ÉTALE COHOMOLOGY

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1. Introduction

These are the notes of a course on étale cohomology taught by Johan de Jong at Columbia University in the Fall of 2009. The original note takers were Thibaut Pugin, Zachary Maddock and Min Lee. Over time we will add references to background material in the rest of the stacks project and provide rigorous proofs of all the statements.

2. Which sections to skip on a first reading?

We want to use the material in this chapter for the development of theory related to algebraic spaces, Deligne-Mumford stacks, algebraic stacks, etc. Thus we have added some pretty technical material to the original exposition of étale cohomology for schemes. The reader can recognize this material by the frequency of the word “topos”, or by discussions related to set theory, or by proofs dealing with very general properties of morphisms of schemes. Some of these discussions can be skipped on a first reading.

In particular, we suggest that the reader skip the following sections:
3. Prologue

These lectures are about another cohomology theory. The first thing to remark is that the Zariski topology is not entirely satisfactory. One of the main reasons that it fails to give the results that we would want is that if $X$ is a complex variety and $\mathcal{F}$ is a constant sheaf then

$$H^i(X, \mathcal{F}) = 0, \quad \text{for all } i > 0.$$ 

The reason for that is the following. In an irreducible scheme (a variety in particular), any two nonempty open subsets meet, and so the restriction mappings of a constant sheaf are surjective. We say that the sheaf is flasque. In this case, all higher Čech cohomology groups vanish, and so do all higher Zariski cohomology groups. In other words, there are “not enough” open sets in the Zariski topology to detect this higher cohomology.

On the other hand, if $X$ is a smooth projective complex variety, then

$$H^{2 \dim X}(\mathcal{F}(\mathbb{C}), \Lambda) = \Lambda \quad \text{for } \Lambda = \mathbb{Z}, \mathbb{Z}/n\mathbb{Z},$$

where $\mathcal{F}(\mathbb{C})$ means the set of complex points of $X$. This is a feature that would be nice to replicate in algebraic geometry. In positive characteristic in particular.

4. The étale topology

It is very hard to simply “add” extra open sets to refine the Zariski topology. One efficient way to define a topology is to consider not only open sets, but also some schemes that lie over them. To define the étale topology, one considers all morphisms $\varphi : U \to X$ which are étale. If $X$ is a smooth projective variety over $\mathbb{C}$, then this means

1. $U$ is a disjoint union of smooth varieties, and
2. $\varphi$ is (analytically) locally an isomorphism.

The word “analytically” refers to the usual (transcendental) topology over $\mathbb{C}$. So the second condition means that the derivative of $\varphi$ has full rank everywhere (and in particular all the components of $U$ have the same dimension as $X$).

A double cover – loosely defined as a finite degree 2 map between varieties – for example

$$\text{Spec}(\mathbb{C}[t]) \longrightarrow \text{Spec}(\mathbb{C}[t]), \quad t \mapsto t^2$$
will not be an étale morphism if it has a fibre consisting of a single point. In the example this happens when \( t = 0 \). For a finite map between varieties over \( \mathbb{C} \) to be étale all the fibers should have the same number of points. Removing the point \( t = 0 \) from the source of the map in the example will make the morphism étale. But we can remove other points from the source of the morphism also, and the morphism will still be étale. To consider the étale topology, we have to look at all such morphisms. Unlike the Zariski topology, these need not be merely be open subsets of \( X \), even though their images always are.

**Definition 4.1.** A family of morphisms \( \{ \varphi_i : U_i \to X \}_{i \in I} \) is called an étale covering if each \( \varphi_i \) is an étale morphism and their images cover \( X \), i.e., \( X = \bigcup_{i \in I} \varphi_i(U_i) \).

This “defines” the étale topology. In other words, we can now say what the sheaves are. An étale sheaf \( F \) of sets (resp. abelian groups, vector spaces, etc) on \( X \) is the data:

1. for each étale morphism \( \varphi : U \to X \) a set (resp. abelian group, vector space, etc) \( F(U) \),
2. for each pair \( U, U' \) of étale schemes over \( X \), and each morphism \( U \to U' \) over \( X \) (which is automatically étale) a restriction map \( \rho_U^{U'} : F(U') \to F(U) \)

These data have to satisfy the condition that \( \rho_U^{U} = \text{id} \) in case of the identity morphism \( U \to U \) and that \( \rho_U^{U'} \circ \rho_U^{U''} = \rho_U^{U''} \) when we have morphisms \( U \to U' \to U'' \) of schemes étale over \( X \) as well as the following sheaf axiom:

\[ (*) \quad \text{for every étale covering } \{ \varphi_i : U_i \to X \}_{i \in I}, \text{ the diagram} \]

\[
\begin{array}{ccc}
\emptyset & \longrightarrow & F(U) \\
\downarrow & & \downarrow \\
\Pi_{i \in I} F(U_i) & \longrightarrow & \Pi_{i,j \in I} F(U_i \times_U U_j)
\end{array}
\]

is exact in the category of sets (resp. abelian groups, vector spaces, etc).

**Remark 4.2.** In the last statement, it is essential not to forget the case where \( i = j \) which is in general a highly nontrivial condition (unlike in the Zariski topology). In fact, frequently important coverings have only one element.

Since the identity is an étale morphism, we can compute the global sections of an étale sheaf, and cohomology will simply be the corresponding right-derived functors. In other words, once more theory has been developed and statements have been made precise, there will be no obstacle to defining cohomology.

## 5. Feats of the étale topology

For a natural number \( n \in \mathbb{N} = \{ 1, 2, 3, 4, \ldots \} \) it is true that

\[
H^2_{\text{étale}}(\mathbb{P}^1_\mathbb{C}, \mathbb{Z}/n\mathbb{Z}) = \mathbb{Z}/n\mathbb{Z}.
\]

More generally, if \( X \) is a complex variety, then its étale Betti numbers with coefficients in a finite field agree with the usual Betti numbers of \( X(\mathbb{C}) \), i.e.,

\[
\dim_{\mathbb{F}_q} H^2_{\text{étale}}(X, \mathbb{F}_q) = \dim_{\mathbb{F}_q} H^2_{\text{Betti}}(X(\mathbb{C}), \mathbb{F}_q).
\]

This is extremely satisfactory. However, these equalities only hold for torsion coefficients, not in general. For integer coefficients, one has

\[
H^2_{\text{étale}}(\mathbb{P}^1_\mathbb{C}, \mathbb{Z}) = 0.
\]

There are ways to get back to nontorsion coefficients from torsion ones by a limit procedure which we will come to shortly.
6. A computation

How do we compute the cohomology of $\mathbb{P}^1_C$ with coefficients $\Lambda = \mathbb{Z}/n\mathbb{Z}$? We use Čech cohomology. A covering of $\mathbb{P}^1_C$ is given by the two standard opens $U_0, U_1$, which are both isomorphic to $\mathbb{A}^1_C$, and which intersection is isomorphic to $\mathbb{A}^1_C \setminus \{0\} = \mathbb{G}_m$. It turns out that the Mayer-Vietoris sequence holds in étale cohomology. This gives an exact sequence

$$H^{i-1}_\text{étale}(U_0 \cap U_1, \Lambda) \to H^i_\text{étale}(\mathbb{P}^1_C, \Lambda) \to H^i_\text{étale}(U_0, \Lambda) \oplus H^i_\text{étale}(U_1, \Lambda) \to H^i_\text{étale}(U_0 \cap U_1, \Lambda).$$

To get the answer we expect, we would need to show that the direct sum in the third term vanishes. In fact, it is true that, as for the usual topology,

$$H^q_\text{étale}(\mathbb{A}^1_C, \Lambda) = 0 \quad \text{for} \quad q \geq 1,$$

and

$$H^q_\text{étale}(\mathbb{A}^1_C \setminus \{0\}, \Lambda) = \begin{cases} \Lambda & \text{if } q = 1, \\ 0 & \text{for } q \geq 2. \end{cases}$$

These results are already quite hard (what is an elementary proof?). Let us explain how we would compute this once the machinery of étale cohomology is at our disposal.

**Higher cohomology.** This is taken care of by the following general fact: if $X$ is an affine curve over $C$, then

$$H^q_\text{étale}(X, \mathbb{Z}/n\mathbb{Z}) = 0 \quad \text{for} \quad q \geq 2.$$

This is proved by considering the generic point of the curve and doing some Galois cohomology. So we only have to worry about the cohomology in degree 1.

**Cohomology in degree 1.** We use the following identifications:

$$H^1_\text{étale}(X, \mathbb{Z}/n\mathbb{Z}) = \left\{ \text{sheaves of sets } \mathcal{F} \text{ on the étale site } X_\text{étale, endowed with an action } \mathbb{Z}/n\mathbb{Z} \times \mathcal{F} \to \mathcal{F} \text{ such that } \mathcal{F} \text{ is a } \mathbb{Z}/n\mathbb{Z}-\text{torsor.} \right\} \cong \left\{ \text{morphisms } Y \to X \text{ which are finite étale together with a free } \mathbb{Z}/n\mathbb{Z} \text{ action such that } X = Y/(\mathbb{Z}/n\mathbb{Z}) \right\} \cong \left\{ \text{morphisms } \bar{Y} \to \bar{X} \right\}.$$

The first identification is very general (it is true for any cohomology theory on a site) and has nothing to do with the étale topology. The second identification is a consequence of descent theory. The last set describes a collection of geometric objects on which we can get our hands.

The curve $\mathbb{A}^1_C$ has no nontrivial finite étale covering and hence $H^1_\text{étale}(\mathbb{A}^1_C, \mathbb{Z}/n\mathbb{Z}) = 0$. This can be seen either topologically or by using the argument in the next paragraph.

Let us describe the finite étale coverings $\varphi : Y \to \mathbb{A}^1_C \setminus \{0\}$. It suffices to consider the case where $Y$ is connected, which we assume. We are going to find out what $Y$ can be by applying the Riemann-Hurwitz formula (of course this is a bit silly, and you can go ahead and skip the next section if you like). Say that this morphism is $n$ to 1, and consider a projective compactification

$$Y' \xrightarrow{\varphi} \bar{Y} \xleftarrow{\bar{\varphi}} \mathbb{A}^1_C \setminus \{0\} \xrightarrow{\varphi} \mathbb{P}^1_C.$$
Even though \( \varphi \) is étale and does not ramify, \( \bar{\varphi} \) may ramify at 0 and \( \infty \). Say that the preimages of 0 are the points \( y_1, \ldots, y_r \) with indices of ramification \( e_1, \ldots, e_r \), and that the preimages of \( \infty \) are the points \( y'_1, \ldots, y'_s \) with indices of ramification \( d_1, \ldots, d_s \). In particular, \( \sum e_i = n = \sum d_j \). Applying the Riemann-Hurwitz formula, we get

\[
2g_Y - 2 = -2n + \sum (e_i - 1) + \sum (d_j - 1)
\]

and therefore \( g_Y = 0 \), \( r = s = 1 \) and \( e_1 = d_1 = n \). Hence \( Y \cong \mathbb{A}^1_{\mathbb{C}} \setminus \{0\} \), and it is easy to see that \( \varphi(z) = \lambda z^n \) for some \( \lambda \in \mathbb{C}^* \). After reparametrizing \( Y \) we may assume \( \lambda = 1 \). Thus our covering is given by taking the \( n \)th root of the coordinate on \( \mathbb{A}^1_{\mathbb{C}} \setminus \{0\} \).

Remember that we need to classify the coverings of \( \mathbb{A}^1_{\mathbb{C}} \setminus \{0\} \) together with free \( \mathbb{Z}/n\mathbb{Z} \)-actions on them. In our case any such action corresponds to an automorphism of \( Y \) sending \( z \) to \( \zeta_n z \), where \( \zeta_n \) is a primitive \( n \)th root of unity. There are \( \phi(n) \) such actions (here \( \phi(n) \) means the Euler function). Thus there are exactly \( \phi(n) \) connected finite étale coverings with a given free \( \mathbb{Z}/n\mathbb{Z} \)-action, each corresponding to a primitive \( n \)th root of unity. We leave it to the reader to see that the disconnected finite étale degree \( n \) coverings of \( \mathbb{A}^1_{\mathbb{C}} \setminus \{0\} \) with a given free \( \mathbb{Z}/n\mathbb{Z} \)-action correspond one-to-one with \( n \)th roots of 1 which are not primitive. In other words, this computation shows that

\[
H^1_{\text{étale}}(\mathbb{A}^1_{\mathbb{C}} \setminus \{0\}, \mathbb{Z}/n\mathbb{Z}) = \text{Hom}(\mu_n(\mathbb{C}), \mathbb{Z}/n\mathbb{Z}) \cong \mathbb{Z}/n\mathbb{Z}.
\]

The first identification is canonical, the second isn’t, see Remark 85.5. Since the proof of Riemann-Hurwitz does not use the computation of cohomology, the above actually constitutes a proof (provided we fill in the details on vanishing, etc).

### 7. Nontorsion coefficients

To study nontorsion coefficients, one makes the following definition:

\[
H^i_{\text{étale}}(X, \mathbb{Q}_\ell) := (\lim_n H^i_{\text{étale}}(X, \mathbb{Z}/\ell^n\mathbb{Z})) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell.
\]

The symbol \( \lim_n \) denotes the limit of the system of cohomology groups \( H^i_{\text{étale}}(X, \mathbb{Z}/\ell^n\mathbb{Z}) \) indexed by \( n \), see Categories, Section 21. Thus we will need to study systems of sheaves satisfying some compatibility conditions.

### 8. Sheaf theory

At this point we start talking about sites and sheaves in earnest. There is an amazing amount of useful abstract material that could fit in the next few sections. Some of this material is worked out in earlier chapters, such as the chapter on sites, modules on sites, and cohomology on sites. We try to refrain from adding too much material here, just enough so the material later in this chapter makes sense.

### 9. Presheaves

A reference for this section is Sites, Section 2.

**Definition 9.1.** Let \( \mathcal{C} \) be a category. A presheaf of sets (respectively, an abelian presheaf) on \( \mathcal{C} \) is a functor \( \mathcal{C}^{\op} \to \text{Sets} \) (resp. \( \text{Ab} \)).
Terminology. If \( U \in \text{Ob}(\mathcal{C}) \), then elements of \( \mathcal{F}(U) \) are called sections of \( \mathcal{F} \) over \( U \). For \( \varphi : V \to U \) in \( \mathcal{C} \), the map \( \mathcal{F}(\varphi) : \mathcal{F}(U) \to \mathcal{F}(V) \) is called the restriction map and is often denoted \( s \mapsto s|_V \) or sometimes \( s \mapsto \varphi^*s \). The notation \( s|_V \) is ambiguous since the restriction map depends on \( \varphi \), but it is a standard abuse of notation. We also use the notation \( \Gamma(U, \mathcal{F}) = \mathcal{F}(U) \).

Saying that \( \mathcal{F} \) is a functor means that if \( W \to V \to U \) are morphisms in \( \mathcal{C} \) and \( s \in \Gamma(U, \mathcal{F}) \) then \( (s|_V)|_W = s|_W \), with the abuse of notation just seen. Moreover, the restriction mappings corresponding to the identity morphisms \( \text{id}_U : U \to U \) are the identity.

The category of presheaves of sets (respectively of abelian presheaves) on \( \mathcal{C} \) is denoted \( \text{PSh}(\mathcal{C}) \) (resp. \( \text{PAb}(\mathcal{C}) \)). It is the category of functors from \( \mathcal{C}^{\text{opp}} \) to \( \text{Sets} \) (resp. \( \text{Ab} \)), which is to say that the morphisms of presheaves are natural transformations of functors. We only consider the categories \( \text{PSh}(\mathcal{C}) \) and \( \text{PAb}(\mathcal{C}) \) when the category \( \mathcal{C} \) is small. (Our convention is that a category is small unless otherwise mentioned, and if it isn’t small it should be listed in Categories, Remark 2.2.)

Example 9.2. Given an object \( X \in \text{Ob}(\mathcal{C}) \), we consider the functor
\[
\begin{align*}
h_X : \quad \mathcal{C}^{\text{opp}} & \longrightarrow \text{Sets} \\
U & \mapsto h_X(U) = \text{Mor}_\mathcal{C}(U, X) \\
V \xrightarrow{\varphi} U & \mapsto \varphi \circ - : h_X(U) \to h_X(V).
\end{align*}
\]
It is a presheaf, called the representable presheaf associated to \( X \).

It is not true that representable presheaves are sheaves in every topology on every site.

Lemma 9.3 (Yoneda). Let \( \mathcal{C} \) be a category, and \( X, Y \in \text{Ob}(\mathcal{C}) \). There is a natural bijection
\[
\begin{align*}
\text{Mor}_\mathcal{C}(X, Y) & \longrightarrow \text{Mor}_{\text{PSh}(\mathcal{C})}(h_X, h_Y) \\
\psi & \mapsto h_\psi = \psi \circ - : h_X \to h_Y.
\end{align*}
\]

Proof. See Categories, Lemma 3.5. \( \square \)

10. Sites

Definition 10.1. Let \( \mathcal{C} \) be a category. A family of morphisms with fixed target \( U = \{ \varphi_i : U_i \to U \}_{i \in I} \) is the data of
\begin{enumerate}
\item an object \( U \in \mathcal{C} \),
\item a set \( I \) (possibly empty), and
\item for all \( i \in I \), a morphism \( \varphi_i : U_i \to U \) of \( \mathcal{C} \) with target \( U \).
\end{enumerate}

There is a notion of a morphism of families of morphisms with fixed target. A special case of that is the notion of a refinement. A reference for this material is Sites, Section 8.

Definition 10.2. A site \( \mathcal{C} \) consists of a category \( \mathcal{C} \) and a set \( \text{Cov}(\mathcal{C}) \) consisting of families of morphisms with fixed target called coverings, such that
\begin{enumerate}
\item (isomorphism) if \( \varphi : V \to U \) is an isomorphism in \( \mathcal{C} \), then \( \{ \varphi : V \to U \} \) is a covering,
\end{enumerate}

\( ^1 \)What we call a site is called a category endowed with a pretopology in [AGV71, Expos´e II, D´efinition 1.3]. In [Art62] it is called a category with a Grothendieck topology.
(2) (locality) if \( \{ \varphi_i : U_i \to U \}_{i \in I} \) is a covering and for all \( i \in I \) we are given a covering \( \{ \psi_{ij} : U_{ij} \to U_i \}_{j \in I_i} \), then
\[
\{ \varphi_i \circ \psi_{ij} : U_{ij} \to U \}_{(i,j) \in \prod_{i \in I} I_i}
\]
is also a covering, and

(3) (base change) if \( \{ U_i \to U \}_{i \in I} \) is a covering and \( V \to U \) is a morphism in \( \mathcal{C} \), then
(a) for all \( i \in I \) the fibre product \( U_i \times_U V \) exists in \( \mathcal{C} \), and
(b) \( \{ U_i \times_U V \to V \}_{i \in I} \) is a covering.

For us the category underlying a site is always “small”, i.e., its collection of objects form a set, and the collection of coverings of a site is a set as well (as in the definition above). We will mostly, in this chapter, leave out the arguments that cut down the collection of objects and coverings to a set. For further discussion, see Sites, Remark 6.3.

03NI **Example 10.3.** If \( X \) is a topological space, then it has an associated site \( X_{\text{Zar}} \) defined as follows: the objects of \( X_{\text{Zar}} \) are the open subsets of \( X \), the morphisms between these are the inclusion mappings, and the coverings are the usual topological (surjective) coverings. Observe that if \( U, V \subset W \subset X \) are open subsets then \( U \times_W V = U \cap V \) exists: this category has fiber products. All the verifications are trivial and everything works as expected.

11. Sheaves

03NJ

03NK **Definition 11.1.** A presheaf \( \mathcal{F} \) of sets (resp. abelian presheaf) on a site \( \mathcal{C} \) is said to be a separated presheaf if for all coverings \( \{ \varphi_i : U_i \to U \}_{i \in I} \in \text{Cov}(\mathcal{C}) \) the map
\[
\mathcal{F}(U) \to \prod_{i \in I} \mathcal{F}(U_i)
\]
is injective. Here the map is \( s \to (s|_{U_i})_{i \in I} \). The presheaf \( \mathcal{F} \) is a sheaf if for all coverings \( \{ \varphi_i : U_i \to U \}_{i \in I} \in \text{Cov}(\mathcal{C}) \), the diagram
\[
\mathcal{F}(U) \to \prod_{i \in I} \mathcal{F}(U_i) \cong \prod_{i,j \in I} \mathcal{F}(U_i \times_U U_j),
\]
where the first map is \( s \to (s|_{U_i})_{i \in I} \) and the two maps on the right are \( (s_i)_{i \in I} \to (s_i|_{U_i \times_U U_j}) \) and \( (s_i)_{i \in I} \to (s_j|_{U_i \times_U U_j}) \), is an equalizer diagram in the category of sets (resp. abelian groups).

03NM **Remark 11.2.** For the empty covering (where \( I = \emptyset \)), this implies that \( \mathcal{F}(\emptyset) \) is an empty product, which is a final object in the corresponding category (a singleton, for both \( \text{Sets} \) and \( \text{Ab} \)).

03NN **Example 11.3.** Working this out for the site \( X_{\text{Zar}} \) associated to a topological space, see Example 10.3 gives the usual notion of sheaves.

03NO **Definition 11.4.** We denote \( \text{Sh}(\mathcal{C}) \) (resp. \( \text{Ab}(\mathcal{C}) \)) the full subcategory of \( \text{PSh}(\mathcal{C}) \) (resp. \( \text{PAb}(\mathcal{C}) \)) whose objects are sheaves. This is the category of sheaves of sets (resp. abelian sheaves) on \( \mathcal{C} \).
12. The example of G-sets

Let \( G \) be a group and define a site \( \mathcal{T}_G \) as follows: the underlying category is the category of \( G \)-sets, i.e., its objects are sets endowed with a left \( G \)-action and the morphisms are equivariant maps; and the coverings of \( \mathcal{T}_G \) are the families \( \{ \varphi_i : U_i \to U \}_{i \in I} \) satisfying \( U = \bigcup_{i \in I} \varphi_i(U_i) \).

There is a special object in the site \( \mathcal{T}_G \), namely the \( G \)-set \( G \) endowed with its natural action by left translations. We denote it \( G^G \). Observe that there is a natural group isomorphism

\[
\rho : \quad G^{\text{opp}} \to \text{Aut}_{G\text{-Sets}}(G^G)
\]

\[
g \mapsto (h \mapsto hg).
\]

In particular, for any presheaf \( F \), the set \( F(G^G) \) inherits a \( G \)-action via \( \rho \). (Note that by contravariance of \( F \), the set \( F(G^G) \) is again a left \( G \)-set.) In fact, the functor

\[
\text{Sh}(\mathcal{T}_G) \to G\text{-Sets}
\]

\[
\mathcal{F} \mapsto F(G^G)
\]

is an equivalence of categories. Its quasi-inverse is the functor \( X \mapsto h_X \). Without giving the complete proof (which can be found in Sites, Section 9) let us try to explain why this is true.

1. If \( S \) is a \( G \)-set, we can decompose it into orbits \( S = \bigsqcup_{i \in I} O_i \). The sheaf axiom for the covering \( \{ O_i \to S \}_{i \in I} \) says that

\[
F(S) \xrightarrow{\text{equalizer}} \prod_{i \in I} F(O_i) \xrightarrow{\text{fib prod}} \prod_{i,j \in I} F(O_i \times_S O_j)
\]

is an equalizer. Observing that fibered products in \( G\text{-Sets} \) are induced from fibered products in \( \text{Sets} \), and using the fact that \( F(\emptyset) \) is a \( G \)-singleton, we get that

\[
\prod_{i,j \in I} F(O_i \times_S O_j) = \prod_{i \in I} F(O_i)
\]

and the two maps above are in fact the same. Therefore the sheaf axiom merely says that \( F(S) = \prod_{i \in I} F(O_i) \).

2. If \( S \) is the \( G \)-set \( S = G/H \) and \( F \) is a sheaf on \( \mathcal{T}_G \), then we claim that

\[
F(G/H) = F(G^G)^H
\]

and in particular \( F(\{e\}) = F(G^G)^G \). To see this, let’s use the sheaf axiom for the covering \( \{ G \to G/H \} \) of \( S \). We have

\[
G^G \times_{G/H} G^G \cong G \times H
\]

\[
(g_1, g_2) \mapsto (g_1, g_1 g_2^{-1})
\]

is a disjoint union of copies of \( G^G \) (as a \( G \)-set). Hence the sheaf axiom reads

\[
F(G/H) \xrightarrow{\text{equalizer}} F(G^G) \xrightarrow{\text{fib prod}} \prod_{h \in H} F(G^G)
\]

where the two maps on the right are \( s \mapsto (s)_{h \in H} \) and \( s \mapsto (hs)_{h \in H} \). Therefore \( F(G/H) = F(G^G)^H \) as claimed.

This doesn’t quite prove the claimed equivalence of categories, but it shows at least that a sheaf \( F \) is entirely determined by its sections over \( G^G \). Details (and set theoretical remarks) can be found in Sites, Section 9.
13. Sheafification

Definition 13.1. Let $\mathcal{F}$ be a presheaf on the site $\mathcal{C}$ and $\mathcal{U} = \{U_i \to U\} \in \text{Cov}(\mathcal{C})$. We define the zeroth Čech cohomology group of $\mathcal{F}$ with respect to $\mathcal{U}$ by

$$\check{H}^0(\mathcal{U}, \mathcal{F}) = \left\{ (s_i)_{i \in I} \in \prod_{i \in I} \mathcal{F}(U_i) \text{ such that } s_i|_{U_i \times_U U_j} = s_j|_{U_i \times_U U_j} \right\}.$$ 

There is a canonical map $\mathcal{F}(U) \to \check{H}^0(\mathcal{U}, \mathcal{F})$, $s \mapsto (s|_{U_i})_{i \in I}$. We say that a morphism of coverings from a covering $\mathcal{V} = \{V_j \to V\}_{j \in J}$ to $\mathcal{U}$ is a triple $(\chi, \alpha, \chi_j)$, where $\chi : V \to U$ is a morphism, $\alpha : J \to I$ is a map of sets, and for all $j \in J$ the morphism $\chi_j$ fits into a commutative diagram

$$V_j \xrightarrow{(\chi_j, \psi_j)} U_{\alpha(j)} \xrightarrow{\chi} U.$$ 

Given the data $\chi, \alpha, \{\chi_j\}_{i \in J}$ we define

$$\check{H}^0(\mathcal{U}, \mathcal{F}) \to \check{H}^0(\mathcal{V}, \mathcal{F}) \quad (s_i)_{i \in I} \mapsto (\chi_j^* (s_{\alpha(j)}))_{j \in J}.$$ 

We then claim that

1. the map is well-defined, and
2. depends only on $\chi$ and is independent of the choice of $\alpha, \{\chi_j\}_{i \in J}$.

We omit the proof of the first fact. To see part (2), consider another triple $(\psi, \beta, \psi_j)$ with $\chi = \psi$. Then we have the commutative diagram

$$V_j \xrightarrow{\chi_j^*} U_{\alpha(j)} \xrightarrow{\chi} U.$$ 

Given a section $s \in \mathcal{F}(U)$, its image in $\mathcal{F}(V_j)$ under the map given by $\chi, \alpha, \{\chi_j\}_{i \in J}$ is $\chi_j^* s_{\alpha(j)}$, and its image under the map given by $\psi, \beta, \{\psi_j\}_{i \in J}$ is $\psi_j^* s_{\beta(j)}$. These two are equal since by assumption $s \in \check{H}(\mathcal{U}, \mathcal{F})$ and hence both are equal to the pullback of the common value

$$s_{\alpha(j)}|_{U_{\alpha(j)} \times_U U_{\beta(j)}} = s_{\beta(j)}|_{U_{\alpha(j)} \times_U U_{\beta(j)}}$$ 

pulled back by the map $(\chi_j, \psi_j)$ in the diagram.

Theorem 13.2. Let $\mathcal{C}$ be a site and $\mathcal{F}$ a presheaf on $\mathcal{C}$.

1. The rule

$$U \mapsto \mathcal{F}^+(U) := \text{colim}_U \text{ covering of } U \check{H}^0(\mathcal{U}, \mathcal{F})$$

is a presheaf. And the colimit is a directed one.

2. There is a canonical map of presheaves $\mathcal{F} \to \mathcal{F}^+.$
(3) If \( F \) is a separated presheaf then \( F^+ \) is a sheaf and the map in \( (2) \) is injective.

(4) \( F^+ \) is a separated presheaf.

(5) \( F^\# = (F^+)^+ \) is a sheaf, and the canonical map induces a functorial isomorphism

\[
\text{Hom}_{\text{Sh}(\mathcal{C})}(F, \mathcal{G}) = \text{Hom}_{\text{PSh}(\mathcal{C})}(F^\#, \mathcal{G})
\]

for any \( \mathcal{G} \in \text{Sh}(\mathcal{C}) \).

**Proof.** See Sites, Theorem 10.10. \( \square \)

In other words, this means that the natural map \( F \to F^\# \) is a left adjoint to the forgetful functor \( \text{Sh}(\mathcal{C}) \to \text{PSh}(\mathcal{C}) \).

### 14. Cohomology

The following is the basic result that makes it possible to define cohomology for abelian sheaves on sites.

**Theorem 14.1.** The category of abelian sheaves on a site is an abelian category which has enough injectives.

**Proof.** See Modules on Sites, Lemma 3.1 and Injectives, Theorem 7.4. \( \square \)

So we can define cohomology as the right-derived functors of the sections functor: if \( U \in \text{Ob}(\mathcal{C}) \) and \( F \in \text{Ab}(\mathcal{C}) \),

\[
H^p(U, F) := R^p\Gamma(U, F) = H^p(\Gamma(U, \mathcal{I}^*))
\]

where \( F \to \mathcal{I}^* \) is an injective resolution. To do this, we should check that the functor \( \Gamma(U, -) \) is left exact. This is true and is part of why the category \( \text{Ab}(\mathcal{C}) \) is abelian, see Modules on Sites, Lemma 3.1. For more general discussion of cohomology on sites (including the global sections functor and its right derived functors), see Cohomology on Sites, Section 3.

### 15. The fpqc topology

Before doing étale cohomology we study a bit the fpqc topology, since it works well for quasi-coherent sheaves.

**Definition 15.1.** Let \( T \) be a scheme. An fpqc covering of \( T \) is a family \( \{ \varphi_i : T_i \to T \}_{i \in I} \) such that

1. each \( \varphi_i \) is a flat morphism and \( \bigcup_{i \in I} \varphi_i(T_i) = T \), and
2. for each affine open \( U \subset T \) there exists a finite set \( K \), a map \( i : K \to I \) and affine opens \( U_{i(k)} \subset T_{i(k)} \) such that \( U = \bigcup_{k \in K} \varphi_{i(k)}(U_{i(k)}) \).

**Remark 15.2.** The first condition corresponds to fp, which stands for fidèlement plat, faithfully flat in French, and the second to qc, quasi-compact. The second part of the first condition is unnecessary when the second condition holds.

**Example 15.3.** Examples of fpqc coverings.

1. Any Zariski open covering of \( T \) is an fpqc covering.
2. A family \( \{ \text{Spec}(B) \to \text{Spec}(A) \} \) is an fpqc covering if and only if \( A \to B \) is a faithfully flat ring map.
3. If \( f : X \to Y \) is flat, surjective and quasi-compact, then \( \{ f : X \to Y \} \) is an fpqc covering.
The morphism $\varphi: \coprod_{x \in A^1_k} \text{Spec}(O_{A^1_k,x}) \to A^1_k$, where $k$ is a field, is flat and surjective. It is not quasi-compact, and in fact the family $\{\varphi\}$ is not a fpqc covering.

Write $A^2_k = \text{Spec}(k[[x,y]])$. Denote $i_x: D(x) \to A^2_k$ and $i_y: D(y) \to A^2_k$ the standard opens. Then the families $\{i_x, i_y, \text{Spec}(k[[x,y]]) \to A^2_k\}$ and $\{i_x, i_y, \text{Spec}(O_{A^2_k,0}) \to A^2_k\}$ are fpqc coverings.

**Lemma 15.4.** The collection of fpqc coverings on the category of schemes satisfies the axioms of site.

**Proof.** See Topologies, Lemma 8.7. □

It seems that this lemma allows us to define the fpqc site of the category of schemes. However, there is a set theoretical problem that comes up when considering the fpqc topology, see Topologies, Section 8. It comes from our requirement that sites are “small”, but that no small category of schemes can contain a cofinal system of fpqc coverings of a given nonempty scheme. Although this does not strictly speaking prevent us from defining “partial” fpqc sites, it does not seem prudent to do so. The work-around is to allow the notion of a sheaf for the fpqc topology (see below) but to prohibit considering the category of all fpqc sheaves.

**Definition 15.5.** Let $S$ be a scheme. The category of schemes over $S$ is denoted $\text{Sch}/S$. Consider a functor $F: (\text{Sch}/S)^{op} \to \text{Sets}$, in other words a presheaf of sets. We say $F$ satisfies the sheaf property for the fpqc topology if for every fpqc covering $\{U_i \to U\}_{i \in I}$ of schemes over $S$ the diagram (11.1.1) is an equalizer diagram.

We similarly say that $F$ satisfies the sheaf property for the Zariski topology if for every open covering $U = \bigcup_{i \in I} U_i$ the diagram (11.1.1) is an equalizer diagram. See Schemes, Definition 15.3. Clearly, this is equivalent to saying that for every scheme $T$ over $S$ the restriction of $F$ to the opens of $T$ is a (usual) sheaf.

**Lemma 15.6.** Let $F$ be a presheaf on $\text{Sch}/S$. Then $F$ satisfies the sheaf property for the fpqc topology if and only if

1. $F$ satisfies the sheaf property with respect to the Zariski topology, and
2. for every faithfully flat morphism $\text{Spec}(B) \to \text{Spec}(A)$ of affine schemes over $S$, the sheaf axiom holds for the covering $\{\text{Spec}(B) \to \text{Spec}(A)\}$.

Namely, this means that $\xymatrix{ F(\text{Spec}(A)) \ar[r] & F(\text{Spec}(B)) & F(\text{Spec}(B \otimes_A B)) \ar[l] }$ is an equalizer diagram.

**Proof.** See Topologies, Lemma 8.13. □

An alternative way to think of a presheaf $F$ on $\text{Sch}/S$ which satisfies the sheaf condition for the fpqc topology is as the following data:

1. for each $T/S$, a usual (i.e., Zariski) sheaf $F_T$ on $T_{\text{Zar}}$,
2. for every map $f: T' \to T$ over $S$, a restriction mapping $f^{-1}F_T \to F_{T'}$ such that
   a. the restriction mappings are functorial,
   b. if $f: T' \to T$ is an open immersion then the restriction mapping $f^{-1}F_T \to F_{T'}$ is an isomorphism, and
(c) for every faithfully flat morphism \( \text{Spec}(B) \to \text{Spec}(A) \) over \( S \), the diagram
\[
\begin{array}{ccc}
\mathcal{F}_{\text{Spec}(A)}(\text{Spec}(A)) & \longrightarrow & \mathcal{F}_{\text{Spec}(B)}(\text{Spec}(B)) \\
\downarrow & & \downarrow \\
\mathcal{F}_{\text{Spec}(B \otimes_A B)}(\text{Spec}(B \otimes_A B))
\end{array}
\]
is an equalizer.

Data (1) and (2) and conditions (a), (b) give the data of a presheaf on \( \text{Sch}/S \) satisfying the sheaf condition for the Zariski topology. By Lemma 15.6 condition (c) then suffices to get the sheaf condition for the fpqc topology.

**Example 15.7.** Consider the presheaf
\[
\mathcal{F} : \ (\text{Sch}/S)^{\text{opp}} \longrightarrow \text{Ab} \\
T/S \quad \longmapsto \quad \Gamma(T, \Omega_{T/S}).
\]
The compatibility of differentials with localization implies that \( \mathcal{F} \) is a sheaf on the Zariski site. However, it does not satisfy the sheaf condition for the fpqc topology. Namely, consider the case \( S = \text{Spec}(\mathbb{F}_p) \) and the morphism
\[
\varphi : V = \text{Spec}(\mathbb{F}_p[v]) \to U = \text{Spec}(\mathbb{F}_p[u])
\]
given by mapping \( u \) to \( v^p \). The family \( \{ \varphi \} \) is an fpqc covering, yet the restriction mapping \( \mathcal{F}(U) \to \mathcal{F}(V) \) sends the generator \( du \) to \( d(v^p) = 0 \), so it is the zero map, and the diagram
\[
\begin{array}{ccc}
\mathcal{F}(U) & \longrightarrow & \mathcal{F}(V) \\
\downarrow & & \downarrow \\
\mathcal{F}(V \times_U V)
\end{array}
\]
is not an equalizer. We will see later that \( \mathcal{F} \) does in fact give rise to a sheaf on the \( \acute{e} \text{tale} \) and smooth sites.

**Lemma 15.8.** Any representable presheaf on \( \text{Sch}/S \) satisfies the sheaf condition for the fpqc topology.

**Proof.** See Descent, Lemma 9.3.

We will return to this later, since the proof of this fact uses descent for quasi-coherent sheaves, which we will discuss in the next section. A fancy way of expressing the lemma is to say that the fpqc topology is weaker than the canonical topology, or that the fpqc topology is subcanonical. In the setting of sites this is discussed in Sites, Section 13.

**Remark 15.9.** The fpqc is the finest topology that we will see. Hence any presheaf satisfying the sheaf condition for the fpqc topology will be a sheaf in the subsequent sites (\( \acute{e} \text{tale}, \text{smooth}, \) etc). In particular representable presheaves will be sheaves on the \( \acute{e} \text{tale} \) site of a scheme for example.

**Example 15.10.** Let \( S \) be a scheme. Consider the additive group scheme \( G_{a,S} = A^1_S \) over \( S \), see Groupoids, Example 5.3. The associated representable presheaf is given by
\[
h_{G_{a,S}}(T) = \text{Mor}_S(T, G_{a,S}) = \Gamma(T, \mathcal{O}_T).
\]
By the above we now know that this is a presheaf of sets which satisfies the sheaf condition for the fpqc topology. On the other hand, it is clearly a presheaf of rings as well. Hence we can think of this as a functor
\[
\mathcal{O} : \ (\text{Sch}/S)^{\text{opp}} \longrightarrow \text{Rings} \\
T/S \quad \longmapsto \quad \Gamma(T, \mathcal{O}_T)
\]
which satisfies the sheaf condition for the fpqc topology. Correspondingly there is a notion of \( \mathcal{O} \)-module, and so on and so forth.
16. Faithfully flat descent

**Definition 16.1.** Let \( \mathcal{U} = \{ t_i : T_i \to T \}_{i \in I} \) be a family of morphisms of schemes with fixed target. A descent datum for quasi-coherent sheaves with respect to \( \mathcal{U} \) is a family \( (\mathcal{F}_i, \varphi_{ij})_{i, j \in I} \) where

1. for all \( i \), \( \mathcal{F}_i \) is a quasi-coherent sheaf on \( T_i \), and
2. for all \( i, j \in I \) the map \( \varphi_{ij} : \text{pr}_0^* \mathcal{F}_i \cong \text{pr}_1^* \mathcal{F}_j \) is an isomorphism on \( T_i \times_T T_j \) such that the diagrams commute on \( T_i \times_T T_j \times_T T_k \).

This descent datum is called effective if there exist a quasi-coherent sheaf \( \mathcal{F} \) over \( T \) and \( \mathcal{O}_T \)-module isomorphisms \( \varphi_i : t_i^* \mathcal{F} \cong \mathcal{F}_i \) satisfying the cocycle condition, namely

\[
\varphi_{ij} = \text{pr}_0^* (\varphi_j) \circ \text{pr}_0^*(\varphi_i)^{-1}.
\]

In this and the next section we discuss some ingredients of the proof of the following theorem, as well as some related material.

**Theorem 16.2.** If \( V = \{ T_i \to T \}_{i \in I} \) is an fpqc covering, then all descent data for quasi-coherent sheaves with respect to \( V \) are effective.

**Proof.** See Descent, Proposition 5.2. \( \square \)

In other words, the fibered category of quasi-coherent sheaves is a stack on the fpqc site. The proof of the theorem is in two steps. The first one is to realize that for Zariski coverings this is easy (or well-known) using standard glueing of sheaves (see Sheaves, Section 33) and the locality of quasi-coherence. The second step is the case of an fpqc covering of the form \( \{ \text{Spec}(B) \to \text{Spec}(A) \} \) where \( A \to B \) is a faithfully flat ring map. This is a lemma in algebra, which we now present.

**Descent of modules.** If \( A \to B \) is a ring map, we consider the complex

\[
(B/A)_\bullet : B \to B \otimes_A B \to B \otimes_A B \otimes_A B \to \ldots
\]

where \( B \) is in degree 0, \( B \otimes_A B \) in degree 1, etc, and the maps are given by

\[
\begin{align*}
b & \mapsto 1 \otimes b - b \otimes 1, \\
b_0 \otimes b_1 & \mapsto 1 \otimes b_0 \otimes b_1 - b_0 \otimes 1 \otimes b_1 + b_0 \otimes b_1 \otimes 1,
\end{align*}
\]

etc.

**Lemma 16.3.** If \( A \to B \) is faithfully flat, then the complex \( (B/A)_\bullet \) is exact in positive degrees, and \( H^0((B/A)_\bullet) = A \).

**Proof.** See Descent, Lemma 3.6. \( \square \)

Grothendieck proves this in three steps. Firstly, he assumes that the map \( A \to B \) has a section, and constructs an explicit homotopy to the complex where \( A \) is the only nonzero term, in degree 0. Secondly, he observes that to prove the result,
it suffices to do so after a faithfully flat base change \( A \to A' \), replacing \( B \) with \( B' = B \otimes_A A' \). Thirdly, he applies the faithfully flat base change \( A \to A' \), remark that the map \( A' = B \to B' = B \otimes_A B \) has a natural section.

The same strategy proves the following lemma.

**Lemma 16.4.** If \( A \to B \) is faithfully flat and \( M \) is an \( A \)-module, then the complex \((B/A)_\bullet \otimes_A M\) is exact in positive degrees, and \( H^0((B/A)_\bullet \otimes_A M) = M \).

**Proof.** See Descent, Lemma 3.6. \( \square \)

**Definition 16.5.** Let \( A \to B \) be a ring map and \( N \) a \( B \)-module. A descent datum for \( N \) with respect to \( A \to B \) is an isomorphism \( \varphi : N \otimes_A B \cong B \otimes_A N \) of \( B \otimes_A B \)-modules such that the diagram of \( B \otimes_A B \otimes_A B \)-modules

\[
\begin{array}{ccc}
N \otimes_A B & \xrightarrow{\varphi_{02}} & B \otimes_A N \otimes_A B \\
\downarrow{\varphi_{01}} & & \downarrow{\varphi_{12}} \\
B \otimes_A B & \otimes_A N & \end{array}
\]

commutes.

If \( N' = B \otimes_A M \) for some \( A \)-module \( M \), then it has a canonical descent datum given by the map

\[
\varphi_{\text{can}} : N' \otimes_A B \to B \otimes_A N' \\
b_0 \otimes m \otimes b_1 \mapsto b_0 \otimes b_1 \otimes m.
\]

**Definition 16.6.** A descent datum \((N, \varphi)\) is called effective if there exists an \( A \)-module \( M \) such that \((N, \varphi) \cong (B \otimes_A M, \varphi_{\text{can}})\), with the obvious notion of isomorphism of descent data.

Theorem 16.2 is a consequence the following result.

**Theorem 16.7.** If \( A \to B \) is faithfully flat then descent data with respect to \( A \to B \) are effective.

**Proof.** See Descent, Proposition 3.9. See also Descent, Remark 3.11 for an alternative view of the proof. \( \square \)

**Remarks 16.8.** The results on descent of modules have several applications:

1. The exactness of the Čech complex in positive degrees for the covering \( \{\text{Spec}(B) \to \text{Spec}(A)\} \) where \( A \to B \) is faithfully flat. This will give some vanishing of cohomology.
2. If \((N, \varphi)\) is a descent datum with respect to a faithfully flat map \( A \to B \), then the corresponding \( A \)-module is given by

\[
M = \ker \left( \begin{array}{c}
N \\
\downarrow{n} \\
\end{array} \right. \begin{array}{c}
\to \\
\mapsto \\
\end{array} \begin{array}{c}
B \otimes_A N \\
1 \otimes n - \varphi(n \otimes 1) \\
\end{array} \right).
\]

See Descent, Proposition 3.9.
17. Quasi-coherent sheaves

We can apply the descent of modules to study quasi-coherent sheaves.

**Proposition 17.1.** For any quasi-coherent sheaf $\mathcal{F}$ on $S$ the presheaf

$$
\mathcal{F}^a : \text{Sch}/S \to \text{Ab} \quad (f : T \to S) \mapsto \Gamma(T, f^* \mathcal{F})
$$

is an $\mathcal{O}$-module which satisfies the sheaf condition for the fpqc topology.

**Proof.** This is proved in Descent, Lemma 7.1. We indicate the proof here. As established in Lemma 15.6, it is enough to check the sheaf property on Zariski coverings and faithfully flat morphisms of affine schemes. The sheaf property for Zariski coverings is standard scheme theory, since $\Gamma(U, i^* \mathcal{F}) = \mathcal{F}(U)$ when $i : U \hookrightarrow S$ is an open immersion.

For $\{\text{Spec}(B) \to \text{Spec}(A)\}$ with $A \to B$ faithfully flat and $\mathcal{F}|_{\text{Spec}(A)} = \tilde{M}$ this corresponds to the fact that $M = H^0((B/A)\bullet \otimes_A M)$, i.e., that

$$
0 \to M \to B \otimes_A M \to B \otimes_A B \otimes_A M
$$

is exact by Lemma 16.4.

There is an abstract notion of a quasi-coherent sheaf on a ringed site. We briefly introduce this here. For more information please consult Modules on Sites, Section 23. Let $\mathcal{C}$ be a category, and let $U$ be an object of $\mathcal{C}$. Then $\mathcal{C}/U$ indicates the category of objects over $U$, see Categories, Example 2.13. If $\mathcal{C}$ is a site, then $\mathcal{C}/U$ is a site as well, namely the coverings of $V/U$ are families $\{V_i/U \to V/U\}$ of morphisms of $\mathcal{C}/U$ with fixed target such that $\{V_i \to V\}$ is a covering of $\mathcal{C}$. Moreover, given any sheaf $\mathcal{F}$ on $\mathcal{C}$ the restriction $\mathcal{F}|_{\mathcal{C}/U}$ (defined in the obvious manner) is a sheaf as well. See Sites, Section 24 for details.

**Definition 17.2.** Let $\mathcal{C}$ be a ringed site, i.e., a site endowed with a sheaf of rings $\mathcal{O}$. A sheaf of $\mathcal{O}$-modules $\mathcal{F}$ on $\mathcal{C}$ is called quasi-coherent if for all $U \in \text{Ob}(\mathcal{C})$ there exists a covering $\{U_i \to U\}_{i \in I}$ of $\mathcal{C}$ such that the restriction $\mathcal{F}|_{\mathcal{C}/U_i}$ is isomorphic to the cokernel of an $\mathcal{O}$-linear map of free $\mathcal{O}$-modules

$$
\bigoplus_{k \in K} \mathcal{O}|_{\mathcal{C}/U_i} \longrightarrow \bigoplus_{l \in L} \mathcal{O}|_{\mathcal{C}/U_i}.
$$

The direct sum over $K$ is the sheaf associated to the presheaf $V \mapsto \bigoplus_{k \in K} \mathcal{O}(V)$ and similarly for the other.

Although it is useful to be able to give a general definition as above this notion is not well behaved in general.

**Remark 17.3.** In the case where $\mathcal{C}$ has a final object, e.g., $S$, it suffices to check the condition of the definition for $U = S$ in the above statement. See Modules on Sites, Lemma 23.3.

**Theorem 17.4** (Meta theorem on quasi-coherent sheaves). Let $S$ be a scheme. Let $\mathcal{C}$ be a site. Assume that

1. the underlying category $\mathcal{C}$ is a full subcategory of $\text{Sch}/S$,
2. any Zariski covering of $T \in \text{Ob}(\mathcal{C})$ can be refined by a covering of $\mathcal{C}$,
3. $S/S$ is an object of $\mathcal{C}$,
4. every covering of $\mathcal{C}$ is an fpqc covering of schemes.
Then the presheaf $\mathcal{O}$ is a sheaf on $\mathcal{C}$ and any quasi-coherent $\mathcal{O}$-module on $(\mathcal{C}, \mathcal{O})$ is of the form $\mathcal{F}^a$ for some quasi-coherent sheaf $\mathcal{F}$ on $\mathcal{S}$.

**Proof.** After some formal arguments this is exactly Theorem 16.2. Details omitted. In Descent, Proposition 7.11 we prove a more precise version of the theorem for the big Zariski, fppf, étale, smooth, and syntomic sites of $\mathcal{S}$, as well as the small Zariski and étale sites of $\mathcal{S}$. □

In other words, there is no difference between quasi-coherent modules on the scheme $\mathcal{S}$ and quasi-coherent $\mathcal{O}$-modules on sites $\mathcal{C}$ as in the theorem. More precise statements for the big and small sites $(\text{Sch}/\mathcal{S})_{\text{fppf}}, \mathcal{S}_{\text{étale}},$ etc can be found in Descent, Section 7. In this chapter we will sometimes refer to a “site as in Theorem 17.4” in order to conveniently state results which hold in any of those situations.

18. Čech cohomology

Our next goal is to use descent theory to show that $H^i(\mathcal{C}, \mathcal{F}^a) = H^i_{\text{ét}}(\mathcal{S}, \mathcal{F})$ for all quasi-coherent sheaves $\mathcal{F}$ on $\mathcal{S}$, and any site $\mathcal{C}$ as in Theorem 17.4. To this end, we introduce Čech cohomology on sites. See [Art62] and Cohomology on Sites, Sections 9, 10 and 11 for more details.

**Definition 18.1.** Let $\mathcal{C}$ be a category, $\mathcal{U} = \{U_i \rightarrow U\}_{i \in I}$ a family of morphisms of $\mathcal{C}$ with fixed target, and $\mathcal{F} \in \text{PAb}(\mathcal{C})$ an abelian presheaf. We define the Čech complex $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})$ by

$$\prod_{i_0 \in I} \mathcal{F}(U_{i_0}) \rightarrow \prod_{i_0, i_1 \in I} \mathcal{F}(U_{i_0} \times_U U_{i_1}) \rightarrow \prod_{i_0, i_1, i_2 \in I} \mathcal{F}(U_{i_0} \times_U U_{i_1} \times_U U_{i_2}) \rightarrow \ldots$$

where the first term is in degree 0, and the maps are the usual ones. Again, it is essential to allow the case $i_0 = i_1$ etc. The Čech cohomology groups are defined by

$$\check{H}^p(\mathcal{U}, \mathcal{F}) = H^p(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})).$$

**Lemma 18.2.** The functor $\check{\mathcal{C}}^\bullet(\mathcal{U}, -)$ is exact on the category $\text{PAb}(\mathcal{C})$.

In other words, if $0 \rightarrow \mathcal{F}_1 \rightarrow \mathcal{F}_2 \rightarrow \mathcal{F}_3 \rightarrow 0$ is a short exact sequence of presheaves of abelian groups, then

$$0 \rightarrow \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}_1) \rightarrow \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}_2) \rightarrow \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}_3) \rightarrow 0$$

is a short exact sequence of complexes.

**Proof.** This follows at once from the definition of a short exact sequence of presheaves. Namely, as the category of abelian presheaves is the category of functors on some category with values in $\text{Ab}$, it is automatically an abelian category: a sequence $\mathcal{F}_1 \rightarrow \mathcal{F}_2 \rightarrow \mathcal{F}_3$ is exact in $\text{PAb}$ if and only if for all $U \in \text{Ob}(\mathcal{C})$, the sequence $\mathcal{F}_1(U) \rightarrow \mathcal{F}_2(U) \rightarrow \mathcal{F}_3(U)$ is exact in $\text{Ab}$. So the complex above is merely a product of short exact sequences in each degree. See also Cohomology on Sites, Lemma 10.1.

This shows that $H^\bullet(\mathcal{U}, -)$ is a $\delta$-functor. We now proceed to show that it is a universal $\delta$-functor. We thus need to show that it is an effaceable functor. We start by recalling the Yoneda lemma.
Let $\mathcal{C}$ be a category. Given a presheaf of sets $\mathcal{G}$, we define the **free abelian presheaf** on $\mathcal{G}$, denoted $\mathcal{Z}_\mathcal{G}$, by the rule

$$\mathcal{Z}_\mathcal{G}(U) = \mathcal{Z}[\mathcal{G}(U)]$$

for $U \in \text{Ob}(\mathcal{C})$ with restriction maps induced by the restriction maps of $\mathcal{G}$. In the special case $\mathcal{G} = h_U$, we write simply $\mathcal{Z}_U = \mathcal{Z}_{h_U}$.

The functor $\mathcal{G} \mapsto \mathcal{Z}_\mathcal{G}$ is left adjoint to the forgetful functor $\text{PAb}(\mathcal{C}) \to \text{PSh}(\mathcal{C})$. Thus, for any presheaf $\mathcal{F}$, there is a canonical isomorphism

$$\text{Hom}_{\text{PAb}(\mathcal{C})}(\mathcal{Z}_U, \mathcal{F}) = \text{Hom}_{\text{PSh}(\mathcal{C})}(h_U, \mathcal{F}) = \mathcal{F}(U)$$

the last equality by the Yoneda lemma. In particular, we have the following result.

This follows from the formula above. See Cohomology on Sites, Lemma 10.3.

This reduces us to studying only the complex in the first argument of the last $\text{Hom}$.

The complex of abelian presheaves

$$\mathcal{Z}_{\mathcal{U}}^\bullet : \bigoplus_{i_0 \in I} \mathcal{Z}_{U_{i_0}} \leftarrow \bigoplus_{i_0, i_1 \in I} \mathcal{Z}_{U_{i_0} \times_U U_{i_1}} \leftarrow \bigoplus_{i_0, i_1, i_2 \in I} \mathcal{Z}_{U_{i_0} \times_U U_{i_1} \times_U U_{i_2}} \leftarrow \ldots$$

is exact in all degrees except 0 in $\text{PAb}(\mathcal{C})$.

For any $V \in \text{Ob}(\mathcal{C})$ the complex of abelian groups $\mathcal{Z}_{\mathcal{V}}^\bullet(V)$ is

$$\mathcal{Z}_{\mathcal{V}}^\bullet(V) = \bigoplus_{\varphi : V \to U} (\mathcal{Z}[\mathcal{S}_\varphi] \leftarrow \mathcal{Z}[\mathcal{S}_\varphi \times \mathcal{S}_\varphi] \leftarrow \mathcal{Z}[\mathcal{S}_\varphi \times \mathcal{S}_\varphi \times \mathcal{S}_\varphi] \leftarrow \ldots).$$

Using this notation we introduce the free abelian presheaf on a presheaf of sets.

This follows from the formula above. See Cohomology on Sites, Lemma 10.3.

This reduces us to studying only the complex in the first argument of the last $\text{Hom}$.

The complex of abelian presheaves

$$\mathcal{Z}_{\mathcal{U}}^\bullet : \bigoplus_{i_0 \in I} \mathcal{Z}_{U_{i_0}} \leftarrow \bigoplus_{i_0, i_1 \in I} \mathcal{Z}_{U_{i_0} \times_U U_{i_1}} \leftarrow \bigoplus_{i_0, i_1, i_2 \in I} \mathcal{Z}_{U_{i_0} \times_U U_{i_1} \times_U U_{i_2}} \leftarrow \ldots$$

is exact in all degrees except 0 in $\text{PAb}(\mathcal{C})$.

For any $V \in \text{Ob}(\mathcal{C})$ the complex of abelian groups $\mathcal{Z}_{\mathcal{V}}^\bullet(V)$ is

$$\mathcal{Z}_{\mathcal{V}}^\bullet(V) = \bigoplus_{\varphi : V \to U} (\mathcal{Z}[\mathcal{S}_\varphi] \leftarrow \mathcal{Z}[\mathcal{S}_\varphi \times \mathcal{S}_\varphi] \leftarrow \mathcal{Z}[\mathcal{S}_\varphi \times \mathcal{S}_\varphi \times \mathcal{S}_\varphi] \leftarrow \ldots).$$
Thus it suffices to show that for each \( S = S_\varphi \), the complex

\[
\mathbb{Z}[S] \leftarrow \mathbb{Z}[S \times S] \leftarrow \mathbb{Z}[S \times S \times S] \leftarrow \ldots
\]

is exact in negative degrees. To see this, we can give an explicit homotopy. Fix \( s \in S \) and define \( K : \eta_{(s_0, \ldots, s_p)} \mapsto \eta_{(s_0, \ldots, s_p)} \). One easily checks that \( K \) is a nullhomotopy for the operator

\[
\delta : \eta_{(s_0, \ldots, s_p)} \mapsto \sum_{i=0}^{p} (-1)^i \eta_{(s_0, \ldots, \hat{s}_i, \ldots, s_p)}.
\]

See Cohomology on Sites, Lemma 10.4 for more details.

\[\square\]

**Lemma 18.7.** Let \( C \) be a category. If \( I \) is an injective object of \( \mathcal{P}Ab(C) \) and \( \mathcal{U} \) is a family of morphisms with fixed target in \( C \), then \( \check{H}^p(\mathcal{U}, I) = 0 \) for all \( p > 0 \).

**Proof.** The Čech complex is the result of applying the functor \( \text{Hom}_{\mathcal{P}Ab(C)}(-, I) \) to the complex \( \mathbb{Z}^*_\mathcal{U} \), i.e.,

\[
\check{H}^p(\mathcal{U}, I) = H^p(\text{Hom}_{\mathcal{P}Ab(C)}(\mathbb{Z}^*_\mathcal{U}, I)).
\]

But we have just seen that \( \mathbb{Z}^*_\mathcal{U} \) is exact in negative degrees, and the functor \( \text{Hom}_{\mathcal{P}Ab(C)}(-, I) \) is exact, hence \( \check{H}^p(\mathcal{U}, I) \) is exact in positive degrees.

\[\square\]

**Theorem 18.8.** On \( \mathcal{P}Ab(C) \) the functors \( \check{H}^p(\mathcal{U}, -) \) are the right derived functors of \( \check{H}^0(\mathcal{U}, -) \).

**Proof.** By the Lemma 18.7, the functors \( \check{H}^p(\mathcal{U}, -) \) are universal \( \delta \)-functors since they are effaceable. So are the right derived functors of \( \check{H}^0(\mathcal{U}, -) \). Since they agree in degree 0, they agree by the universal property of universal \( \delta \)-functors. For more details see Cohomology on Sites, Lemma 10.6.

\[\square\]

**Remark 18.9.** Observe that all of the preceding statements are about presheaves so we haven’t made use of the topology yet.

### 19. The Čech-to-cohomology spectral sequence

This spectral sequence is fundamental in proving foundational results on cohomology of sheaves.

**Lemma 19.1.** The forgetful functor \( \text{Ab}(C) \to \mathcal{P}Ab(C) \) transforms injectives into injectives.

**Proof.** This is formal using the fact that the forgetful functor has a left adjoint, namely sheafification, which is an exact functor. For more details see Cohomology on Sites, Lemma 11.1.

**Theorem 19.2.** Let \( C \) be a site. For any covering \( \mathcal{U} = \{U_i \to U\}_{i \in I} \) of \( U \in \text{Ob}(C) \) and any abelian sheaf \( F \) on \( C \) there is a spectral sequence

\[
E^p_q = \check{H}^p(\mathcal{U}, \check{H}^q(F)) \Rightarrow H^{p+q}(U, F),
\]

where \( \check{H}^q(F) \) is the abelian presheaf \( V \mapsto H^q(V, F) \).
Proof. Choose an injective resolution $\mathcal{F} \to \mathcal{I}^\bullet$ in $\text{Ab}(\mathcal{C})$, and consider the double complex $\check{C}^\bullet(\mathcal{U}, I^\bullet)$ and the maps

$$
\begin{array}{c}
\Gamma(U, I^\bullet) \\
\uparrow \\
\check{C}^\bullet(\mathcal{U}, \mathcal{I})
\end{array}
$$

Here the horizontal map is the natural map $\Gamma(U, I^\bullet) \to \check{C}^0(U, I^\bullet)$ to the left column, and the vertical map is induced by $\mathcal{F} \to \mathcal{I}^0$ and lands in the bottom row. By assumption, $\mathcal{I}^\bullet$ is a complex of injectives in $\text{Ab}(\mathcal{C})$, hence by Lemma 19.1, it is a complex of injectives in $\text{PAb}(\mathcal{C})$. Thus, the rows of the double complex are exact in positive degrees (Lemma 18.7), and the kernel of the horizontal map is equal to $\Gamma(U, I^\bullet)$, since $\mathcal{I}^\bullet$ is a complex of sheaves. In particular, the cohomology of the total complex is the standard cohomology of the global sections functor $H^0(U, \mathcal{F})$.

For the vertical direction, the $q$th cohomology group of the $p$th column is

$$
\prod_{i_0, \ldots, i_p} H^q(U_{i_0} \times U \ldots \times U_{i_p}, \mathcal{F}) = \prod_{i_0, \ldots, i_p} H^q(\mathcal{F})(U_{i_0} \times U \ldots \times U_{i_p})
$$

in the entry $E_1^{p,q}$. So this is a standard double complex spectral sequence, and the $E_2$-page is as prescribed. For more details see Cohomology on Sites, Lemma 11.6. □

Remark 19.3. This is a Grothendieck spectral sequence for the composition of functors

$$
\text{Ab}(\mathcal{C}) \longrightarrow \text{PAb}(\mathcal{C}) \longrightarrow \text{Ab}.
$$

20. Big and small sites of schemes

Let $S$ be a scheme. Let $\tau$ be one of the topologies we will be discussing. Thus $\tau \in \{\text{fppf}, \text{syntomic}, \text{smooth}, \text{étale}, \text{Zariski}\}$. Of course if you are only interested in the étale topology, then you can simply assume $\tau = \text{étale}$ throughout. Moreover, we will discuss étale morphisms, étale coverings, and étale sites in more detail starting in Section 25. In order to proceed with the discussion of cohomology of quasi-coherent sheaves it is convenient to introduce the big $\tau$-site and in case $\tau \in \{\text{étale}, \text{Zariski}\}$, the small $\tau$-site of $S$. In order to do this we first introduce the notion of a $\tau$-covering.

Definition 20.1. (See Topologies, Definitions 7.1, 6.1, 5.1, 4.1, and 3.1) Let $\tau \in \{\text{fppf}, \text{syntomic}, \text{smooth}, \text{étale}, \text{Zariski}\}$. A family of morphisms of schemes $\{f_i : T_i \to T\}_{i \in I}$ with fixed target is called a $\tau$-covering if and only if each $f_i$ is flat of finite presentation, syntomic, smooth, étale, resp. an open immersion, and we have $\bigcup f_i(T_i) = T$.

It turns out that the class of all $\tau$-coverings satisfies the axioms (1), (2) and (3) of Definition 10.2 (our definition of a site), see Topologies, Lemmas 7.3, 6.3, 5.3, 4.3 and 3.2. In order to be able to compare any of these new topologies to the fppf topology and to use the preceding results on descent on modules we single out a special class of $\tau$-coverings of affine schemes called standard coverings.
Let \( S \) (see Topologies, Definitions 7.5, 6.5, 5.5, 4.5, and 3.4.) Let \( T \) be an affine scheme. A standard \( \tau \)-covering of \( T \) is a family \( \{ f_j : U_j \to T \}_{j=1, \ldots, m} \) with each \( U_j \) is affine, and each \( f_j \) flat and of finite presentation, standard syntomic, standard smooth, étale, resp. the immersion of a standard principal open in \( T \) and \( T' = \bigcup f_j(U_j) \).

**Lemma 20.3.** Let \( \tau \in \{ \text{fppf, syntomic, smooth, étale, Zariski} \} \). Any \( \tau \)-covering of an affine scheme can be refined by a standard \( \tau \)-covering.

**Proof.** See Topologies, Lemmas 7.4, 6.4, 5.4, 4.4, and 3.3 \( \square \)

Finally, we come to our definition of the sites we will be working with. This is actually somewhat involved since, contrary to what happens in [AGV71], we do not want to choose a universe. Instead we pick a “partial universe” (which is a suitably large set as in Sets, Section 5), and consider all schemes contained in this set. Of course we make sure that our favorite base scheme \( S \) is contained in the partial universe. Having picked the underlying category we pick a suitably large set of \( \tau \)-coverings which turns this into a site. The details are in the chapter on topologies on schemes; there is a lot of freedom in the choices made, but in the end the actual choices made will not affect the étale (or other) cohomology of \( S \) (just as in [AGV71] the actual choice of universe doesn’t matter at the end). Moreover, the way the material is written the reader who is happy using strongly inaccessible cardinals (i.e., universes) can do so as a substitute.

**Definition 20.4.** Let \( S \) be a scheme. Let \( \tau \in \{ \text{fppf, syntomic, smooth, étale, Zariski} \} \).

1. A big \( \tau \)-site of \( S \) is any of the sites \( (\text{Sch}/S)_{\tau} \) constructed as explained above and in more detail in Topologies, Definitions 7.8, 6.8, 5.8, 4.8, and 3.7.
2. If \( \tau \in \{ \text{étale, Zariski} \} \), then the small \( \tau \)-site of \( S \) is the full subcategory \( S_{\tau} \) of \( (\text{Sch}/S)_{\tau} \) whose objects are schemes \( T \) over \( S \) whose structure morphism \( T \to S \) is étale, resp. an open immersion. A covering in \( S_{\tau} \) is a covering \( \{ U_j \to U \} \) in \( (\text{Sch}/S)_{\tau} \) such that \( U \) is an object of \( S_{\tau} \).

The underlying category of the site \( (\text{Sch}/S)_{\tau} \) has reasonable “closure” properties, i.e., given a scheme \( T \) in it any locally closed subscheme of \( T \) is isomorphic to an object of \( (\text{Sch}/S)_{\tau} \). Other such closure properties are: closed under fibre products of schemes, taking countable disjoint unions, taking finite type schemes over a given scheme, given an affine scheme \( \text{Spec}(R) \) one can complete, localize, or take the quotient of \( R \) by an ideal while staying inside the category, etc. On the other hand, for example arbitrary disjoint unions of schemes in \( (\text{Sch}/S)_{\tau} \) will take you outside of it. Also note that, given an object \( T \) of \( (\text{Sch}/S)_{\tau} \), there will exist \( \tau \)-coverings \( \{ T_i \to T \}_{i \in I} \) (as in Definition 20.1) which are not coverings in \( (\text{Sch}/S)_{\tau} \) for example because the schemes \( T_i \) are not objects of the category \( (\text{Sch}/S)_{\tau} \). But our choice of the sites \( (\text{Sch}/S)_{\tau} \) is such that there always does exist a covering \( \{ U_j \to T \}_{j \in J} \) of \( (\text{Sch}/S)_{\tau} \) which refines the covering \( \{ T_i \to T \}_{i \in I} \), see Topologies, Lemmas 7.7, 6.7, 5.7, 4.7, and 3.6. We will mostly ignore these issues in this chapter.

If \( \mathcal{F} \) is a sheaf on \( (\text{Sch}/S)_{\tau} \) or \( S_{\tau} \), then we denote

\[
H^p_{\tau}(U, \mathcal{F}), \text{ in particular } H^p_{\tau}(S, \mathcal{F})
\]

the cohomology groups of \( \mathcal{F} \) over the object \( U \) of the site, see Section 13. Thus we have \( H^p_{\text{fppf}}(S, \mathcal{F}) \), \( H^p_{\text{syntomic}}(S, \mathcal{F}) \), \( H^p_{\text{smooth}}(S, \mathcal{F}) \), \( H^p_{\text{étale}}(S, \mathcal{F}) \), and \( H^p_{\text{Zar}}(S, \mathcal{F}) \).
The last two are potentially ambiguous since they might refer to either the big or small étale or Zariski site. However, this ambiguity is harmless by the following lemma.

**Lemma 20.5.** Let $\tau \in \{\text{étale, Zariski}\}$. If $\mathcal{F}$ is an abelian sheaf defined on $(\text{Sch}/S)_{\tau}$, then the cohomology groups of $\mathcal{F}$ over $S$ agree with the cohomology groups of $\mathcal{F}|_{S_{\tau}}$ over $S$.

**Proof.** By Topologies, Lemmas 3.13 and 4.13 the functors $S_{\tau} \to (\text{Sch}/S)_{\tau}$ satisfy the hypotheses of Sites, Lemma 20.8. Hence our lemma follows from Cohomology on Sites, Lemma 8.2. $\square$

For completeness we state and prove the invariance under choice of partial universe of the cohomology groups we are considering. We will prove invariance of the small étale topos in Lemma 21.3 below. For notation and terminology used in this lemma we refer to Topologies, Section 10.

**Lemma 20.6.** Let $\tau \in \{\text{fppf, syntomic, smooth, étale, Zariski}\}$. Let $S$ be a scheme. Let $(\text{Sch}/S)_{\tau}$ and $(\text{Sch}'/S)_{\tau}$ be two big $\tau$-sites of $S$, and assume that the first is contained in the second. In this case

1. for any abelian sheaf $\mathcal{F}'$ defined on $(\text{Sch}'/S)_{\tau}$ and any object $U$ of $(\text{Sch}/S)_{\tau}$ we have
   $$H^p_{\tau}(U, \mathcal{F}'|_{(\text{Sch}/S)_{\tau}}) = H^p_{\tau}(U, \mathcal{F}')$$
   In words: the cohomology of $\mathcal{F}'$ over $U$ computed in the bigger site agrees with the cohomology of $\mathcal{F}'$ restricted to the smaller site over $U$.

2. for any abelian sheaf $\mathcal{F}$ on $(\text{Sch}/S)_{\tau}$ there is an abelian sheaf $\mathcal{F}'$ on $(\text{Sch}'/S)_{\tau}$ whose restriction to $(\text{Sch}/S)_{\tau}$ is isomorphic to $\mathcal{F}$.

**Proof.** By Topologies, Lemma 10.2 the inclusion functor $(\text{Sch}/S)_{\tau} \to (\text{Sch}'/S)_{\tau}$ satisfies the assumptions of Sites, Lemma 20.8. This implies (2) and (1) follows from Cohomology on Sites, Lemma 8.2. $\square$

21. The étale topos

A *topos* is the category of sheaves of sets on a site, see Sites, Definition 16.1. Hence it is customary to refer to the use the phrase “étale topos of a scheme” to refer to the category of sheaves on the small étale site of a scheme. Here is the formal definition.

**Definition 21.1.** Let $S$ be a scheme.

1. The *étale topos*, or the *small étale topos* of $S$ is the category $\text{Sh}(S_{\text{étale}})$ of sheaves of sets on the small étale site of $S$.

2. The *Zariski topos*, or the *small Zariski topos* of $S$ is the category $\text{Sh}(S_{\text{Zar}})$ of sheaves of sets on the small Zariski site of $S$.

3. For $\tau \in \{\text{fppf, syntomic, smooth, étale, Zariski}\}$ a *big $\tau$-topos* is the category of sheaves of set on a big $\tau$-topos of $S$.

Note that the small Zariski topos of $S$ is simply the category of sheaves of sets on the underlying topological space of $S$, see Topologies, Lemma 8.1.1. Whereas the small étale topos does not depend on the choices made in the construction of the small étale site, in general the big topos do depend on those choices.
Here is a lemma, which is one of many possible lemmas expressing the fact that it doesn’t matter too much which site we choose to define the small étale topos of a scheme.

**Lemma 21.2.** Let $S$ be a scheme. Let $S_{affine, étale}$ denote the full subcategory of $S_{étale}$ whose objects are those $U/S \in \text{Ob}(S_{étale})$ with $U$ affine. A covering of $S_{affine, étale}$ will be a standard étale covering, see Topologies, Definition 4.7. Then restriction

$$\mathcal{F} \mapsto \mathcal{F}|_{S_{affine, étale}}$$

defines an equivalence of topoi $\text{Sh}(S_{étale}) \cong \text{Sh}(S_{affine, étale})$.

**Proof.** This you can show directly from the definitions, and is a good exercise. But it also follows immediately from Sites, Lemma 28.1 by checking that the inclusion functor $S_{affine, étale} \to S_{étale}$ is a special cocontinuous functor (see Sites, Definition 28.2).

**Lemma 21.3.** Let $S$ be a scheme. The étale topos of $S$ is independent (up to canonical equivalence) of the construction of the small étale site in Definition 20.4.

**Proof.** We have to show, given two big étale sites $\text{Sch}_{étale}$ and $\text{Sch}'_{étale}$ containing $S$, then $\text{Sh}(S_{étale}) \cong \text{Sh}(S'_{étale})$ with obvious notation. By Topologies, Lemma 10.1 we may assume $\text{Sch}_{étale} \subset \text{Sch}'_{étale}$. By Sets, Lemma 9.9 any affine scheme étale over $S$ is isomorphic to an object of both $\text{Sch}_{étale}$ and $\text{Sch}'_{étale}$. Thus the induced functor $S_{affine, étale} \to S'_{affine, étale}$ is an equivalence. Moreover, it is clear that both this functor and a quasi-inverse map transform standard étale coverings into standard étale coverings. Hence the result follows from Lemma 21.2.

22. Cohomology of quasi-coherent sheaves

We start with a simple lemma (which holds in greater generality than stated). It says that the Čech complex of a standard covering is equal to the Čech complex of an fpqc covering of the form $\{\text{Spec}(B) \to \text{Spec}(A)\}$ with $A \to B$ faithfully flat.

**Lemma 22.1.** Let $\tau \in \{\text{fppf, syntomic, smooth, étale, Zariski}\}$. Let $S$ be a scheme. Let $\mathcal{F}$ be an abelian sheaf on $(\text{Sch}/S)_\tau$, or on $S_\tau$ in case $\tau = \text{étale}$, and let $U = \{U_i \to U\}_{i \in I}$ be a standard $\tau$-covering of this site. Let $V = \coprod_{i \in I} U_i$. Then

1. $V$ is an affine scheme,
2. $\mathcal{V} = \{V \to U\}$ is a $\tau$-covering and an fpqc covering,
3. the Čech complexes $\check{C}^\bullet(\mathcal{U}, \mathcal{F})$ and $\check{C}^\bullet(\mathcal{V}, \mathcal{F})$ agree.

**Proof.** As the covering is a standard $\tau$-covering each of the schemes $U_i$ is affine and $I$ is a finite set. Hence $V$ is an affine scheme. It is clear that $V \to U$ is flat and surjective, hence $\mathcal{V}$ is an fpqc covering, see Example 15.3. Note that $U$ is a refinement of $\mathcal{V}$ and hence there is a map of Čech complexes $\check{C}^\bullet(\mathcal{V}, \mathcal{F}) \to \check{C}^\bullet(\mathcal{U}, \mathcal{F})$, see Cohomology on Sites, Equation (9.2.1). Next, we observe that if $T = \coprod_{j \in J} T_j$ is a disjoint union of schemes in the site on which $\mathcal{F}$ is defined then the family of morphisms with fixed target $\{T_j \to T\}_{j \in J}$ is a Zariski covering, and so

$$\mathcal{F}(T) = \mathcal{F} \left( \coprod_{j \in J} T_j \right) = \coprod_{j \in J} \mathcal{F}(T_j)$$
by the sheaf condition of $\mathcal{F}$. This implies the map of Čech complexes above is an isomorphism in each degree because
\[
V \times_U \cdots \times_U V = \coprod_{i_0, \ldots, i_p} U_{i_0} \times_U \cdots \times_U U_{i_p}
\]
as schemes.

Note that Equality \((22.1.1)\) is false for a general presheaf. Even for sheaves it does not hold on any site, since coproducts may not lead to coverings, and may not be disjoint. But it does for all the usual ones (at least all the ones we will study).

\begin{remark}{\textbf{Remark 22.2.}}\end{remark}
In the statement of Lemma \ref{lemma:covering} the covering $U$ is a refinement of $V$ but not the other way around. Coverings of the form \{$V \to U\}$ do not form an initial subcategory of the category of all coverings of $U$. Yet it is still true that we can compute Čech cohomology $H^n(U, \mathcal{F})$ (which is defined as the colimit over the opposite of the category of coverings $U$ of $U$ of the Čech cohomology groups of $\mathcal{F}$ with respect to $U$) in terms of the coverings \{$V \to U\}. We will formulate a precise lemma (it only works for sheaves) and add it here if we ever need it.

\begin{lemma}{\textbf{Lemma 22.3 (Locality of cohomology).}}\end{lemma}
Let $\mathcal{C}$ be a site, $\mathcal{F}$ an abelian sheaf on $\mathcal{C}$, $U$ an object of $\mathcal{C}$, $p > 0$ an integer and $\xi \in H^p(U, \mathcal{F})$. Then there exists a covering $U = \{U_i \to U\}_{i \in I}$ of $U$ in $\mathcal{C}$ such that $\xi|_{U_i} = 0$ for all $i \in I$.

\begin{proof}
Choose an injective resolution $\mathcal{I} \to \mathcal{F}$. Then $\xi$ is represented by a cocycle $\xi \in \mathcal{I}^p(U)$ with $d^p(\xi) = 0$. By assumption, the sequence $\mathcal{I}^{p-1} \to \mathcal{I}^p \to \mathcal{I}^{p+1}$ is exact in $\text{Ab}(\mathcal{C})$, which means that there exists a covering $U = \{U_i \to U\}_{i \in I}$ such that $\xi|_{U_i} = d^p-1(\xi_i)$ for some $\xi_i \in \mathcal{I}^{p-1}(U_i)$. Since the cohomology class $\xi|_{U_i}$ is represented by the cocycle $\xi|_{U_i}$ which is a coboundary, it vanishes. For more details see Cohomology on Sites, Lemma 8.3. \qed
\end{proof}

\begin{theorem}{\textbf{Theorem 22.4.}}\end{theorem}
Let $S$ be a scheme and $\mathcal{F}$ a quasi-coherent $\mathcal{O}_S$-module. Let $\mathcal{C}$ be either $(\text{Sch}/S)_\tau$ for $\tau \in \{\text{fpqc, syntomic, smooth, étale, Zariski}\}$ or $S_{\text{êtale}}$. Then
\[
H^p(S, \mathcal{F}) = H^p_S(S, \mathcal{F}^a)
\]
for all $p \geq 0$ where
\begin{enumerate}
\item the left hand side indicates the usual cohomology of the sheaf $\mathcal{F}$ on the underlying topological space of the scheme $S$, and
\item the right hand side indicates cohomology of the abelian sheaf $\mathcal{F}^a$ (see Proposition \ref{prop:abelianization}) on the site $\mathcal{C}$.
\end{enumerate}

\begin{proof}
We are going to show that $H^p(U, f^*\mathcal{F}) = H^p_S(U, \mathcal{F}^a)$ for any object $f : U \to S$ of the site $\mathcal{C}$. The result is true for $p = 0$ by the sheaf property.

Assume that $U$ is affine. Then we want to prove that $H^p_S(U, \mathcal{F}^a) = 0$ for all $p > 0$. We use induction on $p$.

\begin{enumerate}
\item $p = 1$ Pick $\xi \in H^1_U(U, \mathcal{F}^a)$. By Lemma \ref{lemma:covering} there exists an fpqc covering $U = \{U_i \to U\}_{i \in I}$ such that $\xi|_{U_i} = 0$ for all $i \in I$. Up to refining $U$, we may assume that $U$ is a standard $\tau$-covering. Applying the spectral sequence of Theorem \ref{thm:spectral_sequence}, we see that $\xi$ comes from a cohomology class $\xi \in H^1(U, \mathcal{F}^a)$. Consider the covering $V = \coprod_{i \in I} U_i \to U$. By Lemma \ref{lemma:covering} $H^p(V, \mathcal{F}^a) = H^p(U, \mathcal{F}^a)$. On the other hand, since $V$ is a covering of the form $\{\text{Spec}(B) \to \text{Spec}(A)\}$ and $f^*\mathcal{F} = \mathcal{M}$ for some $A$-module $M$, we see the Čech complex $\check{C}^\bullet(V, \mathcal{F})$ is none other than the complex $H^p(U, \mathcal{F}^a)$.
\end{enumerate}
Now by Lemma 16.4, $H^p(B/A, \otimes_A M) = 0$ for $p > 0$, hence $\tilde{\chi} = 0$ and so $\chi = 0$.

$p > 1$ Pick $\chi \in H^p(U, F^n)$. By Lemma 22.3, there exists an fpqc covering $U = \{U_i \to U\}_{i \in I}$ such that $\chi|_{U_i} = 0$ for all $i \in I$. Up to refining $U$, we may assume that $U$ is a standard $\tau$-covering. We apply the spectral sequence of Theorem 19.2. Observe that the intersections $U_i \times_U \ldots \times_U U_{i_p}$ are affine, so that by induction hypothesis the cohomology groups $E^{p,q}_2 = \check{H}^p(U, H^q(F^a))$ vanish for all $0 < q < p$. We see that $\chi$ must come from a $\tilde{\chi} \in \check{H}^p(U, F^n)$. Replacing $U$ with the covering $V$ containing only one morphism and using Lemma 16.4 again, we see that the Čech cohomology class $\tilde{\chi}$ must be zero, hence $\chi = 0$.

Next, assume that $U$ is separated. Choose an affine open covering $U = \bigcup_{i \in I} U_i$ of $U$. The family $U = \{U_i \to U\}_{i \in I}$ is then an fpqc covering, and all the intersections $U_{i_0} \times_U \ldots \times_U U_{i_p}$ are affine since $U$ is separated. So all rows of the spectral sequence of Theorem 19.2 are zero, except the zeroth row. Therefore

$$H^p(U, F^n) = \check{H}^p(U, F^n) = \check{H}^p(U, F) = H^p(U, F)$$

where the last equality results from standard scheme theory, see Cohomology of Schemes, Lemma 2.6.

