

# DEFORMATION PROBLEMS

0DVK

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## 1. Introduction

0DVL The goal of this chapter is to work out examples of the general theory developed in the chapters Formal Deformation Theory, Deformation Theory, The Cotangent Complex.

Section 3 of the paper [Sch68] by Schlessinger discusses some examples as well.

## 2. Examples of deformation problems

06LA List of things that should go here:

- (1) Deformations of schemes:
  - (a) The Rim-Schlessinger condition.
  - (b) Computing the tangent space.
  - (c) Computing the infinitesimal deformations.
  - (d) The deformation category of an affine hypersurface.
- (2) Deformations of sheaves (for example fix  $X/S$ , a finite type point  $s$  of  $S$ , and a quasi-coherent sheaf  $\mathcal{F}_s$  over  $X_s$ ).

- (3) Deformations of algebraic spaces (very similar to deformations of schemes; maybe even easier?).
- (4) Deformations of maps (eg morphisms between schemes; you can fix both or one of the target and/or source).
- (5) Add more here.

### 3. General outline

0DVM This section lays out the procedure for discussing the next few examples.

Step I. For each section we fix a Noetherian ring  $\Lambda$  and we fix a finite ring map  $\Lambda \rightarrow k$  where  $k$  is a field. As usual we let  $\mathcal{C}_\Lambda = \mathcal{C}_{\Lambda,k}$  be our base category, see Formal Deformation Theory, Definition 3.1.

Step II. In each section we define a category  $\mathcal{F}$  cofibred in groupoids over  $\mathcal{C}_\Lambda$ . Occassionally we will consider instead a functor  $F : \mathcal{C}_\Lambda \rightarrow Sets$ .

Step III. We explain to what extent  $\mathcal{F}$  satisfies the Rim-Schlessinger condition (RS) discussed in Formal Deformation Theory, Section 16. Similarly, we may discuss to what extent our  $\mathcal{F}$  satisfies (S1) and (S2) or to what extent  $F$  satisfies the corresponding Schlessinger's conditions (H1) and (H2). See Formal Deformation Theory, Section 10.

Step IV. Let  $x_0$  be an object of  $\mathcal{F}(k)$ , in other words an object of  $\mathcal{F}$  over  $k$ . In this chapter we will use the notation

$$\mathcal{D}ef_{x_0} = \mathcal{F}_{x_0}$$

to denote the predeformation category constructed in Formal Deformation Theory, Remark 6.4. If  $\mathcal{F}$  satisfies (RS), then  $\mathcal{D}ef_{x_0}$  is a deformation category (Formal Deformation Theory, Lemma 16.11) and satisfies (S1) and (S2) (Formal Deformation Theory, Lemma 16.6). If (S1) and (S2) are satisfied, then an important question is whether the tangent space

$$T\mathcal{D}ef_{x_0} = T_{x_0}\mathcal{F} = T\mathcal{F}_{x_0}$$

(see Formal Deformation Theory, Remark 12.5 and Definition 12.1) is finite dimensional. Namely, this insures that  $\mathcal{D}ef_{x_0}$  has a versal formal object (Formal Deformation Theory, Lemma 13.4).

Step V. If  $\mathcal{F}$  passes Step IV, then the next question is whether the  $k$ -vector space

$$\text{Inf}(\mathcal{D}ef_{x_0}) = \text{Inf}_{x_0}(\mathcal{F})$$

of infinitesimal automorphisms of  $x_0$  is finite dimensional. Namely, if true, this implies that  $\mathcal{D}ef_{x_0}$  admits a presentation by a smooth prorepresentable groupoid in functors on  $\mathcal{C}_\Lambda$ , see Formal Deformation Theory, Theorem 26.4.

### 4. Finite projective modules

0DVN This section is just a warmup. Of course finite projective modules should not have any "moduli".

0D3I **Example 4.1** (Finite projective modules). Let  $\mathcal{F}$  be the category defined as follows

- (1) an object is a pair  $(A, M)$  consisting of an object  $A$  of  $\mathcal{C}_\Lambda$  and a finite projective  $A$ -module  $M$ , and

- (2) a morphism  $(f, g) : (B, N) \rightarrow (A, M)$  consists of a morphism  $f : B \rightarrow A$  in  $\mathcal{C}_\Lambda$  together with a map  $g : N \rightarrow M$  which is  $f$ -linear and induces an isomorphism  $N \otimes_{B, f} A \cong M$ .

The functor  $p : \mathcal{F} \rightarrow \mathcal{C}_\Lambda$  sends  $(A, M)$  to  $A$  and  $(f, g)$  to  $f$ . It is clear that  $p$  is cofibred in groupoids. Given a finite dimensional  $k$ -vector space  $V$ , let  $x_0 = (k, V)$  be the corresponding object of  $\mathcal{F}(k)$ . We set

$$\mathcal{D}ef_V = \mathcal{F}_{x_0}$$

Since every finite projective module over a local ring is finite free (Algebra, Lemma 77.2) we see that

$$\begin{array}{l} \text{isomorphism classes} \\ \text{of objects of } \mathcal{F}(A) \end{array} = \coprod_{n \geq 0} \{*\}$$

Although this means that the deformation theory of  $\mathcal{F}$  is essentially trivial, we still work through the steps outlined in Section 3 to provide an easy example.

0DVP **Lemma 4.2.** *Example 4.1 satisfies the Rim-Schlessinger condition (RS). In particular,  $\mathcal{D}ef_V$  is a deformation category for any finite dimensional vector space  $V$  over  $k$ .*

**Proof.** Let  $A_1 \rightarrow A$  and  $A_2 \rightarrow A$  be morphisms of  $\mathcal{C}_\Lambda$ . Assume  $A_2 \rightarrow A$  is surjective. According to Formal Deformation Theory, Lemma 16.4 it suffices to show that the functor  $\mathcal{F}(A_1 \times_A A_2) \rightarrow \mathcal{F}(A_1) \times_{\mathcal{F}(A)} \mathcal{F}(A_2)$  is an equivalence of categories.

Thus we have to show that the category of finite projective modules over  $A_1 \times_A A_2$  is equivalent to the fibre product of the categories of finite projective modules over  $A_1$  and  $A_2$  over the category of finite projective modules over  $A$ . This is a special case of More on Algebra, Lemma 6.9. We recall that the inverse functor sends the triple  $(M_1, M_2, \varphi)$  where  $M_1$  is a finite projective  $A_1$ -module,  $M_2$  is a finite projective  $A_2$ -module, and  $\varphi : M_1 \otimes_{A_1} A \rightarrow M_2 \otimes_{A_2} A$  is an isomorphism of  $A$ -module, to the finite projective  $A_1 \times_A A_2$ -module  $M_1 \times_\varphi M_2$ .  $\square$

0DVQ **Lemma 4.3.** *In Example 4.1 let  $V$  be a finite dimensional  $k$ -vector space. Then*

$$T\mathcal{D}ef_V = (0) \quad \text{and} \quad \text{Inf}(\mathcal{D}ef_V) = \text{End}_k(V)$$

*are finite dimensional.*

**Proof.** With  $\mathcal{F}$  as in Example 4.1 set  $x_0 = (k, V) \in \text{Ob}(\mathcal{F}(k))$ . Recall that  $T\mathcal{D}ef_V = T_{x_0}\mathcal{F}$  is the set of isomorphism classes of pairs  $(x, \alpha)$  consisting of an object  $x$  of  $\mathcal{F}$  over the dual numbers  $k[\epsilon]$  and a morphism  $\alpha : x \rightarrow x_0$  of  $\mathcal{F}$  lying over  $k[\epsilon] \rightarrow k$ .

Up to isomorphism, there is a unique pair  $(M, \alpha)$  consisting of a finite projective module  $M$  over  $k[\epsilon]$  and  $k[\epsilon]$ -linear map  $\alpha : M \rightarrow V$  which induces an isomorphism  $M \otimes_{k[\epsilon]} k \rightarrow V$ . For example, if  $V = k^{\oplus n}$ , then we take  $M = k[\epsilon]^{\oplus n}$  with the obvious map  $\alpha$ .

Similarly,  $\text{Inf}(\mathcal{D}ef_V) = \text{Inf}_{x_0}(\mathcal{F})$  is the set of automorphisms of the trivial deformation  $x'_0$  of  $x_0$  over  $k[\epsilon]$ . See Formal Deformation Theory, Definition 19.2 for details.

Given  $(M, \alpha)$  as in the second paragraph, we see that an element of  $\text{Inf}_{x_0}(\mathcal{F})$  is an automorphism  $\gamma : M \rightarrow M$  with  $\gamma \bmod \epsilon = \text{id}$ . Then we can write  $\gamma = \text{id}_M + \epsilon\psi$

where  $\psi : M/\epsilon M \rightarrow M/\epsilon M$  is  $k$ -linear. Using  $\alpha$  we can think of  $\psi$  as an element of  $\text{End}_k(V)$  and this finishes the proof.  $\square$

## 5. Representations of a group

0DVR The deformation theory of representations can be very interesting.

0D3J **Example 5.1** (Representations of a group). Let  $\Gamma$  be a group. Let  $\mathcal{F}$  be the category defined as follows

- (1) an object is a triple  $(A, M, \rho)$  consisting of an object  $A$  of  $\mathcal{C}_\Lambda$ , a finite projective  $A$ -module  $M$ , and a homomorphism  $\rho : \Gamma \rightarrow \text{GL}_A(M)$ , and
- (2) a morphism  $(f, g) : (B, N, \tau) \rightarrow (A, M, \rho)$  consists of a morphism  $f : B \rightarrow A$  in  $\mathcal{C}_\Lambda$  together with a map  $g : N \rightarrow M$  which is  $f$ -linear and  $\Gamma$ -equivariant and induces an isomorphism  $N \otimes_{B, f} A \cong M$ .

The functor  $p : \mathcal{F} \rightarrow \mathcal{C}_\Lambda$  sends  $(A, M, \rho)$  to  $A$  and  $(f, g)$  to  $f$ . It is clear that  $p$  is cofibred in groupoids. Given a finite dimensional  $k$ -vector space  $V$  and a representation  $\rho_0 : \Gamma \rightarrow \text{GL}_k(V)$ , let  $x_0 = (k, V, \rho_0)$  be the corresponding object of  $\mathcal{F}(k)$ . We set

$$\text{Def}_{V, \rho_0} = \mathcal{F}_{x_0}$$

Since every finite projective module over a local ring is finite free (Algebra, Lemma 77.2) we see that

$$\begin{array}{l} \text{isomorphism classes} \\ \text{of objects of } \mathcal{F}(A) \end{array} = \coprod_{n \geq 0} \begin{array}{l} \text{GL}_n(A)\text{-conjugacy classes of} \\ \text{homomorphisms } \rho : \Gamma \rightarrow \text{GL}_n(A) \end{array}$$

This is already more interesting than the discussion in Section 4.

0DVS **Lemma 5.2.** *Example 5.1 satisfies the Rim-Schlessinger condition (RS). In particular,  $\text{Def}_{V, \rho_0}$  is a deformation category for any finite dimensional representation  $\rho_0 : \Gamma \rightarrow \text{GL}_k(V)$ .*

**Proof.** Let  $A_1 \rightarrow A$  and  $A_2 \rightarrow A$  be morphisms of  $\mathcal{C}_\Lambda$ . Assume  $A_2 \rightarrow A$  is surjective. According to Formal Deformation Theory, Lemma 16.4 it suffices to show that the functor  $\mathcal{F}(A_1 \times_A A_2) \rightarrow \mathcal{F}(A_1) \times_{\mathcal{F}(A)} \mathcal{F}(A_2)$  is an equivalence of categories.

Consider an object

$$((A_1, M_1, \rho_1), (A_2, M_2, \rho_2), (\text{id}_A, \varphi))$$

of the category  $\mathcal{F}(A_1) \times_{\mathcal{F}(A)} \mathcal{F}(A_2)$ . Then, as seen in the proof of Lemma 4.2, we can consider the finite projective  $A_1 \times_A A_2$ -module  $M_1 \times_\varphi M_2$ . Since  $\varphi$  is compatible with the given actions we obtain

$$\rho_1 \times \rho_2 : \Gamma \longrightarrow \text{GL}_{A_1 \times_A A_2}(M_1 \times_\varphi M_2)$$

Then  $(M_1 \times_\varphi M_2, \rho_1 \times \rho_2)$  is an object of  $\mathcal{F}(A_1 \times_A A_2)$ . This construction determines a quasi-inverse to our functor.  $\square$

0DVT **Lemma 5.3.** *In Example 5.1 let  $\rho_0 : \Gamma \rightarrow \text{GL}_k(V)$  be a finite dimensional representation. Then*

$$T\text{Def}_{V, \rho_0} = \text{Ext}_{k[\Gamma]}^1(V, V) = H^1(\Gamma, \text{End}_k(V)) \quad \text{and} \quad \text{Inf}(\text{Def}_{V, \rho_0}) = H^0(\Gamma, \text{End}_k(V))$$

*Thus  $\text{Inf}(\text{Def}_{V, \rho_0})$  is always finite dimensional and  $T\text{Def}_{V, \rho_0}$  is finite dimensional if  $\Gamma$  is finitely generated.*

**Proof.** We first deal with the infinitesimal automorphisms. Let  $M = V \otimes_k k[\epsilon]$  with induced action  $\rho'_0 : \Gamma \rightarrow \mathrm{GL}_n(M)$ . Then an infinitesimal automorphism, i.e., an element of  $\mathrm{Inf}(\mathcal{D}ef_{V, \rho_0})$ , is given by an automorphism  $\gamma = \mathrm{id} + \epsilon\psi : M \rightarrow M$  as in the proof of Lemma 4.3, where moreover  $\psi$  has to commute with the action of  $\Gamma$  (given by  $\rho_0$ ). Thus we see that

$$\mathrm{Inf}(\mathcal{D}ef_{V, \rho_0}) = H^0(\Gamma, \mathrm{End}_k(V))$$

as predicted in the lemma.

Next, let  $(k[\epsilon], M, \rho)$  be an object of  $\mathcal{F}$  over  $k[\epsilon]$  and let  $\alpha : M \rightarrow V$  be a  $\Gamma$ -equivariant map inducing an isomorphism  $M/\epsilon M \rightarrow V$ . Since  $M$  is free as a  $k[\epsilon]$ -module we obtain an extension of  $\Gamma$ -modules

$$0 \rightarrow V \rightarrow M \xrightarrow{\alpha} V \rightarrow 0$$

We omit the detailed construction of the map on the left. Conversely, if we have an extension of  $\Gamma$ -modules as above, then we can use this to make a  $k[\epsilon]$ -module structure on  $M$  and get an object of  $\mathcal{F}(k[\epsilon])$  together with a map  $\alpha$  as above. It follows that

$$T\mathcal{D}ef_{V, \rho_0} = \mathrm{Ext}_{k[\Gamma]}^1(V, V)$$

as predicted in the lemma. This is equal to  $H^1(\Gamma, \mathrm{End}_k(V))$  by Étale Cohomology, Lemma 56.4.

