the blow up of $Y$ in the origin. Denote $E \subset X$ the exceptional divisor. Then we can factor $f$ as

$$X \xrightarrow{a} \mathbb{P}_Y^1 \xrightarrow{b} Y$$

This gives a factorization $a = c \circ b$ where $b$ is the twisted inverse image for $p$ and $c$ is the twisted inverse image for $i$. Denote $\mathcal{O}(n)$ the Serre twist of the structure sheaf on $\mathbb{P}_Y^1$ and denote $\mathcal{O}_X(n)$ its restriction to $X$. Note that $X \subset \mathbb{P}_Y^1$ is cut out by a degree one equation, hence $\mathcal{O}(X) = \mathcal{O}(1)$. By Lemma 22.5 we have $b(\mathcal{O}_Y) = \mathcal{O}(-2)[1]$. By Lemma 20.12 we have

$$a(\mathcal{O}_Y) = c(b(\mathcal{O}_Y)) = c(\mathcal{O}(-2)[1]) = R\text{Hom}(\mathcal{O}_X, \mathcal{O}(-2)[1]) = \mathcal{O}_X(-1)$$

Last equality by Lemma 33.2. Let $Y' = \text{Spec}(k)$ be the origin in $Y$. The restriction of $a(\mathcal{O}_Y)$ to $X' = E = \mathbb{P}_k^1$ is an invertible sheaf of degree $-1$ placed in cohomological degree 0. But on the other hand, $a'(\mathcal{O}_{\text{Spec}(k)}) = \mathcal{O}_E(-2)[1]$ which is an invertible
Let $\mathcal{F}$ be a scheme of degree $-2$ placed in cohomological degree $-1$, so different. In this example (4) is the only hypothesis of Lemma \textit{23.1} which is violated.

### 23. Compactifications

We interrupt the flow of the arguments for a little bit of geometry.

Let $S$ be a quasi-compact and quasi-separated scheme. We will say a scheme $X$ over $S$ has a compactification over $S$ if there exists an open immersion $X \to \overline{X}$ into a scheme $\overline{X}$ proper over $S$. If $X$ has a compactification over $S$, then $X \to S$ is separated and of finite type. It is a theorem of Nagata (see [Lüt93], [Con07], [Nag56], [Nag57], [Nag62], and [Nag63]) that the converse is true as well (we will give a precise statement and a proof if we ever need this result).

Let $S$ be a scheme and $X$ a scheme over $S$. The category of compactifications of $X$ is the category whose objects are open immersions $j : X \to \overline{X}$ over $S$ with $\overline{X} \to S$ proper and whose morphisms $(j : X \to \overline{X}) \to (j' : X \to \overline{X})$ are morphisms $f : \overline{X} \to \overline{X}$ such that $f \circ j = j$.

**Lemma 23.1.** Let $S$ be a quasi-compact and quasi-separated scheme. Let $X$ be a compactifiable scheme over $S$. The category of compactifications of $X$ over $S$ is cofiltered.

**Proof.** We have to check conditions (1), (2), (3) of Categories, Definition \textit{20.1}. Condition (1) holds exactly because we assumed that $X$ is compactifiable. Let $j_i : X \to \overline{X}_i$, $i = 1, 2$ be two compactifications. Then we can consider the scheme-theoretic closure $\overline{X}$ of $(j_1, j_2) : X \to \overline{X}_1 \times_S \overline{X}_2$. This determines a third compactification $j : X \to \overline{X}$ which dominates both $j_i$:

$$
\begin{array}{ccc}
(X, \overline{X}_1) & \leftarrow & (X, \overline{X}) \\
\overline{X} & \longrightarrow & (X, \overline{X}_2)
\end{array}
$$

Thus (2) holds. Let $f_1, f_2 : \overline{X}_1 \to \overline{X}_2$ be two morphisms between compactifications $j_i : X \to \overline{X}_i$, $i = 1, 2$. Let $X \subseteq \overline{X}_1$ be the equalizer of $f_1$ and $f_2$. As $\overline{X}_2 \to S$ is separated, we see that $\overline{X}$ is a closed subscheme of $\overline{X}_1$ and hence proper over $S$. Moreover, we obtain an open immersion $X \to \overline{X}$ because $f_1|_X = f_2|_X = \text{id}_X$. The morphism $(X \to \overline{X}) \to (j_1 : X \to \overline{X}_1)$ given by the closed immersion $\overline{X} \to \overline{X}_1$ equalizes $f_1$ and $f_2$ which proves condition (3) and finishes the proof.

We can also consider the category of all compactifications (for varying $X$). It turns out that this category, localized at the set of morphisms which induce an isomorphism on the interior is equivalent to the category of compactifiable schemes over $S$.

**Lemma 23.2.** Let $S$ be a quasi-compact and quasi-separated scheme. Let $f : X \to Y$ be a morphism of schemes over $S$ with $Y$ separated and of finite type over $S$ and $X$ compactifiable over $S$. Then $X$ has a compactification over $Y$.

**Proof.** Let $f : X \to Y$ be a morphism of schemes over $S$ with $Y$ separated and of finite type over $S$. Let $j : X \to \overline{X}$ be a compactification of $X$ over $S$. Then we let $\overline{X}^\prime$ be the scheme theoretic image of $(j, f) : X \to \overline{X} \times_S Y$. The morphism $\overline{X}^\prime \to Y$ is proper because $\overline{X} \times_S Y \to Y$ is proper as a base change of $\overline{X} \to S$. On the other hand, since $Y$ is separated over $S$, the morphism $(1, f) : X \to X \times_S Y$ is a closed immersion (Schemes, Lemma \textit{21.11}) and hence $X \to \overline{X}^\prime$ is an open immersion.
Let $S$ be a quasi-compact and quasi-separated scheme. We define the category of compactifications to be the category whose objects are pairs $(X, \overline{X})$ where $\overline{X}$ is a scheme proper over $S$ and $X \subset \overline{X}$ is a quasi-compact open and whose morphisms are commutative diagrams

$$
\begin{array}{ccc}
X & \to & Y \\
\downarrow & & \downarrow \\
\overline{X} & \to & \overline{Y}
\end{array}
$$

of morphisms of schemes over $S$.

**Lemma 23.3.** Let $S$ be a quasi-compact and quasi-separated scheme. The collection of morphisms $(u, \pi) : (X', \overline{X}') \to (X, \overline{X})$ such that $u$ is an isomorphism forms a right multiplicative system (Categories, Definition 26.1) of arrows in the category of compactifications.

**Proof.** Axiom RMS1 is trivial to verify. Let us check RMS2 holds. Suppose given a diagram

$$
\begin{array}{ccc}
(Y', \overline{Y'}) & \to & (X', \overline{X}') \\
\downarrow & & \downarrow \\
(Y, \overline{Y}) & \to & (X, \overline{X})
\end{array}
$$

with $u : X' \to X$ an isomorphism. Then we let $Y' = Y \times_X X'$ with the projection map $v : Y' \to Y$ (an isomorphism). We also set $Y' = Y \times_{\overline{X}} \overline{X}'$ with the projection map $\overline{v} : Y' \to Y$ It is clear that $Y' \to Y'$ is an open immersion. The diagram

$$
\begin{array}{ccc}
(Y', \overline{Y}') & \to & (X', \overline{X}') \\
\downarrow & & \downarrow \\
(Y, \overline{Y}) & \to & (X, \overline{X})
\end{array}
$$

shows that axiom RMS2 holds.

Let us check RMS3 holds. Suppose given a pair of morphisms $(f, \overline{f}), (g, \overline{g}) : (X, \overline{X}) \to (Y, \overline{Y})$ of compactifications and a morphism $(v, \overline{v}) : (Y, \overline{Y}) \to (Y', \overline{Y'})$ such that $v$ is an isomorphism and such that $(v, \overline{v}) \circ (f, \overline{f}) = (v, \overline{v}) \circ (g, \overline{g})$. Then $f = g$. Hence if we let $X' \subset \overline{X}$ be the equalizer of $f$ and $\overline{f}$, then $(u, \pi) : (X, \overline{X}) \to (X, \overline{X})$ will be a morphism of the category of compactifications such that $(f, \overline{f}) \circ (u, \pi) = (g, \overline{g}) \circ (u, \pi)$ as desired.

**Lemma 23.4.** Let $S$ be a quasi-compact and quasi-separated scheme. The functor $(X, \overline{X}) \mapsto X$ defines an equivalence from the category of compactifications localized (Categories, Lemma 26.9) at the right multiplicative system of Lemma 23.3 to the category of compactifyable schemes over $S$.

**Proof.** Denote $\mathcal{C}$ the category of compactifications and denote $Q : \mathcal{C} \to \mathcal{C}'$ the localization functor of Categories, Lemma 26.13. Denote $\mathcal{D}$ the category of compactifyable schemes over $S$. It is clear from the lemma just cited and our choice of multiplicative system that we obtain a functor $\mathcal{C} \to \mathcal{D}$. This functor is clearly essentially surjective. If $f : X \to Y$ is a morphism of compactifyable schemes, then
we choose an open immersion $Y \to \overline{Y}$ into a scheme proper over $S$, and then we choose an embedding $X \to \overline{X}$ into a scheme $\overline{X}$ proper over $Y$ (possible by Lemma \ref{lem:compactification} applied to $X \to \overline{Y}$). This gives a morphism $(X, \overline{X}) \to (Y, \overline{Y})$ of compactifications which produces our given morphism $X \to Y$. Finally, suppose given a pair of morphisms in the localized category with the same source and target: say

$$a = ((f, \overline{f}) : (X', \overline{X'}) \to (Y, \overline{Y}), (u, \overline{u}) : (X', \overline{X'}) \to (X, \overline{X}))$$

and

$$b = ((g, \overline{g}) : (X'', \overline{X''}) \to (Y, \overline{Y}), (v, \overline{v}) : (X'', \overline{X''}) \to (X, \overline{X}))$$

which produce the same morphism $X \to Y$ over $S$, in other words $f \circ v^{-1} = g \circ u^{-1}$. By Categories, Lemma \ref{lem:category} we may assume that $(X', \overline{X'}) = (X'', \overline{X''})$ and $(u, \overline{u}) = (v, \overline{v})$. In this case we can consider the equalizer $\overline{X''} \subset \overline{X'}$ of $\overline{f}$ and $\overline{g}$. The morphism $(w, \overline{w}) : (X', \overline{X''}) \to (X', \overline{X'})$ is in the multiplicative subset and we see that $a = b$ in the localized category by precomposing with $(w, \overline{w})$. \hfill \Box

### 24. Upper shriek functors

In this section, we construct the functors $f^!$ for morphisms between compactifyable schemes over a fixed Noetherian base. As is customary in coherent duality, there are a number of diagrams that have to be shown to be commutative. We suggest the reader, after reading the construction, skips the verification of the lemmas and continues to the next section where we discuss properties of the upper shriek functors.

Given a morphism $f : X \to Y$ of compactifyable schemes over a Noetherian base scheme $S$, we will define an exact functor

$$f^! : D^+_{QCoh}(O_Y) \to D^+_{QCoh}(O_X)$$

of triangulated categories. Namely, we choose a compactification $X \to \overline{X}$ over $Y$ which is possible by Lemma \ref{lem:compactification}. Denote $\overline{f} : \overline{X} \to Y$ the structure morphism. Let $\overline{\pi} : D_{QCoh}(O_Y) \to D_{QCoh}(O_{\overline{X}})$ be the twisted inverse image, i.e., the right adjoint of $R\overline{f}^*$ constructed in Lemma \ref{lem:twisted-inverse-image}. Then we set

$$f^!K = (\overline{\pi}(K))_X$$

for $K \in D^+_{QCoh}(O_Y)$. The result is an object of $D^+_{QCoh}(O_X)$ by Lemma \ref{lem:exactness}.

**Lemma 24.1.** Let $f : X \to Y$ be a morphism between compactifyable schemes over a Noetherian scheme $S$. The functor $f^!$ is, up to canonical isomorphism, independent of the choice of the compactification.

**Proof.** Consider the category of compactifications of $X$ over $Y$, which is cofiltered according to Lemmas \ref{lem:category} and \ref{lem:compactification} To every choice of a compactification

$$j : X \to \overline{X}, \quad \overline{f} : \overline{X} \to Y$$

the construction above associates the functor $j^* \circ \overline{\pi} : D^+_{QCoh}(O_Y) \to D^+_{QCoh}(O_X)$ where $\overline{\pi}$ is the twisted inverse image for $\overline{f}$.