The general case is technical and (to extend the proof as given here) requires a discussion about maps of spectral sequences, so we won’t treat it. It follows from Descent, Proposition 7.10 (whose proof takes a slightly different approach) combined with Cohomology on Sites, Lemma 8.1.

**Remark 22.5.** Comment on Theorem 22.4. Since $S$ is a final object in the category $\mathcal{C}$, the cohomology groups on the right-hand side are merely the right derived functors of the global sections functor. In fact the proof shows that $H^p(U, f^* F) = H^p(U, F^n)$ for any object $f : U \to S$ of the site $\mathcal{C}$.

### 23. Examples of sheaves

Let $S$ and $\tau$ be as in Section 20. We have already seen that any representable presheaf is a sheaf on $(\text{Sch}/S)_\tau$ or $S_\tau$, see Lemma 15.8 and Remark 15.9. Here are some special cases.

**Definition 23.1.** On any of the sites $(\text{Sch}/S)_\tau$ or $S_\tau$ of Section 20

1. The sheaf $T \mapsto \Gamma(T, \mathcal{O}_T)$ is denoted $\mathcal{O}_S$, or $\mathbf{G}_a$, or $\mathbf{G}_{a,S}$ if we want to indicate the base scheme.
2. Similarly, the sheaf $T \mapsto \Gamma(T, \mathcal{O}_T^n)$ is denoted $\mathcal{O}_{S}^n$, or $\mathbf{G}_m$, or $\mathbf{G}_{m,S}$ if we want to indicate the base scheme.
3. The constant sheaf $\mathbb{Z}/n\mathbb{Z}$ on any site is the sheafification of the constant presheaf $U \mapsto \mathbb{Z}/n\mathbb{Z}$.

The first is a sheaf by Theorem 17.4 for example. The second is a sub presheaf of the first, which is easily seen to be a sheaf itself. The third is a sheaf by definition. Note that each of these sheaves is representable. The first and second by the schemes $\mathbf{G}_{a,S}$ and $\mathbf{G}_{a,S}$, see Groupoids, Section 4. The third by the finite étale group scheme $\mathbb{Z}/n\mathbb{Z}_S$ sometimes denoted $(\mathbb{Z}/n\mathbb{Z})_S$ which is just $n$ copies of $S$ endowed
with the obvious group scheme structure over $S$, see Groupoids, Example 5.6 and the following remark.

**Remark 23.2.** Let $G$ be an abstract group. On any of the sites $(\text{Sch}/S)_\tau$ or $S_\tau$ of Section 20 the sheafification $\mathcal{G}$ of the constant presheaf associated to $G$ in the Zariski topology of the site already gives

$$\Gamma(U, \mathcal{G}) = \{\text{Zariski locally constant maps } U \to G\}$$

This Zariski sheaf is representable by the group scheme $G_S$ according to Groupoids, Example 5.6. By Lemma 15.8 any representable presheaf satisfies the sheaf condition for the $\tau$-topology as well, and hence we conclude that the Zariski sheafification $\mathcal{G}$ above is also the $\tau$-sheafification.

**Definition 23.3.** Let $S$ be a scheme. The structure sheaf of $S$ is the sheaf of rings $\mathcal{O}_S$ on any of the sites $S_{\text{Zar}}$, $S_{\text{étale}}$, or $(\text{Sch}/S)_\tau$ discussed above. If there is some possible confusion as to which site we are working on then we will indicate this by using indices. For example we may use $\mathcal{O}_{S_{\text{étale}}}$ to stress the fact that we are working on the small étale site of $S$.

**Remark 23.4.** In the terminology introduced above a special case of Theorem 22.4 is

$$H^p_{\text{fppf}}(X, \mathcal{G}_a) = H^p_{\text{étale}}(X, \mathcal{G}_a) = H^p_{\text{Zar}}(X, \mathcal{G}_a) = H^p(X, \mathcal{O}_X)$$

for all $p \geq 0$. Moreover, we could use the notation $H^p_{\text{fppf}}(X, \mathcal{O}_X)$ to indicate the cohomology of the structure sheaf on the big fppf site of $X$.

### 24. Picard groups

The following theorem is sometimes called “Hilbert 90”.

**Theorem 24.1.** For any scheme $X$ we have canonical identifications

$$H^1_{\text{fppf}}(X, \mathbb{G}_m) = H^1_{\text{syntomic}}(X, \mathbb{G}_m) = H^1_{\text{smooth}}(X, \mathbb{G}_m) = H^1_{\text{étale}}(X, \mathbb{G}_m) = H^1_{2\text{ar}}(X, \mathbb{G}_m) = \text{Pic}(X) = H^1(X, \mathcal{O}_X)$$

**Proof.** Let $\tau$ be one of the topologies considered in Section 20. By Cohomology on Sites, Lemma 7.1 we see that $H^1_\tau(X, \mathcal{O}_m) = H^1_\tau(X, \mathcal{O}_\tau^*) = \text{Pic}(\mathcal{O}_\tau)$ where $\mathcal{O}_\tau$ is the structure sheaf of the site $(\text{Sch}/X)_\tau$. Now an invertible $\mathcal{O}_\tau$-module is a quasi-coherent $\mathcal{O}_\tau$-module. By Theorem 17.4 or the more precise Descent, Proposition 7.11 we see that $\text{Pic}(\mathcal{O}_\tau) = \text{Pic}(X)$. The last equality is proved in the same way. □

### 25. The étale site

At this point we start exploring the étale site of a scheme in more detail. As a first step we discuss a little the notion of an étale morphism.
26. Étale morphisms

03PA For more details, see Morphisms, Section 36 for the formal definition and Étale Morphisms, Sections 11, 12, 13, 14, 16, and 19 for a survey of interesting properties of étale morphisms.

Recall that an algebra $A$ over an algebraically closed field $k$ is smooth if it is of finite type and the module of differentials $\Omega_{A/k}$ is finite locally free of rank equal to the dimension. A scheme $X$ over $k$ is smooth if it is locally of finite type and each affine open is the spectrum of a smooth $k$-algebra. If $k$ is not algebraically closed then an $A$-algebra is said to be a smooth $k$-algebra if $A \otimes_k k$ is a smooth $k$-algebra. A ring map $A \to B$ is smooth if it is flat, finitely presented, and for all primes $p \subset A$ the fibre ring $\kappa(p) \otimes_A B$ is smooth over the residue field $\kappa(p)$. More generally, a morphism of schemes is smooth if it is flat, locally of finite presentation, and the geometric fibers are smooth.

For these facts please see Morphisms, Section 34. Using this we may define an étale morphism as follows.

03PB **Definition 26.1.** A morphism of schemes is étale if it is smooth of relative dimension 0.

In particular, a morphism of schemes $X \to S$ is étale if it is smooth and $\Omega_{X/S} = 0$.

03PC **Proposition 26.2.** Facts on étale morphisms.

1. Let $k$ be a field. A morphism of schemes $U \to \text{Spec}(k)$ is étale if and only if $U \cong \coprod_{i \in I} \text{Spec}(k_i)$ such that for each $i \in I$ the ring $k_i$ is a field which is a finite separable extension of $k$.

2. Let $\varphi : U \to S$ be a morphism of schemes. The following conditions are equivalent:
   a. $\varphi$ is étale,
   b. $\varphi$ is locally finitely presented, flat, and all its fibres are étale,
   c. $\varphi$ is flat, unramified and locally of finite presentation.

3. A ring map $A \to B$ is étale if and only if $B \cong A[x_1, \ldots, x_n]/(f_1, \ldots, f_n)$ such that $\Delta = \det \left( \frac{\partial f_i}{\partial x_j} \right)$ is invertible in $B$.

4. The base change of an étale morphism is étale.

5. Compositions of étale morphisms are étale.

6. Fibre products and products of étale morphisms are étale.

7. An étale morphism has relative dimension 0.

8. Let $Y \to X$ be an étale morphism. If $X$ is reduced (respectively regular) then so is $Y$.

9. Étale morphisms are open.

10. If $X \to S$ and $Y \to S$ are étale, then any $S$-morphism $X \to Y$ is also étale.

**Proof.** We have proved these facts (and more) in the preceding chapters. Here is a list of references: (1) Morphisms, Lemma 36.7. (2) Morphisms, Lemmas 36.8 and 36.16. (3) Algebra, Lemma 141.2. (4) Morphisms, Lemma 36.4. (5) Morphisms, Lemma 36.3. (6) Follows formally from (4) and (5). (7) Morphisms, Lemmas 36.6 and 29.5. (8) See Algebra, Lemmas 157.6 and 157.5. (9) See also more results of this kind in Étale Morphisms, Section 19. (9) See Morphisms, Lemmas 25.9 and 36.12. (10) See Morphisms, Lemma 36.18. □
A ring map $A \to B$ is called standard étale if $B \cong (A[t]/(f))_{g}$ with $f, g \in A[t]$, with $f$ monic, and $df/dt$ invertible in $B$.

It is true that a standard étale ring map is étale. Namely, suppose that $B = (A[t]/(f))_{g}$ with $f, g \in A[t]$, with $f$ monic, and $df/dt$ invertible in $B$. Then $A[t]/(f)$ is a finite free $A$-module of rank equal to the degree of the monic polynomial $f$. Hence $B$, as a localization of this free algebra is finitely presented and flat over $A$. To finish the proof that $B$ is étale it suffices to show that the fibre rings

$$
\kappa(p) \otimes_A B \cong \kappa(p) \otimes_A (A[t]/(f))_g \cong \kappa(p)[t, 1/g]/(\overline{f})
$$

are finite products of finite separable field extensions. Here $\overline{f}, \overline{g} \in \kappa(p)[t]$ are the images of $f$ and $g$. Let

$$
\overline{f} = \overline{f}_1 \cdots \overline{f}_a \overline{f}_{a+1} \cdots \overline{f}_{a+b}
$$

be the factorization of $\overline{f}$ into powers of pairwise distinct irreducible monic factors $\overline{f}_i$ with $e_1, \ldots, e_b > 0$. By assumption $d\overline{f}/dt$ is invertible in $\kappa(p)[t, 1/g]$. Hence we see that at least all the $\overline{f}_i$, $i > a$ are invertible. We conclude that

$$
\kappa(p)[t, 1/g]/(\overline{f}) \cong \prod_{i \in I} \kappa(p)[t]/(\overline{f}_i)
$$

where $I \subset \{1, \ldots, a\}$ is the subset of indices $i$ such that $\overline{f}_i$ does not divide $\overline{g}$. Moreover, the image of $d\overline{f}/dt$ in the factor $\kappa(p)[t]/(\overline{f}_i)$ is clearly equal to a unit times $d\overline{f}_i/dt$. Hence we conclude that $\kappa_i = \kappa(p)[t]/(\overline{f}_i)$ is a finite field extension of $\kappa(p)$ generated by one element whose minimal polynomial is separable, i.e., the field extension $\kappa(p) \subset \kappa_i$ is finite separable as desired.

It turns out that any étale ring map is locally standard étale. To formulate this we introduce the following notation. A ring map $A \to B$ is étale at a prime $q$ of $B$ if there exists $h \in B$, $h \not\in q$ such that $A \to B_h$ is étale. Here is the result.

A ring map $A \to B$ is étale at a prime $q$ if and only if there exists $g \in B$, $g \not\in q$ such that $B_g$ is standard étale over $A$.

Proof. See Algebra, Proposition [41.17] \hfill \Box

27. Étale coverings

We recall the definition.

An étale covering of a scheme $U$ is a family of morphisms of schemes $\{\varphi_i : U_i \to U\}_{i \in I}$ such that

1. each $\varphi_i$ is an étale morphism,
2. the $U_i$ cover $U$, i.e., $U = \bigcup_{i \in I} \varphi_i(U_i)$.

Any étale covering is an fpqc covering.

Proof. (See also Topologies, Lemma [8.6]) Let $\{\varphi_i : U_i \to U\}_{i \in I}$ be an étale covering. Since an étale morphism is flat, and the elements of the covering should cover its target, the property fp (faithfully flat) is satisfied. To check the property qc (quasi-compact), let $V \subset U$ be an affine open, and write $\varphi_i^{-1}(V) = \bigcup_{j \in J_i} V_{ij}$ for some affine opens $V_{ij} \subset U_i$. Since $\varphi_i$ is open (as étale morphisms are open), we see that $V = \bigcup_{i \in I} \bigcup_{j \in J_i} \varphi_i(V_{ij})$ is an open covering of $V$. Further, since $V$ is quasi-compact, this covering has a finite refinement. \hfill \Box
So any statement which is true for fpqc coverings remains true \textit{a fortiori} for étale coverings. For instance, the étale site is subcanonical.

\textbf{Definition 27.3.} (For more details see Section \ref{topologies} or Topologies, Section \ref{topologies}) Let \( S \) be a scheme. The \textit{big étale site over} \( S \) is the site \((\mathbf{Sch}/S)_{\text{étale}}\), see Definition \ref{topologies}. The \textit{small étale site over} \( S \) is the site \( S_{\text{étale}} \), see Definition \ref{topologies}. We define similarly the \textit{big and small Zariski sites} on \( S \), denoted \((\mathbf{Sch}/S)_{\text{Zar}}\) and \( S_{\text{Zar}} \).

Loosely speaking the big étale site of \( S \) is made up out of schemes over \( S \) and coverings the étale coverings. The small étale site of \( S \) is made up out of schemes étale over \( S \) with coverings the étale coverings. Actually any morphism between objects of \( S_{\text{étale}} \) is étale, in virtue of Proposition \ref{topologies}, hence to check that \( \{ U_i \to U \}_{i \in I} \) in \( S_{\text{étale}} \) is a covering it suffices to check that \( \coprod U_i \to U \) is surjective.

The small étale site has fewer objects than the big étale site, it contains only the “opens” of the étale topology on \( S \). It is a full subcategory of the big site, and its topology is induced from the topology on the big site. Hence it is true that the restriction functor from the big étale site to the small one is exact and maps injectives to injectives. This has the following consequence.

\textbf{Proposition 27.4.} Let \( S \) be a scheme and \( F \) an abelian sheaf on \((\mathbf{Sch}/S)_{\text{étale}}\). Then \( F|_{S_{\text{étale}}} \) is a sheaf on \( S_{\text{étale}} \) and \( H^p_{\text{étale}}(S, F|_{S_{\text{étale}}}) = H^p_{\text{étale}}(S, F) \) for all \( p \geq 0 \).

\textbf{Proof.} This is a special case of Lemma \ref{topologies}. \( \square \)

In accordance with the general notation introduced in Section \ref{topologies} we write \( H^p_{\text{étale}}(S, F) \) for the above cohomology group.

### 28. Kummer theory

\textbf{Lemma 28.1.} If \( n \in \mathbb{N} \) then

\[
0 \to \mu_n,S \to \mathbf{G}_m,S \xrightarrow{(\cdot)^n} \mathbf{G}_m,S \to 0
\]

is a short exact sequence of sheaves on both the small and big étale site of \( S \).

\textbf{Proof.} By definition the sheaf \( \mu_n,S \) is the kernel of the map \((\cdot)^n\). Hence it suffices to show that the last map is surjective. Let \( U \) be a scheme over \( S \). Let \( f \in \mathbf{G}_m(U) = \Gamma(U, \mathcal{O}_U^\times) \). We need to show that we can find an étale cover of \( U \) over the members of which the restriction of \( f \) is an \( n \)th power. Set

\[
U' = \text{Spec}_U(\mathcal{O}_U[T]/(T^n - f)) \xrightarrow{\pi} U.
\]
Lemma 28.1 is false when “étale” is replaced with “Zariski”. Since for any \( \text{Spec}(A) \subset U \) be an affine open, and say \( f|_{\text{Spec}(A)} \) corresponds to the unit \( a \in A^* \). Then \( \pi^{-1}(\text{Spec}(A)) = \text{Spec}(B) \) with \( B = A[T]/(T^n - a) \). The ring map \( A \to B \) is finite free of rank \( n \), hence it is faithfully flat, and hence we conclude that \( \text{Spec}(B) \to \text{Spec}(A) \) is surjective. Since this holds for every affine open in \( U \) we conclude that \( \pi \) is surjective. In addition, \( n \) and \( T^{n-1} \) are invertible in \( B \), so \( nT^{n-1} \in B^* \) and the ring map \( A \to B \) is standard étale, in particular étale. Since this holds for every affine open of \( U \) we conclude that \( \pi \) is étale. Hence \( U = \{ \pi : U' \to U \} \) is an étale covering. Moreover, \( f|_{U'} = (f')^n \) where \( f' \) is the class of \( T \) in \( \Gamma(U', \mathcal{O}_{U'}) \), so \( U \) has the desired property. □

Remark 28.2. Lemma 28.1 is false when “étale” is replaced with “Zariski”. Since the étale topology is coarser than the smooth topology, see Topologies, Lemma 5.2 it follows that the sequence is also exact in the smooth topology.

By Theorem 24.1 and Lemma 28.1 and general properties of cohomology we obtain the long exact cohomology sequence

\[
0 \to H^0_{\text{étale}}(S, \mu_{n,S}) \to \Gamma(S, \mathcal{O}_S^*) \xrightarrow{(\cdot)^n} \Gamma(S, \mathcal{O}_S^*) \to H^1_{\text{étale}}(S, \mu_{n,S}) \to \text{Pic}(S) \to \text{Pic}(S) \to \ldots
\]

at least if \( n \) is invertible on \( S \). When \( n \) is not invertible on \( S \) we can apply the following lemma.

Lemma 28.3. For any \( n \in \mathbb{N} \) the sequence

\[
0 \to \mu_{n,S} \to \mathbb{G}_{m,S} \xrightarrow{(\cdot)^n} \mathbb{G}_{m,S} \to 0
\]

is a short exact sequence of sheaves on the site \( (\text{Sch}/S)_{\text{fppf}} \) and \( (\text{Sch}/S)_{\text{syntomic}} \).

Proof. By definition the sheaf \( \mu_{n,S} \) is the kernel of the map \( (\cdot)^n \). Hence it suffices to show that the last map is surjective. Since the syntomic topology is stronger than the fppf topology, see Topologies, Lemma 7.2 it suffices to prove this for the syntomic topology. Let \( U \) be a scheme over \( S \). Let \( f \in \mathbb{G}_{m}(U) = \Gamma(U, \mathcal{O}_U^*) \). We need to show that we can find a syntomic cover of \( U \) over the members of which the restriction of \( f \) is an \( n \)th power. Set

\[
U' = \text{Spec}_{U}(\mathcal{O}_U[T]/(T^n - f)) \xrightarrow{\tilde{\pi}} U.
\]

(See Constructions, Section 3 or 4 for a discussion of the relative spectrum.) Let \( \text{Spec}(A) \subset U \) be an affine open, and say \( f|_{\text{Spec}(A)} \) corresponds to the unit \( a \in A^* \). Then \( \pi^{-1}(\text{Spec}(A)) = \text{Spec}(B) \) with \( B = A[T]/(T^n - a) \). The ring map \( A \to B \) is finite free of rank \( n \), hence it is faithfully flat, and hence we conclude that \( \text{Spec}(B) \to \text{Spec}(A) \) is surjective. Since this holds for every affine open in \( U \) we conclude that \( \pi \) is surjective. In addition, \( B \) is a global relative complete intersection over \( A \), so the ring map \( A \to B \) is standard syntomic, in particular syntomic. Since this holds for every affine open of \( U \) we conclude that \( \pi \) is syntomic. Hence
$U = \{\pi : U' \to U\}$ is a syntomic covering. Moreover, $f|_{U'} = (f')^n$ where $f'$ is the class of $T$ in $\Gamma(U', \mathcal{O}_{U'})$, so $U$ has the desired property. □

**Remark 28.4.** Lemma 28.3 is false for the smooth, étale, or Zariski topology.

By Theorem 24.1 and Lemma 28.3 and general properties of cohomology we obtain the long exact cohomology sequence

$$
0 \to H^0_{fppf}(S, \mu_n, S) \to \Gamma(S, \mathcal{O}_S^*) \xrightarrow{(-)^n} \Gamma(S, \mathcal{O}_S^*) \\
\downarrow \quad \downarrow \\
H^1_{fppf}(S, \mu_n, S) \to \text{Pic}(S) \xrightarrow{(-)^n} \text{Pic}(S) \\
\downarrow \quad \downarrow \\
H^2_{fppf}(S, \mu_n, S) \to \ldots
$$

for any scheme $S$ and any integer $n$. Of course there is a similar sequence with syntomic cohomology.

Let $n \in \mathbb{N}$ and let $S$ be any scheme. There is another more direct way to describe the first cohomology group with values in $\mu_n$. Consider pairs $(L, \alpha)$ where $L$ is an invertible sheaf on $S$ and $\alpha : L^\otimes n \to \mathcal{O}_S$ is a trivialization of the $n$th tensor power of $L$. Let $(L', \alpha')$ be a second such pair. An isomorphism $\varphi : (L, \alpha) \to (L', \alpha')$ is an isomorphism $\varphi : L \to L'$ of invertible sheaves such that the diagram

$$
\begin{array}{ccc}
L^\otimes n & \xrightarrow{\alpha} & \mathcal{O}_S \\
\varphi^\otimes n \downarrow & & \downarrow 1 \\
(L')^\otimes n & \xrightarrow{\alpha'} & \mathcal{O}_S
\end{array}
$$

commutes. Thus we have

$$(28.4.1) \quad \text{Isom}_S((L, \alpha), (L', \alpha')) = \begin{cases} \\
\emptyset & \text{if they are not isomorphic} \\
H^0(S, \mu_n, S) \cdot \varphi & \text{if } \varphi \text{ isomorphism of pairs}
\end{cases}
$$

Moreover, given two pairs $(L, \alpha), (L', \alpha')$ the tensor product

$$
(L, \alpha) \otimes (L', \alpha') = (L \otimes L', \alpha \otimes \alpha')
$$

is another pair. The pair $(\mathcal{O}_S, 1)$ is an identity for this tensor product operation, and an inverse is given by

$$
(L, \alpha)^{-1} = (L^\otimes -1, \alpha^\otimes -1).
$$

Hence the collection of isomorphism classes of pairs forms an abelian group. Note that

$$
(L, \alpha)^\otimes n = (L^\otimes n, \alpha^\otimes n) \xrightarrow{\alpha} (\mathcal{O}_S, 1)
$$

hence every element of this group has order dividing $n$. We warn the reader that this group is in general not the $n$-torsion in Pic($S$).

**Lemma 28.5.** Let $S$ be a scheme. There is a canonical identification

$$H^1_{\text{étale}}(S, \mu_n) = \text{group of pairs } (L, \alpha) \text{ up to isomorphism as above}
$$

if $n$ is invertible on $S$. In general we have

$$H^1_{fppf}(S, \mu_n) = \text{group of pairs } (L, \alpha) \text{ up to isomorphism as above}.$$
Proof. We first prove the second isomorphism. Let \((L, \alpha)\) be a pair as above. Choose an affine open covering \(S = \bigcup U_i\) such that \(L|_{U_i} \cong \mathcal{O}_{U_i}\). Say \(s_i \in L(U_i)\) is a generator. Then \(\alpha(s^n_i) = f_i \in \mathcal{O}_S(U_i)\). Writing \(U_i = \text{Spec}(A_i)\) we see there exists a global relative complete intersection \(A_i \to B_i = \mathbb{A}[T]/(T^n - f_i)\) such that \(f_i\) maps to an \(n\)th power in \(B_i\). In other words, setting \(V_i = \text{Spec}(B_i)\) we obtain a syntomic covering \(V = \{V_i \to S\}_{i \in I}\) and trivializations \(\varphi_i : (L, \alpha)|_{V_i} \to (\mathcal{O}_{V_i}, 1)\).

We will use this result (the existence of the covering \(V\)) to associate to this pair a cohomology class in \(H^1_{\text{syntomic}}(S, \mu_{n,S})\). We give two (equivalent) constructions.

First construction: Using Čech cohomology. Over the double overlaps \(V_i \times_Sh_j\) we have the isomorphism
\[
\mathcal{O}_{V_i \times_Sh_j} \xrightarrow{pr_2^*\varphi_j^{-1}} \mathcal{L}|_{V_i \times_Sh_j, \alpha|_{V_i \times_Sh_j}} \xrightarrow{pr_1^*\varphi_i} \mathcal{O}_{V_i \times_Sh_j}, 1
\]
of pairs. By (28.4.1) this is given by an element \(\zeta_{ij} \in \mu_n(V_i \times_Sh_j)\). We omit the verification that these \(\zeta_{ij}\) give a 1-cocycle, i.e., give an element \((\zeta_{ij})_{ij} \in \check{C}(V, \mu_n)\) with \(d(\zeta_{ij}) = 0\). Thus its class is an element in \(H^1(V, \mu_n)\) and by Theorem 19.2 it maps to a cohomology class in \(H^1_{\text{syntomic}}(S, \mu_{n,S})\).

Second construction: Using torsors. Consider the presheaf \(\mu_n(L, \alpha) : U \mapsto \text{Isom}_U((\mathcal{O}_U, 1), (L, \alpha)|_U)\) on \((\text{Sch}/S)_{\text{syntomic}}\). We may view this as a subsheaf of \(\text{Hom}_\mathcal{O}(\mathcal{O}, L)\) (internal hom sheaf, see Modules on Sites, Section 27). Since the conditions defining this subsheaf are local, we see that it is a sheaf. By (28.4.1) this sheaf has a free action of the sheaf \(\mu_{n,S}\). Hence the only thing we have to check is that it locally has sections. This is true because of the existence of the trivializing cover \(V\). Hence \(\mu_n(L, \alpha)\) is a \(\mu_{n,S}\)-torsor and by Cohomology on Sites, Lemma 5.3 we obtain a corresponding element of \(H^1_{\text{syntomic}}(S, \mu_{n,S})\).

Ok, now we have to still show the following

1. The two constructions give the same cohomology class.
2. Isomorphic pairs give rise to the same cohomology class.
3. The cohomology class of \((L, \alpha) \otimes (L', \alpha')\) is the sum of the cohomology classes of \((L, \alpha)\) and \((L', \alpha')\).
4. If the cohomology class is trivial, then the pair is trivial.
5. Any element of \(H^1_{\text{syntomic}}(S, \mu_{n,S})\) is the cohomology class of a pair.

We omit the proof of (1). Part (2) is clear from the second construction, since isomorphic torsors give the same cohomology classes. Part (3) is clear from the first construction, since the resulting Čech classes add up. Part (4) is clear from the second construction since a torsor is trivial if and only if it has a global section, see Cohomology on Sites, Lemma 5.2.

Part (5) can be seen as follows (although a direct proof would be preferable). Suppose \(\xi \in H^1_{\text{syntomic}}(S, \mu_{n,S})\). Then \(\xi\) maps to an element \(\overline{\xi} \in H^1_{\text{syntomic}}(S, \mathcal{G}_{m,S})\) with \(n\xi = 0\). By Theorem 24.1 we see that \(\overline{\xi}\) corresponds to an invertible sheaf \(\mathcal{L}\) whose \(n\)th tensor power is isomorphic to \(\mathcal{O}_S\). Hence there exists a pair \((L, \alpha')\) whose cohomology class \(\xi'\) has the same image \(\overline{\xi}\) in \(H^1_{\text{syntomic}}(S, \mathcal{G}_{m,S})\). Thus it suffices to show that \(\xi - \xi'\) is the class of a pair. By construction, and the long exact
cohomology sequence above, we see that \( \xi - \xi' = \partial(f) \) for some \( f \in H^0(S, \mathcal{O}_S^\times) \). Consider the pair \((\mathcal{O}_S, f)\). We omit the verification that the cohomology class of this pair is \( \partial(f) \), which finishes the proof of the first identification (with fppf replaced with syntomic).

To see the first, note that if \( n \) is invertible on \( S \), then the covering \( \mathcal{V} \) constructed in the first part of the proof is actually an étale covering (compare with the proof of Lemma \[28.1\]). The rest of the proof is independent of the topology, apart from the very last argument which uses that the Kummer sequence is exact, i.e., uses Lemma \[28.1\].

\[ \square \]

29. Neighborhoods, stalks and points

03PN We can associate to any geometric point of \( S \) a stalk functor which is exact. A map of sheaves on \( S_{\text{étale}} \) is an isomorphism if and only if it is an isomorphism on all these stalks. A complex of abelian sheaves is exact if and only if the complex of stalks is exact at all geometric points. Altogether this means that the small étale site of a scheme \( S \) has enough points. It also turns out that any point of the small étale topos of \( S \) (an abstract notion) is given by a geometric point. Thus in some sense the small étale topos of \( S \) can be understood in terms of geometric points and neighbourhoods.

**Definition 29.1.** Let \( S \) be a scheme.

1. A geometric point of \( S \) is a morphism \( \text{Spec}(k) \rightarrow S \) where \( k \) is algebraically closed. Such a point is usually denoted \( \pi \), i.e., by an overlined small case letter. We often use \( \pi \) to denote the scheme \( \text{Spec}(k) \) as well as the morphism, and we use \( \kappa(\pi) \) to denote \( k \).
2. We say \( \pi \) lies over \( s \) to indicate that \( s \in S \) is the image of \( \pi \).
3. An étale neighborhood of a geometric point \( \pi \) of \( S \) is a commutative diagram

\[
\begin{array}{ccc}
U & \rightarrow & S \\
\downarrow \varphi & & \downarrow \pi \\
\pi & \rightarrow & S
\end{array}
\]

where \( \varphi \) is an étale morphism of schemes. We write \((U, \pi) \rightarrow (S, \pi)\).
4. A morphism of étale neighborhoods \((U, \pi) \rightarrow (U', \pi')\) is an \( S \)-morphism \( h : U \rightarrow U' \) such that \( \pi' = h \circ \pi \).

**Remark 29.2.** Since \( U \) and \( U' \) are étale over \( S \), any \( S \)-morphism between them is also étale, see Proposition \[26.2\]. In particular all morphisms of étale neighborhoods are étale.

**Remark 29.3.** Let \( S \) be a scheme and \( s \in S \) a point. In More on Morphisms, Definition \[27.1\] we defined the notion of an étale neighbourhood \((U, u) \rightarrow (S, s)\) of \((S, s)\). If \( \pi \) is a geometric point of \( S \) lying over \( s \), then any étale neighbourhood \((U, \pi) \rightarrow (S, \pi)\) gives rise to an étale neighbourhood \((U, u) \rightarrow (S, s)\) by taking \( u \in U \) to be the unique point of \( U \) such that \( \pi \) lies over \( u \). Conversely, given an étale neighbourhood \((U, u) \) of \((S, s)\) the residue field extension \( \kappa(s) \subset \kappa(u) \) is finite separable (see Proposition \[26.2\]) and hence we can find an embedding \( \kappa(u) \subset \kappa(\pi) \) over \( \kappa(s) \). In other words, we can find a geometric point \( \pi \) of \( U \) lying over \( u \) such that \((U, \pi)\) is an étale neighbourhood of \((S, \pi)\). We will use these observations to go between the two types of étale neighbourhoods.
Lemma 29.4. Let $S$ be a scheme, and let $\overline{\sigma}$ be a geometric point of $S$. The category of étale neighborhoods is cofiltered. More precisely:

1. Let $(U_i, \overline{\pi}_i)_{i=1,2}$ be two étale neighborhoods of $\overline{\sigma}$ in $S$. Then there exists a third étale neighborhood $(U, \overline{\pi})$ and morphisms $(U, \overline{\pi}) \to (U_i, \overline{\pi}_i)$, $i = 1, 2$.
2. Let $h_1, h_2 : (U, \overline{\pi}) \to (U', \overline{\pi}')$ be two morphisms between étale neighborhoods of $\overline{\sigma}$. Then there exists an étale neighborhood $(U'', \overline{\pi}'')$ and a morphism $h : (U'', \overline{\pi}'') \to (U, \overline{\pi})$ which equals $h_1$ and $h_2$, i.e., such that $h_1 \circ h = h_2 \circ h$.

Proof. For part (1), consider the fibre product $U = U_1 \times_S U_2$. It is étale over both $U_1$ and $U_2$ because étale morphisms are preserved under base change, see Proposition 29.2. The map $\overline{\sigma} \to U$ defined by $(\overline{\pi}_1, \overline{\pi}_2)$ gives it the structure of an étale neighborhood mapping to both $U_1$ and $U_2$. For part (2), define $U''$ as the fibre product

$$
\begin{array}{ccc}
U'' & \longrightarrow & U \\
\downarrow & & \downarrow_{(h_1, h_2)} \\
U' & \longrightarrow & U' \times_S U'.
\end{array}
$$

Since $\overline{\pi}$ and $\overline{\pi}'$ agree over $S$ with $\overline{\sigma}$, we see that $\overline{\pi}'' = (\overline{\pi}, \overline{\pi}')$ is a geometric point of $U''$. In particular $U'' \neq \emptyset$. Moreover, since $U'$ is étale over $S$, so is the fibre product $U' \times_S U'$ (see Proposition 26.2). Hence the vertical arrow $(h_1, h_2)$ is étale by Remark 29.2 above. Therefore $U''$ is étale over $U'$ by base change, and hence also étale over $S$ (because compositions of étale morphisms are étale). Thus $(U'', \overline{\pi}'')$ is a solution to the problem. □

Lemma 29.5. Let $S$ be a scheme. Let $\overline{\sigma}$ be a geometric point of $S$. Let $(U, \overline{\pi})$ be an étale neighborhood of $\overline{\sigma}$. Let $U = \{ \varphi_i : U_i \to U \}_{i \in I}$ be an étale covering. Then there exist $i \in I$ and $\overline{\pi}_i : \overline{\pi} \to U_i$ such that $\varphi_i : (U_i, \overline{\pi}_i) \to (U, \overline{\pi})$ is a morphism of étale neighborhoods.

Proof. As $U = \bigcup_{i \in I} \varphi_i(U_i)$, the fibre product $\overline{\pi} \times_{\pi, U, \varphi_i} U_i$ is not empty for some $i$. Then look at the cartesian diagram

$$
\begin{array}{ccc}
\overline{\pi} \times_{\pi, U, \varphi_i} U_i & \longrightarrow & U_i \\
\downarrow_{\varphi_i} & & \downarrow_{\varphi_i} \\
\text{Spec}(k) = \overline{\pi} & \longrightarrow & U.
\end{array}
$$

The projection $\text{pr}_1$ is the base change of an étale morphisms so it is étale, see Proposition 26.2. Therefore, $\overline{\pi} \times_{\pi, U, \varphi_i} U_i$ is a disjoint union of finite separable extensions of $k$, by Proposition 26.2. Here $\overline{\pi} = \text{Spec}(k)$. But $k$ is algebraically closed, so all these extensions are trivial, and there exists a section $\sigma$ of $\text{pr}_1$. The composition $\text{pr}_2 \circ \sigma$ gives a map compatible with $\overline{\pi}$. □

Definition 29.6. Let $S$ be a scheme. Let $\mathcal{F}$ be a presheaf on $S^{\text{étale}}$. Let $\overline{\sigma}$ be a geometric point of $S$. The stalk of $\mathcal{F}$ at $\overline{\sigma}$ is

$$
\mathcal{F}_{\overline{\sigma}} = \colim_{(U, \overline{\pi})} \mathcal{F}(U)
$$

where $(U, \overline{\pi})$ runs over all étale neighborhoods of $\overline{\sigma}$ in $S$.

By Lemma 29.4, this colimit is over a filtered index category, namely the opposite of the category of étale neighbourhoods. In other words, an element of $\mathcal{F}_{\overline{\sigma}}$
Let $S$ be a scheme. Let $\overline{\pi}$ be a geometric point of $S$. Consider the functor
\[ u : S_{\text{étale}} \to \text{Sets}, \]
\[ U \mapsto |U_{\overline{\pi}}| = \{ \overline{\pi} \text{ such that } (U, \overline{\pi}) \text{ is an étale neighbourhood of } \overline{\pi} \}. \]

Here $|U_{\overline{\pi}}|$ denotes the underlying set of the geometric fibre. Then $u$ defines a point $p$ of the site $S_{\text{étale}}$ (Sites, Definition 31.2) and its associated stalk functor $F \mapsto F_p$ (Sites, Equation 31.1.1) is the functor $F \mapsto F_{\overline{\pi}}$ defined above.

Proof. In the proof of Lemma 29.5 we have seen that the scheme $U_{\overline{\pi}}$ is a disjoint union of schemes isomorphic to $\overline{\pi}$. Thus we can also think of $|U_{\overline{\pi}}|$ as the set of geometric points of $U$ lying over $\overline{\pi}$, i.e., as the collection of morphisms $\overline{\pi} : \overline{\pi} \to U$ fitting into the diagram of Definition 29.1. From this it follows that $u(S)$ is a singleton, and that $u(U \times_V W) = u(U) \times_{u(V)} u(W)$ whenever $U \to V$ and $W \to V$ are morphisms in $S_{\text{étale}}$. And, given a covering $\{U_i \to U\}_{i \in I}$ in $S_{\text{étale}}$ we see that \[ \prod u(U_i) \to u(U) \] is surjective by Lemma 29.7. Hence Sites, Proposition 32.2 applies, so $p$ is a point of the site $S_{\text{étale}}$. Finally, the functor $F \mapsto F_{\overline{\pi}}$ is given by exactly the same colimit as the functor $F \mapsto F_p$ associated to $p$ in Sites, Equation 31.1.1 which proves the final assertion.

Remark 29.8. Let $S$ be a scheme and let $\overline{\pi} : \text{Spec}(k) \to S$ and $\overline{\pi'} : \text{Spec}(k') \to S$ be two geometric points of $S$. A morphism $a : \overline{\pi} \to \overline{\pi'}$ of geometric points is simply a morphism $a : \text{Spec}(k) \to \text{Spec}(k')$ such that $a \circ \overline{\pi} = \overline{\pi'}$. Given such a morphism we obtain a functor from the category of étale neighbourhoods of $\overline{\pi'}$ to the category of étale neighbourhoods of $\overline{\pi}$ by the rule $(U, \overline{\pi}) \mapsto (U, \overline{\pi'} \circ a)$. Hence we obtain a canonical map
\[ F_{\overline{\pi'}} = \colim_{(U, \overline{\pi})} F(U) \to \colim_{(U, \overline{\pi})} F(U) = F_{\overline{\pi}} \]
from Categories, Lemma 14.7. Using the description of elements of stalks as triples this maps the element of $F_{\overline{\pi'}}$ represented by the triple $(U, \overline{\pi'}, \sigma)$ to the element of $F_{\overline{\pi}}$ represented by the triple $(U, \overline{\pi'} \circ a, \sigma)$. Since the functor above is clearly an equivalence we conclude that this canonical map is an isomorphism of stalk functors.

Let us make sure we have the map of stalks corresponding to a pointing in the correct direction. Note that the above means, according to Sites, Definition 36.2 that $a$ defines a morphism $a : p \to p'$ between the points $p, p'$ of the site $S_{\text{étale}}$ associated to $\overline{\pi}, \overline{\pi'}$ by Lemma 29.7. There are more general morphisms of points (corresponding to specializations of points of $S$) which we will describe later, and which will not be isomorphisms (insert future reference here).

Lemma 29.9. Let $S$ be a scheme. Let $\overline{\pi}$ be a geometric point of $S$.

(1) The stalk functor $\text{Ab}(S_{\text{étale}}) \to \text{Ab}$, $F \mapsto F_{\overline{\pi}}$ is exact.

(2) We have $(\mathcal{F}^*)_p = F_{\overline{\pi}}$ for any presheaf of sets $\mathcal{F}$ on $S_{\text{étale}}$.

(3) The functor $\text{Ab}(S_{\text{étale}}) \to \text{Ab}$, $F \mapsto F_{\overline{\pi}}$ is exact.
(4) Similarly the functors $\text{PSh}(S_{\text{étale}}) \to \text{Sets}$ and $\text{Sh}(S_{\text{étale}}) \to \text{Sets}$ given by the stalk functor $\mathcal{F} \mapsto \mathcal{F}_s$ are exact (see Categories, Definition 23.1) and commute with arbitrary colimits.

**Proof.** Before we indicate how to prove this by direct arguments we note that the result follows from the general material in Modules on Sites, Section 35. This is true because $\mathcal{F} \mapsto \mathcal{F}_s$ comes from a point of the small étale site of $S$, see Lemma 29.7. We will only give a direct proof of (1), (2) and (3), and omit a direct proof of (4).

Exactness as a functor on $\text{PAb}(S_{\text{étale}})$ is formal from the fact that directed colimits commute with all colimits and with finite limits. The identification of the stalks in (2) is via the map

$$\kappa : \mathcal{F}_s \to (\mathcal{F}^\#)_s$$

induced by the natural morphism $\mathcal{F} \to \mathcal{F}^\#$, see Theorem 13.2. We claim that this map is an isomorphism of abelian groups. We will show injectivity and omit the proof of surjectivity.

Let $\sigma \in \mathcal{F}_s$. There exists an étale neighborhood $(U, \pi) \to (S, \pi)$ such that $\sigma$ is the image of some section $s \in \mathcal{F}(U)$. If $\kappa(\sigma) = 0$ in $(\mathcal{F}^\#)_s$ then there exists a morphism of étale neighborhoods $(U', \pi') \to (U, \pi)$ such that $s|_{U'}$ is zero in $\mathcal{F}^\#(U')$. It follows there exists an étale covering $\{U'_i \to U\}_{i \in I}$ such that $s|_{U'_i} = 0$ in $\mathcal{F}(U'_i)$ for all $i$. By Lemma 29.7 there exist $i \in I$ and a morphism $\pi'_i : \pi \to U'_i$ such that $(U'_i, \pi'_i) \to (U', \pi') \to (U, \pi)$ are morphisms of étale neighborhoods. Hence $\sigma = 0$ since $(U'_i, \pi'_i) \to (U, \pi)$ is a morphism of étale neighbourhoods such that we have $s|_{U'_i} = 0$. This proves $\kappa$ is injective.

To show that the functor $\text{Ab}(S_{\text{étale}}) \to \text{Ab}$ is exact, consider any short exact sequence in $\text{Ab}(S_{\text{étale}})$: $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$. This gives us the exact sequence of presheaves

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to \mathcal{H}/\mathcal{G} \to 0,$$

where $\mathcal{H}/\mathcal{G}$ denotes the quotient in $\text{PAb}(S_{\text{étale}})$. Taking stalks at $\pi$, we see that $(\mathcal{H}/\mathcal{G})_s = (\mathcal{H}/\mathcal{G})_s = 0$, since the sheafification of $\mathcal{H}/\mathcal{G}$ is 0. Therefore,

$$0 \to \mathcal{F}_s \to \mathcal{G}_s \to \mathcal{H}_s \to \mathcal{H}_s/\mathcal{G}_s = 0 \in (\mathcal{H}/\mathcal{G})_\pi$$

is exact, since taking stalks is exact as a functor from presheaves. □

**Theorem 29.10.** Let $S$ be a scheme. A map $a : \mathcal{F} \to \mathcal{G}$ of sheaves of sets is injective (resp. surjective) if and only if the map on stalks $a_\pi : \mathcal{F}_s \to \mathcal{G}_s$ is injective (resp. surjective) for all geometric points of $S$. A sequence of abelian sheaves on $S_{\text{étale}}$ is exact if and only if it is exact on all stalks at geometric points of $S$.

**Proof.** The necessity of exactness on stalks follows from Lemma 29.9. For the converse, it suffices to show that a map of sheaves is surjective (respectively injective) if and only if it is surjective (respectively injective) on all stalks. We prove this in the case of surjectivity, and omit the proof in the case of injectivity.

Let $\alpha : \mathcal{F} \to \mathcal{G}$ be a map of abelian sheaves such that $\mathcal{F}_s \to \mathcal{G}_s$ is surjective for all geometric points. Fix $U \in \text{Ob}(S_{\text{étale}})$ and $s \in \mathcal{G}(U)$. For every $u \in U$ choose some $\pi \to U$ lying over $u$ and an étale neighborhood $(V_u, \pi_u) \to (U, \pi)$ such that $s|_{V_u} = \alpha(s_{V_u})$ for some $s_{V_u} \in \mathcal{F}(V_u)$. This is possible since $\alpha$ is surjective on stalks. Then $\{V_u \to U\}_{u \in U}$ is an étale covering on which the restrictions of $s$ are in the image of the map $\alpha$. Thus, $\alpha$ is surjective, see Sites, Section 12. □
04HU Remarks 29.11. On points of the geometric sites.

(1) Theorem 29.10 says that the family of points of $S_{\text{étale}}$ given by the geometric points of $S$ (Lemma 29.7) is conservative, see Sites, Definition 37.1. In particular $S_{\text{étale}}$ has enough points.

(2) Suppose $\mathcal{F}$ is a sheaf on the big étale site of $S$. Let $T \to S$ be an object of the big étale site of $S$, and let $\mathfrak{t}$ be a geometric point of $T$. Then we define $\mathcal{F}_{\mathfrak{t}}$ as the stalk of the restriction $\mathcal{F}|_{T_{\text{étale}}}$ of $\mathcal{F}$ to the small étale site of $T$. In other words, we can define the stalk of $\mathcal{F}$ at any geometric point of any scheme $T/S \in \text{Ob}((\text{Sch}/S)_{\text{étale}})$.

(3) The big étale site of $S$ also has enough points, by considering all geometric points of all objects of this site, see (2).

The following lemma should be skipped on a first reading.

04HU Lemma 29.12. Let $S$ be a scheme.

(1) Let $p$ be a point of the small étale site $S_{\text{étale}}$ of $S$ given by a functor $u : S_{\text{étale}} \to \text{Sets}$. Then there exists a geometric point $\pi$ of $S$ such that $p$ is isomorphic to the point of $S_{\text{étale}}$ associated to $\pi$ in Lemma 29.7.

(2) Let $p : \text{Sh}(\text{pt}) \to \text{Sh}(S_{\text{étale}})$ be a point of the small étale topos of $S$. Then $p$ comes from a geometric point of $S$, i.e., the stalk functor $\mathcal{F} \mapsto \mathcal{F}_p$ is isomorphic to a stalk functor as defined in Definition 29.6.

Proof. By Sites, Lemma 31.7 there is a one to one correspondence between points of the site and points of the associated topos, hence it suffices to prove (1). By Sites, Proposition 32.2 the functor $u$ has the following properties: (a) $u(S) = \{\ast\}$, (b) $u(U \times_V W) = u(U) \times_{u(V)} u(W)$, and (c) if $\{U_i \to U\}$ is an étale covering, then $\prod u(U_i) \to u(U)$ is surjective. In particular, if $U' \subset U$ is an open subscheme, then $u(U') \subset u(U)$. Moreover, by Sites, Lemma 31.7 we can write $u(U) = p^{-1}(h_U^\#)$, in other words $u(U)$ is the stalk of the representable sheaf $h_U$. If $U = V \amalg W$, then we see that $h_U = (h_V \amalg h_W)^\#$ and we get $u(U) = u(V) \amalg u(W)$ since $p^{-1}$ is exact.

Consider the restriction of $u$ to $S_{\text{Zar}}$. By Sites, Examples 32.4 and 32.5 there exists a unique point $s \in S$ such that for $S' \subset S$ open we have $u(S') = \{\ast\}$ if $s \in S'$ and $u(S') = \emptyset$ if $s \notin S'$. Note that if $\varphi : U \to S$ is an object of $S_{\text{étale}}$ then $\varphi(U) \subset S$ is open (see Proposition 26.2) and $\{U \to \varphi(U)\}$ is an étale covering. Hence we conclude that $u(U) = \emptyset \iff s \in \varphi(U)$.

Pick a geometric point $\pi : \pi \to S$ lying over $s$, see Definition 29.1 for customary abuse of notation. Suppose that $\varphi : U \to S$ is an object of $S_{\text{étale}}$ with $U$ affine. Note that $\varphi$ is separated, and that the fibre $U_s$ of $\varphi$ over $s$ is an affine scheme over $\text{Spec}(k(s))$ which is the spectrum of a finite product of finite separable extensions $k_i$ of $k(s)$. Hence we may apply Étale Morphisms, Lemma 18.2 to get an étale neighbourhood $(V, \pi)$ of $(S, \pi)$ such that

$$U \times_S V = U_1 \amalg \ldots \amalg U_n \amalg W$$

with $U_i \to V$ an isomorphism and $W$ having no point lying over $\pi$. Thus we conclude that

$$u(U) \times u(V) = u(U \times_S V) = u(U_1) \amalg \ldots \amalg u(U_n) \amalg u(W)$$

and of course also $u(U_i) = u(V)$. After shrinking $V$ a bit we can assume that $V$ has exactly one point lying over $s$, and hence $W$ has no point lying over $s$. By the
above this then gives \( u(W) = \emptyset \). Hence we obtain
\[
 u(U) \times u(V) = u(U_1) \amalg \ldots \amalg u(U_n) = \coprod_{i=1, \ldots, n} u(V)
\]
Note that \( u(V) \neq \emptyset \) as \( s \) is in the image of \( V \to S \). In particular, we see that in this situation \( u(U) \) is a finite set with \( n \) elements.

Consider the limit
\[
 \lim_{\text{limit}} (u(V))
\]
over the category of étale neighbourhoods \((V, \overline{V})\) of \( S \). It is clear that we get the same value when taking the limit over the subcategory of \((V, \overline{V})\) with \( V \) affine. By the previous paragraph (applied with the roles of \( V \) and \( U \) switched) we see that in this case \( u(V) \) is always a finite nonempty set. Moreover, the limit is cofiltered, see Lemma \ref{affine-limit-cofiltered}. Hence by Categories, Section \ref{categories-cofiltered} the limit is nonempty. Pick an element \( x \) from this limit. This means we obtain a \( x_{V, \overline{V}} \in u(V) \) for every étale neighbourhood \((V, \overline{V})\) of \((S, \overline{S})\) such that for every morphism of étale neighbourhoods \( \varphi : (V', \overline{V'}) \to (V, \overline{V}) \) we have \( u(\varphi)(x_{V', \overline{V'}}) = x_{V, \overline{V}} \).

We will use the choice of \( x \) to construct a functorial bijective map
\[
c : |U_S| \to u(U)
\]
for \( U \in \text{Ob}(S_{\text{étale}}) \) which will conclude the proof. See Lemma \ref{affine-limit-cofiltered} and its proof for a description of \(|U_S|\). First we claim that it suffices to construct the map for \( U \) affine. We omit the proof of this claim. Assume \( U \to S \) in \( S_{\text{étale}} \) with \( U \) affine, and let \( \overline{\pi} : \overline{S} \to U \) be an element of \(|U_S|\). Choose a \((V, \overline{V})\) such that \( U \times_S V \) decomposes as in the third paragraph of the proof. Then the pair \((\overline{\pi}, \overline{V})\) gives a geometric point of \( U \times_S V \) lying over \( \overline{V} \) and determines one of the components \( U_i \) of \( U \times_S V \).

More precisely, there exists a section \( \sigma : V \to U \times_S V \) of the projection \( \text{pr}_1 \) such that \((\overline{\pi}, \overline{V}) = (\sigma, \overline{V}) \). Set \( c(\overline{V}) = u(\text{pr}_1)(u(\sigma)(x_{V, \overline{V}})) \in u(U) \). We have to check this is independent of the choice of \((V, \overline{V})\). By Lemma \ref{affine-limit-cofiltered} the category of étale neighbourhoods is cofiltered. Hence it suffice to show that given a morphism of étale neighbourhood \( \varphi : (V', \overline{V'}) \to (V, \overline{V}) \) and a choice of a section \( \sigma' : V' \to U \times_S V' \) of the projection such that \((\overline{\pi}, \overline{V'}) = (\sigma', \overline{V'}) \) we have \( u(\sigma')(x_{V', \overline{V'}}) = u(\sigma)(x_{V, \overline{V}}) \).

Consider the diagram
\[
\begin{array}{ccc}
V' & \xrightarrow{\varphi} & V \\
\downarrow{\sigma'} & & \downarrow{\sigma} \\
U \times_S V' & \xrightarrow{1 \times \varphi} & U \times_S V
\end{array}
\]
Now, it may not be the case that this diagram commutes. The reason is that the schemes \( V' \) and \( V \) may not be connected, and hence the decompositions used to construct \( \sigma' \) and \( \sigma \) above may not be unique. But we do know that \( \sigma \circ \varphi \circ \overline{\sigma'} = (1 \times \varphi) \circ \sigma \circ \overline{\sigma} \) by construction. Hence, since \( U \times_S V \) is étale over \( S \), there exists an open neighbourhood \( V'' \subset V' \) of \( \overline{V'} \) such that the diagram does commute when restricted to \( V'' \), see Morphisms, Lemma \ref{morphisms-cofiltered}. This means we may extend the diagram above to
\[
\begin{array}{ccc}
V'' & \xrightarrow{\varphi} & V \\
\downarrow{{\sigma'}|_{V''}} & & \downarrow{\sigma} \\
U \times_S V'' & \xrightarrow{1 \times \varphi} & U \times_S V
\end{array}
\]
such that the left square and the outer rectangle commute. Since \( u \) is a functor this implies that \( x_{V'} \pi' \) maps to the same element in \( u(U \times_S V) \) no matter which route we take through the diagram. On the other hand, it maps to the elements \( x_{V'} \pi' \) and \( x_{V} \pi \) in \( u(V') \) and \( u(V) \). This implies the desired equality \( u(\sigma')(x_{V'} \pi') = u(\sigma)(x_{V} \pi) \).

In a similar manner one proves that the construction \( c : [U_\pi] \to u(U) \) is functorial in \( U \); details omitted. And finally, by the results of the third paragraph it is clear that the map \( c \) is bijective which ends the proof of the lemma. \( \square \)

### 30. Points in other topologies

In this section we briefly discuss the existence of points for some sites other than the étale site of a scheme. We refer to Sites, Section \( \text{[37]} \) and Topologies, Section \( \text{[2]} \) ff for the terminology used in this section. All of the geometric sites have enough points.

**Lemma 30.1.** Let \( S \) be a scheme. All of the following sites have enough points \( S_{\text{Zar}}, S_{\text{étale}}, (\text{Sch}/S)_{\text{Zar}}, (\text{Aff}/S)_{\text{Zar}}, (\text{Sch}/S)_{\text{étale}}, (\text{Sch}/S)_{\text{smooth}}, (\text{Aff}/S)_{\text{smooth}}, (\text{Sch}/S)_{\text{syntomic}}, (\text{Aff}/S)_{\text{syntomic}}, (\text{Sch}/S)_{fppf}, \) and \( (\text{Aff}/S)_{fppf} \).

**Proof.** For each of the big sites the associated topos is equivalent to the topos defined by the site \( (\text{Aff}/S)_\tau \), see Topologies, Lemmas \( \text{[3.10, 4.11, 5.9, 6.9, 7.11]} \). The result for the sites \( (\text{Aff}/S)_\tau \) follows immediately from Deligne’s result applied to \( S_{\text{affine, étale}} \). \( \square \)

The lemma above guarantees the existence of points, but it doesn’t tell us what these points look like. We can explicitly construct some points as follows. Suppose \( \pi : \text{Spec}(k) \to S \) is a geometric point with \( k \) algebraically closed. Consider the functor

\[
\begin{align*}
u & : (\text{Sch}/S)_{fppf} \to \text{Sets}, \quad u(U) = U(k) = \text{Mor}_S(\text{Spec}(k), U).
\end{align*}
\]

Note that \( U \to U(k) \) commutes with direct limits as \( S(k) = \{ \pi \} \) and \( (U_1 \times_U U_2)(k) = U_1(k) \times_U U_2(k) \). Moreover, if \( \{ U_i \to U \} \) is an fppf covering, then \( \coprod U_i(k) \to U(k) \) is surjective. By Sites, Proposition \( \text{[32.2]} \) we see that \( u \) defines a point \( p \) of \( (\text{Sch}/S)_{fppf} \) with stalks

\[
F_p = \text{colim}_{(U,x)} F(U)
\]

where the colimit is over pairs \( U \to S, x \in U(k) \) as usual. But... this category has an initial object, namely \( (\text{Spec}(k), \text{id}) \), hence we see that

\[
F_p = F(\text{Spec}(k))
\]

which isn’t terribly interesting! In fact, in general these points won’t form a conservative family of points. A more interesting type of point is described in the following remark.

**Remark 30.2.** Let \( S = \text{Spec}(A) \) be an affine scheme. Let \( (p,u) \) be a point of the site \( (\text{Aff}/S)_{fppf} \), see Sites, Sections \( \text{[31 and 32]} \). Let \( B = \mathcal{O}_p \) be the stalk of the structure sheaf at the point \( p \). Recall that

\[
B = \text{colim}_{(U,x)} \mathcal{O}(U) = \text{colim}_{(\text{Spec}(C), x_C)} C
\]

This is discussed in Sites, Section \( \text{[31]} \).
where \( x_C \in u(\text{Spec}(C)) \). It can happen that \( \text{Spec}(B) \) is an object of \((\text{Aff}/S)_{fppf}\) and that there is an element \( x_B \in u(\text{Spec}(B)) \) mapping to the compatible system \( x_C \). In this case the system of neighbourhoods has an initial object and it follows that \( F_p = F(\text{Spec}(B)) \) for any sheaf \( F \) on \((\text{Aff}/S)_{fppf}\). It is straightforward to see that if \( F \mapsto F(\text{Spec}(B)) \) defines a point of \( \text{Sh}((\text{Aff}/S)_{fppf}) \), then \( B \) has to be a local \( A \)-algebra such that for every faithfully flat, finitely presented ring map \( B \to B' \) there is a section \( B' \to B \). Conversely, for any such \( A \)-algebra \( B \) the functor \( F \mapsto F(\text{Spec}(B)) \) is the stalk functor of a point. Details omitted. It is not clear what a general point of the site \((\text{Aff}/S)_{fppf}\) looks like.

### 31. Supports of abelian sheaves

#### Lemma 31.1.

Let \( S \) be a scheme. Let \( F \) be a subsheaf of the final object of the étale topos of \( S \) (see Sites, Example 10.2). Then there exists a unique open \( W \subset S \) such that \( F = h_W \).

**Proof.** The condition means that \( F(U) \) is a singleton or empty for all \( \varphi : U \to S \) in \( \text{Ob}(\text{S}_\text{étale}) \). In particular local sections always glue. If \( F(U) \neq \emptyset \), then \( \varphi \) is an étale covering. Hence we can take \( W = \bigcup_{\varphi : U \to S, F(U) \neq \emptyset} \varphi(U) \).

#### Lemma 31.2.

Let \( S \) be a scheme. Let \( F \) be an abelian sheaf on \( \text{S}_\text{étale} \). Let \( \sigma \in F(U) \) be a local section. There exists an open subset \( W \subset U \) such that

1. \( W \subset U \) is the largest Zariski open subset of \( U \) such that \( \sigma|_W = 0 \),
2. for every \( \varphi : V \to U \) in \( \text{S}_\text{étale} \) we have \( \sigma|_V = 0 \iff \varphi(V) \subset W \),
3. for every geometric point \( \overline{\pi} \) of \( U \) we have \( (U, \overline{\pi}, \sigma) = 0 \) in \( F_\overline{\pi} \iff \overline{\pi} \in W \).

**Proof.** Since \( F \) is a sheaf in the étale topology the restriction of \( F \) to \( U_{\text{Zar}} \) is a sheaf on \( U \) in the Zariski topology. Hence there exists a Zariski open \( W \) having property (1), see Modules, Lemma 5.2. Let \( \varphi : V \to U \) be an arrow of \( S_\text{étale} \). Note that \( \varphi(V) \subset U \) is an open subset and that \( \{ V \to \varphi(V) \} \) is an étale covering. Hence if \( \sigma|_V = 0 \), then by the sheaf condition for \( F \) we see that \( \sigma|_{\varphi(V)} = 0 \). This proves (2). To prove (3) we have to show that if \( (U, \overline{\pi}, \sigma) \) defines the zero element of \( F_\overline{\pi} \), then \( \overline{\pi} \in W \). This is true because the assumption means there exists a morphism of étale neighbourhoods \( (V, \overline{\pi}) \to (U, \overline{\pi}) \) such that \( \sigma|_V = 0 \). Hence by (2) we see that \( V \to U \) maps into \( W \), and hence \( \overline{\pi} \in W \).

Let \( S \) be a scheme. Let \( s \in S \). Let \( F \) be a sheaf on \( S_\text{étale} \). By Remark 29.8 the isomorphism class of the stalk of the sheaf \( F \) at a geometric points lying over \( s \) is well defined.

#### Definition 31.3.

Let \( S \) be a scheme. Let \( F \) be an abelian sheaf on \( S_\text{étale} \).

1. The support of \( F \) is the set of points \( s \in S \) such that \( F_s \neq 0 \) for any (some) geometric point \( \overline{\pi} \) lying over \( s \).
(2) Let $\sigma \in \mathcal{F}(U)$ be a section. The support of $\sigma$ is the closed subset $U \setminus W$, where $W \subset U$ is the largest open subset of $U$ on which $\sigma$ restricts to zero (see Lemma 31.2).

In general the support of an abelian sheaf is not closed. For example, suppose that $S = \text{Spec}(A[1])$. Let $i_t : \text{Spec}(C) \to S$ be the inclusion of the point $t \in C$. We will see later that $F_t = i_{t,*}(\mathbb{Z}/2\mathbb{Z})$ is an abelian sheaf whose support is exactly $\{t\}$, see Section 47. Then

\[ \bigoplus_{n \in \mathbb{N}} F_n \]

is an abelian sheaf with support $\{1, 2, 3, \ldots\} \subset S$. This is true because taking stalks commutes with colimits, see Lemma 29.9. Thus an example of an abelian sheaf whose support is not closed. Here are some basic facts on supports of sheaves and sections.

\underline{Lemma 31.4.} Let $S$ be a scheme. Let $F$ be an abelian sheaf on $S_{\text{etale}}$. Let $U \in \text{Ob}(S_{\text{etale}})$ and $\sigma \in \mathcal{F}(U)$.

1. The support of $\sigma$ is closed in $U$.
2. The support of $\sigma + \sigma'$ is contained in the union of the supports of $\sigma, \sigma' \in \mathcal{F}(U)$.
3. If $\varphi : \mathcal{F} \to \mathcal{G}$ is a map of abelian sheaves on $S_{\text{etale}}$, then the support of $\varphi(\sigma)$ is contained in the support of $\sigma \in \mathcal{F}(U)$.
4. The support of $\mathcal{F}$ is the union of the images of the supports of all local sections of $\mathcal{F}$.
5. If $\mathcal{F} \to \mathcal{G}$ is surjective then the support of $\mathcal{G}$ is a subset of the support of $\mathcal{F}$.
6. If $\mathcal{F} \to \mathcal{G}$ is injective then the support of $\mathcal{F}$ is a subset of the support of $\mathcal{G}$.

\textbf{Proof.} Part (1) holds by definition. Parts (2) and (3) hold because they hold for the restriction of $\mathcal{F}$ and $\mathcal{G}$ to $U_{\text{Zar}}$, see Modules, Lemma 5.2. Part (4) is a direct consequence of Lemma 31.2 part (3). Parts (5) and (6) follow from the other parts. \qed

\underline{Lemma 31.5.} The support of a sheaf of rings on $S_{\text{etale}}$ is closed.

\textbf{Proof.} This is true because (according to our conventions) a ring is 0 if and only if $1 = 0$, and hence the support of a sheaf of rings is the support of the unit section. \qed

\section{32. Henselian rings}

We begin by stating a theorem which has already been used many times in the stacks project. There are many versions of this result; here we just state the algebraic version.

\underline{Theorem 32.1.} Let $A \to B$ be finite type ring map and $p \subset A$ a prime ideal. Then there exist an étale ring map $A \to A'$ and a prime $p' \subset A'$ lying over $p$ such that

1. $\kappa(p) = \kappa(p')$,
2. $B \otimes_A A' = B_1 \times \ldots \times B_r \times C$,
3. $A' \to B_i$ is finite and there exists a unique prime $q_i \subset B_i$ lying over $p'$, and
4. all irreducible components of the fibre $\text{Spec}(C \otimes_{A'} \kappa(p'))$ of $C$ over $p'$ have dimension at least 1.
Proof. See Algebra, Lemma [141.23] or see [GD67, Théorème 18.12.1]. For a slew of versions in terms of morphisms of schemes, see More on Morphisms, Section 30.

Recall Hensel’s lemma. There are many versions of this lemma. Here are two:

(f) if \( f \in \mathbb{Z}_p[T] \) monic and \( f \mod p = g_0h_0 \) with \( \gcd(g_0, h_0) = 1 \) then \( f \) factors as \( f = gh \) with \( \overline{g} = g_0 \) and \( \overline{h} = h_0 \).

(r) if \( f \in \mathbb{Z}_p[T] \), monic \( a_0 \in \mathbb{F}_p \), \( \bar{f}(a_0) = 0 \) but \( \bar{f}'(a_0) \neq 0 \) then there exists \( a \in \mathbb{Z}_p \) with \( f(a) = 0 \) and \( \bar{a} = a_0 \).

Both versions are true (we will see this later). The first version asks for lifts of factorizations into coprime parts, and the second version asks for lifts of simple roots modulo the maximal ideal. It turns out that requiring these conditions for a general local ring are equivalent, and are equivalent to many other conditions. We use the root lifting property as the definition of a henselian local ring as it is often the easiest one to check.

Definition 32.2. (See Algebra, Definition [148.1]) A local ring \((R, m, \kappa)\) is called henselian if for all \( f \in R[T] \) monic, for all \( a_0 \in \kappa \) such that \( \bar{f}(a_0) = 0 \) and \( \bar{f}'(a_0) \neq 0 \), there exists an \( a \in R \) such that \( f(a) = 0 \) and \( a \mod m = a_0 \).

A good example of henselian local rings to keep in mind is complete local rings. Recall (Algebra, Definition [154.1]) that a complete local ring is a local ring \((R, m)\) such that \( R \cong \lim_n R/m^n \), i.e., it is complete and separated for the \( m \)-adic topology.

Theorem 32.3. Complete local rings are henselian.

Proof. Newton’s method. See Algebra, Lemma [148.9] □

Theorem 32.4. Let \((R, m, \kappa)\) be a local ring. The following are equivalent:

1. \( R \) is henselian,
2. for any \( f \in R[T] \) and any factorization \( \bar{f} = g_0h_0 \) in \( \kappa[T] \) with \( \gcd(g_0, h_0) = 1 \), there exists a factorization \( f = gh \) in \( R[T] \) with \( \bar{g} = g_0 \) and \( \bar{h} = h_0 \),
3. any finite \( R \)-algebra \( S \) is isomorphic to a finite product of finite local rings,
4. any finite type \( R \)-algebra \( A \) is isomorphic to a product \( A \cong A' \times C \) where \( A' \cong A_1 \times \ldots \times A_r \) is a product of finite local \( R \)-algebras and all the irreducible components of \( C \otimes_R \kappa \) have dimension at least 1,
5. if \( A \) is an étale \( R \)-algebra and \( n \) is a maximal ideal of \( A \) lying over \( m \) such that \( \kappa \cong A/n \), then there exists an isomorphism \( \varphi : A \cong R \times A' \) such that \( \varphi(n) = m \times A' \subset R \times A' \).

Proof. This is just a subset of the results from Algebra, Lemma [148.3] Note that part (5) above corresponds to part (8) of Algebra, Lemma [148.3] but is formulated slightly differently. □

Lemma 32.5. If \( R \) is henselian and \( A \) is a finite \( R \)-algebra, then \( A \) is a finite product of henselian local rings.

Proof. See Algebra, Lemma [148.4] □

Definition 32.6. A local ring \( R \) is called strictly henselian if it is henselian and its residue field is separably closed.
Example 32.7. In the case $R = \mathbb{C}[t]$, the étale $R$-algebras are finite products of the trivial extension $R \to R$ and the extensions $R \to R[X, X^{-1}]/(X^n - t)$. The latter ones factor through the open $D(t) \subset \text{Spec}(R)$, so any étale covering can be refined by the covering $\{\text{id} : \text{Spec}(R) \to \text{Spec}(R)\}$. We will see below that this is a somewhat general fact on étale coverings of spectra of henselian rings. This will show that higher étale cohomology of the spectrum of a strictly henselian ring is zero.

Theorem 32.8. Let $(R, m, \kappa)$ be a local ring and $\kappa \subset \kappa^{\text{sep}}$ a separable algebraic closure. There exist canonical flat local ring maps $R \to R^h \to R^{sh}$ where

1. $R^h, R^{sh}$ are filtered colimits of étale $R$-algebras,
2. $R^h$ is henselian, $R^{sh}$ is strictly henselian,
3. $mR^h$ (resp. $mR^{sh}$) is the maximal ideal of $R^h$ (resp. $R^{sh}$), and
4. $\kappa = R^h/mR^h$, and $\kappa^{\text{sep}} = R^{sh}/mR^{sh}$ as extensions of $\kappa$.

Proof. The structure of $R^h$ and $R^{sh}$ is described in Algebra, Lemmas 150.1 and 150.2.

The rings constructed in Theorem 32.8 are called respectively the henselization and the strict henselization of the local ring $R$, see Algebra, Definition 150.3. Many of the properties of $R$ are reflected in its (strict) henselization, see More on Algebra, Section 36.

33. Stalks of the structure sheaf

In this section we identify the stalk of the structure sheaf at a geometric point with the strict henselization of the local ring at the corresponding “usual” point.

Lemma 33.1. Let $S$ be a scheme. Let $\overline{s}$ be a geometric point of $S$ lying over $s \in S$. Let $\kappa = \kappa(s)$ and let $\kappa \subset \kappa^{\text{sep}} \subset \kappa(\overline{s})$ denote the separable algebraic closure of $\kappa$ in $\kappa(\overline{s})$. Then there is a canonical identification

$$(\mathcal{O}_{S,s})^{\text{sh}} \cong \mathcal{O}_{S,\overline{s}}$$

where the left hand side is the strict henselization of the local ring $\mathcal{O}_{S,s}$ as described in Theorem 32.8 and right hand side is the stalk of the structure sheaf $\mathcal{O}_{S}$ on $S_{\text{étale}}$ at the geometric point $\overline{s}$.

Proof. Let $\text{Spec}(A) \subset S$ be an affine neighbourhood of $s$. Let $p \subset A$ be the prime ideal corresponding to $s$. With these choices we have canonical isomorphisms $\mathcal{O}_{S,s} = A_p$ and $\kappa(s) = \kappa(p)$. Thus we have $\kappa(p) \subset \kappa^{\text{sep}} \subset \kappa(\overline{s})$. Recall that

$$\mathcal{O}_{S,\overline{s}} = \varinjlim_{(U, \pi)} \mathcal{O}(U)$$

where the limit is over the étale neighbourhoods of $(S, \overline{s})$. A cofinal system is given by those étale neighbourhoods $(U, \pi)$ such that $U$ is affine and $U \to S$ factors through $\text{Spec}(A)$. In other words, we see that

$$\mathcal{O}_{S,\overline{s}} = \varinjlim_{(B, q, \phi)} B$$

where the colimit is over étale $A$-algebras $B$ endowed with a prime $q$ lying over $p$ and a $\kappa(p)$-algebra map $\phi : \kappa(q) \to \kappa(\overline{s})$. Note that since $\kappa(q)$ is finite separable over $\kappa(p)$ the image of $\phi$ is contained in $\kappa^{\text{sep}}$. Via these translations the result of the lemma is equivalent to the result of Algebra, Lemma 150.13. □
**Definition 33.2.** Let $S$ be a scheme. Let $\pi$ be a geometric point of $S$ lying over the point $s \in S$.

1. The étale local ring of $S$ at $\pi$ is the stalk of the structure sheaf $\mathcal{O}_S$ on $S_{\text{étale}}$ at $\pi$. We sometimes call this the strict henselization of $\mathcal{O}_{S,s}$ relative to the geometric point $\pi$. Notation used: $\mathcal{O}_{S,\pi} = \mathcal{O}^{sh}_{S,s}$.

2. The henselization of $\mathcal{O}_{S,s}$ is the henselization of the local ring of $S$ at $s$.

   See Algebra, Definition 150.3 and Theorem 32.8. Notation: $\mathcal{O}^h_{S,s}$.

3. The strict henselization of $S$ at $s$ is the scheme $\text{Spec}(\mathcal{O}^{sh}_{S,s})$.

4. The henselization of $S$ at $s$ is the scheme $\text{Spec}(\mathcal{O}^h_{S,s})$.

**Lemma 33.3.** Let $S$ be a scheme. Let $s \in S$. Then we have

\[
\mathcal{O}^h_{S,s} = \colim (U,u) \mathcal{O}(U)
\]

where the colimit is over the filtered category of étale neighbourhoods $(U,u)$ of $(S,s)$ such that $\kappa(s) = \kappa(u)$.

**Proof.** This lemma is a copy of More on Morphisms, Lemma 27.3. \qed

**Remark 33.4.** Let $S$ be a scheme. Let $s \in S$. If $S$ is locally noetherian then $\mathcal{O}^h_{S,s}$ is also noetherian and it has the same completion:

\[
\widehat{\mathcal{O}_{S,s}} \cong \widehat{\mathcal{O}^h_{S,s}}.
\]

In particular, $\mathcal{O}_{S,s} \subset \mathcal{O}^h_{S,s} \subset \widehat{\mathcal{O}_{S,s}}$. The henselization of $\mathcal{O}_{S,s}$ is in general much smaller than its completion and inherits many of its properties. For example, if $\mathcal{O}_{S,s}$ is reduced, then so is $\mathcal{O}^h_{S,s}$, but this is not true for the completion in general.

Insert future references here.

**Lemma 33.5.** Let $S$ be a scheme. The small étale site $S_{\text{étale}}$ endowed with its structure sheaf $\mathcal{O}_S$ is a locally ringed site, see Modules on Sites, Definition 39.4.

**Proof.** This follows because the stalks $\mathcal{O}^h_{S,s} = \mathcal{O}_{S,\pi}$ are local, and because $S_{\text{étale}}$ has enough points, see Lemma 33.1, Theorem 29.10 and Remarks 29.11. See Modules on Sites, Lemmas 39.2 and 39.3 for the fact that this implies the small étale site is locally ringed. \qed

### 34. Functoriality of small étale topos

**04IO** So far we haven’t yet discussed the functoriality of the étale site, in other words what happens when given a morphism of schemes. A precise formal discussion can be found in Topologies, Section 4. In this and the next sections we discuss this material briefly specifically in the setting of small étale sites.

Let $f : X \rightarrow Y$ be a morphism of schemes. We obtain a functor

\[
u : Y_{\text{étale}} \longrightarrow X_{\text{étale}}, \quad V/Y \mapsto X \times_Y V/X.
\]

This functor has the following important properties

1. $\nu(\text{final object}) = \text{final object},$

2. $\nu$ preserves fibre products,

3. if $\{V_j \rightarrow V\}$ is a covering in $Y_{\text{étale}}$, then $\{\nu(V_j) \rightarrow \nu(V)\}$ is a covering in $X_{\text{étale}}.$
Each of these is easy to check (omitted). As a consequence we obtain what is called a morphism of sites

$$f_{small} : X_{étale} \to Y_{étale},$$

see Sites, Definition 15.1 and Sites, Proposition 15.6. It is not necessary to know about the abstract notion in detail in order to work with étale sheaves and étale cohomology. It usually suffices to know that there are functors $f_{small,*}$ (pushforward) and $f_{small}^{-1}$ (pullback) on étale sheaves, and to know some of their simple properties. We will discuss these properties in the next sections, but we will sometimes refer to the more abstract material for proofs since that is often the natural setting to prove them.

35. Direct images

03PV Let us define the pushforward of a presheaf.

03PW Definition 35.1. Let $f : X \to Y$ be a morphism of schemes. Let $\mathcal{F}$ a presheaf of sets on $X_{étale}$. The direct image, or pushforward of $\mathcal{F}$ (under $f$) is

$$f_* \mathcal{F} : Y_{étale}^{opp} \to Sets, \quad (V/Y) \mapsto \mathcal{F}(X \times_Y V/X).$$

We sometimes write $f_* = f_{small,*}$ to distinguish from other direct image functors (such as usual Zariski pushforward or $f_{big,*}$).

This is a well-defined étale presheaf since the base change of an étale morphism is again étale. A more categorical way of saying this is that $f_* \mathcal{F}$ is the composition of functors $\mathcal{F} \circ u$ where $u$ is as in Equation (34.0.1). This makes it clear that the construction is functorial in the presheaf $\mathcal{F}$ and hence we obtain a functor

$$f_* = f_{small,*} : PSh(X_{étale}) \to PSh(Y_{étale})$$

Note that if $\mathcal{F}$ is a presheaf of abelian groups, then $f_* \mathcal{F}$ is also a presheaf of abelian groups and we obtain

$$f_* = f_{small,*} : PAb(X_{étale}) \to PAb(Y_{étale})$$

as before (i.e., defined by exactly the same rule).

03PX Remark 35.2. We claim that the direct image of a sheaf is a sheaf. Namely, if $\{V_i \to V\}$ is an étale covering in $Y_{étale}$ then $\{X \times_Y V_i \to X \times_Y V\}$ is an étale covering in $X_{étale}$. Hence the sheaf condition for $\mathcal{F}$ with respect to $\{X \times_Y V_i \to X \times_Y V\}$ is equivalent to the sheaf condition for $f_* \mathcal{F}$ with respect to $\{V_i \to V\}$. Thus if $\mathcal{F}$ is a sheaf, so is $f_* \mathcal{F}$.

03PY Definition 35.3. Let $f : X \to Y$ be a morphism of schemes. Let $\mathcal{F}$ a sheaf of sets on $X_{étale}$. The direct image, or pushforward of $\mathcal{F}$ (under $f$) is

$$f_* \mathcal{F} : Y_{étale}^{opp} \to Sets, \quad (V/Y) \mapsto \mathcal{F}(X \times_Y V/X)$$

which is a sheaf by Remark 35.2. We sometimes write $f_* = f_{small,*}$ to distinguish from other direct image functors (such as usual Zariski pushforward or $f_{big,*}$).

The exact same discussion as above applies and we obtain functors

$$f_* = f_{small,*} : Sh(X_{étale}) \to Sh(Y_{étale})$$

and

$$f_* = f_{small,*} : Ab(X_{étale}) \to Ab(Y_{étale})$$

called direct image again.
The functor $f_*$ on abelian sheaves is left exact. (See Homology, Section 4 for what it means for a functor between abelian categories to be left exact.) Namely, if $0 \to F_1 \to F_2 \to F_3$ is exact on $X_{\text{étale}}$, then for every $U/X \in \text{Ob}(X_{\text{étale}})$ the sequence of abelian groups $0 \to f_*(U) \to f_*(U) \to f_*(U)$ is exact. Hence for every $V/Y \in \text{Ob}(Y_{\text{étale}})$ the sequence of abelian groups $0 \to f_*(V) \to f_*(V) \to f_*(V)$ is exact, because this is the previous sequence with $U = X \times_Y V$.

**Definition 35.4.** Let $f : X \to Y$ be a morphism of schemes. The right derived functors $(R^p f_*)_{p \geq 1}$ of $f_* : \text{Ab}(X_{\text{étale}}) \to \text{Ab}(Y_{\text{étale}})$ are called higher direct images.

The higher direct images and their derived category variants are discussed in more detail in (insert future reference here).

### 36. Inverse image

In this section we briefly discuss pullback of sheaves on the small étale sites. The precise construction of this is in Topologies, Section 4.

**Definition 36.1.** Let $f : X \to Y$ be a morphism of schemes. The inverse image, or pullback functors are the functors

$$f^{-1} = f_{\text{small}}^{-1} : \text{Sh}(Y_{\text{étale}}) \longrightarrow \text{Sh}(X_{\text{étale}})$$

and

$$f^{-1} = f_{\text{small},*}^{-1} : \text{Ab}(Y_{\text{étale}}) \longrightarrow \text{Ab}(X_{\text{étale}})$$

which are left adjoint to $f_* = f_{\text{small},*}$. Thus $f^{-1}$ thus characterized by the fact that

$$\text{Hom}_{\text{Sh}(X_{\text{étale}})}(f^{-1} G, F) = \text{Hom}_{\text{Sh}(Y_{\text{étale}})}(G, f_* F)$$

functorially, for any $F \in \text{Sh}(X_{\text{étale}})$ and $G \in \text{Sh}(Y_{\text{étale}})$. We similarly have

$$\text{Hom}_{\text{Ab}(X_{\text{étale}})}(f^{-1} G, F) = \text{Hom}_{\text{Ab}(Y_{\text{étale}})}(G, f_* F)$$

for $F \in \text{Ab}(X_{\text{étale}})$ and $G \in \text{Ab}(Y_{\text{étale}})$.

It is not trivial that such an adjoint exists. On the other hand, it exists in a fairly general setting, see Remark 36.3 below. The general machinery shows that $f^{-1} G$ is the sheaf associated to the presheaf

$$(36.1.1) \quad U/X \mapsto \text{colim}_{U \to X \times_Y V} G(V/Y)$$

where the colimit is over the category of pairs $(V/Y, \varphi : U/X \to X \times_Y V/X)$. To see this apply Sites, Proposition 15.6 to the functor $u$ of Equation (34.0.1) and use the description of $u_* = (u_y)^#$ in Sites, Sections 14 and 5. We will occasionally use this formula for the pullback in order to prove some of its basic properties.

**Lemma 36.2.** Let $f : X \to Y$ be a morphism of schemes.

1. The functor $f^{-1} : \text{Ab}(Y_{\text{étale}}) \to \text{Ab}(X_{\text{étale}})$ is exact.
2. The functor $f^{-1} : \text{Sh}(Y_{\text{étale}}) \to \text{Sh}(X_{\text{étale}})$ is exact, i.e., it commutes with finite limits and colimits, see Categories, Definition 23.4.

---

2We use the notation $f^{-1}$ for pullbacks of sheaves of sets or sheaves of abelian groups, and we reserve $f^*$ for pullbacks of sheaves of modules via a morphism of ringed sites/topoi.
(3) Let \( x \to X \) be a geometric point. Let \( \mathcal{G} \) be a sheaf on \( Y_{\text{étale}} \). Then there is a canonical identification

\[
(f^{-1}\mathcal{G})_{\overline{x}} = \mathcal{G}_{\overline{y}}.
\]

where \( \overline{y} = f \circ \overline{x} \).