The statement on dimensions follows from Étale Cohomology, Lemma 56.5.  $\square$

## 6. Continuous representations

0DVU A very interesting thing one can do is to take an infinite Galois group and study the deformation theory of its representations, see [Maz89].

0D3K **Example 6.1** (Representations of a topological group). Let  $\Gamma$  be a topological group. Let  $\mathcal{F}$  be the category defined as follows

- (1) an object is a triple  $(A, M, \rho)$  consisting of an object  $A$  of  $\mathcal{C}_\Lambda$ , a finite projective  $A$ -module  $M$ , and a continuous homomorphism  $\rho : \Gamma \rightarrow \mathrm{GL}_A(M)$  where  $\mathrm{GL}_A(M)$  is given the discrete topology<sup>1</sup>, and
- (2) a morphism  $(f, g) : (B, N, \tau) \rightarrow (A, M, \rho)$  consists of a morphism  $f : B \rightarrow A$  in  $\mathcal{C}_\Lambda$  together with a map  $g : N \rightarrow M$  which is  $f$ -linear and  $\Gamma$ -equivariant and induces an isomorphism  $N \otimes_{B, f} A \cong M$ .

The functor  $p : \mathcal{F} \rightarrow \mathcal{C}_\Lambda$  sends  $(A, M, \rho)$  to  $A$  and  $(f, g)$  to  $f$ . It is clear that  $p$  is cofibred in groupoids. Given a finite dimensional  $k$ -vector space  $V$  and a continuous representation  $\rho_0 : \Gamma \rightarrow \mathrm{GL}_k(V)$ , let  $x_0 = (k, V, \rho_0)$  be the corresponding object of  $\mathcal{F}(k)$ . We set

$$\mathcal{D}ef_{V, \rho_0} = \mathcal{F}_{x_0}$$

Since every finite projective module over a local ring is finite free (Algebra, Lemma 77.2) we see that

$$\begin{array}{l} \text{isomorphism classes} \\ \text{of objects of } \mathcal{F}(A) \end{array} = \coprod_{n \geq 0} \begin{array}{l} \mathrm{GL}_n(A)\text{-conjugacy classes of} \\ \text{continuous homomorphisms } \rho : \Gamma \rightarrow \mathrm{GL}_n(A) \end{array}$$

<sup>1</sup>An alternative would be to require the  $A$ -module  $M$  with  $G$ -action given by  $\rho$  is an  $A$ - $G$ -module as defined in Étale Cohomology, Definition 56.1. However, since  $M$  is a finite  $A$ -module, this is equivalent.

0DVV **Lemma 6.2.** *Example 6.1 satisfies the Rim-Schlessinger condition (RS). In particular,  $\mathcal{D}ef_{V,\rho_0}$  is a deformation category for any finite dimensional continuous representation  $\rho_0 : \Gamma \rightarrow GL_k(V)$ .*

**Proof.** The proof is exactly the same as the proof of Lemma 5.2.  $\square$

0DVW **Lemma 6.3.** *In Example 6.1 let  $\rho_0 : \Gamma \rightarrow GL_k(V)$  be a finite dimensional continuous representation. Then*

$$T\mathcal{D}ef_{V,\rho_0} = H^1(\Gamma, End_k(V)) \quad \text{and} \quad Inf(\mathcal{D}ef_{V,\rho_0}) = H^0(\Gamma, End_k(V))$$

*Thus  $Inf(\mathcal{D}ef_{V,\rho_0})$  is always finite dimensional and  $T\mathcal{D}ef_{V,\rho_0}$  is finite dimensional if  $\Gamma$  is topologically finitely generated.*

**Proof.** The proof is exactly the same as the proof of Lemma 5.3.  $\square$

## 7. Graded algebras

0DVX We will use the example in this section in the proof that the stack of polarized proper schemes is an algebraic stack. For this reason we will consider commutative graded algebras whose homogeneous parts are finite projective modules (sometimes called “locally finite”).

0D3L **Example 7.1** (Graded algebras). Let  $\mathcal{F}$  be the category defined as follows

- (1) an object is a pair  $(A, P)$  consisting of an object  $A$  of  $\mathcal{C}_\Lambda$  and a graded  $A$ -algebra  $P$  such that  $P_d$  is a finite projective  $A$ -module for all  $d \geq 0$ , and
- (2) a morphism  $(f, g) : (B, Q) \rightarrow (A, P)$  consists of a morphism  $f : B \rightarrow A$  in  $\mathcal{C}_\Lambda$  together with a map  $g : Q \rightarrow P$  which is  $f$ -linear and induces an isomorphism  $Q \otimes_{B,f} A \cong P$ .

The functor  $p : \mathcal{F} \rightarrow \mathcal{C}_\Lambda$  sends  $(A, P)$  to  $A$  and  $(f, g)$  to  $f$ . It is clear that  $p$  is cofibred in groupoids. Given a graded  $k$ -algebra  $P$  with  $\dim_k(P_d) < \infty$  for all  $d \geq 0$ , let  $x_0 = (k, P)$  be the corresponding object of  $\mathcal{F}(k)$ . We set

$$\mathcal{D}ef_P = \mathcal{F}_{x_0}$$

0DVY **Lemma 7.2.** *Example 7.1 satisfies the Rim-Schlessinger condition (RS). In particular,  $\mathcal{D}ef_P$  is a deformation category for any graded  $k$ -algebra  $P$ .*

**Proof.** Let  $A_1 \rightarrow A$  and  $A_2 \rightarrow A$  be morphisms of  $\mathcal{C}_\Lambda$ . Assume  $A_2 \rightarrow A$  is surjective. According to Formal Deformation Theory, Lemma 16.4 it suffices to show that the functor  $\mathcal{F}(A_1 \times_A A_2) \rightarrow \mathcal{F}(A_1) \times_{\mathcal{F}(A)} \mathcal{F}(A_2)$  is an equivalence of categories.

Consider an object

$$((A_1, P_1), (A_2, P_2), (\text{id}_A, \varphi))$$

of the category  $\mathcal{F}(A_1) \times_{\mathcal{F}(A)} \mathcal{F}(A_2)$ . Then we consider  $P_1 \times_\varphi P_2$ . Since  $\varphi : P_1 \otimes_{A_1} A \rightarrow P_2 \otimes_{A_2} A$  is an isomorphism of graded algebras, we see that the graded pieces of  $P_1 \times_\varphi P_2$  are finite projective  $A_1 \times_A A_2$ -modules, see proof of Lemma 4.2. Thus  $P_1 \times_\varphi P_2$  is an object of  $\mathcal{F}(A_1 \times_A A_2)$ . This construction determines a quasi-inverse to our functor and the proof is complete.  $\square$

0DVZ **Lemma 7.3.** *In Example 7.1 let  $P$  be a graded  $k$ -algebra. Then*

$$T\mathcal{D}ef_P \quad \text{and} \quad Inf(\mathcal{D}ef_P) = Der_k(P, P)$$

*are finite dimensional if  $P$  is finitely generated over  $k$ .*

**Proof.** We first deal with the infinitesimal automorphisms. Let  $Q = P \otimes_k k[\epsilon]$ . Then an element of  $\text{Inf}(\mathcal{D}ef_P)$  is given by an automorphism  $\gamma = \text{id} + \epsilon\delta : Q \rightarrow Q$  as above where now  $\delta : P \rightarrow P$ . The fact that  $\gamma$  is graded implies that  $\delta$  is homogeneous of degree 0. The fact that  $\gamma$  is  $k$ -linear implies that  $\delta$  is  $k$ -linear. The fact that  $\gamma$  is multiplicative implies that  $\delta$  is a  $k$ -derivation. Conversely, given a  $k$ -derivation  $\delta : P \rightarrow P$  homogeneous of degree 0, we obtain an automorphism  $\gamma = \text{id} + \epsilon\delta$  as above. Thus we see that

$$\text{Inf}(\mathcal{D}ef_P) = \text{Der}_k(P, P)$$

as predicted in the lemma. Clearly, if  $P$  is generated in degrees  $P_i$ ,  $0 \leq i \leq N$ , then  $\delta$  is determined by the linear maps  $\delta_i : P_i \rightarrow P_i$  for  $0 \leq i \leq N$  and we see that

$$\dim_k \text{Der}_k(P, P) < \infty$$

as desired.

To finish the proof of the lemma we show that there is a finite dimensional deformation space. To do this we choose a presentation

$$k[X_1, \dots, X_n]/(F_1, \dots, F_m) \longrightarrow P$$

of graded  $k$ -algebras where  $\deg(X_i) = d_i$  and  $F_j$  is homogeneous of degree  $e_j$ . Let  $Q$  be any graded  $k[\epsilon]$ -algebra finite free in each degree which comes with an isomorphism  $\alpha : Q/\epsilon Q \rightarrow P$  so that  $(Q, \alpha)$  defines an element of  $T\mathcal{D}ef_P$ . Choose a homogeneous element  $q_i \in Q$  of degree  $d_i$  mapping to the image of  $X_i$  in  $P$ . Then we obtain

$$k[\epsilon][X_1, \dots, X_n] \longrightarrow Q, \quad X_i \longmapsto q_i$$

and since  $P = Q/\epsilon Q$  this map is surjective by Nakayama's lemma. A small diagram chase shows we can choose homogeneous elements  $F_{\epsilon, j} \in k[\epsilon][X_1, \dots, X_n]$  of degree  $e_j$  mapping to zero in  $Q$  and mapping to  $F_j$  in  $k[X_1, \dots, X_n]$ . Then

$$k[\epsilon][X_1, \dots, X_n]/(F_{\epsilon, 1}, \dots, F_{\epsilon, m}) \longrightarrow Q$$

is a presentation of  $Q$  by flatness of  $Q$  over  $k[\epsilon]$ . Write

$$F_{\epsilon, j} = F_j + \epsilon G_j$$

There is some ambiguity in the vector  $(G_1, \dots, G_m)$ . First, using different choices of  $F_{\epsilon, j}$  we can modify  $G_j$  by an arbitrary element of degree  $e_j$  in the kernel of  $k[X_1, \dots, X_n] \rightarrow P$ . Hence, instead of  $(G_1, \dots, G_m)$ , we remember the element

$$(g_1, \dots, g_m) \in P_{e_1} \oplus \dots \oplus P_{e_m}$$

where  $g_j$  is the image of  $G_j$  in  $P_{e_j}$ . Moreover, if we change our choice of  $q_i$  into  $q_i + \epsilon p_i$  with  $p_i$  of degree  $d_i$  then a computation (omitted) shows that  $g_j$  changes into

$$g_j^{new} = g_j - \sum_{i=1}^n p_i \partial F_j / \partial X_i$$

We conclude that the isomorphism class of  $Q$  is determined by the image of the vector  $(G_1, \dots, G_m)$  in the  $k$ -vector space

$$W = \text{Coker}(P_{d_1} \oplus \dots \oplus P_{d_n} \xrightarrow{\left(\frac{\partial F_j}{\partial X_i}\right)} P_{e_1} \oplus \dots \oplus P_{e_m})$$

In this way we see that we obtain an injection

$$T\mathcal{D}ef_P \longrightarrow W$$

Since  $W$  visibly has finite dimension, we conclude that the lemma is true.  $\square$

## 8. Rings

0DY0 The deformation theory of rings is the same as the deformation theory of affine schemes. For rings and schemes when we talk about deformations it means we are thinking about *flat* deformations.

0DY1 **Example 8.1** (Rings). Let  $\mathcal{F}$  be the category defined as follows

- (1) an object is a pair  $(A, P)$  consisting of an object  $A$  of  $\mathcal{C}_\Lambda$  and a flat  $A$ -algebra  $P$ , and
- (2) a morphism  $(f, g) : (B, Q) \rightarrow (A, P)$  consists of a morphism  $f : B \rightarrow A$  in  $\mathcal{C}_\Lambda$  together with a map  $g : Q \rightarrow P$  which is  $f$ -linear and induces an isomorphism  $Q \otimes_{B, f} A \cong P$ .

The functor  $p : \mathcal{F} \rightarrow \mathcal{C}_\Lambda$  sends  $(A, P)$  to  $A$  and  $(f, g)$  to  $f$ . It is clear that  $p$  is cofibred in groupoids. Given a  $k$ -algebra  $P$ , let  $x_0 = (k, P)$  be the corresponding object of  $\mathcal{F}(k)$ . We set

$$\mathcal{D}ef_P = \mathcal{F}_{x_0}$$

0DY2 **Lemma 8.2.** *Example 8.1 satisfies the Rim-Schlessinger condition (RS). In particular,  $\mathcal{D}ef_P$  is a deformation category for any  $k$ -algebra  $P$ .*

**Proof.** Let  $A_1 \rightarrow A$  and  $A_2 \rightarrow A$  be morphisms of  $\mathcal{C}_\Lambda$ . Assume  $A_2 \rightarrow A$  is surjective. According to Formal Deformation Theory, Lemma 16.4 it suffices to show that the functor  $\mathcal{F}(A_1 \times_A A_2) \rightarrow \mathcal{F}(A_1) \times_{\mathcal{F}(A)} \mathcal{F}(A_2)$  is an equivalence of categories. This is a special case of More on Algebra, Lemma 7.7.  $\square$

0DY3 **Lemma 8.3.** *In Example 8.1 let  $P$  be a  $k$ -algebra. Then*

$$T\mathcal{D}ef_P = \text{Ext}_P^1(NL_{P/k}, P) \quad \text{and} \quad \text{Inf}(\mathcal{D}ef_P) = \text{Der}_k(P, P)$$

**Proof.** Recall that  $\text{Inf}(\mathcal{D}ef_P)$  is the set of automorphisms of the trivial deformation  $P[\epsilon] = P \otimes_k k[\epsilon]$  of  $P$  to  $k[\epsilon]$  equal to the identity modulo  $\epsilon$ . By Deformation Theory, Lemma 2.1 this is equal to  $\text{Hom}_P(\Omega_{P/k}, P)$  which in turn is equal to  $\text{Der}_k(P, P)$  by Algebra, Lemma 130.3.

