MORE ON FLATNESS

057M

Contents

1. Introduction

057N

This is a chapter of the Stacks Project, version f2cc092b, compiled on Jun 21, 2017.

In this chapter, we discuss some advanced results on flat modules and flat morphisms of schemes. Most of these results can be found in the paper [GR71] by Raynaud and Gruson.

Before reading this chapter we advise the reader to take a look at the following results (this list also serves as a pointer to previous results):

(1) General discussion on flat modules in Algebra, Section 38.
(2) The relationship between Tor-groups and flatness, see Algebra, Section 74.
(3) Criteria for flatness, see Algebra, Section 98 (Noetherian case), Algebra, Section 100 (Artinian case), Algebra, Section 127 (non-Noetherian case), and finally More on Morphisms, Section 16.
(4) Generic flatness, see Algebra, Section 117 and Morphisms, Section 26.
(5) Openness of the flat locus, see Algebra, Section 128 and More on Morphisms, Section 15.
(6) Flattening, see More on Algebra, Sections 14, 15, 16, 17, and 18.
(7) Additional results in More on Algebra, Sections 19, 20, 23, and 24.

2. Lemmas on étale localization

05FM In this section we list some lemmas on étale localization which will be useful later in this chapter. Please skip this section on a first reading.

057R **Lemma 2.1.** *Let $i : Z \to X$ be a closed immersion of affine schemes. Let $Z' \to Z$ be an étale morphism with Z' affine. Then there exists an étale morphism $X' \to X$ with X' affine such that $Z' \cong Z \times_X X'$ as schemes over Z.*

Proof. See Algebra, Lemma 141.10. $\square$

05H2 **Lemma 2.2.** *Let*

$$\begin{tikzcd} X & X' \arrow[l] \arrow[d] \\ S \arrow[u, leftarrow] & S' \arrow[l] \end{tikzcd}$$

be a commutative diagram of schemes with $X' \to X$ and $S' \to S$ étale. Let $s' \in S'$ be a point. Then

$$X' \times_{S'} \mathrm{Spec}(\mathcal{O}_{S',s'}) \longrightarrow X \times_S \mathrm{Spec}(\mathcal{O}_{S',s'})$$

is étale.

Proof. This is true because $X' \to X_{S'}$ is étale as a morphism of schemes étale over X, see Morphisms, Lemma 34.18 and the base change of an étale morphism is étale, see Morphisms, Lemma 34.4. $\square$

05B9 **Lemma 2.3.** *Let $X \to T \to S$ be morphisms of schemes with $T \to S$ étale. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $x \in X$ be a point. Then*

$$\mathcal{F} \text{ flat over } S \text{ at } x \Leftrightarrow \mathcal{F} \text{ flat over } T \text{ at } x$$

In particular $\mathcal{F}$ is flat over S if and only if $\mathcal{F}$ is flat over T.

Proof. As an étale morphism is a flat morphism (see Morphisms, Lemma 34.12) the implication "$\Leftarrow$" follows from Algebra, Lemma 38.4. For the converse assume that $\mathcal{F}$ is flat at x over S. Denote $\tilde{x} \in X \times_S T$ the point lying over x in X and over the image of x in T in T. Then $(X \times_S T \to X)^*\mathcal{F}$ is flat at $\tilde{x}$ over T via

$\mathrm{pr}_2 : X \times_S T \to T$, see Morphisms, Lemma 24.6. The diagonal $\Delta_{T/S} : T \to T \times_S T$ is an open immersion; combine Morphisms, Lemmas 33.13 and 34.5. So X is identified with open subscheme of $X \times_S T$, the restriction of pr_2 to this open is the given morphism $X \to T$, the point $\tilde{x}$ corresponds to the point x in this open, and $(X \times_S T \to X)^*\mathcal{F}$ restricted to this open is $\mathcal{F}$. Whence we see that $\mathcal{F}$ is flat at x over T. □

05BA **Lemma 2.4.** *Let $T \to S$ be an étale morphism. Let $t \in T$ with image $s \in S$. Let M be a $\mathcal{O}_{T,t}$-module. Then*

$$M \text{ flat over } \mathcal{O}_{S,s} \Leftrightarrow M \text{ flat over } \mathcal{O}_{T,t}.$$

Proof. We may replace S by an affine neighbourhood of s and after that T by an affine neighbourhood of t. Set $\mathcal{F} = (\mathrm{Spec}(\mathcal{O}_{T,t}) \to T)_*\widetilde{M}$. This is a quasi-coherent sheaf (see Schemes, Lemma 24.1 or argue directly) on T whose stalk at t is M (details omitted). Apply Lemma 2.3. □

05VL **Lemma 2.5.** *Let S be a scheme and $s \in S$ a point. Denote $\mathcal{O}^h_{S,s}$ (resp. $\mathcal{O}^{sh}_{S,s}$) the henselization (resp. strict henselization), see Algebra, Definition 150.3. Let M^{sh} be a $\mathcal{O}^{sh}_{S,s}$-module. The following are equivalent*

1. *M^{sh} is flat over $\mathcal{O}_{S,s}$,*
2. *M^{sh} is flat over $\mathcal{O}^h_{S,s}$, and*
3. *M^{sh} is flat over $\mathcal{O}^{sh}_{S,s}$.*

If $M^{sh} = M^h \otimes_{\mathcal{O}^h_{S,s}} \mathcal{O}^{sh}_{S,s}$ this is also equivalent to

4. *M^h is flat over $\mathcal{O}_{S,s}$, and*
5. *M^h is flat over $\mathcal{O}^h_{S,s}$.*

If $M^h = M \otimes_{\mathcal{O}_{S,s}} \mathcal{O}^h_{S,s}$ this is also equivalent to

6. *M is flat over $\mathcal{O}_{S,s}$.*

Proof. By More on Algebra, Lemma 42.1 the local ring maps $\mathcal{O}_{S,s} \to \mathcal{O}^h_{S,s} \to \mathcal{O}^{sh}_{S,s}$ are faithfully flat. Hence (3) $\Rightarrow$ (2) $\Rightarrow$ (1) and (5) $\Rightarrow$ (4) follow from Algebra, Lemma 38.4. By faithful flatness the equivalences (6) $\Leftrightarrow$ (5) and (5) $\Leftrightarrow$ (3) follow from Algebra, Lemma 38.8. Thus it suffices to show that (1) $\Rightarrow$ (2) $\Rightarrow$ (3) and (4) $\Rightarrow$ (5). To prove these we may assume S is an affine scheme.

Assume (1). By Lemma 2.4 we see that M^{sh} is flat over $\mathcal{O}_{T,t}$ for any étale neighbourhood $(T, t) \to (S, s)$. Since $\mathcal{O}^h_{S,s}$ and $\mathcal{O}^{sh}_{S,s}$ are directed colimits of local rings of the form $\mathcal{O}_{T,t}$ (see Algebra, Lemmas 150.7 and 150.13) we conclude that M^{sh} is flat over $\mathcal{O}^h_{S,s}$ and $\mathcal{O}^{sh}_{S,s}$ by Algebra, Lemma 38.6. Thus (1) implies (2) and (3). Of course this implies also (2) $\Rightarrow$ (3) by replacing $\mathcal{O}_{S,s}$ by $\mathcal{O}^h_{S,s}$. The same argument applies to prove (4) $\Rightarrow$ (5). □

0DK0 **Lemma 2.6.** *Let S be a scheme and $s \in S$ a point. Denote $\mathcal{O}^h_{S,s}$ (resp. $\mathcal{O}^{sh}_{S,s}$) the henselization (resp. strict henselization), see Algebra, Definition 150.3. Let M^{sh} be an object of $D(\mathcal{O}^{sh}_{S,s})$. Let $a, b \in \mathbf{Z}$. The following are equivalent*

1. *M^{sh} has tor amplitude in $[a, b]$ over $\mathcal{O}_{S,s}$,*
2. *M^{sh} has tor amplitude in $[a, b]$ over $\mathcal{O}^h_{S,s}$, and*
3. *M^{sh} has tor amplitude in $[a, b]$ over $\mathcal{O}^{sh}_{S,s}$.*

If $M^{sh} = M^h \otimes^{\mathbf{L}}_{\mathcal{O}^h_{S,s}} \mathcal{O}^{sh}_{S,s}$ for $M^h \in D(\mathcal{O}^h_{S,s})$ this is also equivalent to

(4) *M^h has tor amplitude in $[a, b]$ over $\mathcal{O}_{S,s}$, and*
(5) *M^h has tor amplitude in $[a, b]$ over $\mathcal{O}^h_{S,s}$.*

If $M^h = M \otimes^{\mathbf{L}}_{\mathcal{O}_{S,s}} \mathcal{O}^h_{S,s}$ for $M \in D(\mathcal{O}_{S,s})$ this is also equivalent to

(6) *M has tor amplitude in $[a, b]$ over $\mathcal{O}_{S,s}$.*

Proof. By More on Algebra, Lemma 42.1 the local ring maps $\mathcal{O}_{S,s} \to \mathcal{O}^h_{S,s} \to \mathcal{O}^{sh}_{S,s}$ are faithfully flat. Hence (3) $\Rightarrow$ (2) $\Rightarrow$ (1) and (5) $\Rightarrow$ (4) follow from More on Algebra, Lemma 61.11. By faithful flatness the equivalences (6) $\Leftrightarrow$ (5) and (5) $\Leftrightarrow$ (3) follow from More on Algebra, Lemma 61.17. Thus it suffices to show that (1) $\Rightarrow$ (3), (2) $\Rightarrow$ (3), and (4) $\Rightarrow$ (5).

Assume (1). In particular M^{sh} has vanishing cohomology in degrees $< a$ and $> b$. Hence we can represent M^{sh} by a complex $P^\bullet$ of free $\mathcal{O}^{sh}_{X,x}$-modules with $P^i = 0$ for $i > b$ (see for example the very general Derived Categories, Lemma 16.5). Note that P^n is flat over $\mathcal{O}_{S,s}$ for all n. Consider $\text{Coker}(d^{a-1}_P)$. By More on Algebra, Lemma 61.2 this is a flat $\mathcal{O}_{S,s}$-module. Hence by Lemma 2.5 this is a flat $\mathcal{O}^{sh}_{S,s}$-module. Thus $\tau_{\geq a} P^\bullet$ is a complex of flat $\mathcal{O}^{sh}_{S,s}$-modules representing M^{sh} in $D(\mathcal{O}^{sh}_{S,s}$ and we find that M^{sh} has tor amplitude in $[a, b]$, see More on Algebra, Lemma 61.3. Thus (1) implies (3). Of course this implies also (2) $\Rightarrow$ (3) by replacing $\mathcal{O}_{S,s}$ by $\mathcal{O}^h_{S,s}$. The same argument applies to prove (4) $\Rightarrow$ (5). $\square$

05FN **Lemma 2.7.** *Let $g : T \to S$ be a finite flat morphism of schemes. Let $\mathcal{G}$ be a quasi-coherent $\mathcal{O}_S$-module. Let $t \in T$ be a point with image $s \in S$. Then*

$$t \in \mathit{WeakAss}(g^*\mathcal{G}) \Leftrightarrow s \in \mathit{WeakAss}(\mathcal{G})$$

Proof. The implication "$\Leftarrow$" follows immediately from Divisors, Lemma 6.4. Assume $t \in \text{WeakAss}(g^*\mathcal{G})$. Let $\text{Spec}(A) \subset S$ be an affine open neighbourhood of s. Let $\mathcal{G}$ be the quasi-coherent sheaf associated to the A-module M. Let $\mathfrak{p} \subset A$ be the prime ideal corresponding to s. As g is finite flat we have $g^{-1}(\text{Spec}(A)) = \text{Spec}(B)$ for some finite flat A-algebra B. Note that $g^*\mathcal{G}$ is the quasi-coherent $\mathcal{O}_{\text{Spec}(B)}$-module associated to the B-module $M \otimes_A B$ and $g_*g^*\mathcal{G}$ is the quasi-coherent $\mathcal{O}_{\text{Spec}(A)}$-module associated to the A-module $M \otimes_A B$. By Algebra, Lemma 77.4 we have $B_\mathfrak{p} \cong A_\mathfrak{p}^{\oplus n}$ for some integer $n \geq 0$. Note that $n \geq 1$ as we assumed there exists at least one point of T lying over s. Hence we see by looking at stalks that

$$s \in \text{WeakAss}(\mathcal{G}) \Leftrightarrow s \in \text{WeakAss}(g_*g^*\mathcal{G})$$

Now the assumption that $t \in \text{WeakAss}(g^*\mathcal{G})$ implies that $s \in \text{WeakAss}(g_*g^*\mathcal{G})$ by Divisors, Lemma 6.3 and hence by the above $s \in \text{WeakAss}(\mathcal{G})$. $\square$

05FP **Lemma 2.8.** *Let $h : U \to S$ be an étale morphism of schemes. Let $\mathcal{G}$ be a quasi-coherent $\mathcal{O}_S$-module. Let $u \in U$ be a point with image $s \in S$. Then*

$$u \in \mathit{WeakAss}(h^*\mathcal{G}) \Leftrightarrow s \in \mathit{WeakAss}(\mathcal{G})$$

Proof. After replacing S and U by affine neighbourhoods of s and u we may assume that g is a standard étale morphism of affines, see Morphisms, Lemma 34.14. Thus we may assume $S = \text{Spec}(A)$ and $X = \text{Spec}(A[x, 1/g]/(f))$, where f is monic and f' is invertible in $A[x, 1/g]$. Note that $A[x, 1/g]/(f) = (A[x]/(f))_g$ is also the localization of the finite free A-algebra $A[x]/(f)$. Hence we may think of U as an open subscheme of the scheme $T = \text{Spec}(A[x]/(f))$ which is finite locally free over S. This reduces us to Lemma 2.7 above. $\square$

0CTU **Lemma 2.9.** *Let S be a scheme and $s \in S$ a point. Denote $\mathcal{O}^h_{S,s}$ (resp. $\mathcal{O}^{sh}_{S,s}$) the henselization (resp. strict henselization), see Algebra, Definition 150.3. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_S$-module. The following are equivalent*

(1) *s is a weakly associated point of $\mathcal{F}$,*
(2) *$\mathfrak{m}_s$ is a weakly associated prime of $\mathcal{F}_s$,*
(3) *$\mathfrak{m}^h_s$ is a weakly associated prime of $\mathcal{F}_s \otimes_{\mathcal{O}_{S,s}} \mathcal{O}^h_{S,s}$, and*
(4) *$\mathfrak{m}^{sh}_s$ is a weakly associated prime of $\mathcal{F}_s \otimes_{\mathcal{O}_{S,s}} \mathcal{O}^{sh}_{S,s}$.*

Proof. The equivalence of (1) and (2) is the definition, see Divisors, Definition 5.1. The implications (2) $\Rightarrow$ (3) $\Rightarrow$ (4) follows from Divisors, Lemma 6.4 applied to the flat (More on Algebra, Lemma 42.1) morphisms

$$\mathrm{Spec}(\mathcal{O}^{sh}_{S,s}) \to \mathrm{Spec}(\mathcal{O}^h_{S,s}) \to \mathrm{Spec}(\mathcal{O}_{S,s})$$

and the closed points. To prove (4) $\Rightarrow$ (2) we may replace S by an affine neighbourhood. Suppose that $x \in \mathcal{F}_s \otimes_{\mathcal{O}_{S,s}} \mathcal{O}^{sh}_{S,s}$ is an element whose annihilator has radical equal to $\mathfrak{m}^{sh}_s$. (See Algebra, Lemma 65.2.) Since $\mathcal{O}^{sh}_{S,s}$ is equal to the limit of $\mathcal{O}_{U,u}$ over étale neighbourhoods $f : (U, u) \to (S, s)$ by Algebra, Lemma 150.13 we may assume that x is the image of some $x' \in \mathcal{F}_s \otimes_{\mathcal{O}_{S,s}} \mathcal{O}_{U,u}$. The local ring map $\mathcal{O}_{U,u} \to \mathcal{O}^{sh}_{S,s}$ is faithfully flat (as it is the strict henselization), hence universally injective (Algebra, Lemma 81.11). It follows that the annihilator of x' is the inverse image of the annihilator of x. Hence the radical of this annihilator is equal to $\mathfrak{m}_u$. Thus u is a weakly associated point of $f^*\mathcal{F}$. By Lemma 2.8 we see that s is a weakly associated point of $\mathcal{F}$. □

3. The local structure of a finite type module

057P The key technical lemma that makes a lot of the arguments in this chapter work is the geometric Lemma 3.2.

057Q **Lemma 3.1.** *Let $f : X \to S$ be a finite type morphism of affine schemes. Let $\mathcal{F}$ be a finite type quasi-coherent $\mathcal{O}_X$-module. Let $x \in X$ with image $s = f(x)$ in S. Set $\mathcal{F}_s = \mathcal{F}|_{X_s}$. Then there exist a closed immersion $i : Z \to X$ of finite presentation, and a quasi-coherent finite type $\mathcal{O}_Z$-module $\mathcal{G}$ such that $i_*\mathcal{G} = \mathcal{F}$ and $Z_s = Supp(\mathcal{F}_s)$.*

Proof. Say the morphism $f : X \to S$ is given by the ring map $A \to B$ and that $\mathcal{F}$ is the quasi-coherent sheaf associated to the B-module M. By Morphisms, Lemma 14.2 we know that $A \to B$ is a finite type ring map, and by Properties, Lemma 16.1 we know that M is a finite B-module. In particular the support of $\mathcal{F}$ is the closed subscheme of $\mathrm{Spec}(B)$ cut out by the annihilator $I = \{x \in B \mid xm = 0\ \forall m \in M\}$ of M, see Algebra, Lemma 39.5. Let $\mathfrak{q} \subset B$ be the prime ideal corresponding to x and let $\mathfrak{p} \subset A$ be the prime ideal corresponding to s. Note that $X_s = \mathrm{Spec}(B \otimes_A \kappa(\mathfrak{p}))$ and that $\mathcal{F}_s$ is the quasi-coherent sheaf associated to the $B \otimes_A \kappa(\mathfrak{p})$ module $M \otimes_A \kappa(\mathfrak{p})$. By Morphisms, Lemma 5.3 the support of $\mathcal{F}_s$ is equal to $V(I(B \otimes_A \kappa(\mathfrak{p})))$. Since $B \otimes_A \kappa(\mathfrak{p})$ is of finite type over $\kappa(\mathfrak{p})$ there exist finitely many elements $f_1, \ldots, f_m \in I$ such that

$$I(B \otimes_A \kappa(\mathfrak{p})) = (f_1, \ldots, f_n)(B \otimes_A \kappa(\mathfrak{p})).$$

Denote $i : Z \to X$ the closed subscheme cut out by $(f_1, \ldots, f_m)$, in a formula $Z = \mathrm{Spec}(B/(f_1, \ldots, f_m))$. Since M is annihilated by I we can think of M as an $B/(f_1, \ldots, f_m)$-module. In other words, $\mathcal{F}$ is the pushforward of a finite type module on Z. As $Z_s = \text{Supp}(\mathcal{F}_s)$ by construction, this proves the lemma. $\square$

057S **Lemma 3.2.** *Let $f : X \to S$ be morphism of schemes which is locally of finite type. Let $\mathcal{F}$ be a finite type quasi-coherent $\mathcal{O}_X$-module. Let $x \in X$ with image $s = f(x)$ in S. Set $\mathcal{F}_s = \mathcal{F}|_{X_s}$ and $n = \dim_x(Supp(\mathcal{F}_s))$. Then we can construct*

1. *elementary étale neighbourhoods $g : (X', x') \to (X, x)$, $e : (S', s') \to (S, s)$,*
2. *a commutative diagram*

$$\begin{tikzcd} X \arrow[dd, "f"'] & X' \arrow[l, "g"'] \arrow[dd] & Z' \arrow[l, "i"'] \arrow[d, "\pi"] \\ & & Y' \arrow[d, "h"] \\ S & S' \arrow[l, "e"'] \arrow[r, equal] & S' \end{tikzcd}$$

3. *a point $z' \in Z'$ with $i(z') = x'$, $y' = \pi(z')$, $h(y') = s'$,*
4. *a finite type quasi-coherent $\mathcal{O}_{Z'}$-module $\mathcal{G}$,*

such that the following properties hold

1. *X', Z', Y', S' are affine schemes,*
2. *i is a closed immersion of finite presentation,*
3. *$i_*(\mathcal{G}) \cong g^*\mathcal{F}$,*
4. *π is finite and $\pi^{-1}(\{y'\}) = \{z'\}$,*
5. *the extension $\kappa(s') \subset \kappa(y')$ is purely transcendental,*
6. *h is smooth of relative dimension n with geometrically integral fibres.*

Proof. Let $V \subset S$ be an affine neighbourhood of s. Let $U \subset f^{-1}(V)$ be an affine neighbourhood of x. Then it suffices to prove the lemma for $f|_U : U \to V$ and $\mathcal{F}|_U$. Hence in the rest of the proof we assume that X and S are affine.

First, suppose that $X_s = \text{Supp}(\mathcal{F}_s)$, in particular $n = \dim_x(X_s)$. Apply More on Morphisms, Lemmas 40.2 and 40.3. This gives us a commutative diagram

$$\begin{tikzcd} X \arrow[dd] & X' \arrow[l, "g"'] \arrow[d, "\pi"] \\ & Y' \arrow[d, "h"] \\ S & S' \arrow[l, "e"'] \end{tikzcd}$$

and point $x' \in X'$. We set $Z' = X'$, $i = \text{id}$, and $\mathcal{G} = g^*\mathcal{F}$ to obtain a solution in this case.

In general choose a closed immersion $Z \to X$ and a sheaf $\mathcal{G}$ on Z as in Lemma 3.1. Applying the result of the previous paragraph to $Z \to S$ and $\mathcal{G}$ we obtain a diagram

$$\begin{tikzcd} X & Z \arrow[l] \arrow[dd, "f|_Z"] & Z' \arrow[l, "g"'] \arrow[d, "\pi"] \\ & & Y' \arrow[d, "h"] \\ S \arrow[uu, leftarrow, "f"'] & S \arrow[l, equal] & S' \arrow[l, "e"'] \end{tikzcd}$$

and point $z' \in Z'$ satisfying all the required properties. We will use Lemma 2.1 to embed Z' into a scheme étale over X. We cannot apply the lemma directly as we want X' to be a scheme over S'. Instead we consider the morphisms

$$Z' \longrightarrow Z \times_S S' \longrightarrow X \times_S S'$$

The first morphism is étale by Morphisms, Lemma 34.18. The second is a closed immersion as a base change of a closed immersion. Finally, as X, S, S', Z, Z' are all affine we may apply Lemma 2.1 to get an étale morphism of affine schemes $X' \to X \times_S S'$ such that

$$Z' = (Z \times_S S') \times_{(X \times_S S')} X' = Z \times_X X'.$$

As $Z \to X$ is a closed immersion of finite presentation, so is $Z' \to X'$. Let $x' \in X'$ be the point corresponding to $z' \in Z'$. Then the completed diagram

$$\begin{tikzcd} X \arrow[dd] & X' \arrow[l] \arrow[dd] & Z' \arrow[l, "i"'] \arrow[d, "\pi"] \\ & & Y' \arrow[d, "h"] \\ S & S' \arrow[l, "e"'] & S' \arrow[l, equal] \end{tikzcd}$$

is a solution of the original problem. □

057T **Lemma 3.3.** *Assumptions and notation as in Lemma 3.2. If f is locally of finite presentation then π is of finite presentation. In this case the following are equivalent*

(1) *$\mathcal{F}$ is an $\mathcal{O}_X$-module of finite presentation in a neighbourhood of x,*
(2) *$\mathcal{G}$ is an $\mathcal{O}_{Z'}$-module of finite presentation in a neighbourhood of z', and*
(3) *$\pi_*\mathcal{G}$ is an $\mathcal{O}_{Y'}$-module of finite presentation in a neighbourhood of y'.*

Still assuming f locally of finite presentation the following are equivalent to each other

(a) *$\mathcal{F}_x$ is an $\mathcal{O}_{X,x}$-module of finite presentation,*
(b) *$\mathcal{G}_{z'}$ is an $\mathcal{O}_{Z',z'}$-module of finite presentation, and*
(c) *$(\pi_*\mathcal{G})_{y'}$ is an $\mathcal{O}_{Y',y'}$-module of finite presentation.*

Proof. Assume f locally of finite presentation. Then $Z' \to S$ is locally of finite presentation as a composition of such, see Morphisms, Lemma 20.3. Note that $Y' \to S$ is also locally of finite presentation as a composition of a smooth and an étale morphism. Hence Morphisms, Lemma 20.11 implies π is locally of finite presentation. Since π is finite we conclude that it is also separated and quasi-compact, hence π is actually of finite presentation.

To prove the equivalence of (1), (2), and (3) we also consider: (4) $g^*\mathcal{F}$ is a $\mathcal{O}_{X'}$-module of finite presentation in a neighbourhood of x'. The pullback of a module of finite presentation is of finite presentation, see Modules, Lemma 11.4. Hence (1) $\Rightarrow$ (4). The étale morphism g is open, see Morphisms, Lemma 34.13. Hence for any open neighbourhood $U' \subset X'$ of x', the image $g(U')$ is an open neighbourhood of x and the map $\{U' \to g(U')\}$ is an étale covering. Thus (4) $\Rightarrow$ (1) by Descent, Lemma 7.3. Using Descent, Lemma 7.10 and some easy topological arguments (see More on Morphisms, Lemma 40.4) we see that (4) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3).

To prove the equivalence of (a), (b), (c) consider the ring maps

$$\mathcal{O}_{X,x} \to \mathcal{O}_{X',x'} \to \mathcal{O}_{Z',z'} \leftarrow \mathcal{O}_{Y',y'}$$

The first ring map is faithfully flat. Hence $\mathcal{F}_x$ is of finite presentation over $\mathcal{O}_{X,x}$ if and only if $g^*\mathcal{F}_{x'}$ is of finite presentation over $\mathcal{O}_{X',x'}$, see Algebra, Lemma 82.2. The second ring map is surjective (hence finite) and finitely presented by assumption, hence $g^*\mathcal{F}_{x'}$ is of finite presentation over $\mathcal{O}_{X',x'}$ if and only if $\mathcal{G}_{z'}$ is of finite presentation over $\mathcal{O}_{Z',z'}$, see Algebra, Lemma 35.23. Because π is finite, of finite presentation, and $\pi^{-1}(\{y'\}) = \{x'\}$ the ring homomorphism $\mathcal{O}_{Y',y'} \leftarrow \mathcal{O}_{Z',z'}$ is finite and of finite presentation, see More on Morphisms, Lemma 40.4. Hence $\mathcal{G}_{z'}$ is of finite presentation over $\mathcal{O}_{Z',z'}$ if and only if $\pi_*\mathcal{G}_{y'}$ is of finite presentation over $\mathcal{O}_{Y',y'}$, see Algebra, Lemma 35.23. □

057U **Lemma 3.4.** *Assumptions and notation as in Lemma 3.2. The following are equivalent*

1. *$\mathcal{F}$ is flat over S in a neighbourhood of x,*
2. *$\mathcal{G}$ is flat over S' in a neighbourhood of z', and*
3. *$\pi_*\mathcal{G}$ is flat over S' in a neighbourhood of y'.*

The following are equivalent also

(a) *$\mathcal{F}_x$ is flat over $\mathcal{O}_{S,s}$,*
(b) *$\mathcal{G}_{z'}$ is flat over $\mathcal{O}_{S',s'}$, and*
(c) *$(\pi_*\mathcal{G})_{y'}$ is flat over $\mathcal{O}_{S',s'}$.*

Proof. To prove the equivalence of (1), (2), and (3) we also consider: (4) $g^*\mathcal{F}$ is flat over S in a neighbourhood of x'. We will use Lemma 2.3 to equate flatness over S and S' without further mention. The étale morphism g is flat and open, see Morphisms, Lemma 34.13. Hence for any open neighbourhood $U' \subset X'$ of x', the image $g(U')$ is an open neighbourhood of x and the map $U' \to g(U')$ is surjective and flat. Thus (4) $\Leftrightarrow$ (1) by Morphisms, Lemma 24.12. Note that

$$\Gamma(X', g^*\mathcal{F}) = \Gamma(Z', \mathcal{G}) = \Gamma(Y', \pi_*\mathcal{G})$$

Hence the flatness of $g^*\mathcal{F}$, $\mathcal{G}$ and $\pi_*\mathcal{G}$ over S' are all equivalent (this uses that X', Z', Y', and S' are all affine). Some omitted topological arguments (compare More on Morphisms, Lemma 40.4) regarding affine neighbourhoods now show that (4) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3).

To prove the equivalence of (a), (b), (c) consider the commutative diagram of local ring maps

$$\begin{tikzcd} \mathcal{O}_{X',x'} \arrow[r, "\iota"'] & \mathcal{O}_{Z',z'} & \mathcal{O}_{Y',y'} \arrow[l, "\alpha"'] & \mathcal{O}_{S',s'} \arrow[l, "\beta"'] \\ \mathcal{O}_{X,x} \arrow[u, "\gamma"] & & & \mathcal{O}_{S,s} \arrow[lll, "\varphi"'] \arrow[u, "\epsilon"'] \end{tikzcd}$$

We will use Lemma 2.4 to equate flatness over $\mathcal{O}_{S,s}$ and $\mathcal{O}_{S',s'}$ without further mention. The map γ is faithfully flat. Hence $\mathcal{F}_x$ is flat over $\mathcal{O}_{S,s}$ if and only if $g^*\mathcal{F}_{x'}$ is flat over $\mathcal{O}_{S',s'}$, see Algebra, Lemma 38.9. As $\mathcal{O}_{S',s'}$-modules the modules $g^*\mathcal{F}_{x'}$, $\mathcal{G}_{z'}$, and $\pi_*\mathcal{G}_{y'}$ are all isomorphic, see More on Morphisms, Lemma 40.4. This finishes the proof. □

4. One step dévissage

05H3 In this section we explain what is a one step dévissage of a module. A one step dévissage exist étale locally on base and target. We discuss base change, Zariski shrinking and étale localization of a one step dévissage.

05H4 **Definition 4.1.** Let S be a scheme. Let X be locally of finite type over S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module of finite type. Let $s \in S$ be a point. A *one step dévissage of $\mathcal{F}/X/S$ over s* is given by morphisms of schemes over S

$$X \xleftarrow{i} Z \xrightarrow{\pi} Y$$

and a quasi-coherent $\mathcal{O}_Z$-module $\mathcal{G}$ of finite type such that

(1) X, S, Z and Y are affine,
(2) i is a closed immersion of finite presentation,
(3) $\mathcal{F} \cong i_*\mathcal{G}$,
(4) π is finite, and
(5) the structure morphism $Y \to S$ is smooth with geometrically irreducible fibres of dimension $\dim(\text{Supp}(\mathcal{F}_s))$.

In this case we say $(Z, Y, i, \pi, \mathcal{G})$ is a one step dévissage of $\mathcal{F}/X/S$ over s.

Note that such a one step dévissage can only exist if X and S are affine. In the definition above we only require X to be (locally) of finite type over S and we continue working in this setting below. In [GR71] the authors use consistently the setup where $X \to S$ is locally of finite presentation and $\mathcal{F}$ quasi-coherent $\mathcal{O}_X$-module of finite type. The advantage of this choice is that it "makes sense" to ask for $\mathcal{F}$ to be of finite presentation as an $\mathcal{O}_X$-module, whereas in our setting it "does not make sense". Please see More on Morphisms, Section 47 for a discussion; the observations made there show that in our setup we may consider the condition of $\mathcal{F}$ being "locally of finite presentation relative to S", and we could work consistently with this notion. Instead however, we will rely on the results of Lemma 3.3 and the observations in Remark 6.3 to deal with this issue in an ad hoc fashion whenever it comes up.

05H5 **Definition 4.2.** Let S be a scheme. Let X be locally of finite type over S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module of finite type. Let $x \in X$ be a point with image s in S. A *one step dévissage of $\mathcal{F}/X/S$ at x* is a system $(Z, Y, i, \pi, \mathcal{G}, z, y)$, where $(Z, Y, i, \pi, \mathcal{G})$ is a one step dévissage of $\mathcal{F}/X/S$ over s and

(1) $\dim_x(\text{Supp}(\mathcal{F}_s)) = \dim(\text{Supp}(\mathcal{F}_s))$,

(2) $z \in Z$ is a point with $i(z) = x$ and $\pi(z) = y$,
(3) we have $\pi^{-1}(\{y\}) = \{z\}$,
(4) the extension $\kappa(s) \subset \kappa(y)$ is purely transcendental.

A one step dévissage of $\mathcal{F}/X/S$ at x can only exist if X and S are affine. Condition (1) assures us that $Y \to S$ has relative dimension equal to $\dim_x(\text{Supp}(\mathcal{F}_s))$ via condition (5) of Definition 4.1.

05H6 **Lemma 4.3.** *Let $f : X \to S$ be morphism of schemes which is locally of finite type. Let $\mathcal{F}$ be a finite type quasi-coherent $\mathcal{O}_X$-module. Let $x \in X$ with image $s = f(x)$ in S. Then there exists a commutative diagram of pointed schemes*

$$\begin{tikzcd} (X, x) \arrow[d, "f"'] & (X', x') \arrow[l, "g"'] \arrow[d] \\ (S, s) & (S', s') \arrow[l] \end{tikzcd}$$

such that $(S', s') \to (S, s)$ and $(X', x') \to (X, x)$ are elementary étale neighbourhoods, and such that $g^*\mathcal{F}/X'/S'$ has a one step dévissage at x'.

Proof. This is immediate from Definition 4.2 and Lemma 3.2. □

05H7 **Lemma 4.4.** *Let S, X, $\mathcal{F}$, s be as in Definition 4.1. Let $(Z, Y, i, \pi, \mathcal{G})$ be a one step dévissage of $\mathcal{F}/X/S$ over s. Let $(S', s') \to (S, s)$ be any morphism of pointed schemes. Given this data let X', Z', Y', i', π' be the base changes of X, Z, Y, i, π via $S' \to S$. Let $\mathcal{F}'$ be the pullback of $\mathcal{F}$ to X' and let $\mathcal{G}'$ be the pullback of $\mathcal{G}$ to Z'. If S' is affine, then $(Z', Y', i', \pi', \mathcal{G}')$ is a one step dévissage of $\mathcal{F}'/X'/S'$ over s'.*

Proof. Fibre products of affines are affine, see Schemes, Lemma 17.2. Base change preserves closed immersions, morphisms of finite presentation, finite morphisms, smooth morphisms, morphisms with geometrically irreducible fibres, and morphisms of relative dimension n, see Morphisms, Lemmas 2.4, 20.4, 42.6, 32.5, 28.2, and More on Morphisms, Lemma 25.2. We have $i'_*\mathcal{G}' \cong \mathcal{F}'$ because pushforward along the finite morphism i commutes with base change, see Cohomology of Schemes, Lemma 5.1. We have $\dim(\text{Supp}(\mathcal{F}_s)) = \dim(\text{Supp}(\mathcal{F}'_{s'}))$ by Morphisms, Lemma 27.3 because

$$\text{Supp}(\mathcal{F}_s) \times_s s' = \text{Supp}(\mathcal{F}'_{s'}).$$

This proves the lemma. □

05H8 **Lemma 4.5.** *Let S, X, $\mathcal{F}$, x, s be as in Definition 4.2. Let $(Z, Y, i, \pi, \mathcal{G}, z, y)$ be a one step dévissage of $\mathcal{F}/X/S$ at x. Let $(S', s') \to (S, s)$ be a morphism of pointed schemes which induces an isomorphism $\kappa(s) = \kappa(s')$. Let $(Z', Y', i', \pi', \mathcal{G}')$ be as constructed in Lemma 4.4 and let $x' \in X'$ (resp. $z' \in Z'$, $y' \in Y'$) be the unique point mapping to both $x \in X$ (resp. $z \in Z$, $y \in Y$) and $s' \in S'$. If S' is affine, then $(Z', Y', i', \pi', \mathcal{G}', z', y')$ is a one step dévissage of $\mathcal{F}'/X'/S'$ at x'.*

Proof. By Lemma 4.4 $(Z', Y', i', \pi', \mathcal{G}')$ is a one step dévissage of $\mathcal{F}'/X'/S'$ over s'. Properties (1) – (4) of Definition 4.2 hold for $(Z', Y', i', \pi', \mathcal{G}', z', y')$ as the assumption that $\kappa(s) = \kappa(s')$ insures that the fibres $X'_{s'}$, $Z'_{s'}$, and $Y'_{s'}$ are isomorphic to X_s, Z_s, and Y_s. □

05H9 **Definition 4.6.** Let S, X, $\mathcal{F}$, x, s be as in Definition 4.2. Let $(Z, Y, i, \pi, \mathcal{G}, z, y)$ be a one step dévissage of $\mathcal{F}/X/S$ at x. Let us define a *standard shrinking* of this situation to be given by standard opens $S' \subset S$, $X' \subset X$, $Z' \subset Z$, and $Y' \subset Y$ such that $s \in S'$, $x \in X'$, $z \in Z'$, and $y \in Y'$ and such that

$$(Z', Y', i|_{Z'}, \pi|_{Z'}, \mathcal{G}|_{Z'}, z, y)$$

is a one step dévissage of $\mathcal{F}|_{X'}/X'/S'$ at x.

05HA **Lemma 4.7.** *With assumption and notation as in Definition 4.6 we have:*

05HB (1) *If $S' \subset S$ is a standard open neighbourhood of s, then setting $X' = X_{S'}$, $Z' = Z_{S'}$ and $Y' = Y_{S'}$ we obtain a standard shrinking.*
05HC (2) *Let $W \subset Y$ be a standard open neighbourhood of y. Then there exists a standard shrinking with $Y' = W \times_S S'$.*
05HD (3) *Let $U \subset X$ be an open neighbourhood of x. Then there exists a standard shrinking with $X' \subset U$.*

Proof. Part (1) is immediate from Lemma 4.5 and the fact that the inverse image of a standard open under a morphism of affine schemes is a standard open, see Algebra, Lemma 16.4.

Let $W \subset Y$ as in (2). Because $Y \to S$ is smooth it is open, see Morphisms, Lemma 32.10. Hence we can find a standard open neighbourhood S' of s contained in the image of W. Then the fibres of $W_{S'} \to S'$ are nonempty open subschemes of the fibres of $Y \to S$ over S' and hence geometrically irreducible too. Setting $Y' = W_{S'}$ and $Z' = \pi^{-1}(Y')$ we see that $Z' \subset Z$ is a standard open neighbourhood of z. Let $\overline{h} \in \Gamma(Z, \mathcal{O}_Z)$ be a function such that $Z' = D(\overline{h})$. As $i : Z \to X$ is a closed immersion, we can find a function $h \in \Gamma(X, \mathcal{O}_X)$ such that $i^\sharp(h) = \overline{h}$. Take $X' = D(h) \subset X$. In this way we obtain a standard shrinking as in (2).

Let $U \subset X$ be as in (3). We may after shrinking U assume that U is a standard open. By More on Morphisms, Lemma 40.4 there exists a standard open $W \subset Y$ neighbourhood of y such that $\pi^{-1}(W) \subset i^{-1}(U)$. Apply (2) to get a standard shrinking X', S', Z', Y' with $Y' = W_{S'}$. Since $Z' \subset \pi^{-1}(W) \subset i^{-1}(U)$ we may replace X' by $X' \cap U$ (still a standard open as U is also standard open) without violating any of the conditions defining a standard shrinking. Hence we win. $\square$

05HE **Lemma 4.8.** *Let S, X, $\mathcal{F}$, x, s be as in Definition 4.2. Let $(Z, Y, i, \pi, \mathcal{G}, z, y)$ be a one step dévissage of $\mathcal{F}/X/S$ at x. Let*

$$\begin{tikzcd} (Y, y) \arrow[d] & (Y', y') \arrow[l] \arrow[d] \\ (S, s) & (S', s') \arrow[l] \end{tikzcd}$$

be a commutative diagram of pointed schemes such that the horizontal arrows are elementary étale neighbourhoods. Then there exists a commutative diagram

$$\begin{tikzcd} & & (X'', x'') \arrow[dll] \arrow[d] & (Z'', z'') \arrow[l] \arrow[dll] \arrow[d] \\ (X, x) \arrow[d] & (Z, z) \arrow[l] \arrow[d] & (S'', s'') \arrow[dll] & (Y'', y'') \arrow[l] \arrow[dll] \\ (S, s) & (Y, y) \arrow[l] & & \end{tikzcd}$$

of pointed schemes with the following properties:

(1) $(S'', s'') \to (S', s')$ *is an elementary étale neighbourhood and the morphism* $S'' \to S$ *is the composition* $S'' \to S' \to S$,
(2) Y'' *is an open subscheme of* $Y' \times_{S'} S''$,
(3) $Z'' = Z \times_Y Y''$,
(4) $(X'', x'') \to (X, x)$ *is an elementary étale neighbourhood, and*
(5) $(Z'', Y'', i'', \pi'', \mathcal{G}'', z'', y'')$ *is a one step dévissage at* x'' *of the sheaf* $\mathcal{F}''$.

Here $\mathcal{F}''$ *(resp.* $\mathcal{G}''$*) is the pullback of* $\mathcal{F}$ *(resp.* $\mathcal{G}$*) via the morphism* $X'' \to X$ *(resp.* $Z'' \to Z$*) and* $i'' : Z'' \to X''$ *and* $\pi'' : Z'' \to Y''$ *are as in the diagram.*

Proof. Let $(S'', s'') \to (S', s')$ be any elementary étale neighbourhood with S'' affine. Let $Y'' \subset Y' \times_{S'} S''$ be any affine open neighbourhood containing the point $y'' = (y', s'')$. Then we obtain an affine (Z'', z'') by (3). Moreover $Z_{S''} \to X_{S''}$ is a closed immersion and $Z'' \to Z_{S''}$ is an étale morphism. Hence Lemma 2.1 applies and we can find an étale morphism $X'' \to X_{S'}$ of affines such that $Z'' \cong X'' \times_{X_{S'}} Z_{S'}$. Denote $i'' : Z'' \to X''$ the corresponding closed immersion. Setting $x'' = i''(z'')$ we obtain a commutative diagram as in the lemma. Properties (1), (2), (3), and (4) hold by construction. Thus it suffices to show that (5) holds for a suitable choice of $(S'', s'') \to (S', s')$ and Y''.

We first list those properties which hold for any choice of $(S'', s'') \to (S', s')$ and Y'' as in the first paragraph. As we have $Z'' = X'' \times_X Z$ by construction we see that $i''_*\mathcal{G}'' = \mathcal{F}''$ (with notation as in the statement of the lemma), see Cohomology of Schemes, Lemma 5.1. Set $n = \dim(\text{Supp}(\mathcal{F}_s)) = \dim_x(\text{Supp}(\mathcal{F}_s))$. The morphism $Y'' \to S''$ is smooth of relative dimension n (because $Y' \to S'$ is smooth of relative dimension n as the composition $Y' \to Y_{S'} \to S'$ of an étale and smooth morphism of relative dimension n and because base change preserves smooth morphisms of relative dimension n). We have $\kappa(y'') = \kappa(y)$ and $\kappa(s) = \kappa(s'')$ hence $\kappa(y'')$ is a purely transcendental extension of $\kappa(s'')$. The morphism of fibres $X''_{s''} \to X_s$ is an étale morphism of affine schemes over $\kappa(s) = \kappa(s'')$ mapping the point x'' to the point x and pulling back $\mathcal{F}_s$ to $\mathcal{F}''_{s''}$. Hence

$$\dim(\text{Supp}(\mathcal{F}''_{s''})) = \dim(\text{Supp}(\mathcal{F}_s)) = n = \dim_x(\text{Supp}(\mathcal{F}_s)) = \dim_{x''}(\text{Supp}(\mathcal{F}''_{s''}))$$

because dimension is invariant under étale localization, see Descent, Lemma 18.2. As $\pi'' : Z'' \to Y''$ is the base change of π we see that π'' is finite and as $\kappa(y) = \kappa(y'')$ we see that $\pi^{-1}(\{y''\}) = \{z''\}$.

At this point we have verified all the conditions of Definition 4.1 except we have not verified that $Y'' \to S''$ has geometrically irreducible fibres. Of course in general this is not going to be true, and it is at this point that we will use that $\kappa(s) \subset \kappa(y)$ is purely transcendental. Namely, let $T \subset Y'_{s'}$ be the irreducible component of $Y'_{s'}$ containing $y' = (y, s')$. Note that T is an open subscheme of $Y'_{s'}$ as this is a smooth scheme over $\kappa(s')$. By Varieties, Lemma 7.14 we see that T is geometrically connected because $\kappa(s') = \kappa(s)$ is algebraically closed in $\kappa(y') = \kappa(y)$. As T is smooth we see that T is geometrically irreducible. Hence More on Morphisms, Lemma 39.3 applies and we can find an elementary étale morphism $(S'', s'') \to (S', s')$ and an affine open $Y'' \subset Y'_{S''}$ such that all fibres of $Y'' \to S''$ are geometrically irreducible and such that $T = Y''_{s''}$. After shrinking (first Y'' and then S'') we may assume that both Y'' and S'' are affine. This finishes the proof of the lemma. □

05HF **Lemma 4.9.** *Let S, X, $\mathcal{F}$, s be as in Definition 4.1. Let $(Z, Y, i, \pi, \mathcal{G})$ be a one step dévissage of $\mathcal{F}/X/S$ over s. Let $\xi \in Y_s$ be the (unique) generic point. Then there exists an integer $r > 0$ and an $\mathcal{O}_Y$-module map*

$$\alpha : \mathcal{O}_Y^{\oplus r} \longrightarrow \pi_*\mathcal{G}$$

such that

$$\alpha : \kappa(\xi)^{\oplus r} \longrightarrow (\pi_*\mathcal{G})_\xi \otimes_{\mathcal{O}_{Y,\xi}} \kappa(\xi)$$

is an isomorphism. Moreover, in this case we have

$$\dim(\mathit{Supp}(\mathrm{Coker}(\alpha)_s)) < \dim(\mathit{Supp}(\mathcal{F}_s)).$$

Proof. By assumption the schemes S and Y are affine. Write $S = \mathrm{Spec}(A)$ and $Y = \mathrm{Spec}(B)$. As π is finite the $\mathcal{O}_Y$-module $\pi_*\mathcal{G}$ is a finite type quasi-coherent $\mathcal{O}_Y$-module. Hence $\pi_*\mathcal{G} = \widetilde{N}$ for some finite B-module N. Let $\mathfrak{p} \subset B$ be the prime ideal corresponding to ξ. To obtain α set $r = \dim_{\kappa(\mathfrak{p})} N \otimes_B \kappa(\mathfrak{p})$ and pick $x_1, \ldots, x_r \in N$ which form a basis of $N \otimes_B \kappa(\mathfrak{p})$. Take $\alpha : B^{\oplus r} \to N$ to be the map given by the formula $\alpha(b_1, \ldots, b_r) = \sum b_i x_i$. It is clear that $\alpha : \kappa(\mathfrak{p})^{\oplus r} \to N \otimes_B \kappa(\mathfrak{p})$ is an isomorphism as desired. Finally, suppose α is any map with this property. Then $N' = \mathrm{Coker}(\alpha)$ is a finite B-module such that $N' \otimes \kappa(\mathfrak{p}) = 0$. By Nakayama's lemma (Algebra, Lemma 19.1) we see that $N'_\mathfrak{p} = 0$. Since the fibre Y_s is geometrically irreducible of dimension n with generic point ξ and since we have just seen that ξ is not in the support of $\mathrm{Coker}(\alpha)$ the last assertion of the lemma holds. $\square$

5. Complete dévissage

05HG In this section we explain what is a complete dévissage of a module and prove that such exist. The material in this section is mainly bookkeeping.

