MODULI OF CURVES

0DMG

Contents

1. Introduction

0DMH In this chapter we discuss some of the familiar moduli stacks of curves. A reference is the celebrated article of Deligne and Mumford, see [DM69].

2. Conventions and abuse of language

0DMI We continue to use the conventions and the abuse of language introduced in Properties of Stacks, Section 2. Unless otherwise mentioned our base scheme will be $\text{Spec}(\mathbf{Z})$.

3. The stack of curves

0DMJ This section is the continuation of Quot, Section 15. Let $\mathcal{C}urves$ be the stack whose category of sections over a scheme S is the category of families of curves over S. It is somewhat important to keep in mind that a *family of curves* is a morphism $f : X \to S$ where X is an algebraic space (!) and f is flat, proper, of finite presentation and of relative dimension ≤ 1. We already know that $\mathcal{C}urves$ is an

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algebraic stack over $\mathbf{Z}$, see Quot, Theorem 15.11. If we did not allow algebraic spaces in the definition of our stack, then this theorem would be false.

Often base change is denoted by a subscript, but we cannot use this notation for $\mathcal{C}urves$ because $\mathcal{C}urves_S$ is our notation for the fibre category over S. This is why in Quot, Remark 15.5 we used $B\text{-}\mathcal{C}urves$ for the base change

$$B\text{-}\mathcal{C}urves = \mathcal{C}urves \times B$$

to the algebraic space B. The product on the right is over the final object, i.e., over $\mathrm{Spec}(\mathbf{Z})$. The object on the left is the stack classifying families of curves on the category of schemes over B. In particular, if k is a field, then

$$k\text{-}\mathcal{C}urves = \mathcal{C}urves \times \mathrm{Spec}(k)$$

is the moduli stack classifying families of curves on the category of schemes over k. Before we continue, here is a sanity check.

0DMK **Lemma 3.1.** *Let $T \to B$ be a morphism of algebraic spaces. The category*

$$\mathrm{Mor}_B(T, B\text{-}\mathcal{C}urves) = \mathrm{Mor}(T, \mathcal{C}urves)$$

is the category of families of curves over T.

Proof. A family of curves over T is a morphism $f : X \to T$ of algebraic spaces, which is flat, proper, of finite presentation, and has relative dimension ≤ 1 (Morphisms of Spaces, Definition 32.2). This is exactly the same as the definition in Quot, Situation 15.1 except that T the base is allowed to be an algebraic space. Our default base category for algebraic stacks/spaces is the category of schemes, hence the lemma does not follow immediately from the definitions. Having said this, we encourage the reader to skip the proof.

By the product description of $B\text{-}\mathcal{C}urves$ given above, it suffices to prove the lemma in the absolute case. Choose a scheme U and a surjective étale morphism $p : U \to T$. Let $R = U \times_T U$ with projections $s, t : R \to U$.

Let $v : T \to \mathcal{C}urves$ be a morphism. Then $v \circ p$ corresponds to a family of curves $X_U \to U$. The canonical 2-morphism $v \circ p \circ t \to v \circ p \circ s$ is an isomorphism $\varphi : X_U \times_{U,s} R \to X_U \times_{U,t} R$. This isomorphism satisfies the cocycle condition on $R \times_{s,t} R$. By Bootstrap, Lemma 11.2 we obtain a morphism of algebraic spaces $X \to T$ whose pullback to U is equal to X_U compatible with φ. Since $\{U \to T\}$ is an étale covering, we see that $X \to T$ is flat, proper, of finite presentation by Descent on Spaces, Lemmas 10.13, 10.19, and 10.12. Also $X \to T$ has relative dimension ≤ 1 because this is an étale local property. Hence $X \to T$ is a family of curves over T.

Conversely, let $X \to T$ be a family of curves. Then the base change X_U determines a morphism $w : U \to \mathcal{C}urves$ and the canonical isomorphism $X_U \times_{U,s} R \to X_U \times_{U,t} R$ determines a 2-arrow $w \circ s \to w \circ t$ satisfying the cocycle condition. Thus a morphism $v : T = [U/R] \to \mathcal{C}urves$ by the universal property of the quotient $[U/R]$, see Groupoids in Spaces, Lemma 22.2. (Actually, it is much easier in this case to go back to before we introduced our abuse of language and direct construct the functor $Sch/T \to \mathcal{C}urves$ which "is" the morphsim $T \to \mathcal{C}urves$.)

We omit the verification that the constructions given above extend to morphisms between objects and are mutually quasi-inverse. $\square$

4. The stack of polarized curves

0DPY In this section we work out some of the material discussed in Quot, Remark 15.13. Consider the 2-fibre product

$$\begin{tikzcd} \mathit{Curves} \times_{\mathit{Spaces}'_{fp,flat,proper}} \mathit{Polarized} \arrow[r] \arrow[d] & \mathit{Polarized} \arrow[d] \\ \mathit{Curves} \arrow[r] & \mathit{Spaces}'_{fp,flat,proper} \end{tikzcd}$$

We denote this 2-fibre product by

$$\mathit{PolarizedCurves} = \mathit{Curves} \times_{\mathit{Spaces}'_{fp,flat,proper}} \mathit{Polarized}$$

This fibre product parametrizes polarized curves, i.e., families of curves endowed with a relatively ample invertible sheaf. More precisely, an object of *PolarizedCurves* is a pair $(X \to S, \mathcal{L})$ where

(1) $X \to S$ is a morphism of schemes which is proper, flat, of finite presentation, and has relative dimension ≤ 1, and
(2) $\mathcal{L}$ is an invertible $\mathcal{O}_X$-module which is relatively ample on X/S.