Recall that  $T\mathcal{D}ef_P$  is the set of isomorphism classes of flat deformations  $Q$  of  $P$  to  $k[\epsilon]$ , more precisely, the set of isomorphism classes of  $\mathcal{D}ef_P(k[\epsilon])$ . Recall that a  $k[\epsilon]$ -algebra  $Q$  with  $Q/\epsilon Q = P$  is flat over  $k[\epsilon]$  if and only if

$$0 \rightarrow P \xrightarrow{\epsilon} Q \rightarrow P \rightarrow 0$$

is exact. This is proven in More on Morphisms, Lemma 10.1 and more generally in Deformation Theory, Lemma 5.2. Thus we may apply Deformation Theory, Lemma 2.3 to see that the set of isomorphism classes of such deformations is equal to  $\text{Ext}_P^1(NL_{P/k}, P)$ .  $\square$

0DZL **Lemma 8.4.** *In Example 8.1 let  $P$  be a smooth  $k$ -algebra. Then  $T\mathcal{D}ef_P = (0)$ .*

**Proof.** By Lemma 8.3 we have to show  $\text{Ext}_P^1(NL_{P/k}, P) = (0)$ . Since  $k \rightarrow P$  is smooth  $NL_{P/k}$  is quasi-isomorphic to the complex consisting of a finite projective  $P$ -module placed in degree 0.  $\square$

0DY4 **Lemma 8.5.** *In Lemma 8.3 if  $P$  is a finite type  $k$ -algebra, then*

- (1)  $\text{Inf}(\mathcal{D}ef_P)$  is finite dimensional if and only if  $\dim(P) = 0$ , and
- (2)  $T\mathcal{D}ef_P$  is finite dimensional if  $\text{Spec}(P) \rightarrow \text{Spec}(k)$  is smooth except at a finite number of points.



**Proof.** Proof of (1). We view  $\text{Der}_k(P, P)$  as a  $P$ -module. If it has finite dimension over  $k$ , then it has finite length as a  $P$ -module, hence it is supported in finitely many closed points of  $\text{Spec}(P)$  (Algebra, Lemma 51.11). Since  $\text{Der}_k(P, P) = \text{Hom}_P(\Omega_{P/k}, P)$  we see that  $\text{Der}_k(P, P)_{\mathfrak{p}} = \text{Der}_k(P_{\mathfrak{p}}, P_{\mathfrak{p}})$  for any prime  $\mathfrak{p} \subset P$  (this uses Algebra, Lemmas 130.8, 130.15, and 10.2). Let  $\mathfrak{p}$  be a minimal prime ideal of  $P$  corresponding to an irreducible component of dimension  $d > 0$ . Then  $P_{\mathfrak{p}}$  is an Artinian local ring essentially of finite type over  $k$  with residue field and  $\Omega_{P_{\mathfrak{p}}/k}$  is nonzero for example by Algebra, Lemma 138.3. Any nonzero finite module over an Artinian local ring has both a sub and a quotient module isomorphic to the residue field. Thus we find that  $\text{Der}_k(P_{\mathfrak{p}}, P_{\mathfrak{p}}) = \text{Hom}_{P_{\mathfrak{p}}}(\Omega_{P_{\mathfrak{p}}/k}, P_{\mathfrak{p}})$  is nonzero too. Combining all of the above we find that (1) is true.

Proof of (2). For a prime  $\mathfrak{p}$  of  $P$  we will use that  $NL_{P_{\mathfrak{p}}/k} = (NL_{P/k})_{\mathfrak{p}}$  (Algebra, Lemma 132.13) and we will use that  $\text{Ext}_P^1(NL_{P/k}, P)_{\mathfrak{p}} = \text{Ext}_{P_{\mathfrak{p}}}^1(NL_{P_{\mathfrak{p}}/k}, P_{\mathfrak{p}})$  (More on Algebra, Remark 60.20). Given a prime  $\mathfrak{p} \subset P$  then  $k \rightarrow P$  is smooth at  $\mathfrak{p}$  if and only if  $(NL_{P/k})_{\mathfrak{p}}$  is quasi-isomorphic to a finite projective module placed in degree 0 (this follows immediately from the definition of a smooth ring map but it also follows from the stronger Algebra, Lemma 135.12).

Assume that  $P$  is smooth over  $k$  at all but finitely many primes. Then these “bad” primes are maximal ideals  $\mathfrak{m}_1, \dots, \mathfrak{m}_n \subset P$  by Algebra, Lemma 60.3 and the fact that the “bad” primes form a closed subset of  $\text{Spec}(P)$ . For  $\mathfrak{p} \notin \{\mathfrak{m}_1, \dots, \mathfrak{m}_n\}$  we have  $\text{Ext}_P^1(NL_{P/k}, P)_{\mathfrak{p}} = 0$  by the results above. Thus  $\text{Ext}_P^1(NL_{P/k}, P)$  is a finite  $P$ -module whose support is contained in  $\{\mathfrak{m}_1, \dots, \mathfrak{m}_n\}$ . By Algebra, Proposition 62.6 for example, we find that the dimension over  $k$  of  $\text{Ext}_P^1(NL_{P/k}, P)$  is a finite integer combination of  $\dim_k \kappa(\mathfrak{m}_i)$  and hence finite by the Hilbert Nullstellensatz (Algebra, Theorem 33.1).  $\square$

0DYS **Lemma 8.6.** *In Example 8.1 let  $P$  be a  $k$ -algebra. Let  $S \subset P$  be a multiplicative subset. There is a natural functor*

$$\mathcal{D}ef_P \longrightarrow \mathcal{D}ef_{S^{-1}P}$$

*of deformation categories.*

**Proof.** Given a deformation of  $P$  we can take the localization of it to get a deformation of the localization; this is clear and we encourage the reader to skip the proof. More precisely, let  $(A, Q) \rightarrow (k, P)$  be a morphism in  $\mathcal{F}$ , i.e., an object of  $\mathcal{D}ef_P$ . Let  $S_Q \subset Q$  be the inverse image of  $S$ . Then Hence  $(A, S_Q^{-1}Q) \rightarrow (k, S^{-1}P)$  is the desired object of  $\mathcal{D}ef_{S^{-1}P}$ .  $\square$

0DYT **Lemma 8.7.** *In Example 8.1 let  $P$  be a  $k$ -algebra. Let  $J \subset P$  be an ideal. Denote  $(P^h, J^h)$  the henselization of the pair  $(P, J)$ . There is a natural functor*

$$\mathcal{D}ef_P \longrightarrow \mathcal{D}ef_{P^h}$$

*of deformation categories.*

**Proof.** Given a deformation of  $P$  we can take the henselization of it to get a deformation of the henselization; this is clear and we encourage the reader to skip the proof. More precisely, let  $(A, Q) \rightarrow (k, P)$  be a morphism in  $\mathcal{F}$ , i.e., an object of  $\mathcal{D}ef_P$ . Denote  $J_Q \subset Q$  the inverse image of  $J$  in  $Q$ . Let  $(Q^h, J_Q^h)$  be the henselization of the pair  $(Q, J_Q)$ . Recall that  $Q \rightarrow Q^h$  is flat (More on Algebra, Lemma 10.16) and hence  $Q^h$  is flat over  $A$ . By More on Algebra, Lemma 10.19 we

see that the map  $Q^h \rightarrow P^h$  induces an isomorphism  $Q^h \otimes_A k = Q^h \otimes_Q P = P^h$ . Hence  $(A, Q^h) \rightarrow (k, P^h)$  is the desired object of  $\mathcal{D}ef_{P^h}$ .  $\square$

0DYU **Lemma 8.8.** *In Example 8.1 let  $P$  be a  $k$ -algebra. Assume  $P$  is a local ring and let  $P^{sh}$  be a strict henselization of  $P$ . There is a natural functor*

$$\mathcal{D}ef_P \longrightarrow \mathcal{D}ef_{P^{sh}}$$

*of deformation categories.*

**Proof.** Given a deformation of  $P$  we can take the strict henselization of it to get a deformation of the strict henselization; this is clear and we encourage the reader to skip the proof. More precisely, let  $(A, Q) \rightarrow (k, P)$  be a morphism in  $\mathcal{F}$ , i.e., an object of  $\mathcal{D}ef_P$ . Since the kernel of the surjection  $Q \rightarrow P$  is nilpotent, we find that  $Q$  is a local ring with the same residue field as  $P$ . Let  $Q^{sh}$  be the strict henselization of  $Q$ . Recall that  $Q \rightarrow Q^{sh}$  is flat (More on Algebra, Lemma 42.1) and hence  $Q^{sh}$  is flat over  $A$ . By Algebra, Lemma 150.16 we see that the map  $Q^{sh} \rightarrow P^{sh}$  induces an isomorphism  $Q^{sh} \otimes_A k = Q^{sh} \otimes_Q P = P^{sh}$ . Hence  $(A, Q^{sh}) \rightarrow (k, P^{sh})$  is the desired object of  $\mathcal{D}ef_{P^{sh}}$ .  $\square$

0DYV **Lemma 8.9.** *In Example 8.1 let  $P$  be a  $k$ -algebra. Assume  $P$  Noetherian and let  $J \subset P$  be an ideal. Denote  $P^\wedge$  the  $J$ -adic completion. There is a natural functor*

$$\mathcal{D}ef_P \longrightarrow \mathcal{D}ef_{P^\wedge}$$

*of deformation categories.*

**Proof.** Given a deformation of  $P$  we can take the completion of it to get a deformation of the completion; this is clear and we encourage the reader to skip the proof. More precisely, let  $(A, Q) \rightarrow (k, P)$  be a morphism in  $\mathcal{F}$ , i.e., an object of  $\mathcal{D}ef_P$ . Observe that  $Q$  is a Noetherian ring: the kernel of the surjective ring map  $Q \rightarrow P$  is nilpotent and finitely generated and  $P$  is Noetherian; apply Algebra, Lemma 96.5. Denote  $J_Q \subset Q$  the inverse image of  $J$  in  $Q$ . Let  $Q^\wedge$  be the  $J_Q$ -adic completion of  $Q$ . Recall that  $Q \rightarrow Q^\wedge$  is flat (Algebra, Lemma 96.2) and hence  $Q^\wedge$  is flat over  $A$ . The induced map  $Q^\wedge \rightarrow P^\wedge$  induces an isomorphism  $Q^\wedge \otimes_A k = Q^\wedge \otimes_Q P = P^\wedge$  by Algebra, Lemma 96.1 for example. Hence  $(A, Q^\wedge) \rightarrow (k, P^\wedge)$  is the desired object of  $\mathcal{D}ef_{P^\wedge}$ .  $\square$

0DY5 **Lemma 8.10.** *In Lemma 8.3 if  $P = k[[x_1, \dots, x_n]]/(f)$  for some nonzero  $f \in (x_1, \dots, x_n)^2$ , then*

- (1)  $\text{Inf}(\mathcal{D}ef_P)$  is finite dimensional if and only if  $n = 1$ , and
- (2)  $T\mathcal{D}ef_P$  is finite dimensional if

$$\sqrt{(f, \partial f / \partial x_1, \dots, \partial f / \partial x_n)} = (x_1, \dots, x_n)$$

**Proof.** Proof of (1). Consider the derivations  $\partial / \partial x_i$  of  $k[[x_1, \dots, x_n]]$  over  $k$ . Write  $f_i = \partial f / \partial x_i$ . The derivation

$$\theta = \sum h_i \partial / \partial x_i$$

of  $k[[x_1, \dots, x_n]]$  induces a derivation of  $P = k[[x_1, \dots, x_n]]/(f)$  if and only if  $\sum h_i f_i \in (f)$ . Moreover, the induced derivation of  $P$  is zero if and only if  $h_i \in (f)$  for  $i = 1, \dots, n$ . Thus we find

$$\text{Ker}((f_1, \dots, f_n) : P^{\oplus n} \longrightarrow P) \subset \text{Der}_k(P, P)$$

The left hand side is a finite dimensional  $k$ -vector space only if  $n = 1$ ; we omit the proof. We also leave it to the reader to see that the right hand side has finite dimension if  $n = 1$ . This proves (1).

Proof of (2). Let  $Q$  be a flat deformation of  $P$  over  $k[\epsilon]$  as in the proof of Lemma 8.3. Choose lifts  $q_i \in Q$  of the image of  $x_i$  in  $P$ . Then  $Q$  is a complete local ring with maximal ideal generated by  $q_1, \dots, q_n$  and  $\epsilon$  (small argument omitted). Thus we get a surjection

$$k[\epsilon][[x_1, \dots, x_n]] \longrightarrow Q, \quad x_i \longmapsto q_i$$

Choose an element of the form  $f + \epsilon g \in k[\epsilon][[x_1, \dots, x_n]]$  mapping to zero in  $Q$ . Observe that  $g$  is well defined modulo  $(f)$ . Since  $Q$  is flat over  $k[\epsilon]$  we get

$$Q = k[\epsilon][[x_1, \dots, x_n]]/(f + \epsilon g)$$

Finally, if we changing the choice of  $q_i$  amounts to changing the coordinates  $x_i$  into  $x_i + \epsilon h_i$  for some  $h_i \in k[[x_1, \dots, x_n]]$ . Then  $f + \epsilon g$  changes into  $f + \epsilon(g + \sum h_i f_i)$  where  $f_i = \partial f / \partial x_i$ . Thus we see that the isomorphism class of the deformation  $Q$  is determined by an element of

$$k[[x_1, \dots, x_n]]/(f, \partial f / \partial x_1, \dots, \partial f / \partial x_n)$$

This has finite dimension over  $k$  if and only if its support is the closed point of  $k[[x_1, \dots, x_n]]$  if and only if  $\sqrt{(f, \partial f / \partial x_1, \dots, \partial f / \partial x_n)} = (x_1, \dots, x_n)$ .  $\square$

## 9. Schemes

0DY6 The deformation theory of schemes.

0DY7 **Example 9.1** (Schemes). Let  $\mathcal{F}$  be the category defined as follows

- (1) an object is a pair  $(A, X)$  consisting of an object  $A$  of  $\mathcal{C}_\Lambda$  and a scheme  $X$  flat over  $A$ , and
- (2) a morphism  $(f, g) : (B, Y) \rightarrow (A, X)$  consists of a morphism  $f : B \rightarrow A$  in  $\mathcal{C}_\Lambda$  together with a morphism  $g : X \rightarrow Y$  such that

$$\begin{array}{ccc} X & \xrightarrow{\quad g \quad} & Y \\ \downarrow & & \downarrow \\ \text{Spec}(A) & \xrightarrow{\quad f \quad} & \text{Spec}(B) \end{array}$$

is a cartesian commutative diagram of schemes.