(4) For any \( V \to Y \) étale we have \( f^{-1}h_V = h_{X \times_Y V} \).

**Proof.** The exactness of \( f^{-1} \) on sheaves of sets is a consequence of Sites, Proposition \( 15.6 \) applied to our functor \( u \) of Equation \( (34.0.1) \). In fact the exactness of pullback is part of the definition of of a morphism of topoi (or sites if you like). Thus we see (2) holds. It implies part (1) since given an abelian sheaf \( \mathcal{G} \) on \( Y_{\text{étale}} \) the underlying sheaf of sets of \( f^{-1}\mathcal{F} \) is the same as \( f^{-1} \) of the underlying sheaf of sets of \( \mathcal{F} \), see Sites, Section \( 43 \). See also Modules on Sites, Lemma \( 30.2 \). In the literature (1) and (2) are sometimes deduced from (3) via Theorem \( 29.10 \).

Part (3) is a general fact about stalks of pullbacks, see Sites, Lemma \( 33.1 \). We will also prove (3) directly as follows. Note that by Lemma \( 29.9 \) taking stalks commutes with sheafification. Now recall that \( f^{-1}\mathcal{G} \) is the sheaf associated to the presheaf

\[
U \mapsto \text{colim}_{U \to X \times_Y V} \mathcal{G}(V),
\]

see Equation \( (36.1.1) \). Thus we have

\[
(f^{-1}\mathcal{G})_{\overline{x}} = \text{colim}_{(U,\pi)} f^{-1}\mathcal{G}(U) = \text{colim}_{(U,\pi)} \text{colim}_{a:U \to X \times_Y V} \mathcal{G}(V) = \text{colim}_{(V,\pi)} \mathcal{G}(V) = \mathcal{G}_{\overline{y}}
\]

in the third equality the pair \( (U,\pi) \) and the map \( a : U \to X \times_Y V \) corresponds to the pair \( (V,a \circ \pi) \).

Part (4) can be proved in a similar manner by identifying the colimits which define \( f^{-1}h_V \). Or you can use Yoneda’s lemma (Categories, Lemma \( 3.5 \)) and the functorial equalities

\[
\text{Mor}_{\text{Sh}(X_{\text{étale}})}(f^{-1}h_V, \mathcal{F}) = \text{Mor}_{\text{Sh}(Y_{\text{étale}})}(h_V, f_*\mathcal{F}) = f_*\mathcal{F}(V) = \mathcal{F}(X \times_Y V)
\]

combined with the fact that representable presheaves are sheaves. See also Sites, Lemma \( 14.5 \) for a completely general result. \( \square \)

The pair of functors \( (f_*, f^{-1}) \) define a morphism of small étale topoi

\[
f_{\text{small}} : \text{Sh}(X_{\text{étale}}) \to \text{Sh}(Y_{\text{étale}})
\]

Many generalities on cohomology of sheaves hold for topoi and morphisms of topoi. We will try to point out when results are general and when they are specific to the étale topos.

**Remark 36.3.** More generally, let \( \mathcal{C}_1, \mathcal{C}_2 \) be sites, and assume they have final objects and fibre products. Let \( u : \mathcal{C}_2 \to \mathcal{C}_1 \) be a functor satisfying:

1. If \( \{V_i \to V\} \) is a covering of \( \mathcal{C}_2 \), then \( \{u(V_i) \to V\} \) is a covering of \( \mathcal{C}_1 \) (we say that \( u \) is *continuous*), and
2. \( u \) commutes with finite limits (i.e., \( u \) is left exact, i.e., \( u \) preserves fibre products and final objects).

\( 03Q2 \)
Then one can define \( f_* : \text{Sh}(\mathcal{C}_1) \to \text{Sh}(\mathcal{C}_2) \) by \( f_* \mathcal{F}(V) = \mathcal{F}(u(V)) \). Moreover, there exists an exact functor \( f^{-1} \) which is left adjoint to \( f_* \), see Sites, Definition 15.1 and Proposition 15.6. Warning: It is not enough to require simply that \( u \) is continuous and commutes with fibre products in order to get a morphism of topoi.

### 37. Functoriality of big topoi

Given a morphism of schemes \( f : X \to Y \) there are a whole host of morphisms of topoi associated to \( f \), see Topologies, Section 9 for a list. Perhaps the most used ones are the morphisms of topoi

\[ f_{\text{big}} = f_{\text{big}, \tau} : \text{Sh}(\mathcal{S}(\text{Sch}/X), \tau) \to \text{Sh}(\mathcal{S}(\text{Sch}/Y), \tau) \]

where \( \tau \in \{ \text{Zariski, étale, smooth, syntomic, fppf} \} \). These each correspond to a continuous functor

\[ (\text{Sch}/Y, \tau) \to (\text{Sch}/X, \tau), \ V/Y \mapsto X \times_Y V/X \]

which preserves final objects, fibre products and covering, and hence defines a morphism of sites

\[ f_{\text{big}} : (\mathcal{S}(\text{Sch}/X), \tau) \to (\mathcal{S}(\text{Sch}/Y), \tau). \]

See Topologies, Sections 3, 4, 5, 6, and 7. In particular, pushforward along \( f_{\text{big}} \) is given by the rule

\[ (f_{\text{big}}_* \mathcal{F})(V/Y) = \mathcal{F}(X \times_Y V/X) \]

It turns out that these morphisms of topoi have an inverse image functor \( f_{\text{big}}^{-1} \) which is very easy to describe. Namely, we have

\[ (f_{\text{big}}^{-1} \mathcal{G})(U/X) = \mathcal{G}(U/Y) \]

where the structure morphism of \( U/Y \) is the composition of the structure morphism \( U \to X \) with \( f \), see Topologies, Lemmas 3.15, 4.15, 5.10, 6.10, and 7.12.

### 38. Functoriality and sheaves of modules

In this section we are going to reformulate some of the material explained in Descent, Section 7 in the setting of étale topologies. Let \( f : X \to Y \) be a morphism of schemes. We have seen above, see Sections 34, 35, and 36 that this induces a morphism \( f_{\text{small}} \) of small étale sites. In Descent, Remark 7.4 we have seen that \( f \) also induces a natural map

\[ f^\sharp_{\text{small}} : \mathcal{O}_{Y_{\text{étale}}} \to f_{\text{small}, \ast} \mathcal{O}_{X_{\text{étale}}} \]

of sheaves of rings on \( Y_{\text{étale}} \) such that \( (f_{\text{small}}, f^\sharp_{\text{small}}) \) is a morphism of ringed sites. See Modules on Sites, Definition 6.1 for the definition of a morphism of ringed sites. Let us just recall here that \( f^\sharp_{\text{small}} \) is defined by the compatible system of maps

\[ \text{pr}^\sharp_V : \mathcal{O}(V) \to \mathcal{O}(X \times_Y V) \]

for \( V \) varying over the objects of \( Y_{\text{étale}} \).

It is clear that this construction is compatible with compositions of morphisms of schemes. More precisely, if \( f : X \to Y \) and \( g : Y \to Z \) are morphisms of schemes, then we have

\[ (g_{\text{small}}, g^\sharp_{\text{small}}) \circ (f_{\text{small}}, f^\sharp_{\text{small}}) = ((g \circ f)_{\text{small}}, (g \circ f)^\sharp_{\text{small}}) \]
as morphisms of ringed topoi. Moreover, by Modules on Sites, Definition 13.1 we see that given a morphism \( f : X \to Y \) of schemes we get well defined pullback and direct image functors

\[
\begin{align*}
 f^*_\text{small} : \text{Mod}(\mathcal{O}_{\text{\acute{e}tale}}) & \to \text{Mod}(\mathcal{O}_{\text{\acute{e}tale}}), \\
f_{\text{small},*} : \text{Mod}(\mathcal{O}_{\text{\acute{e}tale}}) & \to \text{Mod}(\mathcal{O}_{\text{\acute{e}tale}})
\end{align*}
\]

which are adjoint in the usual way. If \( g : Y \to Z \) is another morphism of schemes, then we have \( (g \circ f)^*_{\text{small}} = f^*_{\text{small}} \circ g^*_{\text{small}} \) and \( (g \circ f)_{\text{small},*} = g_{\text{small},*} \circ f_{\text{small},*} \) because of what we said about compositions.

There is quite a bit of difference between the category of all \( \mathcal{O}_X \) modules on \( X \) and the category between all \( \mathcal{O}_{X,\text{\acute{e}tale}} \)-modules on \( X_{\text{\acute{e}tale}} \). But the results of Descent, Section 7 tell us that there is not much difference between considering quasi-coherent modules on \( S \) and quasi-coherent modules on \( S_{\text{\acute{e}tale}} \). (We have already seen this in Theorem 17.4 for example.) In particular, if \( f : X \to Y \) is any morphism of schemes, then the pullback functors \( f^*_{\text{small}} \) and \( f_* \) match for quasi-coherent sheaves, see Descent, Proposition 7.14. Moreover, the same is true for pushforward provided \( f \) is quasi-compact and quasi-separated, see Descent, Lemma 7.15.

A few words about functoriality of the structure sheaf on big sites. Let \( f : X \to Y \) be a morphism of schemes. Choose any of the topologies \( \tau \in \{ \text{Zariski, \acute{e}tale, smooth, syntomic, fppf} \} \). Then the morphism \( f_{\text{big}} : (\text{Sch}/X)_\tau \to (\text{Sch}/Y)_\tau \) becomes a morphism of ringed sites by a map

\[
f^*_\text{big} : \mathcal{O}_Y \to f_{\text{big},*} \mathcal{O}_X
\]

see Descent, Remark 7.4. In fact it is given by the same construction as in the case of small sites explained above.

### 39. Comparing big and small topoi

Let \( X \) be a scheme. In Topologies, Lemma 4.13 we have introduced comparison morphisms \( \pi_X : (\text{Sch}/X)_{\acute{e}tale} \to X_{\acute{e}tale} \) and \( i_X : \text{Sh}(X_{\acute{e}tale}) \to \text{Sh}((\text{Sch}/X)_{\acute{e}tale}) \) with \( \pi_X \circ i_X = \text{id} \) and \( \pi_{X,*} = i_X^{-1} \). In Descent, Remark 7.4 we have extended these to a morphism of ringed sites

\[
\pi_X : ((\text{Sch}/X)_{\acute{e}tale}, \mathcal{O}) \to (X_{\acute{e}tale}, \mathcal{O}_X)
\]

and a morphism of ringed topoi

\[
i_X : (\text{Sh}(X_{\acute{e}tale}), \mathcal{O}_X) \to (\text{Sh}((\text{Sch}/X)_{\acute{e}tale}), \mathcal{O})
\]

Note that the restriction \( i_X^{-1} = \pi_{X,*} \) (see Topologies, Definition 4.14) transforms \( \mathcal{O} \) into \( \mathcal{O}_X \). Hence \( i_X^{-1} \mathcal{F} = i_X^{-1} \mathcal{F} \) for any \( \mathcal{O} \)-module \( \mathcal{F} \) on \( (\text{Sch}/X)_{\acute{e}tale} \). In particular \( i_X^{-1} \) is exact. This functor is often denoted \( \mathcal{F} \to \mathcal{F}|_{X_{\acute{e}tale}} \).

#### Lemma 39.1

Let \( X \) be a scheme.

1. \( \mathcal{T}|_{X_{\acute{e}tale}} \) is injective in \( \text{Ab}(X_{\acute{e}tale}) \) for \( \mathcal{T} \) injective in \( \text{Ab}((\text{Sch}/X)_{\acute{e}tale}) \), and
2. \( \mathcal{T}|_{X_{\acute{e}tale}} \) is injective in \( \text{Mod}(X_{\acute{e}tale}, \mathcal{O}_X) \) for \( \mathcal{T} \) injective in \( \text{Mod}((\text{Sch}/X)_{\acute{e}tale}, \mathcal{O}) \).

#### Proof

This follows formally from the fact that the restriction functor \( \pi_{X,*} = i_X^{-1} \) is an exact left adjoint of \( i_{X,*} \), see Homology, Lemma 25.1.

\( \square \)
Let $f : X \to Y$ be a morphism of schemes. The commutative diagram of Topologies, Lemma 4.16 (3) leads to a commutative diagram of ringed sites

\[
\begin{array}{ccc}
(T_{\text{étale}}, \mathcal{O}_T) & \leftarrow & ((\text{Sch}/T)_{\text{étale}}, \mathcal{O}) \\
\downarrow & & \downarrow \\
(S_{\text{étale}}, \mathcal{O}_S) & \leftarrow & ((\text{Sch}/S)_{\text{étale}}, \mathcal{O})
\end{array}
\]

as one easily sees by writing out the definitions of $f^\sharp_{\text{small}}$, $f^\sharp_{\text{big}}$, $\pi^\sharp_S$, and $\pi^\sharp_T$. In particular this means that

\[
(39.1.1) \quad (f^\sharp_{\text{big}}, \ast)_Y = (f^\sharp_{\text{small}}, \ast)|_{X_{\text{étale}}}
\]

for any sheaf $\mathcal{F}$ on $(\text{Sch}/X)_{\text{étale}}$ and if $\mathcal{F}$ is a sheaf of $\mathcal{O}$-modules, then (39.1.1) is an isomorphism of $\mathcal{O}_Y$-modules on $Y_{\text{étale}}$.

**Lemma 39.2.** Let $f : X \to Y$ be a morphism of schemes.

1. For any $\mathcal{F} \in \text{Ab}(\text{Sch}/X)_{\text{étale}}$ we have

\[
(Rf^\big, \ast)_Y = (Rf^\small, \ast)(\mathcal{F}|_{X_{\text{étale}}})
\]

in $D(Y_{\text{étale}})$.

2. For any object $\mathcal{F}$ of $\text{Mod}(\text{Sch}/X)_{\text{étale}}, \mathcal{O})$ we have

\[
(Rf^\big, \ast)_Y = (Rf^\small, \ast)(\mathcal{F}|_{X_{\text{étale}}})
\]

in $D(\text{Mod}(Y_{\text{étale}}, \mathcal{O}_Y))$.

**Proof.** Follows immediately from Lemma 39.1 and (39.1.1) on choosing an injective resolution of $\mathcal{F}$. \qed

40. Comparing topologies

**Lemma 40.1.** Let $S$ be a scheme. Let $\mathcal{F}$ be a sheaf of sets on $S_{\text{étale}}$. Let $s, t \in \mathcal{F}(S)$. Then there exists an open $W \subset S$ characterized by the following property: A morphism $f : T \to S$ factors through $W$ if and only if $s|_T = t|_T$ (restriction is pullback by $f^\small$).

**Proof.** Consider the presheaf which assigns to $U \in \text{Ob}(S_{\text{étale}})$ the emptyset if $s|_U \neq t|_U$ and a singleton else. It is clear that this is a subsheaf of the final object of $\text{Sh}(S_{\text{étale}})$. By Lemma 31.1 we find an open $W \subset S$ representing this presheaf. For a geometric point $\pi$ of $S$ we see that $\pi \in W$ if and only if the stalks of $s$ and $t$ at $\pi$ agree. By the description of stalks of pullbacks in Lemma 36.2 we see that $W$ has the desired property. \qed

**Lemma 40.2.** Let $S$ be a scheme. Let $\tau \in \{\text{Zariski, étale}\}$. Consider the morphism

\[
\pi : (\text{Sch}/S)_\tau \to S_\tau
\]

of Topologies, Lemma 3.13 or 4.13. Let $\mathcal{F}$ be a sheaf on $S_\tau$. Then $\pi^{-1}\mathcal{F}$ is given by the rule

\[
\pi^{-1}\mathcal{F}(T) = \Gamma(T, f^{-1}_{\small}(\mathcal{F}))
\]

where $f : T \to S$. Moreover, $\pi^{-1}\mathcal{F}$ satisfies the sheaf condition with respect to fpqc coverings.
Proof. Observe that we have a morphism $i_f : \text{Sh}(T,\tau) \to \text{Sh}(\text{Sch}/S,\tau)$ such that
\[ \pi \circ i_f = f_{\text{small}} \] as morphisms $T \to S$, see Topologies, Lemmas 3.12, 3.16, and 4.12. Since pullback is transitive we see that $i_f^{-1}\pi^{-1}\mathcal{F} = f_{\text{small}}^{-1}\mathcal{F}$ as desired.

Let $\{ g_i : T_i \to T \}_{i \in I}$ be an fpqc covering. The final statement means the following: Given a sheaf $\mathcal{G}$ on $T_\tau$ and given sections $s_i \in \Gamma(T_i, g_i^{-1}\mathcal{G})$ whose pullbacks to $T_i \times_T T_j$ agree, there is a unique section $s$ of $\mathcal{G}$ over $T$ whose pullback to $T_i$ agrees with $s_i$.

Let $V \to T$ be an object of $T_\tau$ and let $t \in \mathcal{G}(V)$. For every $i$ there is a largest open $W_i \subset T_i \times_T V$ such that the pullbacks of $s_i$ and $t$ agree as sections of the pullback of $\mathcal{G}$ to $W_i \subset T_i \times_T V$, see Lemma 40.1. Because $s_i$ and $s_j$ agree over $T_i \times_T T_j$ we find that $W_i$ and $W_j$ pullback to the same open over $T_i \times_T T_j \times_T V$. By Descent, Lemma 9.2 we find an open $W \subset V$ whose inverse image to $T_i \times_T V$ recovers $W_i$.

By construction of $g_i^{-1}\mathcal{G}$ there exists a $\tau$-covering $\{ T_{ij} \to T_i \}_{j \in J_i}$, for each $j$ an open immersion or étale morphism $V_{ij} \to T$, a section $t_{ij} \in \mathcal{G}(V_{ij})$, and commutative diagrams

\[
\begin{array}{ccc}
T_{ij} & \longrightarrow & V_{ij} \\
\downarrow & & \downarrow \\
T_i & \longrightarrow & T
\end{array}
\]

such that $s_i|_{T_{ij}}$ is the pullback of $t_{ij}$. In other words, after replacing the covering $\{ T_i \to T \}$ by $\{ T_{ij} \to T \}$ we may assume there are factorizations $T_i \to V_i \to T$ with $V_i \in \text{Ob}(T_\tau)$ and sections $t_i \in \mathcal{G}(V_i)$ pulling back to $s_i$ over $T_i$. By the result of the previous paragraph we find opens $W_i \subset V_i$ such that $t_i|_{W_i}$ “agrees with” every $s_j$ over $T_j \times_T W_i$. Note that $T_i \to V_i$ factors through $W_i$. Hence $\{ W_i \to T \}$ is a $\tau$-covering and the lemma is proven. \hfill \Box

0A3H Lemma 40.3. Let $S$ be a scheme. Let $f : T \to S$ be a morphism such that
(1) $f$ is flat and quasi-compact, and
(2) the geometric fibres of $f$ are connected.

Let $\mathcal{F}$ be a sheaf on $S_{\text{étale}}$. Then $\Gamma(S, \mathcal{F}) = \Gamma(T, f_{\text{small}}^{-1}\mathcal{F})$.

Proof. There is a canonical map $\Gamma(S, \mathcal{F}) \to \Gamma(T, f_{\text{small}}^{-1}\mathcal{F})$. Since $f$ is surjective (because its fibres are connected) we see that this map is injective.

To show that the map is surjective, let $\alpha \in \Gamma(T, f_{\text{small}}^{-1}\mathcal{F})$. Since $\{ T \to S \}$ is an fpqc covering we can use Lemma 40.2 to see that suffices to prove that $\alpha$ pulls back to the same section over $T \times_T T'$ by the two projections. Let $\tau \to S$ be a geometric point. It suffices to show the agreement holds over $(T \times_S T)_\tau$ as every geometric point of $T \times_S T$ is contained in one of these geometric fibres. In other words, we are trying to show that $\alpha|_{T_\tau}$ pulls back to the same section over $(T \times_S T)_\tau$ by the two projections $T_\tau \times_{T_\tau} T_\tau$. However, since $\mathcal{F}|_{T_\tau}$ is the pullback of $\mathcal{F}|_\tau$ it is a constant sheaf with value $\mathcal{F}_\tau$. Since $T_\tau$ is connected by assumption, any section of a constant sheaf is constant and this proves what we want. \hfill \Box

0A3I Lemma 40.4. Let $k \subset K$ be an extension of fields with $k$ separably algebraically closed. Let $S$ be a scheme over $k$. Denote $p : S_K = S \times_{\text{Spec}(k)} \text{Spec}(K) \to S$ the projection. Let $\mathcal{F}$ be a sheaf on $S_{\text{étale}}$. Then $\Gamma(S, \mathcal{F}) = \Gamma(S_K, p_{\text{small}}^{-1}\mathcal{F})$. 


In this section we prove that the rule which associates to a scheme its locally ringed small étale topos is fully faithful in a suitable sense, see Theorem 41.5.

**Lemma 41.1.** Let \( f : X \to Y \) be a morphism of schemes. The morphism of ringed sites \( (f_{small}, f_{small}^!) \) associated to \( f \) is a morphism of locally ringed sites, see Modules on Sites, Definition 39.9.

**Proof.** Note that the assertion makes sense since we have seen that \((X_{\text{étale}}, \mathcal{O}_{X_{\text{étale}}})\) and \((Y_{\text{étale}}, \mathcal{O}_{Y_{\text{étale}}})\) are locally ringed sites, see Lemma 33.5. Moreover, we know that \( X_{\text{étale}} \) has enough points, see Theorem 29.10 and Remarks 29.11. Hence it suffices to prove that \((f_{small}, f_{small}^!)\) satisfies condition (3) of Modules on Sites, Lemma 39.8. To see this take a point \( p \) of \( X_{\text{étale}} \). By Lemma 29.12 \( p \) corresponds to a geometric point \( \overline{p} \) of \( X \). By Lemma 36.2 the point \( q = f_{small} \circ p \) corresponds to the geometric point \( \overline{q} = f \circ \overline{p} \) of \( Y \). Hence the assertion we have to prove is that the induced map of stalks

\[ \mathcal{O}_{Y, \overline{q}} \longrightarrow \mathcal{O}_{X, \overline{p}} \]

is a local ring map. Suppose that \( a \in \mathcal{O}_{Y, \overline{q}} \) is an element of the left hand side which maps to an element of the maximal ideal of the right hand side. Suppose that \( a \) is the equivalence class of a triple \((V, \overline{p}, a)\) with \( V \to Y \) étale, \( \overline{p} \to V \) over \( Y \), and \( a \in \mathcal{O}(V) \). It maps to the equivalence class of \((X \times_{Y} V, \overline{p} \times_{V} \overline{q}, \text{pr}_{V}^{*}(a))\) in the local ring \( \mathcal{O}_{X, \overline{p}} \). But it is clear that being in the maximal ideal means that pulling back \( \text{pr}_{V}^{*}(a) \) to an element of \( \kappa(\overline{p}) \) gives zero. Hence also pulling back \( a \) to \( \kappa(\overline{p}) \) is zero. Which means that \( a \) lies in the maximal ideal of \( \mathcal{O}_{Y, \overline{q}} \). \( \square \)

**Lemma 41.2.** Let \( X, Y \) be schemes. Let \( f : X \to Y \) be a morphism of schemes. Let \( t \) be a 2-morphism from \((f_{small}, f_{small}^!)\) to itself, see Modules on Sites, Definition 39.7. Then \( t = id \).

**Proof.** This means that \( t : f_{small}^{-1} \longrightarrow f_{small}^{-1} \) is a transformation of functors such that the diagram

\[ \begin{array}{ccc}
  f_{small}^{-1} \mathcal{O}_{Y} & \xrightarrow{t} & f_{small}^{-1} \mathcal{O}_{Y} \\
  \downarrow f_{small}^{-1} & & \downarrow f_{small}^{-1} \\
  f_{small}^{!} \mathcal{O}_{X} & \xrightarrow{t} & f_{small}^{!} \mathcal{O}_{X}
\end{array} \]

is commutative. Suppose \( V \to Y \) is étale with \( V \) affine. By Morphisms, Lemma 39.2 we may choose an immersion \( i : V \to \mathbb{A}^{n}_{Y} \) over \( Y \). In terms of sheaves this means that \( i \) induces an injection \( h_{i} : h_{V} \to \prod_{j=1}^{n} \mathcal{O}_{Y} \) of sheaves. The base change \( i' \) of \( i \) to \( X \) is an immersion (Schemes, Lemma 18.2). Hence \( i' : X \times_{Y} V \to \mathbb{A}^{n}_{X} \) is an immersion, which in turn means that \( h_{i'} : h_{X \times_{Y} V} \to \prod_{j=1}^{n} \mathcal{O}_{X} \) is an injection of sheaves. Via the identification \( f_{small}^{!} h_{V} = h_{X \times_{Y} V} \) of Lemma 36.2 the map \( h_{i'} \) is equal to

\[ f_{small}^{-1} h_{V} \xrightarrow{f_{small}^{-1} h_{i}} \prod_{j=1}^{n} f_{small}^{-1} \mathcal{O}_{Y} \xrightarrow{f_{small}^{!}} \prod_{j=1}^{n} \mathcal{O}_{X} \]
(verification omitted). This means that the map \( t : f_{small}^{-1}h_V \to f_{small}^{-1}h_V \) fits into the commutative diagram

\[
\begin{array}{c}
\begin{array}{ccc}
f_{small}^{-1}h_V & \xrightarrow{f^{-1}h_f} & \prod_{j=1,...,n} f_{small}^{-1}O_Y \\
\downarrow t & & \downarrow \prod_{j=1,...,n} f^j \\
f_{small}^{-1}h_V & \xrightarrow{f^{-1}h_f} & \prod_{j=1,...,n} f_{small}^{-1}O_Y \\
\end{array}
\end{array}
\]

The commutativity of the right square holds by our assumption on \( t \) explained above. Since the composition of the horizontal arrows is injective by the discussion above we conclude that the left vertical arrow is the identity map as well. Any sheaf of sets on \( Y_{\text{étale}} \) admits a surjection from a (huge) coproduct of sheaves of sets of the form \( h_V \) with \( V \) affine (combine Lemma 21.2 with Sites, Lemma 13.5). Thus we conclude that \( t : f_{small}^{-1} \to f_{small}^{-1} \) is the identity transformation as desired. □

**Lemma 41.3.** Let \( X, Y \) be schemes. Any two morphisms \( a, b : X \to Y \) of schemes for which there exists a 2-isomorphism \((a_{small},a_{small}^\sharp) \cong (b_{small},b_{small}^\sharp)\) in the 2-category of ringed topoi are equal.

**Proof.** Let us argue this carefully since it is a bit confusing. Let \( t : a_{small}^{-1} \to b_{small}^{-1} \) be the 2-isomorphism. Consider any open \( V \subset Y \). Note that \( h_V \) is a subsheaf of the final sheaf \( * \). Thus both \( a_{small}^{-1}h_V = h_{a^{-1}(V)} \) and \( b_{small}^{-1}h_V = h_{b^{-1}(V)} \) are subsheaves of the final sheaf. Thus the isomorphism

\[
t : a_{small}^{-1}h_V = h_{a^{-1}(V)} \to b_{small}^{-1}h_V = h_{b^{-1}(V)}
\]

has to be the identity, and \( a^{-1}(V) = b^{-1}(V) \). It follows that \( a \) and \( b \) are equal on underlying topological spaces. Next, take a section \( f \in O_Y(V) \). This determines and is determined by a map of sheaves of sets \( f : h_V \to O_Y \). Pull this back and apply \( t \) to get a commutative diagram

\[
\begin{array}{ccc}
h_{b^{-1}(V)} & \xrightarrow{b_{small}^{-1}} & b_{small}^{-1}h_V \\
\downarrow a_{small}^{-1} & & \downarrow t \\
\prod_{j=1,...,n} a_{small}^{-1}O_Y & \xrightarrow{a_{small}^{-1}} & \prod_{j=1,...,n} b_{small}^{-1}O_Y \\
\end{array}
\]

where the triangle is commutative by definition of a 2-isomorphism in Modules on Sites, Section [[S] □ Above we have seen that the composition of the top horizontal arrows comes from the identity \( a^{-1}(V) = b^{-1}(V) \). Thus the commutativity of the diagram tells us that \( a_{small}^\sharp(f) = b_{small}^\sharp(f) \) in \( O_X(a^{-1}(V)) = O_X(b^{-1}(V)) \). Since this holds for every open \( V \) and every \( f \in O_Y(V) \) we conclude that \( a = b \) as morphisms of schemes. □

**Lemma 41.4.** Let \( X, Y \) be affine schemes. Let

\[
(g, g^\#) : (Sh(X_{\text{étale}}), O_X) \longrightarrow (Sh(Y_{\text{étale}}), O_Y)
\]
be a morphism of locally ringed topoi. Then there exists a unique morphism of schemes \( f : X \to Y \) such that \( (g, g^\sharp) \) is \( 2 \)-isomorphic to \( (f_{\text{small}}, f^\sharp_{\text{small}}) \), see Modules on Sites, Definition 14.11.

**Proof.** In this proof we write \( \mathcal{O}_X \) for the structure sheaf of the small \( \acute{e}tale \) site \( X_{\acute{e}tale} \), and similarly for \( \mathcal{O}_Y \). Say \( Y = \text{Spec}(B) \) and \( X = \text{Spec}(A) \). Since \( B = \Gamma(Y_{\acute{e}tale}, \mathcal{O}_Y) \), \( A = \Gamma(X_{\acute{e}tale}, \mathcal{O}_X) \) we see that \( g^\sharp \) induces a ring map \( \varphi : B \to A \). Let \( f = \text{Spec}(\varphi) : X \to Y \) be the corresponding morphism of affine schemes. We will show this \( f \) does the job.

Let \( V \to Y \) be an affine scheme \( \acute{e}tale \) over \( Y \). Thus we may write \( V = \text{Spec}(C) \) with \( C \) an \( \acute{e}tale \) \( B \)-algebra. We can write

\[
C = B[x_1, \ldots, x_n]/(P_1, \ldots, P_n)
\]

with \( P_i \) polynomials such that \( \Delta = \det(\partial P_i/\partial x_j) \) is invertible in \( C \), see for example Algebra, Lemma 141.2. If \( T \) is a scheme over \( Y \), then a \( T \)-valued point of \( V \) is given by \( n \) sections of \( \Gamma(T, \mathcal{O}_T) \) which satisfy the polynomial equations \( P_1 = 0, \ldots, P_n = 0 \). In other words, the sheaf \( h^\sharp_V \) on \( \acute{e}tale \) \( X \) is the equalizer of the two maps

\[
\prod_{i=1}^n \mathcal{O}_Y \xrightarrow{\alpha} \prod_{j=1}^n \mathcal{O}_Y
\]

where \( b(h_1, \ldots, h_n) = 0 \) and \( a(h_1, \ldots, h_n) = (P_1(h_1, \ldots, h_n), \ldots, P_n(h_1, \ldots, h_n)) \).

Since \( g^{-1} \) is exact we conclude that the top row of the following solid commutative diagram is an equalizer diagram as well:

\[
\begin{array}{ccc}
g^{-1}h^\sharp_V & \longrightarrow & \prod_{i=1}^n g^{-1} \mathcal{O}_Y \\
\downarrow & & \downarrow \Pi g^\sharp \\
h_{X \times_Y V} & \longrightarrow & \prod_{i=1}^n \mathcal{O}_X
\end{array}
\]

Here \( b ' \) is the zero map and \( a ' \) is the map defined by the images \( P_i ' = \varphi(P_i) \in A[x_1, \ldots, x_n] \) via the same rule \( a ' (h_1, \ldots, h_n) = (P_1'(h_1, \ldots, h_n), \ldots, P_n'(h_1, \ldots, h_n)) \), that \( a \) was defined by. The commutativity of the diagram follows from the fact that \( \varphi = g^\sharp \) on global sections. The lower row is an equalizer diagram also, by exactly the same arguments as before since \( X \times_Y V \) is the affine scheme \( \text{Spec}(A \otimes_B C) \) and \( A \otimes_B C = A[x_1, \ldots, x_n]/(P_1', \ldots, P_n') \). Thus we obtain a unique dotted arrow \( g^{-1}h^\sharp_V \to h_{X \times_Y V} \) fitting into the diagram.

We claim that the map of sheaves \( g^{-1}h^\sharp_V \to h_{X \times_Y V} \) is an isomorphism. Since the small \( \acute{e}tale \) site of \( X \) has enough points (Theorem 29.10) it suffices to prove this on stalks. Hence let \( \mathfrak{p} \) be a geometric point of \( X \), and denote \( p \) the associate point of the small \( \acute{e}tale \) topos of \( X \). Set \( q = g \circ p \). This is a point of the small \( \acute{e}tale \) topos of \( Y \). By Lemma 29.12 we see that \( q \) corresponds to a geometric point \( \mathfrak{p} \) of \( Y \). Consider the map of stalks

\[
(g^\sharp)_p : \mathcal{O}_{Y, \mathfrak{p}} = \mathcal{O}_{Y, q} = (g^{-1} \mathcal{O}_Y)_p \longrightarrow \mathcal{O}_{X, p} = \mathcal{O}_{X, \mathfrak{p}}
\]

Since \( (g, g^\sharp) \) is a morphism of locally ringed topoi \( (g^\sharp)_p \) is a local ring homomorphism of strictly henselian local rings. Applying localization to the big commutative diagram above and Algebra, Lemma 148.12 we conclude that \( (g^{-1}h^\sharp)_p \to (h_{X \times_Y V})_p \) is an isomorphism as desired.
We claim that the isomorphisms \( g^{-1}h_V \rightarrow h_{X \times_Y V} \) are functorial. Namely, suppose that \( V_1 \rightarrow V_2 \) is a morphism of affine schemes étale over \( Y \). Write \( V_i = \text{Spec}(C_i) \) with
\[
C_i = B[x_{i,1}, \ldots, x_{i,n_i}] / (P_{i,1}, \ldots, P_{i,n_i})
\]
The morphism \( V_1 \rightarrow V_2 \) is given by a \( B \)-algebra map \( C_2 \rightarrow C_1 \) which in turn is given by some polynomials \( Q_j \in B[x_{1,1}, \ldots, x_{1,n_1}] \) for \( j = 1, \ldots, n_2 \). Then it is an easy matter to show that the diagram of sheaves
\[
\begin{array}{c}
\begin{array}{ccc}
h_{V_1} & \longrightarrow & \prod_{i=1}^{n_1} O_Y \\
\downarrow & & \downarrow Q_{1}, \ldots, Q_{n_2} \\
h_{V_2} & \longrightarrow & \prod_{i=1}^{n_2} O_Y
\end{array}
\end{array}
\]
is commutative, and pulling back to \( X_{\text{étale}} \) we obtain the solid commutative diagram
\[
\begin{array}{c}
\begin{array}{ccc}
g^{-1}h_{V_1} & \longrightarrow & \prod_{i=1}^{n_1} g^{-1} O_Y \\
\downarrow & & \downarrow Q_{1}, \ldots, Q_{n_2} \\
g^\sharp & \longrightarrow & \prod_{i=1}^{n_2} g^{-1} O_Y \\
\downarrow & & \downarrow g^\sharp \\
h_{X \times_Y V_1} & \longrightarrow & \prod_{i=1}^{n_1} O_X \\
\downarrow & & \downarrow Q^{'1}, \ldots, Q^{'n_2} \\
h_{X \times_Y V_2} & \longrightarrow & \prod_{i=1}^{n_2} O_X
\end{array}
\end{array}
\]
where \( Q^{'j} \in A[x_{1,1}, \ldots, x_{1,n_1}] \) is the image of \( Q_j \) via \( \varphi \). Since the dotted arrows exist, make the two squares commute, and the horizontal arrows are injective we see that the whole diagram commutes. This proves functoriality (and also that the construction of \( g^{-1}h_V \rightarrow h_{X \times_Y V} \) is independent of the choice of the presentation, although we strictly speaking do not need to show this).

At this point we are able to show that \( f_{\text{small},*} \cong g_* \). Namely, let \( F \) be a sheaf on \( X_{\text{étale}} \). For every \( V \in \text{Ob}(X_{\text{étale}}) \) affine we have
\[
(g_* F)(V) = \text{Mor}_\text{Sh}(Y_{\text{étale}})(h_V, g_* F) \\
= \text{Mor}_\text{Sh}(X_{\text{étale}})(g^{-1}h_V, F) \\
= \text{Mor}_\text{Sh}(X_{\text{étale}})(h_{X \times_Y V}, F) \\
= F(X \times_Y V) \\
= f_{\text{small},*} F(V)
\]
where in the third equality we use the isomorphism \( g^{-1}h_V \cong h_{X \times_Y V} \) constructed above. These isomorphisms are clearly functorial in \( F \) and functorial in \( V \) as the isomorphisms \( g^{-1}h_V \cong h_{X \times_Y V} \) are functorial. Now any sheaf on \( Y_{\text{étale}} \) is determined by the restriction to the subcategory of affine schemes (Lemma \ref{etale-cohomology}), and hence we obtain an isomorphism of functors \( f_{\text{small},*} \cong g_* \) as desired.

Finally, we have to check that, via the isomorphism \( f_{\text{small},*} \cong g_* \) above, the maps \( f_{\text{small}}^\sharp \) and \( g^\sharp \) agree. By construction this is already the case for the global sections.
of \( \mathcal{O}_Y \), i.e., for the elements of \( B \). We only need to check the result on sections over an affine \( V \) étale over \( Y \) (by Lemma 21.2 again). Writing \( V = \text{Spec}(C) \), \( C = B[x_i]/(P_j) \) as before it suffices to check that the coordinate functions \( x_i \) are mapped to the same sections of \( \mathcal{O}_X \) over \( X \times_Y V \). And this is exactly what it means that the diagram

\[
\begin{array}{ccc}
g^{-1}h_V & \longrightarrow & \prod_{i=1,\ldots,n} g^{-1}\mathcal{O}_Y \\
\downarrow & & \downarrow \Pi g^* \\
h_{X \times_Y V} & \longrightarrow & \prod_{i=1,\ldots,n} \mathcal{O}_X
\end{array}
\]

commutes. Thus the lemma is proved. \( \square \)

Here is a version for general schemes.

**Theorem 41.5.** Let \( X, Y \) be schemes. Let

\[
(g, g^\#) : (\text{Sh}(X_{\text{étale}}), \mathcal{O}_X) \longrightarrow (\text{Sh}(Y_{\text{étale}}), \mathcal{O}_Y)
\]

be a morphism of locally ringed topoi. Then there exists a unique morphism of schemes \( f : X \rightarrow Y \) such that \((g, g^\#)\) is isomorphic to \((f_{\text{small}}, f^\#_{\text{small}})\). In other words, the construction

\[
\text{Sch} \longrightarrow \text{Locally ringed topoi}, \quad X \longrightarrow (X_{\text{étale}}, \mathcal{O}_X)
\]

is fully faithful (morphisms up to 2-isomorphisms on the right hand side).

**Proof.** You can prove this theorem by carefully adjusting the arguments of the proof of Lemma 41.4 to the global setting. However, we want to indicate how we can glue the result of that lemma to get a global morphism due to the rigidity provided by the result of Lemma 41.2. Unfortunately, this is a bit messy.

Let us prove existence when \( Y \) is affine. In this case choose an affine open covering \( X = \bigcup U_i \). For each \( i \) the inclusion morphism \( j_i : U_i \rightarrow X \) induces a morphism of locally ringed topoi \((j_i, f_{i,\text{small}}, f^\#_{i,\text{small}}) : (\text{Sh}(U_i_{\text{étale}}), \mathcal{O}_{U_i}) \rightarrow (\text{Sh}(X_{\text{étale}}), \mathcal{O}_X)\) by Lemma 41.1. We can compose this with \((g, g^\#)\) to obtain a morphism of locally ringed topoi

\[
(g, g^\#) \circ (j_i, f_{i,\text{small}}, f^\#_{i,\text{small}}) : (\text{Sh}(U_i_{\text{étale}}), \mathcal{O}_{U_i}) \rightarrow (\text{Sh}(X_{\text{étale}}), \mathcal{O}_X)
\]

see Modules on Sites, Lemma 39.10. By Lemma 41.4 there exists a unique morphism of schemes \( f_i : U_i \rightarrow Y \) and a 2-isomorphism

\[
t_i : (f_i, f_{i,\text{small}}, f^\#_{i,\text{small}}) \longrightarrow (g, g^\#) \circ (j_i, f_{i,\text{small}}, f^\#_{i,\text{small}}).
\]

Set \( U_{i,i'} = U_i \cap U_{i'} \), and denote \( j_{i,i'} : U_{i,i'} \rightarrow U_i \) the inclusion morphism. Since we have \( j_{i} \circ j_{i,i'} = j_{i'} \circ j_{i',i} \) we see that

\[
(g, g^\#) \circ (j_{i,i'}, f_{i,i'}, f^\#_{i,i'}) = (g, g^\#) \circ (j_{i,i'}, f_{i,i'}, f^\#_{i,i'})
\]

Hence by uniqueness (see Lemma 41.3) we conclude that \( f_i \circ j_{i,i'} = f_{i'} \circ j_{i',i} \), in other words the morphisms of schemes \( f_i = f \circ j_i \) are the restrictions of a global
morphism of schemes $f : X \to Y$. Consider the diagram of 2-isomorphisms (where we drop the components $\sharp$ to ease the notation)

$$
\begin{array}{ccc}
  g \circ j_{i,small} \circ j_{i',small} & \overset{t_i \cdot \text{id}_{j_{i',small}}}{\longrightarrow} & f_{\text{small}} \circ j_{i,small} \circ j_{i',small} \\
  \downarrow & & \downarrow \\
  g \circ j_{i',small} \circ j_{i',i,small} & \overset{t_{i',i} \cdot \text{id}_{j_{i',i,small}}}{\longrightarrow} & f_{\text{small}} \circ j_{i',small} \circ j_{i',i,small}
\end{array}
$$

The notation $\ast$ indicates horizontal composition, see Categories, Definition \[28.1\] in general and Sites, Section \[35\] for our particular case. By the result of Lemma \[41.2\] this diagram commutes. Hence for any sheaf $\mathcal{G}$ on $Y_{\text{etale}}$, the isomorphisms $t_i : f_{\text{small}}^{-1} \mathcal{G}_{|U_i} \to g^{-1} \mathcal{G}_{|U_i}$ agree over $U_{i,i'}$ and we obtain a global isomorphism $t : f_{\text{small}}^{-1} \mathcal{G} \to g^{-1} \mathcal{G}$. It is clear that this isomorphism is functorial in $\mathcal{G}$ and is compatible with the maps $f_{\text{small}}^i$ and $g^i$ (because it is compatible with these maps locally). This proves the theorem in case $Y$ is affine.

In the general case, let $V \subset Y$ be an affine open. Then $h_V$ is a subsheaf of the final sheaf $\ast$ on $Y_{\text{etale}}$. As $g$ is exact we see that $g^{-1} h_V$ is a subsheaf of the final sheaf on $X_{\text{etale}}$. Hence by Lemma \[31.1\] there exists an open subscheme $W \subset X$ such that $g^{-1} h_V = h_W$. By Modules on Sites, Lemma \[39.12\] there exists a commutative diagram of morphisms of locally ringed topoi

$$
\begin{array}{ccc}
  (\text{Sh}(W_{\text{etale}}, \mathcal{O}_W)) & \longrightarrow & (\text{Sh}(X_{\text{etale}}, \mathcal{O}_X)) \\
  g' \downarrow & & \downarrow g \\
  (\text{Sh}(V_{\text{etale}}, \mathcal{O}_V)) & \longrightarrow & (\text{Sh}(Y_{\text{etale}}, \mathcal{O}_Y))
\end{array}
$$

where the horizontal arrows are the localization morphisms (induced by the inclusion morphisms $V \to Y$ and $W \to X$) and where $g'$ is induced from $g$. By the result of the preceding paragraph we obtain a morphism of schemes $f' : W \to V$ and a 2-isomorphism $t : (f'_{\text{small}})(f'_{\text{small}})^\sharp \to (g', (g')^\sharp)$. Exactly as before these morphisms $f'$ (for varying affine opens $V \subset Y$) agree on overlaps by uniqueness, so we get a morphism $f : X \to Y$. Moreover, the 2-isomorphisms $t$ are compatible on overlaps by Lemma \[41.2\] again and we obtain a global 2-isomorphism $(f'_{\text{small}}, (f'_{\text{small}})^\sharp) \to (g, (g)^\sharp)$. as desired. Some details omitted.

\[\square\]

**42. Push and pull**

04C6 Let $f : X \to Y$ be a morphism of schemes. Here is a list of conditions we will consider in the following:

(A) For every étale morphism $U \to X$ and $u \in U$ there exist an étale morphism $V \to Y$ and a disjoint union decomposition $X \times_Y V = W \amalg W'$ and a morphism $h : W \to U$ over $X$ with $u$ in the image of $h$.

(B) For every $V \to Y$ étale, and every étale covering $\{U_i \to X \times_Y V\}$ there exists an étale covering $\{V_j \to V\}$ such that for each $j$ we have $X \times_Y V_j = \coprod W_{ij}$ where $W_{ij} \to X \times_Y V$ factors through $U_i \to X \times_Y V$ for some $i$.

(C) For every $U \to X$ étale, there exists a $V \to Y$ étale and a surjective morphism $X \times_Y V \to U$ over $X$.

It turns out that each of these properties has meaning in terms of the behaviour of the functor $f_{\text{small}} \ast$. We will work this out in the next few sections.
43. Property (A)

Let $f : X \to Y$ be a morphism of schemes. Assume (A).

Lemma 43.1. Let $f : X \to Y$ be a morphism of schemes. Assume (A).

1. $f_{\text{small},*} : \text{Ab}(X_{\text{étale}}) \to \text{Ab}(Y_{\text{étale}})$ reflects injections and surjections,
2. $f^{-1}_{\text{small}}f_{\text{small},*} \mathcal{F} \to \mathcal{F}$ is surjective for any abelian sheaf $\mathcal{F}$ on $X_{\text{étale}},$
3. $f_{\text{small},*} : \text{Ab}(X_{\text{étale}}) \to \text{Ab}(Y_{\text{étale}})$ is faithful.

Proof. Let $\mathcal{F}$ be an abelian sheaf on $X_{\text{étale}}$. Let $U$ be an object of $X_{\text{étale}}$. By assumption we can find a covering $\{W_i \to U\}$ in $X_{\text{étale}}$ such that each $W_i$ is an open and closed subscheme of $X \times Y V_i$ for some object $V_i$ of $Y_{\text{étale}}$. The sheaf condition shows that

$$\mathcal{F}(U) \subset \prod_i \mathcal{F}(W_i)$$

and that $\mathcal{F}(W_i)$ is a direct summand of $\mathcal{F}(X \times Y V_i) = f_{\text{small},*}\mathcal{F}(V_i)$. Hence it is clear that $f_{\text{small},*}$ reflects injections.

Next, suppose that $a : \mathcal{G} \to \mathcal{F}$ is a map of abelian sheaves such that $f_{\text{small},*}a$ is surjective. Let $s \in \mathcal{F}(U)$ with $U$ as above. With $W_i, V_i$ as above we see that it suffices to show that $s|_{W_i}$ is étale locally the image of a section of $\mathcal{G}$ under $a$. Since $\mathcal{F}(W_i)$ is a direct summand of $\mathcal{F}(X \times Y V_i)$ it suffices to show that for any $V \in \text{Ob}(Y_{\text{étale}})$ any element $s \in \mathcal{F}(X \times Y V)$ is étale locally on $X \times Y V$ the image of a section of $\mathcal{G}$ under $a$. Since $\mathcal{F}(X \times Y V) = f_{\text{small},*}\mathcal{F}(V)$ we see by assumption that there exists a covering $\{V_j \to V\}$ such that $s$ is the image of $s_j \in f_{\text{small},*}\mathcal{G}(V_j) = \mathcal{G}(X \times Y V_j)$. This proves $f_{\text{small},*}$ reflects surjections.

Parts (2), (3) follow formally from part (1), see Modules on Sites, Lemma [15.1]

Lemma 43.2. Let $f : X \to Y$ be a separated locally quasi-finite morphism of schemes. Then property (A) above holds.

Proof. Let $U \to X$ be an étale morphism and $u \in U$. The geometric statement (A) reduces directly to the case where $U$ and $Y$ are affine schemes. Denote $x \in X$ and $y \in Y$ the images of $u$. Since $X \to Y$ is locally quasi-finite, and $U \to X$ is locally quasi-finite (see Morphisms, Lemma [36.6]) we see that $U \to Y$ is locally quasi-finite (see Morphisms, Lemma [20.12]). Moreover both $X \to Y$ and $U \to Y$ are separated. Thus More on Morphisms, Lemma [30.3] applies to both morphisms. This means we may pick an étale neighbourhood $(V,v) \to (Y,y)$ such that

$$X \times_Y V = W \amalg R, \quad U \times_Y V = W' \amalg R'$$

and points $w \in W$, $w' \in W'$ such that

1. $W, R$ are open and closed in $X \times Y V$,
2. $W', R'$ are open and closed in $U \times Y V$,
3. $W \to V$ and $W' \to V$ are finite,
4. $w, w'$ map to $v$,
5. $\kappa(v) \subset \kappa(w)$ and $\kappa(v) \subset \kappa(w')$ are purely inseparable, and
6. no other point of $W$ or $W'$ maps to $v$. 

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Here is a commutative diagram

$$
\begin{array}{ccc}
U & \leftarrow & U \times_Y V \\
\downarrow & & \downarrow \\
X & \leftarrow & X \times_Y V \\
\downarrow & & \downarrow \\
Y & \leftarrow & V
\end{array}
$$

After shrinking $V$ we may assume that $W'$ maps into $W$: just remove the image the inverse image of $R$ in $W'$; this is a closed set (as $W' \to V$ is finite) not containing $v$. Then $W' \to W$ is finite because both $W \to V$ and $W' \to V$ are finite. Hence $W' \to W$ is finite étale, and there is exactly one point in the fibre over $w$ with $\kappa(w) = \kappa(w')$. Hence $W' \to W$ is an isomorphism in an open neighbourhood $W^0$ of $w$, see Étale Morphisms, Lemma 14.2. Since $W \to V$ is finite the image of $W \setminus W^0$ is a closed subset $T$ of $V$ not containing $v$. Thus after replacing $V$ by $V \setminus T$ we may assume that $W' \to W$ is an isomorphism. Now the decomposition $X \times_Y V = W \amalg R$ and the morphism $W \to U$ are as desired and we win. □

**Lemma 43.3.** Let $f : X \to Y$ be an integral morphism of schemes. Then property (A) holds.

**Proof.** Let $U \to X$ be étale, and let $u \in U$ be a point. We have to find $V \to Y$ étale, a disjoint union decomposition $X \times_Y V = W \amalg W'$ and an $X$-morphism $W \to U$ with $u$ in the image. We may shrink $U$ and $Y$ and assume $U$ and $Y$ are affine. In this case also $X$ is affine, since an integral morphism is affine by definition. Write $Y = \text{Spec}(A)$, $X = \text{Spec}(B)$ and $U = \text{Spec}(C)$. Then $A \to B$ is an integral ring map, and $B \to C$ is an étale ring map. By Algebra, Lemma 141.3 we can find a finite $A$-subalgebra $B' \subset B$ and an étale ring map $B' \to C'$ such that $C = B \otimes_{B'} C'$. Thus the question reduces to the étale morphism $U' = \text{Spec}(C') \to X' = \text{Spec}(B')$ over the finite morphism $X' \to Y$. In this case the result follows from Lemma 43.2. □

**Lemma 43.4.** Let $f : X \to Y$ be a morphism of schemes. Denote $f_{\text{small}} : \text{Sh}(X_{\text{étale}}) \to \text{Sh}(Y_{\text{étale}})$ the associated morphism of small étale topoi. Assume at least one of the following

1. $f$ is integral, or
2. $f$ is separated and locally quasi-finite.

Then the functor $f_{\text{small},*} : \text{Ab}(X_{\text{étale}}) \to \text{Ab}(Y_{\text{étale}})$ has the following properties

1. the map $f_{\text{small}}^{-1}\mathcal{F} \to \mathcal{F}$ is always surjective,
2. $f_{\text{small},*}$ is faithful, and
3. $f_{\text{small},*}$ reflects injections and surjections.

**Proof.** Combine Lemmas 43.2, 43.3 and 43.1. □

44. Property (B)

Please see Section 42 for the definition of property (B).
Lemma 44.1. Let \( f : X \to Y \) be a morphism of schemes. Assume (B) holds. Then the functor \( f_{\text{small}}^* : \text{Sh}(X_{\text{étale}}) \to \text{Sh}(Y_{\text{étale}}) \) transforms surjections into surjections.

Proof. This follows from Sites, Lemma 40.2. \( \square \)

Lemma 44.2. Let \( f : X \to Y \) be a morphism of schemes. Suppose
1. \( V \to Y \) is an étale morphism of schemes,
2. \( \{U_i \to X \times_Y V\} \) is an étale covering, and
3. \( v \in V \) is a point.

Assume that for any such data there exists an étale neighbourhood \((V', v') \to (V, v)\), a disjoint union decomposition \( X \times_Y V' = \bigsqcup W'_i \), and morphisms \( W'_i \to U_i \) over \( X \times_Y V \). Then property (B) holds.

Proof. Omitted. \( \square \)

Lemma 44.3. Let \( f : X \to Y \) be a finite morphism of schemes. Then property (B) holds.

Proof. Consider \( V \to Y \) étale, \( \{U_i \to X \times_Y V\} \) an étale covering, and \( v \in V \). We have to find a \( V' \to V \) and decomposition and maps as in Lemma 44.2. We may shrink \( V \) and \( Y \), hence we may assume that \( V \) and \( Y \) are affine. Since \( X \) is finite over \( Y \), this also implies that \( X \) is affine. During the proof we may (finitely often) replace \((V, v)\) by an étale neighbourhood \((V', v')\) and correspondingly the covering \( \{U_i \to X \times_Y V\} \) by \( \{V' \times_Y U_i \to X \times_Y V'\} \).

Since \( X \times_Y V \to V \) is finite there exist finitely many (pairwise distinct) points \( x_1, \ldots, x_n \in X \times_Y V \) mapping to \( v \). We may apply More on Morphisms, Lemma 30.5 to \( X \times_Y V \to V \) and the points \( x_1, \ldots, x_n \) lying over \( v \) and find an étale neighbourhood \((V', v') \to (V, v)\). This gives produces an étale neighbourhood \((V', v') \to (V, v)\) and decompositions
\[
X \times_Y V' = R \amalg \bigsqcup T_a
\]
with \( T_a \to V' \) finite with exactly one point \( p_a \) lying over \( v' \) and moreover \( \kappa(v') \subset \kappa(p_a) \) purely inseparable, and such that \( R \to V' \) has empty fibre over \( v' \). Because \( X \to Y \) is finite, also \( R \to V' \) is finite. Hence after shrinking \( V' \) we may assume that \( R = \emptyset \). Thus we may assume that \( X \times_Y V = X_1 \amalg \ldots \amalg X_n \) with exactly one point \( x_l \in X_l \) lying over \( v \) with moreover \( \kappa(v) \subset \kappa(x_l) \) purely inseparable. Note that this property is preserved under refinement of the étale neighbourhood \((V, v)\).

For each \( l \) choose an \( i_l \) and a point \( u_l \in U_{i_l} \) mapping to \( x_l \). Now we apply property (A) for the finite morphism \( X \times_Y V \to V \) and the étale morphisms \( U_{i_l} \to X \times_Y V \) and the points \( u_l \). This is permissible by Lemma 43.3. This gives produces an étale neighbourhood \((V', v') \to (V, v)\) and decompositions
\[
X \times_Y V' = W_l \amalg R_l
\]
and $X$-morphisms $a_i : W_i \to U_i$ whose image contains $u_i$. Here is a picture:

\[
\begin{array}{cccccc}
W_i & \to & W_i \times \mathbb{R} \times U \times Y & \to & X \times Y & \to & X \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
V' & \to & V & \to & Y \\
\end{array}
\]

After replacing $(V,v)$ by $(V',v')$ we conclude that each $x_i$ is contained in an open and closed neighbourhood $W_i$ such that the inclusion morphism $W_i \to X \times Y$ factors through $U_i \to X \times Y$ for some $i$. Replacing $W_i$ by $W_i \cap X_i$ we see that these open and closed sets are disjoint and moreover that $\{x_1, \ldots, x_n\} \subset W_1 \cup \ldots \cup W_n$. Since $X \times Y \to V$ is finite we may shrink $V$ and assume that $X \times Y \to V = W_1 \cdots W_n$ as desired.

\begin{lemma}
Let $f : X \to Y$ be an integral morphism of schemes. Then property (B) holds.
\end{lemma}

\begin{proof}
Consider $V \to Y$ étale, $\{U_i \to X \times Y \}_{i=1}^n$ an étale covering, and $v \in V$. We have to find a $V' \to V$ and decomposition and maps as in Lemma 44.2. We may shrink $V'$ and $Y$, hence we may assume that $V$ and $Y$ are affine. Since $X$ is integral over $Y$, this also implies that $X$ and $X \times Y$ are affine. We may refine the covering $\{U_i \to X \times Y \}$, and hence we may assume that $\{U_i \to X \times Y \}_{i=1}^n$ is a standard étale covering. Write $Y = \text{Spec}(A)$, $X = \text{Spec}(B)$, $V = \text{Spec}(C)$, and $U_i = \text{Spec}(B_i)$. Then $A \to B$ is an integral ring map, and $B \otimes_A C \to B_i$ are étale ring maps. By Algebra, Lemma 141.3 we can find a finite $A$-subalgebra $B' \subset B$ and an étale ring map $B' \otimes_A C \to B_i$ for $i = 1, \ldots, n$ such that $B_i = B \otimes_{B'} B'_i$. Thus the question reduces to the étale covering $\{\text{Spec}(B'_i) \to X' \times Y \}_{i=1}^{n}$ with $X' = \text{Spec}(B')$ finite over $Y$. In this case the result follows from Lemma 44.3.
\end{proof}

\begin{lemma}
Let $f : X \to Y$ be a morphism of schemes. Assume $f$ is integral (for example finite). Then
\begin{enumerate}
\item $f_{\text{small,}*}$ transforms surjections into surjections (on sheaves of sets and on abelian sheaves),
\item $f_{\text{small}}^{-1} \mathcal{F} \to \mathcal{F}$ is surjective for any abelian sheaf $\mathcal{F}$ on $X_{\text{étale}}$,
\item $f_{\text{small,}*} : \text{Ab}(X_{\text{étale}}) \to \text{Ab}(Y_{\text{étale}})$ is faithful and reflects injections and surjections, and
\item $f_{\text{small,}*} : \text{Ab}(X_{\text{étale}}) \to \text{Ab}(Y_{\text{étale}})$ is exact.
\end{enumerate}
\end{lemma}

\begin{proof}
Parts (2), (3) we have seen in Lemma 43.4. Part (1) follows from Lemmas 44.4 and 44.1. Part (4) is a consequence of part (1), see Modules on Sites, Lemma 15.2.
\end{proof}

45. Property (C)

Please see Section 42 for the definition of property (C).

\begin{lemma}
Let $f : X \to Y$ be a morphism of schemes. Assume (C) holds. Then the functor $f_{\text{small,}*} : \text{Sh}(X_{\text{étale}}) \to \text{Sh}(Y_{\text{étale}})$ reflects injections and surjections.
\end{lemma}
Proof. Follows from Sites, Lemma 40.4. We omit the verification that property (C) implies that the functor $Y_{\text{étale}} \to X_{\text{étale}}, V \mapsto X \times_Y V$ satisfies the assumption of Sites, Lemma 40.4. \hfill \square

04DU  **Remark 45.2.** Property (C) holds if $f : X \to Y$ is an open immersion. Namely, if $U \in \text{Ob}(X_{\text{étale}})$, then we can view $U$ also as an object of $Y_{\text{étale}}$ and $U \times_Y X = U$. Hence property (C) does not imply that $f_{\text{small,}*}$ is exact as this is not the case for open immersions (in general).

04DV  **Lemma 45.3.** Let $f : X \to Y$ be a morphism of schemes. Assume that for any $V \to Y$ étale we have that

1. $X \times_Y V \to V$ has property (C), and
2. $X \times_Y V \to V$ is closed.

Then the functor $Y_{\text{étale}} \to X_{\text{étale}}, V \mapsto X \times_Y V$ is almost cocontinuous, see Sites, Definition 41.3.

Proof. Let $V \to Y$ be an object of $Y_{\text{étale}}$ and let $\{U_i \to X \times_Y V\}_{i \in I}$ be a covering of $X_{\text{étale}}$. By assumption (1) for each $i$ we can find an étale morphism $h_i : V_i \to V$ and a surjective morphism $X \times_Y V_i \to U_i$ over $X \times_Y V$. Note that $\bigcup h_i(V_i) \subset V$ is an open set containing the closed set $Z = \text{Im}(X \times_Y V \to V)$. Let $h_0 : V_0 = V \setminus Z \to V$ be the open immersion. It is clear that $\{V_i \to V\}_{i \in I \cup \{0\}}$ is an étale covering such that for each $i \in I \cup \{0\}$ we have either $V_i \times_Y X = \emptyset$ (namely if $i = 0$), or $V_i \times_Y X \to V \times_Y X$ factors through $U_i \to X \times_Y V$ (if $i \neq 0$). Hence the functor $Y_{\text{étale}} \to X_{\text{étale}}$ is almost cocontinuous. \hfill \square

04DW  **Lemma 45.4.** Let $f : X \to Y$ be an integral morphism of schemes which defines a homeomorphism of $X$ with a closed subset of $Y$. Then property (C) holds.

Proof. Let $g : U \to X$ be an étale morphism. We need to find an object $V \to Y$ of $Y_{\text{étale}}$ and a surjective morphism $X \times_Y V \to U$ over $X$. Suppose that for every $u \in U$ we can find an object $V_u \to Y$ of $Y_{\text{étale}}$ and a morphism $h_u : X \times_Y V_u \to U$ over $X$ with $u \in \text{Im}(h_u)$. Then we can take $V = \coprod V_u$ and $h = \coprod h_u$ and we win. Hence given a point $u \in U$ we find a pair $(V_u, h_u)$ as above. To do this we may shrink $U$ and assume that $U$ is affine. In this case $g : U \to X$ is locally quasi-finite. Let $g^{-1}(g(\{u\})) = \{u, u_2, \ldots, u_n\}$. Since there are no specializations $u_i \rightsquigarrow u$ we may replace $U$ by an affine neighbourhood so that $g^{-1}(g(\{u\})) = \{u\}$.

The image $g(U) \subset X$ is open, hence $f(g(U))$ is locally closed in $Y$. Choose an open $V \subset Y$ such that $f(g(U)) = f(X) \cap V$. It follows that $g$ factors through $X \times_Y V$ and that the resulting $(U \to X \times_Y V)$ is an étale covering. Since $f$ has property (B), see Lemma 44.4, we see that there exists an étale covering $\{V_j \to V\}$ such that $X \times_Y V_j \to X \times_Y V$ factor through $U$. This implies that $V' = \coprod V_j$ is étale over $Y$ and that there is a morphism $h : X \times_Y V' \to U$ whose image surjects onto $g(U)$. Since $u$ is the only point in its fibre it must be in the image of $h$ and we win. \hfill \square

We urge the reader to think of the following lemma as a way station on the journey towards the ultimate truth regarding $f_{\text{small,}*}$ for integral universally injective morphisms.

---

3A way station is a place where people stop to eat and rest when they are on a long journey.
Lemma 45.5. Let \( f : X \to Y \) be a morphism of schemes. Assume that \( f \) is universally injective and integral (for example a closed immersion). Then

1. \( \mathcal{f}_{\text{small,*}} : \mathcal{Sh}(X_{\text{\acute{e}tale}}) \to \mathcal{Sh}(Y_{\text{\acute{e}tale}}) \) reflects injections and surjections,
2. \( \mathcal{f}_{\text{small,*}} : \mathcal{Sh}(X_{\text{\acute{e}tale}}) \to \mathcal{Sh}(Y_{\text{\acute{e}tale}}) \) commutes with pushouts and coequalizers (and more generally finite connected colimits),
3. \( \mathcal{f}_{\text{small,*}} \) transforms surjections into surjections (on sheaves of sets and on abelian sheaves),
4. the map \( f^{-1}\mathcal{f}_{\text{small,*}} \mathcal{F} \to \mathcal{F} \) is surjective for any sheaf (of sets or of abelian groups) \( \mathcal{F} \) on \( X_{\text{\acute{e}tale}} \),
5. the functor \( \mathcal{f}_{\text{small,*}} \) is faithful (on sheaves of sets and on abelian sheaves),
6. \( \mathcal{f}_{\text{small,*}} : \text{Ab}(X_{\text{\acute{e}tale}}) \to \text{Ab}(Y_{\text{\acute{e}tale}}) \) is exact, and
7. the functor \( Y_{\text{\acute{e}tale}} \to X_{\text{\acute{e}tale}}, V \mapsto X \times_Y V \) is almost cocontinuous.

Proof. By Lemmas 43.3, 44.4 and 45.4 we know that the morphism \( f \) has properties (A), (B), and (C). Moreover, by Lemma 45.3 we know that the functor \( Y_{\text{\acute{e}tale}} \to X_{\text{\acute{e}tale}} \) is almost cocontinuous. Now we have

1. property (C) implies (1) by Lemma 45.1,
2. almost continuous implies (2) by Sites, Lemma 41.6,
3. property (B) implies (3) by Lemma 44.1.

Properties (4), (5), and (6) follow formally from the first three, see Sites, Lemma 40.1 and Modules on Sites, Lemma 15.2. Property (7) we saw above. □

46. Topological invariance of the small étale site

In the following theorem we show that the small étale site is a topological invariant in the following sense: If \( f : X \to Y \) is a morphism of schemes which is a universal homeomorphism, then \( X_{\text{\acute{e}tale}} \cong Y_{\text{\acute{e}tale}} \) as sites. This improves the result of Étale Morphisms, Theorem 15.2. We first prove the result for morphisms and then we state the result for categories.

Theorem 46.1. Let \( X \) and \( Y \) be two schemes over a base scheme \( S \). Let \( S' \to S \) be a universal homeomorphism. Denote \( X' \) (resp. \( Y' \)) the base change to \( S' \). If \( X \) is étale over \( S \), then the map

\[
\text{Mor}_S(Y, X) \to \text{Mor}_{S'}(Y', X')
\]

is bijective.

Proof. After base changing via \( Y \to S \), we may assume that \( Y = S \). Thus we may and do assume both \( X \) and \( Y \) are étale over \( S \). In other words, the theorem states that the base change functor is a fully faithful functor from the category of schemes étale over \( S \) to the category of schemes étale over \( S' \).

Consider the forgetful functor

\[
(46.1.1) \quad \text{descent data } (X', \phi') \to \text{ schemes } X' \text{ étale over } S' \text{ with } X' \text{ étale over } S'
\]

We claim this functor is an equivalence. On the other hand, the functor

\[
(46.1.2) \quad \text{ schemes } X \text{ étale over } S \to \text{ descent data } (X', \phi') \text{ relative to } S'/S \text{ with } X' \text{ étale over } S'
\]

is fully faithful by Étale Morphisms, Lemma 20.3. Thus the claim implies the theorem.
Proof of the claim. Recall that a universal homeomorphism is the same thing as an integral, universally injective, surjective morphism, see Morphisms, Lemma \[44.3\]. In particular, the diagonal \( \Delta : S' \to S' \times_S S' \) is a thickening by Morphisms, Lemma \[11.2\]. Thus by Étale Morphisms, Theorem \[15.1\] we see that given \( X' \to S' \) étale there is a unique isomorphism

\[ \varphi' : X' \times_S S' \to S' \times_S X' \]

of schemes étale over \( S' \times_S S' \) which pulls back under \( \Delta \) to \( \text{id} : X' \to X' \) over \( S' \). Since \( S' \to S' \times_S S' \times_S S' \) is a thickening as well (it is bijective and a closed immersion) we conclude that \((X', \varphi')\) is a descent datum relative to \( S'/S \). The canonical nature of the construction of \( \varphi' \) shows that it is compatible with morphisms between schemes étale over \( S' \). In other words, we obtain a quasi-inverse \( X' \mapsto (X', \varphi') \) of the functor \[46.1.1\]. This proves the claim and finishes the proof of the theorem. \(\square\)

**Theorem 46.2.** Let \( f : X \to Y \) be a morphism of schemes. Assume \( f \) is integral, universally injective and surjective (i.e., \( f \) is a universal homeomorphism, see Morphisms, Lemma \[44.3\]). The functor

\[ V \mapsto V_X = X \times_Y V \]

defines an equivalence of categories

\(\{\text{schemes } V \text{ étale over } Y\} \leftrightarrow \{\text{schemes } U \text{ étale over } X\}\)

We give two proofs. The first uses effectivity of descent for quasi-compact, separated, étale morphisms relative to surjective integral morphisms. The second uses the material on properties (A), (B), and (C) discussed earlier in the chapter.

**First proof.** By Theorem \[46.1\] we see that the functor is fully faithful. It remains to show that the functor is essentially surjective. Let \( U \to X \) be an étale morphism of schemes.

Suppose that the result holds if \( U \) and \( Y \) are affine. In that case, we choose an affine open covering \( U = \bigcup U_i \) such that each \( U_i \) maps into an affine open of \( Y \). By assumption (affine case) we can find étale morphisms \( V_i \to Y \) such that \( X \times_Y V_i \cong U_i \) as schemes over \( X \). Let \( V_{i,i'} \subset V_i \) be the open subscheme whose underlying topological space corresponds to \( U_i \cap U_{i'} \). Because we have isomorphisms

\[ X \times_Y V_{i,i'} \cong U_i \cap U_{i'} \cong X \times_Y V_{i',i} \]

as schemes over \( X \) we see by fully faithfulness that we obtain isomorphisms \( \theta_{i,i'} : V_{i,i'} \to V_{i',i} \) of schemes over \( Y \). We omit the verification that these isomorphisms satisfy the cocycle condition of Schemes, Section \[14\]. Applying Schemes, Lemma \[14.2\] we obtain a scheme \( V \to Y \) by glueing the schemes \( V_i \) along the identifications \( \theta_{i,i'} \). It is clear that \( V \to Y \) is étale and \( X \times_Y V \cong U \) by construction.

Thus it suffices to show the lemma in case \( U \) and \( Y \) are affine. Recall that in the proof of Theorem \[46.1\] we showed that \( U \) comes with a unique descent datum \((U, \varphi)\) relative to \( X/Y \). By Étale Morphisms, Proposition \[20.6\] (which applies because \( U \to X \) is quasi-compact and separated as well as étale by our reduction to the affine case) there exists an étale morphism \( V \to Y \) such that \( X \times_Y V \cong U \) and the proof is complete. \(\square\)
**Second proof.** By Theorem 46.1 we see that the functor is fully faithful. It remains to show that the functor is essentially surjective. Let \( U \to X \) be an étale morphism of schemes.

Suppose that the result holds if \( U \) and \( Y \) are affine. In that case, we choose an affine open covering \( U = \bigcup U_i \) such that each \( U_i \) maps into an affine open of \( Y \). By assumption (affine case) we can find étale morphisms \( V_i \to Y \) such that \( X \times_Y V_i \cong U_i \) as schemes over \( X \). Let \( V_{i,i'} \subset V_i \) be the open subscheme whose underlying topological space corresponds to \( U_i \cap U_{i'} \). Because we have isomorphisms

\[
X \times_Y V_{i,i'} \cong U_i \cap U_{i'} \cong X \times_Y V_{i,i'}
\]

as schemes over \( X \) we see by fully faithfulness that we obtain isomorphisms \( \theta_{i,i'} : V_{i,i'} \to V_{i,i}' \) of schemes over \( Y \). We omit the verification that these isomorphisms satisfy the cocycle condition of Schemes, Section 14. Applying Schemes, Lemma 14.2 we obtain a scheme \( V \to Y \) by glueing the schemes \( V_i \) along the identifications \( \theta_{i,i'} \). It is clear that \( V \to Y \) is étale and \( X \times_Y V \cong U \) by construction.

Thus it suffices to prove that the functor

\[
\{ \text{affine schemes } V \text{ étale over } Y \} \leftrightarrow \{ \text{affine schemes } U \text{ étale over } X \}
\]

is essentially surjective when \( X \) and \( Y \) are affine.

Let \( U \to X \) be an affine scheme étale over \( X \). We have to find \( V \to Y \) étale (and affine) such that \( X \times_Y V \) is isomorphic to \( U \) over \( X \). Note that an étale morphism of affines has universally bounded fibres, see Morphisms, Lemmas 36.6 and 51.8. Hence we can do induction on the integer \( n \) bounding the degree of the fibres of \( U \to X \). See Morphisms, Lemma 51.7 for a description of this integer in the case of an étale morphism. If \( n = 1 \), then \( U \to X \) is an open immersion (see Étale Morphisms, Theorem 14.1), and the result is clear. Assume \( n > 1 \).

By Lemma 45.4 there exists an étale morphism of schemes \( W \to Y \) and a surjective morphism \( W_X \to U \) over \( X \). As \( U \) is quasi-compact we may replace \( W \) by a disjoint union of finitely many affine opens of \( W \), hence we may assume that \( W \) is affine as well. Here is a diagram

\[
\begin{array}{ccc}
U & \xleftarrow{\pi} & U \times_Y W \\
\downarrow & & \downarrow \\
X & \xleftarrow{\phi} & W_X \\
Y & \xleftarrow{\pi} & W
\end{array}
\]

The disjoint union decomposition arises because by construction the étale morphism of affine schemes \( U \times_Y W \to W_X \) has a section. OK, and now we see that the morphism \( R \to X \times_Y W \) is an étale morphism of affine schemes whose fibres have degree universally bounded by \( n-1 \). Hence by induction assumption there exists a scheme \( V' \to W \) étale such that \( R \cong W_X \times_W V' \). Taking \( V'' = W \amalg V' \) we find a scheme \( V'' \) étale over \( W \) whose base change to \( W_X \) is isomorphic to \( U \times_Y W \) over \( X \times_Y W \).

At this point we can use descent to find \( V \) over \( Y \) whose base change to \( X \) is isomorphic to \( U \) over \( X \). Namely, by the fully faithfulness of the functor (46.2.1).
corresponding to the universal homeomorphism $X \times_Y (W \times_Y W) \to (W \times_Y W)$ there exists a unique isomorphism $\varphi : V'' \times_Y W \to W \times_Y V''$ whose base change to $X \times_Y (W \times_Y W)$ is the canonical descent datum for $U \times_Y W$ over $X \times_Y W$. In particular $\varphi$ satisfies the cocycle condition. Hence by Descent, Lemma 33.1 we see that $\varphi$ is effective (recall that all schemes above are affine). Thus we obtain $V \to Y$ and an isomorphism $V'' \cong W \times_Y V$ such that the canonical descent datum on $W \times_Y V/W/Y$ agrees with $\varphi$. Note that $V \to Y$ is étale, by Descent, Lemma 19.27. Moreover, there is an isomorphism $V_X \cong U$ which comes from descending the isomorphism

$$V_X \times_X W_X = X \times_Y V \times_Y W = (X \times_Y W) \times_W (W \times_Y V) \cong W_X \times_W V'' \cong U \times_Y W$$

which we have by construction. Some details omitted. □

**Remark 46.3.** In the situation of Theorem 46.2 it is also true that $V \to V_X$ induces an equivalence between those étale morphisms $V \to Y$ with $V$ affine and those étale morphisms $U \to X$ with $U$ affine. This follows for example from Limits, Proposition 10.2.

**Proposition 46.4** (Topological invariance of étale cohomology). Let $X_0 \to X$ be a universal homeomorphism of schemes (for example the closed immersion defined by a nilpotent sheaf of ideals). Then

1. the étale sites $X_{\text{ét}}$ and $(X_0)_{\text{ét}}$ are isomorphic,
2. the étale topoi $\mathcal{Sh}(X_{\text{ét}})$ and $\mathcal{Sh}((X_0)_{\text{ét}})$ are equivalent, and
3. $H^q_{\text{ét}}(X, F) = H^q_{\text{ét}}(X_0, F|_{X_0})$ for all $q$ and for any abelian sheaf $F$ on $X_{\text{ét}}$.

**Proof.** The equivalence of categories $X_{\text{ét}} \to (X_0)_{\text{ét}}$ is given by Theorem 46.2. We omit the proof that under this equivalence the étale coverings correspond. Hence (1) holds. Parts (2) and (3) follow formally from (1). □

### 47. Closed immersions and pushforward

Before stating and proving Proposition 47.4 in its correct generality we briefly state and prove it for closed immersions. Namely, some of the preceding arguments are quite a bit easier to follow in the case of a closed immersion and so we repeat them here in their simplified form.

In the rest of this section $i : Z \to X$ is a closed immersion. The functor

$$\text{Sch}/X \longrightarrow \text{Sch}/Z, \quad U \mapsto U_Z = Z \times_X U$$

will be denoted $U \mapsto U_Z$ as indicated. Since being a closed immersion is preserved under arbitrary base change the scheme $U_Z$ is a closed subscheme of $U$.

**Lemma 47.1.** Let $i : Z \to X$ be a closed immersion of schemes. Let $U, U'$ be schemes étale over $X$. Let $h : U_Z \to U'_Z$ be a morphism over $Z$. Then there exists a diagram

$$U \xleftarrow{a} W \xrightarrow{b} U'$$

such that $a_Z : W_Z \to U_Z$ is an isomorphism and $h = b_Z \circ (a_Z)^{-1}$.

**Proof.** Consider the scheme $M = U \times_Y U'$. The graph $\Gamma_h \subset M_Z$ of $h$ is open. This is true for example as $\Gamma_h$ is the image of a section of the étale morphism $\text{pr}_{1,Z} : M_Z \to U_Z$, see Étale Morphisms, Proposition 6.1. Hence there exists an
open subscheme $W \subset M$ whose intersection with the closed subset $M_Z$ is $\Gamma_h$. Set $a = \text{pr}_1|_W$ and $b = \text{pr}_2|_W$.  \hfill \square

\textbf{Lemma 47.2.} Let $i : Z \to X$ be a closed immersion of schemes. Let $V \to Z$ be an étale morphism of schemes. There exist étale morphisms $U_i \to X$ and morphisms $U_i, Z \to V$ such that $\{U_i, Z \to V\}$ is a Zariski covering of $V$.

\textbf{Proof.} Since we only have to find a Zariski covering of $V$ consisting of schemes of the form $U_Z$ with $U$ étale over $X$, we may Zariski localize on $X$ and $V$. Hence we may assume $X$ and $V$ affine. In the affine case this is Algebra, Lemma [141.11]. \hfill \square

If $x : \text{Spec}(k) \to X$ is a geometric point of $X$, then either $x$ factors (uniquely) through the closed subscheme $Z$, or $Z_x = \emptyset$. If $x$ factors through $Z$ we say that $x$ is a geometric point of $Z$ (because it is) and we use the notation “$x \in Z$” to indicate this.

\textbf{Lemma 47.3.} Let $i : Z \to X$ be a closed immersion of schemes. Let $G$ be a sheaf of sets on $Z_{\text{étale}}$. Let $x$ be a geometric point of $X$. Then

$$(i_{\text{small}}, \ast)_{\mathfrak{p}} = \begin{cases} \ast & \text{if } \mathfrak{p} \notin Z \\ F_{\mathfrak{p}} & \text{if } \mathfrak{p} \in Z \end{cases}$$

where $\ast$ denotes a singleton set.

\textbf{Proof.} Note that $i_{\text{small}}, \ast|_{U_{\text{étale}}} = \ast$ is the final object in the category of étale sheaves on $U$, i.e., the sheaf which associates a singleton set to each scheme étale over $U$. This explains the value of $(i_{\text{small}}, \ast)_{\mathfrak{p}}$ if $\mathfrak{p} \notin Z$.

Next, suppose that $\mathfrak{p} \in Z$. Note that

$$(i_{\text{small}}, \ast)_{\mathfrak{p}} = \text{colim}_{(U, \mathfrak{p})} G(U_Z)$$

and on the other hand

$$G_{\mathfrak{p}} = \text{colim}_{(V, \mathfrak{p})} G(V).$$

Let $C_1 = \{(U, \mathfrak{p})\}^{\text{opp}}$ be the opposite of the category of étale neighbourhoods of $\mathfrak{p}$ in $X$, and let $C_2 = \{(V, \mathfrak{p})\}^{\text{opp}}$ be the opposite of the category of étale neighbourhoods of $\mathfrak{p}$ in $Z$. The canonical map

$$G_{\mathfrak{p}} \to (i_{\text{small}}, \ast)_{\mathfrak{p}}$$

corresponds to the functor $F : C_1 \to C_2, F(U, \mathfrak{p}) = (U_Z, \mathfrak{p})$. Now Lemmas 47.2 and 47.1 imply that $C_1$ is cofinal in $C_2$, see Categories, Definition [17.1]. Hence it follows that the displayed arrow is an isomorphism, see Categories, Lemma [17.2]. \hfill \square

\textbf{Proposition 47.4.} Let $i : Z \to X$ be a closed immersion of schemes.

1. The functor

$$i_{\text{small}}, \ast : \text{Sh}(Z_{\text{étale}}) \to \text{Sh}(X_{\text{étale}})$$

is fully faithful and its essential image is those sheaves of sets $\mathcal{F}$ on $X_{\text{étale}}$ whose restriction to $X \setminus Z$ is isomorphic to $\ast$, and

2. the functor

$$i_{\text{small}}, \ast : \text{Ab}(Z_{\text{étale}}) \to \text{Ab}(X_{\text{étale}})$$

is fully faithful and its essential image is those abelian sheaves on $X_{\text{étale}}$ whose support is contained in $Z$.

In both cases $i_{\text{small}}^{-1}$ is a left inverse to the functor $i_{\text{small}}, \ast$. 