05HH **Definition 5.1.** Let S be a scheme. Let X be locally of finite type over S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module of finite type. Let $s \in S$ be a point. A *complete dévissage of $\mathcal{F}/X/S$ over s* is given by a diagram

$$\begin{tikzcd}
X & Z_1 \arrow[l, "i_1"'] \arrow[d, "\pi_1"] & & & \\
& Y_1 & Z_2 \arrow[l, "i_2"'] \arrow[d, "\pi_2"] & & \\
& & Y_2 & Z_3 \arrow[l] \arrow[d] & \\
& & & \ldots & \ldots \arrow[l] \arrow[d] \\
& & & & Y_n
\end{tikzcd}$$

of schemes over S, finite type quasi-coherent $\mathcal{O}_{Z_k}$-modules $\mathcal{G}_k$, and $\mathcal{O}_{Y_k}$-module maps

$$\alpha_k : \mathcal{O}_{Y_k}^{\oplus r_k} \longrightarrow \pi_{k,*}\mathcal{G}_k, \quad k = 1, \ldots, n$$

satisfying the following properties:

(1) $(Z_1, Y_1, i_1, \pi_1, \mathcal{G}_1)$ is a one step dévissage of $\mathcal{F}/X/S$ over s,

(2) the map α_k induces an isomorphism

$$\kappa(\xi_k)^{\oplus r_k} \longrightarrow (\pi_{k,*}\mathcal{G}_k)_{\xi_k} \otimes_{\mathcal{O}_{Y_k,\xi_k}} \kappa(\xi_k)$$

where $\xi_k \in (Y_k)_s$ is the unique generic point,

(3) for $k = 2, \ldots, n$ the system $(Z_k, Y_k, i_k, \pi_k, \mathcal{G}_k)$ is a one step dévissage of $\mathrm{Coker}(\alpha_{k-1})/Y_{k-1}/S$ over s,

(4) $\mathrm{Coker}(\alpha_n) = 0$.

In this case we say that $(Z_k, Y_k, i_k, \pi_k, \mathcal{G}_k, \alpha_k)_{k=1,\ldots,n}$ is a complete dévissage of $\mathcal{F}/X/S$ over s.

05HI **Definition 5.2.** Let S be a scheme. Let X be locally of finite type over S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module of finite type. Let $x \in X$ be a point with image $s \in S$. A *complete dévissage of $\mathcal{F}/X/S$ at x* is given by a system

$$(Z_k, Y_k, i_k, \pi_k, \mathcal{G}_k, \alpha_k, z_k, y_k)_{k=1,\ldots,n}$$

such that $(Z_k, Y_k, i_k, \pi_k, \mathcal{G}_k, \alpha_k)$ is a complete dévissage of $\mathcal{F}/X/S$ over s, and such that

(1) $(Z_1, Y_1, i_1, \pi_1, \mathcal{G}_1, z_1, y_1)$ is a one step dévissage of $\mathcal{F}/X/S$ at x,

(2) for $k = 2, \ldots, n$ the system $(Z_k, Y_k, i_k, \pi_k, \mathcal{G}_k, z_k, y_k)$ is a one step dévissage of $\mathrm{Coker}(\alpha_{k-1})/Y_{k-1}/S$ at y_{k-1}.

Again we remark that a complete dévissage can only exist if X and S are affine.

05HJ **Lemma 5.3.** *Let S, X, $\mathcal{F}$, s be as in Definition 5.1. Let $(S', s') \to (S, s)$ be any morphism of pointed schemes. Let $(Z_k, Y_k, i_k, \pi_k, \mathcal{G}_k, \alpha_k)_{k=1,\ldots,n}$ be a complete dévissage of $\mathcal{F}/X/S$ over s. Given this data let $X', Z'_k, Y'_k, i'_k, \pi'_k$ be the base changes of X, Z_k, Y_k, i_k, π_k via $S' \to S$. Let $\mathcal{F}'$ be the pullback of $\mathcal{F}$ to X' and let $\mathcal{G}'_k$ be the pullback of $\mathcal{G}_k$ to Z'_k. Let α'_k be the pullback of α_k to Y'_k. If S' is affine, then $(Z'_k, Y'_k, i'_k, \pi'_k, \mathcal{G}'_k, \alpha'_k)_{k=1,\ldots,n}$ is a complete dévissage of $\mathcal{F}'/X'/S'$ over s'.*

Proof. By Lemma 4.4 we know that the base change of a one step dévissage is a one step dévissage. Hence it suffices to prove that formation of $\mathrm{Coker}(\alpha_k)$ commutes with base change and that condition (2) of Definition 5.1 is preserved by base change. The first is true as $\pi'_{k,*}\mathcal{G}'_k$ is the pullback of $\pi_{k,*}\mathcal{G}_k$ (by Cohomology of Schemes, Lemma 5.1) and because $\otimes$ is right exact. The second because by the same token we have

$$(\pi_{k,*}\mathcal{G}_k)_{\xi_k} \otimes_{\mathcal{O}_{Y_k,\xi_k}} \kappa(\xi_k) \otimes_{\kappa(\xi_k)} \kappa(\xi'_k) \cong (\pi'_{k,*}\mathcal{G}'_k)_{\xi'_k} \otimes_{\mathcal{O}_{Y'_k,\xi'_k}} \kappa(\xi'_k)$$

with obvious notation. □

05HK **Lemma 5.4.** *Let S, X, $\mathcal{F}$, x, s be as in Definition 5.2. Let $(S', s') \to (S, s)$ be a morphism of pointed schemes which induces an isomorphism $\kappa(s) = \kappa(s')$. Let $(Z_k, Y_k, i_k, \pi_k, \mathcal{G}_k, \alpha_k, z_k, y_k)_{k=1,\ldots,n}$ be a complete dévissage of $\mathcal{F}/X/S$ at x. Let $(Z'_k, Y'_k, i'_k, \pi'_k, \mathcal{G}'_k, \alpha'_k)_{k=1,\ldots,n}$ be as constructed in Lemma 5.3 and let $x' \in X'$ (resp. $z'_k \in Z'$, $y'_k \in Y'$) be the unique point mapping to both $x \in X$ (resp. $z_k \in Z_k$, $y_k \in Y_k$) and $s' \in S'$. If S' is affine, then $(Z'_k, Y'_k, i'_k, \pi'_k, \mathcal{G}'_k, \alpha'_k, z'_k, y'_k)_{k=1,\ldots,n}$ is a complete dévissage of $\mathcal{F}'/X'/S'$ at x'.*

Proof. Combine Lemma 5.3 and Lemma 4.5. □

05HL **Definition 5.5.** Let S, X, $\mathcal{F}$, x, s be as in Definition 5.2. Consider a complete dévissage $(Z_k, Y_k, i_k, \pi_k, \mathcal{G}_k, \alpha_k, z_k, y_k)_{k=1,\ldots,n}$ of $\mathcal{F}/X/S$ at x. Let us define a *standard shrinking* of this situation to be given by standard opens $S' \subset S$, $X' \subset X$, $Z'_k \subset Z_k$, and $Y'_k \subset Y_k$ such that $s_k \in S'$, $x_k \in X'$, $z_k \in Z'$, and $y_k \in Y'$ and such that

$$(Z'_k, Y'_k, i'_k, \pi'_k, \mathcal{G}'_k, \alpha'_k, z_k, y_k)_{k=1,\ldots,n}$$

is a one step dévissage of $\mathcal{F}'/X'/S'$ at x where $\mathcal{G}'_k = \mathcal{G}_k|_{Z'_k}$ and $\mathcal{F}' = \mathcal{F}|_{X'}$.

05HM **Lemma 5.6.** *With assumption and notation as in Definition 5.5 we have:*

05HN (1) *If $S' \subset S$ is a standard open neighbourhood of s, then setting $X' = X_{S'}$, $Z'_k = Z_{S'}$ and $Y'_k = Y_{S'}$ we obtain a standard shrinking.*
05HP (2) *Let $W \subset Y_n$ be a standard open neighbourhood of y. Then there exists a standard shrinking with $Y'_n = W \times_S S'$.*
05HQ (3) *Let $U \subset X$ be an open neighbourhood of x. Then there exists a standard shrinking with $X' \subset U$.*

Proof. Part (1) is immediate from Lemmas 5.4 and 4.7.

Proof of (2). For convenience denote $X = Y_0$. We apply Lemma 4.7 (2) to find a standard shrinking S', Y'_{n-1}, Z'_n, Y'_n of the one step dévissage of $\text{Coker}(\alpha_{n-1})/Y_{n-1}/S$ at y_{n-1} with $Y'_n = W \times_S S'$. We may repeat this procedure and find a standard shrinking $S'', Y''_{n-2}, Z''_{n-1}, Y''_{n-1}$ of the one step dévissage of $\text{Coker}(\alpha_{n-2})/Y_{n-2}/S$ at y_{n-2} with $Y''_{n-1} = Y'_{n-1} \times_S S''$. We may continue in this manner until we obtain $S^{(n)}, Y_0^{(n)}, Z_1^{(n)}, Y_1^{(n)}$. At this point it is clear that we obtain our desired standard shrinking by taking $S^{(n)}$, $X^{(n)}$, $Z_k^{(n-k)} \times_S S^{(n)}$, and $Y_k^{(n-k)} \times_S S^{(n)}$ with the desired property.

Proof of (3). We use induction on the length of the complete dévissage. First we apply Lemma 4.7 (3) to find a standard shrinking S', X', Z'_1, Y'_1 of the one step dévissage of $\mathcal{F}/X/S$ at x with $X' \subset U$. If $n = 1$, then we are done. If $n > 1$, then by induction we can find a standard shrinking S'', Y''_1, Z''_k, and Y''_k of the complete dévissage $(Z_k, Y_k, i_k, \pi_k, \mathcal{G}_k, \alpha_k, z_k, y_k)_{k=2,\ldots,n}$ of $\text{Coker}(\alpha_1)/Y_1/S$ at x such that $Y''_1 \subset Y'_1$. Using Lemma 4.7 (2) we can find $S''' \subset S'$, $X''' \subset X'$, Z'''_1 and $Y'''_1 = Y''_1 \times_S S'''$ which is a standard shrinking. The solution to our problem is to take

$$S''', X''', Z'''_1, Y'''_1, Z''_2 \times_S S''', Y''_2 \times_S S''', \ldots, Z''_n \times_S S''', Y''_n \times_S S'''$$

This ends the proof of the lemma. $\square$

05HR **Proposition 5.7.** *Let S be a scheme. Let X be locally of finite type over S. Let $x \in X$ be a point with image $s \in S$. There exists a commutative diagram*

$$\begin{tikzcd}
(X, x) \arrow[d] & (X', x') \arrow[l, "g"'] \arrow[d] \\
(S, s) & (S', s') \arrow[l]
\end{tikzcd}$$

of pointed schemes such that the horizontal arrows are elementary étale neighbourhoods and such that $g^*\mathcal{F}/X'/S'$ has a complete dévissage at x.

Proof. We prove this by induction on the integer $d = \dim_x(\text{Supp}(\mathcal{F}_s))$. By Lemma 4.3 there exists a diagram

$$\begin{tikzcd} (X, x) \arrow[d] & (X', x') \arrow[l, "g"'] \arrow[d] \\ (S, s) & (S', s') \arrow[l] \end{tikzcd}$$

of pointed schemes such that the horizontal arrows are elementary étale neighbourhoods and such that $g^*\mathcal{F}/X'/S'$ has a one step dévissage at x'. The local nature of the problem implies that we may replace $(X, x) \to (S, s)$ by $(X', x') \to (S', s')$. Thus after doing so we may assume that there exists a one step dévissage $(Z_1, Y_1, i_1, \pi_1, \mathcal{G}_1)$ of $\mathcal{F}/X/S$ at x.

We apply Lemma 4.9 to find a map

$$\alpha_1 : \mathcal{O}_{Y_1}^{\oplus r_1} \longrightarrow \pi_{1,*}\mathcal{G}_1$$

which induces an isomorphism of vector spaces over $\kappa(\xi_1)$ where $\xi_1 \in Y_1$ is the unique generic point of the fibre of Y_1 over s. Moreover $\dim_{y_1}(\text{Supp}(\text{Coker}(\alpha_1)_s)) < d$. It may happen that the stalk of $\text{Coker}(\alpha_1)_s$ at y_1 is zero. In this case we may shrink Y_1 by Lemma 4.7 (2) and assume that $\text{Coker}(\alpha_1) = 0$ so we obtain a complete dévissage of length zero.

Assume now that the stalk of $\text{Coker}(\alpha_1)_s$ at y_1 is not zero. In this case, by induction, there exists a commutative diagram

05HS (5.7.1)
$$\begin{tikzcd} (Y_1, y_1) \arrow[d] & (Y_1', y_1') \arrow[l, "h"'] \arrow[d] \\ (S, s) & (S', s') \arrow[l] \end{tikzcd}$$

of pointed schemes such that the horizontal arrows are elementary étale neighbourhoods and such that $h^* \text{Coker}(\alpha_1)/Y_1'/S'$ has a complete dévissage

$$(Z_k, Y_k, i_k, \pi_k, \mathcal{G}_k, \alpha_k, z_k, y_k)_{k=2,\ldots,n}$$

at y_1'. (In particular $i_2 : Z_2 \to Y_1'$ is a closed immersion into Y_2'.) At this point we apply Lemma 4.8 to $S, X, \mathcal{F}, x, s$, the system $(Z_1, Y_1, i_1, \pi_1, \mathcal{G}_1)$ and diagram (5.7.1). We obtain a diagram

$$\begin{tikzcd} & & (X'', x'') \arrow[dll] \arrow[d] & (Z_1'', z_1'') \arrow[l] \arrow[dll] \arrow[d] \\ (X, x) \arrow[d] & (Z_1, z_1) \arrow[l] \arrow[d] & (S'', s'') \arrow[dll] & (Y_1'', y_1'') \arrow[l] \arrow[dll] \\ (S, s) & (Y_1, y_1) \arrow[l] & & \end{tikzcd}$$

with all the properties as listed in the referenced lemma. In particular $Y_1'' \subset Y_1' \times_{S'} S''$. Set $X_1 = Y_1' \times_{S'} S''$ and let $\mathcal{F}_1$ denote the pullback of $\text{Coker}(\alpha_1)$. By Lemma 5.4 the system

05HT (5.7.2)
$$(Z_k \times_{S'} S'', Y_k \times_{S'} S'', i_k'', \pi_k'', \mathcal{G}_k'', \alpha_k'', z_k'', y_k'')_{k=2,\ldots,n}$$

is a complete dévissage of $\mathcal{F}_1$ to X_1. Again, the nature of the problem allows us to replace $(X, x) \to (S, s)$ by $(X'', x'') \to (S'', s'')$. In this we see that we may assume:

(a) There exists a one step dévissage $(Z_1, Y_1, i_1, \pi_1, \mathcal{G}_1)$ of $\mathcal{F}/X/S$ at x,
(b) there exists an $\alpha_1 : \mathcal{O}_{Y_1}^{\oplus r_1} \to \pi_{1,*}\mathcal{G}_1$ such that $\alpha \otimes \kappa(\xi_1)$ is an isomorphism,
(c) $Y_1 \subset X_1$ is open, $y_1 = x_1$, and $\mathcal{F}_1|_{Y_1} \cong \mathrm{Coker}(\alpha_1)$, and
(d) there exists a complete dévissage $(Z_k, Y_k, i_k, \pi_k, \mathcal{G}_k, \alpha_k, z_k, y_k)_{k=2,\ldots,n}$ of $\mathcal{F}_1/X_1/S$ at x_1.

To finish the proof all we have to do is shrink the one step dévissage and the complete dévissage such that they fit together to a complete dévissage. (We suggest the reader do this on their own using Lemmas 4.7 and 5.6 instead of reading the proof that follows.) Since $Y_1 \subset X_1$ is an open neighbourhood of x_1 we may apply Lemma 5.6 (3) to find a standard shrinking $S', X_1', Z_2', Y_2', \ldots, Y_n'$ of the datum (d) so that $X_1' \subset Y_1$. Note that X_1' is also a standard open of the affine scheme Y_1. Next, we shrink the datum (a) as follows: first we shrink the base S to S', see Lemma 4.7 (1) and then we shrink the result to S'', X'', Z_1'', Y_1'' using Lemma 4.7 (2) such that eventually $Y_1'' = X_1' \times_S S''$ and $S'' \subset S'$. Then we see that

$$Z_1'', Y_1'', Z_2' \times_{S'} S'', Y_2' \times_{S'} S'', \ldots, Y_n' \times_{S'} S''$$

gives the complete dévissage we were looking for. $\square$

Some more bookkeeping gives the following consequence.

05HU **Lemma 5.8.** *Let $X \to S$ be a finite type morphism of schemes. Let $\mathcal{F}$ be a finite type quasi-coherent $\mathcal{O}_X$-module. Let $s \in S$ be a point. There exists an elementary étale neighbourhood $(S', s') \to (S, s)$ and étale morphisms $h_i : Y_i \to X_{S'}$, $i = 1, \ldots, n$ such that for each i there exists a complete dévissage of $\mathcal{F}_i/Y_i/S'$ over s', where $\mathcal{F}_i$ is the pullback of $\mathcal{F}$ to Y_i and such that $X_s = (X_{S'})_{s'} \subset \bigcup h_i(Y_i)$.*

Proof. For every point $x \in X_s$ we can find a diagram

$$\begin{tikzcd} (X, x) \arrow[d] & (X', x') \arrow[l, "g"'] \arrow[d] \\ (S, s) & (S', s') \arrow[l] \end{tikzcd}$$

of pointed schemes such that the horizontal arrows are elementary étale neighbourhoods and such that $g^*\mathcal{F}/X'/S'$ has a complete dévissage at x'. As $X \to S$ is of finite type the fibre X_s is quasi-compact, and since each $g : X' \to X$ as above is open we can cover X_s by a finite union of $g(X'_{s'})$. Thus we can find a finite family of such diagrams

$$\begin{tikzcd} (X, x) \arrow[d] & (X_i', x_i') \arrow[l, "g_i"'] \arrow[d] \\ (S, s) & (S_i', s_i') \arrow[l] \end{tikzcd} \quad i = 1, \ldots, n$$

such that $X_s = \bigcup g_i(X_i')$. Set $S' = S_1' \times_S \ldots \times_S S_n'$ and let $Y_i = X_i \times_{S_i'} S'$ be the base change of X_i' to S'. By Lemma 5.3 we see that the pullback of $\mathcal{F}$ to Y_i has a complete dévissage over s and we win. $\square$

6. Translation into algebra

05HV It may be useful to spell out algebraically what it means to have a complete dévissage. We introduce the following notion (which is not that useful so we give it an impossibly long name).

05HW **Definition 6.1.** Let $R \to S$ be a ring map. Let $\mathfrak q$ be a prime of S lying over the prime $\mathfrak p$ of R. A *elementary étale localization of the ring map $R \to S$ at $\mathfrak q$* is given by a commutative diagram of rings and accompanying primes

$$\begin{tikzcd} S \arrow[r] & S' \\ R \arrow[u] \arrow[r] & R' \arrow[u] \end{tikzcd} \qquad \begin{tikzcd} \mathfrak q \arrow[r, dash] \arrow[d, dash] & \mathfrak q' \arrow[d, dash] \\ \mathfrak p \arrow[r, dash] & \mathfrak p' \end{tikzcd}$$

such that $R \to R'$ and $S \to S'$ are étale ring maps and $\kappa(\mathfrak p) = \kappa(\mathfrak p')$ and $\kappa(\mathfrak q) = \kappa(\mathfrak q')$.

05HX **Definition 6.2.** Let $R \to S$ be a finite type ring map. Let $\mathfrak r$ be a prime of R. Let N be a finite S-module. A *complete dévissage of $N/S/R$ over $\mathfrak r$* is given by R-algebra maps

$$\begin{tikzcd}[column sep=small] & A_1 & & A_2 & & \ldots & & A_n & \\ S \arrow[ur] & & B_1 \arrow[ul] \arrow[ur] & & \ldots \arrow[ul] \arrow[ur] & & \ldots \arrow[ul] \arrow[ur] & & B_n \arrow[ul] \end{tikzcd}$$

finite A_i-modules M_i and B_i-module maps $\alpha_i : B_i^{\oplus r_i} \to M_i$ such that

(1) $S \to A_1$ is surjective and of finite presentation,
(2) $B_i \to A_{i+1}$ is surjective and of finite presentation,
(3) $B_i \to A_i$ is finite,
(4) $R \to B_i$ is smooth with geometrically irreducible fibres,
(5) $N \cong M_1$ as S-modules,
(6) $\text{Coker}(\alpha_i) \cong M_{i+1}$ as B_i-modules,
(7) $\alpha_i : \kappa(\mathfrak p_i)^{\oplus r_i} \to M_i \otimes_{B_i} \kappa(\mathfrak p_i)$ is an isomorphism where $\mathfrak p_i = \mathfrak r B_i$, and
(8) $\text{Coker}(\alpha_n) = 0$.

In this situation we say that $(A_i, B_i, M_i, \alpha_i)_{i=1,\ldots,n}$ is a complete dévissage of $N/S/R$ over $\mathfrak r$.

05HY **Remark 6.3.** Note that the R-algebras B_i for all i and A_i for $i \geq 2$ are of finite presentation over R. If S is of finite presentation over R, then it is also the case that A_1 is of finite presentation over R. In this case all the ring maps in the complete dévissage are of finite presentation. See Algebra, Lemma 6.2. Still assuming S of finite presentation over R the following are equivalent

(1) M is of finite presentation over S,
(2) M_1 is of finite presentation over A_1,
(3) M_1 is of finite presentation over B_1,
(4) each M_i is of finite presentation both as an A_i-module and as a B_i-module.

The equivalences (1) $\Leftrightarrow$ (2) and (2) $\Leftrightarrow$ (3) follow from Algebra, Lemma 35.23. If M_1 is finitely presented, so is $\text{Coker}(\alpha_1)$ (see Algebra, Lemma 5.3) and hence M_2, etc.

05HZ **Definition 6.4.** Let $R \to S$ be a finite type ring map. Let $\mathfrak q$ be a prime of S lying over the prime $\mathfrak r$ of R. Let N be a finite S-module. A *complete dévissage of $N/S/R$ at $\mathfrak q$* is given by a complete dévissage $(A_i, B_i, M_i, \alpha_i)_{i=1,\ldots,n}$ of $N/S/R$ over $\mathfrak r$ and prime ideals $\mathfrak q_i \subset B_i$ lying over $\mathfrak r$ such that

(1) $\kappa(\mathfrak r) \subset \kappa(\mathfrak q_i)$ is purely transcendental,
(2) there is a unique prime $\mathfrak q'_i \subset A_i$ lying over $\mathfrak q_i \subset B_i$,
(3) $\mathfrak q = \mathfrak q'_1 \cap S$ and $\mathfrak q_i = \mathfrak q'_{i+1} \cap A_i$,
(4) $R \to B_i$ has relative dimension $\dim_{\mathfrak q_i}(\text{Supp}(M_i \otimes_R \kappa(\mathfrak r)))$.

05I0 **Remark 6.5.** Let $A \to B$ be a finite type ring map and let N be a finite B-module. Let $\mathfrak q$ be a prime of B lying over the prime $\mathfrak r$ of A. Set $X = \text{Spec}(B)$, $S = \text{Spec}(A)$ and $\mathcal F = \widetilde N$ on X. Let x be the point corresponding to $\mathfrak q$ and let $s \in S$ be the point corresponding to $\mathfrak p$. Then

(1) if there exists a complete dévissage of $\mathcal F/X/S$ over s then there exists a complete dévissage of $N/B/A$ over $\mathfrak p$, and
(2) there exists a complete dévissage of $\mathcal F/X/S$ at x if and only if there exists a complete dévissage of $N/B/A$ at $\mathfrak q$.

There is just a small twist in that we omitted the condition on the relative dimension in the formulation of "a complete dévissage of $N/B/A$ over $\mathfrak p$" which is why the implication in (1) only goes in one direction. The notion of a complete dévissage at $\mathfrak q$ does have this condition built in. In any case we will only use that existence for $\mathcal F/X/S$ implies the existence for $N/B/A$.

05I1 **Lemma 6.6.** *Let $R \to S$ be a finite type ring map. Let M be a finite S-module. Let $\mathfrak q$ be a prime ideal of S. There exists an elementary étale localization $R' \to S', \mathfrak q', \mathfrak p'$ of the ring map $R \to S$ at $\mathfrak q$ such that there exists a complete dévissage of $(M \otimes_S S')/S'/R'$ at $\mathfrak q'$.*

Proof. This is a reformulation of Proposition 5.7 via Remark 6.5 $\square$

7. Localization and universally injective maps

05DD

05DE **Lemma 7.1.** *Let $R \to S$ be a ring map. Let N be a S-module. Assume*

(1) *R is a local ring with maximal ideal $\mathfrak m$,*
(2) *$\overline{S} = S/\mathfrak m S$ is Noetherian, and*
(3) *$\overline{N} = N/\mathfrak m_R N$ is a finite $\overline{S}$-module.*

Let $\Sigma \subset S$ be the multiplicative subset of elements which are not a zerodivisor on $\overline{N}$. Then $\Sigma^{-1}S$ is a semi-local ring whose spectrum consists of primes $\mathfrak q \subset S$ contained in an element of $Ass_S(\overline{N})$. Moreover, any maximal ideal of $\Sigma^{-1}S$ corresponds to an associated prime of $\overline{N}$ over $\overline{S}$.

Proof. Note that $\text{Ass}_S(\overline{N}) = \text{Ass}_{\overline{S}}(\overline{N})$, see Algebra, Lemma 62.14. This is a finite set by Algebra, Lemma 62.5. Say $\{\mathfrak q_1, \ldots, \mathfrak q_r\} = \text{Ass}_S(\overline{N})$. We have $\Sigma = S \setminus (\bigcup \mathfrak q_i)$ by Algebra, Lemma 62.9. By the description of $\text{Spec}(\Sigma^{-1}S)$ in Algebra, Lemma 16.5 and by Algebra, Lemma 14.2 we see that the primes of $\Sigma^{-1}S$ correspond to the primes of S contained in one of the $\mathfrak q_i$. Hence the maximal ideals of $\Sigma^{-1}S$ correspond one-to-one with the maximal (w.r.t. inclusion) elements of the set $\{\mathfrak q_1, \ldots, \mathfrak q_r\}$. This proves the lemma. $\square$

05DF **Lemma 7.2.** *Assumption and notation as in Lemma 7.1. Assume moreover that*

1. *S is local and $R \to S$ is a local homomorphism,*
2. *S is essentially of finite presentation over R,*
3. *N is finitely presented over S, and*
4. *N is flat over R.*

Then each $s \in \Sigma$ defines a universally injective R-module map $s : N \to N$, and the map $N \to \Sigma^{-1}N$ is R-universally injective.

Proof. By Algebra, Lemma 127.4 the sequence $0 \to N \to N \to N/sN \to 0$ is exact and N/sN is flat over R. This implies that $s : N \to N$ is universally injective, see Algebra, Lemma 38.12. The map $N \to \Sigma^{-1}N$ is universally injective as the directed colimit of the maps $s : N \to N$. $\square$

05DG **Lemma 7.3.** *Let $R \to S$ be a ring map. Let N be an S-module. Let $S \to S'$ be a ring map. Assume*

1. *$R \to S$ is a local homomorphism of local rings*
2. *S is essentially of finite presentation over R,*
3. *N is of finite presentation over S,*
4. *N is flat over R,*
5. *$S \to S'$ is flat, and*
6. *the image of $\mathrm{Spec}(S') \to \mathrm{Spec}(S)$ contains all primes $\mathfrak{q}$ of S lying over $\mathfrak{m}_R$ such that $\mathfrak{q}$ is an associated prime of $N/\mathfrak{m}_R N$.*

Then $N \to N \otimes_S S'$ is R-universally injective.

Proof. Set $N' = N \otimes_R S'$. Consider the commutative diagram

$$\begin{tikzcd} N \arrow[r] \arrow[d] & N' \arrow[d] \\ \Sigma^{-1}N \arrow[r] & \Sigma^{-1}N' \end{tikzcd}$$

where $\Sigma \subset S$ is the set of elements which are not a zerodivisor on $N/\mathfrak{m}_R N$. If we can show that the map $N \to \Sigma^{-1}N'$ is universally injective, then $N \to N'$ is too (see Algebra, Lemma 81.10).

By Lemma 7.1 the ring $\Sigma^{-1}S$ is a semi-local ring whose maximal ideals correspond to associated primes of $N/\mathfrak{m}_R N$. Hence the image of $\mathrm{Spec}(\Sigma^{-1}S') \to \mathrm{Spec}(\Sigma^{-1}S)$ contains all these maximal ideals by assumption. By Algebra, Lemma 38.16 the ring map $\Sigma^{-1}S \to \Sigma^{-1}S'$ is faithfully flat. Hence $\Sigma^{-1}N \to \Sigma^{-1}N'$, which is the map

$$N \otimes_S \Sigma^{-1}S \longrightarrow N \otimes_S \Sigma^{-1}S'$$

is universally injective, see Algebra, Lemmas 81.11 and 81.8. Finally, we apply Lemma 7.2 to see that $N \to \Sigma^{-1}N$ is universally injective. As the composition of universally injective module maps is universally injective (see Algebra, Lemma 81.9) we conclude that $N \to \Sigma^{-1}N'$ is universally injective and we win. $\square$

05DH **Lemma 7.4.** *Let $R \to S$ be a ring map. Let N be an S-module. Let $S \to S'$ be a ring map. Assume*

1. *$R \to S$ is of finite presentation and N is of finite presentation over S,*
2. *N is flat over R,*
3. *$S \to S'$ is flat, and*

(4) *the image of* $\mathrm{Spec}(S') \to \mathrm{Spec}(S)$ *contains all primes* $\mathfrak{q}$ *such that* $\mathfrak{q}$ *is an associated prime of* $N \otimes_R \kappa(\mathfrak{p})$ *where* $\mathfrak{p}$ *is the inverse image of* $\mathfrak{q}$ *in* R.

Then $N \to N \otimes_S S'$ *is* R*-universally injective.*

Proof. By Algebra, Lemma 81.12 it suffices to show that $N_\mathfrak{q} \to (N \otimes_R S')_\mathfrak{q}$ is a $R_\mathfrak{p}$-universally injective for any prime $\mathfrak{q}$ of S lying over $\mathfrak{p}$ in R. Thus we may apply Lemma 7.3 to the ring maps $R_\mathfrak{p} \to S_\mathfrak{q} \to S'_\mathfrak{q}$ and the module $N_\mathfrak{q}$. □

The reader may want to compare the following lemma to Algebra, Lemmas 98.1 and 127.4 and the results of Section 25. In each case the conclusion is that the map $u : M \to N$ is universally injective with flat cokernel.

05FQ **Lemma 7.5.** *Let* $(R, \mathfrak{m})$ *be a local ring. Let* $u : M \to N$ *be an* R*-module map. If* M *is a projective* R*-module,* N *is a flat* R*-module, and* $\overline{u} : M/\mathfrak{m}M \to N/\mathfrak{m}N$ *is injective then* u *is universally injective.*

Proof. By Algebra, Theorem 84.4 the module M is free. If we show the result holds for every finitely generated direct summand of M, then the lemma follows. Hence we may assume that M is finite free. Write $N = \mathrm{colim}_i N_i$ as a directed colimit of finite free modules, see Algebra, Theorem 80.4. Note that $u : M \to N$ factors through N_i for some i (as M is finite free). Denote $u_i : M \to N_i$ the corresponding R-module map. As $\overline{u}$ is injective we see that $\overline{u_i} : M/\mathfrak{m}M \to N_i/\mathfrak{m}N_i$ is injective and remains injective on composing with the maps $N_i/\mathfrak{m}N_i \to N_{i'}/\mathfrak{m}N_{i'}$ for all $i' \geq i$. As M and $N_{i'}$ are finite free over the local ring R this implies that $M \to N_{i'}$ is a split injection for all $i' \geq i$. Hence for any R-module Q we see that $M \otimes_R Q \to N_{i'} \otimes_R Q$ is injective for all $i' \geq i$. As $- \otimes_R Q$ commutes with colimits we conclude that $M \otimes_R Q \to N_{i'} \otimes_R Q$ is injective as desired. □

05FR **Lemma 7.6.** *Assumption and notation as in Lemma 7.1. Assume moreover that* N *is projective as an* R*-module. Then each* $s \in \Sigma$ *defines a universally injective* R*-module map* $s : N \to N$*, and the map* $N \to \Sigma^{-1}N$ *is* R*-universally injective.*

Proof. Pick $s \in \Sigma$. By Lemma 7.5 the map $s : N \to N$ is universally injective. The map $N \to \Sigma^{-1}N$ is universally injective as the directed colimit of the maps $s : N \to N$. □

8. Completion and Mittag-Leffler modules

05DI

05DJ **Lemma 8.1.** *Let* R *be a ring. Let* $I \subset R$ *be an ideal. Let* A *be a set. Assume* R *is Noetherian and complete with respect to* I*. The completion* $(\bigoplus_{\alpha \in A} R)^\wedge$ *is flat and Mittag-Leffler.*

Proof. By More on Algebra, Lemma 25.1 the map $(\bigoplus_{\alpha \in A} R)^\wedge \to \prod_{\alpha \in A} R$ is universally injective. Thus, by Algebra, Lemmas 81.7 and 88.7 it suffices to show that $\prod_{\alpha \in A} R$ is flat and Mittag-Leffler. By Algebra, Proposition 89.5 (and Algebra, Lemma 89.4) we see that $\prod_{\alpha \in A} R$ is flat. Thus we conclude because a product of copies of R is Mittag-Leffler, see Algebra, Lemma 90.3. □

05DK **Lemma 8.2.** *Let* R *be a ring. Let* $I \subset R$ *be an ideal. Let* M *be an* R*-module. Assume*

(1) R *is Noetherian and* I*-adically complete,*
(2) M *is flat over* R*, and*

(3) *M/IM is a projective R/I-module.*

Then the I-adic completion $M^\wedge$ is a flat Mittag-Leffler R-module.

Proof. Choose a surjection $F \to M$ where F is a free R-module. By Algebra, Lemma 96.9 the module $M^\wedge$ is a direct summand of the module $F^\wedge$. Hence it suffices to prove the lemma for F. In this case the lemma follows from Lemma 8.1. $\square$

In Lemmas 8.3 and 8.4 the assumption that S be Noetherian holds if $R \to S$ is of finite type, see Algebra, Lemma 30.1.

05DL **Lemma 8.3.** *Let R be a ring. Let $I \subset R$ be an ideal. Let $R \to S$ be a ring map, and N an S-module. Assume*

(1) *R is a Noetherian ring,*
(2) *S is a Noetherian ring,*
(3) *N is a finite S-module, and*
(4) *for any finite R-module Q, any $\mathfrak q \in Ass_S(Q \otimes_R N)$ satisfies $IS + \mathfrak q \neq S$.*

Then the map $N \to N^\wedge$ of N into the I-adic completion of N is universally injective as a map of R-modules.

Proof. We have to show that for any finite R-module Q the map $Q \otimes_R N \to Q \otimes_R N^\wedge$ is injective, see Algebra, Theorem 81.3. As there is a canonical map $Q \otimes_R N^\wedge \to (Q \otimes_R N)^\wedge$ it suffices to prove that the canonical map $Q \otimes_R N \to (Q \otimes_R N)^\wedge$ is injective. Hence we may replace N by $Q \otimes_R N$ and it suffices to prove the injectivity for the map $N \to N^\wedge$.

Let $K = \text{Ker}(N \to N^\wedge)$. It suffices to show that $K_\mathfrak q = 0$ for $\mathfrak q \in \text{Ass}(N)$ as N is a submodule of $\prod_{\mathfrak q \in \text{Ass}(N)} N_\mathfrak q$, see Algebra, Lemma 62.19. Pick $\mathfrak q \in \text{Ass}(N)$. By the last assumption we see that there exists a prime $\mathfrak q' \supset IS + \mathfrak q$. Since $K_\mathfrak q$ is a localization of $K_{\mathfrak q'}$ it suffices to prove the vanishing of $K_{\mathfrak q'}$. Note that $K = \bigcap I^n N$, hence $K_{\mathfrak q'} \subset \bigcap I^n N_{\mathfrak q'}$. Hence $K_{\mathfrak q'} = 0$ by Algebra, Lemma 50.4. $\square$

05DM **Lemma 8.4.** *Let R be a ring. Let $I \subset R$ be an ideal. Let $R \to S$ be a ring map, and N an S-module. Assume*

(1) *R is a Noetherian ring,*
(2) *S is a Noetherian ring,*
(3) *N is a finite S-module,*
(4) *N is flat over R, and*
(5) *for any prime $\mathfrak q \subset S$ which is an associated prime of $N \otimes_R \kappa(\mathfrak p)$ where $\mathfrak p = R \cap \mathfrak q$ we have $IS + \mathfrak q \neq S$.*

Then the map $N \to N^\wedge$ of N into the I-adic completion of N is universally injective as a map of R-modules.

Proof. This follows from Lemma 8.3 because Algebra, Lemma 64.5 and Remark 64.6 guarantee that the set of associated primes of tensor products $N \otimes_R Q$ are contained in the set of associated primes of the modules $N \otimes_R \kappa(\mathfrak p)$. $\square$

9. Projective modules

05DN The following lemma can be used to prove projectivity by Noetherian induction on the base, see Lemma 9.2.

05DP **Lemma 9.1.** *Let R be a ring. Let $I \subset R$ be an ideal. Let $R \to S$ be a ring map, and N an S-module. Assume*

(1) *R is Noetherian and I-adically complete,*
(2) *$R \to S$ is of finite type,*
(3) *N is a finite S-module,*
(4) *N is flat over R,*
(5) *N/IN is projective as a R/I-module, and*
(6) *for any prime $\mathfrak{q} \subset S$ which is an associated prime of $N \otimes_R \kappa(\mathfrak{p})$ where $\mathfrak{p} = R \cap \mathfrak{q}$ we have $IS + \mathfrak{q} \neq S$.*

Then N is projective as an R-module.

Proof. By Lemma 8.4 the map $N \to N^\wedge$ is universally injective. By Lemma 8.2 the module $N^\wedge$ is Mittag-Leffler. By Algebra, Lemma 88.7 we conclude that N is Mittag-Leffler. Hence N is countably generated, flat and Mittag-Leffler as an R-module, whence projective by Algebra, Lemma 92.1. $\square$

05FS **Lemma 9.2.** *Let R be a ring. Let $R \to S$ be a ring map. Assume*

(1) *R is Noetherian,*
(2) *$R \to S$ is of finite type and flat, and*
(3) *every fibre ring $S \otimes_R \kappa(\mathfrak{p})$ is geometrically integral over $\kappa(\mathfrak{p})$.*

Then S is projective as an R-module.

Proof. Consider the set

$$\{I \subset R \mid S/IS \text{ not projective as } R/I\text{-module}\}$$

We have to show this set is empty. To get a contradiction assume it is nonempty. Then it contains a maximal element I. Let $J = \sqrt{I}$ be its radical. If $I \neq J$, then S/JS is projective as a R/J-module, and S/IS is flat over R/I and J/I is a nilpotent ideal in R/I. Applying Algebra, Lemma 76.6 we see that S/IS is a projective R/I-module, which is a contradiction. Hence we may assume that I is a radical ideal. In other words we are reduced to proving the lemma in case R is a reduced ring and S/IS is a projective R/I-module for every nonzero ideal I of R.

Assume R is a reduced ring and S/IS is a projective R/I-module for every nonzero ideal I of R. By generic flatness, Algebra, Lemma 117.1 (applied to a localization R_g which is a domain) or the more general Algebra, Lemma 117.7 there exists a nonzero $f \in R$ such that S_f is free as an R_f-module. Denote $R^\wedge = \lim R/(f^n)$ the (f)-adic completion of R. Note that the ring map

$$R \longrightarrow R_f \times R^\wedge$$

is a faithfully flat ring map, see Algebra, Lemma 96.2. Hence by faithfully flat descent of projectivity, see Algebra, Theorem 94.5 it suffices to prove that $S \otimes_R R^\wedge$ is a projective $R^\wedge$-module. To see this we will use the criterion of Lemma 9.1. First of all, note that $S/fS = (S \otimes_R R^\wedge)/f(S \otimes_R R^\wedge)$ is a projective $R/(f)$-module and that $S \otimes_R R^\wedge$ is flat and of finite type over $R^\wedge$ as a base change of such. Next, suppose that $\mathfrak{p}^\wedge$ is a prime ideal of $R^\wedge$. Let $\mathfrak{p} \subset R$ be the corresponding prime of R. As $R \to S$ has geometrically integral fibre rings, the same is true for the fibre rings of any base change. Hence $\mathfrak{q}^\wedge = \mathfrak{p}^\wedge(S \otimes_R R^\wedge)$, is a prime ideals lying over $\mathfrak{p}^\wedge$ and it is the unique associated prime of $S \otimes_R \kappa(\mathfrak{p}^\wedge)$. Thus we win if $f(S \otimes_R R^\wedge) + \mathfrak{q}^\wedge \neq S \otimes_R R^\wedge$. This is true because $\mathfrak{p}^\wedge + fR^\wedge \neq R^\wedge$ as f lies in the radical of the f-adically complete ring $R^\wedge$ and because $R^\wedge \to S \otimes_R R^\wedge$ is surjective on spectra as its fibres are nonempty (irreducible spaces are nonempty). $\square$

05FT **Lemma 9.3.** *Let R be a ring. Let $R \to S$ be a ring map. Assume*

1. *$R \to S$ is of finite presentation and flat, and*
2. *every fibre ring $S \otimes_R \kappa(\mathfrak{p})$ is geometrically integral over $\kappa(\mathfrak{p})$.*

Then S is projective as an R-module.

Proof. We can find a cocartesian diagram of rings

$$\begin{tikzcd} S_0 \arrow[r] & S \\ R_0 \arrow[u] \arrow[r] & R \arrow[u] \end{tikzcd}$$

such that R_0 is of finite type over $\mathbf{Z}$, the map $R_0 \to S_0$ is of finite type and flat with geometrically integral fibres, see More on Morphisms, Lemmas 30.4, 30.6, 30.7, and 30.11. By Lemma 9.2 we see that S_0 is a projective R_0-module. Hence $S = S_0 \otimes_{R_0} R$ is a projective R-module, see Algebra, Lemma 93.1. $\square$

05FU **Remark 9.4.** Lemma 9.3 is a key step in the development of results in this chapter. The analogue of this lemma in [GR71] is [GR71, I Proposition 3.3.1]: If $R \to S$ is smooth with geometrically integral fibres, then S is projective as an R-module. This is a special case of Lemma 9.3, but as we will later improve on this lemma anyway, we do not gain much from having a stronger result at this point. We briefly sketch the proof of this as it is given in [GR71].

1. First reduce to the case where R is Noetherian as above.
2. Since projectivity descends through faithfully flat ring maps, see Algebra, Theorem 94.5 we may work locally in the fppf topology on R, hence we may assume that $R \to S$ has a section $\sigma : S \to R$. (Just by the usual trick of base changing to S.) Set $I = \mathrm{Ker}(S \to R)$.
3. Localizing a bit more on R we may assume that I/I^2 is a free R-module and that the completion $S^\wedge$ of S with respect to I is isomorphic to $R[[t_1, \ldots, t_n]]$, see Morphisms, Lemma 32.20. Here we are using that $R \to S$ is smooth.
4. To prove that S is projective as an R-module, it suffices to prove that S is flat, countably generated and Mittag-Leffler as an R-module, see Algebra, Lemma 92.1. The first two properties are evident. Thus it suffices to prove that S is Mittag-Leffler as an R-module. By Algebra, Lemma 90.4 the module $R[[t_1, \ldots, t_n]]$ is Mittag-Leffler over R. Hence Algebra, Lemma 88.7 shows that it suffices to show that the $S \to S^\wedge$ is universally injective as a map of R-modules.
5. Apply Lemma 7.4 to see that $S \to S^\wedge$ is R-universally injective. Namely, as $R \to S$ has geometrically integral fibres, any associated point of any fibre ring is just the generic point of the fibre ring which is in the image of $\mathrm{Spec}(S^\wedge) \to \mathrm{Spec}(S)$.

There is an analogy between the proof as sketched just now, and the development of the arguments leading to the proof of Lemma 9.3. In both a completion plays an essential role, and both times the assumption of having geometrically integral fibres assures one that the map from S to the completion of S is R-universally injective.

10. Flat finite type modules, Part I

05I2 In some cases given a ring map $R \to S$ of finite presentation and a finite S-module N the flatness of N over R implies that N is of finite presentation. In this section we prove this is true "pointwise". We remark that the first proof of Proposition 10.3 uses the geometric results of Section 3 but not the existence of a complete dévissage.

05I3 **Lemma 10.1.** *Let $(R, \mathfrak{m})$ be a local ring. Let $R \to S$ be a finitely presented flat ring map with geometrically integral fibres. Write $\mathfrak{p} = \mathfrak{m}S$. Let $\mathfrak{q} \subset S$ be a prime ideal lying over $\mathfrak{m}$. Let N be a finite S-module. There exist $r \geq 0$ and an S-module map*

$$\alpha : S^{\oplus r} \longrightarrow N$$

such that $\alpha : \kappa(\mathfrak{p})^{\oplus r} \to N \otimes_S \kappa(\mathfrak{p})$ is an isomorphism. For any such α the following are equivalent:

1. *$N_{\mathfrak{q}}$ is R-flat,*
2. *α is R-universally injective and $\mathrm{Coker}(\alpha)_{\mathfrak{q}}$ is R-flat,*
3. *α is injective and $\mathrm{Coker}(\alpha)_{\mathfrak{q}}$ is R-flat,*
4. *$\alpha_{\mathfrak{p}}$ is an isomorphism and $\mathrm{Coker}(\alpha)_{\mathfrak{q}}$ is R-flat, and*
5. *$\alpha_{\mathfrak{q}}$ is injective and $\mathrm{Coker}(\alpha)_{\mathfrak{q}}$ is R-flat.*

Proof. To obtain α set $r = \dim_{\kappa(\mathfrak{p})} N \otimes_S \kappa(\mathfrak{p})$ and pick $x_1, \ldots, x_r \in N$ which form a basis of $N \otimes_S \kappa(\mathfrak{p})$. Define $\alpha(s_1, \ldots, s_r) = \sum s_i x_i$. This proves the existence.