The functor  $p : \mathcal{F} \rightarrow \mathcal{C}_\Lambda$  sends  $(A, X)$  to  $A$  and  $(f, g)$  to  $f$ . It is clear that  $p$  is cofibred in groupoids. Given a scheme  $X$  over  $k$ , let  $x_0 = (k, X)$  be the corresponding object of  $\mathcal{F}(k)$ . We set

$$\text{Def}_X = \mathcal{F}_{x_0}$$

0DY8 **Lemma 9.2.** *Example 9.1 satisfies the Rim-Schlessinger condition (RS). In particular,  $\text{Def}_X$  is a deformation category for any scheme  $X$  over  $k$ .*

**Proof.** Let  $A_1 \rightarrow A$  and  $A_2 \rightarrow A$  be morphisms of  $\mathcal{C}_\Lambda$ . Assume  $A_2 \rightarrow A$  is surjective. According to Formal Deformation Theory, Lemma 16.4 it suffices to

show that the functor  $\mathcal{F}(A_1 \times_A A_2) \rightarrow \mathcal{F}(A_1) \times_{\mathcal{F}(A)} \mathcal{F}(A_2)$  is an equivalence of categories. Observe that

$$\begin{array}{ccc} \mathrm{Spec}(A) & \longrightarrow & \mathrm{Spec}(A_2) \\ \downarrow & & \downarrow \\ \mathrm{Spec}(A_1) & \longrightarrow & \mathrm{Spec}(A_1 \times_A A_2) \end{array}$$

is a pushout diagram as in More on Morphisms, Lemma 14.7. Thus the lemma is a special case of More on Morphisms, Lemma 14.10.  $\square$

0DY9 **Lemma 9.3.** *In Example 9.1 let  $X$  be a scheme over  $k$ . Then*

$$\mathrm{Inf}(\mathrm{Def}_X) = \mathrm{Ext}_{\mathcal{O}_X}^0(NL_{X/k}, \mathcal{O}_X) = \mathrm{Hom}_{\mathcal{O}_X}(\Omega_{X/k}, \mathcal{O}_X) = \mathrm{Der}_k(\mathcal{O}_X, \mathcal{O}_X)$$

and

$$T\mathrm{Def}_X = \mathrm{Ext}_{\mathcal{O}_X}^1(NL_{X/k}, \mathcal{O}_X)$$

**Proof.** Recall that  $\mathrm{Inf}(\mathrm{Def}_X)$  is the set of automorphisms of the trivial deformation  $X' = X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k[\epsilon])$  of  $X$  to  $k[\epsilon]$  equal to the identity modulo  $\epsilon$ . By Deformation Theory, Lemma 8.1 this is equal to  $\mathrm{Ext}_{\mathcal{O}_X}^0(NL_{X/k}, \mathcal{O}_X)$ . The equality  $\mathrm{Ext}_{\mathcal{O}_X}^0(NL_{X/k}, \mathcal{O}_X) = \mathrm{Hom}_{\mathcal{O}_X}(\Omega_{X/k}, \mathcal{O}_X)$  follows from More on Morphisms, Lemma 13.3. The equality  $\mathrm{Hom}_{\mathcal{O}_X}(\Omega_{X/k}, \mathcal{O}_X) = \mathrm{Der}_k(\mathcal{O}_X, \mathcal{O}_X)$  follows from Morphisms, Lemma 31.2.

Recall that  $T_{x_0}\mathrm{Def}_X$  is the set of isomorphism classes of flat deformations  $X'$  of  $X$  to  $k[\epsilon]$ , more precisely, the set of isomorphism classes of  $\mathrm{Def}_X(k[\epsilon])$ . Thus the second statement of the lemma follows from Deformation Theory, Lemma 8.1.  $\square$

0DYA **Lemma 9.4.** *In Lemma 9.3 if  $X$  is proper over  $k$ , then  $\mathrm{Inf}(\mathrm{Def}_X)$  and  $T\mathrm{Def}_X$  are finite dimensional.*

**Proof.** By the lemma we have to show  $\mathrm{Ext}_{\mathcal{O}_X}^1(NL_{X/k}, \mathcal{O}_X)$  and  $\mathrm{Ext}_{\mathcal{O}_X}^0(NL_{X/k}, \mathcal{O}_X)$  are finite dimensional. By More on Morphisms, Lemma 13.4 and the fact that  $X$  is Noetherian, we see that  $NL_{X/k}$  has coherent cohomology sheaves zero except in degrees 0 and  $-1$ . By Derived Categories of Schemes, Lemma 10.4 the displayed Ext-groups are finite  $k$ -vector spaces and the proof is complete.  $\square$

0DYW **Lemma 9.5.** *In Example 9.1 let  $X$  be a scheme over  $k$ . Let  $U \subset X$  be an open subscheme. There is a natural functor*

$$\mathrm{Def}_X \longrightarrow \mathrm{Def}_U$$

*of deformation categories.*

**Proof.** Given a deformation of  $X$  we can take the corresponding open of it to get a deformation of  $U$ . We omit the details.  $\square$

0DYX **Lemma 9.6.** *In Example 9.1 let  $X = \mathrm{Spec}(P)$  be an affine scheme over  $k$ . With  $\mathrm{Def}_P$  as in Example 8.1 there is a natural equivalence*

$$\mathrm{Def}_X \longrightarrow \mathrm{Def}_P$$

*of deformation categories.*

**Proof.** The functor sends  $(A, Y)$  to  $\Gamma(Y, \mathcal{O}_Y)$ . This works because any deformation of  $X$  is affine by More on Morphisms, Lemma 2.3.  $\square$

0DZM **Lemma 9.7.** *In Example 9.1 let  $X$  be a scheme over  $k$ . Let  $p \in X$  be a point. With  $\mathcal{D}ef_{\mathcal{O}_{X,p}}$  as in Example 8.1 there is a natural functor*

$$\mathcal{D}ef_X \longrightarrow \mathcal{D}ef_{\mathcal{O}_{X,p}}$$

*of deformation categories.*

**Proof.** Choose an affine open  $U = \text{Spec}(P) \subset X$  containing  $p$ . Then  $\mathcal{O}_{X,p}$  is a localization of  $P$ . We combine the functors from Lemmas 9.5, 9.6, and 8.6.  $\square$

0DYY **Situation 9.8.** Let  $\Lambda \rightarrow k$  be as in Section 3. Let  $X$  be a scheme over  $k$  which has an affine open covering  $X = U_1 \cup U_2$  with  $U_{12} = U_1 \cap U_2$  affine too. Write  $U_1 = \text{Spec}(P_1)$ ,  $U_2 = \text{Spec}(P_2)$  and  $U_{12} = \text{Spec}(P_{12})$ . Let  $\mathcal{D}ef_X$ ,  $\mathcal{D}ef_{U_1}$ ,  $\mathcal{D}ef_{U_2}$ , and  $\mathcal{D}ef_{U_{12}}$  be as in Example 9.1 and let  $\mathcal{D}ef_{P_1}$ ,  $\mathcal{D}ef_{P_2}$ , and  $\mathcal{D}ef_{P_{12}}$  be as in Example 8.1.

0DYZ **Lemma 9.9.** *In Situation 9.8 there is an equivalence*

$$\mathcal{D}ef_X = \mathcal{D}ef_{P_1} \times_{\mathcal{D}ef_{P_{12}}} \mathcal{D}ef_{P_2}$$

*of deformation categories, see Examples 9.1 and 8.1.*

**Proof.** It suffices to show that the functors of Lemma 9.5 define an equivalence

$$\mathcal{D}ef_X \longrightarrow \mathcal{D}ef_{U_1} \times_{\mathcal{D}ef_{U_{12}}} \mathcal{D}ef_{U_2}$$

because then we can apply Lemma 9.6 to translate into rings. To do this we construct a quasi-inverse. Denote  $F_i : \mathcal{D}ef_{U_i} \rightarrow \mathcal{D}ef_{U_{12}}$  the functor of Lemma 9.5. An object of the RHS is given by an  $A$  in  $\mathcal{C}_\Lambda$ , objects  $(A, V_1) \rightarrow (k, U_1)$  and  $(A, V_2) \rightarrow (k, U_2)$ , and a morphism

$$g : F_1(A, V_1) \rightarrow F_2(A, V_2)$$

Now  $F_i(A, V_i) = (A, V_{i,3-i})$  where  $V_{i,3-i} \subset V_i$  is the open subscheme whose base change to  $k$  is  $U_{12} \subset U_i$ . The morphism  $g$  defines an isomorphism  $V_{1,2} \rightarrow V_{2,1}$  of schemes over  $A$  compatible with  $\text{id} : U_{12} \rightarrow U_{12}$  over  $k$ . Thus  $(\{1, 2\}, V_i, V_{i,3-i}, g, g^{-1})$  is a glueing data as in Schemes, Section 14. Let  $Y$  be the glueing, see Schemes, Lemma 14.1. Then  $Y$  is a scheme over  $A$  and the compatibilities mentioned above show that there is a canonical isomorphism  $Y \times_{\text{Spec}(A)} \text{Spec}(k) = X$ . Thus  $(A, Y) \rightarrow (k, X)$  is an object of  $\mathcal{D}ef_X$ . We omit the verification that this construction is a functor and is quasi-inverse to the given one.  $\square$

## 10. Morphisms of Schemes

0E3S The deformation theory of morphisms of schemes. Of course this is just an example of deformations of diagrams of schemes.

0E3T **Example 10.1** (Morphisms of schemes). Let  $\mathcal{F}$  be the category defined as follows

- (1) an object is a pair  $(A, X \rightarrow Y)$  consisting of an object  $A$  of  $\mathcal{C}_\Lambda$  and a morphism  $X \rightarrow Y$  of schemes over  $A$  with both  $X$  and  $Y$  flat over  $A$ , and
- (2) a morphism  $(f, g, h) : (A', X' \rightarrow Y') \rightarrow (A, X \rightarrow Y)$  consists of a morphism  $f : A' \rightarrow A$  in  $\mathcal{C}_\Lambda$  together with morphisms of schemes  $g : X \rightarrow X'$  and

$h : Y \rightarrow Y'$  such that

$$\begin{array}{ccc} X & \xrightarrow{g} & X' \\ \downarrow & & \downarrow \\ Y & \xrightarrow{h} & Y' \\ \downarrow & & \downarrow \\ \text{Spec}(A) & \xrightarrow{f} & \text{Spec}(A') \end{array}$$

is a commutative diagram of schemes where both squares are cartesian.

The functor  $p : \mathcal{F} \rightarrow \mathcal{C}_\Lambda$  sends  $(A, X \rightarrow Y)$  to  $A$  and  $(f, g, h)$  to  $f$ . It is clear that  $p$  is cofibred in groupoids. Given a morphism of schemes  $X \rightarrow Y$  over  $k$ , let  $x_0 = (k, X \rightarrow Y)$  be the corresponding object of  $\mathcal{F}(k)$ . We set

$$\mathcal{D}ef_{X \rightarrow Y} = \mathcal{F}_{x_0}$$

0E3U **Lemma 10.2.** *Example 10.1 satisfies the Rim-Schlessinger condition (RS). In particular,  $\mathcal{D}ef_{X \rightarrow Y}$  is a deformation category for any morphism of schemes  $X \rightarrow Y$  over  $k$ .*

**Proof.** Let  $A_1 \rightarrow A$  and  $A_2 \rightarrow A$  be morphisms of  $\mathcal{C}_\Lambda$ . Assume  $A_2 \rightarrow A$  is surjective. According to Formal Deformation Theory, Lemma 16.4 it suffices to show that the functor  $\mathcal{F}(A_1 \times_A A_2) \rightarrow \mathcal{F}(A_1) \times_{\mathcal{F}(A)} \mathcal{F}(A_2)$  is an equivalence of categories. Observe that

$$\begin{array}{ccc} \text{Spec}(A) & \longrightarrow & \text{Spec}(A_2) \\ \downarrow & & \downarrow \\ \text{Spec}(A_1) & \longrightarrow & \text{Spec}(A_1 \times_A A_2) \end{array}$$

is a pushout diagram as in More on Morphisms, Lemma 14.7. Thus the lemma follows immediately from More on Morphisms, Lemma 14.10 as this describes the category of schemes flat over  $A_1 \times_A A_2$  as the fibre product of the category of schemes flat over  $A_1$  with the category of schemes flat over  $A_2$  over the category of schemes flat over  $A$ .  $\square$

0E3V **Lemma 10.3.** *In Example 9.1 let  $X \rightarrow Y$  be a morphism of schemes over  $k$ . There is a canonical exact sequence of  $k$ -vector spaces*

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{Inf}(\mathcal{D}ef_{X \rightarrow Y}) & \longrightarrow & \text{Inf}(\mathcal{D}ef_X \times \mathcal{D}ef_Y) & \longrightarrow & \text{Der}_k(\mathcal{O}_Y, f_* \mathcal{O}_X) \\ & & & & & & \swarrow \\ & & T\mathcal{D}ef_{X \rightarrow Y} & \longleftarrow & T(\mathcal{D}ef_X \times \mathcal{D}ef_Y) & \longrightarrow & \text{Ext}_{\mathcal{O}_X}^1(Lf^* NL_{Y/k}, \mathcal{O}_X) \end{array}$$

**Proof.** The obvious map of deformation categories  $\mathcal{D}ef_{X \rightarrow Y} \rightarrow \mathcal{D}ef_X \times \mathcal{D}ef_Y$  gives two of the arrows in the exact sequence of the lemma. Denote  $f : X \rightarrow Y$  the given morphism. Recall that  $\text{Inf}(\mathcal{D}ef_{X \rightarrow Y})$  is the set of automorphisms of the trivial deformation

$$f' : X' = X \times_{\text{Spec}(k)} \text{Spec}(k[\epsilon]) \xrightarrow{f \times \text{id}} Y' = Y \times_{\text{Spec}(k)} \text{Spec}(k[\epsilon])$$

of  $X \rightarrow Y$  to  $k[\epsilon]$  equal to the identity modulo  $\epsilon$ . This is clearly the same thing as pairs  $(\alpha, \beta) \in \text{Inf}(\mathcal{D}ef_X \times \mathcal{D}ef_Y)$  of infinitesimal automorphisms of  $X$  and  $Y$

compatible with  $f'$ , i.e., such that  $f' \circ \alpha = \beta \circ f'$ . By Deformation Theory, Lemma 7.1 for an arbitrary pair  $(\alpha, \beta)$  the difference between the morphism  $f' : X' \rightarrow Y'$  and the morphism  $\beta^{-1} \circ f' \circ \alpha : X' \rightarrow Y'$  defines an element in

$$\mathrm{Der}_k(\mathcal{O}_Y, f_*\mathcal{O}_X) = \mathrm{Hom}_{\mathcal{O}_Y}(\Omega_{Y/k}, f_*\mathcal{O}_X)$$

