1. Introduction

In this chapter we prove some results in commutative algebra which are less elementary than those in the first chapter on commutative algebra, see Algebra, Section 1. A reference is [Mat70].
2. Advice for the reader

More than in the chapter on commutative algebra, each of the sections in this chapter stands on its own. Starting with Section 45 we freely use the (unbounded) derived category of modules over rings and all the machinery that comes with it.

3. A comment on the Artin-Rees property

Some of this material is taken from [CJ02]. A general discussion with additional references can be found in [EH05, Section 1].

Let $A$ be a Noetherian ring and let $I \subset A$ be an ideal. Given a homomorphism $f : M \to N$ of finite $A$-modules there exists a $c \geq 0$ such that

$$f(M) \cap I^n N \subset f(I^{n-c}M)$$

for all $n \geq c$, see Algebra, Lemma 49.3. In this situation we will say $c$ works for $f$ in the Artin-Rees lemma.

**Lemma 3.1.** Let $A$ be a Noetherian ring. Let $I \subset A$ be an ideal contained in the Jacobson radical of $A$. Let

$$S : L \xrightarrow{f} M \xrightarrow{g} N \quad \text{and} \quad S' : L \xrightarrow{f'} M \xrightarrow{g'} N$$

be two complexes of finite $A$-modules as shown. Assume that

1. $c$ works in the Artin-Rees lemma for $f$ and $g$,
2. the complex $S$ is exact, and
3. $f' = f \mod I^{c+1}M$ and $g' = g \mod I^{c+1}N$.

Then $c$ works in the Artin-Rees lemma for $g'$ and the complex $S'$ is exact.

**Proof.** We first show that $g'(L) \cap I^n M \subset g'(I^{n-c}L)$ for $n \geq c$. Let $a$ be an element of $M$ such that $g'(a) \in I^n N$. We want to adjust $a$ by an element of $f'(L)$, i.e., without changing $g'(a)$, so that $a \in I^{n-c}M$. Assume that $a \in I^r M$, where $r < n - c$. Then

$$g(a) = g'(a) + (g - g')(a) \in I^n N + I^{r+c+1}N = I^{r+c+1}N.$$

By Artin-Rees for $g$ we have $g(a) \in g(I^{r+1}M)$. Say $g(a) = g(a_1)$ with $a_1 \in I^{r+1}M$. Since the sequence $S$ is exact, $a - a_1 \in f(L)$. Accordingly, we write $a = f(b) + a_1$ for some $b \in L$. Then $f(b) = a - a_1 \in I^r M$. Artin-Rees for $f$ shows that if $r \geq c$, we may replace $b$ by an element of $I^{r+c}L$. Then in all cases, $a = f'(b) + a_2$, where $a_2 = (f - f')(b) + a_1 \in I^{r+1}M$. (Namely, either $c \geq r$ and $(f - f')(b) \in I^{r+1}M$ by assumption, or $c < r$ and $b \in I^{r-c}$, whence again $(f - f')(b) \in I^{r+1}I^{r-c}M = I^{r+1}M$.) So we can adjust $a$ by the element $f'(b) \in f'(L)$ to increase $r$ by 1.

In fact, the argument above shows that $(g')^{-1}(I^n M) \subset f'(L) + I^{n-c}M$ for all $n \geq c$. Hence $S'$ is exact because

$$(g')^{-1}(0) = (g')^{-1}(\bigcap I^n N) \subset \bigcap f'(L) + I^{n-c}M = f'(L)$$

as $I \subset \text{rad}(A)$, see Algebra, Lemma 49.5.

Given an ideal $I \subset A$ of a ring $A$ and an $A$-module $M$ we set

$$\text{Gr}_I(M) = \bigoplus I^n M / I^{n+1}M.$$

We think of this as a graded $\text{Gr}_I(A)$-module.
Lemma 3.2. Assumptions as in Lemma 3.1. Let $Q = \text{Coker}(g)$ and $Q' = \text{Coker}(g')$. Then $\text{Gr}_I(Q) \cong \text{Gr}_I(Q')$ as graded $\text{Gr}_I(A)$-modules.

Proof. In degree $n$ we have $\text{Gr}_I(Q)_n = I^n N / (I^{n+1} N + g(M) \cap I^n N)$ and similarly for $Q'$. We claim that

$$g(M) \cap I^n N \subset I^{n+1} N + g'(M) \cap I^n N.$$  

By symmetry (the proof of the claim will only use that $c$ works for $g$ which also holds for $g'$ by the lemma) this will imply that

$$I^{n+1} N + g(M) \cap I^n N = I^{n+1} N + g'(M) \cap I^n N$$

whence $\text{Gr}_I(Q)_n$ and $\text{Gr}_I(Q')_n$ agree as subquotients of $N$, implying the lemma. Observe that the claim is clear for $n \leq c$ as $f = f' \mod I^{c+1} N$. If $n > c$, then suppose $b \in g(M) \cap I^n N$. Write $b = g(a)$ for $a \in I^{n-c} M$. Set $b' = g'(a)$. We have

$$b - b' = (g - g')(a) \in I^{n+1} N$$

as desired. \qed

Lemma 3.3. Let $A \to B$ be a flat map of Noetherian rings. Let $I \subseteq A$ be an ideal. Let $f : M \to N$ be a homomorphism of finite $A$-modules. Assume that $c$ works for $f$ in the Artin-Rees lemma. Then $c$ works for $f \otimes 1 : M \otimes_A B \to N \otimes_A B$ in the Artin-Rees lemma for the ideal $IB$.

Proof. Note that

$$(f \otimes 1)(M) \cap I^n N \otimes_A B = (f \otimes 1)((f \otimes 1)^{-1}(I^n N \otimes_A B))$$

On the other hand,

$$(f \otimes 1)^{-1}(I^n N \otimes_A B) = \text{Ker}(M \otimes_A B \to N \otimes_A B / (I^n N \otimes_A B)) = \text{Ker}(M \otimes_A B \to (N/I^n N) \otimes_A B)$$

As $A \to B$ is flat taking kernels and cokernels commutes with tensoring with $B$, whence this is equal to $f^{-1}(I^n N) \otimes_A B$. By assumption $f^{-1}(I^n N)$ is contained in $\text{Ker}(f) + I^{n-c} M$. Thus the lemma holds. \qed

4. Fibre products of rings

Fibre products of rings have to do with pushouts of schemes. A special case of pushouts of schemes is discussed in More on Morphisms, Section 11.

Lemma 4.1. Let $R$ be a ring. Let $A \to B$ and $C \to B$ be $R$-algebra maps. Assume

1. $R$ is Noetherian,
2. $A$, $B$, $C$ are of finite type over $R$,
3. $A \to B$ is surjective, and
4. $B$ is finite over $C$.

Then $A \times_B C$ is of finite type over $R$.

Proof. Set $D = A \times_B C$. There is a commutative diagram

$$
\begin{array}{ccc}
0 & \longrightarrow & I & \longrightarrow & A & \longrightarrow & B & \longrightarrow & 0 \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
0 & \longrightarrow & I & \longrightarrow & D & \longrightarrow & C & \longrightarrow & 0
\end{array}
$$

with exact rows. Choose $y_1, \ldots, y_n \in B$ which are generators for $B$ as a $C$-module. Choose $x_i \in A$ mapping to $y_i$. Then $1, x_1, \ldots, x_n$ are generators for $A$ as a $D$-module. The map $D \to A \times C$ is injective, and the ring $A \times C$ is finite as a
D-module (because it is the direct sum of the finite D-modules $A$ and $C$). Hence the lemma follows from the Artin-Tate lemma (Algebra, Lemma \[49.7\]). \hfill \Box

**Lemma 4.2.** Let $R$ be a Noetherian ring. Let $I$ be a finite set. Suppose given a cartesian diagram

\[
P \longrightarrow \prod A_i \\
Q \cong \prod B_i
\]

with $\psi_i$ and $\varphi_i$ surjective, and $Q, A_i, B_i$ of finite type over $R$. Then $P$ is of finite type over $R$.

**Proof.** Follows from Lemma 4.1 and induction on the size of $I$. Namely, let $I = I' \amalg \{i_0\}$. Let $P'$ be the ring defined by the diagram of the lemma using $I'$. Then $P'$ is of finite type by the lemma. Finally, $P$ sits in a fibre product diagram

\[
P \longrightarrow A_{i_0} \\
P' \longrightarrow B_{i_0}
\]

to which the lemma applies. \hfill \Box

**Lemma 4.3.** Suppose given a cartesian diagram of rings

\[
\begin{array}{ccc}
B & \longrightarrow & R \\
| & \downarrow s & \downarrow t \\
B' & \longrightarrow & R',
\end{array}
\]

i.e., $B' = B \times_R R'$. If $h \in B'$ corresponds to $g \in B$ and $f \in R'$ such that $s(g) = t(f)$, then the diagram

\[
\begin{array}{ccc}
B_g & \longrightarrow & R_{s(g)} = R_{t(f)} \\
| & \downarrow s & \downarrow t \\
(B'h) & \longrightarrow & (R')_f
\end{array}
\]

is cartesian too.

**Proof.** Note that $B' = \{(b,r') \in B \times R' \mid s(b) = t(r')\}$. So $h = (g, f) \in B'$. First we show that $(B')_h$ maps injectively into $B_g \times (R')_f$. Namely, suppose that $(x, y)/h^n$ maps to zero. This means that $g^N x = 0$ for some $N$ and $f^M y$ is zero for some $M$. Thus $h^{\max(N, M)}(x, y) = 0$ in $B'$ and hence $(x, y)/h^n = 0$ in $B'_h$. Next, suppose that $x/g^n$ and $y/f^m$ are elements which map to the same element of $R_{s(g)}$. This means that $s(g)^N (t(f)^m s(x) - s(g)^N t(y)) = 0$ in $R'$ for some $N \gg 0$. We can rewrite this as $s(g^{n+N} x) = t(f^{n+N} y)$. Hence we see that the pair $(x/g^n, y/f^m)$ is the image of the element $(g^{n+N} x, f^{n+N} y)/h^{n+m+N}$ of $(B')_h$. \hfill \Box

**Situation 4.4.** In the following we will consider ring maps

\[
B \longrightarrow A \longleftarrow A'
\]
where we assume $A' \to A$ is surjective with kernel $I$. In this situation we set $B' = B \times_A A'$ to obtain a cartesian square

\[
\begin{array}{ccc}
A & \to & A' \\
\uparrow & & \uparrow \\
B & \to & B'
\end{array}
\]

We’d like to understand $B'$-modules in terms of modules over $A'$, $A$, and $B$. In order to do this we consider the functor (where the fibre product of categories as constructed in Categories, Example 30.3)

\[(4.4.1) \quad \text{Mod}_B', \to \text{Mod}_B \times_{\text{Mod}_A} \text{Mod}_{A'}' \mapsto (L' \otimes_{B'} B, L' \otimes_{B'} A', \text{can})
\]

where $\text{can}$ is the canonical identification $L' \otimes_{B'} B \otimes_B A = L' \otimes_{B'} A' \otimes_{A'} A$. In the following we will write $(N, M', \varphi)$ for an object of the right hand side, i.e., $N$ is a $B$-module, $M'$ is an $A'$-module and $\varphi : N \otimes_B A \to M' \otimes_{A'} A$ is an isomorphism. However, it is often more convenient think of $\varphi$ as a $B$-linear map $\varphi : N \to M'/IM'$ which induces an isomorphism $N \otimes_B A \to M' \otimes_{A'} A = M'/IM'$.

**Lemma 4.5.** In Situation (4.4) the functor (4.4.1) has a right adjoint, namely the functor

\[F : (N, M', \varphi) \mapsto N \times_{\varphi, M} M'
\]

where $M = M'/IM'$. Moreover, the composition of $F$ with (4.4.1) is the identity functor on $\text{Mod}_B \times_{\text{Mod}_A} \text{Mod}_{A'}$. In other words, setting $N' = N \times_{\varphi, M} M'$ we have $N' \otimes_{B'} B = N$ and $N' \otimes_{B'} A' = M'$.

**Proof.** The adjointness statement is that for a $B'$-module $L'$ and a triple $(N, M', \varphi)$ we have

\[
\text{Hom}_B(L', N \times_{\varphi, M} M') = \text{Hom}_B(L' \otimes_{B'} B, N) \times_{\text{Hom}_A(L' \otimes_{B'} A, M)} \text{Hom}_{A'}(L' \otimes_{B'} A', M')
\]

This follows from Algebra, Lemma 13.3 and the fact that an element of the left hand side is given by a pair of $B'$-linear maps $L' \to N$ and $L' \to M'$ agreeing as maps to $M$. To prove the final assertion, recall that $B' = B \times_A A'$ and $N' = N \times_{\varphi, M} M'$ and extend these equalities to

\[
\begin{array}{ccc}
A & \to & A' & \to & I \\
\uparrow & & \uparrow & & \uparrow \\
B & \to & B' & \to & J \\
\uparrow & & \uparrow & & \uparrow \\
M & \leftrightarrow & M' & \leftrightarrow & K \\
\uparrow & & \uparrow & & \uparrow \\
N & \leftrightarrow & N' & \leftrightarrow & L
\end{array}
\]

where $I, J, K, L$ are the kernels of the horizontal maps of the original diagrams. We present the proof as a sequence of observations:

1. $K = IM'$ (see statement lemma),
2. $B' \to B$ is surjective with kernel $J$ and $J \to I$ is bijective,
3. $N' \to N$ is surjective with kernel $L$ and $L \to K$ is bijective,
4. $JN' \subseteq L$,
5. $\text{Im}(N \to M)$ generates $M$ as an $A$-module (because $N \otimes_B A = M$),
6. $\text{Im}(N' \to M')$ generates $M'$ as an $A'$-module (because it holds modulo $K$ and $L$ maps isomorphically to $K$),
7. $JN' = L$ (because $L \cong K = IM'$ is generated by images of elements $xn'$ with $x \in I$ and $n' \in N'$ by the previous statement),
8. $N' \otimes_{B'} B = N$ (because $N = N'/L, B = B'/J$, and the previous statement),
there is a map \( \gamma : N' \otimes_{B'} A' \to M' \),

(10) \( \gamma \) is surjective (see above),

(11) the kernel of the composition \( N' \otimes_{B'} A' \to M' \to M \) is generated by elements \( l \otimes 1 \) and \( n' \otimes x \) with \( l \in K, n' \in N', x \in I \) (because \( M = N \otimes_B A \) by assumption and because \( N' \to N \) and \( A' \to A \) are surjective with kernels \( L \) and \( I \)),

(12) any element of \( N' \otimes_{B'} A' \) in the submodule generated by the elements \( l \otimes 1 \) and \( n' \otimes x \) with \( l \in L, n' \in N', x \in I \) can be written as \( l \otimes 1 \) for some \( l \in L \) (because \( J \) maps isomorphically to \( I \) we see that \( n' \otimes x = n'x \otimes 1 \) in \( N' \otimes_{B'} A' \); similarly \( x \sigma \otimes \sigma' = n' \otimes x \sigma' = n'(x \sigma') \otimes 1 \) in \( N' \otimes_{B'} A' \) when \( n' \in N', x \in J \) and \( \sigma' \in A' \); since we have seen that \( JN' = L \) this proves the assertion),

(13) the kernel of \( \gamma \) is zero (because by (10) and (11) any element of the kernel is of the form \( l \otimes 1 \) with \( l \in L \) which is mapped to \( l \in K \subset M' \) by \( \gamma \)).

This finishes the proof. \( \square \)

**Lemma 4.6.** In the situation of Lemma 4.5 for a \( B' \)-module \( L' \) the adjunction map

\[
L' \longrightarrow (L' \otimes_{B'} B) \times_{(L' \otimes_{B'} A')} (L' \otimes_{B'} A')
\]

is surjective but in general not injective.

**Proof.** As in the proof of Lemma 4.5 let \( J \subset B' \) be the kernel of the map \( B' \to B \). Then \( L' \otimes_{B'} B = L'/JL' \). Hence to prove surjectivity it suffices to show that elements of the form \((0, z)\) of the fibre product are in the image of the map of the lemma. The kernel of the map \( L' \otimes_{B'} A' \to L' \otimes_{B'} A \) is the image of \( L' \otimes_{B'} I \to L' \otimes_{B'} A' \). Since the map \( J \to I \) induced by \( B' \to A' \) is an isomorphism the composition

\[
L' \otimes_{B'} J \to L' \to (L' \otimes_{B'} B) \times_{(L' \otimes_{B'} A')} (L' \otimes_{B'} A')
\]

induces a surjection of \( L' \otimes_{B'} J \) onto the set of elements of the form \((0, z)\). To see the map is not injective in general we present a simple example. Namely, take a field \( k \), set \( B' = k[x, y]/(xy), A = B'/x, B = B'/y, A = B'/x, y \) and \( L = B'/x - y \). In that case the class of \( x \) in \( L' \) is nonzero but is mapped to zero under the displayed arrow. \( \square \)

**Lemma 4.7.** In Situation 4.4 let \( (N_1, M_1, \varphi_1) \to (N_2, M_2, \varphi_2) \) be a morphism of \( \text{Mod}_B \times \text{Mod}_A \) with \( N_1 \to N_2 \) and \( M_1' \to M_2' \) surjective. Then

\[
N_1 \times_{M_1} M_1' \to N_2 \times_{M_2} M_2'
\]

is surjective.

**Proof.** Pick \( (x_2, y_2) \in N_2 \times_{M_2} M_2' \). Choose \( x_1 \in N_1 \) mapping to \( x_2 \). Since \( M_1' \to M_1 \) is surjective we can find \( y_1 \in M_1' \) mapping to \( \varphi_1(x_1) \). Then \( (x_1, y_1) \) maps to \( (x_2, y_2') \) in \( N_2 \times_{M_2} M_2' \). Thus it suffices to show that elements of the form \((0, y_2)\) are in the image of the map. Here we see that \( y_2 \in IM_2' \). Write \( y_2 = \sum t_i y_{2,i} \) with \( t_i \in I \). Choose \( y_{1,i} \in M_1' \) mapping to \( y_{2,i} \). Then \( y_1 = \sum t_i y_{1,i} \in IM_1' \) and the element \((0, y_1)\) does the job. \( \square \)

**Situation 4.8.** Let \( A, A', B, B', I \) be as in Situation 4.4. Let \( B' \to D' \) be a ring map. Set \( D = D' \otimes_{B'} B, C' = D' \otimes_{B'} A' \), and \( C = D' \otimes_{B'} A \). This leads to a big
of rings. Observe that we do not assume that the map $D' \to D \times_C C'$ is an isomorphism. In this situation we have the functor

$$(4.8.1) \quad \text{Mod}_{D'} \to \text{Mod}_D \times_{\text{Mod}_C} \text{Mod}_{C'}, \quad L' \mapsto (L' \otimes_{D'} D, L' \otimes_{D'} C', \text{can})$$

analogous to $(4.4.1)$. Note that $L' \otimes_{D'} D = L \otimes_{D'} (D' \otimes_{B'} B) = L \otimes_{B'} B$ and similarly $L' \otimes_{D'} C' = L \otimes_{D'} (D' \otimes_{B'} A') = L \otimes_{B'} A'$ hence the diagram

is commutative. In the following we will write $(N, M, \varphi)$ for an object of $\text{Mod}_D \times_{\text{Mod}_C} \text{Mod}_{C'}$, i.e., $N$ is a $D$-module, $M$ is an $C'$-module and $\varphi : N \otimes_B A \to M' \otimes_{A'} A$ is an isomorphism of $C$-modules. However, it is often more convenient think of $\varphi$ as a $D$-linear map $\varphi : N \to M'/IM'$ which induces an isomorphism $N \otimes_B A \to M' \otimes_{A'} A = M'/IM'$.

**Lemma 4.9.** In Situation $(4.8)$ the functor $(4.8.1)$ has a right adjoint, namely the functor

$$F : (N, M', \varphi) \mapsto N \times_{\varphi, M} M'$$

where $M = M'/IM'$. Moreover, the composition of $F$ with $(4.8.1)$ is the identity functor on $\text{Mod}_D \times_{\text{Mod}_C} \text{Mod}_{C'}$. In other words, setting $N' = N \times_{\varphi, M} M'$ we have $N' \otimes_{D'} D = N$ and $N' \otimes_{D'} C' = M'$.

**Proof.** The adjointness statement is that for a $D'$-module $L'$ and a triple $(N, M', \varphi)$ we have

$$\text{Hom}_{D'}(L', N \times_{\varphi, M} M') = \text{Hom}_D(L' \otimes_{D'} D, N) \times_{\text{Hom}_C(L' \otimes_{D'} C, M)} \text{Hom}_{C'}(L' \otimes_{D'} C', M')$$

This follows from Algebra, Lemma $13.3$ and the fact that an element of the left hand side is given by a pair of $D'$-linear maps $L' \to N$ and $L' \to M'$ agreeing as maps to $M$. The final assertion follows from the corresponding assertion of Lemma 4.5.

**Lemma 4.10.** In Situation $(4.8)$ the map $J D' \to IC'$ is surjective where $J = \text{Ker}(B' \to B)$.

**Proof.** Since $C' = D' \otimes_{B'} A'$ we have that $IC'$ is the image of $D' \otimes_{B'} I = C' \otimes_{A'} I \to C'$. As the ring map $B' \to A'$ induces an isomorphism $J \to I$ the lemma follows.
Lemma 4.11. Let $A, A', B, B', C, C', D, D', I, M', M, N, \varphi$ be as in Lemma 4.9. If $N$ finite over $D$ and $M'$ finite over $C'$, then $N' = N \times_M M'$ is finite over $D'$.

Proof. We will use the results of Lemma 4.9 without further mention. Choose generators $x_1, \ldots, x_r$ of $N$ over $B$ and generators $y_1, \ldots, y_s$ of $M'$ over $A'$. Using that $N = N' \otimes_{D'} D$ and $D' \rightarrow D$ is surjective we can find $u_1, \ldots, u_r \in N'$ mapping to $x_1, \ldots, x_r$ in $N$. Using that $M' = N' \otimes_{D'} C'$ we can find $v_1, \ldots, v_t \in N'$ such that $y_j = \sum v_j \otimes e'_{ij}$ for some $e'_{ij} \in C'$. In particular we see that the images $\bar{v}_j$ of the $v_j$ generate $M'$ over $C'$. We claim that $u_1, \ldots, u_r, v_1, \ldots, v_t$ generate $N'$ as a $D'$-module. Namely, pick $\xi \in N'$. We first choose $d'_1, \ldots, d'_r \in D'$ such that $\xi$ and $\sum d'_i u_i$ map to the same element of $N$. This is possible because $D' \rightarrow D$ is surjective and $x_1, \ldots, x_r$ generate $N$. The difference $\xi - \sum d'_i u_i$ is of the form $(0, \theta)$ for some $\theta$ in $IM'$. Say $\theta = \sum t_j \bar{v}_j$ with $t_j \in IC'$. By Lemma 4.10 we can choose $s_j \in JD'$ mapping to $t_j$. Because $N' = N \times_M M'$ it follows that $\xi = \sum b'_i u_i + \sum s_j v_j$ as desired.


1. Let $(N, M', \varphi)$ be an object of $\text{Mod}_D \times_{\text{Mod}_C} \text{Mod}_{C'}$. If $M'$ is flat over $A'$ and $N'$ is flat over $B'$, then $N' = N \times_M M'$ is flat over $B'$.

2. If $L'$ is a $D'$-module flat over $B'$, then $L' = (L \otimes_{D'} D) \times_{(L \otimes_{D'} C)} (L \otimes_{D'} C')$.

3. The category of $D'$-modules flat over $B'$ is equivalent to the categories of objects $(N, M', \varphi)$ of $\text{Mod}_D \times_{\text{Mod}_C} \text{Mod}_{C'}$ with $N$ flat over $B$ and $M'$ flat over $A'$.

Proof. Proof of (1). Let $J \subseteq B'$ be an ideal. We have to show that $J \otimes_{B'} N' \rightarrow N'$ is injective, see Algebra, Lemma 38.4. We know that

$$J/(J \cap I) \otimes_{B'} N' = J/(J \cap I) \otimes_B N \rightarrow N$$

is injective as $N$ is flat over $B$. As $J \cap I \rightarrow J \rightarrow J/(J \cap I) \rightarrow 0$ is exact, we conclude that it suffices to show that $(J \cap I) \otimes_{B'} N' \rightarrow N'$ is injective. Thus we may assume that $J \subseteq I$; in particular we can think of $J$ as an $A'$-module and an ideal of $A'$ and

$$J \otimes_{B'} N' = J \otimes_{A'} A' \otimes_{B'} N' = J \otimes_{A'} M'$$

which maps injectively into $M'$ by our assumption that $M'$ is flat over $A'$. We conclude that $J \otimes_{B'} N' \rightarrow N'$ is injective and hence the first map is injective as desired.

Proof of (2). This follows by tensoring the short exact sequence $0 \rightarrow B' \rightarrow B \oplus A' \rightarrow A \rightarrow 0$ with $L'$ over $B'$ and using that $L' \otimes_{D'} D = L' \otimes_{B'} B, L' \otimes_{D'} C = L' \otimes_{B'} A'$, and $L' \otimes_{D'} C = L' \otimes_{B'} A$, see discussion in Situation 4.8.

Proof of (3). Immediate consequence of (1) and (2).


1. $N$ is finitely presented over $D$ and flat over $B$,

2. $M'$ finitely presented over $C'$ and flat over $A'$, and

3. the ring map $B' \rightarrow D'$ factors as $B' \rightarrow D'' \rightarrow D'$ with $B' \rightarrow D''$ flat and $D'' \rightarrow D'$ of finite presentation,

then $N' = N \times_M M'$ is finitely presented over $D'$. 


On the other hand, since $N$ and $K$ are finite, we conclude from Lemma 4.11 that the displayed map is an isomorphism. By Algebra, Lemma 5.3 the modules $N'$ and $D$ by $D'' \otimes_{B'} B$, etc. Since $D''$ is flat over $B'$, it follows that we may assume that $B' \to D'$ is flat.

Assume $B' \to D'$ is flat. By Lemma 4.11 the module $N'$ is finite over $D'$. Choose a surjection $(D')^{\oplus n} \to N'$ with kernel $K'$. By base change we obtain maps $D^{\oplus n} \to N$, $(C')^{\oplus n} \to M'$, and $C^{\oplus n} \to M$ with kernels $K_D$, $K_{C'}$, and $K_C$. There is a canonical map

$$K' \to K_D \times_{K_C} K_{C'}.$$

On the other hand, since $N' = N \times_M M'$ and $D' = D \times_C C'$ (by Lemma 4.12), there is also a canonical map $K_D \times_{K_C} K_{C'} \to K'$ inverse to the displayed arrow. Hence the displayed map is an isomorphism. By Algebra, Lemma 5.3 the modules $K_D$ and $K_{C'}$ are finite. We conclude from Lemma 4.11 that $K'$ is a finite $D'$-module provided that $K_D \to K_C$ and $K_{C'} \to K_C$ induce isomorphisms $K_D \otimes_B A = K_C = K_{C'} \otimes_A A$. This is true because the flatness assumptions implies the sequences

$$0 \to K_D \to D^{\oplus n} \to N \to 0 \quad \text{and} \quad 0 \to K_{C'} \to (C')^{\oplus n} \to M' \to 0$$

stay exact upon tensoring, see Algebra, Lemma 38.11.

**Lemma 4.14.** Let $A, A', B, B', I$ be as in Situation 4.4. Let $(D, C', \varphi)$ be a system consisting of an $A$-algebra $D$, an $A'$-algebra $C'$, and an isomorphism $D \otimes_B A \to C'/IC = C$. Set $D' = D \times_C C'$ (as in Lemma 4.5). Then

1. $B' \to D'$ is finite type if and only if $B \to D$ and $A' \to C'$ are finite type,
2. $B' \to D'$ is flat if and only if $B \to D$ and $A' \to C'$ are flat,
3. $B' \to D'$ is flat and of finite presentation if and only if $B \to D$ and $A' \to C'$ are flat and of finite presentation,
4. $B' \to D'$ is smooth if and only if $B \to D$ and $A' \to C'$ are smooth,
5. $B' \to D'$ is étale if and only if $B \to D$ and $A' \to C'$ are étale.

Moreover, if $D'$ is a flat $B'$-algebra, then $D' \to (D' \otimes_{B'} B) \times_{(D' \otimes_{B'} A)} (D' \otimes_{B'} A')$ is an isomorphism. In this way the category of flat $B'$-algebras is equivalent to the categories of systems $(D, C', \varphi)$ as above with $D$ flat over $B$ and $C'$ flat over $A'$.

**Proof.** The implication “$\Rightarrow$” follows from Algebra, Lemmas 13.2, 38.6, 133.3, and 139.3 because we have $D' \otimes_{B'} B = D$ and $D' \otimes_{B'} A' = C'$ by Lemma 4.5. Thus it suffices to prove the implications in the other direction.

Ad (1). Assume $D$ of finite type over $B$ and $C'$ of finite type over $A'$. We will use the results of Lemma 4.5 without further mention. Choose generators $x_1, \ldots, x_r$ of $D$ over $B$ and generators $y_1, \ldots, y_s$ of $C'$ over $A'$. Using that $N = N' \otimes_B B$ and $B' \to B$ is surjective we can find $v_1, \ldots, v_r \in D'$ mapping to $x_1, \ldots, x_r$ in $D$. Using that $C' = D' \otimes_{B'} A'$ we can find $v_1, \ldots, v_r \in D'$ such that $y_i = \sum v_j \otimes a_{ij}'$ for some $a_{ij}' \in A'$. In particular, the images of $v_j$ in $C'$ generate $C'$ as an $A'$-algebra.
Set $N = r + t$ and consider the cube of rings

\[
\begin{array}{ccc}
A[x_1, \ldots, x_N] & \xrightarrow{A} & A'[x_1, \ldots, x_N] \\
\uparrow & & \uparrow \\
B[x_1, \ldots, x_N] & \xrightarrow{B} & B'[x_1, \ldots, x_N] \\
\downarrow & & \downarrow \\
N & & N
\end{array}
\]

Observe that the back square is cartesian as well. Consider the ring map

\[
B'[x_1, \ldots, x_N] \to D', \quad x_i \mapsto u_i \quad \text{and} \quad x_{r+j} \mapsto v_j.
\]

Then we see that the induced maps $B[x_1, \ldots, x_N] \to D$ and $A'[x_1, \ldots, x_N] \to C'$ are surjective, in particular finite. We conclude from Lemma 4.11 that $B'[x_1, \ldots, x_N] \to D'$ is finite, which implies that $D'$ is of finite type over $B'$ for example by Algebra, Lemma 6.2.

Ad (2). The implication “$\Leftarrow$” follows from Lemma 4.12. Moreover, the final statement follows from the final statement of Lemma 4.12.

Ad (3). Assume $B \to D$ and $A' \to C'$ are flat and of finite presentation. The flatness of $B' \to D'$ we've seen in (2). We know $B' \to D'$ is of finite type by (1). Choose a surjection $B'[x_1, \ldots, x_N] \to D'$. By Algebra, Lemma 6.3 the ring $D$ is of finite presentation as a $B[x_1, \ldots, x_N]$-module and the ring $C'$ is of finite presentation as a $A'[x_1, \ldots, x_N]$-module. By Lemma 4.13 we see that $D'$ is of finite presentation as a $B'[x_1, \ldots, x_N]$-module, i.e., $B' \to D'$ is of finite presentation.

Ad (4). Assume $B \to D$ and $A' \to C'$ smooth. By (3) we see that $B' \to D'$ is flat and of finite presentation. By Algebra, Lemma 6.3 it suffices to check that $D' \otimes_B k$ is smooth for any field $k$ over $B'$. If the composition $J \to B' \to k$ is zero, then $B' \to k$ factors as $B' \to B \to k$ and we see that $D' \otimes_B k = D' \otimes_B B \otimes_B k = D \otimes_B k$ is smooth as $B \to D$ is smooth. If the composition $J \to B' \to k$ is nonzero, then there exists an $h \in J$ which does not map to zero in $k$. Then $B' \to k$ factors as $B' \to B_h' \to k$. Observe that $h$ maps to zero in $B$, hence $B_h = 0$. Thus by Lemma 4.3 we have $B_h' = A'_h$ and we get

\[
D' \otimes_B k = D' \otimes_B B_h' \otimes_B k = C_h' \otimes_A k
\]

is smooth as $A' \to C'$ is smooth.

Ad (5). Assume $B \to D$ and $A' \to C'$ are étale. By (4) we see that $B' \to D'$ is smooth. As we can read off whether or not a smooth map is étale from the dimension of fibres we see that (5) holds (argue as in the proof of (4) to identify fibres – some details omitted). □

Remark 4.15. In Situation 4.8 assume $B' \to D'$ is of finite presentation and suppose we are given a $D'$-module $L'$. We claim there is a bijective correspondence between
More on Algebra

12

12 MORE ON ALGEBRA

M

(1) surjections of $D'$-modules $L' \to Q'$ with $Q'$ of finite presentation over $D'$
and flat over $B'$, and

(2) pairs of surjections of modules $(L' \otimes_{D'} D \to Q_1, L' \otimes_{D'} C' \to Q_2)$ with
(a) $Q_1$ of finite presentation over $D$ and flat over $B$, 
(b) $Q_2$ of finite presentation over $C'$ and flat over $A'$,
(c) $Q_1 \otimes_D C = Q_2 \otimes_{C'} C$ as quotients of $L' \otimes_{D'} C$.

The correspondence between these is given by $Q \mapsto (Q_1, Q_2)$ with $Q_1 = Q \otimes_{D'} D$ and
$Q_2 = Q \otimes_{D'} C'$. And for the converse we use $Q = Q_1 \times_{Q_2} Q_2$ where $Q_2$ the
common quotient $Q_1 \otimes_D C = Q_2 \otimes_{C'} C$ of $L' \otimes_{D'} C$. As quotient map we use

$L' \rightarrow (L' \otimes_{D'} D) \times (L' \otimes_{D'} C') \rightarrow Q_1 \times_{Q_2} Q_2 = Q$

where the first arrow is surjective by Lemma 4.6 and the second by Lemma 4.7.
The claim follows by Lemmas 4.12 and 4.13.

5. Fitting ideals

The fitting ideals of a finite module are the ideals determined by the construction
of Lemma 5.2.

**Lemma 5.1.** Let $R$ be a ring. Let $A$ be an $n \times m$ matrix with coefficients in $R$.
Let $I_r(A)$ be the ideal generated by the $r \times r$-minors of $A$ with the convention that
$I_0(A) = R$ and $I_r(A) = 0$ if $r > \min(n, m)$. Then

(1) $I_0(A) \supset I_1(A) \supset I_2(A) \ldots$,
(2) if $B$ is an $(n + n') \times m$ matrix, and $A$ is the first $n$ rows of $B$, then
$I_{r+n'}(B) \subset I_r(A),$
(3) if $C$ is an $n \times n$ matrix then $I_r(CA) \subset I_r(A),$
(4) If $A$ is a block matrix

$$
\begin{pmatrix}
A_1 & 0 \\
0 & A_2
\end{pmatrix}
$$

then $I_r(A) = \sum_{r_1+r_2=r} I_{r_1}(A_1)I_{r_2}(A_2)$.
(5) Add more here.

**Proof.** Omitted. (Hint: Use that a determinant can be computed by expanding
along a column or a row.) \(\square\)

**Lemma 5.2.** Let $R$ be a ring. Let $M$ be a finite $R$-module. Choose a presentation

$$
\bigoplus_{j \in J} R \rightarrow R^{\oplus n} \rightarrow M \rightarrow 0.
$$

of $M$. Let $A = (a_{ij})_{i=1,\ldots,n,j \in J}$ be the matrix of the map $\bigoplus_{j \in J} R \rightarrow R^{\oplus n}$. The
ideal $\text{Fit}_k(M)$ generated by the $(n-k) \times (n-k)$ minors of $A$ is independent of the
choice of the presentation.

**Proof.** Let $K \subset R^{\oplus n}$ be the kernel of the surjection $R^{\oplus n} \rightarrow M$. Pick $z_1, \ldots, z_{n-k} \in K$ and write $z_j = (z_{1j}, \ldots, z_{nj})$. Another description of the ideal $\text{Fit}_k(M)$ is that
it is the ideal generated by the $(n-k) \times (n-k)$ minors of all the matrices $(z_{ij})$ we
obtain in this way.

Suppose we change the surjection into the surjection $R^{\oplus n+n'} \rightarrow M$ with kernel $K'$
where we use the original map on the first $n$ standard basis elements of $R^{\oplus n+n'}$
and 0 on the last $n'$ basis vectors. Then the corresponding ideals are the same.
Namely, if $z_1, \ldots, z_{n-k} \in K$ as above, let $z'_j = (z_{1j}, \ldots, z_{nj}, 0, \ldots, 0) \in K'$ for
Let $R^\oplus m \to M$ be another surjection with kernel $L$. By the previous paragraph we may assume $m = n$. By Algebra, Lemma 5.2 we can choose a map $R^\oplus n \to R^\oplus m$ commuting with the surjections to $M$. Let $C = (c_{ij})$ be the matrix of this map (it is a square matrix as $n = m$). Then given $z_1, \ldots, z_{n-k} \in K$ as above we get $Cz_1, \ldots, Cz_{n-k} \in L$. By Lemma 5.1 we get one of the inclusions. By symmetry we get the other.

**Definition 5.3.** Let $R$ be a ring. Let $M$ be a finite $R$-module. Let $k \geq 0$. The $k$th fitting ideal of $M$ is the ideal $\text{Fit}_k(M)$ constructed in Lemma 5.2. Set $\text{Fit}_{-1}(M) = 0$.

Since the fitting ideals are the ideals of minors of a big matrix (numbered in reverse ordering from the ordering in Lemma 5.1) we see that

$$0 = \text{Fit}_{-1}(M) \subset \text{Fit}_{0}(M) \subset \text{Fit}_{1}(M) \subset \ldots \subset \text{Fit}_{t}(M) = R$$

for some $t \gg 0$. Here are some basic properties of fitting ideals.

**Lemma 5.4.** Let $R$ be a ring. Let $M$ be a finite $R$-module.

1. If $M$ can be generated by $n$ elements, then $\text{Fit}_n(M) = R$.
2. Given a second finite $R$-module $M'$ we have
   $$\text{Fit}_k(M \oplus M') = \sum_{k+k' = t} \text{Fit}_k(M)\text{Fit}_{k'}(M')$$
3. If $R \to R'$ is a ring map, then $\text{Fit}_k(M \otimes_R R')$ is the ideal of $R'$ generated by the image of $\text{Fit}_k(M)$.
4. If $M$ is an $R$-module of finite presentation, then $\text{Fit}_k(M)$ is a finitely generated ideal.
5. If $M \to M'$ is a surjection, then $\text{Fit}_k(M) \subset \text{Fit}_k(M')$.
6. Add more here.

**Proof.** Part (1) follows from the fact that $I_0(A) = R$ in Lemma 5.1. Part (2) follows from the corresponding statement in Lemma 5.1. Part (3) follows from the fact that $\otimes_R R'$ is right exact, so the base change of a presentation of $M$ is a presentation of $M \otimes_R R'$. Proof of (4). Let $R^\oplus m \xrightarrow{A} R^\oplus n \to M \to 0$ be a presentation. Then $\text{Fit}_k(M)$ is the ideal generated by the $n-k \times n-k$ minors of the matrix $A$. Part (5) is immediate from the definition. □

**Example 5.5.** Let $R$ be a ring. The fitting ideals of the finite free module $M = R^\oplus n$ are $\text{Fit}_k(M) = 0$ for $k < n$ and $\text{Fit}_k(M) = R$ for $k \geq n$.

**Lemma 5.6.** Let $R$ be a ring. Let $M$ be a finite $R$-module. Let $k \geq 0$. Let $\mathfrak{p}$ be a prime ideal with $\text{Fit}_k(M) \not\subset \mathfrak{p}$. Then there exists an $f \in R, f \not\in \mathfrak{p}$ such that $M_f$ can be generated by $k$ elements over $R_f$. 
Proof. By Nakayama’s lemma (Algebra, Lemma \[19.1\]) we see that \( M \) can be generated by \( k \) elements over \( R \) for some \( f \in R, f \not\in p \) if \( M \otimes_R \kappa(p) \) can be generated by \( k \) elements. This reduces the problem to the case where \( R \) is a field and \( p = (0) \). In this case the result follows from Example \[5.5\].

\[\square\]

Lemma 5.7. Let \( R \) be a ring. Let \( M \) be a finite \( R \)-module. Let \( r \geq 0 \). The following are equivalent

1. \( M \) is finite locally free of rank \( k \) (Algebra, Definition \[76.1\]),
2. \( \text{Fit}_{r-1}(M) = 0 \) and \( \text{Fit}_r(M) = R \), and
3. \( \text{Fit}_k(M) = 0 \) for \( k < r \) and \( \text{Fit}_k(M) = R \) for \( k \geq r \).

Proof. It is immediate that (2) is equivalent to (3) because the fitting ideals form an increasing sequence of ideals. Since the formation of \( \text{Fit}_k(M) \) commutes with base change (Lemma \[5.4\]) we see that (1) implies (2) by Example \[5.5\] and glueing results (Algebra, Section \[23\]). Conversely, assume (2). By Lemma \[5.6\] we may assume that \( M \) is generated by \( r \) elements. Thus a presentation \( \bigoplus_{j \in J} R \to R^r \to M \to 0 \). But now the assumption that \( \text{Fit}_{r-1}(M) = 0 \) implies that all entries of the matrix of the map \( \bigoplus_{j \in J} R \to R^r \) are zero. Thus \( M \) is free.

\[\square\]

6. Lifting

In this section we collection some lemmas concerning lifting statements of the following kind: If \( A \) is a ring and \( I \subset A \) is an ideal, and \( \xi \) is some kind of structure over \( A/I \), then we can lift \( \xi \) to a similar kind of structure \( \xi' \) over \( A \) or over some étale extension of \( A \). Here are some types of structure for which we have already proved some results:

1. idempotents, see Algebra, Lemmas \[31.5\] and \[31.6\]
2. projective modules, see Algebra, Lemma \[75.4\]
3. basis elements, see Algebra, Lemmas \[98.1\] and \[98.3\]
4. ring maps, i.e., proving certain algebras are formally smooth, see Algebra, Lemma \[134.4\] Proposition \[134.13\] and Lemma \[134.16\]
5. syntonic ring maps, see Algebra, Lemma \[132.18\]
6. smooth ring maps, see Algebra, Lemma \[133.19\]
7. étale ring maps, see Algebra, Lemma \[139.11\]
8. factoring polynomials, see Algebra, Lemma \[139.20\]
9. Algebra, Section \[146\] discusses henselian local rings.

The interested reader will find more results of this nature in Smoothing Ring Maps, Section \[4\] in particular Smoothing Ring Maps, Proposition \[4.2\].

Let \( A \) be a ring and let \( I \subset A \) be an ideal. Let \( \xi \) be some kind of structure over \( A/I \). In the following lemmas we look for étale ring maps \( A \to A' \) which induce isomorphisms \( A/I \to A'/IA' \) and objects \( \xi' \) over \( A' \) lifting \( \xi \). A general remark is that given étale ring maps \( A \to A' \to A'' \) such that \( A/I \cong A'/IA' \) and \( A'/IA' \cong A''/IA'' \) the composition \( A \to A'' \) is also étale (Algebra, Lemma \[139.3\]) and also satisfies \( A/I \cong A''/IA'' \). We will frequently use this in the following lemmas without further mention. Here is a trivial example of the type of result we are looking for.

Lemma 6.1. Let \( A \) be a ring, let \( I \subset A \) be an ideal, let \( \pi \in A/I \) be an invertible element. There exists an étale ring map \( A \to A' \) which induces an isomorphism \( A/I \to A'/IA' \) and an invertible element \( u' \in A' \) lifting \( \pi \).
Proof. Choose any lift \( f \in A \) of \( \overline{x} \) and set \( A' = A_f \) and \( x \) the image of \( f \) in \( A' \). □

Lemma 6.2. Let \( A \) be a ring, let \( I \subset A \) be an ideal, let \( \overline{e} \in A/I \) be an idempotent. There exists an étale ring map \( A \to A' \) which induces an isomorphism \( A/I \to A'/IA' \) and an idempotent \( e' \in A' \) lifting \( \overline{e} \).

Proof. Choose any lift \( x \in A \) of \( \overline{e} \). Set

\[
A' = A[t]/(t^2 - t) \left[ \frac{1}{t - 1 + x} \right].
\]

The ring map \( A \to A' \) is étale because \((2t - 1)dt = 0\) and \((2t - 1)(2t - 1) = 1\) which is invertible. We have \( A'/IA' = A[I][t]/(t^2 - t)[\frac{1}{t - 1 + x}] \cong A/I \) the last map sending \( t \) to \( \overline{e} \) which works as \( \overline{e} \) is a root of \( t^2 - t \). This also shows that setting \( e' \) equal to the class of \( t \) in \( A' \) works.

Lemma 6.3. Let \( A \) be a ring, let \( I \subset A \) be an ideal. Let \( \text{Spec}(A/I) = \coprod_{j \in J} U_j \) be a finite disjoint open covering. Then there exists an étale ring map \( A \to A' \) which induces an isomorphism \( A/I \to A'/IA' \) and a finite disjoint open covering \( \text{Spec}(A') = \coprod_{j \in J} U'_j \) lifting the given covering.

Proof. This follows from Lemma 6.2 and the fact that open and closed subsets of Spectra correspond to idempotents, see Algebra, Lemma 20.3. □

Lemma 6.4. Let \( A \to B \) be a ring map and \( J \subset B \) an ideal. If \( A \to B \) is étale at every prime of \( V(J) \), then there exists a \( g \in B \) mapping to an invertible element of \( B/J \) such that \( A' = B_g \) is étale over \( A \).

Proof. The set of points of \( \text{Spec}(B) \) where \( A \to B \) is not étale is a closed subset of \( \text{Spec}(B) \), see Algebra, Definition 139.1. Write this as \( V(J') \) for some ideal \( J' \subset B \). Then \( V(J') \cap V(J) = \emptyset \) hence \( J + J' = B \) by Algebra, Lemma 16.2. Write \( 1 = f + g \) with \( f \in J \) and \( g \in J' \). Then \( g \) works. □

Next we have three lemmas saying we can lift factorizations of polynomials.

Lemma 6.5. Let \( A \) be a ring, let \( I \subset A \) be an ideal. Let \( f \in A[x] \) be a monic polynomial. Let \( \overline{f} = \overline{g} \overline{h} \) be a factorization of \( f \) in \( A/I[x] \) such that \( \overline{g} \) and \( \overline{h} \) are monic and generate the unit ideal in \( A/I[x] \). Then there exists an étale ring map \( A \to A' \) which induces an isomorphism \( A/I \to A'/IA' \) and a factorization \( f = g'h' \) in \( A'[x] \) with \( g', h' \) monic lifting the given factorization over \( A/I \).