Proof. Let’s discuss the case of sheaves of sets. For any sheaf $\mathcal{G}$ on $Z$ the morphism $i_{\text{small}}^{-1}i_{\text{small}}^*\mathcal{G} \to \mathcal{G}$ is an isomorphism by Lemma 47.3 (and Theorem 29.10). This implies formally that $i_{\text{small}}^*$ is fully faithful, see Sites, Lemma 40.1. It is clear that $i_{\text{small}}^{-1}i_{\text{small}}^*\mathcal{G}|_{\text{etale}} \cong *$ where $U = X \setminus Z$. Conversely, suppose that $\mathcal{F}$ is a sheaf of sets on $X$ such that $\mathcal{F}|_{\text{etale}} \cong *$. Consider the adjunction mapping $\mathcal{F} \to i_{\text{small}}^{-1}i_{\text{small}}^*\mathcal{F}$.

Combining Lemmas 47.3 and 36.2 we see that it is an isomorphism. This finishes the proof of (1). The proof of (2) is identical. □

48. Integral universally injective morphisms

04FY Here is the general version of Proposition 47.4.

04FZ Proposition 48.1. Let $f : X \to Y$ be a morphism of schemes which is integral and universally injective.

(1) The functor $f_{\text{small}}^* : \text{Sh}(X_{\text{etale}}) \to \text{Sh}(Y_{\text{etale}})$ is fully faithful and its essential image is those sheaves of sets $\mathcal{F}$ on $Y_{\text{etale}}$ whose restriction to $Y \setminus f(X)$ is isomorphic to $*$, and

(2) the functor $f_{\text{small}}^* : \text{Ab}(X_{\text{etale}}) \to \text{Ab}(Y_{\text{etale}})$ is fully faithful and its essential image is those abelian sheaves on $Y_{\text{etale}}$ whose support is contained in $f(X)$.

In both cases $f_{\text{small}}^{-1}$ is a left inverse to the functor $f_{\text{small}}^*$.

Proof. We may factor $f$ as $X \xrightarrow{h} Z \xrightarrow{i} Y$ where $h$ is integral, universally injective and surjective and $i : Z \to Y$ is a closed immersion. Apply Proposition 47.3 to $i$ and apply Theorem 46.2 to $h$. □

49. Big sites and pushforward

04E2 In this section we prove some technical results on $f_{\text{big}}^*$ for certain types of morphisms of schemes.

04C7 Lemma 49.1. Let $\tau \in \{\text{Zariski, étale, smooth, syntomic, fppf}\}$. Let $f : X \to Y$ be a monomorphism of schemes. Then the canonical map $f_{\text{big}}^{-1}f_{\text{big}}^*\mathcal{F} \to \mathcal{F}$ is an isomorphism for any sheaf $\mathcal{F}$ on $(\text{Sch}/X)_\tau$.

Proof. In this case the functor $(\text{Sch}/X)_\tau \to (\text{Sch}/Y)_\tau$ is continuous, cocontinuous and fully faithful. Hence the result follows from Sites, Lemma 20.7. □

04C8 Remark 49.2. In the situation of Lemma 49.1 it is true that the canonical map $\mathcal{F} \to f_{\text{big}}^{-1}f_{\text{big}}^*\mathcal{F}$ is an isomorphism for any sheaf of sets $\mathcal{F}$ on $(\text{Sch}/X)_\tau$. The proof is the same. This also holds for sheaves of abelian groups. However, note that the functor $f_{\text{big}}$ for sheaves of abelian groups is defined in Modules on Sites, Section 16 and is in general different from $f_{\text{big}}$ on sheaves of sets. The result for sheaves of abelian groups follows from Modules on Sites, Lemma 16.4.
Lemma 49.3. Let \( f : X \to Y \) be a closed immersion of schemes. Let \( U \to X \) be a syntomic (resp. smooth, resp. \( \acute{e} \)tale) morphism. Then there exist syntomic (resp. smooth, resp. \( \acute{e} \)tale) morphisms \( V_i \to Y \) and morphisms \( V_i \times_Y X \to U \) such that \( \{ V_i \times_Y X \to U \} \) is a Zariski covering of \( U \).

Proof. Let us prove the lemma when \( \tau = \text{syntomic} \). The question is local on \( U \). Thus we may assume that \( U \) is an affine scheme mapping into an affine of \( Y \). Hence we reduce to proving the following case: \( Y = \text{Spec}(A), \ X = \text{Spec}(A/I), \) and \( U = \text{Spec}(B) \), where \( A/I \to \mathcal{B} \) be a syntomic ring map. By Algebra, Lemma 134.18, we can find elements \( g_i \in \mathcal{B} \) such that \( \mathcal{B}/g_i = A_i/I A_i \) for certain syntomic ring maps \( A \to A_i \). This proves the lemma in the syntomic case. The proof of the smooth case is the same except it uses Algebra, Lemma 135.19. In the \( \acute{e} \)tale case use Algebra, Lemma 141.11. \( \square \)

Lemma 49.4. Let \( f : X \to Y \) be a closed immersion of schemes. Let \( \{ U_i \to X \} \) be a syntomic (resp. smooth, resp. \( \acute{e} \)tale) covering. There exists a syntomic (resp. smooth, resp. \( \acute{e} \)tale) covering \( \{ V_j \to Y \} \) such that for each \( j \), either \( V_j \times_Y X = \emptyset \), or the morphism \( V_j \times_Y X \to X \) factors through \( U_i \) for some \( i \).

Proof. For each \( i \) we can choose syntomic (resp. smooth, resp. \( \acute{e} \)tale) morphisms \( \varphi_{ij} : V_{ij} \to Y \) and morphisms \( V_{ij} \times_Y X \to U_i \) over \( X \), such that \( \{ \varphi_{ij} : V_{ij} \times_Y X \to U_i \} \) are Zariski coverings, see Lemma 49.3. This in particular implies that \( \{ \varphi_{ij} \} \) contains the closed subset \( f(X) \). Hence the family of syntomic (resp. smooth, resp. \( \acute{e} \)tale) maps \( \varphi_{ij} \) together with the open immersion \( Y \setminus f(X) \to Y \) forms the desired syntomic (resp. smooth, resp. \( \acute{e} \)tale) covering of \( Y \). \( \square \)

Lemma 49.5. Let \( f : X \to Y \) be a closed immersion of schemes. Let \( \tau \in \{ \text{syntomic, smooth, } \acute{e} \)tale\} \). The functor \( V \mapsto X \times_Y V \) defines an almost cocontinuous functor (see Sites, Definition 41.3) \( (\mathcal{S}ch/Y)_\tau \to (\mathcal{S}ch/X)_\tau \) between big \( \tau \) sites.

Proof. We have to show the following: given a morphism \( V \to Y \) and any syntomic (resp. smooth, resp. \( \acute{e} \)tale) covering \( \{ U_i \to X \times_Y V \} \), there exists a smooth (resp. smooth, resp. \( \acute{e} \)tale) covering \( \{ V_j \to Y \} \) such that for each \( j \), either \( X \times_Y V_j \) is empty, or \( X \times_Y V_j \to Z \times_Y V \) factors through one of the \( U_i \). This follows on applying Lemma 49.4 above to the closed immersion \( X \times_Y V \to V \). \( \square \)

Lemma 49.6. Let \( f : X \to Y \) be a closed immersion of schemes. Let \( \tau \in \{ \text{syntomic, smooth, } \acute{e} \)tale\}.

1. The pushforward \( f_{\text{big}*} : \text{Sh}((\mathcal{S}ch/X)_\tau) \to \text{Sh}((\mathcal{S}ch/Y)_\tau) \) commutes with coequalizers and pushouts.

2. The pushforward \( f_{\text{big}*} : \text{Ab}((\mathcal{S}ch/X)_\tau) \to \text{Ab}((\mathcal{S}ch/Y)_\tau) \) is exact.

Proof. This follows from Sites, Lemma 41.6 Modules on Sites, Lemma 15.3 and Lemma 49.5 above. \( \square \)

Remark 49.7. In Lemma 49.6 the case \( \tau = \text{fppf} \) is missing. The reason is that given a ring \( A \), an ideal \( f \) and a faithfully flat, finitely presented ring map \( A/I \to \mathcal{B} \), there is no reason to think that one can find any flat finitely presented ring map \( A \to B \) with \( B/IB \neq 0 \) such that \( A/I \to B/IB \) factors through \( \mathcal{B} \). Hence the proof of Lemma 49.5 does not work for the fppf topology. In fact it is likely false that \( f_{\text{big}*} : \text{Ab}((\mathcal{S}ch/X)_{\text{fppf}}) \to \text{Ab}((\mathcal{S}ch/Y)_{\text{fppf}}) \) is exact when \( f \) is a closed immersion. If you know an example, please email stacks.project@gmail.com.
50. Exactness of big lower shriek

This is just the following technical result. Note that the functor $f_{\text{big}}$ has nothing whatsoever to do with cohomology with compact support in general.

**Lemma 50.1.** Let $\tau \in \{\text{Zariski, étale, smooth, syntomic, fppf}\}$. Let $f : X \to Y$ be a morphism of schemes. Let

$$f_{\text{big}} : \text{Sh}(\mathbf{Sch}/X)_{\tau} \to \text{Sh}(\mathbf{Sch}/Y)_{\tau}$$

be the corresponding morphism of topoi as in Topologies, Lemma \textit{3.15}, \textit{4.15}, \textit{5.10}, \textit{6.10}, or \textit{7.12}.

1. The functor $f^{-1}_{\text{big}} : \text{Ab}(\mathbf{Sch}/Y)_{\tau} \to \text{Ab}(\mathbf{Sch}/X)_{\tau}$ has a left adjoint $f_{\text{big}}! : \text{Ab}(\mathbf{Sch}/X)_{\tau} \to \text{Ab}(\mathbf{Sch}/Y)_{\tau}$ which is exact.

2. The functor $f_{\text{big}}^* : \text{Mod}(\mathbf{Sch}/Y)_{\tau, \mathcal{O}} \to \text{Mod}(\mathbf{Sch}/X)_{\tau, \mathcal{O}}$ has a left adjoint $f_{\text{big}}! : \text{Mod}(\mathbf{Sch}/X)_{\tau, \mathcal{O}} \to \text{Mod}(\mathbf{Sch}/Y)_{\tau, \mathcal{O}}$ which is exact.

Moreover, the two functors $f_{\text{big}}!$ agree on underlying sheaves of abelian groups.

**Proof.** Recall that $f_{\text{big}}$ is the morphism of topoi associated to the continuous and cocontinuous functor $u : (\mathbf{Sch}/X)_{\tau} \to (\mathbf{Sch}/Y)_{\tau}$, $U/X \mapsto U/Y$. Moreover, we have $f_{\text{big}}^{-1}\mathcal{O} = \mathcal{O}$. Hence the existence of $f_{\text{big}}!$ follows from Modules on Sites, Lemma \textit{16.2}, respectively Modules on Sites, Lemma \textit{40.1}. Note that if $U$ is an object of $(\mathbf{Sch}/X)_{\tau}$ then the functor $u$ induces an equivalence of categories

$$u' : (\mathbf{Sch}/X)_{\tau}/U \to (\mathbf{Sch}/Y)_{\tau}/U$$

because both sides of the arrow are equal to $(\mathbf{Sch}/U)_{\tau}$. Hence the agreement of $f_{\text{big}}!$ on underlying abelian sheaves follows from the discussion in Modules on Sites, Remark \textit{40.2}. The exactness of $f_{\text{big}}!$ follows from Modules on Sites, Lemma \textit{16.2} as the functor $u$ above which commutes with fibre products and equalizers. $\Box$

Next, we prove a technical lemma that will be useful later when comparing sheaves of modules on different sites associated to algebraic stacks.

**Lemma 50.2.** Let $X$ be a scheme. Let $\tau \in \{\text{Zariski, étale, smooth, syntomic, fppf}\}$. Let $\mathcal{C}_1 \subset \mathcal{C}_2 \subset (\mathbf{Sch}/X)_{\tau}$ be full subcategories with the following properties:

1. For an object $U/X$ of $\mathcal{C}_1$,
   a) if $\{U_i \to U\}$ is a covering of $(\mathbf{Sch}/X)_{\tau}$, then $U_i/X$ is an object of $\mathcal{C}_1$,
   b) $U \times \mathbf{A}^1/X$ is an object of $\mathcal{C}_1$.

2. $X/X$ is an object of $\mathcal{C}_1$.

We endow $\mathcal{C}_1$ with the structure of a site whose coverings are exactly those coverings $\{U_i \to U\}$ of $(\mathbf{Sch}/X)_{\tau}$ with $U \in \text{Ob}(\mathcal{C}_1)$. Then

a) The functor $\mathcal{C}_1 \to \mathcal{C}_2$ is fully faithful, continuous, and cocontinuous.

Denote $g : \text{Sh}(\mathcal{C}_1) \to \text{Sh}(\mathcal{C}_2)$ the corresponding morphism of topoi. Denote $\mathcal{O}_t$ the restriction of $\mathcal{O}$ to $\mathcal{C}_1$. Denote $g_t$ the functor of Modules on Sites, Definition \textit{16.1}.

b) The canonical map $g_t\mathcal{O}_1 \to \mathcal{O}_2$ is an isomorphism.
**Proof.** Assertion (a) is immediate from the definitions. In this proof all schemes are schemes over $X$ and all morphisms of schemes are morphisms of schemes over $X$. Note that $g^{-1}$ is given by restriction, so that for an object $U$ of $C_1$ we have $\mathcal{O}_1(U) = \mathcal{O}_2(U) = \mathcal{O}(U)$. Recall that $g_!\mathcal{O}_1$ is the sheaf associated to the presheaf $g_!\mathcal{O}_1$ which associates to $V$ in $C_2$ the group

$$\text{colim}_{V \to U} \mathcal{O}(U)$$

where $U$ runs over the objects of $C_1$ and the colimit is taken in the category of abelian groups. Below we will use frequently that if $V \to U \to U'$

are morphisms with $U, U' \in \text{Ob}(C_1)$ and if $f' \in \mathcal{O}(U')$ restricts to $f \in \mathcal{O}(U)$, then $(V \to U, f)$ and $(V \to U', f')$ define the same element of the colimit. Also, $g_!\mathcal{O}_1 \to \mathcal{O}_2$ maps the element $(V \to U, f)$ simply to the pullback of $f$ to $V$.

**Surjectivity.** Let $V$ be a scheme and let $h \in \mathcal{O}(V)$. Then we obtain a morphism $V \to X \times \mathbb{A}^1$ induced by $h$ and the structure morphism $V \to X$. Writing $\mathbb{A}^1 = \text{Spec}(\mathbb{Z}[x])$ we see the element $x \in \mathcal{O}(X \times \mathbb{A}^1)$ pulls back to $h$. Since $X \times \mathbb{A}^1$ is an object of $C_1$ by assumptions (1)(b) and (2) we obtain the desired surjectivity.

**Injectivity.** Let $V$ be a scheme. Let $s = \sum_{i=1, \ldots, n}(V \to U_i, f_i)$ be an element of the colimit displayed above. For any $i$ we can use the morphism $f_i : U_i \to X \times \mathbb{A}^1$ to see that $(V \to U_i, f_i)$ defines the same element of the colimit as $(f_i : V \to X \times \mathbb{A}^1, x)$. Then we can consider

$$f_1 \times \ldots \times f_n : V \to X \times \mathbb{A}^n$$

and we see that $s$ is equivalent in the colimit to

$$\sum_{i=1, \ldots, n}(f_1 \times \ldots \times f_n : V \to X \times \mathbb{A}^n, x_i) = (f_1 \times \ldots \times f_n : V \to X \times \mathbb{A}^n, x_1 + \ldots + x_n)$$

Now, if $x_1 + \ldots + x_n$ restricts to zero on $V$, then we see that $f_1 \times \ldots \times f_n$ factors through $X \times \mathbb{A}^{n-1} = V(x_1 + \ldots + x_n)$. Hence we see that $s$ is equivalent to zero in the colimit. \hfill \Box

## 51. Étale cohomology

In the following sections we prove some basic results on étale cohomology. Here is an example of something we know for cohomology of topological spaces which also holds for étale cohomology.

**Lemma 51.1** (Mayer-Vietoris for étale cohomology). Let $X$ be a scheme. Suppose that $X = U \cup V$ is a union of two opens. For any abelian sheaf $\mathcal{F}$ on $X_{\text{étale}}$ there exists a long exact cohomology sequence

$$0 \to H^0_{\text{étale}}(X, \mathcal{F}) \to H^0_{\text{étale}}(U, \mathcal{F}) \oplus H^0_{\text{étale}}(V, \mathcal{F}) \to H^0_{\text{étale}}(U \cap V, \mathcal{F}) \to \ldots$$

This long exact sequence is functorial in $\mathcal{F}$.

**Proof.** Observe that if $\mathcal{I}$ is an injective abelian sheaf, then

$$0 \to \mathcal{I}(X) \to \mathcal{I}(U) \oplus \mathcal{I}(V) \to \mathcal{I}(U \cap V) \to 0$$

is exact. This is true in the first and middle spots as $\mathcal{I}$ is a sheaf. It is true on the right, because $\mathcal{I}(U) \to \mathcal{I}(U \cap V)$ is surjective by Cohomology on Sites, Lemma 12.6. Another way to prove it would be to show that the cokernel of the map
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\( \mathcal{I}(U) \oplus \mathcal{I}(V) \rightarrow \mathcal{I}(U \cap V) \) is the first Čech cohomology group of \( \mathcal{I} \) with respect to the covering \( X = U \cup V \) which vanishes by Lemmas \([18,7]\) and \([19,1]\). Thus, if \( \mathcal{F} \rightarrow \mathcal{I}^* \) is an injective resolution, then

\[
0 \rightarrow \mathcal{I}^*(X) \rightarrow \mathcal{I}^*(U) \oplus \mathcal{I}^*(V) \rightarrow \mathcal{I}^*(U \cap V) \rightarrow 0
\]

is a short exact sequence of complexes and the associated long exact cohomology sequence is the sequence of the statement of the lemma. □

52. Colimits

03Q4 We recall that if \((\mathcal{F}_i, \phi_{ij})\) is a diagram of sheaves on a site \( \mathcal{C} \) its colimit (in the category of sheaves) is the sheafification of the presheaf \( U \mapsto \text{colim}_i \mathcal{F}_i(U) \). See Sites, Lemma \([10,13]\). If the system is directed, \( U \) is a quasi-compact object of \( \mathcal{C} \) which has a cofinal system of coverings by quasi-compact objects, then \( \mathcal{F}(U) = \text{colim} \mathcal{F}_i(U) \), see Sites, Lemma \([11,2]\). See Cohomology on Sites, Lemma \([16,1]\) for a result dealing with higher cohomology groups of colimits of abelian sheaves.

We first state and prove a very general result on colimits and cohomology and then we explain what it means in some special cases.

09YQ \textbf{Theorem} 52.1. Let \( X = \text{lim}_{i \in I} X_i \) be a limit of a directed system of schemes with affine transition morphisms \( f_{ij} : X_i \rightarrow X_j \). We assume that \( X_i \) is quasi-compact and quasi-separated for all \( i \in I \). Assume given

1. an abelian sheaf \( \mathcal{F}_i \) on \( (X_i)_{\text{étale}} \) for all \( i \in I \),
2. for \( i' \geq i \) a map \( \varphi_{i'i} : f_{i'i}^{-1} \mathcal{F}_i \rightarrow \mathcal{F}_{i'} \) of abelian sheaves on \( (X_i')_{\text{étale}} \) such that \( \varphi_{ij} = \varphi_{i'i} \circ f_{ij}^{-1} \varphi_{i'i} \) whenever \( i'' \geq i' \geq i \). Denote \( f_i : X \rightarrow X_i \) the projection and set \( \mathcal{F} = \text{colim} f_i^{-1} \mathcal{F}_i \). Then

\[
\text{colim}_{i \in I} H^p_{\text{étale}}(X, \mathcal{F}_i) = H^p_{\text{étale}}(X, \mathcal{F}).
\]

for all \( p \geq 0 \).

\textbf{Proof.} Let us use the affine étale sites of \( X \) and \( X_i \) as introduced in Lemma \([21,2]\). We claim that

\[
X_{\text{affine, étale}} = \text{colim}(X_i)_{\text{affine, étale}}
\]
as sites (see Sites, Lemma \([11,0]\)). If we prove this, then the theorem follows from Cohomology on Sites, Lemma \([16,2]\). The category of schemes of finite presentation over \( X \) is the colimit of the categories of schemes of finite presentation over \( X_i \), see Limits, Lemma \([9,1]\). The same holds for the subcategories of affine objects étale over \( X \) by Limits, Lemmas \([3,10]\) and \([7,8]\). Finally, if \( \{U^j \rightarrow U\} \) is a covering of \( X_{\text{affine, étale}} \) and if \( U^j_i \rightarrow U_i \) is morphism of affine schemes étale over \( X_i \) whose base change to \( X \) is \( U^j \rightarrow U \), then we see that the base change of \( \{U^j_i \rightarrow U_i\} \) to some \( X_{i'} \) is a covering for \( i' \) large enough, see Limits, Lemma \([7,11]\). □

The following two results are special cases of the theorem above.

03Q5 \textbf{Lemma} 52.2. Let \( X \) be a quasi-compact and quasi-separated scheme. Let \( (\mathcal{F}_i, \phi_{ij}) \) be a system of abelian sheaves on \( X_{\text{étale}} \) over the partially ordered set \( I \). If \( I \) is directed then

\[
\text{colim}_{i \in I} H^p_{\text{étale}}(X, \mathcal{F}_i) = H^p_{\text{étale}}(X, \text{colim}_{i \in I} \mathcal{F}_i).
\]

\textbf{Proof.} This is a special case of Theorem 52.1. We also sketch a direct proof. We prove it for all \( X \) at the same time, by induction on \( p \).
(1) For any quasi-compact and quasi-separated scheme $X$ and any étale covering $\mathcal{U}$ of $X$, show that there exists a refinement $\mathcal{V} = \{V_j \to X\}_{j \in J}$ with $J$ finite and each $V_j$ quasi-compact and quasi-separated such that all $V_j \times_X \cdots \times_X V_{j'}$ are also quasi-compact and quasi-separated.

(2) Using the previous step and the definition of colimits in the category of sheaves, show that the theorem holds for $p = 0$ and all $X$.

(3) Using the locality of cohomology (Lemma 22.3), the Čech-to-cohomology spectral sequence (Theorem 19.2) and the fact that the induction hypothesis applies to all $V_{j} \times_X \cdots \times_X V_{j'}$ in the above situation, prove the induction step $p \to p + 1$.

\[ \square \]

**Lemma 52.3.** Let $A$ be a ring, $(I, \leq)$ a directed poset and $(B_i, \varphi_{ij})$ a system of $A$-algebras. Set $B = \text{colim}_{i \in I} B_i$. Let $X \to \text{Spec}(A)$ be a quasi-compact and quasi-separated morphism of schemes. Let $\mathcal{F}$ be an abelian sheaf on $X_{\text{étale}}$. Denote $Y_i = X \times_{\text{Spec}(A)} \text{Spec}(B_i)$, $Y = X \times_{\text{Spec}(A)} \text{Spec}(B)$, $\mathcal{G}_i = (Y_i \to X)^{-1} \mathcal{F}$ and $\mathcal{G} = (Y \to X)^{-1} \mathcal{F}$. Then

\[ H^p_{\text{étale}}(Y, \mathcal{G}) = \text{colim}_{i \in I} H^p_{\text{étale}}(X_i, \mathcal{G}_i). \]

**Proof.** This is a special case of Theorem 52.1. We also outline a direct proof as follows.

1. Given $V \to Y$ étale with $V$ quasi-compact and quasi-separated, there exist $i \in I$ and $V_i \to Y_i$ such that $V = V_i \times_Y Y$. If all the schemes considered were affine, this would correspond to the following algebra statement: if $B = \text{colim}_{i \in I} B_i$ and $B \to C$ is étale, then there exist $i \in I$ and $B_i \to C_i$ étale such that $C \cong B \otimes_{B_i} C_i$. This is proved in Algebra, Lemma 141.3.

2. In the situation of (1) show that $\mathcal{G}(V) = \text{colim}_{i \geq i'} \mathcal{G}_{i'}(V_{i'})$ where $V_{i'}$ is the base change of $V_i$ to $Y_{i'}$.

3. By (1), we see that for every étale covering $\mathcal{V} = \{V_j \to Y\}_{j \in J}$ with $J$ finite and the $V_j$s quasi-compact and quasi-separated, there exists $i \in I$ and an étale covering $\mathcal{V}_i = \{V_{j_i} \to Y_i\}_{j_i \in J}$ such that $V \cong V_i \times_Y Y$.

4. Show that (2) and (3) imply

\[ H^*(V, \mathcal{G}) = \text{colim}_{i \in I} H^*(V_i, \mathcal{G}_i). \]

5. Cleverly use the Čech-to-cohomology spectral sequence (Theorem 19.2).

\[ \square \]

**Lemma 52.4.** Let $f : X \to Y$ be a morphism of schemes and $\mathcal{F} \in \text{Ab}(X_{\text{étale}})$. Then $R^p f_* \mathcal{F}$ is the sheaf associated to the presheaf

\[ (V \to Y) \mapsto H^p_{\text{étale}}(X \times_Y V, \mathcal{F}|_{X \times_Y V}). \]

**Proof.** This lemma is valid for topological spaces, and the proof in this case is the same. See Cohomology on Sites, Lemma 8.4 for details.

\[ \square \]

**Lemma 52.5.** Let $S$ be a scheme. Let $X = \text{lim}_{i \in I} X_i$ be a limit of a directed system of schemes over $S$ with affine transition morphisms $f_{i'i} : X_{i'} \to X_i$. We assume the structure morphism $g_i : X_i \to S$ is quasi-compact and quasi-separated for all $i \in I$ and we set $g : X \to S$. Assume given

1. an abelian sheaf $\mathcal{F}_i$ on $(X_i)_{\text{étale}}$ for all $i \in I$,
2. for $i' \geq i$ a map $\varphi_{i'i} : f_{i'i}^{-1} \mathcal{F}_i \to \mathcal{F}_{i'}$ of abelian sheaves on $(X_{i'})_{\text{étale}}$

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such that $\varphi_{v^i} = \varphi_{v^{i'}} \circ f_{v^i}^{-1} \varphi_{v^i}$ whenever $v'' \geq i' \geq i$. Denote $f_i : X \to X_i$ the projection and set $\mathcal{F} = \colim f_i^{-1} \mathcal{F}_i$. Then
\[
\colim_{i \in I} R^p g_i_* \mathcal{F}_i = R^p g_* \mathcal{F}
\]
for all $p \geq 0$.

**Proof.** Recall (Lemma 52.4) that $R^p g_i_* \mathcal{F}_i$ is the sheaf associated to the presheaf $U \mapsto H^p_{\text{étale}}(U \times_S X_i, \mathcal{F}_i)$ and similarly for $R^p g_* \mathcal{F}$. Moreover, the colimit of a system of sheaves is the sheafification of the colimit on the level of presheaves. Note that every object of $S_{\text{étale}}$ has a covering by quasi-compact and quasi-separated objects (e.g., affine schemes). Moreover, if $U$ is a quasi-compact and quasi-separated object, then we have
\[
\colim H^p_{\text{étale}}(U \times_S X_i, \mathcal{F}_i) = H^p_{\text{étale}}(U \times_S X, \mathcal{F})
\]
by Theorem 52.1. Thus the lemma follows. $\square$

### 53. Stalks of higher direct images

**Theorem 53.1.** Let $f : X \to S$ be a quasi-compact and quasi-separated morphism of schemes, $\mathcal{F}$ an abelian sheaf on $X_{\text{étale}}$, and $\overline{s}$ a geometric point of $S$. Then
\[
(R^p f_* \mathcal{F})_{\overline{s}} = H^p_{\text{étale}}(X \times_S \text{Spec}(\mathcal{O}_{S, \overline{s}}), \mathcal{F})
\]
where $p : X \times_S \text{Spec}(\mathcal{O}_{S, \overline{s}}) \to X$ is the projection.

**Proof.** Let $\mathcal{I}$ be the category of étale neighborhoods of $\overline{s}$ on $S$. By Lemma 52.4 we have
\[
(R^p f_* \mathcal{F})_{\overline{s}} = \colim_{(V, \overline{v}) \in \mathcal{I}_{\text{opp}}} H^p(X \times_S V, \mathcal{F}|_{X \times_S V}).
\]
We may replace $\mathcal{I}$ by the initial subcategory consisting of affine étale neighbourhoods of $\overline{s}$. Observe that
\[
\text{Spec}(\mathcal{O}_{S, \overline{s}}) = \lim_{(V, \overline{v}) \in \mathcal{I}} V
\]
by Lemma 33.1 and Limits, Lemma 2.1. Since fibre products commute with limits we also obtain
\[
X \times_S \text{Spec}(\mathcal{O}_{S, \overline{s}}) = \lim_{(V, \overline{v}) \in \mathcal{I}} X \times_S V
\]
We conclude by Lemma 52.3. $\square$

### 54. The Leray spectral sequence

**Lemma 54.1.** Let $f : X \to Y$ be a morphism and $\mathcal{I}$ an injective object of $\text{Ab}(X_{\text{étale}})$. Let $V \in \text{Ob}(Y_{\text{étale}})$. Then
\begin{enumerate}
  
  
  \item for any covering $\mathcal{V} = \{V_j \to V\}_{j \in J}$ we have $H^p(\mathcal{V}, f_* \mathcal{I}) = 0$ for all $p \geq 0$,
  
  \item $f_* \mathcal{I}$ is acyclic for the functor $f_!(V, -)$, and
  
  \item if $g : Y \to Z$, then $f_* \mathcal{I}$ is acyclic for $g_*$.\n\end{enumerate}
Proof. Observe that \( \check{C}^\bullet(V, f_*I) = \check{C}^\bullet(V \times_Y X, I) \) which has vanishing higher cohomology groups by Lemma 18.7. This proves (1). The second statement follows as a sheaf which has vanishing higher Čech cohomology groups for any covering has vanishing higher cohomology groups. This a wonderful exercise in using the Čech-to-cohomology spectral sequence, but see Cohomology on Sites, Lemma 11.9 for details and a more precise and general statement. Part (3) is a consequence of (2) and the description of \( R^p g_* \) in Lemma 52.4. □

Using the formalism of Grothendieck spectral sequences, this gives the following.

**Proposition 54.2** (Leray spectral sequence). Let \( f : X \to Y \) be a morphism of schemes and \( \mathcal{F} \) an étale sheaf on \( X \). Then there is a spectral sequence

\[
E_2^{p,q} = H^p_{\text{étale}}(Y, R^q f_* \mathcal{F}) \Rightarrow H^{p+q}_{\text{étale}}(X, \mathcal{F}).
\]

**Proof.** See Lemma 54.1 and see Derived Categories, Section 22. □

55. Vanishing of finite higher direct images

The next goal is to prove that the higher direct images of a finite morphism of schemes vanish.

**Lemma 55.1.** Let \( R \) be a strictly henselian local ring. Set \( S = \text{Spec}(R) \) and let \( \overline{\pi} \) be its closed point. Then the global sections functor \( \Gamma(S, -) : \text{Ab}(S_{\text{étale}}) \to \text{Ab} \) is exact. In fact we have \( \Gamma(S, \mathcal{F}) = \mathcal{F}_{\overline{\pi}} \) for any sheaf of sets \( \mathcal{F} \). In particular

\[
\forall p \geq 1, \quad H^p_{\text{étale}}(S, \mathcal{F}) = 0
\]

for all \( \mathcal{F} \in \text{Ab}(S_{\text{étale}}) \).

**Proof.** If we show that \( \Gamma(S, \mathcal{F}) = \mathcal{F}_{\overline{\pi}} \) the \( \Gamma(S, -) \) is exact as the stalk functor is exact. Let \( (U, \overline{u}) \) be an étale neighbourhood of \( \overline{\pi} \). Pick an affine open neighborhood \( \text{Spec}(A) \) of \( \overline{\pi} \) in \( U \). Then \( R \to A \) is étale and \( \kappa(\overline{\pi}) = \kappa(\overline{u}) \). By Theorem 32.4 we see that \( A \cong R \times A' \) as an \( R \)-algebra compatible with maps to \( \kappa(\overline{\pi}) = \kappa(\overline{u}) \). Hence we get a section

\[
\text{Spec}(A) \longrightarrow U \quad \downarrow \quad S
\]

It follows that in the system of étale neighbourhoods of \( \overline{\pi} \) the identity map \( (S, \overline{\pi}) \to (S, \overline{\pi}) \) is cofinal. Hence \( \Gamma(S, \mathcal{F}) = \mathcal{F}_{\overline{\pi}} \). The final statement of the lemma follows as the higher derived functors of an exact functor are zero, see Derived Categories, Lemma 17.9. □

**Proposition 55.2.** Let \( f : X \to Y \) be a finite morphism of schemes.

1. For any geometric point \( \overline{y} : \text{Spec}(k) \to Y \) we have

\[
(f_* \mathcal{F})_{\overline{y}} = \prod_{\overline{\tau} : \text{Spec}(k) \to X, \ f(\overline{\tau}) = \overline{y}} \mathcal{F}_{\overline{\tau}}.
\]

for \( \mathcal{F} \) in \( \text{Sh}(X_{\text{étale}}) \) and

\[
(f_* \mathcal{F})_{\overline{y}} = \bigoplus_{\overline{\tau} : \text{Spec}(k) \to X, \ f(\overline{\tau}) = \overline{y}} \mathcal{F}_{\overline{\tau}}.
\]

for \( \mathcal{F} \) in \( \text{Ab}(X_{\text{étale}}) \).

2. For any \( q \geq 1 \) we have \( R^q f_* \mathcal{F} = 0 \).
Let $X_{\eta}^h$ denote the fiber product $X \times_Y \text{Spec}(\mathcal{O}^h_{Y, \eta})$. By Theorem \ref{derived categories}, the stalk of $R^q f_* \mathcal{F}$ at $\eta$ is computed by $H^q_{\text{étale}}(X_{\eta}^h, \mathcal{F})$. Since $f$ is finite, $X_{\eta}^h$ is finite over $\text{Spec}(\mathcal{O}^h_{Y, \eta})$, thus $X_{\eta}^h = \text{Spec}(A)$ for some ring $A$ finite over $\mathcal{O}^h_{Y, \eta}$. Since the latter is strictly henselian, Lemma \ref{Henselian rings} implies that $A$ is a finite product of henselian local rings $A = A_1 \times \ldots \times A_r$. Since the residue field of $\mathcal{O}^h_{Y, \eta}$ is separably closed the same is true for each $A_i$. Hence $A_i$ is strictly henselian. This implies that $X_{\eta}^h = \prod_{i=1}^r \text{Spec}(A_i)$. The vanishing of Lemma \ref{vanishing} implies that $(R^q f_* \mathcal{F})_{\eta} = 0$ for $q > 0$ which implies (2) by Theorem \ref{vanishing theorem}. Part (1) follows from the corresponding statement of Lemma \ref{vanishing}.

\begin{lemma}
Consider a cartesian square

$$
\begin{array}{ccc}
X' & \xrightarrow{g} & X \\
\downarrow{f'} & & \downarrow{f} \\
Y' & \xrightarrow{g} & Y
\end{array}
$$

of schemes with $f$ a finite morphism. For any sheaf of sets $\mathcal{F}$ on $X_{\text{ étale}}$, we have $f'_* (g')^{-1} \mathcal{F} = g^{-1} f_* \mathcal{F}$.
\end{lemma}

\begin{proof}
In great generality there is a pullback map $g^{-1} f_* \mathcal{F} \to f'_* (g')^{-1} \mathcal{F}$, see Sites, Section \ref{pullbacks}. To check this map is an isomorphism it suffices to check on stalks (Theorem \ref{vanishing theorem}). This is clear from the description of stalks in Proposition \ref{stalks} and Lemma \ref{stalks}.
\end{proof}

The following lemma is a case of cohomological descent dealing with étale sheaves and finite surjective morphisms. We will significantly generalize this result once we prove the proper base change theorem.

\begin{lemma}
Let $f : X \to Y$ be a surjective finite morphism of schemes. Set $f_n : X_n \to Y$ equal to the $(n+1)$-fold fibre product of $X$ over $Y$. For $\mathcal{F} \in \text{Ab}(Y_{\text{ étale}})$ set $\mathcal{F}_n = f_n^* f_{n+1}^{-1} \mathcal{F}$. There is an exact sequence

$$0 \to \mathcal{F} \to \mathcal{F}_0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \ldots$$

on $X_{\text{ étale}}$. Moreover, there is a spectral sequence

$$E_1^{p, q} = H^3_{\text{ étale}}(X_p, f_p^{-1} \mathcal{F})$$

converging to $H^{p+q}(Y_{\text{ étale}}, \mathcal{F})$. This spectral sequence is functorial in $\mathcal{F}$.
\end{lemma}

\begin{proof}
If we prove the first statement of the lemma, then we obtain a spectral sequence with $E_1^{0, q} = H^q_{\text{ étale}}(Y, \mathcal{F})$ converging to $H^{p+q}(Y_{\text{ étale}}, \mathcal{F})$, see Derived Categories, Lemma \ref{spectral sequence}. On the other hand, since $R^i f_{p,*} f_p^{-1} \mathcal{F} = 0$ for $i > 0$ (Proposition \ref{vanishing theorem}) we get

$$H^q_{\text{ étale}}(X_p, f_p^{-1} \mathcal{F}) = H^q_{\text{ étale}}(Y, f_{p,*} f_p^{-1} \mathcal{F}) = H^q_{\text{ étale}}(Y, \mathcal{F}_p)$$

by Proposition \ref{vanishing theorem} and we get the spectral sequence of the lemma.

To prove the first statement of the lemma, observe that $X_n$ forms a simplicial scheme over $Y$, see Simplicial, Example \ref{simplicial schemes}. Observe moreover, that for each of the projections $d_j : X_{n+1} \to X_n$ there is a map $d_j^{-1} f_n^{-1} \mathcal{F} \to f_{n+1}^{-1} \mathcal{F}$. These maps induce maps

$$\delta_j : \mathcal{F}_n \to \mathcal{F}_{n+1}.$$
for $j = 0, \ldots, n + 1$. We use the alternating sum of these maps to define the differentials $\mathcal{F}_n \to \mathcal{F}_{n+1}$. Similarly, there is a canonical augmentation $\mathcal{F} \to \mathcal{F}_0$, namely this is just the canonical map $\mathcal{F} \to f_* f^{-1} \mathcal{F}$. To check that this sequence of sheaves is an exact complex it suffices to check on stalks at geometric points (Theorem 29.10). Thus we let $\mathfrak{g}: \text{Spec}(k) \to Y$ be a geometric point. Let $E = \{ \pi: \text{Spec}(k) \to X \mid f(\pi) = \mathfrak{g} \}$. Then $E$ is a finite nonempty set and we see that

$$(\mathcal{F}_n)_{\mathfrak{g}} = \bigoplus_{e \in E^{n+1}} \mathcal{F}_e$$

by Proposition 55.2 and Lemma 36.2. Thus we have to see that given an abelian group $M$ the sequence

$$0 \to M \to \bigoplus_{e \in E} M \to \bigoplus_{e \in E^2} M \to \ldots$$

is exact. Here the first map is the diagonal map and the map $\bigoplus_{e \in E^{n+1}} M \to \bigoplus_{e \in E^{n+2}} M$ is the alternating sum of the maps induced by the $(n + 2)$ projections $E^{n+2} \to E^{n+1}$. This can be shown directly or deduced by applying Simplicial, Lemma 26.9 to the map $E \to \{\ast\}$.

\[ \square \]

Remark 55.5. In the situation of Lemma 55.4 if $\mathcal{G}$ is a sheaf of sets on $Y_{\text{étale}}$, then we have

$$\Gamma(Y, \mathcal{G}) = \text{Equalizer}(\Gamma(X_0, f_0^{-1} \mathcal{G}) \longrightarrow \Gamma(X_1, f_1^{-1} \mathcal{G}))$$

This is proved in exactly the same way, by showing that the sheaf $\mathcal{G}$ is the equalizer of the two maps $f_0, f_0^{-1} \mathcal{G} \to f_1, f_1^{-1} \mathcal{G}$.

Here is a fun generalization of Lemma 55.1.

Lemma 55.7. Let $S$ be an affine scheme such that (1) all points are closed, and (2) all residue fields are separably algebraically closed. Then for any abelian sheaf $\mathcal{F}$ on $S_{\text{étale}}$ we have $H^i(S_{\text{étale}}, \mathcal{F}) = 0$ for $i > 0$.

Proof. Condition (1) implies that the underlying topological space of $S$ is profinite, see Algebra, Lemma 25.5. Thus the higher cohomology groups of an abelian sheaf on the topological space $S$ (i.e., Zariski cohomology) is trivial, see Cohomology, Lemma 23.3. The local rings are strictly henselian by Algebra, Lemma 148.10. Thus étale cohomology of $S$ is computed by Zariski cohomology by Lemma 55.6 and the proof is done. ✔
56. Galois action on stalks

In this section we define an action of the absolute Galois group of a residue field of a point \( s \) of \( S \) on the stalk functor at any geometric point lying over \( s \).

Galois action on stalks. Let \( S \) be a scheme. Let \( \overline{s} \) be a geometric point of \( S \). Let \( \sigma \in \text{Aut}(\kappa(\overline{s})/\kappa(s)) \). Define an action of \( \sigma \) on the stalk \( F_\overline{s} \) of a sheaf \( F \) as follows

\[
(F_\overline{s} \longrightarrow F_\overline{s}) \quad (U, \overline{\pi}, t) \mapsto (U, \overline{\pi} \circ \text{Spec}(\sigma), t).
\]

where we use the description of elements of the stalk in terms of triples as in the discussion following Definition 29.6. This is a left action, since if \( \sigma_1 \in \text{Aut}(\kappa(\overline{s})/\kappa(s)) \) then

\[
\sigma_1 \cdot (\sigma_2 \cdot (U, \overline{\pi}, t)) = (\sigma_1 \circ \sigma_2 \circ \text{Spec}(\sigma_2), t) = (\sigma_1 \circ \text{Spec}(\sigma_2), t) = (\sigma_1 \circ \sigma_2, (U, \overline{\pi}, t))
\]

It is clear that this action is functorial in the sheaf \( F \). We note that we could have defined this action by referring directly to Remark 29.8.

Definition 56.1. Let \( S \) be a scheme. Let \( \overline{s} \) be a geometric point lying over the point \( s \) of \( S \). Let \( \kappa(s) \subset \kappa(s)_{\text{sep}} \subset \kappa(\overline{s}) \) denote the separable algebraic closure of \( \kappa(s) \) in the algebraically closed field \( \kappa(\overline{s}) \).

1. In this situation the absolute Galois group of \( \kappa(s) \) is \( \text{Gal}(\kappa(s)_{\text{sep}}/\kappa(s)) \). It is sometimes denoted \( \text{Gal}_{\kappa(s)} \).
2. The geometric point \( \overline{s} \) is called algebraic if \( \kappa(s) \subset \kappa(\overline{s}) \) is an algebraic closure of \( \kappa(s) \).

Example 56.2. The geometric point \( \text{Spec}(\mathbb{C}) \to \text{Spec}(\mathbb{Q}) \) is not algebraic.

Let \( \kappa(s) \subset \kappa(s)_{\text{sep}} \subset \kappa(\overline{s}) \) be as in the definition. Note that as \( \kappa(\overline{s}) \) is algebraically closed the map

\[
\text{Aut}(\kappa(\overline{s})/\kappa(s)) \longrightarrow \text{Gal}(\kappa(s)_{\text{sep}}/\kappa(s)) = \text{Gal}_{\kappa(s)}
\]

is surjective. Suppose \( (U, \overline{\pi}) \) is an étale neighbourhood of \( \overline{s} \), and say \( \overline{\pi} \) lies over the point \( u \) of \( U \). Since \( U \to S \) is étale, the residue field extension \( \kappa(s) \subset \kappa(u) \) is finite separable. This implies the following:

1. If \( \sigma \in \text{Aut}(\kappa(\overline{s})/\kappa(s)_{\text{sep}}) \) then \( \sigma \) acts trivially on \( F_\overline{s} \).
2. More precisely, the action of \( \text{Aut}(\kappa(\overline{s})/\kappa(s)) \) determines and is determined by an action of the absolute Galois group \( \text{Gal}_{\kappa(s)} \) on \( F_\overline{s} \).
3. Given \( (U, \overline{\pi}, t) \) representing an element \( \xi \) of \( F_\overline{s} \), any element of \( \text{Gal}(\kappa(s)_{\text{sep}}/K) \) acts trivially, where \( \kappa(s) \subset K \subset \kappa(s)_{\text{sep}} \) is the image of \( \overline{\pi}^t : \kappa(u) \to \kappa(\overline{s}) \).

Altogether we see that \( F_\overline{s} \) becomes a \( \text{Gal}_{\kappa(s)} \)-set (see Fundamental Groups, Definition 2.1). Hence we may think of the stalk functor as a functor

\[
\text{Sh}(S_{\text{étale}}) \longrightarrow \text{Gal}_{\kappa(s)} \cdot \text{Sets}, \quad F \mapsto F_\overline{s}
\]

and from now on we usually do think about the stalk functor in this way.
**Theorem 56.3.** Let $S = \text{Spec}(K)$ with $K$ a field. Let $\overline{s}$ be a geometric point of $S$. Let $G = \text{Gal}_{s}(K)$ denote the absolute Galois group. Taking stalks induces an equivalence of categories

$$\text{Sh}(S_{\text{étale}}) \to G\text{-Sets}, \quad F \mapsto F_{\overline{s}}.$$ 

**Proof.** Let us construct the inverse to this functor. In Fundamental Groups, Lemma 2.2 we have seen that given a $G$-set $M$ there exists an étale morphism $X \to \text{Spec}(K)$ such that $\text{Mor}_{K}(\text{Spec}(K^{\text{sep}}), X)$ is isomorphic to $M$ as a $G$-set. Consider the sheaf $F$ on $\text{Spec}(K)_{\text{étale}}$ defined by the rule $U \mapsto \text{Mor}_{K}(U, X)$. This is a sheaf as the étale topology is subcanonical. Then we see that $F_{\overline{s}} = \text{Mor}_{K}(\text{Spec}(K^{\text{sep}}), X) = M$ as $G$-sets (details omitted). This gives the inverse of the functor and we win. 

**Remark 56.4.** Another way to state the conclusion of Theorem 56.3 and Fundamental Groups, Lemma 2.2 is to say that every sheaf on $\text{Spec}(K)_{\text{étale}}$ is representable by a scheme $X_{\text{étale}}$ over $\text{Spec}(K)$. This does not mean that every sheaf is representable in the sense of Sites, Definition 13.3. The reason is that in our construction of $\text{Spec}(K)_{\text{étale}}$ we chose a sufficiently large set of schemes étale over $\text{Spec}(K)$, whereas sheaves on $\text{Spec}(K)_{\text{étale}}$ form a proper class.

**Lemma 56.5.** Assumptions and notations as in Theorem 56.3. There is a functorial bijection 

$$\Gamma(S, F) = (F_{\overline{s}})^{G}.$$ 

**Proof.** We can prove this using formal arguments and the result of Theorem 56.3 as follows. Given a sheaf $F$ corresponding to the $G$-set $M = F_{\overline{s}}$ we have

$$\Gamma(S, F) = \text{Mor}_{\text{Sh}(S_{\text{étale}})}(h_{\text{Spec}(K)}, F) = \text{Mor}_{G\text{-Sets}}(\{\ast\}, M) = M^{G}.$$ 

Here the first identification is explained in Sites, Sections 2 and 13, the second results from Theorem 56.3 and the third is clear. We will also give a direct proof. 

Suppose that $t \in \Gamma(S, F)$ is a global section. Then the triple $(S, \overline{s}, t)$ defines an element of $F_{\overline{s}}$ which is clearly invariant under the action of $G$. Conversely, suppose that $(U, \overline{u}, t)$ defines an element of $F_{\overline{s}}$ which is invariant. Then we may shrink $U$ and assume $U = \text{Spec}(L)$ for some finite separable field extension of $K$, see Proposition 26.2. In this case the map $F(U) \to F_{\overline{s}}$ is injective, because for any morphism of étale neighbourhoods $(U', \overline{u}') \to (U, \overline{u})$ the restriction map $F(U) \to F(U')$ is injective since $U' \to U$ is a covering of $S_{\text{étale}}$. After enlarging $L$ a bit we may assume $K \subset L$ is a finite Galois extension. At this point we use that 

$$\text{Spec}(L) \times_{\text{Spec}(K)} \text{Spec}(L) = \coprod_{\sigma \in \text{Gal}(L/K)} \text{Spec}(L)$$ 

where the maps $\text{Spec}(L) \to \text{Spec}(L \otimes_{K} L)$ come from the ring maps $a \otimes b \mapsto a\sigma(b)$. Hence we see that the condition that $(U, \overline{u}, t)$ is invariant under all of $G$ implies that $t \in F(\text{Spec}(L))$ maps to the same element of $F(\text{Spec}(L) \times_{\text{Spec}(K)} \text{Spec}(L))$ via restriction by either projection (this uses the injectivity mentioned above; details omitted). Hence the sheaf condition of $F$ for the étale covering

---

4For the doubting Thomases out there.
{Spec(L) \to \text{Spec}(K)} kicks in and we conclude that \( t \) comes from a unique section of \( \mathcal{F} \) over Spec(\( K \)). □

Remark 56.6. Let \( S \) be a scheme and let \( \pi : \text{Spec}(k) \to S \) be a geometric point of \( S \). By definition this means that \( k \) is algebraically closed. In particular the absolute Galois group of \( k \) is trivial. Hence by Theorem 56.3 the category of sheaves on Spec(\( k \)\)\(^{\text{\acute{e}tale}} \)) is equivalent to the category of sets. The equivalence is given by taking sections over Spec(\( k \)). This finally provides us with an alternative definition of the stalk functor. Namely, the functor

\[ \text{Sh}(S^{\text{\acute{e}tale}}) \to \text{Sets}, \quad \mathcal{F} \mapsto \mathcal{F}_\pi \]

is isomorphic to the functor

\[ \text{Sh}(S^{\text{\acute{e}tale}}) \to \text{Sh}(\text{Spec}(k)^{\text{\acute{e}tale}}) = \text{Sets}, \quad \mathcal{F} \mapsto \pi^* \mathcal{F} \]

To prove this rigorously one can use Lemma 36.2 part (3) with \( f = \pi \). Moreover, having said this the general case of Lemma 36.2 part (3) follows from functoriality of pullbacks.

57. Group cohomology

Notation. If we write \( H^i(G, M) \) we will mean that \( G \) is a topological group and \( M \) a discrete \( G \)-module with continuous \( G \)-action. This includes the case of an abstract group \( G \), which simply means that \( G \) is viewed as a topological group with the discrete topology. When the module has a nondiscrete topology, we will use the notation \( H^i_{\text{cont}}(G, M) \) to indicate the cohomology theory discussed in [Tat76].

Definition 57.1. Let \( G \) be a topological group. A \( G \)-module, sometime called a discrete \( G \)-module, is an abelian group \( M \) endowed with a left action \( a : G \times M \to M \) by group homomorphisms such that \( a \) is continuous when \( M \) is given the discrete topology and \( G \times M \) the product topology. A morphism of \( G \)-modules \( f : M \to N \) is simply any \( G \)-equivariant homomorphism from \( M \) to \( N \). The category of \( G \)-modules is denoted \( \text{Mod}_G \).

The condition that \( a : G \times M \to M \) is continuous is equivalent with the condition that the stabilizer of any \( x \in M \) is open in \( G \). If \( G \) is an abstract group then this corresponds to the notion of an abelian group endowed with a \( G \)-action provided we endow \( G \) with the discrete topology.

The category \( \text{Mod}_G \) has enough injectives, see Injectives, Lemma 3.1. Consider the left exact functor

\[ \text{Mod}_G \to \text{Ab}, \quad M \mapsto M^G = \{ x \in M \mid g \cdot x = x \ \forall g \in G \} \]

We sometimes denote \( M^G = H^0(G, M) \) and sometimes we write \( M^G = \Gamma_G(M) \).

This functor has a total right derived functor \( R\Gamma_G(M) \) and \( i \)th right derived functor \( R^i\Gamma_G(M) = H^i(G, M) \) for any \( i \geq 0 \).

Definition 57.2. Let \( G \) be a topological group. Let \( M \) be a \( G \)-module.

(1) The right derived functors \( H^i(G, M) \) are called the continuous group cohomology groups of \( M \).

(2) If \( G \) is an abstract group endowed with the discrete topology then the \( H^i(G, M) \) are called the group cohomology groups of \( M \).

(3) If \( G \) is a Galois group, then the groups \( H^i(G, M) \) are called the Galois cohomology groups of \( M \).
If $G$ is the absolute Galois group of a field $K$, then the groups $H^i(G, M)$ are sometimes called the Galois cohomology groups of $K$ with coefficients in $M$. In this case we sometimes write $H^i(K, M)$ instead of $H^i(G, M)$.

We can compute continuous group cohomology by the complex of inhomogeneous cochains. In fact, we can define this when $M$ is an arbitrary topological abelian group endowed with a continuous $G$-action. Namely, we consider the complex

$$C^\bullet_{\text{cont}}(G, M) : M \to \text{Maps}_{\text{cont}}(G, M) \to \text{Maps}_{\text{cont}}(G \times G, M) \to \ldots$$

where the boundary map is defined for $n \geq 1$ by the rule

$$d(f)(g_1, \ldots, g_{n+1}) = g_1(f(g_2, \ldots, g_{n+1})) + \sum_{j=1}^{n} (-1)^j f(g_1, \ldots, g_j g_{j+1}, \ldots, g_{n+1}) + (-1)^{n+1} f(g_1, \ldots, g_n)$$

and for $n = 0$ sends $m \in M$ to the map $g \mapsto g(m) - m$. We define

$$H^i_{\text{cont}}(G, M) = H^i(C^\bullet_{\text{cont}}(G, M))$$

Since the terms of the complex involve continuous maps from $G$ and self products of $G$ into the topological module $M$, it is not clear that this turns a short exact sequence of topological modules into a long exact cohomology sequence. (One difficulty is that the category of topological abelian groups isn’t an abelian category!) However, this is true when the topology on the modules is discrete. In fact, if $M$ is a $G$-module as in Definition 57.1, then there is a canonical isomorphism

$$H^i(G, M) = H^i_{\text{cont}}(G, M)$$

of cohomology groups.

### 58. Cohomology of a point

As a consequence of the discussion in the preceding sections we obtain the equivalence of étale cohomology of the spectrum of a field with Galois cohomology.

**Lemma 58.1.** Let $S = \text{Spec}(K)$ with $K$ a field. Let $\pi$ be a geometric point of $S$. Let $G = \text{Gal}(\kappa(\pi)/K)$ denote the absolute Galois group. The stalk functor induces an equivalence of categories

$$\text{Ab}(S_{\text{étale}}) \longrightarrow \text{Mod}_G, \quad \mathcal{F} \mapsto \mathcal{F}_\pi.$$

**Proof.** In Theorem 56.3 we have seen the equivalence between sheaves of sets and $G$-sets. The current lemma follows formally from this as an abelian sheaf is just a sheaf of sets endowed with a commutative group law, and a $G$-module is just a $G$-set endowed with a commutative group law. □

**Lemma 58.2.** Notation and assumptions as in Lemma 58.1. Let $\mathcal{F}$ be an abelian sheaf on $\text{Spec}(K)_{\text{étale}}$ which corresponds to the $G$-module $M$. Then

1. in $D(\text{Ab})$ we have a canonical isomorphism $R\Gamma(S, \mathcal{F}) = R\Gamma_G(M)$,
2. $H^0_{\text{étale}}(S, \mathcal{F}) = M^G$, and
3. $H^q_{\text{étale}}(S, \mathcal{F}) = H^q(G, M)$.

**Proof.** Combine Lemma 58.1 with Lemma 56.5. □

**Example 58.3.** Sheaves on $\text{Spec}(K)_{\text{étale}}$. Let $G = \text{Gal}(K^{\text{sep}}/K)$ be the absolute Galois group of $K$. 
(1) The constant sheaf $\mathbb{Z}/n\mathbb{Z}$ corresponds to the module $\mathbb{Z}/n\mathbb{Z}$ with trivial $G$-action,
(2) the sheaf $\mathbb{G}_m|_{\text{Spec}(K)}$ étale corresponds to $(K^{\text{sep}})^*$ with its $G$-action,
(3) the sheaf $\mathbb{G}_a|_{\text{Spec}(K^{\text{sep}})}$ corresponds to $(K^{\text{sep}}, +)$ with its $G$-action, and
(4) the sheaf $\mu_n|_{\text{Spec}(K^{\text{sep}})}$ corresponds to $\mu_n(K^{\text{sep}})$ with its $G$-action.

By Remark 23.4 and Theorem 24.1 we have the following identifications for cohomology groups:

$$H^0_{\text{étale}}(S, \mathbb{G}_m) = \Gamma(S, \mathcal{O}_S^*)$$

$$H^1_{\text{étale}}(S, \mathbb{G}_m) = H^1_\text{Zar}(S, \mathcal{O}_S^*) = \text{Pic}(S)$$

$$H^i_{\text{étale}}(S, \mathbb{G}_a) = H^i_\text{Zar}(S, \mathcal{O}_S)$$

Also, for any quasi-coherent sheaf $\mathcal{F}$ on $S_{\text{étale}}$ we have

$$H^i(S_{\text{étale}}, \mathcal{F}) = H^i_\text{Zar}(S, \mathcal{F}),$$

see Theorem 22.4. In particular, this gives the following sequence of equalities

$$0 = \text{Pic}(\text{Spec}(K)) = H^1_{\text{étale}}(\text{Spec}(K)_{\text{étale}}, \mathbb{G}_m) = H^1(G, (K^{\text{sep}})^*)$$

which is none other than Hilbert’s 90 theorem. Similarly, for $i \geq 1$,

$$0 = H^i(\text{Spec}(K), \mathcal{O}) = H^i_{\text{étale}}(\text{Spec}(K)_{\text{étale}}, \mathbb{G}_a) = H^i(G, K^{\text{sep}})$$

where the $K^{\text{sep}}$ indicates $K^{\text{sep}}$ as a Galois module with addition as group law. In this way we may consider the work we have done so far as a complicated way of computing Galois cohomology groups.

59. Cohomology of curves

03R0 The next task at hand is to compute the étale cohomology of a smooth curve over an algebraically closed field with torsion coefficients, and in particular show that it vanishes in degree at least 3. To prove this, we will compute cohomology at the generic point, which amounts to some Galois cohomology.

60. Brauer groups

03R1 Brauer groups of fields are defined using finite central simple algebras. In this section we review the relevant facts about Brauer groups, most of which are discussed in the chapter Brauer Groups, Section 1. For other references, see [Ser62], [Ser97] or [Wei48].

03R2 **Theorem 60.1.** Let $K$ be a field. For a unital, associative (not necessarily commutative) $K$-algebra $A$ the following are equivalent

1. $A$ is finite central simple $K$-algebra,
2. $A$ is a finite dimensional $K$-vector space, $K$ is the center of $A$, and $A$ has no nontrivial two-sided ideal,
3. there exists $d \geq 1$ such that $A \otimes_K K \cong \text{Mat}(d \times d, K)$,
4. there exists $d \geq 1$ such that $A \otimes_K K^{\text{sep}} \cong \text{Mat}(d \times d, K^{\text{sep}})$,
5. there exists $d \geq 1$ and a finite Galois extension $K \subset K'$ such that $A \otimes_K K' \cong \text{Mat}(d \times d, K')$,
6. there exist $n \geq 1$ and a finite central skew field $D$ over $K$ such that $A \cong \text{Mat}(n \times n, D)$.

The integer $d$ is called the degree of $A$. 
Proof. This is a copy of Brauer Groups, Lemma 8.6 \(\square\)

03R4 **Lemma 60.2.** Let \(A\) be a finite central simple algebra over \(K\). Then
\[
A \otimes_K A^{\text{opp}} \longrightarrow \text{End}_K(A)
\]
\[a \otimes a' \longmapsto (x \mapsto axa')\]
is an isomorphism of algebras over \(K\).

**Proof.** See Brauer Groups, Lemma 4.10 \(\square\)

03R3 **Definition 60.3.** Two finite central simple algebras \(A_1\) and \(A_2\) over \(K\) are called **similar**, or **equivalent** if there exist \(m, n \geq 1\) such that \(\text{Mat}(n \times n, A_1) \cong \text{Mat}(m \times m, A_2)\). We write \(A_1 \sim A_2\).

By Brauer Groups, Lemma 5.1 this is an equivalence relation.

03R5 **Definition 60.4.** Let \(K\) be a field. The **Brauer group** of \(K\) is the set \(\text{Br}(K)\) of similarity classes of finite central simple algebras over \(K\), endowed with the group law induced by tensor product (over \(K\)). The class of \(A\) in \(\text{Br}(K)\) is denoted by \([A]\). The neutral element is \([\text{Mat}(d \times d, K)]\) for any \(d \geq 1\).

The previous lemma implies that inverses exist and that \(-[A] = [A^{\text{opp}}]\). The Brauer group of a field is always torsion. In fact, we will see that \([A]\) has order \(\text{deg}(A)\) for any finite central simple algebra \(A\) (see Lemma 61.2). In general the Brauer group is not finitely generated, for example the Brauer group of a non-Archimedean local field is \(\mathbb{Q}/\mathbb{Z}\). The Brauer group of \(\mathbb{C}(x, y)\) is uncountable.

03R6 **Lemma 60.5.** Let \(K\) be a field and let \(K^{\text{sep}}\) be a separable algebraic closure. Then the set of isomorphism classes of central simple algebras of degree \(d\) over \(K\) is in bijection with the non-abelian cohomology \(H^1(\text{Gal}(K^{\text{sep}}/K), \text{PGL}_d(K^{\text{sep}}))\).

**Sketch of proof.** The Skolem-Noether theorem (see Brauer Groups, Theorem 6.1) implies that for any field \(L\) the group \(\text{Aut}_L(\text{Mat}_d(L))\) equals \(\text{PGL}_d(L)\). By Theorem 60.1 we see that central simple algebras of degree \(d\) correspond to forms of the \(K\)-algebra \(\text{Mat}_d(K)\). Combined we see that isomorphism classes of degree \(d\) central simple algebras correspond to elements of \(H^1(\text{Gal}(K^{\text{sep}}/K), \text{PGL}_d(K^{\text{sep}}))\).

For more details on twisting, see for example [SI86].

If \(A\) is a finite central simple algebra of degree \(d\) over a field \(K\), we denote \(\xi_A\) the corresponding cohomology class in \(H^1(\text{Gal}(K^{\text{sep}}/K), \text{PGL}_d(K^{\text{sep}}))\). Consider the short exact sequence
\[
1 \rightarrow (K^{\text{sep}})^* \rightarrow \text{GL}_d(K^{\text{sep}}) \rightarrow \text{PGL}_d(K^{\text{sep}}) \rightarrow 1,
\]
which gives rise to a long exact cohomology sequence (up to degree 2) with coboundary map
\[
\delta_d : H^1(\text{Gal}(K^{\text{sep}}/K), \text{PGL}_d(K^{\text{sep}})) \longrightarrow H^2(\text{Gal}(K^{\text{sep}}/K), (K^{\text{sep}})^*).
\]
Explicitly, this is given as follows: if \(\xi\) is a cohomology class represented by the 1-cocycle \((g_\sigma)\), then \(\delta_d(\xi)\) is the class of the 2-cocycle
\[
(\sigma, \tau) \mapsto g_\sigma^{-1} g_\sigma \tau g_\tau^{-1} \in (K^{\text{sep}})^*
\]
where \(g_\sigma \in \text{GL}_d(K^{\text{sep}})\) is a lift of \(g_\sigma\). Using this we can make explicit the map
\[
\delta : \text{Br}(K) \longrightarrow H^2(\text{Gal}(K^{\text{sep}}/K), (K^{\text{sep}})^*), \quad [A] \mapsto \delta_{\text{deg}A}(\xi_A)
\]
as follows. Assume $A$ has degree $d$ over $K$. Choose an isomorphism $\varphi : \text{Mat}_d(K^{\text{sep}}) \to A \otimes_K K^{\text{sep}}$. For $\sigma \in \text{Gal}(K^{\text{sep}}/K)$ choose an element $\tilde{g}_\sigma \in \text{GL}_d(K^{\text{sep}})$ such that $\varphi^{-1} \circ \sigma(\varphi)$ is equal to the map $x \mapsto \tilde{g}_\sigma x \tilde{g}_\sigma^{-1}$. The class in $H^2$ is defined by the two cocycles $(60.5.1)$.

**Theorem 60.6.** Let $K$ be a field with separable algebraic closure $K^{\text{sep}}$. The map $\delta : \text{Br}(K) \to H^2(\text{Gal}(K^{\text{sep}}/K), (K^{\text{sep}})^*)$ defined above is a group isomorphism.

**Sketch of proof.** In the abelian case ($d = 1$), one has the identification

$$H^1(\text{Gal}(K^{\text{sep}}/K), \text{GL}_d(K^{\text{sep}})) = H^1_{\text{étale}}(\text{Spec}(K), \text{GL}_d(\mathcal{O}))$$

the latter of which is trivial by fpqc descent. If this were true in the non-abelian case, this would readily imply injectivity of $\delta$. (See [Del77].) Rather, to prove this, one can reinterpret $\delta([A])$ as the obstruction to the existence of a $K$-vector space $V$ with a left $A$-module structure and such that $\dim_K V = \deg A$. In the case where $V$ exists, one has $A \cong \text{End}_K(V)$. For surjectivity, pick a cohomology class $\xi \in H^2(\text{Gal}(K^{\text{sep}}/K), (K^{\text{sep}})^*)$, then there exists a finite Galois extension $K \subset K' \subset K^{\text{sep}}$ such that $\xi$ is the image of some $\xi' \in H^2(\text{Gal}(K'/K), (K')^*)$. Then write down an explicit central simple algebra over $K$ using the data $K', \xi'$.

### 61. The Brauer group of a scheme

**Lemma 61.1.** Let $S$ be a scheme. An $\mathcal{O}_S$-algebra $A$ is called Azumaya if it is étale locally a matrix algebra, i.e., if there exists an étale covering $U = \{\varphi_i : U_i \to S\}_{i \in I}$ such that $\varphi_i^* A \cong \text{Mat}_{d_i}(\mathcal{O}_{U_i})$ for some $d_i \geq 1$. Two such $A$ and $B$ are called equivalent if there exist finite locally free $\mathcal{O}_S$-modules $\mathcal{F}$ and $\mathcal{G}$ which have positive rank at every $s \in S$ such that

$$A \otimes_{\mathcal{O}_S} \text{Hom}_{\mathcal{O}_S}(\mathcal{F}, \mathcal{F}) \cong B \otimes_{\mathcal{O}_S} \text{Hom}_{\mathcal{O}_S}(\mathcal{G}, \mathcal{G})$$

as $\mathcal{O}_S$-algebras. The Brauer group of $S$ is the set $\text{Br}(S)$ of equivalence classes of Azumaya $\mathcal{O}_S$-algebras with the operation induced by tensor product (over $\mathcal{O}_S$).

**Proof.** Fix an isomorphism $\text{Hom}_{\mathcal{O}_S}(\mathcal{F}, \mathcal{F}) \to \text{Hom}_{\mathcal{O}_S}(\mathcal{G}, \mathcal{G})$. Consider the sheaf $\mathcal{L} \subseteq \text{Hom}(\mathcal{F}, \mathcal{G})$ generated as an $\mathcal{O}_S$-module by the local isomorphisms $\varphi : \mathcal{F} \to \mathcal{G}$ such that conjugation by $\varphi$ is the given isomorphism of endomorphism algebras. A local calculation (reducing to the case that $\mathcal{F}$ and $\mathcal{G}$ are finite free and $S$ is affine) shows that $\mathcal{L}$ is invertible. Another local calculation shows that the evaluation map

$$\mathcal{F} \otimes_{\mathcal{O}_S} \mathcal{L} \to \mathcal{G}$$

is an isomorphism. □

The argument given in the proof of the following lemma can be found in [Sal81].

**Lemma 61.2.** Let $S$ be a scheme. Let $A$ be an Azumaya algebra which is locally free of rank $d^2$ over $S$. Then the class of $A$ in the Brauer group of $S$ is annihilated by $d$. 

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**Argument taken from [Sal81].**
Étale cohomology

In this section we will use the following result from Galois cohomology to get vanishing of higher étale cohomology groups over the spectrum of a field.

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In this section we will use the following result from Galois cohomology to get vanishing of higher étale cohomology groups over the spectrum of a field.

03R8

**Proposition 62.1.** Let $K$ be a field with separable algebraic closure $K^{sep}$. Assume that for any finite extension $K'$ of $K$ we have $Br(K') = 0$. Then

(1) $H^q(Gal(K^{sep}/K), (K^{sep})^*) = 0$ for all $q \geq 1$, and

(2) $H^q(Gal(K^{sep}/K), M) = 0$ for any torsion $Gal(K^{sep}/K)$-module $M$ and any $q \geq 2$.

**Proof.** Omitted.
03R9 **Definition 62.2.** A field $K$ is called $C_r$ if for every $0 < d^r < n$ and every $f \in K[T_1, \ldots, T_n]$ homogeneous of degree $d^r$, there exist $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\alpha_i \in K$ not all zero, such that $f(\alpha) = 0$. Such an $\alpha$ is called a nontrivial solution of $f$.

03RA **Example 62.3.** An algebraically closed field is $C_r$.

In fact, we have the following simple lemma.

03RB **Lemma 62.4.** Let $k$ be an algebraically closed field. Let $f_1, \ldots, f_s \in k[T_1, \ldots, T_n]$ be homogeneous polynomials of degree $d_1, \ldots, d_s$ with $d_i > 0$. If $s < n$, then $f_1 = \ldots = f_s = 0$ have a common nontrivial solution.

**Proof.** This follows from dimension theory, for example in the form of Varieties, Lemma 27.2 applied $s - 1$ times. $\square$

The following result computes the Brauer group of $C_1$ fields.

03RC **Theorem 62.5.** Let $K$ be a $C_1$ field. Then $Br(K) = 0$.

**Proof.** Let $D$ be a finite dimensional division algebra over $K$ with center $K$. We have seen that

$$D \otimes_K K^{*\text{sep}} \cong \text{Mat}_d(K^{*\text{sep}})$$

uniquely up to inner isomorphism. Hence the determinant $\det : \text{Mat}_d(K^{*\text{sep}}) \to K^{*\text{sep}}$ is Galois invariant and descends to a homogeneous degree $d$ map

$$\det = N_{\text{red}} : D \to K$$

called the reduced norm. Since $K$ is $C_1$, if $d > 1$, then there exists a nonzero $x \in D$ with $N_{\text{red}}(x) = 0$. This clearly implies that $x$ is not invertible, which is a contradiction. Hence $Br(K) = 0$. $\square$

03RE **Definition 62.6.** Let $k$ be a field. A variety is separated, integral scheme of finite type over $k$. A curve is a variety of dimension 1.

03RD **Theorem 62.7** (Tsen’s theorem). The function field of a variety of dimension $r$ over an algebraically closed field $k$ is $C_r$.

**Proof.** For projective space one can show directly that the field $k(x_1, \ldots, x_r)$ is $C_r$ (exercise).

General case. Without loss of generality, we may assume $X$ to be projective. Let $f \in K[T_1, \ldots, T_n]$ with $0 < d^r < n$. Say the coefficients of $f$ are in $\Gamma(X, \mathcal{O}_X(H))$ for some ample $H \subset X$. Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ with $\alpha_i \in \Gamma(X, \mathcal{O}_X(eH))$. Then $f(\alpha) \in \Gamma(X, \mathcal{O}_X((de+1)H))$. Consider the system of equations $f(\alpha) = 0$. Then by asymptotic Riemann-Roch (Varieties, Proposition 34.13) there exists a $c > 0$ such that

- the number of variables is $n \dim_K \Gamma(X, \mathcal{O}_X(eH)) \sim ne^r c$, and
- the number of equations is $\dim_K \Gamma(X, \mathcal{O}_X((de+1)H)) \sim (de+1)^r c$.

Since $n > d^r$, there are more variables than equations. The equations are homogeneous hence there is a solution by Lemma 62.4. $\square$

03RF **Lemma 62.8.** Let $C$ be a curve over an algebraically closed field $k$. Then the Brauer group of the function field of $C$ is zero: $Br(k(C)) = 0$.

**Proof.** This is clear from Tsen’s theorem, Theorem 62.7. $\square$
Let $k$ be an algebraically closed field and $k \subset K$ a field extension of transcendence degree 1. Then for all $q \geq 1$, $H^q_{\text{étale}}(\text{Spec}(K), G_m) = 0$.

Proof. Recall that $H^q_{\text{étale}}(\text{Spec}(K), G_m) = H^q(\text{Gal}(K^{sep}/K), (K^{sep})^\times)$ by Lemma 58.2. Thus by Proposition 62.1 it suffices to show that if $K \subset K'$ is a finite field extension, then $\text{Br}(K') = 0$. Now observe that $K' = \text{colim} K''$, where $K''$ runs over the finitely generated subextensions of $k$ contained in $K'$ of transcendence degree 1. Note that $\text{Br}(K') = \text{colim} \text{Br}(K'')$ which reduces us to a finitely generated field extension $K''/k$ of transcendence degree 1. Such a field is the function field of a curve over $k$, hence has trivial Brauer group by Lemma 62.8.\hfill\Box

### 63. Higher vanishing for the multiplicative group

In this section, we fix an algebraically closed field $k$ and a smooth curve $X$ over $k$. We denote $i_x : x \to X$ the inclusion of a closed point of $X$ and $j : \eta \to X$ the inclusion of the generic point. We also denote $X_0$ the set of closed points of $X$.

#### Theorem 63.1 (The Fundamental Exact Sequence)

There is a short exact sequence of étale sheaves on $X$

$$0 \to G_{m,X} \to j_* G_{m,\eta} \to \bigoplus_{x \in X_0} i_x Z \to 0.$$  

Proof. Let $\varphi : U \to X$ be an étale morphism. Then by properties of étale morphisms (Proposition 26.2), $U = \bigsqcup U_i$ where each $U_i$ is a smooth curve mapping to $X$. The above sequence for $X$ is a product of the corresponding sequences for each $U_i$, so it suffices to treat the case where $U$ is connected, hence irreducible.

In this case, there is a well known exact sequence

$$1 \to \Gamma(U, O_U^\times) \to k(U)^* \to \bigoplus_{y \in U^0} Z y.$$  

This amounts to a sequence

$$0 \to \Gamma(U, O_U^\times) \to \Gamma(\eta \times_X U, O_{\eta \times_X U}^\times) \to \bigoplus_{x \in X_0} \Gamma(x \times_X U, \mathbb{Z})$$  

which, unfolding definitions, is nothing but a sequence

$$0 \to G_m(U) \to j_* G_{m,\eta}(U) \to \left( \bigoplus_{x \in X_0} i_x Z \right)(U).$$  

This defines the maps in the Fundamental Exact Sequence and shows it is exact except possibly at the last step. To see surjectivity, let us recall that if $U$ is a nonsingular curve and $D$ is a divisor on $U$, then there exists a Zariski open covering \{U_j \to U\} of $U$ such that $D|_{U_j} = \text{div}(f_j)$ for some $f_j \in k(U)^*$.\hfill\Box

#### Lemma 63.2

For any $q \geq 1$, $R^q j_* G_{m,\eta} = 0$.

Proof. We need to show that $(R^q j_* G_{m,\eta})_{\bar{x}} = 0$ for every geometric point $\bar{x}$ of $X$.

Assume that $\bar{x}$ lies over a closed point $x$ of $X$. Let $\text{Spec}(A)$ be an affine open neighbourhood of $x$ in $X$, and $K$ the fraction field of $A$. Then

$$\text{Spec}(O_{X,\bar{x}}^{sh}) \times_A \eta = \text{Spec}(O_{X,\bar{x}}^{sh} \otimes_A K).$$

The ring $O_{X,\bar{x}}^{sh} \otimes_A K$ is a localization of the discrete valuation ring $O_{X,\bar{x}}^{sh}$, so it is either $O_{X,\bar{x}}^{sh}$ again, or its fraction field $K_{\bar{x}}^{sh}$. But since some local uniformizer gets inverted, it must be the latter. Hence

$$\langle R^q j_* G_{m,\eta} \rangle_{\bar{x}} = H^q_{\text{étale}}(\text{Spec} K_{\bar{x}}^{sh}, G_m).$$
Now recall that \( O_{X,\bar{x}}^{\text{etale}} = \text{colim}_{(U,a) \to \bar{x}} O(U) = \text{colim}_{A \to B} B \) where \( A \to B \) is étale, hence \( K_{\bar{x}}^{\text{etale}} \) is an algebraic extension of \( K = k(X) \), and we may apply Lemma 63.4 to get the vanishing.

Assume that \( \bar{x} = \bar{\eta} \) lies over the generic point \( \eta \) of \( X \) (in fact, this case is superfluous). Then \( O_{X,\bar{\eta}} = \kappa(\eta)_{\text{sep}} \) and thus
\[
(R^q j_* G_{m,\eta}) \bar{\eta} = H^q_{\text{étale}}(\text{Spec}(\kappa(\eta)_{\text{sep}}) \times_X \eta, G_m) \\
= H^q_{\text{étale}}(\text{Spec}(\kappa(\eta)_{\text{sep}}), G_m) \\
= 0 \quad \text{for } q \geq 1
\]
since the corresponding Galois group is trivial.

\[\square\]

**Lemma 63.3.** For all \( p \geq 1 \), \( H^p_{\text{étale}}(X, j_* G_{m,\eta}) = 0 \).

**Proof.** The Leray spectral sequence reads
\[
E_2^{p,q} = H^p_{\text{étale}}(X, R^q j_* G_{m,\eta}) \Rightarrow H^{p+q}_{\text{étale}}(\eta, G_{m,\eta}),
\]
which vanishes for \( p + q \geq 1 \) by Lemma 62.9. Taking \( q = 0 \), we get the desired vanishing.

\[\square\]

**Lemma 63.4.** For all \( q \geq 1 \), \( H^q_{\text{étale}}(X, \bigoplus_{x \in X_0} i_x \mathbb{Z}) = 0 \).

**Proof.** For \( X \) quasi-compact and quasi-separated, cohomology commutes with colimits, so it suffices to show the vanishing of \( H^q_{\text{étale}}(X, \bigoplus_{x \in X_0} i_x \mathbb{Z}) \). But then the inclusion \( i_x \) of a closed point is finite so \( R^p i_* \mathbb{Z} = 0 \) for all \( p \geq 1 \) by Proposition 55.2. Applying the Leray spectral sequence, we see that \( H^q_{\text{étale}}(X, i_x \mathbb{Z}) = H^q_{\text{étale}}(x, \mathbb{Z}) \). Finally, since \( x \) is the spectrum of an algebraically closed field, all higher cohomology on \( x \) vanishes.

\[\square\]

Concluding this series of lemmata, we get the following result.

**Theorem 63.5.** Let \( X \) be a smooth curve over an algebraically closed field. Then \( H^q_{\text{étale}}(X, G_m) = 0 \) for all \( q \geq 2 \).

**Proof.** See discussion above.

\[\square\]

We also get the cohomology long exact sequence
\[
0 \to H^0_{\text{étale}}(X, G_m) \to H^0_{\text{étale}}(X, j_* G_{m,\eta}) \to H^0_{\text{étale}}(X, \bigoplus_{x \in X_0} i_x \mathbb{Z}) \to H^1_{\text{étale}}(X, G_m) \to 0
\]
although this is the familiar
\[
0 \to H^0_{\text{zar}}(X, O_X^*) \to k(X)^* \to \text{Div}(X) \to \text{Pic}(X) \to 0.
\]

### 64. The Artin-Schreier sequence

Let \( p \) be a prime number. Let \( S \) be a scheme in characteristic \( p \). The Artin-Schreier sequence is the short exact sequence
\[
0 \to \mathbb{Z}/p\mathbb{Z} \to G_{a,S} \xrightarrow{F-1} G_{a,S} \to 0
\]
where \( F - 1 \) is the map \( x \mapsto x^p - x \).

**Lemma 64.1.** Let \( p \) be a prime. Let \( S \) be a scheme of characteristic \( p \).

(1) If \( S \) is affine, then \( H^q_{\text{étale}}(S, \mathbb{Z}/p\mathbb{Z}) = 0 \) for all \( q \geq 2 \).

(2) If \( S \) is a quasi-compact and quasi-separated scheme of dimension \( d \), then \( H^q_{\text{étale}}(S, \mathbb{Z}/p\mathbb{Z}) = 0 \) for all \( q \geq 2 + d \).
Proof. Recall that the étale cohomology of the structure sheaf is equal to its cohomology on the underlying topological space (Theorem 22.4). The first statement follows from the Artin-Schreier exact sequence and the vanishing of cohomology of the structure sheaf on an affine scheme (Cohomology of Schemes, Lemma 2.2). The second statement follows by the same argument from the vanishing of Cohomology, Proposition 23.4 and the fact that $S$ is a spectral space (Properties, Lemma 64.2).

**Lemma 64.2.** Let $k$ be an algebraically closed field of characteristic $p > 0$. Let $V$ be a finite dimensional $k$-vector space. Let $F : V \to V$ be a Frobenius linear map, i.e., an additive map such that $F(\lambda v) = \lambda^p F(v)$ for all $\lambda \in k$ and $v \in V$. Then $F - 1 : V \to V$ is surjective with kernel a finite dimensional $F_p$-vector space of dimension $\leq \dim_k(V)$.

**Proof.** If $F = 0$, then the statement holds. If we have a filtration of $V$ by $F$-stable subvector spaces such that the statement holds for each graded piece, then it holds for $(V,F)$. Combining these two remarks we may assume the kernel of $F$ is zero.