Equality by More on Morphisms, Lemma 13.3. This defines the last top horizontal arrow and shows exactness in the first two places. For the map

$$\mathrm{Der}_k(\mathcal{O}_Y, f_*\mathcal{O}_X) \rightarrow T\mathrm{Def}_{X \rightarrow Y}$$

we interpret elements of the source as morphisms  $f_\epsilon : X' \rightarrow Y'$  over  $\mathrm{Spec}(k[\epsilon])$  equal to  $f$  modulo  $\epsilon$  using Deformation Theory, Lemma 7.1. We send  $f_\epsilon$  to the isomorphism class of  $(f_\epsilon : X' \rightarrow Y')$  in  $T\mathrm{Def}_{X \rightarrow Y}$ . Note that  $(f_\epsilon : X' \rightarrow Y')$  is isomorphic to the trivial deformation  $(f' : X' \rightarrow Y')$  exactly when  $f_\epsilon = \beta^{-1} \circ f \circ \alpha$  for some pair  $(\alpha, \beta)$  which implies exactness in the third spot. Clearly, if some first order deformation  $(f_\epsilon : X_\epsilon \rightarrow Y_\epsilon)$  maps to zero in  $T(\mathrm{Def}_X \times \mathrm{Def}_Y)$ , then we can choose isomorphisms  $X' \rightarrow X_\epsilon$  and  $Y' \rightarrow Y_\epsilon$  and we conclude we are in the image of the south-west arrow. Therefore we have exactness at the fourth spot. Finally, given two first order deformations  $X_\epsilon, Y_\epsilon$  of  $X, Y$  there is an obstruction in

$$\mathrm{ob}(X_\epsilon, Y_\epsilon) \in \mathrm{Ext}_{\mathcal{O}_X}^1(Lf^*NL_{Y/k}, \mathcal{O}_X)$$

which vanishes if and only if  $f : X \rightarrow Y$  lifts to  $X_\epsilon \rightarrow Y_\epsilon$ , see Deformation Theory, Lemma 7.1. This finishes the proof.  $\square$

0E3W **Lemma 10.4.** *In Lemma 10.3 if  $X$  and  $Y$  are both proper over  $k$ , then  $\mathrm{Inf}(\mathrm{Def}_{X \rightarrow Y})$  and  $T\mathrm{Def}_{X \rightarrow Y}$  are finite dimensional.*

**Proof.** Omitted. Hint: argue as in Lemma 9.4 and use the exact sequence of the lemma.  $\square$

0E3X **Lemma 10.5.** *In Example 9.1 let  $f : X \rightarrow Y$  be a morphism of schemes over  $k$ . If  $f_*\mathcal{O}_X = \mathcal{O}_Y$  and  $R^1f_*\mathcal{O}_X = 0$ , then the morphism of deformation categories*

$$\mathrm{Def}_{X \rightarrow Y} \rightarrow \mathrm{Def}_X$$

*is an equivalence.*

This is discussed in [Vak06, Section 5.3] and [Ran89, Theorem 3.3].

**Proof.** We construct a quasi-inverse to the forgetful functor of the lemma. Namely, suppose that  $(A, U)$  is an object of  $\mathrm{Def}_X$ . The given map  $X \rightarrow U$  is a finite order thickening and we can use it to identify the underlying topological spaces of  $U$  and  $X$ , see More on Morphisms, Section 2. Thus we may and do think of  $\mathcal{O}_U$  as a sheaf of  $A$ -algebras on  $X$ ; moreover the fact that  $U \rightarrow \mathrm{Spec}(A)$  is flat, means that  $\mathcal{O}_U$  is flat as a sheaf of  $A$ -modules. In particular, we have a filtration

$$0 = \mathfrak{m}_A^n \mathcal{O}_U \subset \mathfrak{m}_A^{n-1} \mathcal{O}_U \subset \dots \subset \mathfrak{m}_A^2 \mathcal{O}_U \subset \mathfrak{m}_A \mathcal{O}_U \subset \mathcal{O}_U$$

with subquotients equal to  $\mathcal{O}_X \otimes_k \mathfrak{m}_A^i / \mathfrak{m}_A^{i+1}$  by flatness, see More on Morphisms, Lemma 10.1 or the more general Deformation Theory, Lemma 5.2. Set

$$\mathcal{O}_V = f_*\mathcal{O}_U$$

viewed as sheaf of  $A$ -algebras on  $Y$ . Since  $R^1f_*\mathcal{O}_X = 0$  we find by the description above that  $R^1f_*(\mathfrak{m}_A^i \mathcal{O}_U / \mathfrak{m}_A^{i+1} \mathcal{O}_U) = 0$  for all  $i$ . This implies that the sequences

$$0 \rightarrow (f_*\mathcal{O}_X) \otimes_k \mathfrak{m}_A^i / \mathfrak{m}_A^{i+1} \rightarrow f_*(\mathcal{O}_U / \mathfrak{m}_A^{i+1} \mathcal{O}_U) \rightarrow f_*(\mathcal{O}_U / \mathfrak{m}_A^i \mathcal{O}_U) \rightarrow 0$$

are exact for all  $i$ . Reading the references given above backwards (and using induction) we find that  $\mathcal{O}_V$  is a flat sheaf of  $A$ -algebras with  $\mathcal{O}_V/\mathfrak{m}_A\mathcal{O}_V = \mathcal{O}_Y$ . Using More on Morphisms, Lemma 2.2 we find that  $(Y, \mathcal{O}_V)$  is a scheme, call it  $V$ . The equality  $\mathcal{O}_V = f_*\mathcal{O}_U$  defines a morphism of ringed spaces  $U \rightarrow V$  which is easily seen to be a morphism of schemes. This finishes the proof by the flatness already established.  $\square$

## 11. Algebraic spaces

0E3Y The deformation theory of algebraic spaces.

0E3Z **Example 11.1** (Algebraic spaces). Let  $\mathcal{F}$  be the category defined as follows

- (1) an object is a pair  $(A, X)$  consisting of an object  $A$  of  $\mathcal{C}_\Lambda$  and an algebraic space  $X$  flat over  $A$ , and
- (2) a morphism  $(f, g) : (B, Y) \rightarrow (A, X)$  consists of a morphism  $f : B \rightarrow A$  in  $\mathcal{C}_\Lambda$  together with a morphism  $g : X \rightarrow Y$  of algebraic spaces over  $\Lambda$  such that

$$\begin{array}{ccc} X & \xrightarrow{g} & Y \\ \downarrow & & \downarrow \\ \text{Spec}(A) & \xrightarrow{f} & \text{Spec}(B) \end{array}$$

is a cartesian commutative diagram of algebraic spaces.

The functor  $p : \mathcal{F} \rightarrow \mathcal{C}_\Lambda$  sends  $(A, X)$  to  $A$  and  $(f, g)$  to  $f$ . It is clear that  $p$  is cofibred in groupoids. Given an algebraic space  $X$  over  $k$ , let  $x_0 = (k, X)$  be the corresponding object of  $\mathcal{F}(k)$ . We set

$$\mathcal{D}ef_X = \mathcal{F}_{x_0}$$

0E40 **Lemma 11.2.** *Example 11.1 satisfies the Rim-Schlessinger condition (RS). In particular,  $\mathcal{D}ef_X$  is a deformation category for any algebraic space  $X$  over  $k$ .*

**Proof.** Let  $A_1 \rightarrow A$  and  $A_2 \rightarrow A$  be morphisms of  $\mathcal{C}_\Lambda$ . Assume  $A_2 \rightarrow A$  is surjective. According to Formal Deformation Theory, Lemma 16.4 it suffices to show that the functor  $\mathcal{F}(A_1 \times_A A_2) \rightarrow \mathcal{F}(A_1) \times_{\mathcal{F}(A)} \mathcal{F}(A_2)$  is an equivalence of categories. Observe that

$$\begin{array}{ccc} \text{Spec}(A) & \longrightarrow & \text{Spec}(A_2) \\ \downarrow & & \downarrow \\ \text{Spec}(A_1) & \longrightarrow & \text{Spec}(A_1 \times_A A_2) \end{array}$$

is a pushout diagram as in Pushouts of Spaces, Lemma 2.3. Thus the lemma is a special case of Pushouts of Spaces, Lemma 2.8.  $\square$

0E41 **Lemma 11.3.** *In Example 11.1 let  $X$  be an algebraic space over  $k$ . Then*

$$\text{Inf}(\mathcal{D}ef_X) = \text{Ext}_{\mathcal{O}_X}^0(NL_{X/k}, \mathcal{O}_X) = \text{Hom}_{\mathcal{O}_X}(\Omega_{X/k}, \mathcal{O}_X) = \text{Der}_k(\mathcal{O}_X, \mathcal{O}_X)$$

and

$$T\mathcal{D}ef_X = \text{Ext}_{\mathcal{O}_X}^1(NL_{X/k}, \mathcal{O}_X)$$



**Proof.** Recall that  $\text{Inf}(\mathcal{D}ef_X)$  is the set of automorphisms of the trivial deformation  $X' = X \times_{\text{Spec}(k)} \text{Spec}(k[\epsilon])$  of  $X$  to  $k[\epsilon]$  equal to the identity modulo  $\epsilon$ . By Deformation Theory, Lemma 14.2 this is equal to  $\text{Ext}_{\mathcal{O}_X}^0(NL_{X/k}, \mathcal{O}_X)$ . The equality  $\text{Ext}_{\mathcal{O}_X}^0(NL_{X/k}, \mathcal{O}_X) = \text{Hom}_{\mathcal{O}_X}(\Omega_{X/k}, \mathcal{O}_X)$  follows from More on Morphisms of Spaces, Lemma 21.4. The equality  $\text{Hom}_{\mathcal{O}_X}(\Omega_{X/k}, \mathcal{O}_X) = \text{Der}_k(\mathcal{O}_X, \mathcal{O}_X)$  follows from More on Morphisms of Spaces, Definition 7.2 and Modules on Sites, Definition 32.3.

Recall that  $T_{x_0}\mathcal{D}ef_X$  is the set of isomorphism classes of flat deformations  $X'$  of  $X$  to  $k[\epsilon]$ , more precisely, the set of isomorphism classes of  $\mathcal{D}ef_X(k[\epsilon])$ . Thus the second statement of the lemma follows from Deformation Theory, Lemma 14.2.  $\square$

0E42 **Lemma 11.4.** *In Lemma 11.3 if  $X$  is proper over  $k$ , then  $\text{Inf}(\mathcal{D}ef_X)$  and  $T\mathcal{D}ef_X$  are finite dimensional.*

**Proof.** By the lemma we have to show  $\text{Ext}_{\mathcal{O}_X}^1(NL_{X/k}, \mathcal{O}_X)$  and  $\text{Ext}_{\mathcal{O}_X}^0(NL_{X/k}, \mathcal{O}_X)$  are finite dimensional. By More on Morphisms of Spaces, Lemma 21.5 and the fact that  $X$  is Noetherian, we see that  $NL_{X/k}$  has coherent cohomology sheaves zero except in degrees 0 and  $-1$ . By Derived Categories of Spaces, Lemma 8.4 the displayed Ext-groups are finite  $k$ -vector spaces and the proof is complete.  $\square$

## 12. Deformations of completions

0DZ0 In this section we compare the deformation problem posed by an algebra and its completion. We first discuss “liftability”.

0DZ1 **Lemma 12.1.** *Let  $A' \rightarrow A$  be a surjection of rings with nilpotent kernel. Let  $A' \rightarrow P'$  be a flat ring map. Set  $P = P' \otimes_{A'} A$ . Let  $M$  be an  $A$ -flat  $P$ -module. Then the following are equivalent*

- (1) *there is an  $A'$ -flat  $P'$ -module  $M'$  with  $M' \otimes_{P'} P = M$ , and*
- (2) *there is an object  $K' \in D^-(P')$  with  $K' \otimes_{P'}^{\mathbf{L}} P = M$ .*

**Proof.** Suppose that  $M'$  is as in (1). Then

$$M = M' \otimes_{P'} P = M' \otimes_{A'} A = M' \otimes_A^{\mathbf{L}} A = M' \otimes_{P'}^{\mathbf{L}} P$$

The first two equalities are clear, the third holds because  $M'$  is flat over  $A'$ , and the fourth holds by More on Algebra, Lemma 57.2. Thus (2) holds. Conversely, suppose  $K'$  is as in (2). We may and do assume  $M$  is nonzero. Let  $t$  be the largest integer such that  $H^t(K')$  is nonzero (exists because  $M$  is nonzero). Then  $H^t(K') \otimes_{P'} P = H^t(K' \otimes_{P'}^{\mathbf{L}} P)$  is zero if  $t > 0$ . Since the kernel of  $P' \rightarrow P$  is nilpotent this implies  $H^t(K') = 0$  by Nakayama’s lemma a contradiction. Hence  $t = 0$  (the case  $t < 0$  is absurd as well). Then  $M' = H^0(K')$  is a  $P'$ -module such that  $M = M' \otimes_{P'} P$  and the spectral sequence for Tor gives an injective map

$$\text{Tor}_1^{P'}(M', P) \rightarrow H^{-1}(M' \otimes_{P'}^{\mathbf{L}} P) = 0$$

By the reference on derived base change above  $0 = \text{Tor}_1^{P'}(M', P) = \text{Tor}_1^{A'}(M', A)$ . We conclude that  $M'$  is  $A'$ -flat by Algebra, Lemma 98.8.  $\square$

0DZ2 **Lemma 12.2.** *Consider a commutative diagram of Noetherian rings*

$$\begin{array}{ccccc} A' & \longrightarrow & P' & \longrightarrow & Q' \\ \downarrow & & \downarrow & & \downarrow \\ A & \longrightarrow & P & \longrightarrow & Q \end{array}$$

*with cartesian squares, with flat horizontal arrows, and with surjective vertical arrows whose kernels are nilpotent. Let  $J' \subset P'$  be an ideal such that  $P'/J' = Q'/J'Q'$ . Let  $M$  be an  $A$ -flat  $P$ -module. Assume for all  $g \in J'$  there exists an  $A'$ -flat  $(P')_g$ -module lifting  $M_g$ . Then the following are equivalent*

- (1)  $M$  has an  $A'$ -flat lift to a  $P'$ -module, and
- (2)  $M \otimes_P Q$  has an  $A'$ -flat lift to a  $Q'$ -module.