Proof. Say \( \deg(\overline{g}) = n \) and \( \deg(\overline{h}) = m \) so that \( \deg(f) = n + m \). Write \( f = x^{n+m} + \sum \alpha_i x^{n+m-i} \) for some \( \alpha_1, \ldots, \alpha_{n+m} \in A \). Consider the ring map

\[
R = \mathbb{Z}[a_1, \ldots, a_{n+m}] \to S = \mathbb{Z}[b_1, \ldots, b_n, c_1, \ldots, c_m]
\]

of Algebra, Example 139.13 Let \( R \to A \) be the ring map which sends \( a_i \) to \( \alpha_i \). Set

\[
B = A \otimes_R S
\]

By construction the image of \( f \) in \( B[x] \) factors. Write \( \overline{g} = x^n + \sum \overline{\beta}_i x^{n-i} \) and \( \overline{h} = x^m + \sum \overline{\gamma}_i x^{m-i} \). The \( A \)-algebra map

\[
B \to A/I, \quad 1 \otimes b_i \mapsto \overline{\beta}_i, \quad 1 \otimes c_i \mapsto \overline{\gamma}_i
\]

maps the factorization of \( f \) over \( B \) to the given factorization over \( A/I \). The displayed map is surjective; denote \( J \subset B \) its kernel. From the discussion in Algebra, Example 139.13 it is clear that \( A \to B \) is étale at all points of \( V(J) \subset \text{Spec}(B) \). Choose \( g \in B \) as in Lemma 6.4 and set \( A' = B_g \). □
The assumption on the leading coefficient in the following lemma will be removed in Lemma \[6.7\]

**Lemma 6.6.** Let \( A \) be a ring, let \( I \subset A \) be an ideal. Let \( f \in A[x] \) be a monic polynomial. Let \( \mathcal{F} = \pi \mathcal{H} \) be a factorization of \( f \) in \( A/I[x] \) and assume

1. the leading coefficient of \( \pi \) is an invertible element of \( A/I \), and
2. \( \pi, \mathcal{H} \) generate the unit ideal in \( A/I[x] \).

Then there exists an étale ring map \( A \to A' \) which induces an isomorphism \( A/I \to A'/IA' \) and a factorization \( f = g'h' \) in \( A'[x] \) lifting the given factorization over \( A/I \).

**Proof.** Applying Lemma \[6.1\] we may assume that the leading coefficient of \( \pi \) is the reduction of an invertible element \( u \in A \). Then we may replace \( \pi \) by \( \pi^{-1} \pi \) and \( \mathcal{H} \) by \( \mathcal{H} \). Thus we may assume that \( \pi \) is monic. Since \( f \) is monic we conclude that \( \pi \) is monic too. In this case the result follows from Lemma \[6.5\].

**Lemma 6.7.** Let \( A \) be a ring, let \( I \subset A \) be an ideal. Let \( f \in A[x] \) be a monic polynomial. Let \( \mathcal{F} = \pi \mathcal{H} \) be a factorization of \( f \) in \( A/I[x] \) and assume that \( \pi, \mathcal{H} \) generate the unit ideal in \( A/I[x] \). Then there exists an étale ring map \( A \to A' \) which induces an isomorphism \( A/I \to A'/IA' \) and a factorization \( f = g'h' \) in \( A'[x] \) lifting the given factorization over \( A/I \).

**Proof.** Say \( f = x^d + a_1 x^{d-1} + \ldots + a_d \) has degree \( d \). Write \( \pi = \sum b_j x^j \) and \( \mathcal{H} = \sum c_j x^j \). Then we see that \( 1 = \sum b_j c_{d-j} \). It follows that \( \text{Spec}(A/I) \) is covered by the standard opens \( D(b_j c_{d-j}) \). However, each point \( p \) of \( \text{Spec}(A/I) \) is contained in at most one of these as by looking at the induced factorization of \( f \) over the field \( \kappa(p) \) we see that \( \deg(\pi \mod p) + \deg(\mathcal{H} \mod p) = d \). Hence our open covering is a disjoint open covering. Applying Lemma \[6.3\] (and replacing \( A \) by \( A' \)) we see that we may assume there is a corresponding disjoint open covering of \( \text{Spec}(A) \). This disjoint open covering corresponds to a product decomposition of \( A \), see Algebra, Lemma \[22.3\]. It follows that

\[
A = A_0 \times \ldots \times A_d, \quad I = I_0 \times \ldots \times I_d,
\]

where the image of \( \pi \), resp. \( \mathcal{H} \) in \( A_j/I_j \) has degree \( j \), resp. \( d-j \) with invertible leading coefficient. Clearly, it suffices to prove the result for each factor \( A_j \) separately. Hence the lemma follows from Lemma \[6.6\].

**Lemma 6.8.** Let \( R \to S \) be a ring map. Let \( I \subset R \) be an ideal of \( R \) and let \( J \subset S \) be an ideal of \( S \). If the closure of the image of \( V(J) \) in \( \text{Spec}(R) \) is disjoint from \( V(I) \), then there exists an element \( f \in R \) which maps to 1 in \( R/I \) and to an element of \( J \) in \( S \).

**Proof.** Let \( I' \subset R \) be an ideal such that \( V(I') \) is the closure of the image of \( V(J) \). Then \( V(I) \cap V(I') = \emptyset \) by assumption and hence \( I + I' = R \) by Algebra, Lemma \[16.2\]. Write \( 1 = g + f \) with \( g \in I \) and \( f \in I' \). We have \( V(f') = V(J) \) where \( f' \) is the image of \( f \) in \( S \). Hence \( (f')^n \in J \) for some \( n \), see Algebra, Lemma \[16.2\]. Replacing \( f \) by \( f^n \) we win.

**Lemma 6.9.** Let \( A \) be a ring, let \( I \subset A \) be an ideal. Let \( A \to B \) be an integral ring map. Let \( \pi \in B/IB \) be an idempotent. Then there exists an étale ring map \( A \to A' \) which induces an isomorphism \( A/I \to A'/IA' \) and an idempotent \( e' \in B \otimes_A A' \) lifting \( \pi \).
Proof. Choose an element \( y \in B \) lifting \( \varpi \). Then \( z = y^2 - y \) is an element of \( IB \).

By Algebra, Lemma \[37.4\] there exist a monic polynomial \( g(x) = x^d + \sum a_j x^j \) of degree \( d \) with \( a_j \in I \) such that \( g(z) = 0 \) in \( B \). Hence \( f(x) = g(x^2 - x) \in A[x] \) is a monic polynomial such that \( f(x) \equiv x^d(x - 1)^d \mod I \) and such that \( f(y) = 0 \) in \( B \). By Lemma \[6.6\] we can find an étale ring map \( A \to A' \) which induces an isomorphism \( A/I \to A'/IA' \) and such that \( f = gh \) in \( A[x] \) with \( g(x) = x^d \mod IA' \) and \( h(x) = (x - 1)^d \mod IA' \). After replacing \( A \) by \( A' \) we may assume that the factorization is defined over \( A \). In that case we see that \( b_1 = g(y) \in B \) is a lift of \( \varpi^d = \varpi \) and \( b_2 = h(y) \in B \) is a lift of \( (\varpi - 1)^d = (-1)^d(1 - \varpi)^d = (-1)^d(1 - \varpi) \) and moreover \( b_1b_2 = 0 \). Thus \( (b_1, b_2)B/IB = B/IB \) and \( V(b_1, b_2) \subset \text{Spec}(B) \) is disjoint from \( V(IB) \). Since \( \text{Spec}(B) \to \text{Spec}(A) \) is closed (see Algebra, Lemmas \[35.20\] and \[40.6\]) we can find an \( a \in A \) which maps to an invertible element of \( A/I \) whose image in \( B \) lies in \( (b_1, b_2) \), see Lemma \[6.8\] After replacing \( A \) by the localization \( A_a \) we get that \( (b_1, b_2) = B \). Then \( \text{Spec}(B) = D(b_1) \cap D(b_2) \); disjoint union because \( b_1b_2 = 0 \). Let \( e \in B \) be the idempotent corresponding to the open and closed subset \( D(b_1) \), see Algebra, Lemma \[20.3\]. Since \( b_1 \) is a lift of \( \varpi \) and \( b_2 \) is a lift of \( \pm(1 - \varpi) \) we conclude that \( e \) is a lift of \( \varpi \) by the uniqueness statement in Algebra, Lemma \[20.3\]. □

Lemma 6.10. Let \( A \) be a ring, let \( I \subset A \) be an ideal. Let \( \overline{P} \) be finite projective \( A/I \)-module. Then there exists an étale ring map \( A \to A' \) which induces an isomorphism \( A/I \to A'/IA' \) and a finite projective \( A' \)-module \( P' \) lifting \( \overline{P} \).

Proof. We can choose an integer \( n \) and a direct sum decomposition \( (A/I) \oplus^n = \overline{P} \oplus K \) for some \( R/I \)-module \( K \). Choose a lift \( \varphi : A \oplus^n \to A' \) of the projector \( \overline{p} \) associated to the direct summand \( \overline{P} \). Let \( f \in A[x] \) be the characteristic polynomial of \( \varphi \). Set \( B = A[x]/(f) \). By Cayley-Hamilton (Algebra, Lemma \[15.1\]) there is a map \( B \to \text{End}_A(A \oplus^n) \) mapping \( x \) to \( \varphi \). For every prime \( p \supset I \) the image of \( f \) in \( \kappa(p) \) is \( (x - 1)^{x^n} \) where \( r \) is the dimension of \( \overline{P} \otimes_{A/I} \kappa(p) \). Hence \( (x - 1)^{x^n} \) maps to zero in \( B \otimes_A \kappa(p) \) for all \( p \supset I \). Hence the image of \( (x - 1)^{x^n} \) in \( B \) is contained in
\[
\bigcup_{p \not\supset I} pB = \bigcup_{p \not\supset I} p \bigcap \bigcap_{p \not\supset I} B = \sqrt{IB}
\]
the first equality because \( B \) is a free \( A \)-module and the second by Algebra, Lemma \[16.2\] Thus \( (x - 1)^{x^n} \) is contained in \( IB \) for some \( N \). It follows that \( x^N + (1 - x)^N \) is a unit in \( B/IB \) and that
\[
\varpi = \text{image of } \frac{x^N}{x^N + (1 - x)^N} \text{ in } B/IB
\]
is an idempotent as both assertions hold in \( Z[x]/(x^n(x - 1)^N) \). The image of \( \varpi \) in \( \text{End}_{A/I}(A \oplus^n) \) is
\[
\overline{p}^N + (1 - \overline{p})^N = \overline{p}
\]
as \( \overline{p} \) is an idempotent. After replacing \( A \) by an étale extension \( A' \) as in the lemma, we may assume there exists an idempotent \( e \in B \) which maps to \( \varpi \) in \( B/IB \), see Lemma \[6.9\] Then the image of \( e \) under the map
\[
B = A[x]/(f) \longrightarrow \text{End}_A(A \oplus^n).
\]
is an idempotent element \( p \) which lifts \( \overline{p} \). Setting \( P = \text{Im}(p) \) we win. □
Lemma 6.11. Let $A$ be a ring. Let $0 \to K \to A^{\oplus m} \to M \to 0$ be a sequence of $A$-modules. Consider the $A$-algebra $C = \text{Sym}_A^d(M)$ with its presentation $\alpha : A[y_1, \ldots, y_m] \to C$ coming from the surjection $A^{\oplus m} \to M$. Then

$$NL(\alpha) = (K \otimes A C \to \bigoplus_{j=1,\ldots,m} C d y_j)$$

(see Algebra, Section 130) in particular $\Omega_{C/A} = M \otimes_A C$.

Proof. Let $J = \text{Ker}(\alpha)$. The lemma asserts that $J/J^2 \cong K \otimes_A C$. Note that $\alpha$ is a homomorphism of graded algebras. We will prove that in degree $d$ we have $(J/J^2)_d = K \otimes_A C_{d-1}$. Note that

$$J_d = \text{Ker}(\text{Sym}_A^d(A^{\oplus m}) \to \text{Sym}_A^d(M)) = \text{Im}(K \otimes_A \text{Sym}_A^{d-1}(A^{\oplus m}) \to \text{Sym}_A^d(A^{\oplus m})),\text{ see Algebra, Lemma 12.2}.$$ It follows that $(J^2)_d = \sum_{a+b=d} J_a \cdot J_b$ is the image of

$$K \otimes_A K \otimes_A \text{Sym}_A^{d-2}(A^{\oplus m}) \to \text{Sym}_A^d(A^{\oplus m}).$$

The cokernel of the map $K \otimes_A \text{Sym}_A^{d-2}(A^{\oplus m}) \to \text{Sym}_A^{d-1}(A^{\oplus m})$ is $\text{Sym}_A^{d-1}(M)$ by the lemma referenced above. Hence it is clear that $(J/J^2)_d = J_d/(J^2)_d$ is equal to

$$\text{Coker}(K \otimes_A K \otimes_A \text{Sym}_A^{d-2}(A^{\oplus m}) \to K \otimes_A \text{Sym}_A^{d-1}(A^{\oplus m})) = K \otimes_A \text{Sym}_A^{d-1}(M) = K \otimes_A C_{d-1}$$

as desired. \hfill \Box

Lemma 6.12. Let $A$ be a ring. Let $M$ be an $A$-module. Then $C = \text{Sym}_A^d(M)$ is smooth over $A$ if and only if $M$ is a finite projective $A$-module.

Proof. Let $\sigma : C \to A$ be the projection onto the degree 0 part of $C$. Then $J = \text{Ker}(\sigma)$ is the part of degree $> 0$ and we see that $J/J^2 = M$ as an $A$-module. Hence if $A \to C$ is smooth then $M$ is a finite projective $A$-module by Algebra, Lemma 135.4.

Conversely, assume that $M$ is finite projective and choose a surjection $A^{\oplus n} \to M$ with kernel $K$. Of course the sequence $0 \to K \to A^{\oplus n} \to M \to 0$ is split as $M$ is projective. In particular we see that $K$ is a finite $A$-module and hence $C$ is of finite presentation over $A$ as $C$ is a quotient of $A[x_1, \ldots, x_n]$ by the ideal generated by $K \subset \bigoplus A x_i$. The computation of Lemma 6.11 shows that $N_{C/A}$ is homotopy equivalent to $(K \to M) \otimes_A C$. Hence $N_{C/A}$ is quasi-isomorphic to $C \otimes_A M$ placed in degree 0 which means that $C$ is smooth over $A$ by Algebra, Definition 133.1. \hfill \Box

Lemma 6.13. Let $A$ be a ring, let $I \subset A$ be an ideal. Consider a commutative diagram

$$\begin{array}{ccc}
B & \longrightarrow & A/I \\
\downarrow & & \downarrow \\
A & \longrightarrow & A/I
\end{array}$$

where $B$ is a smooth $A$-algebra. Then there exists an étale ring map $A \to A'$ which induces an isomorphism $A/I \to A'/IA'$ and an $A$-algebra map $B \to A'$ lifting the ring map $B \to A/I$. 

\hfill \Box
In this section a pair \((A, I)\) where \(A\) is a ring and \(I \subset A\) is an ideal. A morphism of pairs \((A, I) \to (B, J)\) is a ring map \(\varphi : A \to B\) with \(\varphi (I) \subset J\). As in Section \[\text{6}\] given an object \(\xi\) over \(A\) we denote \(\bar{\xi}\) the “base change” of \(\xi\) to an object over \(A/I\) (provided this makes sense).

**Definition 7.1.** A henselian pair is a pair \((A, I)\) satisfying

1. \(I\) is contained in the Jacobson radical of \(A\), and

**Proof.** Let \(J \subset B\) be the kernel of \(B \to A/I\) so that \(B/J = A/I\). By Algebra, Lemma \[\text{135.3}\] the sequence

\[0 \to I/I^2 \to J/J^2 \to \Omega_{B/A} \otimes_B B/J \to 0\]

is split exact. Thus \(\mathcal{P} = J/(J^2 + IB) = \Omega_{B/A} \otimes_B B/J\) is a finite projective \(A/I\)-module. Choose an integer \(n\) and a direct sum decomposition \(A/I^\oplus n = \mathcal{P} \oplus K\). By Lemma \[\text{13.10}\] we can find an étale ring map \(A \to A'\) which induces an isomorphism \(A/I \to A'/IA'\) and a finite projective \(A\)-module \(K\) which lifts \(K\). We may and do replace \(A\) by \(A'\). Set \(B' = B \otimes_A \Sym^n_K(K)\). Since \(A \to \Sym^n_K(K)\) is smooth by Lemma \[\text{0.12}\] we see that \(B \to B'\) is smooth which in turn implies that \(A \to B'\) is smooth (see Algebra, Lemmas \[\text{133.4}\] and \[\text{133.13}\]). Moreover the section \(\Sym^n_K(K) \to A\) determines a section \(B' \to B\) and we let \(B' \to A/I\) be the composition \(B' \to B \to A/I\). Let \(J' \subset B'\) be the kernel of \(B' \to A/I\). We have \(JB' \subset J'\) and \(B \otimes_A K \subset J'\). These maps combine to give an isomorphism

\[(A/I)^\oplus n \cong J/J^2 \oplus K \to J'/(J')^2 + IB'\]

Thus, after replacing \(B\) by \(B'\) we may assume that \(J/(J^2 + IB) = \Omega_{B/A} \otimes_B B/J\) is a free \(A/I\)-module of rank \(n\).

In this case, choose \(f_1, \ldots, f_n \in J\) which map to a basis of \(J/(J^2 + IB)\). Consider the finitely presented \(A\)-algebra \(C = B/(f_1, \ldots, f_n)\). Note that we have an exact sequence

\[0 \to H_1(L_{C/A}) \to (f_1, \ldots, f_n)/(f_1, \ldots, f_n)^2 \to \Omega_{B/A} \otimes_B C \to \Omega_{C/A} \to 0\]

see Algebra, Lemma \[\text{130.4}\] (note that \(H_1(L_{B/A}) = 0\) and that \(\Omega_{B/A}\) is finite projective, in particular flat so the Tor group vanishes). For any prime \(\mathfrak{q} \supset J\) of \(B\) the module \(\Omega_{B/A, \mathfrak{q}}\) is free of rank \(n\) because \(\Omega_{B/A}\) is finite projective and because \(\Omega_{B/A} \otimes_B B/J\) is free of rank \(n\). By our choice of \(f_1, \ldots, f_n\) the map

\[\left( (f_1, \ldots, f_n)/(f_1, \ldots, f_n)^2 \right)_\mathfrak{q} \to \Omega_{B/A, \mathfrak{q}}\]

is surjective modulo \(I\). Hence we see that this map of modules over the local ring \(C_\mathfrak{q}\) has to be an isomorphism. Thus \(H_1(L_{C/A})_\mathfrak{q} = 0\) and \(\Omega_{C/A, \mathfrak{q}} = 0\). By Algebra, Lemma \[\text{133.12}\] we see that \(A \to C\) is smooth at the prime \(\mathfrak{q}\) of \(C\) corresponding to \(\mathfrak{q}\). Since \(\Omega_{C/A, \mathfrak{q}} = 0\) it is actually étale at \(\mathfrak{q}\). Thus \(A \to C\) is étale at all primes of \(C\) containing \(JC\). By Lemma \[\text{6.4}\] we can find an \(f \in C\) mapping to an invertible element of \(CJC\) such that \(A \to C_f\) is étale. By our choice of \(f\) it is still true that \(C_f/JC_f = A/I\). The map \(C_f/IC_f \to A/I\) is surjective and étale by Algebra, Lemma \[\text{139.9}\] Hence \(A/I\) is isomorphic to the localization of \(C_f/IC_f\) at some element \(g \in C\), see Algebra, Lemma \[\text{139.10}\]. Set \(A' = C_{fg}\) to conclude the proof. \(\square\)

**7. Henselian pairs**

Some of the results of Section \[\text{6}\] may be viewed as results about henselian pairs. In this section a pair \((A, I)\) is a pair \((A, I)\) of rings and \(I \subset A\) is an ideal. A morphism of pairs \((A, I) \to (B, J)\) is a ring map \(\varphi : A \to B\) with \(\varphi (I) \subset J\). As in Section \[\text{6}\] given an object \(\xi\) over \(A\) we denote \(\bar{\xi}\) the “base change” of \(\xi\) to an object over \(A/I\) (provided this makes sense).

**Definition 7.1.** A henselian pair is a pair \((A, I)\) satisfying

1. \(I\) is contained in the Jacobson radical of \(A\), and
(2) for any monic polynomial $f \in A[T]$ and factorization $\overline{f} = g_0h_0$ with $g_0, h_0 \in A/I[T]$ monic generating the unit ideal in $A/I[T]$, there exists a factorization $f = gh$ in $A[T]$ with $g, h$ monic and $g_0 = \overline{g}$ and $h_0 = \overline{h}$.

Observe that if $A$ is a local ring and $I = \mathfrak{m}$ is the maximal ideal, then $(A, I)$ is a henselian pair if and only if $A$ is a henselian local ring, see Algebra, Lemma 146.3. In Lemma 7.7 we give a number of equivalent characterizations of henselian pairs (and we will add more as time goes on).

**Lemma 7.2.** Let $(A, I)$ be a pair with $I$ locally nilpotent. Then the functor $B \mapsto B/IB$ induces an equivalence between the category of étale algebras over $A$ and the category of étale algebras over $A/I$. Moreover, the pair is henselian.

**Proof.** Essential surjectivity holds by Algebra, Lemma 139.11. If $B, B'$ are étale over $A$ and $B/IB \to B'/IB'$ is a morphism of $A/I$-algebras, then we can lift this by Algebra, Lemma 134.16. Finally, suppose that $f, g : B \to B'$ are two $A$-algebra maps with $f \mod I = g \mod I$. Choose an idempotent $e \in B \otimes_A B$ generating the kernel of the multiplication map $B \otimes_A B \to B$, see Algebra, Lemmas 145.4 and 145.3 (to see that étale is unramified). Then $(f \otimes g)(e) \in IB$. Since $IB$ is locally nilpotent (Algebra, Lemma 31.2) this implies $(f \otimes g)(e) = 0$ by Algebra, Lemma 31.5. Thus $f = g$.

It is clear that $I$ is contained in the radical of $A$. Let $f \in A[T]$ be a monic polynomial and let $\overline{f} = g_0h_0$ be a factorization of $\overline{f} = f \mod I$ with $g_0, h_0 \in A/I[T]$ monic generating the unit ideal in $A/I[T]$. By Lemma 6.5 there exists an étale ring map $A \to A'$ which induces an isomorphism $A/I \to A'/IA'$ such that the factorization lifts to a factorization into monic polynomials over $A'$. By the above we have $A = A'$ and the factorization is over $A$.

**Lemma 7.3.** Let $(A, I)$ be a pair. If $A$ is $I$-adically complete, then the pair is henselian.

**Proof.** By Algebra, Lemma 94.11 the ideal $I$ is contained in the radical of $A$. Let $f \in A[T]$ be a monic polynomial and let $\overline{f} = g_0h_0$ be a factorization of $\overline{f} = f \mod I$ with $g_0, h_0 \in A/I[T]$ monic generating the unit ideal in $A/I[T]$. By Lemma 7.2 we can successively lift this factorization to $f \mod I^n = g_nh_n$ with $g_n, h_n$ monic in $A/I^n[T]$ for all $n \geq 1$. As $A = \varprojlim A/I^n$ this finishes the proof.

**Lemma 7.4.** Let $(A, I)$ be a pair. If $I$ is contained in the Jacobson radical of $A$, then the map from idempotents of $A$ to idempotents of $A/I$ is injective.

**Proof.** An idempotent of a local ring is either 0 or 1. Thus an idempotent is determined by the set of maximal ideals where it vanishes, by Algebra, Lemma 23.1.

**Lemma 7.5.** Let $(A, I)$ be a pair. Let $A \to B$ be an integral ring map such that $B/IB = C_1 \times C_2$ as $A/I$-algebra with $A/I \to C_1$ injective. Any element $b \in B$ mapping to $(0, 1)$ in $B/IB$ is the zero of a monic polynomial $f \in A[T]$ with $f \mod I = gT^n$ and $g(0)$ a unit in $A/I$.

**Proof.** Let $b \in B$ map to $(0, 1)$ in $C_1 \times C_2$. Let $J \subset A[T]$ be the kernel of the map $A[T] \to B$, $T \mapsto b$. Since $B$ is integral over $A$, it is integral over $A[T]$. Hence the image of $\text{Spec}(B)$ in $\text{Spec}(A[T])$ is closed by Algebra, Lemmas 40.6 and 35.20. Hence this image is equal to $V(J) = \text{Spec}(A[T]/J)$ by Algebra, Lemma 29.5. Intersecting
with the inverse image of $V(I)$ our choice of $b$ shows we have $V(J + IA[T]) \subset V(T^2 - T)$. Hence there exists an $n \geq 1$ and $g \in J$ with $g \mod IA[T] = (T^2 - T)^n$. On the other hand, as $A \to B$ is integral there exists a monic polynomial $h \in J$. Note that $h(0) \mod I$ maps to zero under the composition $A[T] \to B \to B/IB \to C_1$. Since $A/I \to C_1$ is injective we conclude $h \mod IA[T] = h_0T$ for some $h_0 \in A/I[T]$. Set 

$$f = g + h^m$$

for $m > n$. If $m$ is large enough, this is a monic polynomial and 

$$f \mod IA[T] = (T^2 - T)^n + h_0^mT^m = T^n((T - 1)^n + h_0^mT^{m-n})$$

and hence the desired conclusion. 

\[ \square \]

**Lemma 7.6.** Let $(A, I)$ be a pair. Let $A \to B$ be a finite type ring map such that $B/IB = C_1 \times C_2$ with $A/I \to C_1$ finite. Let $B'$ be the integral closure of $A$ in $B$. Then we can write $B'/IB' = C_1 \times C_2'$ such that the map $B'/IB' \to B/IB$ preserves product decompositions and there exists a $g \in B'$ mapping to $(1, 0)$ in $C_1 \times C_2'$ with $B_g \to B_g$ an isomorphism.

**Proof.** Observe that $A \to B$ is quasi-finite at every prime of the closed subset $T = \text{Spec}(C_1) \subset \text{Spec}(B)$ (this follows by looking at fibre rings, see Algebra, Definition 119.3). Consider the diagram of topological spaces

\[
\begin{array}{ccc}
\text{Spec}(B) & \xrightarrow{\phi} & \text{Spec}(B') \\
\psi & \downarrow & \psi'
\end{array}
\]

By Algebra, Theorem 120.13 for every $p \in T$ there is a $h_p \in B'$, $h_p \notin p$ such that $B'_h \to B_h$ is an isomorphism. The union $U = \bigcup D(h_p)$ gives an open $U \subset \text{Spec}(B')$ such that $\phi^{-1}(U) \to U$ is a homeomorphism and $T \subset \phi^{-1}(U)$. Since $T$ is open in $\psi^{-1}(V(I))$ we conclude that $\phi(T)$ is open in $U \cap (\psi')^{-1}(V(I))$. Thus $\phi(T)$ is open in $(\psi')^{-1}(V(I))$. On the other hand, since $C_1$ is finite over $A/I$ it is finite over $B'$. Hence $\phi(T)$ is a closed subset of $\text{Spec}(B')$ by Algebra, Lemmas 40.6 and 35.20. We conclude that $\text{Spec}(B'/IB') \supset \phi(T)$ is open and closed. By Algebra, Lemma 22.3 we get a corresponding product decomposition $B'/IB' = C_1' \times C_2'$. The map $B'/IB' \to B/IB$ maps $C_1'$ into $C_1$ and $C_2'$ into $C_2$ as one sees by looking at what happens on spectra (hint: the inverse image of $\phi(T)$ is exactly $T$; some details omitted). Pick a $g \in B'$ mapping to $(1, 0)$ in $C_1' \times C_2'$ such that $D(g) \subset U$; this is possible because $\text{Spec}(C_1')$ and $\text{Spec}(C_2')$ are disjoint and closed in $\text{Spec}(B')$ and $\text{Spec}(C_1')$ is contained in $U$. Then $B'_g \to B_g$ defines a homeomorphism on spectra and an isomorphism on local rings (by our choice of $U$ above). Hence it is an isomorphism, as follows for example from Algebra, Lemma 23.1. Finally, it follows that $C_1' = C_1$ and the proof is complete. 

\[ \square \]

**Lemma 7.7.** Let $(A, I)$ be a pair. The following are equivalent

$(1)$ $(A, I)$ is a henselian pair,

$(2)$ given an étale ring map $A \to A'$ and an $A$-algebra map $\sigma : A' \to A/I$, there exists an $A$-algebra map $A' \to A$ lifting $\sigma$ ,

$(3)$ for any finite $A$-algebra $B$ the map $B \to B/IB$ induces a bijection on idempotents, and
(4) for any integral $A$-algebra $B$ the map $B \to B/IB$ induces a bijection on idempotents.

**Proof.** Assume (2) holds. Then $I$ is contained in the Jacobson radical of $A$, since otherwise there would be a nonunit $f \in A$ not contained in $I$ and the map $A \to A_f$ would contradict (2). Hence $IB \subseteq B$ is contained in the Jacobson radical of $B$ for $B$ integral over $A$ because $\text{Spec}(B) \to \text{Spec}(A)$ is closed by Algebra, Lemmas 40.6 and 35.20. Thus the map from idempotents of $B$ to idempotents of $B/IB$ is injective by Lemma 7.4. On the other hand, since (2) holds, every idempotent of $B$ lifts to an idempotent of $B/IB$ by Lemma 6.9. In this way we see that (2) implies (4).

The implication $(4) \Rightarrow (3)$ is trivial.

Assume (3). Let $m$ be a maximal ideal and consider the finite map $A \to B = A/(I \cap m)$. The condition that $B \to B/IB$ induces a bijection on idempotents implies that $I \subseteq m$ (if not, then $B = A/I \times A/m$ and $B/IB = A/I$). Thus we see that $I$ is contained in the Jacobson radical of $A$. Let $f \in A[T]$ be monic and suppose given a factorization $\overline{f} = g_0 h_0$ with $g_0, h_0 \in A/I[T]$ monic. Set $B = A[T]/(f)$. Let $\overline{e}$ be the nontrivial idempotent of $B/IB$ corresponding to the decomposition

$$B/IB = A/I[T]/(g_0) \times A[T]/(h_0)$$

of $A$-algebras. Let $e \in B$ be an idempotent lifting $\overline{e}$ which exists as we assumed (3). This gives a product decomposition

$$B = eB \times (1 - e)B$$

Note that $B$ is free of rank $\deg(f)$ as an $A$-module. Hence $eB$ and $(1 - e)B$ are finite locally free $A$-modules. However, since $eB$ and $(1 - e)B$ have constant rank $\deg(g_0)$ and $\deg(h_0)$ over $A/I$ we find that the same is true over $\text{Spec}(A)$. We conclude that

$$f = \det_A(T : B \to B) = \det_A(T : eB \to eB) \det_A(T : (1 - e)B \to (1 - e)B)$$

is a factorization into monic polynomials reducing to the given factorization modulo $I$. Thus (3) implies (1).

Assume (1). Let $A \to A'$ be an étale ring map and let $\sigma : A' \to A/I$ be an $A$-algebra map. This implies that $A''/IA'' = A/I \times C$ for some ring $C$. Let $A'' \subseteq A'$ be the integral closure of $A$ in $A'$. By Lemma 7.6 we can write $A''/IA'' = A/I \times C'$ such that $A''/IA''$ maps $A/I$ isomorphically to $A'/IA'$ and $C'$ to $C$ and such that there exists a $a \in A''$ mapping to $(1, 0)$ in $A/I \times C'$ such that $A''/I = A'/I$. By Lemma 7.5 we see that $a$ satisfies a monic polynomial $f \in A[T]$ whose reduction modulo $I$ factors as $\overline{f} = g_0 T^n$ where $T, g_0$ generate the unit ideal in $A/I[T]$. Thus by assumption we can factor $f$ as $f = gh$ where $g$ is a monic lift of $g_0$ and $h$ is a monic lift of $T^n$. Because $I$ is contained in the Jacobson radical of $A$, we find that $g$ and $h$ generate the unit ideal in $A[T]$ (details omitted; hint: use that $A[T]/(g, h)$ is finite over $A$). Thus $A[T]/(f) = A[T]/(h) \times A[T]/(g)$ and we find a corresponding product decomposition $A'' = A''_1 \times A''_2$. By construction we have $A''_1/IA''_1 = A/I$ and $A''_2/IA''_2 = C'$. Since $A''_1$ is integral over $A$ and $I$ is contained in the Jacobson radical of $A$ we see that $a$ maps to an invertible element of $A''_1$. Hence $A'' = A''_1 \times (A''_2)_A$. It follows that $A \to A''$ is integral as well as étale, hence finite locally free. However, $A''_1/IA''_1 = A/I$ thus $A''_1$ has rank 1 as an $A$-module along $V(I)$. Since $I$ is contained in the Jacobson radical of $A$ we conclude that
$A''_1$ has rank 1 everywhere and it follows that $A \to A''_1$ is an isomorphism. Thus $A' \to A''_1 \cong A''_a \to (A''_a)_a = A''_1 = A$ is the desired lift of $\sigma$. In this way we see that (1) implies (2). \hfill \square

**Lemma 7.8.** Let $A$ be a ring. Let $I, J \subseteq A$ be ideals with $V(I) = V(J)$. Then $(A, I)$ is henselian if and only if $(A, J)$ is henselian.

**Proof.** For any integral ring map $A \to B$ we see that $V(IB) = V(JB)$. Hence idempotents of $B/IB$ and $B/JB$ are in bijective correspondence (Algebra, Lemma 20.3). It follows that $B \to B/IB$ induces a bijection on sets of idempotents if and only if $B \to B/JB$ induces a bijection on sets of idempotents. Thus we conclude by Lemma 7.7. \hfill \square

**Lemma 7.9.** Let $(A, I)$ be a henselian pair and let $A \to B$ be an integral ring map. Then $(B, IB)$ is a henselian pair.

**Proof.** Immediate from the fourth characterization of henselian pairs in Lemma 7.7 and the fact that the composition of integral ring maps is integral. \hfill \square

**Lemma 7.10.** Let $J$ be a set and let $\{(A_j, I_j)\}_{j \in J}$ be a collection of pairs. Then $(\prod_{j \in J} A_j, \prod_{j \in J} I_j)$ is Henselian if and only if so is each $(A_j, I_j)$.

**Proof.** For every $j \in J$, the projection $\prod_{j \in J} A_j \to A_j$ is an integral ring map, so Lemma 7.9 proves that each $(A_j, I_j)$ is Henselian if $(\prod_{j \in J} A_j, \prod_{j \in J} I_j)$ is Henselian.

Conversely, suppose that each $(A_j, I_j)$ is a Henselian pair. Then every $1 + x$ with $x \in \prod_{j \in J} I_j$ is a unit in $\prod_{j \in J} A_j$ because it is so componentwise by Algebra, Lemma 18.1 and Definition 7.1. Thus, by Algebra, Lemma 18.1 again, $\prod_{j \in J} I_j$ is contained in the Jacobson radical of $\prod_{j \in J} A_j$. Continuing to work componentwise, it likewise follows that for every monic $f \in (\prod_{j \in J} A_j)[T]$ and every factorization $f = gh$ with monic $g, h \in (\prod_{j \in J} A_j)[T] = (\prod_{j \in J} A_j/I_j)[T]$ that generate the unit ideal in $(\prod_{j \in J} A_j/I_j)[T]$, there exists a factorization $f = gh$ in $(\prod_{j \in J} A_j/T)$ with $g, h$ monic and reducing to $g_0, h_0$. In conclusion, according to Definition 7.1 $(\prod_{j \in J} A_j, \prod_{j \in J} I_j)$ is a Henselian pair. \hfill \square

**Lemma 7.11.** Let $(A, I)$ be a henselian pair. Let $p \subseteq A$ be a prime ideal. Then $V(p + I)$ is connected.

**Proof.** By Lemma 7.9 we see that $(A/p, I + p/p)$ is a henselian pair. Thus it suffices to prove: If $(A, I)$ is a henselian pair and $A$ is a domain, then Spec$(A/I) = V(I)$ is connected. If not, then $A/I$ has a nontrivial idempotent, whence by Lemma 7.7, $A$ has a nontrivial idempotent. This is a contradiction. \hfill \square

**Lemma 7.12.** Let $(A, I)$ be a henselian pair. The functor $B \to B/IB$ determines an equivalence between finite étale $A$-algebras and finite étale $A/I$-algebras.

**Proof.** Let $B, B'$ be two $A$-algebras finite étale over $A$. Then $B' \to B'' = B \otimes_A B'$ is finite étale as well (Algebra, Lemmas 139.3 and 35.11). Now we have 1-to-1 correspondences between

1. $A$-algebra maps $B \to B'$,
2. sections of $B' \to B''$, and
3. idempotents $e$ of $B''$ such that $B' \to B'' \to eB''$ is an isomorphism.
The bijection between (2) and (3) sends $\sigma : B'' \to B'$ to $e$ such that $(1 - e)$ is the idempotent that generates the kernel of $\sigma$ which exists by Algebra, Lemmas 139.9 and 139.10. There is a similar correspondence between $A/I$-algebra maps $B/IB \to B'/IB'$ and idempotents $\tau$ of $B''/IB''$ such that $B'/IB' \to B''/IB'' \to \tau(B''/IB'')$ is an isomorphism. However every idempotent $e$ of $B''$ lifts uniquely to an idempotent $\bar{e}$ of $B''$ (Lemma 7.7). Moreover, if $B''/IB'' \to \tau(B''/IB'')$ is an isomorphism, then $B' \to eB''$ is an isomorphism too by Nakayama’s lemma (Algebra, Lemma 19.1). In this way we see that the functor is fully faithful.

Essential surjectivity. Let $A/I \to C$ be a finite étale map. By Algebra, Lemma 139.11 there exists an étale map $A \to B$ such that $B/IB \cong C$. Let $B'$ be the integral closure of $A/I$ in $B$. By Lemma 7.6 we have $B'/IB' = C \times C'$ for some ring $C'$ and $B'_g \cong B_g$ for some $g \in B'$ mapping to $(1,0) \in C \times C'$. Since idempotents lift (Lemma 7.7) we get $B' = B'_1 \times B'_2$ with $C = B'_1/IB'_1$ and $C' = B'_2/IB'_2$. The image of $g$ in $B'_1$ is invertible and $(B'_2)_g = 0$ because $IB'$ is contained in the Jacobson radical of $B'$ (for example because $(B',IB')$ is a henselian pair by Lemma 7.9). We conclude that $B'_1 = B_g$ is finite étale over $A$ and the proof is done.

Lemma 7.13. The inclusion functor
\[ \text{category of henselian pairs} \to \text{category of pairs} \]
has a left adjoint $(A,I) \mapsto (A^h,I^h)$.

Proof. Let $(A,I)$ be a pair. Consider the category $C$ consisting of étale ring maps $A \to B$ such that $A/I \to B/IB$ is an isomorphism. We will show that the category $C$ is directed and that $A^h = \colim_{B \in C} B$ with ideal $I^h = IA^h$ gives the desired adjoint.

We first prove that $C$ is directed (Categories, Definition 19.1). It is nonempty because $\id : A \to A$ is an object. If $B$ and $B'$ are two objects of $C$, then $B'' = B \otimes_A B'$ is an object of $C$ (use Algebra, Lemma 139.3) and there are morphisms $B \to B''$ and $B' \to B''$. Suppose that $f,g : B \to B''$ are two maps between objects of $C$. Then a coequalizer is $B' \otimes_{f,g} B''$ which is étale over $A$ by Algebra, Lemmas 139.3 and 139.9. Thus the category $C$ is directed.

Since $B/IB = A/I$ for all objects $B$ of $C$ we see that $A^h/I^h = A^h/IA^h = \colim B/IB = \colim A/I = A/I$.

Next, we show that $A^h = \colim_{B \in C} B$ with $I^h = IA^h$ is a henselian pair. To do this we will verify condition (2) of Lemma 7.7. Namely, suppose given an étale ring map $A^h \to A'$ and an $A^h$-algebra map $\sigma : A' \to A^h/I^h$. Then there exists a $B \in C$ and an étale ring map $B \to B'$ such that $A' = B' \otimes_A A^h$. See Algebra, Lemma 139.3. Since $A^h/I^h = A/IB$, the map $\sigma$ induces an $A$-algebra map $s : B' \to A/I$. Then $B'/IB' = A/I \times C$ as $A/I$-algebra, where $C$ is the kernel of the map $B'/IB' \to A/I$ induced by $s$. Let $g \in B'$ map to $(1, 0) \in A/I \times C$. Then $B \to B'_g$ is étale and $A/I \to B'_g/IB'_g$ is an isomorphism, i.e., $B'_g$ is an object of $C$. Thus we obtain a canonical map $B'_g \to A^h$ such that
commute. This induces a map $A' = B' \otimes_B A^h \to A^h$ compatible with $\sigma$ as desired. Let $(A, I) \to (A', I')$ be a morphism of pairs with $(A', I')$ henselian. We will show there is a unique factorization $A \to A^h \to A'$ which will finish the proof. Namely, for each $A \to B$ in $\mathcal{C}$ the ring map $A' \to B' = A' \otimes_A B$ is étale and induces an isomorphism $A'/I' \to B'/I'B'$. Hence there is a section $\sigma_B : B' \to A'$ by Lemma 7.7. Given a morphism $B_1 \to B_2$ in $\mathcal{C}$ we claim the diagram

$$
\begin{array}{ccc}
B'_1 & \longrightarrow & B'_2 \\
\sigma_{B_1} \downarrow & & \downarrow \sigma_{B_2} \\
A' & & 
\end{array}
$$

commutes. This follows once we prove that for every $B$ in $\mathcal{C}$ the section $\sigma_B$ is the unique $A'$-algebra map $B' \to A'$. We have $B' \otimes_A B' = B' \times R$ for some ring $R$, see Algebra, Lemma 145.4. In our case $R/I'R = 0$ as $B'/I'B' = A'/I'$. Thus given two $A'$-algebra maps $\sigma_B, \sigma'_B : B' \to A'$ then $e = (\sigma_B \otimes \sigma'_B)(0,1) \in A'$ is an idempotent contained in $I'$. We conclude that $e = 0$ by Lemma 7.4. Hence $\sigma_B = \sigma'_B$ as desired. Using the commutativity we obtain

$$
A^h = \text{colim}_{B \in \mathcal{C}} B \to \text{colim}_{B \in \mathcal{C}} A' \otimes_A B \overset{\text{colim } \sigma_B}{\longrightarrow} A'
$$

as desired. The uniqueness of the maps $\sigma_B$ also guarantees that this map is unique. Hence $(A, I) \mapsto (A^h, I^h)$ is the desired adjoint. \hfill \Box

**Lemma 7.14.** The functor of Lemma 7.13 associates to a local ring $(A, \mathfrak{m})$ its henselization.

**Proof.** First proof: in the proof of Algebra, Lemma 146.16 it is shown that the henselization of $A$ is given by the the colimit used to construct $A^h$ in Lemma 7.13. Second proof: Both the henselization $S$ and the ring $A^h$ of Lemma 7.13 are filtered colimits of étale $A$-algebras, henselian, and have residue fields equal to $\kappa(\mathfrak{m})$. Hence they are canonically isomorphic by Algebra, Lemma 146.15. \hfill \Box

**Lemma 7.15.** Let $(A, I)$ be a pair. Let $(A^h, I^h)$ be as in Lemma 7.13. Then $A \to A^h$ is flat, $I^h = IA^h$ and $A/I^n \to A^h/I^nA^h$ is an isomorphism for all $n$.

**Proof.** In the proof of Lemma 7.13 we have seen that $A^h$ is a filtered colimit of étale $A$-algebras $B$ such that $A/I \to B/I^2B$ is an isomorphism and we have seen that $I^h = IA^h$. As an étale ring map is flat (Algebra, Lemma 139.3) we conclude that $A \to A^h$ is flat by Algebra, Lemma 38.2. Since each $A \to B$ is flat we find that the maps $A/I^n \to B/I^nB$ are isomorphisms as well (for example by Algebra, Lemma 98.3). Taking the colimit we find that $A/I^n = A^h/I^nA^h$ as desired. \hfill \Box

**Lemma 7.16.** Let $(A, I)$ be a pair with $A$ Noetherian. Let $(A^h, I^h)$ be as in Lemma 7.13. Then the map of $I$-adic completions

$$
A^\wedge \to (A^h)^\wedge
$$

is an isomorphism. Moreover, $A^h$ is Noetherian, the maps $A \to A^h \to A^\wedge$ are flat, and $A^h \to A^\wedge$ is faithfully flat.

**Proof.** The first statement is an immediate consequence of Lemma 7.15 and in fact holds without assuming $A$ is Noetherian. In the proof of Lemma 7.13 we have seen that $A^h$ is a filtered colimit of étale $A$-algebras $B$ such that $A/I \to B/I^2B$ is an isomorphism. For each such $A \to B$ the induced map $A^\wedge \to B^\wedge$ is an
isomorphism (see proof of Lemma 7.15). By Algebra, Lemma 94.3 the ring map $B \to A^\wedge = B^\wedge = (A^h)^\wedge$ is flat for each $B$. Thus $A^h \to A^\wedge = (A^h)^\wedge$ is flat by Algebra, Lemma 38.5. Since $I^h = IA^h$ is contained in the radical ideal of $A^h$ and since $A^h \to A^\wedge$ induces an isomorphism $A^h/I^h \to A/I$ we see that $A^h \to A^\wedge$ is faithfully flat by Algebra, Lemma 38.14. By Algebra, Lemma 94.10 the ring $A^\wedge$ is Noetherian. Hence we conclude that $A^h$ is Noetherian by Algebra, Lemma 153.1.

**Lemma 7.17.** Let $(A, I) = \text{colim}(A_i, I_i)$ be a colimit of pairs. The functor of Lemma 7.15 gives $A^h = \text{colim} A_i^h$ and $I^h = \text{colim} I_i^h$.

**Proof.** This is true for any left adjoint, see Categories, Lemma 24.4.

### 8. Auto-associated rings

Some of this material is in [Laz69].

**Definition 8.1.** A ring $R$ is said to be auto-associated if $R$ is local and its maximal ideal $m$ is weakly associated to $R$.

**Lemma 8.2.** An auto-associated ring $R$ has the following property: (P) Every proper finitely generated ideal $I \subset R$ has a nonzero annihilator.

**Proof.** By assumption there exists a nonzero element $x \in R$ such that for every $f \in m$ we have $f^n x = 0$. Say $I = (f_1, \ldots, f_r)$. Then $x$ is in the kernel of $R \to \bigoplus R_{f_i}$. Hence we see that there exists a nonzero $y \in R$ such that $f_i y = 0$ for all $i$, see Algebra, Lemma 22.4. As $y \in \text{Ann}_R(I)$ we win.

**Lemma 8.3.** Let $R$ be a ring having property (P) of Lemma 8.2. Let $u : N \to M$ be a homomorphism of projective $R$-modules. Then $u$ is universally injective if and only if $u$ is injective.