Suppose given a morphism $g : \overline{X}_1 \to \overline{X}_2$ between compactifications $j_i : X \to \overline{X}_i$ over $Y$. Namely, let $\overline{\pi}_i$ be the twisted inverse image for $g$. Then $\overline{\pi}_0 \circ \overline{\pi}_2 = \overline{\pi}_1$ because these functors are adjoint to $R\overline{f}_2^* \circ Rg_* = R(\overline{f}_2 \circ g)_*$. By \ref{lem:adjunction} we have a canonical transformation

$$j_1^* \circ \overline{\pi} \to j_2^*$$
of functors $D_{QCoh}^+(\mathcal{O}_X) \to D_{QCoh}^+(\mathcal{O}_X)$ which is an isomorphism by Lemma 20.9.

The composition

$$j_1^* \circ \overline{a}_1 \longrightarrow j_1^* \circ \overline{a}_2 \longrightarrow j_2^* \circ \overline{a}_2$$

is an isomorphism of functors which we will denote by $\alpha_g$.

To finish the proof, since the category of compactifications over $X$ over $Y$ is cofiltered, it suffices to show compositions of morphisms of compactifications of $X$ over $Y$ are turned into compositions of isomorphisms of functors. To do this, suppose that $j_3 : X \to \overline{X}_3$ is a third compactification and that $h : X_2 \to \overline{X}_3$ is a morphism of compactifications. Let $\overline{d}$ be the twisted inverse image for $h$. Then $\overline{d} \circ \overline{a}_3 = \overline{a}_2$ and there is a canonical transformation

$$j_2^* \circ \overline{d} \longrightarrow j_3^*$$

of functors $D_{QCoh}^+(\mathcal{O}_{\overline{X}_3}) \to D_{QCoh}^+(\mathcal{O}_X)$ for the same reasons as above. Denote $\overline{\pi}$ the twisted inverse image for $h \circ g$. There is a canonical transformation

$$j_1^* \circ \overline{\pi} \longrightarrow j_3^*$$

of functors $D_{QCoh}^+(\mathcal{O}_{\overline{X}_3}) \to D_{QCoh}^+(\mathcal{O}_X)$ given by (20.6.1). Spelling things out we have to show that the composition

$$\alpha_h \circ \alpha_g : j_1^* \circ \overline{a}_1 \longrightarrow j_1^* \circ \overline{a}_2 \longrightarrow j_2^* \circ \overline{a}_2 \longrightarrow j_2^* \circ \overline{d} \circ \overline{a}_3 \longrightarrow j_3^* \circ \overline{a}_3$$

is the same as the composition

$$\alpha_{h \circ g} : j_1^* \circ \overline{a}_1 \longrightarrow j_1^* \circ \overline{a}_3 \longrightarrow j_3^* \circ \overline{a}_3$$

We split this into two parts. The first is to show that the diagram

$$\begin{array}{ccc}
\overline{a}_1 & \longrightarrow & \overline{a}_2 \\
\downarrow & & \downarrow \\
\overline{\pi} \circ \overline{a}_3 & \longrightarrow & \overline{\pi} \circ \overline{d} \circ \overline{a}_3
\end{array}$$

commutes where the lower horizontal arrow comes from the identification $\overline{\pi} = \overline{\pi} \circ \overline{d}$. This is true because the corresponding diagram of total direct image functors

$$\begin{array}{ccc}
Rj_{1,*} & \longrightarrow & Rg_* \circ Rj_{2,*} \\
\downarrow & & \downarrow \\
R(h \circ g)_* \circ Rj_{3,*} & \longrightarrow & Rg_* \circ Rh_* \circ Rj_{3,*}
\end{array}$$

is commutative (insert future reference here). The second part is to show that the composition

$$j_1^* \circ \overline{\pi} \circ \overline{d} \longrightarrow j_2^* \circ \overline{d} \longrightarrow j_3^*$$

is equal to the map

$$j_1^* \circ \overline{\pi} \longrightarrow j_3^*$$

via the identification $\overline{\pi} = \overline{\pi} \circ \overline{d}$. This was proven in Lemma 21.2 (note that in the current case the morphisms $f', g'$ of that lemma are equal to $\text{id}_X$). □

---

$^2$Namely, if $\alpha, \beta : F \to G$ are morphisms of functors and $\gamma : G \to H$ is an isomorphism of functors such that $\gamma \circ \alpha = \gamma \circ \beta$, then we conclude $\alpha = \beta$. 
Lemma 24.2. Let $f: X \to Y$ and $g: Y \to Z$ be composable morphisms between compactifyable schemes over a Noetherian scheme $S$. Then there is a canonical isomorphism $(g \circ f)^! \to f^! \circ g^!$.

Proof. Choose a compactification $i: Y \to \overline{Y}$ of $Y$ over $Z$. Choose a compactification $X \to \overline{X}$ of $X$ over $\overline{Y}$. This uses Lemma 23.2 twice. Let $\overline{\pi}$ be the twisted inverse image for $\overline{X} \to \overline{Y}$ and let $\overline{b}$ be the twisted inverse image for $\overline{Y} \to Z$. Then $\pi \circ \overline{b}$ is the twisted inverse image for the composition $\overline{X} \to Z$. Hence $g^! = j_Y^* \circ \overline{b}$ and $(g \circ f)^! = (X \to \overline{X})^* \circ \pi \circ \overline{b}$. Let $U$ be the inverse image of $Y$ in $X$ so that we get the commutative diagram

$\diagram
X \ar[r]^j \ar[d] & U \ar[r]^{j'} \ar[d] & \overline{X} \ar[d] \\
Y \ar[r]_i & \overline{Y} \\
Z \ar[u] & \enddiagram$

Let $\pi'$ be the twisted inverse image for $U \to Y$. Then $f^! = j^* \circ \pi'$. We obtain

$\gamma: (j')^* \circ \pi' \to \pi' \circ j_Y^*$

by (20.6.1) and we can use it to define

$(g \circ f)^! = j_X^* \circ \pi \circ \overline{b} = j^* \circ (j')^* \circ \pi \circ \overline{b} \to j^* \circ \pi' \circ j_Y^* \circ \overline{b} = f^! \circ g^!$

which is an isomorphism on objects of $D_{Qcoh}(\mathcal{O}_Z)$ by Lemma 20.9. To finish the proof we show that this isomorphism is independent of choices made.

Suppose we have two diagrams

$\diagram
X \ar[r]^{j_1} \ar[d] & U_1 \ar[r]^{j'_1} \ar[d] & \overline{X}_1 \ar[d] \\
Y \ar[r]_{i_1} & \overline{Y}_1 \\
Z \ar[u] & \enddiagram \quad \text{and} \quad \diagram
X \ar[r]^{j_2} \ar[d] & U_2 \ar[r]^{j'_2} \ar[d] & \overline{X}_2 \ar[d] \\
Y \ar[r]_{i_2} & \overline{Y}_2 \\
Z \ar[u] & \enddiagram$

We can first choose a compactification $i: Y \to \overline{Y}$ of $Y$ over $Z$ which dominates both $\overline{Y}_1$ and $\overline{Y}_2$, see Lemma 23.1 By Lemma 23.3 and Categories, Lemmas 26.10 and 26.11 we can choose a compactification $X \to \overline{X}$ of $X$ over $\overline{Y}$ with morphisms $X \to \overline{X}_1$ and $X \to \overline{X}_2$ and such that the composition $X \to \overline{Y} \to \overline{Y}_1$ is equal to the composition $X \to \overline{X}_1 \to \overline{Y}_1$ and such that the composition $X \to \overline{Y} \to \overline{Y}_2$ is equal to the composition $X \to \overline{X}_2 \to \overline{Y}_2$. Thus we see that it suffices to compare the
maps determined by our diagrams when we have a commutative diagram as follows

\[
\begin{array}{ccc}
X & \xrightarrow{j_1} & U_1 \\
\downarrow & & \downarrow \\
Y & \xrightarrow{i_1} & Y_1 \\
\downarrow & & \downarrow \\
Z & \xrightarrow{i_2} & Y_2 \\
\end{array} \quad \begin{array}{ccc}
X & \xrightarrow{j_2} & U_2 \\
\downarrow & & \downarrow \\
Y & \xrightarrow{i_2} & Y_2 \\
\downarrow & & \downarrow \\
Z & \xrightarrow{i_2} & Y_2 \\
\end{array}
\]

We use \( \pi_i, \pi'_i, \pi, \) and \( \pi' \) for the twisted inverse image for \( X_i \to Y_i, U_i \to Y, \)
\( X_1 \to X_2, \) and \( U_1 \to U_2. \) Each of the squares

\[
\begin{array}{ccc}
X & \xrightarrow{A} & U_1 \\
\downarrow & & \downarrow \\
Y & \xrightarrow{B} & Y_1 \\
\downarrow & & \downarrow \\
Z & \xrightarrow{E} & Y_2 \\
\end{array} \quad \begin{array}{ccc}
X & \xrightarrow{B} & U_2 \\
\downarrow & & \downarrow \\
Y & \xrightarrow{C} & Y_2 \\
\downarrow & & \downarrow \\
Z & \xrightarrow{E} & Y_2 \\
\end{array}
\]

gives rise to a base change map (20.6.1) as follows

\[
\begin{align*}
\gamma_A : & \ j_1^* \circ \pi' \to j_2^* \\
\gamma_B : & \ (j_2')^* \circ \pi_2' \to \pi_2' \circ i_2^* \\
\gamma_C : & \ (j_1')^* \circ \pi_1 \to \pi'_1 \circ i_1^* \\
\gamma_D : & \ i_1^* \circ \pi \to i_2^* \\
\gamma_E : & \ (j_1' \circ j_1)^* \circ \pi_1 \circ \pi_2' \\
\end{align*}
\]

Denote \( f_1^* = j_1^* \circ \pi_1, f_2^* = j_2^* \circ \pi_2', g_1^* = i_1^* \circ \pi_1, g_2^* = i_2^* \circ \pi_2', \) and \( (g \circ f)_1^* = (j_1' \circ j_2)^* \circ \pi_1 \circ \pi_2' \). The construction given in the first paragraph of the proof and in Lemma 24.1 uses

(1) \( \gamma_C \) for the map \( (g \circ f)_1 \to f_1^* \circ g_1^* \),
(2) \( \gamma_B \) for the map \( (g \circ f)_2 \to f_2^* \circ g_2^* \),
(3) \( \gamma_A \) for the map \( f_1^* \to f_2^* \),
(4) \( \gamma_D \) for the map \( g_1^* \to g_2^* \), and
(5) \( \gamma_E \) for the map \( (g \circ f)_1 \to (g \circ f)_2 \).

We have to show that the diagram

\[
\begin{array}{ccc}
(g \circ f)_1 & \xrightarrow{\gamma_E} & (g \circ f)_2 \\
\downarrow \gamma_C & & \downarrow \gamma_B \\
 f_1^* \circ g_1^* & \xrightarrow{\gamma_A \circ \gamma_D} & f_2^* \circ g_2^*
\end{array}
\]

is commutative. We will use Lemmas 21.2 and 21.3 and with (abuse of) notation as in Remark 21.4 (in particular dropping * products with identity transformations.
from the notation). We can write $\gamma_E = \gamma_A \circ \gamma_F$ where

$$
\begin{array}{c}
U_1 \longrightarrow X_1 \\
\downarrow^F \\
U_2 \longrightarrow X_2
\end{array}
$$

Thus we see that

$$
\gamma_B \circ \gamma_E = \gamma_B \circ \gamma_A \circ \gamma_F = \gamma_A \circ \gamma_B \circ \gamma_F
$$

the last equality because the two squares $A$ and $B$ only intersect in one point (similar to the last argument in Remark 21.4). Thus it suffices to prove that $\gamma_D \circ \gamma_C = \gamma_B \circ \gamma_F$. Since both of these are equal to the map (20.6.1) for the square

$$
\begin{array}{c}
U_1 \longrightarrow X_1 \\
\downarrow^F \\
Y \longrightarrow Y_2
\end{array}
$$

we conclude.

\[\square\]

**Lemma 24.3.** Let $S$ be a Noetherian scheme. The constructions of Lemmas 24.1 and 24.2 define a pseudo functor from the category of compactifyable schemes over $S$ into the 2-category of categories (see Categories, Definition 28.5).