Fix an α. The most interesting implication is (1) $\Rightarrow$ (2) which we prove first. Assume (1). Because $S/\mathfrak{m}S$ is a domain with fraction field $\kappa(\mathfrak{p})$ we see that $(S/\mathfrak{m}S)^{\oplus r} \to N_{\mathfrak{p}}/\mathfrak{m}N_{\mathfrak{p}} = N \otimes_S \kappa(\mathfrak{p})$ is injective. Hence by Lemmas 7.5 and 9.3. the map $S^{\oplus r} \to N_{\mathfrak{p}}$ is R-universally injective. It follows that $S^{\oplus r} \to N$ is R-universally injective, see Algebra, Lemma 81.10. Then also the localization $\alpha_{\mathfrak{q}}$ is R-universally injective, see Algebra, Lemma 81.13. We conclude that $\mathrm{Coker}(\alpha)_{\mathfrak{q}}$ is R-flat by Algebra, Lemma 81.7.

The implication (2) $\Rightarrow$ (3) is immediate. If (3) holds, then $\alpha_{\mathfrak{p}}$ is injective as a localization of an injective module map. By Nakayama's lemma (Algebra, Lemma 19.1) $\alpha_{\mathfrak{p}}$ is surjective too. Hence (3) $\Rightarrow$ (4). If (4) holds, then $\alpha_{\mathfrak{p}}$ is an isomorphism, so α is injective as $S_{\mathfrak{q}} \to S_{\mathfrak{p}}$ is injective. Namely, elements of $S \setminus \mathfrak{p}$ are nonzerodivisors on S by a combination of Lemmas 7.6 and 9.3. Hence (4) $\Rightarrow$ (5). Finally, if (5) holds, then $N_{\mathfrak{q}}$ is R-flat as an extension of flat modules, see Algebra, Lemma 38.13. Hence (5) $\Rightarrow$ (1) and the proof is finished. $\square$

05I4 **Lemma 10.2.** *Let $(R, \mathfrak{m})$ be a local ring. Let $R \to S$ be a ring map of finite presentation. Let N be a finite S-module. Let $\mathfrak{q}$ be a prime of S lying over $\mathfrak{m}$. Assume that $N_{\mathfrak{q}}$ is flat over R, and assume there exists a complete dévissage of $N/S/R$ at $\mathfrak{q}$. Then N is a finitely presented S-module, free as an R-module, and there exists an isomorphism*

$$N \cong B_1^{\oplus r_1} \oplus \ldots \oplus B_n^{\oplus r_n}$$

as R-modules where each B_i is a smooth R-algebra with geometrically irreducible fibres.

Proof. Let $(A_i, B_i, M_i, \alpha_i, \mathfrak{q}_i)_{i=1,\ldots,n}$ be the given complete dévissage. We prove the lemma by induction on n. Note that N is finitely presented as an S-module if and only if M_1 is finitely presented as an B_1-module, see Remark 6.3. Note that $N_\mathfrak{q} \cong (M_1)_{\mathfrak{q}_1}$ as R-modules because (a) $N_\mathfrak{q} \cong (M_1)_{\mathfrak{q}'_1}$ where $\mathfrak{q}'_1$ is the unique prime in A_1 lying over $\mathfrak{q}_1$ and (b) $(A_1)_{\mathfrak{q}'_1} = (A_1)_{\mathfrak{q}_1}$ by Algebra, Lemma 40.11, so (c) $(M_1)_{\mathfrak{q}'_1} \cong (M_1)_{\mathfrak{q}_1}$. Hence $(M_1)_{\mathfrak{q}_1}$ is a flat R-module. Thus we may replace (S, N) by (B_1, M_1) in order to prove the lemma. By Lemma 10.1 the map $\alpha_1 : B_1^{\oplus r_1} \to M_1$ is R-universally injective and $\mathrm{Coker}(\alpha_1)_\mathfrak{q}$ is R-flat. Note that $(A_i, B_i, M_i, \alpha_i, \mathfrak{q}_i)_{i=2,\ldots,n}$ is a complete dévissage of $\mathrm{Coker}(\alpha_1)/B_1/R$ at $\mathfrak{q}_1$. Hence the induction hypothesis implies that $\mathrm{Coker}(\alpha_1)$ is finitely presented as a B_1-module, free as an R-module, and has a decomposition as in the lemma. This implies that M_1 is finitely presented as a B_1-module, see Algebra, Lemma 5.3. It further implies that $M_1 \cong B_1^{\oplus r_1} \oplus \mathrm{Coker}(\alpha_1)$ as R-modules, hence a decomposition as in the lemma. Finally, B_1 is projective as an R-module by Lemma 9.3 hence free as an R-module by Algebra, Theorem 84.4. This finishes the proof. □

05I5 **Proposition 10.3.** *Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Let $x \in X$ with image $s \in S$. Assume that*

1. *f is locally of finite presentation,*
2. *$\mathcal{F}$ is of finite type, and*
3. *$\mathcal{F}$ is flat at x over S.*

Then there exists an elementary étale neighbourhood $(S', s') \to (S, s)$ and an open subscheme

$$V \subset X \times_S \mathrm{Spec}(\mathcal{O}_{S',s'})$$

which contains the unique point of $X \times_S \mathrm{Spec}(\mathcal{O}_{S',s'})$ mapping to x such that the pullback of $\mathcal{F}$ to V is an $\mathcal{O}_V$-module of finite presentation and flat over $\mathcal{O}_{S',s'}$.

First proof. This proof is longer but does not use the existence of a complete dévissage. The problem is local around x and s, hence we may assume that X and S are affine. During the proof we will finitely many times replace S by an elementary étale neighbourhood of (S, s). The goal is then to find (after such a replacement) an open $V \subset X \times_S \mathrm{Spec}(\mathcal{O}_{S,s})$ containing x such that $\mathcal{F}|_V$ is flat over S and finitely presented. Of course we may also replace S by $\mathrm{Spec}(\mathcal{O}_{S,s})$ at any point of the proof, i.e., we may assume S is a local scheme. We will prove the proposition by induction on the integer $n = \dim_x(\mathrm{Supp}(\mathcal{F}_s))$.

We can choose

1. elementary étale neighbourhoods $g : (X', x') \to (X, x)$, $e : (S', s') \to (S, s)$,
2. a commutative diagram

$$\begin{tikzcd}
X \arrow[ddd, "f"'] & X' \arrow[l, "g"'] \arrow[ddd] & Z' \arrow[l, "i"'] \arrow[d, "\pi"] \\
 & & Y' \arrow[dd, "h"] \\
 & & \\
S & S' \arrow[l, "e"'] \arrow[r, equal] & S'
\end{tikzcd}$$

3. a point $z' \in Z'$ with $i(z') = x'$, $y' = \pi(z')$, $h(y') = s'$,
4. a finite type quasi-coherent $\mathcal{O}_{Z'}$-module $\mathcal{G}$,

as in Lemma 3.2. We are going to replace S by $\mathrm{Spec}(\mathcal{O}_{S',s'})$, see remarks in first paragraph of the proof. Consider the diagram

$$\begin{tikzcd} X_{\mathcal{O}_{S',s'}} \arrow[rdd, "f"'] & X'_{\mathcal{O}_{S',s'}} \arrow[l, "g"] \arrow[dd] & Z'_{\mathcal{O}_{S',s'}} \arrow[l, "i"] \arrow[d, "\pi"] \\ & & Y'_{\mathcal{O}_{S',s'}} \arrow[ld, "h"] \\ & \mathrm{Spec}(\mathcal{O}_{S',s'}) & \end{tikzcd}$$

Here we have base changed the schemes X', Z', Y' over S' via $\mathrm{Spec}(\mathcal{O}_{S',s'}) \to S'$ and the scheme X over S via $\mathrm{Spec}(\mathcal{O}_{S',s'}) \to S$. It is still the case that g is étale, see Lemma 2.2. After replacing X by $X_{\mathcal{O}_{S',s'}}$, X' by $X'_{\mathcal{O}_{S',s'}}$, Z' by $Z'_{\mathcal{O}_{S',s'}}$, and Y' by $Y'_{\mathcal{O}_{S',s'}}$ we may assume we have a diagram as Lemma 3.2 where in addition $S = S'$ is a local scheme with closed point s. By Lemmas 3.3 and 3.4 the result for $Y' \to S$, the sheaf $\pi_*\mathcal{G}$, and the point y' implies the result for $X \to S$, $\mathcal{F}$ and x. Hence we may assume that S is local and $X \to S$ is a smooth morphism of affines with geometrically irreducible fibres of dimension n.

The base case of the induction: $n = 0$. As $X \to S$ is smooth with geometrically irreducible fibres of dimension 0 we see that $X \to S$ is an open immersion, see Descent, Lemma 22.2. As S is local and the closed point is in the image of $X \to S$ we conclude that $X = S$. Thus we see that $\mathcal{F}$ corresponds to a finite flat $\mathcal{O}_{S,s}$ module. In this case the result follows from Algebra, Lemma 77.4 which tells us that $\mathcal{F}$ is in fact finite free.

The induction step. Assume the result holds whenever the dimension of the support in the closed fibre is $< n$. Write $S = \mathrm{Spec}(A)$, $X = \mathrm{Spec}(B)$ and $\mathcal{F} = \widetilde{N}$ for some B-module N. Note that A is a local ring; denote its maximal ideal $\mathfrak{m}$. Then $\mathfrak{p} = \mathfrak{m}B$ is the unique minimal prime lying over $\mathfrak{m}$ as $X \to S$ has geometrically irreducible fibres. Finally, let $\mathfrak{q} \subset B$ be the prime corresponding to x. By Lemma 10.1 we can choose a map

$$\alpha : B^{\oplus r} \to N$$

such that $\kappa(\mathfrak{p})^{\oplus r} \to N \otimes_B \kappa(\mathfrak{p})$ is an isomorphism. Moreover, as $N_{\mathfrak{q}}$ is A-flat the lemma also shows that α is injective and that $\mathrm{Coker}(\alpha)_{\mathfrak{q}}$ is A-flat. Set $Q = \mathrm{Coker}(\alpha)$. Note that the support of $Q/\mathfrak{m}Q$ does not contain $\mathfrak{p}$. Hence it is certainly the case that $\dim_{\mathfrak{q}}(\mathrm{Supp}(Q/\mathfrak{m}Q)) < n$. Combining everything we know about Q we see that the induction hypothesis applies to Q. It follows that there exists an elementary étale morphism $(S', s) \to (S, s)$ such that the conclusion holds for $Q \otimes_A A'$ over $B \otimes_A A'$ where $A' = \mathcal{O}_{S',s'}$. After replacing A by A' we have an exact sequence

$$0 \to B^{\oplus r} \to N \to Q \to 0$$

(here we use that α is injective as mentioned above) of finite B-modules and we also get an element $g \in B$, $g \notin \mathfrak{q}$ such that Q_g is finitely presented over B_g and flat over A. Since localization is exact we see that

$$0 \to B_g^{\oplus r} \to N_g \to Q_g \to 0$$

is still exact. As B_g and Q_g are flat over A we conclude that N_g is flat over A, see Algebra, Lemma 38.13, and as B_g and Q_g are finitely presented over B_g the same holds for N_g, see Algebra, Lemma 5.3. □

Second proof. We apply Proposition 5.7 to find a commutative diagram

$$\begin{tikzcd} (X, x) \arrow[d] & (X', x') \arrow[l, "g"'] \arrow[d] \\ (S, s) & (S', s') \arrow[l] \end{tikzcd}$$

of pointed schemes such that the horizontal arrows are elementary étale neighbourhoods and such that $g^*\mathcal{F}/X'/S'$ has a complete dévissage at x. (In particular S' and X' are affine.) By Morphisms, Lemma 24.12 we see that $g^*\mathcal{F}$ is flat at x' over S and by Lemma 2.3 we see that it is flat at x' over S'. Via Remark 6.5 we deduce that

$$\Gamma(X', g^*\mathcal{F})/\Gamma(X', \mathcal{O}_{X'})/\Gamma(S', \mathcal{O}_{S'})$$

has a complete dévissage at the prime of $\Gamma(X', \mathcal{O}_{X'})$ corresponding to x'. We may base change this complete dévissage to the local ring $\mathcal{O}_{S',s'}$ of $\Gamma(S', \mathcal{O}_{S'})$ at the prime corresponding to s'. Thus Lemma 10.2 implies that

$$\Gamma(X', \mathcal{F}') \otimes_{\Gamma(S', \mathcal{O}_{S'})} \mathcal{O}_{S',s'}$$

is flat over $\mathcal{O}_{S',s'}$ and of finite presentation over $\Gamma(X', \mathcal{O}_{X'}) \otimes_{\Gamma(S', \mathcal{O}_{S'})} \mathcal{O}_{S',s'}$. In other words, the restriction of $\mathcal{F}$ to $X' \times_{S'} \mathrm{Spec}(\mathcal{O}_{S',s'})$ is of finite presentation and flat over $\mathcal{O}_{S',s'}$. Since the morphism $X' \times_{S'} \mathrm{Spec}(\mathcal{O}_{S',s'}) \to X \times_S \mathrm{Spec}(\mathcal{O}_{S',s'})$ is étale (Lemma 2.2) its image $V \subset X \times_S \mathrm{Spec}(\mathcal{O}_{S',s'})$ is an open subscheme, and by étale descent the restriction of $\mathcal{F}$ to V is of finite presentation and flat over $\mathcal{O}_{S',s'}$. (Results used: Morphisms, Lemma 34.13, Descent, Lemma 7.3, and Morphisms, Lemma 24.12.) □

05M9 **Lemma 10.4.** *Let $f : X \to S$ be a morphism of schemes which is locally of finite type. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module of finite type. Let $s \in S$. Then the set*

$$\{x \in X_s \mid \mathcal{F} \text{ flat over } S \text{ at } x\}$$

is open in the fibre X_s.

Proof. Suppose $x \in U$. Choose an elementary étale neighbourhood $(S', s') \to (S, s)$ and open $V \subset X \times_S \mathrm{Spec}(\mathcal{O}_{S',s'})$ as in Proposition 10.3. Note that $X_{s'} = X_s$ as $\kappa(s) = \kappa(s')$. If $x' \in V \cap X_{s'}$, then the pullback of $\mathcal{F}$ to $X \times_S S'$ is flat over S' at x'. Hence $\mathcal{F}$ is flat at x' over S, see Morphisms, Lemma 24.12. In other words $X_s \cap V \subset U$ is an open neighbourhood of x in U. □

05KT **Lemma 10.5.** *Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Let $x \in X$ with image $s \in S$. Assume that*

1. *f is locally of finite type,*
2. *$\mathcal{F}$ is of finite type, and*
3. *$\mathcal{F}$ is flat at x over S.*

Then there exists an elementary étale neighbourhood $(S', s') \to (S, s)$ and an open subscheme

$$V \subset X \times_S \mathrm{Spec}(\mathcal{O}_{S',s'})$$

which contains the unique point of $X \times_S \mathrm{Spec}(\mathcal{O}_{S',s'})$ mapping to x such that the pullback of $\mathcal{F}$ to V is flat over $\mathcal{O}_{S',s'}$.

Proof. (The only difference between this and Proposition 10.3 is that we do not assume f is of finite presentation.) The question is local on X and S, hence we may assume X and S are affine. Write $X = \mathrm{Spec}(B)$, $S = \mathrm{Spec}(A)$ and write $B = A[x_1, \ldots, x_n]/I$. In other words we obtain a closed immersion $i : X \to \mathbf{A}^n_S$. Denote $t = i(x) \in \mathbf{A}^n_S$. We may apply Proposition 10.3 to $\mathbf{A}^n_S \to S$, the sheaf $i_*\mathcal{F}$ and the point t. We obtain an elementary étale neighbourhood $(S', s') \to (S, s)$ and an open subscheme

$$W \subset \mathbf{A}^n_{\mathcal{O}_{S',s'}}$$

such that the pullback of $i_*\mathcal{F}$ to W is flat over $\mathcal{O}_{S',s'}$. This means that $V := W \cap \big(X \times_S \mathrm{Spec}(\mathcal{O}_{S',s'})\big)$ is the desired open subscheme. □

05KU **Lemma 10.6.** *Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Let $s \in S$. Assume that*

1. *f is of finite presentation,*
2. *$\mathcal{F}$ is of finite type, and*
3. *$\mathcal{F}$ is flat over S at every point of the fibre X_s.*

Then there exists an elementary étale neighbourhood $(S', s') \to (S, s)$ and an open subscheme

$$V \subset X \times_S \mathrm{Spec}(\mathcal{O}_{S',s'})$$

which contains the fibre $X_s = X \times_S s'$ such that the pullback of $\mathcal{F}$ to V is an $\mathcal{O}_V$-module of finite presentation and flat over $\mathcal{O}_{S',s'}$.

Proof. For every point $x \in X_s$ we can use Proposition 10.3 to find an elementary étale neighbourhood $(S_x, s_x) \to (S, s)$ and an open $V_x \subset X \times_S \mathrm{Spec}(\mathcal{O}_{S_x,s_x})$ such that $x \in X_s = X \times_S s_x$ is contained in V_x and such that the pullback of $\mathcal{F}$ to V_x is an $\mathcal{O}_{V_x}$-module of finite presentation and flat over $\mathcal{O}_{S_x,s_x}$. In particular we may view the fibre $(V_x)_{s_x}$ as an open neighbourhood of x in X_s. Because X_s is quasi-compact we can find a finite number of points $x_1, \ldots, x_n \in X_s$ such that X_s is the union of the $(V_{x_i})_{s_{x_i}}$. Choose an elementary étale neighbourhood $(S', s') \to (S, s)$ which dominates each of the neighbourhoods (S_{x_i}, s_{x_i}), see More on Morphisms, Lemma 31.4. Set $V = \bigcup V_i$ where V_i is the inverse images of the open V_{x_i} via the morphism

$$X \times_S \mathrm{Spec}(\mathcal{O}_{S',s'}) \longrightarrow X \times_S \mathrm{Spec}(\mathcal{O}_{S_{x_i},s_{x_i}})$$

By construction V contains X_s and by construction the pullback of $\mathcal{F}$ to V is an $\mathcal{O}_V$-module of finite presentation and flat over $\mathcal{O}_{S',s'}$. □

05KV **Lemma 10.7.** *Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Let $s \in S$. Assume that*

1. *f is of finite type,*
2. *$\mathcal{F}$ is of finite type, and*
3. *$\mathcal{F}$ is flat over S at every point of the fibre X_s.*

Then there exists an elementary étale neighbourhood $(S', s') \to (S, s)$ and an open subscheme

$$V \subset X \times_S \mathrm{Spec}(\mathcal{O}_{S',s'})$$

which contains the fibre $X_s = X \times_S s'$ such that the pullback of $\mathcal{F}$ to V is flat over $\mathcal{O}_{S',s'}$.

Proof. (The only difference between this and Lemma 10.6 is that we do not assume f is of finite presentation.) For every point $x \in X_s$ we can use Lemma 10.5 to find an elementary étale neighbourhood $(S_x, s_x) \to (S, s)$ and an open $V_x \subset X \times_S \mathrm{Spec}(\mathcal{O}_{S_x,s_x})$ such that $x \in X_s = X \times_S s_x$ is contained in V_x and such that the pullback of $\mathcal{F}$ to V_x is flat over $\mathcal{O}_{S_x,s_x}$. In particular we may view the fibre $(V_x)_{s_x}$ as an open neighbourhood of x in X_s. Because X_s is quasi-compact we can find a finite number of points $x_1, \ldots, x_n \in X_s$ such that X_s is the union of the $(V_{x_i})_{s_{x_i}}$. Choose an elementary étale neighbourhood $(S', s') \to (S, s)$ which dominates each of the neighbourhoods (S_{x_i}, s_{x_i}), see More on Morphisms, Lemma 31.4. Set $V = \bigcup V_i$ where V_i is the inverse images of the open V_{x_i} via the morphism

$$X \times_S \mathrm{Spec}(\mathcal{O}_{S',s'}) \longrightarrow X \times_S \mathrm{Spec}(\mathcal{O}_{S_{x_i},s_{x_i}})$$

By construction V contains X_s and by construction the pullback of $\mathcal{F}$ to V is flat over $\mathcal{O}_{S',s'}$. □

05I6 **Lemma 10.8.** *Let S be a scheme. Let X be locally of finite type over S. Let $x \in X$ with image $s \in S$. If X is flat at x over S, then there exists an elementary étale neighbourhood $(S', s') \to (S, s)$ and an open subscheme*

$$V \subset X \times_S \mathrm{Spec}(\mathcal{O}_{S',s'})$$

which contains the unique point of $X \times_S \mathrm{Spec}(\mathcal{O}_{S',s'})$ mapping to x such that $V \to \mathrm{Spec}(\mathcal{O}_{S',s'})$ is flat and of finite presentation.

Proof. The question is local on X and S, hence we may assume X and S are affine. Write $X = \mathrm{Spec}(B)$, $S = \mathrm{Spec}(A)$ and write $B = A[x_1, \ldots, x_n]/I$. In other words we obtain a closed immersion $i : X \to \mathbf{A}^n_S$. Denote $t = i(x) \in \mathbf{A}^n_S$. We may apply Proposition 10.3 to $\mathbf{A}^n_S \to S$, the sheaf $\mathcal{F} = i_*\mathcal{O}_X$ and the point t. We obtain an elementary étale neighbourhood $(S', s') \to (S, s)$ and an open subscheme

$$W \subset \mathbf{A}^n_{\mathcal{O}_{S',s'}}$$

such that the pullback of $i_*\mathcal{O}_X$ is flat and of finite presentation. This means that $V := W \cap \big(X \times_S \mathrm{Spec}(\mathcal{O}_{S',s'})\big)$ is the desired open subscheme. □

05I7 **Lemma 10.9.** *Let $f : X \to S$ be a morphism which is locally of finite presentation. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module of finite type. If $x \in X$ and $\mathcal{F}$ is flat at x over S, then $\mathcal{F}_x$ is an $\mathcal{O}_{X,x}$-module of finite presentation.*

Proof. Let $s = f(x)$. By Proposition 10.3 there exists an elementary étale neighbourhood $(S', s') \to (S, s)$ such that the pullback of $\mathcal{F}$ to $X \times_S \mathrm{Spec}(\mathcal{O}_{S',s'})$ is of finite presentation in a neighbourhood of the point $x' \in X_{s'} = X_s$ corresponding to x. The ring map

$$\mathcal{O}_{X,x} \longrightarrow \mathcal{O}_{X \times_S \mathrm{Spec}(\mathcal{O}_{S',s'}),x'} = \mathcal{O}_{X \times_S S',x'}$$

is flat and local as a localization of an étale ring map. Hence $\mathcal{F}_x$ is of finite presentation over $\mathcal{O}_{X,x}$ by descent, see Algebra, Lemma 82.2 (and also that a flat local ring map is faithfully flat, see Algebra, Lemma 38.17). □

05I8 **Lemma 10.10.** *Let $f : X \to S$ be a morphism which is locally of finite type. Let $x \in X$ with image $s \in S$. If f is flat at x over S, then $\mathcal{O}_{X,x}$ is essentially of finite presentation over $\mathcal{O}_{S,s}$.*

Proof. We may assume X and S affine. Write $X = \mathrm{Spec}(B)$, $S = \mathrm{Spec}(A)$ and write $B = A[x_1, \ldots, x_n]/I$. In other words we obtain a closed immersion $i : X \to \mathbf{A}^n_S$. Denote $t = i(x) \in \mathbf{A}^n_S$. We may apply Lemma 10.9 to $\mathbf{A}^n_S \to S$, the sheaf $\mathcal{F} = i_*\mathcal{O}_X$ and the point t. We conclude that $\mathcal{O}_{X,x}$ is of finite presentation over $\mathcal{O}_{\mathbf{A}^n_S,t}$ which implies what we want. $\square$

11. Extending properties from an open

0B47 In this section we collect a number of results of the form: If $f : X \to S$ is a flat morphism of schemes and f satisfies some property over a dense open of S, then f satisfies the same property over all of S.

081N **Lemma 11.1.** *Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $U \subset S$ be open. Assume*

1. *f is locally of finite presentation,*
2. *$\mathcal{F}$ is of finite type and flat over S,*
3. *$U \subset S$ is retrocompact and scheme theoretically dense,*
4. *$\mathcal{F}|_{f^{-1}U}$ is of finite presentation.*

Then $\mathcal{F}$ is of finite presentation.

Proof. The problem is local on X and S, hence we may assume X and S affine. Write $S = \mathrm{Spec}(A)$ and $X = \mathrm{Spec}(B)$. Let N be a finite B-module such that $\mathcal{F}$ is the quasi-coherent sheaf associated to N. We have $U = D(f_1) \cup \ldots \cup D(f_n)$ for some $f_i \in A$, see Algebra, Lemma 28.1. As U is schematically dense the map $A \to A_{f_1} \times \ldots \times A_{f_n}$ is injective. Pick a prime $\mathfrak{q} \subset B$ lying over $\mathfrak{p} \subset A$ corresponding to $x \in X$ mapping to $s \in S$. By Lemma 10.9 the module $N_\mathfrak{q}$ is of finite presentation over $B_\mathfrak{q}$. Choose a surjection $\varphi : B^{\oplus m} \to N$ of B-modules. Choose $k_1, \ldots, k_t \in \mathrm{Ker}(\varphi)$ and set $N' = B^{\oplus m}/\sum Bk_j$. There is a canonical surjection $N' \to N$ and N is the filtered colimit of the B-modules N' constructed in this manner. Thus we see that we can choose $k_1, \ldots, k_t$ such that (a) $N'_{f_i} \cong N_{f_i}$, $i = 1, \ldots, n$ and (b) $N'_\mathfrak{q} \cong N_\mathfrak{q}$. This in particular implies that $N'_\mathfrak{q}$ is flat over A. By openness of flatness, see Algebra, Theorem 128.4 we conclude that there exists a $g \in B$, $g \notin \mathfrak{q}$ such that N'_g is flat over A. Consider the commutative diagram

$$\begin{tikzcd} N'_g \arrow[r] \arrow[d] & N_g \arrow[d] \\ \prod N'_{gf_i} \arrow[r] & \prod N_{gf_i} \end{tikzcd}$$

The bottom arrow is an isomorphism by choice of $k_1, \ldots, k_t$. The left vertical arrow is an injective map as $A \to \prod A_{f_i}$ is injective and N'_g is flat over A. Hence the top horizontal arrow is injective, hence an isomorphism. This proves that N_g is of finite presentation over B_g. We conclude by applying Algebra, Lemma 23.2. $\square$

081P **Lemma 11.2.** *Let $f : X \to S$ be a morphism of schemes. Let $U \subset S$ be open. Assume*

1. *f is locally of finite type and flat,*
2. *$U \subset S$ is retrocompact and scheme theoretically dense,*
3. *$f|_{f^{-1}U} : f^{-1}U \to U$ is locally of finite presentation.*

Then f is of locally of finite presentation.

Proof. The question is local on X and S, hence we may assume X and S affine. Choose a closed immersion $i : X \to \mathbf{A}^n_S$ and apply Lemma 11.1 to $i_*\mathcal{O}_X$. Some details omitted. $\square$

081L **Lemma 11.3.** *Let $f : X \to S$ be a morphism of schemes which is flat and locally of finite type. Let $U \subset S$ be a dense open such that $X_U \to U$ has relative dimension $\leq e$, see Morphisms, Definition 28.1. If also either*

(1) *f is locally of finite presentation, or*
(2) *$U \subset S$ is retrocompact,*

then f has relative dimension $\leq e$.

Proof. Proof in case (1). Let $W \subset X$ be the open subscheme constructed and studied in More on Morphisms, Lemmas 20.7 and 20.9. Note that every generic point of every fibre is contained in W, hence it suffices to prove the result for W. Since $W = \bigcup_{d \geq 0} U_d$, it suffices to prove that $U_d = \emptyset$ for $d > e$. Since f is flat and locally of finite presentation it is open hence $f(U_d)$ is open (Morphisms, Lemma 24.9). Thus if U_d is not empty, then $f(U_d) \cap U \neq \emptyset$ as desired.

Proof in case (2). We may replace S by its reduction. Then U is scheme theoretically dense. Hence f is locally of finite presentation by Lemma 11.2. In this way we reduce to case (1). $\square$

0B48 **Lemma 11.4.** *Let $f : X \to S$ be a morphism of schemes which is flat and proper. Let $U \subset S$ be a dense open such that $X_U \to U$ is finite. If also either f is locally of finite presentation or $U \subset S$ is retrocompact, then f is finite.*

Proof. By Lemma 11.3 the fibres of f have dimension zero. Hence f is quasi-finite (Morphisms, Lemma 28.5) whence has finite fibres (Morphisms, Lemma 19.10). Hence f is finite by More on Morphisms, Lemma 38.4. $\square$

081M **Lemma 11.5.** *Let $f : X \to S$ be a morphism of schemes and $U \subset S$ an open. If*

(1) *f is separated, locally of finite type, and flat,*
(2) *$f^{-1}(U) \to U$ is an isomorphism, and*
(3) *$U \subset S$ is retrocompact and scheme theoretically dense,*

then f is an open immersion.

Proof. By Lemma 11.2 the morphism f is locally of finite presentation. The image $f(X) \subset S$ is open (Morphisms, Lemma 24.9) hence we may replace S by $f(X)$. Thus we have to prove that f is an isomorphism. We may assume S is affine. We can reduce to the case that X is quasi-compact because it suffices to show that any quasi-compact open $X' \subset X$ whose image is S maps isomorphically to S. Thus we may assume f is quasi-compact. All the fibers of f have dimension 0, see Lemma 11.3. Hence f is quasi-finite, see Morphisms, Lemma 28.5. Let $s \in S$. Choose an elementary étale neighbourhood $g : (T, t) \to (S, s)$ such that $X \times_S T = V \amalg W$ with $V \to T$ finite and $W_t = \emptyset$, see More on Morphisms, Lemma 36.6. Denote $\pi : V \amalg W \to T$ the given morphism. Since π is flat and locally of finite presentation, we see that $\pi(V)$ is open in T (Morphisms, Lemma 24.9). After shrinking T we may assume that $T = \pi(V)$. Since f is an isomorphism over U we see that π is an isomorphism over $g^{-1}U$. Since $\pi(V) = T$ this implies that $\pi^{-1}g^{-1}U$ is contained in V. By Morphisms, Lemma 24.14 we see that $\pi^{-1}g^{-1}U \subset V \amalg W$ is scheme theoretically dense. Hence we deduce that $W = \emptyset$. Thus $X \times_S T = V$ is finite over

T. This implies that f is finite (after replacing S by an open neighbourhood of s), for example by Descent, Lemma 20.23. Then f is finite locally free (Morphisms, Lemma 45.2) and after shrinking S to a smaller open neighbourhood of s we see that f is finite locally free of some degree d (Morphisms, Lemma 45.5). But $d = 1$ as is clear from the fact that the degree is 1 over the dense open U. Hence f is an isomorphism. □

12. Flat finitely presented modules

05I9 In some cases given a ring map $R \to S$ of finite presentation and a finitely presented S-module N the flatness of N over R implies that N is projective as an R-module, at least after replacing S by an étale extension. In this section we collect a some results of this nature.

05IA **Lemma 12.1.** *Let R be a ring. Let $R \to S$ be a finitely presented flat ring map with geometrically integral fibres. Let $\mathfrak{q} \subset S$ be a prime ideal lying over the prime $\mathfrak{r} \subset R$. Set $\mathfrak{p} = \mathfrak{r}S$. Let N be a finitely presented S-module. There exists $r \geq 0$ and an S-module map*

$$\alpha : S^{\oplus r} \longrightarrow N$$

such that $\alpha : \kappa(\mathfrak{p})^{\oplus r} \to N \otimes_S \kappa(\mathfrak{p})$ is an isomorphism. For any such α the following are equivalent:

1. *$N_\mathfrak{q}$ is R-flat,*
2. *there exists an $f \in R$, $f \notin \mathfrak{r}$ such that $\alpha_f : S_f^{\oplus r} \to N_f$ is R_f-universally injective and a $g \in S$, $g \notin \mathfrak{q}$ such that $\mathrm{Coker}(\alpha)_g$ is R-flat,*
3. *$\alpha_\mathfrak{r}$ is $R_\mathfrak{r}$-universally injective and $\mathrm{Coker}(\alpha)_\mathfrak{q}$ is R-flat*
4. *$\alpha_\mathfrak{r}$ is injective and $\mathrm{Coker}(\alpha)_\mathfrak{q}$ is R-flat,*
5. *$\alpha_\mathfrak{p}$ is an isomorphism and $\mathrm{Coker}(\alpha)_\mathfrak{q}$ is R-flat, and*
6. *$\alpha_\mathfrak{q}$ is injective and $\mathrm{Coker}(\alpha)_\mathfrak{q}$ is R-flat.*

Proof. To obtain α set $r = \dim_{\kappa(\mathfrak{p})} N \otimes_S \kappa(\mathfrak{p})$ and pick $x_1, \ldots, x_r \in N$ which form a basis of $N \otimes_S \kappa(\mathfrak{p})$. Define $\alpha(s_1, \ldots, s_r) = \sum s_i x_i$. This proves the existence.

Fix a choice of α. We may apply Lemma 10.1 to the map $\alpha_\mathfrak{r} : S_\mathfrak{r}^{\oplus r} \to N_\mathfrak{r}$. Hence we see that (1), (3), (4), (5), and (6) are all equivalent. Since it is also clear that (2) implies (3) we see that all we have to do is show that (1) implies (2).

Assume (1). By openness of flatness, see Algebra, Theorem 128.4, the set

$$U_1 = \{\mathfrak{q}' \subset S \mid N_{\mathfrak{q}'} \text{ is flat over } R\}$$

is open in $\mathrm{Spec}(S)$. It contains $\mathfrak{q}$ by assumption and hence $\mathfrak{p}$. Because $S^{\oplus r}$ and N are finitely presented S-modules the set

$$U_2 = \{\mathfrak{q}' \subset S \mid \alpha_{\mathfrak{q}'} \text{ is an isomorphism}\}$$

is open in $\mathrm{Spec}(S)$, see Algebra, Lemma 78.2. It contains $\mathfrak{p}$ by (5). As $R \to S$ is finitely presented and flat the map $\Phi : \mathrm{Spec}(S) \to \mathrm{Spec}(R)$ is open, see Algebra, Proposition 40.8. For any prime $\mathfrak{r}' \in \Phi(U_1 \cap U_2)$ we see that there exists a prime $\mathfrak{q}'$ lying over $\mathfrak{r}'$ such that $N_{\mathfrak{q}'}$ is flat and such that $\alpha_{\mathfrak{q}'}$ is an isomorphism, which implies that $\alpha \otimes \kappa(\mathfrak{p}')$ is an isomorphism where $\mathfrak{p}' = \mathfrak{r}'S$. Thus $\alpha_{\mathfrak{r}'}$ is $R_{\mathfrak{r}'}$-universally injective by the implication (1) ⇒ (3). Hence if we pick $f \in R$, $f \notin \mathfrak{r}$ such that $D(f) \subset \Phi(U_1 \cap U_2)$ then we conclude that α_f is R_f-universally injective, see Algebra, Lemma 81.12. The same reasoning also shows that for any $\mathfrak{q}' \in U_1 \cap \Phi^{-1}(\Phi(U_1 \cap U_2))$

the module $\mathrm{Coker}(\alpha)_{\mathfrak q'}$ is R-flat. Note that $\mathfrak q \in U_1 \cap \Phi^{-1}(\Phi(U_1 \cap U_2))$. Hence we can find a $g \in S$, $g \notin \mathfrak q$ such that $D(g) \subset U_1 \cap \Phi^{-1}(\Phi(U_1 \cap U_2))$ and we win. □

05IB **Lemma 12.2.** *Let $R \to S$ be a ring map of finite presentation. Let N be a finitely presented S-module flat over R. Let $\mathfrak r \subset R$ be a prime ideal. Assume there exists a complete dévissage of $N/S/R$ over $\mathfrak r$. Then there exists an $f \in R$, $f \notin \mathfrak r$ such that*

$$N_f \cong B_1^{\oplus r_1} \oplus \ldots \oplus B_n^{\oplus r_n}$$

as R-modules where each B_i is a smooth R_f-algebra with geometrically irreducible fibres. Moreover, N_f is projective as an R_f-module.

Proof. Let $(A_i, B_i, M_i, \alpha_i)_{i=1,\ldots,n}$ be the given complete dévissage. We prove the lemma by induction on n. Note that the assertions of the lemma are entirely about the structure of N as an R-module. Hence we may replace N by M_1, and we may think of M_1 as a B_1-module. See Remark 6.3 in order to see why M_1 is of finite presentation as a B_1-module. By Lemma 12.1 we may, after replacing R by R_f for some $f \in R$, $f \notin \mathfrak r$, assume the map $\alpha_1 : B_1^{\oplus r_1} \to M_1$ is R-universally injective. Since M_1 and $B_1^{\oplus r_1}$ are R-flat and finitely presented as B_1-modules we see that $\mathrm{Coker}(\alpha_1)$ is R-flat (Algebra, Lemma 81.7) and finitely presented as a B_1-module. Note that $(A_i, B_i, M_i, \alpha_i)_{i=2,\ldots,n}$ is a complete dévissage of $\mathrm{Coker}(\alpha_1)$. Hence the induction hypothesis implies that, after replacing R by R_f for some $f \in R$, $f \notin \mathfrak r$, we may assume that $\mathrm{Coker}(\alpha_1)$ has a decomposition as in the lemma and is projective. In particular $M_1 = B_1^{\oplus r_1} \oplus \mathrm{Coker}(\alpha_1)$. This proves the statement regarding the decomposition. The statement on projectivity follows as B_1 is projective as an R-module by Lemma 9.3. □

05IC **Remark 12.3.** There is a variant of Lemma 12.2 where we weaken the flatness condition by assuming only that N is flat at some given prime $\mathfrak q$ lying over $\mathfrak r$ but where we strengthen the dévissage condition by assuming the existence of a complete dévissage *at* $\mathfrak q$. Compare with Lemma 10.2.

The following is the main result of this section.

05ID **Proposition 12.4.** *Let $f : X \to S$ be a morphism of schemes. Let $\mathcal F$ be a quasi-coherent sheaf on X. Let $x \in X$ with image $s \in S$. Assume that*

1. *f is locally of finite presentation,*
2. *$\mathcal F$ is of finite presentation, and*
3. *$\mathcal F$ is flat at x over S.*

Then there exists a commutative diagram of pointed schemes

$$\begin{tikzcd} (X,x) \arrow[d] & (X',x') \arrow[l, "g"'] \arrow[d] \\ (S,s) & (S',s') \arrow[l] \end{tikzcd}$$

whose horizontal arrows are elementary étale neighbourhoods such that X', S' are affine and such that $\Gamma(X', g^*\mathcal F)$ is a projective $\Gamma(S', \mathcal O_{S'})$-module.

Proof. By openness of flatness, see More on Morphisms, Theorem 15.1 we may replace X by an open neighbourhood of x and assume that $\mathcal F$ is flat over S. Next, we apply Proposition 5.7 to find a diagram as in the statement of the proposition such that $g^*\mathcal F/X'/S'$ has a complete dévissage over s'. (In particular S' and X'

are affine.) By Morphisms, Lemma 24.12 we see that $g^*\mathcal{F}$ is flat over S and by Lemma 2.3 we see that it is flat over S'. Via Remark 6.5 we deduce that

$$\Gamma(X', g^*\mathcal{F})/\Gamma(X', \mathcal{O}_{X'})/\Gamma(S', \mathcal{O}_{S'})$$

has a complete dévissage over the prime of $\Gamma(S', \mathcal{O}_{S'})$ corresponding to s'. Thus Lemma 12.2 implies that the result of the proposition holds after replacing S' by a standard open neighbourhood of s'. $\square$

In the rest of this section we prove a number of variants on this result. The first is a "global" version.

05KW **Lemma 12.5.** *Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Let $s \in S$. Assume that*

(1) *f is of finite presentation,*
(2) *$\mathcal{F}$ is of finite presentation, and*
(3) *$\mathcal{F}$ is flat over S at every point of the fibre X_s.*

Then there exists an elementary étale neighbourhood $(S', s') \to (S, s)$ and a commutative diagram of schemes

$$\begin{tikzcd} X \arrow[d] & X' \arrow[l, "g"'] \arrow[d] \\ S & S' \arrow[l] \end{tikzcd}$$

such that g is étale, $X_s \subset g(X')$, the schemes X', S' are affine, and such that $\Gamma(X', g^*\mathcal{F})$ is a projective $\Gamma(S', \mathcal{O}_{S'})$-module.

Proof. For every point $x \in X_s$ we can use Proposition 12.4 to find a commutative diagram

$$\begin{tikzcd} (X, x) \arrow[d] & (Y_x, y_x) \arrow[l, "g_x"'] \arrow[d] \\ (S, s) & (S_x, s_x) \arrow[l] \end{tikzcd}$$

whose horizontal arrows are elementary étale neighbourhoods such that Y_x, S_x are affine and such that $\Gamma(Y_x, g_x^*\mathcal{F})$ is a projective $\Gamma(S_x, \mathcal{O}_{S_x})$-module. In particular $g_x(Y_x) \cap X_s$ is an open neighbourhood of x in X_s. Because X_s is quasi-compact we can find a finite number of points $x_1, \ldots, x_n \in X_s$ such that X_s is the union of the $g_{x_i}(Y_{x_i}) \cap X_s$. Choose an elementary étale neighbourhood $(S', s') \to (S, s)$ which dominates each of the neighbourhoods (S_{x_i}, s_{x_i}), see More on Morphisms, Lemma 31.4. We may also assume that S' is affine. Set $X' = \coprod Y_{x_i} \times_{S_{x_i}} S'$ and endow it with the obvious morphism $g : X' \to X$. By construction $g(X')$ contains X_s and

$$\Gamma(X', g^*\mathcal{F}) = \bigoplus \Gamma(Y_{x_i}, g_{x_i}^*\mathcal{F}) \otimes_{\Gamma(S_{x_i}, \mathcal{O}_{S_{x_i}})} \Gamma(S', \mathcal{O}_{S'}).$$

This is a projective $\Gamma(S', \mathcal{O}_{S'})$-module, see Algebra, Lemma 93.1. $\square$

The following two lemmas are reformulations of the results above in case $\mathcal{F} = \mathcal{O}_X$.

05IE **Lemma 12.6.** *Let $f : X \to S$ be locally of finite presentation. Let $x \in X$ with image $s \in S$. If f is flat at x over S, then there exists a commutative diagram of* *pointed schemes*

$$\begin{tikzcd} (X, x) \arrow[d] & (X', x') \arrow[l, "g"'] \arrow[d] \\ (S, s) & (S', s') \arrow[l] \end{tikzcd}$$

whose horizontal arrows are elementary étale neighbourhoods such that X', S' are affine and such that $\Gamma(X', \mathcal{O}_{X'})$ is a projective $\Gamma(S', \mathcal{O}_{S'})$-module.

Proof. This is a special case of Proposition 12.4. $\square$

05KX **Lemma 12.7.** *Let $f : X \to S$ be of finite presentation. Let $s \in S$. If X is flat over S at all points of X_s, then there exists an elementary étale neighbourhood $(S', s') \to (S, s)$ and a commutative diagram of schemes*

$$\begin{tikzcd} X \arrow[d] & X' \arrow[l, "g"'] \arrow[d] \\ S & S' \arrow[l] \end{tikzcd}$$

with g étale, $X_s \subset g(X')$, such that X', S' are affine, and such that $\Gamma(X', \mathcal{O}_{X'})$ is a projective $\Gamma(S', \mathcal{O}_{S'})$-module.

Proof. This is a special case of Lemma 12.5. $\square$

The following lemmas explain consequences of Proposition 12.4 in case we only assume the morphism and the sheaf are of finite type (and not necessarily of finite presentation).

05KY **Lemma 12.8.** *Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Let $x \in X$ with image $s \in S$. Assume that*

1. *f is locally of finite presentation,*
2. *$\mathcal{F}$ is of finite type, and*
3. *$\mathcal{F}$ is flat at x over S.*

Then there exists an elementary étale neighbourhood $(S', s') \to (S, s)$ and a commutative diagram of pointed schemes

$$\begin{tikzcd} (X, x) \arrow[d] & (X', x') \arrow[l, "g"'] \arrow[d] \\ (S, s) & (\mathrm{Spec}(\mathcal{O}_{S', s'}), s') \arrow[l] \end{tikzcd}$$

such that $X' \to X \times_S \mathrm{Spec}(\mathcal{O}_{S', s'})$ is étale, $\kappa(x) = \kappa(x')$, the scheme X' is affine of finite presentation over $\mathcal{O}_{S', s'}$, the sheaf $g^*\mathcal{F}$ is of finite presentation over $\mathcal{O}_{X'}$, and such that $\Gamma(X', g^*\mathcal{F})$ is a free $\mathcal{O}_{S', s'}$-module.

Proof. To prove the lemma we may replace (S, s) by any elementary étale neighbourhood, and we may also replace S by $\mathrm{Spec}(\mathcal{O}_{S,s})$. Hence by Proposition 10.3 we may assume that $\mathcal{F}$ is finitely presented and flat over S in a neighbourhood of x. In this case the result follows from Proposition 12.4 because Algebra, Theorem 84.4 assures us that projective = free over a local ring. $\square$

05KZ **Lemma 12.9.** *Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Let $x \in X$ with image $s \in S$. Assume that*

(1) *f is locally of finite type,*
(2) *$\mathcal{F}$ is of finite type, and*
(3) *$\mathcal{F}$ is flat at x over S.*

Then there exists an elementary étale neighbourhood $(S', s') \to (S, s)$ and a commutative diagram of pointed schemes

$$\begin{tikzcd} (X, x) \arrow[d] & (X', x') \arrow[l, "g"'] \arrow[d] \\ (S, s) & (\mathrm{Spec}(\mathcal{O}_{S',s'}), s') \arrow[l] \end{tikzcd}$$

such that $X' \to X \times_S \mathrm{Spec}(\mathcal{O}_{S',s'})$ is étale, $\kappa(x) = \kappa(x')$, the scheme X' is affine, and such that $\Gamma(X', g^*\mathcal{F})$ is a free $\mathcal{O}_{S',s'}$-module.