**Proof.** Assume $u$ is injective. Our goal is to show $u$ is universally injective. First we choose a module $Q$ such that $N \oplus Q$ is free. On considering the map $N \oplus Q \to M \oplus Q$ we see that it suffices to prove the lemma in case $N$ is free. In this case $N$ is a directed colimit of finite free $R$-modules. Thus we reduce to the case that $N$ is a finite free $R$-module, say $N = R^{\oplus n}$. We prove the lemma by induction on $n$. The case $n = 0$ is trivial.

Let $u : R^{\oplus n} \to M$ be an injective module map with $M$ projective. Choose an $R$-module $Q$ such that $M \oplus Q$ is free. After replacing $u$ by the composition $R^{\oplus n} \to M \to M \oplus Q$ we see that we may assume that $M$ is free. Then we can find a direct summand $R^{\oplus m} \subset M$ such that $u(R^{\oplus n}) \subset R^{\oplus m}$. Hence we may assume that $M = R^{\oplus m}$. In this case $u$ is given by a matrix $A = (a_{ij})$ so that $u(x_1, \ldots, x_n) = (\sum x_i a_{i1}, \ldots, \sum x_i a_{im})$. As $u$ is injective, in particular $u(x, 0, \ldots, 0) = (xa_{11}, xa_{12}, \ldots, xa_{1m}) \neq 0$ if $x \neq 0$, and as $R$ has property (P) we see that $a_{11}R + a_{12}R + \ldots + a_{1m}R = R$. Hence see that $R(a_{11}, \ldots, a_{1m}) \subset R^{\oplus m}$ is a direct summand of $R^{\oplus m}$, in particular $R^{\oplus m}/R(a_{11}, \ldots, a_{1m})$ is a projective $R$-module. We get a commutative diagram

$$
\begin{array}{cccccc}
0 & \to & R & \to & R^{\oplus n} & \to & R^{\oplus n-1} & \to & 0 \\
\downarrow 1 & & \downarrow u & & \downarrow \ & & \\
0 & \to & R & \to & R^{\oplus m} & \to & R^{\oplus m}/R(a_{11}, \ldots, a_{1m}) & \to & 0
\end{array}
$$
with split exact rows. Thus the right vertical arrow is injective and we may apply the induction hypothesis to conclude that the right vertical arrow is universally injective. It follows that the middle vertical arrow is universally injective. □

**Lemma 8.4.** Let $R$ be a ring. The following are equivalent

1. $R$ has property $(P)$ of Lemma 8.3.
2. any injective map of projective $R$-modules is universally injective,
3. if $u : N \to M$ is injective and $N, M$ are finite projective $R$-modules then $\text{Coker}(u)$ is a finite projective $R$-module,
4. if $N \subset M$ and $N, M$ are finite projective as $R$-modules, then $N$ is a direct summand of $M$, and
5. any injective map $R \to R^{\oplus n}$ is a split injection.

**Proof.** The implication $(1) \Rightarrow (2)$ is Lemma 8.3. It is clear that $(3)$ and $(4)$ are equivalent. We have $(2) \Rightarrow (3)$, $(4)$ by Algebra, Lemma 80.4. Part $(5)$ is a special case of $(4)$. Assume $(5)$. Let $I = (a_1, \ldots, a_n)$ be a proper finitely generated ideal of $R$. As $I \neq R$ we see that $R \to R^{\oplus n}, x \mapsto (xa_1, \ldots, xa_n)$ is not a split injection. Hence it has a nonzero kernel and we conclude that $\text{Ann}_R(I) \neq 0$. Thus (1) holds. □

**Example 8.5.** If the equivalent conditions of Lemma 8.4 hold, then it is not always the case that every injective map of free $R$-modules is a split injection. For example suppose that $R = k[x_1, x_2, x_3, \ldots]/(x_1^2)$. This is an auto-associated ring. Consider the map of free $R$-modules

$$u : \bigoplus_{i \geq 1} R e_i \to \bigoplus_{i \geq 1} R f_i, \quad e_i \mapsto f_i - x_1 f_{i+1}.$$ 

For any integer $n$ the restriction of $u$ to $\bigoplus_{i=1, \ldots, n} R e_i$ is injective as the images $u(e_1), \ldots, u(e_n)$ are $R$-linearly independent. Hence $u$ is injective and hence universally injective by the lemma. Since $u \otimes \text{id}_k$ is bijective we see that if $u$ were a split injection then $u$ would be surjective. But $u$ is not surjective because the inverse image of $f_1$ would be the element

$$\sum_{i \geq 0} x_1 \ldots x_i e_{i+1} = e_1 + x_1 e_2 + x_1 x_2 e_3 + \ldots$$

which is not an element of the direct sum. A side remark is that $\text{Coker}(u)$ is a flat (because $u$ is universally injective), countably generated $R$-module which is not projective (as $u$ is not split), hence not Mittag-Leffler (see Algebra, Lemma 91.1).

9. **Flattening stratification**

Let $R \to S$ be a ring map and let $N$ be an $S$-module. For any $R$-algebra $R'$ we can consider the base changes $S' = S \otimes_R R'$ and $M' = M \otimes_R R'$. We say $R \to R'$ flattens $M$ if the module $M'$ is flat over $R'$. We would like to understand the structure of the collection of ring maps $R \to R'$ which flatten $M$. In particular we would like to know if there exists a universal flattening $R \to R_{\text{univ}}$ of $M$, i.e., a ring map $R \to R_{\text{univ}}$ which flattens $M$ and has the property that any ring map $R \to R'$ which flattens $M$ factors through $R \to R_{\text{univ}}$. It turns out that such a universal solution usually does not exist.

We will discuss universal flattenings and flattening stratifications in a scheme theoretic setting $\mathcal{F}/X/S$ in More on Flatness, Section 20. If the universal flattening
$R \to R_{\text{univ}}$ exists then the morphism of schemes $\text{Spec}(R_{\text{univ}}) \to \text{Spec}(R)$ is the universal flattening of the quasi-coherent module $M$ on $\text{Spec}(S)$.

In this and the next few sections we prove some basic algebra facts related to this. The most basic result is perhaps the following.

**Lemma 9.1.** Let $R$ be a ring. Let $M$ be an $R$-module. Let $I_1, I_2$ be ideals of $R$. If $M/I_1 M$ is flat over $R/I_1$ and $M/I_2 M$ is flat over $R/I_2$, then $M/(I_1 \cap I_2) M$ is flat over $R/(I_1 \cap I_2)$.

**Proof.** By replacing $R$ with $R/(I_1 \cap I_2)$ and $M$ by $M/(I_1 \cap I_2) M$ we may assume that $I_1 \cap I_2 = 0$. Let $J \subset R$ be an ideal. To prove that $M$ is flat over $R$ we have to show that $J \otimes_R M \to M$ is injective, see Algebra, Lemma 38.4. By flatness of $M/I_1 M$ over $R/I_1$ the map

$$J/(J \cap I_1) \otimes_R M = (J + I_1)/I_1 \otimes_{R/I_1} M/I_1 M \to M/I_1 M$$

is injective. As $0 \to (J \cap I_1) \to J \to J/(J \cap I_1) \to 0$ is exact we obtain a diagram

$$
\begin{array}{ccc}
J \cap I_1 \otimes_R M & \to & J \otimes_R M & \to & J/(J \cap I_1) \otimes_R M & \to & 0 \\
\downarrow & & & & & & \\
M & \to & M & \to & M/I_1 M \\
\end{array}
$$

hence it suffices to show that $(J \cap I_1) \otimes_R M \to M$ is injective. Since $I_1 \cap I_2 = 0$ the ideal $J \cap I_1$ maps isomorphically to an ideal $J' \subset R/I_2$ and we see that $(J \cap I_1) \otimes_R M = J' \otimes_{R/I_2} M/I_2 M$. By flatness of $M/I_2 M$ over $R/I_2$ the map $J' \otimes_{R/I_2} M/I_2 M \to M/I_2 M$ is injective, which clearly implies that $(J \cap I_1) \otimes_R M \to M$ is injective. □

10. Flattening over an Artinian ring

A universal flattening exists when the base ring is an Artinian local ring. It exists for an arbitrary module. Hence, as we will see later, a flattening stratification exists when the base scheme is the spectrum of an Artinian local ring.

**Lemma 10.1.** Let $R$ be an Artinian ring. Let $M$ be an $R$-module. Then there exists a smallest ideal $I \subset R$ such that $M/IM$ is flat over $R/I$.

**Proof.** This follows directly from Lemma 9.1 and the Artinian property. □

This ideal has the following universal property.

**Lemma 10.2.** Let $R$ be an Artinian ring. Let $M$ be an $R$-module. Let $I \subset R$ be the smallest ideal $I \subset R$ such that $M/IM$ is flat over $R/I$. Then $I$ has the following universal property: For every ring map $\varphi : R \to R'$ we have

$$R' \otimes_R M \text{ is flat over } R' \iff \varphi(I) = 0.$$ 

**Proof.** Note that $I$ exists by Lemma 10.1. The implication $\iff$ follows from Algebra, Lemma 10.6. Let $\varphi : R \to R'$ be such that $M \otimes_R R'$ is flat over $R'$. Let $J = \text{Ker}(\varphi)$. By Algebra, Lemma 10.7 and as $R' \otimes_R M = R' \otimes_{R/I} M/IM$ is flat over $R'$ we conclude that $M/IM$ is flat over $R/J$. Hence $I \subset J$ as desired. □
11. Flattening over a closed subset of the base

Let $R \to S$ be a ring map. Let $I \subset R$ be an ideal. Let $M$ be an $S$-module. In the following we will consider the following condition

\[(11.0.1) \quad \forall q \in V(IS) \subset \text{Spec}(S) : M_q \text{ is flat over } R.\]

Geometrically, this means that $M$ is flat over $R$ along the inverse image of $V(I)$ in $\text{Spec}(S)$. If $R$ and $S$ are Noetherian rings and $M$ is a finite $S$-module, then \[(11.0.1)\] is equivalent to the condition that $M/I^nM$ is flat over $R/I^n$ for all $n \geq 1$, see Algebra, Lemma \[96.11\]

**Lemma 11.1.** Let $R \to S$ be a ring map. Let $I \subset R$ be an ideal. Let $M$ be an $S$-module. Let $R \to R'$ be a ring map and $IR' \subset I' \subset R'$ an ideal. If \[(11.0.1)\] holds for $(R \to S, I, M)$, then \[(11.0.1)\] holds for $(R' \to S \otimes_R R', I', M \otimes_R R')$.

**Proof.** Assume \[(11.0.1)\] holds for $(R \to S, I \subset R, M)$. Let $I'(S \otimes_R R') \subset q'$ be a prime of $S \otimes_R R'$. Let $q \subset S$ be the corresponding prime of $S$. Then $IS \subset q$. Note that $(M \otimes_R R')_{q'}$ is a localization of the base change $M_q \otimes_R R'$. Hence $(M \otimes_R R')_{q'}$ is flat over $R'$ as a localization of a flat module, see Algebra, Lemmas \[38.6\] and \[38.19\]

**Lemma 11.2.** Let $R \to S$ be a ring map. Let $I \subset R$ be an ideal. Let $M$ be an $S$-module. Let $R \to R'$ be a ring map and $IR' \subset I' \subset R'$ an ideal such that

1. the map $V(I') \to V(I)$ induced by $\text{Spec}(R') \to \text{Spec}(R)$ is surjective, and
2. $R'_p$ is flat over $R$ for all primes $p' \in V(I')$.

If \[(11.0.1)\] holds for $(R' \to S \otimes_R R', I', M \otimes_R R')$, then \[(11.0.1)\] holds for $(R \to S, I, M)$.

**Proof.** Assume \[(11.0.1)\] holds for $(R' \to S \otimes_R R', IR', M \otimes_R R')$. Pick a prime $IS \subset q \subset S$. Let $I \subset p \subset R$ be the corresponding prime of $R$. By assumption there exists a prime $p' \in V(I')$ lying over $p$ and $R_p \to R'_p$ is flat. Choose a prime $\overline{q} \subset \kappa(q) \otimes_{\kappa(p)} \kappa(p')$ which corresponds to a prime $q' \subset S \otimes_R R'$ which lies over $q$ and over $p'$. Note that $(S \otimes_R R')_{q'}$ is a localization of $S_q \otimes_{R_p} R'_p$. By assumption the module $(M \otimes_R R')_{q'}$ is flat over $R'_p$. Hence Algebra, Lemma \[97.1\] implies that $M_q$ is flat over $R_p$ which is what we wanted to prove.

**Lemma 11.3.** Let $R \to S$ be a ring map of finite presentation. Let $M$ be an $S$-module of finite presentation. Let $R' = \text{colim}_{\lambda \in \Lambda} R_\lambda$ be a directed colimit of $R$-algebras. Let $I_\lambda \subset R_\lambda$ be ideals such that $I_\lambda R_\mu \subset I_\mu$ for all $\mu \geq \lambda$ and set $I' = \text{colim}_\lambda I_\lambda$. If \[(11.0.1)\] holds for $(R' \to S \otimes_R R', I', M \otimes_R R')$, then there exists $\lambda \in \Lambda$ such that \[(11.0.1)\] holds for $(R_\lambda \to S \otimes_R R_\lambda, I_\lambda, M \otimes_R R_\lambda)$.

**Proof.** We are going to write $S_\lambda = S \otimes_R R_\lambda$, $S' = S \otimes_R R'$, $M_\lambda = M \otimes_R R_\lambda$, and $M' = M \otimes_R R'$. The base change $S'$ is of finite presentation over $R'$ and $M'$ is of finite presentation over $S'$ and similarly for the versions with subscript $\lambda$, see Algebra, Lemma \[13.2\]. By Algebra, Theorem \[126.3\] the set $U' = \{q' \in \text{Spec}(S') \mid M'_{q'} \text{ is flat over } R'\}$ is open in $\text{Spec}(S')$. Note that $V(I'S')$ is a quasi-compact space which is contained in $U'$ by assumption. Hence there exist finitely many $g'_j \in S'$, $j = 1, \ldots, m$ such that $D(g'_j) \subset U'$ and such that $V(I'S') \subset \bigcup D(g'_j)$. Note that in particular $(M')_{g'_j}$ is a flat module over $R'$. 
We are going to pick increasingly large elements $\lambda \in \Lambda$. First we pick it large enough so that we can find $g_{j,\lambda} \in S_{\lambda}$ mapping to $g_j$. The inclusion $V(I' S') \subseteq \bigcup D(g_j')$ means that $I' S' + (g_1', \ldots, g_m') = S'$ which can be expressed as $1 = \sum z_n h_n + \sum f_j g_j'$ for some $z_n \in I'$, $h_n, f_j \in S'$. After increasing $\lambda$ we may assume such an equation holds in $S_{\lambda}$. Hence we may assume that $V(I_{\lambda} S_{\lambda}) \subseteq \bigcup D(g_{j,\lambda})$. By Algebra, Lemma 157.1 we see that for some sufficiently large $\lambda$ the modules $(M_{\lambda})_{g_{j,\lambda}}$ are flat over $R_{\lambda}$. In particular the module $M_{\lambda}$ is flat over $R_{\lambda}$ at all the primes lying over the ideal $I_{\lambda}$.

12. Flattening over a closed subsets of source and base

In this section we slightly generalize the discussion in Section 11. We strongly suggest the reader first read and understand that section.

**Situation** 12.1. Let $R \to S$ be a ring map. Let $J \subseteq S$ be an ideal. Let $M$ be an $S$-module.

In this situation, given an $R$-algebra $R'$ and an ideal $I' \subseteq R'$ we set $S' = S \otimes_R R'$ and $M' = M \otimes_R R'$. We will consider the condition

(12.1.1) $$\forall q' \in V(I' S' + JS') \subseteq \Spec(S') : M'_q' \text{ is flat over } R'.$$

Geometrically, this means that $M'$ is flat over $R'$ along the intersection of the inverse image of $V(I')$ with the inverse image of $V(J)$. Since $(R \to S, J, M)$ are fixed, condition (12.1.1) only depends on the pair $(R', I')$ where $R'$ is viewed as an $R$-algebra.

**Lemma 12.2.** In Situation 12.1 let $R' \to R''$ be an $R$-algebra map. Let $I' \subseteq R'$ and $I'R'' \subseteq I'' \subseteq R''$ be ideals. If (12.1.1) holds for $(R', I')$, then (12.1.1) holds for $(R'', I'')$.

**Proof.** Assume (12.1.1) holds for $(R', I')$. Let $I'' S'' + JS'' \subseteq q''$ be a prime of $S''$. Let $q' \subseteq S$ be the corresponding prime of $S'$. Then both $I'S' + q'$ and $JS' \subseteq q'$ because the corresponding conditions hold for $q''$. Note that $(M'')_{q''}$ is a localization of the base change $M'_q \otimes_R R''$. Hence $(M'')_{q''}$ is flat over $R''$ as a localization of a flat module, see Algebra, Lemmas 38.6 and 38.19.

**Lemma 12.3.** In Situation 12.1 let $R' \to R''$ be an $R$-algebra map. Let $I' \subseteq R'$ and $I'R'' \subseteq I'' \subseteq R''$ be ideals. Assume

1. the map $V(I'') \to V(I')$ induced by $\Spec(R'') \to \Spec(R')$ is surjective, and
2. $R''_{p''}$ is flat over $R'$ for all primes $p'' \in V(I'')$.

If (12.1.1) holds for $(R'', I'')$, then (12.1.1) holds for $(R', I')$.

**Proof.** Assume (12.1.1) holds for $(R'', I'')$. Pick a prime $I'S' + JS' \subseteq q' \subseteq S'$. Let $I' \subseteq p' \subseteq R'$ be the corresponding prime of $R'$. By assumption there exists a prime $p'' \subseteq V(I'')$ of $R''$ lying over $p'$ and $R''_{p''}$ is flat. Choose a prime $q'' \subseteq \kappa(p') \otimes_{\kappa(p')} \kappa(p'')$. This corresponds to a prime $q'' \subseteq S'' = S' \otimes_R R''$ which lies over $q'$ and over $p''$. In particular we see that $I'' S'' \subseteq q''$ and that $JS'' \subseteq q''$. Note that $(S' \otimes_R R'')_{q''}$ is a localization of $S'_q \otimes_{R'_p} R''_{p''}$. By assumption the module $(M' \otimes_{R'} R'')_{q''}$ is flat over $R''_{p''}$. Hence Algebra, Lemma 97.1 implies that $M'_q$ is flat over $R'_p$ which is what we wanted to prove.
Lemma 12.4. In Situation\ref{situation-infinite-1} assume $R \to S$ is essentially of finite presentation and $M$ is an $S$-module of finite presentation. Let $R' = \text{colim}_\lambda R_\lambda$ be a directed colimit of $R$-algebras. Let $I_\lambda \subset R_\lambda$ be ideals such that $I_\lambda R_\mu \subset I_\mu$ for all $\mu \geq \lambda$ and set $I' = \text{colim}_\lambda I_\lambda$. If \ref{equation-fin-gen-flat} holds for $(R', I')$, then there exists a $\lambda \in \Lambda$ such that \ref{equation-fin-gen-flat} holds for $(R_\lambda, I_\lambda)$.

Proof. We first prove the lemma in case $R \to S$ is of finite presentation and then we explain what needs to be changed in the general case. We are going to write $S_\lambda = S \otimes_R R_\lambda$, $S' = S \otimes_R R'$, $M_\lambda = M \otimes_R R_\lambda$, and $M' = M \otimes_R R'$. The base change $S'$ is of finite presentation over $R'$ and $M'$ is of finite presentation over $S'$ and similarly for the versions with subscript $\lambda$, see Algebra, Lemma \ref{lemma-finite-presentation-base-change}. By Algebra, Theorem \ref{theorem-limits-flat} the set

$$U' = \{ q' \in \text{Spec}(S') \mid M'_q \text{ is flat over } R' \}$$

is open in $\text{Spec}(S')$. Note that $V(I'S' + JS')$ is a quasi-compact space which is contained in $U'$ by assumption. Hence there exist finitely many $g'_j \in S'$, $j = 1, \ldots, m$ such that $D(g'_j) \subset U'$ and such that $V(I'S' + JS') \subset \bigcup D(g'_j)$. Note that in particular $(M')_{g'_j}$ is a flat module over $R'$.

We are going to pick increasingly large elements $\lambda \in \Lambda$. First we pick it large enough so that we can find $g_{j, \lambda} \in S_\lambda$ mapping to $g'_j$. The inclusion $V(I'S' + JS') \subset \bigcup D(g_{j, \lambda})$ means that $I'S' + JS' + (g_{1, \lambda}, \ldots, g_{m, \lambda}) = S'$ which can be expressed as

$$1 = \sum y_k k_t + \sum z_h h_s + \sum f_j g'_j$$

for some $z_s \in I'$, $y_t \in J$, $k_t, h_s, f_j \in S'$. After increasing $\lambda$ we may assume such an equation holds in $S_\lambda$. Hence we may assume that $V(I_\lambda S_\lambda + JS_\lambda) \subset \bigcup D(g_{j, \lambda})$. By Algebra, Lemma \ref{lemma-finite-presentation-flat} we see that for some sufficiently large $\lambda$ the modules $(M_\lambda)_{g_{j, \lambda}}$ are flat over $R_\lambda$. In particular the module $M_\lambda$ is flat over $R_\lambda$ at all the primes corresponding to points of $V(I_\lambda S_\lambda + JS_\lambda)$.

In the case that $S$ is essentially of finite presentation, we can write $S = \Sigma^{-1}C$ where $R \to C$ is of finite presentation and $\Sigma \subset C$ is a multiplicative subset. We can also write $M = \Sigma^{-1}N$ for some finitely presented $C$-module $N$, see Algebra, Lemma \ref{lemma-finite-presentation}. At this point we introduce $C_\lambda$, $C'$, $N_\lambda$, $N'$. Then in the discussion above we obtain an open $U' \subset \text{Spec}(C')$ over which $N'$ is flat over $R'$. The assumption that \ref{equation-fin-gen-flat} is true means that $V(I'S' + JS')$ maps into $U'$, because for a prime $q \subset S'$, corresponding to a prime $r' \subset C'$ we have $M'_q = N'_q$. Thus we can find $g'_j \in C'$ such that $\bigcup D(g'_j)$ contains the image of $V(I'S' + JS')$. The rest of the proof is exactly the same as before.

Lemma 12.5. In Situation\ref{situation-infinite-1} Let $I \subset R$ be an ideal. Assume

1. $R$ is a Noetherian ring,
2. $S$ is a Noetherian ring,
3. $M$ is a finite $S$-module, and
4. for each $n \geq 1$ and any prime $q \in V(J + IS)$ the module $(M/I^nM)_q$ is flat over $R/I^n$.

Then \ref{equation-fin-gen-flat} holds for $(R, I)$, i.e., for every prime $q \in V(J + IS)$ the localization $M_q$ is flat over $R$.

Proof. Let $q \in V(J + IS)$. Then Algebra, Lemma \ref{lemma-flat-loc} applied to $R \to S_q$ and $M_q$ implies that $M_q$ is flat over $R$. \qed
13. Flattening over a Noetherian complete local ring

The following three lemmas give a completely algebraic proof of the existence of the “local” flattening stratification when the base is a complete local Noetherian ring $R$ and the given module is finite over a finite type $R$-algebra $S$.

**Lemma 13.1.** Let $R \to S$ be a ring map. Let $M$ be an $S$-module. Assume

(1) $(R, \mathfrak{m})$ is a complete local Noetherian ring,
(2) $S$ is a Noetherian ring, and
(3) $M$ is finite over $S$.

Then there exists an ideal $I \subset \mathfrak{m}$ such that

(1) $(M/IM)_q$ is flat over $R/I$ for all primes $q$ of $S/IS$ lying over $\mathfrak{m}$, and
(2) if $J \subset R$ is an ideal such that $(M/JM)_q$ is flat over $R/J$ for all primes $q$ lying over $\mathfrak{m}$, then $I \subset J$.

In other words, $I$ is the smallest ideal of $R$ such that (11.0.1) holds for $(\overline{R} \to \overline{S}, \overline{\mathfrak{m}}, \overline{M})$ where $\overline{R} = R/I$, $\overline{S} = S/IS$, $\overline{\mathfrak{m}} = \mathfrak{m}/I$ and $\overline{M} = M/IM$.

**Proof.** Let $J \subset R$ be an ideal. Apply Algebra, Lemma 11.11 to the module $M/JM$ over the ring $R/J$. Then we see that $(M/JM)_q$ is flat over $R/J$ for all primes $q$ of $S/JS$ if and only if $M/(J + \mathfrak{m}^n)M$ is flat over $R/(J + \mathfrak{m}^n)$ for all $n \geq 1$. We will use this remark below.

For every $n \geq 1$ the local ring $R/\mathfrak{m}^n$ is Artinian. Hence, by Lemma 10.1 there exists a smallest ideal $I_n \supset \mathfrak{m}^n$ such that $M/I_n M$ is flat over $R/I_n$. It is clear that $I_{n+1} + \mathfrak{m}^n$ contains $I_n$ and applying Lemma 9.1 we see that $I_n = I_{n+1} + \mathfrak{m}^n$. Since $R = \lim_n R/\mathfrak{m}^n$ we see that $I = \lim_n I_n/\mathfrak{m}^n$ is an ideal in $R$ such that $I_n = I + \mathfrak{m}^n$ for all $n \geq 1$. By the initial remarks of the proof we see that $I$ verifies (1) and (2). Some details omitted. \]

**Lemma 13.2.** With notation $R \to S$, $M$, and $I$ and assumptions as in Lemma 13.1. Consider a local homomorphism of local rings $\varphi : (R, \mathfrak{m}) \to (R', \mathfrak{m}')$ such that $R'$ is Noetherian. Then the following are equivalent

(1) condition (11.0.1) holds for $(R' \to S \otimes_R R', \mathfrak{m}', M \otimes_R R')$, and
(2) $\varphi(I) = 0$.

**Proof.** The implication (2) $\Rightarrow$ (1) follows from Lemma 11.1. Let $\varphi : R \to R'$ be as in the lemma satisfying (1). We have to show that $\varphi(I) = 0$. This is equivalent to the condition that $\varphi(I)R' = 0$. By Artin-Rees in the Noetherian local ring $R'$ (see Algebra, Lemma 49.4) this is equivalent to the condition that $\varphi(I)R' + (\mathfrak{m}')^n = (\mathfrak{m}')^n$ for all $n > 0$. Hence this is equivalent to the condition that the composition $\varphi_n : R \to R' \to R'/(\mathfrak{m}')^n$ annihilates $I$ for each $n$. Now assumption (1) for $\varphi$ implies assumption (1) for $\varphi_n$ by Lemma 11.1. This reduces us to the case where $R'$ is Artinian local.

Assume $R'$ Artinian. Let $J = \ker(\varphi)$. We have to show that $I \subset J$. By the construction of $I$ in Lemma 13.1 it suffices to show that $(M/JM)_q$ is flat over $R/J$ for every prime $q$ of $S/JS$ lying over $\mathfrak{m}$. As $R'$ is Artinian, condition (1) signifies that $M \otimes_R R'$ is flat over $R'$. As $R'$ is Artinian and $R/J \to R'$ is a local injective ring map, it follows that $R/J$ is Artinian too. Hence the flatness of $M \otimes_R R' = M/JM \otimes_{R/J} R'$ over $R'$ implies that $M/JM$ is flat over $R/J$ by Algebra, Lemma 98.7. This concludes the proof. \]
Lemma 13.3. With notation \( R \to S, M, \) and \( I \) and assumptions as in Lemma 13.1. In addition assume that \( R \to S \) is of finite type. Then for any local homomorphism of local rings \( \varphi : (R, \mathfrak{m}) \to (R', \mathfrak{m}') \) the following are equivalent

1. condition \((11.0.1)\) holds for \((R' \to S \otimes_R R', M \otimes_R R')\), and
2. \( \varphi(I) = 0 \).

Proof. The implication \((2) \Rightarrow (1)\) follows from Lemma 11.1. Let \( \varphi : R \to R' \) be as in the lemma satisfying \((1)\). As \( R \) is Noetherian we see that \( R \to S \) is of finite presentation and \( M \) is an \( S\)-module of finite presentation. Write \( R' = \text{colim}_\lambda R_\lambda \) as a directed colimit of local \( R\)-subalgebras \( R_\lambda \subset R' \), with maximal ideals \( \mathfrak{m}_\lambda = R_\lambda \cap \mathfrak{m}' \) such that each \( R_\lambda \) is essentially of finite type over \( R \). By Lemma 11.3 we see that condition \((11.0.1)\) holds for \((R_\lambda \to S \otimes_R R_\lambda, \mathfrak{m}_\lambda \otimes_R R_\lambda)\) for some \( \lambda \). Hence Lemma 13.2 applies to the ring map \( R \to R_\lambda \) and we see that \( I \) maps to zero in \( R_\lambda \), a fortiori it maps to zero in \( R' \).

\[ \square \]

14. Descent flatness along integral maps

First a few simple lemmas.

Lemma 14.1. Let \( R \) be a ring. Let \( P(T) \) be a monic polynomial with coefficients in \( R \). If there exists an \( \alpha \in R \) such that \( P(\alpha) = 0 \), then \( P(T) = (T - \alpha)Q(T) \) for some monic polynomial \( Q(T) \in R[T] \).

Proof. By induction on the degree of \( P \). If \( \deg(P) = 1 \), then \( P(T) = T - \alpha \) and the result is true. If \( \deg(P) > 1 \), then we can write \( P(T) = (T - \alpha)Q(T) + r \) for some polynomial \( Q \in R[T] \) of degree \( < \deg(P) \) and some \( r \in R \) by long division. By assumption \( 0 = P(\alpha) = (\alpha - \alpha)Q(\alpha) + r = r \) and we conclude that \( r = 0 \) as desired.

\[ \square \]

Lemma 14.2. Let \( R \) be a ring. Let \( P(T) \) be a monic polynomial with coefficients in \( R \). There exists a finite free ring map \( R \to R' \) such that \( P(T) = (T - \alpha)Q(T) \) for some \( \alpha \in R' \) and some monic polynomial \( Q(T) \in R'[T] \).

Proof. Write \( P(T) = T^d + a_1 T^{d-1} + \ldots + a_0 \). Set \( R' = R[x]/(x^d + a_1 x^{d-1} + \ldots + a_0) \). Set \( \alpha \) equal to the congruence class of \( x \). Then it is clear that \( P(\alpha) = 0 \). Thus we win by Lemma 14.1.

\[ \square \]

Lemma 14.3. Let \( R \to S \) be a finite ring map. There exists a finite free ring extension \( R \subset R' \) such that \( S \otimes_R R' \) is a quotient of a ring of the form

\[ R'[T_1, \ldots, T_n]/(P_1(T_1), \ldots, P_n(T_n)) \]

with \( P_i(T) = \prod_{j=1}^{d_i}(T - \alpha_{ij}) \) for some \( \alpha_{ij} \in R' \).

Proof. Let \( x_1, \ldots, x_n \in S \) be generators of \( S \) over \( R \). For each \( i \) we can choose a monic polynomial \( P_i(T) \in R[T] \) such that \( P_i(x_i) = 0 \) in \( S \), see Algebra, Lemma 35.3. Say \( \deg(P_i) = d_i \). By Lemma 14.2 (applied \( \sum d_i \) times) there exists a finite free ring extension \( R \subset R' \) such that each \( P_i \) splits completely:

\[ P_i(T) = \prod_{j=1}^{d_i}(T - \alpha_{ij}) \]

for certain \( \alpha_{ik} \in R' \). Let \( R'[T_1, \ldots, T_n] \to S \otimes_R R' \) be the \( R'\)-algebra map which maps \( T_i \) to \( x_i \otimes 1 \). As this maps \( P_i(T_i) \) to zero, this induces the desired surjection.

\[ \square \]
Lemma 14.4. Let $R$ be a ring. Let $S = R[T_1, \ldots, T_n]/J$. Assume $J$ contains elements of the form $P(T) = \prod_{i=1}^{k} (T - \alpha_{ij})$ for some $\alpha_{ij} \in R$. For $k = (k_1, \ldots, k_n)$ with $1 \leq k_i \leq d_i$ consider the ring map
$$\Phi_k : R[T_1, \ldots, T_n] \to R, \quad T_i \mapsto \alpha_{ik_i}.$$ 
Set $J_k = \Phi_k(J)$. Then the image of $\text{Spec}(S) \to \text{Spec}(R)$ is equal to $V(\bigcap J_k)$. 

Proof. This lemma proves itself. Hint: $V(\bigcap J_k) = \bigcup V(J_k)$. □

The following result is due to Ferrand, see [Fer69].

Lemma 14.5. Let $R \to S$ be a finite injective homomorphism of Noetherian rings. Let $M$ be an $R$-module. If $M \otimes_R S$ is a flat $S$-module, then $M$ is a flat $R$-module.

Proof. Let $M$ be an $R$-module such that $M \otimes_R S$ is flat over $S$. By Algebra, Lemma [38.7] in order to prove that $M$ is flat we may replace $R$ by any faithfully flat ring extension. By Lemma [14.3] we can find a finite locally free ring extension $R \subset R'$ such that $S' = S \otimes_R R' = R'[T_1, \ldots, T_n]/J$ for some ideal $J \subset R'[T_1, \ldots, T_n]$ which contains the elements of the form $P_t(T)$ with $P_t(T) = \prod_{i=1}^{k} (T - \alpha_{ij})$ for some $\alpha_{ij} \in R'$. Note that $R'$ is Noetherian and that $R' \subset S'$ is a finite extension of rings. Hence we may replace $R$ by $R'$ and assume that $S$ has a presentation as in Lemma [14.3]. Note that $\text{Spec}(S) \to \text{Spec}(R)$ is surjective, see Algebra, Lemma [35.15]. Thus, using Lemma [14.4] we conclude that $I = \bigcap J_k$ is an ideal such that $V(I) = \text{Spec}(R)$. This means that $I \subset \sqrt{(0)}$, and since $R$ is Noetherian that $I$ is nilpotent. The maps $\Phi_k$ induce commutative diagrams

$$\begin{array}{ccc}
S & \longrightarrow & R/J_k \\
\downarrow & & \downarrow \\
R & \longrightarrow & R/J_k
\end{array}$$

from which we conclude that $M/J_k M$ is flat over $R/J_k$. By Lemma [9.1] we see that $M/IM$ is flat over $R/I$. Finally, applying Algebra, Lemma [98.5] we conclude that $M$ is flat over $R$. □

Lemma 14.6. Let $R \to S$ be an injective integral ring map. Let $M$ be a finitely presented module over $R[x_1, \ldots, x_n]$. If $M \otimes_R S$ is flat over $S$, then $M$ is flat over $R$.

Proof. Choose a presentation
$$R[x_1, \ldots, x_n]^{\oplus t} \to R[x_1, \ldots, x_n]^{\oplus r} \to M \to 0.$$ 
Let’s say that the first map is given by the $r \times t$-matrix $T = (f_{ij})$ with $f_{ij} \in R[x_1, \ldots, x_n]$. Write $f_{ij} = \sum f_{ij, t} x^t$ with $f_{ij, t} \in R$ (multi-index notation). Consider diagrams

$$\begin{array}{ccc}
R & \longrightarrow & S \\
\uparrow & & \uparrow \\
R_{\lambda} & \longrightarrow & S_{\lambda}
\end{array}$$

where $R_{\lambda}$ is a finitely generated $\mathbb{Z}$-subalgebra of $R$ containing all $f_{ij, t}$ and $S_{\lambda}$ is a finite $R_{\lambda}$-subalgebra of $S$. Let $M_{\lambda}$ be the finite $R_{\lambda}[x_1, \ldots, x_n]$-module defined by a presentation as above, using the same matrix $T$ but now viewed as a matrix over
Note that $S$ is the directed colimit of the $S_\lambda$ (details omitted). By Algebra, Lemma 157.1 we see that for some $\lambda$ the module $M_\lambda \otimes_{R_\lambda} S_\lambda$ is flat over $S_\lambda$. By Lemma 14.5 we conclude that $M_\lambda$ is flat over $R_\lambda$. Since $M = M_\lambda \otimes_{R_\lambda} R$ we win by Algebra, Lemma 38.6.

15. Torsion free modules

In this section we discuss torsion free modules and the relationship with flatness (especially over dimension 1 rings).

**Definition 15.1.** Let $R$ be a domain. Let $M$ be an $R$-module.

1. We say an element $x \in M$ is torsion if there exists a nonzero $f \in R$ such that $fx = 0$.
2. We say $M$ is torsion free if the only torsion element of $M$ is 0.

Let $R$ be a domain and let $S = R \setminus \{0\}$ be the multiplicative set of nonzero elements of $R$. Then an $R$-module $M$ is torsion free if and only if $M \to S^{-1}M$ is injective. In other words, if and only if the map $M \to M \otimes_R K$ is injective where $K = S^{-1}R$ is the fraction field of $R$.

**Lemma 15.2.** Let $R$ be a domain. Let $M$ be an $R$-module. The set of torsion elements of $M$ forms a submodule $M_{\text{tors}} \subset M$. The quotient module $M/M_{\text{tors}}$ is torsion free.

**Proof.** Omitted.

**Lemma 15.3.** Let $R$ be a domain. Let $M$ be a torsion free $R$-module. For any multiplicative set $S \subset R$ the module $S^{-1}M$ is a torsion free $S^{-1}R$-module.

**Proof.** Omitted.

**Lemma 15.4.** Let $R \to R'$ be a flat homomorphism of domains. If $M$ is a torsion free $R$-module, then $M \otimes_R R'$ is a torsion free $R'$-module.

**Proof.** If $M$ is torsion free, then $M \subset M \otimes_R K$ is injective where $K$ is the fraction field of $R$. Since $R'$ is flat over $R$ we see that $M \otimes_R R' \to (M \otimes_R K) \otimes_R R'$ is injective. Since $M \otimes_R K$ is isomorphic to a direct sum of copies of $K$, it suffices to see that $K \otimes_R R'$ is torsion free. This is true because it is a localization of $R'$.

**Lemma 15.5.** Let $R$ be a domain. Let 0 $\to$ $M$ $\to$ $M'$ $\to$ $M''$ $\to$ 0 be a short exact sequence of $R$-modules. If $M$ and $M''$ are torsion free, then $M'$ is torsion free.

**Proof.** Omitted.

**Lemma 15.6.** Let $R$ be a domain. Let $M$ be an $R$-module. Then $M$ is torsion free if and only if $M_m$ is a torsion free $R_m$-module for all maximal ideals $m$ of $R$.


**Lemma 15.7.** Let $R$ be a domain. Let $M$ be a finite $R$-module. Then $M$ is torsion free if and only if $M$ is a submodule of a finite free module.

**Proof.** If $M$ is a submodule of $R^{\oplus n}$, then $M$ is torsion free. For the converse, assume $M$ is torsion free. Let $K$ be the fraction field of $R$. Then $M \otimes_R K$ is a finite dimensional $K$-vector space. Choose a basis $e_1, \ldots, e_r$ for this vector space. Let $x_1, \ldots, x_n$ be generators of $M$. Write $x_i = \sum (a_{ij}/b_{ij}) e_j$ for some $a_{ij}, b_{ij} \in R$. Then $x_i$ is a finite sum of elements of the form $x_j b_{ij}^{-1}$, which shows that $x_i$ is torsion free. Therefore, $M$ is torsion free. The converse follows from the fact that $M \subset R^{\oplus n}$ is a torsion free module. □
with \( b_{ij} \neq 0 \). Set \( b = \prod_{i,j} b_{ij} \). Since \( M \) is torsion free the map \( M \to M \otimes_R K \) is injective and the image is contained in \( R^{\oplus r} = Re_1/b \oplus \ldots \oplus Re_r/b \).

**Lemma 15.8.** Let \( R \) be a Noetherian domain. Let \( M \) be a nonzero finite \( R \)-module. The following are equivalent

1. \( M \) is torsion free,
2. \( M \) is a submodule of a finite free module,
3. \( (0) \) is the only associated prime of \( M \),
4. \( (0) \) is in the support of \( M \) and \( M \) has property \((S_1)\), and
5. \( (0) \) is in the support of \( M \) and \( M \) has no embedded associated prime.

**Proof.** We have seen the equivalence of (1) and (2) in Lemma 15.7. We have seen the equivalence of (4) and (5) in Algebra, Lemma 147.2. The equivalence between (3) and (5) is immediate from the definition. A localization of a torsion free module is torsion free (Lemma 15.3), hence it is clear that a \( M \) has no associated primes different from \( (0) \). Thus (1) implies (5). Conversely, assume (5). If \( M \) has torsion, then there exists an embedding \( R/I \subset M \) for some nonzero ideal \( I \) of \( R \). Hence \( M \) has an associated prime different from \( (0) \) (see Algebra, Lemmas 62.3 and 62.7). This is an embedded associated prime which contradicts the assumption. \(\square\)

**Lemma 15.9.** Let \( R \) be a domain. Any flat \( R \)-module is torsion free.

**Proof.** If \( x \in R \) is nonzero, then \( x: R \to R \) is injective, and hence if \( M \) is flat over \( R \), then \( x: M \to M \) is injective. Thus if \( M \) is flat over \( R \), then \( M \) is torsion free. \(\square\)

**Lemma 15.10.** Let \( A \) be a valuation ring. An \( A \)-module \( M \) is flat over \( A \) if and only if \( M \) is torsion free.

**Proof.** The implication “flat \(\Rightarrow\) torsion free” is Lemma 15.9. For the converse, assume \( M \) is torsion free. By the equational criterion of flatness (see Algebra, Lemma 38.10), we have to show that every relation in \( M \) is trivial. To do this assume that \( \sum_{i=1,\ldots,n} a_i x_i = 0 \) with \( x_i \in M \) and \( f_i \in A \). After renumbering we may assume that \( v(a_1) \leq v(a_i) \) for all \( i \). Hence we can write \( a_i = a_i' a_1 \) for some \( a_i' \in A \). Note that \( a_1' = 1 \). As \( A \) is torsion free we see that \( x_1 = -\sum_{i \geq 2} a_i' x_i \). Thus, if we choose \( y_i = x_i, i = 2,\ldots,n \) then

\[
x_1 = \sum_{j \geq 2} -a_j' y_j, \quad x_i = y_i, (i \geq 2) \quad 0 = a_1 \cdot (-a_j') + a_j \cdot 1 (j \geq 2)
\]

shows that the relation was trivial (to be explicit the elements \( a_{ij} \) are defined by setting \( a_{1j} = -a_j' \) and \( a_{ij} = \delta_{ij} \) for \( i,j \geq 2 \)). \(\square\)

**Lemma 15.11.** Let \( A \) be a Dedekind domain (for example a PID or a discrete valuation ring).

1. An \( A \)-module is flat if and only if it is torsion free.
2. A finite torsion free \( A \)-module is finite locally free.
3. A finite torsion free \( A \)-module is finite free if \( A \) is a PID or a discrete valuation ring.

**Proof.** Proof of (1). Since a PID is a Dedekind domain (Algebra, Lemma 117.12), it suffices to prove this for Dedekind domains. By Lemma 15.6 and Algebra, Lemma 38.19 it suffices to check the statement over \( A_m \) for \( m \subset A \) maximal. Since \( A_m \) is a discrete valuation ring (Algebra, Lemma 117.14) we win by Lemma 15.10.
Proof of (2). Follows from Algebra, Lemma 15.7 and (1).

Proof of (3). If $A$ is a discrete valuation ring this follows from (2) and the definitions. Let $A$ be a PID and let $M$ be a finite torsion free module. By Lemma 15.7 we see that $M \subset A^{\oplus n}$ for some $n$. We argue that $M$ is free by induction on $M$. The case $n = 1$ expresses exactly the fact that $A$ is a PID. If $n > 1$ let $M' \subset R^{\oplus n-1}$ be the image of the projection onto the last $n - 1$ summands of $R^{\oplus n}$. By induction we see that $M$ is an extension of finite free $R$-modules, whence finite free. □

Lemma 15.12. Let $R$ be a domain. Let $M$, $N$ be $R$-modules. If $N$ is torsion free, so is $\text{Hom}_R(M,N)$.

Proof. Choose a surjection $\bigoplus_{i \in I} R \to M$. Then $\text{Hom}_R(M,N) \subset \prod_{i \in I} N$. □

16. Reflexive modules

Here is our definition.

Definition 16.1. Let $R$ be a domain. We say an $R$-module $M$ is reflexive if the natural map

$$j : M \to \text{Hom}_R(\text{Hom}_R(M,R),R)$$

which sends $m \in M$ to the map sending $\varphi \in \text{Hom}_R(M,R)$ to $\varphi(m) \in R$ is an isomorphism.

We can make this definition for more general rings, but already the definition above has drawbacks. It would be wise to restrict to Noetherian domains and finite torsion free modules and (perhaps) impose some regularity conditions on $R$ (e.g., $R$ is normal).

Lemma 16.2. Let $R$ be a domain and let $M$ be an $R$-module.

(1) If $M$ is reflexive, then $M$ is torsion free.

(2) If $M$ is finite, then $j : M \to \text{Hom}_R(\text{Hom}_R(M,R),R)$ is injective if and only if $M$ is torsion free

Proof. Follows immediately from Lemmas 15.12 and 15.7 □

Lemma 16.3. Let $R$ be a Noetherian domain. Let $M$ be a finite $R$-module. Then $M$ is reflexive if and only if $M_m$ is a reflexive $R_m$-module for all maximal ideals $m$ of $R$.

Proof. Omitted. Hint: Use Algebra, Lemmas 23.1 and 10.2 □

Lemma 16.4. Let $R$ be a Noetherian domain. Let $M$ be a finite $R$-module. The following are equivalent

(1) $M$ is reflexive,

(2) there exists a short exact sequence $0 \to M \to F \to N \to 0$ with $F$ finite free and $N$ torsion free.

Proof. We will use without further mention that $\text{Hom}_R(N,N')$ is a finite $R$-module for any finite $R$-modules $N$ and $N'$, see Algebra, Lemma 10.2. Given an exact sequence $0 \to M \to F \to N \to 0$ as in (2) we take duals to get an exact sequence $\text{Hom}_R(M,R) \leftarrow \text{Hom}_R(F,R) \leftarrow \text{Hom}_R(N,R) \leftarrow 0$
Dualizing again we obtain a commutative diagram

\[
\begin{array}{ccc}
\text{Hom}_R(\text{Hom}_R(M, R), R) & \longrightarrow & \text{Hom}_R(\text{Hom}_R(F, R), R) \\
\uparrow & & \uparrow \\
M & \longrightarrow & F \\
\downarrow & & \downarrow \\
\text{Hom}_R(\text{Hom}_R(N, R), R) & \longrightarrow & N
\end{array}
\]

We do not know the top row is exact. But we do know the middle arrow is an isomorphism as \( F \) is finite free and hence reflexive. Moreover, if \( S = R \setminus \{0\} \), then inverting \( S \) commutes with taking \( \text{Hom}_R \) for finite \( R \)-modules, see Algebra, Lemma 10.2. Since \( S^{-1}M \) and \( S^{-1}N \) are finite free over the fraction field \( K = S^{-1}R \) of \( R \), we find that the vertical maps are isomorphisms after inverting \( S \). Since \( \text{Hom}_R(\text{Hom}_R(M, R), R) \) is torsion free (Lemma 15.12), it follows in particular that the left top horizontal arrow is injective. Since \( N \) is torsion free the right vertical arrow is injective (Lemma 16.2). Now a diagram chase shows that \( M \) is reflexive.