A morphism $(X' \to S', \mathcal{L}') \to (X \to S, \mathcal{L})$ between objects of *PolarizedCurves* is given by a triple (f, g, φ) where $f : X' \to X$ and $g : S' \to S$ are morphisms of schemes which fit into a commutative diagram

$$\begin{tikzcd} X' \arrow[r, "f"'] \arrow[d] & X \arrow[d] \\ S' \arrow[r, "g"] & S \end{tikzcd}$$

inducing an isomorphism $X' \to S' \times_S X$, in other words, the diagram is cartesian, and $\varphi : f^*\mathcal{L} \to \mathcal{L}'$ is an isomorphism. Composition is defined in the obvious manner.

0DPZ **Lemma 4.1.** *The morphism PolarizedCurves $\to$ Polarized is an open and closed immersion.*

Proof. This is true because the 1-morphism $\mathit{Curves} \to \mathit{Spaces}'_{fp,flat,proper}$ is representable by open and closed immersions, see Quot, Lemma 15.12. $\square$

0DQ0 **Lemma 4.2.** *The morphism PolarizedCurves $\to$ Curves is smooth and surjective.*

Proof. Surjective. Given a field k and a proper algebraic space X over k of dimension ≤ 1, i.e., an object of *Curves* over k. By Spaces over Fields, Lemma 6.3 the algebraic space X is a scheme. Hence X is a proper scheme of dimension ≤ 1 over k. By Varieties, Lemma 41.4 we see that X is H-projective over κ. In particular, there exists an ample invertible $\mathcal{O}_X$-module $\mathcal{L}$ on X. Then $(X, \mathcal{L})$ is an object of *PolarizedCurves* over k which maps to X.

Smooth. Let $X \to S$ be an object of *Curves*, i.e., a morphism $S \to \mathit{Curves}$. It is clear that

$$\mathit{PolarizedCurves} \times_{\mathit{Curves}} S \subset \mathcal{P}\mathit{ic}_{X/S}$$

is the substack of objects $(T/S, \mathcal{L}/X_T)$ such that $\mathcal{L}$ is ample on X_T/T. This is an open substack by Descent on Spaces, Lemma 12.2. Since $\mathcal{P}\mathit{ic}_{X/S} \to S$ is smooth by Moduli Stacks, Lemma 8.5 we win. $\square$

5. Properties of the stack of curves

0DSP The following lemma isn't true for moduli of surfaces, see Remark 5.2.

0DSQ **Lemma 5.1.** *The diagonal of $\mathcal{C}urves$ is separated and of finite presentation.*

Proof. Recall that $\mathcal{C}urves$ is a limit preserving algebraic stack, see Quot, Lemma 15.6. By Limits of Stacks, Lemma 3.6 this implies that $\Delta : \mathcal{P}olarized \to \mathcal{P}olarized \times \mathcal{P}olarized$ is limit preserving. Hence Δ is locally of finite presentation by Limits of Stacks, Proposition 3.8.

Let us prove that Δ is separated. To see this, it suffices to show that given a scheme U and two objects $Y \to U$ and $X \to U$ of $\mathcal{C}urves$ over U, the algebraic space

$$Isom_U(Y, X)$$

is separated. This we have seen in Moduli Stacks, Lemmas 10.2 and 10.3 that the target is a separated algebraic space.

To finish the proof we show that Δ is quasi-compact. Since Δ is representable by algebraic spaces, it suffice to check the base change of Δ by a surjective smooth morphism $U \to \mathcal{C}urves \times \mathcal{C}urves$ is quasi-compact (see for example Properties of Stacks, Lemma 3.3). We choose $U = \coprod U_i$ to be a disjoint union of affine opens with a surjective smooth morphism

$$U \longrightarrow PolarizedCurves \times PolarizedCurves$$

Then $U \to \mathcal{C}urves \times \mathcal{C}urves$ will be surjective and smooth since $PolarizedCurves \to \mathcal{C}urves$ is surjective and smooth by Lemma 4.2. Since $PolarizedCurves$ is limit preserving (by Artin's Axioms, Lemma 11.2 and Quot, Lemmas 15.6, 14.7, and 13.5), we see that $PolarizedCurves \to \mathrm{Spec}(\mathbf{Z})$ is locally of finite presentation, hence $U_i \to \mathrm{Spec}(\mathbf{Z})$ is locally of finite presentation (Limits of Stacks, Proposition 3.8 and Morphisms of Stacks, Lemmas 26.2 and 32.5). In particular, U_i is Noetherian affine. This reduces us to the case discussed in the next paragraph.

In this paragraph, given a Noetherian affine scheme U and two objects $(Y, \mathcal{N})$ and $(X, \mathcal{L})$ of $PolarizedCurves$ over U, we show the algebraic space

$$Isom_U(Y, X)$$

is quasi-compact. Since the connected components of U are open and closed we may replace U by these. Thus we may and do assume U is connected. Let $u \in U$ be a point. Let Q, P be the Hilbert polynomials of these families, i.e.,

$$Q(n) = \chi(Y_u, \mathcal{N}_u^{\otimes n}) \quad \text{and} \quad P(n) = \chi(X_u, \mathcal{L}_u^{\otimes n})$$

see Varieties, Lemma 43.1. Since U is connected and since the functions $u \mapsto \chi(Y_u, \mathcal{N}_u^{\otimes n})$ and $u \mapsto \chi(X_u, \mathcal{L}_u^{\otimes n})$ are locally constant (see Derived Categories of Schemes, Lemma 28.2) we see that we get the same Hilbert polynomial in every point of U. Set

$$\mathcal{M} = \mathrm{pr}_1^*\mathcal{N} \otimes_{\mathcal{O}_{Y \times_U X}} \mathrm{pr}_2^*\mathcal{L}$$

on $Y \times_U X$. Given $(f, \varphi) \in \mathit{Isom}_U(Y, X)(T)$ for some scheme T over U then for every $t \in T$ we have