**Proof.** Let  $I = \text{Ker}(A' \rightarrow A)$ . By induction on the integer  $n > 1$  such that  $I^n = 0$  we reduce to the case where  $I$  is an ideal of square zero; details omitted. We translate the condition of liftability of  $M$  into the problem of finding an object of  $D^-(P')$  as in Lemma 12.1. The obstruction to doing this is the element

$$\omega(M) \in \text{Ext}_P^2(M, M \otimes_P^{\mathbf{L}} IP) = \text{Ext}_P^2(M, M \otimes_P IP)$$

constructed in Deformation Theory, Lemma 15.1. The equality in the displayed formula holds as  $M \otimes_P^{\mathbf{L}} IP = M \otimes_P IP$  since  $M$  and  $P$  are  $A$ -flat<sup>2</sup>. The obstruction for lifting  $M \otimes_P Q$  is similarly the element

$$\omega(M \otimes_P Q) \in \text{Ext}_Q^2(M \otimes_P Q, (M \otimes_P Q) \otimes_Q IQ)$$

which is the image of  $\omega(M)$  by the functoriality of the construction  $\omega(-)$  of Deformation Theory, Lemma 15.1. By More on Algebra, Lemma 85.2 we have

$$\text{Ext}_Q^2(M \otimes_P Q, (M \otimes_P Q) \otimes_Q IQ) = \text{Ext}_P^2(M, M \otimes_P IP) \otimes_P Q$$

here we use that  $P$  is Noetherian and  $M$  finite. Our assumption on  $P' \rightarrow Q'$  guarantees that for an  $P$ -module  $E$  the map  $E \rightarrow E \otimes_P Q$  is bijective on  $J'$ -power torsion, see More on Algebra, Lemma 78.3. Thus we conclude that it suffices to show  $\omega(M)$  is  $J'$ -power torsion. In other words, it suffices to show that  $\omega(M)$  dies in

$$\text{Ext}_P^2(M, M \otimes_P IP)_g = \text{Ext}_{P_g}^2(M_g, M_g \otimes_{P_g} IP_g)$$

for all  $g \in J'$ . However, by the compatibility of formation of  $\omega(M)$  with base change again, we conclude that this is true as  $M_g$  is assumed to have a lift (of course you have to use the whole string of equivalences again).  $\square$

0DZ3 **Lemma 12.3.** *Let  $A' \rightarrow A$  be a surjective map of Noetherian rings with nilpotent kernel. Let  $A \rightarrow B$  be a finite type flat ring map. Let  $\mathfrak{b} \subset B$  be an ideal such that  $\text{Spec}(B) \rightarrow \text{Spec}(A)$  is syntomic on the complement of  $V(\mathfrak{b})$ . Then  $B$  has a flat lift to  $A'$  if and only if the  $\mathfrak{b}$ -adic completion  $B^\wedge$  has a flat lift to  $A'$ .*

**Proof.** Choose an  $A$ -algebra surjection  $P = A[x_1, \dots, x_n] \rightarrow B$ . Let  $\mathfrak{p} \subset P$  be the inverse image of  $\mathfrak{b}$ . Set  $P' = A'[x_1, \dots, x_n]$  and denote  $\mathfrak{p}' \subset P'$  the inverse image of  $\mathfrak{p}$ . (Of course  $\mathfrak{p}$  and  $\mathfrak{p}'$  do not designate prime ideals here.) We will denote  $P^\wedge$  and  $(P')^\wedge$  the respective completions.

<sup>2</sup>Choose a resolution  $F_\bullet \rightarrow I$  by free  $A$ -modules. Since  $A \rightarrow P$  is flat,  $P \otimes_A F_\bullet$  is a free resolution of  $IP$ . Hence  $M \otimes_P^{\mathbf{L}} IP$  is represented by  $M \otimes_P P \otimes_A F_\bullet = M \otimes_A F_\bullet$ . This only has cohomology in degree 0 as  $M$  is  $A$ -flat.

Suppose  $A' \rightarrow B'$  is a flat lift of  $A \rightarrow B$ , in other words,  $A' \rightarrow B'$  is flat and there is an  $A'$ -algebra isomorphism  $B = B' \otimes_{A'} A$ . Then we can choose an  $A'$ -algebra map  $P' \rightarrow B'$  lifting the given surjection  $P \rightarrow B$ . By Nakayama's lemma (Algebra, Lemma 19.1) we find that  $B'$  is a quotient of  $P'$ . In particular, we find that we can endow  $B'$  with an  $A'$ -flat  $P'$ -module structure lifting  $B$  as an  $A'$ -flat  $P'$ -module. Conversely, if we can lift  $B$  to a  $P'$ -module  $M'$  flat over  $A'$ , then  $M'$  is a cyclic module  $M' \cong P'/J'$  (using Nakayama again) and setting  $B' = P'/J'$  we find a flat lift of  $B$  as an algebra.

Set  $C = B^\wedge$  and  $\mathfrak{c} = \mathfrak{b}C$ . Suppose that  $A' \rightarrow C'$  is a flat lift of  $A \rightarrow C$ . Then  $C'$  is complete with respect to the inverse image  $\mathfrak{c}'$  of  $\mathfrak{c}$  (Algebra, Lemma 96.10). We choose an  $A'$ -algebra map  $P' \rightarrow C'$  lifting the  $A$ -algebra map  $P \rightarrow C$ . These maps pass through completions to give surjections  $P^\wedge \rightarrow C$  and  $(P')^\wedge \rightarrow C'$  (for the second again using Nakayama's lemma). In particular, we find that we can endow  $C'$  with an  $A'$ -flat  $(P')^\wedge$ -module structure lifting  $C$  as an  $A'$ -flat  $P^\wedge$ -module. Conversely, if we can lift  $C$  to a  $(P')^\wedge$ -module  $N'$  flat over  $A'$ , then  $N'$  is a cyclic module  $N' \cong (P')^\wedge/\tilde{J}$  (using Nakayama again) and setting  $C' = (P')^\wedge/\tilde{J}$  we find a flat lift of  $C$  as an algebra.

Observe that  $P' \rightarrow (P')^\wedge$  is a flat ring map which induces an isomorphism  $P'/\mathfrak{p}' = (P')^\wedge/\mathfrak{p}'(P')^\wedge$ . We conclude that our lemma is a consequence of Lemma 12.2 provided we can show that  $B_g$  lifts to an  $A'$ -flat  $P'_g$ -module for  $g \in \mathfrak{p}'$ . However, the ring map  $A \rightarrow B_g$  is syntomic and hence lifts to an  $A'$ -flat algebra  $B'$  by Smoothing Ring Maps, Proposition 3.2. Since  $A' \rightarrow P'_g$  is smooth, we can lift  $P_g \rightarrow B_g$  to a surjective map  $P'_g \rightarrow B'$  as before and we get what we want.  $\square$

Notation. Let  $A \rightarrow B$  be a ring map. Let  $N$  be a  $B$ -module. We denote  $\text{Exal}_A(B, N)$  the set of isomorphism classes of extensions

$$0 \rightarrow N \rightarrow C \rightarrow B \rightarrow 0$$

of  $A$ -algebras such that  $N$  is an ideal of square zero in  $C$ . Given a second such  $0 \rightarrow N \rightarrow C' \rightarrow B \rightarrow 0$  an isomorphism is a  $A$ -algebra isomorphism  $C \rightarrow C'$  such that the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & N & \longrightarrow & C & \longrightarrow & B \longrightarrow 0 \\ & & \text{id} \downarrow & & \downarrow & & \text{id} \downarrow \\ 0 & \longrightarrow & N & \longrightarrow & C' & \longrightarrow & B \longrightarrow 0 \end{array}$$

commutes. The assignment  $N \mapsto \text{Exal}_A(B, N)$  is a functor which transforms products into products. Hence this is an additive functor and  $\text{Exal}_A(B, N)$  has a natural  $B$ -module structure. In fact, by Deformation Theory, Lemma 2.3 we have  $\text{Exal}_A(B, N) = \text{Ext}_B^1(NL_{B/A}, N)$ .

0DZ4 **Lemma 12.4.** *Let  $k$  be a field. Let  $B$  be a finite type  $k$ -algebra. Let  $J \subset B$  be an ideal such that  $\text{Spec}(B) \rightarrow \text{Spec}(k)$  is smooth on the complement of  $V(J)$ . Let  $N$  be a finite  $B$ -module. Then there is a canonical bijection*

$$\text{Exal}_k(B, N) \rightarrow \text{Exal}_k(B^\wedge, N^\wedge)$$

Here  $B^\wedge$  and  $N^\wedge$  are the  $J$ -adic completions.

**Proof.** The map is given by completion: given  $0 \rightarrow N \rightarrow C \rightarrow B \rightarrow 0$  in  $\text{Exal}_k(B, N)$  we send it to the completion  $C^\wedge$  of  $C$  with respect to the inverse image of  $J$ . Compare with the proof of Lemma 8.9.

Since  $k \rightarrow B$  is of finite presentation the complex  $NL_{B/k}$  can be represented by a complex  $N^{-1} \rightarrow N^0$  where  $N^i$  is a finite  $B$ -module, see Algebra, Section 132 and in particular Algebra, Lemma 132.2. As  $B$  is Noetherian, this means that  $NL_{B/k}$  is pseudo-coherent. For  $g \in J$  the  $k$ -algebra  $B_g$  is smooth and hence  $(NL_{B/k})_g = NL_{B_g/k}$  is quasi-isomorphic to a finite projective  $B$ -module sitting in degree 0. Thus  $\text{Ext}_B^i(NL_{B/k}, N)_g = 0$  for  $i \geq 1$  and any  $B$ -module  $N$ . By More on Algebra, Lemma 86.5 we conclude that

$$\text{Ext}_B^1(NL_{B/k}, N) \longrightarrow \lim_n \text{Ext}_B^1(NL_{B/k}, N/J^n N)$$

is an isomorphism for any finite  $B$ -module  $N$ .

Injectivity of the map. Suppose that  $0 \rightarrow N \rightarrow C \rightarrow B \rightarrow 0$  is in  $\text{Exal}_k(B, N)$  and maps to zero in  $\text{Exal}_k(B^\wedge, N^\wedge)$ . Choose a splitting  $C^\wedge = B^\wedge \oplus N^\wedge$ . Then the induced map  $C \rightarrow C^\wedge \rightarrow N^\wedge$  gives maps  $C \rightarrow N/J^n N$  for all  $n$ . Hence we see that our element is in the kernel of the maps

$$\text{Ext}_B^1(NL_{B/k}, N) \rightarrow \text{Ext}_B^1(NL_{B/k}, N/J^n N)$$

for all  $n$ . By the previous paragraph we conclude that our element is zero.

Surjectivity of the map. Let  $0 \rightarrow N^\wedge \rightarrow C' \rightarrow B^\wedge \rightarrow 0$  be an element of  $\text{Exal}_k(B^\wedge, N^\wedge)$ . Pulling back by  $B \rightarrow B^\wedge$  we get an element  $0 \rightarrow N^\wedge \rightarrow C'' \rightarrow B \rightarrow 0$  in  $\text{Exal}_k(B, N^\wedge)$ . We have

$$\text{Ext}_B^1(NL_{B/k}, N^\wedge) = \text{Ext}_B^1(NL_{B/k}, N) \otimes_B B^\wedge = \text{Ext}_B^1(NL_{B/k}, N)$$

The first equality as  $N^\wedge = N \otimes_B B^\wedge$  (Algebra, Lemma 96.1) and More on Algebra, Remark 60.19. The second equality because  $\text{Ext}_B^1(NL_{B/k}, N)$  is  $J$ -power torsion (see above),  $B \rightarrow B^\wedge$  is flat and induces an isomorphism  $B/J \rightarrow B^\wedge/JB^\wedge$ , and More on Algebra, Lemma 78.3. Thus we can find a  $C \in \text{Exal}_k(B, N)$  mapping to  $C''$  in  $\text{Exal}_k(B, N^\wedge)$ . Thus

$$0 \rightarrow N^\wedge \rightarrow C' \rightarrow B^\wedge \rightarrow 0 \quad \text{and} \quad 0 \rightarrow N^\wedge \rightarrow C^\wedge \rightarrow B^\wedge \rightarrow 0$$

are two elements of  $\text{Exal}_k(B^\wedge, N^\wedge)$  mapping to the same element of  $\text{Exal}_k(B, N^\wedge)$ . Taking the difference we get an element  $0 \rightarrow N^\wedge \rightarrow C' \rightarrow B^\wedge \rightarrow 0$  of  $\text{Exal}_k(B^\wedge, N^\wedge)$  whose image in  $\text{Exal}_k(B, N^\wedge)$  is zero. This means there exists

$$\begin{array}{ccccccc} 0 & \longrightarrow & N^\wedge & \longrightarrow & C' & \longrightarrow & B^\wedge \longrightarrow 0 \\ & & & & \uparrow & \nearrow & \\ & & & & B & & \end{array}$$

Let  $J' \subset C'$  be the inverse image of  $JB^\wedge \subset B^\wedge$ . To finish the proof it suffices to note that  $\sigma$  is continuous for the  $J$ -adic topology on  $B$  and the  $J'$ -adic topology on  $C'$  and that  $C'$  is  $J'$ -adically complete by Algebra, Lemma 96.10 (here we also use that  $C'$  is Noetherian; small detail omitted). Namely, this means that  $\sigma$  factors through the completion  $B^\wedge$  and  $C' = 0$  in  $\text{Exal}_k(B^\wedge, N^\wedge)$ .  $\square$

0DZ5 **Lemma 12.5.** *In Example 8.1 let  $P$  be a  $k$ -algebra. Let  $J \subset P$  be an ideal. Denote  $P^\wedge$  the  $J$ -adic completion. If*

- (1)  $k \rightarrow P$  is of finite type, and

(2)  $\text{Spec}(P) \rightarrow \text{Spec}(k)$  is smooth on the complement of  $V(J)$ .  
 then the functor between deformation categories of Lemma 8.9

$$\mathcal{D}ef_P \longrightarrow \mathcal{D}ef_{P^\wedge}$$

is smooth and induces an isomorphism on tangent spaces.