Assume \( M \) is reflexive. Choose a presentation \( R^{\oplus m} \to R^{\oplus n} \to \text{Hom}_R(M, R) \to 0 \). Dualizing and using reflexivity we get an exact sequence

\[
0 \to \text{Hom}_R(\text{Hom}_R(M, R), R) \to R^{\oplus n} \to N \to 0
\]

with \( N = \text{Im}(R^{\oplus n} \to R^{\oplus m}) \) a torsion free module. \( \square \)

**Lemma 16.5.** Let \( R \) be a Noetherian domain. Let \( M \) be a finite \( R \)-module. Let \( N \) be a reflexive \( R \)-module. Then \( \text{Hom}_R(M, N) \) is reflexive.

**Proof.** Choose a presentation \( R^{\oplus m} \to R^{\oplus n} \to M \to 0 \). Then we obtain

\[
0 \to \text{Hom}_R(M, N) \to N^{\oplus n} \to N' \to 0
\]

with \( N' = \text{Im}(N^{\oplus n} \to N^{\oplus m}) \) torsion free. Choose a sequence \( 0 \to N \to F \to N'' \to 0 \) with \( N'' \) torsion free as in Lemma 16.4. We obtain an injective map \( \delta : \text{Hom}_R(M, N) \to F^{\oplus n} \). A snake lemma argument shows there is a short exact sequence

\[
0 \to N' \to \text{Coker}(\delta) \to (N'')^{\oplus n} \to 0
\]

Thus \( \text{Coker}(\delta) \) is an extension of torsion free modules, hence torsion free (Lemma 15.5). \( \square \)

**Definition 16.6.** Let \( R \) be a Noetherian domain. Let \( M \) be a finite \( R \)-module. The module \( M^{**} = \text{Hom}_R(\text{Hom}_R(M, R), R) \) is called the reflexive hull of \( M \).

This makes sense because the reflexive hull is reflexive by Lemma 16.5. The assignment \( M \mapsto M^{**} \) is a functor. If \( \varphi : M \to N \) is an \( R \)-module map into a reflexive \( R \)-module \( N \), then \( \varphi \) factors \( M \to M^{**} \to N \) through the reflexive hull of \( M \). Another way to say this is that taking the reflexive hull is the left adjoint to the inclusion functor

finite reflexive modules \( \subset \) finite modules

over a Noetherian domain \( R \).

**Lemma 16.7.** Let \( R \) be a Noetherian local ring. Let \( M, N \) be finite \( R \)-modules.

1. If \( N \) has depth \( \geq 1 \), then \( \text{Hom}_R(M, N) \) has depth \( \geq 1 \).
2. If \( N \) has depth \( \geq 2 \), then \( \text{Hom}_R(M, N) \) has depth \( \geq 2 \).
Choose a presentation $R^\oplus m \to R^\oplus n \to M \to 0$. Dualizing we get an exact sequence

$$0 \to \text{Hom}_R(M, N) \to N^\oplus n \to N' \to 0$$

with $N' = \text{Im}(N^\oplus n \to N^\oplus m)$. A submodule of a module with depth $\geq 1$ has depth $\geq 1$; this follows immediately from the definition. Thus part (1) is clear. For (2) note that here the assumption and the previous remark implies $N'$ has depth $\geq 1$. The module $N^\oplus n$ has depth $\geq 2$. From Algebra, Lemma 70.6 we conclude $\text{Hom}_R(M, N)$ has depth $\geq 2$. □

Lemma 16.8. Let $R$ be a Noetherian ring. Let $M, N$ be finite $R$-modules.

1. If $N$ has property $(S_1)$, then $\text{Hom}_R(M, N)$ has property $(S_1)$.
2. If $N$ has property $(S_2)$, then $\text{Hom}_R(M, N)$ has property $(S_2)$.
3. If $R$ is a domain, $N$ is torsion free and $(S_2)$, then $\text{Hom}_R(M, N)$ is torsion free and has property $(S_2)$.

Proof. Since localizing at primes commutes with taking $\text{Hom}_R$ for finite $R$-modules (Algebra, Lemma 69.9) parts (1) and (2) follow immediately from Lemma 16.7. Part (3) follows from (2) and Lemma 15.12. □

Lemma 16.9. Let $R$ be a Noetherian ring. Let $\varphi : M \to N$ be a map of $R$-modules. Assume that for every prime $p$ of $R$ at least one of the following happens

1. $M_p \to N_p$ is injective, or
2. $p \notin \text{Ass}(M)$.

Then $\varphi$ is injective.

Proof. Let $p$ be an associated prime of $\text{Ker}(\varphi)$. Then there exists an element $x \in M_p$ which is in the kernel of $M_p \to N_p$ and is annihilated by $p R_p$ (Algebra, Lemma 62.14). This is impossible in all three cases. Hence $\text{Ass}(\text{Ker}(\varphi)) = \emptyset$ and we conclude $\text{Ker}(\varphi) = 0$ by Algebra, Lemma 62.7. □

Lemma 16.10. Let $R$ be a Noetherian ring. Let $\varphi : M \to N$ be a map of $R$-modules. Assume $M$ is finite and that for every prime $p$ of $R$ one of the following happens

1. $M_p \to N_p$ is an isomorphism, or
2. $\text{depth}(M_p) \geq 2$ and $p \notin \text{Ass}(N)$.

Then $\varphi$ is an isomorphism.

Proof. By Lemma 16.9 we see that $\varphi$ is injective. Let $N' \subset N$ be an finitely generated $R$-module containing the image of $M$. Then $\text{Ass}(N_p) = \emptyset$ implies $\text{Ass}(N'_p) = \emptyset$. Hence the assumptions of the lemma hold for $M \to N'$. In order to prove that $\varphi$ is an isomorphism, it suffices to prove the same thing for every such $N' \subset N$. Thus we may assume $N$ is a finite $R$-module. In this case, $p \notin \text{Ass}(N) \Rightarrow \text{depth}(N_p) \geq 1$, see Algebra, Lemma 62.17. Consider the short exact sequence

$$0 \to M \to N \to Q \to 0$$

defining $Q$. Looking at the conditions we see that either $Q_p = 0$ in case (1) or $\text{depth}(Q_p) \geq 1$ in case (2) by Algebra, Lemma 70.6. This implies that $Q$ does not have any associated primes, hence $Q = 0$ by Algebra, Lemma 62.7. □
Lemma 16.11. Let \( R \) be a Noetherian domain. Let \( \varphi : M \to N \) be a map of \( R \)-modules. Assume \( M \) is finite, \( N \) is torsion free, and that for every prime \( p \) of \( R \) one of the following happens

1. \( M_p \to N_p \) is an isomorphism, or
2. \( \text{depth}(M_p) \geq 2 \).

Then \( \varphi \) is an isomorphism.

Proof. This is a special case of Lemma 16.10. \( \square \)

Lemma 16.12. Let \( R \) be a Noetherian domain. Let \( M \) be a finite \( R \)-module. The following are equivalent

1. \( M \) is reflexive,
2. for every prime \( p \) of \( R \) one of the following happens
   a. \( M_p \) is a reflexive \( R_p \)-module, or
   b. \( \text{depth}(R_p) \geq 2 \) and \( \text{depth}(M_p) \geq 2 \).

Proof. If (1) is true, then (2) holds by Lemmas \[16.3\] and \[16.7\]. Conversely, assume (2) is true. Set \( N = \text{Hom}_R(\text{Hom}_R(M, R), R) \) so that \( N_p = \text{Hom}_{R_p}(\text{Hom}_{R_p}(M_p, R_p), R_p) \) (Algebra, Lemma \[10.2\]) for every prime \( p \) of \( R \). We apply Lemma 16.11 to the map \( j : M \to N \). This is allowed because \( M \) is finite, \( N \) is torsion free by Lemma \[15.12\], in case (2)(a) the map \( M_p \to N_p \) is an isomorphism, and in case (2)(b) we have \( \text{depth}(M_p) \geq 2 \). \( \square \)

Lemma 16.13. Let \( R \) be a Noetherian normal domain with fraction field \( K \). Let \( M \) be a finite \( R \)-module. The following are equivalent

1. \( M \) is reflexive,
2. \( M \) is torsion free and has property \((S_2)\),
3. \( M \) is torsion free and \( M = \bigcap_{\text{height}(p)=1} M_p \) where the intersection happens in \( M \otimes_R K \).

Proof. By Algebra, Lemma \[147.4\] we see that \( R \) satisfies \((R_1)\) and \((S_2)\). Observe that in all three cases \( M \) is a torsion free module (Lemma \[16.2\]). Let \( p \) be a prime of height 1, hence \( R_p \) is a discrete valuation ring by \((R_1)\). By Lemma \[16.11\] we see that \( M_p \) is finite free, in particular reflexive. The same is true for \( M_{(0)} \), since \( R \) is normal, we have \( \text{depth}(R_p) \geq 2 \) for every prime of heigth \( \geq 2 \) by \((S_2)\) for \( R \). Thus Lemma \[16.12\] applies to show the equivalence of (1) and (2).

Assume the equivalent conditions (1) and (2) hold and let \( M' = \bigcap_{\text{height}(p)=1} M_p \). Then \( M' \) is torsion free, \( M \subseteq M' \) and \( M_p = M'_p \) for every prime of height 1. Since we’ve seen \( M \) has depth \( \geq 2 \) at primes of height \( > 1 \), we see that \( M \to M' \) is an isomorphism by Lemma \[16.11\].

Assume (3). The map \( M \to M^{**} \) induces an isomorphism at all the primes \( p \) of height 1, because \( M_p \) is finite free as we’ve seen above. Thus the condition \( M = \bigcap_{\text{height}(p)=1} M_p \) implies that \( M = M^{**} \) and we win. \( \square \)

Lemma 16.14. Let \( R \) be a Noetherian normal domain. Let \( M \) be a finite \( R \)-module. Then the reflexive hull of \( M \) is the intersection

\[
M^{**} = \bigcap_{\text{height}(p)=1} M_p/(M_p)_{\text{tors}} = \bigcap_{\text{height}(p)=1} (M/M_{\text{tors}})_p
\]
Applying our hypothesis on $M'$ we conclude that $u(x)$ does not map to zero under the map

$$IN = I \otimes_A N \xrightarrow{\chi \otimes 1} A/m \otimes N \cong N/mN$$

and we conclude. \qed

**Lemma 17.4.** Let $A$ be a ring. Let $M$ be a flat Mittag-Leffler module. Then every element of $M$ has a content ideal.

**Proof.** This is a special case of Algebra, Lemma \ref{flat-mittag-leffler}. \qed

**18. Flatness and finiteness conditions**

In this section we discuss some implications of the type “flat + finite type $\Rightarrow$ finite presentation”. We will revisit this result in the chapter on flatness, see More on Flatness, Section \ref{mof-flatness}. A first result of this type was proved in Algebra, Lemma \ref{flat-finite-type}. 

Lemma 18.1. Let $R$ be a ring. Let $S = R[x_1, \ldots, x_n]$ be a polynomial ring over $R$. Let $M$ be an $S$-module. Assume

1. there exist finitely many primes $p_1, \ldots, p_m$ of $R$ such that the map $R \to \prod R_{p_j}$ is injective,
2. $M$ is a finite $S$-module,
3. $M$ flat over $R$, and
4. for every prime $p$ of $R$ the module $M_p$ is of finite presentation over $S_p$.

Then $M$ is of finite presentation over $S$.

Proof. Choose a presentation

$$0 \to K \to S^{\oplus r} \to M \to 0$$

of $M$ as an $S$-module. Let $q$ be a prime ideal of $S$ lying over a prime $p$ of $R$. By assumption there exist finitely many elements $k_1, \ldots, k_t \in K$ such that if we set $K' = \sum Sk_j \subset K$ then $K'_p = K_p$ and $K'_{p_j} = K_{p_j}$ for $j = 1, \ldots, m$. Setting $M' = S^{\oplus r}/K'$ we deduce that in particular $M'_q = M_q$. By openness of flatness, see Algebra, Theorem 126.4 we conclude that there exists a $g \in S$, $g \notin q$ such that $M'_g$ is flat over $R$. Thus $M'_g \to M_g$ is a surjective map of flat $R$-modules. Consider the commutative diagram

$$
\begin{array}{ccc}
M'_g & \longrightarrow & M_g \\
\downarrow & & \downarrow \\
\prod (M'_g)_{p_j} & \longrightarrow & \prod (M_g)_{p_j}
\end{array}
$$

The bottom arrow is an isomorphism by choice of $k_1, \ldots, k_t$. The left vertical arrow is an injective map as $R \to \prod R_{p_j}$ is injective and $M'_g$ is flat over $R$. Hence the top horizontal arrow is injective, hence an isomorphism. This proves that $M_g$ is of finite presentation over $S_g$. We conclude by applying Algebra, Lemma 23.2. □

Lemma 18.2. Let $R \to S$ be a ring homomorphism. Assume

1. there exist finitely many primes $p_1, \ldots, p_m$ of $R$ such that the map $R \to \prod R_{p_j}$ is injective,
2. $R \to S$ is of finite type,
3. $S$ flat over $R$, and
4. for every prime $p$ of $R$ the ring $S_p$ is of finite presentation over $R_p$.

Then $S$ is of finite presentation over $R$.

Proof. By assumption $S$ is a quotient of a polynomial ring over $R$. Thus the result follows directly from Lemma 18.1. □

Lemma 18.3. Let $R$ be a ring. Let $S = R[x_1, \ldots, x_n]$ be a graded polynomial algebra over $R$, i.e., $\deg(x_i) > 0$ but not necessarily equal to 1. Let $M$ be a graded $S$-module. Assume

1. $R$ is a local ring,
2. $M$ is a finite $S$-module, and
3. $M$ is flat over $R$.

Then $M$ is finitely presented as an $S$-module.
Proof. Let \( M = \bigoplus M_d \) be the grading on \( M \). Pick homogeneous generators \( m_1, \ldots, m_r \in M \) of \( M \). Say \( \deg(m_i) = d_i \in \mathbb{Z} \). This gives us a presentation
\[
0 \to K \to \bigoplus_{i=1,\ldots,r} S(-d_i) \to M \to 0
\]
which in each degree \( d \) leads to the short exact sequence
\[
0 \to K_d \to \bigoplus_{i=1,\ldots,r} S_{d-d_i} \to M_d \to 0.
\]
By assumption each \( M_d \) is a flat \( R \)-module. By Algebra, Lemma [76.4] this implies each \( M_d \) is a finite free \( R \)-module. Hence we see each \( K_d \) is a finite \( R \)-module. Also each \( K_d \) is flat over \( R \) by Algebra, Lemma [38.12]. Hence we conclude that each \( K_d \) is finite free by Algebra, Lemma [76.4] again.

Let \( m \) be the maximal ideal of \( R \). By the flatness of \( M \) over \( R \) the short exact sequences above remain short exact after tensoring with \( \kappa = \kappa(m) \). As the ring \( S \otimes_R \kappa \) is Noetherian we see that there exist homogeneous elements \( k_1, \ldots, k_t \in K \) such that the images \( \mathcal{E}_j \) generate \( K \otimes_R \kappa \) over \( S \otimes_R \kappa \). Say \( \deg(k_j) = e_j \). Thus for any \( d \) the map
\[
\bigoplus_{j=1,\ldots,t} S_{d-e_j} \longrightarrow K_d
\]
becomes surjective after tensoring with \( \kappa \). By Nakayama’s lemma (Algebra, Lemma [19.1]) this implies the map is surjective over \( R \). Hence \( K \) is generated by \( k_1, \ldots, k_t \) over \( S \) and we win.

Lemma 18.4. Let \( R \) be a ring. Let \( S = \bigoplus_{n \geq 0} S_n \) be a graded \( R \)-algebra. Let \( M = \bigoplus_{d \in \mathbb{Z}} M_d \) be a graded \( S \)-module. Assume \( S \) is finitely generated as an \( R \)-algebra, assume \( S_0 \) is a finite \( R \)-algebra, and assume there exist finitely many primes \( p_j, i = 1, \ldots, m \) such that \( R \to \prod R_{p_j} \) is injective.

1. If \( S \) is flat over \( R \), then \( S \) is a finitely presented \( R \)-algebra.
2. If \( M \) is flat as an \( R \)-module and finite as an \( S \)-module, then \( M \) is finitely presented as an \( S \)-module.

Proof. As \( S \) is finitely generated as an \( R \)-algebra, it is finitely generated as an \( S_0 \) algebra, say by homogeneous elements \( t_1, \ldots, t_n \in S \) of degrees \( d_1, \ldots, d_n > 0 \). Set \( P = R[x_1, \ldots, x_n] \) with \( \deg(x_i) = d_i \). The ring map \( P \to S, x_i \to t_i \) is finite as \( S_0 \) is a finite \( R \)-module. To prove (1) it suffices to prove that \( S \) is a finitely presented \( P \)-module. To prove (2) it suffices to prove that \( M \) is a finitely presented \( P \)-module. Thus it suffices to prove that if \( S = P \) is a graded polynomial ring and \( M \) is a finite \( S \)-module flat over \( R \), then \( M \) is finitely presented as an \( S \)-module. By Lemma [18.3] we see \( M_p \) is a finitely presented \( S_p \)-module for every prime \( p \) of \( R \). Thus the result follows from Lemma [18.1].

Remark 18.5. Let \( R \) be a ring. When does \( R \) satisfy the condition mentioned in Lemmas [18.1] [18.2] and [18.4]? This holds if
1. \( R \) is local,
2. \( R \) is Noetherian,
3. \( R \) is a domain,
4. \( R \) is a reduced ring with finitely many minimal primes, or
5. \( R \) has finitely many weakly associated primes, see Algebra, Lemma [65.16].

Thus these lemmas hold in all cases listed above.

The following lemma will be improved on in More on Flatness, Proposition [12.9]
Lemma 18.6. Let $A$ be a valuation ring. Let $A \to B$ be a ring map of finite type. Let $M$ be a finite $B$-module.

1. If $B$ is flat over $A$, then $B$ is a finitely presented $A$-algebra.
2. If $M$ is flat as an $A$-module, then $M$ is finitely presented as a $B$-module.

Proof. We are going to use that an $A$-module is flat if and only if it is torsion free, see Lemma [15.10]. By Algebra, Lemma [56.10] we can find a graded $A$-algebra $S$ with $S_0 = A$ and generated by finitely many elements in degree 1, an element $f \in S_1$ and a finite graded $S$-module $N$ such that $B \cong S(f)$ and $M \cong N(f)$. If $M$ is torsion free, then we can take $N$ torsion free by replacing it by $N/N_{tors}$, see Lemma [15.2]. Similarly, if $B$ is torsion free, then we can take $S$ torsion free by replacing it by $S/S_{tors}$. Hence in case (1), we may apply Lemma [18.4] to see that $S$ is a finitely presented $A$-algebra, which implies that $B = S(f)$ is a finitely presented $A$-algebra. To see (2) we may first replace $S$ by a graded polynomial ring, and then we may apply Lemma [18.3] to conclude. □

19. Blowing up and flatness

In this section we begin our discussion of results of the form: “After a blow up the strict transform becomes flat”. More results of this type may be found in More on Flatness, Section [27].

Definition 19.1. Let $R$ be a domain. Let $M$ be an $R$-module. Let $R \subset R'$ be an extension of domains. The strict transform of $M$ along $R \to R'$ is the torsion free $R'$-module

$$M' = (M \otimes_R R')/(M \otimes_R R')_{tors}.$$

The following is a very weak version of flattening by blowing up, but it is already sometimes a useful result.

Lemma 19.2. Let $(R, \mathfrak{m})$ be a local domain with fraction field $K$. Let $S$ be a finite type $R$-algebra. Let $M$ be a finite $S$-module. For every valuation ring $A \subset K$ dominating $R$ there exists an ideal $I \subset \mathfrak{m}$ and a nonzero element $a \in I$ such that

1. $I$ is finitely generated,
2. $A$ has center on $R^*[1/a]$,
3. the fibre ring of $R \to R^*[1/a]$ at $\mathfrak{m}$ is not zero, and
4. the strict transform $S_{I,a}$ of $S$ along $R \to R^*[1/a]$ is flat and of finite presentation over $R$, and the strict transform $M_{I,a}$ of $M$ along $R \to R^*[1/a]$ is flat over $R$ and finitely presented over $S_{I,a}$.

Proof. Note that the assertion makes sense as $R^*[1/a]$ is a domain, and $R \to R^*[1/a]$ is injective, see Algebra, Lemmas [56.4] and [56.6]. Before we start the proof of the Lemma, note that there is no loss in generality assuming that $S = R[x_1, \ldots, x_n]$ is a polynomial ring over $R$. We also fix a presentation

$$0 \to K \to S^{\oplus r} \to M \to 0.$$

Let $M_A$ be the strict transform of $M$ along $R \to A$. It is a finite module over $S_A = A[x_1, \ldots, x_n]$. By Lemma [15.10] we see that $M_A$ is flat over $A$. By Lemma [18.6] we see that $M_A$ is finitely presented. Hence there exist finitely many elements $k_1, \ldots, k_t \in S_A^{\oplus r}$ which generate the kernel of the presentation $S_A^{\oplus r} \to M_A$ as an

\footnote{This is somewhat nonstandard notation.}
$S_A$-module. For any choice of $a \in I \subset \mathfrak{m}$ satisfying (1), (2), and (3) we denote $M_{I,a}$ the strict transform of $M$ along $R \to R[\frac{I}{a}]$. It is a finite module over $S_{I,a} = R[\frac{I}{a}][x_1, \ldots, x_n]$. By Algebra, Lemma 56.7 we have $A = \text{colim}_{I,a} R[\frac{I}{a}]$. This implies that $S_A = \text{colim}_{I,a} S_{I,a}$ and $M_A = \text{colim}_{I,a} M_{I,a}$. Thus we may choose $a \in I \subset R$ such that $k_1, \ldots, k_t$ are elements of $S_{I,a}^{\text{gr}}$ and map to zero in $M_{I,a}$. For any such pair $(I,a)$ we set

$$M'_{I,a} = S_{I,a}^{\text{gr}} / \sum S_{I,a} k_j.$$ 

Since $M_A = \text{colim}_{I,a} M'_{I,a}$ we see that also $M_A = \text{colim}_{I,a} M_{I,a}$. At this point we may apply Algebra, Lemma 157.1 (3) to conclude that $M'_{I,a}$ is flat for some pair $(I,a)$. (This lemma does not apply a priori to the system $M_{I,a}$ as the transition maps may not satisfy the assumptions of the lemma.) Since flatness implies torsion free (Lemma 15.9), we also conclude that $M'_{I,a} = M_{I,a}$ for such a pair and we win. \hfill $\Box$

20. Completion and flatness

In this section we discuss when the completion of a “big” flat module is flat.

**Lemma 20.1.** Let $R$ be a ring. Let $I \subset R$ be an ideal. Let $A$ be a set. Assume $R$ is Noetherian and complete with respect to $I$. There is a canonical map

$$\left( \bigoplus_{\alpha \in A} R \right)^{\wedge} \to \prod_{\alpha \in A} R$$

from the $I$-adic completion of the direct sum into the product which is universally injective.

**Proof.** By definition an element $x$ of the left hand side is $x = (x_n)$ where $x_n = (x_{n,\alpha}) \in \bigoplus_{\alpha \in A} R/I^n$ such that $x_{n,\alpha} = x_{n+1,\alpha} \mod I^n$. As $R = R^{\wedge}$ we see that for any $\alpha$ there exists a $y_\alpha \in R$ such that $x_{n,\alpha} = y_\alpha \mod I^n$. Note that for each $n$ there are only finitely many $\alpha$ such that the elements $x_{n,\alpha}$ are nonzero. Conversely, given $(y_\alpha) \in \prod_{\alpha \in A} R$ such that for each $n$ there are only finitely many $\alpha$ such that $y_\alpha \mod I^n$ is nonzero, then this defines an element of the left hand side. Hence we can think of an element of the left hand side as infinite “convergent sums” $\sum_\alpha y_\alpha$ with $y_\alpha \in R$ such that for each $n$ there are only finitely many $y_\alpha$ which are nonzero modulo $I^n$. The displayed map maps this element to the element $(y_\alpha)$ in the product. In particular the map is injective.

Let $Q$ be a finite $R$-module. We have to show that the map

$$Q \otimes_R \left( \bigoplus_{\alpha \in A} R \right)^{\wedge} \to Q \otimes_R \left( \prod_{\alpha \in A} R \right)$$

is injective, see Algebra, Theorem 80.3. Choose a presentation $R^{\oplus k} \to R^{\oplus m} \to Q \to 0$ and denote $q_1, \ldots, q_m \in Q$ the corresponding generators for $Q$. By Artin-Rees (Algebra, Lemma 49.2) there exists a constant $c$ such that $\text{im}(R^{\oplus k} \to R^{\oplus m}) \cap (I^N)^{\oplus m} \subset \text{im}((I^{N-c})^{\oplus k} \to R^{\oplus m})$. Let us contemplate the diagram

$$\begin{array}{ccc}
\bigoplus_{i=1}^k (\bigoplus_{\alpha \in A} R)^{\wedge} & \to & \bigoplus_{j=1}^m (\bigoplus_{\alpha \in A} R)^{\wedge} \\
\downarrow & & \downarrow \\
\bigoplus_{i=1}^k (\prod_{\alpha \in A} R) & \to & \bigoplus_{j=1}^m (\prod_{\alpha \in A} R) \\
& & \downarrow \\
& & Q \otimes_R \left( \prod_{\alpha \in A} R \right) \\
& & \downarrow \\
& & 0 \\
\end{array}$$

$\text{im}(R^{\oplus k} \to R^{\oplus m}) \cap (I^N)^{\oplus m} \subset \text{im}((I^{N-c})^{\oplus k} \to R^{\oplus m})$. Let us contemplate the diagram

$$\begin{array}{ccc}
\bigoplus_{i=1}^k (\bigoplus_{\alpha \in A} R)^{\wedge} & \to & \bigoplus_{j=1}^m (\bigoplus_{\alpha \in A} R)^{\wedge} \\
\downarrow & & \downarrow \\
\bigoplus_{i=1}^k (\prod_{\alpha \in A} R) & \to & \bigoplus_{j=1}^m (\prod_{\alpha \in A} R) \\
& & \downarrow \\
& & Q \otimes_R \left( \prod_{\alpha \in A} R \right) \\
& & \downarrow \\
& & 0 \\
\end{array}$$

is injective, see Algebra, Theorem 80.3. Choose a presentation $R^{\oplus k} \to R^{\oplus m} \to Q \to 0$ and denote $q_1, \ldots, q_m \in Q$ the corresponding generators for $Q$. By Artin-Rees (Algebra, Lemma 49.2) there exists a constant $c$ such that $\text{im}(R^{\oplus k} \to R^{\oplus m}) \cap (I^N)^{\oplus m} \subset \text{im}((I^{N-c})^{\oplus k} \to R^{\oplus m})$. Let us contemplate the diagram

$$\begin{array}{ccc}
\bigoplus_{i=1}^k (\bigoplus_{\alpha \in A} R)^{\wedge} & \to & \bigoplus_{j=1}^m (\bigoplus_{\alpha \in A} R)^{\wedge} \\
\downarrow & & \downarrow \\
\bigoplus_{i=1}^k (\prod_{\alpha \in A} R) & \to & \bigoplus_{j=1}^m (\prod_{\alpha \in A} R) \\
& & \downarrow \\
& & Q \otimes_R \left( \prod_{\alpha \in A} R \right) \\
& & \downarrow \\
& & 0 \\
\end{array}$$
Lemma 20.4. Let \( \prod \) \( \bigoplus \) Algebra, Lemma 94.8). In this case Lemma 20.1 tells us the map \( (\prod) \) \( \bigoplus \) \( R \) we may replace \( R \) that \( \square \) induction. see that \( \prod \) \( \bigoplus \) \( p > c \) there is a constant \( \text{Tor} \) \( A \) \( \text{finite} \) \( \text{Lemma 20.2.} \) Let \( R \) be a ring. Let \( I \subset R \) be an ideal. Let \( A \) be a set. Assume \( R \) is Noetherian. The completion \( (\prod) \) \( \bigoplus \) \( R \) \( ^\wedge \) is a flat \( R \)-module.

**Proof.** Denote \( R \) \( ^\wedge \) the completion of \( R \) with respect to \( I \). As \( R \to R \to R \) is flat by Algebra, Lemma 94.3 it suffices to prove that \( (\prod) \) \( \bigoplus \) \( R \) \( ^\wedge \) is a flat \( R^\wedge \)-module (use Algebra, Lemma 38.3). Since
\[
(\prod) \\bigoplus (\prod) \\bigoplus (\prod)
\]
we may replace \( R \) by \( R \) \( ^\wedge \) and assume that \( R \) is complete with respect to \( I \) (see Algebra, Lemma 94.8). In this case Lemma 20.1 tells us the map \( (\prod) \) \( \bigoplus \) \( R \) \( ^\wedge \) \( \to \) \( \prod \) \( \bigoplus \) \( R \) is universally injective. Thus, by Algebra, Lemma 80.7 it suffices to show that \( \prod \) \( \bigoplus \) \( R \) \( ^\wedge \) is flat. By Algebra, Proposition 88.5 and Algebra, Lemma 88.4 we see that \( \prod \) \( \bigoplus \) \( A \) \( ^\wedge \) is flat.

**Lemma 20.3.** Let \( A \) be a Noetherian ring. Let \( I \) be an ideal of \( A \). Let \( M \) be a finite \( A \)-module. For every \( p > 0 \) there exists a \( c > 0 \) such that \( \text{Tor}^p_\wedge (M, A/I^{n+c}) \to \text{Tor}^p_\wedge (M, A/I^n) \) is zero.

**Proof.** Proof for \( p = 1 \). Choose a short exact sequence \( 0 \to K \to R^\oplus \to A \). Then \( \text{Tor}^1_\wedge (M, A/I^n) \to K \cap (I^n)^\oplus/I^nK \). By Artin-Rees (Algebra, Lemma 49.2) there is a constant \( c > 0 \) such that \( K \cap (I^n)^\oplus \subset I^nK \). Thus the result for \( p = 1 \). For \( p > 1 \) we have \( \text{Tor}^p_\wedge (M, A/I^n) = \text{Tor}^{p-1}_\wedge (K, A/I^n) \). Thus the lemma follows by induction.

**Lemma 20.4.** Let \( A \) be a Noetherian ring. Let \( I \) be an ideal of \( A \). Let \( (M_n) \) be an inverse system of \( A \)-modules such that
\begin{enumerate}
\item \( M_n \) is a flat \( A/I^n \)-module,
\item \( M_{n+1} \to M_n \) is surjective.
\end{enumerate}
Then \( M = \lim M_n \) is a flat \( A \)-module and \( Q \otimes_A M = \lim Q \otimes_A M_n \) for every finite \( A \)-module \( Q \).

**Proof.** We first show that \( Q \otimes_A M = \lim Q \otimes_A M_n \) for every finite \( A \)-module \( Q \). Choose a resolution \( F_2 \to F_1 \to F_0 \to Q \to 0 \) by finite free \( A \)-modules \( F_i \). Then
\[
F_2 \otimes_A M_n \to F_1 \otimes_A M_n \to F_0 \otimes_A M_n
\]
is a chain complex whose homology in degree 0 is \( Q \otimes_A M_n \) and whose homology in degree 1 is
\[
\text{Tor}^1_\wedge (Q, M_n) = \text{Tor}^1_\wedge (Q, A/I^n) \otimes_{A/I^n} M_n
\]
as \( M_n \) is flat over \( A/I^n \). By Lemma 20.3 we see that this system is essentially constant (with value 0). It follows from Homology, Lemma 27.7 that \( \lim Q \otimes_A M \)
$A/I^n = \text{Coker}(\lim_{n} F_1 \otimes_A M_n \to \lim_{n} F_0 \otimes_A M_n)$. Since $F_i$ is finite free this equals \(\text{Coker}(F_1 \otimes_A M \to F_0 \otimes_A M) = Q \otimes_A M\).

Next, let $Q \to Q'$ be an injective map of finite $A$-modules. We have to show that $Q \otimes_A M \to Q' \otimes_A M$ is injective (Algebra, Lemma \[38.4\]). By the above we see
\[
\text{Ker}(Q \otimes_A M \to Q' \otimes_A M) = \text{Ker}(\lim_{n} Q \otimes_A M_n \to \lim_{n} Q' \otimes_A M_n).
\]

For each $n$ we have an exact sequence
\[
\text{Tor}_1^A(Q', M_n) \to \text{Tor}_1^A(Q'', M_n) \to Q \otimes_A M_n \to Q' \otimes_A M_n
\]
where $Q'' = \text{Coker}(Q \to Q')$. Above we have seen that the inverse systems of Tor’s are essentially constant with value $0$. It follows from Homology, Lemma \[27.7\] that the inverse limit of the right most maps is injective. \(\square\)

**Lemma 20.5.** Let $R$ be a ring. Let $I \subset R$ be an ideal. Let $M$ be an $R$-module. Assume

1. $I$ is finitely generated,
2. $R/I$ is Noetherian,
3. $M/IM$ is flat over $R/I$,
4. $\text{Tor}_1^R(M, R/I) = 0$.

Then the $I$-adic completion $R^\wedge$ is a Noetherian ring and $M^\wedge$ is flat over $R^\wedge$.

**Proof.** By Algebra, Lemma \[96.8\] the modules $M/IM^n$ are flat over $R/I^n$ for all $n$. By Algebra, Lemmas \[94.6\] and \[94.7\] we have (a) $R^\wedge$ and $M^\wedge$ are $I$-adically complete and (b) $R/I^n = R^\wedge/I^n R^\wedge$ for all $n$. By Algebra, Lemma \[94.9\] the ring $R^\wedge$ is Noetherian. Applying Lemma \[20.4\] we conclude that $M^\wedge = \lim M/I^n M$ is flat as an $R^\wedge$-module. \(\square\)

21. The Koszul complex

We define the Koszul complex as follows.

**Definition 21.1.** Let $R$ be a ring. Let $\varphi : E \to R$ be an $R$-module map. The **Koszul complex** $K_\bullet(\varphi)$ associated to $\varphi$ is the commutative differential graded algebra defined as follows:

1. the underlying graded algebra is the exterior algebra $K_\bullet(\varphi) = \wedge(E)$,
2. the differential $d : K_\bullet(\varphi) \to K_\bullet(\varphi)$ is the unique derivation such that $d(e) = \varphi(e)$ for all $e \in E = K_1(\varphi)$.

Explicitly, if $e_1 \wedge \ldots \wedge e_n$ is one of the generators of degree $n$ in $K_\bullet(\varphi)$, then
\[
d(e_1 \wedge \ldots \wedge e_n) = \sum_{i=1}^{n} (-1)^{i+1} \varphi(e_i) e_1 \wedge \ldots \wedge \hat{e}_i \wedge \ldots \wedge e_n.
\]

It is straightforward to see that this gives a well defined derivation on the tensor algebra, which annihilates $e \otimes e$ and hence factors through the exterior algebra.

We often assume that $E$ is a finite free module, say $E = R^{\oplus n}$. In this case the map $\varphi$ is given by a sequence of elements $f_1, \ldots, f_n \in R$.

**Definition 21.2.** Let $R$ be a ring and let $f_1, \ldots, f_r \in R$. The **Koszul complex on** $f_1, \ldots, f_r$ is the Koszul complex associated to the map $(f_1, \ldots, f_r) : R^{\oplus r} \to R$. Notation $K_\bullet(f_\bullet)$, $K_\bullet(f_1, \ldots, f_r)$, $K_\bullet(R, f_1, \ldots, f_r)$, or $K_\bullet(R, f_\bullet)$.
Of course, if \( E \) is finite locally free, then \( K_\bullet(\varphi) \) is locally on Spec(\( R \)) isomorphic to a Koszul complex \( K_\bullet(f_1, \ldots, f_r) \). This complex has many interesting formal properties.

**Lemma 21.3.** Let \( \varphi : E \to R \) and \( \varphi : E' \to R \) be an \( R \)-module maps. Let \( \psi : E \to E' \) be an \( R \)-module map such that \( \varphi' \circ \psi = \varphi \). Then \( \psi \) induces a homomorphism of differential graded algebras \( K_\bullet(\varphi) \to K_\bullet(\varphi') \).

**Proof.** This is immediate from the definitions.

**Lemma 21.4.** Let \( f_1, \ldots, f_r \in R \) be a sequence. Let \( (x_{ij}) \) be an invertible \( r \times r \)-matrix with coefficients in \( R \). Then the complexes \( K_\bullet(f_\bullet) \) and

\[
K_\bullet\left( \sum x_{1j} f_j, \sum x_{2j} f_j, \ldots, \sum x_{cj} f_j \right)
\]

are isomorphic.

**Proof.** Set \( g_i = \sum x_{ij} f_j \). The matrix \( (x_{ij}) \) gives an isomorphism \( x : R^{\oplus r} \to R^{\oplus r} \) such that \( (g_1, \ldots, g_r) \circ x = (f_1, \ldots, f_r) \). Hence this follows from the functoriality of the Koszul complex described in Lemma 21.3.

**Lemma 21.5.** Let \( R \) be a ring. Let \( \varphi : E \to R \) be an \( R \)-module map. Let \( e \in E \) with image \( f = \varphi(e) \in R \). Then

\[
f = de + ed
\]

as endomorphisms of \( K_\bullet(\varphi) \).

**Proof.** This is true because \( d(ea) = d(e)a - ed(a) = fa - ed(a) \).

**Lemma 21.6.** Let \( R \) be a ring. Let \( f_1, \ldots, f_r \in R \) be a sequence. Multiplication by \( f_i \) on \( K_\bullet(f_\bullet) \) is homotopic to zero, and in particular the cohomology modules \( H_i(K_\bullet(f_\bullet)) \) are annihilated by the ideal \( (f_1, \ldots, f_r) \).

**Proof.** Special case of Lemma 21.5.

In Derived Categories, Section 9 we defined the cone of a morphism of cochain complexes. The cone \( C(f) \_\bullet \) of a morphism of chain complexes \( f : A_\bullet \to B_\bullet \) is the complex \( C(f)_\bullet \) given by \( C(f)_n = B_n \oplus A_{n-1} \) and differential

\[
d_{C(f),n} = \begin{pmatrix} d_{B,n} & f_{n-1} \\ 0 & -d_{A,n-1} \end{pmatrix}
\]

It comes equipped with canonical morphisms of complexes \( i : B_\bullet \to C(f)_\bullet \) and \( p : C(f)_\bullet \to A_{\bullet-1} \) induced by the obvious maps \( B_n \to C(f)_n \to A_{n-1} \).

**Lemma 21.7.** Let \( R \) be a ring. Let \( \varphi : E \to R \) be an \( R \)-module map. Let \( f \in R \). Set \( E' = E \oplus R \) and define \( \varphi' : E' \to R \) by \( \varphi \) on \( E \) and multiplication by \( f \) on \( R \). The complex \( K_\bullet(\varphi') \) is isomorphic to the cone of the map of complexes

\[
f : K_\bullet(\varphi) \to K_\bullet(\varphi')
\]

**Proof.** Denote \( e_0 \in E' \) the element \( 1 \in R \subset R \oplus E \). By our definition of the cone above we see that

\[
C(f)_n = K_n(\varphi) \oplus K_{n-1}(\varphi') = \bigwedge^n(E) \oplus \bigwedge^{n-1}(E) = \bigwedge^n(E')
\]

where in the last = we map \((0, e_1 \wedge \ldots \wedge e_{n-1})\) to \( e_0 \wedge e_1 \wedge \ldots \wedge e_{n-1} \) in \( \bigwedge^n(E') \). A computation shows that this isomorphism is compatible with differentials. Namely, this is clear for elements of the first summand as \( \varphi'|_E = \varphi \) and \( d_{C(f)} \) restricted to
the first summand is just \( d_{K_\bullet(\varphi)}. \) On the other hand, if \( e_1 \wedge \ldots \wedge e_{n-1} \) is in the first summand, then
\[
d_{C(f)}(0, e_1 \wedge \ldots \wedge e_{n-1}) = fe_1 \wedge \ldots \wedge e_{n-1} - d_{K_\bullet(\varphi)}(e_1 \wedge \ldots \wedge e_{n-1})
\]
and on the other hand
\[
d_{K_\bullet(\varphi')}(e_0 \wedge e_1 \wedge \ldots \wedge e_{n-1})
\]
\[
= \sum_{i=0,\ldots,n-1} (-1)^i \varphi'(e_i) e_0 \wedge \ldots \wedge \widehat{e_i} \wedge \ldots \wedge e_{n-1}
\]
\[
= fe_1 \wedge \ldots \wedge e_{n-1} + \sum_{i=1,\ldots,n-1} (-1)^i \varphi(e_i) e_0 \wedge \ldots \wedge \widehat{e_i} \wedge \ldots \wedge e_{n-1}
\]
\[
= fe_1 \wedge \ldots \wedge e_{n-1} - e_0 \left( \sum_{i=1,\ldots,n-1} (-1)^{i+1} \varphi(e_i) e_0 \wedge \ldots \wedge \widehat{e_i} \wedge \ldots \wedge e_{n-1} \right)
\]
which is the image of the result of the previous computation.

**Lemma 21.8.** Let \( R \) be a ring. Let \( f_1, \ldots, f_r \) be a sequence of elements of \( R \). The complex \( K_\bullet(f_1, \ldots, f_r) \) is isomorphic to the cone of the map of complexes
\[
f_n : K_\bullet(f_1, \ldots, f_{r-1}) \to K_\bullet(f_1, \ldots, f_{r-1}).
\]

**Proof.** Special case of Lemma 21.7.

**Lemma 21.9.** Let \( R \) be a ring. Let \( A_\bullet \) be a complex of \( R \)-modules. Let \( f, g \in R \). Let \( C(f)_\bullet \) be the cone of \( f : A_\bullet \to A_\bullet \). Define similarly \( C(g)_\bullet \) and \( C(fg)_\bullet \). Then \( C(fg)_\bullet \) is homotopy equivalent to the cone of a map
\[
C(f)_\bullet[1] \to C(g)_\bullet.
\]

**Proof.** We first prove this if \( A_\bullet \) is the complex consisting of \( R \) placed in degree 0. In this case the map we use is
\[
\begin{array}{c c c c c}
0 & \rightarrow & 0 & \rightarrow & R \\
\downarrow & & \downarrow & & f \\
0 & \rightarrow & R & \rightarrow & 0 \\
\end{array}
\]
\[
\begin{array}{c c c c c}
0 & \rightarrow & R & \rightarrow & 0 \\
\downarrow & & \downarrow & & 1 \\
0 & \rightarrow & R & \rightarrow & 0 \\
\end{array}
\]
The cone of this is the chain complex consisting of \( R \oplus R \) placed in degrees 1 and 0 and differential \( [21.6.1] \)
\[
\left( \begin{array}{ll}
g & 1 \\
0 & -f \\
\end{array} \right) : R^{\oplus 2} \to R^{\oplus 2}
\]
We leave it to the reader to show this this chain complex is homotopic to the complex \( fg : R \to R \). In general we write \( C(f)_\bullet \) and \( C(g)_\bullet \) as the total complex of the double complexes
\[
(R \xrightarrow{f} R) \otimes_R A_\bullet \quad \text{and} \quad (R \xrightarrow{g} R) \otimes_R A_\bullet
\]
and in this way we deduce the result from the special case discussed above. Some details omitted.

**Lemma 21.10.** Let \( R \) be a ring. Let \( \varphi : E \to R \) be an \( R \)-module map. Let \( f, g \in R \). Set \( E' = E \oplus_R R \) and define \( \varphi'_f, \varphi'_g, \varphi'_{fg} : E' \to R \) by \( \varphi \) on \( E \) and multiplication by \( f, g, fg \) on \( R \). The complex \( K_\bullet(\varphi'_f)[1] \) is isomorphic to the cone of a map of complexes
\[
K_\bullet(\varphi'_f)[1] \to K_\bullet(\varphi'_g).
Let \( \text{Definition 22.1.} \) Please take a look at Algebra, Sections 67, 68, and 70 before looking at this one.

By Lemma 21.7 the complex \( K^\bullet(\varphi') \) is isomorphic to the cone of multiplication by \( f \) on \( K^\bullet(\varphi) \) and similarly for the other two cases. Hence the lemma follows from Lemma 21.9. \( \square \)

**Lemma 21.11.** Let \( R \) be a ring. Let \( f_1, \ldots, f_{r-1} \) be a sequence of elements of \( R \). Let \( f, g \in R \). The complex \( K^\bullet(f_1, \ldots, f_{r-1}, fg) \) is homotopy equivalent to the cone of a map of complexes

\[ K^\bullet(f_1, \ldots, f_{r-1}, f)[1] \to K^\bullet(f_1, \ldots, f_{r-1}, g) \]

**Proof.** Special case of Lemma 21.10. \( \square \)

**Lemma 21.12.** Let \( A \) be a ring. Let \( f_1, \ldots, f_r, g_1, \ldots, g_s \) be elements of \( A \). Then there is an isomorphism of Koszul complexes

\[ K^\bullet(A, f_1, \ldots, f_r, g_1, \ldots, g_s) = \text{Tot}(K^\bullet(A, f_1, \ldots, f_r) \otimes_A K^\bullet(A, g_1, \ldots, g_s)). \]

**Proof.** Omitted. Hint: If \( K^\bullet(A, f_1, \ldots, f_r) \) is generated as a differential graded algebra by \( x_1, \ldots, x_r \) with \( d(x_i) = f_i \) and \( K^\bullet(A, g_1, \ldots, g_s) \) is generated as a differential graded algebra by \( y_1, \ldots, y_s \) with \( d(y_j) = g_j \), then we can think of \( K^\bullet(A, f_1, \ldots, f_r, g_1, \ldots, g_s) \) as the differential graded algebra generated by the sequence of elements \( x_1, \ldots, x_r, y_1, \ldots, y_r \) with \( d(x_i) = f_i \) and \( d(y_j) = g_j \). \( \square \)

**Lemma 21.13.** Let \( R \) be a ring. Let \( f_1, \ldots, f_r \in R \). The extended alternating Čech complex

\[ R \to \prod_{i_0} R_{f_{i_0}} \to \prod_{i_0 < i_1} R_{f_{i_0}f_{i_1}} \to \cdots \to R_{f_1 \cdots f_r} \]

is a colimit of the Koszul complexes \( K(R, f_1^n, \ldots, f_r^n) \).