$$\begin{aligned} \chi(Y_t, (\mathrm{id} \times f)^*\mathcal{M}^{\otimes n}) &= \chi(Y_t, \mathcal{N}_t^{\otimes n} \otimes_{\mathcal{O}_{Y_t}} f_t^*\mathcal{L}_t^{\otimes n}) \\ &= n\deg(\mathcal{N}_t) + n\deg(f_t^*\mathcal{L}_t) + \chi(Y_t, \mathcal{O}_{Y_t}) \\ &= Q(n) + n\deg(\mathcal{L}_t) \\ &= Q(n) + P(n) - P(0) \end{aligned}$$

by Riemann-Roch for proper curves, more precisely by Varieties, Definition 42.1 and Lemma 42.7 and the fact that f_t is an isomorphism. Setting $P'(t) = Q(t) + P(t) - P(0)$ we find

$$\mathit{Isom}_U(Y, X) = \mathit{Isom}_U(Y, X) \cap \mathit{Mor}_U^{P', \mathcal{M}}(Y, X)$$

The intersection is an intersection of open subspaces of $\mathit{Mor}_U(Y, X)$, see Moduli Stacks, Lemma 10.3 and Remark 10.4. Now $\mathit{Mor}_U^{P', \mathcal{M}}(Y, X)$ is a Noetherian algebraic space as it is of finite presentation over U by Moduli Stacks, Lemma 10.5. Thus the intersection is a Noetherian algebraic space too and the proof is finished. □

0DSR **Remark 5.2.** The boundedness argument in the proof of Lemma 5.1 does not work for moduli of surfaces and in fact, the result is wrong, for example because K3 surfaces over fields can have infinite discrete automorphism groups. The "reason" the argument does not work is that on a projective surface S over a field, given ample invertible sheaves $\mathcal{N}$ and $\mathcal{L}$ with Hilbert polynomials Q and P, there is no a priori bound on the Hilbert polynomial of $\mathcal{N} \otimes_{\mathcal{O}_S} \mathcal{L}$. In terms of intersection theory, if H_1, H_2 are ample effective Cartier divisors on S, then there is no (upper) bound on the intersection number $H_1 \cdot H_2$ in terms of $H_1 \cdot H_1$ and $H_2 \cdot H_2$.

0DSS **Lemma 5.3.** *The morphism $\mathit{Curves} \to \mathrm{Spec}(\mathbf{Z})$ is quasi-separated and locally of finite presentation.*

Proof. To check $\mathit{Curves} \to \mathrm{Spec}(\mathbf{Z})$ is quasi-separated we have to show that its diagonal is quasi-compact and quasi-separated. This is immediate from Lemma 5.1. To prove that $\mathit{Curves} \to \mathrm{Spec}(\mathbf{Z})$ is locally of finite presentation, it suffices to show that Curves is limit preserving, see Limits of Stacks, Proposition 3.8. This is Quot, Lemma 15.6. □

6. Open substacks of the stack of curves

0E0E Below we will often characterize an open substack of Curves by a propery P of morphisms of algebraic spaces. To see that P defines an open substack it suffices to check

(o) given a family of curves $f : X \to S$ there exists a largest open subscheme $S' \subset S$ such that $f|_{f^{-1}(S')} : f^{-1}(S') \to S'$ has P and such that formation of S' commutes with arbitrary base change.

Namely, suppose (o) holds. Choose a scheme U and a surjective smooth morphism $m : U \to \mathit{Curves}$. Let $R = U \times_{\mathit{Curves}} U$ and denote $t, s : R \to U$ the projections. Recall that $\mathit{Curves} = [U/R]$ is a presentation, see Algebraic Stacks, Lemma 16.2 and Definition 16.5. By construction of Curves as the stack of curves, the morphism

m is the classifying morphism for a family of curves $C \to U$. The 2-commutativity of the diagram

$$\begin{tikzcd} R \arrow[r, "s"'] \arrow[d, "t"'] & U \arrow[d] \\ U \arrow[r] & \mathcal{C}urves \end{tikzcd}$$

implies that $C \times_{U,s} R \cong C \times_{U,t} R$ (isomorphism of families of curves over R). Let $W \subset U$ be the largest open subscheme such that $f|_{f^{-1}(W)} : f^{-1}(W) \to W$ has P as in (o). Since formation of W commutes with base change according to (o) and by the isomorphism above we find that $s^{-1}(W) = t^{-1}(W)$. Thus $W \subset U$ corresponds to an open substack

$$\mathcal{C}urves^P \subset \mathcal{C}urves$$

according to Properties of Stacks, Lemma 9.7.

Continuing with the setup of the previous paragrpah, we claim the open substack $\mathcal{C}urves^P$ has the following two universal properties:

(1) given a family of curves $X \to S$ the following are equivalent
  (a) the classifying morphism $S \to \mathcal{C}urves$ factors through $\mathcal{C}urves^P$,
  (b) the morphism $X \to S$ has P,
(2) given X a proper scheme over a field k of dimension ≤ 1 the following are equivalent
  (a) the classifying morphism $\mathrm{Spec}(k) \to \mathcal{C}urves$ factors through $\mathcal{C}urves^P$,
  (b) the morphism $X \to \mathrm{Spec}(k)$ has P.

This follows by considering the 2-fibre product

$$\begin{tikzcd} T \arrow[r, "p"'] \arrow[d, "q"'] & U \arrow[d] \\ S \arrow[r] & \mathcal{C}urves \end{tikzcd}$$

Observe that $T \to S$ is surjective and smooth as the base change of $U \to \mathcal{C}urves$. Thus the open $S' \subset S$ given by (o) is determined by its inverse image in T. However, by the invariance under base change of these opens in (o) and because $X \times_S T \cong C \times_U T$ by the 2-commutativity, we find $q^{-1}(S') = p^{-1}(W)$ as opens of T. This immediately implies (1). Part (2) is a special case of (1).