Proof. (The only difference with Lemma 12.8 is that we do not assume f is of finite presentation.) The problem is local on X and S. Hence we may assume X and S are affine, say $X = \mathrm{Spec}(B)$ and $S = \mathrm{Spec}(A)$. Since B is a finite type A-algebra we can find a surjection $A[x_1, \ldots, x_n] \to B$. In other words, we can choose a closed immersion $i : X \to \mathbf{A}^n_S$. Set $t = i(x)$ and $\mathcal{G} = i_*\mathcal{F}$. Note that $\mathcal{G}_t \cong \mathcal{F}_x$ are $\mathcal{O}_{S,s}$-modules. Hence $\mathcal{G}$ is flat over S at t. We apply Lemma 12.8 to the morphism $\mathbf{A}^n_S \to S$, the point t, and the sheaf $\mathcal{G}$. Thus we can find an elementary étale neighbourhood $(S', s') \to (S, s)$ and a commutative diagram of pointed schemes

$$\begin{tikzcd} (\mathbf{A}^n_S, t) \arrow[d] & (Y, y) \arrow[l, "h"'] \arrow[d] \\ (S, s) & (\mathrm{Spec}(\mathcal{O}_{S',s'}), s') \arrow[l] \end{tikzcd}$$

such that $Y \to \mathbf{A}^n_{\mathcal{O}_{S',s'}}$ is étale, $\kappa(t) = \kappa(y)$, the scheme Y is affine, and such that $\Gamma(Y, h^*\mathcal{G})$ is a projective $\mathcal{O}_{S',s'}$-module. Then a solution to the original problem is given by the closed subscheme $X' = Y \times_{\mathbf{A}^n_S} X$ of Y. $\square$

05L0 **Lemma 12.10.** *Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Let $s \in S$. Assume that*

(1) *f is of finite presentation,*
(2) *$\mathcal{F}$ is of finite type, and*
(3) *$\mathcal{F}$ is flat over S at all points of X_s.*

Then there exists an elementary étale neighbourhood $(S', s') \to (S, s)$ and a commutative diagram of schemes

$$\begin{tikzcd} X \arrow[d] & X' \arrow[l, "g"'] \arrow[d] \\ S & \mathrm{Spec}(\mathcal{O}_{S',s'}) \arrow[l] \end{tikzcd}$$

such that $X' \to X \times_S \mathrm{Spec}(\mathcal{O}_{S',s'})$ is étale, $X_s = g((X')_{s'})$, the scheme X' is affine of finite presentation over $\mathcal{O}_{S',s'}$, the sheaf $g^*\mathcal{F}$ is of finite presentation over $\mathcal{O}_{X'}$, and such that $\Gamma(X', g^*\mathcal{F})$ is a free $\mathcal{O}_{S',s'}$-module.

Proof. For every point $x \in X_s$ we can use Lemma 12.8 to find an elementary étale neighbourhood $(S_x, s_x) \to (S, s)$ and a commutative diagram

$$\begin{tikzcd} (X, x) \arrow[d] & (Y_x, y_x) \arrow[l, "g_x"'] \arrow[d] \\ (S, s) & (\mathrm{Spec}(\mathcal{O}_{S_x, s_x}), s_x) \arrow[l] \end{tikzcd}$$

such that $Y_x \to X \times_S \mathrm{Spec}(\mathcal{O}_{S_x, s_x})$ is étale, $\kappa(x) = \kappa(y_x)$, the scheme Y_x is affine of finite presentation over $\mathcal{O}_{S_x, s_x}$, the sheaf $g_x^*\mathcal{F}$ is of finite presentation over $\mathcal{O}_{Y_x}$, and such that $\Gamma(Y_x, g_x^*\mathcal{F})$ is a free $\mathcal{O}_{S_x, s_x}$-module. In particular $g_x((Y_x)_{s_x})$ is an open neighbourhood of x in X_s. Because X_s is quasi-compact we can find a finite number of points $x_1, \ldots, x_n \in X_s$ such that X_s is the union of the $g_{x_i}((Y_{x_i})_{s_{x_i}})$. Choose an elementary étale neighbourhood $(S', s') \to (S, s)$ which dominates each of the neighbourhoods (S_{x_i}, s_{x_i}), see More on Morphisms, Lemma 31.4. Set

$$X' = \coprod Y_{x_i} \times_{\mathrm{Spec}(\mathcal{O}_{S_{x_i}, s_{x_i}})} \mathrm{Spec}(\mathcal{O}_{S', s'})$$

and endow it with the obvious morphism $g : X' \to X$. By construction $X_s = g(X'_{s'})$ and

$$\Gamma(X', g^*\mathcal{F}) = \bigoplus \Gamma(Y_{x_i}, g_{x_i}^*\mathcal{F}) \otimes_{\mathcal{O}_{S_{x_i}, s_{x_i}}} \mathcal{O}_{S', s'}.$$

This is a free $\mathcal{O}_{S', s'}$-module as a direct sum of base changes of free modules. Some minor details omitted. □

05L1 **Lemma 12.11.** *Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Let $s \in S$. Assume that*

1. *f is of finite type,*
2. *$\mathcal{F}$ is of finite type, and*
3. *$\mathcal{F}$ is flat over S at all points of X_s.*

Then there exists an elementary étale neighbourhood $(S', s') \to (S, s)$ and a commutative diagram of schemes

$$\begin{tikzcd} X \arrow[d] & X' \arrow[l, "g"'] \arrow[d] \\ S & \mathrm{Spec}(\mathcal{O}_{S', s'}) \arrow[l] \end{tikzcd}$$

such that $X' \to X \times_S \mathrm{Spec}(\mathcal{O}_{S', s'})$ is étale, $X_s = g((X')_{s'})$, the scheme X' is affine, and such that $\Gamma(X', g^*\mathcal{F})$ is a free $\mathcal{O}_{S', s'}$-module.

Proof. (The only difference with Lemma 12.10 is that we do not assume f is of finite presentation.) For every point $x \in X_s$ we can use Lemma 12.9 to find an elementary étale neighbourhood $(S_x, s_x) \to (S, s)$ and a commutative diagram

$$\begin{tikzcd} (X, x) \arrow[d] & (Y_x, y_x) \arrow[l, "g_x"'] \arrow[d] \\ (S, s) & (\mathrm{Spec}(\mathcal{O}_{S_x, s_x}), s_x) \arrow[l] \end{tikzcd}$$

such that $Y_x \to X \times_S \mathrm{Spec}(\mathcal{O}_{S_x, s_x})$ is étale, $\kappa(x) = \kappa(y_x)$, the scheme Y_x is affine, and such that $\Gamma(Y_x, g_x^*\mathcal{F})$ is a free $\mathcal{O}_{S_x, s_x}$-module. In particular $g_x((Y_x)_{s_x})$ is an open neighbourhood of x in X_s. Because X_s is quasi-compact we can find a finite number of points $x_1, \ldots, x_n \in X_s$ such that X_s is the union of the $g_{x_i}((Y_{x_i})_{s_{x_i}})$. Choose an elementary étale neighbourhood $(S', s') \to (S, s)$ which dominates each of the neighbourhoods (S_{x_i}, s_{x_i}), see More on Morphisms, Lemma 31.4. Set

$$X' = \coprod Y_{x_i} \times_{\mathrm{Spec}(\mathcal{O}_{S_{x_i}, s_{x_i}})} \mathrm{Spec}(\mathcal{O}_{S', s'})$$

and endow it with the obvious morphism $g : X' \to X$. By construction $X_s = g(X'_{s'})$ and

$$\Gamma(X', g^*\mathcal{F}) = \bigoplus \Gamma(Y_{x_i}, g_{x_i}^*\mathcal{F}) \otimes_{\mathcal{O}_{S_{x_i}, s_{x_i}}} \mathcal{O}_{S', s'}.$$

This is a free $\mathcal{O}_{S', s'}$-module as a direct sum of base changes of free modules. $\square$

13. Flat finite type modules, Part II

05IF We will need the following lemma.

0CU6 **Lemma 13.1.** *Let $R \to S$ be a ring map of finite presentation. Let N be a finitely presented S-module. Let $\mathfrak{q} \subset S$ be a prime ideal lying over $\mathfrak{p} \subset R$. Set $\overline{S} = S \otimes_R \kappa(\mathfrak{p})$, $\overline{\mathfrak{q}} = \mathfrak{q}\overline{S}$, and $\overline{N} = N \otimes_R \kappa(\mathfrak{p})$. Then we can find a $g \in S$ with $g \notin \mathfrak{q}$ such that $\overline{g} \in \mathfrak{r}$ for all $\mathfrak{r} \in Ass_{\overline{S}}(\overline{N})$ such that $\mathfrak{r} \not\subset \overline{\mathfrak{q}}$.*

Proof. Namely, if $\mathrm{Ass}_{\overline{S}}(\overline{N}) = \{\mathfrak{r}_1, \ldots, \mathfrak{r}_n\}$ (finiteness by Algebra, Lemma 62.5), then after renumbering we may assume that

$$\mathfrak{r}_1 \subset \overline{\mathfrak{q}}, \ldots, \mathfrak{r}_r \subset \overline{\mathfrak{q}}, \quad \mathfrak{r}_{r+1} \not\subset \overline{\mathfrak{q}}, \ldots, \mathfrak{r}_n \not\subset \overline{\mathfrak{q}}$$

Since $\overline{\mathfrak{q}}$ is a prime ideal we see that the product $\mathfrak{r}_{r+1} \ldots \mathfrak{r}_n$ is not contained in $\overline{\mathfrak{q}}$ and hence we can pick an element a of $\overline{S}$ contained in $\mathfrak{r}_{r+1}, \ldots, \mathfrak{r}_n$ but not in $\overline{\mathfrak{q}}$. If there exists $g \in S$ mapping to a, then g works. In general we can find a nonzero element $\lambda \in \kappa(\mathfrak{p})$ such that λa is the image of a $g \in S$. $\square$

The following lemma has a sligthly stronger variant Lemma 13.4 below.

05IG **Lemma 13.2.** *Let $R \to S$ be a ring map of finite presentation. Let N be a finitely presented S-module which is flat as an R-module. Let M be an R-module. Let $\mathfrak{q}$ be a prime of S lying over $\mathfrak{p} \subset R$. Then*

$$\mathfrak{q} \in WeakAss_S(M \otimes_R N) \Leftrightarrow \Big(\mathfrak{p} \in WeakAss_R(M) \text{ and } \overline{\mathfrak{q}} \in Ass_{\overline{S}}(\overline{N})\Big)$$

Here $\overline{S} = S \otimes_R \kappa(\mathfrak{p})$, $\overline{\mathfrak{q}} = \mathfrak{q}\overline{S}$, and $\overline{N} = N \otimes_R \kappa(\mathfrak{p})$.

Proof. Pick $g \in S$ as in Lemma 13.1. Apply Proposition 12.4 to the morphism of schemes $\mathrm{Spec}(S_g) \to \mathrm{Spec}(R)$, the quasi-coherent module associated to N_g, and the points corresponding to the primes $\mathfrak{q}S_g$ and $\mathfrak{p}$. Translating into algebra we obtain a commutative diagram of rings

$$\begin{tikzcd} S \arrow[r] & S_g \arrow[r] & S' \\ & R \arrow[ul] \arrow[u] \arrow[r] & R' \arrow[u] \end{tikzcd} \qquad \begin{tikzcd} \mathfrak{q} \arrow[r, dash] \arrow[dr, dash] & \mathfrak{q}S_g \arrow[r, dash] \arrow[d, dash] & \mathfrak{q}' \arrow[d, dash] \\ & \mathfrak{p} \arrow[r, dash] & \mathfrak{p}' \end{tikzcd}$$

endowed with primes as shown, the horizontal arrows are étale, and $N \otimes_S S'$ is projective as an R'-module. Set $N' = N \otimes_S S'$, $M' = M \otimes_R R'$, $\overline{S}' = S' \otimes_{R'} \kappa(\mathfrak{q}')$, $\overline{\mathfrak{q}}' = \mathfrak{q}'\overline{S}'$, and

$$\overline{N}' = N' \otimes_{R'} \kappa(\mathfrak{p}') = \overline{N} \otimes_{\overline{S}} \overline{S}'$$

By Lemma 2.8 we have

$$\begin{aligned} \mathrm{WeakAss}_{S'}(M' \otimes_{R'} N') &= (\mathrm{Spec}(S') \to \mathrm{Spec}(S))^{-1}\mathrm{WeakAss}_S(M \otimes_R N) \\ \mathrm{WeakAss}_{R'}(M') &= (\mathrm{Spec}(R') \to \mathrm{Spec}(R))^{-1}\mathrm{WeakAss}_R(M) \\ \mathrm{Ass}_{\overline{S}'}(\overline{N}') &= (\mathrm{Spec}(\overline{S}') \to \mathrm{Spec}(\overline{S}))^{-1}\mathrm{Ass}_{\overline{S}}(\overline{N}) \end{aligned}$$

Use Algebra, Lemma 65.8 for $\overline{N}$ and $\overline{N}'$. In particular we have

$$\begin{aligned} \mathfrak q \in \mathrm{WeakAss}_S(M \otimes_R N) &\Leftrightarrow \mathfrak q' \in \mathrm{WeakAss}_{S'}(M' \otimes_{R'} N') \\ \mathfrak p \in \mathrm{WeakAss}_R(M) &\Leftrightarrow \mathfrak p' \in \mathrm{WeakAss}_{R'}(M') \\ \overline{\mathfrak q} \in \mathrm{Ass}_{\overline{S}}(\overline{N}) &\Leftrightarrow \overline{\mathfrak q}' \in \mathrm{WeakAss}_{\overline{S}'}(\overline{N}') \end{aligned}$$

Our careful choice of g and the formula for $\mathrm{Ass}_{\overline{S}'}(\overline{N}')$ above shows that

0CU7 (13.2.1) $$\text{if } \mathfrak r' \in \mathrm{Ass}_{\overline{S}'}(\overline{N}') \text{ lies over } \mathfrak r \subset \overline{S} \text{ then } \mathfrak r \subset \overline{\mathfrak q}$$

This will be a key observation later in the proof. We will use the characterization of weakly associated primes given in Algebra, Lemma 65.2 without further mention.

Suppose that $\overline{\mathfrak q} \not\in \mathrm{Ass}_{\overline{S}}(\overline{N})$. Then $\overline{\mathfrak q}' \not\in \mathrm{Ass}_{\overline{S}'}(\overline{N}')$. By Algebra, Lemmas 62.9, 62.5, and 14.2 there exists an element $\overline{a}' \in \overline{\mathfrak q}'$ which is not a zerodivisor on $\overline{N}'$. After replacing $\overline{a}'$ by $\lambda\overline{a}'$ for some nonzero $\lambda \in \kappa(\mathfrak p)$ we can find $a' \in \mathfrak q'$ mapping to $\overline{a}'$. By Lemma 7.6 the map $a' : N'_{\mathfrak p'} \to N'_{\mathfrak p'}$ is $R'_{\mathfrak p'}$-universally injective. In particular we see that $a' : M' \otimes_{R'} N' \to M' \otimes_{R'} N'$ is injective after localizing at $\mathfrak p'$ and hence after localizing at $\mathfrak q'$. Clearly this implies that $\mathfrak q' \not\in \mathrm{WeakAss}_{S'}(M' \otimes_{R'} N')$. We conclude that $\mathfrak q \in \mathrm{WeakAss}_S(M \otimes_R N)$ implies $\overline{\mathfrak q} \in \mathrm{Ass}_{\overline{S}}(\overline{N})$.

Assume $\mathfrak q \in \mathrm{WeakAss}_S(M \otimes_R N)$. We want to show $\mathfrak p \in \mathrm{WeakAss}_S(M)$. Let $z \in M \otimes_R N$ be an element such that $\mathfrak q$ is minimal over $J = \mathrm{Ann}_S(z)$. Let $f_i \in \mathfrak p$, $i \in I$ be a set of generators of the ideal $\mathfrak p$. Since $\mathfrak q$ lies over $\mathfrak p$, for every i we can choose an $n_i \geq 1$ and $g_i \in S$, $g_i \not\in \mathfrak q$ with $g_i f_i^{n_i} \in J$, i.e., $g_i f_i^{n_i} z = 0$. Let $z' \in (M' \otimes_{R'} N')_{\mathfrak p'}$ be the image of z. Observe that z' is nonzero because z has nonzero image in $(M \otimes_R N)_{\mathfrak q}$ and because $S_{\mathfrak q} \to S'_{\mathfrak q'}$ is faithfully flat. We claim that $f_i^{n_i} z' = 0$.

Proof of the claim: Let $g'_i \in S'$ be the image of g_i. By the key observation (13.2.1) we find that the image $\overline{g}'_i \in \overline{S}'$ is not contained in $\mathfrak r'$ for any $\mathfrak r' \in \mathrm{Ass}_{\overline{S}'}(\overline{N})$. Hence by Lemma 7.6 we see that $g'_i : N'_{\mathfrak p'} \to N'_{\mathfrak p'}$ is $R'_{\mathfrak p'}$-universally injective. In particular we see that $g'_i : M' \otimes_{R'} N' \to M' \otimes_{R'} N'$ is injective after localizating at $\mathfrak p'$. The claim follows because $g_i f_i^{n_i} z' = 0$.

Our claim shows that the annihilator of z' in $R'_{\mathfrak p'}$ contains the elements $f_i^{n_i}$. As $R \to R'$ is étale we have $\mathfrak p' R'_{\mathfrak p'} = \mathfrak p R'_{\mathfrak p'}$ by Algebra, Lemma 141.5. Hence the annihilator of z' in $R'_{\mathfrak p'}$ has radical equal to $\mathfrak p' R_{\mathfrak p'}$ (here we use z' is not zero). On the other hand

$$z' \in (M' \otimes_{R'} N')_{\mathfrak p'} = M'_{\mathfrak p'} \otimes_{R'_{\mathfrak p'}} N'_{\mathfrak p'}$$

The module $N'_{\mathfrak p'}$ is projective over the local ring $R'_{\mathfrak p'}$ and hence free (Algebra, Theorem 84.4). Thus we can find a finite free direct summand $F' \subset N'_{\mathfrak p'}$ such that $z' \in M'_{\mathfrak p'} \otimes_{R'_{\mathfrak p'}} F'$. If F' has rank n, then we deduce that $\mathfrak p' R'_{\mathfrak p'} \in \mathrm{WeakAss}_{R'_{\mathfrak p'}}({M'_{\mathfrak p'}}^{\oplus n})$. This implies $\mathfrak p' R'_{\mathfrak p'} \in \mathrm{WeakAss}(M'_{\mathfrak p'})$ for example by Algebra, Lemma 65.3. Then

$\mathfrak{p}' \in \text{WeakAss}_{R'}(M')$ which in turn gives $\mathfrak{p} \in \text{WeakAss}_R(M)$. This finishes the proof of the implication "$\Rightarrow$" of the equivalence of the lemma.

Assume that $\mathfrak{p} \in \text{WeakAss}_R(M)$ and $\overline{\mathfrak{q}} \in \text{Ass}_{\overline{S}}(\overline{N})$. We want to show that $\mathfrak{q}$ is weakly associated to $M \otimes_R N$. Note that $\overline{\mathfrak{q}}'$ is a maximal element of $\text{Ass}_{\overline{S}'}(\overline{N}')$. This is a consequence of (13.2.1) and the fact that there are no inclusions among the primes of $\overline{S}'$ lying over $\overline{\mathfrak{q}}$ (as fibres of étale morphisms are discrete Morphisms, Lemma 34.7). Thus, after replacing $R, S, \mathfrak{p}, \mathfrak{q}, M, N$ by $R', S', \mathfrak{p}', \mathfrak{q}', M', N'$ we may assume, in addition to the assumptions of the lemma, that

(1) $\mathfrak{p} \in \text{WeakAss}_R(M)$,
(2) $\overline{\mathfrak{q}} \in \text{Ass}_{\overline{S}}(\overline{N})$,
(3) N is projective as an R-module, and
(4) $\overline{\mathfrak{q}}$ is maximal in $\text{Ass}_{\overline{S}}(\overline{N})$.

There is one more reduction, namely, we may replace R, S, M, N by their localizations at $\mathfrak{p}$. This leads to one more condition, namely,

(5) R is a local ring with maximal ideal $\mathfrak{p}$.

We will finish by showing that (1) – (5) imply $\mathfrak{q} \in \text{WeakAss}(M \otimes_R N)$.

Since R is local and $\mathfrak{p} \in \text{WeakAss}_R(M)$ we can pick a $y \in M$ whose annihilator I has radical equal to $\mathfrak{p}$. Write $\overline{\mathfrak{q}} = (\overline{g}_1, \ldots, \overline{g}_n)$ for some $\overline{g}_i \in \overline{S}$. Choose $g_i \in S$ mapping to $\overline{g}_i$. Then $\mathfrak{q} = \mathfrak{p}S + g_1S + \ldots + g_nS$. Consider the map

$$\Psi : N/IN \longrightarrow (N/IN)^{\oplus n}, \quad z \longmapsto (g_1z, \ldots, g_nz).$$

This is a homomorphism of projective R/I-modules. The local ring R/I is auto-associated (More on Algebra, Definition 13.1) as $\mathfrak{p}/I$ is locally nilpotent. The map $\Psi \otimes \kappa(\mathfrak{p})$ is not injective, because $\overline{\mathfrak{q}} \in \text{Ass}_{\overline{S}}(\overline{N})$. Hence More on Algebra, Lemma 13.4 implies Ψ is not injective. Pick $z \in N/IN$ nonzero in the kernel of Ψ. The annihilator $J = \text{Ann}_S(z)$ contains IS and g_i by construction. Thus $\sqrt{J} \subset S$ contains $\mathfrak{q}$. Let $\mathfrak{s} \subset S$ be a prime minimal over J. Then $\mathfrak{q} \subset \mathfrak{s}$, $\mathfrak{s}$ lies over $\mathfrak{p}$, and $\mathfrak{s} \in \text{WeakAss}_S(N/IN)$. The last fact by definition of weakly associated primes. Apply the "$\Rightarrow$" part of the lemma (which we've already proven) to the ring map $R \to S$ and the modules R/I and N to conclude that $\overline{\mathfrak{s}} \in \text{Ass}_{\overline{S}}(\overline{N})$. Since $\overline{\mathfrak{q}} \subset \overline{\mathfrak{s}}$ the maximality of $\overline{\mathfrak{q}}$, see condition (4) above, implies that $\overline{\mathfrak{q}} = \overline{\mathfrak{s}}$. This shows that $\mathfrak{q} = \mathfrak{s}$ and we conlude what we want. $\square$

05IH **Lemma 13.3.** *Let S be a scheme. Let $f : X \to S$ be locally of finite type. Let $x \in X$ with image $s \in S$. Let $\mathcal{F}$ be a finite type quasi-coherent sheaf on X. Let $\mathcal{G}$ be a quasi-coherent sheaf on S. If $\mathcal{F}$ is flat at x over S, then*

$$x \in \mathit{WeakAss}_X(\mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{G}) \Leftrightarrow s \in \mathit{WeakAss}_S(\mathcal{G}) \text{ and } x \in \mathit{Ass}_{X_s}(\mathcal{F}_s).$$

Proof. In this paragraph we reduce to f being of finite presentation. The question is local on X and S, hence we may assume X and S are affine. Write $X = \text{Spec}(B)$, $S = \text{Spec}(A)$ and write $B = A[x_1, \ldots, x_n]/I$. In other words we obtain a closed immersion $i : X \to \mathbf{A}^n_S$ over S. Denote $t = i(x) \in \mathbf{A}^n_S$. Note that $i_*\mathcal{F}$ is a finite type quasi-coherent sheaf on $\mathbf{A}^n_S$ which is flat at t over S and note that

$$i_*(\mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{G}) = i_*\mathcal{F} \otimes_{\mathcal{O}_{\mathbf{A}^n_S}} p^*\mathcal{G}$$

where $p : \mathbf{A}^n_S \to S$ is the projection. Note that t is a weakly associated point of $i_*(\mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{G})$ if and only if x is a weakly associated point of $\mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{G}$, see Divisors, Lemma 6.3. Similarly $x \in \text{Ass}_{X_s}(\mathcal{F}_s)$ if and only if $t \in \text{Ass}_{\mathbf{A}^n_s}((i_*\mathcal{F})_s)$

(see Algebra, Lemma 62.14). Hence it suffices to prove the lemma in case $X = \mathbf{A}^n_S$. Thus we may assume that $X \to S$ is of finite presentation.

In this paragraph we reduce to $\mathcal{F}$ being of finite presentation and flat over S. Choose an elementary étale neighbourhood $e : (S', s') \to (S, s)$ and an open $V \subset X \times_S \mathrm{Spec}(\mathcal{O}_{S',s'})$ as in Proposition 10.3. Let $x' \in X' = X \times_S S'$ be the unique point mapping to x and s'. Then it suffices to prove the statement for $X' \to S'$, x', s', $(X' \to X)^*\mathcal{F}$, and $e^*\mathcal{G}$, see Lemma 2.8. Let $v \in V$ the unique point mapping to x' and let $s' \in \mathrm{Spec}(\mathcal{O}_{S',s'})$ be the closed point. Then $\mathcal{O}_{V,v} = \mathcal{O}_{X',x'}$ and $\mathcal{O}_{\mathrm{Spec}(\mathcal{O}_{S',s'}),s'} = \mathcal{O}_{S',s'}$ and similarly for the stalks of pullbacks of $\mathcal{F}$ and $\mathcal{G}$. Also $V_{s'} \subset X'_{s'}$ is an open subscheme. Since the condition of being a weakly associated point depend only on the stalk of the sheaf, we may replace $X' \to S'$, x', s', $(X' \to X)^*\mathcal{F}$, and $e^*\mathcal{G}$ by $V \to \mathrm{Spec}(\mathcal{O}_{S',s'})$, v, s', $(V \to X)^*\mathcal{F}$, and $(\mathrm{Spec}(\mathcal{O}_{S',s'}) \to S)^*\mathcal{G}$. Thus we may assume that f is of finite presentation and $\mathcal{F}$ of finite presentation and flat over S.

Assume f is of finite presentation and $\mathcal{F}$ of finite presentation and flat over S. After shrinking X and S to affine neighbourhoods of x and s, this case is handled by Lemma 13.2. □

05II **Lemma 13.4.** *Let $R \to S$ be a ring map which is essentially of finite type. Let N be a localization of a finite S-module flat over R. Let M be an R-module. Then*

$$WeakAss_S(M \otimes_R N) = \bigcup\nolimits_{\mathfrak{p} \in WeakAss_R(M)} Ass_{S \otimes_R \kappa(\mathfrak{p})}(N \otimes_R \kappa(\mathfrak{p}))$$

Proof. This lemma is a translation of Lemma 13.3 into algebra. Details of translation omitted. □

05IJ **Lemma 13.5.** *Let $f : X \to S$ be a morphism which is locally of finite type. Let $\mathcal{F}$ be a finite type quasi-coherent sheaf on X which is flat over S. Let $\mathcal{G}$ be a quasi-coherent sheaf on S. Then we have*

$$WeakAss_X(\mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{G}) = \bigcup\nolimits_{s \in WeakAss_S(\mathcal{G})} Ass_{X_s}(\mathcal{F}_s)$$

Proof. Immediate consequence of Lemma 13.3. □

05IK **Theorem 13.6.** *Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Assume*

1. *$X \to S$ is locally of finite presentation,*
2. *$\mathcal{F}$ is an $\mathcal{O}_X$-module of finite type, and*
3. *the set of weakly associated points of S is locally finite in S.*

Then $U = \{x \in X \mid \mathcal{F} \text{ flat at } x \text{ over } S\}$ is open in X and $\mathcal{F}|_U$ is an $\mathcal{O}_U$-module of finite presentation and flat over S.

Proof. Let $x \in X$ be such that $\mathcal{F}$ is flat at x over S. We have to find an open neighbourhood of x such that $\mathcal{F}$ restricts to a S-flat finitely presented module on this neighbourhood. The problem is local on X and S, hence we may assume that X and S are affine. As $\mathcal{F}_x$ is a finitely presented $\mathcal{O}_{X,x}$-module by Lemma 10.9 we conclude from Algebra, Lemma 125.5 there exists a finitely presented $\mathcal{O}_X$-module $\mathcal{F}'$ and a map $\varphi : \mathcal{F}' \to \mathcal{F}$ which induces an isomorphism $\varphi_x : \mathcal{F}'_x \to \mathcal{F}_x$. In particular we see that $\mathcal{F}'$ is flat over S at x, hence by openness of flatness More on Morphisms, Theorem 15.1 we see that after shrinking X we may assume that $\mathcal{F}'$ is flat over S. As $\mathcal{F}$ is of finite type after shrinking X we may assume that

φ is surjective, see Modules, Lemma 9.4 or alternatively use Nakayama's lemma (Algebra, Lemma 19.1). By Lemma 13.5 we have

$$\text{WeakAss}_X(\mathcal{F}') \subset \bigcup\nolimits_{s \in \text{WeakAss}(S)} \text{Ass}_{X_s}(\mathcal{F}'_s)$$

As WeakAss(S) is finite by assumption and since $\text{Ass}_{X_s}(\mathcal{F}'_s)$ is finite by Divisors, Lemma 2.5 we conclude that $\text{WeakAss}_X(\mathcal{F}')$ is finite. Using Algebra, Lemma 14.2 we may, after shrinking X once more, assume that $\text{WeakAss}_X(\mathcal{F}')$ is contained in the generalization of x. Now consider $\mathcal{K} = \text{Ker}(\varphi)$. We have $\text{WeakAss}_X(\mathcal{K}) \subset \text{WeakAss}_X(\mathcal{F}')$ (by Divisors, Lemma 5.4) but on the other hand, φ_x is an isomorphism, also $\varphi_{x'}$ is an isomorphism for all $x' \leadsto x$. We conclude that $\text{WeakAss}_X(\mathcal{K}) = \emptyset$ whence $\mathcal{K} = 0$ by Divisors, Lemma 5.5. □

05IL **Lemma 13.7.** *Let $R \to S$ be a ring map of finite presentation. Let M be a finite S-module. Assume $\text{WeakAss}_S(S)$ is finite. Then*

$$U = \{\mathfrak{q} \subset S \mid M_\mathfrak{q} \text{ flat over } R\}$$

is open in $\text{Spec}(S)$ and for every $g \in S$ such that $D(g) \subset U$ the localization M_g is a finitely presented S_g-module flat over R.

Proof. Follows immediately from Theorem 13.6. □

05IM **Lemma 13.8.** *Let $f : X \to S$ be a morphism of schemes which is locally of finite type. Assume the set of weakly associated points of S is locally finite in S. Then the set of points $x \in X$ where f is flat is an open subscheme $U \subset X$ and $U \to S$ is flat and locally of finite presentation.*

Proof. The problem is local on X and S, hence we may assume that X and S are affine. Then $X \to S$ corresponds to a finite type ring map $A \to B$. Choose a surjection $A[x_1, \ldots, x_n] \to B$ and consider B as an $A[x_1, \ldots, x_n]$-module. An application of Lemma 13.7 finishes the proof. □

05IN **Lemma 13.9.** *Let $f : X \to S$ be a morphism of schemes which is locally of finite type and flat. If S is integral, then f is locally of finite presentation.*

Proof. Special case of Lemma 13.8. □

053G **Proposition 13.10.** *Let R be a domain. Let $R \to S$ be a ring map of finite type. Let M be a finite S-module.*

(1) *If S is flat over R, then S is a finitely presented R-algebra.*
(2) *If M is flat as an R-module, then M is finitely presented as an S-module.*

Proof. Part (1) is a special case of Lemma 13.9. For Part (2) choose a surjection $R[x_1, \ldots, x_n] \to S$. By Lemma 13.7 we find that M is finitely presented as an $R[x_1, \ldots, x_n]$-module. We conclude by Algebra, Lemma 6.4. □

05IQ **Remark 13.11** (Finite type version of Theorem 13.6)**.** Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Assume

(1) $X \to S$ is locally of finite type,
(2) $\mathcal{F}$ is an $\mathcal{O}_X$-module of finite type, and
(3) the set of weakly associated points of S is locally finite in S.

Then $U = \{x \in X \mid \mathcal{F} \text{ flat at } x \text{ over } S\}$ is open in X and $\mathcal{F}|_U$ is flat over S and locally finitely presented relative to S (see More on Morphisms, Definition 47.1). If we ever need this result in the Stacks project we will convert this remark into a lemma with a proof.

05IR **Remark 13.12** (Algebra version of Remark 13.11)**.** Let $R \to S$ be a ring map of finite type. Let M be a finite S-module. Assume $\text{WeakAss}_R(R)$ is finite. Then

$$U = \{\mathfrak{q} \subset S \mid M_\mathfrak{q} \text{ flat over } R\}$$

is open in $\text{Spec}(S)$ and for every $g \in S$ such that $D(g) \subset U$ the localization M_g is flat over R and an S_g-module finitely presented relative to R (see More on Algebra, Definition 71.2). If we ever need this result in the Stacks project we will convert this remark into a lemma with a proof.

14. Examples of relatively pure modules

05IS In the short section we discuss some examples of results that will serve as motivation for the notion of a *relatively pure module* and the concept of an *impurity* which we will introduce later. Each of the examples is stated as a lemma. Note the similarity with the condition on associated primes to the conditions appearing in Lemmas 7.4, 8.3, 8.4, and 9.1. See also Algebra, Lemma 64.1 for a discussion.

05FV **Lemma 14.1.** *Let R be a local ring with maximal ideal $\mathfrak{m}$. Let $R \to S$ be a ring map. Let N be an S-module. Assume*

(1) *N is projective as an R-module, and*
(2) *$S/\mathfrak{m}S$ is Noetherian and $N/\mathfrak{m}N$ is a finite $S/\mathfrak{m}S$-module.*

Then for any prime $\mathfrak{q} \subset S$ which is an associated prime of $N \otimes_R \kappa(\mathfrak{p})$ where $\mathfrak{p} = R \cap \mathfrak{q}$ we have $\mathfrak{q} + \mathfrak{m}S \neq S$.

Proof. Note that the hypotheses of Lemmas 7.1 and 7.6 are satisfied. We will use the conclusions of these lemmas without further mention. Let $\Sigma \subset S$ be the multiplicative set of elements which are not zerodivisors on $N/\mathfrak{m}N$. The map $N \to \Sigma^{-1}N$ is R-universally injective. Hence we see that any $\mathfrak{q} \subset S$ which is an associated prime of $N \otimes_R \kappa(\mathfrak{p})$ is also an associated prime of $\Sigma^{-1}N \otimes_R \kappa(\mathfrak{p})$. Clearly this implies that $\mathfrak{q}$ corresponds to a prime of $\Sigma^{-1}S$. Thus $\mathfrak{q} \subset \mathfrak{q}'$ where $\mathfrak{q}'$ corresponds to an associated prime of $N/\mathfrak{m}N$ and we win. $\square$

The following lemma gives another (slightly silly) example of this phenomenon.

05IT **Lemma 14.2.** *Let R be a ring. Let $I \subset R$ be an ideal. Let $R \to S$ be a ring map. Let N be an S-module. If N is I-adically complete, then for any R-module M and for any prime $\mathfrak{q} \subset S$ which is an associated prime of $N \otimes_R M$ we have $\mathfrak{q} + IS \neq S$.*

Proof. Let $S^\wedge$ denote the I-adic completion of S. Note that N is an $S^\wedge$-module, hence also $N \otimes_R M$ is an $S^\wedge$-module. Let $z \in N \otimes_R M$ be an element such that $\mathfrak{q} = \text{Ann}_S(z)$. Since $z \neq 0$ we see that $\text{Ann}_{S^\wedge}(z) \neq S^\wedge$. Hence $\mathfrak{q}S^\wedge \neq S^\wedge$. Hence there exists a maximal ideal $\mathfrak{m} \subset S^\wedge$ with $\mathfrak{q}S^\wedge \subset \mathfrak{m}$. Since $IS^\wedge \subset \mathfrak{m}$ by Algebra, Lemma 95.6 we win. $\square$

Note that the following lemma gives an alternative proof of Lemma 14.1 as a projective module over a local ring is free, see Algebra, Theorem 84.4.

05IU **Lemma 14.3.** *Let R be a local ring with maximal ideal $\mathfrak{m}$. Let $R \to S$ be a ring map. Let N be an S-module. Assume N is isomorphic as an R-module to a direct sum of finite R-modules. Then for any R-module M and for any prime $\mathfrak{q} \subset S$ which is an associated prime of $N \otimes_R M$ we have $\mathfrak{q} + \mathfrak{m}S \neq S$.*

Proof. Write $N = \bigoplus_{i \in I} M_i$ with each M_i a finite R-module. Let M be an R-module and let $\mathfrak{q} \subset S$ be an associated prime of $N \otimes_R M$ such that $\mathfrak{q} + \mathfrak{m}S = S$. Let $z \in N \otimes_R M$ be an element with $\mathfrak{q} = \text{Ann}_S(z)$. After modifying the direct sum decomposition a little bit we may assume that $z \in M_1 \otimes_R M$ for some element $1 \in I$. Write $1 = f + \sum x_j g_j$ for some $f \in \mathfrak{q}$, $x_j \in \mathfrak{m}$, and $g_j \in S$. For any $g \in S$ denote g' the R-linear map

$$M_1 \to N \xrightarrow{g} N \to M_1$$

where the first arrow is the inclusion map, the second arrow is multiplication by g and the third arrow is the projection map. Because each $x_j \in R$ we obtain the equality

$$f' + \sum x_j g_j' = \text{id}_{M_1} \in \text{End}_R(M_1)$$

By Nakayama's lemma (Algebra, Lemma 19.1) we see that f' is surjective, hence by Algebra, Lemma 15.4 we see that f' is an isomorphism. In particular the map

$$M_1 \otimes_R M \to N \otimes_R M \xrightarrow{f} N \otimes_R M \to M_1 \otimes_R M$$

is an isomorphism. This contradicts the assumption that $fz = 0$. $\square$

05IV **Lemma 14.4.** *Let R be a henselian local ring with maximal ideal $\mathfrak{m}$. Let $R \to S$ be a ring map. Let N be an S-module. Assume N is countably generated and Mittag-Leffler as an R-module. Then for any R-module M and for any prime $\mathfrak{q} \subset S$ which is an associated prime of $N \otimes_R M$ we have $\mathfrak{q} + \mathfrak{m}S \neq S$.*

Proof. This lemma reduces to Lemma 14.3 by Algebra, Lemma 148.13. $\square$

Suppose $f : X \to S$ is a morphism of schemes and $\mathcal{F}$ is a quasi-coherent module on X. Let $\xi \in \text{Ass}_{X/S}(\mathcal{F})$ and let $Z = \overline{\{\xi\}}$. Picture

$$\begin{tikzcd} \xi \arrow[d, mapsto] & & Z \arrow[r] \arrow[dr] & X \arrow[d, "f"] \\ f(\xi) & & & S \end{tikzcd}$$

Note that $f(Z) \subset \overline{\{f(\xi)\}}$ and that $f(Z)$ is closed if and only if equality holds, i.e., $f(Z) = \overline{\{f(\xi)\}}$. It follows from Lemma 14.1 that if S, X are affine, the fibres X_s are Noetherian, $\mathcal{F}$ is of finite type, and $\Gamma(X, \mathcal{F})$ is a projective $\Gamma(S, \mathcal{O}_S)$-module, then $f(Z) = \overline{\{f(\xi)\}}$ is a closed subset. Slightly different analogous statements holds for the cases described in Lemmas 14.2, 14.3, and 14.4.

15. Impurities

05IW We want to formalize the phenomenon of which we gave examples in Section 14 in terms of specializations of points of $\text{Ass}_{X/S}(\mathcal{F})$. We also want to work locally around a point $s \in S$. In order to do so we make the following definitions.

05FW **Situation 15.1.** Here S, X are schemes and $f : X \to S$ is a finite type morphism. Also, $\mathcal{F}$ is a finite type quasi-coherent $\mathcal{O}_X$-module. Finally s is a point of S.

In this situation consider a morphism $g : T \to S$, a point $t \in T$ with $g(t) = s$, a specialization $t' \rightsquigarrow t$, and a point $\xi \in X_T$ in the base change of X lying over t'. Picture

05IX (15.1.1)
$$
\begin{tikzcd}
\xi \arrow[d, mapsto] & & \\
t' \arrow[r, rightsquigarrow] & t \arrow[r, mapsto] & s
\end{tikzcd}
\qquad
\begin{tikzcd}
X_T \arrow[r] \arrow[d] & X \arrow[d] \\
T \arrow[r, "g"] & S
\end{tikzcd}
$$

Moreover, denote $\mathcal{F}_T$ the pullback of $\mathcal{F}$ to X_T.

05IY **Definition 15.2.** In Situation 15.1 we say a diagram (15.1.1) defines an *impurity of $\mathcal{F}$ above s* if $\xi \in \text{Ass}_{X_T/T}(\mathcal{F}_T)$ and $\overline{\{\xi\}} \cap X_t = \emptyset$. We will indicate this by saying "let $(g : T \to S, t' \rightsquigarrow t, \xi)$ be an impurity of $\mathcal{F}$ above s".

05FX **Lemma 15.3.** *In Situation 15.1. If there exists an impurity of $\mathcal{F}$ above s, then there exists an impurity $(g : T \to S, t' \rightsquigarrow t, \xi)$ of $\mathcal{F}$ above s such that g is locally of finite presentation and t a closed point of the fibre of g above s.*

Proof. Let $(g : T \to S, t' \rightsquigarrow t, \xi)$ be any impurity of $\mathcal{F}$ above s. We apply Limits, Lemma 14.1 to $t \in T$ and $Z = \overline{\{\xi\}}$ to obtain an open neighbourhood $V \subset T$ of t, a commutative diagram

$$
\begin{tikzcd}
V \arrow[r, "a"'] \arrow[d] & T' \arrow[d, "b"] \\
T \arrow[r, "g"] & S,
\end{tikzcd}
$$

and a closed subscheme $Z' \subset X_{T'}$ such that

(1) the morphism $b : T' \to S$ is locally of finite presentation,
(2) we have $Z' \cap X_{a(t)} = \emptyset$, and
(3) $Z \cap X_V$ maps into Z' via the morphism $X_V \to X_{T'}$.

As t' specializes to t we may replace T by the open neighbourhood V of t. Thus we have a commutative diagram

$$
\begin{tikzcd}
X_T \arrow[r] \arrow[d] & X_{T'} \arrow[r] \arrow[d] & X \arrow[d] \\
T \arrow[r, "a"] & T' \arrow[r, "b"] & S
\end{tikzcd}
$$

where $b \circ a = g$. Let $\xi' \in X_{T'}$ denote the image of ξ. By Divisors, Lemma 7.3 we see that $\xi' \in \text{Ass}_{X_{T'}/T'}(\mathcal{F}_{T'})$. Moreover, by construction the closure of $\overline{\{\xi'\}}$ is contained in the closed subset Z' which avoids the fibre $X_{a(t)}$. In this way we see that $(T' \to S, a(t') \rightsquigarrow a(t), \xi')$ is an impurity of $\mathcal{F}$ above s.

Thus we may assume that $g : T \to S$ is locally of finite presentation. Let $Z = \overline{\{\xi\}}$. By assumption $Z_t = \emptyset$. By More on Morphisms, Lemma 22.1 this means that $Z_{t''} = \emptyset$ for t'' in an open subset of $\overline{\{t\}}$. Since the fibre of $T \to S$ over s is a Jacobson scheme, see Morphisms, Lemma 15.10 we find that there exist a closed point $t'' \in \overline{\{t\}}$ such that $Z_{t''} = \emptyset$. Then $(g : T \to S, t' \rightsquigarrow t'', \xi)$ is the desired impurity. $\square$

05IZ **Lemma 15.4.** *In Situation 15.1. Let $(g : T \to S, t' \leadsto t, \xi)$ be an impurity of $\mathcal{F}$ above s. Assume $T = \lim_{i \in I} T_i$ is a directed limit of affine schemes over S. Then for some i the triple $(T_i \to S, t'_i \leadsto t_i, \xi_i)$ is an impurity of $\mathcal{F}$ above s.*

Proof. The notation in the statement means this: Let $p_i : T \to T_i$ be the projection morphisms, let $t_i = p_i(t)$ and $t'_i = p_i(t')$. Finally $\xi_i \in X_{T_i}$ is the image of ξ. By Divisors, Lemma 7.3 it is true that ξ_i is a point of the relative assassin of $\mathcal{F}_{T_i}$ over T_i. Thus the only point is to show that $\overline{\{\xi_i\}} \cap X_{t_i} = \emptyset$ for some i.

First proof. Let $Z_i = \overline{\{\xi_i\}} \subset X_{T_i}$ and $Z = \overline{\{\xi\}} \subset X_T$ endowed with the reduced induced scheme structure. Then $Z = \lim Z_i$ by Limits, Lemma 4.4. Choose a field k and a morphism $\mathrm{Spec}(k) \to T$ whose image is t. Then

$$\emptyset = Z \times_T \mathrm{Spec}(k) = (\lim Z_i) \times_{(\lim T_i)} \mathrm{Spec}(k) = \lim Z_i \times_{T_i} \mathrm{Spec}(k)$$

because limits commute with fibred products (limits commute with limits). Each $Z_i \times_{T_i} \mathrm{Spec}(k)$ is quasi-compact because $X_{T_i} \to T_i$ is of finite type and hence $Z_i \to T_i$ is of finite type. Hence $Z_i \times_{T_i} \mathrm{Spec}(k)$ is empty for some i by Limits, Lemma 4.3. Since the image of the composition $\mathrm{Spec}(k) \to T \to T_i$ is t_i we obtain what we want.

Second proof. Set $Z = \overline{\{\xi\}}$. Apply Limits, Lemma 14.1 to this situation to obtain an open neighbourhood $V \subset T$ of t, a commutative diagram

$$\begin{tikzcd} V \arrow[r, "a"'] \arrow[d] & T' \arrow[d, "b"] \\ T \arrow[r, "g"] & S, \end{tikzcd}$$

and a closed subscheme $Z' \subset X_{T'}$ such that

(1) the morphism $b : T' \to S$ is locally of finite presentation,
(2) we have $Z' \cap X_{a(t)} = \emptyset$, and
(3) $Z \cap X_V$ maps into Z' via the morphism $X_V \to X_{T'}$.

We may assume V is an affine open of T, hence by Limits, Lemmas 4.11 and 4.13 we can find an i and an affine open $V_i \subset T_i$ with $V = f_i^{-1}(V_i)$. By Limits, Proposition 6.1 after possibly increasing i a bit we can find a morphism $a_i : V_i \to T'$ such that $a = a_i \circ f_i|_V$. The induced morphism $X_{V_i} \to X_{T'}$ maps ξ_i into Z'. As $Z' \cap X_{a(t)} = \emptyset$ we conclude that $(T_i \to S, t'_i \leadsto t_i, \xi_i)$ is an impurity of $\mathcal{F}$ above s. $\square$

05J0 **Lemma 15.5.** *In Situation 15.1. If there exists an impurity $(g : T \to S, t' \leadsto t, \xi)$ of $\mathcal{F}$ above s with g quasi-finite at t, then there exists an impurity $(g : T \to S, t' \leadsto t, \xi)$ such that $(T, t) \to (S, s)$ is an elementary étale neighbourhood.*

Proof. Let $(g : T \to S, t' \leadsto t, \xi)$ be an impurity of $\mathcal{F}$ above s such that g is quasi-finite at t. After shrinking T we may assume that g is locally of finite type. Apply More on Morphisms, Lemma 36.1 to $T \to S$ and $t \mapsto s$. This gives us a diagram

$$\begin{tikzcd} T \arrow[d] & T \times_S U \arrow[l] \arrow[d] & V \arrow[l] \arrow[dl] \\ S & U \arrow[l] & \end{tikzcd}$$

where $(U, u) \to (S, s)$ is an elementary étale neighbourhood and $V \subset T \times_S U$ is an open neighbourhood of $v = (t, u)$ such that $V \to U$ is finite and such that v

is the unique point of V lying over u. Since the morphism $V \to T$ is étale hence flat we see that there exists a specialization $v' \leadsto v$ such that $v' \mapsto t'$. Note that $\kappa(t') \subset \kappa(v')$ is finite separable. Pick any point $\zeta \in X_{v'}$ mapping to $\xi \in X_{t'}$. By Divisors, Lemma 7.3 we see that $\zeta \in \text{Ass}_{X_V/V}(\mathcal{F}_V)$. Moreover, the closure $\overline{\{\zeta\}}$ does not meet the fibre X_v as by assumption the closure $\overline{\{\xi\}}$ does not meet X_t. In other words $(V \to S, v' \leadsto v, \zeta)$ is an impurity of $\mathcal{F}$ above S.