Choose a basis $v_1, \ldots, v_n$ of $V$ and write $F(v_i) = \sum a_{ij} v_j$. Observe that $v = \sum \lambda_i v_i$ is in the kernel if and only if $\sum \lambda_i^p a_{ij} v_j = 0$. Since $k$ is algebraically closed this implies the matrix $(a_{ij})$ is invertible. Let $(b_{ij})$ be its inverse. Then to see that $F - 1$ is surjective we pick $w = \sum \mu_i v_i \in V$ and try to solve

$$(F - 1)(\sum \lambda_i v_i) = \sum \lambda_i^p a_{ij} v_j - \sum \lambda_j v_j = \sum \mu_j v_j$$

This is equivalent to

$$\sum \lambda_i^p v_j - \sum b_{ij} \lambda_i v_j = \sum b_{ij} \mu_i v_j$$

in other words

$$\lambda_i^p - \sum b_{ij} \lambda_i = \sum b_{ij} \mu_i, \quad j = 1, \ldots, \dim(V).$$

The algebra

$$A = k[x_1, \ldots, x_n]/(x_j^p - \sum b_{ij} x_i - \sum b_{ij} \mu_i)$$

is standard smooth over $k$ (Algebra, Definition 135.6) because the matrix $(b_{ij})$ is invertible and the partial derivatives of $x_j^p$ are zero. A basis of $A$ over $k$ is the set of monomials $x_1^{e_1} \cdots x_n^{e_n}$ with $e_i < p$, hence $\dim_k(A) = p^n$. Since $k$ is algebraically closed we see that $\text{Spec}(A)$ has exactly $p^n$ points. It follows that $F - 1$ is surjective and every fibre has $p^n$ points, i.e., the kernel of $F - 1$ is a group with $p^n$ elements.

**Lemma 64.3.** Let $X$ be a separated scheme of finite type over a field $k$. Let $\mathcal{F}$ be a coherent sheaf of $\mathcal{O}_X$-modules. Then $\dim_k H^d(X, \mathcal{F}) < \infty$ where $d = \dim(X)$.

**Proof.** We will prove this by induction on $d$. The case $d = 0$ holds because in that case $X$ is the spectrum of a finite dimensional $k$-algebra $A$ (Varieties, Lemma 17.2) and every coherent sheaf $\mathcal{F}$ corresponds to a finite $A$-module $M = H^0(X, \mathcal{F})$ which has $\dim_k M < \infty$.

Assume $d > 0$ and the result has been shown for separated schemes of finite type of dimension $< d$. The scheme $X$ is Noetherian. Consider the property $\mathcal{P}$ of coherent sheaves on $X$ defined by the rule

$$\mathcal{P}(\mathcal{F}) \iff \dim_k H^d(X, \mathcal{F}) < \infty$$
We are going to use the result of Cohomology of Schemes, Lemma 12.4 to prove that $P$ holds for every coherent sheaf on $X$.

Let 

$$0 \to \mathcal{F}_1 \to \mathcal{F} \to \mathcal{F}_2 \to 0$$

be a short exact sequence of coherent sheaves on $X$. Consider the long exact sequence of cohomology

$$H^d(X, \mathcal{F}_1) \to H^d(X, \mathcal{F}) \to H^d(X, \mathcal{F}_2)$$

Thus if $P$ holds for $\mathcal{F}_1$ and $\mathcal{F}_2$, then it holds for $\mathcal{F}$.

Let $Z \subset X$ be an integral closed subscheme. Let $I$ be a coherent sheaf of ideals on $Z$.

To finish the proof we have to show that $H^d(X, i^* I) = H^d(Z, I)$ is finite dimensional. If $\dim(Z) < d$, then the result holds because the cohomology group will be zero (Cohomology, Proposition 21.6). In this way we reduce to the situation discussed in the following paragraph.

Assume $X$ is a variety of dimension $d$ and $\mathcal{F} = I$ is a coherent ideal sheaf. In this case we have a short exact sequence

$$0 \to \mathcal{I} \to \mathcal{O}_X \to i_* \mathcal{O}_Z \to 0$$

where $i : Z \to X$ is the closed subscheme defined by $\mathcal{I}$. By induction hypothesis we see that $H^{d-1}(Z, \mathcal{O}_Z) = H^{d-1}(X, i_* \mathcal{O}_Z)$ is finite dimensional. Thus we see that it suffices to prove the result for the structure sheaf.

We can apply Chow’s lemma (Cohomology of Schemes, Lemma 17.1) to the morphism $X \to \text{Spec}(k)$. Thus we get a diagram

$$\begin{array}{ccc}
X & \xrightarrow{i} & \mathbb{P}^n_k \\
\downarrow \pi & & \downarrow g' \\
\text{Spec}(k) & \xrightarrow{g} & \mathbb{P}^n_k
\end{array}$$

as in the statement of Chow’s lemma. Also, let $U \subset X$ be the dense open subscheme such that $\pi^{-1}(U) \to U$ is an isomorphism. We may assume $X'$ is a variety as well, see Cohomology of Schemes, Remark 17.2. The morphism $i' = (i, \pi) : X' \to \mathbb{P}^n_k$ is a closed immersion (loc. cit.). Hence

$$L = i^* \mathcal{O}_{\mathbb{P}^n_k}(1) \cong (i')^* \mathcal{O}_{\mathbb{P}^n_k}(1)$$

is $\pi$-relatively ample (for example by Morphisms, Lemma 39.7). Hence by Cohomology of Schemes, Lemma 15.2 there exists an $n \geq 0$ such that $R^p \pi_* \mathcal{L} \otimes^n = 0$ for all $p > 0$. Set $\mathcal{G} = \pi_* \mathcal{L} \otimes^n$. Choose any nonzero global section $s$ of $\mathcal{L} \otimes^n$. Since $\mathcal{G} = \pi_* \mathcal{L} \otimes^n$, the section $s$ corresponds to section of $\mathcal{G}$, i.e., a map $\mathcal{O}_X \to \mathcal{G}$. Since $s|_U \neq 0$ as $X'$ is a variety and $L$ invertible, we see that $\mathcal{O}_X|_U \to \mathcal{G}|_U$ is nonzero. As $\mathcal{G}|_U = \mathcal{K}\mathcal{L} \otimes^n|_{\pi^{-1}(U)}$ is invertible we conclude that we have a short exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{G} \to Q \to 0$$

where $Q$ is coherent and supported on a proper closed subscheme of $X$. Arguing as before using our induction hypothesis, we see that it suffices to prove $\dim H^d(X, \mathcal{G}) < \infty$.

By the Leray spectral sequence (Cohomology, Lemma 14.6) we see that $H^d(X, \mathcal{G}) = H^d(X', \mathcal{L} \otimes^n)$. Let $\overline{X}' \subset \mathbb{P}^n_k$ be the closure of $X'$. Then $X'$ is a projective variety.
of dimension $d$ over $k$ and $X' \subset \overline{X}$ is a dense open. The invertible sheaf $\mathcal{L}$ is the restriction of $\mathcal{O}_{\overline{X}}(n)$ to $X$. By Cohomology, Proposition 23.4 the map

$$H^d(\overline{X}', \mathcal{O}_{\overline{X}}(n)) \to H^d(X', \mathcal{L}^\otimes n)$$

is surjective. Since the cohomology group on the left has finite dimension by Cohomology of Schemes, Lemma 14.1 the proof is complete.

Lemma 64.4. Let $X$ be separated of finite type over an algebraically closed field $k$ of characteristic $p > 0$. Then $H^q_{\text{étale}}(X, \mathbb{Z}/p\mathbb{Z}) = 0$ for $q \geq \dim(X) + 1$.

Proof. Let $d = \dim(X)$. By the vanishing established in Lemma 64.1 it suffices to show that $H^{d+1}_{\text{étale}}(X, \mathbb{Z}/p\mathbb{Z}) = 0$. By Lemma 64.3 we see that $H^d(X, \mathcal{O}_X)$ is a finite dimensional $k$-vector space. Hence the long exact cohomology sequence associated to the Artin-Schreier sequence ends with

$$H^d(X, \mathcal{O}_X) \xrightarrow{F-1} H^d(X, \mathcal{O}_X) \to H^{d+1}_{\text{étale}}(X, \mathbb{Z}/p\mathbb{Z}) \to 0$$

By Lemma 64.2 the map $F - 1$ in this sequence is surjective. This proves the lemma.

Lemma 64.5. Let $X$ be a proper scheme over an algebraically closed field $k$ of characteristic $p > 0$. Then

1. $H^q_{\text{étale}}(X, \mathbb{Z}/p\mathbb{Z})$ is a finite $\mathbb{Z}/p\mathbb{Z}$-module for all $q$, and
2. $H^q_{\text{étale}}(X, \mathbb{Z}/p\mathbb{Z}) \to H^q_{\text{étale}}(X_{k'}, \mathbb{Z}/p\mathbb{Z})$ is an isomorphism if $k \subset k'$ is an extension of algebraically closed fields.

Proof. By Cohomology of Schemes, Lemma 18.3 and the comparison of cohomology of Theorem 22.4 the cohomology groups $H^q_{\text{étale}}(X, \mathbb{G}_a) = H^q(X, \mathcal{O}_X)$ are finite dimensional $k$-vector spaces. Hence by Lemma 64.2 the long exact cohomology sequence associated to the Artin-Schreier sequence splits into short exact sequences

$$0 \to H^q_{\text{étale}}(X, \mathbb{Z}/p\mathbb{Z}) \to H^q(X, \mathcal{O}_X) \xrightarrow{F-1} H^q(X, \mathcal{O}_X) \to 0$$

and moreover the $F_p$-dimension of the cohomology groups $H^q_{\text{étale}}(X, \mathbb{Z}/p\mathbb{Z})$ is equal to the $k$-dimension of the vector space $H^q(X, \mathcal{O}_X)$. This proves the first statement. The second statement follows as $H^q(X, \mathcal{O}_X) \otimes_k k' \to H^q(X_{k'}, \mathcal{O}_{X_{k'}})$ is an isomorphism by flat base change (Cohomology of Schemes, Lemma 5.2).

65. Picard groups of curves

Our next step is to use the Kummer sequence to deduce some information about the cohomology group of a curve with finite coefficients. In order to get vanishing in the long exact sequence, we review some facts about Picard groups.

Let $X$ be a smooth projective curve over an algebraically closed field $k$. Let $g = \dim_k H^1(X, \mathcal{O}_X)$ be the genus of $X$. There exists a short exact sequence

$$0 \to \text{Pic}^0(X) \to \text{Pic}(X) \xrightarrow{\text{deg}} \mathbb{Z} \to 0,$$

The abelian group $\text{Pic}^0(X)$ can be identified with $\text{Pic}^0(X) = \text{Pic}^0_{X/k}(k)$, i.e., the $k$-valued points of an abelian variety $\text{Pic}^0_{X/k}$ over $k$ of dimension $g$. Consequently, if $n \in k^*$ then $\text{Pic}^0(X)[n] \cong (\mathbb{Z}/n\mathbb{Z})^{2g}$ as abelian groups. See Picard Schemes of Curves, Section 3 and Groupoids, Section 9. This key fact, namely the description
of the torsion in the Picard group of a smooth projective curve over an algebraically closed field does not appear to have an elementary proof.

**Lemma 65.1.** Let $X$ be a smooth projective curve of genus $g$ over an algebraically closed field $k$ and let $n \geq 1$ be invertible in $k$. Then there are canonical identifications

$$H^q_{\text{étale}}(X, \mu_n) = \begin{cases} 
\mu_n(k) & \text{if } q = 0, \\
Pic^0(X)[n] & \text{if } q = 1, \\
\mathbb{Z}/n\mathbb{Z} & \text{if } q = 2, \\
0 & \text{if } q \geq 3.
\end{cases}$$

Since $\mu_n \cong \mathbb{Z}/n\mathbb{Z}$, this gives (noncanonical) identifications

$$H^q_{\text{étale}}(X, \mathbb{Z}/n\mathbb{Z}) \cong \begin{cases} 
\mathbb{Z}/n\mathbb{Z} & \text{if } q = 0, \\
(\mathbb{Z}/n\mathbb{Z})^{2g} & \text{if } q = 1, \\
\mathbb{Z}/n\mathbb{Z} & \text{if } q = 2, \\
0 & \text{if } q \geq 3.
\end{cases}$$

**Proof.** The Kummer sequence $0 \to \mu_n, X \to G_{m, X} \xrightarrow{(\cdot)^n} G_{m, X} \to 0$ give the long exact cohomology sequence

$$0 \to \mu_n(k) \to k^* \xrightarrow{(\cdot)^n} k^* \xrightarrow{\cdot} H^1_{\text{étale}}(X, \mu_n) \to \text{Pic}(X) \xrightarrow{(\cdot)^n} \text{Pic}(X) \xrightarrow{\cdot} H^2_{\text{étale}}(X, \mu_n) \to 0 \to 0 \ldots$$

The $n$ power map $k^* \to k^*$ is surjective since $k$ is algebraically closed. So we need to compute the kernel and cokernel of the map $n : \text{Pic}(X) \to \text{Pic}(X)$. Consider the commutative diagram with exact rows

$$0 \to \text{Pic}^0(X) \to \text{Pic}(X) \xrightarrow{(\cdot)^n} \mathbb{Z} \xrightarrow{\deg} 0$$

$$0 \to \text{Pic}^0(X) \to \text{Pic}(X) \xrightarrow{(\cdot)^n} \mathbb{Z} \xrightarrow{\deg} 0$$

where the left vertical map is surjective by Groupoids, Proposition 9.10. Applying the snake lemma gives canonical identifications as stated in the lemma.

To get the noncanonical identifications of the lemma we need to compute the kernel of $n : \text{Pic}^0(X) \to \text{Pic}^0(X)$. First we note that the group $\text{Pic}^0(X)$ is the $k$-points of the group scheme $\text{Pic}^0_{X/k}$, see Picard Schemes of Curves, Lemma 6.7. The same lemma tells us that $\text{Pic}^0_{X/k}$ is a $g$-dimensional abelian variety over $k$ as defined in Groupoids, Definition 9.1. Thus we obtain what we want by applying Groupoids, Proposition 9.10. \qed

**Lemma 65.2.** Let $\pi : X \to Y$ be a nonconstant morphism of smooth projective curves over an algebraically closed field $k$ and let $n \geq 1$ be invertible in $k$. The map

$$\pi^* : H^2_{\text{étale}}(Y, \mu_n) \to H^2_{\text{étale}}(X, \mu_n)$$
is given by multiplication by the degree of \( \pi \).

**Proof.** Observe that the statement makes sense as we have identified both cohomology groups \( H^2_{\text{étale}}(Y, \mu_n) \) and \( H^2_{\text{étale}}(X, \mu_n) \) with \( \mathbb{Z}/n\mathbb{Z} \) in Lemma 65.1. In fact, if \( \mathcal{L} \) is a line bundle of degree 1 on \( Y \) with class \([\mathcal{L}] \in H^1_{\text{étale}}(Y, G_m)\), then the coboundary of \([\mathcal{L}]\) is the generator of \( H^2_{\text{étale}}(Y, \mu_n) \). Here the coboundary is the coboundary of the long exact sequence of cohomology associated to the Kummer sequence. Thus the result of the lemma follows from the fact that the degree of the line bundle \( \pi^* \mathcal{L} \) on \( X \) is \( \text{deg}(\pi) \). Some details omitted. \( \square \)

**Lemma 65.3.** Let \( X \) be an affine smooth curve over an algebraically closed field \( k \) and \( n \in k^* \). Then

1. \( H^0_{\text{étale}}(X, \mu_n) = \mu_n(k) \);
2. \( H^1_{\text{étale}}(X, \mu_n) \cong (\mathbb{Z}/n\mathbb{Z})^{2g+r-1} \), where \( r \) is the number of points in \( \bar{X} - X \) for some smooth projective compactification \( \bar{X} \) of \( X \), and
3. for all \( q \geq 2 \), \( H^q_{\text{étale}}(X, \mu_n) = 0 \).

**Proof.** Write \( X = \bar{X} - \{x_1, \ldots, x_r\} \). Then \( \text{Pic}(X) = \text{Pic}(\bar{X})/R \), where \( R \) is the subgroup generated by \( \mathcal{O}_\bar{X}(x_i) \), \( 1 \leq i \leq r \). Since \( r \geq 1 \), we see that \( \text{Pic}^0(X) \to \text{Pic}(X) \) is surjective, hence \( \text{Pic}(X) \) is divisible. Applying the Kummer sequence, we get (1) and (3). For (2), recall that

\[
H^1_{\text{étale}}(X, \mu_n) = \{(\mathcal{L}, \alpha) \mid \mathcal{L} \in \text{Pic}(X), \alpha : \mathcal{L} \otimes \mu_n \to \mathcal{O}_X\}/ \cong \{(\bar{\mathcal{L}}, D, \bar{\alpha}) \}/\bar{R}
\]

where \( \mathcal{L} \in \text{Pic}^0(\bar{X}) \), \( D \) is a divisor on \( \bar{X} \) supported on \( \{x_1, \ldots, x_r\} \) and \( \bar{\alpha} : \bar{\mathcal{L}} \otimes \mu_n \cong \mathcal{O}_{\bar{X}}(D) \) is an isomorphism. Note that \( D \) must have degree 0. Further \( \bar{R} \) is the subgroup of triples of the form \((\mathcal{O}_{\bar{X}}(D'), nD', 1 \otimes \mu_n)\) where \( D' \) is supported on \( \{x_1, \ldots, x_r\} \) and has degree 0. Thus, we get an exact sequence

\[
0 \to H^1_{\text{étale}}(\bar{X}, \mu_n) \to H^1_{\text{étale}}(X, \mu_n) \overset{\sim}{\to} \bigoplus_{i=1}^r \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0
\]

where the middle map sends the class of a triple \((\bar{\mathcal{L}}, D, \bar{\alpha})\) with \( D = \sum_{i=1}^r a_i(x_i) \) to the \( r \)-tuple \( (a_i)_{i=1}^r \). It now suffices to use Lemma 65.1 to count ranks. \( \square \)

**Remark 65.4.** The “natural” way to prove the previous corollary is to excise \( X \) from \( \bar{X} \). This is possible, we just haven’t developed that theory.

**Remark 65.5.** Let \( k \) be an algebraically closed field. Let \( n \) be an integer prime to the characteristic of \( k \). Recall that

\[
G_{m,k} = \mathbb{A}^1_k \setminus \{0\} = \mathbb{P}^1_k \setminus \{0, \infty\}
\]

We claim there is a canonical isomorphism

\[
H^1_{\text{étale}}(G_{m,k}, \mu_n) = \mathbb{Z}/n\mathbb{Z}
\]

What does this mean? This means there is an element \( 1_k \) in \( H^1_{\text{étale}}(G_{m,k}, \mu_n) \) such that for every morphism \( \text{Spec}(k') \to \text{Spec}(k) \) the pullback map on étale cohomology for the map \( G_{m,k'} \to G_{m,k} \) maps \( 1_k \) to \( 1_{k'} \). (In particular this element is fixed under all automorphisms of \( k \).) To see this, consider the \( \mu_n, \mathbb{Z} \)-torsor \( G_{m,k} \to G_{m,k} \), \( x \mapsto x^n \). By the identification of torsors with first cohomology, this pulls back
to give our canonical elements $1_k$. Twisting back we see that there are canonical identifications

$$H^1_{\text{étale}}(G_{m,k}, \mathbb{Z}/n\mathbb{Z}) = \text{Hom}(\mu_n(k), \mathbb{Z}/n\mathbb{Z}),$$

i.e., these isomorphisms are compatible with respect to maps of algebraically closed fields, in particular with respect to automorphisms of $k$.

### 66. Extension by zero

03S2 The general material in Modules on Sites, Section 19 allows us to make the following definition.

**Definition 66.1.** Let $j : U \to X$ be an étale morphism of schemes.

1. The restriction functor $j^{-1} : \text{Sh}(X_{\text{étale}}) \to \text{Sh}(U_{\text{étale}})$ has a left adjoint $j^! : \text{Sh}(U_{\text{étale}}) \to \text{Sh}(X_{\text{étale}})$.
2. The restriction functor $j^{-1} : \text{Ab}(X_{\text{étale}}) \to \text{Ab}(U_{\text{étale}})$ has a left adjoint which is denoted $j_! : \text{Ab}(U_{\text{étale}}) \to \text{Ab}(X_{\text{étale}})$ and called extension by zero.
3. Let $\Lambda$ be a ring. The restriction functor $j^{-1} : \text{Mod}(X_{\text{étale}}, \Lambda) \to \text{Mod}(U_{\text{étale}}, \Lambda)$ has a left adjoint which is denoted $j_! : \text{Mod}(U_{\text{étale}}, \Lambda) \to \text{Mod}(X_{\text{étale}}, \Lambda)$ and called extension by zero.

If $F$ is an abelian sheaf on $X_{\text{étale}}$, then $j_! F \neq j^! F$ in general. On the other hand $j_!$ for sheaves of $\Lambda$-modules agrees with $j_!$ on underlying abelian sheaves (Modules on Sites, Remark 19.5). The functor $j_!$ is characterized by the functorial isomorphism

$$\text{Hom}_X(j_! F, G) = \text{Hom}_U(F, j^{-1} G)$$

for all $F \in \text{Ab}(U_{\text{étale}})$ and $G \in \text{Ab}(X_{\text{étale}})$. Similarly for sheaves of $\Lambda$-modules.

To describe it more explicitly, recall that $j^{-1}$ is just the restriction via the functor $U_{\text{étale}} \to X_{\text{étale}}$. In other words, $j^{-1} G(U') = G(U')$ for $U'$ étale over $U$. For $F \in \text{Ab}(U_{\text{étale}})$ we consider the presheaf

$$j^P_{\text{Sh}} F : X_{\text{étale}} \to \text{Ab}, \quad V \mapsto \bigoplus_{V \to U} F(V)$$

Then $j_! F$ is the sheafification of $j^P_{\text{Sh}} F$.

**Exercise 66.2.** Prove directly that $j_!$ is left adjoint to $j^{-1}$ and that $j_*$ is right adjoint to $j^{-1}$.

**Proposition 66.3.** Let $j : U \to X$ be an étale morphism of schemes. Let $F$ in $\text{Ab}(U_{\text{étale}})$. If $\pi : \text{Spec}(k) \to X$ is a geometric point of $X$, then

$$(j_! F)_{\pi} = \bigoplus_{\pi : \text{Spec}(k) \to U, \, j(\pi) = \pi} F_{\pi}.$$

In particular, $j_!$ is an exact functor.

**Proof.** Exactness of $j_!$ is very general, see Modules on Sites, Lemma 19.3. Of course it does also follow from the description of stalks. The formula for the stalk of $j_! F$ can be deduced directly from the explicit description of $j_!$ given above. On the other hand, we can deduce it from the very general Modules on Sites, Lemma 37.1 and the description of points of the small étale site in terms of geometric points, see Lemma 29.12. \qed
Lemma 66.4 (Extension by zero commutes with base change). Let $f : Y \to X$ be a morphism of schemes. Let $j : V \to X$ be an étale morphism. Consider the fibre product

\[
V' = Y \times_X V \xrightarrow{f'} Y \\
\downarrow \quad \quad \downarrow j \\
V \xrightarrow{j} X
\]

Then we have $j'_! f'^{-1} = f^{-1} j_!$ on abelian sheaves and on sheaves of modules.

Proof. This is true because $j'_! f'^{-1}$ is left adjoint to $f'_*(j')^{-1}$ and $f^{-1} j_!$ is left adjoint to $j^{-1} f_*$. Further $f'_*(j')^{-1} = j^{-1} f_*$ because $f_*$ commutes with étale localization (by construction). In fact, the lemma holds very generally in the setting of a morphism of sites, see Modules on Sites, Lemma 20.1.

Lemma 66.5. Let $j : U \to X$ be finite and étale. Then $j_! = j_*$ on abelian sheaves and sheaves of $\Lambda$-modules.

Proof. We prove this in the case of abelian sheaves. By Modules on Sites, Remark 19.7 there is a natural transformation $j_! \to j_*$. It suffices to check this is an isomorphism étale locally on $X$. Thus we may assume $U \to X$ is a finite disjoint union of isomorphisms, see Étale Morphisms, Lemma 18.3. We omit the proof in this case.

Lemma 66.6. Let $X$ be a scheme. Let $Z \subset X$ be a closed subscheme and let $U \subset X$ be the complement. Denote $i : Z \to X$ and $j : U \to X$ the inclusion morphisms. For every abelian sheaf on $X_{\acute{e}tale}$ there is a canonical short exact sequence

\[
0 \to j_! j^{-1} \mathcal{F} \to \mathcal{F} \to i_* i^{-1} \mathcal{F} \to 0
\]

on $X_{\acute{e}tale}$.

Proof. We obtain the maps by the adjointness properties of the functors involved. For a geometric point $\xi$ in $X$ we have either $\xi \in U$ in which case the map on the left hand side is an isomorphism on stalks and the stalk of $i_* i^{-1} \mathcal{F}$ is zero or $\xi \in Z$ in which case the map on the right hand side is an isomorphism on stalks and the stalk of $j_! j^{-1} \mathcal{F}$ is zero. Here we have used the description of stalks of Lemma 47.3 and Proposition 66.3.

67. Locally constant sheaves

This section is the analogue of Modules on Sites, Section 42 for the étale site.

Definition 67.1. Let $X$ be a scheme. Let $\mathcal{F}$ be a sheaf of sets on $X_{\acute{e}tale}$.

1. Let $E$ be a set. We say $\mathcal{F}$ is the constant sheaf with value $E$ if $\mathcal{F}$ is the sheafification of the presheaf $U \mapsto E$. Notation: $\underline{E}_X$ or $\underline{E}$.
2. We say $\mathcal{F}$ is a constant sheaf if it is isomorphic to a sheaf as in (1).
3. We say $\mathcal{F}$ is locally constant if there exists a covering $\{ U_i \to X \}$ such that $\mathcal{F}|_{U_i}$ is a constant sheaf.
4. We say that $\mathcal{F}$ is finite locally constant if it is locally constant and the values are finite sets.

Let $\mathcal{F}$ be a sheaf of abelian groups on $X_{\acute{e}tale}$. 
(1) Let $A$ be an abelian group. We say $F$ is the constant sheaf with value $A$ if $F$ is the sheafification of the presheaf $U \mapsto A$. Notation: $A_X$ or $A$.

(2) We say $F$ is a constant sheaf if it is isomorphic as an abelian sheaf to a sheaf as in (1).

(3) We say $F$ is locally constant if there exists a covering $\{U_i \to X\}$ such that $F|_{U_i}$ is a constant sheaf.

(4) We say that $F$ is finite locally constant if it is locally constant and the values are finite abelian groups.

Let $\Lambda$ be a ring. Let $F$ be a sheaf of $\Lambda$-modules on $X_{\text{ét}}$.

(1) Let $M$ be a $\Lambda$-module. We say $F$ is the constant sheaf with value $M$ if $F$ is the sheafification of the presheaf $U \mapsto M$. Notation: $M_X$ or $M$.

(2) We say $F$ is a constant sheaf if it is isomorphic as a sheaf of $\Lambda$-modules to a sheaf as in (1).

(3) We say $F$ is locally constant if there exists a covering $\{U_i \to X\}$ such that $F|_{U_i}$ is a constant sheaf.

(4) We say that $F$ is finite locally constant if it is locally constant and the values are finite abelian groups.

095A Lemma 67.2. Let $f : X \to Y$ be a morphism of schemes. If $G$ is a locally constant sheaf of sets, abelian groups, or $\Lambda$-modules on $Y_{\text{ét}}$, the same is true for $f^{-1}G$ on $X_{\text{ét}}$.

Proof. Holds for any morphism of topoi, see Modules on Sites, Lemma 42.2.

095B Lemma 67.3. Let $f : X \to Y$ be a finite étale morphism of schemes. If $F$ is a (finite) locally constant sheaf of sets, (finite) locally constant sheaf of abelian groups, or (finite type) locally constant sheaf of $\Lambda$-modules on $X_{\text{ét}}$, the same is true for $f_* F$ on $Y_{\text{ét}}$.

Proof. The construction of $f_*$ commutes with étale localization. A finite étale morphism is locally isomorphic to a disjoint union of isomorphisms, see Étale Morphisms, Lemma 18.3. Thus the lemma says that if $F_i, i = 1, \ldots, n$ are (finite) locally constant sheaves of sets, then $\prod_{i=1,\ldots,n} F_i$ is too. This is clear. Similarly for sheaves of abelian groups and modules.

03RV Lemma 67.4. Let $X$ be a scheme and $F$ a sheaf of sets on $X_{\text{ét}}$. Then the following are equivalent

1. $F$ is finite locally constant, and
2. $F = h_U$ for some finite étale morphism $U \to X$.

Proof. A finite étale morphism is locally isomorphic to a disjoint union of isomorphisms, see Étale Morphisms, Lemma 18.3. Thus (2) implies (1). Conversely, if $F$ is finite locally constant, then there exists an étale covering $\{X_i \to X\}$ such that $F|_{X_i}$ is representable by $U_i \to X_i$ finite étale. Arguing exactly as in the proof of Descent, Lemma 35.1 we obtain a descent datum for schemes $(U_i, \varphi_{ij})$ relative to $\{X_i \to X\}$ (details omitted). This descent datum is effective for example by Descent, Lemma 33.1 and the resulting morphism of schemes $U \to X$ is finite étale by Descent, Lemmas 19.21 and 19.27.

095C Lemma 67.5. Let $X$ be a scheme.

1. Let $\varphi : F \to G$ be a map of locally constant sheaves of sets on $X_{\text{ét}}$. If $F$ is finite locally constant, there exists an étale covering $\{U_i \to X\}$ such that $\varphi|_{U_i}$ is the map of constant sheaves associated to a map of sets.
(2) Let \( \varphi : \mathcal{F} \to \mathcal{G} \) be a map of locally constant sheaves of abelian groups on \( X_{\text{étale}} \). If \( \mathcal{F} \) is finite locally constant, there exists an étale covering \( \{ U_i \to X \} \) such that \( \varphi|_{U_i} \) is the map of constant abelian sheaves associated to a map of abelian groups.

(3) Let \( \Lambda \) be a ring. Let \( \varphi : \mathcal{F} \to \mathcal{G} \) be a map of locally constant sheaves of \( \Lambda \)-modules on \( X_{\text{étale}} \). If \( \mathcal{F} \) is of finite type, then there exists an étale covering \( \{ U_i \to X \} \) such that \( \varphi|_{U_i} \) is the map of constant sheaves of \( \Lambda \)-modules associated to a map of \( \Lambda \)-modules.

**Proof.** This holds on any site, see Modules on Sites, Lemma 42.3. \( \square \)

**Lemma 67.6.** Let \( X \) be a scheme.

(1) The category of finite locally constant sheaves of sets is closed under finite limits and colimits inside \( \text{Sh}(X_{\text{étale}}) \).

(2) The category of finite locally constant abelian sheaves is a weak Serre subcategory of \( \text{Ab}(X_{\text{étale}}) \).

(3) Let \( \Lambda \) be a Noetherian ring. The category of finite type, locally constant sheaves of \( \Lambda \)-modules on \( X_{\text{étale}} \) is a weak Serre subcategory of \( \text{Mod}(X_{\text{étale}}, \Lambda) \).

**Proof.** This holds on any site, see Modules on Sites, Lemma 42.5. \( \square \)

**Lemma 67.7.** Let \( X \) be a scheme. Let \( \Lambda \) be a ring. The tensor product of two locally constant sheaves of \( \Lambda \)-modules on \( X_{\text{étale}} \) is a locally constant sheaf of \( \Lambda \)-modules.

**Proof.** This holds on any site, see Modules on Sites, Lemma 42.6. \( \square \)

**Lemma 67.8.** Let \( X \) be a connected scheme. Let \( \Lambda \) be a ring and let \( \mathcal{F} \) be a locally constant sheaf of \( \Lambda \)-modules. Then there exists a \( \Lambda \)-module \( M \) and an étale covering \( \{ U_i \to X \} \) such that \( \mathcal{F}|_{U_i} \cong M|_{U_i} \).

**Proof.** Choose an étale covering \( \{ U_i \to X \} \) such that \( \mathcal{F}|_{U_i} \) is constant, say \( \mathcal{F}|_{U_i} \cong M_{i, j} \). Observe that \( U_i \times_X U_j \) is empty if \( M_{i, j} \) is not isomorphic to \( M_{i, j} \). For each \( \Lambda \)-module \( M \) let \( I_M = \{ i \in I \mid M_i \cong M \} \). As étale morphisms are open we see that \( U_M = \bigcup_{i \in I_M} \text{Im}(U_i \to X) \) is an open subset of \( X \). Then \( X = \bigsqcup U_M \) is a disjoint open covering of \( X \). As \( X \) is connected only one \( U_M \) is nonempty and the lemma follows. \( \square \)

### 68. Constructible sheaves

Let \( X \) be a scheme. A constructible locally closed subscheme of \( X \) is a locally closed subscheme \( T \subset X \) such that the underlying topological space of \( T \) is a constructible subset of \( X \). If \( T, T' \subset X \) are locally closed subschemes with the same underlying topological space, then \( T_{\text{étale}} \cong T'_{\text{étale}} \) by the topological invariance of the étale site (Theorem 46.2). Thus in the following definition we may assume are locally closed subschemes are reduced.

**Definition 68.1.** Let \( X \) be a scheme.

(1) A sheaf of sets on \( X_{\text{étale}} \) is constructible if for every affine open \( U \subset X \) there exists a finite decomposition of \( U \) into constructible locally closed subschemes \( U = \bigsqcup U_i \) such that \( \mathcal{F}|_{U_i} \) is finite locally constant for all \( i \).
(2) A sheaf of abelian groups on $X_{\text{étale}}$ is constructible if for every affine open $U \subset X$ there exists a finite decomposition of $U$ into constructible locally closed subschemes $U = \bigsqcup U_i$ such that $\mathcal{F}|_{U_i}$ is finite locally constant for all $i$.

(3) Let $\Lambda$ be a Noetherian ring. A sheaf of $\Lambda$-modules on $X_{\text{étale}}$ is constructible if for every affine open $U \subset X$ there exists a finite decomposition of $U$ into constructible locally closed subschemes $U = \bigsqcup U_i$ such that $\mathcal{F}|_{U_i}$ is of finite type and locally constant for all $i$.

It seems that this is the accepted definition. An alternative, which lends itself more readily to generalizations beyond the étale site of a scheme, would have been to define constructible sheaves by starting with $h_U$, $j_{U!}\mathbb{Z}/n\mathbb{Z}$, and $j_{U!}\Lambda$ where $U$ runs over all quasi-compact and quasi-separated objects of $X_{\text{étale}}$, and then take the smallest full subcategory of $\text{Sh}(X_{\text{étale}})$, $\text{Ab}(X_{\text{étale}})$, and $\text{Mod}(X_{\text{étale}}, \Lambda)$ containing these and closed under finite limits and colimits. It follows from Lemma 68.6 and Lemmas 70.5, 70.7, and 70.6 that this produces the same category if $X$ is quasi-compact and quasi-separated. In general this does not produce the same category however.

A disjoint union decomposition $U = \bigsqcup U_i$ of a scheme by locally closed subschemes will be called a partition of $U$ (compare with Topology, Section 27).

**Lemma 68.2.** Let $X$ be a quasi-compact and quasi-separated scheme. Let $\mathcal{F}$ be a sheaf of sets on $X_{\text{étale}}$. The following are equivalent

1. $\mathcal{F}$ is constructible,
2. there exists an open covering $X = \bigcup U_i$ such that $\mathcal{F}|_{U_i}$ is constructible, and
3. there exists a partition $X = \bigsqcup X_i$ by constructible locally closed subschemes such that $\mathcal{F}|_{X_i}$ is finite locally constant.

A similar statement holds for abelian sheaves and sheaves of $\Lambda$-modules if $\Lambda$ is Noetherian.

**Proof.** It is clear that (1) implies (2).

Assume (2). For every $x \in X$ we can find an $i$ and an affine open neighbourhood $V_x \subset U_i$ of $x$. Hence we can find a finite affine open covering $X = \bigcup V_j$ such that for each $j$ there exists a finite decomposition $V_j = \bigsqcup V_{j,k}$ by locally closed constructible subsets such that $\mathcal{F}|_{V_{j,k}}$ is finite locally constant. By Topology, Lemma 14.5 each $V_{j,k}$ is constructible as a subset of $X$. By Topology, Lemma 27.7 we can find a finite stratification $X = \bigsqcup X_l$ with constructible locally closed strata such that each $V_{j,k}$ is a union of $X_l$. Thus (3) holds.

Assume (3) holds. Let $U \subset X$ be an affine open. Then $U \cap X_i$ is a constructible locally closed subset of $U$ (for example by Properties, Lemma 2.1) and $U = \bigsqcup U \cap X_i$ is a partition of $U$ as in Definition 68.1. Thus (1) holds.

**Lemma 68.3.** Let $X$ be a quasi-compact and quasi-separated scheme. Let $\mathcal{F}$ be a sheaf of sets, abelian groups, $\Lambda$-modules (with $\Lambda$ Noetherian) on $X_{\text{étale}}$. If there exist constructible locally closed subschemes $T_j \subset X$ such that (a) $X = \bigcup T_j$ and (b) $\mathcal{F}|_{T_j}$ is constructible, then $\mathcal{F}$ is constructible.

**Proof.** First, we can assume the covering is finite as $X$ is quasi-compact in the spectral topology (Topology, Lemma 22.2 and Properties, Lemma 2.4). Observe
that each $T_i$ is a quasi-compact and quasi-separated scheme in its own right (because it is constructible in $X$; details omitted). Thus we can find a finite partition $T_i = \bigsqcup T_{i,j}$ into locally closed constructible parts of $T_i$ such that $F|_{T_{i,j}}$ is finite locally constant (Lemma \[68.2\]). By Topology, Lemma \[14.12\] we see that $T_{i,j}$ is a constructible locally closed subscheme of $X$. Then we can apply Topology, Lemma \[27.7\] to $X = \bigcup T_{i,j}$ to find the desired partition of $X$.

\begin{lemma}
Let $X$ be a scheme. Checking constructibility of a sheaf of sets, abelian groups, $\Lambda$-modules (with $\Lambda$ Noetherian) can be done Zariski locally on $X$.
\begin{proof}
The statement means if $X = \bigcup U_i$ is an open covering such that $F|_{U_i}$ is constructible, then $F$ is constructible. If $U \subset X$ is affine open, then $U = \bigcup U \cap U_i$ and $F|_{U \cap U_i}$ is constructible (it is trivial that the restriction of a constructible sheaf to an open is constructible). It follows from Lemma \[68.2\] that $F|_U$ is constructible, i.e., a suitable partition of $U$ exists.
\end{proof}
\end{lemma}

\begin{lemma}
Let $f : X \to Y$ be a morphism of schemes. If $F$ is a constructible sheaf of sets, abelian groups, or $\Lambda$-modules (with $\Lambda$ Noetherian) on $Y_{\text{étale}}$, the same is true for $f^{-1}F$ on $X_{\text{étale}}$.
\begin{proof}
By Lemma \[68.4\] this reduces to the case where $X$ and $Y$ are affine. By Lemma \[68.2\] it suffices to find a finite partition of $X$ by constructible locally closed subschemes such that $f^{-1}F$ is finite locally constant on each of them. To find it we just pull back the partition of $Y$ adapted to $F$ and use Lemma \[67.2\].
\end{proof}
\end{lemma}

\begin{lemma}
Let $X$ be a scheme.
\begin{enumerate}
\item The category of constructible sheaves of sets is closed under finite limits and colimits inside $\text{Sh}(X_{\text{étale}})$.
\item The category of constructible abelian sheaves is a weak Serre subcategory of $\text{Ab}(X_{\text{étale}})$.
\item Let $\Lambda$ be a Noetherian ring. The category of constructible sheaves of $\Lambda$-modules on $X_{\text{étale}}$ is a weak Serre subcategory of $\text{Mod}(X_{\text{étale}}, \Lambda)$.
\end{enumerate}
\begin{proof}
We prove (3). We will use the criterion of Homology, Lemma \[9.3\]. Suppose that $\varphi : F \to G$ is a map of constructible sheaves of $\Lambda$-modules. We have to show that $K = \text{Ker}(\varphi)$ and $Q = \text{Coker}(\varphi)$ are constructible. Similarly, suppose that $0 \to F \to E \to G \to 0$ is a short exact sequence of sheaves of $\Lambda$-modules with $F$, $G$ constructible. We have to show that $E$ is constructible. In both cases we can replace $X$ with the members of an affine open covering. Hence we may assume $X$ is affine. The we may further replace $X$ by the members of a finite partition of $X$ by constructible locally closed subschemes on which $F$ and $G$ are of finite type and locally constant. Thus we may apply Lemma \[67.6\] to conclude.

The proofs of (1) and (2) are very similar and are omitted.
\end{proof}
\end{lemma}

\begin{lemma}
Let $X$ be a scheme. Let $\Lambda$ be a Noetherian ring. The tensor product of two constructible sheaves of $\Lambda$-modules on $X_{\text{étale}}$ is a constructible sheaf of $\Lambda$-modules.
\begin{proof}
The question immediately reduces to the case where $X$ is affine. Since any two partitions of $X$ with constructible locally closed strata have a common refinement of the same type and since pullbacks commute with tensor product we reduce to Lemma \[67.7\].
\end{proof}
\end{lemma}
Let $X$ be a quasi-compact and quasi-separated scheme.

1. Let $F \to G$ be a map of constructible sheaves of sets on $X_{\text{étale}}$. Then the set of points $x \in X$ where $F_x \to G_x$ is surjective, resp. injective, resp. is isomorphic to a given map of sets, is constructible in $X$.

2. Let $F$ be a constructible abelian sheaf on $X_{\text{étale}}$. The support of $F$ is constructible.

3. Let $\Lambda$ be a Noetherian ring. Let $F$ be a constructible sheaf of $\Lambda$-modules on $X_{\text{étale}}$. The support of $F$ is constructible.

Proof. Proof of (1). Let $X = \coprod X_i$ be a partition of $X$ by locally closed constructible subschemes such that both $F$ and $G$ are finite locally constant over the parts (use Lemma 68.2 for both $F$ and $G$ and choose a common refinement). Then apply Lemma 68.9 to the restriction of the map to each part.

The proof of (2) and (3) is omitted. □

The following lemma will turn out to be very useful later on. It roughly says that the category of constructible sheaves has a kind of weak “Noetherian” property.

Let $X$ be a quasi-compact and quasi-separated scheme. Let $F = \colim_{i \in I} F_i$ be a filtered colimit of sheaves of sets, abelian sheaves, or sheaves of modules.

1. If $F$ and $F_i$ are constructible sheaves of sets, then the ind-object $F_i$ is essentially constant with value $F$.

2. If $F$ and $F_i$ are constructible sheaves of abelian groups, then the ind-object $F_i$ is essentially constant with value $F$.

3. Let $\Lambda$ be a Noetherian ring. If $F$ and $F_i$ are constructible sheaves of $\Lambda$-modules, then the ind-object $F_i$ is essentially constant with value $F$.

Proof. Proof of (1). We will use without further mention that finite limits and colimits of constructible sheaves are constructible (Lemma 67.6). For each $i$ let $T_i \subset X$ be the set of points $x \in X$ where $F_i.x \to F.x$ is not surjective. Because $F_i$ and $F$ are constructible $T_i$ is a constructible subset of $X$ (Lemma 68.8). Since the stalks of $F$ are finite and since $F = \colim_{i \in I} F_i$ we see that for all $x \in X$ we have $x \notin T_i$ for $i$ large enough. Since $X$ is a spectral space by Properties, Lemma 2.4 the constructible topology on $X$ is quasi-compact by Topology, Lemma 22.2. Thus $T_i = \emptyset$ for $i$ large enough. Assume now that $F_i \to F$ is surjective for all $i$. Choose $i \in I$. For $i' \geq i$ denote $S_i' \subset X$ the set of points $x$ such that the number of elements in $\text{Im}(F_i.x \to F.x)$ is equal to the number of elements in $\text{Im}(F_i.x \to F_i.x)$. Because $F_i$, $F_i'$ and $F$ are constructible $S_i'$ is a constructible subset of $X$ (details omitted; hint: use Lemma 68.8). Since the stalks of $F_i$ and $F$ are finite and since $F = \colim_{i \geq i} F_i'$ we see that for all $x \in X$ we have $x \notin S_i'$ for $i'$ large enough. By the same argument as above we can find a large $i'$ such that $S_i' = \emptyset$. Thus $F_i \to F_i'$ factors through $F$ as desired.

Proof of (2). Observe that a constructible abelian sheaf is a constructible sheaf of sets. Thus case (2) follows from (1).

Proof of (3). We will use without further mention that the category of constructible sheaves of $\Lambda$-modules is abelian (Lemma 67.6). For each $i$ let $Q_i$ be the cokernel of the map $F_i \to F$. The support $T_i$ of $Q_i$ is a constructible subset of $X$ as $Q_i$ is constructible (Lemma 68.8). Since the stalks of $F$ are finite $\Lambda$-modules and since
$F = \operatorname{colim}_{i \in I} \mathcal{F}_i$ we see that for all $x \in X$ we have $x \notin T_i$ for $i$ large enough. Since $X$ is a spectral space by Properties, Lemma 102.4, the constructible topology on $X$ is quasi-compact by Topology, Lemma 22.2. Thus $T_i = \emptyset$ for $i$ large enough. This proves the first assertion. For the second, assume now that $\mathcal{F}_i \to \mathcal{F}$ is surjective for all $i$. Choose $i \in I$. For $i' \geq i$ denote $\mathcal{K}_{i'}$ the image of $\operatorname{Ker}(\mathcal{F}_i \to \mathcal{F})$ in $\mathcal{F}_{i'}$. The support $S_{i'}$ of $\mathcal{K}_{i'}$ is a constructible subset of $X$ as $\mathcal{K}_{i'}$ is constructible. Since the stalks of $\operatorname{Ker}(\mathcal{F}_i \to \mathcal{F})$ are finite $\Lambda$-modules and since $\mathcal{F} = \operatorname{colim}_{i' \geq i} \mathcal{F}_{i'}$ we see that for all $x \in X$ we have $x \notin S_{i'}$ for $i'$ large enough. By the same argument as above we can find a large $i'$ such that $S_{i'} = \emptyset$. Thus $\mathcal{F}_i \to \mathcal{F}_{i'}$ factors through $\mathcal{F}$ as desired. 

\section{095J Auxiliary lemmas on morphisms}

03S0 \textbf{Lemma 69.1.} Let $U \to X$ be an étale morphism of quasi-compact and quasi-separated schemes (for example an étale morphism of Noetherian schemes). Then there exists a partition $X = \coprod X_i$ by constructible locally closed subschemes such that $X_i \times_X U \to X_i$ is finite étale for all $i$.

\textbf{Proof.} If $U \to X$ is separated, then this is More on Morphisms, Lemma 102.9. In general, we may assume $X$ is affine. Choose a finite affine open covering $U = \bigcup U_j$. Apply the previous case to all the morphisms $U_j \to X$ and $U_j \cap U_{j'} \to X$ and choose a common refinement $X = \coprod X_i$ of the resulting partitions. After refining the partition further we may assume $X_i$ affine as well. Fix $i$ and set $V = U \times_X X_i$. The morphisms $V_j = U_j \times_X X_i \to X_i$ and $V_{j,j'} = (U_j \cap U_{j'}) \times_X X_i \to X_i$ are finite étale. Hence $V_j$ and $V_{j,j'}$ are affine schemes and $V_{j,j'} \subset V_j$ is closed as well as open (since $V_{j,j'} \to X_i$ is proper, so Morphisms, Lemma 102.7 applies). Then $V = \bigcup V_j$ is separated because $\mathcal{O}(V_j) \to \mathcal{O}(V_{j,j'})$ is surjective, see Schemes, Lemma 102.8. Thus the previous case applies to $V \to X_i$ and we can further refine the partition if needed (it actually isn’t but we don’t need this). 

In the Noetherian case one can prove the preceding lemma by Noetherian induction and the following amusing lemma.

03S1 \textbf{Lemma 69.2.} Let $f : X \to Y$ be a morphism of schemes which is quasi-compact, quasi-separated, and locally of finite type. If $\eta$ is a generic point of an irreducible component of $Y$ such that $f^{-1}(\eta)$ is finite, then there exists an open $V \subset Y$ containing $\eta$ such that $f^{-1}(V) \to V$ is finite.

\textbf{Proof.} This is Morphisms, Lemma 102.1. 

The statement of the following lemma can be strengthened a bit.

095K \textbf{Lemma 69.3.} Let $f : Y \to X$ be a quasi-finite and finitely presented morphism of affine schemes.

1. There exists a surjective morphism of affine schemes $X' \to X$ and a closed subscheme $Z' \subset Y' = X' \times_X Y$ such that
   a. $Z' \subset Y'$ is a thickening, and
   b. $Z' \to X'$ is a finite étale morphism.

2. There exists a surjective morphism of affine schemes $X'' \to X$ and a closed subscheme $W' \subset Y'' = X'' \times_X Y$ such that
   a. $W' \subset Y''$ is a thickening, and
   b. $W' \to X''$ is a finite étale morphism.
(2) There exists a finite partition \( X = \coprod X_i \) by locally closed, constructible, affine strata, and surjective finite locally free morphisms \( X'_i \to X_i \) such that the reduction of \( Y'_i = X'_i \times_X Y \to X'_i \) is isomorphic to \( \coprod_{j=1}^{n_i} (X'_i)_{\text{red}} \to (X'_i)_{\text{red}} \) for some \( n_i \).

Proof. Setting \( X' = \coprod X'_i \) we see that (2) implies (1). Write \( X = \text{Spec}(A) \) and \( Y = \text{Spec}(B) \). Write \( A \) as a filtered colimit of finite type \( \mathbf{Z} \)-algebras \( A_i \). Since \( B \) is an \( A \)-algebra of finite presentation, we see that there exists \( 0 \in I \) and a finite type ring map \( A_0 \to B_0 \) such that \( B = \operatorname{colim} B_i \) with \( B_i = A_i \otimes_{A_0} B_0 \), see Algebra, Lemma \( 126.6 \). For \( i \) sufficiently large we see that \( A_i \to B_i \) is quasi-finite, see Limits, Lemma \( 44.2 \). Thus we reduce to the case of finite type algebras over \( \mathbf{Z} \), in particular we reduce to the Noetherian case. (Details omitted.)

Assume \( X \) and \( Y \) Noetherian. In this case any locally closed subset of \( X \) is constructible. By Lemma \( 69.2 \) and Noetherian induction we see that there is a finite partition \( X = \coprod X_i \) of \( X \) by locally closed strata such that \( Y \times_X X_i \to X_i \) is finite. We can refine this partition to get affine strata. Thus after replacing \( X \) by \( X' = \coprod X_i \), we may assume \( Y \to X \) is finite.

Assume \( X \) and \( Y \) Noetherian and \( Y \to X \) finite. Suppose that we can prove (2) after base change by a surjective, flat, quasi-finite morphism \( U \to X \). Thus we have a partition \( U = \coprod U_i \) and finite locally free morphisms \( U'_i \to U_i \) such that \( U'_i \times_X Y \to U'_i \) is isomorphic to \( \coprod_{j=1}^{n_i} (U'_i)_{\text{red}} \to (U'_i)_{\text{red}} \) for some \( n_i \). Then, by the argument in the previous paragraph, we can find a partition \( X = \coprod X_j \) with locally closed affine strata such that \( X_j \times_X U_i \to X_j \) is finite for all \( i, j \). By Morphisms, Lemma \( 45.2 \) each \( X_j \times_X U_i \to X_j \) is finite locally free. Hence \( X_j \times_X U'_i \to X_j \) is finite locally free (Morphisms, Lemma \( 45.3 \)). It follows that \( X = \coprod X_j \) and \( X'_j = \coprod X_j \times_X U'_i \) is a solution for \( Y \to X \). Thus it suffices to prove the result (in the Noetherian case) after a surjective flat quasi-finite base change.

Applying Morphisms, Lemma \( 45.6 \) we see we may assume that \( Y \) is a closed subscheme of an affine scheme \( Z \) which is (set theoretically) a finite union \( Z = \bigcup_{i \in I} Z_i \) of closed subschemes mapping isomorphically to \( X \). In this case we will find a finite partition of \( X = \coprod X_j \) with affine locally closed strata that works (in other words \( X'_j = X_j \)). Set \( T_i = Y \cap Z_i \). This is a closed subscheme of \( X \). As \( X \) is Noetherian we can find a finite partition of \( X = \coprod X_j \) by affine locally closed subschemes, such that each \( X_j \times_X T_i \) is (set theoretically) a union of strata \( X_j \times_X Z_i \). Replacing \( X \) by \( X_j \) we see that we may assume \( I = I_1 \coprod I_2 \) with \( Z_i \subset Y \) for \( i \in I_1 \) and \( Z_i \cap Y = \emptyset \) for \( i \in I_2 \). Replacing \( Z \) by \( \bigcup_{i \in I_1} Z_i \) we see that we may assume \( Y = Z \). Finally, we can replace \( X \) again by the members of a partition as above such that for every \( i, i' \subset I \) the intersection \( Z_i \cap Z_{i'} \) is either empty or (set theoretically) equal to \( Z_i \) and \( Z_{i'} \). This clearly means that \( Y \) is (set theoretically) equal to a disjoint union of the \( Z_i \) which is what we wanted to show.

\[ \Box \]

70. More on constructible sheaves

095M Let \( \Lambda \) be a Noetherian ring. Let \( X \) be a scheme. We often consider \( X_{\text{étale}} \) as a ringed site with sheaf of rings \( \Lambda \). In case of abelian sheaves we often take \( \Lambda = \mathbf{Z}/n\mathbf{Z} \) for a suitable integer \( n \).

03S8 Lemma 70.1. Let \( j : U \to X \) be an étale morphism of quasi-compact and quasi-separated schemes.
(1) The sheaf $h_U$ is a constructible sheaf of sets.

(2) The sheaf $j_!M$ is a constructible abelian sheaf for a finite abelian group $M$.

(3) If $\Lambda$ is a Noetherian ring and $M$ is a finite $\Lambda$-module, then $j_!M$ is a constructible sheaf of $\Lambda$-modules on $X_{\acute{e}tale}$.

**Proof.** By Lemma 69.1 there is a partition $\coprod_i X_i$ such that $\pi_i : j^{-1}(X_i) \to X_i$ is finite étale. The restriction of $h_U$ to $X_i$ is $h_{j^{-1}(X_i)}$ which is finite locally constant by Lemma 67.4. For cases (2) and (3) we note that $j_!(M)|_{X_i} = \pi_!(M) = \pi_*M$ by Lemmas 66.4 and 66.5. Thus it suffices to show the lemma for $\pi : Y \to X$ finite étale. This is Lemma 67.3. □

**Lemma 70.2.** Let $X$ be a quasi-compact and quasi-separated scheme.

(1) Let $\mathcal{F}$ be a sheaf of sets on $X_{\acute{e}tale}$. Then $\mathcal{F}$ is a filtered colimit of constructible sheaves of sets.

(2) Let $\mathcal{F}$ be a torsion abelian sheaf on $X_{\acute{e}tale}$. Then $\mathcal{F}$ is a filtered colimit of constructible abelian sheaves.

(3) Let $\Lambda$ be a Noetherian ring and $\mathcal{F}$ a sheaf of $\Lambda$-modules on $X_{\acute{e}tale}$. Then $\mathcal{F}$ is a filtered colimit of constructible sheaves of $\Lambda$-modules.

**Proof.** Let $\mathcal{B}$ be the collection of quasi-compact and quasi-separated objects of $X_{\acute{e}tale}$. By Modules on Sites, Lemma 29.6 any sheaf of sets is a filtered colimit of sheaves of the form

$$\text{Coequalizer} \left( \coprod_{j=1,\ldots,m} h_{V_j} \rightrightarrows \coprod_{i=1,\ldots,n} j_{U_i} \right)$$

with $V_j$ and $U_i$ quasi-compact and quasi-separated objects of $X_{\acute{e}tale}$. By Lemmas 70.1 and 68.6 these coequalizers are constructible. This proves (1).

Let $\Lambda$ be a Noetherian ring. By Modules on Sites, Lemma 29.6 $\Lambda$-modules $\mathcal{F}$ is a filtered colimit of modules of the form

$$\text{Coker} \left( \bigoplus_{j=1,\ldots,m} j_{V_j!}\Lambda V_j \longrightarrow \bigoplus_{i=1,\ldots,n} j_{U_i!}\Lambda U_i \right)$$

with $V_j$ and $U_i$ quasi-compact and quasi-separated objects of $X_{\acute{e}tale}$. By Lemmas 70.1 and 68.6 these cokernels are constructible. This proves (3).

Proof of (2). First write $\mathcal{F} = \bigcup \mathcal{F}[n]$ where $\mathcal{F}[n]$ is the $n$-torsion subsheaf. Then we can view $\mathcal{F}[n]$ as a sheaf of $\mathbb{Z}/n\mathbb{Z}$-modules and apply (3). □

**Lemma 70.3.** Let $f : X \to Y$ be a surjective morphism of quasi-compact and quasi-separated schemes.

(1) Let $\mathcal{F}$ be a sheaf of sets on $Y_{\acute{e}tale}$. Then $\mathcal{F}$ is constructible if and only if $f^{-1}\mathcal{F}$ is constructible.

(2) Let $\mathcal{F}$ be an abelian sheaf on $Y_{\acute{e}tale}$. Then $\mathcal{F}$ is constructible if and only if $f^{-1}\mathcal{F}$ is constructible.

(3) Let $\Lambda$ be a Noetherian ring. Let $\mathcal{F}$ be sheaf of $\Lambda$-modules on $Y_{\acute{e}tale}$. Then $\mathcal{F}$ is constructible if and only if $f^{-1}\mathcal{F}$ is constructible.

**Proof.** One implication follows from Lemma 68.5. For the converse, assume $f^{-1}\mathcal{F}$ is constructible. Write $\mathcal{F} = \text{colim} \mathcal{F}_i$ as a filtered colimit of constructible sheaves (of sets, abelian groups, or modules) using Lemma 70.2. Since $f^{-1}$ is a left adjoint it commutes with colimits (Categories, Lemma 24.4) and we see that $f^{-1}\mathcal{F} =$
colim $f^{-1} \mathcal{F}_i$. By Lemma 68.9 we see that $f^{-1} \mathcal{F}_i \to f^{-1} \mathcal{F}$ is surjective for all $i$ large enough. Since $f$ is surjective we conclude (by looking at stalks using Lemma 36.2 and Theorem 29.10) that $\mathcal{F}_i \to \mathcal{F}$ is surjective for all $i$ large enough. Thus $\mathcal{F}$ is the quotient of a constructible sheaf $\mathcal{G}$. Applying the argument once more to $\mathcal{G} \times_F \mathcal{G}$ or the kernel of $\mathcal{G} \to \mathcal{F}$ we conclude using that $f^{-1}$ is exact and that the category of constructible sheaves (of sets, abelian groups, or modules) is preserved under finite (co)limits or (co)kernels inside $\text{Sh}(\mathcal{Y}_{\text{etale}})$, $\text{Sh}(\mathcal{X}_{\text{etale}})$, $\text{Ab}(\mathcal{Y}_{\text{etale}})$, $\text{Ab}(\mathcal{X}_{\text{etale}})$, $\text{Mod}(\mathcal{Y}_{\text{etale}}, \Lambda)$, and $\text{Mod}(\mathcal{X}_{\text{etale}}, \Lambda)$, see Lemma 68.6.

**Lemma 70.4.** Let $f : X \to Y$ be a finite étale morphism of schemes. Let $\Lambda$ be a Noetherian ring. If $\mathcal{F}$ is a constructible sheaf of sets, constructible sheaf of abelian groups, or constructible sheaf of $\Lambda$-modules on $X_{\text{etale}}$, the same is true for $f_* \mathcal{F}$ on $Y_{\text{etale}}$.

**Proof.** By Lemma 68.4 it suffices to check this Zariski locally on $Y$ and by Lemma 70.3 we may replace $Y$ by an étale cover (the construction of $f_*$ commutes with étale localization). A finite étale morphism is étale locally isomorphic to a disjoint union of isomorphisms, see Étale Morphisms, Lemma 18.3. Thus, in the case of sheaves of sets, the lemma says that if $\mathcal{F}_i$, $i = 1, \ldots, n$ are constructible sheaves of sets, then $\prod_{i=1}^n \mathcal{F}_i$ is too. This is clear. Similarly for sheaves of abelian groups and modules.

**Lemma 70.5.** Let $X$ be a quasi-compact and quasi-separated scheme. The category of constructible sheaves of sets is the full subcategory of $\text{Sh}(X_{\text{etale}})$ consisting of sheaves $\mathcal{F}$ which are coequalizers

$$\mathcal{F}_1 \rightrightarrows \mathcal{F}_0 \to \mathcal{F}$$

such that $\mathcal{F}_i$, $i = 0, 1$ is a finite coproduct of sheaves of the form $h_U$ with $U$ a quasi-compact and quasi-separated object of $X_{\text{etale}}$.

**Proof.** In the proof of Lemma 70.2 we have seen that sheaves of this form are constructible. For the converse, suppose that for every constructible sheaf of sets $\mathcal{F}$ we can find a surjection $\mathcal{F}_0 \to \mathcal{F}$ with $\mathcal{F}_0$ as in the lemma. Then we find our surjection $\mathcal{F}_1 \to \mathcal{F}_0 \times_F \mathcal{F}_0$ because the latter is constructible by Lemma 68.6.

By Topology, Lemma 27.7 we may choose a finite stratification $X = \bigsqcup_{i \in I} X_i$ such that $\mathcal{F}$ is finite locally constant on each stratum. We will prove the result by induction on the cardinality of $I$. Let $i \in I$ be a minimal element in the partial ordering of $I$. Then $X_i \subset X$ is closed. By induction, there exist finitely many quasi-compact and quasi-separated objects $U_\alpha$ of $(X \setminus X_i)_{\text{etale}}$ and a surjective map $\coprod h_{U_\alpha} \to \mathcal{F}|_{X \setminus X_i}$. These determine a map

$$\coprod h_{U_\alpha} \to \mathcal{F}$$

which is surjective after restricting to $X \setminus X_i$. By Lemma 67.4 we see that $\mathcal{F}|_{X_i} = h_V$ for some scheme $V$ finite étale over $X_i$. Let $\overline{v}$ be a geometric point of $V$ lying over $v \in X_i$. We may think of $\overline{v}$ as an element of the stalk $\mathcal{F}_{\overline{v}} = V_{\overline{v}}$. Thus we can find an étale neighbourhood $(U, \overline{v})$ of $\overline{v}$ and a section $s \in \mathcal{F}(U)$ whose stalk at $\overline{v}$ gives $\overline{v}$. Thinking of $s$ as a map $s : h_U \to \mathcal{F}$, restricting to $X_i$ we obtain a morphism $s|_{X_i} : U \times_{X_i} X_i \to V$ over $X_i$ which maps $\overline{v}$ to $\overline{v}$. Since $V$ is quasi-compact (finite over the closed subscheme $X_i$ of the quasi-compact scheme $X$) a finite number
\(s^{(1)}, \ldots, s^{(m)}\) of these sections of \(\mathcal{F}\) over \(U^{(1)}, \ldots, U^{(m)}\) will determine a jointly surjective map

\[
\prod s^{(j)}|_{X_i} : \prod U^{(j)} \times_X X_i \to V
\]

Then we obtain the surjection

\[
\prod h_{U_i} \amalg \prod h_{U^{(j)}} \to \mathcal{F}
\]

as desired. \(\square\)

**Lemma 70.6.** Let \(X\) be a quasi-compact and quasi-separated scheme. Let \(\Lambda\) be a Noetherian ring. The category of constructible sheaves of \(\Lambda\)-modules is exactly the category of modules of the form

\[
\text{Coker} \left( \bigoplus_{j=1}^{m} j_{V_j!} \Delta_{V_j} \rightarrow \bigoplus_{i=1}^{n} j_{U_i!} \Delta_{U_i} \right)
\]

with \(V_j\) and \(U_i\) quasi-compact and quasi-separated objects of \(X_{\text{étale}}\). In fact, we can even assume \(U_i\) and \(V_j\) affine.

**Proof.** In the proof of Lemma 70.2 we have seen modules of this form are constructible. Since the category of constructible modules is abelian (Lemma 68.6) it suffices to prove that given a constructible module \(\mathcal{F}\) there is a surjection

\[
\bigoplus_{i=1, \ldots, n} j_{U_i!} \Delta_{U_i} \to \mathcal{F}
\]

for some affine objects \(U_i\) in \(X_{\text{étale}}\). By Modules on Sites, Lemma 29.6 there is a surjection

\[
\Psi : \bigoplus_{i \in I} j_{U_i!} \Delta_{U_i} \to \mathcal{F}
\]

with \(U_i\) affine and the direct sum over a possibly infinite index set \(I\). For every finite subset \(I' \subset I\) set

\[
T_{I'} = \text{Supp}(\text{Coker} \left( \bigoplus_{i \in I'} j_{U_i!} \Delta_{U_i} \rightarrow \mathcal{F} \right))
\]

By the very definition of constructible sheaves, the set \(T_{I'}\) is a constructible subset of \(X\). We want to show that \(T_{I'} = \emptyset\) for some \(I'\). Since every stalk \(\mathcal{F}_x\) is a finite type \(\Lambda\)-module and since \(\Psi\) is surjective, for every \(x \in X\) there is an \(I'\) such that \(x \notin T_{I'}\). In other words we have \(\emptyset = \bigcap_{I' \subset I\text{ finite}} T_{I'}\). Since \(X\) is a spectral space by Properties, Lemma 2.4 the constructible topology on \(X\) is quasi-compact by Topology, Lemma 22.2. Thus \(T_{I'} = \emptyset\) for some \(I' \subset I\) finite as desired. \(\square\)

**Lemma 70.7.** Let \(X\) be a quasi-compact and quasi-separated scheme. The category of constructible abelian sheaves is exactly the category of abelian sheaves of the form

\[
\text{Coker} \left( \bigoplus_{j=1}^{m} j_{V_j!} \mathbb{Z}/m_j \mathbb{Z}_{V_j} \rightarrow \bigoplus_{i=1}^{n} j_{U_i!} \mathbb{Z}/n_i \mathbb{Z}_{U_i} \right)
\]

with \(V_j\) and \(U_i\) quasi-compact and quasi-separated objects of \(X_{\text{étale}}\) and \(m_j, n_i\) positive integers. In fact, we can even assume \(U_i\) and \(V_j\) affine.

**Proof.** This follows from Lemma 70.6 applied with \(\Lambda = \mathbb{Z}/n \mathbb{Z}\) and the fact that, since \(X\) is quasi-compact, every constructible abelian sheaf is annihilated by some positive integer \(n\) (details omitted). \(\square\)
09YU Lemma 70.8. Let $X$ be a quasi-compact and quasi-separated scheme. Let $\Lambda$ be a Noetherian ring. Let $\mathcal{F}$ be a constructible sheaf of sets, abelian groups, or $\Lambda$-modules on $X_{\text{\acute{e}tale}}$. Let $\mathcal{G} = \text{colim} \mathcal{G}_i$ be a filtered colimit of sheaves of sets, abelian groups, or $\Lambda$-modules. Then
\[ \text{Mor}(\mathcal{F}, \mathcal{G}) = \text{colim} \text{Mor}(\mathcal{F}, \mathcal{G}_i) \]
in the category of sheaves of sets, abelian groups, or $\Lambda$-modules on $X_{\text{\acute{e}tale}}$.

Proof. The case of sheaves of sets. By Lemma 70.5 it suffices to prove the lemma for $h_U$ where $U$ is a quasi-compact and quasi-separated object of $X_{\text{\acute{e}tale}}$. Recall that $\text{Mor}(h_U, \mathcal{G}) = \mathcal{G}(U)$. Hence the result follows from Sites, Lemma 11.2.

In the case of abelian sheaves or sheaves of modules, the result follows in the same way using Lemmas 70.7 and 70.6. For the case of abelian sheaves, we add that $\text{Mor}(j_U \mathbb{Z}/n \mathbb{Z}, \mathcal{G})$ is equal to the $n$-torsion elements of $\mathcal{G}(U)$. □

09YU Lemma 70.9. Let $f : X \to Y$ be a finite and finitely presented morphism of schemes. Let $\Lambda$ be a Noetherian ring. If $\mathcal{F}$ is a constructible sheaf of sets, abelian groups, or $\Lambda$-modules on $X_{\text{\acute{e}tale}}$, then $f_* \mathcal{F}$ is too.

Proof. It suffices to prove this when $X$ and $Y$ are affine by Lemma 68.4. By Lemmas 55.3 and 70.3 we may base change to any affine scheme surjective over $X$. By Lemma 69.3 this reduces us to the case of a finite étale morphism (because a thickening leads to an equivalence of étale topoi and even small étale sites, see Theorem 46.2). The finite étale case is Lemma 70.4. □

09YU Lemma 70.10. Let $X = \text{lim}_{i \in I} X_i$ be a limit of a directed system of schemes with affine transition morphisms. We assume that $X_i$ is quasi-compact and quasi-separated for all $i \in I$.

(1) The category of constructible sheaves of sets on $X_{\text{\acute{e}tale}}$ is the colimit of the categories of constructible sheaves of sets on $(X_i)_{\text{\acute{e}tale}}$.

(2) The category of constructible abelian sheaves on $X_{\text{\acute{e}tale}}$ is the colimit of the categories of constructible abelian sheaves on $(X_i)_{\text{\acute{e}tale}}$.

(3) Let $\Lambda$ be a Noetherian ring. The category of constructible sheaves of $\Lambda$-modules on $X_{\text{\acute{e}tale}}$ is the colimit of the categories of constructible sheaves of $\Lambda$-modules on $(X_i)_{\text{\acute{e}tale}}$.

Proof. Proof of (1). Denote $f_i : X \to X_i$ the projection maps. There are 3 parts to the proof corresponding to “faithful”, “fully faithful”, and “essentially surjective”.

Faithful. Choose $0 \in I$ and let $\mathcal{F}_0, \mathcal{G}_0$ be constructible sheaves on $X_0$. Suppose that $a, b : \mathcal{F}_0 \to \mathcal{G}_0$ are maps such that $f_0^{-1}a = f_0^{-1}b$. Let $E \subset X_0$ be the set of points $x \in X_0$ such that $a_x = b_x$. By Lemma 68.8 the subset $E \subset X_0$ is constructible. By assumption $X \to X_0$ maps into $E$. By Limits, Lemma 3.7 we find an $i \geq 0$ such that $X_i \to X_0$ maps into $E$. Hence $f_{i0}^{-1}a = f_{i0}^{-1}b$.

Fully faithful. Choose $0 \in I$ and let $\mathcal{F}_0, \mathcal{G}_0$ be constructible sheaves on $X_0$. Suppose that $a : f_0^{-1} \mathcal{F}_0 \to f_0^{-1} \mathcal{G}_0$ is a map. We claim there is an $i$ and a map $a_i : f_{i0}^{-1} \mathcal{F}_0 \to f_{i0}^{-1} \mathcal{G}_0$ which pulls back to $a$ on $X$. By Lemma 70.5 we can replace $\mathcal{F}_0$ by a finite coproduct of sheaves represented by quasi-compact and quasi-separated objects of $(X_0)_{\text{\acute{e}tale}}$. Thus we have to show: If $U_0 \to X_0$ is such an object of $(X_0)_{\text{\acute{e}tale}}$, then
\[ f_0^{-1} \mathcal{G}(U) = \text{colim}_{i \geq 0} f_{i0}^{-1} \mathcal{G}(U_i) \]
where $U = X \times_{X_0} U_0$ and $U_i = X_i \times_{X_0} U_0$. This is a special case of Theorem 52.1. Essentially surjective. We have to show every constructible $\mathcal{F}$ on $X$ is isomorphic to $f_i^{-1}\mathcal{F}$ for some constructible $\mathcal{F}_i$ on $X_i$. Applying Lemma 70.5 and using the results of the previous two paragraphs, we see that it suffices to prove this for $h_U$ for some quasi-compact and quasi-separated object $U$ of $\mathcal{X}_{\text{étale}}$. In this case we have to show that $U$ is the base change of a quasi-compact and quasi-separated scheme étale over $X_i$ for some $i$. This follows from Limits, Lemmas 9.1 and 7.8.

Proof of (3). The argument is very similar to the argument for sheaves of sets, but using Lemma 70.6 instead of Lemma 70.5. Details omitted. Part (2) follows from part (3) because every constructible abelian sheaf over a quasi-compact scheme is a constructible sheaf of $\mathbb{Z}/n\mathbb{Z}$-modules for some $n$. □

Lemma 70.11. Let $X$ be an irreducible scheme with generic point $\eta$.

1. Let $S' \subset S$ be an inclusion of sets. If we have $S' \subset G \subset S$ in $\text{Sh}(\mathcal{X}_{\text{étale}})$ and $S' = \mathcal{G}_{\eta}$, then $G = S'$.

2. Let $A' \subset A$ be an inclusion of abelian groups. If we have $A' \subset G \subset A$ in $\text{Ab}(\mathcal{X}_{\text{étale}})$ and $A' = \mathcal{G}_{\eta}$, then $G = A'$.

3. Let $M' \subset M$ be an inclusion of modules over a ring $\Lambda$. If we have $M' \subset G \subset M$ in $\text{Mod}(\mathcal{X}_{\text{étale}}, \Lambda)$ and $M' = \mathcal{G}_{\eta}$, then $G = M'$.

Proof. This is true because for every étale morphism $U \to X$ with $U \neq \emptyset$ the point $\eta$ is in the image. □

Lemma 70.12. Let $X$ be an integral normal scheme with function field $K$. Let $E$ be a set.

1. Let $g : \text{Spec}(K) \to X$ be the inclusion of the generic point. Then $g_*E = E$.

2. Let $j : U \to X$ be the inclusion of a nonempty open. Then $j_*E = E$.

Proof. Proof of (1). Let $x \in X$ be a point. Let $\mathcal{O}_{X,x}$ be a strict henselization of $\mathcal{O}_{X,x}$. By More on Algebra, Lemma 36.6 we see that $\mathcal{O}_{X,x}$ is a normal domain. Hence $\text{Spec}(K) \times_X \text{Spec}(\mathcal{O}_{X,x})$ is irreducible. It follows that the stalk $(g_*E)_x$ is equal to $E$, see Theorem 53.1.

Proof of (2). Since $g$ factors through $j$ there is a map $j_*E \to g_*E$. This map is injective because for every scheme $V$ étale over $X$ the set $\text{Spec}(K) \times_X V$ is dense in $U \times_X V$. On the other hand, we have a map $E \to j_*E$ and we conclude. □

71. Constructible sheaves on Noetherian schemes

If $X$ is a Noetherian scheme then any locally closed subset is a constructible locally closed subset (Topology, Lemma 15.1). Hence an abelian sheaf $\mathcal{F}$ on $\mathcal{X}_{\text{étale}}$ is constructible if and only if there exists a finite partition $X = \bigsqcup X_i$ such that $\mathcal{F}|_{X_i}$ is finite locally constant. (By convention a partition of a topological space has locally closed parts, see Topology, Section 27) In other words, we can omit the adjective “constructible” in Definition 68.1. Actually, the category of constructible sheaves on Noetherian schemes has some additional properties which we will catalogue in this section.

Proposition 71.1. Let $X$ be a Noetherian scheme. Let $\Lambda$ be a Noetherian ring.

1. Any sub or quotient sheaf of a constructible sheaf of sets is constructible.
The category of constructible abelian sheaves on $X_{\text{étale}}$ is a (strong) Serre subcategory of $\text{Ab}(X_{\text{étale}})$. In particular, every sub and quotient sheaf of a constructible abelian sheaf on $X_{\text{étale}}$ is constructible.

(3) The category of constructible sheaves of $\Lambda$-modules on $X_{\text{étale}}$ is a (strong) Serre subcategory of $\text{Mod}(X_{\text{étale}}, \Lambda)$. In particular, every submodule and quotient module of a constructible sheaf of $\Lambda$-modules on $X_{\text{étale}}$ is constructible.

Proof. Proof of (1). Let $\mathcal{G} \subset \mathcal{F}$ with $\mathcal{F}$ a constructible sheaf of sets on $X_{\text{étale}}$. Let $\eta \in X$ be a generic point of an irreducible component of $X$. By Noetherian induction it suffices to find an open neighbourhood $U$ of $\eta$ such that $\mathcal{G}|_U$ is locally constant. To do this we may replace $X$ by an étale neighbourhood of $\eta$. Hence we may assume $\mathcal{F}$ is constant and $X$ is irreducible.

Say $\mathcal{F} = S$ for some finite set $S$. Then $S' = \mathcal{G}_\pi \subset S$ say $S' = \{s_1, \ldots, s_t\}$. Pick an étale neighbourhood $(U, \pi)$ of $\eta$ and sections $\sigma_1, \ldots, \sigma_t \in \mathcal{G}(U)$ which map to $s_i$ in $\mathcal{G}_\pi \subset S$. Since $\sigma_i$ maps to an element $s_i \in S' \subset S = \Gamma(X, \mathcal{F})$ we see that the two pullbacks of $\sigma_i$ to $U \times_X U$ are the same as sections of $\mathcal{G}$. By the sheaf condition for $\mathcal{G}$ we find that $\sigma_i$ comes from a section of $\mathcal{G}$ over the open $\text{Im}(U \to X)$ of $X$. Shrinking $X$ we may assume $S' \subset \mathcal{G} \subset S$. Then we see that $S' = \mathcal{G}$ by Lemma 70.11.

Let $\mathcal{F} \to \mathcal{Q}$ be a surjection with $\mathcal{F}$ a constructible sheaf of sets on $X_{\text{étale}}$. Then set $\mathcal{G} = \mathcal{F} \times_\mathcal{Q} \mathcal{F}$. By the first part of the proof we see that $\mathcal{G}$ is constructible as a subsheaf of $\mathcal{F} \times \mathcal{F}$. This in turn implies that $\mathcal{Q}$ is constructible, see Lemma 68.6.

Proof of (3). We already know that constructible sheaves of modules form a weak Serre subcategory, see Lemma 68.6. Thus it suffices to show the statement on submodules.

Let $\mathcal{G} \subset \mathcal{F}$ be a submodule of a constructible sheaf of $\Lambda$-modules on $X_{\text{étale}}$. Let $\eta \in X$ be a generic point of an irreducible component of $X$. By Noetherian induction it suffices to find an open neighbourhood $U$ of $\eta$ such that $\mathcal{G}|_U$ is locally constant. To do this we may replace $X$ by an étale neighbourhood of $\eta$. Hence we may assume $\mathcal{F}$ is constant and $X$ is irreducible.

Say $\mathcal{F} = \mathcal{M}$ for some finite $\Lambda$-module $\mathcal{M}$. Then $\mathcal{M}' = \mathcal{G}_\pi \subset \mathcal{M}$. Pick finitely many elements $s_1, \ldots, s_t$ generating $\mathcal{M}'$ as a $\Lambda$-module. (This is possible as $\Lambda$ is Noetherian and $\mathcal{M}$ is finite.) Pick an étale neighbourhood $(U, \pi)$ of $\eta$ and sections $\sigma_1, \ldots, \sigma_t \in \mathcal{G}(U)$ which map to $s_i$ in $\mathcal{G}_\pi \subset \mathcal{M}$. Since $\sigma_i$ maps to an element $s_i \in \mathcal{M}' \subset \mathcal{M} = \Gamma(X, \mathcal{F})$ we see that the two pullbacks of $\sigma_i$ to $U \times_X U$ are the same as sections of $\mathcal{G}$. By the sheaf condition for $\mathcal{G}$ we find that $\sigma_i$ comes from a section of $\mathcal{G}$ over the open $\text{Im}(U \to X)$ of $X$. Shrinking $X$ we may assume $\mathcal{M}' \subset \mathcal{G} \subset \mathcal{M}$. Then we see that $\mathcal{M}' = \mathcal{G}$ by Lemma 70.11.

Proof of (2). This follows in the usual manner from (3). Details omitted.

The following lemma tells us that every object of the abelian category of constructible sheaves on $X$ is “Noetherian”, i.e., satisfies a.c.c. for subobjects.
in the category of sheaves of sets, abelian groups, or \(\Lambda\)-modules. If \(\mathcal{F}\) is constructible, then for some \(n\) we have \(\mathcal{F}_n = \mathcal{F}_{n+1} = \mathcal{F}_{n+2} = \ldots\).

**Proof.** By Proposition 71.1 we see that \(\mathcal{F}_i\) and \(\text{colim} \mathcal{F}_i\) are constructible. Then the lemma follows from Lemma 68.9. \(\square\)

**Lemma 71.3.** Let \(X\) be a Noetherian scheme.

1. Let \(\mathcal{F}\) be a constructible sheaf of sets on \(X_{\text{étale}}\). There exist an injective map of sheaves
   \[
   \mathcal{F} \to \coprod_{i=1,\ldots,n} f_i_* E_i
   \]
   where \(f_i : Y_i \to X\) is a finite morphism and \(E_i\) is a finite set.

2. Let \(\mathcal{F}\) be a constructible abelian sheaf on \(X_{\text{étale}}\). There exist an injective map of abelian sheaves
   \[
   \mathcal{F} \to \bigoplus_{i=1,\ldots,n} f_i_* M_i
   \]
   where \(f_i : Y_i \to X\) is a finite morphism and \(M_i\) is a finite abelian group.

3. Let \(\Lambda\) be a Noetherian ring. Let \(\mathcal{F}\) be a constructible sheaf of \(\Lambda\)-modules on \(X_{\text{étale}}\). There exist an injective map of sheaves of modules
   \[
   \mathcal{F} \to \bigoplus_{i=1,\ldots,n} f_i_* M_i
   \]
   where \(f_i : Y_i \to X\) is a finite morphism and \(M_i\) is a finite \(\Lambda\)-module.

Moreover, we may assume each \(Y_i\) is irreducible, reduced, maps onto an irreducible and reduced closed subscheme \(Z_i \subset X\) such that \(Y_i \to Z_i\) is finite étale over a nonempty open of \(Z_i\).