**Proof.** The transition maps \( K(R, f_1^n, \ldots, f_r^n) \to K(R, f_1^{n+1}, \ldots, f_r^{n+1}) \) are the maps sending \( e_{i_0} \wedge \cdots \wedge e_{i_r} \) to \( f_{i_0} + \cdots + f_{i_r} e_{i_0} \wedge \cdots \wedge e_{i_r} \) where the indices are such that \( \{1, \ldots, r\} = \{i_0, \ldots, i_r\} \). In particular the transition maps are always 1 in degree \( r \) and equal to \( f_1 \cdots f_r \) in degree 0. The terms of the colimit are equal to the terms of the extended alternating Čech complex by Algebra, Lemma 21.10. \( \square \)

22. Koszul regular sequences

Please take a look at Algebra, Sections 67, 68 and 70 before looking at this one.

**Definition 22.1.** Let \( R \) be a ring. Let \( r \geq 0 \) and let \( f_1, \ldots, f_r \in R \) be a sequence of elements. Let \( M \) be an \( R \)-module. The sequence \( f_1, \ldots, f_r \) is called

1. \( M \)-Koszul-regular if \( H_i(K^\bullet(f_1, \ldots, f_r) \otimes_R M) = 0 \) for all \( i \neq 0 \),
2. \( M \)-H\(_1\)-regular if \( H_1(K^\bullet(f_1, \ldots, f_r) \otimes_R M) = 0 \),
3. Koszul-regular if \( H_1(K^\bullet(f_1, \ldots, f_r)) = 0 \) for all \( i \neq 0 \), and
4. \( H_1 \)-Koszul-regular if \( H_1(K^\bullet(f_1, \ldots, f_r)) = 0 \).

We will see in Lemmas 22.2 and 22.5 that for elements \( f_1, \ldots, f_r \) of a ring \( R \) we have the following implications

\[ f_1, \ldots, f_r \text{ is a regular sequence} \Rightarrow f_1, \ldots, f_r \text{ is a Koszul-regular sequence} \]

\[ \Rightarrow f_1, \ldots, f_r \text{ is an } H_1 \text{-regular sequence} \]

\[ \Rightarrow f_1, \ldots, f_r \text{ is a quasi-regular sequence.} \]

In general none of these implications can be reversed, but if \( R \) is a Noetherian local ring and \( f_1, \ldots, f_r \in \mathfrak{m}_R \), then the 4 conditions are all equivalent (Lemma 22.6).
If \( f = f_1 \in R \) is a length 1 sequence then it is clear that the following are all equivalent

1. \( f \) is a regular sequence of length one,
2. \( f \) is a Koszul-regular sequence of length one, and
3. \( f \) is a \( H_1 \)-regular sequence of length one.

It is also clear that these imply that \( f \) is a quasi-regular sequence of length one. But there do exist quasi-regular sequences of length 1 which are not regular sequences. Namely, let

\[
R = k[x, y_0, y_1, \ldots]/(xy_0, xy_1 - y_0, xy_2 - y_1, \ldots)
\]

and let \( f \) be the image of \( x \) in \( R \). Then \( f \) is a zerodivisor, but \( \bigoplus_{n \geq 0} (f^n)/(f^{n+1}) \cong k[x] \) is a polynomial ring.

**Lemma 22.2.** An \( M \)-regular sequence is \( M \)-Koszul-regular. A regular sequence is Koszul-regular.

**Proof.** Let \( R \) be a ring and let \( M \) be an \( R \)-module. It is immediate that an \( M \)-regular sequence of length 1 is \( M \)-Koszul-regular. Let \( f_1, \ldots, f_r \) be an \( M \)-regular sequence. Then \( f_1 \) is a nonzerodivisor on \( M \). Hence

\[
0 \to K_{\bullet}(f_2, \ldots, f_r) \otimes M \xrightarrow{\mathcal{L}_1} K_{\bullet}(f_2, \ldots, f_r) \otimes M \to K_{\bullet}(\overline{f}_2, \ldots, \overline{f}_r) \otimes M/f_1M \to 0
\]

is a short exact sequence of complexes where \( \overline{f}_i \) is the image of \( f_i \) in \( R/(f_1) \). By Lemma [21.8] the complex \( K_{\bullet}(R, f_1, \ldots, f_r) \) is isomorphic to the cone of multiplication by \( f_1 \) on \( K_{\bullet}(f_2, \ldots, f_r) \). Thus \( K_{\bullet}(R, f_1, \ldots, f_r) \otimes M \) is isomorphic to the cone on the first map. Hence \( K_{\bullet}(\overline{f}_2, \ldots, \overline{f}_r) \otimes M/f_1M \) is quasi-isomorphic to \( K_{\bullet}(f_1, \ldots, f_r) \otimes M \). As \( \overline{f}_2, \ldots, \overline{f}_r \) is an \( M/f_1M \)-regular sequence in \( R/(f_1) \) the result follows from the case \( r = 1 \) and induction. □

**Lemma 22.3.** Let \( f_1, \ldots, f_{r-1} \in R \) be a sequence and \( f, g \in R \). Let \( M \) be an \( R \)-module.

1. If \( f_1, \ldots, f_{r-1}, f \) and \( f_1, \ldots, f_{r-1}, g \) are \( M \)-\( H_1 \)-regular then \( f_1, \ldots, f_{r-1}, fg \) is \( M \)-\( H_1 \)-regular too.
2. If \( f_1, \ldots, f_{r-1}, f \) and \( f_1, \ldots, f_{r-1}, f \) are \( M \)-Koszul-regular then \( f_1, \ldots, f_{r-1}, fg \) is \( M \)-Koszul-regular too.

**Proof.** By Lemma [21.11] we have exact sequences

\[
H_i(K_{\bullet}(f_1, \ldots, f_{r-1}, f) \otimes M) \to H_i(K_{\bullet}(f_1, \ldots, f_{r-1}, fg) \otimes M) \to H_i(K_{\bullet}(f_1, \ldots, f_{r-1}, g) \otimes M)
\]

for all \( i \). □

**Lemma 22.4.** Let \( \varphi : R \to S \) be a flat ring map. Let \( f_1, \ldots, f_r \in R \). Let \( M \) be an \( R \)-module and set \( N = M \otimes_R S \).

1. If \( f_1, \ldots, f_r \) in \( R \) an \( M \)-\( H_1 \)-regular sequence, then \( \varphi(f_1), \ldots, \varphi(f_r) \) is an \( N \)-\( H_1 \)-regular sequence in \( S \).
2. If \( f_1, \ldots, f_r \) is an \( M \)-Koszul-regular sequence in \( R \), then \( \varphi(f_1), \ldots, \varphi(f_r) \) is an \( N \)-Koszul-regular sequence in \( S \).

**Proof.** This is true because \( K_{\bullet}(f_1, \ldots, f_r) \otimes_R S = K_{\bullet}(\varphi(f_1), \ldots, \varphi(f_r)) \) and therefore \( (K_{\bullet}(f_1, \ldots, f_r) \otimes_R M) \otimes_R S = K_{\bullet}(\varphi(f_1), \ldots, \varphi(f_r)) \otimes_S N \). □

**Lemma 22.5.** An \( M \)-\( H_1 \)-regular sequence is \( M \)-quasi-regular.
**Proof.** Let $R$ be a ring and let $M$ be an $R$-module. Let $f_1, \ldots, f_r$ be an $M$-$H_1$-regular sequence. Denote $J = (f_1, \ldots, f_r)$. The assumption means that we have an exact sequence

$$\wedge^2(R^r) \otimes M \to R^\oplus_r \otimes M \to JM \to 0$$

where the first arrow is given by $e_i \wedge e_j \otimes m \mapsto (f_i e_j - f_j e_i) \otimes m$. In particular this implies that

$$JM/J^2M = JM \otimes_R R/J = (M/JM)^\oplus_r$$

is a finite free module. To finish the proof we have to prove for every $n \geq 2$ the following: if

$$\xi = \sum_{|I| = n, I = (i_1, \ldots, i_r)} m_I f_{i_1}^{r_1} \cdots f_{i_r}^{r_r} \in J^{n+1}M$$

then $m_I \in JM$ for all $I$. Note that $f_1, \ldots, f_{r-1}, f_r^2$ is an $M$-$H_1$-regular sequence by Lemma 22.3. Hence we see that the required result holds for the multi-index $I = (0, \ldots, 0, n)$. It turns out that we can reduce the general case to this case as follows.

Let $S = R[x_1, x_2, \ldots, x_r, 1/x_r]$. The ring map $R \to S$ is faithfully flat, hence $f_1, \ldots, f_r$ is an $M$-$H_1$-regular sequence in $S$, see Lemma 22.4. By Lemma 21.4 we see that

$$g_1 = f_1 - x_1 x_r f_{r-1} \cdots g_{r-1} = f_{r-1} - x_r f_{r-1} f_r g_r = (1/x_r) f_r$$

is an $M$-$H_1$-regular sequence in $S$. Finally, note that our element $\xi$ can be rewritten

$$\xi = \sum_{|I| = n, I = (i_1, \ldots, i_r)} m_I (g_1 + x_r g_r)^{r_1} \cdots (g_{r-1} + x_r g_r)^{r_{r-1}}(x_r g_r)^{r_r}$$

and the coefficient of $g_n$ in this expression is

$$\sum m_I x_1^{i_1} \cdots x_r^{i_r} \in J(M \otimes_R S).$$

Since the monomials $x_1^{i_1} \cdots x_r^{i_r}$ form part of an $R$-basis of $S$ over $R$ we conclude that $m_I \in J$ for all $I$ as desired. \qed

For nonzero finite modules over Noetherian local rings all of the types of regular sequences introduced so far are equivalent.

**Lemma 22.6.** Let $(R, \mathfrak{m})$ be a Noetherian local ring. Let $M$ be a nonzero finite $R$-module. Let $f_1, \ldots, f_r \in \mathfrak{m}$. The following are equivalent

1. $f_1, \ldots, f_r$ is an $M$-regular sequence,
2. $f_1, \ldots, f_r$ is a $M$-Koszul-regular sequence,
3. $f_1, \ldots, f_r$ is an $M$-$H_1$-regular sequence,
4. $f_1, \ldots, f_r$ is an $M$-quasi-regular sequence.

In particular the sequence $f_1, \ldots, f_r$ is a regular sequence in $R$ if and only if it is a Koszul regular sequence, if and only if it is a $H_1$-regular sequence, if and only if it is a quasi-regular sequence.

**Proof.** The implication (1) ⇒ (2) is Lemma 22.2. The implication (2) ⇒ (3) is immediate. The implication (3) ⇒ (4) is Lemma 22.5. The implication (4) ⇒ (1) is Algebra, Lemma 68.6 \qed

**Lemma 22.7.** Let $A$ be a ring. Let $I \subset A$ be an ideal. Let $g_1, \ldots, g_m$ be a sequence in $A$ whose image in $A/I$ is $H_1$-regular. Then $I \cap (g_1, \ldots, g_m) = I(g_1, \ldots, g_m)$.
Proof. Consider the exact sequence of complexes
\[ 0 \to I \otimes_A K_\bullet(A, g_1, \ldots, g_m) \to K_\bullet(A, g_1, \ldots, g_m) \to K_\bullet(A/I, g_1, \ldots, g_m) \to 0 \]
Since the complex on the right has \( H_1 = 0 \) by assumption we see that
\[ \text{Coker}(I^\oplus m \to I) \to \text{Coker}(A^\oplus m \to A) \]
is injective. This is equivalent to the assertion of the lemma. \( \square \)

Lemma 22.8. Let \( A \) be a ring. Let \( I \subset J \subset A \) be ideals. Assume that \( J/I \subset A/I \)
is generated by an \( H_1 \)-regular sequence. Then \( I \cap J^2 = IJ \).

Proof. To prove this choose \( g_1, \ldots, g_m \in J \) whose images in \( A/I \) form a \( H_1 \)-regular sequence which generates \( J/I \). In particular \( J = I + (g_1, \ldots, g_m) \). Suppose that \( x \in I \cap J^2 \). Because \( x \in J^2 \) we can write
\[ x = \sum a_{ij}g_ig_j + \sum a_jg_j + a \]
with \( a_{ij} \in A, a_j \in I \) and \( a \in I^2 \). Then \( \sum a_{ij}g_ig_j \in I \cap (g_1, \ldots, g_m) \) hence by Lemma 22.7 we see that \( \sum a_{ij}g_ig_j \in (g_1(1, \ldots, g_m)) \). Thus \( x \in IJ \) as desired. \( \square \)

Lemma 22.9. Let \( A \) be a ring. Let \( I \) be an ideal generated by a quasi-regular sequence \( f_1, \ldots, f_n \) in \( A \). Let \( g_1, \ldots, g_m \in A \) be elements whose images \( \overline{g}_1, \ldots, \overline{g}_m \) form an \( H_1 \)-regular sequence in \( A/I \). Then \( f_1, \ldots, f_n, g_1, \ldots, g_m \) is a quasi-regular sequence in \( A \).

Proof. We claim that \( g_1, \ldots, g_m \) forms an \( H_1 \)-regular sequence in \( A/I^d \) for every \( d \). By induction assume that this holds in \( A/I^{d-1} \). We have a short exact sequence of complexes
\[ 0 \to K_\bullet(A, g_\bullet) \otimes_A I^{d-1}/I^d \to K_\bullet(A/I^d, g_\bullet) \to K_\bullet(A/I^{d-1}, g_\bullet) \to 0 \]
Since \( f_1, \ldots, f_n \) is quasi-regular we see that the first complex is a direct sum of copies of \( K_\bullet(A/I, g_1, \ldots, g_m) \) hence acyclic in degree 1. By induction hypothesis the last complex is acyclic in degree 1. Hence also the middle complex is. In particular, the sequence \( g_1, \ldots, g_m \) forms a quasi-regular sequence in \( A/I^d \) for every \( d \geq 1 \), see Lemma 22.5. Now we are ready to prove that \( f_1, \ldots, f_n, g_1, \ldots, g_m \) is a quasi-regular sequence in \( A \). Namely, set \( J = (f_1, \ldots, f_n, g_1, \ldots, g_m) \) and suppose that (with multinomial notation)
\[ \sum_{|N|=|M|=d} a_{N,M} f^N g^M \in J^{d+1} \]
for some \( a_{N,M} \in A \). We have to show that \( a_{N,M} \in J \) for all \( N, M \). Let \( e \in \{0, 1, \ldots, d\} \). Then
\[ \sum_{|N|=d-e, |M|=e} a_{N,M} f^N g^M \in (g_1, \ldots, g_m)^{e+1} + I^{d-e+1} \]
Because \( g_1, \ldots, g_m \) is a quasi-regular sequence in \( A/I^{d-e+1} \) we deduce
\[ \sum_{|N|=d-e} a_{N,M} f^N \in (g_1, \ldots, g_m) + I^{d-e+1} \]
for each \( M \) with \( |M| = e \). By Lemma 22.7 applied to \( I^{d-e}/I^{d-e+1} \) in the ring \( A/I^{d-e+1} \) this implies \( \sum_{|N|=d-e} a_{N,M} f^N \in I^{d-e}(g_1, \ldots, g_m) \). Since \( f_1, \ldots, f_n \) is quasi-regular in \( A \) this implies that \( a_{N,M} \in J \) for each \( N, M \) with \( |N| = d - e \) and \( |M| = e \). This proves the lemma. \( \square \)
**Lemma 22.10.** Let $A$ be a ring. Let $I$ be an ideal generated by an $H_1$-regular sequence $f_1, \ldots, f_n$ in $A$. Let $g_1, \ldots, g_m \in A$ be elements whose images $\overline{g}_1, \ldots, \overline{g}_m$ form an $H_1$-regular sequence in $A/I$. Then $f_1, \ldots, f_n, g_1, \ldots, g_m$ is an $H_1$-regular sequence in $A$.

**Proof.** We have to show that $H_1(A, f_1, \ldots, f_n, g_1, \ldots, g_m) = 0$. To do this consider the commutative diagram

$$
\begin{array}{c}
\wedge^2(A^{\oplus n+m}) 
\xrightarrow{\wedge^2(A/I^{\oplus m})} A^{\oplus n+m} 
\xrightarrow{\wedge^2(A/I^{\oplus m})} A/I 
\xrightarrow{0}
\end{array}
$$

Consider an element $(a_1, \ldots, a_{n+m}) \in A^{\oplus n+m}$ which maps to zero in $A$. Because $\overline{g}_1, \ldots, \overline{g}_m$ form an $H_1$-regular sequence in $A/I$ we see that $(\sigma_{n+1}, \ldots, \sigma_{n+m})$ is the image of some element $\alpha$ of $\wedge^2(A/I^{\oplus m})$. We can lift $\alpha$ to an element $\alpha \in \wedge^2(A^{\oplus n+m})$ and substract the image of it in $A^{\oplus n+m}$ from our element $(a_1, \ldots, a_{n+m})$. Thus we may assume that $a_{n+1}, \ldots, a_{n+m} \in I$. Since $I = (f_1, \ldots, f_n)$ we can modify our element $(a_1, \ldots, a_{n+m})$ by linear combinations of the elements $(0, \ldots, g_j, 0, \ldots, 0, f_i, 0, \ldots, 0)$ in the image of the top left horizontal arrow to reduce to the case that $a_{n+1}, \ldots, a_{n+m}$ are zero. In this case $(a_1, \ldots, a_n, 0, \ldots, 0)$ defines an element of $H_1(A, f_1, \ldots, f_n)$ which we assumed to be zero.

**Lemma 22.11.** Let $A$ be a ring. Let $f_1, \ldots, f_n, g_1, \ldots, g_m \in A$ be an $H_1$-regular sequence. Then the images $\overline{g}_1, \ldots, \overline{g}_m$ in $A/(f_1, \ldots, f_n)$ form an $H_1$-regular sequence.

**Proof.** Set $I = (f_1, \ldots, f_n)$. We have to show that any relation $\sum_{j=1, \ldots, m} a_j \overline{g}_j$ in $A/I$ is a linear combination of trivial relations. Because $I = (f_1, \ldots, f_n)$ we can lift this relation to a relation

$$
\sum_{j=1, \ldots, m} a_j g_j + \sum_{i=1, \ldots, n} b_i f_i = 0
$$

in $A$. By assumption this relation in $A$ is a linear combination of trivial relations. Taking the image in $A/I$ we obtain what we want.

**Lemma 22.12.** Let $A$ be a ring. Let $I$ be an ideal generated by a Koszul-regular sequence $f_1, \ldots, f_n$ in $A$. Let $g_1, \ldots, g_m \in A$ be elements whose images $\overline{g}_1, \ldots, \overline{g}_m$ form a Koszul-regular sequence in $A/I$. Then $f_1, \ldots, f_n, g_1, \ldots, g_m$ is a Koszul-regular sequence in $A$.

**Proof.** Our assumptions say that $K_\bullet(A, f_1, \ldots, f_n)$ is a finite free resolution of $A/I$ and $K_\bullet(A/I, \overline{g}_1, \ldots, \overline{g}_m)$ is a finite free resolution of $A/(f_i, g_j)$ over $A/I$. Then

$$
K_\bullet(A, f_1, \ldots, f_n, g_1, \ldots, g_m) = \text{Tot}(K_\bullet(A, f_1, \ldots, f_n) \otimes_A K_\bullet(A, g_1, \ldots, g_m))
\cong A/I \otimes_A K_\bullet(A, g_1, \ldots, g_m)
= K_\bullet(A/I, \overline{g}_1, \ldots, \overline{g}_m)
\cong A/(f_i, g_j)
$$

The first equality by Lemma 21.12. The first quasi-isomorphism $\cong$ by (the dual of) Homology, Lemma 22.7 as the $q$th row of the double complex $K_\bullet(A, f_1, \ldots, f_n) \otimes_A
$K_\bullet (A, g_1, \ldots, g_m)$ is a resolution of $A/I \otimes_A K_q(A, g_1, \ldots, g_m)$. The second equality is clear. The last quasi-isomorphism by assumption. Hence we win. □

To conclude in the following lemma it is necessary to assume that both $f_1, \ldots, f_n$ and $f_1, \ldots, f_n, g_1, \ldots, g_m$ are Koszul-regular. A counter example to dropping the assumption that $f_1, \ldots, f_n$ is Koszul-regular is Examples, Lemma 13.1

Lemma 22.13. Let $A$ be a ring. Let $f_1, \ldots, f_n, g_1, \ldots, g_m \in A$. If both $f_1, \ldots, f_n$ and $f_1, \ldots, f_n, g_1, \ldots, g_m$ are Koszul-regular sequences in $A$, then $g_1, \ldots, g_m$ in $A/(f_1, \ldots, f_n)$ form a Koszul-regular sequence.

Proof. Set $I = (f_1, \ldots, f_n)$. Our assumptions say that $K_\bullet (A, f_1, \ldots, f_n)$ is a finite free resolution of $A/I$ and $K_\bullet (A, f_1, \ldots, f_n, g_1, \ldots, g_m)$ is a finite free resolution of $A/(f_1, \ldots, f_n)$ over $A$. Then

\[
A/(f_1, g_j) \cong K_\bullet (A, f_1, \ldots, f_n, g_1, \ldots, g_m)
= \text{Tot}(K_\bullet (A, f_1, \ldots, f_n) \otimes_A K_\bullet (A, g_1, \ldots, g_m))
\cong A/I \otimes_A K_\bullet (A, g_1, \ldots, g_m)
= K_\bullet (A/I, \overline{g}_1, \ldots, \overline{g}_m)
\]

The first quasi-isomorphism $\cong$ by assumption. The first equality by Lemma 21.12. The second quasi-isomorphism by (the dual of) Homology, Lemma 22.7 as the $q$th row of the double complex $K_\bullet (A, f_1, \ldots, f_n) \otimes_A K_\bullet (A, g_1, \ldots, g_m)$ is a resolution of $A/I \otimes_A K_q(A, g_1, \ldots, g_m)$. The second equality is clear. Hence we win. □

Lemma 22.14. Let $R$ be a ring. Let $I$ be an ideal generated by $f_1, \ldots, f_r \in R$.

1. If $I$ can be generated by a quasi-regular sequence of length $r$, then $f_1, \ldots, f_r$ is a quasi-regular sequence.
2. If $I$ can be generated by an $H_1$-regular sequence of length $r$, then $f_1, \ldots, f_r$ is an $H_1$-regular sequence.
3. If $I$ can be generated by a Koszul-regular sequence of length $r$, then $f_1, \ldots, f_r$ is a Koszul-regular sequence.

Proof. If $I$ can be generated by a quasi-regular sequence of length $r$, then $I/I^2$ is free of rank $r$ over $R/I$. Since $f_1, \ldots, f_r$ generate by assumption we see that the images $\overline{f}_j$ form a basis of $I/I^2$ over $R/I$. It follows that $f_1, \ldots, f_r$ is a quasi-regular sequence as all this means, besides the freeness of $I/I^2$, is that the maps $\text{Sym}^n_{R/I}(I/I^2) \to I^n/I^{n+1}$ are isomorphisms.

We continue to assume that $I$ can be generated by a quasi-regular sequence, say $g_1, \ldots, g_r$. Write $g_j = \sum a_{ij} f_i$. As $f_1, \ldots, f_r$ is quasi-regular according to the previous paragraph, we see that $\det(a_{ij})$ is invertible mod $I$. The matrix $a_{ij}$ gives a map $R^{\oplus r} \to R^{\oplus r}$ which induces a map of Koszul complexes $\alpha : K_\bullet (R, f_1, \ldots, f_r) \to K_\bullet (R, g_1, \ldots, g_r)$, see Lemma 21.3. This map becomes an isomorphism on inverting $\det(a_{ij})$. Since the cohomology modules of both $K_\bullet (R, f_1, \ldots, f_r)$ and $K_\bullet (R, g_1, \ldots, g_r)$ are annihilated by $I$, see Lemma 21.6, we see that $\alpha$ is a quasi-isomorphism. Hence if $g_1, \ldots, g_r$ is $H_1$-regular, then so is $f_1, \ldots, f_r$. Similarly for Koszul-regular. □

Lemma 22.15. Let $A \to B$ be a ring map. Let $f_1, \ldots, f_r$ be a sequence in $B$ such that $B/(f_1, \ldots, f_r)$ is $A$-flat. Let $A \to A'$ be a ring map. Then the canonical map

\[
H_1(K_\bullet (B, f_1, \ldots, f_r)) \otimes_A A' \to H_1(K_\bullet (B', f_1', \ldots, f_r'))
\]
is surjective, where $B' = B \otimes_A A'$ and $f'_i \in B'$ is the image of $f_i$.

**Proof.** The sequence

$$\wedge^2(B^{\oplus r}) \to B^{\oplus r} \to B \to B/J \to 0$$

is a complex of $A$-modules with $B/J$ flat over $A$ and cohomology group $H_1 = H_1(K_1(B, f_1, \ldots, f_r))$ in the spot $B^{\oplus r}$. If we tensor this with $A'$ we obtain a complex

$$\wedge^2((B')^{\oplus r}) \to (B')^{\oplus r} \to B' \to B'/J' \to 0$$

which is exact at $B'$ and $B'/J'$. In order to compute its cohomology group $H'_1 = H_1(K_1(B', f'_1, \ldots, f'_r))$ at $(B')^{\oplus r}$ we split the first sequence above into short exact sequences $0 \to J \to B \to B/J \to 0$ and $0 \to K \to B^{\oplus r} \to J \to 0$ and $\wedge^2(B^{\oplus r}) \to K \to H_1 \to 0$. Tensoring with $A'$ over $A$ we obtain the exact sequences

$$0 \to J \otimes_A A' \to B \otimes_A A' \to (B/J) \otimes_A A' \to 0$$

$$K \otimes_A A' \to B^{\oplus r} \otimes_A A' \to J \otimes_A A' \to 0$$

$$\wedge^2(B^{\oplus r}) \otimes_A A' \to K \otimes_A A' \to H_1 \otimes_A A' \to 0$$

where the first one is exact as $B/J$ is flat over $A$, see Algebra, Lemma \[38.11\]. Hence we conclude what we want. \[\square\]

**Lemma 22.16.** Let $R$ be a ring. Let $a_1, \ldots, a_n \in R$ be elements such that $R \to R^{\oplus n}, x \mapsto (xa_1, \ldots, xa_n)$ is injective. Then the element $\sum a_it_i$ of the polynomial ring $R[t_1, \ldots, t_n]$ is a nonzerodivisor.

**Proof.** If one of the $a_i$ is a unit this is just the statement that any element of the form $t_1 + a_2t_2 + \ldots + a_nt_n$ is a nonzerodivisor in the polynomial ring over $R$.

Case I: $R$ is Noetherian. Let $q_j, j = 1, \ldots, m$ be the associated primes of $R$. We have to show that each of the maps

$$\sum a_it_i : \text{Sym}^d(R^{\oplus n}) \to \text{Sym}^{d+1}(R^{\oplus n})$$

is injective. As $\text{Sym}^d(R^{\oplus n})$ is a free $R$-module its associated primes are $q_j, j = 1, \ldots, m$. For each $j$ there exists an $i = i(j)$ such that $a_i \notin q_j$ because there exists an $x \in R$ with $q_jx = 0$ but $a_ix \neq 0$ for some $i$ by assumption. Hence $a_i$ is a unit in $R_{q_j}$ and the map is injective after localizing at $q_j$. Thus the map is injective, see Algebra, Lemma \[62.18\].

Case II: $R$ general. We can write $R$ as the union of Noetherian rings $R_\lambda$ with $a_1, \ldots, a_n \in R_\lambda$. For each $R_\lambda$ the result holds, hence the result holds for $R$. \[\square\]

**Lemma 22.17.** Let $R$ be a ring. Let $f_1, \ldots, f_n$ be a Koszul-regular sequence in $R$. Consider the faithfully flat, smooth ring map

$$R \to S = R[[t_{ij} : i \leq j, t_{11}^{-1}, t_{22}^{-1}, \ldots, t_{nn}^{-1}]]$$

For $1 \leq i \leq n$ set

$$g_i = \sum_{i \leq j} t_{ij}f_j \in S.$$ 

Then $g_1, \ldots, g_n$ is a regular sequence in $S$ and $(f_1, \ldots, f_n)S = (g_1, \ldots, g_n)$. 

Proof. The equality of ideals is obvious as the matrix
\[
\begin{pmatrix}
t_{11} & t_{12} & t_{13} & \ldots \\
0 & t_{22} & t_{23} & \ldots \\
0 & 0 & t_{33} & \ldots \\
\vdots & \vdots & \vdots & \ddots \\
\end{pmatrix}
\]
is invertible in \( S \). Because \( f_1, \ldots, f_n \) is a Koszul-regular sequence we see that the kernel of \( R \to R^{\oplus n}, x \mapsto (xf_1, \ldots, xf_n) \) is zero (as it computes the \( n \)th Koszul homology of \( R \) w.r.t. \( f_1, \ldots, f_n \)). Hence by Lemma \[22.16\] we see that \( g_1 = f_1 t_{11} + \ldots + f_n t_{1n} \) is a nonzerodivisor in \( S' = R[t_{11}, t_{12}, \ldots, t_{1n}, t_{11}] \). We see that \( g_1, f_2, \ldots, f_n \) is a Koszul-sequence in \( S' \) by Lemma \[22.4\] and \[22.14\]. We conclude that \( \overline{f_2}, \ldots, \overline{f_n} \) is a Koszul-regular sequence in \( S'/\langle g_2 \rangle \) by Lemma \[22.13\]. Hence by induction on \( n \) we see that the images \( \overline{g_2}, \ldots, \overline{g_n} \) of \( g_2, \ldots, g_n \) in \( S'/\langle g_2 \rangle \)[\( t_{ij} \rangle_{2 \leq i \leq j, t_{22}^{-1}, \ldots, t_{nn}^{-1}} \] form a regular sequence. This in turn means that \( g_1, \ldots, g_n \) forms a regular sequence in \( S \). \( \square \)

23. Regular ideals

We will discuss the notion of a regular ideal sheaf in great generality in Divisors, Section \[14\]. Here we define the corresponding notion in the affine case, i.e., in the case of an ideal in a ring.

Definition 23.1. Let \( R \) be a ring and let \( I \subset R \) be an ideal.

1. We say \( I \) is a regular ideal if for every \( p \in V(I) \) there exists a \( g \in R \), \( g \notin p \) and a regular sequence \( f_1, \ldots, f_r \in R_\mathfrak{g} \) such that \( I_\mathfrak{g} \) is generated by \( f_1, \ldots, f_r \).

2. We say \( I \) is a Koszul-regular ideal if for every \( p \in V(I) \) there exists a \( g \in R \), \( g \notin p \) and a Koszul-regular sequence \( f_1, \ldots, f_r \in R_\mathfrak{g} \) such that \( I_\mathfrak{g} \) is generated by \( f_1, \ldots, f_r \).

3. We say \( I \) is a \( H_1 \)-regular ideal if for every \( p \in V(I) \) there exists a \( g \in R \), \( g \notin p \) and an \( H_1 \)-regular sequence \( f_1, \ldots, f_r \in R_\mathfrak{g} \) such that \( I_\mathfrak{g} \) is generated by \( f_1, \ldots, f_r \).

4. We say \( I \) is a quasi-regular ideal if for every \( p \in V(I) \) there exists a \( g \in R \), \( g \notin p \) and a quasi-regular sequence \( f_1, \ldots, f_r \in R_\mathfrak{g} \) such that \( I_\mathfrak{g} \) is generated by \( f_1, \ldots, f_r \).

It is clear that given \( I \subset R \) we have the implications

\[ I \text{ is a regular ideal } \implies I \text{ is a Koszul-regular ideal} \]
\[ \implies I \text{ is a } H_1 \text{-regular ideal} \]
\[ \implies I \text{ is a quasi-regular ideal} \]

see Lemmas \[22.2\] and \[22.5\]. Such an ideal is always finitely generated.

Lemma 23.2. A quasi-regular ideal is finitely generated.

Proof. Let \( I \subset R \) be a quasi-regular ideal. Since \( V(I) \) is quasi-compact, there exist \( g_1, \ldots, g_m \in R \) such that \( V(I) \subset D(g_1) \cup \ldots \cup D(g_m) \) and such that \( I_\mathfrak{g}_j \) is generated by a quasi-regular sequence \( g_{1j}, \ldots, g_{rj} \in R_\mathfrak{g}_j \). Write \( g_{ij} = g'_{ij}/g'_j \) for some \( g'_{ij} \in I \). Write \( 1 + x = \sum g_j h_j \) for some \( x \in I \) which is possible as \( V(I) \subset D(g_1) \cup \ldots \cup D(g_m) \). Note that Spec(\( R \)) = \( D(g_1) \cup \ldots \cup D(g_m) \cup D(x) \). Then \( I \) is generated by the elements \( g'_j \) and \( x \) as these generate on each of the pieces of the cover, see Algebra, Lemma \[23.2\]. \( \square \)
**Lemma 23.3.** Let $I \subset R$ be a quasi-regular ideal of a ring. Then $I/I^2$ is a finite projective $R/I$-module.

**Proof.** This follows from Algebra, Lemma 76.2 and the definitions. \qed

We prove flat descent for Koszul-regular, $H_1$-regular, quasi-regular ideals.

**Lemma 23.4.** Let $A \to B$ be a faithfully flat ring map. Let $I \subset A$ be an ideal. If $IB$ is a Koszul-regular (resp. $H_1$-regular, resp. quasi-regular) ideal in $B$, then $I$ is a Koszul-regular (resp. $H_1$-regular, resp. quasi-regular) ideal in $A$.

**Proof.** We fix the prime $p \supset I$ throughout the proof. Assume $IB$ is quasi-regular. By Lemma 23.2 $IB$ is a finite module, hence $I$ is a finite $A$-module by Algebra, Lemma 81.2. As $A \to B$ is flat we see that

$$I/I^2 \otimes_{A/I} B/IB = I/I^2 \otimes_A B = IB/(IB)^2.$$ 

As $IB$ is quasi-regular, the $B/IB$-module $IB/(IB)^2$ is finite locally free. Hence $I/I^2$ is finite projective, see Algebra, Proposition 81.3. In particular, after replacing $A$ by $A_f$ for some $f \in A$, $f \notin p$ we may assume that $I/I^2$ is free of rank $r$. Pick $f_1, \ldots, f_r \in I$ which give a basis of $I/I^2$. By Nakayama’s lemma (see Algebra, Lemma 19.1) we see that, after another replacement $A \to A_f$ as above, $I$ is generated by $f_1, \ldots, f_r$.

Proof of the “quasi-regular” case. Above we have seen that $I/I^2$ is free on the $r$-generators $f_1, \ldots, f_r$. To finish the proof in this case we have to show that the maps $\text{Sym}^d(I/I^2) \to I^d/I^{d+1}$ are isomorphisms for each $d \geq 2$. This is clear as the faithfully flat base changes $\text{Sym}^d(IB/(IB)^2) \to (IB)^d/(IB)^{d+1}$ are isomorphisms locally on $B$ by assumption. Details omitted.

Proof of the “$H_1$-regular” and “Koszul-regular” case. Consider the sequence of elements $f_1, \ldots, f_r$ generating $I$ we constructed above. By Lemma 22.14 we see that $f_1, \ldots, f_r$ map to a $H_1$-regular or Koszul-regular sequence in $B_g$ for any $g \in B$ such that $IB$ is generated by an $H_1$-regular or Koszul-regular sequence. Hence $K_*(A, f_1, \ldots, f_r) \otimes_A B_g$ has vanishing $H_i$ or $H_i$, $i > 0$. Since the homology of $K_*(B, f_1, \ldots, f_r) = K_*(A, f_1, \ldots, f_r) \otimes_A B$ is annihilated by $IB$ (see Lemma 21.6) and since $V(IB) \subset \bigcup g$ as above $D(g)$ we conclude that $K_*(A, f_1, \ldots, f_r) \otimes_A B$ has vanishing homology in degree 1 or all positive degrees. Using that $A \to B$ is faithfully flat we conclude that the same is true for $K_*(A, f_1, \ldots, f_r)$.

**Lemma 23.5.** Let $A$ be a ring. Let $I \subset J \subset A$ be ideals. Assume that $J/I \subset A/I$ is a $H_1$-regular ideal. Then $I \cap J^2 = IJ$.

**Proof.** Follows immediately from Lemma 22.8 by localizing. \qed

## 24. Local complete intersection maps

We can use the material above to define a local complete intersection map between rings using presentations by (finite) polynomial algebras.

**Lemma 24.1.** Let $A \to B$ be a finite type ring map. If for some presentation $\alpha : A[x_1, \ldots, x_n] \to B$ the kernel $I$ is a Koszul-regular ideal then for any presentation $\beta : A[y_1, \ldots, y_m] \to B$ the kernel $J$ is a Koszul-regular ideal.
Proof. Choose \( f_j \in A[x_1, \ldots, x_n] \) with \( \alpha(f_j) = \beta(y_j) \) and \( g_i \in A[y_1, \ldots, y_m] \) with \( \beta(g_i) = \alpha(x_i) \). Then we get a commutative diagram

\[
\begin{array}{ccc}
A[x_1, \ldots, x_n, y_1, \ldots, y_m] & \xrightarrow{y_i \mapsto f_j} & A[x_1, \ldots, x_n] \\
\downarrow{\scriptstyle x_i \mapsto g_i} & & \downarrow \\
A[y_1, \ldots, y_m] & \longrightarrow & B
\end{array}
\]

Note that the kernel \( K \) of \( A[x_i, y_j] \to B \) is equal to \( K = (I, y_j - f_j) = (I, x_i - f_i) \). In particular, as \( I \) is finitely generated by Lemma \ref{finite-gen}, we see that \( J = K/(x_i - f_i) \) is finitely generated too.

Pick a prime \( q \subset B \). Since \( I/I^2 \oplus B^{\oplus m} = J/J^2 \oplus B^{\oplus n} \) (Algebra, Lemma \ref{koszul-regular}) we see that

\[ \dim J/J^2 \otimes_B \kappa(q) + n = \dim I/I^2 \otimes_B \kappa(q) + m. \]

Pick \( p_1, \ldots, p_t \in I \) which map to a basis of \( I/I^2 \otimes \kappa(q) = I \otimes A[x_i] \kappa(q) \). Pick \( q_1, \ldots, q_s \in J \) which map to a basis of \( J/J^2 \otimes \kappa(q) = J \otimes A[y_j] \kappa(q) \). So \( s + n = t + m \).

By Nakayama’s lemma there exist \( h \in A[x_i] \) and \( h' \in A[y_j] \) both mapping to a nonzero element of \( \kappa(q) \) such that \( I_h = (p_1, \ldots, p_t) \) in \( A[x_i, 1/h] \) and \( J_{h'} = (q_1, \ldots, q_s) \) in \( A[y_j, 1/h'] \). As \( I \) is Koszul-regular we may also assume that \( I_h \) is generated by a Koszul regular sequence. This sequence must necessarily have length \( t = \dim I/I^2 \otimes_B \kappa(q) \), hence we see that \( p_1, \ldots, p_t \) is a Koszul-regular sequence by Lemma \ref{koszul-regular}.

As also \( y_1 - f_1, \ldots, y_m - f_m \) is a regular sequence we conclude

\[ y_1 - f_1, \ldots, y_m - f_m, p_1, \ldots, p_t \]

is a Koszul-regular sequence in \( A[x_i, y_j, 1/h] \) (see Lemma \ref{koszul-regular}). This sequence generates the ideal \( K_h \). Hence the ideal \( K_{h'} \) is generated by a Koszul-regular sequence of length \( m + t = n + s \). But it is also generated by the sequence

\[ x_1 - g_1, \ldots, x_n - g_n, q_1, \ldots, q_s \]

of the same length which is thus a Koszul-regular sequence by Lemma \ref{koszul-regular}. Finally, by Lemma \ref{koszul-regular} we conclude that the images of \( q_1, \ldots, q_s \) in

\[ A[x_i, y_j, 1/hh']/(x_1 - g_1, \ldots, x_n - g_n) \cong A[y_j, 1/h'] \]

form a Koszul-regular sequence generating \( J_{h''} \). Since \( h'' \) is the image of \( hh' \) it doesn’t map to zero in \( \kappa(q) \) and we win. \( \square \)

This lemma allows us to make the following definition.

**Definition** 24.2. A ring map \( A \to B \) is called a *local complete intersection* if it is of finite type and for some (equivalently any) presentation \( B = A[x_1, \ldots, x_n]/I \) the ideal \( I \) is Koszul-regular.

This notion is local.

**Lemma** 24.3. Let \( R \to S \) be a ring map. Let \( g_1, \ldots, g_m \in S \) generate the unit ideal. If each \( R \to S_{g_j} \) is a local complete intersection so is \( R \to S \).

**Proof.** Let \( S = R[x_1, \ldots, x_n]/I \) be a presentation. Pick \( h_j \in R[x_1, \ldots, x_n] \) mapping to \( g_j \) in \( S \). Then \( R[x_1, \ldots, x_n, x_{n+1}]/(I, x_{n+1}h_j - 1) \) is a presentation of \( S_{g_j} \).

Hence \( I_j = (I, x_{n+1}h_j - 1) \) is a Koszul-regular ideal in \( R[x_1, \ldots, x_n, x_{n+1}] \). Pick a prime \( I \subset q \subset R[x_1, \ldots, x_n] \). Then \( h_j \not\in q \) for some \( j \) and \( q_j = (q, x_{n+1}h_j - 1) \) is a prime ideal of \( V(I_j) \) lying over \( q \). Pick \( f_1, \ldots, f_r \in I \) which map to a basis of
I/I^2 \otimes \kappa(q)$. Then \(x_{n+1}h_j - 1, f_1, \ldots, f_r\) is a sequence of elements of \(I_j\) which map to a basis of \(I_j \otimes \kappa(q_j)\). By Nakayama’s lemma there exists an \(h \in R[x_1, \ldots, x_n, x_{n+1}]\) such that \((I_j)_h\) is generated by \(x_{n+1}h_j - 1, f_1, \ldots, f_r\). We may also assume that \((I_j)_h\) is generated by a Koszul regular sequence of some length \(e\). Looking at the dimension of \(I_j \otimes \kappa(q_j)\) we see that \(e = r + 1\). Hence by Lemma \ref{22.14} we see that \(x_{n+1}h_j - 1, f_1, \ldots, f_r\) is a Koszul-regular sequence generating \((I_j)_h\) for some \(h \in R[x_1, \ldots, x_n, x_{n+1}]\), \(h \not\in q_j\). By Lemma \ref{22.13} we see that \(I_h\) is generated by a Koszul-regular sequence for some \(h' \in R[x_1, \ldots, x_n]\), \(h' \not\in q\) as desired. 

**Lemma 24.4.** Let \(R\) be a ring. Let \(R[x_1, \ldots, x_n]/(f_1, \ldots, f_c)\) be a relative global complete intersection. Then \(f_1, \ldots, f_c\) is a Koszul regular sequence.

**Proof.** Recall that the homology groups \(H_i(K_\bullet(f_\bullet))\) are annihilated by the ideal \((f_1, \ldots, f_c)\). Hence it suffices to show that \(H_i(K_\bullet(f_\bullet))q\) is zero for all primes \(q \subset R[x_1, \ldots, x_n]\) containing \((f_1, \ldots, f_c)\). This follows from Algebra, Lemma \ref{130.13} and the fact that a regular sequence is Koszul regular (Lemma \ref{22.2}).

**Lemma 24.5.** A syntomic ring map is a local complete intersection.

**Proof.** Combine Lemmas \ref{24.4} and \ref{24.3} and Algebra, Lemma \ref{132.15}.

For a local complete intersection \(R \to S\) we have \(H_n(L_{S/R}) = 0\) for \(n \geq 2\). Since we haven’t (yet) defined the full cotangent complex we can’t state and prove this, but we can deduce one of the consequences.

**Lemma 24.6.** Let \(A \to B \to C\) be ring maps. Assume \(B \to C\) is a local complete intersection homomorphism. Choose a presentation \(\alpha : A[x_1, s \in S] \to B\) with kernel \(I\). Choose a presentation \(\beta : B[y_1, \ldots, y_m] \to C\) with kernel \(J\). Let \(\gamma : A[x_1, y_1] \to C\) be the induced presentation of \(C\) with kernel \(K\). Then we get a canonical commutative diagram

\[
\begin{array}{ccccccc}
0 & \longrightarrow & \Omega_{A[x_1]/A} \otimes C & \longrightarrow & \Omega_{A[x_1, y_1]/A} \otimes C & \longrightarrow & \Omega_{B[y_1]/B} \otimes C & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
0 & \longrightarrow & I/I^2 \otimes C & \longrightarrow & K/K^2 & \longrightarrow & J/J^2 & \longrightarrow & 0 \\
\end{array}
\]

with exact rows. In particular, the six term exact sequence of Algebra, Lemma \ref{130.4} can be completed with a zero on the left, i.e., the sequence

\[
0 \to H_1(NL_{B/A} \otimes B C) \to H_1(L_{C/A}) \to H_1(L_{C/B}) \to \Omega_{B/A} \otimes B C \to \Omega_{C/A} \to \Omega_{C/B} \to 0
\]

is exact.

**Proof.** The only thing to prove is the injectivity of the map \(I/I^2 \otimes C \to K/K^2\). By assumption the ideal \(J\) is Koszul-regular. Hence we have \(I A[x_1, y_1] \cap K^2 = I K\) by Lemma \ref{23.5} This means that the kernel of \(K/K^2 \to J/J^2\) is isomorphic to \(I A[x_1, y_1]/I K\). Since \(I/I^2 \otimes A C = I A[x_1, y_1]/I K\) this provides us with the desired injectivity of \(I/I^2 \otimes A C \to K/K^2\) so that the result follows from the snake lemma, see Homology, Lemma \ref{5.17}.

**Lemma 24.7.** Let \(A \to B \to C\) be ring maps. If \(B \to C\) is a filtered colimit of local complete intersection homomorphisms then the conclusion of Lemma \ref{24.6} remains valid.

**Proof.** Follows from Lemma \ref{24.6} and Algebra, Lemma \ref{130.9}.
25. Cartier’s equality and geometric regularity

A reference for this section and the next is [Mat70, Section 39]. In order to comfortably read this section the reader should be familiar with the naive cotangent complex and its properties, see Algebra, Section 130.

Lemma 25.1 (Cartier equality). Let $K/k$ be a finitely generated field extension. Then $\Omega_{K/k}$ and $H_1(L_{K/k})$ are finite dimensional and $\text{trdeg}_k(K) = \dim_K \Omega_{K/k} - \dim_K H_1(L_{K/k})$.

Proof. We can find a global complete intersection $A = k[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ over $k$ such that $K$ is isomorphic to the fraction field of $A$, see Algebra, Lemma 148.11 and its proof. In this case we see that $NL_{K/k}$ is homotopy equivalent to the complex

$$\bigoplus_{j=1,\ldots,c} K \to \bigoplus_{i=1,\ldots,n} K dx_i$$

by Algebra, Lemmas 130.2 and 130.13. The transcendence degree of $K$ over $k$ is the dimension of $A$ (by Algebra, Lemma 113.1) which is $n - c$ and we win. □

Lemma 25.2. Let $K \subset L \subset M$ be field extensions. Then the Jacobi-Zariski sequence

$$0 \to H_1(L_{L/K}) \otimes_L M \to H_1(L_{M/K}) \to H_1(L_{M/L}) \to \Omega_{L/K} \otimes_L M \to \Omega_{M/K} \to \Omega_{M/L} \to 0$$

is exact.

