MODULI OF CURVES

0DMG

Contents

1. Introduction

0DMH In this chapter we discuss some of the familiar moduli stacks of curves. A reference is the celebrated article of Deligne and Mumford, see [DM69].

2. Conventions and abuse of language

0DMI We continue to use the conventions and the abuse of language introduced in Properties of Stacks, Section 2. Unless otherwise mentioned our base scheme will be $\mathrm{Spec}(\mathbf{Z})$.

3. The stack of curves

0DMJ This section is the continuation of Quot, Section 15. Let $\mathcal{C}urves$ be the stack whose category of sections over a scheme S is the category of families of curves over S. We already know that $\mathcal{C}urves$ is an algebraic stack over $\mathbf{Z}$, see Quot, Theorem 15.11.

Often base change is denoted by a subscript, but we cannot use this notation for $\mathcal{C}urves$ because $\mathcal{C}urves_S$ is our notation for the fibre category over S. This is why in Quot, Remark 15.5 we used $B\text{-}\mathcal{C}urves$ for the base change

$$B\text{-}\mathcal{C}urves = \mathcal{C}urves \times B$$

to the algebraic space B. The product on the right is over the final object, i.e., over $\mathrm{Spec}(\mathbf{Z})$. The object on the left is the stack classifying families of curves on the category of schemes over B. In particular, if k is a field, then

$$k\text{-}\mathcal{C}urves = \mathcal{C}urves \times \mathrm{Spec}(k)$$

is the moduli stack classifying families of curves on the category of schemes over k. Before we continue, here is a sanity check.

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0DMK **Lemma 3.1.** *Let $T \to B$ be a morphism of algebraic spaces. The category*

$$\mathrm{Mor}_B(T, B\text{-}\mathcal{C}urves) = \mathrm{Mor}(T, \mathcal{C}urves)$$

is the category of families of curves over T.

Proof. A family of curves over T is a morphism $f : X \to T$ of algebraic spaces, which is flat, proper, of finite presentation, and has relative dimension ≤ 1 (Morphisms of Spaces, Definition 32.2). This is exactly the same as the definition in Quot, Situation 15.1 except that T the base is allowed to be an algebraic space. Our default base category for algebraic stacks/spaces is the category of schemes, hence the lemma does not follow immediately from the definitions. Having said this, we encourage the reader to skip the proof.

By the product description of B-$\mathcal{C}urves$ given above, it suffices to prove the lemma in the absolute case. Choose a scheme U and a surjective étale morphism $p : U \to T$. Let $R = U \times_T U$ with projections $s, t : R \to U$.

Let $v : T \to \mathcal{C}urves$ be a morphism. Then $v \circ p$ corresponds to a family of curves $X_U \to U$. The canonical 2-morphism $v \circ p \circ t \to v \circ p \circ s$ is an isomorphism $\varphi : X_U \times_{U,s} R \to X_U \times_{U,t} R$. This isomorphism satisfies the cocycle condition on $R \times_{s,t} R$. By Bootstrap, Lemma 11.2 we obtain a morphism of algebraic spaces $X \to T$ whose pullback to U is equal to X_U compatible with φ. Since $\{U \to T\}$ is an étale covering, we see that $X \to T$ is flat, proper, of finite presentation by Descent on Spaces, Lemmas 10.13, 10.19, and 10.12. Also $X \to T$ has relative dimension ≤ 1 because this is an étale local property. Hence $X \to T$ is a family of curves over T.

Conversely, let $X \to T$ be a family of curves. Then the base change X_U determines a morphism $w : U \to \mathcal{C}urves$ and the canonical isomorphism $X_U \times_{U,s} R \to X_U \times_{U,t} R$ determines a 2-arrow $w \circ s \to w \circ t$ satisfying the cocycle condition. Thus a morphism $v : T = [U/R] \to \mathcal{C}urves$ by the universal property of the quotient $[U/R]$, see Groupoids in Spaces, Lemma 22.2. (Actually, it is much easier in this case to go back to before we introduced our abuse of language and direct construct the functor $Sch/T \to \mathcal{C}urves$ which "is" the morphsim $T \to \mathcal{C}urves$.)