**Proof.** Proof of (1). Because we have the ascending chain condition for subsheaves of \(\mathcal{F}\) (Lemma 71.2), it suffices to show that for every point \(x \in X\) we can find a map \(\varphi : \mathcal{F} \to f_* E\) where \(f : Y \to X\) is finite and \(E\) is a finite set such that \(\varphi_x : \mathcal{F}_x \to (f_* E)_x\) is injective. (This argument can be avoided by picking a partition of \(X\) as in Lemma 68.2 and constructing a \(Y_i \to X\) for each irreducible component of each part.) Let \(Z \subset X\) be the induced reduced scheme structure (Schemes, Definition 12.5) on \(\{x\}\). Since \(\mathcal{F}\) is constructible, there is a finite separable extension \(\kappa(x) \subset \text{Spec}(K)\) such that \(\mathcal{F}|_{\text{Spec}(K)}\) is the constant sheaf with value \(E\) for some finite set \(E\). Let \(Y \to Z\) be the normalization of \(Z\) in \(\text{Spec}(K)\). By Morphisms, Lemma 48.12 we see that \(Y\) is a normal integral scheme. As \(\kappa(x) \subset K\) is finite, it is clear that \(K\) is the function field of \(Y\). Denote \(g : \text{Spec}(K) \to Y\) the inclusion. The map \(\mathcal{F}|_{\text{Spec}(K)} \to E\) is adjoint to a map \(\mathcal{F}|_Y \to g_* E = E\) (Lemma 70.12). This in turn is adjoint to a map \(\varphi : \mathcal{F} \to f_* E\). Observe that the stalk of \(\varphi\) at a geometric point \(\overline{y} \in Y\) of \(x\) and the commutative diagram

\[
\begin{array}{ccc}
\mathcal{F}_x & \xrightarrow{\varphi} & (\mathcal{F}|_Y)_x \\
\downarrow & & \downarrow \\
(f_* E)_x & \xrightarrow{f_* \varphi} & (E)_y
\end{array}
\]

proves the injectivity. We are not yet done, however, as the morphism \(f : Y \to Z\) is integral but in general not finite. \(\square\)

If \(X\) is a Nagata scheme, for example of finite type over a field, then \(Y \to Z\) is finite.
To fix the problem stated in the last sentence of the previous paragraph, we write $Y = \lim_{i \in I} Y_i$ with $Y_i$ irreducible, integral, and finite over $Z$. Namely, apply Properties, Lemma 22.13 to $f_!O_Y$ viewed as a sheaf of $O_Z$-algebras and apply the functor $\text{Spec}_Z$. Then $f_*E = \colim f_{i,*}E$ by Lemma 32.5. By Lemma 70.8 the map $\mathcal{F} \to f_*E$ factors through $f_{i,*}E$ for some $i$. Since $Y_i \to Z$ is a finite morphism of integral schemes and since the function field extension induced by this morphism is finite separable, we see that the morphism is finite étale over a nonempty open of $Z$ (use Algebra, Lemma 138.9; details omitted). This finishes the proof of (1).

The proofs of (2) and (3) are identical to the proof of (1). □

In the following lemma we use a standard trick to reduce a very general statement to the Noetherian case.

09Z7 Lemma 71.4. Let $X$ be a quasi-compact and quasi-separated scheme.

(1) Let $\mathcal{F}$ be a constructible sheaf of sets on $X_{\text{ét}}$. There exist an injective map of sheaves

$$\mathcal{F} \to \coprod_{i=1,\ldots,n} f_{i,*}E_i$$

where $f_i : Y_i \to X$ is a finite and finitely presented morphism and $E_i$ is a finite set.

(2) Let $\mathcal{F}$ be a constructible abelian sheaf on $X_{\text{ét}}$. There exist an injective map of abelian sheaves

$$\mathcal{F} \to \bigoplus_{i=1,\ldots,n} f_{i,*}M_i$$

where $f_i : Y_i \to X$ is a finite and finitely presented morphism and $M_i$ is a finite abelian group.

(3) Let $\Lambda$ be a Noetherian ring. Let $\mathcal{F}$ be a constructible sheaf of $\Lambda$-modules on $X_{\text{ét}}$. There exist an injective map of sheaves of modules

$$\mathcal{F} \to \bigoplus_{i=1,\ldots,n} f_{i,*}M_i$$

where $f_i : Y_i \to X$ is a finite and finitely presented morphism and $M_i$ is a finite $\Lambda$-module.

Proof. We will reduce this lemma to the Noetherian case by absolute Noetherian approximation. Namely, by Limits, Proposition 4.4 we can write $X = \lim_{t \in T} X_t$ with each $X_t$ of finite type over $\text{Spec}(Z)$ and with affine transition morphisms. By Lemma 70.10 the category of constructible sheaves (of sets, abelian groups, or $\Lambda$-modules) on $X_{\text{ét}}$ is the colimit of the corresponding categories for $X_t$. Thus our constructible sheaf $\mathcal{F}$ is the pullback of a similar constructible sheaf $\mathcal{F}_t$ over $X_t$ for some $t$. Then we apply the Noetherian case (Lemma 71.3) to find an injection

$$\mathcal{F}_t \to \coprod_{i=1,\ldots,n} f_{i,*}E_i \quad \text{or} \quad \mathcal{F}_t \to \bigoplus_{i=1,\ldots,n} f_{i,*}M_i$$

over $X_t$ for some finite morphisms $f_i : Y_i \to X_t$. Since $X_t$ is Noetherian the morphisms $f_i$ are of finite presentation. Since pullback is exact and since formation of $f_{i,*}$ commutes with base change (Lemma 55.3), we conclude. □
72. Cohomology with support in a closed subscheme

Let $X$ be a scheme and let $Z \subset X$ be a closed subscheme. Let $\mathcal{F}$ be an abelian sheaf on $X_{\text{étale}}$. We let

$$\Gamma_Z(X, \mathcal{F}) = \{ s \in \mathcal{F}(X) \mid \text{Supp}(s) \subset Z \}$$

be the sections with support in $Z$ (Definition 31.3). This is a left exact functor which is not exact in general. Hence we obtain a derived functor

$$R\Gamma_Z(X, -) : D(X_{\text{étale}}) \to D(Ab)$$

and cohomology groups with support in $Z$ defined by $H^p_Z(X, \mathcal{F}) = R^p\Gamma_Z(X, \mathcal{F})$.

Let $\mathcal{I}$ be an injective abelian sheaf on $X_{\text{étale}}$. Let $U = X \setminus Z$. Then the restriction map $\mathcal{I}(X) \to \mathcal{I}(U)$ is surjective (Cohomology on Sites, Lemma 12.6) with kernel $\Gamma_Z(X, \mathcal{I})$. It immediately follows that for $K \in D(X_{\text{étale}})$ there is a distinguished triangle

$$R\Gamma_Z(X, K) \to R\Gamma(X, K) \to R\Gamma(U, K) \to R\Gamma_Z(X, K)[1]$$

in $D(Ab)$. As a consequence we obtain a long exact cohomology sequence

$$\ldots \to H^p_Z(X, K) \to H^p(X, K) \to H^p(U, K) \to H^{p+1}_Z(X, K) \to \ldots$$

for any $K$ in $D(X_{\text{étale}})$.

For an abelian sheaf $\mathcal{F}$ on $X_{\text{étale}}$ we can consider the subsheaf of sections with support in $Z$, denoted $\mathcal{H}_Z(\mathcal{F})$, defined by the rule

$$\mathcal{H}_Z(\mathcal{F})(U) = \{ s \in \mathcal{F}(U) \mid \text{Supp}(s) \subset U \times_X Z \}$$

Here we use the support of a section from Definition 31.3. Using the equivalence of Proposition 47.4 we may view $\mathcal{H}_Z(\mathcal{F})$ as an abelian sheaf on $Z_{\text{étale}}$. Thus we obtain a functor

$$Ab(X_{\text{étale}}) \to Ab(Z_{\text{étale}}), \quad \mathcal{F} \mapsto \mathcal{H}_Z(\mathcal{F})$$

which is left exact, but in general not exact.

**Lemma 72.1.** Let $i : Z \to X$ be a closed immersion of schemes. Let $\mathcal{I}$ be an injective abelian sheaf on $X_{\text{étale}}$. Then $\mathcal{H}_Z(\mathcal{I})$ is an injective abelian sheaf on $Z_{\text{étale}}$.

**Proof.** Observe that for any abelian sheaf $\mathcal{G}$ on $Z_{\text{étale}}$ we have

$$\text{Hom}_Z(\mathcal{G}, \mathcal{H}_Z(\mathcal{F})) = \text{Hom}_X(i_*\mathcal{G}, \mathcal{F})$$

because after all any section of $i_*\mathcal{G}$ has support in $Z$. Since $i_*$ is exact (Section 47) and as $\mathcal{I}$ is injective on $X_{\text{étale}}$ we conclude that $\mathcal{H}_Z(\mathcal{I})$ is injective on $Z_{\text{étale}}$. □

Denote

$$R\mathcal{H}_Z : D(X_{\text{étale}}) \to D(Z_{\text{étale}})$$

the derived functor. We set $\mathcal{H}^0_Z(\mathcal{F}) = R^0\mathcal{H}_Z(\mathcal{F})$ so that $\mathcal{H}^0_Z(\mathcal{F}) = \mathcal{H}_Z(\mathcal{F})$. By the lemma above we have a Grothendieck spectral sequence

$$E^{p,q}_2 = H^p(Z, \mathcal{H}^q_Z(\mathcal{F})) \Rightarrow H^{p+q}_Z(X, \mathcal{F})$$

**Lemma 72.2.** Let $i : Z \to X$ be a closed immersion of schemes. Let $\mathcal{G}$ be an injective abelian sheaf on $Z_{\text{étale}}$. Then $\mathcal{H}^p_Z(i_*\mathcal{G}) = 0$ for $p > 0$.

**Proof.** This is true because the functor $i_*$ is exact and transforms injective abelian sheaves into injective abelian sheaves (Cohomology on Sites, Lemma 14.2). □
Lemma 72.3. Let $i: Z \to X$ be a closed immersion of schemes. Let $j: U \to X$ be the inclusion of the complement of $Z$. Let $\mathcal{F}$ be an abelian sheaf on $X_{\text{étale}}$. There is a distinguished triangle

$$i_*RH_Z(\mathcal{F}) \to \mathcal{F} \to Rj_*(\mathcal{F}|_U) \to i_*RH_Z(\mathcal{F})[1]$$

in $D(X_{\text{étale}})$. This produces an exact sequence

$$0 \to i_*\mathcal{H}_Z(\mathcal{F}) \to \mathcal{F} \to j_*\mathcal{H}_Z^1(\mathcal{F}) \to 0$$

and isomorphisms $R^pj_*(\mathcal{F}|_U) \cong i_*\mathcal{H}_Z^{p+1}(\mathcal{F})$ for $p \geq 1$.

Proof. To get the distinguished triangle, choose an injective resolution $\mathcal{F} \to \mathcal{I}^\bullet$. Then we can choose a $K$-injective representative $iD\to I$.

To finish the proof we have to show that

$$0 \to i_*\mathcal{H}_Z(\mathcal{I}^\bullet) \to \mathcal{I}^\bullet \to j_*\mathcal{H}_Z(\mathcal{I}^\bullet|_U) \to 0$$

by the discussion above. Thus the distinguished triangle by Derived Categories, Section 12.

Let $X$ be a scheme and let $Z \subset X$ be a closed subscheme. We denote $D_Z(X_{\text{étale}})$ the strictly full saturated triangulated subcategory of $D(X_{\text{étale}})$ consisting of complexes whose cohomology sheaves are supported on $Z$. Note that $D_Z(X_{\text{étale}})$ only depends on the underlying closed subset of $X$.

Lemma 72.4. Let $i: Z \to X$ be a closed immersion of schemes. The map $R^{\text{small},*} = i^{\text{small},*}: D(Z_{\text{étale}}) \to D(X_{\text{étale}})$ induces an equivalence $D(Z_{\text{étale}}) \to D_Z(X_{\text{étale}})$ with quasi-inverse

$$i^{-1}_{\text{small}}|_{D_Z(X_{\text{étale}})} = RH_Z|_{D_Z(X_{\text{étale}})}$$

Proof. Recall that $i^{-1}_{\text{small}}$ and $i^{\text{small},*}$ is an adjoint pair of exact functors such that $i^{\text{small}}i^{\text{small},*}$ is isomorphic to the identify functor on abelian sheaves. See Proposition 17.4 and Lemma 36.2. Thus $i^{\text{small},*}: D(Z_{\text{étale}}) \to D(Z_{\text{étale}})$ is fully faithful and $i^{-1}_{\text{small}}$ determines a left inverse. On the other hand, suppose that $K$ is an object of $D_Z(X_{\text{étale}})$ and consider the adjunction map $K \to i^{\text{small},*}i^{-1}_{\text{small}}K$. Using exactness of $i^{\text{small},*}$ and $i^{-1}_{\text{small}}$ this induces the adjunction maps $H^n(K) \to i^{-1}_{\text{small}}i^{\text{small}}H^n(K)$ on cohomology sheaves. Since these cohomology sheaves are supported on $Z$ we see these adjunction maps are isomorphisms and we conclude that $D(Z_{\text{étale}}) \to D_Z(X_{\text{étale}})$ is an equivalence.

To finish the proof we have to show that $RH_Z(K) = i^{-1}_{\text{small}}K$ if $K$ is an object of $D_Z(X_{\text{étale}})$. To do this we can use that $K = i^{\text{small},*}i^{-1}_{\text{small}}K$ as we’ve just proved this is the case. Then we can choose a $K$-injective representative $\mathcal{I}^\bullet$ for $i^{-1}_{\text{small}}K$. Since $i^{\text{small},*}$ is the right adjoint to the exact functor $i^{-1}_{\text{small}}$, the complex $i^{\text{small},*}\mathcal{I}^\bullet$ is $K$-injective (Derived Categories, Lemma 29.9). We see that $RH_Z(K)$ is computed by $\mathcal{H}_Z(i^{\text{small},*}\mathcal{I}^\bullet) = \mathcal{I}^\bullet$ as desired.

Lemma 72.5. Let $X$ be a scheme. Let $Z \subset X$ be a closed subscheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module and denote $\mathcal{F}^a$ the associated quasi-coherent sheaf on the small étale site of $X$ (Proposition 17.1). Then

1. $H^2_Z(X, \mathcal{F})$ agrees with $H^2_Z(X_{\text{étale}}, \mathcal{F}^a)$,
2. if the complement of $Z$ is retrocompact in $X$, then $i_*\mathcal{H}_Z^p(\mathcal{F}^a)$ is a quasi-coherent sheaf of $\mathcal{O}_X$-modules equal to $(i_*\mathcal{H}^p(\mathcal{F}))^a$. 
Proof. Let $j : U \to X$ be the inclusion of the complement of $Z$. The statement (1) on cohomology groups follows from the long exact sequences for cohomology with supports and the agreements $H^q(X_{\text{ét}}, \mathcal{F}^a) = H^q(X, \mathcal{F})$ and $H^q(U_{\text{ét}}, \mathcal{F}^a) = H^q(U, \mathcal{F})$, see Theorem 22.4 If $j : U \to X$ is a quasi-compact morphism, i.e., if $U \subset X$ is retrocompact, then $R^q j_*$ transforms quasi-coherent sheaves into quasi-coherent sheaves (Cohomology of Schemes, Lemma 4.5) and commutes with taking associated sheaf on étale sites (Descent, Lemma 7.15). We conclude by applying Lemma 72.3.

73. Affine analog of proper base change

In this section we discuss a result by Ofer Gabber, see [Gab94]. This was also proved by Roland Huber, see [Hub93].

Lemma 73.1. Let $X$ be an integral normal scheme with separably closed function field.

(1) A separated étale morphism $U \to X$ is a disjoint union of open immersions.

(2) All local rings of $X$ are strictly henselian.

Proof. Let $R$ be a normal domain whose fraction field is separably algebraically closed. Let $R \to A$ be an étale ring map. Then $A \otimes_R K$ is as a $K$-algebra a finite product $\prod_{i=1,\ldots,n} K$ of copies of $K$. Let $e_i$, $i = 1, \ldots, n$ be the corresponding idempotents of $A \otimes_R K$. Since $A$ is normal (Algebra, Lemma 157.7) the idempotents $e_i$ are in $A$ (Algebra, Lemma 36.12). Hence $A = \prod A e_i$ and we may assume $A \otimes_R K = K$. Since $A \subset A \otimes_R K = K$ (by flatness of $R \to A$ and since $R \subset K$) we conclude that $A$ is a domain. By the same argument we conclude that $A \otimes_R A \subset (A \otimes_R A) \otimes_R K = K$. It follows that the map $A \otimes_R A \to A$ is injective as well as surjective. Thus $R \to A$ defines an open immersion by Morphisms, Lemma 11.2 and Étale Morphisms, Theorem 14.1.

Let $f : U \to X$ be a separated étale morphism. Let $\eta \in X$ be the generic point and let $f^{-1}(\{\eta\}) = \{\xi_i\}_{i \in I}$. The result of the previous paragraph shows the following: For any affine open $U' \subset U$ whose image in $X$ is contained in an affine we have $U' = \coprod_{i \in I} U'_i$ where $U'_i$ is the set of point of $U'$ which are specializations of $\xi_i$. Moreover, the morphism $U'_i \to X$ is an open immersion. It follows that $U_i = \{\xi_i\}$ is an open and closed subscheme of $U$ and that $U_i \to X$ is locally on the source an isomorphism. By Morphisms, Lemma 9.6 the fact that $U_i \to X$ is separated, implies that $U_i \to X$ is injective and we conclude that $U_i \to X$ is an open immersion, i.e., (1) holds.

Part (2) follows from part (1) and the description of the strict henselization of $\mathcal{O}_{X,x}$ as the local ring at $\mathfrak{p}$ on the étale site of $X$ (Lemma 33.1). It can also be proved directly, see Fundamental Groups, Lemma 11.2.

Lemma 73.2. Let $X$ be an affine integral normal scheme with separably closed function field. Let $Z \subset X$ be a closed subscheme. Let $V \to Z$ be an étale morphism with $V$ affine. Then $V$ is a finite disjoint union of open subschemes of $Z$. If $V \to Z$ is surjective and finite étale, then $V \to Z$ has a section.

Proof. By Algebra, Lemma 141.11 we can lift $V$ to an affine scheme $\tilde{U}$ étale over $X$. Apply Lemma 73.1 to $\tilde{U} \to X$ to get the first statement.
The final statement is a consequence of the first. Let \( V = \coprod_{i=1,...,n} V_i \) be a finite decomposition into open and closed subschemes with \( V_i \to Z \) an open immersion. As \( V \to Z \) is finite we see that \( V_i \to Z \) is also closed. Let \( U_i \subset Z \) be the image. Then we have a decomposition into open and closed subschemes

\[
Z = \coprod_{(A,B)} \bigcap_{i \in A} U_i \cap \bigcap_{i \in B} U_i^c
\]

where the disjoint union is over \( \{1, \ldots, n\} = A \amalg B \) where \( A \) has at least one element. Each of the strata is contained in a single \( U_i \) and we find our section. \( \square \)

09ZB **Lemma 73.3.** Let \( X \) be a normal integral affine scheme with with separably closed function field. Let \( Z \subset X \) be a closed subscheme. For any finite abelian group \( M \) we have \( H^1_{\text{ét}}(Z,\underline{M}) = 0 \).

**Proof.** By Cohomology on Sites, Lemma 5.3 an element of \( H^1_{\text{ét}}(Z,\underline{M}) \) corresponds to a \( M \)-torsor \( \mathcal{F} \) on \( Z_{\text{ét}} \). Such a torsor is clearly a finite locally constant sheaf. Hence \( \mathcal{F} \) is representable by a scheme \( V \) finite étale over \( Z \), Lemma 67.4. Of course \( V \to Z \) is surjective as a torsor is locally trivial. Since \( V \to Z \) has a section by Lemma 73.2 we are done. \( \square \)

09ZC **Lemma 73.4.** Let \( X \) be a normal integral affine scheme with with separably closed function field. Let \( Z \subset X \) be a closed subscheme. For any finite abelian group \( M \) we have \( H^q_{\text{ét}}(Z,\underline{M}) = 0 \) for \( q \geq 1 \).

**Proof.** We have seen that the result is true for \( H^1 \) in Lemma 73.3. We will prove the result for \( q \geq 2 \) by induction on \( q \). Let \( \xi \in H^q_{\text{ét}}(Z,\underline{M}) \).

Let \( X = \text{Spec}(R) \). Let \( I \subset R \) be the set of elements \( f \in R \) such that \( \xi|_{Z \cap D(f)} = 0 \). All local rings of \( Z \) are strictly henselian by Lemma 73.1 and Algebra, Lemma 150.16. Hence étale cohomology on \( Z \) or open subschemes of \( Z \) is equal to Zariski cohomology, see Lemma 55.6. In particular \( \xi \) is Zariski locally trivial. It follows that for every prime \( \mathfrak{p} \) of \( R \) there exists an \( f \in I \) with \( f \not\in \mathfrak{p} \). Thus if we can show that \( I \) is an ideal, then \( 1 \in I \) and we’re done. It is clear that \( f \in I \), \( r \in R \) implies \( rf \in I \). Thus we now assume that \( f, g \in I \) and we show that \( f + g \in I \). Note that

\[
D(f + g) \cap Z = D(f(f + g)) \cap Z \cup D(g(f + g)) \cap Z
\]

By Mayer-Vietoris (Cohomology, Lemma 9.2 which applies as étale cohomology on open subschemes of \( Z \) equals Zariski cohomology) we have an exact sequence

\[
\begin{align*}
H^{q-1}_{\text{ét}}(D(fg(f + g)) \cap Z, \underline{M}) \\
H^q_{\text{ét}}(D(f + g) \cap Z, \underline{M}) \\
H^q_{\text{ét}}(D(f(f + g)) \cap Z, \underline{M}) \oplus H^q_{\text{ét}}(D(g(f + g)) \cap Z, \underline{M})
\end{align*}
\]

and the result follows as the first group is zero by induction. \( \square \)

09ZD **Lemma 73.5.** Let \( X \) be an affine scheme.

1. There exists an integral surjective morphism \( X' \to X \) such that for every closed subscheme \( Z' \subset X' \), every finite abelian group \( M \), and every \( q \geq 1 \) we have \( H^q_{\text{ét}}(Z',\underline{M}) = 0 \).
(2) For any closed subscheme $Z \subset X$, finite abelian group $M$, $q \geq 1$, and $\xi \in H^q_{\text{ét}}(Z, M)$ there exists a finite surjective morphism $X' \to X$ of finite presentation such that $\xi$ pulls back to zero in $H^q_{\text{ét}}(X' \times_X Z, M)$.

Proof. Write $X = \text{Spec}(A)$. Write $A = \mathbb{Z}[x_i]/J$ for some ideal $J$. Let $R$ be the integral closure of $\mathbb{Z}[x_i]$ in an algebraic closure of the fraction field of $\mathbb{Z}[x_i]$. Let $A' = R/JR$ and set $X' = \text{Spec}(A')$. This gives an example as in (1) by Lemma 73.4.

Proof of (2). Let $X' \to X$ be the integral surjective morphism we found above. Certainly, $\xi$ maps to zero in $H^q_{\text{ét}}(X' \times_X Z, M)$. We may write $X'$ as a limit $X' = \lim X'_i$ of schemes finite and of finite presentation over $X$; this is easy to do in our current affine case, but it is a special case of the more general Limits, Lemma 6.2. By Lemma 52.3 we see that $\eta$ maps to zero in $H^q_{\text{ét}}(X'_i \times_X Z, M)$ for some $i$ large enough. \hfill \Box

Lemma 73.6. Let $X$ be an affine scheme. Let $F$ be a torsion abelian sheaf on $X_{\text{ét}}$. Let $Z \subset X$ be a closed subscheme. Let $\xi \in H^q_{\text{ét}}(Z, F|_Z)$ for some $q > 0$. Then there exists an injective map $F \to F'$ of torsion abelian sheaves on $X_{\text{ét}}$ such that the image of $\xi$ in $H^q_{\text{ét}}(Z, F|_Z)$ is zero.

Proof. By Lemmas 70.2 and 52.2 we can find a map $\mathcal{G} \to F$ with $\mathcal{G}$ a constructible abelian sheaf and $\xi$ coming from an element $\zeta$ of $H^q_{\text{ét}}(Z, \mathcal{G}|_Z)$. Suppose we can find an injective map $\mathcal{G} \to \mathcal{G}'$ of torsion abelian sheaves on $X_{\text{ét}}$ such that the image of $\zeta$ in $H^q_{\text{ét}}(Z, \mathcal{G}'|_Z)$ is zero. Then we can take $F'$ to be the pushout

$$F' = \mathcal{G}' \amalg_{\mathcal{G}} F$$

and we conclude the result of the lemma holds. (Observe that restriction to $Z$ is exact, so commutes with finite limits and colimits and moreover it commutes with arbitrary colimits as a left adjoint to pushforward.) Thus we may assume $F$ is constructible.

Assume $F$ is constructible. By Lemma 71.4 it suffices to prove the result when $F$ is of the form $f_*M$ where $M$ is a finite abelian group and $f : Y \to X$ is a finite morphism of finite presentation (such sheaves are still constructible by Lemma 70.9 but we won’t need this). Since formation of $f_*$ commutes with any base change (Lemma 55.3) we see that the restriction of $f_*M$ to $Z$ is equal to the pushforward of $M$ via $Y \times_X Z \to Z$. By the Leray spectral sequence (Proposition 54.2) and vanishing of higher direct images (Proposition 55.2), we find

$$H^q_{\text{ét}}(Z, f_*M|_Z) = H^q_{\text{ét}}(Y \times_X Z, M).$$

By Lemma 73.5 we can find a finite surjective morphism $Y' \to Y$ of finite presentation such that $\xi$ maps to zero in $H^q(Y' \times_X Z, M)$. Denoting $f' : Y' \to X$ the composition $Y' \to Y \to X$ we claim the map

$$f_*M \to f'_*M$$

is injective which finishes the proof by what was said above. To see the desired injectivity we can look at stalks. Namely, if $\mathfrak{f} : \text{Spec}(k) \to X$ is a geometric point, then

$$(f_*M)_\mathfrak{f} = \bigoplus_{f(\mathfrak{f}) = \mathfrak{f}} M$$
by Proposition 55.2 and similarly for the other sheaf. Since \( Y' \to Y \) is surjective and finite we see that the induced map on geometric points lifting \( \mathfrak{f} \) is surjective too and we conclude. □

The lemma above will take care of higher cohomology groups in Gabber’s result. The following lemma will be used to deal with global sections.

**Lemma 73.7.** Let \( X \) be a quasi-compact and quasi-separated scheme. Let \( i : Z \to X \) be a closed immersion. Assume that

1. for any sheaf \( \mathcal{F} \) on \( X_{\mathrm{Zar}} \) the map \( \Gamma(X, \mathcal{F}) \to \Gamma(Z, i^{-1} \mathcal{F}) \) is bijective, and
2. for any finite morphism \( X' \to X \) assumption (1) holds for \( Z \times_X X' \to X' \).

Then for any sheaf \( \mathcal{F} \) on \( X_{\mathrm{etale}} \) we have \( \Gamma(X, \mathcal{F}) = \Gamma(Z, i^{-1}_{\mathrm{sm}} \mathcal{F}) \).

**Proof.** Let \( \mathcal{F} \) be a sheaf on \( X_{\mathrm{etale}} \). There is a canonical (base change) map

\[
i^{-1}(\mathcal{F}|_{X_{\mathrm{Zar}}}) \to (i^{-1}_{\mathrm{sm}} \mathcal{F})|_{Z_{\mathrm{Zar}}}
\]

of sheaves on \( Z_{\mathrm{Zar}} \). This map is injective as can be seen by looking on stalks. The stalk on the left hand side at \( z \in Z \) is the stalk of \( \mathcal{F}|_{X_{\mathrm{Zar}}} \) at \( z \). The stalk on the right hand side is the colimit over all elementary \( \mathrm{etale} \) neighbourhoods \((U, u) \to (X, z)\) such that \( U \times_X Z \to Z \) has a section over a neighbourhood of \( z \). As \( \mathrm{etale} \) morphisms are open, the image of \( U \to X \) is an open neighbourhood of \( z \) in \( X \) and injectivity follows.

It follows from this and assumption (1) that the map \( \Gamma(X, \mathcal{F}) \to \Gamma(Z, i^{-1}_{\mathrm{sm}} \mathcal{F}) \) is injective. By (2) the same thing is true on all \( X' \) finite over \( X \).

Let \( s \in \Gamma(Z, i^{-1}_{\mathrm{sm}} \mathcal{F}) \). By construction of \( i^{-1}_{\mathrm{sm}} \mathcal{F} \) there exists an \( \mathrm{etale} \) covering \( \{V_j \to Z\} \), \( \mathrm{etale} \) morphisms \( U_j \to X \), sections \( s_j \in \mathcal{F}(U_j) \) and morphisms \( V_j \to U_j \) over \( X \) such that \( s|_{V_j} \) is the pullback of \( s_j \). Observe that every closed subscheme \( T \subset X \) meets \( Z \) by assumption (1) applied to the sheaf \( (T \to X)_Z \) for example. Thus we see that \( \bigsqcup U_j \to X \) is surjective. By More on Morphisms, Lemma 31.13 we can find a finite surjective morphism \( X' \to X \) such that \( X' \to X \) Zariski locally factors through \( \bigsqcup U_j \to X \). It follows that \( s|_Z \), Zariski locally comes from a section of \( \mathcal{F}|_{X'} \). In other words, \( s|_Z \) comes from \( t' \in \Gamma(X', \mathcal{F}|_{X'}) \) by assumption (2). By injectivity we conclude that the two pullbacks of \( t' \) to \( X' \times_X X' \) are the same (after all this is true for the pullbacks of \( s \) to \( Z' \times_Z Z' \)). Hence we conclude \( t' \) comes from a section of \( \mathcal{F} \) over \( X \) by Remark 55.5. □

**Lemma 73.8.** Let \( X \) be a topological space and let \( Z \subset X \) be a closed subset. Suppose that for every \( x \in X \) the intersection \( Z \cap \{x\} \) is connected (in particular nonempty). Then for any sheaf \( \mathcal{F} \) on \( X \) we have \( \Gamma(X, \mathcal{F}) = \Gamma(Z, \mathcal{F}|_Z) \).

**Proof.** Let’s view a global section of \( \mathcal{F} \) as an assignment \( x \mapsto s_x \in \mathcal{F}_x \) satisfying the continuity property \((*)\) introduced in Sheaves, Section 17. If \( x \to z \) is a specialization on \( X \), then there is a corresponding map on stalks \( \mathcal{F}_z \to \mathcal{F}_x \). Thus, given a global section \( s = (s_z)_{z \in Z} \) of \( \mathcal{F}|_Z \) we can assign to every \( x \in X \) a value \( s_x \) by choosing a \( z \in Z \cap \{x\} \) and taking the image of \( s_z \). The fact that \( s_x \) is independent of the choice of \( z \) comes from the fact that we assumed \( Z \cap \{x\} \) is connected (details omitted). It is clear that this rule satisfies \((*)\) and provides us with a section \( \tilde{s} \) of \( \mathcal{F} \) over \( X \) which restricts to \( s \). □

**Lemma 73.9.** Let \((A, I)\) be a henselian pair. Set \( X = \text{Spec}(A) \) and \( Z = \text{Spec}(A/I) \). For any sheaf \( \mathcal{F} \) on \( X_{\mathrm{etale}} \) we have \( \Gamma(X, \mathcal{F}) = \Gamma(Z, \mathcal{F}|_Z) \).
Proof. Combine Lemmas 73.7 and 73.8 and More on Algebra, Lemmas 8.9 and 8.11.

Finally, we can state and prove Gabber’s theorem.

**Theorem 73.10 (Gabber).** Let \((A, I)\) be a henselian pair. Set \(X = \text{Spec}(A)\) and \(Z = \text{Spec}(A/I)\). For any torsion abelian sheaf \(F\) on \(X_{\text{étale}}\) we have \(H^q_{\text{étale}}(X, F) = H^q_{\text{étale}}(Z, F|_Z)\).

**Proof.** The result holds for \(q = 0\) by Lemma 73.9. Let \(q \geq 1\). Suppose the result has been shown in all degrees \(< q\). Let \(F\) be a torsion abelian sheaf. Let \(F \to F'\) be an injective map of torsion abelian sheaves (to be chosen later) with cokernel \(Q\) so that we have the short exact sequence

\[
0 \to F \to F' \to Q \to 0
\]

of torsion abelian sheaves on \(X_{\text{étale}}\). This gives a map of long exact cohomology sequences over \(X\) and \(Z\) part of which looks like

\[
\begin{array}{ccccccc}
\ldots & H^{q-1}_{\text{étale}}(X, F') & \longrightarrow & H^q_{\text{étale}}(X, Q) & \longrightarrow & H^q_{\text{étale}}(X, F) & \longrightarrow & H^{q+1}_{\text{étale}}(X, F') \\
& \downarrow & & \downarrow & & \downarrow & & \\
& H^{q-1}_{\text{étale}}(Z, F'|_Z) & \longrightarrow & H^q_{\text{étale}}(Z, Q|_Z) & \longrightarrow & H^q_{\text{étale}}(Z, F|_Z) & \longrightarrow & H^{q+1}_{\text{étale}}(Z, F'|_Z)
\end{array}
\]

Using this commutative diagram of abelian groups with exact rows we will finish the proof.

Injectivity for \(F\). Let \(\xi\) be a nonzero element of \(H^q_{\text{étale}}(X, F)\). By Lemma 73.6 applied with \(Z = X\) (!) we can find \(F \subset F'\) such that \(\xi\) maps to zero to the right. Then \(\xi\) is the image of an element of \(H^{q-1}_{\text{étale}}(X, Q)\) and bijectivity for \(q-1\) implies \(\xi\) does not map to zero in \(H^{q-1}_{\text{étale}}(Z, F|_Z)\).

Surjectivity for \(F\). Let \(\xi\) be an element of \(H^q_{\text{étale}}(Z, F|_Z)\). By Lemma 73.6 applied with \(Z = Z\) we can find \(F \subset F'\) such that \(\xi\) maps to zero to the right. Then \(\xi\) is the image of an element of \(H^{q-1}_{\text{étale}}(Z, Q|_Z)\) and bijectivity for \(q-1\) implies \(\xi\) is in the image of the vertical map. \(\square\)

**Lemma 73.11.** Let \(X\) be a scheme with affine diagonal which can be covered by \(n + 1\) affine opens. Let \(Z \subset X\) be a closed subscheme. Let \(A\) be a torsion sheaf of rings on \(X_{\text{étale}}\) and let \(I\) be an injective sheaf of \(A\)-modules on \(X_{\text{étale}}\). Then \(H^q_{\text{étale}}(Z, I|_Z) = 0\) for \(q > n\).

**Proof.** We will prove this by induction on \(n\). If \(n = 0\), then \(X\) is affine. Say \(X = \text{Spec}(A)\) and \(Z = \text{Spec}(A/I)\). Let \(A^h\) be the filtered colimit of étale \(A\)-algebras \(B\) such that \(A/I \to B/IB\) is an isomorphism. Then \((A^h, IA^h)\) is a henselian pair and \(A/I = A^h/IA^h\), see More on Algebra, Lemma 8.13 and its proof. Set \(X^h = \text{Spec}(A^h)\). By Theorem 73.10 we see that

\[
H^q_{\text{étale}}(Z, I|_Z) = H^q_{\text{étale}}(X^h, I|_{X^h})
\]

By Theorem 52.1 we have

\[
H^q_{\text{étale}}(X^h, I|_{X^h}) = \colim_{A \to B} H^q_{\text{étale}}(\text{Spec}(B), I|_{\text{Spec}(B)})
\]

where the colimit is over the \(A\)-algebras \(B\) as above. Since the morphisms \(\text{Spec}(B) \to \text{Spec}(A)\) are étale, the restriction \(I|_{\text{Spec}(B)}\) is an injective sheaf of \(A|_{\text{Spec}(B)}\) modules
The goal of this section is to prove Theorem 74.12. The proof uses the “méthode de la trace” as explained in [AGV71, Exposé IX, §5].

Let $f : Y \to X$ be an étale morphism of schemes. There are pairs of adjoint functors $(f_!, f^*)$ and $(f^*, f_!)$ between $Ab(X^\text{étale})$ and $Ab(Y^\text{étale})$. The adjunction map $f_! f^* \to \text{id}$ is called restriction. The adjunction map $f_* f^{-1} = f_! f^* \to \text{id}$ is often called the trace map. If $f$ is finite, then $f_* = f_!$ and we can view this as a map $f_* f^{-1} \to \text{id}$.

74. Cohomology of torsion sheaves on curves

03SB The goal of this section is to prove Theorem 74.12. The proof uses the “méthode de la trace” as explained in [AGV71, Exposé IX, §5].

Let $f : Y \to X$ be an étale morphism of schemes. There are pairs of adjoint functors $(f_!, f^*)$ and $(f^*, f_!)$ between $Ab(X^\text{étale})$ and $Ab(Y^\text{étale})$. The adjunction map $f_! f^* \to \text{id}$ is called restriction. The adjunction map $f_* f^{-1} = f_! f^* \to \text{id}$ is often called the trace map. If $f$ is finite, then $f_* = f_!$ and we can view this as a map $f_* f^{-1} \to \text{id}$.

03SE Definition 74.1. Let $f : Y \to X$ be a finite étale morphism of schemes. The map $f_* f^{-1} \to \text{id}$ described above is called the trace.

Let $f : Y \to X$ be a finite étale morphism. The trace map is characterized by the following two properties:

1. It commutes with étale localization and
2. If $Y = \coprod_{i=1}^d X$ then the trace map is the sum map $f_* f^{-1} F = F^\oplus d \to F$.

It follows that if $f$ has constant degree $d$, then the composition

$$F \xrightarrow{f_*} f_* f^{-1} F \xrightarrow{\text{trace}} F$$

is multiplication by $d$. An example of the “méthode de la trace” is the following observation: If $F$ is an abelian sheaf on $X^\text{étale}$ such that multiplication by $d$ is an isomorphism $F \cong F$, and if furthermore $H^q_\text{étale}(Y, f^{-1} F) = 0$ then $H^q_\text{étale}(X, F) = 0$ as well. Indeed, multiplication by $d$ induces an isomorphism on $H^q_\text{étale}(X, F)$ which factors through $H^q_\text{étale}(Y, f^{-1} F) = 0$. This will be used in the proof of Lemma 74.11 below.

0A52 Situation 74.2. Here $k$ is an algebraically closed field, $X$ is a separated, finite type scheme of dimension $\leq 1$ over $k$, and $F$ is a torsion abelian sheaf on $X^\text{étale}$.

In Situation 74.2 we want to prove the following statements

0A53 (1) $H^q_\text{étale}(X, F) = 0$ for $q > 2$,
0A54 (2) $H^q_\text{étale}(X, F) = 0$ for $q > 1$ if $X$ is affine,
0A55 (3) $H^q_\text{étale}(X, F) = 0$ for $q > 1$ if $p = \text{char}(k) > 0$ and $F$ is $p$-power torsion,
0A56 (4) $H^q_\text{étale}(X, F)$ is finite if $F$ is constructible and torsion prime to $\text{char}(k)$,
0A57 (5) $H^q_\text{étale}(X, F)$ is finite if $X$ is proper and $F$ constructible,
0A58 (6) $H^q_\text{étale}(X, F) \to H^q_\text{étale}(X_{k'}, F|_{X_{k'}})$ is an isomorphism for any extension $k \subset k'$ of algebraically closed fields if $F$ is torsion prime to $\text{char}(k)$,
0A59 (7) $H^q_\text{étale}(X, F) \to H^q_\text{étale}(X_{k'}, F|_{X_{k'}})$ is an isomorphism for any extension $k \subset k'$ of algebraically closed fields if $X$ is proper,
(8) $H^2_{\text{étale}}(X, \mathcal{F}) \to H^2_{\text{étale}}(U, \mathcal{F})$ is surjective for all $U \subset X$ open.

Given any Situation 74.2 we will say that “statements (1) – (8) hold” if those statements that apply to the given situation are true. We start the proof with the following consequence of our computation of cohomology with constant coefficients.

Lemma 74.3. In Situation 74.2 assume $X$ is smooth and $\mathcal{F} = \mathbb{Z}/\ell \mathbb{Z}$ for some prime number $\ell$. Then statements (1) – (8) hold for $\mathcal{F}$.

Proof. Since $X$ is smooth, we see that $X$ is a finite disjoint union of smooth curves. We may assume $X$ is a smooth curve.

Case I: $\ell$ different from the characteristic of $k$. This case follows from Lemma 65.1 (projective case) and Lemma 65.3 (affine case). Statement (6) follows from the fact that $g$ and the number of “punctures” $r$ do not change when passing from $k$ to $k'$. Statement (8) follows as $H^2_{\text{étale}}(U, \mathcal{F})$ is zero as soon as $U \neq X$, because then $U$ is affine (Varieties, Lemmas 32.2 and 32.5).

Case II: $\ell$ is equal to the characteristic of $k$. Vanishing by Lemma 64.4. Statements (5) and (7) follow from Lemma 64.5. □

Remark 74.4 (Invariance under extension of algebraically closed ground field). Let $k$ be an algebraically closed field of characteristic $p > 0$. In Section 64 we have seen that there is an exact sequence

$$k[x] \to k[x] \to H^1_{\text{étale}}(\mathbb{A}^1_k, \mathbb{Z}/p\mathbb{Z}) \to 0$$

where the first arrow maps $f(x)$ to $f^p - f$. A set of representatives for the cokernel is formed by the polynomials

$$\sum_{\nu|n} \lambda_n x^n$$

with $\lambda_n \in k$. (If $k$ is not algebraically closed you have to add some constants to this as well.) In particular when $k' \supset k$ is an algebraically closed overfield, then the map

$$H^1_{\text{étale}}(\mathbb{A}^1_k, \mathbb{Z}/p\mathbb{Z}) \to H^1_{\text{étale}}(\mathbb{A}^1_{k'}, \mathbb{Z}/p\mathbb{Z})$$

is not an isomorphism in general. In particular, the map $\pi_1(\mathbb{A}^1_k) \to \pi_1(\mathbb{A}^1_{k'})$ between étale fundamental groups (insert future reference here) is not an isomorphism either. Thus the étale homotopy type of the affine line depends on the algebraically closed ground field. From Lemma 74.3 above we see that this is a phenomenon which only happens in characteristic $p$ with $p$-power torsion coefficients.

Lemma 74.5. Let $k$ be an algebraically closed field. Let $X$ be a separated finite type scheme over $k$ of dimension $\leq 1$. Let $0 \to \mathcal{F}_1 \to \mathcal{F} \to \mathcal{F}_2 \to 0$ be a short exact sequence of torsion abelian sheaves on $X$. If statements (1) – (8) hold for $\mathcal{F}_1$ and $\mathcal{F}_2$, then they hold for $\mathcal{F}$.

Proof. This is mostly immediate from the definitions and the long exact sequence of cohomology. Also observe that $\mathcal{F}$ is constructible (resp. of torsion prime to the characteristic of $k$) if and only if both $\mathcal{F}_1$ and $\mathcal{F}_2$ are constructible (resp. of torsion prime to the characteristic of $k$). See Proposition 71.1. Some details omitted. □

Lemma 74.6. Let $k$ be an algebraically closed field. Let $f : X \to Y$ be a finite morphism of separated finite type schemes over $k$ of dimension $\leq 1$. Let $\mathcal{F}$ be a
torsion abelian sheaf on \( X \). If statements (1) – (8) hold for \( \mathcal{F} \), then they hold for \( f_*\mathcal{F} \).

**Proof.** Follows from the vanishing of the higher direct images \( R^q f_* \) (Proposition 55.2), the Leray spectral sequence (Proposition 54.2), and the fact that formation of \( f_* \) commutes with arbitrary base change (Lemma 55.3). \( \square \)

**Lemma 74.7.** In Situation 74.2 assume \( X \) is smooth. Let \( j : U \to X \) an open immersion. Let \( \ell \) be a prime number. Let \( \mathcal{F} = j_! \mathbb{Z}/\ell \mathbb{Z} \). Then statements (1) – (8) hold for \( \mathcal{F} \).

**Proof.** Consider the short exact sequence

\[
0 \to j_! \mathbb{Z}/\ell \mathbb{Z} \to \mathbb{Z}/\ell \mathbb{Z}_X \to \bigoplus_{x \in X \setminus U} i_x^* \mathbb{Z}/\ell \mathbb{Z} \to 0.
\]

Statements (1) – (8) hold for \( \mathbb{Z}/\ell \mathbb{Z} \) by Lemma 74.3. Since the inclusion morphisms \( i_x : x \to X \) are finite and since \( x \) is the spectrum of an irreducible curve, we see that \( H^q_{\text{étale}}(X, i_x^* \mathbb{Z}/\ell \mathbb{Z}) \) is zero for \( q > 0 \) and equal to \( \mathbb{Z}/\ell \mathbb{Z} \) for \( q = 0 \). Thus we get from the long exact cohomology sequence

\[
0 \to H^0_{\text{étale}}(X, \mathcal{F}) \to H^0(X, \mathbb{Z}/\ell \mathbb{Z}_X) \to \bigoplus_{x \in X \setminus U} \mathbb{Z}/\ell \mathbb{Z} \to 0 \tag{1}
\]

and \( H^q_{\text{étale}}(X, \mathcal{F}) = H^q_{\text{étale}}(X, \mathbb{Z}/\ell \mathbb{Z}_X) \) for \( q \geq 2 \). Each of the statements (1) – (8) follows by inspection. \( \square \)

**Lemma 74.8.** In Situation 74.2 assume \( X \) reduced. Let \( j : U \to X \) an open immersion. Let \( \ell \) be a prime number and \( \mathcal{F} = j_! \mathbb{Z}/\ell \mathbb{Z} \). Then statements (1) – (8) hold for \( \mathcal{F} \).

**Proof.** The difference with Lemma 74.7 is that here we do not assume \( X \) is smooth. Let \( \nu : X' \to X \) be the normalization morphism which is finite as varieties are Nagata schemes. Let \( j' : U' \to X' \) be the inverse image of \( U \). By Lemma 74.7 the result holds for \( j'_! \mathbb{Z}/\ell \mathbb{Z} \). By Lemma 74.6 the result holds for \( \nu_* j'_! \mathbb{Z}/\ell \mathbb{Z} \). In general it won’t be true that \( \nu_* j'_! \mathbb{Z}/\ell \mathbb{Z} \) is equal to \( j_! \mathbb{Z}/\ell \mathbb{Z} \), but there will be a canonical injective map

\[
j_! \mathbb{Z}/\ell \mathbb{Z} \to \nu_* j'_! \mathbb{Z}/\ell \mathbb{Z}
\]

whose cokernel is of the form \( \bigoplus_{x \in Z} i_x^* M_x \), where \( Z \subset X \) is a finite set of closed points and \( M_x \) is a finite dimensional \( \mathbb{F}_\ell \)-vector space for each \( x \in Z \). We obtain a short exact sequence

\[
0 \to j_! \mathbb{Z}/\ell \mathbb{Z} \to \nu_* j'_! \mathbb{Z}/\ell \mathbb{Z} \to \bigoplus_{x \in Z} i_x^* M_x \to 0
\]

and we can argue exactly as in the proof of Lemma 74.7 to finish the argument. Some details omitted. \( \square \)

**Exercise 74.9.** Let \( f : X \to Y \) be a finite étale morphism with \( X \) and \( Y \) irreducible. Then there exists a finite étale Galois morphism \( X' \to Y \) which dominates \( X \) over \( Y \). If you do not want to do the exercise yourself, then you can observe that this follows by combining Fundamental Groups, Lemmas 5.5 and 5.8.
Lemma 74.10. Let $S$ be an irreducible scheme. Let $\ell$ be a prime number. Let $\mathcal{F}$ be a finite locally constant sheaf of $\mathbf{F}_\ell$-vector spaces on $S_{\text{étale}}$. There exists a finite étale morphism $f : T \to S$ of degree prime to $\ell$ such that $f^{-1}\mathcal{F}$ has a finite filtration whose successive quotients are $\mathbf{Z}/\ell\mathbf{Z}$. 

Proof. Since $\mathcal{F}$ is finite locally constant and $S$ is irreducible, we see that $\mathcal{F}$ has constant rank $r$. Let $T \to S$ be a finite étale covering such that $f^{-1}\mathcal{F}$ is isomorphic to $\mathbf{Z}/\ell\mathbf{Z}^r$. We may assume $T$ is irreducible and $T \to S$ is Galois with group $G$. This means simply that we have $G \subset \text{Aut}(T/S)$ and that $G$ maps isomorphically to the Galois group of the field extension in the generic points. Observe that the action of $G$ on $T$ lifts to an action of $G$ on $f^{-1}\mathcal{F} \cong \mathbf{Z}/\ell\mathbf{Z}^r$. Looking at the stalk in the generic point we obtain a representation $\rho : G \to \text{GL}_r(\mathbf{F}_\ell)$. Let $H \subset G$ be an $\ell$-Sylow subgroup. We claim that $T/H \to S$ works. Namely, since $H$ is a finite $\ell$-group, the irreducible constituents of the representation $\rho|_H$ are each trivial of rank $1$. Moreover the degree of $T/H \to S$ is prime to $\ell$. Some details omitted. \hfill $\square$

Lemma 74.11. In Situation 74.2 assume $X$ reduced. Let $j : U \to X$ an open immersion with $U$ irreducible. Let $\ell$ be a prime number. Let $\mathcal{G}$ be a finite locally constant sheaf of $\mathbf{F}_\ell$-vector spaces on $U$. Let $\mathcal{F} = j_!\mathcal{G}$. Then statements (1) – (8) hold for $\mathcal{F}$.

Proof. Let $f : V \to U$ be a finite étale morphism of degree prime to $\ell$ as in Lemma 74.10. The trace map gives maps $$\mathcal{G} \to f_*f^{-1}\mathcal{G} \to \mathcal{G}$$ whose composition is an isomorphism. Hence it suffices to prove the lemma with $\mathcal{F} = j_!f_*f^{-1}\mathcal{G}$. By Zariski’s Main theorem (More on Morphisms, Lemma [31.3]) we can choose a diagram

$$
\begin{array}{ccc}
V & \xrightarrow{j} & Y \\
f \downarrow & & \downarrow \overline{j} \\
U & \xrightarrow{i} & X
\end{array}
$$

with $\overline{j} : Y \to X$ finite and $j'$ an open immersion with dense image. Since $f$ is finite this implies that $V = U \times_X Y$. Hence $j_!f_*f^{-1}\mathcal{G} = \overline{j}_!j'_!f^{-1}\mathcal{G}$ by Lemma [55.3] By Lemma 74.6 it suffices to prove the lemma for $j'_!f^{-1}\mathcal{G}$. The existence of the filtration given by Lemma 74.10, the fact that $j'_!$ is exact, and Lemma 74.5 reduces us to the case $\mathcal{F} = j'_!\mathbf{Z}/\ell\mathbf{Z}$ which is Lemma 74.8. \hfill $\square$

Theorem 74.12. If $k$ is an algebraically closed field, $X$ is a separated, finite type scheme of dimension $\leq 1$ over $k$, and $\mathcal{F}$ is a torsion abelian sheaf on $X_{\text{étale}}$, then

1. $H^q_{\text{étale}}(X, \mathcal{F}) = 0$ for $q > 2$,
2. $H^q_{\text{étale}}(X, \mathcal{F}) = 0$ for $q > 1$ if $X$ is affine,
3. $H^q_{\text{étale}}(X, \mathcal{F}) = 0$ for $q > 1$ if $p = \text{char}(k) > 0$ and $\mathcal{F}$ is $p$-power torsion,
4. $H^q_{\text{étale}}(X, \mathcal{F})$ is finite if $\mathcal{F}$ is constructible and torsion prime to $\text{char}(k)$,
5. $H^q_{\text{étale}}(X, \mathcal{F})$ is finite if $X$ is proper and $\mathcal{F}$ constructible,
6. $H^q_{\text{étale}}(X, \mathcal{F}) \to H^q_{\text{étale}}(X_{k'}, \mathcal{F}|_{X_{k'}})$ is an isomorphism for any extension $k \subset k'$ of algebraically closed fields if $\mathcal{F}$ is torsion prime to $\text{char}(k)$,
7. $H^q_{\text{étale}}(X, \mathcal{F}) \to H^q_{\text{étale}}(X_{k'}, \mathcal{F}|_{X_{k'}})$ is an isomorphism for any extension $k \subset k'$ of algebraically closed fields if $X$ is proper,
8. $H^2_{\text{étale}}(X, \mathcal{F}) \to H^2_{\text{étale}}(U, \mathcal{F})$ is surjective for all $U \subset X$ open.
The “trace method” is very general. For instance, it applies in Situation 74.2 statements 1 – 8 hold. Our first step is to replace $X$ by its reduction, which is permissible by Proposition 46.4. By Lemma 70.2 we can write $\mathcal{F}$ as a filtered colimit of constructible abelian sheaves. Taking cohomology commutes with colimits, see Lemma 52.2. Moreover, pullback via $X_{k'} \to X$ commutes with colimits as a left adjoint. Thus it suffices to prove the statements for a constructible sheaf.

In this paragraph we use Lemma 74.5 without further mention. Writing $\mathcal{F} = \mathcal{F}_1 \oplus \ldots \oplus \mathcal{F}_r$ where $\mathcal{F}_i$ is $\ell_i$-primary for some prime $\ell_i$, we may assume that $\ell^n$ kills $\mathcal{F}$ for some prime $\ell$. Now consider the exact sequence

$$0 \to \mathcal{F}[\ell] \to \mathcal{F} \to \mathcal{F}/[\ell] \to 0.$$ 

Thus we see that it suffices to assume that $\mathcal{F}$ is $\ell$-torsion. This means that $\mathcal{F}$ is a constructible sheaf of $\mathbb{F}_\ell$-vector spaces for some prime number $\ell$.

By definition this means there is a dense open $U \subset X$ such that $\mathcal{F}|_U$ is finite locally constant sheaf of $\mathbb{F}_\ell$-vector spaces. Since $\dim(X) \leq 1$ we may assume, after shrinking $U$, that $U = U_1 \amalg \ldots \amalg U_n$ is a disjoint union of irreducible schemes (just remove the closed points which lie in the intersections of $\geq 2$ components of $U$). Consider the short exact sequence

$$0 \to j_! j^{-1} \mathcal{F} \to \mathcal{F} \to \bigoplus_{x \in Z} i_{x!*} M_x \to 0$$

where $Z = X \setminus U$ and $M_x$ is a finite dimensional $\mathbb{F}_\ell$ vector space, see Lemma 66.6. Since the étale cohomology of $i_{x!*} M_x$ vanishes in degrees $\geq 1$ and is equal to $M_x$ in degree 0 it suffices to prove the theorem for $j_! j^{-1} \mathcal{F}$ (argue exactly as in the proof of Lemma 74.7). Thus we reduce to the case $\mathcal{F} = j_! \mathcal{G}$ where $\mathcal{G}$ is a finite locally constant sheaf of $\mathbb{F}_\ell$-vector spaces on $U$.

Since we chose $U = U_1 \amalg \ldots \amalg U_n$ with $U_i$ irreducible we have

$$j_! \mathcal{G} = j_!(\mathcal{G}|_{U_1}) \oplus \ldots \oplus j_!(\mathcal{G}|_{U_n})$$

where $j_i : U_i \to X$ is the inclusion morphism. The case of $j_!(\mathcal{G}|_{U_i})$ is handled in Lemma 74.11. □

03SH Remarks 74.13. The “trace method” is very general. For instance, it applies in Galois cohomology, and this is essentially how Proposition 62.1 is proved.

03RT Theorem 74.14. Let $X$ be a finite type, dimension 1 scheme over an algebraically closed field $k$. Let $\mathcal{F}$ be a torsion sheaf on $X_{\text{étale}}$. Then

$$H^n_{\text{étale}}(X, \mathcal{F}) = 0, \quad \forall q \geq 3.$$ 

If $X$ affine then also $H^2_{\text{étale}}(X, \mathcal{F}) = 0$.

Proof. If $X$ is separated, this follows immediately from the more precise Theorem 74.12. If $X$ is nonseparated, choose an affine open covering $X = X_1 \cup \ldots \cup X_n$. By induction on $n$ we may assume the vanishing holds over $U = X_1 \cup \ldots \cup X_{n-1}$. Then Mayer-Vietoris (Lemma 51.1) gives

$$H^2_{\text{étale}}(U, \mathcal{F}) \oplus H^2_{\text{étale}}(X_n, \mathcal{F}) \to H^2_{\text{étale}}(U \cap X_n, \mathcal{F}) \to H^3_{\text{étale}}(X, \mathcal{F}) \to 0$$

However, since $U \cap X_n$ is an open of an affine scheme and hence affine by our dimension assumption, the group $H^2_{\text{étale}}(U \cap X_n, \mathcal{F})$ vanishes by Theorem 74.12. □
Lemma 74.15. Let \( k \subset k' \) be an extension of separably closed fields. Let \( X \) be a proper scheme over \( k \) of dimension \( \leq 1 \). Let \( \mathcal{F} \) be a torsion abelian sheaf on \( X \). Then the map \( H^q_{\text{ét}}(X, \mathcal{F}) \to H^q_{\text{ét}}(X_{k'}, \mathcal{F}|_{X_{k'}}) \) is an isomorphism for \( q \geq 0 \).

Proof. We have seen this for algebraically closed fields in Theorem 74.12. Given \( k \subset k' \) as in the statement of the lemma we can choose a diagram

\[
\begin{array}{ccc}
k' & \longrightarrow & k' \\
\uparrow & & \uparrow \\
k & \longrightarrow & k
\end{array}
\]

where \( k \subset \overline{k} \) and \( k' \subset \overline{k'} \) are the algebraic closures. Since \( k \) and \( k' \) are separably closed the field extensions \( k \subset \overline{k} \) and \( k' \subset \overline{k'} \) are algebraic and purely inseparable. In this case the morphisms \( X_{\overline{k}} \to X \) and \( X_{\overline{k'}} \to X_{k'} \) are universal homeomorphisms. Thus the cohomology of \( \mathcal{F} \) may be computed on \( X_{\overline{k}} \) and the cohomology of \( \mathcal{F}|_{X_{\overline{k'}}} \) may be computed on \( X_{\overline{k'}} \), see Proposition 16.4. Hence we deduce the general case from the case of algebraically closed fields. \( \square \)

75. First cohomology of proper schemes

In Fundamental Groups, Section 8 we have seen, in some sense, that taking \( R^1f_*G \) commutes with base change if \( f : X \to Y \) is a proper morphism and \( G \) is a finite group (not necessarily commutative). In this section we deduce a useful consequence of these results.

Lemma 75.1. Let \( A \) be a henselian local ring. Let \( X \) be a proper scheme over \( A \) with closed fibre \( X_0 \). Let \( M \) be a finite abelian group. Then \( H^1_{\text{ét}}(X, M) = H^1_{\text{ét}}(X_0, M) \).

Proof. By Cohomology on Sites, Lemma 5.3 an element of \( H^1_{\text{ét}}(X, M) \) corresponds to a \( M \)-torsor \( \mathcal{F} \) on \( X_{\text{ét}} \). Such a torsor is clearly a finite locally constant sheaf. Hence \( \mathcal{F} \) is representable by a scheme \( V \) finite étale over \( X \), Lemma 67.4. Conversely, a scheme \( V \) finite étale over \( X \) with an \( M \)-action which turns it into an \( M \)-torsor over \( X \) gives rise to a cohomology class. The same translation between cohomology classes over \( X_0 \) and torsors finite étale over \( X_0 \) holds. Thus the lemma is a consequence of the equivalence of categories of Fundamental Groups, Lemma 8.1. \( \square \)

The following technical lemma is a key ingredient in the proof of the proper base change theorem. The argument works word for word for any proper scheme over \( A \) whose special fibre has dimension \( \leq 1 \), but in fact the conclusion will be a consequence of the proper base change theorem and we only need this particular version in its proof.

Lemma 75.2. Let \( A \) be a henselian local ring. Let \( X = \mathbb{P}^1_A \). Let \( X_0 \subset X \) be the closed fibre. Let \( \ell \) be a prime number. Let \( \mathcal{I} \) be an injective sheaf of \( \mathbb{Z}/\ell\mathbb{Z} \)-modules on \( X_{\text{ét}} \). Then \( H^q_{\text{ét}}(X_0, \mathcal{I}|_{X_0}) = 0 \) for \( q > 0 \).

Proof. Observe that \( X \) is a separated scheme which can be covered by 2 affine opens. Hence for \( q > 1 \) this follows from Gabber’s affine variant of the proper base change theorem, see Lemma 73.11. Thus we may assume \( q = 1 \). Let \( \xi \in H^1_{\text{ét}}(X_0, \mathcal{I}|_{X_0}) \). Goal: show that \( \xi = 0 \). By Lemmas 70.2 and 52.2 we can find
a map $\mathcal{F} \to \mathcal{I}$ with $\mathcal{F}$ a constructible sheaf of $\mathbb{Z}/\ell\mathbb{Z}$-modules and $\xi$ coming from an element $\zeta$ of $H^1_{\text{etale}}(X_0, \mathcal{F}|_{X_0})$. Suppose we have an injective map $\mathcal{F} \to \mathcal{F}'$ of sheaves of $\mathbb{Z}/\ell\mathbb{Z}$-modules on $X_{\text{etale}}$. Since $\mathcal{I}$ is injective we can extend the given map $\mathcal{F} \to \mathcal{I}$ to a map $\mathcal{F}' \to \mathcal{I}$. In this situation we may replace $\mathcal{F}$ by $\mathcal{F}'$ and $\zeta$ by the image of $\zeta$ in $H^1_{\text{etale}}(X_0, \mathcal{F}'|_{X_0})$. Also, if $\mathcal{F} = \mathcal{F}_1 \oplus \mathcal{F}_2$ is a direct sum, then we may replace $\mathcal{F}$ by $\mathcal{F}_1$ and $\xi$ by the image of $\zeta$ in $H^1_{\text{etale}}(X_0, \mathcal{F}_1|_{X_0})$.

By Lemma 71.4 and the remarks above we may assume $\mathcal{F}$ is of the form $f_*M$ where $M$ is a finite $\mathbb{Z}/\ell\mathbb{Z}$-module and $f: Y \to X$ is a finite morphism of finite presentation (such sheaves are still constructible by Lemma 70.9 but we don’t need this). Since formation of $f_*$ commutes with any base change (Lemma 55.3) we see that the restriction of $f_*M$ to $X_0$ is equal to the pushforward of $M$ via the induced morphism $Y_0 \to X_0$ of special fibres. By the Leray spectral sequence (Proposition 54.2) and vanishing of higher direct images (Proposition 55.2), we find

$$H^1_{\text{etale}}(X_0, f_*M|_{X_0}) = H^1_{\text{etale}}(Y_0, M).$$

Since $Y \to \text{Spec}(A)$ is proper we can use Lemma 75.1 to see that the $H^1_{\text{etale}}(Y_0, M)$ is equal to $H^1_{\text{etale}}(Y, M)$. Thus we see that our cohomology class $\zeta$ lifts to a cohomology class

$$\tilde{\zeta} \in H^1_{\text{etale}}(Y, M) = H^1_{\text{etale}}(X, f_*M)$$

However, $\tilde{\zeta}$ maps to zero in $H^1_{\text{etale}}(X, \mathcal{I})$ as $\mathcal{I}$ is injective and by commutativity of

$$\begin{CD}
H^1_{\text{etale}}(X, f_*M) @>>> H^1_{\text{etale}}(X, \mathcal{I}) \\
\downarrow @VVV \downarrow \\
H^1_{\text{etale}}(X_0, (f_*M)|_{X_0}) @>>> H^1_{\text{etale}}(X_0, \mathcal{I}|_{X_0})
\end{CD}$$

we conclude that the image $\xi$ of $\zeta$ is zero as well. \qed

76. The proper base change theorem

095S The proper base change theorem is stated and proved in this section. Our approach follows roughly the proof in [AGV71, XII, Theorem 5.1] using Gabber’s ideas (from the affine case) to slightly simplify the arguments.

0A0B Lemma 76.1. Let $(A, I)$ be a henselian pair. Let $f: X \to \text{Spec}(A)$ be a proper morphism of schemes. Let $Z = X \times_{\text{Spec}(A)} \text{Spec}(A/I)$. For any sheaf $\mathcal{F}$ on the topological space associated to $X$ we have $\Gamma(X, \mathcal{F}) = \Gamma(Z, \mathcal{F}|_Z)$.

Proof. We will use Lemma 73.8 to prove this. To do this let $Y \subset X$ be an irreducible closed subscheme. We have to show that $Y \cap Z = Y \times_{\text{Spec}(A)} \text{Spec}(A/I)$ is connected. Thus we may assume that $X$ is irreducible and we have to show that $Z$ is connected. Let $X \to \text{Spec}(B) \to \text{Spec}(A)$ be the Stein factorization of $f$ (More on Morphisms, Theorem 38.4). Then $A \to B$ is integral and $(B, IB)$ is a henselian pair (More on Algebra, Lemma 8.9). Thus we may assume the fibres of $X \to \text{Spec}(A)$ are geometrically connected. On the other hand, the image $T \subset \text{Spec}(A)$ of $f$ is irreducible and closed as $X$ is proper over $A$. Hence $T \cap V(I)$ is connected by More on Algebra, Lemma 8.11. Now $Y \times_{\text{Spec}(A)} \text{Spec}(A/I) \to T \cap V(I)$ is a surjective closed map with connected fibres. The result now follows from Topology, Lemma 6.4. \qed
Lemma 76.2. Let \((A, I)\) be a henselian pair. Let \(f : X \rightarrow \Spec(A)\) be a proper morphism of schemes. Let \(i : Z \rightarrow X\) be the closed immersion of \(X \times_{\Spec(A)} \Spec(A/I)\) into \(X\). For any sheaf \(\mathcal{F}\) on \(X_{\text{étale}}\) we have \(\Gamma(X, \mathcal{F}) = \Gamma(Z, i_{\text{small}}^{-1}\mathcal{F})\).

Proof. This follows from Lemma 73.7 and 76.1 and the fact that any scheme finite over \(X\) is proper over \(\Spec(A)\).

Lemma 76.3. Let \(A\) be a henselian local ring. Let \(f : X \rightarrow \Spec(A)\) be a proper morphism of schemes. Let \(X_0 \subset X\) be the fibre of \(f\) over the closed point. For any sheaf \(\mathcal{F}\) on \(X_{\text{étale}}\) we have \(\Gamma(X, \mathcal{F}) = \Gamma(X_0, \mathcal{F}|_{X_0})\).

Proof. This is a special case of Lemma 76.2.

Let \(f : X \rightarrow S\) be a morphism of schemes. Let \(\pi : \Spec(k) \rightarrow S\) be a geometric point. The fibre of \(f\) at \(\pi\) is the scheme \(X_\pi = \Spec(k) \times_S X\) viewed as a scheme over \(\Spec(k)\). If \(\mathcal{F}\) is a sheaf on \(X_{\text{étale}}\), then denote \(\mathcal{F}_\pi = p_{\text{small}}^{-1}\mathcal{F}\) the pullback of \(\mathcal{F}\) to \((X_\pi)_{\text{étale}}\). In the following we will consider the set \(\Gamma(X_\pi, \mathcal{F}_\pi)\).

Let \(s \in S\) be the image point of \(\pi\). Let \(\kappa(s)^{\text{sep}}\) be the separable algebraic closure of \(\kappa(s)\) in \(k\) as in Definition 56.1. By Lemma 40.4 pullback defines a bijection

\[
\Gamma(X_{\kappa(s)^{\text{sep}}}, p_{\text{sep}}^{-1}\mathcal{F}) \rightarrow \Gamma(X_\pi, \mathcal{F}_\pi)
\]

where \(p_{\text{sep}} : X_{\kappa(s)^{\text{sep}}} = \Spec(\kappa(s)^{\text{sep}}) \times_S X \rightarrow X\) is the projection.

Lemma 76.4. Let \(f : X \rightarrow S\) be a proper morphism of schemes. Let \(\pi : S\) be a geometric point. For any sheaf \(\mathcal{F}\) on \(X_{\text{étale}}\) the canonical map

\[
(f_*\mathcal{F})_\pi \rightarrow \Gamma(X_\pi, \mathcal{F}_\pi)
\]

is bijective.

Proof. By Theorem 53.1 (for sheaves of sets) we have

\[
(f_*\mathcal{F})_\pi = \Gamma(X \times_S \Spec(O^{\text{sh}}_{S, \pi}), p^{-1}_{\text{small}}\mathcal{F})
\]

where \(p : X \times_S \Spec(O^{\text{sh}}_{S, \pi}) \rightarrow X\) is the projection. Since the residue field of the strictly henselian local ring \(O^{\text{sh}}_{S, \pi}\) is \(\kappa(s)^{\text{sep}}\) we conclude from the discussion above the lemma and Lemma 76.3.

Lemma 76.5. Let \(f : X \rightarrow Y\) be a proper morphism of schemes. Let \(g : Y' \subset Y\) be a morphism of schemes. Set \(X' = Y' \times_Y X\) with projections \(f' : X' \rightarrow Y'\) and \(g' : X' \rightarrow X\). Let \(\mathcal{F}\) be any sheaf on \(X_{\text{étale}}\). Then \(g^{-1}f_*\mathcal{F} = f'_*g'_*(g')^{-1}\mathcal{F}\).

Proof. There is a canonical map \(g^{-1}f_*\mathcal{F} \rightarrow f'_*(g')^{-1}\mathcal{F}\). Namely, it is adjoint to the map

\[
f_*\mathcal{F} \rightarrow g_*f'_*(g')^{-1}\mathcal{F} = f_*g'_*(g')^{-1}\mathcal{F}
\]

which is \(f_*\) applied to the canonical map \(\mathcal{F} \rightarrow g'_*(g')^{-1}\mathcal{F}\). To check this map is an isomorphism we can compute what happens on stalks. Let \(y' : \Spec(k) \rightarrow Y'\) be a geometric point with image \(y\) in \(Y\). By Lemma 76.4 the stalks are \(\Gamma(X_{y'}, \mathcal{F}_{y'})\) and \(\Gamma(X_y, \mathcal{F}_y)\) respectively. Here the sheaves \(\mathcal{F}_y\) and \(\mathcal{F}_{y'}\) are the pullbacks of \(\mathcal{F}\) by the projections \(X_y \rightarrow X\) and \(X_{y'} \rightarrow X\). Thus we see that the groups agree by Lemma 40.4. We omit the verification that this isomorphism is compatible with our map.
At this point we start discussing the proper base change theorem. To do so we introduce some notation. Consider a commutative diagram

\[
\begin{array}{ccc}
X' & \xrightarrow{g} & X \\
\downarrow{f'} & & \downarrow{f} \\
Y' & \xrightarrow{g} & Y
\end{array}
\]

of morphisms of schemes. Then we obtain a commutative diagram of sites

\[
\begin{array}{ccc}
X'_{\text{étale}} & \xrightarrow{g_{\text{small}}} & X_{\text{étale}} \\
\downarrow{f'_{\text{small}}} & & \downarrow{f_{\text{small}}} \\
Y'_{\text{étale}} & \xrightarrow{g_{\text{small}}} & Y_{\text{étale}}
\end{array}
\]

For any object \( E \) of \( D(X_{\text{étale}}) \) we obtain a canonical base change map

\[
g_{\text{small}}^{-1}Rf_{\text{small}}^*E \longrightarrow Rf'_{\text{small}}(g'_{\text{small}})^{-1}E
\]

in \( D(Y'_{\text{étale}}) \). See Cohomology on Sites, Remark 19.2 where we use the constant sheaf \( \mathbb{Z} \) as our sheaf of rings. We will usually omit the subscripts \( \text{small} \) in this formula. For example, if \( E = \mathcal{F}[0] \) where \( \mathcal{F} \) is an abelian sheaf on \( X_{\text{étale}} \), the base change map is a map

\[
g_{\text{small}}^{-1}Rf^*\mathcal{F} \longrightarrow Rf'_{\text{small}}(g')^{-1}\mathcal{F}
\]

in \( D(Y'_{\text{étale}}) \).

The map (76.5.2) has no chance of being an isomorphism in the generality given above. The goal is to show it is an isomorphism if the diagram (76.5.1) is cartesian, \( f : X \to Y \) proper, and the cohomology sheaves of \( E \) are torsion. To study this question we introduce the following terminology. Let us say that cohomology commutes with base change for \( f : X \to Y \) if (76.5.3) is an isomorphism for every diagram (76.5.1) where \( X' = Y' \times_Y X \) and every torsion abelian sheaf \( \mathcal{F} \).

Lemma 76.6. Let \( f : X \to Y \) be a proper morphism of schemes. The following are equivalent

1. cohomology commutes with base change for \( f \) (see above),
2. for every prime number \( \ell \) and every injective sheaf of \( \mathbb{Z}/\ell\mathbb{Z} \)-modules \( \mathcal{I} \) on \( X_{\text{étale}} \) and every diagram (76.5.1) where \( X' = Y' \times_Y X \) the sheaves \( R^q f'_{\text{small}}(g')^{-1}\mathcal{I} \) are zero for \( q > 0 \).

Proof. It is clear that (1) implies (2). Conversely, assume (2) and let \( \mathcal{F} \) be an abelian sheaf on \( X_{\text{étale}} \). Let \( Y' \to Y \) be a morphism of schemes and let \( X' = Y' \times_Y X \) with projections \( g' : X' \to X \) and \( f' : X' \to Y' \) as in diagram (76.5.1). We want to show the maps of sheaves

\[
g_{\text{small}}^{-1}R^q f^*\mathcal{F} \longrightarrow R^q f'_{\text{small}}(g')^{-1}\mathcal{F}
\]

are isomorphisms for all \( q \geq 0 \).

For every \( n \geq 1 \), let \( \mathcal{F}[n] \) be the subsheaf of sections of \( \mathcal{F} \) annihilated by \( n \). Then \( \mathcal{F} = \text{colim} \mathcal{F}[n] \). The functors \( g_{\text{small}}^{-1} \) and \( (g')^{-1} \) commute with arbitrary colimits (as left adjoints). Taking higher direct images along \( f \) or \( f' \) commutes with filtered colimits by Lemma 52.3. Hence we see that

\[
g_{\text{small}}^{-1}R^q f^*\mathcal{F} = \text{colim} g_{\text{small}}^{-1}R^q f_*\mathcal{F}[n] \quad \text{and} \quad R^q f'_{\text{small}}(g')^{-1}\mathcal{F} = \text{colim} R^q f'_{\text{small}}(g')^{-1}\mathcal{F}[n]
\]
Let \( Z \) be a prime number. Let \( f : X \to Y \) and \( g : Y \to Z \) be proper morphisms of schemes.

Assume

1. cohomology commutes with base change for \( f \);
2. cohomology commutes with base change for \( g \circ f \), and
3. \( f \) is surjective.

Then cohomology commutes with base change for \( g \).

**Proof.** We will use the equivalence of Lemma 76.6 without further mention. Let \( \ell \) be a prime number. Let \( I \) be an injective sheaf of \( \mathbb{Z}/\ell \mathbb{Z} \)-modules on \( Y_{\text{étale}} \). Choose an injective map of sheaves \( f^{-1} I \to J \) where \( J \) is an injective sheaf of \( \mathbb{Z}/\ell \mathbb{Z} \)-modules on \( Z_{\text{étale}} \). Since \( f \) is surjective the map \( I \to f_* J \) is injective (look at stalks in geometric points). Since \( I \) is injective we see that \( I \) is a direct summand of \( f_* J \). Thus it suffices to prove the desired vanishing for \( f_* J \).

Let \( Z' \to Z \) be a morphism of schemes and set \( Y' = Z' \times_Z Y \) and \( X' = Z' \times_Z X = Y' \times_Y X \). Denote \( a : X' \to X \), \( b : Y' \to Y \), and \( c : Z' \to Z \) the projections. Similarly for \( f' : X' \to Y' \) and \( g' : Y' \to Z' \). By Lemma 76.5, we have \( b^{-1} f_* J = f'_* a^{-1} J \). On the other hand, we know that \( R^q f'_* a^{-1} J \) and \( R^q (g' \circ f')_* a^{-1} J \) are zero for \( q > 0 \). Using the spectral sequence (Cohomology on Sites, Lemma 14.7)

\[
R^p g_* [R^q f'_* a^{-1} J] \Rightarrow R^{p+q} ((g' \circ f')_* a^{-1} J)
\]

we conclude that \( R^p g_* (f^{-1} f_* J) = R^p g_* (f'_* a^{-1} J) = 0 \) for \( p > 0 \) as desired. □

**Lemma 76.8.** Let \( f : X \to Y \) and \( g : Y \to Z \) be proper morphisms of schemes.

Assume

1. cohomology commutes with base change for \( f \), and
2. cohomology commutes with base change for \( g \).

Then cohomology commutes with base change for \( f \circ g \).

**Proof.** We will use the equivalence of Lemma 76.6 without further mention. Let \( \ell \) be a prime number. Let \( I \) be an injective sheaf of \( \mathbb{Z}/\ell \mathbb{Z} \)-modules on \( X_{\text{étale}} \). Then \( f_* I \) is an injective sheaf of \( \mathbb{Z}/\ell \mathbb{Z} \)-modules on \( Y_{\text{étale}} \) (Cohomology on Sites, Lemma 14.2). The result follows formally from this, but we will also spell it out.

Let \( Z' \to Z \) be a morphism of schemes and set \( Y' = Z' \times_Z Y \) and \( X' = Z' \times_Z X = Y' \times_Y X \). Denote \( a : X' \to X \), \( b : Y' \to Y \), and \( c : Z' \to Z \) the projections. Similarly
for $f' : X' \to Y'$ and $g' : Y' \to Z'$. By Lemma 76.5 we have $b^{-1} f_* \mathcal{I} = f'_* a^{-1} \mathcal{I}$. On the other hand, we know that $R^q f'_* a^{-1} \mathcal{I}$ and $R^q (g')_* b^{-1} f_* \mathcal{I}$ are zero for $q > 0$. Using the spectral sequence (Cohomology on Sites, Lemma 14.7

\[ R^p g'_* R^q f'_* a^{-1} \mathcal{I} \Rightarrow R^{p+q} (g' \circ f')_* a^{-1} \mathcal{I} \]

we conclude that $R^p (g' \circ f')_* a^{-1} \mathcal{I} = 0$ for $p > 0$ as desired. □

0A4E **Lemma 76.9.** Let $f : X \to Y$ be a finite morphism of schemes. Then cohomology commutes with base change for $f$.

**Proof.** Observe that a finite morphism is proper, see Morphisms, Lemma 43.10. Moreover, the base change of a finite morphism is finite, see Morphisms, Lemma 43.6. Thus the result follows from Lemma 76.6 combined with Proposition 55.2. □

0A4F **Lemma 76.10.** To prove that cohomology commutes with base change for every proper morphism of schemes it suffices to prove it holds for the morphism $\mathbb{P}^1_S \to S$ for every scheme $S$.

**Proof.** Let $f : X \to Y$ be a proper morphism of schemes. Let $Y = \bigcup Y_i$ be an affine open covering and set $X_i = f^{-1}(Y_i)$. If we can prove cohomology commutes with base change for $X_i \to Y_i$, then cohomology commutes with base change for $f$. Namely, the formation of the higher direct images commutes with Zariski (and even étale) localization on the base, see Lemma 52.4. Thus we may assume $Y$ is affine.

Let $Y$ be an affine scheme and let $X \to Y$ be a proper morphism. By Chow’s lemma there exists a commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{\pi} & X' \\
\downarrow & & \downarrow \\
Y & \leftarrow & \mathbb{P}^n_Y
\end{array}
\]

where $X' \to \mathbb{P}^n_Y$ is an immersion, and $\pi : X' \to X$ is proper and surjective, see Limits, Lemma 11.1. Since $X \to Y$ is proper, we find that $X' \to Y$ is proper (Morphisms, Lemma 41.4). Hence $X' \to \mathbb{P}^n_Y$ is a closed immersion (Morphisms, Lemma 41.7). It follows that $X' \to X \times_Y \mathbb{P}^n_Y = \mathbb{P}^n_X$ is a closed immersion (as an immersion with closed image).

By Lemma 76.7 it suffices to prove cohomology commutes with base change for $\pi$ and $X' \to Y$. These morphisms both factor as a closed immersion followed by a projection $\mathbb{P}^n_S \to S$ (for some $S$). By Lemma 76.9 the result holds for closed immersions (as closed immersions are finite). By Lemma 76.8 it suffices to prove the result for projections $\mathbb{P}^n_S \to S$.

For every $n \geq 1$ there is a finite surjective morphism

\[ \mathbb{P}^1_S \times_S \ldots \times_S \mathbb{P}^1_S \longrightarrow \mathbb{P}^n_S \]

given on coordinates by

\[ ((x_1 : y_1), (x_2 : y_2), \ldots, (x_n : y_n)) \longmapsto (F_0 : \ldots : F_n) \]

where $F_0, \ldots, F_n$ in $x_1, \ldots, y_n$ are the polynomials with integer coefficients such that

\[ \prod (x_i t + y_i) = F_0 t^n + F_1 t^{n-1} + \ldots + F_n \]
Applying Lemmas [76.7, 76.9] and [76.8] one more time we conclude that the lemma is true.

**Theorem 76.11.** Let \( f : X \to Y \) be a proper morphism of schemes. Let \( g : Y' \to Y \) be a morphism of schemes. Set \( X' = Y' \times_Y X \) and consider the cartesian diagram

\[
\begin{array}{ccc}
X' & \xrightarrow{g'} & X \\
\downarrow f' & & \downarrow f \\
Y' & \xrightarrow{g} & Y
\end{array}
\]

Let \( F \) be an abelian torsion sheaf on \( X_{\text{étale}} \). Then the base change map

\[
g^{-1} Rf_* F \to Rf'_*(g')^{-1} F
\]

is an isomorphism.