Proof. Combine Lemma 24.7 with Algebra, Lemma 148.11 □

Lemma 25.3. Given a commutative diagram of fields

$$\begin{array}{ccc}
K & \to & K' \\
\uparrow & & \uparrow \\
k & \to & k'
\end{array}$$

with $k \subset k'$ and $K \subset K'$ finitely generated field extensions the kernel and cokernel of the maps

$$\alpha : \Omega_{K/k} \otimes_K K' \to \Omega_{K'/k'}$$

and

$$\beta : H_1(L_{K/k}) \otimes_K K' \to H_1(L_{K'/k'})$$

are finite dimensional and

$$\dim \text{Ker}(\alpha) - \dim \text{Coker}(\alpha) - \dim \text{Ker}(\beta) + \dim \text{Coker}(\beta) = \text{trdeg}_k(k') - \text{trdeg}_K(K')$$

Proof. The Jacobi-Zariski sequences for $k \subset k' \subset K'$ and $k \subset K \subset K'$ are

$$0 \to H_1(L_{k'/k}) \otimes K' \to H_1(L_{K'/k}) \to H_1(L_{K'/k'}) \to \Omega_{k'/k} \otimes K' \to \Omega_{K'/k} \to \Omega_{K'/k} \to 0$$

and

$$0 \to H_1(L_{K/k}) \otimes K' \to H_1(L_{K'/k}) \to H_1(L_{K'/k'}) \to \Omega_{K/k} \otimes K' \to \Omega_{K'/k} \to \Omega_{K'/k} \to 0$$

By Lemma 25.1 the vector spaces $\Omega_{k'/k}$, $\Omega_{K'/K}$, $H_1(L_{K'/K})$, and $H_1(L_{k'/k})$ are finite dimensional and the alternating sum of their dimensions is $\text{trdeg}_k(k') - \text{trdeg}_K(K')$. The lemma follows. □
26. Geometric regularity

Let $k$ be a field. Let $(A, m, K)$ be a Noetherian local $k$-algebra. The Jacobi-Zariski sequence (Algebra, Lemma 130.4) is a canonical exact sequence

$$H_1(L_{K/k}) \to m/m^2 \to \Omega_{A/k} \otimes_A K \to \Omega_{K/k} \to 0$$

because $H_3(L_{K/A}) = m/m^2$ by Algebra, Lemma 130.6. We will show that exactness on the left of this sequence characterizes whether or not a regular local ring $A$ is geometrically regular over $k$. We will link this to the notion of formal smoothness in Section 30.

**Proposition 26.1.** Let $k$ be a field of characteristic $p > 0$. Let $(A, m, K)$ be a Noetherian local $k$-algebra. The following are equivalent

1. $A$ is geometrically regular over $k$,
2. for all $k \subset k' \subset k^{1/p}$ finite over $k$ the ring $A \otimes_k k'$ is regular,
3. $A$ is regular and the canonical map $H_1(L_{K/k}) \to m/m^2$ is injective, and
4. $A$ is regular and the map $\Omega_{k/F_p} \otimes_k K \to \Omega_{A/F_p} \otimes_A K$ is injective.

**Proof.** Proof of (3) $\Rightarrow$ (1). Assume (3). Let $k \subset k'$ be a finite purely inseparable extension. Set $A' = A \otimes_k k'$. This is a local ring with maximal ideal $m'$. Set $K' = A'/m'$. We get a commutative diagram

$$0 \longrightarrow H_1(L_{K/k}) \otimes K' \longrightarrow m/m^2 \otimes K' \longrightarrow \Omega_{A/k} \otimes_A K' \longrightarrow \Omega_{K/k} \otimes K' \longrightarrow 0$$

with exact rows. The third vertical arrow is an isomorphism by base change for modules of differentials (Algebra, Lemma 128.12). Thus $\alpha$ is surjective. By Lemma 25.3 we have

$$\dim \ker(\alpha) - \dim \ker(\beta) + \dim \coker(\beta) = 0$$

(and these dimensions are all finite). A diagram chase shows that $\dim m'/m'^2 \leq \dim m/m^2$. However, since $A \to A'$ is finite flat we see that $\dim(A) = \dim(A')$, see Algebra, Lemma 109.6. Hence $A'$ is regular by definition.

Equivalence of (3) and (4). Consider the Jacobi-Zariski sequences for rows of the commutative diagram

$$\begin{array}{ccc} F_p & \longrightarrow & A \\
\uparrow & & \uparrow \\
F_p & \longrightarrow & K \\
\end{array}$$

$$\begin{array}{ccc} K & \longrightarrow & K \\
\uparrow & & \uparrow \\
F_p & \longrightarrow & k \\
\end{array}$$

to get a commutative diagram

$$0 \longrightarrow m/m^2 \longrightarrow \Omega_{A/F_p} \otimes_A K \longrightarrow \Omega_{K/F_p} \longrightarrow 0$$

$$0 \longrightarrow H_1(L_{K/k}) \longrightarrow \Omega_{k/F_p} \otimes_k K \longrightarrow \Omega_{K/F_p} \longrightarrow 0$$

with exact rows. We have used that $H_1(L_{K/A}) = m/m^2$ and that $H_1(L_{K/F_p}) = 0$ as $K/F_p$ is separable, see Algebra, Proposition 148.9. Thus it is clear that the kernels of $H_1(L_{K/k}) \to m/m^2$ and $\Omega_{k/F_p} \otimes_k K \to \Omega_{A/F_p} \otimes_A K$ have the same dimension.
Proof of (2) ⇒ (4) following Faltings, see [Fal78]. Let \( a_1, \ldots, a_n \in k \) be elements such that \( d_1, \ldots, d_n \) are linearly independent in \( \Omega_{k/F_p} \). Consider the field extension \( k' = k(a_1^{1/p}, \ldots, a_n^{1/p}) \). By Algebra, Lemma 148.3 we see that \( k' = k[x_1, \ldots, x_n]/(x_i^p - a_i, x_j^p - a_j) \). In particular we see that the naive cotangent complex of \( k'/k \) is homotopic to the complex \( \bigoplus_{i=1}^n k' \rightarrow \bigoplus_{i=1}^n k' \) with the zero differential as \( d(x_i^p - a_i) = 0 \) in \( \Omega_{k[x_1, \ldots, x_n]/k} \). Set \( A' = A \otimes_k k' \) and \( K' = A'/m' \) as above. By Algebra, Lemma 130.8 we see that \( NL_{A'/A} \) is homotopy equivalent to the complex \( \bigoplus_{i=1}^n A' \rightarrow \bigoplus_{i=1}^n A' \) with the zero differential, i.e., \( H_1(L_{A'/A}) \) and \( A_{A/A} \) are free of rank \( n \). The Jacobi-Zariski sequence for \( F_p \rightarrow A \rightarrow A' \) is

\[
H_1(L_{A'/A}) \rightarrow \Omega_{A/F_p} \otimes A' \rightarrow \Omega_{A'/F_p} \rightarrow \Omega_{A'/A} \rightarrow 0
\]

Using the presentation \( A[x_1, \ldots, x_n] \rightarrow A' \) with kernel \( (x_i^p - a_i) \) we see, unwinding the maps in Algebra, Lemma 130.4 that the \( j \)th basis vector of \( H_1(L_{A'/A}) \) maps to \( d a_j \otimes 1 \) in \( \Omega_{A/F_p} \otimes A' \). As \( \Omega_{A'/A} \) is free (hence flat) we get on tensoring with \( K' \) an exact sequence

\[
K^{\otimes n} \rightarrow \Omega_{A/F_p} \otimes A' K' \xrightarrow{\beta} \Omega_{A'/F_p} \otimes A' K' \rightarrow K^{\otimes n} \rightarrow 0
\]

We conclude that the elements \( d a_j \otimes 1 \) generate \( \text{Ker}(\beta) \) and we have to show that are linearly independent, i.e., we have to show \( \text{dim}(\text{Ker}(\beta)) = n \). Consider the following big diagram

\[
\begin{array}{ccccccccc}
0 & \rightarrow & m'/m'^2 & \rightarrow & \Omega_{A'/F_p} \otimes K' & \rightarrow & \Omega_{K'/F_p} & \rightarrow & 0 \\
& & \downarrow{\alpha} & & \downarrow{\beta} & & \downarrow{\gamma} & & \\
0 & \rightarrow & m/m^2 \otimes K' & \rightarrow & \Omega_{A/F_p} \otimes K' & \rightarrow & \Omega_{K/F_p} & \otimes K' & \rightarrow 0
\end{array}
\]

By Lemma 25.1 and the Jacobi-Zariski sequence for \( F_p \rightarrow K \rightarrow K' \) we see that the kernel and cokernel of \( \gamma \) have the same finite dimension. By assumption \( A' \) is regular (and of the same dimension as \( A \), see above) hence the kernel and cokernel of \( \alpha \) have the same dimension. It follows that the kernel and cokernel of \( \beta \) have the same dimension which is what we wanted to show.

The implication (1) ⇒ (2) is trivial. This finishes the proof of the proposition. \( \square \)

**Lemma 26.2.** Let \( k \) be a field of characteristic \( p > 0 \). Let \( (A, m, K) \) be a Noetherian local \( k \)-algebra. Assume \( A \) is geometrically regular over \( k \). Let \( k \subset F \subset K \) be a finitely generated subextension. Let \( \varphi : k[y_1, \ldots, y_m] \rightarrow A \) be a \( k \)-algebra map such that \( y_i \) maps to an element of \( F \) in \( K \) and such that \( d y_1, \ldots, d y_m \) map to a basis of \( \Omega_{F/k} \). Set \( p = \varphi^{-1}(m) \). Then

\[
k[y_1, \ldots, y_m]_p \rightarrow A
\]

is flat and \( A/pA \) is regular.

**Proof.** Set \( A_0 = k[y_1, \ldots, y_m]_p \) with maximal ideal \( m_0 \) and residue field \( K_0 \). Note that \( \Omega_{A_0/k} \) is free of rank \( m \) and \( \Omega_{A_0/k} \otimes K_0 = \Omega_{K_0/k} \) is an isomorphism. It is clear that \( A_0 \) is geometrically regular over \( k \). Hence \( H_1(L_{K_0/k}) \rightarrow m_0/m_0^2 \) is an
isomorphism, see Proposition 26.1. Now consider
\[
H_1(L_{K_0/k}) \otimes K \rightarrow m_0/m_0^2 \otimes K
\]
\[
H_1(L_{K/k}) \rightarrow m/m^2
\]
Since the left vertical arrow is injective by Lemma 25.2 and the lower horizontal by Proposition 26.1 we conclude that the right vertical one is too. Hence a regular system of parameters in \( A_0 \) maps to part of a regular system of parameters in \( A \). We win by Algebra, Lemmas 125.2 and 103.3.

27. Topological rings and modules

Let’s quickly discuss some properties of topological abelian groups. An abelian group \( M \) is a \textit{topological abelian group} if \( M \) is endowed with a topology such that addition \( M \times M \rightarrow M, (x, y) \mapsto x + y \) and inverse \( M \rightarrow M, x \mapsto -x \) are continuous. A \textit{homomorphism of topological abelian groups} is just a homomorphism of abelian groups which is continuous. The category of commutative topological groups is additive and has kernels and cokernels, but is not abelian (as the axiom Im = Coim doesn’t hold). If \( N \subset M \) is a subgroup, then we think of \( N \) and \( M/N \) as topological groups also, namely using the induced topology on \( N \) and the quotient topology on \( M/N \) (i.e., such that \( M \rightarrow M/N \) is submersive). Note that if \( N \subset M \) is an open subgroup, then the topology on \( M/N \) is discrete.

We say the topology on \( M \) is \textit{linear} if there exists a fundamental system of neighbourhoods of 0 consisting of subgroups. If so then these subgroups are also open. An example is the following. Let \( I \) be a directed partially ordered set and let \( G_i \) be an inverse system of (discrete) abelian groups over \( I \). Then
\[
G = \lim_{i \in I} G_i
\]
with the inverse limit topology is linearly topologized with a fundamental system of neighbourhoods of 0 given by \( \operatorname{Ker}(G \rightarrow G_i) \). Conversely, let \( M \) be a linearly topologized abelian group. Choose any fundamental system of open subgroups \( U_i \subset M, i \in I \) (i.e., the \( U_i \) form a fundamental system of open neighbourhoods and each \( U_i \) is a subgroup of \( M \)). Setting \( i \geq i' \Leftrightarrow U_i \subset U_{i'} \) we see that \( I \) is a directed partially ordered set. We obtain a homomorphism of linearly topologized abelian groups
\[
c : M \rightarrow \lim_{i \in I} M/U_i.
\]
It is clear that \( M \) is separated (as a topological space) if and only if \( c \) is injective. We say that \( M \) is \textit{complete} if \( c \) is an isomorphism.\footnote{We include being separated as part of being complete as we’d like to have a unique limits in complete groups. There is a definition of completeness for any topological group, agreeing, modulo the separation issue, with this one in our special case.} We leave it to the reader to check that this condition is independent of the choice of fundamental system of open subgroups \( \{U_i\}_{i \in I} \) chosen above. In fact the topological abelian group \( M^\wedge = \lim_{i \in I} M/U_i \) is independent of this choice and is sometimes called the \textit{completion} of \( M \). Any \( G = \lim G_i \) as above is complete, in particular, the completion \( M^\wedge \) is always complete.

\textbf{Definition 27.1 (Topological rings).} Let \( R \) be a ring and let \( M \) be an \( R \)-module.
(1) We say \( R \) is a topological ring if \( R \) is endowed with a topology such that both addition and multiplication are continuous as maps \( R \times R \to R \) where \( R \times R \) has the product topology. In this case we say \( M \) is a topological module if \( M \) is endowed with a topology such that addition \( M \times M \to M \) and scalar multiplication \( R \times M \to M \) are continuous.

(2) A homomorphism of topological modules is just a continuous \( R \)-module map. A homomorphism of topological rings is a ring homomorphism which is continuous for the given topologies.

(3) We say \( M \) is linearly topologized if \( 0 \) has a fundamental system of neighbourhoods consisting of submodules. We say \( R \) is linearly topologized if \( 0 \) has a fundamental system of neighbourhoods consisting of ideals.

(4) If \( R \) is linearly topologized, we say that \( I \subset R \) is an ideal of definition if \( I \) is open and if every neighbourhood of \( 0 \) contains \( I^n \) for some \( n \).

(5) If \( R \) is linearly topologized, we say that \( R \) is pre-admissible if \( R \) has an ideal of definition.

(6) If \( R \) is linearly topologized, we say that \( R \) is admissible if it is pre-admissible and complete.

(7) If \( R \) is linearly topologized, we say that \( R \) is pre-adic if there exists an ideal of definition \( I \) such that \( \{I^n\}_{n \geq 0} \) forms a fundamental system of neighbourhoods of \( 0 \).

(8) If \( R \) is linearly topologized, we say that \( R \) is adic if \( R \) is pre-adic and complete.

Note that a (pre)adic topological ring is the same thing as a (pre)admissible topological ring which has an ideal of definition \( I \) such that \( I^n \) is open for all \( n \geq 1 \).

Let \( R \) be a ring and let \( M \) be an \( R \)-module. Let \( I \subset R \) be an ideal. Then we can consider the linear topology on \( R \) which has \( \{I^n\}_{n \geq 0} \) as a fundamental system of neighbourhoods of \( 0 \). This topology is called the \( I \)-adic topology: \( R \) is a pre-adic topological ring in the \( I \)-adic topology. Moreover, the linear topology on \( M \) which has \( \{I^nM\}_{n \geq 0} \) as a fundamental system of open neighbourhoods of \( 0 \) turns \( M \) into a topological \( R \)-module. This is called the \( I \)-adic topology on \( M \). We see that \( M \) is \( I \)-adically complete (as defined in Algebra, Definition 94.5) if and only \( M \) is complete in the \( I \)-adic topology. In particular, we see that \( R \) is \( I \)-adically complete if and only if \( R \) is an adic topological ring in the \( I \)-adic topology.

As a special case, note that the discrete topology is the 0-adic topology and that any ring in the discrete topology is adic.

**Lemma 27.2.** Let \( \varphi : R \to S \) be a ring map. Let \( I \subset R \) and \( J \subset S \) be ideals and endow \( R \) with the \( I \)-adic topology and \( S \) with the \( J \)-adic topology. Then \( \varphi \) is a homomorphism of topological rings if and only if \( \varphi(I^n) \subset J \) for some \( n \geq 1 \).

**Proof.** Omitted.

---

\(^3\)By our conventions this includes separated.

\(^4\)Thus the \( I \)-adic topology is sometimes called the \( I \)-pre-adic topology.

\(^5\)It may happen that the \( I \)-adic completion \( M^\wedge \) is not \( I \)-adically complete, even though \( M^\wedge \) is always complete with respect to the limit topology. If \( I \) is finitely generated then the \( I \)-adic topology and the limit topology on \( M^\wedge \) agree, see Algebra, Lemma 94.7 and its proof.
28. Formally smooth maps of topological rings

There is a version of formal smoothness which applies to homomorphisms of topological rings.

**Definition 28.1.** Let \( R \to S \) be a homomorphism of topological rings with \( R \) and \( S \) linearly topologized. We say \( S \) is **formally smooth over** \( R \) if for every commutative solid diagram

\[
\begin{array}{ccc}
S & \longrightarrow & A/J \\
\downarrow & & \downarrow \\
R & \longrightarrow & A
\end{array}
\]

of homomorphisms of topological rings where \( A \) is a discrete ring and \( J \subset A \) is an ideal of square zero, a dotted arrow exists which makes the diagram commute.

We will mostly use this notion when given ideals \( m \subset R \) and \( n \subset S \) and we endow \( R \) with the \( m \)-adic topology and \( S \) with the \( n \)-adic topology. Continuity of \( \varphi : R \to S \) holds if and only if \( \varphi(m^m) \subset n \) for some \( m \geq 1 \), see Lemma 27.2. It turns out that in this case only the topology on \( S \) is relevant.

**Lemma 28.2.** Let \( \varphi : R \to S \) be a ring map.

1. If \( R \to S \) is formally smooth in the sense of Algebra, Definition 134.1, then \( R \to S \) is formally smooth for any linear topology on \( R \) and any pre-adic topology on \( S \) such that \( R \to S \) is continuous.
2. Let \( n \subset S \) and \( m \subset R \) ideals such that \( \varphi \) is continuous for the \( m \)-adic topology on \( R \) and the \( n \)-adic topology on \( S \). Then the following are equivalent
   a. \( \varphi \) is formally smooth for the \( m \)-adic topology on \( R \) and the \( n \)-adic topology on \( S \), and
   b. \( \varphi \) is formally smooth for the discrete topology on \( R \) and the \( n \)-adic topology on \( S \).

**Proof.** Assume \( R \to S \) is formally smooth in the sense of Algebra, Definition 134.1. If \( S \) has a pre-adic topology, then there exists an ideal \( n \subset S \) such that \( S \) has the \( n \)-adic topology. Suppose given a solid commutative diagram as in Definition 28.1. Continuity of \( S \to A/J \) means that \( n^k \) maps to zero in \( A/J \) for some \( k \geq 1 \), see Lemma 27.2. We obtain a ring map \( \psi : S \to A \) from the assumed formal smoothness of \( S \) over \( R \). Then \( \psi(n^k) \subset J \) hence \( \psi(n^{2k}) = 0 \) as \( J^2 = 0 \). Hence \( \psi \) is continuous by Lemma 27.2. This proves (1).

The proof of (2)(b) \( \Rightarrow \) (2)(a) is the same as the proof of (1). Assume (2)(a). Suppose given a solid commutative diagram as in Definition 28.1 where we use the discrete topology on \( R \). Since \( \varphi \) is continuous we see that \( \varphi(m^m) \subset n \) for some \( m \geq 1 \). As \( S \to A/J \) is continuous we see that \( n^k \) maps to zero in \( A/J \) for some \( k \geq 1 \). Hence \( m^{nk} \) maps into \( J \) under the map \( R \to A \). Thus \( m^{2nk} \) maps to zero in \( A \) and we see that \( R \to A \) is continuous in the \( m \)-adic topology. Thus (2)(a) gives a dotted arrow as desired.  

**Definition 28.3.** Let \( R \to S \) be a ring map. Let \( n \subset S \) be an ideal. If the equivalent conditions (2)(a) and (2)(b) of Lemma 28.2 hold, then we say \( R \to S \) is **formally smooth for the** \( n \)-adic topology.

This property is inherited by the completions.
Lemma 28.4. Let \((R, \mathfrak{m})\) and \((S, \mathfrak{n})\) be rings endowed with finitely generated ideals. Endow \(R\) and \(S\) with the \(\mathfrak{m}\)-adic and \(\mathfrak{n}\)-adic topologies. Let \(R \to S\) be a homomorphism of topological rings. The following are equivalent

1. \(R \to S\) is formally smooth for the \(\mathfrak{n}\)-adic topology,
2. \(R \to S^\wedge\) is formally smooth for the \(\mathfrak{n}^\wedge\)-adic topology,
3. \(R^\wedge \to S^\wedge\) is formally smooth for the \(\mathfrak{n}^\wedge\)-adic topology.

Here \(R^\wedge\) and \(S^\wedge\) are the \(\mathfrak{m}\)-adic and \(\mathfrak{n}\)-adic completions of \(R\) and \(S\).

Proof. The assumption that \(\mathfrak{m}\) is finitely generated implies that \(R^\wedge\) is \(\mathfrak{m}^\wedge\)-adically complete, that \(\mathfrak{m}^\wedge R^\wedge = \mathfrak{m}^\wedge\) and that \(R^\wedge / \mathfrak{m}^n R^\wedge = R^\wedge / \mathfrak{m}^n\), see Algebra, Lemma [94.7] and its proof. Similarly for \((S, \mathfrak{n})\). Thus it is clear that diagrams as in Definition 28.1 for the cases (1), (2), and (3) are in 1-to-1 correspondence. □

The advantage of working with adic rings is that one gets a stronger lifting property.

Lemma 28.5. Let \(R \to S\) be a ring map. Let \(n\) be an ideal of \(S\). Assume that \(R \to S\) is formally smooth in the \(n\)-adic topology. Consider a solid commutative diagram

\[
\begin{array}{ccc}
S & \xrightarrow{\psi} & A/J \\
\downarrow & & \downarrow \\
R & \xrightarrow{\psi} & A \\
\end{array}
\]

of homomorphisms of topological rings where \(A\) is adic and \(A/J\) is the quotient (as topological ring) of \(A\) by a closed ideal \(J \subset A\) such that \(J^t\) is contained in an ideal of definition of \(A\) for some \(t \geq 1\). Then there exists a dotted arrow in the category of topological rings which makes the diagram commute.

Proof. Let \(I \subset A\) be an ideal of definition so that \(I \supset J^t\) for some \(n\). Then \(A = \lim A/I^n\) and \(A/J = \lim A/J + I^n\) because \(J\) is assumed closed. Consider the following diagram of discrete \(R\) algebras \(A_{n,m} = A/J^n + I^m\):

\[
\begin{array}{ccc}
A/J^3 + I^3 & \longrightarrow & A/J^2 + I^3 \\
\downarrow & & \downarrow \\
A/J^3 + I^2 & \longrightarrow & A/J^2 + I^2 \\
\downarrow & & \downarrow \\
A/J^3 + I & \longrightarrow & A/J^2 + I \\
\end{array}
\]

Note that each of the commutative squares defines a surjection

\[
A_{n+1,m+1} \longrightarrow A_{n+1,m} \times_{A_{n,m}} A_{n,m+1}
\]

of \(R\)-algebras whose kernel has square zero. We will inductively construct \(R\)-algebra maps \(\varphi_{n,m} : S \to A_{n,m}\). Namely, we have the maps \(\varphi_{1,m} = \psi \mod J + I^m\). Note that each of these maps is continuous as \(\psi\) is. We can inductively choose the maps \(\varphi_{n,1}\) by starting with our choice of \(\varphi_{1,1}\) and lifting up, using the formal smoothness of \(S\) over \(R\), along the right column of the diagram above. We construct the remaining maps \(\varphi_{n,m}\) by induction on \(n + m\). Namely, we choose \(\varphi_{n+1,m+1}\) by lifting the pair \((\varphi_{n+1,m}, \varphi_{n,m+1})\) along the displayed surjection above (again using the formal smoothness of \(S\) over \(R\)). In this way all of the maps \(\varphi_{n,m}\)
are compatible with the transition maps of the system. As $J^t \subset I$ we see that for example $\varphi_n = \varphi_{nt,n} \mod I^n$ induces a map $S \to A/I^n$. Taking the limit $\varphi = \lim \varphi_n$ we obtain a map $S \to A = \lim A/I^n$. The composition into $A/J$ agrees with $\psi$ as we have seen that $A/J = \lim A/J + I^n$. Finally we show that $\varphi$ is continuous. Namely, we know that $\psi(n^r) \subset J + I^r/J$ for some $r$ by our assumption that $\psi$ is a morphism of topological rings, see Lemma 27.2. Hence $\varphi(n^r) \subset J + I$ hence $\varphi(n^r) \subset I$ as desired.

Lemma 28.6. Let $R \to S$ be a ring map. Let $n \subset n' \subset S$ be ideals. If $R \to S$ is formally smooth for the $n$-adic topology, then $R \to S$ is formally smooth for the $n'$-adic topology.

Proof. Omitted. □

Lemma 28.7. A composition of formally smooth continuous homomorphisms of linearly topologized rings is formally smooth.

Proof. Omitted. (Hint: This is completely formal, and follows from considering a suitable diagram.) □

Lemma 28.8. Let $R, S$ be rings. Let $n \subset S$ be an ideal. Let $R \to S$ be formally smooth for the $n$-adic topology. Let $R \to R'$ be any ring map. Then $R' \to S' = S \otimes_R R'$ is formally smooth in the $n' = nS'$-adic topology.

Proof. Let a solid diagram

\[
\begin{array}{c}
S \\ R
\end{array} \longrightarrow \begin{array}{c} S' \\ R'
\end{array} \longrightarrow \begin{array}{c} A/J \\ A
\end{array}
\]

as in Definition 28.1 be given. Then the composition $S \to S' \to A/J$ is continuous. By assumption the longer dotted arrow exists. By the universal property of tensor product we obtain the shorter dotted arrow. □

We have seen descent for formal smoothness along faithfully flat ring maps in Algebra, Lemma 134.15. Something similar holds in the current setting of topological rings. However, here we just prove the following very simple and easy to prove version which is already quite useful.

Lemma 28.9. Let $R, S$ be rings. Let $n \subset S$ be an ideal. Let $R \to R'$ be a ring map. Set $S' = S \otimes_R R'$ and $n' = nS$. If

1. the map $R \to R'$ embeds $R$ as a direct summand of $R'$ as an $R$-module, and
2. $R' \to S'$ is formally smooth for the $n'$-adic topology,

then $R \to S$ is formally smooth in the $n$-adic topology.

Proof. Let a solid diagram

\[
\begin{array}{c}
S \\ R
\end{array} \longrightarrow \begin{array}{c} A/J \\ A
\end{array}
\]
as in Definition 28.1 be given. Set $A' = A \otimes_R R'$ and $J' = \text{Im}(J \otimes_R R' \to A')$. The base change of the diagram above is the diagram

$$
\begin{array}{ccc}
S' & \longrightarrow & A'/J' \\
\downarrow \psi' & & \downarrow \\
R' & \longrightarrow & A'
\end{array}
$$

with continuous arrows. By condition (2) we obtain the dotted arrow $\psi' : S' \to A'$. Using condition (1) choose a direct summand decomposition $R' = R \oplus C$ as $R$-modules. (Warning: $C$ isn’t an ideal in $R'$.) Then $A' = A \oplus A \otimes_R C$. Set

$$J'' = \text{Im}(J \otimes_R C \to A \otimes_R C) \subset J' \subset A'.
$$

Then $J' = J \oplus J''$ as $A$-modules. The image of the composition $\psi : S \to A'$ of $\psi'$ with $S \to S'$ is contained in $A + J' = A \oplus J''$. However, in the ring $A + J' = A \oplus J''$ the $A$-submodule $J''$ is an ideal! (Use that $J^2 = 0$.) Hence the composition $S \to A + J' \to (A + J')/J'' = A$ is the arrow we were looking for.

The following lemma will be improved on in Section 30.

**Lemma 28.10.** Let $k$ be a field and let $(A, m, K)$ be a Noetherian local $k$-algebra. If $k \to A$ is formally smooth for the $m$-adic topology, then $A$ is a regular local ring.

**Proof.** Let $k_0 \subset k$ be the prime field. Then $k_0$ is perfect, hence $k/k_0$ is separable, hence formally smooth by Algebra, Lemma 148.7. By Lemmas 28.2 and 28.7 we see that $k_0 \to A$ is formally smooth for the $m$-adic topology on $A$. Hence we may assume $k = \mathbb{Q}$ or $k = \mathbb{F}_p$.

By Algebra, Lemmas 94.4 and 107.9 it suffices to prove the completion $A^\wedge$ is regular. By Lemma 28.4 we may replace $A$ by $A^\wedge$. Thus we may assume that $A$ is a Noetherian complete local ring. By the Cohen structure theorem (Algebra, Theorem 150.8) there exist a map $K \to A$. As $k$ is the prime field we see that $K \to A$ is a $k$-algebra map.

Let $x_1, \ldots, x_n \in m$ be elements whose images form a basis of $m/m^2$. Set $T = K[[X_1, \ldots, X_n]]$. Note that $A/m^2 \cong K[x_1, \ldots, x_n]/(x_ix_j)$ and $T/m_T^2 \cong K[X_1, \ldots, X_n]/(X_iX_j)$.

Let $A/m^2 \to T/m_T^2$ be the local $K$-algebra isomorphism given by mapping the class of $x_i$ to the class of $X_i$. Denote $f_1 : A \to T/m_T^2$ the composition of this isomorphism with the quotient map $A \to A/m^2$. The assumption that $k \to A$ is formally smooth in the $m$-adic topology means we can lift $f_1$ to a map $f_2 : A \to T/m_T^2$, then to a map $f_3 : A \to T/m_T^3$, and so on, for all $n \geq 1$. Warning: the maps $f_n$ are continuous $k$-algebra maps and may not be $K$-algebra maps. We get an induced map $f : A \to T = \lim T/m_T^n$ of local $k$-algebras. By our choice of $f_1$, the map $f$ induces an isomorphism $m/m^2 \to m_T/m_T^2$, hence each $f_n$ is surjective and we conclude $f$ is surjective as $A$ is complete. This implies $\dim(A) \geq \dim(T) = n$. Hence $A$ is regular by definition. (It also follows that $f$ is an isomorphism.)

The following result will be improved on in Section 30.
Lemma 28.11. Let $k$ be a field. Let $(A, m, K)$ be a regular local $k$-algebra such that $K/k$ is separable. Then $k \to A$ is formally smooth in the $m$-adic topology.

Proof. It suffices to prove that the completion of $A$ is formally smooth over $k$, see Lemma 28.4. Hence we may assume that $A$ is a complete local regular $k$-algebra with residue field $K$ separable over $k$. Since $K$ is formally smooth over $k$ by Algebra, Proposition 148.9 we can successively find maps

$$\cdots \to A/m^3 \to A/m^2 \to K$$

of $k$-algebras. Since $A$ is complete this defines a $k$-algebra map $K \to A$. Pick $a_1, \ldots, a_n \in m$ which map to a $K$-basis of $m/m^2$. Consider the $K$-algebra map

$$c : K[[x_1, \ldots, x_n]] \to A$$

which maps $x_i$ to $a_i$ (existence of $c$ follows from the universal property of the power series ring). By construction the maps $K[[x_1, \ldots, x_n]] \to A/m^c$ are surjective for all $c \geq 1$. Since $K[[x_1, \ldots, x_n]]$ is complete we see that $c$ is surjective. Since $\dim(A) = n$ as $A$ is regular and since $K[[x_1, \ldots, x_n]]$ is a domain of dimension $n$ we see that the kernel of $c$ is zero. Hence $c$ is an isomorphism.

We win because the power series ring $K[[x_1, \ldots, x_n]]$ is formally smooth over $k$. Namely, $K$ is formally smooth over $k$ and $K[x_1, \ldots, x_n]$ is formally smooth over $K$ as a polynomial algebra. Hence $K[x_1, \ldots, x_n]$ is formally smooth over $k$ by Algebra, Lemma 134.3. It follows that $k \to K[x_1, \ldots, x_n]$ is formally smooth for the $(x_1, \ldots, x_n)$-adic topology by Lemma 28.2. Finally, it follows that $k \to K[[x_1, \ldots, x_n]]$ is formally smooth for the $(x_1, \ldots, x_n)$-adic topology by Lemma 28.4.

Lemma 28.12. Let $A \to B$ be a finite type ring map with $A$ Noetherian. Let $q \subset B$ be a prime ideal lying over $p \subset A$. The following are equivalent

(1) $A \to B$ is smooth at $q$, and
(2) $A_p \to B_q$ is formally smooth in the $q$-adic topology.

Proof. The implication (2) $\Rightarrow$ (1) follows from Algebra, Lemma 137.2. Conversely, if $A \to B$ is smooth at $q$, then $A \to B_q$ is smooth for some $g \in B$, $g \notin q$. Then $A \to B_g$ is formally smooth by Algebra, Proposition 134.13. Hence $A_p \to B_q$ is formally smooth as localization preserves formal smoothness (for example by the criterion of Algebra, Proposition 134.8 and the fact that the cotangent complex behaves well with respect to localization, see Algebra, Lemmas 130.11 and 130.13). Finally, Lemma 28.2 implies that $A_p \to B_q$ is formally smooth in the $q$-adic topology.

29. Some results on power series rings

Questions on formally smooth maps between Noetherian local rings can often be reduced to questions on maps between power series rings. In this section we prove some helper lemmas to facilitate this kind of argument.

Lemma 29.1. Let $K$ be a field of characteristic $0$ and $A = K[[x_1, \ldots, x_n]]$. Let $L$ be a field of characteristic $p > 0$ and $B = L[[x_1, \ldots, x_n]]$. Let $\Lambda$ be a Cohen ring. Let $C = \Lambda[[x_1, \ldots, x_n]]$. 
(1) \( \mathbb{Q} \to A \) is formally smooth in the \( m \)-adic topology.

(2) \( F_p \to B \) is formally smooth in the \( m \)-adic topology.

(3) \( \mathbb{Z} \to C \) is formally smooth in the \( m \)-adic topology.

**Proof.** By the universal property of power series rings it suffices to prove:

(1) \( \mathbb{Q} \to K \) is formally smooth.

(2) \( F_p \to L \) is formally smooth.

(3) \( \mathbb{Z} \to \Lambda \) is formally smooth in the \( m \)-adic topology.

The first two are Algebra, Proposition 148.9. The third follows from Algebra, Lemma 150.7 since for any test diagram as in Definition 28.1 some power of \( p \) will be zero in \( A/J \) and hence some power of \( p \) will be zero in \( A \). \( \square \)

**Lemma 29.2.** Let \( K \) be a field and \( A = K[[x_1, \ldots, x_n]] \). Let \( \Lambda \) be a Cohen ring and let \( B = \Lambda[[z_1, \ldots, z_r, y_1, \ldots, y_{n-r}]] \).

(1) If \( y_1, \ldots, y_n \in A \) is a regular system of parameters then \( K[[y_1, \ldots, y_n]] \to A \) is an isomorphism.

(2) If \( z_1, \ldots, z_r \in A \) form part of a regular system of parameters for \( A \), then \( r \leq n \) and \( A/(z_1, \ldots, z_r) \cong K[[y_1, \ldots, y_{n-r}]] \).

(3) If \( p, y_1, \ldots, y_n \in B \) is a regular system of parameters then \( \Lambda[[y_1, \ldots, y_n]] \to B \) is an isomorphism.

(4) If \( p, z_1, \ldots, z_r \in B \) form part of a regular system of parameters for \( B \), then \( r \leq n \) and \( B/(z_1, \ldots, z_r) \cong \Lambda[[y_1, \ldots, y_{n-r}]] \).

**Proof.** Proof of (1). Set \( A' = K[[y_1, \ldots, y_n]] \). It is clear that the map \( A' \to A \) induces an isomorphism \( A'/m^2_{A'} \to A/m^2_A \) for all \( n \geq 1 \). Since \( A \) and \( A' \) are both complete we deduce that \( A' \to A \) is an isomorphism. Proof of (2). Extend \( z_1, \ldots, z_r \) to a regular system of parameters \( z_1, \ldots, z_r, y_1, \ldots, y_{n-r} \) of \( A \). Consider the map \( A' = K[[z_1, \ldots, z_r, y_1, \ldots, y_{n-r}]] \to A \). This is an isomorphism by (1). Hence (2) follows as it is clear that \( A'/((z_1, \ldots, z_r)) \cong K[[y_1, \ldots, y_{n-r}]] \). The proofs of (3) and (4) are exactly the same as the proofs of (1) and (2). \( \square \)

**Lemma 29.3.** Let \( A \to B \) be a local homomorphism of Noetherian complete local rings. Then there exists a commutative diagram

\[
\begin{array}{ccc}
S & \longrightarrow & B \\
\uparrow & & \uparrow \\
R & \longrightarrow & A
\end{array}
\]

with the following properties:

(1) the horizontal arrows are surjective,

(2) if the characteristic of \( A/m_A \) is zero, then \( S \) and \( R \) are power series rings over fields,

(3) if the characteristic of \( A/m_A \) is \( p > 0 \), then \( S \) and \( R \) are power series rings over Cohen rings, and

(4) \( R \to S \) maps a regular system of parameters of \( R \) to part of a regular system of parameters of \( S \).

In particular \( R \to S \) is flat (see Algebra, Lemma 125.2) with regular fibre \( S/m_RS \) (see Algebra, Lemma 103.3).
Proof. Use the Cohen structure theorem (Algebra, Theorem 150.8) to choose a surjection \( S \to B \) as in the statement of the lemma where we choose \( S \) to be a power series over a Cohen ring if the residue characteristic is \( p > 0 \) and a power series over a field else. Let \( J \subset S \) be the kernel of \( S \to B \). Next, choose a surjection \( R = \Lambda[[x_1, \ldots, x_n]] \to A \) where we choose \( \Lambda \) to be a Cohen ring if the residue characteristic of \( A \) is \( p > 0 \) and \( \Lambda \) equal to the residue field of \( A \) otherwise. We lift the composition \( \Lambda[[x_1, \ldots, x_n]] \to A \to B \) to a map \( \varphi : R \to S \). This is possible because \( \Lambda[[x_1, \ldots, x_n]] \) is formally smooth over \( \mathbf{Z} \) in the \( \mathfrak{m} \)-adic topology (see Lemma 29.1) by an application of Lemma 28.5. Finally, we replace \( \varphi \) by the map \( \varphi' : R = \Lambda[[x_1, \ldots, x_n]] \to S' = S[[y_1, \ldots, y_n]] \) with \( \varphi'|_{\Lambda} = \varphi|_{\Lambda} \) and \( \varphi'(x_i) = \varphi(x_i) + y_i \). We also replace \( S \to B \) by the map \( S' \to B \) which maps \( y_i \) to zero. After this replacement it is clear that a regular system of parameters of \( R \) maps to part of a regular sequence in \( S' \) and we win. \( \square \)

There should be an elementary proof of the following lemma.

Lemma 29.4. Let \( S \to R \) and \( S' \to R \) be surjective maps of complete Noetherian local rings. Then \( S \times_R S' \) is a complete Noetherian local ring.

Proof. Let \( k \) be the residue field of \( R \). If the characteristic of \( k \) is \( p > 0 \), then we denote \( \Lambda \) a Cohen ring (Algebra, Definition 150.5) with residue field \( k \) (Algebra, Lemma 150.6). If the characteristic of \( k \) is 0 we set \( \Lambda = k \). Choose a surjection \( \Lambda[[x_1, \ldots, x_n]] \to R \) (as in the Cohen structure theorem, see Algebra, Theorem 150.8) and lift this to maps \( \Lambda[[x_1, \ldots, x_n]] \to S \) and \( \varphi : \Lambda[[x_1, \ldots, x_n]] \to S \) and \( \varphi : \Lambda[[x_1, \ldots, x_n]] \to S' \) using Lemmas 29.1 and 28.5. Next, choose \( f_1, \ldots, f_m \in S \) generating the kernel of \( S \to R \) and \( f_1', \ldots, f_{m'} \in S' \) generating the kernel of \( S' \to R \). Then the map

\[
\Lambda[[x_1, \ldots, x_n, y_1, \ldots, y_m, z_1, \ldots, z_{m'}]] \to S \times_R S',
\]

which sends \( x_i \) to \( (\varphi(x_i), \varphi'(x_i)) \) and \( y_j \) to \( (f_j, 0) \) and \( z_j' \) to \( (0, f_j') \) is surjective. Thus \( S \times_R S' \) is a quotient of a complete local ring, whence complete. \( \square \)

30. Geometric regularity and formal smoothness

In this section we combine the results of the previous sections to prove the following characterization of geometrically regular local rings over fields. We then recycle some of our arguments to prove a characterization of formally smooth maps in the \( \mathfrak{m} \)-adic topology between Noetherian local rings.

Theorem 30.1. Let \( k \) be a field. Let \( (A, \mathfrak{m}, K) \) be a Noetherian local \( k \)-algebra. If the characteristic of \( k \) is zero then the following are equivalent

(1) \( A \) is a regular local ring, and
(2) \( k \to A \) is formally smooth in the \( \mathfrak{m} \)-adic topology.

If the characteristic of \( k \) is \( p > 0 \) then the following are equivalent

(1) \( A \) is geometrically regular over \( k \),
(2) \( k \to A \) is formally smooth in the \( \mathfrak{m} \)-adic topology.
(3) for all \( k' \subset k^{1/p} \) finite over \( k \) the ring \( A \otimes_k k' \) is regular,
(4) \( A \) is regular and the canonical map \( H_1(L_{K/k}) \to \mathfrak{m}/\mathfrak{m}^2 \) is injective, and
(5) \( A \) is regular and the map \( \Omega_{k/F_p} \otimes_k K \to \Omega_{A/F_p} \otimes_A K \) is injective.
Proof. If the characteristic of $k$ is zero, then the equivalence of (1) and (2) follows from Lemmas 28.10 and 28.11.

If the characteristic of $k$ is $p > 0$, then it follows from Proposition 26.1 that (1), (3), (4), and (5) are equivalent. Assume (2) holds. By Lemma 28.8 we see that $k' \rightarrow A' = A \otimes_k k'$ is formally smooth for the $m'$-adic topology. Hence if $k \subset k'$ is finite purely inseparable, then $A'$ is a regular local ring by Lemma 28.10.

Thus we see that (1) holds.

Finally, we will prove that (5) implies (2). Choose a solid diagram

$$
\begin{array}{ccc}
A & \longrightarrow & B/J \\
\downarrow \psi & & \downarrow \pi \\
k & \longrightarrow & B
\end{array}
$$

as in Definition 28.1. As $J^2 = 0$ we see that $J$ has a canonical $B/J$ module structure and via $\psi$ an $A$-module structure. As $\psi$ is continuous for the $m$-adic topology we see that $m^nJ = 0$ for some $n$. Hence we can filter $J$ by $B/J$-submodules $0 \subset J_1 \subset J_2 \subset \ldots \subset J_n = J$ such that each quotient $J_{i+1}/J_i$ is annihilated by $m$.

Considering the sequence of ring maps $B \rightarrow B/J_1 \rightarrow B/J_2 \rightarrow \ldots \rightarrow B/J$ we see that it suffices to prove the existence of the dotted arrow when $J$ is annihilated by $m$, i.e., when $J$ is a $K$-vector space.

Assume given a diagram as above such that $J$ is annihilated by $m$. By Lemma 28.11 we see that $F_p \rightarrow A$ is formally smooth in the $m$-adic topology. Hence we can find a ring map $\psi : A \rightarrow B$ such that $\pi \circ \psi = \psi$. Then $\psi \circ i, \varphi : k \rightarrow B$ are two maps whose compositions with $\pi$ are equal. Hence $D = \psi \circ i - \varphi : k \rightarrow J$ is a derivation. By Algebra, Lemma 128.3 we can write $D = \xi \circ d$ for some $k$-linear map $\xi : \Omega_{k/F_p} \rightarrow J$. Using the $K$-vector space structure on $J$ we extend $\xi$ to a $K$-linear map $\xi' : \Omega_{k/F_p} \otimes_k K \rightarrow J$. Using (5) we can find a $K$-linear map $\xi'' : \Omega_A/F_p \otimes_A K$ whose restriction to $\Omega_{k/F_p} \otimes_k K$ is $\xi'$.

Write

$$
D' : A \rightarrow \Omega_A/F_p \rightarrow \Omega_A/F_p \otimes_A K \xrightarrow{\xi''} J.
$$

Finally, set $\psi' = \psi - D' : A \rightarrow B$. The reader verifies that $\psi'$ is a ring map such that $\pi \circ \psi' = \psi$ and such that $\psi' \circ i = \varphi$ as desired.

\[\square\]

**Example 30.2.** Let $k$ be a field of characteristic $p > 0$. Suppose that $a \in k$ is an element which is not a $p$th power. A standard example of a geometrically regular local $k$-algebra whose residue field is purely inseparable over $k$ is the ring

$$
A = k[x, y]/(x, y^p - a)/\langle y^p - a - x \rangle
$$

Namely, $A$ is a localization of a smooth algebra over $k$ hence $k \rightarrow A$ is formally smooth, hence $k \rightarrow A$ is formally smooth for the $m$-adic topology. A closely related example is the following. Let $k = F_p(s)$ and $K = F_p(t)^{\text{perf}}$. We claim the ring map

$$
k \rightarrow A = K[[x]], \quad s \mapsto t + x
$$

is formally smooth for the $(x)$-adic topology on $A$. Namely, $\Omega_{k/F_p}$ is 1-dimensional with basis $ds$. It maps to the element $dx + dt = dx$ in $\Omega_{A/F_p}$. We leave it to the reader to show that $\Omega_{A/F_p}$ is free on $dx$ as an $A$-module. Hence we see that condition (5) of Theorem 30.1 holds and we conclude that $k \rightarrow A$ is formally smooth in the $(x)$-adic topology.
Lemma 30.3. Let $A \to B$ be a local homomorphism of Noetherian local rings. Assume $A \to B$ is formally smooth in the $\mathfrak{m}_B$-adic topology. Then $A \to B$ is flat.