We omit the verification that the constructions given above extend to morphisms between objects and are mutually quasi-inverse. $\square$

4. The stack of polarized curves

0DPY In this section we work out some of the material discussed in Quot, Remark 15.13. Consider the 2-fibre product

$$\begin{tikzcd}
\mathcal{C}urves \times_{\mathcal{S}paces'_{fp,flat,proper}} \mathcal{P}olarized \arrow[r] \arrow[d] & \mathcal{P}olarized \arrow[d] \\
\mathcal{C}urves \arrow[r] & \mathcal{S}paces'_{fp,flat,proper}
\end{tikzcd}$$

We denote this 2-fibre product by

$$\mathcal{P}olarized\mathcal{C}urves = \mathcal{C}urves \times_{\mathcal{S}paces'_{fp,flat,proper}} \mathcal{P}olarized$$

This fibre product parametrizes polarized curves, i.e., families of curves endowed with a relatively ample invertible sheaf. More precisely, an object of *PolarizedCurves* is a pair $(X \to S, \mathcal{L})$ where

(1) $X \to S$ is a morphism of schemes which is proper, flat, of finite presentation, and has relative dimension ≤ 1, and
(2) $\mathcal{L}$ is an invertible $\mathcal{O}_X$-module which is relatively ample on X/S.

A morphism $(X' \to S', \mathcal{L}') \to (X \to S, \mathcal{L})$ between objects of *PolarizedCurves* is given by a triple (f, g, φ) where $f : X' \to X$ and $g : S' \to S$ are morphisms of schemes which fit into a commutative diagram

$$\begin{tikzcd} X' \arrow[r, "f"'] \arrow[d] & X \arrow[d] \\ S' \arrow[r, "g"] & S \end{tikzcd}$$

inducing an isomorphism $X' \to S' \times_S X$, in other words, the diagram is cartesian, and $\varphi : f^*\mathcal{L} \to \mathcal{L}'$ is an isomorphism. Composition is defined in the obvious manner.

0DPZ **Lemma 4.1.** *The morphism PolarizedCurves $\to \mathcal{P}olarized$ is an open and closed immersion.*

Proof. This is true because the 1-morphism $\mathcal{C}urves \to \mathcal{S}paces'_{fp,flat,proper}$ is representable by open and closed immersions, see Quot, Lemma 15.12. $\square$

0DQ0 **Lemma 4.2.** *The morphism PolarizedCurves $\to \mathcal{C}urves$ is smooth and surjective.*

Proof. Surjective. Given a field k and a proper algebraic space X over k of dimension ≤ 1, i.e., an object of $\mathcal{C}urves$ over k. By Spaces over Fields, Lemma 6.3 the algebraic space X is a scheme. Hence X is a proper scheme of dimension ≤ 1 over k. By Varieties, Lemma 41.4 we see that X is H-projective over κ. In particular, there exists an ample invertible $\mathcal{O}_X$-module $\mathcal{L}$ on X. Then $(X, \mathcal{L})$ is an object of *PolarizedCurves* over k which maps to X.

Smooth. Let $X \to S$ be an object of $\mathcal{C}urves$, i.e., a morphism $S \to \mathcal{C}urves$. It is clear that

$$PolarizedCurves \times_{\mathcal{C}urves} S \subset \mathcal{P}ic_{X/S}$$

is the substack of objects $(T/S, \mathcal{L}/X_T)$ such that $\mathcal{L}$ is ample on X_T/T. This is an open substack by Descent on Spaces, Lemma 12.2. Since $\mathcal{P}ic_{X/S} \to S$ is smooth by Moduli Stacks, Lemma 8.5 we win. $\square$

5. Properties of the stack of curves

0DSP The following lemma isn't true for moduli of surfaces, see Remark 5.2.

0DSQ **Lemma 5.1.** *The diagonal of $\mathcal{C}urves$ is separated and of finite presentation.*

Proof. Recall that $\mathcal{C}urves$ is a limit preserving algebraic stack, see Quot, Lemma 15.6. By Limits of Stacks, Lemma 3.6 this implies that $\Delta : \mathcal{P}olarized \to \mathcal{P}olarized \times \mathcal{P}olarized$ is limit preserving. Hence Δ is locally of finite presentation by Limits of Stacks, Proposition 3.8.