Next, let $u' \in U'$ be the image of v' and let $\theta \in X_U$ be the image of ζ. Then $\theta \mapsto u'$ and $u' \leadsto u$. By Divisors, Lemma 7.3 we see that $\theta \in \text{Ass}_{X_U/U}(\mathcal{F})$. Moreover, as $\pi : X_V \to X_U$ is finite we see that $\pi\big(\overline{\{\zeta\}}\big) = \overline{\{\pi(\zeta)\}}$. Since v is the unique point of V lying over u we see that $X_u \cap \overline{\{\pi(\zeta)\}} = \emptyset$ because $X_v \cap \overline{\{\zeta\}} = \emptyset$. In this way we conclude that $(U \to S, u' \leadsto u, \theta)$ is an impurity of $\mathcal{F}$ above s and we win. $\square$

05J1 **Lemma 15.6.** *In Situation 15.1. Assume that S is locally Noetherian. If there exists an impurity of $\mathcal{F}$ above s, then there exists an impurity $(g : T \to S, t' \leadsto t, \xi)$ of $\mathcal{F}$ above s such that g is quasi-finite at t.*

Proof. We may replace S by an affine neighbourhood of s. By Lemma 15.3 we may assume that we have an impurity $(g : T \to S, t' \leadsto t, \xi)$ of such that g is locally of finite type and t a closed point of the fibre of g above s. We may replace T by the reduced induced scheme structure on $\overline{\{t'\}}$. Let $Z = \overline{\{\xi\}} \subset X_T$. By assumption $Z_t = \emptyset$ and the image of $Z \to T$ contains t'. By More on Morphisms, Lemma 23.1 there exists a nonempty open $V \subset Z$ such that for any $w \in f(V)$ any generic point ξ' of V_w is in $\text{Ass}_{X_T/T}(\mathcal{F}_T)$. By More on Morphisms, Lemma 22.2 there exists a nonempty open $W \subset T$ with $W \subset f(V)$. By More on Morphisms, Lemma 44.7 there exists a closed subscheme $T' \subset T$ such that $t \in T'$, $T' \to S$ is quasi-finite at t, and there exists a point $z \in T' \cap W$, $z \leadsto t$ which does not map to s. Choose any generic point ξ' of the nonempty scheme V_z. Then $(T' \to S, z \leadsto t, \xi')$ is the desired impurity. $\square$

In the following we will use the henselization $S^h = \text{Spec}(\mathcal{O}_{S,s}^h)$ of S at s, see Étale Cohomology, Definition 33.2. Since $S^h \to S$ maps to closed point of S^h to s and induces an isomorphism of residue fields, we will indicate $s \in S^h$ this closed point also. Thus $(S^h, s) \to (S, s)$ is a morphism of pointed schemes.

05J2 **Lemma 15.7.** *In Situation 15.1. If there exists an impurity $(S^h \to S, s' \leadsto s, \xi)$ of $\mathcal{F}$ above s then there exists an impurity $(T \to S, t' \leadsto t, \xi)$ of $\mathcal{F}$ above s where $(T, t) \to (S, s)$ is an elementary étale neighbourhood.*

Proof. We may replace S by an affine neighbourhood of s. Say $S = \text{Spec}(A)$ and s corresponds to the prime $\mathfrak{p} \subset A$. Then $\mathcal{O}_{S,s}^h = \text{colim}_{(T,t)} \Gamma(T, \mathcal{O}_T)$ where the limit is over the opposite of the cofiltered category of affine elementary étale neighbourhoods (T, t) of (S, s), see More on Morphisms, Lemma 31.5 and its proof. Hence $S^h = \lim_i T_i$ and we win by Lemma 15.4. $\square$

05J3 **Lemma 15.8.** *In Situation 15.1 the following are equivalent*

1. *there exists an impurity $(S^h \to S, s' \leadsto s, \xi)$ of $\mathcal{F}$ above s where S^h is the henselization of S at s,*
2. *there exists an impurity $(T \to S, t' \leadsto t, \xi)$ of $\mathcal{F}$ above s such that $(T, t) \to (S, s)$ is an elementary étale neighbourhood, and*

(3) *there exists an impurity* $(T \to S, t' \leadsto t, \xi)$ *of* $\mathcal{F}$ *above* s *such that* $T \to S$ *is quasi-finite at* t.

Proof. As an étale morphism is locally quasi-finite it is clear that (2) implies (3). We have seen that (3) implies (2) in Lemma 15.5. We have seen that (1) implies (2) in Lemma 15.7. Finally, if $(T \to S, t' \leadsto t, \xi)$ is an impurity of $\mathcal{F}$ above s such that $(T, t) \to (S, s)$ is an elementary étale neighbourhood, then we can choose a factorization $S^h \to T \to S$ of the structure morphism $S^h \to S$. Choose any point $s' \in S^h$ mapping to t' and choose any $\xi' \in X_{s'}$ mapping to $\xi \in X_{t'}$. Then $(S^h \to S, s' \leadsto s, \xi')$ is an impurity of $\mathcal{F}$ above s. We omit the details. $\square$

16. Relatively pure modules

05BB The notion of a module pure relative to a base was introduced in [GR71].

05J4 **Definition 16.1.** Let $f : X \to S$ be a morphism of schemes which is of finite type. Let $\mathcal{F}$ be a finite type quasi-coherent $\mathcal{O}_X$-module.

(1) Let $s \in S$. We say $\mathcal{F}$ is *pure along* X_s if there is no impurity $(g : T \to S, t' \leadsto t, \xi)$ of $\mathcal{F}$ above s with $(T, t) \to (S, s)$ an elementary étale neighbourhood.
(2) We say $\mathcal{F}$ is *universally pure along* X_s if there does not exist any impurity of $\mathcal{F}$ above s.
(3) We say that X is *pure along* X_s if $\mathcal{O}_X$ is pure along X_s.
(4) We say $\mathcal{F}$ is *universally* S*-pure*, or *universally pure relative to* S if $\mathcal{F}$ is universally pure along X_s for every $s \in S$.
(5) We say $\mathcal{F}$ is S*-pure*, or *pure relative to* S if $\mathcal{F}$ is pure along X_s for every $s \in S$.
(6) We say that X is S*-pure* or *pure relative to* S if $\mathcal{O}_X$ is pure relative to S.

We intentionally restrict ourselves here to morphisms which are of finite type and not just morphisms which are locally of finite type, see Remark 16.2 for a discussion. In the situation of the definition Lemma 15.8 tells us that the following are equivalent

(1) $\mathcal{F}$ is pure along X_s,
(2) there is no impurity $(g : T \to S, t' \leadsto t, \xi)$ with g quasi-finite at t,
(3) there does not exist any impurity of the form $(S^h \to S, s' \leadsto s, \xi)$, where S^h is the henselization of S at s.

If we denote $X^h = X \times_S S^h$ and $\mathcal{F}^h$ the pullback of $\mathcal{F}$ to X^h, then we can formulate the last condition in the following more positive way:

(4) All points of $\text{Ass}_{X^h/S^h}(\mathcal{F}^h)$ specialize to points of X_s.

In particular, it is clear that $\mathcal{F}$ is pure along X_s if and only if the pullback of $\mathcal{F}$ to $X \times_S \text{Spec}(\mathcal{O}_{S,s})$ is pure along X_s.

05J5 **Remark 16.2.** Let $f : X \to S$ be a morphism which is locally of finite type and $\mathcal{F}$ a quasi-coherent finite type $\mathcal{O}_X$-module. In this case it is still true that (1) and (2) above are equivalent because the proof of Lemma 15.5 does not use that f is quasi-compact. It is also clear that (3) and (4) are equivalent. However, we don't know if (1) and (3) are equivalent. In this case it may sometimes be more convenient to define purity using the equivalent conditions (3) and (4) as is done in [GR71]. On the other hand, for many applications it seems that the correct notion is really that of being universally pure.

A natural question to ask is if the property of being pure relative to the base is preserved by base change, i.e., if being pure is the same thing as being universally pure. It turns out that this is true over Noetherian base schemes (see Lemma 16.5), or if the sheaf is flat (see Lemmas 18.3 and 18.4). It is not true in general, even if the morphism and the sheaf are of finite presentation, see Examples, Section 33 for a counter example. First we match our usage of "universally" to the usual notion.

05J6 **Lemma 16.3.** *Let $f : X \to S$ be a morphism of schemes which is of finite type. Let $\mathcal{F}$ be a finite type quasi-coherent $\mathcal{O}_X$-module. Let $s \in S$. The following are equivalent*

1. *$\mathcal{F}$ is universally pure along X_s, and*
2. *for every morphism of pointed schemes $(S', s') \to (S, s)$ the pullback $\mathcal{F}_{S'}$ is pure along $X_{s'}$.*

In particular, $\mathcal{F}$ is universally pure relative to S if and only if every base change $\mathcal{F}_{S'}$ of $\mathcal{F}$ is pure relative to S'.

Proof. This is formal. $\square$

05J7 **Lemma 16.4.** *Let $f : X \to S$ be a morphism of schemes which is of finite type. Let $\mathcal{F}$ be a finite type quasi-coherent $\mathcal{O}_X$-module. Let $s \in S$. Let $(S', s') \to (S, s)$ be a morphism of pointed schemes. If $S' \to S$ is quasi-finite at s' and $\mathcal{F}$ is pure along X_s, then $\mathcal{F}_{S'}$ is pure along $X_{s'}$.*

Proof. It $(T \to S', t' \rightsquigarrow t, \xi)$ is an impurity of $\mathcal{F}_{S'}$ above s' with $T \to S'$ quasi-finite at t, then $(T \to S, t' \to t, \xi)$ is an impurity of $\mathcal{F}$ above s with $T \to S$ quasi-finite at t, see Morphisms, Lemma 19.12. Hence the lemma follows immediately from the characterization (2) of purity given following Definition 16.1. $\square$

05J8 **Lemma 16.5.** *Let $f : X \to S$ be a morphism of schemes which is of finite type. Let $\mathcal{F}$ be a finite type quasi-coherent $\mathcal{O}_X$-module. Let $s \in S$. If $\mathcal{O}_{S,s}$ is Noetherian then $\mathcal{F}$ is pure along X_s if and only if $\mathcal{F}$ is universally pure along X_s.*

Proof. First we may replace S by $\mathrm{Spec}(\mathcal{O}_{S,s})$, i.e., we may assume that S is Noetherian. Next, use Lemma 15.6 and characterization (2) of purity given in discussion following Definition 16.1 to conclude. $\square$

Purity satisfies flat descent.

05J9 **Lemma 16.6.** *Let $f : X \to S$ be a morphism of schemes which is of finite type. Let $\mathcal{F}$ be a finite type quasi-coherent $\mathcal{O}_X$-module. Let $s \in S$. Let $(S', s') \to (S, s)$ be a morphism of pointed schemes. Assume $S' \to S$ is flat at s'.*

1. *If $\mathcal{F}_{S'}$ is pure along $X_{s'}$, then $\mathcal{F}$ is pure along X_s.*
2. *If $\mathcal{F}_{S'}$ is universally pure along $X_{s'}$, then $\mathcal{F}$ is universally pure along X_s.*

Proof. Let $(T \to S, t' \rightsquigarrow t, \xi)$ be an impurity of $\mathcal{F}$ above s. Set $T_1 = T \times_S S'$, and let t_1 be the unique point of T_1 mapping to t and s'. Since $T_1 \to T$ is flat at t_1, see Morphisms, Lemma 24.7, there exists a specialization $t'_1 \rightsquigarrow t_1$ lying over $t' \rightsquigarrow t$, see Algebra, Section 40. Choose a point $\xi_1 \in X_{t'_1}$ which corresponds to a generic point of $\mathrm{Spec}(\kappa(t'_1) \otimes_{\kappa(t')} \kappa(\xi))$, see Schemes, Lemma 17.5. By Divisors, Lemma 7.3 we see that $\xi_1 \in \mathrm{Ass}_{X_{T_1}/T_1}(\mathcal{F}_{T_1})$. As the Zariski closure of $\{\xi_1\}$ in X_{T_1} maps into the Zariski closure of $\{\xi\}$ in X_T we conclude that this closure is disjoint from X_{t_1}. Hence $(T_1 \to S', t'_1 \rightsquigarrow t_1, \xi_1)$ is an impurity of $\mathcal{F}_{S'}$ above s'. In other words we have proved the contrapositive to part (2) of the lemma. Finally, if $(T, t) \to (S, s)$ is an elementary étale neighbourhood, then $(T_1, t_1) \to (S', s')$ is an elementary étale neighbourhood too, and in this way we see that (1) holds. $\square$

05K1 **Lemma 16.7.** *Let $i : Z \to X$ be a closed immersion of schemes of finite type over a scheme S. Let $s \in S$. Let $\mathcal{F}$ be a finite type, quasi-coherent sheaf on Z. Then $\mathcal{F}$ is (universally) pure along Z_s if and only if $i_*\mathcal{F}$ is (universally) pure along X_s.*

Proof. This follows from Divisors, Lemma 8.3. $\square$

17. Examples of relatively pure sheaves

05K2 Here are some example cases where it is possible to see what purity means.

05K3 **Lemma 17.1.** *Let $f : X \to S$ be a morphism of schemes which is of finite type. Let $\mathcal{F}$ be a finite type quasi-coherent $\mathcal{O}_X$-module.*

1. *If the support of $\mathcal{F}$ is proper over S, then $\mathcal{F}$ is universally pure relative to S.*
2. *If f is proper, then $\mathcal{F}$ is universally pure relative to S.*
3. *If f is proper, then X is universally pure relative to S.*

Proof. First we reduce (1) to (2). Namely, let $Z \subset X$ be the scheme theoretic support of $\mathcal{F}$. Let $i : Z \to X$ be the corresponding closed immersion and write $\mathcal{F} = i_*\mathcal{G}$ for some finite type quasi-coherent $\mathcal{O}_Z$-module $\mathcal{G}$, see Morphisms, Section 5. In case (1) $Z \to S$ is proper by assumption. Thus by Lemma 16.7 case (1) reduces to case (2).

Assume f is proper. Let $(g : T \to S, t' \rightsquigarrow t, \xi)$ be an impurity of $\mathcal{F}$ above $s \in S$. Since f is proper, it is universally closed. Hence $f_T : X_T \to T$ is closed. Since $f_T(\xi) = t'$ this implies that $t \in f(\overline{\{\xi\}})$ which is a contradiction. $\square$

05K4 **Lemma 17.2.** *Let $f : X \to S$ be a separated, finite type morphism of schemes. Let $\mathcal{F}$ be a finite type, quasi-coherent $\mathcal{O}_X$-module. Assume that $\mathrm{Supp}(\mathcal{F}_s)$ is finite for every $s \in S$. Then the following are equivalent*

1. *$\mathcal{F}$ is pure relative to S,*
2. *the scheme theoretic support of $\mathcal{F}$ is finite over S, and*
3. *$\mathcal{F}$ is universally pure relative to S.*

In particular, given a quasi-finite separated morphism $X \to S$ we see that X is pure relative to S if and only if $X \to S$ is finite.

Proof. Let $Z \subset X$ be the scheme theoretic support of $\mathcal{F}$, see Morphisms, Definition 5.5. Then $Z \to S$ is a separated, finite type morphism of schemes with finite fibres. Hence it is separated and quasi-finite, see Morphisms, Lemma 19.10. By Lemma 16.7 it suffices to prove the lemma for $Z \to S$ and the sheaf $\mathcal{F}$ viewed as a finite type quasi-coherent module on Z. Hence we may assume that $X \to S$ is separated and quasi-finite and that $\mathrm{Supp}(\mathcal{F}) = X$.

It follows from Lemma 17.1 and Morphisms, Lemma 42.10 that (2) implies (3). Trivially (3) implies (1). Assume (1) holds. We will prove that (2) holds. It is clear that we may assume S is affine. By More on Morphisms, Lemma 38.3 we can find a diagram

$$\begin{tikzcd} X \arrow[rr, "j"] \arrow[dr, "f"'] & & T \arrow[dl, "\pi"] \\ & S & \end{tikzcd}$$

with π finite and j a quasi-compact open immersion. If we show that j is closed, then j is a closed immersion and we conclude that $f = \pi \circ j$ is finite. To show that j is closed it suffices to show that specializations lift along j, see Schemes, Lemma 19.8. Let $x \in X$, set $t' = j(x)$ and let $t' \rightsquigarrow t$ be a specialization. We have to show $t \in j(X)$. Set $s' = f(x)$ and $s = \pi(t)$ so $s' \rightsquigarrow s$. By More on Morphisms, Lemma 36.4 we can find an elementary étale neighbourhood $(U, u) \to (S, s)$ and a decomposition

$$T_U = T \times_S U = V \amalg W$$

into open and closed subschemes, such that $V \to U$ is finite and there exists a unique point v of V mapping to u, and such that v maps to t in T. As $V \to T$ is étale, we can lift generalizations, see Morphisms, Lemmas 24.8 and 34.12. Hence there exists a specialization $v' \rightsquigarrow v$ such that v' maps to $t' \in T$. In particular we see that $v' \in X_U \subset T_U$. Denote $u' \in U$ the image of t'. Note that $v' \in \text{Ass}_{X_U/U}(\mathcal{F})$ because $X_{u'}$ is a finite discrete set and $X_{u'} = \text{Supp}(\mathcal{F}_{u'})$. As $\mathcal{F}$ is pure relative to S we see that v' must specialize to a point in X_u. Since v is the only point of V lying over u (and since no point of W can be a specialization of v') we see that $v \in X_u$. Hence $t \in X$. □

05K5 **Lemma 17.3.** *Let $f : X \to S$ be a finite type, flat morphism of schemes with geometrically integral fibres. Then X is universally pure over S.*

Proof. Let $\xi \in X$ with $s' = f(\xi)$ and $s' \rightsquigarrow s$ a specialization of S. If ξ is an associated point of $X_{s'}$, then ξ is the unique generic point because $X_{s'}$ is an integral scheme. Let ξ_0 be the unique generic point of X_s. As $X \to S$ is flat we can lift $s' \rightsquigarrow s$ to a specialization $\xi' \rightsquigarrow \xi_0$ in X, see Morphisms, Lemma 24.8. The $\xi \rightsquigarrow \xi'$ because ξ is the generic point of $X_{s'}$ hence $\xi \rightsquigarrow \xi_0$. This means that $(\text{id}_S, s' \to s, \xi)$ is not an impurity of $\mathcal{O}_X$ above s. Since the assumption that f is finite type, flat with geometrically integral fibres is preserved under base change, we see that there doesn't exist an impurity after any base change. In this way we see that X is universally S-pure. □

05K6 **Lemma 17.4.** *Let $f : X \to S$ be a finite type, affine morphism of schemes. Let $\mathcal{F}$ be a finite type quasi-coherent $\mathcal{O}_X$-module such that $f_*\mathcal{F}$ is locally projective on S, see Properties, Definition 21.1. Then $\mathcal{F}$ is universally pure over S.*

Proof. After reducing to the case where S is the spectrum of a henselian local ring this follows from Lemma 14.1. □

18. A criterion for purity

05L2 We first prove that given a flat family of finite type quasi-coherent sheaves the points in the relative assassin specialize to points in the relative assassins of nearby fibres (if they specialize at all).

05L3 **Lemma 18.1.** *Let $f : X \to S$ be a morphism of schemes of finite type. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module of finite type. Let $s \in S$. Assume that $\mathcal{F}$ is* *flat over S at all points of X_s. Let $x' \in Ass_{X/S}(\mathcal{F})$ with $f(x') = s'$ such that $s' \rightsquigarrow s$ is a specialization in S. If x' specializes to a point of X_s, then $x' \rightsquigarrow x$ with $x \in Ass_{X_s}(\mathcal{F}_s)$.*

Proof. Let $x' \rightsquigarrow t$ be a specialization with $t \in X_s$. We may replace X by an affine neighbourhood of t and S by an affine neighbourhood of s. Choose a closed immersion $i : X \to \mathbf{A}^n_S$. Then it suffices to prove the lemma for the module $i_*\mathcal{F}$ on $\mathbf{A}^n_S$ and the point $i(x')$. Hence we may assume $X \to S$ is of finite presentation.

Let $x' \rightsquigarrow t$ be a specialization with $t \in X_s$. Set $A = \mathcal{O}_{S,s}$, $B = \mathcal{O}_{X,t}$, and $N = \mathcal{F}_t$. Note that B is essentially of finite presentation over A and that N is a finite B-module flat over A. Also N is a finitely presented B-module by Lemma 10.9. Let $\mathfrak{q}' \subset B$ be the prime ideal corresponding to x' and let $\mathfrak{p}' \subset A$ be the prime ideal corresponding to s'. The assumption $x' \in \text{Ass}_{X/S}(\mathcal{F})$ means that $\mathfrak{q}'$ is an associated prime of $N \otimes_A \kappa(\mathfrak{p}')$. Let $\Sigma \subset B$ be the multiplicative subset of elements which are not zerodivisors on $N/\mathfrak{m}_A N$. By Lemma 7.2 the map $N \to \Sigma^{-1}N$ is universally injective. In particular, we see that $N \otimes_A \kappa(\mathfrak{p}') \to \Sigma^{-1}N \otimes_A \kappa(\mathfrak{p}')$ is injective which implies that $\mathfrak{q}'$ is an associated prime of $\Sigma^{-1}N \otimes_A \kappa(\mathfrak{p}')$ and hence $\mathfrak{q}'$ is in the image of $\text{Spec}(\Sigma^{-1}B) \to \text{Spec}(B)$. Thus Lemma 7.1 implies that $\mathfrak{q}' \subset \mathfrak{q}$ for some prime $\mathfrak{q} \in \text{Ass}_B(N/\mathfrak{m}_A N)$ (which in particular implies that $\mathfrak{m}_A = A \cap \mathfrak{q}$). If $x \in X_s$ denotes the point corresponding to $\mathfrak{q}$, then $x \in \text{Ass}_{X_s}(\mathcal{F}_s)$ and $x' \rightsquigarrow x$ as desired. $\square$

05L4 **Lemma 18.2.** *Let $f : X \to S$ be a morphism of schemes of finite type. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module of finite type. Let $s \in S$. Let $(S', s') \to (S, s)$ be an elementary étale neighbourhood and let*

$$\begin{tikzcd} X \arrow[d] & X' \arrow[l, "g"'] \arrow[d] \\ S & S' \arrow[l] \end{tikzcd}$$

be a commutative diagram of morphisms of schemes. Assume

1. *$\mathcal{F}$ is flat over S at all points of X_s,*
2. *$X' \to S'$ is of finite type,*
3. *$g^*\mathcal{F}$ is pure along $X'_{s'}$,*
4. *$g : X' \to X$ is étale, and*
5. *$g(X')$ contains $Ass_{X_s}(\mathcal{F}_s)$.*

In this situation $\mathcal{F}$ is pure along X_s if and only if the image of $X' \to X \times_S S'$ contains the points of $Ass_{X \times_S S'/S'}(\mathcal{F} \times_S S')$ lying over points in S' which specialize to s'.

Proof. Since the morphism $S' \to S$ is étale, we see that if $\mathcal{F}$ is pure along X_s, then $\mathcal{F} \times_S S'$ is pure along X_s, see Lemma 16.4. Since purity satisfies flat descent, see Lemma 16.6, we see that if $\mathcal{F} \times_S S'$ is pure along $X_{s'}$, then $\mathcal{F}$ is pure along X_s. Hence we may replace S by S' and assume that $S = S'$ so that $g : X' \to X$ is an étale morphism between schemes of finite type over S. Moreover, we may replace S by $\text{Spec}(\mathcal{O}_{S,s})$ and assume that S is local.

First, assume that $\mathcal{F}$ is pure along X_s. In this case every point of $\text{Ass}_{X/S}(\mathcal{F})$ specializes to a point of X_s by purity. Hence by Lemma 18.1 we see that every point of $\text{Ass}_{X/S}(\mathcal{F})$ specializes to a point of $\text{Ass}_{X_s}(\mathcal{F}_s)$. Thus every point of $\text{Ass}_{X/S}(\mathcal{F})$ is in the image of g (as the image is open and contains $\text{Ass}_{X_s}(\mathcal{F}_s)$).

Conversely, assume that $g(X')$ contains $\text{Ass}_{X/S}(\mathcal{F})$. Let $S^h = \text{Spec}(\mathcal{O}_{S,s}^h)$ be the henselization of S at s. Denote $g^h : (X')^h \to X^h$ the base change of g by $S^h \to S$, and denote $\mathcal{F}^h$ the pullback of $\mathcal{F}$ to X^h. By Divisors, Lemma 7.3 and Remark 7.4 the relative assassin $\text{Ass}_{X^h/S^h}(\mathcal{F}^h)$ is the inverse image of $\text{Ass}_{X/S}(\mathcal{F})$ via the projection $X^h \to X$. As we have assumed that $g(X')$ contains $\text{Ass}_{X/S}(\mathcal{F})$ we conclude that the base change $g^h((X')^h) = g(X') \times_S S^h$ contains $\text{Ass}_{X^h/S^h}(\mathcal{F}^h)$. In this way we reduce to the case where S is the spectrum of a henselian local ring. Let $x \in \text{Ass}_{X/S}(\mathcal{F})$. To finish the proof of the lemma we have to show that x specializes to a point of X_s, see criterion (4) for purity in discussion following Definition 16.1. By assumption there exists a $x' \in X'$ such that $g(x') = x$. As $g : X' \to X$ is étale, we see that $x' \in \text{Ass}_{X'/S}(g^*\mathcal{F})$, see Lemma 2.8 (applied to the morphism of fibres $X'_w \to X_w$ where $w \in S$ is the image of x'). Since $g^*\mathcal{F}$ is pure along X'_s we see that $x' \rightsquigarrow y$ for some $y \in X'_s$. Hence $x = g(x') \rightsquigarrow g(y)$ and $g(y) \in X_s$ as desired. $\square$

05L5 **Lemma 18.3.** *Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $s \in S$. Assume*

1. *f is of finite type,*
2. *$\mathcal{F}$ is of finite type,*
3. *$\mathcal{F}$ is flat over S at all points of X_s, and*
4. *$\mathcal{F}$ is pure along X_s.*

Then $\mathcal{F}$ is universally pure along X_s.

Proof. We first make a preliminary remark. Suppose that $(S', s') \to (S, s)$ is an elementary étale neighbourhood. Denote $\mathcal{F}'$ the pullback of $\mathcal{F}$ to $X' = X \times_S S'$. By the discussion following Definition 16.1 we see that $\mathcal{F}'$ is pure along $X'_{s'}$. Moreover, $\mathcal{F}'$ is flat over S' along $X'_{s'}$. Then it suffices to prove that $\mathcal{F}'$ is universally pure along $X'_{s'}$. Namely, given any morphism $(T, t) \to (S, s)$ of pointed schemes the fibre product $(T', t') = (T \times_S S', (t, s'))$ is flat over (T, t) and hence if $\mathcal{F}_{T'}$ is pure along $X_{t'}$ then $\mathcal{F}_T$ is pure along X_t by Lemma 16.6. Thus during the proof we may always replace (s, S) by an elementary étale neighbourhood. We may also replace S by $\text{Spec}(\mathcal{O}_{S,s})$ due to the local nature of the problem.

Choose an elementary étale neighbourhood $(S', s') \to (S, s)$ and a commutative diagram

$$\begin{tikzcd} X \arrow[d] & X' \arrow[l, "g"'] \arrow[d] \\ S & \text{Spec}(\mathcal{O}_{S',s'}) \arrow[l] \end{tikzcd}$$

such that $X' \to X \times_S \text{Spec}(\mathcal{O}_{S',s'})$ is étale, $X_s = g((X')_{s'})$, the scheme X' is affine, and such that $\Gamma(X', g^*\mathcal{F})$ is a free $\mathcal{O}_{S',s'}$-module, see Lemma 12.11. Note that $X' \to \text{Spec}(\mathcal{O}_{S',s'})$ is of finite type (as a quasi-compact morphism which is the composition of an étale morphism and the base change of a finite type morphism). By our preliminary remarks in the first paragraph of the proof we may replace S

by $\text{Spec}(\mathcal{O}_{S',s'})$. Hence we may assume there exists a commutative diagram

$$\begin{tikzcd} X \arrow[dr] & & X' \arrow[ll, "g"'] \arrow[dl] \\ & S & \end{tikzcd}$$

of schemes of finite type over S, where g is étale, $X_s \subset g(X')$, with S local with closed point s, with X' affine, and with $\Gamma(X', g^*\mathcal{F})$ a free $\Gamma(S, \mathcal{O}_S)$-module. Note that in this case $g^*\mathcal{F}$ is universally pure over S, see Lemma 17.4.

In this situation we apply Lemma 18.2 to deduce that $\text{Ass}_{X/S}(\mathcal{F}) \subset g(X')$ from our assumption that $\mathcal{F}$ is pure along X_s and flat over S along X_s. By Divisors, Lemma 7.3 and Remark 7.4 we see that for any morphism of pointed schemes $(T, t) \to (S, s)$ we have

$$\text{Ass}_{X_T/T}(\mathcal{F}_T) \subset (X_T \to X)^{-1}(\text{Ass}_{X/S}(\mathcal{F})) \subset g(X') \times_S T = g_T(X'_T).$$

Hence by Lemma 18.2 applied to the base change of our displayed diagram to (T, t) we conclude that $\mathcal{F}_T$ is pure along X_t as desired. □

05L6 **Lemma 18.4.** *Let $f : X \to S$ be a finite type morphism of schemes. Let $\mathcal{F}$ be a finite type quasi-coherent $\mathcal{O}_X$-module. Assume $\mathcal{F}$ is flat over S. In this case $\mathcal{F}$ is pure relative to S if and only if $\mathcal{F}$ is universally pure relative to S.*

Proof. Immediate consequence of Lemma 18.3 and the definitions. □

05MA **Lemma 18.5.** *Let I be a directed set. Let $(S_i, g_{ii'})$ be an inverse system of affine schemes over I. Set $S = \lim_i S_i$ and $s \in S$. Denote $g_i : S \to S_i$ the projections and set $s_i = g_i(s)$. Suppose that $f : X \to S$ is a morphism of finite presentation, $\mathcal{F}$ a quasi-coherent $\mathcal{O}_X$-module of finite presentation which is pure along X_s and flat over S at all points of X_s. Then there exists an $i \in I$, a morphism of finite presentation $X_i \to S_i$, a quasi-coherent $\mathcal{O}_{X_i}$-module $\mathcal{F}_i$ of finite presentation which is pure along $(X_i)_{s_i}$ and flat over S_i at all points of $(X_i)_{s_i}$ such that $X \cong X_i \times_{S_i} S$ and such that the pullback of $\mathcal{F}_i$ to X is isomorphic to $\mathcal{F}$.*

Proof. Let $U \subset X$ be the set of points where $\mathcal{F}$ is flat over S. By More on Morphisms, Theorem 15.1 this is an open subscheme of X. By assumption $X_s \subset U$. As X_s is quasi-compact, we can find a quasi-compact open $U' \subset U$ with $X_s \subset U'$. By Limits, Lemma 10.1 we can find an $i \in I$ and a morphism of finite presentation $f_i : X_i \to S_i$ whose base change to S is isomorphic to f_i. Fix such a choice and set $X_{i'} = X_i \times_{S_i} S_{i'}$. Then $X = \lim_{i'} X_{i'}$ with affine transition morphisms. By Limits, Lemma 10.2 we can, after possible increasing i assume there exists a quasi-coherent $\mathcal{O}_{X_i}$-module $\mathcal{F}_i$ of finite presentation whose base change to S is isomorphic to $\mathcal{F}$. By Limits, Lemma 4.11 after possibly increasing i we may assume there exists an open $U'_i \subset X_i$ whose inverse image in X is U'. Note that in particular $(X_i)_{s_i} \subset U'_i$. By Limits, Lemma 10.4 (after increasing i once more) we may assume that $\mathcal{F}_i$ is flat on U'_i. In particular we see that $\mathcal{F}_i$ is flat along $(X_i)_{s_i}$.

Next, we use Lemma 12.5 to choose an elementary étale neighbourhood $(S'_i, s'_i) \to (S_i, s_i)$ and a commutative diagram of schemes

$$\begin{tikzcd} X_i \arrow[d] & X'_i \arrow[l, "g_i"'] \arrow[d] \\ S_i & S'_i \arrow[l] \end{tikzcd}$$

such that g_i is étale, $(X_i)_{s_i} \subset g_i(X'_i)$, the schemes X'_i, S'_i are affine, and such that $\Gamma(X'_i, g_i^*\mathcal{F}_i)$ is a projective $\Gamma(S'_i, \mathcal{O}_{S'_i})$-module. Note that $g_i^*\mathcal{F}_i$ is universally pure over S'_i, see Lemma 17.4. We may base change the diagram above to a diagram with morphisms $(S'_{i'}, s'_{i'}) \to (S_{i'}, s_{i'})$ and $g_{i'} : X'_{i'} \to X_{i'}$ over $S_{i'}$ for any $i' \geq i$ and we may base change the diagram to a diagram with morphisms $(S', s') \to (S, s)$ and $g : X' \to X$ over S.

At this point we can use our criterion for purity. Set $W'_i \subset X_i \times_{S_i} S'_i$ equal to the image of the étale morphism $X'_i \to X_i \times_{S_i} S'_i$. For every $i' \geq i$ we have similarly the image $W'_{i'} \subset X_{i'} \times_{S_{i'}} S'_{i'}$ and we have the image $W' \subset X \times_S S'$. Taking images commutes with base change, hence $W'_{i'} = W'_i \times_{S'_i} S'_{i'}$ and $W' = W_i \times_{S'_i} S'$. Because $\mathcal{F}$ is pure along X_s the Lemma 18.2 implies that

05MB (18.5.1) $$f^{-1}(\mathrm{Spec}(\mathcal{O}_{S',s'})) \cap \mathrm{Ass}_{X \times_S S'/S'}(\mathcal{F} \times_S S') \subset W'$$

By More on Morphisms, Lemma 23.5 we see that

$$E = \{t \in S' \mid \mathrm{Ass}_{X_t}(\mathcal{F}_t) \subset W'\} \quad \text{and} \quad E_{i'} = \{t \in S'_{i'} \mid \mathrm{Ass}_{X_t}(\mathcal{F}_{i',t}) \subset W'_{i'}\}$$

are locally constructible subsets of S' and $S'_{i'}$. By More on Morphisms, Lemma 23.4 we see that $E_{i'}$ is the inverse image of E_i under the morphism $S'_{i'} \to S'_i$ and that E is the inverse image of E_i under the morphism $S' \to S'_i$. Thus Equation (18.5.1) is equivalent to the assertion that $\mathrm{Spec}(\mathcal{O}_{S',s'})$ maps into E_i. As $\mathcal{O}_{S',s'} = \mathrm{colim}_{i' \geq i} \mathcal{O}_{S'_{i'}, s'_{i'}}$ we see that $\mathrm{Spec}(\mathcal{O}_{S'_{i'}, s'_{i'}})$ maps into E_i for some $i' \geq i$, see Limits, Lemma 4.10. Then, applying Lemma 18.2 to the situation over $S_{i'}$, we conclude that $\mathcal{F}_{i'}$ is pure along $(X_{i'})_{s_{i'}}$. $\square$

05MC **Lemma 18.6.** *Let $f : X \to S$ be a morphism of finite presentation. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module of finite presentation flat over S. Then the set*

$$U = \{s \in S \mid \mathcal{F} \textit{ is pure along } X_s\}$$

is open in S.

Proof. Let $s \in U$. Using Lemma 12.5 we can find an elementary étale neighbourhood $(S', s') \to (S, s)$ and a commutative diagram

$$\begin{tikzcd} X \arrow[d] & X' \arrow[l, "g"'] \arrow[d] \\ S & S' \arrow[l] \end{tikzcd}$$

such that g is étale, $X_s \subset g(X')$, the schemes X', S' are affine, and such that $\Gamma(X', g^*\mathcal{F})$ is a projective $\Gamma(S', \mathcal{O}_{S'})$-module. Note that $g^*\mathcal{F}$ is universally pure over S', see Lemma 17.4. Set $W' \subset X \times_S S'$ equal to the image of the étale morphism $X' \to X \times_S S'$. Note that W is open and quasi-compact over S'. Set

$$E = \{t \in S' \mid \mathrm{Ass}_{X_t}(\mathcal{F}_t) \subset W'\}.$$

By More on Morphisms, Lemma 23.5 E is a constructible subset of S'. By Lemma 18.2 we see that $\mathrm{Spec}(\mathcal{O}_{S',s'}) \subset E$. By Morphisms, Lemma 21.4 we see that E contains an open neighbourhood V' of s'. Applying Lemma 18.2 once more we see that for any point s_1 in the image of V' in S the sheaf $\mathcal{F}$ is pure along X_{s_1}. Since $S' \to S$ is étale the image of V' in S is open and we win. $\square$

19. How purity is used

05L7 Here are some examples of how purity can be used. The first lemma actually uses a slightly weaker form of purity.

05L8 **Lemma 19.1.** *Let $f : X \to S$ be a morphism of finite type. Let $\mathcal{F}$ be a quasi-coherent sheaf of finite type on X. Assume S is local with closed point s. Assume $\mathcal{F}$ is pure along X_s and that $\mathcal{F}$ is flat over S. Let $\varphi : \mathcal{F} \to \mathcal{G}$ of quasi-coherent $\mathcal{O}_X$-modules. Then the following are equivalent*

1. *the map on stalks φ_x is injective for all $x \in Ass_{X_s}(\mathcal{F}_s)$, and*
2. *φ is injective.*

Proof. Let $\mathcal{K} = \mathrm{Ker}(\varphi)$. Our goal is to prove that $\mathcal{K} = 0$. In order to do this it suffices to prove that $\mathrm{WeakAss}_X(\mathcal{K}) = \emptyset$, see Divisors, Lemma 5.5. We have $\mathrm{WeakAss}_X(\mathcal{K}) \subset \mathrm{WeakAss}_X(\mathcal{F})$, see Divisors, Lemma 5.4. As $\mathcal{F}$ is flat we see from Lemma 13.5 that $\mathrm{WeakAss}_X(\mathcal{F}) \subset \mathrm{Ass}_{X/S}(\mathcal{F})$. By purity any point x' of $\mathrm{Ass}_{X/S}(\mathcal{F})$ is a generalization of a point of X_s, and hence is the specialization of a point $x \in \mathrm{Ass}_{X_s}(\mathcal{F}_s)$, by Lemma 18.1. Hence the injectivity of φ_x implies the injectivity of $\varphi_{x'}$, whence $\mathcal{K}_{x'} = 0$. $\square$

05MD **Proposition 19.2.** *Let $f : X \to S$ be an affine, finitely presented morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module of finite presentation, flat over S. Then the following are equivalent*

1. *$f_*\mathcal{F}$ is locally projective on S, and*
2. *$\mathcal{F}$ is pure relative to S.*

In particular, given a ring map $A \to B$ of finite presentation and a finitely presented B-module N flat over A we have: N is projective as an A-module if and only if $\widetilde{N}$ on $\mathrm{Spec}(B)$ *is pure relative to* $\mathrm{Spec}(A)$.

Proof. The implication (1) $\Rightarrow$ (2) is Lemma 17.4. Assume $\mathcal{F}$ is pure relative to S. Note that by Lemma 18.3 this implies $\mathcal{F}$ remains pure after any base change. By Descent, Lemma 7.7 it suffices to prove $f_*\mathcal{F}$ is fpqc locally projective on S. Pick $s \in S$. We will prove that the restriction of $f_*\mathcal{F}$ to an étale neighbourhood of s is locally projective. Namely, by Lemma 12.5, after replacing S by an affine elementary étale neighbourhood of s, we may assume there exists a diagram

$$\begin{tikzcd} X \arrow[dr] & & X' \arrow[ll, "g"'] \arrow[dl] \\ & S & \end{tikzcd}$$

of schemes affine and of finite presentation over S, where g is étale, $X_s \subset g(X')$, and with $\Gamma(X', g^*\mathcal{F})$ a projective $\Gamma(S, \mathcal{O}_S)$-module. Note that in this case $g^*\mathcal{F}$ is universally pure over S, see Lemma 17.4. Hence by Lemma 18.2 we see that the open $g(X')$ contains the points of $\mathrm{Ass}_{X/S}(\mathcal{F})$ lying over $\mathrm{Spec}(\mathcal{O}_{S,s})$. Set

$$E = \{t \in S \mid \mathrm{Ass}_{X_t}(\mathcal{F}_t) \subset g(X')\}.$$

By More on Morphisms, Lemma 23.5 E is a constructible subset of S. We have seen that $\text{Spec}(\mathcal{O}_{S,s}) \subset E$. By Morphisms, Lemma 21.4 we see that E contains an open neighbourhood of s. Hence after replacing S by an affine neighbourhood of s we may assume that $\text{Ass}_{X/S}(\mathcal{F}) \subset g(X')$. By Lemma 7.4 this means that

$$\Gamma(X, \mathcal{F}) \longrightarrow \Gamma(X', g^*\mathcal{F})$$

is $\Gamma(S, \mathcal{O}_S)$-universally injective. By Algebra, Lemma 88.7 we conclude that $\Gamma(X, \mathcal{F})$ is Mittag-Leffler as an $\Gamma(S, \mathcal{O}_S)$-module. Since $\Gamma(X, \mathcal{F})$ is countably generated and flat as a $\Gamma(S, \mathcal{O}_S)$-module, we conclude it is projective by Algebra, Lemma 92.1. $\square$

We can use the proposition to improve some of our earlier results. The following lemma is an improvement of Proposition 12.4.

05ME **Lemma 19.3.** *Let $f : X \to S$ be a morphism which is locally of finite presentation. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module which is of finite presentation. Let $x \in X$ with $s = f(x) \in S$. If $\mathcal{F}$ is flat at x over S there exists an affine elementary étale neighbourhood $(S', s') \to (S, s)$ and an affine open $U' \subset X \times_S S'$ which contains $x' = (x, s')$ such that $\Gamma(U', \mathcal{F}|_{U'})$ is a projective $\Gamma(S', \mathcal{O}_{S'})$-module.*

Proof. During the proof we may replace X by an open neighbourhood of x and we may replace S by an elementary étale neighbourhood of s. Hence, by openness of flatness (see More on Morphisms, Theorem 15.1) we may assume that $\mathcal{F}$ is flat over S. We may assume S and X are affine. After shrinking X some more we may assume that any point of $\text{Ass}_{X_s}(\mathcal{F}_s)$ is a generalization of x. This property is preserved on replacing (S, s) by an elementary étale neighbourhood. Hence we may apply Lemma 12.5 to arrive at the situation where there exists a diagram

$$\begin{tikzcd} X \arrow[dr] & & X' \arrow[ll, "g"'] \arrow[dl] \\ & S & \end{tikzcd}$$

of schemes affine and of finite presentation over S, where g is étale, $X_s \subset g(X')$, and with $\Gamma(X', g^*\mathcal{F})$ a projective $\Gamma(S, \mathcal{O}_S)$-module. Note that in this case $g^*\mathcal{F}$ is universally pure over S, see Lemma 17.4.

Let $U \subset g(X')$ be an affine open neighbourhood of x. We claim that $\mathcal{F}|_U$ is pure along U_s. If we prove this, then the lemma follows because $\mathcal{F}|_U$ will be pure relative to S after shrinking S, see Lemma 18.6, whereupon the projectivity follows from Proposition 19.2. To prove the claim we have to show, after replacing (S, s) by an arbitrary elementary étale neighbourhood, that any point ξ of $\text{Ass}_{U/S}(\mathcal{F}|_U)$ lying over some $s' \in S$, $s' \rightsquigarrow s$ specializes to a point of U_s. Since $U \subset g(X')$ we can find a $\xi' \in X'$ with $g(\xi') = \xi$. Because $g^*\mathcal{F}$ is pure over S, using Lemma 18.1, we see there exists a specialization $\xi' \rightsquigarrow x'$ with $x' \in \text{Ass}_{X'_s}(g^*\mathcal{F}_s)$. Then $g(x') \in \text{Ass}_{X_s}(\mathcal{F}_s)$ (see for example Lemma 2.8 applied to the étale morphism $X'_s \to X_s$ of Noetherian schemes) and hence $g(x') \rightsquigarrow x$ by our choice of X above! Since $x \in U$ we conclude that $g(x') \in U$. Thus $\xi = g(\xi') \rightsquigarrow g(x') \in U_s$ as desired. $\square$

The following lemma is an improvement of Lemma 12.9.

05MF **Lemma 19.4.** *Let $f : X \to S$ be a morphism which is locally of finite type. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module which is of finite type. Let $x \in X$ with $s = f(x) \in S$. If $\mathcal{F}$ is flat at x over S there exists an affine elementary étale* *neighbourhood* $(S', s') \to (S, s)$ *and an affine open* $U' \subset X \times_S \mathrm{Spec}(\mathcal{O}_{S',s'})$ *which contains* $x' = (x, s')$ *such that* $\Gamma(U', \mathcal{F}|_{U'})$ *is a free* $\mathcal{O}_{S',s'}$*-module.*

Proof. The question is Zariski local on X and S. Hence we may assume that X and S are affine. Then we can find a closed immersion $i : X \to \mathbf{A}^n_S$ over S. It is clear that it suffices to prove the lemma for the sheaf $i_*\mathcal{F}$ on $\mathbf{A}^n_S$ and the point $i(x)$. In this way we reduce to the case where $X \to S$ is of finite presentation. After replacing S by $\mathrm{Spec}(\mathcal{O}_{S',s'})$ and X by an open of $X \times_S \mathrm{Spec}(\mathcal{O}_{S',s'})$ we may assume that $\mathcal{F}$ is of finite presentation, see Proposition 10.3. In this case we may appeal to Lemma 19.3 and Algebra, Theorem 84.4 to conclude. $\square$

05U7 **Lemma 19.5.** *Let* $A \to B$ *be a local ring map of local rings which is essentially of finite type. Let* N *be a finite* B*-module which is flat as an* A*-module. If* A *is henselian, then* N *is a filtered colimit*

$$N = \mathrm{colim}_i\, F_i$$

of free A*-modules* F_i *such that all transition maps* $u_i : F_i \to F_{i'}$ *of the system induce injective maps* $\overline{u}_i : F_i/\mathfrak{m}_A F_i \to F_{i'}/\mathfrak{m}_A F_{i'}$. *Also,* N *is a Mittag-Leffler* A*-module.*

Proof. We can find a morphism of finite type $X \to S = \mathrm{Spec}(A)$ and a point $x \in X$ lying over the closed point s of S and a finite type quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ such that $\mathcal{F}_x \cong N$ as an A-module. After shrinking X we may assume that each point of $\mathrm{Ass}_{X_s}(\mathcal{F}_s)$ specializes to x. By Lemma 19.4 we see that there exists a fundamental system of affine open neighbourhoods $U_i \subset X$ of x such that $\Gamma(U_i, \mathcal{F})$ is a free A-module F_i. Note that if $U_{i'} \subset U_i$, then

$$F_i/\mathfrak{m}_A F_i = \Gamma(U_{i,s}, \mathcal{F}_s) \longrightarrow \Gamma(U_{i',s}, \mathcal{F}_s) = F_{i'}/\mathfrak{m}_A F_{i'}$$

is injective because a section of the kernel would be supported at a closed subset of X_s not meeting x which is a contradiction to our choice of X above. Since the maps $F_i \to F_{i'}$ are A-universally injective (Lemma 7.5) it follows that N is Mittag-Leffler by Algebra, Lemma 88.8. $\square$

The following lemma should be skipped if reading through for the first time.