Proof. We may assume that $A$ and $B$ a Noetherian complete local rings by Lemma 28.4 and Algebra, Lemma 94.10 (this also uses Algebra, Lemma 38.8 and 94.4 to see that flatness of the map on completions implies flatness of $A \to B$). Choose a commutative diagram

$$
\begin{array}{ccc}
S & \to & B \\
\uparrow & & \uparrow \\
R & \to & A
\end{array}
$$

as in Lemma 29.3 with $R \to S$ flat. Let $I \subset R$ be the kernel of $R \to A$. Because $B$ is formally smooth over $A$ we see that the $A$-algebra map $S/IS \to B$ has a section, see Lemma 28.5. Hence $B$ is a direct summand of the flat $A$-module $S/IS$ (by base change of flatness, see Algebra, Lemma 38.6), whence flat. □

Proposition 30.4. Let $A \to B$ be a local homomorphism of Noetherian local rings. Let $k$ be the residue field of $A$ and $\overline{B} = B \otimes_A k$ the special fibre. The following are equivalent

1. $A \to B$ is flat and $\overline{B}$ is geometrically regular over $k$,
2. $A \to B$ is flat and $k \to \overline{B}$ is formally smooth in the $\mathfrak{m}_{\overline{B}}$-adic topology, and
3. $A \to B$ is formally smooth in the $\mathfrak{m}_B$-adic topology.

Proof. The equivalence of (1) and (2) follows from Theorem 30.1.

Assume (3). By Lemma 30.3 we see that $A \to B$ is flat. By Lemma 28.8 we see that $k \to \overline{B}$ is formally smooth in the $\mathfrak{m}_{\overline{B}}$-adic topology. Thus (2) holds.

Assume (2). Lemma 28.4 tells us formal smoothness is preserved under completion. The same is true for flatness by Algebra, Lemma 94.4. Hence we may replace $A$ and $B$ by their respective completions and assume that $A$ and $B$ are Noetherian complete local rings. In this case choose a diagram

$$
\begin{array}{ccc}
S & \to & B \\
\uparrow & & \uparrow \\
R & \to & A
\end{array}
$$

as in Lemma 29.3. We will use all of the properties of this diagram without further mention. Fix a regular system of parameters $t_1, \ldots, t_d$ of $R$ with $t_1 = p$ in case the characteristic of $k$ is $p > 0$. Set $S = S \otimes_R k$. Consider the short exact sequence

$$
0 \to J \to S \to B \to 0
$$

Since $B$ is flat over $A$ we see that $J \otimes_R k$ is the kernel of $S \to \overline{B}$. As $\overline{B}$ and $S$ are regular we see that $J \otimes_R k$ is generated by elements $\overline{x}_1, \ldots, \overline{x}_r$ which form part of a regular system of parameters of $S$, see Algebra, Lemma 103.4. Lift these elements to $x_1, \ldots, x_r \in J$. Then $t_1, \ldots, t_d, x_1, \ldots, x_r$ is part of a regular system of parameters for $S$. Hence $S/(x_1, \ldots, x_r)$ is a power series ring over a field (if the characteristic of $k$ is zero) or a power series ring over a Cohen ring (if the characteristic of $k$ is $p > 0$), see Lemma 29.2. Moreover, it is still the case that $R \to S/(x_1, \ldots, x_r)$
maps $t_1, \ldots, t_d$ to a part of a regular system of parameters of $S/(x_1, \ldots, x_r)$. In other words, we may replace $S$ by $S/(x_1, \ldots, x_r)$ and assume we have a diagram

$$
\begin{array}{ccc}
S & \longrightarrow & B \\
\uparrow & & \uparrow \\
R & \longrightarrow & A
\end{array}
$$

as in Lemma 29.3 with moreover $\overline{S} = \overline{B}$. In this case the map

$$S \otimes_R A \longrightarrow B$$

is an isomorphism as it is surjective and an isomorphism on special fibres, see Algebra, Lemma 96.1. Thus by Lemma 28.8 it suffices to show that $R$ is an isomorphism as it is surjective and an isomorphism on special fibres, see Algebra, Lemma 128.3. We can write $\overline{S}$ formally smooth in the $m_S$-adic topology. Of course, since $\overline{S} = \overline{B}$, we have that $\overline{S}$ is formally smooth over $k = R/m_R$.

Choose elements $y_1, \ldots, y_m \in S$ such that $t_1, \ldots, t_d, y_1, \ldots, y_m$ is a regular system of parameters for $S$. If the characteristic of $k$ is zero, choose a coefficient field $K \subset S$ and if the characteristic of $k$ is $p > 0$ choose a Cohen ring $\Lambda \subset S$ with residue field $K$. At this point the map $K[[t_1, \ldots, t_d, y_1, \ldots, y_m]] \to S$ (characteristic zero case) or $\Lambda[[t_2, \ldots, t_d, y_1, \ldots, y_m]] \to S$ (characteristic $p > 0$ case) is an isomorphism, see Lemma 29.2. From now on we think of $S$ as the above power series ring.

The rest of the proof is analogous to the argument in the proof of Theorem 30.1. Choose a solid diagram

$$
\begin{array}{ccc}
S & \longrightarrow & N/J \\
\downarrow \psi & & \downarrow \pi \\
R & \longrightarrow & N
\end{array}
$$

as in Definition 28.1. As $J^2 = 0$ we see that $J$ has a canonical $N/J$ module structure and via $\psi$ a $S$-module structure. As $\psi$ is continuous for the $m_S$-adic topology we see that $m_S^n J = 0$ for some $n$. Hence we can filter $J$ by $N/J$-submodules $0 \subset J_1 \subset J_2 \subset \ldots \subset J_n = J$ such that each quotient $J_{i+1}/J_i$ is annihilated by $m_S$. Considering the sequence of ring maps $N \to N/J_1 \to N/J_2 \to \ldots \to N/J$ we see that it suffices to prove the existence of the dotted arrow when $J$ is annihilated by $m_S$, i.e., when $J$ is a $K$-vector space.

Assume given a diagram as above such that $J$ is annihilated by $m_S$. As $Q \to S$ (characteristic zero case) or $\mathbf{Z} \to S$ (characteristic $p > 0$ case) is formally smooth in the $m_S$-adic topology (see Lemma 29.1), we can find a ring map $\psi : S \to N$ such that $\pi \circ \psi = \psi$. Since $S$ is a power series ring in $t_1, \ldots, t_d$ (characteristic zero) or $t_2, \ldots, t_d$ (characteristic $p > 0$) over a subring, it follows from the universal property of power series rings that we can change our choice of $\psi$ so that $\psi(t_i)$ equals $\varphi(t_i)$ (automatic for $t_1 = p$ in the characteristic $p$ case). Then $\psi \circ i$ and $\varphi : R \to N$ are two maps whose compositions with $\pi$ are equal and which agree on $t_1, \ldots, t_d$. Hence $D = \psi \circ i - \varphi : R \to J$ is a derivation which annihilates $t_1, \ldots, t_d$. By Algebra, Lemma 128.3 we can write $D = \xi \circ \delta$ for some $R$-linear map $\xi : \Omega_{R/\mathbf{Z}} \to J$ which annihilates $dt_1, \ldots, dt_d$ (by construction) and $m_R \Omega_{R/\mathbf{Z}}$ (as $J$ is annihilated by $m_R$). Hence $\xi$ factors as a composition

$$\Omega_{R/\mathbf{Z}} \to \Omega_{k/\mathbf{Z}} \xrightarrow{\xi} J$$
where \( \xi' \) is \( k \)-linear. Using the \( K \)-vector space structure on \( J \) we extend \( \xi' \) to a \( K \)-linear map

\[
\xi'' : \Omega_{k/Z} \otimes_k K \rightarrow J.
\]

Using that \( \mathcal{S}/k \) is formally smooth we see that

\[
\Omega_{k/Z} \otimes_k K \rightarrow \Omega_{\mathcal{S}/Z} \otimes S K
\]

is injective by Theorem 30.1 (this is true also in the characteristic zero case as it is even true that \( \Omega_{k/Z} \rightarrow \Omega_{K/Z} \) is injective in characteristic zero, see Algebra, Proposition 148.9). Hence we can find a \( K \)-linear map \( \xi''' : \Omega_{\mathcal{S}/Z} \otimes S K \rightarrow J \) whose restriction to \( \Omega_{k/Z} \otimes_k K \) is \( \xi'' \).

Write

\[
D' : S \to \Omega_{\mathcal{S}/Z} \rightarrow \Omega_{\mathcal{S}/Z} \otimes S K \xrightarrow{\xi''} J.
\]

Finally, set \( \psi' = \psi - D' : S \rightarrow N \). The reader verifies that \( \psi' \) is a ring map such that \( \pi \circ \psi' = \bar{\psi} \) and such that \( \psi' \circ i = \varphi \) as desired.

As an application of the result above we prove that deformations of formally smooth algebras are unobstructed.

**Lemma 30.5.** Let \( A \) be a Noetherian complete local ring with residue field \( k \). Let \( B \) be a Noetherian complete local \( k \)-algebra. Assume \( k \rightarrow B \) is formally smooth in the \( m_B \)-adic topology. Then there exists a Noetherian complete local ring \( C \) and a local homomorphism \( A \rightarrow C \) which is formally smooth in the \( m_C \)-adic topology such that \( C \otimes_A k \cong B \).

**Proof.** Choose a diagram

\[
\begin{array}{ccc}
S & \rightarrow & B \\
\uparrow & & \uparrow \\
R & \rightarrow & A
\end{array}
\]

as in Lemma 29.3 Let \( t_1, \ldots, t_d \) be a regular system of parameters for \( R \) with \( t_1 = p \) in case the characteristic of \( k \) is \( p > 0 \). As \( B \) and \( \mathcal{S} = S \otimes_A k \) are regular we see that \( \text{Ker}(\mathcal{S} \rightarrow B) \) is generated by elements \( x_1, \ldots, x_r \) which form part of a regular system of parameters of \( \mathcal{S} \), see Algebra, Lemma 103.4. Lift these elements to \( x_1, \ldots, x_r \in S \). Then \( t_1, \ldots, t_d, x_1, \ldots, x_r \) is part of a regular system of parameters for \( S \). Hence \( S/(x_1, \ldots, x_r) \) is a power series ring over a field (if the characteristic of \( k \) is zero) or a power series ring over a Cohen ring (if the characteristic of \( k \) is \( p > 0 \), see Lemma 29.2). Moreover, it is still the case that \( R \rightarrow S/(x_1, \ldots, x_r) \) maps \( t_1, \ldots, t_d \) to a part of a regular system of parameters of \( S/(x_1, \ldots, x_r) \). In other words, we may replace \( S \) by \( S/(x_1, \ldots, x_r) \) and assume we have a diagram

\[
\begin{array}{ccc}
S & \rightarrow & B \\
\uparrow & & \uparrow \\
R & \rightarrow & A
\end{array}
\]

as in Lemma 29.3 with moreover \( \mathcal{S} = B \). In this case \( R \rightarrow S \) is formally smooth in the \( m_S \)-adic topology by Proposition 30.4. Hence the base change \( C = S \otimes_R A \) is formally smooth over \( A \) in the \( m_C \)-adic topology by Lemma 28.8. \( \square \)
Remark 30.6. The assertion of Lemma 30.5 is quite strong. Namely, suppose that we have a diagram

![Diagram](image)

of local homomorphisms of Noetherian complete local rings where \( A \to A' \) induces an isomorphism of residue fields \( k = A/\mathfrak{m}_A = A'/\mathfrak{m}_{A'} \) and with \( B \otimes_{A'} k \) formally smooth over \( k \). Then we can extend this to a commutative diagram

![Diagram](image)

of local homomorphisms of Noetherian complete local rings where \( A \to C \) is formally smooth in the \( \mathfrak{m}_{C'} \)-adic topology and where \( C \otimes_A A' = B \otimes_{A'} k \). Denote \( C \otimes_A A' \) the completion of \( C \otimes_A A' \) with respect to the ideal \( C \otimes_A \mathfrak{m}_{A'} \). Note that \( C \otimes_A A' \) is a Noetherian complete local ring (see Algebra, Lemma 94.9) which is flat over \( A' \) (see Algebra, Lemma 96.1). We have moreover

1. \( C \otimes_A A' \to B \) is surjective,
2. if \( A \to A' \) is surjective, then \( C \to B \) is surjective,
3. if \( A \to A' \) is finite, then \( C \to B \) is finite, and
4. if \( A' \to B \) is flat, then \( C \otimes_A A' \cong B \).

Namely, by Nakayama’s lemma for nilpotent ideals (see Algebra, Lemma 19.1) we see that \( C \otimes_A A' \to B \) implies that \( C \otimes_A A'/\mathfrak{m}_{A'} \to B/\mathfrak{m}_{A'} \) is surjective for all \( n \). This proves (1). Parts (2) and (3) follow from part (1). Part (4) follows from Algebra, Lemma 96.1.

31. Regular ring maps

Let \( k \) be a field. Recall that a Noetherian \( k \)-algebra \( A \) is said to be geometrically regular over \( k \) if and only if \( A \otimes_k k' \) is regular for all finite purely inseparable extensions \( k' \) of \( k \), see Algebra, Definition 155.2. Moreover, if this is the case then \( A \otimes_k k' \) is regular for every finitely generated field extension \( k \subset k' \), see Algebra, Lemma 155.1. We use this notion in the following definition.

Definition 31.1. A ring map \( R \to \Lambda \) is regular if it is flat and for every prime \( \mathfrak{p} \subset R \) the fibre ring

\[
\Lambda \otimes_R \kappa(\mathfrak{p}) = \Lambda_{\mathfrak{p}}/\mathfrak{p}\Lambda_{\mathfrak{p}}
\]

is Noetherian and geometrically regular over \( \kappa(\mathfrak{p}) \).

If \( R \to \Lambda \) is a ring map with \( \Lambda \) Noetherian, then the fibre rings are always Noetherian.

Lemma 31.2 (Regular is a local property). Let \( R \to \Lambda \) be a ring map with \( \Lambda \) Noetherian. The following are equivalent

1. \( R \to \Lambda \) is regular,
2. \( R_{\mathfrak{p}} \to \Lambda_{\mathfrak{q}} \) is regular for all \( \mathfrak{q} \subset \Lambda \) lying over \( \mathfrak{p} \subset R \), and
For any finite type ring map $R \to \Lambda$ be a regular ring map. For any finite type ring map $R \to R'$ the base change $R' \to \Lambda \otimes_R R'$ is regular too.

Proof. Flatness is preserved under any base change, see Algebra, Lemma 38.6. Consider a prime $\mathfrak{p}' \subset R'$ lying over $\mathfrak{p} \subset R$. The residue field extension $\kappa(\mathfrak{p}) \subset \kappa(\mathfrak{p}')$ is finitely generated as $R'$ is of finite type over $R$. Hence the fibre ring

$$(\Lambda \otimes_R R') \otimes_{R'} \kappa(\mathfrak{p}') = \Lambda \otimes_R \kappa(\mathfrak{p}) \otimes_{\kappa(\mathfrak{p})} \kappa(\mathfrak{p}')$$

is Noetherian by Algebra, Lemma 30.7 and the assumption on the fibre rings of $R \to \Lambda$. Geometric regularity of the fibres is preserved by Algebra, Lemma 155.1.

Lemma 31.4 (Composition of regular maps). Let $A \to B \to C$ be regular ring maps. If the fibre rings of $A \to C$ are Noetherian, then $A \to C$ is regular.

Proof. Let $\mathfrak{p} \subset A$ be a prime. Let $\kappa(\mathfrak{p}) \subset k$ be a finite purely inseparable extension. We have to show that $C \otimes_A k$ is regular. By Lemma 31.3 we may assume that $A = k$ and we reduce to proving that $C$ is regular. The assumption is that $B$ is regular and that $B \to C$ is flat with regular fibres. Then $C$ is regular by Algebra, Lemma 109.8. Some details omitted.

Lemma 31.5. Let $R$ be a ring. Let $(A_i, \varphi_{ii'})$ be a directed system of smooth $R$-algebras. Set $\Lambda = \text{colim} A_i$. If the fibre rings $\Lambda \otimes_R \kappa(\mathfrak{p})$ are Noetherian for all $\mathfrak{p} \subset R$, then $R \to \Lambda$ is regular.

Proof. Note that $\Lambda$ is flat over $R$ by Algebra, Lemmas 38.2 and 133.10. Let $\kappa(\mathfrak{p}) \subset k$ be a finite purely inseparable extension. Note that

$$\Lambda \otimes_R \kappa(\mathfrak{p}) \otimes_{\kappa(\mathfrak{p})} k = \Lambda \otimes_R k = \text{colim} A_i \otimes_R k$$

is a colimit of smooth $k$-algebras, see Algebra, Lemma 133.4. Since each local ring of a smooth $k$-algebra is regular by Algebra, Lemma 136.3 we conclude that all local rings of $\Lambda \otimes_R k$ are regular by Algebra, Lemma 103.8. This proves the lemma.

Let’s see when a field extension defines a regular ring map.

Lemma 31.6. Let $k \subset K$ be a field extension. Then $k \to K$ is a regular ring map if and only if $K$ is a separable field extension of $k$.

Proof. If $k \to K$ is regular, then $K$ is geometrically reduced over $k$, hence $K$ is separable over $k$ by Algebra, Proposition 148.9. Conversely, if $K/k$ is separable, then $K$ is a colimit of smooth $k$-algebras, see Algebra, Lemma 148.11 hence is regular by Lemma 31.5.

Lemma 31.7. Let $A \to B \to C$ be ring maps. If $A \to C$ is regular and $B \to C$ is flat and surjective on spectra, then $A \to B$ is regular.

Proof. By Algebra, Lemma 38.9 we see that $A \to B$ is flat. Let $\mathfrak{p} \subset A$ be a prime. The ring map $B \otimes_A \kappa(\mathfrak{p}) \to C \otimes_A \kappa(\mathfrak{p})$ is flat and surjective on spectra. Hence $B \otimes_A \kappa(\mathfrak{p})$ is geometrically regular by Algebra, Lemma 155.3.
32. Ascending properties along regular ring maps

This section is the analogue of Algebra, Section 152 but where the ring map $R \to S$ is regular.

**Lemma 32.1.** Let $\varphi : R \to S$ be a ring map. Assume

1. $\varphi$ is regular,
2. $S$ is Noetherian, and
3. $R$ is Noetherian and reduced.

Then $S$ is reduced.

**Proof.** For Noetherian rings being reduced is the same as having properties $(S_1)$ and $(R_0)$, see Algebra, Lemma 147.3 Hence we may apply Algebra, Lemmas 152.4 and 152.5.

33. Permanence of properties under completion

Given a Noetherian local ring $A$ we denote $A^\wedge$ the completion of $A$ with respect to its maximal ideal. We will use without further mention that $A^\wedge$ is a Noetherian complete local ring (Algebra, Lemmas 94.10 and 94.7) and that $A \to A^\wedge$ is flat (Algebra, Lemma 94.3).

**Lemma 33.1.** Let $A$ be a Noetherian local ring. Then $\dim(A) = \dim(A^\wedge)$.

**Proof.** See for example Algebra, Lemma 109.7.

**Lemma 33.2.** Let $A$ be a Noetherian local ring. Then $\depth(A) = \depth(A^\wedge)$.

**Proof.** See Algebra, Lemma 152.2.

**Lemma 33.3.** Let $A$ be a Noetherian local ring. Then $A$ is Cohen-Macaulay if and only if $A^\wedge$ is so.

**Proof.** A local ring $A$ is Cohen-Macaulay if and only $\dim(A) = \depth(A)$. As both of these invariants are preserved under completion (Lemmas 33.1 and 33.2) the claim follows.

**Lemma 33.4.** Let $A$ be a Noetherian local ring. Then $A$ is regular if and only if $A^\wedge$ is so.

**Proof.** If $A^\wedge$ is regular, then $A$ is regular by Algebra, Lemma 107.9 Assume $A$ is regular. Let $m$ be the maximal ideal of $A$. Then $\dim_k(m/m^2) = \dim(A) = \dim(A^\wedge)$ (Lemma 33.1). On the other hand, $mA^\wedge$ is the maximal ideal of $A^\wedge$ and hence $mA^\wedge$ is generated by at most $\dim(A^\wedge)$ elements. Thus $A^\wedge$ is regular. (You can also use Algebra, Lemma 109.8.)

**Lemma 33.5.** Let $A$ be a Noetherian local ring. Then $A$ is a discrete valuation ring if and only if $A^\wedge$ is so.

**Proof.** This follows from Lemmas 33.1 and 33.4 and Algebra, Lemma 116.6.

**Lemma 33.6.** Let $A$ be a Noetherian local ring.

1. If $A^\wedge$ is reduced, then so is $A$.
2. In general $A$ reduced does not imply $A^\wedge$ is reduced.
3. If $A$ is Nagata, then $A$ is reduced if and only if $A^\wedge$ is reduced.
Proof. As \( A \to A^\wedge \) is faithfully flat we have (1) by Algebra, Lemma \[153.2\] For (2) see Algebra, Example \[116.4\] (there are also examples in characteristic zero, see Algebra, Remark \[116.5\]). For (3) see Algebra, Lemmas \[151.28\] and \[151.25\]. \( \square \)

**Lemma 33.7.** Let \( A \to B \) be a flat local homomorphism of Noetherian local rings such that \( m_A B = m_B \) and \( \kappa(m_A) = \kappa(m_B) \). Then \( A \to B \) induces an isomorphism \( A^\wedge \to B^\wedge \) of completions.

**Proof.** By Algebra, Lemma \[94.18\] we see that \( B^\wedge \) is the \( m_A \)-adic completion of \( B \) and that \( A^\wedge \to B^\wedge \) is finite. Since \( A \to B \) is flat we have \( \text{Tor}_1^B(B, \kappa(m_A)) = 0 \). Hence we see that \( B^\wedge \) is flat over \( A^\wedge \) by Lemma \[20.5\]. Thus \( B^\wedge \) is a free \( A^\wedge \)-module by Algebra, Lemma \[76.4\]. Since \( A^\wedge \to B^\wedge \) induces an isomorphism \( \kappa(m_A) = A^\wedge/m_A A^\wedge \to B^\wedge/m_A B^\wedge = B^\wedge/m_B B^\wedge = \kappa(m_B) \) by our assumptions (and Algebra, Lemmas \[94.6\] and \[94.7\]), we see that \( B^\wedge \) is free of rank 1. Thus \( A^\wedge \to B^\wedge \) is an isomorphism. \( \square \)

### 34. Permanence of properties under étale maps

In this section we consider an étale ring map \( \varphi : A \to B \) and we study which properties of \( A \) are inherited by \( B \) and which properties of the local ring of \( B \) at \( q \) are inherited by the local ring of \( A \) at \( p = \varphi^{-1}(q) \). Basically, this section reviews and collects earlier results and does not add any new material.

We will use without further mention that an étale ring map is flat (Algebra, Lemma \[139.3\]) and that a flat local homomorphism of local rings is faithfully flat (Algebra, Lemma \[38.16\]).

**Lemma 34.1.** If \( A \to B \) is an étale ring map and \( q \) is a prime of \( B \) lying over \( p \subset A \), then \( A_p \) is Noetherian if and only if \( B_q \) is Noetherian.

**Proof.** Since \( A_p \to B_q \) is faithfully flat we see that \( B_q \) Noetherian implies that \( A_p \) is Noetherian, see Algebra, Lemma \[153.1\]. Conversely, if \( A_p \) is Noetherian, then \( B_q \) is Noetherian as it is a localization of a finite type \( A_p \)-algebra. \( \square \)

**Lemma 34.2.** If \( A \to B \) is an étale ring map and \( q \) is a prime of \( B \) lying over \( p \subset A \), then \( \dim(A_p) = \dim(B_q) \).

**Proof.** Namely, because \( A_p \to B_q \) is flat we have going down, and hence the inequality \( \dim(A_p) \leq \dim(B_q) \), see Algebra, Lemma \[109.1\]. On the other hand, suppose that \( q_0 \subset q_1 \subset \ldots \subset q_n \) is a chain of primes in \( B_q \). Then the corresponding sequence of primes \( p_0 \subset p_1 \subset \ldots \subset p_n \) (with \( p_i = q_i \cap A_p \)) is chain also (i.e., no equalities in the sequence) as an étale ring map is quasi-finite (see Algebra, Lemma \[139.6\]) and a quasi-finite ring map induces a map of spectra with discrete fibres (by definition). This means that \( \dim(A_p) \geq \dim(B_q) \) as desired. \( \square \)

**Lemma 34.3.** If \( A \to B \) is an étale ring map and \( q \) is a prime of \( B \) lying over \( p \subset A \), then \( A_p \) is regular if and only if \( B_q \) is regular.

**Proof.** By Lemma \[34.1\] we may assume both \( A_p \) and \( B_q \) are Noetherian in order to prove the equivalence. Let \( x_1, \ldots, x_t \in pA_p \) be a minimal set of generators. As \( A_p \to B_q \) is faithfully flat we see that the images \( y_1, \ldots, y_t \) in \( B_q \) form a minimal system of generators for \( pB_q = qB_q \) (Algebra, Lemma \[139.5\]). Regularity of \( A_p \) by \( \varphi \), definition means \( t = \dim(A_p) \) and similarly for \( B_q \). Hence the lemma follows from the equality \( \dim(A_p) = \dim(B_q) \) of Lemma \[34.2\]. \( \square \)
Lemma 34.4. If $A \to B$ is an étale ring map and $A$ is a Dedekind domain, then $B$ is a finite product of Dedekind domains. In particular, the localizations $B_q$ for $q \subset B$ maximal are discrete valuation rings.

Proof. The statement on the local rings follows from Lemmas 34.2 and 34.3 and Algebra, Lemma 116.6. It follows that $B$ is a Noetherian normal ring of dimension 1. By Algebra, Lemma 36.14 we conclude that $B$ is a finite product of normal domains of dimension 1. These are Dedekind domains by Algebra, Lemma 117.14. □

35. Permanence of properties under henselization

Given a local ring $R$ we denote $R^h$, resp. $R^{sh}$ the henselization, resp. strict henselization of $R$, see Algebra, Definition 146.18. Many of the properties of $R$ are reflected in $R^h$ and $R^{sh}$ as we will show in this section.

Lemma 35.1. Let $(R, \mathfrak{m}, \kappa)$ be a local ring. Then we have the following

1. $R \to R^h \to R^{sh}$ are faithfully flat ring maps,
2. $\mathfrak{m}R^h = \mathfrak{m}^h$ and $\mathfrak{m}R^{sh} = \mathfrak{m}^h R^{sh} = \mathfrak{m}^{sh}$,
3. $R/\mathfrak{m}^n = R^h/\mathfrak{m}^h R^h$ for all $n$,
4. there exist elements $x_i \in R^{sh}$ such that $R^{sh}/\mathfrak{m}^n R^{sh}$ is a free $R/\mathfrak{m}^n$-module on $x_i \mod \mathfrak{m}^n R^{sh}$.

Proof. By construction $R^h$ is a colimit of étale $R$-algebras, see Algebra, Lemma 146.16. Since étale ring maps are flat (Algebra, Lemma 139.3) we see that $R^h$ is flat over $R$ by Algebra, Lemma 38.2. As a flat local ring homomorphism is faithfully flat (Algebra, Lemma 38.16) we see that $R \to R^h$ is faithfully flat. The ring map $R^h \to R^{sh}$ is a colimit of finite étale ring maps, see proof of Algebra, Lemma 146.17. Hence the same arguments as above show that $R^h \to R^{sh}$ is faithfully flat.

Part (2) follows from Algebra, Lemmas 146.16 and 146.17. Part (3) follows from Algebra, Lemma 98.1 because $R/\mathfrak{m} \to R^h/\mathfrak{m} R^h$ is a isomorphism and $R/\mathfrak{m}^n \to R^h/\mathfrak{m}^n R^h$ is flat as a base change of the flat ring map $R \to R^h$ (Algebra, Lemma 38.6). Let $\kappa^{sep}$ be the residue field of $R^{sh}$ (it is a separable algebraic closure of $\kappa$). Choose $x_i \in R^{sh}$ mapping to a basis of $\kappa^{sep}$ as a $\kappa$-vector space. Then (4) follows from Algebra, Lemma 98.1 in exactly the same way as above. □

Lemma 35.2. Let $(R, \mathfrak{m}, \kappa)$ be a local ring. Then

1. $R \to R^h$, $R^h \to R^{sh}$, and $R \to R^{sh}$ are formally étale,
2. $R \to R^h$, $R^h \to R^{sh}$, resp. $R \to R^{sh}$ are formally smooth in the $\mathfrak{m}^h$, $\mathfrak{m}^{sh}$, resp. $\mathfrak{m}^{sh}$-topology.

Proof. Part (1) follows from the fact that $R^h$ and $R^{sh}$ are directed colimits of étale algebras (by construction), that étale algebras are formally étale (Algebra, Lemma 144.2), and that colimits of formally étale algebras are formally étale (Algebra, Lemma 144.3). Part (2) follows from the fact that a formally étale ring map is formally smooth and Lemma 28.2. □

Lemma 35.3. Let $R$ be a local ring. The following are equivalent

1. $R$ is Noetherian,
2. $R^h$ is Noetherian, and
3. $R^{sh}$ is Noetherian.

In this case we have
\begin{proof}
Since \( R \to R^h \to R^{sh} \) are faithfully flat (Lemma \ref{flat}), we see that \( R^h \) or \( R^{sh} \) being Noetherian implies that \( R \) is Noetherian, see Algebra, Lemma \ref{Noetherian}. In the rest of the proof we assume \( R \) is Noetherian.

As \( m \subset R \) is finitely generated it follows that \( m^h = mR^h \) and \( m^{sh} = mR^{sh} \) are finitely generated, see Lemma \ref{finitely-generated}. Hence \((R^h)^{\wedge}\) and \((R^{sh})^{\wedge}\) are Noetherian by Algebra, Lemma \ref{Noetherian-completion}. This proves (a).

Note that (b) is immediate from Lemma \ref{finitely-generated} In particular we see that \((R^h)^{\wedge}\) is flat over \( R \), see Algebra, Lemma \ref{flat}.

Next, we show that \( R^h \to (R^h)^{\wedge} \) is flat. Write \( R^h = \text{colim}_i R_i \) as a directed colimit of localizations of étale \( R \)-algebras. By Algebra, Lemma \ref{flat-localization} if \((R^h)^{\wedge}\) is flat over each \( R_i \), then \( R^h \to (R^h)^{\wedge} \) is flat. Note that \( R^h = R_i^{sh} \) (by construction). Hence \( R^h = (R^h)^{\wedge} \) by part (b) is flat over \( R_i \) as desired. To finish the proof of (c) we show that \( R^{sh} \to (R^{sh})^{\wedge} \) is flat. To do this, by a limit argument as above, it suffices to show that \((R^{sh})^{\wedge}\) is flat over \( R \). Note that it follows from Lemma \ref{finitely-generated} that \((R^{sh})^{\wedge}\) is the completion of a free \( R \)-module. By Lemma \ref{completion-flat}, we see this is flat over \( R \) as desired. This finishes the proof of (c).

At this point we know (c) is true and that \((R^h)^{\wedge}\) and \((R^{sh})^{\wedge}\) are Noetherian. It follows from Algebra, Lemma \ref{Noetherian} that \( R^h \) and \( R^{sh} \) are Noetherian.

Part (d) follows from Lemma \ref{completion-flat} and Lemma \ref{flat-localization}. \qed

\begin{lemma}
Let \( R \) be a local ring. The following are equivalent: \( R \) is reduced, the henselization \( R^h \) of \( R \) is reduced, and the strict henselization \( R^{sh} \) of \( R \) is reduced.
\end{lemma}

\begin{proof}
The ring maps \( R \to R^h \to R^{sh} \) are faithfully flat. Hence one direction of the implications follows from Algebra, Lemma \ref{faithful-flat}. Conversely, assume \( R \) is reduced. Since \( R^h \) and \( R^{sh} \) are filtered colimits of étale, hence smooth \( R \)-algebras, the result follows from Algebra, Lemma \ref{henselization-flat} \qed
\end{proof}

\begin{lemma}
Let \( R \) be a local ring. Let nil(\( R \)) denote the ideal of nilpotent elements of \( R \). Then nil(\( R \))\( R^h = \text{nil}(R^h) \) and nil(\( R \))\( R^{sh} = \text{nil}(R^{sh}) \).
\end{lemma}

\begin{proof}
Note that nil(\( R \)) is the biggest ideal consisting of nilpotent elements such that the quotient \( R/\text{nil}(R) \) is reduced. Note that nil(\( R \))\( R^h \) consists of nilpotent elements by Algebra, Lemma \ref{nilpotent}. Also, note that \( R^h/\text{nil}(R)R^h \) is the henselization of \( R/\text{nil}(R) \) by Algebra, Lemma \ref{henselization-flat}. Hence \( R^h/\text{nil}(R)R^h \) is reduced by Lemma \ref{henselization-flat}. We conclude that nil(\( R \))\( R^h = \text{nil}(R^h) \) as desired. Similarly for the strict henselization but using Algebra, Lemma \ref{henselization-flat} \qed
\end{proof}

\begin{lemma}
Let \( R \) be a local ring. The following are equivalent: \( R \) is a normal domain, the henselization \( R^h \) of \( R \) is a normal domain, and the strict henselization \( R^{sh} \) of \( R \) is a normal domain.
\end{lemma}

\begin{proof}
A preliminary remark is that a local ring is normal if and only if it is a normal domain (see Algebra, Definition \ref{normal-domain}). The ring maps \( R \to R^h \to R^{sh} \) are faithfully flat. Hence one direction of the implications follows from Algebra, Lemma \ref{faithful-flat}.
\end{proof}
Conversely, assume $R$ is normal. Since $R^h$ and $R^{sh}$ are filtered colimits of étale, hence smooth $R$-algebras, the result follows from Algebra, Lemma 152.7. □

**Lemma 35.7.** Given any local ring $R$ we have $\dim(R) = \dim(R^h) = \dim(R^{sh}).$

**Proof.** Since $R \to R^{sh}$ is faithfully flat (Lemma 35.1) we see that $\dim(R^{sh}) \geq \dim(R)$ by going down, see Algebra, Lemma 109.1. For the converse, we write $R^h = \colim_i R_i$ as a directed colimit of local rings $R_i$ each of which is a localization of an étale $R$-algebra. Now if $q_0 \subset q_1 \subset \ldots \subset q_n$ is a chain of prime ideals in $R^{sh}$, then for some sufficiently large $i$ the sequence

$$R_i \cap q_0 \subset R_i \cap q_1 \subset \ldots \subset R_i \cap q_n$$

is a chain of primes in $R_i$. Thus we see that $\dim(R^{sh}) \leq \sup_i \dim(R_i)$. But by the result of Lemma 35.2 we have $\dim(R_i) = \dim(R)$ for each $i$ and we win. □

**Lemma 35.8.** Given a Noetherian local ring $R$ we have $\depth(R) = \depth(R^h) = \depth(R^{sh}).$

**Proof.** By Lemma 35.3 we know that $R^h$ and $R^{sh}$ are Noetherian. Hence the lemma follows from Algebra, Lemma 152.2. □

**Lemma 35.9.** Let $R$ be a Noetherian local ring. The following are equivalent: $R$ is Cohen-Macaulay, the henselization $R^h$ of $R$ is Cohen-Macaulay, and the strict henselization $R^{sh}$ of $R$ is Cohen-Macaulay.

**Proof.** By Lemma 35.3 we know that $R^h$ and $R^{sh}$ are Noetherian, hence the lemma makes sense. Since we have $\depth(R) = \depth(R^h) = \depth(R^{sh})$ and $\dim(R) = \dim(R^h) = \dim(R^{sh})$ by Lemmas 35.8 and 35.7 we conclude. □

**Lemma 35.10.** Let $R$ be a Noetherian local ring. The following are equivalent: $R$ is a regular local ring, the henselization $R^h$ of $R$ is a regular local ring, and the strict henselization $R^{sh}$ of $R$ is a regular local ring.

**Proof.** By Lemma 35.3 we know that $R^h$ and $R^{sh}$ are Noetherian, hence the lemma makes sense. Let $m$ be the maximal ideal of $R$. Let $x_1, \ldots, x_t \in m$ be a minimal system of generators of $m$, i.e., such that the images in $m/m^2$ form a basis over $\kappa = R/m$. Because $R \to R^h$ and $R \to R^{sh}$ are faithfully flat, it follows that the images $x_1^h, \ldots, x_t^h$ in $R^h$, resp. $x_1^{sh}, \ldots, x_t^{sh}$ in $R^{sh}$ are a minimal system of generators for $m^h = mR^h$, resp. $m^{sh} = mR^{sh}$. Regularity of $R$ by definition means $t = \dim(R)$ and similarly for $R^h$ and $R^{sh}$. Hence the lemma follows from the equality of dimensions $\dim(R) = \dim(R^h) = \dim(R^{sh})$ of Lemma 35.7. □

**Lemma 35.11.** Let $R$ be a Noetherian local ring. Then $R$ is a discrete valuation ring if and only if $R^h$ is a discrete valuation ring if and only if $R^{sh}$ is a discrete valuation ring.

**Proof.** This follows from Lemmas 35.7 and 35.10 and Algebra, Lemma 116.6. □

**Lemma 35.12.** Let $A$ be a ring. Let $B$ be a filtered colimit of étale $A$-algebras. Let $p$ be a prime of $A$. If $B$ is Noetherian, then there are finitely many primes $q_1, \ldots, q_t$ lying over $p$, we have $B \otimes_A \kappa(p) = \prod \kappa(q_i)$, and each of the field extensions $\kappa(p) \subset \kappa(q_i)$ is separable algebraic.
Proof. Write $B$ as a filtered colimit $B = \text{colim} B_i$ with $A \to B_i$ étale. Then on the one hand $B \otimes_A \kappa(p) = \text{colim} B_i \otimes_A \kappa(p)$ is a filtered colimit of étale $\kappa(p)$-algebras, and on the other hand it is Noetherian. An étale $\kappa(p)$-algebra is a finite product of finite separable field extensions (Algebra, Lemma \[139.4\]). Hence there are no nontrivial specializations between the primes (which are all maximal and minimal primes) of the algebras $B_i \otimes_A \kappa(p)$ and hence there are no nontrivial specializations between the primes of $B \otimes_A \kappa(p)$. Thus $B \otimes_A \kappa(p)$ is reduced and has finitely many primes which all minimal. Thus it is a finite product of fields (use Algebra, Lemma \[24.4\] or Algebra, Proposition \[59.6\]). Each of these fields is a colimit of finite separable extensions and hence the final statement of the lemma follows. 

\[\square\]

Lemma 35.13. Let $R$ be a Noetherian local ring. Let $p \subset R$ be a prime. Then

$$R^h \otimes_R \kappa(p) = \prod_{i=1, \ldots, t} \kappa(q_i) \quad \text{resp.} \quad R^{sh} \otimes_R \kappa(p) = \prod_{i=1, \ldots, s} \kappa(r_i)$$

where $q_1, \ldots, q_t$, resp. $r_1, \ldots, r_s$ are the primes of $R^h$, resp. $R^{sh}$ lying over $p$. Moreover, the field extensions $\kappa(p) \subset \kappa(q_i)$ resp. $\kappa(p) \subset \kappa(q_i)$ are separable algebraic.

Proof. This can be deduced from the more general Lemma \[35.12\] using that the henselization and strict henselization are Noetherian (as we've seen above). But we also give a direct proof as follows.

We will use without further mention the results of Lemmas \[35.1\] and \[35.3\]. Note that $R^h/pR^h$, resp. $R^{sh}/pR^{sh}$ is the henselization, resp. strict henselization of $R/p$, see Algebra, Lemma \[146.24\] resp. Algebra, Lemma \[146.30\]. Hence we may replace $R$ by $R/p$ and assume that $R$ is a Noetherian local domain and that $p = (0)$. Since $R^h$, resp. $R^{sh}$ is Noetherian, it has finitely many minimal primes $q_1, \ldots, q_t$, resp. $r_1, \ldots, r_s$. Since $R \to R^h$, resp. $R \to R^{sh}$ is flat these are exactly the primes lying over $p = (0)$ (by going down). Finally, as $R$ is a domain, we see that $R^h$, resp. $R^{sh}$ is reduced, see Lemma \[35.4\]. Thus we see that $R^h \otimes_R f.f.(R) = R^h \otimes_R \kappa(p)$ resp. $R^{sh} \otimes_R f.f.(R) = R^{sh} \otimes_R \kappa(p)$ is a reduced Noetherian ring with finitely many primes, all of which are minimal (and hence maximal). Thus these rings are Artinian and are products of their localizations at maximal ideals, each necessarily a field (see Algebra, Proposition \[59.6\] and Algebra, Lemma \[24.1\]).

The final statement follows from the fact that $R \to R^h$, resp. $R \to R^{sh}$ is a colimit of étale ring maps and hence the induced residue field extensions are colimits of finite separable extensions, see Algebra, Lemma \[139.5\]. \[\square\]

36. Field extensions, revisited

In this section we study some peculiarities of field extensions in characteristic $p > 0$.

Definition 36.1. Let $p$ be a prime number. Let $k \to K$ be an extension of fields of characteristic $p$. Denote $kK^p$ the compositum of $k$ and $K^p$ in $K$.

1. A subset $\{x_i\} \subset K$ is called $p$-independent over $k$ if the elements $x^p = \prod x_i^{e_i}$ where $0 \leq e_i < p$ are linearly independent over $kK^p$.

2. A subset $\{x_i\}$ of $K$ is called a $p$-basis of $K$ over $k$ if the elements $x^p$ form a basis of $K$ over $kK^p$.

This is related to the notion of a $p$-basis of a $\mathbf{F}_p$-algebra which we will discuss later (insert future reference here).
Lemma 36.2. Let $k \subset K$ be a field extension. Assume $k$ has characteristic $p > 0$.
Let $\{x_i\}$ be a subset of $K$. The following are equivalent

1. the elements $\{x_i\}$ are $p$-independent over $k$, and
2. the elements $dx_i$ are $K$-linearly independent in $\Omega_{K/k}$.

Any $p$-independent collection can be extended to a $p$-basis of $K$ over $k$. In particular, the field $K$ has a $p$-basis over $k$. Moreover, the following are equivalent:

(a) $\{x_i\}$ is a $p$-basis of $K$ over $k$, and
(b) $dx_i$ is a basis of the $K$-vector space $\Omega_{K/k}$.

Proof. Assume (2) and suppose that $\sum a_E x^E = 0$ is a linear relation with $a_E \in kK^p$. Let $\theta_i : K \to K$ be a $k$-derivation such that $\theta_i(x_j) = \delta_{ij}$ (Kronecker delta). Note that any $k$-derivation of $K$ annihilates $kK^p$. Applying $\theta_i$ to the given relation we obtain new relations

$$\sum_{E,E_i > 0} e_i a_E x_1^{e_1} \cdots x_i^{e_i-1} \cdots x_n^n = 0$$

Hence if we pick $\sum a_E x^E$ as the relation with minimal total degree $|E| = \sum e_i$ for some $a_E \neq 0$, then we get a contradiction. Hence (2) holds.

If $\{x_i\}$ is a $p$-basis for $K$ over $k$, then $K \cong kK^p[X_i]/(X_i^p - x_i^p)$. Hence we see that $dx_i$ forms a basis for $\Omega_{K/k}$ over $K$. Thus (a) implies (b).

Let $\{x_i\}$ be a $p$-independent subset of $K$ over $k$. An application of Zorn’s lemma shows that we can enlarge this to a maximal $p$-independent subset of $K$ over $k$. We claim that any maximal $p$-independent subset $\{x_i\}$ of $K$ is a $p$-basis of $K$ over $k$.

The claim will imply that (1) implies (2) and establish the existence of $p$-bases. To prove the claim let $L$ be the subfield of $K$ generated by $kK^p$ and the $x_i$. We have to show that $L = K$. If $x \in K$ but $x \not\in L$, then $x^p \in L$ and $L(x) \cong L[z]/(z^p - x)$. Hence $\{x_i\} \cup \{x\}$ is $p$-independent over $k$, a contradiction.

Finally, we have to show that (b) implies (a). By the equivalence of (1) and (2) we see that $\{x_i\}$ is a maximal $p$-independent subset of $K$ over $k$. Hence by the claim above it is a $p$-basis.

Lemma 36.3. Let $k \subset K$ be a field extension. Let $\{K_\alpha\}_{\alpha \in A}$ be a collection of subfields of $K$ with the following properties

1. $k \subset K_\alpha$ for all $\alpha \in A$,
2. $k = \bigcap_{\alpha \in A} K_\alpha$,
3. for $\alpha, \alpha' \in A$ there exists an $\alpha'' \in A$ such that $K_{\alpha''} \subset K_\alpha \cap K_{\alpha'}$.

Then for $n \geq 1$ and $V \subset K^{\otimes n}$ a $K$-vector space we have $V \cap k^{\otimes n} \neq 0$ if and only if $V \cap K_\alpha^{\otimes n} \neq 0$ for all $\alpha \in A$.

Proof. By induction on $n$. The case $n = 1$ follows from the assumptions. Assume the result proven for subspaces of $K^{\otimes n-1}$. Assume that $V \subset K^{\otimes n}$ has nonzero intersection with $K_\alpha^{\otimes n}$ for all $\alpha \in A$. If $V \cap 0 \oplus K_\alpha^{\otimes n-1}$ is nonzero then we win. Hence we may assume this is not the case. By induction hypothesis we can find an $\alpha$ such that $V \cap 0 \oplus K_\alpha^{\otimes n-1}$ is zero. Let $v = (x_1, \ldots, x_n) \in V \cap K_\alpha$ be a nonzero element. By our choice of $\alpha$ we see that $x_1$ is not zero. Replace $v$ by $x_1^{-1}v$ so that $v = (1, x_2, \ldots, x_n)$. Note that if $v' = (x'_1, \ldots, x'_n) \in V \cap K_\alpha$, then $v' - x'_1v = 0$ by our choice of $\alpha$. Hence we see that $V \cap K_\alpha^{\otimes n} = K_\alpha v$. If we choose some $\alpha'$ such
that \( K_{\alpha'} \subset K_\alpha \), then we see that necessarily \( v \in V \cap K_\alpha^{\oplus n} \) (by the same arguments applied to \( \alpha' \)). Hence

\[
x_2, \ldots, x_n \in \bigcap_{\alpha' \in A, K_{\alpha'} \subset K_\alpha} K_{\alpha'}
\]

which equals \( k \) by (2) and (3).

Lemma 36.4. Let \( K \) be a field of characteristic \( p \). Let \( \{ K_\alpha \}_{\alpha \in A} \) be a collection of subfields of \( K \) with the following properties

1. \( K^p \subset K_\alpha \) for all \( \alpha \in A \),
2. \( K^p = \bigcap_{\alpha \in A} K_\alpha \),
3. for \( \alpha, \alpha' \in A \) there exists an \( \alpha'' \in A \) such that \( K_{\alpha''} \subset K_\alpha \cap K_{\alpha'} \).

Then

1. the intersection of the kernels of the maps \( \Omega_{K/F_p} \rightarrow \Omega_{K/K_\alpha} \) is zero,
2. for any finite extension \( K \subset L \) we have \( L^p = \bigcap_{\alpha \in A} L^p K_\alpha \).