**Proof.** We know that  $\mathcal{D}ef_P$  and  $\mathcal{D}ef_{P^\wedge}$  are deformation categories by Lemma 8.2. Thus it suffices to check our functor identifies tangent spaces and a correspondence between liftability, see Formal Deformation Theory, Lemma 20.3. The property on liftability is proven in Lemma 12.3 and the isomorphism on tangent spaces is the special case of Lemma 12.4 where  $N = B$ .  $\square$

### 13. Deformations of localizations

0DZ6 In this section we compare the deformation problem posed by an algebra and its localization at a multiplicative subset. We first discuss “liftability”.

0DZ7 **Lemma 13.1.** *Let  $A' \rightarrow A$  be a surjective map of Noetherian rings with nilpotent kernel. Let  $A \rightarrow B$  be a finite type flat ring map. Let  $S \subset B$  be a multiplicative subset such that if  $\text{Spec}(B) \rightarrow \text{Spec}(A)$  is not syntomic at  $\mathfrak{q}$ , then  $S \cap \mathfrak{q} = \emptyset$ . Then  $B$  has a flat lift to  $A'$  if and only if  $S^{-1}B$  has a flat lift to  $A'$ .*

**Proof.** This proof is the same as the proof of Lemma 12.3 but easier. We suggest the reader to skip the proof. Choose an  $A$ -algebra surjection  $P = A[x_1, \dots, x_n] \rightarrow B$ . Let  $S_P \subset P$  be the inverse image of  $S$ . Set  $P' = A'[x_1, \dots, x_n]$  and denote  $S_{P'} \subset P'$  the inverse image of  $S_P$ .

Suppose  $A' \rightarrow B'$  is a flat lift of  $A \rightarrow B$ , in other words,  $A' \rightarrow B'$  is flat and there is an  $A$ -algebra isomorphism  $B = B' \otimes_{A'} A$ . Then we can choose an  $A'$ -algebra map  $P' \rightarrow B'$  lifting the given surjection  $P \rightarrow B$ . By Nakayama’s lemma (Algebra, Lemma 19.1) we find that  $B'$  is a quotient of  $P'$ . In particular, we find that we can endow  $B'$  with an  $A'$ -flat  $P'$ -module structure lifting  $B$  as an  $A$ -flat  $P$ -module. Conversely, if we can lift  $B$  to a  $P'$ -module  $M'$  flat over  $A'$ , then  $M'$  is a cyclic module  $M' \cong P'/J'$  (using Nakayama again) and setting  $B' = P'/J'$  we find a flat lift of  $B$  as an algebra.

Set  $C = S^{-1}B$ . Suppose that  $A' \rightarrow C'$  is a flat lift of  $A \rightarrow C$ . Elements of  $C'$  which map to invertible elements of  $C$  are invertible. We choose an  $A'$ -algebra map  $P' \rightarrow C'$  lifting the  $A$ -algebra map  $P \rightarrow C$ . By the remark above these maps pass through localizations to give surjections  $S_P^{-1}P \rightarrow C$  and  $S_{P'}^{-1}P' \rightarrow C'$  (for the second use Nakayama’s lemma). In particular, we find that we can endow  $C'$  with an  $A'$ -flat  $S_{P'}^{-1}P'$ -module structure lifting  $C$  as an  $A$ -flat  $S_P^{-1}P$ -module. Conversely, if we can lift  $C$  to a  $S_{P'}^{-1}P'$ -module  $N'$  flat over  $A'$ , then  $N'$  is a cyclic module  $N' \cong S_{P'}^{-1}P'/\tilde{J}$  (using Nakayama again) and setting  $C' = S_{P'}^{-1}P'/\tilde{J}$  we find a flat lift of  $C$  as an algebra.

The syntomic locus of a morphism of schemes is open by definition. Let  $J_B \subset B$  be an ideal cutting out the set of points in  $\text{Spec}(B)$  where  $\text{Spec}(B) \rightarrow \text{Spec}(A)$  is not syntomic. Denote  $J_P \subset P$  and  $J_{P'} \subset P'$  the corresponding ideals. Observe that  $P' \rightarrow S_{P'}^{-1}P'$  is a flat ring map which induces an isomorphism  $P'/J_{P'} = S_{P'}^{-1}P'/J_{P'}S_{P'}^{-1}P'$  by our assumption on  $S$  in the lemma, namely, the assumption in the lemma is exactly that  $B/J_B = S^{-1}(B/J_B)$ . We conclude that our lemma is a consequence of Lemma 12.2 provided we can show that  $B_g$  lifts to an  $A'$ -flat

$P'_g$ -module for  $g \in J_B$ . However, the ring map  $A \rightarrow B_g$  is syntomic and hence lifts to an  $A'$ -flat algebra  $B'$  by Smoothing Ring Maps, Proposition 3.2. Since  $A' \rightarrow P'_g$  is smooth, we can lift  $P_g \rightarrow B_g$  to a surjective map  $P'_g \rightarrow B'$  as before and we get what we want.  $\square$

0DZ8 **Lemma 13.2.** *Let  $k$  be a field. Let  $B$  be a finite type  $k$ -algebra. Let  $S \subset B$  be a multiplicative subset ideal such that if  $\text{Spec}(B) \rightarrow \text{Spec}(k)$  is not smooth at  $\mathfrak{q}$  then  $S \cap \mathfrak{q} = \emptyset$ . Let  $N$  be a finite  $B$ -module. Then there is a canonical bijection*

$$\text{Exal}_k(B, N) \rightarrow \text{Exal}_k(S^{-1}B, S^{-1}N)$$

**Proof.** This proof is the same as the proof of Lemma 12.4 but easier. We suggest the reader to skip the proof. The map is given by localization: given  $0 \rightarrow N \rightarrow C \rightarrow B \rightarrow 0$  in  $\text{Exal}_k(B, N)$  we send it to the localization  $S_C^{-1}C$  of  $C$  with respect to the inverse image  $S_C \subset C$  of  $S$ . Compare with the proof of Lemma 8.6.

The smooth locus of a morphism of schemes is open by definition. Let  $J \subset B$  be an ideal cutting out the set of points in  $\text{Spec}(B)$  where  $\text{Spec}(B) \rightarrow \text{Spec}(A)$  is not smooth. Since  $k \rightarrow B$  is of finite presentation the complex  $NL_{B/k}$  can be represented by a complex  $N^{-1} \rightarrow N^0$  where  $N^i$  is a finite  $B$ -module, see Algebra, Section 132 and in particular Algebra, Lemma 132.2. As  $B$  is Noetherian, this means that  $NL_{B/k}$  is pseudo-coherent. For  $g \in J$  the  $k$ -algebra  $B_g$  is smooth and hence  $(NL_{B/k})_g = NL_{B_g/k}$  is quasi-isomorphic to a finite projective  $B$ -module sitting in degree 0. Thus  $\text{Ext}_B^i(NL_{B/k}, N)_g = 0$  for  $i \geq 1$  and any  $B$ -module  $N$ . Finally, we have

$$\text{Ext}_{S^{-1}B}^1(NL_{S^{-1}B/k}, S^{-1}N) = \text{Ext}_B^1(NL_{B/k}, N) \otimes_B S^{-1}B = \text{Ext}_B^1(NL_{B/k}, N)$$

The first equality by More on Algebra, Lemma 85.2 and Algebra, Lemma 132.13. The second because  $\text{Ext}_B^1(NL_{B/k}, N)$  is  $J$ -power torsion and elements of  $S$  act invertibly on  $J$ -power torsion modules. This concludes the proof by the description of  $\text{Exal}_A(B, N)$  as  $\text{Ext}_B^1(NL_{B/A}, N)$  given just above Lemma 12.4.  $\square$

0DZ9 **Lemma 13.3.** *In Example 8.1 let  $P$  be a  $k$ -algebra. Let  $S \subset P$  be a multiplicative subset. If*

- (1)  $k \rightarrow P$  is of finite type, and
- (2)  $\text{Spec}(P) \rightarrow \text{Spec}(k)$  is smooth at all points of  $V(g)$  for all  $g \in S$ .

*then the functor between deformation categories of Lemma 8.6*

$$\mathcal{D}ef_P \longrightarrow \mathcal{D}ef_{S^{-1}P}$$

*is smooth and induces an isomorphism on tangent spaces.*

**Proof.** We know that  $\mathcal{D}ef_P$  and  $\mathcal{D}ef_{S^{-1}P}$  are deformation categories by Lemma 8.2. Thus it suffices to check our functor identifies tangent spaces and a correspondence between liftability, see Formal Deformation Theory, Lemma 20.3. The property on liftability is proven in Lemma 13.1 and the isomorphism on tangent spaces is the special case of Lemma 13.2 where  $N = B$ .  $\square$

## 14. Deformations of henselizations

0DZA In this section we compare the deformation problem posed by an algebra and its completion. We first discuss “liftability”.

0DZB **Lemma 14.1.** *Let  $A' \rightarrow A$  be a surjective map of Noetherian rings with nilpotent kernel. Let  $A \rightarrow B$  be a finite type flat ring map. Let  $\mathfrak{b} \subset B$  be an ideal such that  $\text{Spec}(B) \rightarrow \text{Spec}(A)$  is syntomic on the complement of  $V(\mathfrak{b})$ . Let  $(B^h, \mathfrak{b}^h)$  be the henselization of the pair  $(B, \mathfrak{b})$ . Then  $B$  has a flat lift to  $A'$  if and only if  $B^h$  has a flat lift to  $A'$ .*

**First proof.** This proof is a cheat. Namely, if  $B$  has a flat lift  $B'$ , then taking the henselization  $(B')^h$  we obtain a flat lift of  $B^h$  (compare with the proof of Lemma 8.7). Conversely, suppose that  $C'$  is an  $A'$ -flat lift of  $(B')^h$ . Then let  $\mathfrak{c}' \subset C'$  be the inverse image of the ideal  $\mathfrak{b}^h$ . Then the completion  $(C')^\wedge$  of  $C'$  with respect to  $\mathfrak{c}'$  is a lift of  $B^\wedge$  (details omitted). Hence we see that  $B$  has a flat lift by Lemma 12.3.  $\square$

**Second proof.** Choose an  $A$ -algebra surjection  $P = A[x_1, \dots, x_n] \rightarrow B$ . Let  $\mathfrak{p} \subset P$  be the inverse image of  $\mathfrak{b}$ . Set  $P' = A'[x_1, \dots, x_n]$  and denote  $\mathfrak{p}' \subset P'$  the inverse image of  $\mathfrak{p}$ . (Of course  $\mathfrak{p}$  and  $\mathfrak{p}'$  do not designate prime ideals here.) We will denote  $P^h$  and  $(P')^h$  the respective henselizations. We will use that taking henselizations is functorial and that the henselization of a quotient is the corresponding quotient of the henselization, see More on Algebra, Lemmas 10.13 and 10.19.

Suppose  $A' \rightarrow B'$  is a flat lift of  $A \rightarrow B$ , in other words,  $A' \rightarrow B'$  is flat and there is an  $A$ -algebra isomorphism  $B = B' \otimes_{A'} A$ . Then we can choose an  $A'$ -algebra map  $P' \rightarrow B'$  lifting the given surjection  $P \rightarrow B$ . By Nakayama's lemma (Algebra, Lemma 19.1) we find that  $B'$  is a quotient of  $P'$ . In particular, we find that we can endow  $B'$  with an  $A'$ -flat  $P'$ -module structure lifting  $B$  as an  $A$ -flat  $P$ -module. Conversely, if we can lift  $B$  to a  $P'$ -module  $M'$  flat over  $A'$ , then  $M'$  is a cyclic module  $M' \cong P'/J'$  (using Nakayama again) and setting  $B' = P'/J'$  we find a flat lift of  $B$  as an algebra.

Set  $C = B^h$  and  $\mathfrak{c} = \mathfrak{b}C$ . Suppose that  $A' \rightarrow C'$  is a flat lift of  $A \rightarrow C$ . Then  $C'$  is henselian with respect to the inverse image  $\mathfrak{c}'$  of  $\mathfrak{c}$  (by More on Algebra, Lemma 10.11 and the fact that the kernel of  $C' \rightarrow C$  is nilpotent). We choose an  $A'$ -algebra map  $P' \rightarrow C'$  lifting the  $A$ -algebra map  $P \rightarrow C$ . These maps pass through henselizations to give surjections  $P^h \rightarrow C$  and  $(P')^h \rightarrow C'$  (for the second again using Nakayama's lemma). In particular, we find that we can endow  $C'$  with an  $A'$ -flat  $(P')^h$ -module structure lifting  $C$  as an  $A$ -flat  $P^h$ -module. Conversely, if we can lift  $C$  to a  $(P')^h$ -module  $N'$  flat over  $A'$ , then  $N'$  is a cyclic module  $N' \cong (P')^h/\tilde{J}$  (using Nakayama again) and setting  $C' = (P')^h/\tilde{J}$  we find a flat lift of  $C$  as an algebra.

Observe that  $P' \rightarrow (P')^h$  is a flat ring map which induces an isomorphism  $P'/\mathfrak{p}' = (P')^h/\mathfrak{p}'(P')^h$  (More on Algebra, Lemma 10.16). We conclude that our lemma is a consequence of Lemma 12.2 provided we can show that  $B_g$  lifts to an  $A'$ -flat  $P'_g$ -module for  $g \in \mathfrak{p}'$ . However, the ring map  $A \rightarrow B_g$  is syntomic and hence lifts to an  $A'$ -flat algebra  $B'$  by Smoothing Ring Maps, Proposition 3.2. Since  $A' \rightarrow P'_g$  is smooth, we can lift  $P_g \rightarrow B_g$  to a surjective map  $P'_g \rightarrow B'$  as before and we get what we want.  $\square$

0DZC **Lemma 14.2.** *Let  $k$  be a field. Let  $B$  be a finite type  $k$ -algebra. Let  $J \subset B$  be an ideal such that  $\text{Spec}(B) \rightarrow \text{Spec}(k)$  is smooth on the complement of  $V(J)$ . Let  $N$*

be a finite  $B$ -module. Then there is a canonical bijection

$$\text{Exal}_k(B, N) \rightarrow \text{Exal}_k(B^h, N^h)$$

Here  $(B^h, J^h)$  is the henselization of  $(B, J)$  and  $N^h = N \otimes_B B^h$ .

**Proof.** This proof is the same as the proof of Lemma 12.4 but easier. We suggest the reader to skip the proof. The map is given by henselization: given  $0 \rightarrow N \rightarrow C \rightarrow B \rightarrow 0$  in  $\text{Exal}_k(B, N)$  we send it to the henselization  $C^h$  of  $C$  with respect to the inverse image  $J_C \subset C$  of  $J$ . Compare with the proof of Lemma 8.7.