Given two properties P and Q of morphisms of algebraic spaces, supposing we already have esthablished $\mathcal{C}urves^Q$ is an open substack of $\mathcal{C}urves$, then we can use exactly the same method to prove openness of $\mathcal{C}urves^{Q,P} \subset \mathcal{C}urves^Q$. We omit a precise explanation.

7. Curves with finite reduced automorphism groups

0DST Let X be a proper scheme over a field k of dimension ≤ 1, i.e., an object of $\mathcal{C}urves$ over k. By Lemma 5.1 the automorphism group algebraic space $\mathit{Aut}(X)$ is finite type and separated over k. In particular, $\mathit{Aut}(X)$ is a group scheme, see More on Groupoids in Spaces, Lemma 10.2. If the characteristic of k is zero, then $\mathit{Aut}(X)$ is reduced and even smooth over k (Groupoids, Lemma 8.2). However, in general $\mathit{Aut}(X)$ is not reduced, even if X is geometrically reduced.

0DSU **Example 7.1** (Non-reduced automorphism group)**.** Let k be an algebraically closed field of characteristic 2. Set $Y = Z = \mathbf{P}^1_k$. Choose three pairwise distinct k-valued points a, b, c in $\mathbf{A}^1_k$. Thinking of $\mathbf{A}^1_k \subset \mathbf{P}^1_k = Y = Z$ as an open subschemes, we get a closed immersion

$$T = \mathrm{Spec}(k[t]/(t-a)^2) \amalg \mathrm{Spec}(k[t]/(t-b)^2) \amalg \mathrm{Spec}(k[t]/(t-c)^2) \longrightarrow \mathbf{P}^1_k$$

Let X be the pushout in the diagram

$$\begin{tikzcd} T \arrow[r] \arrow[d] & Y \arrow[d] \\ Z \arrow[r] & X \end{tikzcd}$$

Let $U \subset X$ be the affine open part which is the image of $\mathbf{A}^1_k \amalg \mathbf{A}^1_k$. Then we have an equalizer diagram

$$\mathcal{O}_X(U) \longrightarrow k[t] \times k[t] \rightrightarrows k[t]/(t-a)^2 \times k[t]/(t-b)^2 \times k[t]/(t-c)^2$$

Over the dual numbers $A = k[\epsilon]$ we have a nontrivial automorphism of this equalizer diagram sending t to $t + \epsilon$. We leave it to the reader to see that this automorphism extends to an automorphism of X over A. On the other hand, the reader easily shows that the automorphism group of X over k is finite. Thus $Aut(X)$ must be non-reduced.

Let X be a proper scheme over a field k of dimension ≤ 1, i.e., an object of $\mathcal{C}urves$ over k. If $Aut(X)$ is geometrically reduced, then it need not be the case that it has dimension 0, even if X is smooth and geometrically connected.

0DSV **Example 7.2** (Smooth positive dimensional automorphism group)**.** Let k be an algebraically closed field. If X is a smooth genus 0, resp. 1 curve, then the automorphism group has dimension 3, resp. 1. Namely, in the genus 0 case we have $X \cong \mathbf{P}^1_k$ by Algebraic Curves, Proposition 8.4. Since

$$Aut(\mathbf{P}^1_k) = \mathrm{PGL}_{2,k}$$

as functors we see that the dimension is 3. On the other hand, if the genus of X is 1, then we see that the map $X = \underline{\mathrm{Hilb}}^1_{X/k} \to \underline{\mathrm{Pic}}^1_{X/k}$ is an isomorphism, see Picard Schemes of Curves, Lemma 6.7 and Algebraic Curves, Theorem 2.6. Thus X has the structure of an abelian variety (since $\underline{\mathrm{Pic}}^1_{X/k} \cong \underline{\mathrm{Pic}}^0_{X/k}$). In particular the (co)tangent bundle of X are trivial (Groupoids, Lemma 6.3). We conclude that $\dim_k H^0(X, T_X) = 1$ hence $\dim Aut(X) \leq 1$. On the other hand, the translations (viewing X as a group scheme) provide a 1-dimensional piece of $\mathrm{Aut}(X)$ and we conlude its dimension is indeed 1.

It turns out that there is an open substack of $\mathcal{C}urves$ parametrizing curves whose automorphism group is geometrically reduced and finite. Here is a precise statement.

0DSW **Lemma 7.3.** *There exist an open substack $\mathcal{C}urves^{DM} \subset \mathcal{C}urves$ with the following properties*

1. *$\mathcal{C}urves^{DM} \subset \mathcal{C}urves$ is the maximal open substack which is DM,*
2. *given a family of curves $X \to S$ the following are equivalent*
   - (a) *the classifying morphism $S \to \mathcal{C}urves$ factors through $\mathcal{C}urves^{DM}$,*
   - (b) *the group algebraic space $Aut_S(X)$ is unramified over S,*

(3) *given X a proper scheme over a field k of dimension ≤ 1 the following are equivalent*
  (a) *the classifying morphism* $\mathrm{Spec}(k) \to \mathcal{C}urves$ *factors through* $\mathcal{C}urves^{DM}$,
  (b) $Aut(X)$ *is geometrically reduced over k and has dimension* 0,
  (c) $Aut(X) \to \mathrm{Spec}(k)$ *is unramified.*

Proof. The existence of an open substack with property (1) is Morphisms of Stacks, Lemma 22.1. The points of this open substack are characterized by (3)(c) by Morphisms of Stacks, Lemma 22.2. The equivalence of (3)(b) and (3)(c) is the statement that an algebraic space G which is locally of finite type, geometrically reduced, and of dimension 0 over a field k, is unramified over k. First, G is a scheme by Spaces over Fields, Lemma 6.1. Then we can take an affine open in G and observe that it will be proper over k and apply Varieties, Lemma 9.3. Minor details omitted.