0ASX **Lemma 19.6.** *Let* $A \to B$ *be a local ring map of local rings which is essentially of finite type. Let* N *be a finite* B*-module which is flat as an* A*-module. If* A *is a valuation ring, then any element of* N *has a content ideal* $I \subset A$ *(More on Algebra, Definition 22.1).*

Proof. Let $A \subset A^h$ be the henselization. Let B' be the localization of $B \otimes_A A^h$ at the maximal ideal $\mathfrak{m}_B \otimes A^h + B \otimes \mathfrak{m}_{A^h}$. Then $B \to B'$ is flat, hence faithfully flat. Let $N' = N \otimes_B B'$. Let $x \in N$ and let $x' \in N'$ be the image. We claim that for an ideal $I \subset A$ we have $x \in IN \Leftrightarrow x' \in IN'$. Namely, $N/IN \to N'/IN'$ is the tensor product of $B \to B'$ with N/IN and $B \to B'$ is universally injective by Algebra, Lemma 81.11. By More on Algebra, Lemma 96.5 and Algebra, Lemma 49.17 the map $A \to A^h$ defines an inclusion preserving bijection $I \mapsto IA^h$ on sets of ideals. We conclude that x has a content ideal in A if and only if x' has a content ideal in A^h. The assertion for $x' \in N'$ follows from Lemma 19.5 and Algebra, Lemma 88.6. $\square$

20. Flattening functors

05MG Let S be a scheme. Recall that a functor $F : (Sch/S)^{opp} \to Sets$ is called limit preserving if for every directed inverse system $\{T_i\}_{i \in I}$ of affine schemes with limit T we have $F(T) = \text{colim}_i F(T_i)$.

05MH **Situation 20.1.** Let $f : X \to S$ be a morphism of schemes. Let $u : \mathcal{F} \to \mathcal{G}$ be a homomorphism of quasi-coherent $\mathcal{O}_X$-modules. For any scheme T over S we will denote $u_T : \mathcal{F}_T \to \mathcal{G}_T$ the base change of u to T, in other words, u_T is the pullback of u via the projection morphism $X_T = X \times_S T \to X$. In this situation we can consider the functor

05MI
$$(20.1.1) \quad F_{iso} : (Sch/S)^{opp} \longrightarrow Sets, \quad T \longrightarrow \begin{cases} \{*\} & \text{if} \quad u_T \text{ is an isomorphism,} \\ \emptyset & \text{else.} \end{cases}$$

There are variants F_{inj}, F_{surj}, F_{zero} where we ask that u_T is injective, surjective, or zero.

05MJ **Lemma 20.2.** *In Situation 20.1.*

(1) *Each of the functors F_{iso}, F_{inj}, F_{surj}, F_{zero} satisfies the sheaf property for the fpqc topology.*
(2) *If f is quasi-compact and $\mathcal{G}$ is of finite type, then F_{surj} is limit preserving.*
(3) *If f is quasi-compact and $\mathcal{F}$ of finite type, then F_{zero} is limit preserving.*
(4) *If f is quasi-compact, $\mathcal{F}$ is of finite type, and $\mathcal{G}$ is of finite presentation, then F_{iso} is limit preserving.*

Proof. Let $\{T_i \to T\}_{i \in I}$ be an fpqc covering of schemes over S. Set $X_i = X_{T_i} = X \times_S T_i$ and $u_i = u_{T_i}$. Note that $\{X_i \to X_T\}_{i \in I}$ is an fpqc covering of X_T, see Topologies, Lemma 9.7. In particular, for every $x \in X_T$ there exists an $i \in I$ and an $x_i \in X_i$ mapping to x. Since $\mathcal{O}_{X_T,x} \to \mathcal{O}_{X_i,x_i}$ is flat, hence faithfully flat (see Algebra, Lemma 38.17) we conclude that $(u_i)_{x_i}$ is injective, surjective, bijective, or zero if and only if $(u_T)_x$ is injective, surjective, bijective, or zero. Whence part (1) of the lemma.

Proof of (2). Assume f quasi-compact and $\mathcal{G}$ of finite type. Let $T = \lim_{i \in I} T_i$ be a directed limit of affine S-schemes and assume that u_T is surjective. Set $X_i = X_{T_i} = X \times_S T_i$ and $u_i = u_{T_i} : \mathcal{F}_i = \mathcal{F}_{T_i} \to \mathcal{G}_i = \mathcal{G}_{T_i}$. To prove part (2) we have to show that u_i is surjective for some i. Pick $i_0 \in I$ and replace I by $\{i \mid i \geq i_0\}$. Since f is quasi-compact the scheme X_{i_0} is quasi-compact. Hence we may choose affine opens $W_1, \ldots, W_m \subset X$ and an affine open covering $X_{i_0} = U_{1,i_0} \cup \ldots \cup U_{m,i_0}$ such that U_{j,i_0} maps into W_j under the projection morphism $X_{i_0} \to X$. For any $i \in I$ let $U_{j,i}$ be the inverse image of U_{j,i_0}. Setting $U_j = \lim_i U_{j,i}$ we see that $X_T = U_1 \cup \ldots \cup U_m$ is an affine open covering of X_T. Now it suffices to show, for a given $j \in \{1, \ldots, m\}$ that $u_i|_{U_{j,i}}$ is surjective for some $i = i(j) \in I$. Using Properties, Lemma 16.1 this translates into the following algebra problem: Let A be a ring and let $u : M \to N$ be an A-module map. Suppose that $R = \text{colim}_{i \in I} R_i$ is a directed colimit of A-algebras. If N is a finite A-module and if $u \otimes 1 : M \otimes_A R \to N \otimes_A R$ is surjective, then for some i the map $u \otimes 1 : M \otimes_A R_i \to N \otimes_A R_i$ is surjective. This is Algebra, Lemma 126.5 part (2).

Proof of (3). Exactly the same arguments as given in the proof of (2) reduces this to the following algebra problem: Let A be a ring and let $u : M \to N$ be an A-module map. Suppose that $R = \text{colim}_{i \in I} R_i$ is a directed colimit of A-algebras. If M is a finite A-module and if $u \otimes 1 : M \otimes_A R \to N \otimes_A R$ is zero, then for some i the map $u \otimes 1 : M \otimes_A R_i \to N \otimes_A R_i$ is zero. This is Algebra, Lemma 126.5 part (1).

Proof of (4). Assume f quasi-compact and $\mathcal{F}, \mathcal{G}$ of finite presentation. Arguing in exactly the same manner as in the previous paragraph (using in addition also Properties, Lemma 16.2) part (3) translates into the following algebra statement: Let A be a ring and let $u : M \to N$ be an A-module map. Suppose that $R = \mathrm{colim}_{i \in I} R_i$ is a directed colimit of A-algebras. Assume M is a finite A-module, N is a finitely presented A-module, and $u \otimes 1 : M \otimes_A R \to N \otimes_A R$ is an isomorphism. Then for some i the map $u \otimes 1 : M \otimes_A R_i \to N \otimes_A R_i$ is an isomorphism. This is Algebra, Lemma 126.5 part (3). ☐

05MK **Situation 20.3.** Let $(A, \mathfrak{m}_A)$ be a local ring. Denote $\mathcal{C}$ the category whose objects are A-algebras A' which are local rings such that the algebra structure $A \to A'$ is a local homomorphism of local rings. A morphism between objects A', A'' of $\mathcal{C}$ is a local homomorphism $A' \to A''$ of A-algebras. Let $A \to B$ be a local ring map of local rings and let M be a B-module. If A' is an object of $\mathcal{C}$ we set $B' = B \otimes_A A'$ and we set $M' = M \otimes_A A'$ as a B'-module. Given $A' \in \mathrm{Ob}(\mathcal{C})$, consider the condition

05ML $$\forall \mathfrak{q} \in V(\mathfrak{m}_{A'}B' + \mathfrak{m}_B B') \subset \mathrm{Spec}(B') : M'_{\mathfrak{q}} \text{ is flat over } A'. \quad (20.3.1)$$

Note the similarity with More on Algebra, Equation (17.1.1). In particular, if $A' \to A''$ is a morphism of $\mathcal{C}$ and (20.3.1) holds for A', then it holds for A'', see More on Algebra, Lemma 17.2. Hence we obtain a functor

05MM $$F_{lf} : \mathcal{C} \longrightarrow \mathit{Sets}, \quad A' \longrightarrow \begin{cases} \{*\} & \text{if (20.3.1) holds,} \\ \emptyset & \text{else.} \end{cases} \quad (20.3.2)$$

05MN **Lemma 20.4.** *In Situation 20.3.*

(1) *If $A' \to A''$ is a flat morphism in $\mathcal{C}$ then $F_{lf}(A') = F_{lf}(A'')$.*
(2) *If $A \to B$ is essentially of finite presentation and M is a B-module of finite presentation, then F_{lf} is limit preserving: If $\{A_i\}_{i \in I}$ is a directed system of objects of $\mathcal{C}$, then $F_{lf}(\mathrm{colim}_i A_i) = \mathrm{colim}_i F_{lf}(A_i)$.*

Proof. Part (1) is a special case of More on Algebra, Lemma 17.3. Part (2) is a special case of More on Algebra, Lemma 17.4. ☐

05P4 **Lemma 20.5.** *In Situation 20.3. Let $B \to C$ is a local map of local A-algebras and N a C-module. Denote $F'_{lf} : \mathcal{C} \to \mathit{Sets}$ the functor associated to the pair (C, N). If $M \cong N$ as B-modules and $B \to C$ is finite, then $F_{lf} = F'_{lf}$.*

Proof. Let A' be an object of $\mathcal{C}$. Set $C' = C \otimes_A A'$ and $N' = N \otimes_A A'$ similarly to the definitions of B', M' in Situation 20.3. Note that $M' \cong N'$ as B'-modules. The assumption that $B \to C$ is finite has two consequences: (a) $\mathfrak{m}_C = \sqrt{\mathfrak{m}_B C}$ and (b) $B' \to C'$ is finite. Consequence (a) implies that

$$V(\mathfrak{m}_{A'}C' + \mathfrak{m}_C C') = (\mathrm{Spec}(C') \to \mathrm{Spec}(B'))^{-1} V(\mathfrak{m}_{A'}B' + \mathfrak{m}_B B').$$

Suppose $\mathfrak{q} \subset V(\mathfrak{m}_{A'}B' + \mathfrak{m}_B B')$. Then $M'_{\mathfrak{q}}$ is flat over A' if and only if the $C'_{\mathfrak{q}}$-module $N'_{\mathfrak{q}}$ is flat over A' (because these are isomorphic as A'-modules) if and only if for every maximal ideal $\mathfrak{r}$ of $C'_{\mathfrak{q}}$ the module $N'_{\mathfrak{r}}$ is flat over A' (see Algebra, Lemma 38.19). As $B'_{\mathfrak{q}} \to C'_{\mathfrak{q}}$ is finite by (b), the maximal ideals of $C'_{\mathfrak{q}}$ correspond exactly to the primes of C' lying over $\mathfrak{q}$ (see Algebra, Lemma 35.22) and these primes are all contained in $V(\mathfrak{m}_{A'}C' + \mathfrak{m}_C C')$ by the displayed equation above. Thus the result of the lemma holds. ☐

05P5 **Lemma 20.6.** *In Situation 20.3 suppose that $B \to C$ is a flat local homomorphism of local rings. Set $N = M \otimes_B C$. Denote $F'_{lf} : \mathcal{C} \to Sets$ the functor associated to the pair (C, N). Then $F_{lf} = F'_{lf}$.*

Proof. Let A' be an object of $\mathcal{C}$. Set $C' = C \otimes_A A'$ and $N' = N \otimes_A A' = M' \otimes_{B'} C'$ similarly to the definitions of B', M' in Situation 20.3. Note that

$$V(\mathfrak{m}_{A'}B' + \mathfrak{m}_B B') = \mathrm{Spec}(\kappa(\mathfrak{m}_B) \otimes_A \kappa(\mathfrak{m}_{A'}))$$

and similarly for $V(\mathfrak{m}_{A'}C' + \mathfrak{m}_C C')$. The ring map

$$\kappa(\mathfrak{m}_B) \otimes_A \kappa(\mathfrak{m}_{A'}) \longrightarrow \kappa(\mathfrak{m}_C) \otimes_A \kappa(\mathfrak{m}_{A'})$$

is faithfully flat, hence $V(\mathfrak{m}_{A'}C' + \mathfrak{m}_C C') \to V(\mathfrak{m}_{A'}B' + \mathfrak{m}_B B')$ is surjective. Finally, if $\mathfrak{r} \in V(\mathfrak{m}_{A'}C' + \mathfrak{m}_C C')$ maps to $\mathfrak{q} \in V(\mathfrak{m}_{A'}B' + \mathfrak{m}_B B')$, then $M'_{\mathfrak{q}}$ is flat over A' if and only if $N'_{\mathfrak{r}}$ is flat over A' because $B' \to C'$ is flat, see Algebra, Lemma 38.9. The lemma follows formally from these remarks. $\square$

05MP **Situation 20.7.** Let $f : X \to S$ be a smooth morphism with geometrically irreducible fibres. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module of finite type. For any scheme T over S we will denote $\mathcal{F}_T$ the base change of $\mathcal{F}$ to T, in other words, $\mathcal{F}_T$ is the pullback of $\mathcal{F}$ via the projection morphism $X_T = X \times_S T \to X$. Note that $X_T \to T$ is smooth with geometrically irreducible fibres, see Morphisms, Lemma 32.5 and More on Morphisms, Lemma 25.2. Let $p \geq 0$ be an integer. Given a point $t \in T$ consider the condition

05MQ (20.7.1) $$\mathcal{F}_T \text{ is free of rank } p \text{ in a neighbourhood of } \xi_t$$

where ξ_t is the generic point of the fibre X_t. This condition for all $t \in T$ is stable under base change, and hence we obtain a functor

05MR (20.7.2) $$H_p : (Sch/S)^{opp} \longrightarrow Sets, \quad T \longrightarrow \begin{cases} \{*\} & \text{if } \mathcal{F}_T \text{ satisfies (20.7.1) } \forall t \in T, \\ \emptyset & \text{else.} \end{cases}$$

05MS **Lemma 20.8.** *In Situation 20.7.*

1. *The functor H_p satisfies the sheaf property for the fpqc topology.*
2. *If $\mathcal{F}$ is of finite presentation, then functor H_p is limit preserving.*

Proof. Let $\{T_i \to T\}_{i \in I}$ be an fpqc[1] covering of schemes over S. Set $X_i = X_{T_i} = X \times_S T_i$ and denote $\mathcal{F}_i$ the pullback of $\mathcal{F}$ to X_i. Assume that $\mathcal{F}_i$ satisfies (20.7.1) for all i. Pick $t \in T$ and let $\xi_t \in X_T$ denote the generic point of X_t. We have to show that $\mathcal{F}$ is free in a neighbourhood of ξ_t. For some $i \in I$ we can find a $t_i \in T_i$ mapping to t. Let $\xi_i \in X_i$ denote the generic point of X_{t_i}, so that ξ_i maps to ξ_t. The fact that $\mathcal{F}_i$ is free of rank p in a neighbourhood of ξ_i implies that $(\mathcal{F}_i)_{x_i} \cong \mathcal{O}_{X_i,x_i}^{\oplus p}$ which implies that $\mathcal{F}_{T,\xi_t} \cong \mathcal{O}_{X_T,\xi_t}^{\oplus p}$ as $\mathcal{O}_{X_T,\xi_t} \to \mathcal{O}_{X_i,x_i}$ is flat, see for example Algebra, Lemma 77.5. Thus there exists an affine neighbourhood U of ξ_t in X_T and a surjection $\mathcal{O}_U^{\oplus p} \to \mathcal{F}_U = \mathcal{F}_T|_U$, see Modules, Lemma 9.4. After shrinking T we may assume that $U \to T$ is surjective. Hence $U \to T$ is a smooth morphism of affines with geometrically irreducible fibres. Moreover, for every $t' \in T$ we see that the induced map

$$\alpha : \mathcal{O}_{U,\xi_{t'}}^{\oplus p} \longrightarrow \mathcal{F}_{U,\xi_{t'}}$$

[1]It is quite easy to show that H_p is a sheaf for the fppf topology using that flat morphisms of finite presentation are open. This is all we really need later on. But it is kind of fun to prove directly that it also satisfies the sheaf condition for the fpqc topology.

is an isomorphism (since by the same argument as before the module on the right is free of rank p). It follows from Lemma 10.1 that

$$\Gamma(U, \mathcal{O}_U^{\oplus p}) \otimes_{\Gamma(T,\mathcal{O}_T)} \mathcal{O}_{T,t'} \longrightarrow \Gamma(U, \mathcal{F}_U) \otimes_{\Gamma(T,\mathcal{O}_T)} \mathcal{O}_{T,t'}$$

is injective for every $t' \in T$. Hence we see the surjection α is an isomorphism. This finishes the proof of (1).

Assume that $\mathcal{F}$ is of finite presentation. Let $T = \lim_{i\in I} T_i$ be a directed limit of affine S-schemes and assume that $\mathcal{F}_T$ satisfies (20.7.1). Set $X_i = X_{T_i} = X \times_S T_i$ and denote $\mathcal{F}_i$ the pullback of $\mathcal{F}$ to X_i. Let $U \subset X_T$ denote the open subscheme of points where $\mathcal{F}_T$ is flat over T, see More on Morphisms, Theorem 15.1. By assumption every generic point of every fibre is a point of U, i.e., $U \to T$ is a smooth surjective morphism with geometrically irreducible fibres. We may shrink U a bit and assume that U is quasi-compact. Using Limits, Lemma 4.11 we can find an $i \in I$ and a quasi-compact open $U_i \subset X_i$ whose inverse image in X_T is U. After increasing i we may assume that $\mathcal{F}_i|_{U_i}$ is flat over T_i, see Limits, Lemma 10.4. In particular, $\mathcal{F}_i|_{U_i}$ is finite locally free hence defines a locally constant rank function $\rho : U_i \to \{0, 1, 2, \ldots\}$. Let $(U_i)_p \subset U_i$ denote the open and closed subset where ρ has value p. Let $V_i \subset T_i$ be the image of $(U_i)_p$; note that V_i is open and quasi-compact. By assumption the image of $T \to T_i$ is contained in V_i. Hence there exists an $i' \geq i$ such that $T_{i'} \to T_i$ factors through V_i by Limits, Lemma 4.11. Then $\mathcal{F}_{i'}$ satisfies (20.7.1) as desired. Some details omitted. □

0CWF **Lemma 20.9.** *Let $f : X \to S$ be a morphism of schemes which is locally of finite type. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module of finite type. Let $n \geq 0$. The following are equivalent*

1. *for $s \in S$ the closed subset $Z \subset X_s$ of points where $\mathcal{F}$ is not flat over S (see Lemma 10.4) satisfies $\dim(Z) < n$, and*
2. *for $x \in X$ such that $\mathcal{F}$ is not flat at x over S we have $trdeg_{\kappa(f(x))}(\kappa(x)) < n$.*

If this is true, then it remains true after any base change.

Proof. Let $x \in X$ be a point over $s \in S$. Then the dimension of the closure of $\{x\}$ in X_s is $\text{trdeg}_{\kappa(s)}(\kappa(x))$ by Varieties, Lemma 20.3. Conversely, if $Z \subset X_s$ is a closed subset of dimension d, then there exists a point $x \in Z$ with $\text{trdeg}_{\kappa(s)}(\kappa(x)) = d$ (same reference). Therefore the equivalence of (1) and (2) holds (even fibre by fibre). The statement on base change follows from Morphisms, Lemmas 24.6 and 27.3. □

0CWG **Definition 20.10.** Let $f : X \to S$ be a morphism of schemes which is locally of finite type. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module of finite type. Let $n \geq 0$. We say $\mathcal{F}$ *is flat over S in dimensions $\geq n$* if the equivalent conditions of Lemma 20.9 are satisfied.

05MT **Situation 20.11.** Let $f : X \to S$ be a morphism of schemes which is locally of finite type. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module of finite type. For any scheme T over S we will denote $\mathcal{F}_T$ the base change of $\mathcal{F}$ to T, in other words, $\mathcal{F}_T$ is the pullback of $\mathcal{F}$ via the projection morphism $X_T = X \times_S T \to X$. Note that $X_T \to T$ is of finite type and that $\mathcal{F}_T$ is an $\mathcal{O}_{X_T}$-module of finite type (Morphisms, Lemma 14.4 and Modules, Lemma 9.2). Let $n \geq 0$. By Definition 20.10 and Lemma 20.9

we obtain a functor
(20.11.1)

05MU $$F_n : (Sch/S)^{opp} \longrightarrow Sets, \quad T \longrightarrow \begin{cases} \{*\} & \text{if } \mathcal{F}_T \text{ is flat over } T \text{ in } \dim \geq n, \\ \emptyset & \text{else.} \end{cases}$$

05MV **Lemma 20.12.** *In Situation 20.11.*

(1) *The functor F_n satisfies the sheaf property for the fpqc topology.*
(2) *If f is quasi-compact and locally of finite presentation and $\mathcal{F}$ is of finite presentation, then the functor F_n is limit preserving.*

Proof. Let $\{T_i \to T\}_{i \in I}$ be an fpqc covering of schemes over S. Set $X_i = X_{T_i} = X \times_S T_i$ and denote $\mathcal{F}_i$ the pullback of $\mathcal{F}$ to X_i. Assume that $\mathcal{F}_i$ is flat over T_i in dimensions $\geq n$ for all i. Let $t \in T$. Choose an index i and a point $t_i \in T_i$ mapping to t. Consider the cartesian diagram

$$\begin{tikzcd} X_{\mathrm{Spec}(\mathcal{O}_{T,t})} \arrow[d] & X_{\mathrm{Spec}(\mathcal{O}_{T_i,t_i})} \arrow[l] \arrow[d] \\ \mathrm{Spec}(\mathcal{O}_{T,t}) & \mathrm{Spec}(\mathcal{O}_{T_i,t_i}) \arrow[l] \end{tikzcd}$$

As the lower horizontal morphism is flat we see from More on Morphisms, Lemma 15.2 that the set $Z_i \subset X_{t_i}$ where $\mathcal{F}_i$ is not flat over T_i and the set $Z \subset X_t$ where $\mathcal{F}_T$ is not flat over T are related by the rule $Z_i = Z_{\kappa(t_i)}$. Hence we see that $\mathcal{F}_T$ is flat over T in dimensions $\geq n$ by Morphisms, Lemma 27.3.

Assume that f is quasi-compact and locally of finite presentation and that $\mathcal{F}$ is of finite presentation. In this paragraph we first reduce the proof of (2) to the case where f is of finite presentation. Let $T = \lim_{i \in I} T_i$ be a directed limit of affine S-schemes and assume that $\mathcal{F}_T$ is flat in dimensions $\geq n$. Set $X_i = X_{T_i} = X \times_S T_i$ and denote $\mathcal{F}_i$ the pullback of $\mathcal{F}$ to X_i. We have to show that $\mathcal{F}_i$ is flat in dimensions $\geq n$ for some i. Pick $i_0 \in I$ and replace I by $\{i \mid i \geq i_0\}$. Since T_{i_0} is affine (hence quasi-compact) there exist finitely many affine opens $W_j \subset S$, $j = 1, \ldots, m$ and an affine open overing $T_{i_0} = \bigcup_{j=1,\ldots,m} V_{j,i_0}$ such that $T_{i_0} \to S$ maps V_{j,i_0} into W_j. For $i \geq i_0$ denote $V_{j,i}$ the inverse image of V_{j,i_0} in T_i. If we can show, for each j, that there exists an i such that $\mathcal{F}_{V_{j,i_0}}$ is flat in dimensions $\geq n$, then we win. In this way we reduce to the case that S is affine. In this case X is quasi-compact and we can choose a finite affine open covering $X = W_1 \cup \ldots \cup W_m$. In this case the result for $(X \to S, \mathcal{F})$ is equivalent to the result for $(\coprod W_j, \coprod \mathcal{F}|_{W_j})$. Hence we may assume that f is of finite presentation.

Assume f is of finite presentation and $\mathcal{F}$ is of finite presentation. Let $U \subset X_T$ denote the open subscheme of points where $\mathcal{F}_T$ is flat over T, see More on Morphisms, Theorem 15.1. By assumption the dimension of every fibre of $Z = X_T \setminus U$ over T has dimension $\leq n$. By Limits, Lemma 16.3 we can find a closed subscheme $Z \subset Z' \subset X_T$ such that $\dim(Z'_t) < n$ for all $t \in T$ and such that $Z' \to X_T$ is of finite presentation. By Limits, Lemmas 10.1 and 8.5 there exists an $i \in I$ and a closed subscheme $Z'_i \subset X_i$ of finite presentation whose base change to T is Z'. By Limits, Lemma 16.1 we may assume all fibres of $Z'_i \to T_i$ have dimension $< n$. By Limits, Lemma 10.4 we may assume that $\mathcal{F}_i|_{X_i \setminus T'_i}$ is flat over T_i. This implies that $\mathcal{F}_i$ is flat in dimensions $\geq n$; here we use that $Z' \to X_T$ is of finite presentation, and hence the complement $X_T \setminus Z'$ is quasi-compact! Thus part (2) is proved and the proof of the lemma is complete. $\square$

05MW **Situation 20.13.** Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. For any scheme T over S we will denote $\mathcal{F}_T$ the base change of $\mathcal{F}$ to T, in other words, $\mathcal{F}_T$ is the pullback of $\mathcal{F}$ via the projection morphism $X_T = X \times_S T \to X$. Since the base change of a flat module is flat we obtain a functor

05MX (20.13.1) $$F_{flat} : (Sch/S)^{opp} \longrightarrow Sets, \quad T \longrightarrow \begin{cases} \{*\} & \text{if } \mathcal{F}_T \text{ is flat over } T, \\ \emptyset & \text{else.} \end{cases}$$

05MY **Lemma 20.14.** *In Situation 20.13.*

(1) *The functor F_{flat} satisfies the sheaf property for the fpqc topology.*
(2) *If f is quasi-compact and locally of finite presentation and $\mathcal{F}$ is of finite presentation, then the functor F_{flat} is limit preserving.*

Proof. Part (1) follows from the following statement: If $T' \to T$ is a surjective flat morphism of schemes over S, then $\mathcal{F}_{T'}$ is flat over T' if and only if $\mathcal{F}_T$ is flat over T, see More on Morphisms, Lemma 15.2. Part (2) follows from Limits, Lemma 10.4 after reducing to the case where X and S are affine (compare with the proof of Lemma 20.12). $\square$

21. Flattening stratifications

052F Just the definitions and an important baby case.

05P6 **Definition 21.1.** Let $X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. We say that the *universal flattening of $\mathcal{F}$ exists* if the functor F_{flat} defined in Situation 20.13 is representable by a scheme S' over S. We say that the *universal flattening of X exists* if the universal flattening of $\mathcal{O}_X$ exists.

Note that if the universal flattening S'[2] of $\mathcal{F}$ exists, then the morphism $S' \to S$ is a monomorphism of schemes such that $\mathcal{F}_{S'}$ is flat over S' and such that a morphism $T \to S$ factors through S' if and only if $\mathcal{F}_T$ is flat over T.

We define (compare with Topology, Remark 28.5) a (locally finite, scheme theoretic) *stratification* of a scheme S to be given by closed subschemes $Z_i \subset S$ indexed by a partially ordered set I such that $S = \bigcup Z_i$ (set theoretically), such that every point of S has a neighbourhood meeting only a finite number of Z_i, and such that

$$Z_i \cap Z_j = \bigcup\nolimits_{k \leq i,j} Z_k.$$

Setting $S_i = Z_i \setminus \bigcup_{j<i} Z_j$ the actual stratification is the decomposition $S = \coprod S_i$ into locally closed subschemes. We often only indicate the strata S_i and leave the construction of the closed subschemes Z_i to the reader. Given a stratification we obtain a monomorphism

$$S' = \coprod\nolimits_{i \in I} S_i \longrightarrow S.$$

We will call this the *monomorphism associated to the stratification.* With this terminology we can define what it means to have a flattening stratification.

[2]The scheme S' is sometimes called the *universal flatificator*. In [GR71] it is called the *platificateur universel*. Existence of the universal flattening should not be confused with the type of results discussed in More on Algebra, Section 24.

05P7 **Definition 21.2.** Let $X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. We say that $\mathcal{F}$ has a *flattening stratification* if the functor F_{flat} defined in Situation 20.13 is representable by a monomorphism $S' \to S$ associated to a stratification of S by locally closed subschemes. We say that X has a *flattening stratification* if $\mathcal{O}_X$ has a flattening stratification.

When a flattening stratification exists, it is often important to understand the index set labeling the strata and its partial ordering. This often has to do with ranks of modules, as in the case of the stratification of a scheme by the Fitting ideals of a finite type quasi-coherent module discussed in Divisors, Section 9.

We end this section showing that if we do not insist on a canonical stratification, then we can use generic flatness to construct some stratification such that our sheaf is flat over the strata.

0ASY **Lemma 21.3** (Generic flatness stratification)**.** *Let $f : X \to S$ be a morphism of finite presentation between quasi-compact and quasi-separated schemes. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module of finite presentation. Then there exists a $t \geq 0$ and closed subschemes*

$$S \supset S_0 \supset S_1 \supset \ldots \supset S_t = \emptyset$$

such that $S_i \to S$ is defined by a finite type ideal sheaf, $S_0 \subset S$ is a thickening, and $\mathcal{F}$ pulled back to $X \times_S (S_i \setminus S_{i+1})$ is flat over $S_i \setminus S_{i+1}$.

Proof. We can find a cartesian diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & X_0 \arrow[d] \\ S \arrow[r] & S_0 \end{tikzcd}$$

and a finitely presented $\mathcal{O}_{X_0}$-module $\mathcal{F}_0$ which pulls back to $\mathcal{F}$ such that X_0 and S_0 are of finite type over $\mathbf{Z}$. See Limits, Proposition 5.4 and Lemmas 10.1 and 10.2. Thus we may assume X and S are of finite type over $\mathbf{Z}$ and $\mathcal{F}$ is a coherent $\mathcal{O}_X$-module.

Assume X and S are of finite type over $\mathbf{Z}$ and $\mathcal{F}$ is a coherent $\mathcal{O}_X$-module. In this case every quasi-coherent ideal is of finite type, hence we do not have to check the condition that S_i is cut out by a finite type ideal. Set $S_0 = S_{red}$ equal to the reduction of S. By generic flatness as stated in Morphisms, Proposition 26.2 there is a dense open $U_0 \subset S_0$ such that $\mathcal{F}$ pulled back to $X \times_S U_0$ is flat over U_0. Let $S_1 \subset S_0$ be the reduced closed subscheme whose underlying closed subset is $S \setminus U_0$. We continue in this way, provided $S_1 \neq \emptyset$, to find $S_0 \supset S_1 \supset \ldots$. Because S is Noetherian any descending chain of closed subsets stabilizes hence we see that $S_t = \emptyset$ for some $t \geq 0$. $\square$

22. Flattening stratification over an Artinian ring

05PA A flatting stratification exists when the base scheme is the spectrum of an Artinian ring.

05PB **Lemma 22.1.** *Let S be the spectrum of an Artinian ring. For any scheme X over S, and any quasi-coherent $\mathcal{O}_X$-module there exists a universal flattening. In fact the universal flattening is given by a closed immersion $S' \to S$, and hence is a flattening stratification for $\mathcal{F}$ as well.*

Proof. Choose an affine open covering $X = \bigcup U_i$. Then F_{flat} is the product of the functors associated to each of the pairs $(U_i, \mathcal{F}|_{U_i})$. Hence it suffices to prove the result for each $(U_i, \mathcal{F}|_{U_i})$. In the affine case the lemma follows immediately from More on Algebra, Lemma 15.2. $\square$

23. Flattening a map

05PC Theorem 23.3 is the key to further flattening statements.

05PD **Lemma 23.1.** *Let S be a scheme. Let $g : X' \to X$ be a flat morphism of schemes over S with X locally of finite type over S. Let $\mathcal{F}$ be a finite type quasi-coherent $\mathcal{O}_X$-module which is flat over S. If $Ass_{X/S}(\mathcal{F}) \subset g(X')$ then the canonical map*

$$\mathcal{F} \longrightarrow g_*g^*\mathcal{F}$$

is injective, and remains injective after any base change.

Proof. The final assertion means that $\mathcal{F}_T \to (g_T)_*g_T^*\mathcal{F}_T$ is injective for any morphism $T \to S$. The assumption $\text{Ass}_{X/S}(\mathcal{F}) \subset g(X')$ is preserved by base change, see Divisors, Lemma 7.3 and Remark 7.4. The same holds for the assumption of flatness and finite type. Hence it suffices to prove the injectivity of the displayed arrow. Let $\mathcal{K} = \text{Ker}(\mathcal{F} \to g_*g^*\mathcal{F})$. Our goal is to prove that $\mathcal{K} = 0$. In order to do this it suffices to prove that $\text{WeakAss}_X(\mathcal{K}) = \emptyset$, see Divisors, Lemma 5.5. We have $\text{WeakAss}_X(\mathcal{K}) \subset \text{WeakAss}_X(\mathcal{F})$, see Divisors, Lemma 5.4. As $\mathcal{F}$ is flat we see from Lemma 13.5 that $\text{WeakAss}_X(\mathcal{F}) \subset \text{Ass}_{X/S}(\mathcal{F})$. By assumption any point x of $\text{Ass}_{X/S}(\mathcal{F})$ is the image of some $x' \in X'$. Since g is flat the local ring map $\mathcal{O}_{X,x} \to \mathcal{O}_{X',x'}$ is faithfully flat, hence the map

$$\mathcal{F}_x \longrightarrow g^*\mathcal{F}_{x'} = \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{X',x'}$$

is injective (see Algebra, Lemma 81.11). This implies that $\mathcal{K}_x = 0$ as desired. $\square$

05PE **Lemma 23.2.** *Let A be a ring. Let $u : M \to N$ be a surjective map of A-modules. If M is projective as an A-module, then there exists an ideal $I \subset A$ such that for any ring map $\varphi : A \to B$ the following are equivalent*

1. *$u \otimes 1 : M \otimes_A B \to N \otimes_A B$ is an isomorphism, and*
2. *$\varphi(I) = 0$.*

Proof. As M is projective we can find a projective A-module C such that $F = M \oplus C$ is a free R-module. By replacing u by $u \oplus 1 : F = M \oplus C \to N \oplus C$ we see that we may assume M is free. In this case let I be the ideal of A generated by coefficients of all the elements of $\text{Ker}(u)$ with respect to some (fixed) basis of M. The reason this works is that, since u is surjective and $\otimes_A B$ is right exact, $\text{Ker}(u \otimes 1)$ is the image of $\text{Ker}(u) \otimes_A B$ in $M \otimes_A B$. $\square$

05PF **Theorem 23.3.** *In Situation 20.1 assume*

1. *f is of finite presentation,*
2. *$\mathcal{F}$ is of finite presentation, flat over S, and pure relative to S, and*
3. *u is surjective.*

Then F_{iso} is representable by a closed immersion $Z \to S$. Moreover $Z \to S$ is of finite presentation if $\mathcal{G}$ is of finite presentation.

Proof. We will use without further mention that $\mathcal{F}$ is universally pure over S, see Lemma 18.3. By Lemma 20.2 and Descent, Lemmas 34.2 and 36.1 the question is local for the étale topology on S. Hence it suffices to prove, given $s \in S$, that there exists an étale neighbourhood of (S, s) so that the theorem holds.

Using Lemma 12.5 and after replacing S by an elementary étale neighbourhood of s we may assume there exists a commutative diagram

$$\begin{tikzcd} X \arrow[dr] & & X' \arrow[ll, "g"'] \arrow[dl] \\ & S & \end{tikzcd}$$

of schemes of finite presentation over S, where g is étale, $X_s \subset g(X')$, the schemes X' and S are affine, $\Gamma(X', g^*\mathcal{F})$ a projective $\Gamma(S, \mathcal{O}_S)$-module. Note that $g^*\mathcal{F}$ is universally pure over S, see Lemma 17.4. Hence by Lemma 18.2 we see that the open $g(X')$ contains the points of $\text{Ass}_{X/S}(\mathcal{F})$ lying over $\text{Spec}(\mathcal{O}_{S,s})$. Set

$$E = \{t \in S \mid \text{Ass}_{X_t}(\mathcal{F}_t) \subset g(X')\}.$$

By More on Morphisms, Lemma 23.5 E is a constructible subset of S. We have seen that $\text{Spec}(\mathcal{O}_{S,s}) \subset E$. By Morphisms, Lemma 21.4 we see that E contains an open neighbourhood of s. Hence after replacing S by a smaller affine neighbourhood of s we may assume that $\text{Ass}_{X/S}(\mathcal{F}) \subset g(X')$.

Since we have assumed that u is surjective we have $F_{iso} = F_{inj}$. From Lemma 23.1 it follows that $u : \mathcal{F} \to \mathcal{G}$ is injective if and only if $g^*u : g^*\mathcal{F} \to g^*\mathcal{G}$ is injective, and the same remains true after any base change. Hence we have reduced to the case where, in addition to the assumptions in the theorem, $X \to S$ is a morphism of affine schemes and $\Gamma(X, \mathcal{F})$ is a projective $\Gamma(S, \mathcal{O}_S)$-module. This case follows immediately from Lemma 23.2.

To see that Z is of finite presentation if $\mathcal{G}$ is of finite presentation, combine Lemma 20.2 part (4) with Limits, Remark 6.2. ☐

07AI **Lemma 23.4.** *Let $f : X \to S$ be a morphism of schemes which is of finite presentation, flat, and pure. Let Y be a closed subscheme of X. Let $F = f_*Y$ be the Weil restriction functor of Y along f, defined by*

$$F : (Sch/S)^{opp} \to Sets, \quad T \mapsto \begin{cases} \{*\} & \text{if} \quad Y_T \to X_T \text{ is an isomorphism,} \\ \emptyset & \text{else.} \end{cases}$$

Then F is representable by a closed immersion $Z \to S$. Moreover $Z \to S$ is of finite presentation if $Y \to S$ is.

Proof. Let $\mathcal{I}$ be the ideal sheaf defining Y in X and let $u : \mathcal{O}_X \to \mathcal{O}_X/\mathcal{I}$ be the surjection. Then for an S-scheme T, the closed immersion $Y_T \to X_T$ is an isomorphism if and only if u_T is an isomorphism. Hence the result follows from Theorem 23.3. ☐

24. Flattening in the local case

05MZ In this section we start applying the earlier material to obtain a shadow of the flattening stratification.

05PG **Theorem 24.1.** *In Situation 20.3 assume A is henselian, B is essentially of finite type over A, and M is a finite B-module. Then there exists an ideal $I \subset A$ such that A/I corepresents the functor F_{lf} on the category $\mathcal{C}$. In other words given a local homomorphism of local rings $\varphi : A \to A'$ with $B' = B \otimes_A A'$ and $M' = M \otimes_A A'$ the following are equivalent:*

1. *$\forall \mathfrak{q} \in V(\mathfrak{m}_{A'}B' + \mathfrak{m}_B B') \subset \mathrm{Spec}(B') : M'_{\mathfrak{q}}$ is flat over A', and*
2. *$\varphi(I) = 0$.*

If B is essentially of finite presentation over A and M of finite presentation over B, then I is a finitely generated ideal.

Proof. Choose a finite type ring map $A \to C$ and a finite C-module N and a prime $\mathfrak{q}$ of C such that $B = C_{\mathfrak{q}}$ and $M = N_{\mathfrak{q}}$. In the following, when we say "the theorem holds for $(N/C/A, \mathfrak{q})$ we mean that it holds for $(A \to B, M)$ where $B = C_{\mathfrak{q}}$ and $M = N_{\mathfrak{q}}$. By Lemma 20.6 the functor F_{lf} is unchanged if we replace B by a local ring flat over B. Hence, since A is henselian, we may apply Lemma 6.6 and assume that there exists a complete dévissage of $N/C/A$ at $\mathfrak{q}$.

Let $(A_i, B_i, M_i, \alpha_i, \mathfrak{q}_i)_{i=1,\ldots,n}$ be such a complete dévissage of $N/C/A$ at $\mathfrak{q}$. Let $\mathfrak{q}'_i \subset A_i$ be the unique prime lying over $\mathfrak{q}_i \subset B_i$ as in Definition 6.4. Since $C \to A_1$ is surjective and $N \cong M_1$ as C-modules, we see by Lemma 20.5 it suffices to prove the theorem holds for $(M_1/A_1/A, \mathfrak{q}'_1)$. Since $B_1 \to A_1$ is finite and $\mathfrak{q}_1$ is the only prime of B_1 over $\mathfrak{q}'_1$ we see that $(A_1)_{\mathfrak{q}'_1} \to (B_1)_{\mathfrak{q}_1}$ is finite (see Algebra, Lemma 40.11 or More on Morphisms, Lemma 40.4). Hence by Lemma 20.5 it suffices to prove the theorem holds for $(M_1/B_1/A, \mathfrak{q}_1)$.

At this point we may assume, by induction on the length n of the dévissage, that the theorem holds for $(M_2/B_2/A, \mathfrak{q}_2)$. (If $n = 1$, then $M_2 = 0$ which is flat over A.) Reversing the last couple of steps of the previous paragraph, using that $M_2 \cong \mathrm{Coker}(\alpha_2)$ as B_1-modules, we see that the theorem holds for $(\mathrm{Coker}(\alpha_1)/B_1/A, \mathfrak{q}_1)$.

Let A' be an object of $\mathcal{C}$. At this point we use Lemma 10.1 to see that if $(M_1 \otimes_A A')_{\mathfrak{q}'}$ is flat over A' for a prime $\mathfrak{q}'$ of $B_1 \otimes_A A'$ lying over $\mathfrak{m}_{A'}$, then $(\mathrm{Coker}(\alpha_1) \otimes_A A')_{\mathfrak{q}'}$ is flat over A'. Hence we conclude that F_{lf} is a subfunctor of the functor F'_{lf} associated to the module $\mathrm{Coker}(\alpha_1)_{\mathfrak{q}_1}$ over $(B_1)_{\mathfrak{q}_1}$. By the previous paragraph we know F'_{lf} is corepresented by A/J for some ideal $J \subset A$. Hence we may replace A by A/J and assume that $\mathrm{Coker}(\alpha_1)_{\mathfrak{q}_1}$ is flat over A.

Since $\mathrm{Coker}(\alpha_1)$ is a B_1-module for which there exist a complete dévissage of $N_1/B_1/A$ at $\mathfrak{q}_1$ and since $\mathrm{Coker}(\alpha_1)_{\mathfrak{q}_1}$ is flat over A by Lemma 10.2 we see that $\mathrm{Coker}(\alpha_1)$ is free as an A-module, in particular flat as an A-module. Hence Lemma 10.1 implies $F_{lf}(A')$ is nonempty if and only if $\alpha \otimes 1_{A'}$ is injective. Let $N_1 = \mathrm{Im}(\alpha_1) \subset M_1$ so that we have exact sequences

$$0 \to N_1 \to M_1 \to \mathrm{Coker}(\alpha_1) \to 0 \quad \text{and} \quad B_1^{\oplus r_1} \to N_1 \to 0$$

The flatness of $\mathrm{Coker}(\alpha_1)$ implies the first sequence is universally exact (see Algebra, Lemma 81.5). Hence $\alpha \otimes 1_{A'}$ is injective if and only if $B_1^{\oplus r_1} \otimes_A A' \to N_1 \otimes_A A'$ is an isomorphism. Finally, Theorem 23.3 applies to show this functor is corepresentable by A/I for some ideal I and we conclude F_{lf} is corepresentable by A/I also.

To prove the final statement, suppose that $A \to B$ is essentially of finite presentation and M of finite presentation over B. Let $I \subset A$ be the ideal such that F_{lf} is corepresented by A/I. Write $I = \bigcup I_\lambda$ where I_λ ranges over the finitely generated ideals contained in I. Then, since $F_{lf}(A/I) = \{*\}$ we see that $F_{lf}(A/I_\lambda) = \{*\}$ for some λ, see Lemma 20.4 part (2). Clearly this implies that $I = I_\lambda$. □

05PH **Remark 24.2.** Here is a scheme theoretic reformulation of Theorem 24.1. Let $(X, x) \to (S, s)$ be a morphism of pointed schemes which is locally of finite type. Let $\mathcal{F}$ be a finite type quasi-coherent $\mathcal{O}_X$-module. Assume S henselian local with closed point s. There exists a closed subscheme $Z \subset S$ with the following property: for any morphism of pointed schemes $(T, t) \to (S, s)$ the following are equivalent

1. $\mathcal{F}_T$ is flat over T at all points of the fibre X_t which map to $x \in X_s$, and
2. $\mathrm{Spec}(\mathcal{O}_{T,t}) \to S$ factors through Z.

Moreover, if $X \to S$ is of finite presentation at x and $\mathcal{F}_x$ of finite presentation over $\mathcal{O}_{X,x}$, then $Z \to S$ is of finite presentation.

At this point we can obtain some very general results completely for free from the result above. Note that perhaps the most interesting case is when $E = X_s$!

05PI **Lemma 24.3.** *Let S be the spectrum of a henselian local ring with closed point s. Let $X \to S$ be a morphism of schemes which is locally of finite type. Let $\mathcal{F}$ be a finite type quasi-coherent $\mathcal{O}_X$-module. Let $E \subset X_s$ be a subset. There exists a closed subscheme $Z \subset S$ with the following property: for any morphism of pointed schemes $(T, t) \to (S, s)$ the following are equivalent*

1. *$\mathcal{F}_T$ is flat over T at all points of the fibre X_t which map to a point of $E \subset X_s$, and*
2. *$\mathrm{Spec}(\mathcal{O}_{T,t}) \to S$ factors through Z.*

Moreover, if $X \to S$ is locally of finite presentation, $\mathcal{F}$ is of finite presentation, and $E \subset X_s$ is closed and quasi-compact, then $Z \to S$ is of finite presentation.