**Proof.** To show this we have to prove given morphisms $f : X \to Y$, $g : Y \to Z$, $h : Z \to T$ that

$$
\begin{array}{c}
(h \circ g \circ f)^! \longrightarrow f^! \circ (h \circ g)^! \\
\downarrow_{\gamma_{A+B}} \quad \quad \downarrow_{\gamma_C} \\
(g \circ f)^! \circ h^! \quad \gamma_A \quad f^! \circ g^! \circ h^!
\end{array}
$$

is commutative (for the meaning of the $\gamma$’s, see below). To do this we choose a compactification $\overline{Z}$ of $Z$ over $T$, then a compactification $\overline{Y}$ of $Y$ over $\overline{Z}$, and then a compactification $\overline{X}$ of $X$ over $\overline{Y}$. This uses Lemma 23.2 thrice. Let $W \subset \overline{Y}$ be the inverse image of $Z$ under $\overline{Y} \to \overline{Z}$ and let $U \subset V \subset X$ be the inverse images of $Y \subset W$ under $\overline{X} \to \overline{Y}$. This produces the following diagram

$$
\begin{array}{c}
X \longrightarrow U \longrightarrow V \longrightarrow X \\
\downarrow^f \\
Y \longrightarrow Y \longrightarrow W \longrightarrow Y \\
\downarrow^g \\
Z \longrightarrow Z \longrightarrow Z \longrightarrow Z \\
\downarrow^h \\
T \longrightarrow T \longrightarrow T \longrightarrow T
\end{array}
$$

Without introducing tons of notation but arguing exactly as in the proof of Lemma 24.2 we see that the maps in the first displayed diagram use the maps (20.6.1) for the rectangles $A+B$, $B+C$, $A$, and $C$ as indicated. Since by Lemmas 21.2 and 21.3
we have $\gamma_{A+B} = \gamma_A \circ \gamma_B$ and $\gamma_{B+C} = \gamma_C \circ \gamma_B$ we conclude that the desired equality holds provided $\gamma_A \circ \gamma_C = \gamma_C \circ \gamma_A$. This is true because the two squares $A$ and $C$ only intersect in one point (similar to the last argument in Remark 21.4).

25. Properties of upper shriek functors

Here are some properties of the upper shriek functors.

**Lemma 25.1.** Let $S$ be a Noetherian scheme. Let $Y$ be a compactifyable scheme over $S$ and let $j : X \to Y$ be an open immersion. Then there is a canonical isomorphism $j^! = j^*$ of functors.

**Proof.** In this case we may choose $X = Y$ as our compactification. Then the twisted inverse image for $\text{id} : Y \to Y$ is the identity functor and hence $j^! = j^*$ by definition.

**Lemma 25.2.** Let $S$ be a Noetherian scheme. Let $Y$ be a compactifyable scheme over $S$ and let $f : X = \mathbb{A}^1_Y \to Y$ be the projection. Then there is a (noncanonical) isomorphism $f^!(-) \cong Lf^*(-)[1]$ of functors.

**Proof.** Since $X = \mathbb{A}^1_Y \subset \mathbb{P}^1_Y$ and since $\mathcal{O}_{\mathbb{P}^1_Y}(-2)|_X \cong \mathcal{O}_X$ this follows from Lemmas 25.3 and 22.3.

**Lemma 25.3.** Let $S$ be a Noetherian scheme. Let $Y$ be a compactifyable scheme over $S$ and let $i : X \to Y$ be a closed immersion. Then there is a canonical isomorphism $i^!(-) = R\mathcal{H}om(\mathcal{O}_X, -)$ of functors.

**Proof.** This is a restatement of Lemma 20.12.

**Lemma 25.4.** Let $S$ be a Noetherian scheme. Let $f : X \to Y$ be a morphism of compactifyable schemes over $S$. Then $f^!$ maps $D^+_c(\mathcal{O}_Y)$ into $D^+_c(\mathcal{O}_X)$.

**Proof.** The question is local on $X$ hence we may assume that $X$ and $Y$ are affine schemes. In this case we can factor $f : X \to Y$ as

$X \xrightarrow{i} \mathbb{A}^n_Y \to \mathbb{A}^{n-1}_Y \to \ldots \to \mathbb{A}^1_Y \to Y$

where $i$ is a closed immersion. The lemma follows from By Lemmas 25.2, 15.9, 14.5 and induction.

**Lemma 25.5.** Let $S$ be a Noetherian scheme. Let $f : X \to Y$ be a morphism of compactifyable schemes over $S$. If $K$ is a dualizing complex for $Y$, then $f^!K$ is a dualizing complex for $X$.

**Proof.** The question is local on $X$ hence we may assume that $X$ and $Y$ are affine schemes mapping into an affine open of $S$. In this case we can factor $f : X \to Y$ as

$X \xrightarrow{i} \mathbb{A}^n_Y \to \mathbb{A}^{n-1}_Y \to \ldots \to \mathbb{A}^1_Y \to Y$

where $i$ is a closed immersion. By Lemmas 25.2 and 15.9 and induction we see that the $p^!K$ is a dualizing complex on $\mathbb{A}^n_Y$ where $p : \mathbb{A}^n_Y \to Y$ is the projection. Similarly, by Lemmas 15.8, 14.4 and 25.3 we see that $i^!$ transforms dualizing complexes into dualizing complexes.
Lemma 25.6. Let $S$ be a Noetherian scheme. Let $f : X \to Y$ be a morphism of compactifyable schemes over $S$. Let $K$ be a dualizing complex on $Y$. Set $D_Y(M) = R\mathcal{H}om_{\mathcal{O}_Y}(M, K)$ for $M \in D_{\text{Coh}}(\mathcal{O}_Y)$ and $D_X(E) = R\mathcal{H}om_{\mathcal{O}_X}(E, f^!K)$ for $E \in D_{\text{Coh}}(\mathcal{O}_X)$. Then there is a canonical isomorphism

$$f^!M \to D_X(Lf^*D_Y(M))$$

for $M \in D_{\text{Coh}}^+(\mathcal{O}_Y)$.

Proof. Choose compactification $j : X \subset \overline{X}$ of $X$ over $Y$ (Lemma 23.2). Let $a$ be the twisted inverse image for $\overline{X} \to Y$. Set $D_{\overline{X}}(E) = R\mathcal{H}om_{\mathcal{O}_{\overline{X}}}(E, a(K))$ for $E \in D_{\text{Coh}}(\mathcal{O}_{\overline{X}})$. Since formation of $R\mathcal{H}om$ commutes with restriction to opens and since $f^! = j^* \circ a$ we see that it suffices to prove that there is a canonical isomorphism

$$a(M) \to D_{\overline{X}}(Lf^*D_Y(M))$$

for $M \in D_{\text{Coh}}(\mathcal{O}_Y)$. For $F \in D_{Q\text{Coh}}(\mathcal{O}_X)$ we have

$$\text{Hom}_{\overline{X}}(F, D_{\overline{X}}(Lf^*D_Y(M))) = \text{Hom}_{\overline{X}}(F \otimes^L_{\mathcal{O}_X} Lf^*D_Y(M), a(K))$$

$$= \text{Hom}_Y(Rf_*(F \otimes^L_{\mathcal{O}_X} Lf^*D_Y(M)), K)$$

$$= \text{Hom}_Y(Rf_*(F) \otimes^L_{\mathcal{O}_Y} D_Y(M), K)$$

$$= \text{Hom}_Y(Rf_*(F), D_Y(D_Y(M)))$$

$$= \text{Hom}_Y(Rf_*(F), M)$$

$$= \text{Hom}_{\overline{X}}(F, a(M))$$

The first equality by Cohomology, Lemma 34.2. The second by definition of the twisted inverse image. The third by Derived Categories of Schemes, Lemma 17.1. The fourth by Cohomology, Lemma 34.2 and the definition of $D_Y$. The fifth by Lemma 19.4. The final equality by definition of the twisted inverse image. Hence we see that $a(M) = D_{\overline{X}}(Lf^*D_Y(M))$ by Yoneda’s lemma. \qed

26. A duality theory

Recall that a dualizing complex on a Noetherian scheme $S$, is an object of $D(\mathcal{O}_S)$ which affine locally gives a dualizing complex for the corresponding rings, see Definition 19.2.

Situation 26.1. Here $S$ is a Noetherian scheme and $\omega_S^\bullet$ is a dualizing complex.

For $(S, \omega_S^\bullet)$ as in Situation 26.1 We summarize the most important points of the results obtained above:

1. the functors $f^!$ for morphisms between compactifyable schemes over $S$ turn $D_{Q\text{Coh}}^+$ into a pseudo functor,
2. $\omega_X^\bullet = (X \to S)^!\omega_S^\bullet$ is a dualizing complex for $X$ over $S$ compactifyable,
3. the functor $D_X = R\mathcal{H}om(-, \omega_X^\bullet)$ defines an involution of $D_{\text{Coh}}(\mathcal{O}_X)$ switching $D_{\text{Coh}}^+(\mathcal{O}_X)$ and $D_{\text{Coh}}(\mathcal{O}_X)$ and fixing $D_{\text{Coh}}^b(\mathcal{O}_X)$,
4. $\omega_X^\bullet = \omega_Y^\bullet f^!$ for $f : X \to Y$ between compactifyable schemes over $S$,
5. $f^!M = D_X(Lf^*D_Y(M))$ canonically for $M \in D_{\text{Coh}}^+(\mathcal{O}_Y)$, and
6. if in addition $f$ is proper then $f^!$ is the twisted inverse image for $f$, there is a canonical isomorphism

$$Rf_* R\mathcal{H}om_{\mathcal{O}_X}(K, f^!M) \to R\mathcal{H}om_{\mathcal{O}_Y}(Rf_* K, M)$$
for all $K \in D_{QCoh}(\mathcal{O}_X)$ and $M \in D_{QCoh}^+(\mathcal{O}_Y)$, and most importantly

$$Rf_* R\mathcal{H}om_{\mathcal{O}_X}(K, \omega^*_X) = R\mathcal{H}om_{\mathcal{O}_Y}(Rf_* K, \omega^*_Y)$$

See Lemmas [24.3 25.5 19.4 23.2 25.6] and [20.10].

We have obtained our functors by a very abstract procedure which finally rests on invoking an existence theorem (Derived Categories, Proposition 35.2). This means we have no explicit description of the functors $f^!$. This can sometimes be a problem. However, as we will see, often it is enough to know the existence of a dualizing complex and the duality isomorphism to pin down what it is more exactly.

### 27. Glueing dualizing complexes

We will now use glueing of dualizing complexes to get a theory which works for all finite type schemes over $S$ given a pair $(S, \omega^*_S)$ as in Situation [26.1]. This is similar to [Har66, Remark on page 310].

Let $X$ be a scheme of finite type over $S$. Let $\mathcal{U} : X = \bigcup_{i=1}^n U_i$ be a finite open covering of $X$ by quasi-compact compactifyable schemes over $S$. Every affine scheme of finite type over $S$ is compactifyable over $S$ by Morphisms, Lemma [40.3] hence such open coverings certainly exist. For each $i, j, k \in \{1, \ldots, n\}$ the schemes $p_i : U_i \to S$, $p_{ij} : U_i \cap U_j \to S$, and $p_{ijk} : U_i \cap U_j \cap U_k \to S$ are compactifyable. From such an open covering we obtain

1. $\omega^*_i = p^!_i \omega^*_S$ as in Section [26].
2. For each $i, j$ a canonical isomorphism $\varphi_{ij} : \omega^*_i |_{U_i \cap U_j} \to \omega^*_j |_{U_i \cap U_j}$, and
3. For each $i, j, k$ we have

$$\varphi_{ik} |_{U_i \cap U_j \cap U_k} = \varphi_{jk} |_{U_j \cap U_k |_{U_i \cap U_j \cap U_k}} \circ \varphi_{ij} |_{U_i \cap U_j \cap U_k}$$

in $D(\mathcal{O}_{U_i \cap U_j \cap U_k})$.

Here, in (2) we use that $(U_i \cap U_j \to U_j)^!$ is given by restriction (Lemma [25.1]) and that we have canonical isomorphisms

$$(U_i \cap U_j \to U_j)^! \circ p^!_i = (U_i \cap U_j \to U_j)^! \circ p^!_{ij}$$

by Lemma [24.2] and to get (3) we use that the upper shriek functors form a pseudo functor by Lemma [24.3].

In the situation just described a dualizing complex normalized relative to $\omega^*_S$ and $\mathcal{U}$ is a pair $(K, \alpha_i)$ where $K \in D(\mathcal{O}_X)$ and $\alpha_i : K |_{U_i} \to \omega^*_i$ are isomorphisms such that $\varphi_{ij}$ is given by $\alpha_j |_{U_i \cap U_j} \circ \alpha_i^{-1} |_{U_i \cap U_j}$. Since being a dualizing complex on a scheme is a local property we see that dualizing complexes normalized relative to $\omega^*_S$ and $\mathcal{U}$ are indeed dualizing complexes.