Part (2) is true because (3) holds. Namely, the morphism $Aut_S(X) \to S$ is locally of finite type. Thus we can check whether $Aut_S(X) \to S$ is unramified at all points of $Aut_S(X)$ by checking on fibres at points of the scheme S, see Morphisms of Spaces, Lemma 37.10. But after base change to a point of S we fall back into the equivalence of (3)(a) and (3)(c). □

8. Cohen-Macaulay curves

0E0H There is an open substack of $\mathcal{C}urves$ parametrizing the Cohen-Macaulay "curves".

0E0I **Lemma 8.1.** *There exist an open substack* $\mathcal{C}urves^{CM} \subset \mathcal{C}urves$ *such that*
(1) *given a family of curves $X \to S$ the following are equivalent*
  (a) *the classifying morphism* $S \to \mathcal{C}urves$ *factors through* $\mathcal{C}urves^{CM}$,
  (b) *the morphism $X \to S$ is Cohen-Macaulay,*
(2) *given a scheme X proper over a field k with* $\dim(X) \leq 1$ *the following are equivalent*
  (a) *the classifying morphism* $\mathrm{Spec}(k) \to \mathcal{C}urves$ *factors through* $\mathcal{C}urves^{CM}$,
  (b) *X is Cohen-Macaulay.*

Proof. Let $f : X \to S$ be a family of curves. By More on Morphisms of Spaces, Lemma 26.7 the set

$$W = \{x \in |X| : f \text{ is Cohen-Macaulay at } x\}$$

is open in $|X|$ and formation of this open commutes with arbitrary base change. Since f is proper the subset

$$S' = S \setminus f(|X| \setminus W)$$

of S is open and $X \times_S S' \to S'$ is Cohen-Macaulay. Moreover, formation of S' commutes with arbitrary base change because this is true for W Thus we get the open substack with the desired properties by the method discussed in Section 6. □

0E1F **Lemma 8.2.** *There exist an open substack* $\mathcal{C}urves^{CM,1} \subset \mathcal{C}urves$ *such that*
(1) *given a family of curves $X \to S$ the following are equivalent*
  (a) *the classifying morphism* $S \to \mathcal{C}urves$ *factors through* $\mathcal{C}urves^{CM,1}$,
  (b) *the morphism $X \to S$ is Cohen-Macaulay and has relative dimension* 1 *(Morphisms of Spaces, Definition 32.2),*

(2) *given a scheme X proper over a field k with $\dim(X) \leq 1$ the following are equivalent*
   (a) *the classifying morphism $\mathrm{Spec}(k) \to \mathcal{C}urves$ factors through $\mathcal{C}urves^{CM,1}$,*
   (b) *X is Cohen-Macaulay and every irreducible component of X has dimension 1.*

Proof. By Lemma 8.1 it is clear that we have $\mathcal{C}urves^{CM,1} \subset \mathcal{C}urves^{CM}$ if it exists. Let $f : X \to S$ be a family of curves such that f is a Cohen-Macaulay morphism. By More on Morphisms of Spaces, Lemma 26.8 we have a decomposition

$$X = X_0 \amalg X_1$$

by open and closed subspaces such that $X_0 \to S$ has relative dimension 0 and $X_1 \to S$ has relative dimension 1. Since f is proper the subset

$$S' = S \setminus f(|X_0|)$$

of S is open and $X \times_S S' \to S'$ is Cohen-Macaulay and has relative dimension 1. Moreover, formation of S' commutes with arbitrary base change because this is true for the decomposition above (as relative dimension behaves well with respect to base change, see Morphisms of Spaces, Lemma 33.3). Thus we get the open substack with the desired properties by the method discussed in Section 6. □

9. Geometrically reduced curves

0E0F There is an open substack of $\mathcal{C}urves$ parametrizing the geometrically reduced "curves".

0E0G **Lemma 9.1.** *There exist an open substack $\mathcal{C}urves^{geomred} \subset \mathcal{C}urves$ such that*

(1) *given a family of curves $X \to S$ the following are equivalent*
   (a) *the classifying morphism $S \to \mathcal{C}urves$ factors through $\mathcal{C}urves^{geomred}$,*
   (b) *the fibres of the morphism $X \to S$ are geometrically reduced (More on Morphisms of Spaces, Definition 29.2),*

(2) *given a scheme X proper over a field k with $\dim(X) \leq 1$ the following are equivalent*
   (a) *the classifying morphism $\mathrm{Spec}(k) \to \mathcal{C}urves$ factors through $\mathcal{C}urves^{geomred}$,*
   (b) *X is geometrically reduced over k.*

Proof. Let $f : X \to S$ be a family of curves. By More on Morphisms of Spaces, Lemma 29.6 the set

$$E = \{s \in S : \text{the fibre of } X \to S \text{ at } s \text{ is geometrically reduced}\}$$

is open in S. Formation of this open commutes with arbitrary base change by More on Morphisms of Spaces, Lemma 29.3. Thus we get the open substack with the desired properties by the method discussed in Section 6. □

0E1G **Lemma 9.2.** *We have $\mathcal{C}urves^{geomred} \subset \mathcal{C}urves^{CM}$ as open substacks of $\mathcal{C}urves$.*

Proof. This is true because a reduced Noetherian scheme of dimension ≤ 1 is Cohen-Macaulay. See Algebra, Lemma 151.3. □

10. Geometrically reduced and connected curves

0E1H There is an open substack of $\mathcal{C}urves$ parametrizing the geometrically reduced and connected "curves". We will get rid of 0-dimensional objects right away.