Since  $k \rightarrow B$  is of finite presentation the complex  $NL_{B/k}$  can be represented by a complex  $N^{-1} \rightarrow N^0$  where  $N^i$  is a finite  $B$ -module, see Algebra, Section 132 and in particular Algebra, Lemma 132.2. As  $B$  is Noetherian, this means that  $NL_{B/k}$  is pseudo-coherent. For  $g \in J$  the  $k$ -algebra  $B_g$  is smooth and hence  $(NL_{B/k})_g = NL_{B_g/k}$  is quasi-isomorphic to a finite projective  $B$ -module sitting in degree 0. Thus  $\text{Ext}_B^i(NL_{B/k}, N)_g = 0$  for  $i \geq 1$  and any  $B$ -module  $N$ . Finally, we have

$$\begin{aligned} \text{Ext}_{B^h}^1(NL_{B^h/k}, N^h) &= \text{Ext}_{B^h}^1(NL_{B/k} \otimes_B B^h, N \otimes_B B^h) \\ &= \text{Ext}_B^1(NL_{B/k}, N) \otimes_B B^h \\ &= \text{Ext}_B^1(NL_{B/k}, N) \end{aligned}$$

The first equality by More on Algebra, Lemma 30.8 (or rather its analogue for henselizations of pairs). The second by More on Algebra, Lemma 85.2. The third because  $\text{Ext}_B^1(NL_{B/k}, N)$  is  $J$ -power torsion, the map  $B \rightarrow B^h$  is flat and induces an isomorphism  $B/J \rightarrow B^h/JB^h$  (More on Algebra, Lemma 10.16), and More on Algebra, Lemma 78.3. This concludes the proof by the description of  $\text{Exal}_A(B, N)$  as  $\text{Ext}_B^1(NL_{B/A}, N)$  given just above Lemma 12.4.  $\square$

0DZD **Lemma 14.3.** *In Example 8.1 let  $P$  be a  $k$ -algebra. Let  $J \subset P$  be an ideal. Denote  $(P^h, J^h)$  the henselization of the pair  $(P, J)$ . If*

- (1)  $k \rightarrow P$  is of finite type, and
- (2)  $\text{Spec}(P) \rightarrow \text{Spec}(k)$  is smooth on the complement of  $V(J)$ ,

then the functor between deformation categories of Lemma 8.7

$$\mathcal{D}ef_P \longrightarrow \mathcal{D}ef_{P^h}$$

is smooth and induces an isomorphism on tangent spaces.

**Proof.** We know that  $\mathcal{D}ef_P$  and  $\mathcal{D}ef_{P^h}$  are deformation categories by Lemma 8.2. Thus it suffices to check our functor identifies tangent spaces and a correspondence between liftability, see Formal Deformation Theory, Lemma 20.3. The property on liftability is proven in Lemma 14.1 and the isomorphism on tangent spaces is the special case of Lemma 14.2 where  $N = B$ .  $\square$

## 15. Application to isolated singularities

0DZE We apply the discussion above to study the deformation theory of a finite type algebra with finitely many singular points.

0DZF **Lemma 15.1.** *In Example 8.1 let  $P$  be a  $k$ -algebra. Assume that  $k \rightarrow P$  is of finite type and that  $\text{Spec}(P) \rightarrow \text{Spec}(k)$  is smooth except at the maximal ideals*



$\mathfrak{m}_1, \dots, \mathfrak{m}_n$  of  $P$ . Let  $P_{\mathfrak{m}_i}, P_{\mathfrak{m}_i}^h, P_{\mathfrak{m}_i}^\wedge$  be the local ring, henselization, completion. Then the maps of deformation categories

$$\mathcal{D}ef_P \rightarrow \prod \mathcal{D}ef_{P_{\mathfrak{m}_i}} \rightarrow \prod \mathcal{D}ef_{P_{\mathfrak{m}_i}^h} \rightarrow \prod \mathcal{D}ef_{P_{\mathfrak{m}_i}^\wedge}$$

are smooth and induce isomorphisms on their finite dimensional tangent spaces.

**Proof.** The tangent space is finite dimensional by Lemma 8.5. The functors between the categories are constructed in Lemmas 8.6, 8.7, and 8.9 (we omit some verifications of the form: the completion of the henselization is the completion).

Set  $J = \mathfrak{m}_1 \cap \dots \cap \mathfrak{m}_n$  and apply Lemma 12.5 to get that  $\mathcal{D}ef_P \rightarrow \mathcal{D}ef_{P^\wedge}$  is smooth and induces an isomorphism on tangent spaces where  $P^\wedge$  is the  $J$ -adic completion of  $P$ . However, since  $P^\wedge = \prod P_{\mathfrak{m}_i}^\wedge$  we see that the map  $\mathcal{D}ef_P \rightarrow \prod \mathcal{D}ef_{P_{\mathfrak{m}_i}^\wedge}$  is smooth and induces an isomorphism on tangent spaces.

Let  $(P^h, J^h)$  be the henselization of the pair  $(P, J)$ . Then  $P^h = \prod P_{\mathfrak{m}_i}^h$  (look at idempotents and use More on Algebra, Lemma 10.8). Hence we can apply Lemma 14.3 to conclude as in the case of completion.

To get the the final case it suffices to show that  $\mathcal{D}ef_{P_{\mathfrak{m}_i}} \rightarrow \mathcal{D}ef_{P_{\mathfrak{m}_i}^\wedge}$  is smooth and induce isomorphisms on tangent spaces for each  $i$  separately. To do this, we may replace  $P$  by a principal localization whose only singular point is a maximal ideal  $\mathfrak{m}$  (corresponding to  $\mathfrak{m}_i$  in the original  $P$ ). Then we can apply Lemma 13.3 with multiplicative subset  $S = P \setminus \mathfrak{m}$  to conclude. Minor details omitted.  $\square$

## 16. Unobstructed deformation problems

0DZG Let  $p : \mathcal{F} \rightarrow \mathcal{C}_\Lambda$  be a category cofibred in groupoids. Recall that we say  $\mathcal{F}$  is *smooth* or *unobstructed* if  $p$  is smooth. This means that given a surjection  $\varphi : A' \rightarrow A$  in  $\mathcal{C}_\Lambda$  and  $x \in \text{Ob}(\mathcal{F}(A))$  there exists a morphism  $f : x' \rightarrow x$  in  $\mathcal{F}$  with  $p(f) = \varphi$ . See Formal Deformation Theory, Section 9. In this section we give some geometrically meaningful examples.

0DZH **Lemma 16.1.** *In Example 8.1 let  $P$  be a local complete intersection over  $k$  (Algebra, Definition 133.1). Then  $\mathcal{D}ef_P$  is unobstructed.*

**Proof.** Let  $(A, Q) \rightarrow (k, P)$  be an object of  $\mathcal{D}ef_P$ . Then we see that  $A \rightarrow Q$  is a syntomic ring map by Algebra, Definition 134.1. Hence for any surjection  $A' \rightarrow A$  in  $\mathcal{C}_\Lambda$  we see that there is a morphism  $(A', Q') \rightarrow (A, Q)$  lifting  $A' \rightarrow A$  by Smoothing Ring Maps, Proposition 3.2. This proves the lemma.  $\square$

0DZN **Lemma 16.2.** *In Situation 9.8 if  $U_{12} \rightarrow \text{Spec}(k)$  is smooth, then the morphism*

$$\mathcal{D}ef_X \longrightarrow \mathcal{D}ef_{U_1} \times \mathcal{D}ef_{U_2} = \mathcal{D}ef_{P_1} \times \mathcal{D}ef_{P_2}$$

*is smooth. If in addition  $U_1$  is a local complete intersection over  $k$ , then*

$$\mathcal{D}ef_X \longrightarrow \mathcal{D}ef_{U_2} = \mathcal{D}ef_{P_2}$$

*is smooth.*

**Proof.** The equality signs hold by Lemma 9.6. Let us think of  $\mathcal{C}_\Lambda$  as a deformation category over  $\mathcal{C}_\Lambda$  as in Formal Deformation Theory, Section 9. Then

$$\mathcal{D}ef_{P_1} \times \mathcal{D}ef_{P_2} = \mathcal{D}ef_{P_1} \times_{\mathcal{C}_\Lambda} \mathcal{D}ef_{P_2},$$

see Formal Deformation Theory, Remarks 5.2 (14). Using Lemma 9.9 the first statement is that the functor

$$\mathcal{D}ef_{P_1} \times_{\mathcal{D}ef_{P_{12}}} \mathcal{D}ef_{P_2} \longrightarrow \mathcal{D}ef_{P_1} \times_{\mathcal{C}_\Lambda} \mathcal{D}ef_{P_2}$$

is smooth. This follows from Formal Deformation Theory, Lemma 20.2 as long as we can show that  $T\mathcal{D}ef_{P_{12}} = (0)$ . This vanishing follows from Lemma 8.4 as  $P_{12}$  is smooth over  $k$ . For the second statement it suffices to show that  $\mathcal{D}ef_{P_1} \rightarrow \mathcal{C}_\Lambda$  is smooth, see Formal Deformation Theory, Lemma 8.7. In other words, we have to show  $\mathcal{D}ef_{P_1}$  is unobstructed, which is Lemma 16.1.  $\square$

0DZP **Lemma 16.3.** *In Example 9.1 let  $X$  be a scheme over  $k$ . Assume*

- (1)  $X$  is separated, finite type over  $k$  and  $\dim(X) \leq 1$ ,
- (2)  $X \rightarrow \text{Spec}(k)$  is smooth except at the closed points  $p_1, \dots, p_n \in X$ .

Let  $\mathcal{O}_{X,p_1}$ ,  $\mathcal{O}_{X,p_1}^h$ ,  $\mathcal{O}_{X,p_1}^\wedge$  be the local ring, henselization, completion. Consider the maps of deformation categories

$$\mathcal{D}ef_X \longrightarrow \prod \mathcal{D}ef_{\mathcal{O}_{X,p_i}} \longrightarrow \prod \mathcal{D}ef_{\mathcal{O}_{X,p_i}^h} \longrightarrow \prod \mathcal{D}ef_{\mathcal{O}_{X,p_i}^\wedge}$$

The first arrow is smooth and the second and third arrows are smooth and induce isomorphisms on tangent spaces.

**Proof.** Choose an affine open  $U_2 \subset X$  containing  $p_1, \dots, p_n$  and the generic point of every irreducible component of  $X$ . This is possible by Varieties, Lemma 41.3 and Properties, Lemma 29.5. Then  $X \setminus U_2$  is finite and we can choose an affine open  $U_1 \subset X \setminus \{p_1, \dots, p_n\}$  such that  $X = U_1 \cup U_2$ . Set  $U_{12} = U_1 \cap U_2$ . Then  $U_1$  and  $U_{12}$  are smooth affine schemes over  $k$ . We conclude that

$$\mathcal{D}ef_X \longrightarrow \mathcal{D}ef_{U_2}$$

is smooth by Lemma 16.2. Applying Lemmas 9.6 and 15.1 we win.  $\square$

0DZQ **Lemma 16.4.** *In Example 9.1 let  $X$  be a scheme over  $k$ . Assume*

- (1)  $X$  is separated, finite type over  $k$  and  $\dim(X) \leq 1$ ,
- (2)  $X$  is a local complete intersection over  $k$ , and
- (3)  $X \rightarrow \text{Spec}(k)$  is smooth except at finitely many points.

Then  $\mathcal{D}ef_X$  is unobstructed.

**Proof.** Let  $p_1, \dots, p_n \in X$  be the points where  $X \rightarrow \text{Spec}(k)$  isn't smooth. Choose an affine open  $U_2 \subset X$  containing  $p_1, \dots, p_n$  and the generic point of every irreducible component of  $X$ . This is possible by Varieties, Lemma 41.3 and Properties, Lemma 29.5. Then  $X \setminus U_2$  is finite and we can choose an affine open  $U_1 \subset X \setminus \{p_1, \dots, p_n\}$  such that  $X = U_1 \cup U_2$ . Set  $U_{12} = U_1 \cap U_2$ . Then  $U_1$  and  $U_{12}$  are smooth affine schemes over  $k$ . We conclude that

$$\mathcal{D}ef_X \longrightarrow \mathcal{D}ef_{U_2}$$

is smooth by Lemma 16.2. Applying Lemmas 9.6 and 16.1 we win.  $\square$

## 17. Other chapters

Preliminaries

- (1) Introduction
- (2) Conventions

- (3) Set Theory
- (4) Categories
- (5) Topology
- (6) Sheaves on Spaces

- (7) Sites and Sheaves
- (8) Stacks
- (9) Fields
- (10) Commutative Algebra
- (11) Brauer Groups
- (12) Homological Algebra
- (13) Derived Categories
- (14) Simplicial Methods
- (15) More on Algebra
- (16) Smoothing Ring Maps
- (17) Sheaves of Modules
- (18) Modules on Sites
- (19) Injectives
- (20) Cohomology of Sheaves
- (21) Cohomology on Sites
- (22) Differential Graded Algebra
- (23) Divided Power Algebra
- (24) Hypercoverings
- Schemes
  - (25) Schemes
  - (26) Constructions of Schemes
  - (27) Properties of Schemes
  - (28) Morphisms of Schemes
  - (29) Cohomology of Schemes
  - (30) Divisors
  - (31) Limits of Schemes
  - (32) Varieties
  - (33) Topologies on Schemes
  - (34) Descent
  - (35) Derived Categories of Schemes
  - (36) More on Morphisms
  - (37) More on Flatness
  - (38) Groupoid Schemes
  - (39) More on Groupoid Schemes
  - (40) Étale Morphisms of Schemes
- Topics in Scheme Theory
  - (41) Chow Homology
  - (42) Intersection Theory
  - (43) Picard Schemes of Curves
  - (44) Adequate Modules
  - (45) Dualizing Complexes
  - (46) Duality for Schemes
  - (47) Discriminants and Differents
  - (48) Local Cohomology
  - (49) Algebraic Curves
  - (50) Resolution of Surfaces
  - (51) Semistable Reduction
  - (52) Fundamental Groups of Schemes
- (53) Étale Cohomology
- (54) Crystalline Cohomology
- (55) Pro-étale Cohomology
- Algebraic Spaces
  - (56) Algebraic Spaces
  - (57) Properties of Algebraic Spaces
  - (58) Morphisms of Algebraic Spaces
  - (59) Decent Algebraic Spaces
  - (60) Cohomology of Algebraic Spaces
  - (61) Limits of Algebraic Spaces
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