**Lemma 27.1.** In Situation [26.1] let $X$ be a scheme of finite type over $S$ and let $\mathcal{U}$ be a finite open covering of $X$ by compactifyable schemes. If there exists a dualizing complex normalized relative to $\omega^*_S$ and $\mathcal{U}$, then it is unique up to unique isomorphism.

**Proof.** If $(K, \alpha_i)$ and $(K', \alpha'_i)$ are two, then we consider $L = R\mathcal{H}om(K, K')$. By Lemma [19.5] and its proof, this is an invertible object of $D(\mathcal{O}_X)$. Using $\alpha_i$ and $\alpha'_i$ we obtain an isomorphism

$$\alpha^*_i \otimes \alpha'^*_i : L |_{U_i} \longrightarrow R\mathcal{H}om(\omega^*_i, \omega^*_i) = \mathcal{O}_{U_i}[0]$$
This already implies that $L = H^0(L)[0]$ in $D(O_X)$. Moreover, $H^0(L)$ is an invertible sheaf with given trivializations on the opens $U_i$ of $X$. Finally, the condition that $\alpha_j|_{U_i \cap U_j} \circ \alpha_i^{-1}|_{U_i \cap U_j}$ and $\alpha_j'|_{U_i \cap U_j} \circ (\alpha_i')^{-1}|_{U_i \cap U_j}$ both give $\varphi_{ij}$ implies that the transition maps are $1$ and we get an isomorphism $H^0(L) = O_X$. □

**Lemma 27.2.** In Situation 26.1 let $X$ be a scheme of finite type over $S$ and let $U, V$ be two finite open coverings of $X$ by compactifyable schemes. Assume that $V$ If there exists a dualizing complex normalized relative to $\omega_S$ and $U$, then there exists a dualizing complex normalized relative to $\omega_S$ and $V$ and these complexes are canonically isomorphic.

**Proof.** It suffices to prove this when $U$ is given by the opens $U_1, \ldots, U_n$ and $V$ by the opens $U_1', \ldots, U_{n+1}$. In fact, we may and do even assume $m = 1$. To go from a dualizing complex $(K, \alpha_i)$ normalized relative to $\omega_S$ and $V$ to a dualizing complex normalized relative to $\omega_S$ and $U$ is achieved by forgetting about $\alpha_i$ for $i = n + 1$. Conversely, let $(K, \alpha_i)$ be a dualizing complex normalized relative to $\omega_S$ and $U$. To finish the proof we need to construct a map $\alpha_{n+1} : K|_{U_{n+1}} \to \omega_{n+1}^*$ satisfying the desired conditions. To do this we observe that $U_{n+1} = \bigcup U_i \cap U_{n+1}$ is an open covering. It is clear that $(K|_{U_{n+1}}, \alpha|_{U_i \cap U_{n+1}})$ is a dualizing complex normalized relative to $\omega_S$ and the covering $U_{n+1} = \bigcup U_i \cap U_{n+1}$. On the other hand, by condition (3) the pair $(\omega_{n+1}^*|_{U_{n+1}}, \varphi_{n+1})$ is another dualizing complex normalized relative to $\omega_S$ and the covering $U_{n+1} = \bigcup U_i \cap U_{n+1}$. By Lemma 27.1 we obtain a unique isomorphism

$$\alpha_{n+1} : K|_{U_{n+1}} \to \omega_{n+1}^*$$

compatible with the given local isomorphisms. It is a pleasant exercise to show that this means it satisfies the required property. □

**Lemma 27.3.** In Situation 26.1 let $X$ be a scheme of finite type over $S$ and let $U$ be two a finite open covering of $X$ by compactifyable schemes. Then there exists a dualizing complex normalized relative to $\omega_S$ and $U$.

**Proof.** Say $U : X = \bigcup_{i=1,\ldots,n} U_i$. We prove the lemma by induction on $n$. The base case $n = 1$ is immediate. Assume $n > 1$. Set $X' = U_1 \cup \ldots \cup U_{n-1}$ and let $(K', \alpha'_i)_{i=1,\ldots,n-1}$ be a dualizing complex normalized relative to $\omega_S$ and $U' : X' = \bigcup_{i=1,\ldots,n-1} U_i$. It is clear that $(K'|_{X \cap U_i}, \alpha'|_{U_i \cap U'})$ is a dualizing complex normalized relative to $\omega_S$ and the covering $X' \cap U_n = \bigcup_{i=1,\ldots,n-1} U_i \cap U_n$. On the other hand, by condition (3) the pair $(\omega_{n}^*|_{X \cap U_i}, \varphi_{ni})$ is another dualizing complex normalized relative to $\omega_S$ and the covering $X' \cap U_n = \bigcup_{i=1,\ldots,n-1} U_i \cap U_n$. By Lemma 27.1 we obtain a unique isomorphism

$$\epsilon : K'|_{X \cap U_n} \to \omega_{n}^*|_{X \cap U_n}$$

compatible with the given local isomorphisms. By Cohomology, Lemma 30.10 we obtain $K \in D(O_X)$ together with isomorphisms $\beta : K|_{X'} \to K'$ and $\gamma : K|_{U^n} \to \omega_{n}^*$ such that $\epsilon = \gamma|_{X \cap U_n} \circ \beta|_{X \cap U_n}^{-1}$. Then we define

$$\alpha_i = \alpha'_i \circ \beta|_{U_i}, i = 1, \ldots, n - 1, \text{ and } \alpha_n = \gamma$$

We still need to verify that $\varphi_{ij}$ is given by $\alpha_j|_{U_i \cap U_j} \circ \alpha_i^{-1}|_{U_i \cap U_j}$. For $i, j \leq n - 1$ this follows from the corresping condition for $\alpha'_i$. For $i = j = n$ it is clear as well. If $i < j = n$, then we get

$$\alpha_n|_{U_i \cap U_n} \circ \alpha_i^{-1}|_{U_i \cap U_n} = \gamma|_{U_i \cap U_n} \circ \beta^{-1}|_{U_i \cap U_n} \circ (\alpha'_i)^{-1}|_{U_i \cap U_n} = \epsilon|_{U_i \cap U_n} \circ (\alpha'_i)^{-1}|_{U_i \cap U_n}$$
This is equal to \( \alpha_{ni} \) exactly because \( \epsilon \) is the unique map compatible with the maps \( \alpha'_i \) and \( \alpha_{ni} \).

Let \((S, \omega^*_S)\) be as in Situation \ref{duality-criteria}. The upshot of the lemmas above is that given any scheme \( X \) of finite type over \( S \), there is a pair \((K, \alpha_U)\) given up to unique isomorphism, consisting of an object \( K \in D(O_X) \) and isomorphisms \( \alpha_U : K|_U \to \omega^*_U \) for every open subscheme \( U \subset X \) which has a compactification over \( S \) and where \( \omega^*_U \) is as in Section \ref{compactification-over-S} such that, if \( \mathcal{U} : X = \bigcup U_i \) is a finite open covering by opens which are compactifyable over \( S \), then \((K, \alpha_{U_i})\) is a dualizing complex normalized relative to \( \omega^*_S \) and \( \mathcal{U} \). Namely, uniqueness up to unique isomorphism by Lemma \ref{duality-criteria}, existence for one open covering by Lemma \ref{existence-for-one-open-cov}, and the fact that \( K \) then works for all open coverings is Lemma \ref{existence-for-any-open-cov}.

**Definition** \ref{duality-criteria}. Let \( S \) be a Noetherian scheme and let \( \omega^*_S \) be a dualizing complex on \( S \). Let \( X \) be a scheme of finite type over \( S \). The complex \( K \) constructed above is called the **dualizing complex normalized relative to** \( \omega^*_S \) and is denoted \( \omega^*_X \).

As the terminology suggest, a dualizing complex normalized relative to \( \omega^*_S \) is not just an object of the derived category of \( X \) but comes equipped with the local isomorphisms described above. This does not conflict with setting \( \omega^*_X = p^! \omega^*_S \) where \( p : X \to S \) is the structure morphism if \( X \) has a compactification over \( S \) (see Section \ref{compactification-over-S}). More generally we have the following sanity check.

**Lemma** \ref{existence-for-any-open-cov}. Let \((S, \omega^*_S)\) be as in Situation \ref{duality-criteria}. Let \( f : X \to Y \) be a morphism of finite type schemes over \( S \). Let \( \omega^*_X \) and \( \omega^*_Y \) be dualizing complexes normalized relative to \( \omega^*_S \). Then \( \omega^*_X \) is a dualizing complex normalized relative to \( \omega^*_Y \).

**Proof.** This is just a matter of bookkeeping. Choose a finite affine open covering \( V : Y = \bigcup V_j \). For each \( j \) choose a finite affine open covering \( f^{-1}(V_j) = U_{ji} \). Set \( \mathcal{U} : X = \bigcup U_{ji} \). The schemes \( V_j \) and \( U_{ji} \) are compactifyable over \( S \), hence we have the upper shriek functors for \( q_j : V_j \to S \), \( p_{ji} : U_{ji} \to S \) and \( f_{ji} : U_{ji} \to V_j \) and \( f'_{ji} : U_{ji} \to Y \). Let \((L, \beta_j)\) be a dualizing complex normalized relative to \( \omega^*_S \) and \( V \). Let \((K, \gamma_j)\) be a dualizing complex normalized relative to \( \omega^*_S \) and \( \mathcal{U} \). (In other words, \( L = \omega^*_Y \) and \( K = \omega^*_X \).) We can define

\[
\alpha_{ji} : K|_{U_{ji}} \xrightarrow{\gamma_{ji}} p'_{ji*}\omega^*_S = f'_{ji*}q_{ji*}\omega^*_S = f'_{ji*}\beta_{ji}^{-1}f_{ji*}(L|_{V_j}) = (f'_{ji})^{-1}(L)
\]

To finish the proof we have to show that \( \alpha_{ji}|_{U_{ji} \cap U_{j'i'}} \circ \alpha_{j'i'}^{-1}|_{U_{ji} \cap U_{j'i'}} \) is the canonical isomorphism \((f'_{j'i'})^{-1}(L)|_{U_{ji} \cap U_{j'i'}} \to (f'_{ji})^{-1}(L)|_{U_{ji} \cap U_{j'i'}} \). This is formal and we omit the details.

**Lemma** \ref{existence-for-any-immersion}. Let \((S, \omega^*_S)\) be as in Situation \ref{duality-criteria}. Let \( j : X \to Y \) be an open immersion of schemes of finite type over \( S \). Let \( \omega^*_X \) and \( \omega^*_Y \) be dualizing complexes normalized relative to \( \omega^*_S \). Then there is a canonical isomorphism \( \omega^*_X \to \omega^*_Y|_X \).

**Proof.** Immediate from the construction of normalized dualizing complexes given just above Definition \ref{duality-criteria}.

**Lemma** \ref{existence-for-any-proper-immersion}. Let \((S, \omega^*_S)\) be as in Situation \ref{duality-criteria}. Let \( f : X \to Y \) be a proper morphism of schemes of finite type over \( S \). Let \( \omega^*_X \) and \( \omega^*_Y \) be dualizing complexes normalized relative to \( \omega^*_S \). Let \( a \) be the twisted inverse image for \( f \). Then there is a canonical isomorphism \( a(\omega^*_Y) = \omega^*_X \).
Proof. Let \( p : X \to S \) and \( q : Y \to S \) be the structure morphisms. If \( X \) and \( Y \) are compactifyable over \( S \), then this follows from the fact that \( \omega_X^p = p^! \omega_S^\bullet \), \( \omega_Y^q = q^! \omega_S^\bullet \), \( f^! = a \), and \( f^! \circ q^! = p^! \) (Lemma 24.2). In the general case we first use Lemma 27.5 to reduce to the case \( Y = S \). In this case \( X \) and \( Y \) are compactifyable over \( S \) and we’ve just seen the result. \( \square \)

Let \((S, \omega_S^\bullet)\) be as in Situation 26.1. For a scheme \( X \) of finite type over \( S \) denote \( \omega_X^\bullet \) the dualizing complex for \( X \) normalized relative to \( \omega_S^\bullet \). Define \( D_X(\_)= R\mathcal{H}om_{\mathcal{O}_X}(\_,-\omega_X^\bullet) \) as in Lemma 19.4. Let \( f : X \to Y \) be a morphism of finite type schemes over \( S \). Define

\[
f^!_{\text{new}} = D_X \circ Lf^* \circ D_Y : D^+_{\text{Coh}}(\mathcal{O}_Y) \to D^+_{\text{Coh}}(\mathcal{O}_X)
\]

If \( f : X \to Y \) and \( g : Y \to Z \) are composed morphisms between schemes of finite type over \( S \), define

\[
(g \circ f)^!_{\text{new}} = D_X \circ L(g \circ f)^* \circ D_Z = D_X \circ Lf^* \circ Lg^* \circ D_Z = D_X \circ Lf^* \circ D_Y \circ Lg^* \circ D_Z = f^!_{\text{new}} \circ g^!_{\text{new}}
\]

where the arrow is defined in Lemma 19.4. We collect the results together in the following lemma.