0E1I **Lemma 10.1.** *There exist an open substack* $\mathcal{C}urves^{grc,1} \subset \mathcal{C}urves$ *such that*

1. *given a family of curves $X \to S$ the following are equivalent*
   (a) *the classifying morphism $S \to \mathcal{C}urves$ factors through $\mathcal{C}urves^{grc,1}$,*
   (b) *the geometric fibres of the morphism $X \to S$ are reduced, connected, and have dimension 1,*
2. *given a scheme X proper over a field k with $\dim(X) \leq 1$ the following are equivalent*
   (a) *the classifying morphism $\mathrm{Spec}(k) \to \mathcal{C}urves$ factors through $\mathcal{C}urves^{grc,1}$,*
   (b) *X is geometrically reduced, geometrically connected, and has dimension 1.*

Proof. By Lemmas 9.1, 9.2, 8.1, and 8.2 it is clear that we have

$$\mathcal{C}urves^{geomredcon} \subset \mathcal{C}urves^{geomred} \cap \mathcal{C}urves^{CM,1}$$

if it exists. Let $f : X \to S$ be a family of curves such that f is Cohen-Macaulay, has geometrically reduced fibres, and has relative dimension 1. By More on Morphisms of Spaces, Lemma 34.9 in the Stein factorization

$$X \to T \to S$$

the morphism $T \to S$ is étale. This implies that there is an open and closed subscheme $S' \subset S$ such that $X \times_S S' \to S'$ has geometrically connected fibres (in the decomposition of Morphisms, Lemma 45.5 for the finite locally free morphism $T \to S$ this corresponds to S_1). Formation of this open commutes with arbitrary base change because the number of connected components of geometric fibres is invariant under base change (it is also true that the Stein factorization commutes with base change in our particular case but we don't need this to conclude). Thus we get the open substack with the desired properties by the method discussed in Section 6. $\square$

0E1J **Lemma 10.2.** *Let $f : X \to S$ be a family of curves whose geometric fibres are reduced, connected, and have dimension 1. Then*

1. *$f_*\mathcal{O}_X = \mathcal{O}_S$ and this holds universally,*
2. *$R^1f_*\mathcal{O}_X$ is a finite locally free $\mathcal{O}_S$-module,*
3. *for any morphism $h : S' \to S$ if $f' : X' \to S'$ is the base change, then $h^*(R^1f_*\mathcal{O}_X) = R^1f'_*\mathcal{O}_{X'}$.*

Proof. Part (1) holds by Derived Categories of Spaces, Lemma 26.6. By cohomology and base change (more precisely by Derived Categories of Spaces, Lemma 25.4) we see that $E = Rf_*\mathcal{O}_X$ is a perfect object of the derived category of S and that its formation commutes with arbitrary change of base. By part (1) we can locally on S write $E = \mathcal{O}_S \oplus E'$ in $D(\mathcal{O}_S)$ with $E' = \tau_{\geq 1}E$ a perfect object with tor amplitude in $[1, \infty)$, see More on Algebra, Lemma 69.2. For $s \in S$ we have

$$H^i(E' \otimes^{\mathbf{L}}_{\mathcal{O}_S} \kappa(s)) = H^i(X_s, \mathcal{O}_{X_s}) \text{ for } i \geq 1$$

since formation of E commutes with arbitrary base change. This is zero unless $i = 1$ since X_s is a 1-dimensional Noetherian scheme, see Cohomology, Proposition

21.7. Then $E' = H^1(E')[-1]$ and $H^1(E')$ is finite locally free by More on Algebra, Lemma 68.6. Since everything is compatible with base change we also see that (3) holds. $\square$

0E1K **Lemma 10.3.** *There is a decomposition into open and closed substacks*

$$\mathcal{C}urves^{grc,1} = \coprod\nolimits_{g \geq 0} \mathcal{C}urves_g^{grc,1}$$

where each $\mathcal{C}urves_g^{grc,1}$ is characterized as follows:

1. *given a family of curves $f : X \to S$ the following are equivalent*
   - (a) *the classifying morphism $S \to \mathcal{C}urves$ factors through $\mathcal{C}urves_g^{grc,1}$,*
   - (b) *the geometric fibres of the morphism $f : X \to S$ are reduced, connected, of dimension 1 and $R^1 f_* \mathcal{O}_X$ is a locally free $\mathcal{O}_S$-module of rank g,*
2. *given a scheme X proper over a field k with $\dim(X) \leq 1$ the following are equivalent*
   - (a) *the classifying morphism $\mathrm{Spec}(k) \to \mathcal{C}urves$ factors through $\mathcal{C}urves_g^{grc,1}$,*
   - (b) *X is geometrically reduced, geometrically connected, has dimension 1, and has genus g.*

Proof. The existence of the decomposition into open and closed substacks follows immediately from the discussion in Section 6 and Lemma 10.2. This proves the characterization in (1). The characterization in (2) follows as well since the genus of a geometrically reduced and connected proper 1-dimensional scheme X/k is defined (Algebraic Curves, Definition 6.1 and Varieties, Lemma 9.3) and is equal to $\dim_k H^1(X, \mathcal{O}_X)$. $\square$

11. Gorenstein curves

0E1L There is an open substack of *Curves* parametrizing the Gorenstein "curves".

0E1M **Lemma 11.1.** *There exist an open substack $\mathcal{C}urves^{Gorenstein} \subset \mathcal{C}urves$ such that*

1. *given a family of curves $X \to S$ the following are equivalent*
   - (a) *the classifying morphism $S \to \mathcal{C}urves$ factors through $\mathcal{C}urves^{Gorenstein}$,*
   - (b) *the morphism $X \to S$ is Gorenstein,*
2. *given a scheme X proper over a field k with $\dim(X) \leq 1$ the following are equivalent*
   - (a) *the classifying morphism $\mathrm{Spec}(k) \to \mathcal{C}urves$ factors through $\mathcal{C}urves^{Gorenstein}$,*
   - (b) *X is Gorenstein.*

Proof. Let $f : X \to S$ be a family of curves. By More on Morphisms of Spaces, Lemma 27.7 the set

$$W = \{x \in |X| : f \text{ is Gorenstein at } x\}$$

is open in $|X|$ and formation of this open commutes with arbitrary base change. Since f is proper the subset

$$S' = S \setminus f(|X| \setminus W)$$

of S is open and $X \times_S S' \to S'$ is Gorenstein. Moreover, formation of S' commutes with arbitrary base change because this is true for W Thus we get the open substack with the desired properties by the method discussed in Section 6. $\square$

12. Local complete intersection curves

0E0J There is an open substack of *Curves* parametrizing the local complete intersection "curves".