Let us prove that Δ is separated. To see this, it suffices to show that given a scheme U and two objects $Y \to U$ and $X \to U$ of $\mathcal{C}urves$ over U, the algebraic space

$$Isom_U(Y, X)$$

is separated. This we have seen in Moduli Stacks, Lemmas 10.2 and 10.3 that the target is a separated algebraic space.

To finish the proof we show that Δ is quasi-compact. Since Δ is representable by algebraic spaces, it suffice to check the base change of Δ by a surjective smooth morphism $U \to \mathcal{C}urves \times \mathcal{C}urves$ is quasi-compact (see for example Properties of Stacks, Lemma 3.3). We choose $U = \coprod U_i$ to be a disjoint union of affine opens with a surjective smooth morphism

$$U \longrightarrow \mathit{PolarizedCurves} \times \mathit{PolarizedCurves}$$

Then $U \to \mathcal{C}urves \times \mathcal{C}urves$ will be surjective and smooth since $\mathit{PolarizedCurves} \to \mathcal{C}urves$ is surjective and smooth by Lemma 4.2. Since $\mathit{PolarizedCurves}$ is limit preserving (by Artin's Axioms, Lemma 11.2 and Quot, Lemmas 15.6, 14.7, and 13.5), we see that $\mathit{PolarizedCurves} \to \mathrm{Spec}(\mathbf{Z})$ is locally of finite presentation, hence $U_i \to \mathrm{Spec}(\mathbf{Z})$ is locally of finite presentation (Limits of Stacks, Proposition 3.8 and Morphisms of Stacks, Lemmas 26.2 and 32.5). In particular, U_i is Noetherian affine. This reduces us to the case discussed in the next paragraph.

In this paragraph, given a Noetherian affine scheme U and two objects $(Y, \mathcal{N})$ and $(X, \mathcal{L})$ of $\mathit{PolarizedCurves}$ over U, we show the algebraic space

$$\mathit{Isom}_U(Y, X)$$

is quasi-compact. Since the connected components of U are open and closed we may replace U by these. Thus we may and do assume U is connected. Let $u \in U$ be a point. Let Q, P be the Hilbert polynomials of these families, i.e.,

$$Q(n) = \chi(Y_u, \mathcal{N}_u^{\otimes n}) \quad \text{and} \quad P(n) = \chi(X_u, \mathcal{L}_u^{\otimes n})$$

see Varieties, Lemma 43.1. Since U is connected and since the functions $u \mapsto \chi(Y_u, \mathcal{N}_u^{\otimes n})$ and $u \mapsto \chi(X_u, \mathcal{L}_u^{\otimes n})$ are locally constant (see Derived Categories of Schemes, Lemma 28.2) we see that we get the same Hilbert polynomial in every point of U. Set

$$\mathcal{M} = \mathrm{pr}_1^*\mathcal{N} \otimes_{\mathcal{O}_{Y \times_U X}} \mathrm{pr}_2^*\mathcal{L}$$

on $Y \times_U X$. Given $(f, \varphi) \in \mathit{Isom}_U(Y, X)(T)$ for some scheme T over U then for every $t \in T$ we have

$$\begin{aligned} \chi(Y_t, (\mathrm{id} \times f)^*\mathcal{M}^{\otimes n}) &= \chi(Y_t, \mathcal{N}_t^{\otimes n} \otimes_{\mathcal{O}_{Y_t}} f_t^*\mathcal{L}_t^{\otimes n}) \\ &= n\deg(\mathcal{N}_t) + n\deg(f_t^*\mathcal{L}_t) + \chi(Y_t, \mathcal{O}_{Y_t}) \\ &= Q(n) + n\deg(\mathcal{L}_t) \\ &= Q(n) + P(n) - P(0) \end{aligned}$$

by Riemann-Roch for proper curves, more precisely by Varieties, Definition 42.1 and Lemma 42.7 and the fact that f_t is an isomorphism. Setting $P'(t) = Q(t) + P(t) - P(0)$ we find

$$\mathit{Isom}_U(Y, X) = \mathit{Isom}_U(Y, X) \cap \mathit{Mor}_U^{P', \mathcal{M}}(Y, X)$$