Lemma 27.8. Let \((S, \omega_S^\bullet)\) be as in Situation 26.1. With \( f^!_{\text{new}} \) and \( \omega_X^\bullet \) defined for all (morphisms of) schemes of finite type over \( S \) as above:

1. the functors \( f^!_{\text{new}} \) and the arrows \( (g \circ f)^!_{\text{new}} \to f^!_{\text{new}} \circ g^!_{\text{new}} \) turn \( D^+_{\text{Coh}} \) into a pseudo functor from the category of schemes of finite type over \( S \) into the 2-category of categories,
2. \( \omega_X^\bullet = (X \to S)^!_{\text{new}} \omega_S^\bullet \),
3. the functor \( D_X \) defines an involution of \( D_{\text{Coh}}(\mathcal{O}_X) \) switching \( D^+_{\text{Coh}}(\mathcal{O}_X) \) and \( D^-_{\text{Coh}}(\mathcal{O}_X) \),
4. \( \omega_X^\bullet = f^!_{\text{new}} \omega_S^\bullet \) for \( f : X \to Y \) a morphism of finite type schemes over \( S \),
5. \( f^!_{\text{new}} M = D_X(Lf^* D_Y(M)) \) for \( M \in D^+_{\text{Coh}}(\mathcal{O}_Y) \), and
6. if in addition \( f \) is proper, then \( f^!_{\text{new}} \) is isomorphic to twisted inverse image for \( f \), there is a canonical isomorphism

\[
Rf_* R\mathcal{H}om_{\mathcal{O}_X}(K, f^!_{\text{new}} M) \to R\mathcal{H}om_{\mathcal{O}_Y}(Rf_*K, M)
\]

for all \( K \in D_{\text{QCoh}}(\mathcal{O}_X) \) and \( M \in D^+_{\text{Coh}}(\mathcal{O}_Y) \), and most importantly

\[
Rf_* R\mathcal{H}om_{\mathcal{O}_X}(K, \omega_X^\bullet) = R\mathcal{H}om_{\mathcal{O}_Y}(Rf_*K, \omega_Y^\bullet)
\]

Moreover, if \( X \) is compactifyable over \( S \) then \( \omega_X^\bullet \) is canonically isomorphic to the complex \( \omega_S^\bullet \) of Section 26 and if \( f \) is a morphism between compactifyable schemes over \( S \), then there is a canonical isomorphism \(^3 f^!_{\text{new}} K = f^! K \) for \( K \) in \( D^+_{\text{Coh}} \).

[^3]: We haven’t checked that these are compatible with the isomorphisms \((g \circ f)^! \to f^! \circ g^!\) and \((g \circ f)^!_{\text{new}} \to f^!_{\text{new}} \circ g^!_{\text{new}}\). We will do this here if we need this later.
**Proof.** Let \( f : X \to Y \), \( g : Y \to Z \), \( h : Z \to T \) be morphisms of schemes of finite type over \( S \). We have to show that

\[
\begin{array}{ccc}
(h \circ g \circ f)^!_{\text{new}} & \longrightarrow & f^!_{\text{new}} \circ (h \circ g)^!_{\text{new}} \\
\downarrow & & \downarrow \\
(g \circ f)^!_{\text{new}} \circ h^!_{\text{new}} & \longrightarrow & f^!_{\text{new}} \circ g^!_{\text{new}} \circ h^!_{\text{new}}
\end{array}
\]

is commutative. Let \( \eta_Y : \text{id} \to D^2_Y \) and \( \eta_Z : \text{id} \to D^2_Z \) be the canonical isomorphisms of Lemma \[19.4\] Then, using Categories, Lemma \[27.2\] a computation (omitted) shows that both arrows \((h \circ g \circ f)^!_{\text{new}} \to f^!_{\text{new}} \circ (h \circ g)^!_{\text{new}}\) are given by

\[
1 \ast \eta_Y \ast 1 \ast \eta_Z \ast 1 : D_X \circ Lf^* \circ Lg^* \circ Lh^* \circ DT \longrightarrow D_X \circ Lf^* \circ D^2_Y \circ Lg^* \circ D^2_Z \circ Lh^* \circ DT
\]

This proves (1). Part (2) is immediate from the definition of \((X \to S)^!_{\text{new}}\) and the fact that \( D_S(\omega^*_S) = \mathcal{O}_S \). Part (3) is Lemma \[19.4\] Part (4) follows by the same argument as part (2). Part (5) is the definition of \( f^!_{\text{new}} \).

Proof of (6). Let \( a \) be the twisted inverse image of the proper morphism \( f : X \to Y \) of schemes of finite type over \( S \). The issue is that we do not know \( X \) or \( Y \) is compactifiable over \( S \) (and in general this won’t be true) hence we cannot immediately apply Lemma \[25.6\] to \( f \) over \( S \). To get around this we use the canonical identification \( \omega^*_X = a(\omega^*_Y) \) of Lemma \[27.7\]. Hence \( f^!_{\text{new}} \) is the restriction of \( a \) to \( D^\text{Coh}_Y(\mathcal{O}_Y) \) by Lemma \[25.6\] applied to \( f : X \to Y \) over the base scheme \( Y \)! Thus the result is true by Lemma \[20.10\].

The final assertions follow from the construction of normalized dualizing complexes and the already used Lemma \[25.6\]. \( \square \)

28. **Trace maps**

Let \( f : X \to Y \) be a morphism of quasi-compact and quasi-separated schemes. Let \( a : D_{Q\text{Coh}}(\mathcal{O}_Y) \to D_{Q\text{Coh}}(\mathcal{O}_X) \) be the twisted inverse image of \( f \) as in Lemma \[20.1\]. Thus for \( K \in D_{Q\text{Coh}}(\mathcal{O}_Y) \) we obtain an adjunction map

\[
\text{Tr}_{f,K} : Rf_*a(K) \longrightarrow K
\]

functorial in \( K \) which is sometimes called the *trace map*. This is the map which has the property that the bijection

\[
\text{Hom}_X(L,a(K)) \longrightarrow \text{Hom}_Y(Rf_*L,K)
\]

for \( L \in D_{Q\text{Coh}}(\mathcal{O}_X) \) which defines the twisted inverse image is given by

\[
\varphi \longmapsto \text{Tr}_{f,K} \circ Rf_*\varphi
\]

If \( f \) is a proper morphism of Noetherian schemes, then Lemma \[20.10\] shows that the isomorphism

\[
R\text{Hom}_{\mathcal{O}_X}(L,a(K)) \longrightarrow R\text{Hom}_{\mathcal{O}_Y}(Rf_*L,K)
\]

comes about by composition with \( \text{Tr}_{f,K} \). Every trace map we are going to consider in this section will be a special case of this trace map.

For example, let \( A \to B \) be a ring map. Let \( Y = \text{Spec}(A) \) and \( X = \text{Spec}(B) \) and \( f : X \to Y \) the morphism corresponding to \( A \to B \). As seen in Example \[20.2\] the
twisted inverse image sends an object $K$ of $D(A) = D_{QCoh}(\mathcal{O}_Y)$ to $R\text{Hom}(B, K)$ in $D(B) = D_{QCoh}(\mathcal{O}_X)$. The trace map is the map

$$\text{Tr}_{f, K} : R\text{Hom}(B, K) \rightarrow R\text{Hom}(A, K) = K$$

induced by the $A$-module map $A \rightarrow B$.

If $i : Z \rightarrow X$ is a closed immersion of Noetherian schemes, then similarly to the paragraph above the diagram

$$
\begin{array}{ccc}
  i_*a(K) & \xrightarrow{\text{Tr}_{i, K}} & K \\
  | & | & | \\
  i_*R\text{Hom}(\mathcal{O}_Z, K) & \xrightarrow{R\text{Hom}(i_*\mathcal{O}_Z, K)} & K
\end{array}
$$

is commutative for $K \in D^+_{QCoh}(\mathcal{O}_X)$. Here the horizontal equality sign is Lemma 14.3 and the lower horizontal arrow is induced by the map $\mathcal{O}_X \rightarrow i_*\mathcal{O}_Z$. The commutativity of the diagram is a consequence of Lemma 20.12.

Let $S$ be a Noetherian scheme and let $\omega^\bullet_S$ be a dualizing complex. Let $f : X \rightarrow Y$ be a proper morphism of finite type schemes over $S$. Let $\omega^\bullet_X$ and $\omega^\bullet_Y$ be dualizing complexes normalized relative to $\omega^\bullet_S$. In this situation we have $a(\omega^\bullet_Y) = \omega^\bullet_X$ (Lemma 27.7) and hence the trace map is a canonical arrow

$$\text{Tr}_f : Rf_*\omega^\bullet_X \rightarrow \omega^\bullet_Y$$

which produces the isomorphisms (Lemma 27.8)

$$\text{Hom}_X(L, \omega^\bullet_X) = \text{Hom}_Y(Rf_*L, \omega^\bullet_Y)$$

and

$$Rf_*R\text{Hom}_{\mathcal{O}_X}(L, \omega^\bullet_X) = R\text{Hom}_{\mathcal{O}_Y}(Rf_*L, \omega^\bullet_Y)$$

for $L$ in $D_{QCoh}(\mathcal{O}_X)$.

29. Dualizing modules

If $(A, m, \kappa)$ is a Noetherian local ring and $\omega^\bullet_A$ is a normalized dualizing complex, then we say the module $\omega_A = H^{-\dim(A)}(\omega^\bullet_A)$, described in Lemma 17.5, is a dualizing module for $A$. This module is a canonical module of $A$. It seems generally agreed upon to define a canonical module for a Noetherian local ring $(A, m, \kappa)$ to be a finite $A$-module $K$ such that

$$\text{Hom}_A(K, E) \cong H^m_{\dim(A)}(A)$$

where $E$ is an injective hull of the residue field. A dualizing module is canonical because

$$\text{Hom}_A(H^m_{\dim(A)}(A), E) = (\omega_A)^\wedge$$

by Lemma 18.4 and hence applying $\text{Hom}_A(-, E)$ we get

$$\text{Hom}_A(\omega_A, E) = \text{Hom}_A((\omega_A)^\wedge, E)$$

$$= \text{Hom}_A(\text{Hom}_A(H^m_{\dim(A)}(A), E), E)$$

$$= H^m_{\dim(A)}(A)$$

the first equality because $E$ is $m$-power torsion, the second by the above, and the third by Matlis duality (Proposition 7.8). The utility of the definition of a canonical module given above lies in the fact that it makes sense even if $A$ does not have a dualizing complex.
Let $X$ be a Noetherian scheme and let $\omega^\bullet_X$ be a dualizing complex. Let $n \in \mathbb{Z}$ be the smallest integer such that $H^n(\omega^\bullet_X)$ is nonzero. In other words, $-n$ is the maximal value of the dimension function associated to $\omega^\bullet_X$ (Lemma 19.6). Sometimes $H^n(\omega^\bullet_X)$ is called a dualizing module or dualizing sheaf for $X$ and then it is often denoted by $\omega_X$. We will say “let $\omega_X$ be a dualizing module” to indicate the above.

Care has to be taken when using dualizing modules $\omega_X$ on Noetherian schemes $X$:

1. the integer $n$ may change when passing from $X$ to an open $U$ of $X$ and then it won’t be true that $\omega_X|_U = \omega_U$,
2. the dualizing complex isn’t unique; the dualizing module is only unique up to tensoring by an invertible module.

The second problem will often be irrelevant because we will work with $X$ of finite type over a base change $S$ which is endowed with a fixed dualizing complex $\omega^\bullet_S$ and $\omega^\bullet_X$ will be the dualizing complex normalized relative to $\omega^\bullet_S$. The first problem will not occur if $X$ is equidimensional, more precisely, if the dimension function associated to $\omega^\bullet_X$ (Lemma 19.6) maps every generic point of $X$ to the same integer.