0DZV **Lemma 12.1.** *There exist an open substack $\mathit{Curves}^{lci} \subset \mathit{Curves}$ such that*

1. *given a family of curves $X \to S$ the following are equivalent*
   1. (a) *the classifying morphism $S \to \mathit{Curves}$ factors through Curves^{lci},*
   2. (b) *$X \to S$ is a local complete intersection morphism, and*
   3. (c) *$X \to S$ is a syntomic morphism.*
2. *given X a proper scheme over a field k of dimension ≤ 1 the following are equivalent*
   1. (a) *the classifying morphism $\mathrm{Spec}(k) \to \mathit{Curves}$ factors through Curves^{lci},*
   2. (b) *X is a local complete intersection over k.*

Proof. Recall that being a syntomic morphism is the same as being flat and a local complete intersection morphism, see More on Morphisms of Spaces, Lemma 46.6. Thus (1)(b) is equivalent to (1)(c). In Section 6 we have seen it suffices to show that given a family of curves $f : X \to S$, there is an open subscheme $S' \subset S$ such that $S' \times_S X \to S'$ is a local complete intersection morphism and such that formation of S' commutes with arbitrary base change. This follows from the more general More on Morphisms of Spaces, Lemma 47.7. □

13. Curves with isolated singularities

0E0K We can look at the open substack of *Curves* parametrizing "curves" with only a finite number of singular points (these may correspond to 0-dimensional components in our setup).

0DZW **Lemma 13.1.** *There exist an open substack $\mathit{Curves}^{+} \subset \mathit{Curves}$ such that*

1. *given a family of curves $X \to S$ the following are equivalent*
   1. (a) *the classifying morphism $S \to \mathit{Curves}$ factors through Curves^{+},*
   2. (b) *the singular locus of $X \to S$ endowed with any/some closed subspace structure is finite over S.*
2. *given X a proper scheme over a field k of dimension ≤ 1 the following are equivalent*
   1. (a) *the classifying morphism $\mathrm{Spec}(k) \to \mathit{Curves}$ factors through Curves^{+},*
   2. (b) *$X \to \mathrm{Spec}(k)$ is smooth except at finitely many points.*

Proof. To prove the lemma it suffices to show that given a family of curves $f : X \to S$, there is an open subscheme $S' \subset S$ such that the fibre of $S' \times_S X \to S'$ have property (2). (Formation of the open will automatically commute with base change.) By definition the locus $T \subset |X|$ of points where $X \to S$ is not smooth is closed. Let $Z \subset X$ be the closed subspace given by the reduced induced algebraic space structure on T (Properties of Spaces, Definition 11.6). Now if $s \in S$ is a point where Z_s is finite, then there is an open neighbourhood $U_s \subset S$ of s such that $Z \cap f^{-1}(U_s) \to U_s$ is finite, see More on Morphisms of Spaces, Lemma 33.6. This proves the lemma. □

14. The smooth locus of the stack of curves

0DZT

The morphism

$$\mathcal{C}urves \longrightarrow \mathrm{Spec}(\mathbf{Z})$$

is smooth over a maximal open substack, see Morphisms of Stacks, Lemma 32.6. We want to give a criterion for when a curve is in this locus. We will do this using a bit of deformation theory.

Let k be a field. Let X be a proper scheme of dimension ≤ 1 over k. Choose a Cohen ring Λ for k, see Algebra, Lemma 154.6. Then we are in the situation described in Deformation Problems, Example 9.1 and Lemma 9.2. Thus we obtain a deformation category $\mathcal{D}ef_X$ on the category $\mathcal{C}_\Lambda$ of Artinian local Λ-algebras with residue field k.

0DZU **Lemma 14.1.** *In the situation above the following are equivalent*

1. *the classifying morphism* $\mathrm{Spec}(k) \to \mathcal{C}urves$ *factors through the open where* $\mathcal{C}urves \to \mathrm{Spec}(\mathbf{Z})$ *is smooth,*
2. *the deformation category* $\mathcal{D}ef_X$ *is unobstructed.*

Proof. Since $\mathcal{C}urves \longrightarrow \mathrm{Spec}(\mathbf{Z})$ is locally of finite presentation (Lemma 5.3) formation of the open substack where $\mathcal{C}urves \longrightarrow \mathrm{Spec}(\mathbf{Z})$ commutes with flat base change (Morphisms of Stacks, Lemma 32.6). Since the Cohen ring Λ is flat over $\mathbf{Z}$, we may work over Λ. In other words, we are trying to prove that

$$\Lambda\text{-}\mathcal{C}urves \longrightarrow \mathrm{Spec}(\Lambda)$$

is smooth in an open neighbourhood of the point $x_0 : \mathrm{Spec}(k) \to \Lambda\text{-}\mathcal{C}urves$ defined by X/k if and only if $\mathcal{D}ef_X$ is unobstructed.