The intersection is an intersection of open subspaces of $\mathit{Mor}_U(Y, X)$, see Moduli Stacks, Lemma 10.3 and Remark 10.4. Now $\mathit{Mor}_U^{P', \mathcal{M}}(Y, X)$ is a Noetherian algebraic space as it is of finite presentation over U by Moduli Stacks, Lemma 10.5. Thus the intersection is a Noetherian algebraic space too and the proof is finished. $\square$

0DSR **Remark 5.2.** The boundedness argument in the proof of Lemma 5.1 does not work for moduli of surfaces and in fact, the result is wrong, for example because K3 surfaces over fields can have infinite discrete automorphism groups. The "reason" the argument does not work is that on a projective surface S over a field, given ample invertible sheaves $\mathcal{N}$ and $\mathcal{L}$ with Hilbert polynomials Q and P, there is no a priori bound on the Hilbert polynomial of $\mathcal{N} \otimes_{\mathcal{O}_S} \mathcal{L}$. In terms of intersection theory, if H_1, H_2 are ample effective Cartier divisors on S, then there is no (upper) bound on the intersection number $H_1 \cdot H_2$ in terms of $H_1 \cdot H_1$ and $H_2 \cdot H_2$.

0DSS **Lemma 5.3.** *The morphism $\mathcal{C}urves \to \mathrm{Spec}(\mathbf{Z})$ is quasi-separated and locally of finite presentation.*

Proof. To check $\mathcal{C}urves \to \mathrm{Spec}(\mathbf{Z})$ is quasi-separated we have to show that its diagonal is quasi-compact and quasi-separated. This is immediate from Lemma 5.1. To prove that $\mathcal{C}urves \to \mathrm{Spec}(\mathbf{Z})$ is locally of finite presentation, it suffices to show that $\mathcal{C}urves$ is limit preserving, see Limits of Stacks, Proposition 3.8. This is Quot, Lemma 15.6. $\square$

6. Finite, reduced automorphism groups

0DST Let X be a proper scheme over a field k of dimension ≤ 1, i.e., an object of $\mathcal{C}urves$ over k. By Lemma 5.1 the automorphism group algebraic space $Aut(X)$ is finite type and separated over k. In particular, $Aut(X)$ is a group scheme, see More on Groupoids in Spaces, Lemma 10.2. If the characteristic of k is zero, then $Aut(X)$ is reduced and even smooth over k (Groupoids, Lemma 8.2). However, in general $Aut(X)$ is not reduced, even if X is geometrically reduced.

0DSU **Example 6.1** (Non-reduced automorphism group)**.** Let k be an algebraically closed field of characteristic 2. Set $Y = Z = \mathbf{P}^1_k$. Choose three pairwise distinct k-valued points a, b, c in $\mathbf{A}^1_k$. Thinking of $\mathbf{A}^1_k \subset \mathbf{P}^1_k = Y = Z$ as an open subschemes, we get a closed immersion

$$T = \mathrm{Spec}(k[t]/(t-a)^2) \amalg \mathrm{Spec}(k[t]/(t-b)^2) \amalg \mathrm{Spec}(k[t]/(t-c)^2) \longrightarrow \mathbf{P}^1_k$$

Let X be the pushout in the diagram

$$\begin{tikzcd} T \arrow[r] \arrow[d] & Y \arrow[d] \\ Z \arrow[r] & X \end{tikzcd}$$

Let $U \subset X$ be the affine open part which is the image of $\mathbf{A}^1_k \amalg \mathbf{A}^1_k$. Then we have an equalizer diagram

$$\mathcal{O}_X(U) \longrightarrow k[t] \times k[t] \rightrightarrows k[t]/(t-a)^2 \times k[t]/(t-b)^2 \times k[t]/(t-c)^2$$

Over the dual numbers $A = k[\epsilon]$ we have a nontrivial automorphism of this equalizer diagram sending t to $t + \epsilon$. We leave it to the reader to see that this automorphism extends to an automorphism of X over A. On the other hand, the reader easily shows that the automorphism group of X over k is finite. Thus $Aut(X)$ must be non-reduced.