Proof. For $x \in X_s$ denote $Z_x \subset S$ the closed subscheme we found in Remark 24.2. Then it is clear that $Z = \bigcap_{x \in E} Z_x$ works!

To prove the final statement assume X locally of finite presentation, $\mathcal{F}$ of finite presentation and Z closed and quasi-compact. First, choose finitely many affine opens $W_j \subset X$ such that $E \subset \bigcup W_j$. It clearly suffices to prove the result for each morphism $W_j \to S$ with sheaf $\mathcal{F}|_{X_j}$ and closed subset $E \cap W_j$. Hence we may assume X is affine. In this case, More on Algebra, Lemma 17.4 shows that the functor defined by (1) is "limit preserving". Hence we can show that $Z \to S$ is of finite presentation exactly as in the last part of the proof of Theorem 24.1. □

052G **Remark 24.4.** Tracing the proof of Lemma 24.3 to its origins we find a long and winding road. But if we assume that

1. f is of finite type,
2. $\mathcal{F}$ is a finite type $\mathcal{O}_X$-module,
3. $E = X_s$, and
4. S is the spectrum of a Noetherian complete local ring.

then there is a proof relying completely on more elementary algebra as follows: first we reduce to the case where X is affine by taking a finite affine open cover. In this case Z exists by More on Algebra, Lemma 18.3. The key step in this proof is constructing the closed subscheme Z step by step inside the truncations $\mathrm{Spec}(\mathcal{O}_{S,s}/\mathfrak{m}_s^n)$. This relies on the fact that flattening stratifications always exist when the base is Artinian, and the fact that $\mathcal{O}_{S,s} = \lim \mathcal{O}_{S,s}/\mathfrak{m}_s^n$.

25. Variants of a lemma

0ASZ In this section we discuss variants of Algebra, Lemmas 127.4 and 98.1. The most general version is Proposition 25.13; this was stated as [GR71, Lemma 4.2.2] but the proof in loc.cit. only gives the weaker result as stated in Lemma 25.5. The intricate proof of Proposition 25.13 is due to Ofer Gabber. As we currently have no application for the proposition we encourage the reader to skip to the next section after reading the proof of Lemma 25.5; this lemma will be used in the next section to prove Theorem 26.1.

0AT0 **Situation 25.1.** Let $\varphi : A \to B$ be a local ring homomorphism of local rings which is essentially of finite type. Let M be a flat A-module, N a finite B-module and $u : N \to M$ an A-module map such that $\overline{u} : N/\mathfrak{m}_A N \to M/\mathfrak{m}_A M$ is injective.

In this situation it is our goal to show that u is A-universally injective, N is of finite presentation over B, and N is flat as an A-module. If this is true, we will say *the lemma holds* in the given situation.

0AT1 **Lemma 25.2.** *If in Situation 25.1 the ring A is Noetherian then the lemma holds.*

Proof. Applying Algebra, Lemma 98.1 we see that u is injective and that $N/u(M)$ is flat over A. Then u is A-universally injective (Algebra, Lemma 38.12) and N is A-flat (Algebra, Lemma 38.13). Since B is Noetherian in this case we see that N is of finite presentation. $\square$

0AT2 **Lemma 25.3.** *Let A_0 be a local ring. If the lemma holds for every Situation 25.1 with $A = A_0$, with B a localization of a polynomial algebra over A, and N of finite presentation over B, then the lemma holds for every Situation 25.1 with $A = A_0$.*

Proof. Let $A \to B$, $u : N \to M$ be as in Situation 25.1. Write $B = C/I$ where C is the localization of a polynomial algebra over A at a prime. If we can show that N is finitely presented as a C-module, then a fortiori this shows that N is finitely presented as a B-module (see Algebra, Lemma 6.4). Hence we may assume that B is the localization of a polynomial algebra. Next, write $N = B^{\oplus n}/K$ for some submodule $K \subset B^{\oplus n}$. Since $B/\mathfrak{m}_A B$ is Noetherian (as it is essentially of finite type over a field), there exist finitely many elements $k_1, \ldots, k_s \in K$ such that for $K' = \sum Bk_i$ and $N' = B^{\oplus n}/K'$ the canonical surjection $N' \to N$ induces an isomorphism $N'/\mathfrak{m}_A N' \cong N/\mathfrak{m}_A N$. Now, if the lemma holds for the composition $u' : N' \to M$, then u' is injective, hence $N' = N$ and $u' = u$. Thus the lemma holds for the original situation. $\square$

0AT3 **Lemma 25.4.** *If in Situation 25.1 the ring A is henselian then the lemma holds.*

Proof. It suffices to prove this when B is essentially of finite presentation over A and N is of finite presentation over B, see Lemma 25.3. Let us temporarily make the additional assumption that N is flat over A. Then N is a filtered colimit $N = \text{colim}_i F_i$ of free A-modules F_i such that the transition maps $u_{ii'} : F_i \to F_{i'}$ are injective modulo $\mathfrak{m}_A$, see Lemma 19.5. Each of the compositions $u_i : F_i \to M$ is A-universally injective by Lemma 7.5 wherefore $u = \text{colim}\, u_i$ is A-universally injective as desired.

Assume A is a henselian local ring, B is essentially of finite presentation over A, N of finite presentation over B. By Theorem 24.1 there exists a finitely generated ideal $I \subset A$ such that N/IN is flat over A/I and such that N/I^2N is not flat over

A/I^2 unless $I = 0$. The result of the previous paragraph shows that the lemma holds for u mod $I : N/IN \to M/IM$ over A/I. Consider the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & M \otimes_A I/I^2 \arrow[r] & M/I^2M \arrow[r] & M/IM \arrow[r] & 0 \\ & N \otimes_A I/I^2 \arrow[r] \arrow[u, "u"] & N/I^2N \arrow[r] \arrow[u, "u"] & N/IN \arrow[r] \arrow[u, "u"] & 0 \end{tikzcd}$$

whose rows are exact by right exactness of $\otimes$ and the fact that M is flat over A. Note that the left vertical arrow is the map $N/IN \otimes_{A/I} I/I^2 \to M/IM \otimes_{A/I} I/I^2$, hence is injective. A diagram chase shows that the lower left arrow is injective, i.e., $\text{Tor}^1_{A/I^2}(I/I^2, M/I^2) = 0$ see Algebra, Remark 74.9. Hence N/I^2N is flat over A/I^2 by Algebra, Lemma 98.8 a contradiction unless $I = 0$. ☐

The following lemma discusses the special case of Situation 25.1 where M has a B-module structure and u is B-linear. This is the case most often used in practice and it is significantly easier to prove than the general case.

0AT4 **Lemma 25.5.** *Let $A \to B$ be a local ring homomorphism of local rings which is essentially of finite type. Let $u : N \to M$ be a B-module map. If N is a finite B-module, M is flat over A, and $\overline{u} : N/\mathfrak{m}_AN \to M/\mathfrak{m}_AM$ is injective, then u is A-universally injective, N is of finite presentation over B, and N is flat over A.*

Proof. Let $A \to A^h$ be the henselization of A. Let B' be the localization of $B \otimes_A A^h$ at the maximal ideal $\mathfrak{m}_B \otimes A^h + B \otimes \mathfrak{m}_{A^h}$. Since $B \to B'$ is flat (hence faithfully flat, see Algebra, Lemma 38.17), we may replace $A \to B$ with $A^h \to B'$, the module M by $M \otimes_B B'$, the module N by $N \otimes_B B'$, and u by $u \otimes \text{id}_{B'}$, see Algebra, Lemmas 82.2 and 38.9. Thus we may assume that A is a henselian local ring. In this case our lemma follows from the more general Lemma 25.4. ☐

0AT5 **Lemma 25.6.** *If in Situation 25.1 the ring A is a valuation ring then the lemma holds.*

Proof. Recall that an A-module is flat if and only if it is torsion free, see More on Algebra, Lemma 20.10. Let $T \subset N$ be the A-torsion. Then $u(T) = 0$ and N/T is A-flat. Hence N/T is finitely presented over B, see More on Algebra, Lemma 23.6. Thus T is a finite B-module, see Algebra, Lemma 5.3. Since N/T is A-flat we see that $T/\mathfrak{m}_AT \subset N/\mathfrak{m}_AN$, see Algebra, Lemma 38.12. As $\overline{u}$ is injective but $u(T) = 0$, we conclude that $T/\mathfrak{m}_AT = 0$. Hence $T = 0$ by Nakayama's lemma, see Algebra, Lemma 19.1. At this point we have proved two out of the three assertions (N is A-flat and of finite presentation over B) and what is left is to show that u is universally injective.

By Algebra, Theorem 81.3 it suffices to show that $N \otimes_A Q \to M \otimes_A Q$ is injective for every finitely presented A-module Q. By More on Algebra, Lemma 97.3 we may assume $Q = A/(a)$ with $a \in \mathfrak{m}_A$ nonzero. Thus it suffices to show that $N/aN \to M/aM$ is injective. Let $x \in N$ with $u(x) \in aM$. By Lemma 19.6 we know that x has a content ideal $I \subset A$. Since I is finitely generated (More on Algebra, Lemma 22.2) and A is a valuation ring, we have $I = (b)$ for some b (by Algebra, Lemma 49.15). By More on Algebra, Lemma 22.3 the element $u(x)$ has content ideal I as well. Since $u(x) \in aM$ we see that $(b) \subset (a)$ by More on Algebra, Definition 22.1. Since $x \in bN$ we conclude $x \in aN$ as desired. ☐

Consider the following situation

0AT6 (25.6.1)
$$A \to B \text{ of finite presentation, } S \subset B \text{ a multiplicative subset, and } \\ N \text{ a finitely presented } S^{-1}B\text{-module}$$

In this situation a *pure spreadout* is an affine open $U \subset \mathrm{Spec}(B)$ with $\mathrm{Spec}(S^{-1}B) \subset U$ and a finitely presented $\mathcal{O}(U)$-module N' extending N such that N' is A-projective and $N' \to N = S^{-1}N'$ is A-universally injective.

In (25.6.1) if $A \to A_1$ is a ring map, then we can base change: take $B_1 = B \otimes_A A_1$, let $S_1 \subset B_1$ be the image of S, and let $N_1 = N \otimes_A A_1$. This works because $S_1^{-1}B_1 = S^{-1}B \otimes_A A_1$. We will use this without further mention in the following.

0AT7 **Lemma 25.7.** *In (25.6.1) if there exists a pure spreadout, then*

(1) *elements of N have content ideals in A, and*
(2) *if $u : N \to M$ is a morphism to a flat A-module M such that $N/\mathfrak{m}N \to M/\mathfrak{m}M$ is injective for all maximal ideals $\mathfrak{m}$ of A, then u is A-universally injective.*

Proof. Choose U, N' as in the definition of a pure spreadout. Any element $x' \in N'$ has a content ideal in A because N' is A-projective (this can easily be seen directly, but it also follows from More on Algebra, Lemma 22.4 and Algebra, Example 90.1). Since $N' \to N$ is A-universally injective, we see that the image $x \in N$ of any $x' \in N'$ has a content ideal in A (it is the same as the content ideal of x'). For a general $x \in N$ we choose $s \in S$ such that sx is in the image of $N' \to N$ and we use that x and sx have the same content ideal.

Let $u : N \to M$ be as in (2). To show that u is A-universally injective, we may replace A by a localization at a maximal ideal (small detail omitted). Assume A is local with maximal ideal $\mathfrak{m}$. Pick $s \in S$ and consider the composition

$$N' \to N \xrightarrow{1/s} N \xrightarrow{u} M$$

Each of these maps is injective modulo $\mathfrak{m}$, hence the composition is A-universally injective by Lemma 7.5. Since $N = \mathrm{colim}_{s \in S}(1/s)N'$ we conclude that u is A-inversally injective as a colimit of universally injective maps. □

0AT8 **Lemma 25.8.** *In (25.6.1) for every $\mathfrak{p} \in \mathrm{Spec}(A)$ there is a finitely generated ideal $I \subset \mathfrak{p}A_\mathfrak{p}$ such that over $A_\mathfrak{p}/I$ we have a pure spreadout.*

Proof. We may replace A by $A_\mathfrak{p}$. Thus we may assume A is local and $\mathfrak{p}$ is the maximal ideal $\mathfrak{m}$ of A. We may write $N = S^{-1}N'$ for some finitely presented B-module N' by clearing denominators in a presentation of N over $S^{-1}B$. Since $B/\mathfrak{m}B$ is Noetherian, the kernel K of $N'/\mathfrak{m}N' \to N/\mathfrak{m}N$ is finitely generated. Thus we can pick $s \in S$ such that K is annihilated by s. After replacing B by B_s which is allowed as it just means passing to an affine open subscheme of $\mathrm{Spec}(B)$, we find that the elements of S are injective on $N'/\mathfrak{m}N'$. At this point we choose a local subring $A_0 \subset A$ essentially of finite type over $\mathbf{Z}$, a finite type ring map $A_0 \to B_0$ such that $B = A \otimes_{A_0} B_0$, and a finite B_0-module N'_0 such that $N' = B \otimes_{B_0} N'_0 = A \otimes_{A_0} N'_0$. We claim that $I = \mathfrak{m}_{A_0}A$ works. Namely, we have

$$N'/IN' = N'_0/\mathfrak{m}_{A_0}N'_0 \otimes_{\kappa_{A_0}} A/I$$

which is free over A/I. Multiplication by the elements of S is injective after dividing out by the maximal ideal, hence $N'/IN' \to N/IN$ is universally injective for example by Lemma 7.6. □

0AT9 **Lemma 25.9.** *In (25.6.1) assume N is A-flat, M is a flat A-module, and $u : N \to M$ is an A-module map such that $u \otimes id_{\kappa(\mathfrak p)}$ is injective for all $\mathfrak p \in \text{Spec}(A)$. Then u is A-universally injective.*

Proof. By Algebra, Lemma 81.14 it suffices to check that $N/IN \to M/IM$ is injective for every ideal $I \subset A$. After replacing A by A/I we see that it suffices to prove that u is injective.

Proof that u is injective. Let $x \in N$ be a nonzero element of the kernel of u. Then there exists a weakly associated prime $\mathfrak p$ of the module Ax, see Algebra, Lemma 65.4. Replacing A by $A_{\mathfrak p}$ we may assume A is local and we find a nonzero element $y \in Ax$ whose annihilator has radical equal to $\mathfrak m_A$, see Algebra, Lemma 65.2. Thus $\text{Supp}(y) \subset \text{Spec}(S^{-1}B)$ is nonempty and contained in the closed fibre of $\text{Spec}(S^{-1}B) \to \text{Spec}(A)$. Let $I \subset \mathfrak m_A$ be a finitely generated ideal so that we have a pure spreadout over A/I, see Lemma 25.8. Then $I^n y = 0$ for some n. Now $y \in \text{Ann}_M(I^n) = \text{Ann}_A(I^n) \otimes_R N$ by flatness. Thus, to get the desired contradiction, it suffices to show that

$$\text{Ann}_A(I^n) \otimes_R N \longrightarrow \text{Ann}_A(I^n) \otimes_R M$$

is injective. Since N and M are flat and since $\text{Ann}_A(I^n)$ is annihilated by I^n, it suffices to show that $Q \otimes_A N \to Q \otimes_A M$ is injective for every A-module Q annihilated by I. This holds by our choice of I and Lemma 25.7 part (2). $\square$

0ATA **Lemma 25.10.** *Let A be a local domain. Let S be a set of finitely generated ideals of A. Assume that S is closed under products and such that $\bigcap_{I \in S} V(I)$ is the complement of the generic point of $\text{Spec}(A)$. Then $\bigcap_{I \in S} I = (0)$.*

Proof. Let $f \in A$ be nonzero. Then $V(f) \subset \bigcup_{I \in S} V(I)$. Since the constructible topology on $V(f)$ is quasi-compact (Topology, Lemma 23.2 and Algebra, Lemma 25.2) we find that $V(f) \subset V(I_1) \cup \ldots \cup V(I_n)$ for some $I_j \in S$. Because $I_1 \ldots I_n \in S$ we see that $V(f) \subset V(I)$ for some I. As I is finitely generated this implies that $I^m \subset (f)$ for some m and since S is closed under products we see that $I \subset (f^2)$ for some $I \in S$. Then it is not possible to have $f \in I$. $\square$

0ATB **Lemma 25.11.** *Let A be a local ring. Let $I, J \subset A$ be ideals. If J is finitely generated and $I \subset J^n$ for all $n \geq 1$, then $V(I)$ contains the closed points of $\text{Spec}(A) \setminus V(J)$.*

Proof. Let $\mathfrak p \subset A$ be a closed point of $\text{Spec}(A) \setminus V(J)$. We want to show that $I \subset \mathfrak p$. If not, then some $f \in I$ maps to a nonzero element of $A/\mathfrak p$. Note that $V(J) \cap \text{Spec}(A/\mathfrak p)$ is the set of non-generic points. Hence by Lemma 25.10 applied to the collection of ideals $J^n A/\mathfrak p$ we conclude that the image of f is zero in $A/\mathfrak p$. $\square$

0ATC **Lemma 25.12.** *Let A be a local ring. Let $I \subset A$ be an ideal. Let $U \subset \text{Spec}(A)$ be quasi-compact open. Let M be an A-module. Assume that*

1. *M/IM is flat over A/I,*
2. *M is flat over U,*

Then M/I_2M is flat over A/I_2 where $I_2 = \text{Ker}(I \to \Gamma(U, I/I^2))$.

Proof. It suffices to show that $M \otimes_A I/I_2 \to IM/I_2M$ is injective, see Algebra, Lemma 98.9. This is true over U by assumption (2). Thus it suffices to show that $M \otimes_A I/I_2$ injects into its sections over U. We have $M \otimes_A I/I_2 = M/IM \otimes_A I/I_2$ and M/IM is a filtered colimit of finite free A/I-modules (Algebra, Theorem 80.4).

Hence it suffices to show that I/I_2 injects into its sections over U, which follows from the construction of I_2. $\square$

05U9 **Proposition 25.13.** *Let $A \to B$ be a local ring homomorphism of local rings which is essentially of finite type. Let M be a flat A-module, N a finite B-module and $u : N \to M$ an A-module map such that $\overline{u} : N/\mathfrak{m}_A N \to M/\mathfrak{m}_A M$ is injective. Then u is A-universally injective, N is of finite presentation over B, and N is flat over A.*

Proof. We may assume that B is the localization of a finitely presented A-algebra B_0 and that N is the localization of a finitely presented B_0-module M_0, see Lemma 25.3. By Lemma 21.3 there exists a "generic flatness stratification" for $\widetilde{M_0}$ on $\mathrm{Spec}(B_0)$ over $\mathrm{Spec}(A)$. Translating back to N we find a sequence of closed subschemes

$$S = \mathrm{Spec}(A) \supset S_0 \supset S_1 \supset \ldots \supset S_t = \emptyset$$

with $S_i \subset S$ cut out by a finitely generated ideal of A such that the pullback of $\widetilde{N}$ to $\mathrm{Spec}(B) \times_S (S_i \setminus S_{i+1})$ is flat over $S_i \setminus S_{i+1}$. We will prove the proposition by induction on t (the base case $t = 1$ will be proved in parallel with the other steps). Let $\mathrm{Spec}(A/J_i)$ be the scheme theoretic closure of $S_i \setminus S_{i+1}$.

Claim 1. N/J_iN is flat over A/J_i. This is immediate for $i = t - 1$ and follows from the induction hypothesis for $i > 0$. Thus we may assume $t > 1$, $S_{t-1} \neq \emptyset$, and $J_0 = 0$ and we have to prove that N is flat. Let $J \subset A$ be the ideal defining S_1. By induction on t again, we also have flatness modulo powers of J. Let A^h be the henselization of A and let B' be the localization of $B \otimes_A A^h$ at the maximal ideal $\mathfrak{m}_B \otimes A^h + B \otimes \mathfrak{m}_{A^h}$. Then $B \to B'$ is faithfully flat. Set $N' = N \otimes_B B'$. Note that N' is A^h-flat if and only if N is A-flat. By Theorem 24.1 there is a smallest ideal $I \subset A^h$ such that N'/IN' is flat over A^h/I, and I is finitely generated. By the above $I \subset J^nA^h$ for all $n \geq 1$. Let $S_i^h \subset \mathrm{Spec}(A^h)$ be the inverse image of $S_i \subset \mathrm{Spec}(A)$. By Lemma 25.11 we see that $V(I)$ contains the closed points of $U = \mathrm{Spec}(A^h) - S_1^h$. By construction N' is A^h-flat over U. By Lemma 25.12 we see that N'/I_2N' is flat over A/I_2, where $I_2 = \mathrm{Ker}(I \to \Gamma(U, I/I^2))$. Hence $I = I_2$ by minimality of I. This implies that $I = I^2$ locally on U, i.e., we have $I\mathcal{O}_{U,u} = (0)$ or $I\mathcal{O}_{U,u} = (1)$ for all $u \in U$. Since $V(I)$ contains the closed points of U we see that $I = 0$ on U. Since $U \subset \mathrm{Spec}(A^h)$ is scheme theoretically dense (because replaced A by A/J_0 in the beginning of this paragraph), we see that $I = 0$. Thus N' is A^h-flat and hence Claim 1 holds.

We return to the situation as laid out before Claim 1. With A^h the henselization of A, with B' the localization of $B \otimes_A A^h$ at the maximal ideal $\mathfrak{m}_B \otimes A^h + B \otimes \mathfrak{m}_{A^h}$, and with $N' = N \otimes_B B'$ we now see that the flattening ideal $I \subset A^h$ of Theorem 24.1 is nilpotent. If $nil(A^h)$ denotes the ideal of nilpotent elements, then $nil(A^h) = nil(A)A^h$ (More on Algebra, Lemma 42.5). Hence there exists a finitely generated nilpotent ideal $I_0 \subset A$ such that N/I_0N is flat over A/I_0.

Claim 2. For every prime ideal $\mathfrak{p} \subset A$ the map $\kappa(\mathfrak{p}) \otimes_A N \to \kappa(\mathfrak{p}) \otimes_A M$ is injective. We say $\mathfrak{p}$ is bad it this is false. Suppose that C is a nonempty chain of bad primes and set $\mathfrak{p}^* = \bigcup_{\mathfrak{p} \in C} \mathfrak{p}$. By Lemma 25.8 there is a finitely generated ideal $\mathfrak{a} \subset \mathfrak{p}^*A_{\mathfrak{p}^*}$ such that there is a pure spreadout over $V(\mathfrak{a})$. If $\mathfrak{p}^*$ were good, then it would follow from Lemma 25.7 that the points of $V(\mathfrak{a})$ are good. However, since $\mathfrak{a}$ is finitely generated and since $\mathfrak{p}^*A_{\mathfrak{p}^*} = \bigcup_{\mathfrak{p} \in C} A_{\mathfrak{p}^*}$ we see that $V(\mathfrak{a})$ contains a $\mathfrak{p} \in C$, contradiction. Hence $\mathfrak{p}^*$ is bad. By Zorn's lemma, if there exists a bad prime, there exists a maximal one, say $\mathfrak{p}$. In other words, we may assume every $\mathfrak{p}' \supset \mathfrak{p}$, $\mathfrak{p}' \neq \mathfrak{p}$ is good. In this case we see that for every $f \in A$, $f \notin \mathfrak{p}$ the map $u \otimes \mathrm{id}_{A/(\mathfrak{p}+f)}$ is universally injective, see Lemma 25.9. Thus it suffices to show that $N/\mathfrak{p}N$ is separated for the topology defined by the submodules $f(N/\mathfrak{p}N)$. Since $B \to B'$ is faithfully flat, it is enough to prove the same for the module $N'/\mathfrak{p}N'$. By Lemma 19.5 and More on Algebra, Lemma 22.4 elements of $N'/\mathfrak{p}N'$ have content ideals in $A^h/\mathfrak{p}A^h$. Thus it suffices to show that $\bigcap_{f \in A, f \notin \mathfrak{p}} f(A^h/\mathfrak{p}A^h) = 0$. Then it suffices to show the same for $A^h/\mathfrak{q}A^h$ for every prime $\mathfrak{q} \subset A^h$ minimal over $\mathfrak{p}A^h$. Because $A \to A^h$ is the henselization, every $\mathfrak{q}$ contracts to $\mathfrak{p}$ and every $\mathfrak{q}' \supset \mathfrak{q}$, $\mathfrak{q}' \neq \mathfrak{q}$ contracts to a prime $\mathfrak{p}'$ which strictly contains $\mathfrak{p}$. Thus we get the vanishing of the intersections from Lemma 25.10.

At this point we can put everything together. Namely, using Claim 1 and Claim 2 we see that $N/I_0N \to M/I_0M$ is A/I_0-universally injective by Lemma 25.9. Then the diagrams

$$\begin{tikzcd} N \otimes_A (I_0^n/I_0^{n+1}) \arrow[r] \arrow[d] & M \otimes_A (I_0^n/I_0^{n+1}) \arrow[d, equal] \\ I_0^nN/I_0^{n+1}N \arrow[r] & I_0^nM/I_0^{n+1}M \end{tikzcd}$$

show that the left vertical arrows are injective. Hence by Algebra, Lemma 98.9 we see that N is flat. In a similar way the universal injectivity of u can be reduced (even without proving flatness of N first) to the one modulo I_0. This finishes the proof. $\square$

26. Flat finite type modules, Part III

05U8 The following result is one of the main results of this chapter.

05UA **Theorem 26.1.** *Let $f : X \to S$ be locally of finite type. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module of finite type. Let $x \in X$ with image $s \in S$. The following are equivalent*

1. *$\mathcal{F}$ is flat at x over S, and*
2. *for every $x' \in Ass_{X_s}(\mathcal{F}_s)$ which specializes to x we have that $\mathcal{F}$ is flat at x' over S.*

Proof. It is clear that (1) implies (2) as $\mathcal{F}_{x'}$ is a localization of $\mathcal{F}_x$ for every point which specializes to x. Set $A = \mathcal{O}_{S,s}$, $B = \mathcal{O}_{X,x}$ and $N = \mathcal{F}_x$. Let $\Sigma \subset B$ be the multiplicative subset of B of elements which act as nonzerodivisors on $N/\mathfrak{m}_AN$. Assumption (2) implies that $\Sigma^{-1}N$ is A-flat by the description of $\mathrm{Spec}(\Sigma^{-1}N)$ in Lemma 7.1. On the other hand, the map $N \to \Sigma^{-1}N$ is injective modulo $\mathfrak{m}_A$ by construction. Hence applying Lemma 25.5 we win. $\square$

Now we apply this directly to obtain the following useful results.

05UB **Lemma 26.2.** *Let S be a local scheme with closed point s. Let $f : X \to S$ be locally of finite type. Let $\mathcal{F}$ be a finite type quasi-coherent $\mathcal{O}_X$-module. Assume that*

1. *every point of $Ass_{X/S}(\mathcal{F})$ specializes to a point of the closed fibre X_s*[3]*,*
2. *$\mathcal{F}$ is flat over S at every point of X_s.*

Then $\mathcal{F}$ is flat over S.

[3]For example this holds if f is finite type and $\mathcal{F}$ is pure along X_s, or if f is proper.

Proof. This is immediate from the fact that it suffices to check for flatness at points of the relative assassin of $\mathcal{F}$ over S by Theorem 26.1. $\square$

27. Universal flattening

05PS If $f : X \to S$ is a proper, finitely presented morphism of schemes then one can find a universal flattening of f. In this section we discuss this and some of its variants.

05UC **Lemma 27.1.** *In Situation 20.7. For each $p \geq 0$ the functor H_p (20.7.2) is representable by a locally closed immersion $S_p \to S$. If $\mathcal{F}$ is of finite presentation, then $S_p \to S$ is of finite presentation.*

Proof. For each S we will prove the statement for all $p \geq 0$ concurrently. The functor H_p is a sheaf for the fppf topology by Lemma 20.8. Hence combining Descent, Lemma 36.1, More on Morphisms, Lemma 46.1 , and Descent, Lemma 21.1 we see that the question is local for the étale topology on S. In particular, the question is Zariski local on S.

For $s \in S$ denote ξ_s the unique generic point of the fibre X_s. Note that for every $s \in S$ the restriction $\mathcal{F}_s$ of $\mathcal{F}$ is locally free of some rank $p(s) \geq 0$ in some neighbourhood of ξ_s. (As X_s is irreducible and smooth this follows from generic flatness for $\mathcal{F}_s$ over X_s, see Algebra, Lemma 117.1 although this is overkill.) For future reference we note that

$$p(s) = \dim_{\kappa(\xi_s)}(\mathcal{F}_{\xi_s} \otimes_{\mathcal{O}_{X,\xi_s}} \kappa(\xi_s)).$$

In particular $H_{p(s)}(s)$ is nonempty and $H_q(s)$ is empty if $q \neq p(s)$.

Let $U \subset X$ be an open subscheme. As $f : X \to S$ is smooth, it is open. It is immediate from (20.7.2) that the functor H_p for the pair $(f|_U : U \to f(U), \mathcal{F}|_U)$ and the functor H_p for the pair $(f|_{f^{-1}(f(U))}, \mathcal{F}|_{f^{-1}(f(U))})$ are the same. Hence to prove the existence of S_p over $f(U)$ we may always replace X by U.

Pick $s \in S$. There exists an affine open neighbourhood U of ξ_s such that $\mathcal{F}|_U$ can be generated by at most $p(s)$ elements. By the arguments above we see that in order to prove the statement for $H_{p(s)}$ in an neighbourhood of s we may assume that $\mathcal{F}$ is generated by $p(s)$ elements, i.e., that there exists a surjection

$$u : \mathcal{O}_X^{\oplus p(s)} \longrightarrow \mathcal{F}$$

In this case it is clear that $H_{p(s)}$ is equal to F_{iso} (20.1.1) for the map u (this follows immediately from Lemma 19.1 but also from Lemma 12.1 after shrinking a bit more so that both S and X are affine.) Thus we may apply Theorem 23.3 to see that $H_{p(s)}$ is representable by a closed immersion in a neighbourhood of s.

The result follows formally from the above. Namely, the arguments above show that locally on S the function $s \mapsto p(s)$ is bounded. Hence we may use induction on $p = \max_{s \in S} p(s)$. The functor H_p is representable by a closed immersion $S_p \to S$ by the above. Replace S by $S \setminus S_p$ which drops the maximum by at least one and we win by induction hypothesis.

Assume $\mathcal{F}$ is of finite presentation. Then $S_p \to S$ is locally of finite presentation by Lemma 20.8 part (2) combined with Limits, Remark 6.2. Then we redo the induction argument in the paragraph to see that each S_p is quasi-compact when S is affine: first if $p = \max_{s \in S} p(s)$, then $S_p \subset S$ is closed (see above) hence quasi-compact. Then $U = S \setminus S_p$ is quasi-compact open in S because $S_p \to S$ is a closed immersion of finite presentation (see discussion in Morphisms, Section 21 for example). Then $S_{p-1} \to U$ is a closed immersion of finite presentation, and so S_{p-1} is quasi-compact and $U' = S \setminus (S_p \cup S_{p-1})$ is quasi-compact. And so on. $\square$

05UD **Lemma 27.2.** *In Situation 20.11. Let $h : X' \to X$ be an étale morphism. Set $\mathcal{F}' = h^*\mathcal{F}$ and $f' = f \circ h$. Let F'_n be (20.11.1) associated to $(f' : X' \to S, \mathcal{F}')$. Then F_n is a subfunctor of F'_n and if $h(X') \supset Ass_{X/S}(\mathcal{F})$, then $F_n = F'_n$.*

Proof. Let $T \to S$ be any morphism. Then $h_T : X'_T \to X_T$ is étale as a base change of the étale morphism g. For $t \in T$ denote $Z \subset X_t$ the set of points where $\mathcal{F}_T$ is not flat over T, and similarly denote $Z' \subset X'_t$ the set of points where $\mathcal{F}'_T$ is not flat over T. As $\mathcal{F}'_T = h_T^*\mathcal{F}_T$ we see that $Z' = h_t^{-1}(Z)$, see Morphisms, Lemma 24.12. Hence $Z' \to Z$ is an étale morphism, so $\dim(Z') \leq \dim(Z)$ (for example by Descent, Lemma 18.2 or just because an étale morphism is smooth of relative dimension 0). This implies that $F_n \subset F'_n$.

Finally, suppose that $h(X') \supset \text{Ass}_{X/S}(\mathcal{F})$ and that $T \to S$ is a morphism such that $F'_n(T)$ is nonempty, i.e., such that $\mathcal{F}'_T$ is flat in dimensions $\geq n$ over T. Pick a point $t \in T$ and let $Z \subset X_t$ and $Z' \subset X'_t$ be as above. To get a contradiction assume that $\dim(Z) \geq n$. Pick a generic point $\xi \in Z$ corresponding to a component of dimension $\geq n$. Let $x \in \text{Ass}_{X_t}(\mathcal{F}_t)$ be a generalization of ξ. Then x maps to a point of $\text{Ass}_{X/S}(\mathcal{F})$ by Divisors, Lemma 7.3 and Remark 7.4. Thus we see that x is in the image of h_T, say $x = h_T(x')$ for some $x' \in X'_T$. But $x' \notin Z'$ as $x \leadsto \xi$ and $\dim(Z') < n$. Hence $\mathcal{F}'_T$ is flat over T at x' which implies that $\mathcal{F}_T$ is flat at x over T (by Morphisms, Lemma 24.12). Since this holds for every such x we conclude that $\mathcal{F}_T$ is flat over T at ξ by Theorem 26.1 which is the desired contradiction. $\square$

05UE **Lemma 27.3.** *Assume that $X \to S$ is a smooth morphism of affine schemes with geometrically irreducible fibres of dimension d and that $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_X$-module of finite presentation. Then $F_d = \coprod_{p=0,\ldots,c} H_p$ for some $c \geq 0$ with F_d as in (20.11.1) and H_p as in (20.7.2).*

Proof. As X is affine and $\mathcal{F}$ is quasi-coherent of finite presentation we know that $\mathcal{F}$ can be generated by $c \geq 0$ elements. Then $\dim_{\kappa(x)}(\mathcal{F}_x \otimes \kappa(x))$ in any point $x \in X$ never exceeds c. In particular $H_p = \emptyset$ for $p > c$. Moreover, note that there certainly is an inclusion $\coprod H_p \to F_d$. Having said this the content of the lemma is that, if a base change $\mathcal{F}_T$ is flat in dimensions $\geq d$ over T and if $t \in T$, then $\mathcal{F}_T$ is free of some rank r in an open neighbourhood $U \subset X_T$ of the unique generic point ξ of X_t. Namely, then H_r contains the image of U which is an open neighbourhood of t. The existence of U follows from More on Morphisms, Lemma 16.7. $\square$

05UF **Lemma 27.4.** *In Situation 20.11. Let $s \in S$ let $d \geq 0$. Assume*

1. *there exists a complete dévissage of $\mathcal{F}/X/S$ over some point $s \in S$,*
2. *X is of finite presentation over S,*
3. *$\mathcal{F}$ is an $\mathcal{O}_X$-module of finite presentation, and*
4. *$\mathcal{F}$ is flat in dimensions $\geq d + 1$ over S.*

Then after possibly replacing S by an open neighbourhood of s the functor F_d (20.11.1) is representable by a monomorphism $Z_d \to S$ of finite presentation.

Proof. A preliminary remark is that X, S are affine schemes and that it suffices to prove F_d is representable by a monomorphism of finite presentation $Z_d \to S$ on the category of affine schemes over S. (Of course we do not require Z_d to be affine.)

Hence throughout the proof of the lemma we work in the category of affine schemes over S.

Let $(Z_k, Y_k, i_k, \pi_k, \mathcal{G}_k, \alpha_k)_{k=1,\ldots,n}$ be a complete dévissage of $\mathcal{F}/X/S$ over s, see Definition 5.1. We will use induction on the length n of the dévissage. Recall that $Y_k \to S$ is smooth with geometrically irreducible fibres, see Definition 4.1. Let d_k be the relative dimension of Y_k over S. Recall that $i_{k,*}\mathcal{G}_k = \mathrm{Coker}(\alpha_k)$ and that i_k is a closed immersion. By the definitions referenced above we have $d_1 = \dim(\mathrm{Supp}(\mathcal{F}_s))$ and

$$d_k = \dim(\mathrm{Supp}(\mathrm{Coker}(\alpha_{k-1})_s)) = \dim(\mathrm{Supp}(\mathcal{G}_{k,s}))$$

for $k = 2, \ldots, n$. It follows that $d_1 > d_2 > \ldots > d_n \geq 0$ because α_k is an isomorphism in the generic point of $(Y_k)_s$.

Note that i_1 is a closed immersion and $\mathcal{F} = i_{1,*}\mathcal{G}_1$. Hence for any morphism of schemes $T \to S$ with T affine, we have $\mathcal{F}_T = i_{1,T,*}\mathcal{G}_{1,T}$ and $i_{1,T}$ is still a closed immersion of schemes over T. Thus $\mathcal{F}_T$ is flat in dimensions $\geq d$ over T if and only if $\mathcal{G}_{1,T}$ is flat in dimensions $\geq d$ over T. Because $\pi_1 : Z_1 \to Y_1$ is finite we see in the same manner that $\mathcal{G}_{1,T}$ is flat in dimensions $\geq d$ over T if and only if $\pi_{1,T,*}\mathcal{G}_{1,T}$ is flat in dimensions $\geq d$ over T. The same arguments work for "flat in dimensions $\geq d + 1$" and we conclude in particular that $\pi_{1,*}\mathcal{G}_1$ is flat over S in dimensions $\geq d + 1$ by our assumption on $\mathcal{F}$.

Suppose that $d_1 > d$. It follows from the discussion above that in particular $\pi_{1,*}\mathcal{G}_1$ is flat over S at the generic point of $(Y_1)_s$. By Lemma 12.1 we may replace S by an affine neighbourhood of s and assume that α_1 is S-universally injective. Because α_1 is S-universally injective, for any morphism $T \to S$ with T affine, we have a short exact sequence

$$0 \to \mathcal{O}_{Y_{1,T}}^{\oplus r_1} \to \pi_{1,T,*}\mathcal{G}_{1,T} \to \mathrm{Coker}(\alpha_1)_T \to 0$$

and still the first arrow is T-universally injective. Hence the set of points of $(Y_1)_T$ where $\pi_{1,T,*}\mathcal{G}_{1,T}$ is flat over T is the same as the set of points of $(Y_1)_T$ where $\mathrm{Coker}(\alpha_1)_T$ is flat over S. In this way the question reduces to the sheaf $\mathrm{Coker}(\alpha_1)$ which has a complete dévissage of length $n - 1$ and we win by induction.

If $d_1 < d$ then F_d is represented by S and we win.

The last case is the case $d_1 = d$. This case follows from a combination of Lemma 27.3 and Lemma 27.1. □

05UG **Theorem 27.5.** *In Situation 20.11. Assume moreover that f is of finite presentation, that $\mathcal{F}$ is an $\mathcal{O}_X$-module of finite presentation, and that $\mathcal{F}$ is pure relative to S. Then F_n is representable by a monomorphism $Z_n \to S$ of finite presentation.*

Proof. The functor F_n is a sheaf for the fppf topology by Lemma 20.12. Observe that a monomorphism of finite presentation is separated and quasi-finite (Morphisms, Lemma 19.15). Hence combining Descent, Lemma 36.1, More on Morphisms, Lemma 46.1 , and Descent, Lemmas 20.31 and 20.13 we see that the question is local for the étale topology on S.

In particular the situation is local for the Zariski topology on S and we may assume that S is affine. In this case the dimension of the fibres of f is bounded above, hence we see that F_n is representable for n large enough. Thus we may use descending induction on n. Suppose that we know F_{n+1} is representable by a monomorphism

$Z_{n+1} \to S$ of finite presentation. Consider the base change $X_{n+1} = Z_{n+1} \times_S X$ and the pullback $\mathcal{F}_{n+1}$ of $\mathcal{F}$ to X_{n+1}. The morphism $Z_{n+1} \to S$ is quasi-finite as it is a monomorphism of finite presentation, hence Lemma 16.4 implies that $\mathcal{F}_{n+1}$ is pure relative to Z_{n+1}. Since F_n is a subfunctor of F_{n+1} we conclude that in order to prove the result for F_n it suffices to prove the result for the corresponding functor for the situation $\mathcal{F}_{n+1}/X_{n+1}/Z_{n+1}$. In this way we reduce to proving the result for F_n in case $S_{n+1} = S$, i.e., we may assume that $\mathcal{F}$ is flat in dimensions $\geq n+1$ over S.

Fix n and assume $\mathcal{F}$ is flat in dimensions $\geq n+1$ over S. To finish the proof we have to show that F_n is representable by a monomorphism $Z_n \to S$ of finite presentation. Since the question is local in the étale topology on S it suffices to show that for every $s \in S$ there exists an elementary étale neighbourhood $(S', s') \to (S, s)$ such that the result holds after base change to S'. Thus by Lemma 5.8 we may assume there exist étale morphisms $h_j : Y_j \to X$, $j = 1, \ldots, m$ such that for each j there exists a complete dévissage of $\mathcal{F}_j/Y_j/S$ over s, where $\mathcal{F}_j$ is the pullback of $\mathcal{F}$ to Y_j and such that $X_s \subset \bigcup h_j(Y_j)$. Note that by Lemma 27.2 the sheaves $\mathcal{F}_j$ are still flat over in dimensions $\geq n+1$ over S. Set $W = \bigcup h_j(Y_j)$, which is a quasi-compact open of X. As $\mathcal{F}$ is pure along X_s we see that

$$E = \{t \in S \mid \text{Ass}_{X_t}(\mathcal{F}_t) \subset W\}.$$

contains all generalizations of s. By More on Morphisms, Lemma 23.5 E is a constructible subset of S. We have seen that $\text{Spec}(\mathcal{O}_{S,s}) \subset E$. By Morphisms, Lemma 21.4 we see that E contains an open neighbourhood of s. Hence after shrinking S we may assume that $E = S$. It follows from Lemma 27.2 that it suffices to prove the lemma for the functor F_n associated to $X = \coprod Y_j$ and $\mathcal{F} = \coprod \mathcal{F}_j$. If $F_{j,n}$ denotes the functor for $Y_j \to S$ and the sheaf $\mathcal{F}_i$ we see that $F_n = \prod F_{j,n}$. Hence it suffices to prove each $F_{j,n}$ is representable by some monomorphism $Z_{j,n} \to S$ of finite presentation, since then

$$Z_n = Z_{1,n} \times_S \ldots \times_S Z_{m,n}$$

Thus we have reduced the theorem to the special case handled in Lemma 27.4. $\square$

We make explicit what the theorem means in terms of universal flattenings in the following lemma.

05UH **Lemma 27.6.** *Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module.*

1. *If f is of finite presentation, $\mathcal{F}$ is an $\mathcal{O}_X$-module of finite presentation, and $\mathcal{F}$ is pure relative to S, then there exists a universal flattening $S' \to S$ of $\mathcal{F}$. Moreover $S' \to S$ is a monomorphism of finite presentation.*
2. *If f is of finite presentation and X is pure relative to S, then there exists a universal flattening $S' \to S$ of X. Moreover $S' \to S$ is a monomorphism of finite presentation.*
3. *If f is proper and of finite presentation and $\mathcal{F}$ is an $\mathcal{O}_X$-module of finite presentation, then there exists a universal flattening $S' \to S$ of $\mathcal{F}$. Moreover $S' \to S$ is a monomorphism of finite presentation.*
4. *If f is proper and of finite presentation then there exists a universal flattening $S' \to S$ of X.*

Proof. These statements follow immediately from Theorem 27.5 applied to $F_0 = F_{flat}$ and the fact that if f is proper then $\mathcal{F}$ is automatically pure over the base, see Lemma 17.1. □

28. Grothendieck's Existence Theorem, IV

0CTB This section continues the discussion in Cohomology of Schemes, Sections 23, 24, and 26. We will work in the following situation.

0CTC **Situation 28.1.** Here we have an inverse system of rings (A_n) with surjective transition maps whose kernels are locally nilpotent. Set $A = \lim A_n$. We have a scheme X separated and of finite presentation over A. We set $X_n = X \times_{\mathrm{Spec}(A)} \mathrm{Spec}(A_n)$ and we view it as a closed subscheme of X. We assume further given a system $(\mathcal{F}_n, \varphi_n)$ where $\mathcal{F}_n$ is a finitely presented $\mathcal{O}_{X_n}$-module, flat over A_n, with support proper over A_n, and

$$\varphi_n : \mathcal{F}_n \otimes_{\mathcal{O}_{X_n}} \mathcal{O}_{X_{n-1}} \longrightarrow \mathcal{F}_{n-1}$$

is an isomorphism (notation using the equivalence of Morphisms, Lemma 4.1).

Our goal is to see if we can find a quasi-coherent sheaf $\mathcal{F}$ on X such that $\mathcal{F}_n = \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_{X_n}$ for all n.

0CTD **Lemma 28.2.** *In Situation 28.1 consider*

$$K = R\lim\nolimits_{D_{QCoh}(\mathcal{O}_X)}(\mathcal{F}_n) = DQ_X(R\lim\nolimits_{D(\mathcal{O}_X)} \mathcal{F}_n)$$

Then K is in $D^b_{QCoh}(\mathcal{O}_X)$ and in fact K has nonzero cohomology sheaves only in degrees ≥ 0.

Proof. Special case of Derived Categories of Schemes, Example 20.5. □

0CTE **Lemma 28.3.** *In Situation 28.1 let K be as in Lemma 28.2. For any perfect object E of $D(\mathcal{O}_X)$ we have*

1. *$M = R\Gamma(X, K \otimes^{\mathbf{L}} E)$ is a perfect object of $D(A)$ and there is a canonical isomorphism $R\Gamma(X_n, \mathcal{F}_n \otimes^{\mathbf{L}} E|_{X_n}) = M \otimes_A^{\mathbf{L}} A_n$ in $D(A_n)$,*
2. *$N = R\mathrm{Hom}_X(E, K)$ is a perfect object of $D(A)$ and there is a canonical isomorphism $R\mathrm{Hom}_{X_n}(E|_{X_n}, \mathcal{F}_n) = N \otimes_A^{\mathbf{L}} A_n$ in $D(A_n)$.*

In both statements $E|_{X_n}$ denotes the derived pullback of E to X_n.