**Example 29.1.** Say $S = \text{Spec}(A)$ with $(A, m, \kappa)$ a local Noetherian ring, and $\omega^\bullet_S$ corresponds to a normalized dualizing complex $\omega^\bullet_A$. Then if $f : X \to S$ is proper over $S$ and $\omega^\bullet_X = f^! \omega^\bullet_S$ the coherent sheaf

$$\omega_X = H^{-\dim(X)}(\omega^\bullet_X)$$

is a dualizing module and is often called the dualizing module of $X$ (with $S$ and $\omega^\bullet_S$ being understood). We will see that this has good properties.

**Example 29.2.** Say $X$ is an equidimensional scheme of finite type over a field $k$. Then it is customary to take $\omega^\bullet_X$ the dualizing complex normalized relative to $k[0]$ and to refer to

$$\omega_X = H^{-\dim(X)}(\omega^\bullet_X)$$

as the dualizing module of $X$.

**Lemma 29.3.** Let $X$ be a connected Noetherian scheme and let $\omega_X$ be a dualizing module on $X$. The support of $\omega_X$ is the union of the irreducible components of maximal dimension with respect to any dimension function and $\omega_X$ is a coherent $\mathcal{O}_X$-module having property $(S_2)$.

**Proof.** By our conventions discussed above there exists a dualizing complex $\omega^\bullet_X$ such that $\omega_X$ is the leftmost nonvanishing cohomology sheaf. Since $X$ is connected, any two dimension functions differ by a constant (Topology, Lemma 19.6). Hence we may use the dimension function associated to $\omega^\bullet_X$ (Lemma 19.6). With these remarks in place, the lemma now follows from Lemma 17.5 and the definitions (in particular Cohomology of Schemes, Definition 11.1).

To say a bit more about dualizing modules we need a bit more information about how the dimension functions change when passing to a scheme of finite type over another.

**Lemma 29.4.** Let $(A, m, \kappa)$ be a Noetherian local ring. Let $\omega^\bullet_A$ be a normalized dualizing complex. Let $X$ be a scheme of finite type over $A$ and let $\omega^\bullet_X$ be the dualizing complex normalized relative to $\omega^\bullet_A$. If $x \in X$ is a closed point lying over the closed point $s$ of $S = \text{Spec}(A)$, then $\omega^\bullet_{X,x}$ is a normalized dualizing complex over $\mathcal{O}_{X,x}$.
Proof. We may replace $X$ by an affine neighbourhood of $x$, hence we may and do assume that $f : X \to S = \text{Spec}(A)$ is compactifiable. Then $\omega_X^* = f^* \omega_S^*$. We have to show that $R\text{Hom}_{O_X} (\kappa(x), \omega_{X,x}^*)$ is sitting in degree 0. Let $i_x : x \to X$ denote the inclusion morphism which is a closed immersion as $x$ is a closed point. Hence $R\text{Hom}_{O_X} (\kappa(x), \omega_{X,x}^*)$ represents $i_x^! \omega_X^*$ by Lemma 25.3. Since $x$ lives over the closed point we see that $A \to \kappa(x)$ factors through $\kappa$ and since $x$ is a closed point of $X$ we see that $\kappa \subset \kappa(x)$ is a finite extension (Morphisms, Lemma 21.3). Thus we get a commutative diagram

$$
\begin{array}{ccc}
x & \longrightarrow & X \\
\pi \downarrow & & \downarrow f \\
S & \longrightarrow & S \\
\end{array}
$$

with $\pi$ finite. We conclude that $i_x^! \omega_X^* = i_x^! f^! \omega_S^* = \pi^! i_x^! \omega_S^*$

Since $\omega_X^*$ is normalized and $s$ is the closed point we see that $i_x^! \omega_S^* = \kappa[0]$. We have

$$R\pi_* (\pi^! \kappa[0]) = R\text{Hom} (R\pi_*(\kappa(x)[0]), \kappa[0]) = R\text{Hom}_x (\kappa(x), \kappa)$$

The first equality by Lemma 20.10 applied with $L = \kappa(x)[0]$. The second equality holds because $\pi_*$ is exact. Thus $\pi^! \kappa[0]$ is supported in degree 0 and we win. □

Lemma 29.5. Let $S$ be a Noetherian scheme and let $\omega_S^*$ be a dualizing complex. Let $f : X \to S$ be of finite type and let $\omega_X^*$ be the dualizing complex normalized relative to $\omega_S^*$. For all $x \in X$

$$\delta_X(x) - \delta_S(f(x)) = \text{trdeg}_{\kappa(f(x))} (\kappa(x))$$

where $\delta_S$, resp. $\delta_X$ is the dimension function of $\omega_S^*$, resp. $\omega_X^*$, see Lemma 19.0

Proof. We may replace $X$ by an affine neighbourhood of $x$. Hence we may and do assume there is a compactification $X \subset \ol{X}$ over $S$. Then we may replace $X$ by $\ol{X}$ and assume that $X$ is proper over $S$. We may also assume $X$ is connected by replacing $X$ by the connected component of $X$ containing $x$. Next, recall that both $\delta_X$ and the function $x \mapsto \delta_S(f(x)) + \text{trdeg}_{\kappa(f(x))} (\kappa(x))$ are dimension functions on $X$, see Morphisms, Lemma 31.2. By Topology, Lemma 19.3 we see that the difference is locally constant, hence constant as $X$ is connected. Thus it suffices to prove equality in any point of $X$. By Properties, Lemma 5.8 the scheme $X$ has a closed point $x$. Since $X \to S$ is proper the image $s$ of $x$ is closed in $S$. Thus we may apply Lemma 29.4 to conclude.

Lemma 29.6. Let $X/A$ with $\omega_X^*$ and $\omega_X^*$ be as in Example 29.7. Then

1. $H^i(\omega_X^*) \neq 0 \Rightarrow i \in \{- \dim(X), \ldots, 0\}$,
2. the dimension of the support of $H^i(\omega_X^*)$ is at most $-i$,
3. $\text{Supp}(\omega_X^*)$ is the union of the components of dimension $\dim(X)$, and
4. $\omega_X^*$ has property (S2).

Proof. Let $\delta_X$ and $\delta_S$ be the dimension functions associated to $\omega_X^*$ and $\omega_S^*$ as in Lemma 29.5. As $X$ is proper over $A$, every closed subscheme of $X$ contains a closed point $x$ which maps to the closed point $s \in S$ and $\delta_X(x) = \delta_S(s) = 0$. Hence $\delta_X(\xi) = \dim(\xi)$ for any point $\xi \in X$. Hence we can check each of the statements of the lemma by looking at what happens over $\text{Spec}(O_{X,x})$ in which case the result
follows from Lemmas 16.4 and 17.5. Some details omitted. The last two statements can also be deduced from Lemma 29.3.

**Lemma 29.7.** Let $X/A$ with dualizing module $\omega_X$ be as in Example 29.1. Let $d = \dim(X_s)$ be the dimension of the closed fibre. If $\dim(X) = d + \dim(A)$, then the dualizing module $\omega_X$ represents the functor

$$F \mapsto \text{Hom}_A(H^d(X,F),\omega_A)$$
onumber

on the category of coherent $O_X$-modules.

**Proof.** We have

$$\text{Hom}_X(F,\omega_X) = \text{Ext}^{-\dim(X)}_X(F,\omega_X^\bullet)$$

$$= \text{Hom}_X(F[\dim(X)],\omega_X)$$

$$= \text{Hom}_X(F[\dim(X)], f^!(\omega_A))$$

$$= \text{Hom}_S(Rf_*F[\dim(X)],\omega_A)$$

$$= \text{Hom}_A(H^d(X,F),\omega_A)$$

The first equality because $H^i(\omega_X^\bullet) = 0$ for $i < -\dim(X)$, see Lemma 29.6 and Derived Categories, Lemma 27.3. The second equality is follows from the definition of Ext groups. The third equality is our choice of $\omega_X^\bullet$. The fourth equality holds because $f^!$ is the twisted inverse image for $f$, see Section 26. The final equality holds because $R^if_*F$ is zero for $i > d$ (Cohomology of Schemes, Lemma 18.9) and $H^j(\omega_A)$ is zero for $j < -\dim(A)$.

Duality takes a particularly simple form for Cohen-Macaulay schemes.

**Lemma 30.1.** Let $(A,m,\kappa)$ be a Noetherian local ring with normalized dualizing complex $\omega_A^\bullet$. Then depth$(A)$ is equal to the smallest integer $\delta \geq 0$ such that $H^{-\delta}(\omega_A^\bullet) \neq 0$.

**Proof.** This follows immediately from Lemma 16.4. Here are two other ways to see that it is true.

First alternative. By Nakayama’s lemma we see that $\delta$ is the smallest integer such that $\text{Hom}_A(H^{-\delta}(\omega_A^\bullet),\kappa) \neq 0$. In other words, it is the smallest integer such that $\text{Ext}^{-\delta}_A(\omega_A^\bullet,\kappa)$ is nonzero. Using Lemma 15.2 and the fact that $\omega_A^\bullet$ is normalized this is equal to the smallest integer such that $\text{Ext}^{\delta}_A(\kappa,A)$ is nonzero. This is equal to the depth of $A$ by Algebra, Lemma 70.5.

Second alternative. By the local duality theorem (in the form of Lemma 18.4) $\delta$ is the smallest integer such that $H^\delta_m(A)$ is nonzero. This is equal to the depth of $A$ by Lemma 9.1.

**Lemma 30.2.** Let $(A,m,\kappa)$ be a Noetherian local ring with normalized dualizing complex $\omega_A^\bullet$ and dualizing module $\omega_A = H^{-\dim(A)}(\omega_A^\bullet)$. The following are equivalent

1. $A$ is Cohen-Macaulay,
2. $\omega_A^\bullet$ is concentrated in a single degree, and
3. $\omega_A^\bullet = \omega_A[\dim(A)]$.

In this case $\omega_A$ is a maximal Cohen-Macaulay module.
Proof. Follows immediately from Lemma 16.5 □

Lemma 30.3. Let $X$ be a connected Cohen-Macaulay scheme. If $\omega^*_X$ is a dualizing complex on $X$, then there is an integer $n$ and a coherent Cohen-Macaulay $O_X$-module $\omega_X$ such that $\omega^*_X = \omega_X[-n]$.

Proof. By definition and Lemma 15.6 for every $x \in X$ the complex $\omega^*_{X,x}$ is a dualizing complex over $O_{X,x}$. Let $n_x$ be the unique integer such that $H^{n_x}(\omega^*_{X,x})$ is nonzero, see Lemma 30.2. For an affine neighbourhood $U \subset X$ of $x$ we have $\omega^*_{X,x}|_U$ is in $D^b_{\text{Coh}}(O_U)$ hence there are finitely many nonzero coherent modules $H^i(\omega^*_{X,x})|_U$. Thus after shrinking $U$ we may assume only $H^{n_x}$ is nonzero, see Modules, Lemma 9.5. In this way we see that the map $x \mapsto n_x$ is locally constant. Since $X$ is connected it is constant, say equal to $n$. Setting $\omega_X = H^n(\omega^*_X)$ we see that the lemma holds because $\omega_X$ is Cohen-Macaulay by Lemma 30.2 (and Cohomology of Schemes, Definition 11.2). □


(1) Let $A$ be a Noetherian ring. If there exists a finite $A$-module $\omega_A$ such that $\omega_A[0]$ is a dualizing complex, then $A$ is Cohen-Macaulay.

(2) Let $X$ be a locally Noetherian scheme. If there exists a coherent sheaf $\omega_X$ such that $\omega_X[0]$ is a dualizing complex on $X$, then $X$ is a Cohen-Macaulay scheme.

Proof. Part (2) follows from part (1) and our definitions. To see (1) we may replace $A$ by the localization at a prime (use Lemma 15.6 and Algebra, Definition 101.6). In this case the result follows immediately from Lemma 30.2 □

31. Gorenstein schemes

So far, the only explicit dualizing complex we seen is $\kappa$ on $\kappa$ for a field $\kappa$, see proof of Lemma 15.11. By Proposition 15.10 this means that any finite type algebra over a field has a dualizing complex. However, it turns out that there are Noetherian (local) rings which do not have a dualizing complex. Namely, we have seen that a ring which has a dualizing complex is universally catenary (Lemma 17.4) but there are examples of Noetherian local rings which are not catenary, see Examples, Section 16.