**Proof.** In the terminology introduced above, this means that cohomology commutes with base change for every proper morphism of schemes. By Lemma [76.10] it suffices to prove that cohomology commutes with base change for the morphism \( \mathbf{P}^1_S \to S \) for every scheme \( S \).

Let \( S \) be the spectrum of a strictly henselian local ring with closed point \( s \). Set \( X = \mathbf{P}^1_S \) and \( X_0 = X_s = \mathbf{P}^1_\kappa(t) \). Let \( F \) be a sheaf of \( \mathbf{Z}/\ell\mathbf{Z} \)-modules on \( X_{\text{étale}} \). The key to our proof is that

\[
H^q_{\text{étale}}(X, F) = H^q_{\text{étale}}(X_0, F|_{X_0}).
\]

Namely, choose a resolution \( F \to \mathcal{I}^* \) by injective sheaves of \( \mathbf{Z}/\ell\mathbf{Z} \)-modules. Then \( \mathcal{I}^*|_{X_0} \) is a resolution of \( F|_{X_0} \) by right \( H^0_{\text{étale}}(X_0, -) \)-acyclic objects, see Lemma [75.2]. Leray’s acyclicity lemma tells us the right hand side is computed by the complex \( H^0_{\text{étale}}(X_0, \mathcal{I}^*_X|_{X_0}) \) which is equal to \( H^0_{\text{étale}}(X, \mathcal{I}^*_X) \) by Lemma [76.3]. This complex computes the left hand side.

Assume \( S \) is general and \( F \) is a sheaf of \( \mathbf{Z}/\ell\mathbf{Z} \)-modules on \( X_{\text{étale}} \). Let \( \tilde{s} : \text{Spec}(k) \to S \) be a geometric point of \( S \) lying over \( s \in S \). We have

\[
(R^q f_* F)_{\tilde{s}} = H^q_{\text{étale}}(\mathbf{P}^1_{\kappa(t), \text{étale}}, F|_{\mathbf{P}^1_{\kappa(t), \text{étale}}}) = H^q_{\text{étale}}(\mathbf{P}^1_{\kappa(s), \text{étale}}, F|_{\mathbf{P}^1_{\kappa(s), \text{étale}}})
\]

where \( \kappa(s)_{\text{sep}} \) is the residue field of \( \mathcal{O}_{S, \tilde{s}} \), i.e., the separable algebraic closure of \( \kappa(s) \) in \( k \). The first equality by Theorem [53.1] and the second equality by the displayed formula in the previous paragraph.

Finally, consider any morphism of schemes \( g : T \to S \) where \( S \) and \( F \) are as above. Set \( f' : \mathbf{P}^1_T \to T \) the projection and let \( g' : \mathbf{P}^1_T \to \mathbf{P}^1_T \) the morphism induced by \( g \). Consider the base change map

\[
g^{-1} R^q f_* F \to R^q f'_*(g')^{-1} F
\]

Let \( \tilde{t} \) be a geometric point of \( T \) with image \( \tilde{s} = g(\tilde{t}) \). By our discussion above the map on stalks at \( \tilde{t} \) is the map

\[
H^q_{\text{étale}}(\mathbf{P}^1_{\kappa(s)_{\text{sep}}}, F|_{\mathbf{P}^1_{\kappa(s)_{\text{sep}}}}) \to H^q_{\text{étale}}(\mathbf{P}^1_{\kappa(t)_{\text{sep}}}, F|_{\mathbf{P}^1_{\kappa(t)_{\text{sep}}}})
\]

Since \( \kappa(s)_{\text{sep}} \subset \kappa(t)_{\text{sep}} \) this map is an isomorphism by Lemma [74.15]. This proves cohomology commutes with base change for \( \mathbf{P}^1_S \to S \) and sheaves of \( \mathbf{Z}/\ell\mathbf{Z} \)-modules. In particular, for an injective sheaf of \( \mathbf{Z}/\ell\mathbf{Z} \)-modules the higher
direct images of any base change are zero. In other words, condition (2) of Lemma 76.6 holds and the proof is complete. □

77. Applications of proper base change

As an application of the proper base change theorem we obtain the following.

095U Lemma 77.1. Let $f : X \to Y$ be a proper morphism of schemes all of whose fibres have dimension $\leq n$. Then for any abelian torsion sheaf $\mathcal{F}$ on $X_{\text{étale}}$ we have $R^q f_* \mathcal{F} = 0$ for $q > 2n$.

Proof. Omitted. Hints: By the proper base change theorem it suffices to prove that for a proper scheme $X$ over an algebraically closed field, the étale cohomology of $\mathcal{F}$ vanishes above $2 \dim X$. By the proper base change theorem and dévissage (using Chow’s lemma for example) one can reduce to the case where the dimension of $X$ is 1. The case of curves is Theorem 74.14. See also Remarks 74.15. □

03V Lemma 77.2. Let $f : X \to Y$ be a morphism of finite type with $Y$ quasi-compact. Then the dimension of the fibres of $f$ is bounded.

Proof. By Morphisms, Lemma 28.4 the set $U_n \subset X$ of points where the dimension of the fibre is $\leq n$ is open. Since $f$ is of finite type, every point is contained in some $U_n$. Since $Y$ is quasi-compact and $f$ is of finite type, we see that $X$ is quasi-compact. Hence $X = U_n$ for some $n$. □

78. The trace formula

A typical course in étale cohomology would normally state and prove the proper and smooth base change theorems, purity and Poincaré duality. All of these can be found in [Del77, Arcata]. Instead, we are going to study the trace formula for the frobenius, following the account of Deligne in [Del77, Rapport]. We will only look at dimension 1, but using proper base change this is enough for the general case. Since all the cohomology groups considered will be étale, we drop the subscript $\text{étale}$. Let us now describe the formula we are after. Let $X$ be a finite type scheme of dimension 1 over a finite field $k$, $\ell$ a prime number and $\mathcal{F}$ a constructible, flat $\mathbb{Z}/\ell^n\mathbb{Z}$ sheaf. Then

\[
\sum_{x \in X(k)} \text{Tr}(\text{Frob}|\mathcal{F}_x) = \sum_{i=0}^{2} (-1)^i \text{Tr}(\pi_X^*|H^i_c(X \otimes_k \bar{k}, \mathcal{F}))
\]

as elements of $\mathbb{Z}/\ell^n\mathbb{Z}$. As we will see, this formulation is slightly wrong as stated. Let us nevertheless describe the symbols that occur therein.

79. Frobenii

In this section we will prove a “baffling” theorem. A topological analogue of the baffling theorem is the following.

Exercise 79.1. Let $X$ be a topological space and $g : X \to X$ a continuous map such that $g^{-1}(U) = U$ for all opens $U$ of $X$. Then $g$ induces the identity on cohomology on $X$ (for any coefficients).

We now turn to the statement for the étale site.
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Lemma 79.2. Let \( X \) be a scheme and \( g : X \to X \) a morphism. Assume that for all \( \varphi : U \to X \) étale, there is an isomorphism
\[
\begin{array}{ccc}
U & \overset{\sim}{\longrightarrow} & U \times_{\varphi, X, g} X \\
\downarrow{\varphi} & & \downarrow{pr_2} \\
X & \overset{\varphi}{\longrightarrow} & X
\end{array}
\]
functorial in \( U \). Then \( g \) induces the identity on cohomology (for any sheaf).

Proof. The proof is formal and without difficulty. \( \square \)

Definition 79.3. Let \( X \) be a scheme in characteristic \( p \). The absolute frobenius of \( X \) is the morphism \( F_X : X \to X \) which is the identity on the induced topological space, and which takes a function to its \( p \)th power. Thus \( F_X^p : \mathcal{O}_X \to \mathcal{O}_X \) is given by \( g \mapsto g^p \).

Theorem 79.4 (The Baffling Theorem). Let \( X \) be a scheme in characteristic \( p > 0 \). Then the absolute frobenius induces (by pullback) the trivial map on cohomology, i.e., for all integers \( j \geq 0 \),
\[
F_X^* : H^j(X, \mathbb{Z}/n\mathbb{Z}) \to H^j(X, \mathbb{Z}/n\mathbb{Z})
\]
is the identity.

This theorem is purely formal. It is a good idea, however, to review how to compute the pullback of a cohomology class. Let us simply say that in the case where cohomology agrees with \( \acute{\text{C}} \)ech cohomology, it suffices to pull back (using the fiber products on a site) the \( \acute{\text{C}} \)ech cocycles. The general case is quite technical, see Hypercoverings, Theorem 9.1. To prove the theorem, we merely verify that the assumption of Lemma 79.2 holds for the frobenius.

Proof of Theorem 79.4. We need to verify the existence of a functorial isomorphism as above. For an étale morphism \( \varphi : U \to S \), consider the diagram
\[
\begin{array}{ccc}
U & \overset{\sim}{\longrightarrow} & U \times_{\varphi, X, F_X} X \\
\downarrow{\varphi} & & \downarrow{pr_2} \\
X & \overset{\varphi}{\longrightarrow} & X
\end{array}
\]

The dotted arrow is an étale morphism which induces an isomorphism on the underlying topological spaces, so it is an isomorphism. \( \square \)

Definition 79.5. Let \( k \) be a finite field with \( q = p^f \) elements. Let \( X \) be a scheme over \( k \). The geometric frobenius of \( X \) is the morphism \( \pi_X : X \to X \) over \( \text{Spec}(k) \) which equals \( F_X^f \).

Since \( \pi_X \) is a morphism over \( k \), we can base change it to any scheme over \( k \). In particular we can base change it to the algebraic closure \( \bar{k} \) and get a morphism \( \pi_X : X_{\bar{k}} \to X_{\bar{k}} \). Using \( \pi_X \) also for this base change should not be confusing as \( X_{\bar{k}} \) does not have a geometric frobenius of its own.
Lemma 79.6. Let $F$ be a sheaf on $X_{\text{étale}}$. Then there are canonical isomorphisms $\pi_{X}^{-1}F \cong F$ and $F \cong \pi_{X}^{*}F$.

This is false for the fppf site.

Proof. Let $\varphi : U \to X$ be étale. Recall that $\pi_{X}^{*}F(U) = F(U \times_{\varphi,X,\pi_{X}} X)$. Since $\pi_{X} = F_{U}^{1}$, it follows from the proof of Theorem 79.4 that there is a functorial isomorphism

$$U \xrightarrow{\gamma_{U}} U \times_{\varphi,X,\pi_{X}} X \xrightarrow{pr_{2}} X$$

where $\gamma_{U} = (\varphi, F_{U}^{1})$. Now we define an isomorphism $F(U) \to \pi_{X}^{*}F(U) = F(U \times_{\varphi,X,\pi_{X}} X)$ by taking the restriction map of $F$ along $\gamma_{U}^{-1}$. The other isomorphism is analogous. \[\square\]

Remark 79.7. It may or may not be the case that $F_{U}^{1}$ equals $\pi_{U}$.

We continue discussion cohomology of sheaves on our scheme $X$ over the finite field $k$ with $q = p^{f}$ elements. Fix an algebraic closure $\bar{k}$ of $k$ and write $G_{k} = \text{Gal}(\bar{k}/k)$ for the absolute Galois group of $k$. Let $F$ be an abelian sheaf on $X_{\text{étale}}$. We will define a left $G_{k}$-module structure cohomology group $H^{j}(X_{\bar{k}}, F|_{X_{\bar{k}}})$ as follows: if $\sigma \in G_{k}$, the diagram

$$X_{\bar{k}} \xrightarrow{\Spec(\sigma) \times \id_{X}} X_{\bar{k}} \xrightarrow{\Spec(\sigma) \times \id_{X}} X$$

commutes. Thus we can set, for $\xi \in H^{j}(X_{\bar{k}}, F|_{X_{\bar{k}}})$

$$\sigma \cdot \xi := (\Spec(\sigma) \times \id_{X})^{*}(\Spec(\sigma) \times \id_{X})^{-1}F|_{X_{\bar{k}}} = H^{j}(X_{\bar{k}}, F|_{X_{\bar{k}}})$$

where the last equality follows from the commutativity of the previous diagram. This endows the latter group with the structure of a $G_{k}$-module.

Lemma 79.8. In the situation above denote $\alpha : X \to \Spec(k)$ the structure morphism. Consider the stalk $(R^{i}\alpha_{*}F)_{\Spec(\bar{k})}$ endowed with its natural Galois action as in Section 56. Then the identification

$$(R^{i}\alpha_{*}F)_{\Spec(\bar{k})} \cong H^{j}(X_{\bar{k}}, F|_{X_{\bar{k}}})$$

from Theorem 53.1 is an isomorphism of $G_{k}$-modules.

A similar result holds comparing $(R^{i}\alpha_{*}F)_{\Spec(\bar{k})}$ with $H^{j}_{\text{fppf}}(X_{\bar{k}}, F|_{X_{\bar{k}}})$.

Proof. Omitted. \[\square\]

Definition 79.9. The arithmetic frobenius is the map $\text{frob}_{k} : \bar{k} \to \bar{k}, x \mapsto x^{q}$ of $G_{k}$.

Theorem 79.10. Let $F$ be an abelian sheaf on $X_{\text{étale}}$. Then for all $j \geq 0$, $\text{frob}_{k}$ acts on the cohomology group $H^{j}(X_{\bar{k}}, F|_{X_{\bar{k}}})$ as the inverse of the map $\pi_{X}^{-1}$.
The map \( \pi_X \) is defined by the composition
\[
H^j(X_k, \mathcal{F}|_{X_k}) \xrightarrow{\pi_X} H^j(X_k, (\pi_X^{-1}\mathcal{F})|_{X_k}) \cong H^j(X_k, \mathcal{F}|_{X_k}).
\]
where the last isomorphism comes from the canonical isomorphism \( \pi_X^{-1}\mathcal{F} \cong \mathcal{F} \) of Lemma 79.6.

**Proof.** The composition \( X_k \xrightarrow{\text{Spec}(\text{frob}_k)} X_k \xrightarrow{\pi_X} X_k \) is equal to \( F^j_{X_k} \), hence the result follows from the baffling theorem suitably generalized to nontrivial coefficients. Note that the previous composition commutes in the sense that \( F^j_{X_k} = \pi_X \circ \text{Spec}(\text{frob}_k) = \text{Spec}(\text{frob}_k) \circ \pi_X \). \( \square \)

**Definition 79.11.** If \( x \in X(k) \) is a rational point and \( \bar{x} : \text{Spec}(\bar{k}) \to X \) the geometric point lying over \( x \), we let \( \pi_x : \mathcal{F}_{\bar{x}} \to \mathcal{F}_{\bar{x}} \) denote the action by \( \text{frob}_{\bar{k}}^{-1} \) and call it the geometric frobenius.

We can now make a more precise statement (albeit a false one) of the trace formula (78.0.1). Let \( X \) be a finite type scheme of dimension 1 over a finite field \( k \), \( \ell \) a prime number and \( \mathcal{F} \) a constructible, flat \( \mathcal{Z}/\ell^n\mathcal{Z} \) sheaf. Then

\[
\sum_{x \in X(k)} \text{Tr}(\pi_x|_{\mathcal{F}_{\bar{x}}}) = \sum_{i=0}^{2} (-1)^i \text{Tr}(\pi_X^*|H^i_j(X_k, \mathcal{F}))
\]
as elements of \( \mathcal{Z}/\ell^n\mathcal{Z} \). The reason this equation is wrong is that the trace in the right-hand side does not make sense for the kind of sheaves considered. Before addressing this issue, we try to motivate the appearance of the geometric frobenius (apart from the fact that it is a natural morphism!).

Let us consider the case where \( X = \mathbb{P}^1_k \) and \( \mathcal{F} = \mathcal{Z}/\ell\mathcal{Z} \). For any point, the Galois module \( \mathcal{F}_{\bar{x}} \) is trivial, hence for any morphism \( \varphi \) acting on \( \mathcal{F}_{\bar{x}} \), the left-hand side is
\[
\sum_{x \in X(k)} \text{Tr}(\varphi|_{\mathcal{F}_{\bar{x}}}) = \#\mathbb{P}^1_k(k) = q + 1.
\]

Now \( \mathbb{P}^1_k \) is proper, so compactly supported cohomology equals standard cohomology, and so for a morphism \( \pi : \mathbb{P}^1_k \to \mathbb{P}^1_k \), the right-hand side equals
\[
\text{Tr}(\pi^*|H^0(\mathbb{P}^1_k, \mathcal{Z}/\ell\mathcal{Z})) + \text{Tr}(\pi^*|H^2(\mathbb{P}^1_k, \mathcal{Z}/(\ell^n\mathcal{Z}))).
\]
The Galois module \( H^0(\mathbb{P}^1_k, \mathcal{Z}/\ell\mathcal{Z}) = \mathcal{Z}/\ell\mathcal{Z} \) is trivial, since the pullback of the identity is the identity. Hence the first trace is 1, regardless of \( \pi \). For the second trace, we need to compute the pullback \( \pi^* : H^2(\mathbb{P}^1_k, \mathcal{Z}/(\ell^n\mathcal{Z})) \) for a map \( \pi : \mathbb{P}^1_k \to \mathbb{P}^1_k \). This is a good exercise and the answer is multiplication by the degree of \( \pi \) (for a proof see Lemma 65.2). In other words, this works as in the familiar situation of complex cohomology. In particular, if \( \pi \) is the geometric frobenius we get
\[
\text{Tr}(\pi_X^*|H^2(\mathbb{P}^1_k, \mathcal{Z}/(\ell\mathcal{Z}))) = q
\]
and if \( \pi \) is the arithmetic frobenius then we get
\[
\text{Tr}(\text{frob}_k^*|H^2(\mathbb{P}^1_k, \mathcal{Z}/(\ell\mathcal{Z}))) = q^{-1}.
\]
The latter option is clearly wrong.

\footnote{This notation is not standard. This operator is denoted \( F_x \) in \([\text{Del}77]\). We will likely change this notation in the future.}
03SY **Remark 79.12.** The computation of the degrees can be done by lifting (in some obvious sense) to characteristic 0 and considering the situation with complex coefficients. This method almost never works, since lifting is in general impossible for schemes which are not projective space.

The question remains as to why we have to consider compactly supported cohomology. In fact, in view of Poincaré duality, it is not strictly necessary for smooth varieties, but it involves adding in certain powers of $q$. For example, let us consider the case where $X = \mathbb{A}^1_k$ and $F = \mathbb{Z}/\ell \mathbb{Z}$. The action on stalks is again trivial, so we only need look at the action on cohomology. But then $\pi_X^*$ acts as the identity on $H^0(\mathbb{A}^1_k, \mathbb{Z}/\ell \mathbb{Z})$ and as multiplication by $q$ on $H^2_c(\mathbb{A}^1_k, \mathbb{Z}/\ell \mathbb{Z})$.

80. Traces

03SZ We now explain how to take the trace of an endomorphism of a module over a noncommutative ring. Fix a finite ring $\Lambda$ with cardinality prime to $p$. Typically, $\Lambda$ is the group ring $(\mathbb{Z}/\ell^n \mathbb{Z})[G]$ for some finite group $G$. By convention, all the $\Lambda$-modules considered will be left $\Lambda$-modules.

We introduce the following notation: We set $\Lambda^\natural$ to be the quotient of $\Lambda$ by its additive subgroup generated by the commutators (i.e., the elements of the form $ab - ba$, $a, b \in \Lambda$). Note that $\Lambda^\natural$ is not a ring.

For instance, the module $(\mathbb{Z}/\ell^n \mathbb{Z})[G]^\natural$ is the dual of the class functions, so

$$(\mathbb{Z}/\ell^n \mathbb{Z})[G]^\natural = \bigoplus_{\text{conjugacy classes of } G} \mathbb{Z}/\ell^n \mathbb{Z}.$$ 

For a free $\Lambda$-module, we have $\text{End}_\Lambda(\Lambda^{\otimes m}) = \text{Mat}_n(\Lambda)$. Note that since the modules are left modules, representation of endomorphism by matrices is a right action: if $a \in \text{End}(\Lambda^{\otimes m})$ has matrix $A$ and $v \in \Lambda$, then $a(v) = vA$.

03T0 **Definition 80.1.** The trace of the endomorphism $a$ is the sum of the diagonal entries of a matrix representing it. This defines an additive map $\text{Tr} : \text{End}_\Lambda(\Lambda^{\otimes m}) \to \Lambda^\natural$.

03T1 **Exercise 80.2.** Given maps

$$\Lambda^{\otimes n} \xrightarrow{a} \Lambda^{\otimes n} \xrightarrow{b} \Lambda^{\otimes m}$$

show that $\text{Tr}(ab) = \text{Tr}(ba)$.

We extend the definition of the trace to a finite projective $\Lambda$-module $P$ and an endomorphism $\varphi$ of $P$ as follows. Write $P$ as the summand of a free $\Lambda$-module, i.e., consider maps $P \xrightarrow{a} \Lambda^{\otimes n} \xrightarrow{b} P$ with

1. $\Lambda^{\otimes n} = \text{Im}(a) \oplus \text{Ker}(b)$; and
2. $b \circ a = \text{id}_P$.

Then we set $\text{Tr}(\varphi) = \text{Tr}(a\varphi b)$. It is easy to check that this is well-defined, using the previous exercise.
81. Why derived categories?

With this definition of the trace, let us now discuss another issue with the formula as stated. Let $C$ be a smooth projective curve over $k$. Then there is a correspondence between finite locally constant sheaves $F$ on $C_{\text{ étale}}$ which stalks are isomorphic to $(\mathbb{Z}/\ell^n\mathbb{Z})^\oplus m$ on the one hand, and continuous representations $\rho : \pi_1(C, \bar{c}) \to \text{GL}_m(\mathbb{Z}/\ell^n\mathbb{Z})$ (for some fixed choice of $\bar{c}$) on the other hand. We denote $\mathcal{F}_\rho$ the sheaf corresponding to $\rho$. Then $H^2(C_k, \mathcal{F}_\rho)$ is the group of coinvariants for the action of $\rho(\pi_1(C, \bar{c}))$ on $(\mathbb{Z}/\ell^n\mathbb{Z})^\oplus m$, and there is a short exact sequence

$$0 \to \pi_1(C_k, \bar{c}) \to \pi_1(C, \bar{c}) \to G_k \to 0.$$ 

For instance, let $\mathbb{Z} = \mathbb{Z}[\sigma]$ act on $\mathbb{Z}/\ell^n\mathbb{Z}$ via $\sigma(x) = (1 + \ell)x$. The coinvariants are $(\mathbb{Z}/\ell^n\mathbb{Z})_{\sigma} = \mathbb{Z}/\ell\mathbb{Z}$, which is not a flat $\mathbb{Z}/\ell\mathbb{Z}$-module. Hence we cannot take the trace of some action on $H^2(C_k, \mathcal{F}_\rho)$, at least not in the sense of the previous section.

In fact, our goal is to consider a trace formula for $\ell$-adic coefficients. But $Q_\ell = \mathbb{Z}[1/\ell]$ and $\mathbb{Z}_\ell = \lim \mathbb{Z}/\ell^n\mathbb{Z}$, and even for a flat $\mathbb{Z}/\ell^n\mathbb{Z}$ sheaf, the individual cohomology groups may not be flat, so we cannot compute traces. One possible remedy is consider the total derived complex $R\Gamma(C_k, \mathcal{F}_\rho)$ in the derived category $D(\mathbb{Z}/\ell^n\mathbb{Z})$ and show that it is a perfect object, which means that it is quasi-isomorphic to a finite complex of finite free module. For such complexes, we can define the trace, but this will require an account of derived categories.

82. Derived categories

To set up notation, let $\mathcal{A}$ be an abelian category. Let $\text{Comp}(\mathcal{A})$ be the abelian category of complexes in $\mathcal{A}$. Let $K(\mathcal{A})$ be the category of complexes up to homotopy, with objects equal to complexes in $\mathcal{A}$ and objects equal to homotopy classes of morphisms of complexes. This is not an abelian category. Loosely speaking, $D(\mathcal{A})$ is defined to be the category obtained by inverting all quasi-isomorphisms in $\text{Comp}(\mathcal{A})$ or, equivalently, in $K(\mathcal{A})$. Moreover, we can define $\text{Comp}^+(\mathcal{A}), K^+(\mathcal{A}), D^+(\mathcal{A})$ analogously using only bounded below complexes. Similarly, we can define $\text{Comp}^- (\mathcal{A}), K^- (\mathcal{A}), D^- (\mathcal{A})$ using bounded above complexes, and we can define $\text{Comp}^b (\mathcal{A}), K^b (\mathcal{A}), D^b (\mathcal{A})$ using bounded complexes.

\textbf{Remark 82.1.} Notes on derived categories.

1. There are some set-theoretical problems when $\mathcal{A}$ is somewhat arbitrary, which we will happily disregard.

2. The categories $K(\mathcal{A})$ and $D(\mathcal{A})$ may be endowed with the structure of triangulated category, but we will not need these structures in the following discussion.

3. The categories $\text{Comp}(\mathcal{A})$ and $K(\mathcal{A})$ can also be defined when $\mathcal{A}$ is an additive category.

The homology functor $H^i : \text{Comp}(\mathcal{A}) \to \mathcal{A}$ taking a complex $K^\bullet \mapsto H^i(K^\bullet)$ extends to functors $H^i : K(\mathcal{A}) \to \mathcal{A}$ and $H^i : D(\mathcal{A}) \to \mathcal{A}$.

\textbf{Lemma 82.2.} An object $E$ of $D(\mathcal{A})$ is contained in $D^+(\mathcal{A})$ if and only if $H^i(E) = 0$ for all $i \leq 0$. Similar statements hold for $D^-$ and $D^+$.

Lemma 82.3. Morphisms between objects in the derived category.

1. Let $I^\bullet \in \text{Comp}^+(\mathcal{A})$ with $I^n$ injective for all $n \in \mathbb{Z}$. Then
   \[
   \text{Hom}_{D(\mathcal{A})}(K^\bullet, I^\bullet) = \text{Hom}_{K(\mathcal{A})}(K^\bullet, I^\bullet).
   \]

2. Let $P^\bullet \in \text{Comp}^-(\mathcal{A})$ with $P^n$ projective for all $n \in \mathbb{Z}$. Then
   \[
   \text{Hom}_{D(\mathcal{A})}(P^\bullet, K^\bullet) = \text{Hom}_{K(\mathcal{A})}(P^\bullet, K^\bullet).
   \]

3. If $\mathcal{A}$ has enough injectives and $\mathcal{I} \subset \mathcal{A}$ is the additive subcategory of injectives, then $D^+(\mathcal{A}) \cong K^+(\mathcal{I})$ (as triangulated categories).

4. If $\mathcal{A}$ has enough projectives and $\mathcal{P} \subset \mathcal{A}$ is the additive subcategory of projectives, then $D^-(\mathcal{A}) \cong K^-(\mathcal{P})$.

Proof. Omitted. □

Definition 82.4. Let $F : \mathcal{A} \to \mathcal{B}$ be a left exact functor and assume that $\mathcal{A}$ has enough injectives. We define the total right derived functor of $F$ as the functor $RF : D^+(\mathcal{A}) \to D^+(\mathcal{B})$ fitting into the diagram

\[
D^+(\mathcal{A}) \xrightarrow{RF} D^+(\mathcal{B}) \quad \xrightarrow{F} \quad K^+(\mathcal{I}) \xrightarrow{F} K^+(\mathcal{B}).
\]

This is possible since the left vertical arrow is invertible by the previous lemma. Similarly, let $G : \mathcal{A} \to \mathcal{B}$ be a right exact functor and assume that $\mathcal{A}$ has enough projectives. We define the total right derived functor of $G$ as the functor $LG : D^-(\mathcal{A}) \to D^-(\mathcal{B})$ fitting into the diagram

\[
D^-(\mathcal{A}) \xrightarrow{LG} D^-(\mathcal{B}) \quad \xleftarrow{G} \quad K^-(\mathcal{P}) \xleftarrow{G} K^-(\mathcal{B}).
\]

This is possible since the left vertical arrow is invertible by the previous lemma.

Remark 82.5. In these cases, it is true that $R^iF(K^\bullet) = H^i(RF(K^\bullet))$, where the left hand side is defined to be $i$th homology of the complex $F(K^\bullet)$.

83. Filtered derived category

Definition 83.1. Let $\mathcal{A}$ be an abelian category.

1. Let Fil($\mathcal{A}$) be the category of filtered objects $(A, F)$ of $\mathcal{A}$, where $F$ is a filtration of the form
   \[
   A \supset \ldots \supset F^n A \supset F^{n+1} A \supset \ldots \supset 0.
   \]
   This is an additive category.

2. We denote Fil$^f$($\mathcal{A}$) the full subcategory of Fil($\mathcal{A}$) whose objects $(A, F)$ have finite filtration. This is also an additive category.

3. An object $I \in$ Fil$^f$($\mathcal{A}$) is called filtered injective (respectively projective) provided that $\text{gr}^p(I) = \text{gr}_F^p(I) = F^p I/F^{p+1} I$ is injective (resp. projective) in $\mathcal{A}$ for all $p$. 
(4) The category of complexes \( \text{Comp}(\text{Fil}^f(A)) \supset \text{Comp}^+(\text{Fil}^f(A)) \) and its homotopy category \( K(\text{Fil}^f(A)) \supset K^+(\text{Fil}^f(A)) \) are defined as before.

(5) A morphism \( \alpha : K^\bullet \rightarrow L^\bullet \) of complexes in \( \text{Comp}(\text{Fil}^f(A)) \) is called a filtered quasi-isomorphism provided that

\[
gr^p(\alpha) : \text{gr}^p(K^\bullet) \rightarrow \text{gr}^p(L^\bullet)
\]

is a quasi-isomorphism for all \( p \in \mathbb{Z} \).

(6) We define \( DF(A) \) (resp. \( DF^+(A) \)) by inverting the filtered quasi-isomorphisms in \( K(\text{Fil}^f(A)) \) (resp. \( K^+(\text{Fil}^f(A)) \)).

\[\text{Lemma 83.2.} \quad \text{If } A \text{ has enough injectives, then } DF^+(A) \cong K^+(\text{Fil}^f(A)) \text{, where } I \text{ is the full additive subcategory of } \text{Fil}^f(A) \text{ consisting of filtered injective objects. Similarly, if } A \text{ has enough projectives, then } DF^-(A) \cong K^-(\text{Fil}^f(A)) \text{, where } P \text{ is the full additive subcategory of } \text{Fil}^f(A) \text{ consisting of filtered projective objects.} \]

\[\text{Proof.} \quad \text{Omitted.} \quad \Box\]

**84. Filtered derived functors**

And then there are the filtered derived functors.

**Definition 84.1.** Let \( T : A \rightarrow B \) be a left exact functor and assume that \( A \) has enough injectives. Define \( RT : DF^+(A) \rightarrow DF^+(B) \) to fit in the diagram

\[
\begin{array}{ccc}
DF^+(A) & \xrightarrow{RT} & DF^+(B) \\
\uparrow & & \uparrow \\
K^+(I) & \xrightarrow{T} & K^+(\text{Fil}^f(B)).
\end{array}
\]

This is well-defined by the previous lemma. Let \( G : A \rightarrow B \) be a right exact functor and assume that \( A \) has enough projectives. Define \( LG : DF^+(A) \rightarrow DF^+(B) \) to fit in the diagram

\[
\begin{array}{ccc}
DF^-(A) & \xrightarrow{LG} & DF^-(B) \\
\uparrow & & \uparrow \\
K^-(P) & \xrightarrow{G} & K^-(\text{Fil}^f(B)).
\end{array}
\]

Again, this is well-defined by the previous lemma. The functors \( RT \), resp. \( LG \), are called the filtered derived functor of \( T \), resp. \( G \).

**Proposition 84.2.** In the situation above, we have

\[
gr^p \circ RT = RT \circ gr^p
\]

where the \( RT \) on the left is the filtered derived functor while the one on the right is the total derived functor. That is, there is a commuting diagram

\[
\begin{array}{ccc}
DF^+(A) & \xrightarrow{RT} & DF^+(B) \\
gr^p & & gr^p \\
D^+(A) & \xrightarrow{RT} & D^+(B).
\end{array}
\]

**Proof.** Omitted. \( \Box \)
Given $K^\bullet \in DF^+(B)$, we get a spectral sequence
\[ E_1^{p,q} = H^{p+q}(\text{gr}^p K^\bullet) \Rightarrow H^{p+q}(\text{forget filt}(K^\bullet)). \]

85. Application of filtered complexes

Let $\mathcal{A}$ be an abelian category with enough injectives, and $0 \to L \to M \to N \to 0$ a short exact sequence in $\mathcal{A}$. Consider $\tilde{M} \in \text{Fil}^f(\mathcal{A})$ to be $M$ along with the filtration defined by
\[ F_1^0 M = L, \quad F_n^0 M = M \text{ for } n \leq 0, \quad F_n^0 M = 0 \text{ for } n \geq 2. \]

By definition, we have
\[ \text{forget filt}(\tilde{M}) = M, \quad \text{gr}^0(\tilde{M}) = N, \quad \text{gr}^1(\tilde{M}) = L \]
and $\text{gr}^n(\tilde{M}) = 0$ for all other $n \neq 0, 1$. Let $T : \mathcal{A} \to \mathcal{B}$ be a left exact functor. Assume that $\mathcal{A}$ has enough injectives. Then $RT(\tilde{M}) \in DF^+(B)$ is a filtered complex with
\[ \text{gr}^p(RT(\tilde{M})) \cong \begin{cases} 0 & \text{if } p \neq 0, 1, \\ RT(N) & \text{if } p = 0, \\ RT(L) & \text{if } p = 1. \end{cases} \]
and $\text{forget filt}(RT(\tilde{M})) \cong RT(M)$. The spectral sequence applied to $RT(\tilde{M})$ gives
\[ E_1^{p,q} = R^{p+q}T(\text{gr}^p(\tilde{M})) \Rightarrow R^{p+q}T(\text{forget filt}(\tilde{M})). \]

Unwinding the spectral sequence gives us the long exact sequence
\[
\begin{array}{c}
0 \to T(L) \to T(M) \to T(N) \\
\downarrow \quad \downarrow R^1T(L) \to R^1T(M) \to \ldots
\end{array}
\]

This will be used as follows. Let $X/k$ be a scheme of finite type. Let $F$ be a flat constructible $\mathbb{Z}/\ell^n\mathbb{Z}$-module. Then we want to show that the trace
\[ \text{Tr}((\pi_X^*)^*R\Gamma_c(X_{\overline{k}}, F)) \in \mathbb{Z}/\ell^n\mathbb{Z} \]
is additive on short exact sequences. To see this, it will not be enough to work with $R\Gamma_c(X_{\overline{k}}, -) \in D^+(\mathbb{Z}/\ell^n\mathbb{Z})$, but we will have to use the filtered derived category.

86. Perfectness

Let $\Lambda$ be a (possibly noncommutative) ring, $\text{Mod}_\Lambda$ the category of left $\Lambda$-modules, $K(\Lambda) = K(\text{Mod}_\Lambda)$ its homotopy category, and $D(\Lambda) = D(\text{Mod}_\Lambda)$ the derived category.

Definition 86.1. We denote by $K_{perf}(\Lambda)$ the category whose objects are bounded complexes of finite projective $\Lambda$-modules, and whose morphisms are morphisms of complexes up to homotopy. The functor $K_{perf}(\Lambda) \to D(\Lambda)$ is fully faithful (Derived Categories, Lemma 19.8). Denote $D_{perf}(\Lambda)$ its essential image. An object of $D(\Lambda)$ is called perfect if it is in $D_{perf}(\Lambda)$.

Proposition 86.2. Let $K \in D_{perf}(\Lambda)$ and $f \in \text{End}_{D(\Lambda)}(K)$. Then the trace $\text{Tr}(f) \in \Lambda^2$ is well defined.
Proof. We will use Derived Categories, Lemma 19.8 without further mention in this proof. Let $P^\bullet$ be a bounded complex of finite projective $\Lambda$-modules and let $\alpha : P^\bullet \to K$ be an isomorphism in $D(\Lambda)$. Then $\alpha^{-1} \circ f \circ \alpha$ corresponds to a morphism of complexes $f^\bullet : P^\bullet \to P^\bullet$ well defined up to homotopy. Set

$$\text{Tr}(f) = \sum_i (-1)^i \text{Tr}(f^i : P^i \to P^i) \in \Lambda^2.$$ 

Given $P^\bullet$ and $\alpha$, this is independent of the choice of $f^\bullet$. Namely, any other choice is of the form $\tilde{f}^\bullet = f^\bullet + dh + hd$ for some $h^i : P^i \to P^{i-1}$ $(i \in \mathbb{Z})$. But

$$\text{Tr}(dh) = \sum_i (-1)^i \text{Tr}(P^i \xrightarrow{dh} P^i)$$

$$= \sum_i (-1)^i \text{Tr}(P^{i-1} \xrightarrow{hd} P^{i-1})$$

$$= -\sum_i (-1)^{i-1} \text{Tr}(P^{i-1} \xrightarrow{hd} P^{i-1})$$

$$= -\text{Tr}(hd)$$

and so $\sum_i (-1)^i \text{Tr}((dh+hd)|_{P^i}) = 0$. Furthermore, this is independent of the choice of $(P^\bullet, \alpha)$: suppose $(Q^\bullet, \beta)$ is another choice. The compositions

$$Q^\bullet \xrightarrow{\beta} K \xrightarrow{\alpha^{-1}} P^\bullet$$

and

$$P^\bullet \xrightarrow{\alpha} K \xrightarrow{\beta^{-1}} Q^\bullet$$

are representable by morphisms of complexes $\gamma_1^\bullet$ and $\gamma_2^\bullet$ respectively, such that $\gamma_2^\bullet \circ \gamma_1^\bullet$ is homotopic to the identity. Thus, the morphism of complexes $\gamma_2^\bullet \circ f^\bullet \circ \gamma_1^\bullet : Q^\bullet \to Q^\bullet$ represents the morphism $\beta^{-1} \circ f \circ \beta$ in $D(\Lambda)$. Now

$$\text{Tr}(\gamma_2^\bullet \circ f^\bullet \circ \gamma_1^\bullet|_{Q^\bullet}) = \text{Tr}(\gamma_1^\bullet \circ \gamma_2^\bullet \circ f^\bullet|_{P^\bullet})$$

$$= \text{Tr}(f^\bullet|_{P^\bullet})$$

by the fact that $\gamma_1^\bullet \circ \gamma_2^\bullet$ is homotopic to the identity and the independence of the choice of $f^\bullet$ we saw above. \qed

87. Filtrations and perfect complexes

We now present a filtered version of the category of perfect complexes. An object $(M,F)$ of $\text{Fil}^f(\text{Mod}_\Lambda)$ is called filtered finite projective if for all $p$, $\text{gr}_p^F(M)$ is finite and projective. We then consider the homotopy category $KF_{perf}(\Lambda)$ of bounded complexes of filtered finite projective objects of $\text{Fil}^f(\text{Mod}_\Lambda)$. We have a diagram of categories

$$\begin{array}{ccc}
KF(\Lambda) & \supset & KF_{perf}(\Lambda) \\
\downarrow & & \downarrow \\
DF(\Lambda) & \supset & DF_{perf}(\Lambda)
\end{array}$$

where the vertical functor on the right is fully faithful and the category $DF_{perf}(\Lambda)$ is its essential image, as before.

Lemma 87.1 (Additivity). Let $K \in DF_{perf}(\Lambda)$ and $f \in \text{End}_{DF}(K)$. Then

$$\text{Tr}(f|_K) = \sum_{p \in \mathbb{Z}} \text{Tr}(f|_{\text{gr}^p K}).$$

Proof. By Proposition 86.2 we may assume we have a bounded complex $P^\bullet$ of filtered finite projectives of $\text{Fil}^f(\text{Mod}_\Lambda)$ and a map $f^\bullet : P^\bullet \to P^\bullet$ in $\text{Comp}(\text{Fil}^f(\text{Mod}_\Lambda))$. So the lemma follows from the following result, which proof is left to the reader. \qed
Lemma 87.2. Let $P \in \text{Fil}^f(\text{Mod}_\Lambda)$ be filtered finite projective, and $f : P \to P$ an endomorphism in $\text{Fil}^f(\text{Mod}_\Lambda)$. Then

$$\text{Tr}(f|_P) = \sum_p \text{Tr}(f|_{\text{gr}^p(P)}).$$

Proof. Omitted. □

88. Characterizing perfect objects

For the commutative case see More on Algebra, Sections 54, 55, and 61.

Definition 88.1. Let $\Lambda$ be a (possibly noncommutative) ring. An object $K \in D(\Lambda)$ has finite Tor-dimension if there exist $a, b \in \mathbb{Z}$ such that for any right $\Lambda$-module $N$, we have $H^i(N \otimes^L \Lambda K) = 0$ for all $i \notin [a, b]$.

This in particular means that $K \in D^b(\Lambda)$ as we see by taking $N = \Lambda$.

Lemma 88.2. Let $\Lambda$ be a left noetherian ring and $K \in D(\Lambda)$. Then $K$ is perfect if and only if the two following conditions hold:

1. $K$ has finite Tor-dimension, and
2. for all $i \in \mathbb{Z}$, $H^i(K)$ is a finite $\Lambda$-module.

Proof. See More on Algebra, Lemma 61.2 for the proof in the commutative case. □

The reader is strongly urged to try and prove this. The proof relies on the fact that a finite module on a finitely left-presented ring is flat if and only if it is projective.

Remark 88.3. A variant of this lemma is to consider a Noetherian scheme $X$ and the category $D_{\text{perf}}(\mathcal{O}_X)$ of complexes which are locally quasi-isomorphic to a finite complex of finite locally free $\mathcal{O}_X$-modules. Objects $K$ of $D_{\text{perf}}(\mathcal{O}_X)$ can be characterized by having coherent cohomology sheaves and bounded tor dimension.

89. Complexes with constructible cohomology

Let $\Lambda$ be a ring. Let $X$ be a scheme. Let $K(X, \Lambda)$ the homotopy category of sheaves of $\Lambda$-modules on $X_{\text{\acute{e}tale}}$. Denote $D(X, \Lambda)$ the corresponding derived category. We denote by $D^b(X, \Lambda)$ (respectively $D^+, D^-$) the full subcategory of bounded (resp. above, below) complexes in $D(X, \Lambda)$.

Definition 89.1. Let $X$ be a scheme. Let $\Lambda$ be a Noetherian ring. We denote $D_c(X, \Lambda)$ the full subcategory of $D(X, \Lambda)$ of complexes whose cohomology sheaves are constructible sheaves of $\Lambda$-modules.

This definition makes sense by Lemma 68.6 and Derived Categories, Section 13. Thus we see that $D_c(X, \Lambda)$ is a strictly full, saturated triangulated subcategory of $D(X, \Lambda)$.

Lemma 89.2. Let $\Lambda$ be a Noetherian ring. If $j : U \to X$ is an étale morphism of schemes, then

1. $K|_U \in D_c(U, \Lambda)$ if $K \in D_c(X, \Lambda)$, and
2. $j_* M \in D_c(X, \Lambda)$ if $M \in D_c(U, \Lambda)$ and the morphism $j$ is quasi-compact and quasi-separated.

Proof. The first assertion is clear. The second follows from the fact that $j_!$ is exact and Lemma 70.1. □
095Y **Lemma 89.3.** Let \( \Lambda \) be a Noetherian ring. Let \( f : X \to Y \) be a morphism of schemes. If \( K \in D_c(Y, \Lambda) \) then \( Lf^*K \in D_c(X, \Lambda) \).

**Proof.** This follows as \( f^{-1} = f^* \) is exact and Lemma 68.5.

095Z **Lemma 89.4.** Let \( X \) be a quasi-compact and quasi-separated scheme. Let \( \Lambda \) be a Noetherian ring. Let \( K \in D(X, \Lambda) \) and \( b \in \mathbf{Z} \) such that \( H^b(K) \) is constructible. Then there exist a sheaf \( \mathcal{F} \) which is a finite direct sum of \( j_{U!}\Lambda \) with \( U \in \text{Ob}(X_{\text{étale}}) \) affine and a map \( \mathcal{F}[-b] \to K \) in \( D(X, \Lambda) \) inducing a surjection \( \mathcal{F} \to H^b(K) \).

**Proof.** Represent \( K \) by a complex \( \mathcal{K}^\bullet \) of sheaves of \( \Lambda \)-modules. Consider the surjection

\[
\text{Ker}(\mathcal{K}^b \to \mathcal{K}^{b+1}) \twoheadrightarrow H^b(K)
\]

By Modules on Sites, Lemma 29.5 we may choose a surjection \( \bigoplus_{i \in I} j_{U!}\Lambda \to \text{Ker}(\mathcal{K}^b \to \mathcal{K}^{b+1}) \) with \( U_i \) affine. For \( I' \subset I \) finite, denote \( \mathcal{H}_{I'} \subset H^b(K) \) the image of \( \bigoplus_{i \in I'} j_{U!}\Lambda \). By Lemma 68.9 we see that \( \mathcal{H}_{I'} = H^b(K) \) for some \( I' \subset I \) finite. The lemma follows taking \( \mathcal{F} = \bigoplus_{i \in I'} j_{U!}\Lambda \).

0960 **Lemma 89.5.** Let \( X \) be a quasi-compact and quasi-separated scheme. Let \( \Lambda \) be a Noetherian ring. Let \( K \in D_c(X, \Lambda) \). Then the following are equivalent

1. \( K \) is in \( D_c(X, \Lambda) \),
2. \( K \) can be represented by a bounded above complex whose terms are finite direct sums of \( j_{U!}\Lambda \) with \( U \in \text{Ob}(X_{\text{étale}}) \) affine,
3. \( K \) can be represented by a bounded above complex of flat constructible sheaves of \( \Lambda \)-modules.

**Proof.** It is clear that (2) implies (3) and that (3) implies (1). Assume \( K \) is in \( D_c(X, \Lambda) \). Say \( H^i(K) = 0 \) for \( i > b \). By induction on \( a \) we will construct a complex \( \mathcal{F}^a \to \ldots \to \mathcal{F}^b \) such that each \( \mathcal{F}^i \) is a finite direct sum of \( j_{U!}\Lambda \) with \( U \in \text{Ob}(X_{\text{étale}}) \) affine and a map \( \mathcal{F}^* \to K \) which induces an isomorphism \( H^i(\mathcal{F}^*) \to H^i(K) \) for \( i > a \) and a surjection \( H^a(\mathcal{F}^*) \to H^a(K) \). For \( a = b \) this can be done by Lemma 89.4. Given such a datum choose a distinguished triangle

\[
\mathcal{F}^* \to K \to L \to \mathcal{F}^*[1]
\]

Then we see that \( H^i(L) = 0 \) for \( i \geq a \). Choose \( \mathcal{F}^{a-1}[-a+1] \to L \) as in Lemma 89.4. The composition \( \mathcal{F}^{a-1}[-a+1] \to L \to \mathcal{F}^* \) corresponds to a map \( \mathcal{F}^{a-1} \to \mathcal{F}^a \) such that the composition with \( \mathcal{F}^a \to \mathcal{F}^{a+1} \) is zero. By TR4 we obtain a map

\[
(\mathcal{F}^{a-1} \to \ldots \to \mathcal{F}^b) \to K
\]

in \( D(X, \Lambda) \). This finishes the induction step and the proof of the lemma.

0961 **Lemma 89.6.** Let \( X \) be a scheme. Let \( \Lambda \) be a Noetherian ring. Let \( K, L \in D_c^-(X, \Lambda) \). Then \( K \otimes_{\Lambda} L \) is in \( D_c^-(X, \Lambda) \).

**Proof.** This follows from Lemmas 89.5 and 68.7.

03TQ **Definition 89.7.** Let \( X \) be a scheme. Let \( \Lambda \) be a Noetherian ring. We denote \( D_{\text{ctf}}(X, \Lambda) \) the full subcategory of \( D_c(X, \Lambda) \) consisting of objects having locally finite tor dimension.
This is a strictly full, saturated triangulated subcategory of $D_c(X, \Lambda)$ and $D(X, \Lambda)$.

By our conventions, see Cohomology on Sites, Definition 36.1 we see that

$$D_{ctf}(X, \Lambda) \subset D^b(X, \Lambda)$$

if $X$ is quasi-compact. A good way to think about objects of $D_{ctf}(X, \Lambda)$ is given in Lemma 89.9.

Remark 89.8. The situation with objects of $D_{ctf}(X, \Lambda)$ is different from $D_{perf}(\mathcal{O}_X)$ in Remark 88.3. Namely, it can happen that a complex of $\mathcal{O}_X$-modules is locally quasi-isomorphic to a finite complex of finite locally free $\mathcal{O}_X$-modules, without being globally quasi-isomorphic to a bounded complex of locally free $\mathcal{O}_X$-modules.

The following lemma shows this does not happen for $D_{ctf}$ on a Noetherian scheme.

Lemma 89.9. Let $\Lambda$ be a Noetherian ring. Let $X$ be a quasi-compact and quasi-separated scheme. Let $K \in D(X, \Lambda)$. The following are equivalent

1. $K \in D_{ctf}(X, \Lambda)$, and
2. $K$ can be represented by a finite complex of constructible flat sheaves of $\Lambda$-modules.

In fact, if $K$ has tor amplitude in $[a, b]$ then we can represent $K$ by a complex $\mathcal{F}^a \to \ldots \to \mathcal{F}^b$ with $\mathcal{F}^b$ a constructible flat sheaf of $\Lambda$-modules.

Proof. It is clear that a finite complex of constructible flat sheaves of $\Lambda$-modules has finite tor dimension. It is also clear that it is an object of $D_c(X, \Lambda)$. Thus we see that (2) implies (1).

Assume (1). Choose $a, b \in \mathbb{Z}$ such that $H^i(K \otimes_\Lambda \mathcal{G}) = 0$ if $i \notin [a, b]$ for all sheaves of $\Lambda$-modules $\mathcal{G}$. We will prove the final assertion holds by induction on $b - a$. If $a = b$, then $K = H^a(K)[-a]$ is a flat constructible sheaf and the result holds. Next, assume $b > a$. Represent $K$ by a complex $K^\bullet$ of sheaves of $\Lambda$-modules. Consider the surjection

$$\text{Ker}(K^b \to K^{b+1}) \to H^b(K)$$

By Lemma 70.6 we can find finitely many affine schemes $U_i$ étale over $X$ and a surjection $\bigoplus j_{U_i!} \Delta_{U_i} \to H^b(K)$. After replacing $U_i$ by standard étale coverings $\{U_{ij} \to U_i\}$ we may assume this surjection lifts to a map $\mathcal{F} = \bigoplus j_{U_i!} \Delta_{U_i} \to \text{Ker}(K^b \to K^{b+1})$. This map determines a distinguished triangle

$$\mathcal{F}[-b] \to K \to L \to \mathcal{F}[-b+1]$$

in $D(X, \Lambda)$. Since $D_{ctf}(X, \Lambda)$ is a triangulated subcategory we see that $L$ is in it too. In fact $L$ has tor amplitude in $[a, b - 1]$ as $\mathcal{F}$ surjects onto $H^b(K)$ (details omitted). By induction hypothesis we can find a finite complex $\mathcal{F}^a \to \ldots \to \mathcal{F}^{b-1}$ of flat constructible sheaves of $\Lambda$-modules representing $L$. The map $L \to \mathcal{F}[-b+1]$ corresponds to a map $\mathcal{F}^{b-1} \to \mathcal{F}^b$. Then it follows from axiom TR3 that $K$ is represented by the complex

$$\mathcal{F}^a \to \ldots \to \mathcal{F}^{b-1} \to \mathcal{F}^b$$

which finishes the proof. □

Remark 89.10. Let $\Lambda$ be a Noetherian ring. Let $X$ be a scheme. For a bounded complex $K^\bullet$ of constructible flat $\Lambda$-modules on $X_{\text{ét}}$ each stalk $K^p_x$ is a finite projective $\Lambda$-module. Hence the stalks of the complex are perfect complexes of $\Lambda$-modules.
Lemma 89.11. Let $\Lambda$ be a Noetherian ring. If $j : U \to X$ is an étale morphism of schemes, then
\begin{enumerate}
\item $K|_U \in D_{ctf}(U, \Lambda)$ if $K \in D_{ctf}(X, \Lambda)$, and
\item $j_! M \in D_{ctf}(X, \Lambda)$ if $M \in D_{ctf}(U, \Lambda)$ and the morphism $j$ is quasi-compact and quasi-separated.
\end{enumerate}

Proof. Perhaps the easiest way to prove this lemma is to reduce to the case where $X$ is affine and then apply Lemma 89.9 to translate it into a statement about finite complexes of flat constructible sheaves of $\Lambda$-modules where the result follows from Lemma 70.1. \hfill $\square$

Lemma 89.13. Let $\Lambda$ be a Noetherian ring. Let $f : X \to Y$ be a morphism of schemes. If $K \in D_{ctf}(Y, \Lambda)$ then $Lf^* K \in D_{ctf}(X, \Lambda)$.

Proof. Apply Lemma 89.9 to reduce this to a question about finite complexes of flat constructible sheaves of $\Lambda$-modules. Then the statement follows as $f^{-1} = f^*$ is exact and Lemma 68.5. \hfill $\square$

Lemma 89.14. Let $X$ be a connected scheme. Let $\Lambda$ be a Noetherian ring. Let $K \in D_{ctf}(X, \Lambda)$ have locally constant cohomology sheaves. Then there exists a finite complex of finite projective $\Lambda$-modules $M^\bullet$ and an étale covering $\{U_i \to X\}$ such that $K|_{U_i} \cong M^\bullet|_{U_i}$ in $D(U_i, \Lambda)$.

Proof. Choose an étale covering $\{U_i \to X\}$ such that $K|_{U_i}$ is constant, say $K|_{U_i} \cong M_i^\bullet|_{U_i}$ for some finite complex of finite $\Lambda$-modules $M_i^\bullet$. See Cohomology on Sites, Lemma 26.1. Observe that $U_i \times_X U_j$ is empty if $M_i^\bullet$ is not isomorphic to $M_j^\bullet$ in $D(\Lambda)$. For each complex of $\Lambda$-modules $M^\bullet$ let $I_{M^\bullet} = \{i \in I \mid M_i^\bullet \cong M^\bullet\} \in D(\Lambda)$.

As étale morphisms are open we see that $U_{M^\bullet} = \bigcup_{i \in I_{M^\bullet}} \text{Im}(U_i \to X)$ is an open subset of $X$. Then $X = \coprod U_{M^\bullet}$ is a disjoint open covering of $X$. As $X$ is connected only one $U_{M^\bullet}$ is nonempty. As $K$ is in $D_{ctf}(X, \Lambda)$ we see that $M^\bullet$ is a perfect complex of $\Lambda$-modules, see More on Algebra, Lemma 89.14. Hence we may assume $M^\bullet$ is a finite complex of finite projective $\Lambda$-modules. \hfill $\square$

90. Cohomology of nice complexes

The following is a special case of a more general result about compactly supported cohomology of objects of $D_{ctf}(X, \Lambda)$.

Proposition 90.1. Let $X$ be a projective curve over a field $k$, $\Lambda$ a finite ring and $K \in D_{ctf}(X, \Lambda)$. Then $R\Gamma(X_\bar{k}, K) \in D_{perf}(\Lambda)$.

Sketch of proof. The first step is to show:
\begin{enumerate}
\item The cohomology of $R\Gamma(X_\bar{k}, K)$ is bounded.
\end{enumerate}

Consider the spectral sequence
\[ H^i(X_\bar{k}, H^j(K)) \Rightarrow H^{i+j}(R\Gamma(X_\bar{k}, K)). \]

Since $K$ is bounded and $\Lambda$ is finite, the sheaves $H^j(K)$ are torsion. Moreover, $X_\bar{k}$ has finite cohomological dimension, so the left-hand side is nonzero for finitely many $i$ and $j$ only. Therefore, so is the right-hand side.
(2) The cohomology groups $H^{i+j}(R\Gamma(X_\bar{k}, K))$ are finite. Since the sheaves $H^i(K)$ are constructible, the groups $H^i(X_\bar{k}, H^j(K))$ are finite (Section 74) so it follows by the spectral sequence again.

(3) $R\Gamma(X_\bar{k}, K)$ has finite Tor-dimension. Let $N$ be a right $\Lambda$-module (in fact, since $\Lambda$ is finite, it suffices to assume that $N$ is finite). By the projection formula (change of module),

$$N \otimes^L_\Lambda R\Gamma(X_\bar{k}, K) = R\Gamma(X_\bar{k}, N \otimes^L_\Lambda K).$$

Therefore,

$$H^i(N \otimes^L_\Lambda R\Gamma(X_\bar{k}, K)) = H^i(R\Gamma(X_\bar{k}, N \otimes^L_\Lambda K)).$$

Now consider the spectral sequence

$$H^i(X_\bar{k}, H^j(N \otimes^L_\Lambda K)) \Rightarrow H^{i+j}(R\Gamma(X_\bar{k}, N \otimes^L_\Lambda K)).$$

Since $K$ has finite Tor-dimension, $H^j(N \otimes^L_\Lambda K)$ vanishes universally for $j$ small enough, and the left-hand side vanishes whenever $i < 0$. Therefore $R\Gamma(X_\bar{k}, K)$ has finite Tor-dimension, as claimed. So it is a perfect complex by Lemma 88.2. □

91. Lefschetz numbers

03TW The fact that the total cohomology of a constructible complex of finite tor dimension is a perfect complex is the key technical reason why cohomology behaves well, and allows us to define rigorously the traces occurring in the trace formula.

03TX Definition 91.1. Let $\Lambda$ be a finite ring, $X$ a projective curve over a finite field $k$ and $K \in D_{ctf}(X, \Lambda)$ (for instance $K = \Lambda$). There is a canonical map $c_K : \pi_X^{-1}K \to K$, and its base change $c_K|_{X_{\bar{k}}}$ induces an action denoted $\pi_X^*$ on the perfect complex $R\Gamma(X_{\bar{k}}, K|_{X_{\bar{k}}})$. The global Lefschetz number of $K$ is the trace $\text{Tr}(\pi_X^*|_{R\Gamma(X_{\bar{k}}, K)})$ of that action. It is an element of $\Lambda^\natural$.

03TY Definition 91.2. With $\Lambda, X, k, K$ as in Definition 91.1. Since $K \in D_{ctf}(X, \Lambda)$, for any geometric point $\bar{x}$ of $X$, the complex $K_{\bar{x}}$ is a perfect complex (in $D_{perf}(\Lambda)$). As we have seen in Section 79, the Frobenius $\pi_X$ acts on $K_{\bar{x}}$. The local Lefschetz number of $K$ is the sum

$$\sum_{x \in X(k)} \text{Tr}(\pi_X|_{K_{\bar{x}}})$$

which is again an element of $\Lambda^\natural$.

At last, we can formulate precisely the trace formula.

03TZ Theorem 91.3 (Lefschetz Trace Formula). Let $X$ be a projective curve over a finite field $k$, $\Lambda$ a finite ring and $K \in D_{ctf}(X, \Lambda)$. Then the global and local Lefschetz numbers of $K$ are equal, i.e.,

$$\text{Tr}(\pi_X^*|_{R\Gamma(X_{\bar{k}}, K)}) = \sum_{x \in X(k)} \text{Tr}(\pi_X|_{K_x})$$

in $\Lambda^\natural$.

Proof. See discussion below. □
We will use, rather than prove, the trace formula. Nevertheless, we will give quite a few details of the proof of the theorem as given in [Del77] (some of the things that are not adequately explained are listed in Section 98).

We only stated the formula for curves, and in some weak sense it is a consequence of the following result.

**Theorem 91.4 (Weil).** Let $C$ be a nonsingular projective curve over an algebraically closed field $k$, and $\varphi : C \to C$ a $k$-endomorphism of $C$ distinct from the identity. Let $V(\varphi) = \Delta_C \cdot \Gamma_\varphi$, where $\Delta_C$ is the diagonal, $\Gamma_\varphi$ is the graph of $\varphi$, and the intersection number is taken on $C \times C$. Let $J = \text{Pic}^0_{C/k}$ be the jacobian of $C$ and denote $\varphi^* : J \to J$ the action induced by $\varphi$ by taking pullbacks. Then

$$V(\varphi) = 1 - \text{Tr}_J(\varphi^*) + \deg \varphi.$$

**Proof.** The number $V(\varphi)$ is the number of fixed points of $\varphi$, it is equal to

$$V(\varphi) = \sum_{c \in |C| : \varphi(c) = c} m_{\text{Fix}(\varphi)}(c)$$

where $m_{\text{Fix}(\varphi)}(c)$ is the multiplicity of $c$ as a fixed point of $\varphi$, namely the order or vanishing of the image of a local uniformizer under $\varphi - \text{id}_C$. Proofs of this theorem can be found in [Lan02] and [Wei48]. □

**Example 91.5.** Let $C = E$ be an elliptic curve and $\varphi = [n]$ be multiplication by $n$. Then $\varphi^* = \varphi^t$ is multiplication by $n$ on the jacobian, so it has trace $2n$ and degree $n^2$. On the other hand, the fixed points of $\varphi$ are the points $p \in E$ such that $np = p$, which is the $(n-1)$-torsion, which has cardinality $(n-1)^2$. So the theorem reads

$$(n-1)^2 = 1 - 2n + n^2.$$

**Jacobians.** We now discuss without proofs the correspondence between a curve and its jacobian which is used in Weil’s proof. Let $C$ be a nonsingular projective curve over an algebraically closed field $k$ and choose a base point $c_0 \in C(k)$. Denote by $A^1(C \times C)$ (or $\text{Pic}(C \times C)$, or $\text{CaCl}(C \times C)$) the abelian group of codimension 1 divisors of $C \times C$. Then

$$A^1(C \times C) = \text{pr}_1^*(A^1(C)) \oplus \text{pr}_2^*(A^1(C)) \oplus R$$

where

$$R = \{ Z \in A^1(C \times C) \mid Z|_{C \times \{c_0\}} \sim \text{rat} 0 \text{ and } Z|_{\{c_0\} \times C} \sim \text{rat} 0 \}.$$ 

In other words, $R$ is the subgroup of line bundles which pull back to the trivial one under either projection. Then there is a canonical isomorphism of abelian groups $R \cong \text{End}(J)$ which maps a divisor $Z$ in $R$ to the endomorphism

$$J \quad \text{to} \quad (\text{pr}_1|_Z )^* (\text{pr}_2|_Z)^*(D).$$
The aforementioned correspondence is the following. We denote by $\sigma$ the automorphism of $C \times C$ that switches the factors.

<table>
<thead>
<tr>
<th>$\text{End}(J)$</th>
<th>$R$</th>
</tr>
</thead>
<tbody>
<tr>
<td>composition of $\alpha, \beta$</td>
<td>$\text{pr}<em>{13 \ast} (\text{pr}</em>{12 \ast}(\alpha) \circ \text{pr}_{23 \ast}(\beta))$</td>
</tr>
<tr>
<td>$\text{id}_J$</td>
<td>$\Delta_C - {c_0} \times C - C \times {c_0}$</td>
</tr>
<tr>
<td>$\varphi^*$</td>
<td>$\Gamma_\varphi - C \times {\varphi(c_0)} - \sum_{c(c) = c_0} {c} \times C$</td>
</tr>
</tbody>
</table>

The trace form

$$\alpha, \beta \mapsto \text{Tr}(\alpha \beta)$$

the Rosati involution

$$\alpha \mapsto \alpha^\dagger$$

positivity of Rosati

$$\text{Tr}(\alpha \alpha^\dagger) > 0$$

In fact, in light of the Kunneth formula, the subgroup $R$ corresponds to the 1,1 hodge classes in $H^1(C) \otimes H^1(C)$.

**Weil’s proof.** Using this correspondence, we can prove the trace formula. We have

$$V(\varphi) = \int_{C \times C} \Gamma_\varphi \cdot \Delta_C$$

$$= \int_{C \times C} \Gamma_\varphi \cdot (\Delta_C - \{c_0\} \times C - C \times \{c_0\}) + \int_{C \times C} \Gamma_\varphi \cdot (\{c_0\} \times C + C \times \{c_0\})$$

Now, on the one hand

$$\int_{C \times C} \Gamma_\varphi \cdot (\{c_0\} \times C + C \times \{c_0\}) = 1 + \deg \varphi$$

and on the other hand, since $R$ is the orthogonal of the ample divisor $\{c_0\} \times C + C \times \{c_0\}$,

$$\int_{C \times C} \Gamma_\varphi \cdot (\Delta_C - \{c_0\} \times C - C \times \{c_0\})$$

$$= \int_{C \times C} \left(\Gamma_\varphi - C \times \{\varphi(c_0)\} - \sum_{c(c) = c_0} \{c\} \times C\right) \cdot (\Delta_C - \{c_0\} \times C - C \times \{c_0\})$$

$$= -\text{Tr}_J(\varphi^* \circ \text{id}_J).$$

Recapitulating, we have

$$V(\varphi) = 1 - \text{Tr}_J(\varphi^*) + \deg \varphi$$

which is the trace formula.
Lemma 91.6. Consider the situation of Theorem 91.4 and let \( \ell \) be a prime number invertible in \( k \). Then
\[
\sum_{i=0}^{2} (-1)^i \text{Tr}(\varphi^*|_{H^i(C,\mathbb{Z}/\ell^n\mathbb{Z})}) = V(\varphi) \mod \ell^n.
\]

Sketch of proof. Observe first that the assumption makes sense because \( H^i(C,\mathbb{Z}/\ell^n\mathbb{Z}) \) is a free \( \mathbb{Z}/\ell^n\mathbb{Z} \)-module for all \( i \). The trace of \( \varphi^* \) on the 0th degree cohomology is 1. The choice of a primitive \( \ell^n \)th root of unity in \( k \) gives an isomorphism
\[
H^i(C,\mathbb{Z}/\ell^n\mathbb{Z}) \cong H^i(C,\mu_{\ell^n})
\]
compatibly with the action of the geometric Frobenius. On the other hand,
\[
H^1(C,\mu_{\ell^n}) = J[\ell^n].
\]
Therefore,
\[
\text{Tr}(\varphi^*|_{H^1(C,\mathbb{Z}/\ell^n\mathbb{Z})}) = \text{Tr}_J(\varphi^*) \mod \ell^n = \text{Tr}_{\mathbb{Z}/\ell^n\mathbb{Z}}(\varphi^*: J[\ell^n] \to J[\ell^n]).
\]
Moreover, \( H^2(C,\mu_{\ell^n}) = \text{Pic}(C)/\ell^n\text{Pic}(C) \cong \mathbb{Z}/\ell^n\mathbb{Z} \) where \( \varphi^* \) is multiplication by \( \deg \varphi \). Hence
\[
\text{Tr}(\varphi^*|_{H^2(C,\mathbb{Z}/\ell^n\mathbb{Z})}) = \deg \varphi.
\]
Thus we have
\[
\sum_{i=0}^{2} (-1)^i \text{Tr}(\varphi^*|_{H^i(C,\mathbb{Z}/\ell^n\mathbb{Z})}) = 1 - \text{Tr}_J(\varphi^*) + \deg \varphi \mod \ell^n
\]
and the corollary follows from Theorem 91.4. \( \square \)

An alternative way to prove this corollary is to show that
\[
X \mapsto H^*(X,\mathbb{Q}_\ell) = \mathbb{Q}_\ell \otimes \lim_n H^*(X,\mathbb{Z}/\ell^n\mathbb{Z})
\]
defines a Weil cohomology theory on smooth projective varieties over \( k \). Then the trace formula
\[
V(\varphi) = \sum_{i=0}^{2} (-1)^i \text{Tr}(\varphi^*|_{H^i(C,\mathbb{Q}_\ell)})
\]
is a formal consequence of the axioms (it’s an exercise in linear algebra, the proof is the same as in the topological case).

92. Preliminaries and sorites

Notation: We fix the notation for this section. We denote by \( A \) a commutative ring, \( \Lambda \) a (possibly noncommutative) ring with a ring map \( A \to \Lambda \) which image lies in the center of \( \Lambda \). We let \( G \) be a finite group, \( \Gamma \) a monoid extension of \( G \) by \( N \), meaning that there is an exact sequence
\[
1 \to G \to \tilde{\Gamma} \to \mathbb{Z} \to 1
\]
and \( \Gamma \) consists of those elements of \( \tilde{\Gamma} \) which image is nonnegative. Finally, we let \( P \) be an \( A[\Gamma] \)-module which is finite and projective as an \( A[G] \)-module, and \( M \) a \( \Lambda[\Gamma] \)-module which is finite and projective as a \( \Lambda \)-module.

Our goal is to compute the trace of \( 1 \in N \) acting over \( \Lambda \) on the coinvariants of \( G \) on \( P \otimes_A M \), that is, the number
\[
\text{Tr}_\Lambda (1; (P \otimes_A M)_G) \in \Lambda^2.
\]
The element \( 1 \in N \) will correspond to the Frobenius.
Lemma 92.1. Let \( e \in G \) denote the neutral element. The map
\[
\Lambda[G] \to \Lambda^2,
\]
factors through \( \Lambda[G]^2 \). We denote \( \varepsilon : \Lambda[G]^2 \to \Lambda^2 \) the induced map.

Proof. We have to show the map annihilates commutators. One has
\[
\left( \sum \lambda_g g \right) \left( \sum \mu_g g \right) - \left( \sum \mu_g g \right) \left( \sum \lambda_g g \right) = \sum_g \left( \lambda_{g_1} \mu_{g_2} - \mu_{g_1} \lambda_{g_2} \right) g
\]
The coefficient of \( e \) is
\[
\sum_g \left( \lambda_{g_1} \mu_{g_2} - \mu_{g_1} \lambda_{g_2} \right)
\]
which is a sum of commutators, hence it is zero in \( \Lambda^2 \).

Definition 92.2. Let \( f : P \to P \) be an endomorphism of a finite projective \( \Lambda[G] \)-module \( P \). We define \( \text{Tr}_A(G)(f; P) := \varepsilon(\text{Tr}_{\Lambda[G]}(f; P)) \) to be the \( G \)-trace of \( f \) on \( P \).

Lemma 92.3. Let \( f : P \to P \) be an endomorphism of the finite projective \( \Lambda[G] \)-module \( P \). Then
\[
\text{Tr}_A(f; P) = \#G \cdot \text{Tr}_A^G(f; P).
\]

Proof. By additivity, reduce to the case \( P = \Lambda[G] \). In that case, \( f \) is given by right multiplication by some element \( \sum \lambda_g g \) of \( \Lambda[G] \). In the basis \( (g)_{g \in G} \), the matrix of \( f \) has coefficient \( \lambda_{g_{2^{-1}} g_1} \) in the \((g_1, g_2)\) position. In particular, all diagonal coefficients are \( \lambda_e \), and there are \( \#G \) such coefficients.

Lemma 92.4. The map \( A \to \Lambda \) defines an \( A \)-module structure on \( \Lambda^2 \).

Proof. This is clear.

Lemma 92.5. Let \( P \) be a finite projective \( \Lambda[G] \)-module and \( M \) a \( \Lambda[G] \)-module, finite projective as a \( \Lambda \)-module. Then \( P \otimes_A M \) is a finite projective \( \Lambda[G] \)-module, for the structure induced by the diagonal action of \( G \).

Note that \( P \otimes_A M \) is naturally a \( \Lambda \)-module since \( M \) is. Explicitly, together with the diagonal action this reads
\[
\left( \sum \lambda_g g \right) (p \otimes m) = \sum gp \otimes \lambda_g gm.
\]

Proof. For any \( \Lambda[G] \)-module \( N \) one has
\[
\text{Hom}_{\Lambda[G]}(P \otimes_A M, N) = \text{Hom}_{\Lambda[G]}(P, \text{Hom}_{\Lambda}(M, N))
\]
where the \( G \)-action on \( \text{Hom}_{\Lambda}(M, N) \) is given by \((g \cdot \varphi)(m) = g \varphi(g^{-1}m)\). Now it suffices to observe that the right-hand side is a composition of exact functors, because of the projectivity of \( P \) and \( M \).

Lemma 92.6. With assumptions as in Lemma 92.5, let \( u \in \text{End}_{\Lambda[G]}(P) \) and \( v \in \text{End}_{\Lambda[G]}(M) \). Then
\[
\text{Tr}_A^G(u \otimes v; P \otimes_A M) = \text{Tr}_A^G(u; P) \cdot \text{Tr}_A(v; M).
\]
Sketch of proof. Reduce to the case $P = A[G]$. In that case, $u$ is right multiplication by some element $a = \sum a_g g$ of $A[G]$, which we write $u = R_a$. There is an isomorphism of $\Lambda[G]$-modules

$$\varphi : A[G] \otimes_A M \cong (A[G] \otimes_A M)'$$

where $(A[G] \otimes_A M)'$ has the module structure given by the left $G$-action, together with the $\Lambda$-linearity on $M$. This transport of structure changes $u \otimes v$ into $\sum g a_g R_g \otimes g^{-1} v$. In other words,

$$\varphi \circ (u \otimes v) \circ \varphi^{-1} = \sum g a_g R_g \otimes g^{-1} v.$$

Working out explicitly both sides of the equation, we have to show

$$\text{Tr}_{\Lambda}^G \left( \sum_g a_g R_g \otimes g^{-1} v \right) = a_e \cdot \text{Tr}_{\Lambda}(v; M).$$

This is done by showing that

$$\text{Tr}_{\Lambda}^G \left( a_g R_g \otimes g^{-1} v \right) = \begin{cases} 0 & \text{if } g \neq e \\ a_e \cdot \text{Tr}_{\Lambda}(v; M) & \text{if } g = e \end{cases}$$

by reducing to $M = \Lambda$. \hfill \Box

Notation: Consider the monoid extension $1 \to G \to \Gamma \to N \to 1$ and let $\gamma \in \Gamma$. Then we write $Z_\gamma = \{ g \in G | g\gamma = \gamma g \}$.

**Lemma 92.7.** Let $P$ be a $\Lambda[\Gamma]$-module, finite and projective as a $\Lambda[G]$-module, and $\gamma \in \Gamma$. Then

$$\text{Tr}_{\Lambda}(\gamma, P) = \#Z_\gamma \cdot \text{Tr}_{\Lambda}^Z(\gamma, P).$$

**Proof.** This follows readily from Lemma 92.3. \hfill \Box

**Lemma 92.8.** Let $P$ be an $A[\Gamma]$-module, finite projective as $A[G]$-module. Let $M$ be a $\Lambda[\Gamma]$-module, finite projective as a $\Lambda$-module. Then

$$\text{Tr}_{\Lambda}^Z(\gamma, P \otimes_A M) = \text{Tr}_{\Lambda}^Z(\gamma, P) \cdot \text{Tr}_{\Lambda}(\gamma, M).$$

**Proof.** This follows directly from Lemma 92.6. \hfill \Box

**Lemma 92.9.** Let $P$ be a $\Lambda[\Gamma]$-module, finite projective as $\Lambda[G]$-module. Then the coinvariants $P_G = \Lambda \otimes_{\Lambda[G]} P$ form a finite projective $\Lambda$-module, endowed with an action of $\Gamma/G = N$. Moreover, we have

$$\text{Tr}_{\Lambda}(1; P_G) = \sum_{\gamma \to 1}^{'} \text{Tr}_{\Lambda}^Z(\gamma, P)$$

where $\sum_{\gamma \to 1}^{'}$ means taking the sum over the $G$-conjugacy classes in $\Gamma$.

**Sketch of proof.** We first prove this after multiplying by $\#G$.

$$\#G \cdot \text{Tr}_{\Lambda}(1; P_G) = \text{Tr}_{\Lambda}(\sum_{\gamma \to 1} \gamma, P_G) = \text{Tr}_{\Lambda}(\sum_{\gamma \to 1} \gamma, P)$$

where the second equality follows by considering the commutative triangle

$$\begin{array}{ccc}
P & \overset{a}{\longrightarrow} & P_G \\
\downarrow{c} & & \downarrow{b} \\
\end{array}$$

$$\begin{array}{ccc}
P^G & \overset{a}{\longrightarrow} & P \\
\end{array}$$
where \( a \) is the canonical inclusion, \( b \) the canonical surjection and \( c = \sum_{\gamma \to 1} \gamma \). Then we have

\[
(\sum_{\gamma \to 1} \gamma)|_P = a \circ c \circ b \quad \text{and} \quad (\sum_{\gamma \to 1} \gamma)|_{P_G} = b \circ a \circ c
\]

hence they have the same trace. We then have

\[
\# G \cdot \text{Tr}_A(1; P_G) = \sum_{\gamma \to 1} \# G \cdot \text{Tr}_A(\gamma, P) = \sum_{\gamma \to 1} \# G \cdot \text{Tr}(\gamma, P).
\]

To finish the proof, reduce to case \( \Lambda \) torsion-free by some universality argument. See [Del77] for details.

03UE **Remark 92.10.** Let us try to illustrate the content of the formula of Lemma 92.8. Suppose that \( \Lambda \), viewed as a trivial \( \Gamma \)-module, admits a finite resolution

\[
0 \to P_r \to \ldots \to P_1 \to P_0 \to \Lambda \to 0
\]

by some \( \Lambda[\Gamma] \)-modules \( P_i \) which are finite and projective as \( \Lambda[\Gamma] \)-modules. In that case

\[
H_*((P_\bullet)_G) = \text{Tor}_A^\Lambda(G, \Lambda) = H_*((G, \Lambda)
\]

and

\[
\text{Tr}_A^\Lambda(\gamma, P_\bullet) = \frac{1}{\# Z_\gamma} \text{Tr}_A(\gamma, P_\bullet) = \frac{1}{\# Z_\gamma} \text{Tr}(\gamma, \Lambda) = \frac{1}{\# Z_\gamma}.
\]

Therefore, Lemma 92.8 says

\[
\text{Tr}_A(1, P_G) = \text{Tr}(1|H_*((G, \Lambda)) = \sum_{\gamma \to 1} \frac{1}{\# Z_\gamma}.
\]

This can be interpreted as a point count on the stack \( BG \). If \( \Lambda = \mathbb{F}_\ell \) with \( \ell \) prime to \( \# G \), then \( H_*((G, \Lambda) \) is \( \mathbb{F}_\ell \) in degree 0 (and 0 in other degrees) and the formula reads

\[
1 = \sum_{\gamma \rightarrow \text{conjugacy classes}} \frac{1}{\# Z_\gamma} \mod \ell.
\]

This is in some sense a “trivial” trace formula for \( G \). Later we will see that [91.3.1] can in some cases be viewed as a highly nontrivial trace formula for a certain type of group, see Section 107.

93. Proof of the trace formula

03UF **Theorem 93.1.** Let \( k \) be a finite field and \( X \) a finite type, separated scheme of dimension at most 1 over \( k \). Let \( \Lambda \) be a finite ring whose cardinality is prime to that of \( k \), and \( K \in D_{ctf}(X, \Lambda) \). Then

\[
\text{Tr}(\pi_X^*|_{K^\times (X_k, K)}) = \sum_{x \in X(k)} \text{Tr}(\pi_x|K_x)
\]

in \( \Lambda^\times \).

Please see Remark 93.2 for some remarks on the statement. Notation: For short, we write

\[
T'(X, K) = \sum_{x \in X(k)} \text{Tr}(\pi_x|K_x)
\]

for the right-hand side of 93.1.1 and

\[
T''(X, K) = \text{Tr}(\pi_x^*|_{K^\times (X_k, K)})
\]

for the left-hand side.
Proof of Theorem 93.1  The proof proceeds in a number of steps.

Step 1. Let \( j : \mathcal{U} \hookrightarrow X \) be an open immersion with complement \( Y = X - \mathcal{U} \) and \( i : Y \hookrightarrow X \). Then \( T''(X, K) = T''(\mathcal{U}, j^{-1} K) + T''(Y, i^{-1} K) \) and \( T'(X, K) = T'(\mathcal{U}, j^{-1} K) + T'(Y, i^{-1} K) \).

This is clear for \( T' \). For \( T'' \) use the exact sequence

\[
0 \to jj^{-1}K \to K \to ii^{-1}K \to 0
\]
to get a filtration on \( K \). This gives rise to an object \( \overline{K} \in DF(X, \Lambda) \) whose graded pieces are \( jj^{-1}K \) and \( ii^{-1}K \), both of which lie in \( D_{ctf}(X, \Lambda) \). Then, by filtered derived abstract nonsense (INSERT REFERENCE), \( R\Gamma_c(X, K) \in DF_{perf}(\Lambda) \), and it comes equipped with \( \pi_X^* \) in \( DF_{perf}(\Lambda) \). By the discussion of traces on filtered complexes (INSERT REFERENCE) we get

\[
\text{Tr}(\pi_X^*|R\Gamma_c(X, K)) = \text{Tr}(\pi_X^*|R\Gamma_c(X, j^{-1}K)) + \text{Tr}(\pi_X^*|R\Gamma_c(X, i^{-1}K)) = T''(U, i^{-1}K) + T''(Y, i^{-1}K).
\]

Step 2. The theorem holds if \( \dim X \leq 0 \).

Indeed, in that case

\[
R\Gamma_c(X, K) = R\Gamma(X, K) = \Gamma(X, K) = \bigoplus_{x \in X} K_x \leftarrow \pi_X^*.
\]

Since the fixed points of \( \pi_X : X, \overline{\kappa} \to X, \overline{\kappa} \) are exactly the points \( \overline{x} \in X, \overline{\kappa} \) which lie over a \( k \)-rational point \( x \in X(k) \) we get

\[
\text{Tr}(\pi_X^*|R\Gamma_c(X, K)) = \sum_{x \in X(k)} \text{Tr}(\pi_{\overline{x}}|_{\kappa_x}).
\]

Step 3. It suffices to prove the equality \( T'(\mathcal{U}, \mathcal{F}) = T''(\mathcal{U}, \mathcal{F}) \) in the case where

- \( \mathcal{U} \) is a smooth irreducible affine curve over \( k \),
- \( \mathcal{U}(k) = \emptyset \),
- \( K = \mathcal{F} \) is a finite locally constant sheaf of \( \Lambda \)-modules on \( \mathcal{U} \) whose stalk(s) are finite projective \( \Lambda \)-modules, and
- \( \Lambda \) is killed by a power of a prime \( \ell \) and \( \ell \in k^* \).