The lemma now follows from Geometry of Stacks, Lemma 2.7 and the equality

$$\mathcal{D}ef_X = \mathcal{F}_{\Lambda\text{-}\mathcal{C}urves,k,x_0}$$

This equality is not completely trivial to esthablish. Namely, on the left hand side we have the deformation category classifying all flat deformations $Y \to \mathrm{Spec}(A)$ of X as a scheme over $A \in \mathrm{Ob}(\mathcal{C}_\Lambda)$. On the right hand side we have the deformation category classifying all flat morphisms $Y \to \mathrm{Spec}(A)$ with special fibre X where Y is an algebraic space and $Y \to \mathrm{Spec}(A)$ is proper, of finite presentation, and of relative dimension ≤ 1. Since A is Artinian, we find that Y is a scheme for example by Spaces over Fields, Lemma 6.3. Thus it remains to show: a flat deformation $Y \to \mathrm{Spec}(A)$ of X as a scheme over an Artinian local ring A with residue field k is proper, of finite presentation, and of relative dimension ≤ 1. Relative dimension is defined in terms of fibres and hence holds automatically for Y/A since it holds for X/k. The morphism $Y \to \mathrm{Spec}(A)$ is proper and locally of finite presentation as this is true for $X \to \mathrm{Spec}(k)$, see More on Morphisms, Lemma 10.3. $\square$

Here is a "large" open of the stack of curves which is contained in the smooth locus.

0DZX **Lemma 14.2.** *The open substack*

$$\mathcal{C}urves^{lci+} = \mathcal{C}urves^{lci} \cap \mathcal{C}urves^{+} \subset \mathcal{C}urves$$

has the following properties

1. $\mathcal{C}urves^{lci+} \to \mathrm{Spec}(\mathbf{Z})$ *is smooth,*
2. *given* X *a proper scheme over a field* k *of dimension* ≤ 1 *the following are equivalent*
   (a) *the classifying morphism* $\mathrm{Spec}(k) \to \mathcal{C}urves$ *factors through* $\mathcal{C}urves^{lci+}$,

(b) *X is a local complete intersection over k and $X \to \mathrm{Spec}(k)$ is smooth except at finitely many points.*

Proof. If we can show that there is an open substack $\mathcal{C}urves^{lci+}$ whose points are characterized by (2), then we see that (1) holds by combining Lemma 14.1 with Deformation Problems, Lemma 14.4. Since

$$\mathcal{C}urves^{lci+} = \mathcal{C}urves^{lci} \cap \mathcal{C}urves^{+}$$

inside $\mathcal{C}urves$, we conclude by Lemmas 12.1 and 13.1. $\square$

15. Smooth curves

0DZY The moduli stack of smooth curves defined as follows.

0DZZ **Lemma 15.1.** *There exist an open substack $\mathcal{C}urves^{smooth} \subset \mathcal{C}urves$ such that*

1. *given a family of curves $f : X \to S$ the following are equivalent*
   (a) *f is smooth of relative dimension 1, and*
   (b) *the classifying morphism $S \to \mathcal{C}urves$ factors through $\mathcal{C}urves^{smooth}$.*
2. *given X a scheme proper over a field k with $\dim(X) \leq 1$ the following are equivalent*
   (a) *the classifying morphism $\mathrm{Spec}(k) \to \mathcal{C}urves$ factors through $\mathcal{C}urves^{smooth}$,*
   (b) *X is smooth.*

Proof. To prove the lemma it suffices to show that given a family of curves $f : X \to S$, there is an open subscheme $S' \subset S$ such that $S' \times_S X \to S'$ is smooth and such that the formation of this open commutes with base change. We know that there is a maximal open $U \subset X$ such that $U \to S$ is smooth and that formation of U commutes with arbitrary base change, see Morphisms of Spaces, Lemma 36.9. If $T = |X| \setminus |U|$ then $f(T)$ is closed in S as f is proper. Setting $S' = S \setminus f(T)$ finishes the proof. $\square$

0E1N **Lemma 15.2.** *The morphism $\mathcal{C}urves^{smooth} \to \mathrm{Spec}(\mathbf{Z})$ is smooth.*

Proof. Follows immediately from the observation that $\mathcal{C}urves^{smooth} \subset \mathcal{C}urves^{lci+}$ and Lemma 14.2. $\square$

16. Nodal curves

0DSX In algebraic geometry a special role is played by nodal curves. We suggest the reader take a brief look at some of the discussion in Algebraic Curves, Sections 16 and 17 and More on Morphisms of Spaces, Section 53.

0DSY **Lemma 16.1.** *There exist an open substack $\mathcal{C}urves^{nodal} \subset \mathcal{C}urves$ such that*

1. *given a family of curves $f : X \to S$ the following are equivalent*
   (a) *f is at-worst-nodal of relative dimension 1, and*
   (b) *the classifying morphism $S \to \mathcal{C}urves$ factors through $\mathcal{C}urves^{nodal}$,*
2. *given X a scheme proper over a field X with $\dim(X) \leq 1$ the following are equivalent*
   (a) *the classifying morphism $\mathrm{Spec}(k) \to \mathcal{C}urves$ factors through $\mathcal{C}urves^{nodal}$,*
   (b) *the singularities of X are at-worst-nodal.*

Proof. In fact, it suffices to show that given a family of curves $f : X \to S$, there is an open subscheme $S' \subset S$ such that $S' \times_S X \to S'$ is at-worst-nodal of relative dimension 1 and such that formation of S' commutes with arbitrary base change. By More on Morphisms of Spaces, Lemma 53.4 there is a maximal open subspace $X' \subset X$ such that $f|_{X'} : X' \to S$ is at-worst-nodal of relative dimension 1. Moreover, formation of X' commutes with base change. Hence we can take

$$S' = S \setminus |f|(|X| \setminus |X'|)$$

This is open because a proper morphism is universally closed by definition. $\square$

0E00 **Lemma 16.2.** *The morphism* $\mathit{Curves}^{nodal} \to \mathrm{Spec}(\mathbf{Z})$ *is smooth.*

Proof. Follows immediately from the observation that $\mathit{Curves}^{nodal} \subset \mathit{Curves}^{lci+}$ and Lemma 14.2. $\square$

17. Other chapters

References

[DM69] Pierre Deligne and David Mumford, *The irreducibility of the space of curves of given genus*, Publ. Math. IHES **36** (1969), 75–110.