Nonetheless many rings in algebraic geometry have dualizing complexes simply because they are quotients of Gorenstein rings. This condition is in fact both necessary and sufficient. That is: a Noetherian ring has dualizing complexes if and only if it is a quotient of a finite dimensional Gorenstein ring. This is Sharp’s conjecture ([Sha79]) which can be found as [Kaw02, Corollary 1.4] in the literature. Returning to our current topic, here is the definition of Gorenstein rings.

Definition 31.1. Gorenstein rings and schemes.

(1) Let $A$ be a Noetherian local ring. We say $A$ is Gorenstein if $A[0]$ is a dualizing complex for $A$.

(2) Let $A$ be a Noetherian ring. We say $A$ is Gorenstein if $A_p$ is Gorenstein for every prime $p$ of $A$.

(3) Let $X$ be a locally Noetherian scheme. We say $X$ is Gorenstein if $O_{X,x}$ is Gorenstein for all $x \in X$. 
This definition makes sense, because if $A[0]$ is a dualizing complex for $A$, then $S^{-1}A[0]$ is a dualizing complex for $S^{-1}A$ by Lemma \[15.6\]. We will see later that a finite dimensional Noetherian ring is Gorenstein if it has finite injective dimension as a module over itself. An example of a Gorenstein ring is a regular ring.

**Lemma 31.2.** A regular local ring is Gorenstein. A regular ring is Gorenstein.

**Proof.** Let $A$ be a regular ring of finite dimension $d$. Then $A$ has finite global dimension $d$, see Algebra, Lemma \[107.8\]. Hence $\text{Ext}_A^{d+1}(M, A) = 0$ for all $A$-modules $M$, see Algebra, Lemma \[106.4\]. Thus $A$ has finite injective dimension as an $A$-module by More on Algebra, Lemma \[56.2\]. It follows that $A[0]$ is a dualizing complex, hence $A$ is Gorenstein by the remark following the definition. □

**Remark 31.3.** Let $(A, m, \kappa)$ be a Noetherian local ring. If $A$ has a dualizing complex, then the formal fibres of $A$ are Gorenstein. To prove this, let $p$ be a prime of $A$. The formal fibre of $A$ at $p$ is isomorphic to the formal fibre of $A/p$ at $(0)$ and $A/p$ has a dualizing complex (Lemma \[15.8\]). Thus it suffices to check the statement when $A$ is a local domain and $p = (0)$. Let $\omega_A^\bullet$ be a dualizing complex for $A$. Then $\omega_A^\bullet \otimes_A A^\wedge$ is a dualizing complex for the completion $A^\wedge$ (Lemma \[16.9\]). Then $\omega_A^\bullet \otimes_A f.f.(A)$ is a dualizing complex for $K = f.f.(A)$ (Lemma \[15.6\]) hence is isomorphic to $K[n]$ for some $n \in \mathbb{Z}$. Similarly, we conclude a dualizing complex for the formal fibre $A^\wedge \otimes_A K$ is

$$\omega_A^\bullet \otimes_A A^\wedge \otimes_A K = (\omega_A^\bullet \otimes_A K) \otimes_K (A^\wedge \otimes_A K) \cong (A^\wedge \otimes_A K)[n]$$

as desired.

### 32. Duality for a finite morphism

In this section work out what some of the results above mean for finite morphisms of schemes.

**Lemma 32.1.** Let $A \to B$ be a finite ring map of Noetherian rings. Let $\omega_A^\bullet$ be a dualizing complex. Then $R\text{Hom}(B, \omega_A^\bullet)$ is a dualizing complex for $B$.

**Proof.** The proof is identical to the proof of Lemma \[15.8\]. □

**Lemma 32.2.** Let $(A, m, \kappa) \to (B, m', \kappa')$ be a finite local map of Noetherian local rings. Let $\omega_A^\bullet$ be a normalized dualizing complex. Then $\omega_B^\bullet = R\text{Hom}_A(B, \omega_A^\bullet)$ is a normalized dualizing complex for $B$.

**Proof.** By Lemma \[32.1\] the complex $\omega_B^\bullet$ is dualizing for $B$. We compute

$$R\text{Hom}_B(\kappa', R\text{Hom}_A(B, \omega_A^\bullet)) = R\text{Hom}_A(\kappa', \omega_A^\bullet) \cong \text{Hom}_A(\kappa', \kappa)[0]$$

which is isomorphic in $D(\kappa')$ to $\kappa'$ placed in degree 0 as desired. The first equality holds by Lemma \[13.1\]. □

Let $f : X \to Y$ be a finite morphism of schemes. Let us denote

$$R \mathcal{H}om(f_* \mathcal{O}_X, -) : D(\mathcal{O}_Y) \to D(f_* \mathcal{O}_X)$$

the functor right adjoint to the restriction functor. It is the right derived functor of the left exact functor $\text{Mod}(\mathcal{O}_Y) \to \text{Mod}(f_* \mathcal{O}_X)$ given by $\mathcal{G} \mapsto \mathcal{H}om_{\mathcal{O}_Y}(f_* \mathcal{O}_X, \mathcal{G})$. See Derived Categories, Lemma \[28.4\].
Lemma 32.3. Let $f : X \to Y$ be a finite pseudo-coherent morphism of schemes (any finite morphism of Noetherian schemes is pseudo-coherent). The functor $R\mathcal{H}om(f_*\mathcal{O}_X,-)$ maps $D^+_{QCoh}(\mathcal{O}_Y)$ into $D^+_{QCoh}(f_*\mathcal{O}_X)$ and the diagram

$$
\begin{array}{ccc}
D^+_{QCoh}(\mathcal{O}_Y) & \xrightarrow{a} & D^+_{QCoh}(\mathcal{O}_X) \\
\downarrow_{R\mathcal{H}om(f_*\mathcal{O}_X,-)} & & \downarrow_{\Phi} \\
D^+_{QCoh}(f_*\mathcal{O}_X) & & 
\end{array}
$$

is commutative, where $a$ is the twisted inverse image of $f$ and $\Phi$ is the equivalence of Derived Categories of Schemes, Lemma 33.3.

Proof. (The parenthetical remark follows from More on Morphisms, Lemma 42.9.)
Since $f$ is pseudo-coherent, the $\mathcal{O}_Y$-module $f_*\mathcal{O}_X$ is pseudo-coherent, see More on Morphisms, Lemma 42.8. Thus $R\mathcal{H}om(f_*\mathcal{O}_X,-)$ maps $D^+_{QCoh}(\mathcal{O}_Y)$ into $D^+_{QCoh}(f_*\mathcal{O}_X)$, see Derived Categories of Schemes, Lemma 10.8. Then $\Phi \circ a$ and $R\mathcal{H}om(f_*\mathcal{O}_X,-)$ agree on $D^+_{QCoh}(\mathcal{O}_Y)$ because these functors are both right adjoint to the restriction functor $D^+_{QCoh}(f_*\mathcal{O}_X) \to D^+_{QCoh}(\mathcal{O}_Y)$.

Remark 32.4. If $f : X \to Y$ is a finite of Noetherian schemes, then the diagram

$$
\begin{array}{ccc}
Rf_*a(K) & \xrightarrow{Tr_{f,K}} & K \\
\downarrow & & \downarrow \\
R\mathcal{H}om(f_*\mathcal{O}_X,K) & \xrightarrow{\Phi} & K
\end{array}
$$

is commutative for $K \in D^+_{QCoh}(\mathcal{O}_Y)$. This follows from Lemma 32.3. The lower horizontal arrow is induced by the map $\mathcal{O}_Y \to f_*\mathcal{O}_X$ and the upper horizontal arrow is the trace map discussed in Section 28.

Remark 32.5. Let $S$ be a Noetherian scheme and let $\omega^*_S$ be a dualizing complex. Let $f : X \to Y$ be a finite morphism between schemes of finite type over $S$. Let $\omega^*_X$ and $\omega^*_Y$ be dualizing complexes normalized relative to $\omega^*_S$. Then we have

$$f_*\omega^*_X = R\mathcal{H}om(f_*\mathcal{O}_X,\omega^*_Y)
$$

in $D^+_{QCoh}(f_*\mathcal{O}_X)$ by Lemmas 32.3 and 27.7 and the trace map of Section 28 is the map

$$Tr_f : Rf_*\omega^*_X = f_*\omega^*_X = R\mathcal{H}om(f_*\mathcal{O}_X,\omega^*_Y) \to \omega^*_Y
$$

which often goes under the name “evaluation at 1”.

33. Effective Cartier divisors

Let $X$ be a scheme and let $i : D \to X$ be the inclusion of an effective Cartier divisor. Denote $\mathcal{N} = i^*\mathcal{O}_X(D)$ the normal sheaf of $i$, see Morphisms, Section 33 and Divisors, Section 11. Recall that $R\mathcal{H}om(\mathcal{O}_D,-)$ denotes the right adjoint to $i_* : D(\mathcal{O}_D) \to D(\mathcal{O}_X)$ and has the property $i_*R\mathcal{H}om(\mathcal{O}_D,-) = R\mathcal{H}om(i_*\mathcal{O}_D,-)$, see Section 11.

Lemma 33.1. As above, let $X$ be a scheme and let $D \subset X$ be an effective Cartier divisor. There is a canonical isomorphism $R\mathcal{H}om(\mathcal{O}_D,\mathcal{O}_X) = \mathcal{N}[-1]$ in $D(\mathcal{O}_D)$. 

\[ \mathcal{N} = i^*\mathcal{O}_X(D) \]
For every object $K$ and from the fact that $D$ is the zero scheme of the canonical section of $O_X(D)$ and from the fact that $\mathcal{N} = i^*O_X(D)$. □

For every object $K$ of $D(O_X)$ there is a canonical map

\begin{equation}
Li^*K \otimes^L_D R\text{Hom}(O_D, O_X) \to R\text{Hom}(O_D, K)
\end{equation}

functorial in $K$ and compatible with distinguished triangles. Namely, this map is adjoint to a map

\[ i_*(Li^*K \otimes^L_D R\text{Hom}(O_D, O_X)) = K \otimes^L_O R\text{Hom}(i_*O_D, O_X) \to K \]

where the equality is Cohomology, Lemma 40.4 and the arrow comes from the canonical map $R\text{Hom}(i_*O_D, O_X) \to O_X$ induced by $O_X \to i_*O_D$.

**Lemma 33.2.** As above, let $X$ be a scheme and let $D \subset X$ be an effective Cartier divisor. Then (33.1.1) combined with Lemma 33.1 defines an isomorphism

\[ Li^*K \otimes^L_D \mathcal{N}[-1] \to R\text{Hom}(O_D, K) \]

functorial in $K$ in $D(O_X)$.

**Proof.** Since $i_*$ is exact and fully faithful on modules, to prove the map is an isomorphism, it suffices to show that it is an isomorphism after applying $i_*$. We will use the short exact sequences $0 \to I \to O_X \to i_*O_D \to 0$ and $0 \to O_X \to O_X(D) \to i_*\mathcal{N} \to 0$ used in the proof of Lemma 33.1 without further mention. By Cohomology, Lemma 40.4 which was used to define the map (33.1.1) the left hand side becomes

\[ K \otimes^L_O i_*\mathcal{N}[-1] = K \otimes^L_O (O_X \to O_X(D)) \]

The right hand side becomes

\[ R\text{Hom}_{O_X}(i_*O_D, K) = R\text{Hom}_{O_X}((I \to O_X), K) = R\text{Hom}_{O_X}((I \to O_X), O_X) \otimes^L_O K \]

the final equality by Cohomology, Lemma 38.10. Since the map comes from the isomorphism

\[ R\text{Hom}_{O_X}((I \to O_X), O_X) = (O_X \to O_X(D)) \]

the lemma is clear. □
34. Riemann-Roch and duality

Let $k$ be a field. Let $X$ be a proper scheme of dimension $\leq 1$ over $k$. In Varieties, Section 28 we have defined the degree of a locally free $\mathcal{O}_X$-module $\mathcal{E}$ of constant rank by the formula

$$\text{deg}(\mathcal{E}) = \chi(X, \mathcal{E}) - \text{rank}(\mathcal{E}) \chi(X, \mathcal{O}_X)$$

see Varieties, Definition 28.1. In the chapter on Chow Homology we defined the first chern class of $\mathcal{E}$ as an operation on cycles (Chow Homology, Section 34) and we proved that

$$\text{deg}(\mathcal{E}) = \text{deg}(c_1(\mathcal{E}) \cap [X])$$

see Chow Homology, Lemma 40.3. Combining the two we obtain a Riemann-Roch formula of the form

$$\chi(X, \mathcal{E}) = \text{deg}(c_1(\mathcal{E}) \cap [X]) + \text{rank}(\mathcal{E}) \chi(X, \mathcal{O}_X)$$

To obtain a true Riemann-Roch theorem we would like to write $\chi(X, \mathcal{O}_X)$ as the degree of a canonical zero cycle on $X$. We refer to [Ful98] for a fully general version of this. We will use duality to get a formula in the case where $X$ is Gorenstein; however, in some sense this is a cheat (for example because this method cannot work in higher dimension).