Proof. Proof of (2). Write $E_n = E|_{X_n}$ and $N_n = R\mathrm{Hom}_{X_n}(E_n, \mathcal{F}_n)$. Recall that $R\mathrm{Hom}_{X_n}(-,-)$ is equal to $R\Gamma(X_n, R\mathcal{H}om(-,-))$, see Cohomology, Section 38. Hence by Derived Categories of Schemes, Lemma 26.7 we see that N_n is a perfect object of $D(A_n)$ whose formation commutes with base change. Thus the maps $N_n \otimes_{A_n}^{\mathbf{L}} A_{n-1} \to N_{n-1}$ coming from φ_n are isomorphisms. By More on Algebra, Lemma 83.3 we find that $R\lim N_n$ is perfect and that its base change back to A_n recovers N_n. On the other hand, the exact functor $R\mathrm{Hom}_X(E, -) : D_{QCoh}(\mathcal{O}_X) \to D(A)$ of triangulated categories commutes with products and hence with derived limits, whence

$$R\mathrm{Hom}_X(E, K) = R\lim R\mathrm{Hom}(E, \mathcal{F}_n) = R\lim R\mathrm{Hom}(E_n, \mathcal{F}_n) = R\lim N_n$$

This proves (2). To see that (1) holds, translate it into (2) using Cohomology, Lemma 43.11. □

0CTF **Lemma 28.4.** *In Situation 28.1 let K be as in Lemma 28.2. Then K is pseudo-coherent relative to A.*

Proof. Combinging Lemma 28.3 and Derived Categories of Schemes, Lemma 30.3 we see that $R\Gamma(X, K \otimes^{\mathbf{L}} E)$ is pseudo-coherent in $D(A)$ for all pseudo-coherent E in $D(\mathcal{O}_X)$. Thus the lemma follows from More on Morphisms, Lemma 57.4. $\square$

0CTG **Lemma 28.5.** *In Situation 28.1 let K be as in Lemma 28.2. For any quasi-compact open $U \subset X$ we have*

$$R\Gamma(U, K) \otimes_A^{\mathbf{L}} A_n = R\Gamma(U_n, \mathcal{F}_n)$$

in $D(A_n)$ where $U_n = U \cap X_n$.

Proof. Fix n. By Derived Categories of Schemes, Lemma 29.3 there exists a system of perfect complexes E_m on X such that $R\Gamma(U, K) = \text{hocolim} R\Gamma(X, K \otimes^{\mathbf{L}} E_m)$. In fact, this formula holds not just for K but for every object of $D_{QCoh}(\mathcal{O}_X)$. Applying this to $\mathcal{F}_n$ we obtain

$$\begin{aligned} R\Gamma(U_n, \mathcal{F}_n) &= R\Gamma(U, \mathcal{F}_n) \\ &= \text{hocolim}_m R\Gamma(X, \mathcal{F}_n \otimes^{\mathbf{L}} E_m) \\ &= \text{hocolim}_m R\Gamma(X_n, \mathcal{F}_n \otimes^{\mathbf{L}} E_m|_{X_n}) \end{aligned}$$

Using Lemma 28.3 and the fact that $- \otimes_A^{\mathbf{L}} A_n$ commutes with homotopy colimits we obtain the result. $\square$

0CTH **Lemma 28.6.** *In Situation 28.1 let K be as in Lemma 28.2. Denote $X_0 \subset X$ the closed subset consisting of points lying over the closed subset* $\text{Spec}(A_1) = \text{Spec}(A_2) = \ldots$ *of* $\text{Spec}(A)$. *There exists an open $W \subset X$ containing X_0 such that*

(1) *$H^i(K)|_W$ is zero unless $i = 0$,*
(2) *$\mathcal{F} = H^0(K)|_W$ is of finite presentation, and*
(3) *$\mathcal{F}_n = \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_{X_n}$.*

Proof. Fix $n \geq 1$. By construction there is a canonical map $K \to \mathcal{F}_n$ in $D_{QCoh}(\mathcal{O}_X)$ and hence a canonical map $H^0(K) \to \mathcal{F}_n$ of quasi-coherent sheaves. This explains the meaning of part (3).

Let $x \in X_0$ be a point. We will find an open neighbourhood W of x such that (1), (2), and (3) are true. Since X_0 is quasi-compact this will prove the lemma. Let $U \subset X$ be an affine open neighbourhood of x. Say $U = \text{Spec}(B)$. Choose a surjection $P \to B$ with P smooth over A. By Lemma 28.4 and the definition of relative pseudo-coherence there exists a bounded above complex $F^\bullet$ of finite free P-modules representing Ri_*K where $i : U \to \text{Spec}(P)$ is the closed immersion induced by the presentation. Let M_n be the B-module corresponding to $\mathcal{F}_n|_U$. By Lemma 28.5

$$H^i(F^\bullet \otimes_A A_n) = \begin{cases} 0 & \text{if} \quad i \neq 0 \\ M_n & \text{if} \quad i = 0 \end{cases}$$

Let i be the maximal index such that F^i is nonzero. If $i \leq 0$, then (1), (2), and (3) are true. If not, then $i > 0$ and we see that the rank of the map

$$F^{i-1} \to F^i$$

in the point x is maximal. Hence in an open neighbourhood of x inside $\text{Spec}(P)$ the rank is maximal. Thus after replacing P by a principal localization we may assume that the displayed map is surjective. Since F^i is finite free we may choose a splitting $F^{i-1} = F' \oplus F^i$. Then we may replace $F^\bullet$ by the complex

$$\ldots \to F^{i-2} \to F' \to 0 \to \ldots$$

and we win by induction on i. $\square$

0CTI **Lemma 28.7.** *In Situation 28.1 let K be as in Lemma 28.2. Let $W \subset X$ be as in Lemma 28.6. Set $\mathcal{F} = H^0(K)|_W$. Then, after possibly shrinking the open W, the support of $\mathcal{F}$ is proper over A.*

Proof. Fix $n \geq 1$. Let $I_n = \mathrm{Ker}(A \to A_n)$. By More on Algebra, Lemma 10.3 the pair (A, I_n) is henselian. Let $Z \subset W$ be the support of $\mathcal{F}$. This is a closed subset as $\mathcal{F}$ is of finite presentation. By part (3) of Lemma 28.6 we see that $Z \times_{\mathrm{Spec}(A)} \mathrm{Spec}(A_n)$ is equal to the support of $\mathcal{F}_n$ and hence proper over $\mathrm{Spec}(A/I)$. By More on Morphisms, Lemma 45.9 we can write $Z = Z_1 \amalg Z_2$ with Z_1, Z_2 open and closed in Z, with Z_1 proper over A, and with $Z_1 \times_{\mathrm{Spec}(A)} \mathrm{Spec}(A/I_n)$ equal to the support of $\mathcal{F}_n$. In other words, Z_2 does not meet X_0. Hence after replacing W by $W \setminus Z_2$ we obtain the lemma. $\square$

0CTJ **Lemma 28.8.** *Let $A = \lim A_n$ be a limit of a system of rings whose transition maps are surjective and with locally nilpotent kernels. Let $S = \mathrm{Spec}(A)$. Let $T \to S$ be a monomorphism which is locally of finite type. If $\mathrm{Spec}(A_n) \to S$ factors through T for all n, then $T = S$.*

Proof. Set $S_n = \mathrm{Spec}(A_n)$. Let $T_0 \subset T$ be the common image of the factorizations $S_n \to T$. Then T_0 is quasi-compact. Let $T' \subset T$ be a quasi-compact open containing T_0. Then $S_n \to T$ factors through T'. If we can show that $T' = S$, then $T' = T = S$. Hence we may assume T is quasi-compact.

Assume T is quasi-compact. In this case $T \to S$ is separated and quasi-finite (Morphisms, Lemma 19.15). Using Zariski's Main Theorem (in the form of More on Morphisms, Lemma 38.3) we choose a factorization $T \to W \to S$ with $W \to S$ finite and $T \to W$ an open immersion. Write $W = \mathrm{Spec}(B)$. The (unique) factorizations $S_n \to T$ may be viewed as morphisms into W and we obtain

$$A \longrightarrow B \longrightarrow \lim A_n = A$$

Consider the morphism $h : S = \mathrm{Spec}(A) \to \mathrm{Spec}(B) = W$ coming from the arrow on the right. Then

$$T \times_{W,h} S$$

is an open subscheme of S containing the image of $S_n \to S$ for all n. To finish the proof it suffices to show that any open $U \subset S$ containing the image of $S_n \to S$ for some $n \geq 1$ is equal to S. This is true because $(A, \mathrm{Ker}(A \to A_n))$ is a henselian pair (More on Algebra, Lemma 10.3) and hence every closed point of S is contained in the image of $S_n \to S$. $\square$

0CTK **Theorem 28.9** (Grothendieck Existence Theorem)**.** *In Situation 28.1 there exists a finitely presented $\mathcal{O}_X$-module $\mathcal{F}$, flat over A, with support proper over A, such that $\mathcal{F}_n = \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_{X_n}$ for all n compatibly with the maps φ_n.*

Proof. Apply Lemmas 28.2, 28.3, 28.4, 28.5, 28.6, and 28.7 to get an open subscheme $W \subset X$ containing all points lying over $\mathrm{Spec}(A_n)$ and a finitely presented $\mathcal{O}_W$-module $\mathcal{F}$ whose support is proper over A with $\mathcal{F}_n = \mathcal{F} \otimes_{\mathcal{O}_W} \mathcal{O}_{X_n}$ for all

$n \geq 1$. (This makes sense as $X_n \subset W$.) By Lemma 17.1 we see that $\mathcal{F}$ is universally pure relative to $\mathrm{Spec}(A)$. By Theorem 27.5 (for explanation, see Lemma 27.6) there exists a universal flattening $S' \to \mathrm{Spec}(A)$ of $\mathcal{F}$ and moreover the morphism $S' \to \mathrm{Spec}(A)$ is a monomorphism of finite presentation. Since the base change of $\mathcal{F}$ to $\mathrm{Spec}(A_n)$ is $\mathcal{F}_n$ we find that $\mathrm{Spec}(A_n) \to \mathrm{Spec}(A)$ factors (uniquely) through S' for each n. By Lemma 28.8 we see that $S' = \mathrm{Spec}(A)$. This means that $\mathcal{F}$ is flat over A. Finally, since the scheme theoretic support Z of $\mathcal{F}$ is proper over $\mathrm{Spec}(A)$, the morphism $Z \to X$ is closed. Hence the pushforward $(W \to X)_*\mathcal{F}$ is supported on W and has all the desired properties. $\square$

29. Grothendieck's Existence Theorem, V

0DIA In this section we prove an analogue for Grothendieck's existence theorem in the derived category, following the method used in Section 28 for quasi-coherent modules. The classical case is discussed in Cohomology of Schemes, Sections 23, 24, and 26. We will work in the following situation.

0DIB **Situation 29.1.** Here we have an inverse system of rings (A_n) with surjective transition maps whose kernels are locally nilpotent. Set $A = \lim A_n$. We have a scheme X proper, flat, and of finite presentation over A. We set $X_n = X \times_{\mathrm{Spec}(A)} \mathrm{Spec}(A_n)$ and we view it as a closed subscheme of X. We assume further given a system (K_n, φ_n) where K_n is a pseudo-coherent object of $D(\mathcal{O}_{X_n})$ and

$$\varphi_n : K_n \longrightarrow K_{n-1}$$

is a map in $D(\mathcal{O}_{X_n})$ which induces an isomorphism $K_n \otimes^{\mathbf{L}}_{\mathcal{O}_{X_n}} \mathcal{O}_{X_{n-1}} \to K_{n-1}$ in $D(\mathcal{O}_{X_{n-1}})$.

More precisely, we should write $\varphi_n : K_n \to Ri_{n-1,*}K_{n-1}$ where $i_{n-1} : X_{n-1} \to X_n$ is the inclusion morphism and in this notation the condition is that the adjoint map $Li^*_{n-1}K_n \to K_{n-1}$ is an isomorphism. Our goal is to find a pseudo-coherent $K \in D(\mathcal{O}_X)$ such that $K_n = K \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{O}_{X_n}$ for all n (with the same abuse of notation).

0DIC **Lemma 29.2.** *In Situation 29.1 consider*

$$K = R\lim_{D_{QCoh}(\mathcal{O}_X)}(K_n) = DQ_X(R\lim_{D(\mathcal{O}_X)} K_n)$$

Then K is in $D^-_{QCoh}(\mathcal{O}_X)$.

Proof. The functor DQ_X exists because X is quasi-compact and quasi-separated, see Derived Categories of Schemes, Lemma 20.1. Since DQ_X is a right adjoint it commutes with products and therefore with derived limits. Hence the equality in the statement of the lemma.

By Derived Categories of Schemes, Lemma 20.4 the functor DQ_X has bounded cohomological dimension. Hence it suffices to show that $R\lim K_n \in D^-(\mathcal{O}_X)$. To see this, let $U \subset X$ be an affine open. Then there is a canonical exact sequence

$$0 \to R^1\lim H^{m-1}(U, K_n) \to H^m(U, R\lim K_n) \to \lim H^m(U, K_n) \to 0$$

by Cohomology, Lemma 32.1. Since U is affine and K_n is pseudo-coherent (and hence has quasi-coherent cohomology sheaves by Derived Categories of Schemes, Lemma 9.1) we see that $H^m(U, K_n) = H^m(K_n)(U)$ by Derived Categories of Schemes, Lemma 3.5. Thus we conclude that it suffices to show that K_n is bounded above independent of n.

Since K_n is pseudo-coherent we have $K_n \in D^-(\mathcal{O}_{X_n})$. Suppose that a_n is maximal such that $H^{a_n}(K_n)$ is nonzero. Of course $a_1 \leq a_2 \leq a_3 \leq \ldots$. Note that $H^{a_n}(K_n)$ is an $\mathcal{O}_{X_n}$-module of finite presentation (Cohomology, Lemma 41.9). We have $H^{a_n}(K_{n-1}) = H^{a_n}(K_n) \otimes_{\mathcal{O}_{X_n}} \mathcal{O}_{X_{n-1}}$. Since $X_{n-1} \to X_n$ is a thickening, it follows from Nakayama's lemma (Algebra, Lemma 19.1) that if $H^{a_n}(K_n) \otimes_{\mathcal{O}_{X_n}} \mathcal{O}_{X_{n-1}}$ is zero, then $H^{a_n}(K_n)$ is zero too. Thus $a_n = a_{n-1}$ for all n and we conclude. $\square$

0DID **Lemma 29.3.** *In Situation 29.1 let K be as in Lemma 29.2. For any perfect object E of $D(\mathcal{O}_X)$ the cohomology*

$$M = R\Gamma(X, K \otimes^{\mathbf{L}} E)$$

is a pseudo-coherent object of $D(A)$ and there is a canonical isomorphism

$$R\Gamma(X_n, K_n \otimes^{\mathbf{L}} E|_{X_n}) = M \otimes_A^{\mathbf{L}} A_n$$

in $D(A_n)$. Here $E|_{X_n}$ denotes the derived pullback of E to X_n.

Proof. Write $E_n = E|_{X_n}$ and $M_n = R\Gamma(X_n, K_n \otimes^{\mathbf{L}} E|_{X_n})$. By Derived Categories of Schemes, Lemma 26.5 we see that M_n is a pseudo-coherent object of $D(A_n)$ whose formation commutes with base change. Thus the maps $M_n \otimes_{A_n}^{\mathbf{L}} A_{n-1} \to M_{n-1}$ coming from φ_n are isomorphisms. By More on Algebra, Lemma 83.1 we find that $R\lim M_n$ is pseudo-coherent and that its base change back to A_n recovers M_n. On the other hand, the exact functor $R\Gamma(X, -) : D_{QCoh}(\mathcal{O}_X) \to D(A)$ of triangulated categories commutes with products and hence with derived limits, whence

$$R\Gamma(X, E \otimes^{\mathbf{L}} K) = R\lim R\Gamma(X, E \otimes^{\mathbf{L}} K_n) = R\lim R\Gamma(X_n, E_n \otimes^{\mathbf{L}} K_n) = R\lim M_n$$

as desired. $\square$

0DIE **Lemma 29.4.** *In Situation 29.1 let K be as in Lemma 29.2. Then K is pseudo-coherent on X.*

Proof. Combinging Lemma 29.3 and Derived Categories of Schemes, Lemma 30.3 we see that $R\Gamma(X, K \otimes^{\mathbf{L}} E)$ is pseudo-coherent in $D(A)$ for all pseudo-coherent E in $D(\mathcal{O}_X)$. Thus it follows from More on Morphisms, Lemma 57.4 that K is pseudo-coherent relative to A. Since X is of flat and of finite presentation over A, this is the same as being pseudo-coherent on X, see More on Morphisms, Lemma 48.18. $\square$

0DIF **Lemma 29.5.** *In Situation 29.1 let K be as in Lemma 29.2. For any quasi-compact open $U \subset X$ we have*

$$R\Gamma(U, K) \otimes_A^{\mathbf{L}} A_n = R\Gamma(U_n, K_n)$$

in $D(A_n)$ where $U_n = U \cap X_n$.

Proof. Fix n. By Derived Categories of Schemes, Lemma 29.3 there exists a system of perfect complexes E_m on X such that $R\Gamma(U, K) = \text{hocolim} R\Gamma(X, K \otimes^{\mathbf{L}} E_m)$. In fact, this formula holds not just for K but for every object of $D_{QCoh}(\mathcal{O}_X)$. Applying this to K_n we obtain

$$\begin{aligned} R\Gamma(U_n, K_n) &= R\Gamma(U, K_n) \\ &= \text{hocolim}_m R\Gamma(X, K_n \otimes^{\mathbf{L}} E_m) \\ &= \text{hocolim}_m R\Gamma(X_n, K_n \otimes^{\mathbf{L}} E_m|_{X_n}) \end{aligned}$$

Using Lemma 29.3 and the fact that $- \otimes_A^{\mathbf{L}} A_n$ commutes with homotopy colimits we obtain the result. ☐

0DIG **Theorem 29.6** (Derived Grothendieck Existence Theorem)**.** *In Situation 29.1 there exists a pseudo-coherent K in $D(\mathcal{O}_X)$ such that $K_n = K \otimes_{\mathcal{O}_X}^{\mathbf{L}} \mathcal{O}_{X_n}$ for all n compatibly with the maps φ_n.*

Proof. Apply Lemmas 29.2, 29.3, 29.4 to get a pseudo-coherent object K of $D(\mathcal{O}_X)$. Choosing affine opens in Lemma 29.5 it follows immediately that K restricts to K_n over X_n. ☐

0DIH **Remark 29.7.** The result in this section can be generalized. It is probably correct if we only assume $X \to \text{Spec}(A)$ to be separated, of finite presentation, and K_n pseudo-coherent relative to A_n supported on a closed subset of X_n proper over A_n. The outcome will be a K which is pseudo-coherent relative to A supported on a closed subset proper over A.. If we ever need this, we will formulate a precise statement and prove it here.

30. Blowing up and flatness

080X In this section we begin our discussion of results of the form: "After a blowup the strict transform becomes flat". We will use the following (more or less standard) notation in this section. If $X \to S$ is a morphism of schemes, $\mathcal{F}$ is a quasi-coherent module on X, and $T \to S$ is a morphism of schemes, then we denote $\mathcal{F}_T$ the pullback of $\mathcal{F}$ to the base change $X_T = X \times_S T$.

080Y **Remark 30.1.** Let S be a quasi-compact and quasi-separated scheme. Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent module on X. Let $U \subset S$ be a quasi-compact open subscheme. Given a U-admissible blowup $S' \to S$ we denote X' the strict transform of X and $\mathcal{F}'$ the strict transform of $\mathcal{F}$ which we think of as a quasi-coherent module on X' (via Divisors, Lemma 30.2). Let P be a property of $\mathcal{F}/X/S$ which is stable under strict transform (as above) for U-admissible blowups. The general problem in this section is: Show (under auxiliary conditions on $\mathcal{F}/X/S$) there exists a U-admissible blowup $S' \to S$ such that the strict transform $\mathcal{F}'/X'/S'$ has P.

The general strategy will be to use that a composition of U-admissible blowups is a U-admissible blowup, see Divisors, Lemma 31.2. In fact, we will make use of the more precise Divisors, Lemma 29.14 and combine it with Divisors, Lemma 30.6. The result is that it suffices to find a sequence of U-admissible blowups

$$S = S_0 \leftarrow S_1 \leftarrow \ldots \leftarrow S_n$$

such that, setting $\mathcal{F}_0 = \mathcal{F}$ and $X_0 = X$ and setting $\mathcal{F}_i/X_i$ equal to the strict transform of $\mathcal{F}_{i-1}/X_{i-1}$, we arrive at $\mathcal{F}_n/X_n/S_n$ with property P.

In particular, choose a finite type quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_S$ such that $V(\mathcal{I}) = S \setminus U$, see Properties, Lemma 24.1. Let $S' \to S$ be the blowup in $\mathcal{I}$ and let $E \subset S'$ be the exceptional divisor (Divisors, Lemma 29.4). Then we see that we've reduced the problem to the case where there exists an effective Cartier divisor $D \subset S$ whose support is $X \setminus U$. In particular we may assume U is scheme theoretically dense in S (Divisors, Lemma 13.4).

Suppose that P is local on S: If $S = \bigcup S_i$ is a finite open covering by quasi-compact opens and P holds for $\mathcal{F}_{S_i}/X_{S_i}/S_i$ then P holds for $\mathcal{F}/X/S$. In this case the general problem above is local on S as well, i.e., if given $s \in S$ we can find a quasi-compact open neighbourhood W of s such that the problem for $\mathcal{F}_W/X_W/W$ is solvable, then the problem is solvable for $\mathcal{F}/X/S$. This follows from Divisors, Lemmas 31.3 and 31.4.

0810 **Lemma 30.2.** *Let R be a ring and let $f \in R$. Let $r \geq 0$ be an integer. Let $R \to S$ be a ring map and let M be an S-module. Assume*

(1) *$R \to S$ is of finite presentation and flat,*
(2) *every fibre ring $S \otimes_R \kappa(\mathfrak{p})$ is geometrically integral over R,*
(3) *M is a finite S-module,*
(4) *M_f is a finitely presented S_f-module,*
(5) *for all $\mathfrak{p} \in R$, $f \notin \mathfrak{p}$ with $\mathfrak{q} = \mathfrak{p}S$ the module $M_{\mathfrak{q}}$ is free of rank r over $S_{\mathfrak{q}}$.*

Then there exists a finitely generated ideal $I \subset R$ with $V(f) = V(I)$ such that for all $a \in I$ with $R' = R[\frac{I}{a}]$ the quotient

$$M' = (M \otimes_R R')/a\text{-power torsion}$$

over $S' = S \otimes_R R'$ satisfies the following: for every prime $\mathfrak{p}' \subset R'$ there exists a $g \in S'$, $g \notin \mathfrak{p}'S'$ such that M'_g is a free S'_g-module of rank r.

Proof. This lemma is a generalization of More on Algebra, Lemma 24.5; we urge the reader to read that proof first. Choose a surjection $S^{\oplus n} \to M$, which is possible by (1). Choose a finite submodule $K \subset \text{Ker}(S^{\oplus n} \to M)$ such that $S^{\oplus n}/K \to M$ becomes an isomorphism after inverting f. This is possible by (4). Set $M_1 = S^{\oplus n}/K$ and suppose we can prove the lemma for M_1. Say $I \subset R$ is the corresponding ideal. Then for $a \in I$ the map

$$M'_1 = (M_1 \otimes_R R')/a\text{-power torsion} \longrightarrow M' = (M \otimes_R R')/a\text{-power torsion}$$

is surjective. It is also an isomorphism after inverting a in R' as $R'_a = R_f$, see Algebra, Lemma 69.4. But a is a nonzerodivisor on M'_1, whence the displayed map is an isomorphism. Thus it suffices to prove the lemma in case M is a finitely presented S-module.

Assume M is a finitely presented S-module satisfying (3). Then $J = \text{Fit}_r(M) \subset S$ is a finitely generated ideal. By Lemma 9.3 we can write S as a direct summand of a free R-module: $\bigoplus_{\alpha \in A} R = S \oplus C$. For any element $h \in S$ writing $h = \sum a_\alpha$ in the decomposition above, we say that the a_α are the coefficients of h. Let $I' \subset R$ be the ideal of coefficients of elements of J. Multiplication by an element of S defines an R-linear map $S \to S$, hence I' is generated by the coefficients of the generators of J, i.e., I' is a finitely generated ideal. We claim that $I = fI'$ works.

We first check that $V(f) = V(I)$. The inclusion $V(f) \subset V(I)$ is clear. Conversely, if $f \notin \mathfrak{p}$, then $\mathfrak{q} = \mathfrak{p}S$ is not an element of $V(J)$ by property (5) and More on Algebra, Lemma 8.6. Hence there is an element of J which does not map to zero in $S \otimes_R \kappa(\mathfrak{p})$. Thus there exists an element of I' which is not contained in $\mathfrak{p}$, so $\mathfrak{p} \notin V(fI') = V(I)$.

Let $a \in I$ and set $R' = R[\frac{I}{a}]$. We may write $a = fa'$ for some $a' \in I'$. By Algebra, Lemmas 69.2 and 69.5 we see that $I'R' = a'R'$ and a' is a nonzerodivisor in R'. Set $S' = S \otimes_S R'$. Every element g of $JS' = \text{Fit}_r(M \otimes_S S')$ can be written as $g = \sum_\alpha c_\alpha$ for some $c_\alpha \in I'R'$. Since $I'R' = a'R'$ we can write $c_\alpha = a'c'_\alpha$ for some $c'_\alpha \in R'$ and $g = (\sum c'_\alpha)a' = g'a'$ in S'. Moreover, there is an $g_0 \in J$ such that $a' = c_\alpha$ for some α. For this element we have $g_0 = g'_0 a'$ in S' where g'_0 is a unit in S'. Let $\mathfrak p' \subset R'$ be a prime ideal and $\mathfrak q' = \mathfrak p' S'$. By the above we see that $JS'_{\mathfrak q'}$ is the principal ideal generated by the nonzerodivisor a'. It follows from More on Algebra, Lemma 8.8 that $M'_{\mathfrak q'}$ can be generated by r elements. Since M' is finite, there exist $m_1, \ldots, m_r \in M'$ and $g \in S'$, $g \not\in \mathfrak q'$ such that the corresponding map $(S')^{\oplus r} \to M'$ becomes surjective after inverting g.

Finally, consider the ideal $J' = \text{Fit}_{k-1}(M')$. Note that $J'S'_g$ is generated by the coefficients of relations between $m_1, \ldots, m_r$ (compatibility of Fitting ideal with base change). Thus it suffices to show that $J' = 0$, see More on Algebra, Lemma 8.7. Since $R'_a = R_f$ (Algebra, Lemma 69.4) and $M'_a = M_f$ we see from (5) that J'_a maps to zero in $S_{\mathfrak q''}$ for any prime $\mathfrak q'' \subset S'$ of the form $\mathfrak q'' = \mathfrak p'' S'$ where $\mathfrak p'' \subset R'_a$. Since $S'_a \subset \prod_{\mathfrak q'' \text{ as above}} S'_{\mathfrak q''}$ (as $(S'_a)_{\mathfrak p''} \subset S'_{\mathfrak q''}$ by Lemma 7.4) we see that $J'R'_a = 0$. Since a is a nonzerodivisor in R' we conclude that $J' = 0$ and we win. $\square$

0811 **Lemma 30.3.** *Let S be a quasi-compact and quasi-separated scheme. Let $X \to S$ be a morphism of schemes. Let $\mathcal F$ be a quasi-coherent module on X. Let $U \subset S$ be a quasi-compact open. Assume*

1. *$X \to S$ is affine, of finite presentation, flat, geometrically integral fibres,*
2. *$\mathcal F$ is a module of finite type,*
3. *$\mathcal F_U$ is of finite presentation,*
4. *$\mathcal F$ is flat over S at all generic points of fibres lying over points of U.*

Then there exists a U-admissible blowup $S' \to S$ and an open subscheme $V \subset X_{S'}$ such that (a) the strict transform $\mathcal F'$ of $\mathcal F$ restricts to a finitely locally free $\mathcal O_V$-module and (b) $V \to S'$ is surjective.

Proof. Given $\mathcal F/X/S$ and $U \subset S$ with hypotheses as in the lemma, denote P the property "$\mathcal F$ is flat over S at all generic points of fibres". It is clear that P is preserved under strict transform, see Divisors, Lemma 30.3 and Morphisms, Lemma 24.6. It is also clear that P is local on S. Hence any and all observations of Remark 30.1 apply to the problem posed by the lemma.

Consider the function $r : U \to \mathbf Z_{\geq 0}$ which assigns to $u \in U$ the integer

$$r(u) = \dim_{\kappa(\xi_u)}(\mathcal F_{\xi_u} \otimes \kappa(\xi_u))$$

where ξ_u is the generic point of the fibre X_u. By More on Morphisms, Lemma 16.7 and the fact that the image of an open in X_S in S is open, we see that $r(u)$ is locally constant. Accordingly $U = U_0 \amalg U_1 \amalg \ldots \amalg U_c$ is a finite disjoint union of open and closed subschemes where r is constant with value i on U_i. By Divisors, Lemma 31.5 we can find a U-admissible blowup to decompose S into the disjoint union of two schemes, the first containing U_0 and the second $U_1 \cup \ldots \cup U_c$. Repeating this $c-1$ more times we may assume that S is a disjoint union $S = S_0 \amalg S_1 \amalg \ldots \amalg S_c$ with $U_i \subset S_i$. Thus we may assume the function r defined above is constant, say with value r.

By Remark 30.1 we see that we may assume that we have an effective Cartier divisor $D \subset S$ whose support is $S \setminus U$. Another application of Remark 30.1 combined with Divisors, Lemma 13.2 tells us we may assume that $S = \text{Spec}(R)$ and $D = \text{Spec}(R/(f))$ for some nonzerodivisor $f \in R$. This case is handled by Lemma 30.2. $\square$

0812 **Lemma 30.4.** *Let $A \to C$ be a finite locally free ring map of rank d. Let $h \in C$ be an element such that C_h is étale over A. Let $J \subset C$ be an ideal. Set $I = Fit_0(C/J)$*

where we think of C/J as a finite A-module. Then $IC_h = JJ'$ for some ideal $J' \subset C_h$. If J is finitely generated so are I and J'.

Proof. We will use basic properties of Fitting ideals, see More on Algebra, Lemma 8.4. Then IC is the Fitting ideal of $C/J \otimes_A C$. Note that $C \to C \otimes_A C$, $c \mapsto 1 \otimes c$ has a section (the multiplication map). By assumption $C \to C \otimes_A C$ is étale at every prime in the image of $\mathrm{Spec}(C_h)$ under this section. Hence the multiplication map $C \otimes_A C_h \to C_h$ is étale in particular flat, see Algebra, Lemma 141.8. Hence there exists a C_h-algebra such that $C \otimes_A C_h \cong C_h \oplus C'$ as C_h-algebras, see Algebra, Lemma 141.9. Thus $(C/J) \otimes_A C_h \cong (C_h/J_h) \oplus C'/I'$ as C_h-modules for some ideal $I' \subset C'$. Hence $IC_h = JJ'$ with $J' = \mathrm{Fit}_0(C'/I')$ where we view C'/J' as a C_h-module. $\square$

0813 **Lemma 30.5.** *Let $A \to B$ be an étale ring map. Let $a \in A$ be a nonzerodivisor. Let $J \subset B$ be a finite type ideal with $V(J) \subset V(aB)$. For every $\mathfrak q \subset B$ there exists a finite type ideal $I \subset A$ with $V(I) \subset V(a)$ and $g \in B$, $g \notin \mathfrak q$ such that $IB_g = JJ'$ for some finite type ideal $J' \subset B_g$.*

Proof. We may replace B by a principal localization at an element $g \in B$, $g \notin \mathfrak q$. Thus we may assume that B is standard étale, see Algebra, Proposition 141.16. Thus we may assume B is a localization of $C = A[x]/(f)$ for some monic $f \in A[x]$ of some degree d. Say $B = C_h$ for some $h \in C$. Choose elements $h_1, \ldots, h_n \in C$ which generate J over B. The condition $V(J) \subset V(aB)$ signifies that $a^m = \sum b_i h_i$ in B for some large m. Set $h_{n+1} = a^m$. As in Lemma 30.4 we take $I = \mathrm{Fit}_0(C/(h_1, \ldots, h_{r+1}))$. Since the module $C/(h_1, \ldots, h_{r+1})$ is annihilated by a^m we see that $a^{dm} \in I$ which implies that $V(I) \subset V(a)$. $\square$

0814 **Lemma 30.6.** *Let S be a quasi-compact and quasi-separated scheme. Let $X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent module on X. Let $U \subset S$ be a quasi-compact open. Assume there exist finitely many commutative diagrams*

$$\begin{tikzcd} & X_i \arrow[r, "j_i"] \arrow[d] & X \arrow[d] \\ S_i^* \arrow[r] & S_i \arrow[r, "e_i"] & S \end{tikzcd}$$

where

1. *$e_i : S_i \to S$ are quasi-compact étale morphisms and $S = \bigcup e_i(S_i)$,*
2. *$j_i : X_i \to X$ are étale morphisms and $X = \bigcup j_i(X_i)$,*
3. *$S_i^* \to S_i$ is an $e_i^{-1}(U)$-admissible blowup such that the strict transform $\mathcal{F}_i^*$ of $j_i^*\mathcal{F}$ is flat over S_i^*.*

Then there exists a U-admissible blowup $S' \to S$ such that the strict transform of $\mathcal{F}$ is flat over S'.

Proof. We claim that the hypotheses of the lemma are preserved under U-admissible blowups. Namely, suppose $b : S' \to S$ is a U-admissible blowup in the quasi-coherent sheaf of ideals $\mathcal{I}$. Moreover, let $S'_i \to S_i$ be the blowup in the quasi-coherent sheaf of ideals $\mathcal{J}_i$. Then the collection of morphisms $e'_i : S'_i = S_i \times_S S' \to S'$ and $j'_i : X'_i = X_i \times_S S' \to X \times_S S'$ satisfy conditions (1), (2), (3) for the strict transform $\mathcal{F}'$ of $\mathcal{F}$ relative to the blowup $S' \to S$. First, observe that S'_i is the blowup of S_i in the pullback of $\mathcal{I}$, see Divisors, Lemma 29.3. Second, consider the blowup $S_i'^* \to S_i'$ of S_i' in the pullback of the ideal $\mathcal{J}_i$. By Divisors, Lemma 29.12 we get a commutative diagram

$$\begin{tikzcd} S_i'^* \arrow[r] \arrow[d] \arrow[dr] & S_i' \arrow[d] \\ S_i^* \arrow[r] & S_i \end{tikzcd}$$

and all the morphisms in the diagram above are blowups. Hence by Divisors, Lemmas 30.3 and 30.6 we see

$$\begin{aligned} & \text{the strict transform of } (j_i')^*\mathcal{F}' \text{ under } S_i'^* \to S_i' \\ = & \text{ the strict transform of } j_i^*\mathcal{F} \text{ under } S_i'^* \to S_i \\ = & \text{ the strict transform of } \mathcal{F}_i' \text{ under } S_i'^* \to S_i' \\ = & \text{ the pullback of } \mathcal{F}_i^* \text{ via } X_i \times_{S_i} S_i'^* \to X_i \end{aligned}$$

which is therefore flat over $S_i'^*$ (Morphisms, Lemma 24.6). Having said this, we see that all observations of Remark 30.1 apply to the problem of finding a U-admissible blowup such that the strict transform of $\mathcal{F}$ becomes flat over the base under assumptions as in the lemma. In particular, we may assume that $S \setminus U$ is the support of an effective Cartier divisor $D \subset S$. Another application of Remark 30.1 combined with Divisors, Lemma 13.2 shows we may assume that $S = \mathrm{Spec}(A)$ and $D = \mathrm{Spec}(A/(a))$ for some nonzerodivisor $a \in A$.

Pick an i and $s \in S_i$. Lemma 30.5 implies we can find an open neighbourhood $s \in W_i \subset S_i$ and a finite type quasi-coherent ideal $\mathcal{I} \subset \mathcal{O}_S$ such that $\mathcal{I} \cdot \mathcal{O}_{W_i} = \mathcal{J}_i\mathcal{J}_i'$ for some finite type quasi-coherent ideal $\mathcal{J}_i' \subset \mathcal{O}_{W_i}$ and such that $V(\mathcal{I}) \subset V(a) = S \setminus U$. Since S_i is quasi-compact we can replace S_i by a finite collection $W_1, \ldots, W_n$ of these opens and assume that for each i there exists a quasi-coherent sheaf of ideals $\mathcal{I}_i \subset \mathcal{O}_S$ such that $\mathcal{I}_i \cdot \mathcal{O}_{S_i} = \mathcal{J}_i\mathcal{J}_i'$ for some finite type quasi-coherent ideal $\mathcal{J}_i' \subset \mathcal{O}_{S_i}$. As in the discussion of the first paragraph of the proof, consider the blowup S' of S in the product $\mathcal{I}_1 \ldots \mathcal{I}_n$ (this blowup is U-admissible by construction). The base change of $S' \to S$ to S_i is the blowup in

$$\mathcal{J}_i \cdot \mathcal{J}_i'\mathcal{I}_1 \ldots \hat{\mathcal{I}}_i \ldots \mathcal{I}_n$$

which factors through the given blowup $S_i^* \to S_i$ (Divisors, Lemma 29.12). In the notation of the diagram above this means that $S_i'^* = S_i'$. Hence after replacing S by S' we arrive in the situation that $j_i^*\mathcal{F}$ is flat over S_i. Hence $j_i^*\mathcal{F}$ is flat over S, see Lemma 2.3. By Morphisms, Lemma 24.12 we see that $\mathcal{F}$ is flat over S. $\square$

0815 **Theorem 30.7.** *Let S be a quasi-compact and quasi-separated scheme. Let X be a scheme over S. Let $\mathcal{F}$ be a quasi-coherent module on X. Let $U \subset S$ be a quasi-compact open. Assume*

1. *X is quasi-compact,*
2. *X is locally of finite presentation over S,*
3. *$\mathcal{F}$ is a module of finite type,*
4. *$\mathcal{F}_U$ is of finite presentation, and*
5. *$\mathcal{F}_U$ is flat over U.*

Then there exists a U-admissible blowup $S' \to S$ such that the strict transform $\mathcal{F}'$ of $\mathcal{F}$ is an $\mathcal{O}_{X \times_S S'}$-module of finite presentation and flat over S'.

Proof. We first prove that we can find a U-admissible blowup such that the strict transform is flat. The question is étale local on the source and the target, see Lemma 30.6 for a precise statement. In particular, we may assume that $S = \mathrm{Spec}(R)$ and $X = \mathrm{Spec}(A)$ are affine. For $s \in S$ write $\mathcal{F}_s = \mathcal{F}|_{X_s}$ (pullback of $\mathcal{F}$ to the fibre). As $X \to S$ is of finite type $d = \max_{s \in S} \dim(\mathrm{Supp}(\mathcal{F}_s))$ is an integer. We will do induction on d.

Let $x \in X$ be a point of X lying over $s \in S$ with $\dim_x(\mathrm{Supp}(\mathcal{F}_s)) = d$. Apply Lemma 3.2 to get $g : X' \to X$, $e : S' \to S$, $i : Z' \to X'$, and $\pi : Z' \to Y'$. Observe that $Y' \to S'$ is a smooth morphism of affines with geometrically irreducible fibres of dimension d. Because the problem is étale local it suffices to prove the theorem for $g^*\mathcal{F}/X'/S'$. Because $i : Z' \to X'$ is a closed immersion of finite presentation (and since strict transform commutes with affine pushforward, see Divisors, Lemma 30.4) it suffices to prove the flattening result for $\mathcal{G}$. Since π is finite (hence also affine) it suffices to prove the flattening result for $\pi_*\mathcal{G}/Y'/S'$. Thus we may assume that $X \to S$ is a smooth morphism of affines with geometrically irreducible fibres of dimension d.

Next, we apply a blow up as in Lemma 30.3. Doing so we reach the situation where there exists an open $V \subset X$ surjecting onto S such that $\mathcal{F}|_V$ is finite locally free. Let $\xi \in X$ be the generic point of X_s. Let $r = \dim_{\kappa(\xi)} \mathcal{F}_\xi \otimes \kappa(\xi)$. Choose a map $\alpha : \mathcal{O}_X^{\oplus r} \to \mathcal{F}$ which induces an isomorphism $\kappa(\xi)^{\oplus r} \to \mathcal{F}_\xi \otimes \kappa(\xi)$. Because $\mathcal{F}$ is locally free over V we find an open neighbourhood W of ξ where α is an isomorphism. Shrink S to an affine open neighbourhood of s such that $W \to S$ is surjective. Say $\mathcal{F}$ is the quasi-coherent module associated to the A-module N. Since $\mathcal{F}$ is flat over S at all generic points of fibres (in fact at all points of W), we see that

$$\alpha_\mathfrak{p} : A_\mathfrak{p}^{\oplus r} \to N_\mathfrak{p}$$

is universally injective for all primes $\mathfrak{p}$ of R, see Lemma 10.1. Hence α is universally injective, see Algebra, Lemma 81.12. Set $\mathcal{H} = \mathrm{Coker}(\alpha)$. By Divisors, Lemma 30.7 we see that, given a U-admissible blowup $S' \to S$ the strict transforms of $\mathcal{F}'$ and $\mathcal{H}'$ fit into an exact sequence

$$0 \to \mathcal{O}_{X \times_S S'}^{\oplus r} \to \mathcal{F}' \to \mathcal{H}' \to 0$$

Hence Lemma 10.1 also shows that $\mathcal{F}'$ is flat at a point x' if and only if $\mathcal{H}'$ is flat at that point. In particular $\mathcal{H}_U$ is flat over U and $\mathcal{H}_U$ is a module of finite presentation. We may apply the induction hypothesis to $\mathcal{H}$ to see that there exists a U-admissible blowup such that the strict transform $\mathcal{H}'$ is flat as desired.

To finish the proof of the theorem we still have to show that $\mathcal{F}'$ is a module of finite presentation (after possibly another U-admissible blowup). This follows from Lemma 11.1 as we can assume $U \subset S$ is scheme theoretically dense (see third paragraph of Remark 30.1). This finishes the proof of the theorem. □

31. Applications

081Q In this section we apply some of the results above.

081R **Lemma 31.1.** *Let S be a quasi-compact and quasi-separated scheme. Let X be a scheme over S. Let $U \subset S$ be a quasi-compact open. Assume*

1. *$X \to S$ is of finite type and quasi-separated, and*

(2) $X_U \to U$ *is flat and locally of finite presentation.*

Then there exists a U-admissible blowup $S' \to S$ such that the strict transform of X is flat and of finite presentation over S'.

Proof. Since $X \to S$ is quasi-compact and quasi-separated by assumption, the strict transform of X with respect to a blowing up $S' \to S$ is also quasi-compact and quasi-separated. Hence to prove the lemma it suffices to find a U-admissible blowup such that the strict transform is flat and locally of finite presentation. Let $X = W_1 \cup \ldots \cup W_n$ be a finite affine open covering. If we can find a U-admissible blowup $S_i \to S$ such that the strict transform of W_i is flat and locally of finite presentation, then there exists a U-admissible blowing up $S' \to S$ dominating all $S_i \to S$ which does the job (see Divisors, Lemma 31.4; see also Remark 30.1). Hence we may assume X is affine.

Assume X is affine. By Morphisms, Lemma 37.2 we can choose an immersion $j : X \to \mathbf{A}^n_S$ over S. Let $V \subset \mathbf{A}^n_S$ be a quasi-compact open subscheme such that j induces a closed immersion $i : X \to V$ over S. Apply Theorem 30.7 to $V \to S$ and the quasi-coherent module $i_*\mathcal{O}_X$ to obtain a U-admissible blowup $S' \to S$ such that the strict transform of $i_*\mathcal{O}_X$ is flat over S' and of finite presentation over $\mathcal{O}_{V \times_S S'}$. Let X' be the strict transform of X with respect to $S' \to S$. Let $i' : X' \to V \times_S S'$ be the induced morphism. Since taking strict transform commutes with pushforward along affine morphisms (Divisors, Lemma 30.4), we see that $i'_*\mathcal{O}_{X'}$ is flat over S and of finite presentation as a $\mathcal{O}_{V \times_S S'}$-module. This implies the lemma. $\square$

0B49 **Lemma 31.2.** *Let S be a quasi-compact and quasi-separated scheme. Let X be a scheme over S. Let $U \subset S$ be a quasi-compact open. Assume*

(1) $X \to S$ *is proper, and*
(2) $X_U \to U$ *is finite locally free.*

Then there exists a U-admissible blowup $S' \to S$ such that the strict transform of X is finite locally free over S'.

Proof. By Lemma 31.1 we may assume that $X \to S$ is flat and of finite presentation. After replacing S by a U-admissible blow up if necessary, we may assume that $U \subset S$ is scheme theoretically dense. Then f is finite by Lemma 11.4. Hence f is finite locally free by Morphisms, Lemma 45.2. $\square$

081S **Lemma 31.3.** *Let $\varphi : X \to S$ be a separated morphism of finite type with S quasi-compact and quasi-separated. Let $U \subset S$ be a quasi-compact open such that $\varphi^{-1}U \to U$ is an isomorphism. Then there exists a U-admissible blowup $S' \to S$ such that the strict transform X' of X is isomorphic to an open subscheme of S'.*

Proof. The discussion in Remark 30.1 applies. Thus we may do a first U-admissible blowup and assume the complement $S \setminus U$ is the support of an effective Cartier divisor D. In particular U is scheme theoretically dense in S. Next, we do another U-admissible blowup to get to the situation where $X \to S$ is flat and of finite presentation, see Lemma 31.1. In this case the result follows from Lemma 11.5. $\square$

The following lemma says that a proper modification can be dominated by a blowup.

081T **Lemma 31.4.** *Let $\varphi : X \to S$ be a proper morphism with S quasi-compact and quasi-separated. Let $U \subset S$ be a quasi-compact open such that $\varphi^{-1}U \to U$ is an isomorphism. Then there exists a U-admissible blowup $S' \to S$ which dominates X, i.e., such that there exists a factorization $S' \to X \to S$ of the blowup morphism.*

Proof. The discussion in Remark 30.1 applies. Thus we may do a first U-admissible blowup and assume the complement $S \setminus U$ is the support of an effective Cartier divisor D. In particular U is scheme theoretically dense in S. Choose another U-admissible blowup $S' \to S$ such that the strict transform X' of X is an open subscheme of S', see Lemma 31.3. Since $X' \to S'$ is proper, and $U \subset S'$ is dense, we see that $X' = S'$. Some details omitted. □

0CP1 **Lemma 31.5.** *Let S be a scheme. Let $U \subset W \subset S$ be open subschemes. Let $f : X \to W$ be a morphism and let $s : U \to X$ be a morphism such that $f \circ s = id_U$. Assume*

1. *f is proper,*
2. *S is quasi-compact and quasi-separated, and*
3. *U and W are quasi-compact.*

Then there exists a U-admissible blowup $b : S' \to S$ and a morphism $s' : b^{-1}(W) \to X$ extending s with $f \circ s' = b|_{b^{-1}(W)}$.

Proof. We may and do replace X by the scheme theoretic image of s. Then $X \to W$ is an isomorphism over U, see Morphisms, Lemma 6.8. By Lemma 31.4 there exists a U-admissible blowup $W' \to W$ and an extension $W' \to X$ of s. We finish the proof by applying Divisors, Lemma 31.3 to extend $W' \to W$ to a U-admissible blowup of S. □

32. Other chapters

References

[GR71] Laurent Gruson and Michel Raynaud, *Critères de platitude et de projectivité*, Invent. math. **13** (1971), 1–89.