Let X be a proper scheme over a field k of dimension ≤ 1, i.e., an object of $\mathcal{C}urves$ over k. If $Aut(X)$ is geometrically reduced, then it need not be the case that it has dimension 0, even if X is smooth and geometrically connected.

0DSV **Example 6.2** (Smooth positive dimensional automorphism group)**.** Let k be an algebraically closed field. If X is a smooth genus 0, resp. 1 curve, then the automorphism group has dimension 3, resp. 1. Namely, in the genus 0 case we have $X \cong \mathbf{P}^1_k$ by Algebraic Curves, Proposition 8.4. Since

$$Aut(\mathbf{P}^1_k) = \text{PGL}_{2,k}$$

as functors we see that the dimension is 3. On the other hand, if the genus of X is 1, then we see that the map $X = \underline{\text{Hilb}}^1_{X/k} \to \underline{\text{Pic}}^1_{X/k}$ is an isomorphism, see Picard Schemes of Curves, Lemma 6.7 and Algebraic Curves, Theorem 2.6. Thus X has the structure of an abelian variety (since $\underline{\text{Pic}}^1_{X/k} \cong \underline{\text{Pic}}^0_{X/k}$). In particular the (co)tangent bundle of X are trivial (Groupoids, Lemma 6.3). We conclude that $\dim_k H^0(X, T_X) = 1$ hence $\dim Aut(X) \leq 1$. On the other hand, the translations (viewing X as a group scheme) provide a 1-dimensional piece of $\text{Aut}(X)$ and we conlude its dimension is indeed 1.

It turns out that there is an open substack of *Curves* parametrizing curves whose automorphism group is geometrically reduced and finite. Here is a precise statement.

0DSW **Lemma 6.3.** *There exist an open substack $Curves^{DM} \subset Curves$ with the following properties*

1. *$Curves^{DM} \subset Curves$ is the maximal open substack which is DM,*
2. *given X a proper scheme over a field k of dimension ≤ 1 the following are equivalent*
   (a) *the classifying morphism $\text{Spec}(k) \to Curves$ factors through $Curves^{DM}$,*
   (b) *$Aut(X)$ is geometrically reduced over k and has dimension 0,*
   (c) *$Aut(X) \to \text{Spec}(k)$ is unramified.*

Proof. The existence of an open substack with property (1) is Morphisms of Stacks, Lemma 22.1. The points of this open substack are characterized by (2)(c) by Morphisms of Stacks, Lemma 22.2. The equivalence of (2)(b) and (2)(c) is the statement that an algebraic space G which is locally of finite type, geometrically reduced, and of dimension 0 over a field k, is unramified over k. First, G is a scheme by Spaces over Fields, Lemma 6.1. Then we can take an affine open in G and observe that it will be proper over k and apply Varieties, Lemma 9.3. Some details omitted. $\square$

7. Nodal curves

0DSX In algebraic geometry a special role is played by nodal curves. We suggest the reader take a brief look at some of the discussion in Algebraic Curves, Sections 16 and 17 and More on Morphisms of Spaces, Section 49.

0DSY **Lemma 7.1.** *There exist an open substack $Curves^{nodal} \subset Curves$ such that for a family of curves $f : X \to S$ the following are equivalent*

1. *f is at-worst-nodal of relative dimension 1, and*
2. *the classifying morphism $S \to Curves$ factors through $Curves^{nodal}$.*

Proof. In fact, it suffices to show that given a family of curves $f : X \to S$, there is an open subscheme $S' \subset S$ such that $S' \times_S X \to S'$ is at-worst-nodal of relative dimension 1 and such that formation of S' commutes with arbitrary base change. By More on Morphisms of Spaces, Lemma 49.4 there is a maximal open subspace $X' \subset X$ such that $f|_{X'} : X' \to S$ is at-worst-nodal of relative dimension 1. Moreover, formation of X' commutes with base change. Hence we can take

$$S' = S \setminus |f|(|X| \setminus |X'|)$$

This is open because a proper morphism is universally closed by definition. □

8. Other chapters

References

[DM69] Pierre Deligne and David Mumford, *The irreducibility of the space of curves of given genus*, Publ. Math. IHES **36** (1969), 75–110.