Indeed, because of Step 2, we can throw out any finite set of points. But we have only finitely many rational points, so we may assume there are none. We may assume that \( \mathcal{U} \) is smooth irreducible and affine by passing to irreducible components and throwing away the bad points if necessary. The assumptions of \( \mathcal{F} \) come from unwinding the definition of \( D_{ctf}(X, \Lambda) \) and those on \( \Lambda \) from considering its primary decomposition.

For the remainder of the proof, we consider the situation

\[
\begin{array}{ccc}
\mathcal{V} & \longrightarrow & Y \\
\downarrow f & & \downarrow j \\
\mathcal{U} & \longrightarrow & X
\end{array}
\]

where \( \mathcal{U} \) is as above, \( f \) is a finite étale Galois covering, \( \mathcal{V} \) is connected and the horizontal arrows are projective completions. Denoting \( G = \text{Aut}(\mathcal{V}|\mathcal{U}) \), we also assume (as we may) that \( f^{-1}\mathcal{F} = \overline{M} \) is constant, where the module \( M = \Gamma(\mathcal{V}, f^{-1}\mathcal{F}) \) is a

\footnote{At this point, there should be an evil laugh in the background.}
\( \Lambda[G] \)-module which is finite and projective over \( \Lambda \). This corresponds to the trivial monoid extension

\[
1 \to G \to \Gamma = G \times \mathbb{N} \to \mathbb{N} \to 1.
\]

In that context, using the reductions above, we need to show that \( T^n(\mathcal{U}, \mathcal{F}) = 0 \).

Step 4. There is a natural action of \( G \) on \( f_*f^{-1}\mathcal{F} \) and the trace map \( f_*f^{-1}\mathcal{F} \to \mathcal{F} \) defines an isomorphism

\[
(f_*f^{-1}\mathcal{F}) \otimes_{\Lambda[G]} \Lambda = (f_*f^{-1}\mathcal{F})_G \cong \mathcal{F}.
\]

To prove this, simply unwind everything at a geometric point.

Step 5. Let \( A = \mathbb{Z}/\ell^n\mathbb{Z} \) with \( n \gg 0 \). Then \( f_*f^{-1}\mathcal{F} \cong (f_*A) \otimes_\Lambda M \) with diagonal \( G \)-action.

Step 6. There is a canonical isomorphism \( (f_*A) \otimes_\Lambda M \otimes_{\Lambda[G]} \Lambda \cong \mathcal{F} \).

In fact, this is a derived tensor product, because of the projectivity assumption on \( \mathcal{F} \).

Step 7. There is a canonical isomorphism

\[
R\Gamma_c(U_k, \mathcal{F}) = (R\Gamma_c(U_k, f_*A) \otimes_\Lambda M) \otimes_{\Lambda[G]} \Lambda,
\]

compatible with the action of \( \pi^*_U \).

This comes from the universal coefficient theorem, i.e., the fact that \( R\Gamma_c \) commutes with \( \otimes^L \), and the flatness of \( \mathcal{F} \) as a \( \Lambda \)-module.

We have

\[
\text{Tr}(\pi^*_U|_{R\Gamma_c(U_k, \mathcal{F})}) = \sum_{g \in G} \text{Tr}^Z_g \left( (g, \pi^*_U)|_{R\Gamma_c(U_k, f_*A) \otimes_\Lambda M} \right)
\]

\[
= \sum_{g \in G} \text{Tr}^Z_g \left( (g, \pi^*_U)|_{R\Gamma_c(U_k, f_*A)} \right) \cdot \text{Tr}_A(g|M)
\]

where \( \Gamma \) acts on \( R\Gamma_c(U_k, \mathcal{F}) \) by \( G \) and \( (e, 1) \) acts via \( \pi^*_U \). So the monoidal extension is given by \( \Gamma = G \times \mathbb{N} \to \mathbb{N}, \gamma \mapsto 1 \). The first equality follows from Lemma \([92.7]\) and the second from Lemma \([92.8]\).

Step 8. It suffices to show that \( \text{Tr}^Z_g \left( (g, \pi^*_U)|_{R\Gamma_c(U_k, f_*A)} \right) \in A \) maps to zero in \( \Lambda \).

Recall that

\[
\#Z_g \cdot \text{Tr}^Z_A \left( (g, \pi^*_U)|_{R\Gamma_c(U_k, f_*A)} \right) = \text{Tr}_A \left( (g, \pi^*_U)|_{R\Gamma_c(U_k, f_*A)} \right)
\]

\[
= \text{Tr}_A \left( (g^{-1}\pi_Y)^*|_{R\Gamma_c(Y \cdot A)} \right).
\]

The first equality is Lemma \([92.7]\), the second is the Leray spectral sequence, using the finiteness of \( f \) and the fact that we are only taking traces over \( A \). Now since \( A = \mathbb{Z}/\ell^n\mathbb{Z} \) with \( n \gg 0 \) and \( \#Z_g = \ell^a \) for some (fixed) \( a \), it suffices to show the following result.

Step 9. We have \( \text{Tr}_A \left( (g^{-1}\pi_Y)^*|_{R\Gamma_c(Y \cdot A)} \right) = 0 \) in \( A \).

By additivity again, we have

\[
\text{Tr}_A \left( (g^{-1}\pi_Y)^*|_{R\Gamma_c(Y \cdot A)} \right) + \text{Tr}_A \left( (g^{-1}\pi_Y)^*|_{R\Gamma_c(Y \cdot Y \cdot A)} \right)
\]

\[
= \text{Tr}_A \left( (g^{-1}\pi_Y)^*|_{R\Gamma_c(Y \cdot A)} \right)
\]
The latter trace is the number of fixed points of $g^{-1} \pi_Y$ on $Y$, by Weil’s trace formula Theorem 91.4. Moreover, by the 0-dimensional case already proven in step 2,

$$\text{Tr}_A ((g^{-1} \pi_Y)^* |_{R\Gamma_c(Y-V)_k, A})$$

is the number of fixed points of $g^{-1} \pi_Y$ on $(Y-V)_k$. Therefore,

$$\text{Tr}_A ((g^{-1} \pi_Y)^* |_{R\Gamma_c(V_k, A)})$$

is the number of fixed points of $g^{-1} \pi_Y$ on $(V-V)_k$. But there are no such points: if $\bar{y} \in Y_k$ is fixed under $g^{-1} \pi_Y$, then $f(\bar{y}) \in X_k$ is fixed under $\pi_X$. But $U$ has no $k$-rational point, so we must have $f(\bar{y}) \in (X-U)_k$ and so $\bar{y} \notin V_k$, a contradiction.

This finishes the proof. $\square$

Remark 93.2. Remarks on Theorem 93.1

(1) This formula holds in any dimension. By a dévissage lemma (which uses proper base change etc.) it reduces to the current statement – in that generality.

(2) The complex $R\Gamma_c(X_k, K)$ is defined by choosing an open immersion $j : X \hookrightarrow \bar{X}$ with $\bar{X}$ projective over $k$ of dimension at most 1 and setting

$$R\Gamma_c(X_k, K) := R\Gamma(\bar{X}_k, j^! K).$$

This is independent of the choice of $\bar{X}$ follows from (insert reference here).

We define $H^i_c(X_k, K)$ to be the $i$th cohomology group of $R\Gamma_c(X_k, K)$.

Remark 93.3. Even though all we did are reductions and mostly algebra, the trace formula Theorem 93.1 is much stronger than Weil’s geometric trace formula (Theorem 91.4) because it applies to coefficient systems (sheaves), not merely constant coefficients.

94. Applications

OK, having indicated the proof of the trace formula, let’s try to use it for something.

95. On $\ell$-adic sheaves

Definition 95.1. Let $X$ be a noetherian scheme. A $\mathbb{Z}_\ell$-sheaf on $X$, or simply a $\ell$-adic sheaf is an inverse system $\{F_n\}_{n \geq 1}$ where

1. $F_n$ is a constructible $\mathbb{Z}/\ell^n\mathbb{Z}$-module on $X_{\text{etale}}$, and
2. the transition maps $F_{n+1} \rightarrow F_n$ induce isomorphisms $F_{n+1} \otimes_{\mathbb{Z}/\ell^{n+1}\mathbb{Z}} \mathbb{Z}/\ell^n\mathbb{Z} \cong F_n$.

We say that $F$ is lisse if each $F_n$ is locally constant. A morphism of such is merely a morphism of inverse systems.

Lemma 95.2. Let $\{G_n\}_{n \geq 1}$ be an inverse system of constructible $\mathbb{Z}/\ell^n\mathbb{Z}$-modules. Suppose that for all $k \geq 1$, the maps

$$\frac{G_{n+1}}{\ell^k G_{n+1}} \rightarrow \frac{G_n}{\ell^k G_n}$$

are isomorphisms for all $n \gg 0$ (where the bound possibly depends on $k$). In other words, assume that the system $\{\frac{G_n}{\ell^k G_n}\}_{n \geq 1}$ is eventually constant, and call $F_k$ the corresponding sheaf. Then the system $\{\bar{F}_k\}_{k \geq 1}$ forms a $\mathbb{Z}_\ell$-sheaf on $X$.

Proof. The proof is obvious. $\square$
Lemma 95.3. The category of \( \mathbb{Z}_\ell \)-sheaves on \( X \) is abelian.

Proof. Let \( \Phi = \{ \varphi_n \}_{n \geq 1} : \{ \mathcal{F}_n \} \to \{ \mathcal{G}_n \} \) be a morphism of \( \mathbb{Z}_\ell \)-sheaves. Set

\[
\operatorname{Coker}(\Phi) = \left\{ \operatorname{Coker} \left( \mathcal{F}_n \xrightarrow{\varphi_n} \mathcal{G}_n \right) \right\}_{n \geq 1}
\]

and \( \operatorname{Ker}(\Phi) \) is the result of Lemma 95.2 applied to the inverse system

\[
\left\{ \bigcap_{m \geq n} \operatorname{Im} (\operatorname{Ker}(\varphi_m) \to \operatorname{Ker}(\varphi_n)) \right\}_{n \geq 1}.
\]

That this defines an abelian category is left to the reader. \( \square \)

Example 95.4. Let \( X = \text{Spec}(\mathbb{C}) \) and \( \Phi : \mathbb{Z}_\ell \to \mathbb{Z}_\ell \) be multiplication by \( \ell \). More precisely,

\[
\Phi = \left\{ \mathbb{Z}/\ell^n \mathbb{Z} \xrightarrow{\ell^n} \mathbb{Z}/\ell^n \mathbb{Z} \right\}_{n \geq 1}.
\]

To compute the kernel, we consider the inverse system

\[
\ldots \to \mathbb{Z}/\ell \mathbb{Z} \xrightarrow{0} \mathbb{Z}/\ell \mathbb{Z} \xrightarrow{0} \mathbb{Z}/\ell \mathbb{Z}.
\]

Since the images are always zero, \( \operatorname{Ker}(\Phi) \) is zero as a system.

Remark 95.5. If \( \mathcal{F} = \{ \mathcal{F}_n \}_{n \geq 1} \) is a \( \mathbb{Z}_\ell \)-sheaf on \( X \) and \( \bar{x} \) is a geometric point then \( M_n = \{ \mathcal{F}_{n,\bar{x}} \} \) is an inverse system of finite \( \mathbb{Z}/\ell^n \mathbb{Z} \)-modules such that \( M_{n+1} \to M_n \) is surjective and \( M_n = M_{n+1}/\ell^n M_{n+1} \). It follows that

\[
M = \lim_n M_n = \lim \mathcal{F}_{n,\bar{x}}
\]

is a finite \( \mathbb{Z}_\ell \)-module. Indeed, \( M/\ell M = M_1 \) is finite over \( \mathbb{F}_\ell \), so by Nakayama \( M \) is finite over \( \mathbb{Z}_\ell \). Therefore, \( M \cong \mathbb{Z}_\ell^{\oplus r} \oplus \oplus_{i=1}^t \mathbb{Z}/\ell^{e_i} \mathbb{Z} \) for some \( r, t \geq 0, e_i \geq 1 \). The module \( M = \mathcal{F}_{\bar{x}} \) is called the stalk of \( \mathcal{F} \) at \( \bar{x} \).

Definition 95.6. A \( \mathbb{Z}_\ell \)-sheaf \( \mathcal{F} \) is torsion if \( \ell^n : \mathcal{F} \to \mathcal{F} \) is the zero map for some \( n \). The abelian category of \( \mathbb{Q}_\ell \)-sheaves on \( X \) is the quotient of the abelian category of \( \mathbb{Z}_\ell \)-sheaves by the Serre subcategory of torsion sheaves. In other words, its objects are \( \mathbb{Z}_\ell \)-sheaves on \( X \), and if \( \mathcal{F}, \mathcal{G} \) are two such, then

\[
\operatorname{Hom}_{\mathbb{Q}_\ell} (\mathcal{F}, \mathcal{G}) = \operatorname{Hom}_{\mathbb{Z}_\ell} (\mathcal{F}, \mathcal{G}) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell.
\]

We denote by \( \mathcal{F} \mapsto \mathcal{F} \otimes \mathbb{Q}_\ell \) the quotient functor (right adjoint to the inclusion). If \( \mathcal{F} = \mathcal{F}' \otimes \mathbb{Q}_\ell \) where \( \mathcal{F}' \) is a \( \mathbb{Z}_\ell \)-sheaf and \( \bar{x} \) is a geometric point, then the stalk of \( \mathcal{F} \) at \( \bar{x} \) is \( \mathcal{F}_{\bar{x}} = \mathcal{F}'_{\bar{x}} \otimes \mathbb{Q}_\ell \).

Remark 95.7. Since a \( \mathbb{Z}_\ell \)-sheaf is only defined on a noetherian scheme, it is torsion if and only if its stalks are torsion.

Definition 95.8. If \( X \) is a separated scheme of finite type over an algebraically closed field \( k \) and \( \mathcal{F} = \{ \mathcal{F}_n \}_{n \geq 1} \) is a \( \mathbb{Z}_\ell \)-sheaf on \( X \), then we define

\[
H^i(X, \mathcal{F}) := \lim_n H^i(X, \mathcal{F}_n) \quad \text{and} \quad H^i_c(X, \mathcal{F}) := \lim_n H^i_c(X, \mathcal{F}_n).
\]

If \( \mathcal{F} = \mathcal{F}' \otimes \mathbb{Q}_\ell \) for a \( \mathbb{Z}_\ell \)-sheaf \( \mathcal{F}' \) then we set

\[
H^i_c(X, \mathcal{F}) := H^i_c(X, \mathcal{F}') \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell.
\]

We call these the \( \ell \)-adic cohomology of \( X \) with coefficients \( \mathcal{F} \).
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\section{L-functions}

\begin{definition} \textbf{96.1.}\ Let $X$ be a scheme of finite type over a finite field $k$. Let $\Lambda$ be a finite ring of order prime to the characteristic of $k$ and $\mathcal{F}$ a constructible flat $\Lambda$-module on $X_{\text{étale}}$. Then we set
\[
L(X, \mathcal{F}) := \prod_{x \in |X|} \det(1 - \pi_x^* T^{\deg x} |_{\mathcal{F}_x})^{-1} \in \Lambda[[T]]
\]
where $|X|$ is the set of closed points of $X$, $\deg x = [\kappa(x) : k]$ and $\bar{x}$ is a geometric point lying over $x$. This definition clearly generalizes to the case where $\mathcal{F}$ is replaced by a $K \in D_{ctf}(X, \Lambda)$. We call this the $L$-function of $\mathcal{F}$.
\end{definition}

\begin{remark} \textbf{96.2.}\ Intuitively, $T$ should be thought of as $T = t^f$ where $p^f = \#k$. The definitions are then independent of the size of the ground field.
\end{remark}

\begin{definition} \textbf{96.3.}\ Now assume that $\mathcal{F}$ is a $\mathbb{Q}_\ell$-sheaf on $X$. In this case we define
\[
L(X, \mathcal{F}) := \prod_{x \in |X|} \det(1 - \pi_x^* T^{\deg x} |_{\mathcal{F}_x})^{-1} \in \mathbb{Q}_\ell[[T]].
\]
Note that this product converges since there are finitely many points of a given degree. We call this the $L$-function of $\mathcal{F}$.
\end{definition}

\section{Cohomological interpretation}

This is how Grothendieck interpreted the $L$-function.

\begin{theorem} \textbf{97.1 (Finite Coefficients).}\ Let $X$ be a scheme of finite type over a finite field $k$. Let $\Lambda$ be a finite ring of order prime to the characteristic of $k$ and $\mathcal{F}$ a constructible flat $\Lambda$-module on $X_{\text{étale}}$. Then
\[
L(X, \mathcal{F}) = \det(1 - \pi_X^* T |_{H^i_c(X_k, \mathcal{F})})^{-1} \in \Lambda[[T]].
\]
\end{theorem}

\begin{proof} Omitted. \hfill \Box \end{proof}

Thus far, we don’t even know whether each cohomology group $H^i_c(X_k, \mathcal{F})$ is free.

\begin{theorem} \textbf{97.2 (Adic sheaves).}\ Let $X$ be a scheme of finite type over a finite field $k$, and $\mathcal{F}$ a $\mathbb{Q}_\ell$-sheaf on $X$. Then
\[
L(X, \mathcal{F}) = \prod_i \det(1 - \pi_X^* T |_{H^i_c(X_k, \mathcal{F})})^{-(-1)^{i+1}} \in \mathbb{Q}_\ell[[T]].
\]
\end{theorem}

\begin{proof} This is sketched below. \hfill \Box \end{proof}

\begin{remark} \textbf{97.3.}\ Since we have only developed some theory of traces and not of determinants, Theorem 97.1 is harder to prove than Theorem 97.2. We will only prove the latter, for the former see [Del77]. Observe also that there is no version of this theorem more general for $\mathbb{Z}_\ell$ coefficients since there is no $\ell$-torsion.

We reduce the proof of Theorem 97.2 to a trace formula. Since $\mathbb{Q}_\ell$ has characteristic 0, it suffices to prove the equality after taking logarithmic derivatives. More
Let equality inversion formula, we see that Theorem 97.2 is a consequence of the following by the same formula again. Now, comparing powers of $T$jective Λ-module $M$ which holds for any commutative ring Λ and any endomorphism $f$ of a finite projective Λ-module $M$. On the other hand, we have

$$ T \frac{d}{dT} \log \left( \det (1 - fT) \right) = \sum_{n \geq 1} \Tr(f^n) T^n $$

where the last equality results from the formula

$$ T \frac{d}{dT} \log \left( \det (1 - fT) \right) = \sum_{n \geq 1} \Tr(f^n) T^n $$

which holds for any commutative ring Λ and any endomorphism $f$ of a finite projective Λ-module $M$. On the other hand, we have

$$ T \frac{d}{dT} \log \left( \prod_i \det (1 - \pi^n T)^{(1-1+i)} \right) = \sum_i (-1)^i \sum_{n \geq 1} \Tr((\pi^n)^* X) T^n $$

by the same formula again. Now, comparing powers of $T$ and using the Mobius inversion formula, we see that Theorem 97.2 is a consequence of the following equality

$$ \sum d \sum_{x \in \mathbb{X}} \Tr((\pi^n/d)^* X) = \sum_i (-1)^i \Tr((\pi^n)^* X) T^n $$

Writing $k_n$ for the degree $n$ extension of $k$, $X_n = X \times_{\text{Spec } k} \text{Spec}(k_n)$ and $nF = F|_{X_n}$, this boils down to

$$ \sum_{x \in X_n(k_n)} \Tr((\pi^n/d)^* X) = \sum_i (-1)^i \Tr((\pi^n)^* X) T^n $$

which is a consequence of Theorem 97.5.

**Theorem 97.4.** Let $X/k$ be as above, let $\Lambda$ be a finite ring with $\# \Lambda \in k^*$ and $K \in D_{ctf}(X, \Lambda)$. Then $R\Gamma_c(X_k, K) \in D_{perf}(\Lambda)$ and

$$ \sum_{x \in X(k)} \Tr(\pi_x | K) = \Tr(\pi_X | R\Gamma_c(X_k, K)). $$

**Proof.** Note that we have already proved this (REFERENCE) when $\dim X \leq 1$. The general case follows easily from that case together with the proper base change theorem. \[\square\]

**Theorem 97.5.** Let $X$ be a separated scheme of finite type over a finite field $k$ and $F$ be a $\mathbb{Q}_\ell$-sheaf on $X$. Then $\dim \mathbb{Q}_\ell H^i_c(X_k, F) < \infty$ for all $i$, and is nonzero for $0 \leq i \leq 2 \dim X$. Furthermore, we have

$$ \sum_{x \in X(k)} \Tr(\pi_X | F) = \sum_i (-1)^i \Tr(\pi_X | H^i_c(X_k, F)). $$
Proof. We explain how to deduce this from Theorem 97.4. We first use some étale cohomology arguments to reduce the proof to an algebraic statement which we subsequently prove.

Let $F$ be as in the theorem. We can write $F$ as $F' \otimes \mathbb{Q}_\ell$ where $F' = \{ F'_n \}$ is a $\mathbb{Z}_\ell$-sheaf without torsion, i.e., $\ell : F' \to F'$ has trivial kernel in the category of $\mathbb{Z}_\ell$-sheaves. Then each $F'_n$ is a flat constructible $\mathbb{Z}/\ell^n\mathbb{Z}$-module on $X_{\text{étale}}$, so $F'_n \in D_{\text{ctf}}(X, \mathbb{Z}/\ell^n\mathbb{Z})$ and $F'_{n+1} \otimes_{\mathbb{Z}/\ell^{n+1}\mathbb{Z}} \mathbb{Z}/\ell^n\mathbb{Z} = F'_n$. Note that the last equality holds also for standard (non-derived) tensor product, since $F'_n$ is flat (it is the same equality). Therefore,

1. the complex $K_n = R\Gamma_c(X_{\overline{k}}, F'_n)$ is perfect, and it is endowed with an endomorphism $\pi_n : K_n \to K_n$ in $D(\mathbb{Z}/\ell^n\mathbb{Z})$,
2. there are identifications $K_{n+1} \otimes_{\mathbb{Z}/\ell^{n+1}\mathbb{Z}} \mathbb{Z}/\ell^n\mathbb{Z} = K_n$

Moreover $\pi$ is an isomorphism. Moreover

\[ \lim_{n} K_{n+1} \otimes_{\mathbb{Z}/\ell^{n+1}\mathbb{Z}} \mathbb{Z}/\ell^n\mathbb{Z} \to K_n \]

is a finite complex of finite free $\mathbb{Z}_\ell$-modules such that the map $K^i_n \to K^{i+1}_n$ has image contained in $\ell K^i_{n+1}$. It is a fact that such a complex is unique up to isomorphism. Moreover $\pi_n$ can be represented by a morphism of complexes $\pi_n^* : K^* \to K^*_n$ (which is unique up to homotopy). By the same token the isomorphism $\phi_n : K_{n+1} \otimes_{\mathbb{Z}/\ell^{n+1}\mathbb{Z}} \mathbb{Z}/\ell^n\mathbb{Z} \to K_n$ is represented by a map of complexes

\[ \phi_n^* : K^*_{n+1} \otimes_{\mathbb{Z}/\ell^{n+1}\mathbb{Z}} \mathbb{Z}/\ell^n\mathbb{Z} \to K^*_n \]

In fact, $\phi_n^*$ is an isomorphism of complexes, thus we see that

1. there exist $a, b \in \mathbb{Z}$ independent of $n$ such that $K^i_n = 0$ for all $i \not\in [a, b]$,
2. the rank of $K^i_n$ is independent of $n$.

Therefore, the module $K^i_\infty = \lim_{n} \{ K^i_n, \phi_n^* \}$ is a finite free $\mathbb{Z}_\ell$-module and $K^*_\infty$ is a finite complex of finite free $\mathbb{Z}_\ell$-modules. By induction on the number of nonzero terms, one can prove that $H^i(K^*_\infty) = \lim_{n} H^i(K^*_n)$ (this is not true for unbounded complexes). We conclude that $H^i_\infty = H^i(K^*_\infty)$ is a finite $\mathbb{Z}_\ell$-module. This proves
To prove the remainder of the lemma, we need to overcome the possible non-commutativity of the diagrams

\[ K_{n+1}^\bullet \xrightarrow{\varphi_n^\bullet} K_n^\bullet \]
\[ \pi_{n+1}^\bullet \downarrow \quad \downarrow \pi_n^\bullet \]
\[ K_{n+1}^\bullet \xrightarrow{\varphi_n^\bullet} K_n^\bullet \]

However, this diagram does commute in the derived category, hence it commutes up to homotopy. We inductively replace \( \pi_n^\bullet \) for \( n \geq 2 \) by homotopic maps of complexes making these diagrams commute. Namely, if \( h^i : K_{n+1}^i \to K_n^{i-1} \) is a homotopy, i.e.,

\[ \pi_n^\bullet \circ \varphi_n^\bullet - \varphi_n^\bullet \circ \pi_{n+1}^\bullet = dh + hd, \]

then we choose \( \tilde{h}^i : K_{n+1}^i \to K_n^{i-1} \) lifting \( h^i \). This is possible because \( K_n^{i-1} \) is free and \( K_n^{i-1} \to K_n^{i-1} \) is surjective. Then replace \( \pi_n^\bullet \) by \( \tilde{\pi}_n^\bullet \) defined by

\[ \tilde{\pi}_{n+1}^\bullet = \pi_{n+1}^\bullet + d\tilde{h} + \tilde{h}d. \]

With this choice of \( \{ \pi_n^\bullet \} \), the above diagrams commute, and the maps fit together to define an endomorphism \( \pi_\infty^\bullet = \lim_n \pi_n^\bullet \) of \( K_\infty^\bullet \). Then part i is clear: the elements \( t_n = \sum (-1)^i \text{Tr} (\pi_n^i|_{K_n^i}) \) fit into an element \( t_\infty \) of \( \mathbf{Z}_\ell \). Moreover

\[ t_\infty = \sum (-1)^i \text{Tr}_{\mathbf{Z}_\ell} (\pi_\infty^i|_{K_\infty^i}) \]
\[ = \sum (-1)^i \text{Tr}_{\mathbf{Q}_\ell} (\pi_\infty^i|_{K_\infty^i} \otimes_{\mathbf{Z}_\ell} \mathbf{Q}_\ell) \]
\[ = \sum (-1)^i \text{Tr}(\pi_\infty^i|_{H^i(K_\infty^\bullet \otimes \mathbf{Q}_\ell)}) \]

where the last equality follows from the fact that \( \mathbf{Q}_\ell \) is a field, so the complex \( K_\infty^\bullet \otimes \mathbf{Q}_\ell \) is quasi-isomorphic to its cohomology \( H^i(K_\infty^\bullet \otimes \mathbf{Q}_\ell) \). The latter is also equal to \( H^i(K_\infty^\bullet \otimes_{\mathbf{Z}_\ell} \mathbf{Q}_\ell) = H^i_{\mathbf{Z}}(K_\infty^\bullet \otimes \mathbf{Q}_\ell) \), which finishes the proof of the lemma, and also that of Theorem 97.5. \( \square \)

98. List of things which we should add above

03V5 What did we skip the proof of in the lectures so far:

1. curves and their Jacobians,
2. proper base change theorem,
3. inadequate discussion of \( Rf_* \),
4. more generally, given \( f : X \to S \) finite type, separated \( S \) quasi-projective, discussion of \( Rf_! \) on étale sheaves.
5. discussion of \( \otimes^L \)
6. discussion of why \( Rf_* \) commutes with \( \otimes^L \)

99. Examples of L-functions

03V6 We use Theorem 97.2 for curves to give examples of \( L \)-functions
100. Constant sheaves

Let $k$ be a finite field, $X$ a smooth, geometrically irreducible curve over $k$ and $\mathcal{F} = \mathcal{O}_x$ the constant sheaf. If $\bar{x}$ is a geometric point of $X$, the Galois module $\mathcal{F}_{\bar{x}} = \mathcal{O}_x$ is trivial, so

$$\det(1 - \pi_x^* T^{\deg x}|_{\mathcal{F}_{\bar{x}}})^{-1} = \frac{1}{1 - T^{\deg x}}.$$  \hspace{1cm} (1)

Applying Theorem 97.2 we get

$$L(X, \mathcal{F}) = \prod_{i=0}^{2} \det(1 - \pi_X^* T|_{H^i_\ell(X_k, \mathcal{O}_x)})^{(-1)^{i+1}}$$

$$= \frac{\det(1 - \pi_X^* T|_{H^0_\ell(X_k, \mathcal{O}_x)})}{\det(1 - \pi_X^* T|_{H^2_\ell(X_k, \mathcal{O}_x)})}.$$  \hspace{1cm} (2)

To compute the latter, we distinguish two cases.

**Projective case.** Assume that $X$ is projective, so $H^i_\ell(X_k, \mathcal{O}_x) = H^i(X_k, \mathcal{O}_x)$, and we have

$$H^i(X_k, \mathcal{O}_x) = \left\{ \begin{array}{ll} \mathcal{O}_x & \text{if } i = 0, \\ \mathcal{O}_x^{2g} & \text{if } i = 1, \\ \mathcal{O}_x & \text{if } i = 2. \end{array} \right. $$

The identification of the action of $\pi_X^*$ on $H^2$ comes from Lemma 65.3 and the fact that the degree of $\pi_X$ is $q = \#(k)$. We do not know much about the action of $\pi_X^*$ on the degree 1 cohomology. Let us call $\alpha_1, \ldots, \alpha_{2g}$ its eigenvalues in $\mathcal{O}_x$. Putting everything together, Theorem 97.2 yields the equality

$$\prod_{x \in X} \frac{1}{1 - T^{\deg x}} = \frac{\det(1 - \pi_X^* T|_{H^1(X_k, \mathcal{O}_x)})}{(1 - T)(1 - qT)} = \frac{(1 - \alpha_1 T) \cdots (1 - \alpha_{2g} T)}{(1 - T)(1 - qT)},$$

from which we deduce the following result.

**Lemma 100.1.** Let $X$ be a smooth, projective, geometrically irreducible curve over a finite field $k$. Then

1. The $L$-function $L(X, \mathcal{O}_x)$ is a rational function.
2. The eigenvalues $\alpha_1, \ldots, \alpha_{2g}$ of $\pi_X^*$ on $H^1(X_k, \mathcal{O}_x)$ are algebraic integers independent of $\ell$.
3. The number of rational points of $X$ on $k_n$, where $[k_n : k] = n$, is
   $$\#X(k_n) = 1 - \sum_{i=1}^{2g} \alpha_i^n + q^n,$$
   (4) for each $i$, $|\alpha_i| < q$.

**Proof.** Part (3) is Theorem 97.5 applied to $\mathcal{F} = \mathcal{O}_x$ on $X \otimes k_n$. For part (4), use the following result. \hfill $\square$

**Exercise 100.2.** Let $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$. Then for any conic sector containing the positive real axis of the form $C_\varepsilon = \{ z \in \mathbb{C} \mid |\arg z| < \varepsilon \}$ with $\varepsilon > 0$, there exists an integer $k \geq 1$ such that $\alpha_1^k, \ldots, \alpha_n^k \in C_\varepsilon$.

Then prove that $|\alpha_i| \leq q$ for all $i$. Then, use elementary considerations on complex numbers to prove (as in the proof of the prime number theorem) that $|\alpha_i| < q$. In fact, the Riemann hypothesis says that for all $|\alpha_i| = \sqrt{q}$ for all $i$. We will come back to this later.
Affine case. Assume now that $X$ is affine, say $X = \bar{X} - \{x_1, \ldots, x_n\}$ where $j : X \hookrightarrow \bar{X}$ is a projective nonsingular completion. Then $H_c^0(X, \mathcal{Q}_\ell) = 0$ and $H_c^2(X, \mathcal{Q}_\ell) = H^2(\bar{X}, \mathcal{Q}_\ell)$ so Theorem 97.2 reads

$$L(X, \mathcal{Q}_\ell) = \prod_{x \in |X|} \frac{1}{1 - T^{\deg x}} = \frac{\det(1 - \pi_x^* T|H^1_j(X, \mathcal{Q}_\ell))}{1 - qT}.$$  

On the other hand, the previous case gives

$$L(X, \mathcal{Q}_\ell) = L(\bar{X}, \mathcal{Q}_\ell) \prod_{i=1}^n (1 - T^{\deg x_i}) = \prod_{i=1}^n (1 - T^{\deg x_i}) \prod_{j=1}^{2g} (1 - \alpha_j T).$$

Therefore, we see that $\dim H_c^1(X, \mathcal{Q}_\ell) = 2g + \sum_{i=1}^n \deg(x_i) - 1$, and the eigenvalues $\alpha_1, \ldots, \alpha_{2g}$ of $\pi_x^*$ acting on the degree 1 cohomology are roots of unity. More precisely, each $x_i$ gives a complete set of $\deg(x_i)$th roots of unity, and one occurrence of 1 is omitted. To see this directly using coherent sheaves, consider the short exact sequence on $\bar{X}$

$$0 \to j^! \mathcal{Q}_\ell \to \mathcal{Q}_\ell \to \bigoplus_{i=1}^n \mathcal{Q}_{\ell, x_i} \to 0.$$

The long exact cohomology sequence reads

$$0 \to \mathcal{Q}_\ell \to \bigoplus_{i=1}^n \mathcal{Q}_{\ell}^{\oplus \deg x_i} \to H^1_c(X, \mathcal{Q}_\ell) \to H^1_c(\bar{X}, \mathcal{Q}_\ell) \to 0$$

where the action of Frobenius on $\bigoplus_{i=1}^n \mathcal{Q}_{\ell}^{\oplus \deg x_i}$ is by cyclic permutation of each term; and $H^2_c(X, \mathcal{Q}_\ell) = H^2_c(\bar{X}, \mathcal{Q}_\ell)$.

101. The Legendre family

03VA Let $k$ be a finite field of odd characteristic, $X = \text{Spec}(k[\lambda, \frac{1}{\lambda(A-1)}])$, and consider the family of elliptic curves $f : E \to X$ on $\mathbb{P}^2_X$ whose affine equation is $y^2 = x(x-1)(x-\lambda)$. We set $\mathcal{F} = Rf^!_\ell \mathcal{Q}_\ell = \{R^1f_\ell \mathbb{Z}/\ell^n \mathbb{Z}\}_{n \geq 1} \otimes \mathcal{Q}_\ell$. In this situation, the following is true

- for each $n \geq 1$, the sheaf $R^1f_\ell \mathbb{Z}/\ell^n \mathbb{Z}$ is finite locally constant – in fact, it is free of rank 2 over $\mathbb{Z}/\ell^n \mathbb{Z}$,
- the system $\{R^1f_\ell \mathbb{Z}/\ell^n \mathbb{Z}\}_{n \geq 1}$ is a lisse $\ell$-adic sheaf, and
- for all $x \in |X|$, $\det(1 - \pi_x T^{\deg x}|_{\mathcal{F}_x}) = (1 - \alpha_x T^{\deg x})(1 - \beta_x T^{\deg x})$ where $\alpha_x, \beta_x$ are the eigenvalues of the geometric Frobenius of $E_x$ acting on $H^1(E^\bullet_x, \mathcal{Q}_\ell)$.

Note that $E_x$ is only defined over $\kappa(x)$ and not over $k$. The proof of these facts uses the proper base change theorem and the local acyclicity of smooth morphisms. For details, see [Del77]. It follows that

$$L(E/X) := L(X, \mathcal{F}) = \prod_{x \in |X|} \frac{1}{(1 - \alpha_x T^{\deg x})(1 - \beta_x T^{\deg x})}.$$
Applying Theorem 97.2 we get
\[
L(E/X) = \prod_{i=0}^{2} \det (1 - \pi_X^* T|_{H^i_c(X,\mathcal{F})})^{(-1)^{i+1}},
\]
and we see in particular that this is a rational function. Furthermore, it is relatively easy to show that \(H^0_c(X,\mathcal{F}) = H^2_c(X,\mathcal{F}) = 0\), so we merely have
\[
L(E/X) = \det (1 - \pi_X^* T|_{H^1_c(X,\mathcal{F})}).
\]
To compute this determinant explicitly, consider the Leray spectral sequence for the proper morphism \(f : E \to X\) over \(\mathbb{Q}_\ell\), namely
\[
H^i_c(X, R^j f_* \mathbb{Q}_\ell) \Rightarrow H^{i+j}_{\acute{e}tale}(E, \mathbb{Q}_\ell)
\]
which degenerates. We have \(f_* \mathbb{Q}_\ell = \mathbb{Q}_\ell\) and \(R^1 f_* \mathbb{Q}_\ell = \mathbb{Q}_\ell(-1)\) is the Tate twist of \(\mathbb{Q}_\ell\), i.e., it is the sheaf \(\mathbb{Q}_\ell\) where the Galois action is given by multiplication by \(\#k(x)\) on the stalk at \(\bar{x}\). It follows that, for all \(n \geq 1\),
\[
\#E(k_n) = \sum_{x \in X(k_n)} (-1)^i \text{Tr}(\pi_X^* |_{H^i_c(E, Q)}),
\]
where the first equality follows from Theorem 97.5, the second one from the Leray spectral sequence and the third one by writing down the higher direct images of \(Q\) under \(f\). Alternatively, we could write
\[
\#E(k_n) = \sum_{x \in X(k_n)} \#E_x(k_n)
\]
and use the trace formula for each curve. We can also find the number of \(k_n\)-rational points simply by counting. The zero section contributes \(q^n - 2\) points (we omit the points where \(\lambda = 0, 1\)) hence
\[
\#E(k_n) = q^n - 2 + \#\{y^2 = x(x-1)(x-\lambda), \lambda \neq 0, 1\}.
\]
Now we have
\[
\#\{y^2 = x(x-1)(x-\lambda), \lambda \neq 0, 1\} = \#\{y^2 = x(x-1)^2\} - \#\{y^2 = x(x-1)^2\}
\]
\[
= \#\{\lambda = \frac{-y^2}{x(x-1)} + x, x \neq 0, 1\} + \#\{y^2 = x(x-1)(x-\lambda), x = 0, 1\} - 2(q^n - \varepsilon_n)
\]
\[
= q^n(q^n - 2) + 2q^n - 2(q^n - \varepsilon_n)
\]
where \(\varepsilon_n = 1\) if \(-1\) is a square in \(k_n\), 0 otherwise, i.e.,
\[
\varepsilon_n = \frac{1}{2} \left(1 + \left(\frac{-1}{k_n}\right)\right) = \frac{1}{2} \left(1 + (-1)^{\frac{n-1}{2}}\right).
\]
Thus \(\#E(k_n) = q^{2n} - q^n - 2 + 2\varepsilon_n\). Comparing with the previous formula, we find
\[
\text{Tr}(\pi_X^* |_{H^1_c(X,\mathcal{F})}) = 2\varepsilon_n = 1 + (-1)^{\frac{n-1}{2}},
\]
which implies, by elementary algebra of complex numbers, that if $-1$ is a square in $k^*_n$, then $\dim H^1_c(X, F) = 2$ and the eigenvalues are 1 and 1. Therefore, in that case we have

$$L(E/X) = (1 - T)^2.$$ 

102. Exponential sums

A standard problem in number theory is to evaluate sums of the form

$$S_{a, b}(p) = \sum_{x \in F_p - \{0, 1\}} e^{2\pi i a x(x - 1)^b/p}.$$ 

In our context, this can be interpreted as a cohomological sum as follows. Consider the base scheme $S = \text{Spec}(F_p[x, 1/(x - 1)])$ and the affine curve $f : X \to \mathbb{P}^1 - \{0, 1, \infty\}$ over $S$ given by the equation $y^{p-1} = x^a(x - 1)^b$. This is a finite étale Galois cover with group $F_p^*$ and there is a splitting

$$f_*(Q^*_l) = \bigoplus_{\chi : F_p^* \to \mathbb{Q}_l^*} \mathcal{F}_\chi$$

where $\chi$ varies over the characters of $F_p^*$ and $\mathcal{F}_\chi$ is a rank 1 lisse $Q^*_l$-sheaf on which $F_p^*$ acts via $\chi$ on stalks. We get a corresponding decomposition

$$H^1_c(X, Q^*_l) = \bigoplus_{\chi} H^1_c(\mathbb{P}^1 \to \{0, 1, \infty\}, \mathcal{F}_\chi)$$

and the cohomological interpretation of the exponential sum is given by the trace formula applied to $\mathcal{F}_\chi$ over $\mathbb{P}^1 - \{0, 1, \infty\}$ for some suitable $\chi$. It reads

$$S_{a, b}(p) = -\text{Tr}(\pi^*_X|_{H^1_c(\mathbb{P}^1 \to \{0, 1, \infty\}, \mathcal{F}_\chi)}).$$

The general yoga of Weil suggests that there should be some cancellation in the sum. Applying (roughly) the Riemann-Hurwitz formula, we see that

$$2g_X - 2 \approx -2(p - 1) + 3(p - 2) \approx p$$

so $g_X \approx p/2$, which also suggests that the $\chi$-pieces are small.

103. Trace formula in terms of fundamental groups

In the following sections we reformulate the trace formula completely in terms of the fundamental group of a curve, except if the curve happens to be $\mathbb{P}^1$.

104. Fundamental groups

This material is discussed in more detail in the chapter on fundamental groups. See Fundamental Groups, Section 1. Let $X$ be a connected scheme and let $\pi : Y \to X$ be a geometric point. Consider the functor

$$F_\pi : \text{ finite étale schemes over } X \to \text{ finite sets}$$

$$Y/X \to F_\pi(Y) = \left\{ \text{ geom points } \pi \text{ of } Y \text{ lying over } \pi \right\} = Y_\pi$$

Set

$$\pi_1(X, \pi) = \text{ Aut}(F_\pi) = \text{ set of automorphisms of the functor } F_\pi$$

Note that for every finite étale $Y \to X$ there is an action

$$\pi_1(X, \pi) \times F_\pi(Y) \to F_\pi(Y)$$
**Étale Cohomology**

**Definition 104.1.** A subgroup of the form \( \text{Stab}(\overline{y} \in F_\pi(Y)) \subset \pi_1(X, \overline{x}) \) is called open.

**Theorem 104.2** (Grothendieck). Let \( X \) be a connected scheme.

1. There is a topology on \( \pi_1(X, \overline{x}) \) such that the open subgroups form a fundamental system of open nbhds of \( e \in \pi_1(X, \overline{x}) \).

2. With topology of (1) the group \( \pi_1(X, \overline{x}) \) is a profinite group.

3. The functor

\[
\begin{array}{ccc}
\text{schemes finite étale over } X & \to & \text{finite discrete continuous } \\
Y/X & \mapsto & F_\pi(Y) \text{ with its natural action}
\end{array}
\]

is an equivalence of categories.

**Proof.** See [Gro71].

**Proposition 104.3.** Let \( X \) be an integral normal Noetherian scheme. Let \( \overline{y} \to X \) be an algebraic geometric point lying over the generic point \( \eta \in X \). Then

\[
\pi_x(X, \overline{\eta}) = \text{Gal}(M/\kappa(\eta))
\]

(\( \kappa(\eta) \), function field of \( X \)) where

\[
\kappa(\overline{\eta}) \supset M \supset \kappa(\eta) = k(X)
\]

is the max sub-extension such that for every finite subextension \( M \supset L \supset \kappa(\eta) \) the normalization of \( X \) in \( L \) is finite étale over \( X \).

**Proof.** Omitted.

**Change of base point.** For any \( \overline{x}_1, \overline{x}_2 \) geom. points of \( X \) there exists an isom. of fibre functions

\[
F_{\overline{x}_1} \cong F_{\overline{x}_2}
\]

(This is a path from \( \overline{x}_1 \) to \( \overline{x}_2 \).) Conjugation by this path gives isom

\[
\pi_1(X, \overline{x}_1) \cong \pi_1(X, \overline{x}_2)
\]

well defined up to inner actions.

**Functoriality.** For any morphism \( X_1 \to X_2 \) of connected schemes any \( \overline{x} \in X_1 \) there is a canonical map

\[
\pi_1(X_1, \overline{x}) \to \pi_1(X_2, \overline{x})
\]

(Why? because the fibre functor ...)

**Base field.** Let \( X \) be a variety over a field \( k \). Then we get

\[
\pi_1(X, \overline{x}) \to \pi_1(\text{Spec}(k), \overline{x}) \cong \text{prop } \text{Gal}(k_{\text{sep}}/k)
\]

This map is surjective if and only if \( X \) is geometrically connected over \( k \). So in the geometrically connected case we get s.e.s. of profinite groups

\[
1 \to \pi_1(X_{\overline{k}}, \overline{x}) \to \pi_1(X, \overline{x}) \to \text{Gal}(k_{\text{sep}}/k) \to 1
\]

(\( \pi_1(X_{\overline{k}}, \overline{x}) \): geometric fundamental group of \( X \), \( \pi_1(X, \overline{x}) \): arithmetic fundamental group of \( X \))

**Comparison.** If \( X \) is a variety over \( \mathbb{C} \) then

\[
\pi_1(X, \overline{x}) = \text{profinite completion of } \pi_1(X(\mathbb{C}))(\text{ usual topology}, x)
\]

(have \( x \in X(\mathbb{C}) \))
**Frobenius.** $X$ variety over $k$, $\#k < \infty$. For any $x \in X$ closed point, let

$$F_x \in \pi_1(x, \bar{x}) = \text{Gal}(\kappa(x)_{\text{sep}}/\kappa(x))$$

be the geometric frobenius. Let $\eta$ be an alg. geom. gen. pt. Then

$$\pi_1(X, \eta) \xleftarrow{\cong} \pi_1(X, \bar{x}) \xleftarrow{\text{functoriality}} \pi_1(x, \bar{x})$$

Easy fact:

$$\pi_1(X, \eta) \xrightarrow{\text{deg}} \pi_1(\text{Spec}(k), \eta)^* = \text{Gal}(k_{\text{sep}}/k) \xrightarrow{||} \mathbb{Z} / F_{\text{Spec}(k)}$$

Recall: $\text{deg}(x) = [\kappa(x) : k]$

**Fundamental groups and lisse sheaves.** Let $X$ be a connected scheme, $\bar{x}$ geom. pt. There are equivalences of categories

$$\begin{array}{ccc}
\text{(}\Lambda \text{ a finite ring)} & \text{fin. loc. const. sheaves of} & \text{finite (discrete)} \Lambda\text{-modules} \\
\text{etale} & \Lambda\text{-modules of} X & \text{with continuous} \pi_1(X, \bar{x})\text{-action} \\
\text{(} \ell \text{ a prime)} & \text{lisse } \ell\text{-adic} & \text{finitely generated } \mathbb{Z}_\ell\text{-modules} \\
& \text{sheaves} & \text{with continuous} \\
& & \pi_1(X, \bar{x})\text{-action where we use } \ell\text{-adic topology on } M \\
\end{array}$$

In particular lisse $\mathbb{Q}_\ell$-sheaves correspond to continuous homomorphisms

$$\pi_1(X, \bar{x}) \to \text{GL}_r(\mathbb{Q}_\ell), \quad r \geq 0$$

Notation: A module with action $(M, \rho)$ corresponds to the sheaf $F_{\rho}$.

**Trace formulas.** $X$ variety over $k$, $\#k < \infty$.

1. A finite ring ($\#\Lambda, \#k = 1$

$$\rho : \pi_1(X, \bar{x}) \to \text{GL}_r(\Lambda)$$

continuous. For every $n \geq 1$ we have

$$\sum_{d|n} d \left( \sum_{\substack{\sigma \in \{1, \ldots, n\} \atop \deg(x) = d}} \text{Tr}(\rho(F_x^{\sigma/d})) \right) = \text{Tr} (\pi_X^{*} |_{R^r_{\text{et}}(X, \bar{x})})$$

2. $\ell \neq \text{char}(k)$ prime, $\rho : \pi_1(X, \bar{x}) \to \text{GL}_r(\mathbb{Q}_\ell)$. For any $n \geq 1$

$$\sum_{d|n} d \left( \sum_{\substack{\sigma \in \{1, \ldots, n\} \atop \deg(x) = d}} \text{Tr}(\rho(F_x^{\sigma/d}) \right) = \sum_{i=0}^{2\dim X} (-1)^i \text{Tr} (\pi_X^i |_{H^i_{\text{et}}(X, \bar{x})})$$

**Weil conjectures.** (Deligne-Weil I, 1974) $X$ smooth proj. over $k$, $\#k = q$, then the eigenvalues of $\pi_X^n$ on $H^i(X_{\bar{\text{et}}}, \mathbb{Q}_\ell)$ are algebraic integers $\alpha$ with $|\alpha| = q^{1/2}$.

**Deligne’s conjectures.** (almost completely proved by Lafforgue + . . .) Let $X$ be a normal variety over $k$ finite

$$\rho : \pi_1(X, \bar{x}) \to \text{GL}_r(\mathbb{Q}_\ell)$$

continuous. Assume: $\rho$ irreducible $\det(\rho)$ of finite order. Then

1. there exists a number field $E$ such that for all $x \in |X|$ (closed points) the char. poly of $\rho(F_x)$ has coefficients in $E$.

2. for any $x \in |X|$ the eigenvalues $\alpha_{x, i}, i = 1, \ldots, r$ of $\rho(F_x)$ have complex absolute value 1. (these are algebraic numbers not necessary integers)
(3) for every finite place $\lambda$ (not dividing $p$), of $E$ (maybe after enlarging $E$ a bit) there exists
\[ \rho_\lambda : \pi_1(X, \pi) \to \text{GL}_r(E_\lambda) \]
compatible with $\rho$. (some char. polys of $F_x$’s)

**Theorem 104.4** (Deligne, Weil II). For a sheaf $\mathcal{F}_\rho$ with $\rho$ satisfying the conclusions of the conjecture above then the eigenvalues of $\pi_\chi$ on $H^i_c(X, \mathcal{F}_\rho)$ are algebraic numbers $\alpha$ with absolute values
\[ |\alpha| = q^{w/2}, \text{ for } w \in \mathbb{Z}, w \leq i \]
Moreover, if $X$ smooth and proj. then $w = i$.

**Proof.** See [Del74]. \(\square\)

### 105. Profinite groups, cohomology and homology

Let $G$ be a profinite group.

**Cohomology.** Consider the category of discrete modules with continuous $G$-action. This category has enough injectives and we can define
\[ H^i(G, M) = R^iH^0(G, M) = R^i(M \mapsto M^G) \]
Also there is a derived version $RH^0(G, -)$.

**Homology.** Consider the category of compact abelian groups with continuous $G$-action. This category has enough projectives and we can define
\[ H_i(G, M) = L_iH_0(G, M) = L_i(M \mapsto M_G) \]
and there is also a derived version.

**Trivial duality.** The functor $M \mapsto M^\wedge = \text{Hom}_{cont}(M, S^1)$ exchanges the categories above and
\[ H^i(G, M)^\wedge = H_i(G, M^\wedge) \]
Moreover, this functor maps torsion discrete $G$-modules to profinite continuous $G$-modules and vice versa, and if $M$ is either a discrete or profinite continuous $G$-module, then $M^\wedge = \text{Hom}(M, \mathbb{Q}/\mathbb{Z})$.

**Notes on Homology.**

(1) If we look at $\Lambda$-modules for a finite ring $\Lambda$ then we can identify
\[ H_i(G, M) = \text{Tor}_i^{\Lambda[[G]]}(M, \Lambda) \]
where $\Lambda[[G]]$ is the limit of the group algebras of the finite quotients of $G$.

(2) If $G$ is a normal subgroup of $\Gamma$, and $\Gamma$ is also profinite then
- $H^0(G, -)$: discrete $\Gamma$-module $\to$ discrete $\Gamma/G$-modules
- $H_0(G, -)$: compact $\Gamma$-modules $\to$ compact $\Gamma/G$-modules

and hence the profinite group $\Gamma/G$ acts on the cohomology groups of $G$ with values in a $\Gamma$-module. In other words, there are derived functors
\[ RH^0(G, -) : D^+(\text{discrete } \Gamma\text{-modules}) \to D^+(\text{discrete } \Gamma/G\text{-modules}) \]
and similarly for $LH_0(G, -)$. 
106. Cohomology of curves, revisited

Let $k$ be a field, $X$ be geometrically connected, smooth curve over $k$. We have the fundamental short exact sequence

$$1 \to \pi_1(X_{\overline{k}}, \eta) \to \pi_1(X, \eta) \to \text{Gal}(k^{sep}/k) \to 1$$

If $\Lambda$ is a finite ring with $\#\Lambda \in k^*$ and $M$ a finite $\Lambda$-module, and we are given $\rho : \pi_1(X, \eta) \to \text{Aut}_\Lambda(M)$ continuous, then $F_\rho$ denotes the associated sheaf on $X_{\text{étale}}$.

**Lemma 106.1.** There is a canonical isomorphism

$$H^2_c(X_{\overline{k}}, F_\rho) = (M)_{\pi_1(X_{\overline{k}}, \eta)}(-1)$$

as $\text{Gal}(k^{sep}/k)$-modules.

Here the subscript $\pi_1(X_{\overline{k}}, \eta)$ indicates co-invariants, and $(-1)$ indicates the Tate twist i.e., $\sigma \in \text{Gal}(k^{sep}/k)$ acts via $\chi_{\text{cycl}}(\sigma)^{-1} \sigma$ on RHS

where $\chi_{\text{cycl}} : \text{Gal}(k^{sep}/k) \to \prod_{l \neq \text{char}(k)} \mathbb{Z}_l^*$

is the cyclotomic character.

Reformulation (Deligne, Weil II, page 338). For any finite locally constant sheaf $F$ on $X$ there is a maximal quotient $F \to F''$ with $F''/X_{\overline{k}}$ a constant sheaf, hence $F'' = (X \to \text{Spec}(k))^{-1} F''$ where $F''$ is a sheaf $\text{Spec}(k)$, i.e., a $\text{Gal}(k^{sep}/k)$-module. Then

$$H^2_c(X_{\overline{k}}, F) \to H^2_c(X_{\overline{k}}, F'') \to F''(-1)$$

is an isomorphism.

**Proof of Lemma 106.1** Let $Y \to X$ be the finite étale Galois covering corresponding to $\text{Ker}(\rho) \subset \pi_1(X, \eta)$. So

$$\text{Aut}(Y/X) = \text{Ind}(\rho)$$

is Galois group. Then $\varphi^* F_\rho = M_Y$ and

$$\varphi_* \varphi^* F_\rho \to F_\rho$$

which gives

$$H^2_c(X_{\overline{k}}, \varphi_* \varphi^* F_\rho) \to H^2_c(X_{\overline{k}}, F_\rho)$$

$$= H^2_c(Y_{\overline{k}}, \varphi^* F_\rho)$$

$$= H^2_c(Y_{\overline{k}}, M) = \bigoplus_{\text{irred. comp. of } M} \mathbb{Z}$$

$$\text{Im}(\rho) \to H^2_c(Y_{\overline{k}}, M) = \bigoplus_{\text{irred. comp. of } M} \to \text{Im}(\rho)\text{ equivalent } H^2_c(X_{\overline{k}}, F_\rho) \to \text{trivial Im}(\rho)\text{ action}$$

irreducible curve $C/\overline{k}$, $H^2_c(C, M) = M$.

Since

$$\text{set of irreducible components of } Y_k = \frac{\text{Im}(\rho)}{\text{Im}(\rho)_{\pi_1(X_{\overline{k}}, \eta)}}$$
We conclude that $H_c^2(X, F_\rho)$ is a quotient of $M_{\pi_1(X, \eta)}$. On the other hand, there is a surjection

$$F_\rho \to F'' = \text{sheaf on } X \text{ associated to } (M)_{\pi_1(X, \eta)} \leftarrow \pi_1(X, \eta)$$

$$H_c^2(X, F_\rho) \to M_{\pi_1(X, \eta)}$$

The twist in Galois action comes from the fact that $H_c^2(X, \mu_n) = \mathbb{Z}/n\mathbb{Z}$. □

**Remark 106.2.** Thus we conclude that if $X$ is also projective then we have functorially in the representation $\rho$ the identifications

$$H^0(X, F_\rho) = M_{\pi_1(X, \eta)}$$

and

$$H_c^2(X, F_\rho) = M_{\pi_1(X, \eta)}(-1)$$

Of course if $X$ is not projective, then $H^0_c(X, F_\rho) = 0$.

**Proposition 106.3.** Let $X/k$ as before but $X_k \neq \mathbb{P}_k^1$. The functors $(M, \rho) \mapsto H_c^{2-i}(X, F_\rho)$ are the left derived functor of $(M, \rho) \mapsto H_c^2(X, F_\rho)$ so

$$H_c^{2-i}(X, F_\rho) = H_i(\pi_1(X, \eta), M)(-1)$$

Moreover, there is a derived version, namely

$$R\Gamma_c(X, F_\rho) = LH_0(\pi_1(X, \eta), M(-1)) = M(-1) \otimes_{\Lambda[[\pi_1(X, \eta)]]} \Lambda$$

in $D(\Lambda[[\hat{\mathbb{Z}}]])$. Similarly, the functors $(M, \rho) \mapsto H^i(X, F_\rho)$ are the right derived functor of $(M, \rho) \mapsto M_{\pi_1(X, \eta)}$ so

$$H^i(X, F_\rho) = H^i(\pi_1(X, \eta), M)$$

Moreover, in this case there is a derived version too.

**Proof.** (Idea) Show both sides are universal $\delta$-functors. □

**Remark 106.4.** By the proposition and Trivial duality then you get

$$H_c^{2-i}(X, F_\rho) \times H^i(X, F_\rho^\wedge(1)) \to \mathbb{Q}/\mathbb{Z}$$

a perfect pairing. If $X$ is projective then this is Poincare duality.

### 107. Abstract trace formula

**Suppose** given an extension of profinite groups,

$$1 \to G \to \Gamma \xrightarrow{\deg} \hat{\mathbb{Z}} \to 1$$

We say $\Gamma$ has an abstract trace formula if and only if there exist

1. an integer $q \geq 1$, and
2. for every $d \geq 1$ a finite set $S_d$ and for each $x \in S_d$ a conjugacy class $F_x \in \Gamma$

with $\deg(F_x) = d$

such that the following hold

1. for all $\ell$ not dividing $q$ have $\text{cd}_\ell(G) < \infty$, and
(2) for all finite rings $\Lambda$ with $q \in \Lambda^*$, for all finite projective $\Lambda$-modules $M$ with continuous $\Gamma$-action, for all $n > 0$ we have

$$\sum_{d|n} d \left( \sum_{x \in S_d} \text{Tr}(F^n x^d | M) \right) = q^n \text{Tr}(F^n | M \otimes_{\Lambda[[G]]} \Lambda)$$

in $\Lambda^\wedge$.

Here $M \otimes_{\Lambda[[G]]} \Lambda = LH_0(G, M)$ denotes derived homology, and $F = 1$ in $\Gamma/G = \hat{\mathbb{Z}}$.

**Remark 107.1.** Here are some observations concerning this notion.

1. If modeling projective curves then we can use cohomology and we don’t need factor $q^n$.
2. The only examples I know are $\Gamma = \pi_1(X, \eta)$ where $X$ is smooth, geometrically irreducible and $K(\pi, 1)$ over finite field. In this case $q = (\#k)^{\dim X}$. Modulo the proposition, we proved this for curves in this course.
3. Given the integer $q$ then the sets $S_d$ are uniquely determined. (You can multiple $q$ by an integer $m$ and then replace $S_d$ by $m^d$ copies of $S_d$ without changing the formula.)

**Example 107.2.** Fix an integer $q \geq 1$

$$1 \rightarrow G = \hat{\mathbb{Z}}^{(q)} \rightarrow \Gamma \rightarrow \hat{\mathbb{Z}} \rightarrow 1$$

with $FXF^{-1} = ux, u \in (\hat{\mathbb{Z}}^{(q)})^*$. Just using the trivial modules $\mathbb{Z}/m\mathbb{Z}$ we see

$$q^n - (qu)^n \equiv \sum_{d|n} d\#S_d$$

in $\mathbb{Z}/m\mathbb{Z}$ for all $(m, q) = 1$ (up to $u \rightarrow u^{-1}$) this implies $qu = a \in \mathbb{Z}$ and $|a| < q$. The special case $a = 1$ does occur with

$$\Gamma = \pi_1^1(\mathbb{G}_m, F, \eta), \ #S_1 = q - 1, \ \text{and} \ #S_2 = (q^2 - 1) - (q - 1)$$

108. Automorphic forms and sheaves

**References:** See especially the amazing papers [Dri83], [Dri84] and [Dri80] by Drinfeld.

**Unramified cusp forms.** Let $k$ be a finite field of characteristic $p$. Let $X$ geometrically irreducible projective smooth curve over $k$. Set $K = k(X)$ equal to the function field of $X$. Let $v$ be a place of $K$ which is the same thing as a closed point $x \in X$. Let $K_v$ be the completion of $K$ at $v$, which is the same thing as the fraction field of the completion of the local ring of $X$ at $x$, i.e., $K_v = f.f.(\hat{O}_{X,x})$. Denote $O_v \subset K_v$ the ring of integers. We further set

$$O = \prod_v O_v \subset A = \prod_v K_v$$

and we let $\Lambda$ be any ring with $p$ invertible in $\Lambda$.

**Definition 108.1.** An unramified cusp form on $GL_2(A)$ with values in $\Lambda^8$ is a function

$$f : GL_2(A) \rightarrow \Lambda$$

---

8This is likely nonstandard notation.
such that
\[(1) \quad f(x\gamma) = f(x) \text{ for all } x \in \text{GL}_2(A) \text{ and all } \gamma \in \text{GL}_2(K)\]
\[(2) \quad f(ux) = f(x) \text{ for all } x \in \text{GL}_2(A) \text{ and all } u \in \text{GL}_2(O)\]
\[(3) \quad \text{for all } x \in \text{GL}_2(A), \int_A \text{ mod } K f \left( x \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} \right) \, dz = 0\]

see [dJ01] Section 4.1 for an explanation of how to make sense out of this for a general ring \( \Lambda \) in which \( p \) is invertible.

**Hecke Operators.** For \( v \) a place of \( K \) and \( f \) an unramified cusp form we set
\[
T_v(f)(x) = \int_{g \in M_v} f(g^{-1}x) \, dg,
\]
and
\[
U_v(f)(x) = f \left( \begin{pmatrix} \pi_v^{-1} & 0 \\ 0 & \pi_v^{-1} \end{pmatrix} x \right)
\]

Notations used: here \( \pi_v \in O_v \) is a uniformizer
\[
M_v = \{ h \in \text{Mat}(2 \times 2, O_v) \mid \det h = \pi_v O_v^* \}
\]
and \( dg = \) is the Haar measure on \( \text{GL}_2(K_v) \) with \( \int_{\text{GL}_2(O_v)} \, dg = 1 \). Explicitly we have
\[
T_v(f)(x) = f \left( \begin{pmatrix} \pi_v^{-1} & 0 \\ 0 & 1 \end{pmatrix} x \right) + \sum_{i=1}^{q_v} f \left( \begin{pmatrix} 1 & 0 \\ -\pi_v^{-1} \lambda_i & \pi_v^{-1} \end{pmatrix} x \right)
\]
with \( \lambda_i \in O_v \) a set of representatives of \( O_v/(\pi_v) = \kappa_v, q_v = \#\kappa_v \).

**Eigenforms.** An eigenform \( f \) is an unramified cusp form such that some value of \( f \) is a unit and \( T_v f = t_v f \) and \( U_v f = u_v f \) for some (uniquely determined) \( t_v, u_v \in \Lambda \).

03VT **Theorem 108.2.** Given an eigenform \( f \) with values in \( \overline{Q}_l \) and eigenvalues \( u_v \in \mathbb{Z}_l^* \) then there exists
\[
\rho : \pi_1(X) \to \text{GL}_2(E)
\]
continuous, absolutely irreducible where \( E \) is a finite extension of \( Q_l \) contained in \( \overline{Q}_l \) such that \( t_v = \text{Tr}(\rho(F_v)) \), and \( u_v = q_v^{-1} \det(\rho(F_v)) \) for all places \( v \).

**Proof.** See [Dri80]. \( \square \)

03VU **Theorem 108.3.** Suppose \( \overline{Q}_l \subset E \) finite, and
\[
\rho : \pi_1(X) \to \text{GL}_2(E)
\]
absolutely irreducible, continuous. Then there exists an eigenform \( f \) with values in \( \overline{Q}_l \) whose eigenvalues \( t_v, u_v \) satisfy the equalities \( t_v = \text{Tr}(\rho(F_v)) \) and \( u_v = q_v^{-1} \det(\rho(F_v)) \).

**Proof.** See [Dri83]. \( \square \)

03VV **Remark 108.4.** We now have, thanks to Lafforgue and many other mathematicians, complete theorems like this two above for \( \text{GL}_n \) and allowing ramification! In other words, the full global Langlands correspondence for \( \text{GL}_n \) is known for function fields of curves over finite fields. At the same time this does not mean there aren’t a lot of interesting questions left to answer about the fundamental groups of curves over finite fields, as we shall see below.
Central character. If $f$ is an eigenform then
$$
\chi_f : \frac{\mathbb{O}^* \backslash \mathbb{A}^*/K^*}{(1, \ldots, \pi_v, 1, \ldots, 1)} \to \Lambda^* \ni u_v^{-1}
$$
is called the central character. If corresponds to the determinant of $\rho$ via normalizations as above. Set
$$
C(\Lambda) = \left\{ \text{unr. cusp forms } f \text{ with coefficients in } \Lambda \text{ such that } U_v f = \varphi_v^{-1} f \forall v \right\}
$$

**Proposition 108.5.** If $\Lambda$ is Noetherian then $C(\Lambda)$ is a finitely generated $\Lambda$-module. Moreover, if $\Lambda$ is a field with prime subfield $F \subset \Lambda$ then
$$
C(\Lambda) = (C(F)) \otimes_F \Lambda
$$
compatibly with $T_v$ acting.

**Proof.** See [dJ01, Proposition 4.7].

This proposition trivially implies the following lemma.

**Lemma 108.6.** Algebraicity of eigenvalues. If $\Lambda$ is a field then the eigenvalues $t_v$ for $f \in C(\Lambda)$ are algebraic over the prime subfield $F \subset \Lambda$.

**Proof.** Follows from Proposition 108.5

Combining all of the above we can do the following very useful trick.

**Lemma 108.7.** Switching $l$. Let $E$ be a number field. Start with
$$
\rho : \pi_1(X) \to \text{SL}_2(\mathbb{E}_\lambda)
$$
absolutely irreducible continuous, where $\lambda$ is a place of $E$ not lying above $p$. Then for any second place $\lambda'$ of $E$ not lying above $p$ there exists a finite extension $E'_{\lambda'}$, and an absolutely irreducible continuous representation
$$
\rho' : \pi_1(X) \to \text{SL}_2(\mathbb{E}'_{\lambda'})
$$
which is compatible with $\rho$ in the sense that the characteristic polynomials of all Frobenii are the same.

Note how this is an instance of Deligne’s conjecture!

**Proof.** To prove the switching lemma use Theorem 108.3 to obtain $f \in C(\mathbb{Q}_l)$ eigenform ass. to $\rho$. Next, use Proposition 108.3 to see that we may choose $f \in C(\mathbb{E}')$ with $E \subset E'$ finite. Next we may complete $E'$ to see that we get $f \in C(\mathbb{E}'_{\lambda'})$ eigenform with $E'_{\lambda'}$, a finite extension of $E_{\lambda'}$. And finally we use Theorem 108.2 to obtain $\rho' : \pi_1(X) \to \text{SL}_2(\mathbb{E}'_{\lambda'})$ abs. irred. and continuous after perhaps enlarging $E'_{\lambda'}$ a bit again. 

Speculation: If for a (topological) ring $\Lambda$ we have
$$
\left( \rho : \pi_1(X) \to \text{SL}_2(\Lambda) \right) \leftrightarrow \text{eigen forms in } C(\Lambda)
$$
then all eigenvalues of $\rho(F_v)$ algebraic (won’t work in an easy way if $\Lambda$ is a finite ring. Based on the speculation that the Langlands correspondence works more generally than just over fields one arrives at the following conjecture.
Conjecture. (See [dJ01]) For any continuous
\[ \rho : \pi_1(X) \to \text{GL}_n(F_l[[t]]) \]
we have \(#(\rho(\pi_1(X_F))) < \infty\).

A rephrasing in the language of sheaves: "For any lisse sheaf of \( F_l((t)) \)-modules the geom monodromy is finite."

03VZ **Theorem 108.8.** The Conjecture holds if \( n \leq 2 \).

**Proof.** See [dJ01]. □

03W0 **Theorem 108.9.** Conjecture holds if \( l > 2n \) modulo some unproven things.

**Proof.** See [Gal07]. □

It turns out the conjecture is useful for something. See work of Drinfeld on Kashiwara's conjectures. But there is also the much more down to earth application as follows.

03W1 **Theorem 108.10.** (See [dJ01, Theorem 3.5]) Suppose \( \rho_0 : \pi_1(X) \to \text{GL}_n(F_l) \)
is a continuous, \( l \neq p \). Assume

1. Conj. holds for \( X \),
2. \( \rho_0|_{\pi_1(X_{k})} \) abs. irred., and
3. \( l \) does not divide \( n \).

Then the universal determination ring \( R_{univ} \) of \( \rho_0 \) is finite flat over \( Z_l \).

Explanation: There is a representation \( \rho_{univ} : \pi_1(X) \to \text{GL}_n(R_{univ}) \) (Univ. Defor ring) \( R_{univ} \) loc. complete, residue field \( F_l \) and \( (R_{univ} \to F_l) \circ \rho_{univ} \cong \rho_0 \). And given any \( R \to F_l \), \( R \) local complete and \( \rho : \pi_1(X) \to \text{GL}_n(R) \) then there exists \( \psi : R_{univ} \to R \) such that \( \psi \circ \rho_{univ} \cong \rho \). The theorem says that the morphism
\[ \text{Spec}(R_{univ}) \longrightarrow \text{Spec}(Z_l) \]
is finite and flat. In particular, such a \( \rho_0 \) lifts to a \( \rho : \pi_1(X) \to \text{GL}_n(Q_l) \).

Notes:
1. The theorem on deformations is easy.
2. Any result towards the conjecture seems hard.
3. It would be interesting to have more conjectures on \( \pi_1(X) \)!

109. Counting points

03W2 Let \( X \) be a smooth, geometrically irreducible, projective curve over \( k \) and \( q = \#k \).

The trace formula gives: there exists algebraic integers \( w_1, \ldots, w_{2g} \) such that
\[ \#X(k_n) = q^n - \sum_{i=1}^{2g} w_i^n + 1. \]

If \( \sigma \in \text{Aut}(X) \) then for all \( i \), there exists \( j \) such that \( \sigma(w_i) = w_j \).

Riemann-Hypothesis. For all \( i \) we have \( |w_i| = \sqrt{q} \).

This was formulated by Emil Artin, in 1924, for hyperelliptic curves. Proved by Weil 1940. Weil gave two proofs
- using intersection theory on \( X \times X \), using the Hodge index theorem, and
There is another proof whose initial idea is due to Stephanov, and which was given by Bombieri: it uses the function field $k(X)$ and its Frobenius operator (1969). The starting point is that given $f \in k(X)$ one observes that $f^q - f$ is a rational function which vanishes in all the $\mathbf{F}_q$-rational points of $X$, and that one can try to use this idea to give an upper bound for the number of points.

110. Precise form of Chebotarev

Let us prove a precise form of Chebotarev for a finite étale Galois covering of curves. Let $\varphi : Y \to X$ be a finite étale Galois covering with group $G$. This corresponds to a homomorphism

$$\pi_1(X) \longrightarrow G = \text{Aut}(Y/X)$$

Assume $Y_{\overline{k}}$ is irreducible. If $C \subset G$ is a conjugacy class then for all $n > 0$, we have

$$|\#\{x \in X(k_n) \mid F_x \in C\} - \frac{|C|}{|G|} \cdot \#X(k_n)| \leq (|C|)(2g - 2)\sqrt{q^n}$$

(Warning: Please check the coefficient $|C|$ on the right hand side carefully before using.)

**Sketch.** Write

$$\varphi_*(\mathbb{Q}_l) = \bigoplus_{\pi \in \hat{G}} F_{\pi}$$

where $\hat{G}$ is the set of isomorphism classes of irreducible representations of $G$ over $\mathbb{Q}_l$. For $\pi \in \hat{G}$ let $\chi_\pi : G \to \mathbb{Q}_l$ be the character of $\pi$. Then

$$H^*(Y_{\overline{k}}, \mathbb{Q}_l) = \bigoplus_{\pi \in \hat{G}} H^*(Y_{\overline{k}}, \mathbb{Q}_l)_\pi = (\varphi \text{ finite }) \bigoplus_{\pi \in \hat{G}} H^*(X_{\overline{k}}, F_\pi)$$

If $\pi \neq 1$ then we have

$$H^0(X_{\overline{k}}, F_\pi) = H^2(X_{\overline{k}}, F_\pi) = 0, \quad \dim H^1(X_{\overline{k}}, F_\pi) = (2g_X - 2)d_\pi^2$$

(can get this from trace formula for acting on ...) and we see that

$$|\sum_{x \in X(k_n)} \chi_\pi(F_x)| \leq (2g_X - 2)d_\pi^2 \sqrt{q^n}$$

Write $1_C = \sum_\pi a_\pi \chi_\pi$, then $a_\pi = \langle 1_C, \chi_\pi \rangle$, and $a_1 = \langle 1_C, \chi_1 \rangle = \frac{|C|}{|G|}$ where

$$\langle f, h \rangle = \frac{1}{|G|} \sum_{g \in G} f(g)h(g)$$

Thus we have the relation

$$\frac{|C|}{|G|} = ||1_C||^2 = \sum |a_\pi|^2$$
Final step:
\[
\# \{ x \in X(k_n) \mid F_x \in C \} = \sum_{x \in X(k_n)} 1_C(x) = \sum_{x \in X(k_n)} \sum_{\pi} a_{\pi} \chi_{\pi}(F_x) = \frac{\# C}{\# G} \# X(k_n) + \sum_{\pi \neq 1} a_{\pi} \sum_{x \in X(k_n)} \chi_{\pi}(F_x)
\]

We can bound the error term by
\[
|E| \leq \sum_{\pi \in G, \pi \neq 1} |a_{\pi}|(2g - 2)d_{\pi}^2 \sqrt{q^n}
\]

By Weil’s conjecture, \( \# X(k_n) \sim q^n \).

111. How many primes decompose completely?

This section gives a second application of the Riemann Hypothesis for curves over a finite field. For number theorists it may be nice to look at the paper by Ihara, entitled “How many primes decompose completely in an infinite unramified Galois extension of a global field?”, see [Iha83]. Consider the fundamental exact sequence
\[
1 \to \pi_1(X_{\overline{k}}) \to \pi_1(X) \xrightarrow{\text{deg}} \hat{\mathbb{Z}} \to 1
\]

**Proposition 111.1.** There exists a finite set \( x_1, \ldots, x_n \) of closed points of \( X \) such that that set of all frobenius elements corresponding to these points topologically generate \( \pi_1(X) \).

Another way to state this is: There exist \( x_1, \ldots, x_n \in |X| \) such that the smallest normal closed subgroup \( \Gamma \) of \( \pi_1(X) \) containing 1 frobenius element for each \( x_i \) is all of \( \pi_1(X) \). i.e., \( \Gamma = \pi_1(X) \).

**Proof.** Pick \( N \gg 0 \) and let
\[
\{ x_1, \ldots, x_n \} = \text{set of all closed points of } X \text{ of degree } \leq N \text{ over } k
\]

Let \( \Gamma \subseteq \pi_1(X) \) be as in the variant statement for these points. Assume \( \Gamma \neq \pi_1(X) \).

Then we can pick a normal open subgroup \( U \) of \( \pi_1(X) \) containing \( \Gamma \) with \( U \neq \pi_1(X) \).

By R.H. for \( X \) our set of points will have some \( x_1 \) of degree \( N \), some \( x_i \) of degree \( N - 1 \). This shows \( \text{deg} : \Gamma \to \hat{\mathbb{Z}} \) is surjective and so the same holds for \( U \). This exactly means if \( Y \to X \) is the finite étale Galois covering corresponding to \( U \), then \( Y_{\overline{k}} \) irreducible. Set \( G = \text{Aut}(Y/X) \). Picture
\[
Y \xrightarrow{G} X, \quad G = \pi_1(X)/U
\]

By construction all points of \( X \) of degree \( \leq N \), split completely in \( Y \). So, in particular
\[
\# Y(k_N) \geq (\# G) \# X(k_N)
\]
Use R.H. on both sides. So you get
\[ q^N + 1 + 2g_Yq^{N/2} \geq \#G\#X(k_N) \geq \#G(q^N + 1 - 2g_Xq^{N/2}) \]
Since \( 2g_Y - 2 = (\#G)(2g_X - 2) \), this means
\[ q^N + 1 + (\#G)(2g_X - 1)q^{N/2} \geq \#G(q^N + 1 - 2g_Xq^{N/2}) \]
Thus we see that \( G \) has to be the trivial group if \( N \) is large enough. \( \square \)

**Weird Question.** Set \( W_X = \deg^{-1}(Z) \subset \pi_1(X) \). Is it true that for some finite set of closed points \( x_1, \ldots, x_n \) of \( X \) the set of all frobenii corresponding to these points algebraically generate \( W_X \)?

By a Baire category argument this translates into the same question for all Frobenii.

### 112. How many points are there really?

If the genus of the curve is large relative to \( q \), then the main term in the formula
\[ \#X(k) = q - \sum \omega_i + 1 \] is not \( q \) but the second term \( \sum \omega_i \) which can (a priori) have size about \( 2g_X \sqrt{q} \). In the paper [VD83] the authors Drinfeld and Vladut show that this maximum is (as predicted by Ihara earlier) actually at most about \( g \sqrt{q} \).

Fix \( q \) and let \( k \) be a field with \( k \) elements. Set
\[ A(q) = \limsup_{g_X \to \infty} \frac{\#X(k)}{g_X} \]
where \( X \) runs over geometrically irreducible smooth projective curves over \( k \). With this definition we have the following results:

- RH \( \Rightarrow \) \( A(q) \leq 2 \sqrt{q} \)
- Ihara \( \Rightarrow \) \( A(q) \leq \sqrt{2q} \)
- DV \( \Rightarrow \) \( A(q) \leq \sqrt{q} - 1 \) (actually this is sharp if \( q \) is a square)

**Proof.** Given \( X \) let \( w_1, \ldots, w_{2g} \) and \( g = g_X \) be as before. Set \( \alpha_i = \frac{w_i}{\sqrt{q}} \), so \( |\alpha_i| = 1 \).

If \( \alpha_i \) occurs then \( \overline{\alpha}_i = \alpha_i^{-1} \) also occurs. Then
\[ N = \#X(k) \leq X(k_r) = q^r + 1 - (\sum_i \alpha_i^r)q^{r/2} \]
Rewriting we see that for every \( r \geq 1 \)
\[ -\sum_i \alpha_i^r \geq Nq^{-r/2} - q^{r/2} - q^{-r/2} \]
Observe that
\[ 0 \leq |\alpha_i^n + \alpha_i^{n-1} + \ldots + \alpha_i + 1|^2 = (n + 1) + \sum_{j=1}^n (n + 1 - j)(\alpha_i^j + \alpha_i^{-j}) \]
So
\[ 2g(n + 1) \geq -\sum_i \left( \sum_{j=1}^n (n + 1 - j)(\alpha_i^j + \alpha_i^{-j}) \right) \]
\[ = -\sum_{j=1}^n (n + 1 - j) \left( \sum_{i} \alpha_i^j + \sum_{i} \alpha_i^{-j} \right) \]
Take half of this to get
\[ g(n + 1) \geq -\sum_{j=1}^{n} (n + 1 - j)(\sum_{i} \alpha_{i}^{j}) \]
\[ \geq N \sum_{j=1}^{n} (n + 1 - j)q^{-j/2} - \sum_{j=1}^{n} (n + 1 - j)(q^{j/2} + q^{-j/2}) \]
This gives
\[ \frac{N}{g} \leq \left( \sum_{j=1}^{n} \frac{n + 1 - j}{n + 1} q^{-j/2} \right)^{-1} \cdot \left( 1 + \frac{1}{g} \sum_{j=1}^{n} \frac{n + 1 - j}{n + 1} (q^{j/2} + q^{-j/2}) \right) \]
Fix \( n \) let \( g \to \infty \)
\[ A(q) \leq \left( \sum_{j=1}^{n} \frac{n + 1 - j}{n + 1} q^{-j/2} \right)^{-1} \]
So
\[ A(q) \leq \lim_{n \to \infty}(\ldots) = \left( \sum_{j=1}^{\infty} q^{-j/2} \right)^{-1} = \sqrt{q} - 1 \]
\[ \Box \]

113. Other chapters

Preliminaries

(1) Introduction
(2) Conventions
(3) Set Theory
(4) Categories
(5) Topology
(6) Sheaves on Spaces
(7) Sites and Sheaves
(8) Stacks
(9) Fields
(10) Commutative Algebra
(11) Brauer Groups
(12) Homological Algebra
(13) Derived Categories
(14) Simplicial Methods
(15) More on Algebra
(16) Smoothing Ring Maps
(17) Sheaves of Modules
(18) Modules on Sites
(19) Injectives
(20) Cohomology of Sheaves
(21) Cohomology on Sites
(22) Differential Graded Algebra
(23) Divided Power Algebra

(24) Hypercoverings

Schemes

(25) Schemes
(26) Constructions of Schemes
(27) Properties of Schemes
(28) Morphisms of Schemes
(29) Cohomology of Schemes
(30) Divisors
(31) Limits of Schemes
(32) Varieties
(33) Topologies on Schemes
(34) Descent

Topics in Scheme Theory

(35) Derived Categories of Schemes
(36) More on Morphisms
(37) More on Flatness
(38) Groupoid Schemes
(39) More on Groupoid Schemes
(40) Étale Morphisms of Schemes
(41) Chow Homology
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References


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