First, let us assume that $X$ is Cohen-Macaulay and equidimensional of dimension 1. Let $\omega_X$ be the dualizing module of $X$, normalized as in Example 29.2. For a coherent sheaf $\mathcal{F}$ on $X$ we have

$$\text{Ext}^i_X(\mathcal{F}, \omega_X[1]) = \text{Hom}_k(H^{-i}(X, \mathcal{F}), k)$$

functorially in $\mathcal{F}$. This holds essentially by definition of the twisted inverse image for the morphism $X \to \text{Spec}(k)$ and our choice of $\omega_X$; the real miracle of duality comes from the fact that a twisted inverse image functor exists and that it transforms dualizing complexes into dualizing complexes. Concretely, this gives

$$\text{Hom}_X(\mathcal{F}, \omega_X) = \text{Hom}_k(H^1(X, \mathcal{F}), k)$$

for $i = -1$ and for $i = 0$ we obtain

$$\text{Ext}^{1}_X(\mathcal{F}, \omega_X) = \text{Hom}_k(H^0(X, \mathcal{F}), k)$$

For all other values of $i$ we obtain $0 = 0$. If $\mathcal{F}$ is locally free, then we have

$$\text{Ext}^{1}_X(\mathcal{F}, \omega_X) = H^1(X, \text{Hom}(\mathcal{F}, \mathcal{O}_X) \otimes_{\mathcal{O}_X} \omega_X) = H^1(X, \text{Hom}(\mathcal{F}, \omega_X))$$

In particular, if we apply this when $\mathcal{F}$ is an invertible $\mathcal{O}_X$-module $\mathcal{L}$, then we obtain canonical isomorphisms

$$H^0(X, \mathcal{L}^{\otimes -1} \otimes \omega_X) = \text{Hom}_k(H^1(X, \mathcal{L}), k)$$

and

$$H^1(X, \mathcal{L}^{\otimes -1} \otimes \omega_X) = \text{Hom}_k(H^0(X, \mathcal{L}), k)$$

Finally setting $\mathcal{L} = \mathcal{O}_X$ we arrive at the realization that

$$\chi(X, \mathcal{O}_X) = \dim H^0(X, \mathcal{O}_X) - \dim H^1(X, \mathcal{O}_X)$$

$$= \dim H^1(X, \omega_X) - \dim H^0(X, \omega_X)$$

$$= -\chi(X, \omega_X)$$

To use this to get a formula for $\chi(X, \mathcal{O}_X)$ we are going to assume that $\omega_X$ is invertible, i.e., that $X$ is Gorenstein.
Thus, let us now assume that $X$ is Gorenstein and equidimensional of dimension 1. In this case $\omega_X$ (normalized as above) is an invertible $O_X$-module and we can put it into our Riemann-Roch formula above to get

$$\chi(X, \omega_X) = \deg(c_1(\omega_X) \cap [X]_1) + \chi(X, O_X)$$

Combined with the above this gives

$$2\chi(X, O_X) = -\deg(c_1(\omega_X) \cap [X]_1) = -\deg(\omega_X)$$

Thus $-1/2$ of the first chern class of $\omega_X$ capped with the cycle $[X]_1$ associated to $X$ is a natural zero cycle on $X$ with half-integer coefficients whose degree is equal to the Euler characteristic of the structure sheaf of $X$. All in all the Riemann-Roch theorem for a Gorenstein curve over $k$ says that

$$\chi(X, E) = \deg(E) - \frac{1}{2}\text{rank}(E)\deg(\omega_X)$$

for a locally free $O_X$-module $E$ of constant rank keeping in mind that $\deg(E)$ and $\deg(\omega_X)$ can be computed using intersection theoretic methods, and keeping in mind that

$$\dim H^i(X, E) = \dim H^{1-i}(X, E \otimes \omega_X)$$

by duality.

### 35. Some vanishing results

In this section we work in the following situation.

**Situation 35.1.** Here $k$ is a field and $X$ is a proper scheme over $k$ which is Cohen-Macaulay, equidimensional of dimension 1, and has $H^0(X, O_X) = k$. Let $\omega_X$ be the dualizing sheaf of $X$ as in Example 29.2.

From the discussion in Section 34 we see that the dualizing sheaf $\omega_X$ on $X$ has nonvanishing $H^1$. It turns out that anything slightly more “positive” than $\omega_X$ has vanishing $H^1$.

**Lemma 35.2.** In Situation 35.1. Given an exact sequence

$$\omega_X \to F \to Q \to 0$$

of coherent $O_X$-modules with $H^1(X, Q) = 0$ (for example if $\dim(\text{Supp}(Q)) = 0$), then either $H^1(X, F) = 0$ or $F = \omega_X \oplus Q$.

**Proof.** (The parenthetical statement follows from Cohomology of Schemes, Lemma 9.10.) Since $H^0(X, O_X) = k$ is dual to $H^1(X, \omega_X)$ (see Section 34) we see that $\dim H^1(X, \omega_X) = 1$. The sheaf $\omega_X$ represents the functor $F \to \text{Hom}_k(H^1(X, F), k)$ on the category of coherent $O_X$-modules (Lemma 29.7). Consider an exact sequence as in the statement of the lemma and assume that $H^1(X, F) \neq 0$. Since $H^1(X, Q) = 0$ we see that $H^1(X, \omega_X) \to H^1(X, F)$ is an isomorphism. By the universal property of $\omega_X$ stated above, we conclude there is a map $F \to \omega_X$ whose action on $H^1$ is the inverse of this isomorphism. The composition $\omega_X \to F \to \omega_X$ is the identity (by the universal property) and the lemma is proved.

**Lemma 35.3.** In Situation 35.1. Given an exact sequence

$$\omega_X \to F \to Q \to 0$$
of coherent $\mathcal{O}_X$-modules with $\dim(Supp(Q)) = 0$ and $\dim_k H^0(X, Q) \geq 2$ and such that there is no nonzero submodule $Q' \subset \mathcal{F}$ such that $\mathcal{Q}' \rightarrow Q$ is injective. Then the submodule of $\mathcal{F}$ generated by global sections surjects onto $Q$.

**Proof.** Let $\mathcal{F}' \subset \mathcal{F}$ be the submodule generated by global sections and the image of $\omega_X \rightarrow \mathcal{F}$. Since $\dim_k H^0(X, Q) \geq 2$ and $\dim_k H^1(X, \omega_X) = \dim_k H^0(X, \mathcal{O}_X) = 1$, we see that $\mathcal{F}' \rightarrow Q$ is not zero and $\omega_X \rightarrow \mathcal{F}'$ is not an isomorphism. Hence $H^1(X, \mathcal{F'}) = 0$ by Lemma \ref{lemma-equidimensional} and our assumption on $\mathcal{F}$. Consider the short exact sequence

$$0 \rightarrow \mathcal{F}' \rightarrow \mathcal{F} \rightarrow Q/\text{Im}(\mathcal{F}' \rightarrow Q) \rightarrow 0$$

If the quotient on the right is nonzero, then we obtain a contradiction because then $H^0(X, \mathcal{F})$ is bigger than $H^0(X, \mathcal{F}')$.

Here is an example global generation statement.

**Lemma 35.4.** In Situation 35.1 assume that $X$ is integral. Let $0 \rightarrow \omega_X \rightarrow \mathcal{F} \rightarrow Q \rightarrow 0$ be a short exact sequence of coherent $\mathcal{O}_X$-modules with $\mathcal{F}$ torsion free, $\dim(Supp(Q)) = 0$, and $\dim_k H^0(X, Q) \geq 2$. Then $\mathcal{F}$ is globally generated.

**Proof.** Consider the submodule $\mathcal{F}'$ generated by the global sections. By Lemma \ref{lemma-algebraically-closed} we see that $\mathcal{F}' \rightarrow Q$ is surjective, in particular $\mathcal{F}' \neq 0$. Since $X$ is a curve, we see that $\mathcal{F}' \subset \mathcal{F}$ is an inclusion of rank 1 sheaves, hence $\mathcal{Q}' = \mathcal{F}/\mathcal{F}'$ is supported in finitely many points. To get a contradiction, assume that $\mathcal{Q}'$ is nonzero. Then we see that $H^1(X, \mathcal{F}') \neq 0$. Then we get a nonzero map $\mathcal{F}' \rightarrow \omega_X$ by the universal property (Lemma \ref{lemma-dualizing}). The image of the composition $\mathcal{F}' \rightarrow \omega_X \rightarrow \mathcal{F}$ is generated by global sections, hence is inside of $\mathcal{F}'$. Thus we get a nonzero self map $\mathcal{F}' \rightarrow \mathcal{F}'$. Since $\mathcal{F}'$ is torsion free of rank 1 on a proper curve this has to be an automorphism (details omitted). But then this implies that $\mathcal{F}'$ is contained in $\omega_X \subset \mathcal{F}$ contradicting the surjectivity of $\mathcal{F}' \rightarrow Q$.

**Lemma 35.5.** In Situation 35.1 Let $\mathcal{L}$ be a very ample invertible $\mathcal{O}_X$-module with $\deg(\mathcal{L}) \geq 2$. Then $\omega_X \otimes_{\mathcal{O}_X} \mathcal{L}$ is globally generated.

**Proof.** Assume $k$ is algebraically closed. Let $x \in X$ be a closed point. Let $C_i \subset X$ be the irreducible components and for each $i$ let $x_i \in C_i$ be the generic point. By Varieties, Lemma \ref{lemma-dualizing} we can choose a section $s \in H^0(X, \mathcal{L})$ such that $s$ vanishes at $x$ but not at $x_i$ for all $i$. The corresponding module map $s : \mathcal{O}_X \rightarrow \mathcal{L}$ is injective with cokernel $Q$ supported in finitely many points and with $H^0(X, Q) \geq 2$. Consider the corresponding exact sequence

$$0 \rightarrow \omega_X \rightarrow \omega_X \otimes \mathcal{L} \rightarrow \omega_X \otimes Q \rightarrow 0$$

By Lemma \ref{lemma-algebraically-closed} we see that the module generated by global sections surjects onto $\omega_X \otimes Q$. Since $x$ was arbitrary this proves the lemma. Some details omitted.

We will reduce the case where $k$ is not algebraically closed, to the algebraically closed field case. We suggest the reader skip the rest of the proof. Choose an algebraic closure $\overline{k}$ of $k$ and consider the base change $X_{\overline{k}}$. Let us check that $X_{\overline{k}} \rightarrow \text{Spec}(\overline{k})$ is an example of Situation 35.1. By flat base change (Cohomology of Schemes, Lemma \ref{lemma-flat-base-change}) we see that $H^0(X_{\overline{k}}, \mathcal{O}) = \overline{k}$. By Varieties, Lemma \ref{lemma-dualizing} we see that $X_{\overline{k}}$ is Cohen-Macaulay. The scheme $X_{\overline{k}}$ is proper over $\overline{k}$ (Morphisms, Lemma \ref{lemma-proper}) and equidimensional of dimension 1 (Morphisms, Lemma \ref{lemma-dualizing}). The pullback of $\omega_X$ to $X_{\overline{k}}$ is the dualizing module of $X_{\overline{k}}$ by Lemma \ref{lemma-dualizing}. The pullback
of $\mathcal{L}$ to $X_\tau$ is very ample (Morphisms, Lemma \ref{morphisms-lemma-lift-ample}). The degree of the pullback of $\mathcal{L}$ to $X_\tau$ is equal to the degree of $\mathcal{L}$ on $X$ (Varieties, Lemma \ref{varieties-lemma-dualizing-pushforward}). Finally, we see that $\omega_X \otimes \mathcal{L}$ is globally generated if and only if its base change is so (Varieties, Lemma \ref{varieties-lemma-base-change}). In this way we see that the result follows from the result in the case of an algebraically closed ground field. \qed

36. Other chapters

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