Proof. Proof of (1). Choose a \( p \)-basis \( \{ x_i \} \) for \( K \) over \( F_p \). Suppose that \( \eta = \sum_{i \in I'} a_i x_i \) maps to zero in \( \Omega_{K/K_\alpha} \) for every \( \alpha \in A \). Here the index set \( I' \) is finite. By Lemma 36.2 this means that for every \( \alpha \) there exists a relation

\[
\sum_E a_{E,\alpha} x_E = 0, \quad a_{E,\alpha} \in K_\alpha
\]

where \( E \) runs over multi-indices \( E = (e_i)_{i \in I'} \) with \( 0 \leq e_i < p \). On the other hand, Lemma 36.2 guarantees there is no such relation \( \sum a_{E} x_E = 0 \) with \( a_E \in K^p \). This is a contradiction by Lemma 36.3.

Proof of (2). Suppose that we have a tower \( K \subset M \subset L \) of finite extensions of fields. Set \( M_\alpha = M^p K_\alpha \) and \( L_\alpha = L^p K_\alpha = L^p M_\alpha \). Then we can first prove that \( M^p = \bigcap_{\alpha \in A} M_\alpha \), and after that prove that \( L^p = \bigcap_{\alpha \in A} L_\alpha \). Hence it suffices to prove (2) for primitive field extensions having no nontrivial subfields. First, assume that \( L = K(\theta) \) is separable over \( K \). Then \( L \) is generated by \( \theta^p \) over \( K \), hence we may assume that \( \theta \in L^p \). In this case we see that

\[
L^p = K^p \oplus K^p \theta \oplus \ldots \oplus K^p \theta^{d-1} \quad \text{and} \quad L^p K_\alpha = K_\alpha \oplus K_\alpha \theta \oplus \ldots \oplus K_\alpha \theta^{d-1}
\]

where \( d = [L : K] \). Thus the conclusion is clear in this case. The other case is where \( L = K(\theta) \) with \( \theta^d = t \in K \), \( t \notin K^p \). In this case we have

\[
L^p = K^p \oplus K^p t \oplus \ldots \oplus K^p t^{p-1} \quad \text{and} \quad L^p K_\alpha = K_\alpha \oplus K_\alpha t \oplus \ldots \oplus K_\alpha t^{p-1}
\]

Again the result is clear.

Lemma 36.5. Let \( K \) be a field of characteristic \( p > 0 \). Let \( n, m \geq 0 \). As \( k' \) ranges through all subfields \( k^p \subset k' \subset k \) with \( [k : k'] < \infty \) the subfields

\[
f.f.(k'[[x_1^p, \ldots, x_n^p]][y_1^p, \ldots, y_m^p]) \subset f.f.(k[[x_1, \ldots, x_d]][y_1, \ldots, y_m])
\]

form a family of subfields as in Lemma 36.4. Moreover, each of the ring extensions \( k'[[x_1^p, \ldots, x_n^p]][y_1^p, \ldots, y_m^p] \subset k[[x_1, \ldots, x_n]][y_1, \ldots, y_m] \) is finite.

Proof. Write \( A = k[[x_1, \ldots, x_n]][y_1, \ldots, y_m] \) and \( A' = k'[[x_1^p, \ldots, x_n^p]][y_1^p, \ldots, y_m^p] \). We also set \( K = f.f.(A) \) and \( K' = f.f.(A') \). The ring extension \( k'[[x_1^p, \ldots, x_n^p]] \subset k[[x_1, \ldots, x_d]] \) is finite by Algebra, Lemma 94.18 which implies that \( A \rightarrow A' \) is finite. For \( f \in A \) we see that \( f^p \in A' \). Hence \( K^p \subset K' \). Any element of \( K' \) can be written as \( a/b^p \) with \( a \in A' \) and \( b \in A \) nonzero. Suppose that \( f/g \in K \), \( f, g \in A \), \( g \neq 0 \) is contained in \( K' \) for every choice of \( k' \). Fix a choice of \( k' \) for the moment.
By the above we see \( f/g^p = a/b^p \) for some \( a \in A' \) and some nonzero \( b \in A \). Hence \( b^f f \in A' \). For any \( A' \)-derivation \( D : A \to A \) we see that \( 0 = D(b^f f) = b^f D(f) \) hence \( D(f) = 0 \) as \( A \) is a domain. Taking \( D = \partial_x \) and \( D = \partial_y \), we conclude that \( f \in k[[x_1^p, \ldots, x_n^p]][[y_1^p, \ldots, y_d^p]] \). Applying a \( k' \)-derivation \( \theta : k \to k \) we similarly conclude that all coefficients of \( f \) are in \( k' \), i.e., \( f \in A' \). Since it is clear that \( A = \bigcap_{k'} A' \) where \( k' \) ranges over all subfields as in the lemma we win. \[ \Box \]

### 37. The singular locus

Let \( R \) be a Noetherian ring. The regular locus \( \text{Reg}(X) \) of \( X = \text{Spec}(R) \) is the set of primes \( p \) such that \( R_p \) is a regular local ring. The singular locus \( \text{Sing}(X) \) of \( X = \text{Spec}(R) \) is the complement \( X \setminus \text{Reg}(X) \), i.e., the set of primes \( p \) such that \( R_p \) is not a regular local ring. By the discussion preceding Algebra, Definition 107.7 we see that \( \text{Reg}(X) \) is stable under generalization In the section we study conditions that guarantee that \( \text{Reg}(X) \) is open.

**Definition 37.1.** Let \( R \) be a Noetherian ring. Let \( X = \text{Spec}(R) \).

1. We say \( R \) is J-0 if \( \text{Reg}(X) \) contains a nonempty open.
2. We say \( R \) is J-1 if \( \text{Reg}(X) \) is open.
3. We say \( R \) is J-2 if any finite type \( R \)-algebra is J-1.

The ring \( \mathbb{Q}[x]/(x^2) \) does not satisfy J-0. On the other hand J-1 implies J-0 for domains and even reduced rings as such a ring is regular at the minimal primes. Here is a characterization of the J-1 property.

**Lemma 37.2.** Let \( R \) be a Noetherian ring. Let \( X = \text{Spec}(R) \). The ring \( R \) is J-1 if and only if \( V(p) \cap \text{Reg}(X) \) contains a nonempty open subset of \( V(p) \) for all \( p \in \text{Reg}(X) \).

**Proof.** This follows immediately from Topology, Lemma 15.5 \[ \Box \]

**Lemma 37.3.** Let \( R \) be a Noetherian ring. Let \( X = \text{Spec}(R) \). Assume that for all \( p \subset R \) the ring \( R/p \) is J-0. Then \( R \) is J-1.

**Proof.** We will show that the criterion of Lemma 37.2 applies. Let \( p \in \text{Reg}(X) \) be a prime of height \( r \). Pick \( f_1, \ldots, f_r \in p \) which map to generators of \( pR_p \). Since \( p \in \text{Reg}(X) \) we see that \( f_1, \ldots, f_r \) maps to a regular sequence in \( R_p \), see Algebra, Lemma 103.3 Thus by Algebra, Lemma 67.6 we see that after replacing \( R \) by \( R_g \) for some \( g \in R, g \not\in p \) the sequence \( f_1, \ldots, f_r \) is a regular sequence in \( R \). Next, let \( p \subset q \) be a prime ideal such that \( (R/q)_q \) is a regular local ring. By the assumption of the lemma there exists a non-empty open subset of \( V(p) \) consisting of such primes, hence it suffices to prove \( R_q \) is regular. Note that \( f_1, \ldots, f_r \) is a regular sequence in \( R_q \) such that \( R_q/(f_1, \ldots, f_r)R_q \) is regular. Hence \( R_q \) is regular by Algebra, Lemma 103.7 \[ \Box \]

**Lemma 37.4.** Let \( R \to S \) be a ring map. Assume that

1. \( R \) is a Noetherian domain,
2. \( R \to S \) is injective and of finite type, and
3. \( S \) is a domain and J-0.

Then \( R \) is J-0.
**Proof.** After replacing $S$ by $S_g$ for some nonzero $g \in S$ we may assume that $S$ is a regular ring. By generic flatness we may assume that also $R \to S$ is faithfully flat, see Algebra, Lemma \[115.1\] Then $R$ is regular by Algebra, Lemma \[153.4\]. □

**Lemma 37.5.** Let $R \to S$ be a ring map. Assume that

1. $R$ is a Noetherian domain and $J$-0,
2. $R \to S$ is injective and of finite type, and
3. $S$ is a domain and $f.f.(R) \to f.f.(S)$ is separable.

Then $S$ is $J$-0.

**Proof.** We may replace $R$ by a principal localization and assume $R$ is a regular ring. By Algebra, Lemma \[136.9\] the ring map $R \to S$ is smooth at $(0)$. Hence after replacing $S$ by a principal localization we may assume that $S$ is smooth over $R$. Then $S$ is regular too, see Algebra, Lemma \[152.8\]. □

**Lemma 37.6.** Let $R$ be a Noetherian ring. The following are equivalent

1. $R$ is $J$-2,
2. every finite type $R$-algebra which is a domain is $J$-0,
3. every finite $R$-algebra is $J$-1,
4. for every prime $p$ and every finite purely inseparable extension $\kappa(p) \subset L$ there exists a finite $R$-algebra $R'$ which is a domain, which is $J$-0, and whose field of fractions is $L$.

**Proof.** It is clear that we have the implications (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (4). Recall that a domain which is $J$-1 is $J$-0. Hence we also have the implications (1) $\Rightarrow$ (3) and (3) $\Rightarrow$ (4).

Let $R \to S$ be a finite type ring map and let’s try to show $S$ is $J$-1. By Lemma \[37.3\] it suffices to prove that $S/q$ is $J$-0 for every prime $q$ of $S$. In this way we see (2) $\Rightarrow$ (1).

Assume (4). We will show that (2) holds which will finish the proof. Let $R \to S$ be a finite type ring map with $S$ a domain. Let $p = \text{Ker}(R \to S)$. Set $K = f.f.(S)$. There exists a diagram of fields

\[
\begin{array}{ccc}
K & \longrightarrow & K' \\
\uparrow & & \uparrow \\
\kappa(p) & \longrightarrow & L
\end{array}
\]

where the horizontal arrows are finite purely inseparable field extensions and where $K'/L$ is separable, see Algebra, Lemma \[41.4\]. Choose $R' \subset L$ as in (4) and let $S'$ be the image of the map $S \otimes_R R' \to K'$. Then $S'$ is a domain whose fraction field is $K'$, hence $S'$ is $J$-0 by Lemma \[37.5\] and our choice of $R'$. Then we apply Lemma \[37.3\] to see that $S$ is $J$-0 as desired. □
38. Regularity and derivations

Let $R \to S$ be a ring map. Let $D : R \to R$ be a derivation. We say that $D$ extends to $S$ if there exists a derivation $D' : S \to S$ such that

\[
\begin{array}{c}
S \\
\downarrow D' \\
R \\
\downarrow D \\
S \\
\end{array}
\]

is commutative.

**Lemma 38.1.** Let $R$ be a ring. Let $D : R \to R$ be a derivation.

1. For any ideal $I \subset R$ the derivation $D$ extends canonically to a derivation $D^\wedge : R^\wedge \to R^\wedge$ on the $I$-adic completion.
2. For any multiplicative subset $S \subset R$ the derivation $D$ extends uniquely to the localization $S^{-1}R$ of $R$.

If $R \subset R'$ is an finite type extension of rings such that $R_g \cong R'_g$ for some nonzero divisor $g \in R$, then $g^N D$ extends to $R'$ for some $N \geq 0$.

**Proof.** Proof of (1). For $n \geq 2$ we have $D(I^n) \subset I^{n-1}$ by the Leibniz rule. Hence $D$ induces maps $D_n : R/I^n \to R/I^{n-1}$. Taking the limit we obtain $D^\wedge$. We omit the verification that $D^\wedge$ is a derivation.

Proof of (2). To extend $D$ to $S^{-1}R$ just set $D(r/s) = D(r)/s - rD(s)/s^2$ and check the axioms.

Proof of the final statement. Let $x_1, \ldots, x_n \in R'$ be generators of $R'$ over $R$. Choose an $N$ such that $g^N x_i \in R$. Consider $g^{N+1} D$. By (2) this extends to $R_g$. Moreover, by the Leibniz rule and our construction of the extension above we have

\[
g^{N+1} D(x_i) = g^{N+1} D(g^{-N} g^N x_i) = -N g^N x_i D(g) + g D(g^N x_i)
\]

and both terms are in $R$. This implies that

\[
g^{N+1} D(x_1^{e_1} \ldots x_n^{e_n}) = \sum e_i x_1^{e_i} \ldots x_i^{e_i-1} \ldots x_n^{e_n} g^{N+1} D(x_i)
\]

is an element of $R'$. Hence every element of $R'$ (which can be written as a sum of monomials in the $x_i$ with coefficients in $R$) is mapped to an element of $R'$ by $g^{N+1} D$ and we win. □

**Lemma 38.2.** Let $R$ be a regular ring. Let $f \in R$. Assume there exists a derivation $D : R \to R$ such that $D(f)$ is a unit of $R/f(f)$. Then $R/(f)$ is regular.

**Proof.** It suffices to prove this when $R$ is a local ring with maximal ideal $m$ and residue field $k$. In this case it suffices to prove that $f \not\in m^2$, see Algebra, Lemma 103.3. However, if $f \in m^2$ then $D(f) \in m$ by the Leibniz rule, a contradiction. □

**Lemma 38.3.** Let $R$ be a regular $F_p$-algebra. Let $f \in R$. Assume there exists a derivation $D : R \to R$ such that $D(f)$ is a unit of $R$. Then $R[z]/(z^p - f)$ is regular.

**Proof.** Apply Lemma 38.2 to the extension of $D$ to $R[z]$ which maps $z$ to zero. □

**Lemma 38.4.** Let $p$ be a prime number. Let $B$ be a domain with $p = 0$ in $B$. Let $f \in B$ be an element which is not a $p$th power in the fraction field of $B$. If $B$ is of finite type over a Noetherian complete local ring, then there exists a derivation $D : B \to B$ such that $D(f)$ is not zero.
Proof. Let $R$ be a Noetherian complete local ring such that there exists a finite type ring map $R \to B$. Of course we may replace $R$ by its image in $B$, hence we may assume $R$ is a domain of characteristic $p > 0$ (as well as Noetherian complete local).

By Algebra, Lemma [150.10] we can write $R$ as a finite extension of $k[[x_1, \ldots, x_n]]$ for some field $k$ and integer $n$. Hence we may replace $R$ by $k[[x_1, \ldots, x_n]]$. Next, we use Algebra, Lemma [112.7] to factor $R \to B$ as

$$R \subset R[y_1, \ldots, y_d] \subset B' \subset B$$

with $B'$ finite over $R[y_1, \ldots, y_d]$ and $B'_g \cong B_g$ for some nonzero $g \in R$. Note that $f' = g^n f \in B'$ for some large integer $N$. It is clear that $f'$ is not a $p$th power in $f.f.(B') = f.f.(B)$. If we can find a derivation $D' : B' \to B'$ with $D'(f') \neq 0$, then Lemma [38.1] guarantees that $D = g^M D'$ extends to $S$ for some $M > 0$. Then $D(f) = g^N D'(f) = g^M D'(g^{-pN} f') = g^M D'(f')$ is nonzero. Thus it suffices to prove the lemma in case $B$ is a finite extension of $A = k[[x_1, \ldots, x_n]][y_1, \ldots, y_m]$.

Note that $df$ is not zero in $\Omega_{f.f.(B)/A}$, see Algebra, Lemma [148.2]. We apply Lemma [36.5] to find a subfield $k' \subset k$ of finite index such that with $A' = k'[x_1^n, \ldots, x_n^n][y_1, \ldots, y_m]$ the element $df$ does not map to zero in $\Omega_{f.f.(B)/f.f.(A')}$. Thus we can choose a $f.f.(A')$-derivation $D' : f.f.(B) \to f.f.(B)$ with $D'(f) \neq 0$.

Since $A' \subset A$ and $A \subset B$ are finite by construction we see that $A' \subset B$ is finite. Choose $b_1, \ldots, b_t \in B$ which generate $A$ as an $A'$-module. Then $D'(b_i) = f_i/g_i$ for some $f_i, g_i \in B$ with $g_i \neq 0$. Setting $D = g_1 \ldots g_tD'$ we win. \hfill $\square$

Lemma 38.5. Let $A$ be a Noetherian complete local domain. Then $A$ is $J$-0.

Proof. By Algebra, Lemma [150.10] we can find a regular subring $A_0 \subset A$ with $A$ finite over $A_0$. If $f.f.(A_0) \subset f.f.(A)$ is separable, then we are done by Lemma [37.3]. If not, then $A_0$ and $A$ have characteristic $p > 0$. For any subextension $f.f.(A_0) \subset M \subset f.f.(A)$ there exists a finite subextension $A_0 \subset B \subset A$ such that $f.f.(B) = M$. Hence, arguing by induction on $[f.f.(A) : f.f.(A_0)]$ we may assume there exists $A_0 \subset B \subset A$ such that $B$ is $J$-0 and $f.f.(B) \subset f.f.(A)$ has no nontrivial subextensions. In this case, if $f.f.(B) \subset f.f.(A)$ is separable, then we see that $A$ is $J$-0 by Lemma [37.5]. If not, then $f.f.(A) = f.f.(B)[z]/(z^p - b)$ for some $b \in B$ which is not a $p$th power in $f.f.(B)$. By Lemma [38.4] we can find a derivation $D : B \to B$ with $D(f) \neq 0$. Applying Lemma [38.3] we see that $A_p$ is regular for any prime $p$ of $A$ lying over a regular prime of $B$ and not containing $D(f)$. As $B$ is $J$-0 we conclude $A$ is too. \hfill $\square$

Proposition 38.6. The following types of rings are $J$-2:

1. fields,
2. Noetherian complete local rings,
3. $\mathbb{Z}$,
4. Dedekind domains with fraction field of characteristic zero,
5. finite type ring extensions of any of the above.

Proof. For fields, $\mathbb{Z}$ and Dedekind domains of characteristic zero you just check condition (4) of Lemma [37.6]. In the case of Noetherian complete local rings, note that if $R \to R'$ is finite and $R$ is a Noetherian complete local ring, then $R'$ is a product of Noetherian complete local rings, see Algebra, Lemma [150.2]. Hence it suffices to prove that a Noetherian complete local ring which is a domain is $J$-0, which is Lemma [38.5]. \hfill $\square$
39. Formal smoothness and regularity

The title of this section refers to Proposition 39.2.

**Lemma 39.1.** Let \( A \to B \) be a local homomorphism of Noetherian local rings. Let \( D : A \to A \) be a derivation. Assume that \( B \) is complete and \( A \to B \) is formally smooth in the \( \mathfrak{m}_B \)-adic topology. Then there exists an extension \( D' : B \to B \) of \( D \).

**Proof.** Denote \( B[\epsilon] = B[x]/(x^2) \) the ring of dual numbers over \( B \). Consider the ring map \( \psi : A \to B[\epsilon], a \mapsto a + \epsilon D(a) \). Consider the commutative diagram

\[
\begin{array}{ccc}
B & \xrightarrow{1} & B \\
\downarrow & & \downarrow \\
A & \xrightarrow{\psi} & B[\epsilon]
\end{array}
\]

By Lemma 28.5 and the assumption of formal smoothness of \( B/A \) we find a map \( \varphi : B \to B[\epsilon] \) fitting into the diagram. Write \( \varphi(b) = b + \epsilon D'(b) \). Then \( D' : B \to B \) is the desired extension. \( \square \)

**Proposition 39.2.** Let \( A \to B \) be a local homomorphism of Noetherian complete local rings. The following are equivalent

1. \( A \to B \) is regular,
2. \( A \to B \) is flat and \( B \) is geometrically regular over \( k \),
3. \( A \to B \) is flat and \( k \to B \) is formally smooth in the \( \mathfrak{m}_B \)-adic topology, and
4. \( A \to B \) is formally smooth in the \( \mathfrak{m}_B \)-adic topology.

**Proof.** We have seen the equivalence of (2), (3), and (4) in Proposition 30.4. It is clear that (1) implies (2). Thus we assume the equivalent conditions (2), (3), and (4) hold and we prove (1).

Let \( \mathfrak{p} \) be a prime of \( A \). We will show that \( B \otimes_A \kappa(\mathfrak{p}) \) is geometrically regular over \( \kappa(\mathfrak{p}) \). By Lemma 28.8 we may replace \( A \) by \( A/\mathfrak{p} \) and \( B \) by \( B/\mathfrak{p}B \). Thus we may assume that \( A \) is a domain and that \( \mathfrak{p} = (0) \).

Choose \( A_0 \subset A \) as in Algebra, Lemma 150.10. We will use all the properties stated in that lemma without further mention. As \( A_0 \to A \) induces an isomorphism on residue fields, and as \( B/\mathfrak{m}_AB \) is geometrically regular over \( A/\mathfrak{m}_A \) we can find a diagram

\[
\begin{array}{ccc}
C & \xrightarrow{} & B \\
\downarrow & & \downarrow \\
A_0 & \xrightarrow{} & A
\end{array}
\]

with \( A_0 \to C \) formally smooth in the \( \mathfrak{m}_C \)-adic topology such that \( B = C \otimes_{A_0} A \), see Remark 30.6. (Completion in the tensor product is not needed as \( A_0 \to A \) is finite, see Algebra, Lemma 94.2.) Hence it suffices to show that \( C \otimes_{A_0} f.f.(A_0) \) is a geometrically regular algebra over \( f.f.(A_0) \).

The upshot of the preceding paragraph is that we may assume that \( A = k[[x_1, \ldots, x_n]] \) where \( k \) is a field or \( A = \Lambda[[x_1, \ldots, x_n]] \) where \( \Lambda \) is a Cohen ring. In this case \( B \) is a regular ring, see Algebra, Lemma 109.8. Hence \( B \otimes_A f.f.(A) \) is a regular ring too and we win if the characteristic of \( f.f.(A) \) is zero.
Thus we are left with the case where $A = k[[x_1, \ldots, x_n]]$ and $k$ is a field of characteristic $p > 0$. Set $K = f.f.(A)$. Let $L \supset K$ be a finite purely inseparable field extension. We will show by induction on $[L : K]$ that $B \otimes_A L$ is regular. The base case is $L = K$ which we’ve seen above. Let $K \subset M \subset L$ be a subfield such that $L$ is a degree $p$ extension of $M$ obtained by adjoining a $p$th root of an element $f \in M$. Let $A'$ be a finite $A$-subalgebra of $M$ with fraction field $M$. Clearing denominators, we may and do assume $f \in A'$. Set $A'' = A'[z]/(z^p - f)$ and note that $A' \subset A''$ is finite and that the fraction field of $A''$ is $L$. By induction we know that $B \otimes_A M$ ring is regular. We have

$$B \otimes_A L = B \otimes_A M[z]/(z^p - f)$$

By Lemma 38.4 we know there exists a derivation $D : A' \to A'$ such that $D(f) \neq 0$. As $A' \to B \otimes_A A'$ is formally smooth in the $m$-adic topology by Lemma 28.9 we can use Lemma 39.1 to extend $D$ to a derivation $D' : B \otimes_A A' \to B \otimes_A A'$. Note that $D'(f) = D(f)$ is a unit in $B \otimes_A M$ as $D(f)$ is not zero in $A' \subset M$. Hence $B \otimes_A L$ is regular by Lemma 38.3 and we win.

□

40. G-rings

Let $A$ be a Noetherian local ring $A$. In Section 33 we have seen that some but not all properties of $A$ are reflected in the completion $A^\wedge$ of $A$. To study this further we introduce some terminology. For a prime $q$ of $A$ the fibre ring

$$(A^\wedge) \otimes_A \kappa(q) = (A^\wedge)_q/\kappa(A^\wedge)_q$$

is called a formal fibre of $A$. We think of the formal fibre as an algebra over $\kappa(q)$. Thus $A \to A^\wedge$ is a regular ring homomorphism if and only if all the formal fibres are geometrically regular algebras.

**Definition 40.1.** A ring $R$ is called a G-ring if $R$ is Noetherian and for every prime $p$ of $R$ the ring map $R_p \to (R_p)^\wedge$ is regular.

By the discussion above we see that $R$ is a G-ring if and only if every local ring $R_p$ has geometrically regular formal fibres. Note that if $Q \subset R$, then it suffices to check the formal fibres are regular. Another way to express the G-ring condition is described in the following lemma.

**Lemma 40.2.** Let $R$ be a Noetherian ring. Then $R$ is a G-ring if and only if for every pair of primes $q \subset p \subset R$ the algebra

$$(R/q_p^\wedge \otimes_{R/q} \kappa(q)$$

is geometrically regular over $\kappa(q)$.

**Proof.** This follows from the fact that

$$R_p^\wedge \otimes_R \kappa(q) = (R/q_p^\wedge \otimes_{R/q} \kappa(q)$$

as algebras over $\kappa(q)$. □

**Lemma 40.3.** Let $R \to R'$ be a finite type map of Noetherian rings and let

$$\begin{array}{ccc}
q' & \rightarrow & p' \\
\downarrow & & \downarrow \\
q & \rightarrow & p \\
\downarrow & & \downarrow \\
R & \rightarrow & R'
\end{array}$$
be primes. Assume $R \to R'$ is quasi-finite at $p'$.

1. If the formal fibre $R^\wedge_p \otimes_R \kappa(q)$ is geometrically regular over $\kappa(q)$, then the formal fibre $R^\wedge_{p'} \otimes_{R'} \kappa(q')$ is geometrically regular over $\kappa(q')$.

2. If the formal fibres of $R_p$ are geometrically regular, then the formal fibres of $R^\wedge_{p'}$ are geometrically regular.

3. If $R \to R'$ is quasi-finite and $R$ is a G-ring, then $R'$ is a G-ring.

**Proof.** It is clear that (1) $\Rightarrow$ (2) $\Rightarrow$ (3). Assume $R^\wedge_p \otimes_R \kappa(q)$ is geometrically regular over $\kappa(q)$. By Algebra, Lemma [121.3] we see that

$$R^\wedge_p \otimes_R R' = (R^\wedge_{p'})^\wedge \times B$$

for some $R^\wedge_p$-algebra $B$. Hence $R^\wedge_{p'} \to (R^\wedge_{p'})^\wedge$ is a factor of a base change of the map $R_p \to R^\wedge_p$. It follows that $(R^\wedge_{p'})^\wedge \otimes_{R'} \kappa(q')$ is a factor of

$$R^\wedge_p \otimes_R R' \otimes_{R'} \kappa(q') = R^\wedge_p \otimes_R \kappa(q) \otimes_{\kappa(q)} \kappa(q').$$

Thus the result follows as extension of base field preserves geometric regularity, see Algebra, Lemma [155.1].

**Lemma 40.4.** Let $R$ be a Noetherian ring. Then $R$ is a G-ring if and only if for every finite free ring map $R \to S$ the formal fibres of $S$ are regular rings.

**Proof.** Assume that for any finite free ring map $R \to S$ the ring $S$ has regular formal fibres. Let $q \subset p \subset R$ be primes and let $\kappa(q) \subset L$ be a finite purely inseparable extension. To show that $R$ is a G-ring it suffices to show that

$$R^\wedge_p \otimes_R \kappa(q) \otimes_{\kappa(q)} L$$

is a regular ring. Choose a finite free extension $R \to R'$ such that $q' = qR'$ is a prime and such that $\kappa(q')$ is isomorphic to $L$ over $\kappa(q)$, see Algebra, Lemma [149.3]

By Algebra, Lemma [94.19] we have

$$R^\wedge_p \otimes_R R' = \prod (R^\wedge_{p'})^\wedge$$

where $p'_i$ are the primes of $R'$ lying over $p$. Thus we have

$$R^\wedge_p \otimes_R \kappa(q) \otimes_{\kappa(q)} L = R^\wedge_p \otimes_R R' \otimes_{R'} \kappa(q') = \prod (R^\wedge_{p'_i})^\wedge \otimes_{R'_{p'_i}} \kappa(q')$$

Our assumption is that the rings on the right are regular, hence the ring on the left is regular too. Thus $R$ is a G-ring. The converse follows from Lemma [10.3].

**Lemma 40.5.** Let $k$ be a field of characteristic $p$. Let $A = k[[x_1, \ldots, x_n]][y_1, \ldots, y_n]$ and denote $K = f.f.(A)$. Let $p \subset A$ be a prime. Then $A^\wedge_p \otimes_A K$ is geometrically regular over $K$.

**Proof.** Let $L \supset K$ be a finite purely inseparable field extension. We will show by induction on $[L : K]$ that $A^\wedge_p \otimes L$ is regular. The base case is $L = K$: as $A$ is regular, $A^\wedge_p$ is regular (Lemma [33.4]), hence the localization $A^\wedge_p \otimes K$ is regular. Let $K \subset M \subset L$ be a subfield such that $L$ is a degree $p$ extension of $M$ obtained by adjoining a $p$th root of an element $f \in M$. Let $B$ be a finite $A$-subalgebra of $M$ with fraction field $M$. Clearing denominators, we may and do assume $f \in B$. Set $C = B[z]/(z^p - f)$ and note that $B \subset C$ is finite and that the fraction field of $C$ is $L$. Since $A \subset B \subset C$ are finite and $L/M/K$ are purely inseparable we see that for
every element of $B$ or $C$ some power of it lies in $A$. Hence there is a unique prime $r \subset B$, resp. $q \subset C$ lying over $p$. Note that

$$A^\wedge_p \otimes_A M = B^\wedge_t \otimes_B M$$

see Algebra, Lemma 94.19. By induction we know that this ring is regular. In the same manner we have

$$A^\wedge_p \otimes_A L = C^\wedge_t \otimes_C L = B^\wedge_t \otimes_B M[z]/(z^p - f)$$

the last equality because the completion of $C = B[z]/(z^p - f)$ equals $B^\wedge_t[z]/(z^p - f)$. By Lemma 38.4 we know there exists a derivation $D : B \to B$ such that $D(f) \neq 0$. In other words, $g = D(f)$ is a unit in $M$. By Lemma 38.1 $D$ extends to a derivation of $B^\wedge_t$, $B^\wedge_t$ and $B^\wedge_t \otimes_B M$ (successively extending through a localization, a completion, and a localization). Since it is an extension we end up with a derivation of $B^\wedge_t \otimes_B M$ which maps $f$ to $g$ and $g$ is a unit of the ring $B^\wedge_t \otimes_B M$. Hence $A^\wedge_p \otimes_A L$ is regular by Lemma 38.3 and we win.

**Proposition 40.6.** A Noetherian complete local ring is a G-ring.

**Proof.** Let $A$ be a Noetherian complete local ring. By Lemma 40.2 it suffices to check that $B = A/q$ has geometrically regular formal fibres over the minimal prime $(0)$ of $B$. Thus we may assume that $A$ is a domain and it suffices to check the condition for the formal fibres over the minimal prime $(0)$ of $A$. Set $K = f.f(A)$.

We can choose a subring $A_0 \subset A$ which is a regular complete local ring such that $A$ is finite over $A_0$, see Algebra, Lemma 150.10 Moreover, we may assume that $A_0$ is a power series ring over a field or a Cohen ring. By Lemma 40.3 we see that it suffices to prove the result for $A_0$.

Assume that $A$ is a power series ring over a field or a Cohen ring. Since $A$ is regular the localizations $A_p$ are regular (see Algebra, Definition 107.7 and the discussion preceding it). Hence the completions $A^\wedge_p$ are regular, see Lemma 33.4. Hence the fibre $A^\wedge_p \otimes_A K$ is, as a localization of $A^\wedge_p$, also regular. Thus we are done if the characteristic of $K$ is 0. The positive characteristic case is the case $A = k[[x_1, \ldots, x_d]]$ which is a special case of Lemma 40.5.

**Lemma 40.7.** Let $R$ be a Noetherian ring. Then $R$ is a G-ring if and only if $R_m$ has geometrically regular formal fibres for every maximal ideal $m$ of $R$.

**Proof.** Assume $R_m \to R^\wedge_m$ is regular for every maximal ideal $m$ of $R$. Let $p$ be a prime of $R$ and choose a maximal ideal $p \subset m$. Since $R_m \to R^\wedge_m$ is faithfully flat we can choose a prime $p'$ if $R^\wedge_m$ lying over $pR_m$. Consider the commutative diagram

$$
\begin{array}{ccc}
R^\wedge_m & \longrightarrow & (R^\wedge_m)_{p'} \\
\uparrow & & \uparrow \\
R_m & \longrightarrow & R_p \\
\downarrow & & \downarrow \\
R^\wedge_p & \longrightarrow & R^\wedge_{p'}
\end{array}
$$

By assumption the ring map $R_m \to R^\wedge_m$ is regular. By Proposition 40.6 $(R^\wedge_m)_{p'} \to (R^\wedge_m)_{p'}$ is regular. Hence $R_m \to (R^\wedge_m)_{p'}$ is regular and since it factors through the localization $R_p$, also the ring map $R_p \to (R^\wedge_m)_{p'}$ is regular. Thus we may apply Lemma 31.7 to see that $R_p \to R^\wedge_p$ is regular.
Lemma 40.8. Let $R$ be a Noetherian local ring ring which is a $G$-ring. Then the henselization $R^h$ and the strict henselization $R^{sh}$ are $G$-rings.

Proof. We will use the criterion of Lemma 40.7. Let $q \subset R^h$ be a prime and set $p = R \cap q$. Set $q_1 = q$ and let $q_2, \ldots, q_t$ be the other primes of $R^h$ lying over $p$, so that $R^h \otimes_R \kappa(p) = \prod_{i=1}^{t} \kappa(q_i)$, see Lemma 35.3. Using that $(R^h)^\wedge = R^\wedge$ (Lemma 35.3) we see

$$\prod_{i=1}^{t} (R^h)^\wedge \otimes_{R^h} \kappa(q_i) = (R^h)^\wedge \otimes_{R^h} (R^h \otimes_R \kappa(p)) = R^\wedge \otimes_R \kappa(p)$$

Hence $(R^h)^\wedge \otimes_{R^h} \kappa(q_i)$ is geometrically regular over $\kappa(p)$ by assumption. Since $\kappa(q_i)$ is separable algebraic over $\kappa(p)$ it follows from Algebra, Lemma 155.6 that $(R^h)^\wedge \otimes_{R^h} \kappa(q_i)$ is geometrically regular over $\kappa(q_i)$.

Let $\tau \subset R^{sh}$ be a prime and set $p = R \cap \tau$. Set $\tau_1 = \tau$ and let $\tau_2, \ldots, \tau_s$ be the other primes of $R^{sh}$ lying over $p$, so that $R^{sh} \otimes_R \kappa(p) = \prod_{i=1}^{s} \kappa(\tau_i)$, see Lemma 35.3. Then we see that

$$\prod_{i=1}^{s} (R^{sh})^\wedge \otimes_{R^{sh}} \kappa(\tau_i) = (R^{sh})^\wedge \otimes_{R^{sh}} (R^{sh} \otimes_R \kappa(p)) = (R^{sh})^\wedge \otimes_R \kappa(p)$$

Note that $R^\wedge \to (R^h)^\wedge$ is formally smooth in the $\mathfrak{m}_{(R^h)^\wedge}$-adic topology, see Lemma 35.3. Hence $R^\wedge \to (R^{sh})^\wedge$ is regular by Proposition 39.2. We conclude that $(R^{sh})^\wedge \otimes_{R^{sh}} \kappa(\tau_i)$ is regular over $\kappa(p)$ by Lemma 31.4 as $R^{\wedge} \otimes_R \kappa(p)$ is regular over $\kappa(p)$ by assumption. Since $\kappa(\tau_i)$ is separable algebraic over $\kappa(p)$ it follows from Algebra, Lemma 155.6 that $(R^{sh})^\wedge \otimes_{R^{sh}} \kappa(\tau_i)$ is geometrically regular over $\kappa(\tau_i)$. □

Lemma 40.9. Let $p$ be a prime number. Let $A$ be a Noetherian complete local domain with fraction field $K$ of characteristic $p$. Let $q \subset A[x]$ be a maximal ideal lying over the maximal ideal of $A$ and let $\tau \subset q$ be a prime lying over $(0) \subset A$. Then $A[x]_q^\wedge \otimes_{A[x]} \kappa(\tau)$ is geometrically regular over $\kappa(\tau)$.

Proof. Note that $K \subset \kappa(\tau)$ is finite. Hence, given a finite purely inseparable extension $\kappa(\tau) \subset L$ there exists a finite extension of Noetherian complete local domains $A \subset B$ such that $\kappa(\tau) \otimes_A B$ surjects onto $L$. Namely, you take $B \subset L$ a finite $A$-subalgebra whose field of fractions is $L$. Denote $\tau' \subset B[x]$ the kernel of the map $B[x] = A[x] \otimes_{A[x]} B \to \kappa(\tau) \otimes_{A[x]} B \to L$ so that $\kappa(\tau') = L$. Then

$$A[x]_q^\wedge \otimes_{A[x]} L = A[x]_q^\wedge \otimes_{A[x]} B[x] \otimes_{B[x]} \kappa(\tau') = \prod B[x]_q^\wedge \otimes_{B[x]} \kappa(\tau')$$

where $q_1, \ldots, q_s$ are the primes of $B[x]$ lying over $q$, see Algebra, Lemma 94.19. Thus we see that it suffices to prove the rings $B[x]_q^\wedge \otimes_{B[x]} \kappa(\tau')$ are regular. This reduces us to showing that $A[x]_q^\wedge \otimes_{A[x]} \kappa(\tau)$ is regular in the special case that $K = \kappa(\tau)$.

Assume $K = \kappa(\tau)$. In this case we see that $xK[x]$ is generated by $x - f$ for some $f \in K$ and

$$A[x]_q^\wedge \otimes_{A[x]} \kappa(\tau) = (A[x]_q^\wedge \otimes_A K)/(x - f)$$

The derivation $D = \frac{d}{dx}$ of $A[x]$ extends to $K[x]$ and maps $x - f$ to a unit of $K[x]$. Moreover $D$ extends to $A[x]_q^\wedge \otimes_A K$ by Lemma 38.1. As $A \to A[x]_q^\wedge$ is formally smooth (see Lemmas 28.2 and 28.4) the ring $A[x]_q^\wedge \otimes_A K$ is regular by Proposition 39.2 (the arguments of the proof of that proposition simplify significantly in this particular case). We conclude by Lemma 38.2. □
Proposition 40.10. Let $R$ be a $G$-ring. If $R \to S$ is essentially of finite type then $S$ is a $G$-ring.

Proof. Since being a $G$-ring is a property of the local rings it is clear that a localization of a $G$-ring is a $G$-ring. Conversely, if every localization at a prime is a $G$-ring, then the ring is a $G$-ring. Thus it suffices to show that $S_q$ is a $G$-ring for every finite type $R$-algebra $S$ and every prime $q$ of $S$. Writing $S$ as a quotient of $R[x_1, \ldots, x_n]$ we see from Lemma 40.3 that it suffices to prove that $R[x_1, \ldots, x_n]$ is a $G$-ring. By induction on $n$ it suffices to prove that $R[x]$ is a $G$-ring. Let $q \subset R[x]$ be a maximal ideal. By Lemma 40.7 it suffices to show that

$$R[x]_q \longrightarrow R[x]_q^\wedge$$

is regular. If $q$ lies over $p \subset R$, then we may replace $R$ by $R_p$. Hence we may assume that $R$ is a Noetherian local $G$-ring with maximal ideal $m$ and that $q \subset R[x]$ lies over $m$. Note that there is a unique prime $q' \subset R^\wedge[x]$ lying over $q$. Consider the diagram

$$
\begin{array}{ccc}
R[x]^\wedge_q & \longrightarrow & (R^\wedge[x]_{q'})^\wedge \\
\uparrow & & \uparrow \\
R[x]_q & \longrightarrow & R^\wedge[x]_{q'}
\end{array}
$$

Since $R$ is a $G$-ring the lower horizontal arrow is regular (as a localization of a base change of the regular ring map $R \to R^\wedge$). Suppose we can prove the right vertical arrow is regular. Then it follows that the composition $R[x]_q \to (R^\wedge[x]_{q'})^\wedge$ is regular, and hence the left vertical arrow is regular by Lemma 33.7. Hence we see that we may assume $R$ is a Noetherian complete local ring and $q$ a prime lying over the maximal ideal of $R$.

Let $R$ be a Noetherian complete local ring and let $q \subset R[x]$ be a maximal ideal lying over the maximal ideal of $R$. Let $r \subset q$ be a prime ideal. We want to show that $R[x]_q^\wedge \otimes_{R[x]} \kappa(r)$ is a geometrically regular algebra over $\kappa(r)$. Set $p = R \cap r$. Then we can replace $R$ by $R/p$ and $q$ and $r$ by their images in $R/p[x]$, see Lemma 40.2. Hence we may assume that $R$ is a domain and that $r \cap R = (0)$.

By Algebra, Lemma 150.10 we can find $R_0 \subset R$ which is regular and such that $R$ is finite over $R_0$. Applying Lemma 40.3 we see that it suffices to prove $R[x]_q^\wedge \otimes_{R[x]} \kappa(r)$ is geometrically regular over $\kappa(r)$ when, in addition to the above, $R$ is a regular complete local ring.

Now $R$ is a regular complete local ring, we have $q \subset r \subset R[x]$, we have $(0) = R \cap r$ and $q$ is a maximal ideal lying over the maximal ideal of $R$. Since $R$ is regular the ring $R[x]$ is regular (Algebra, Lemma 152.8). Hence the localization $R[x]_q$ is regular. Hence the completions $R[x]_q^\wedge$ are regular, see Lemma 33.4. Hence the fibre $R[x]_q^\wedge \otimes_{R[x]} \kappa(r)$ is, as a localization of $R[x]_q^\wedge$, also regular. Thus we are done if the characteristic of $f.f.(R)$ is 0.

If the characteristic of $R$ is positive, then $R = k[[x_1, \ldots, x_n]]$. In this case we split the argument in two subcases:

1. The case $r = (0)$. The result is a direct consequence of Lemma 40.5
2. The case $r \neq (0)$. This is Lemma 40.9

□
Remark 40.11. Let $R$ be a G-ring and let $I \subset R$ be an ideal. In general it is not the case that the $I$-adic completion $R^\wedge$ is a G-ring. An example was given by Nishimura in [Nis81]. A generalization and, in some sense, clarification of this example can be found in the last section of [Dum00].

Proposition 40.12. The following types of rings are G-rings:

1. fields,
2. Noetherian complete local rings,
3. $\mathbb{Z}$,
4. Dedekind domains with fraction field of characteristic zero,
5. finite type ring extensions of any of the above.

Proof. For fields, $\mathbb{Z}$ and Dedekind domains of characteristic zero this follows immediately from the definition and the fact that the completion of a discrete valuation ring is a discrete valuation ring. A Noetherian complete local ring is a G-ring by Proposition 40.6. The statement on finite type overrings is Proposition 40.10.

Lemma 40.13. Let $(A, \mathfrak{m})$ be a henselian local ring. Then $A$ is a filtered colimit of a system of henselian local Noetherian G-rings with local transition maps.

Proof. Write $A = \text{colim} A_i$ as a filtered colimit of finite type $\mathbb{Z}$-algebras. Let $p_i$ be the prime ideal of $A_i$ lying under $\mathfrak{m}$. We may replace $A_i$ by the localization of $A_i$ at $p_i$. Then $A_i$ is a Noetherian local G-ring (Proposition 40.12). By Lemma 7.17 we see that $A = \text{colim} A_i$. By Lemma 40.8 the rings $A_i$ are G-rings.

Lemma 40.14. Let $A$ be a Noetherian G-ring. Let $I \subset A$ be an ideal and let $A^\wedge$ be the completion of $A$ with respect to $I$. Then $A \to A^\wedge$ is regular.

Proof. The ring map $A \to A^\wedge$ is flat by Algebra, Lemma 94.3. The ring $A^\wedge$ is Noetherian by Algebra, Lemma 94.10. Thus it suffices to check the third condition of Lemma 31.2. Let $m' \subset A^\wedge$ be a maximal ideal lying over $m \subset A$. By Algebra, Lemma 94.11 we have $IA^\wedge \subset m'$. Since $A^\wedge/IA^\wedge = A/I$ we see that $I \subset m$, $m/I = m'/IA^\wedge A/m = A^\wedge/m'$. Since $A^\wedge/m'$ is a field, we conclude that $m$ is a maximal ideal as well. Then $A_m \to A^\wedge_{m'}$ is a flat local ring homomorphism of Noetherian local rings which identifies residue fields and such that $mA_m = m'A^\wedge_{m'}$. Thus it induces an isomorphism on complete local rings, see Lemma 33.7. Let $(A^\wedge)_{m'}$ be the completion of $A_m$ with respect to its maximal ideal. The ring map $(A^\wedge)_{m'} \to ((A^\wedge)_{m'})^\wedge = (A_m)^\wedge$ is faithfully flat (Algebra, Lemma 94.4). Thus we can apply Lemma 31.7 to the ring maps $A_m \to (A^\wedge)_{m'} \to (A_m)^\wedge$ to conclude because $A_m \to (A_m)^\wedge$ is regular as $A$ is a G-ring.

Lemma 40.15. Let $A$ be a Noetherian G-ring. Let $I \subset A$ be an ideal. Let $(A^h, I^h)$ be the henselization of the pair $(A, I)$, see Lemma 7.13. Then $A^h$ is a Noetherian G-ring.

Proof. Let $m^h \subset A^h$ be a maximal ideal. We have to show that the map from $A^h_m$ to its completion has geometrically regular fibres, see Lemma 40.7. Let $m$ be the inverse image of $m^h$ in $A$. Note that $I^h \subset m^h$ and hence $I \subset m$ as $(A^h, I^h)$ is a henselian pair. Recall that $A^h$ is Noetherian, $I^h = IA^h$, and that $A \to A^h$ is a flat local ring homomorphism of Noetherian local rings which identifies residue fields and such that $mA^h = m'A^h$. Thus it induces an isomorphism on complete local rings, see Lemma 33.7. Let $(A^h)_{m'}$ be the completion of $A^h_m$ with respect to its maximal ideal. The ring map $(A^h)_{m'} \to ((A^h)_{m'})^\wedge = (A_m)^\wedge$ is faithfully flat (Algebra, Lemma 94.4). Thus we can apply Lemma 31.7 to the ring maps $A_m \to (A^h)_{m'} \to (A_m)^\wedge$ to conclude because $A_m \to (A_m)^\wedge$ is regular as $A$ is a G-ring.
induces an isomorphism on $I$-adic completions, see Lemma 7.16. Then the local homomorphism of Noetherian local rings

$$A_m \to A_{m}^h$$

induces an isomorphism on completions at maximal ideals by Lemma 33.7 (details omitted). Let $q^h$ be a prime of the middle ring lying over $q \subset A_m$. Set $q_1 = q^h$ and let $q_2, \ldots, q_t$ be the other primes of $A^h$ lying over $q$, so that $A^h \otimes_A \kappa(q) = \prod_{i=1, \ldots, t} \kappa(q_i)$; see Lemma 35.12. Using that $(A^h_{m})^\wedge = (A_{m})^\wedge$ as discussed above we see

$$\prod_{i=1, \ldots, t} (A_{m}^h)^\wedge \otimes_{A_{m}^h} \kappa(q_i) = (A_{m}^h)^\wedge \otimes_{A_{m}^h} (A_{m}^h \otimes_{A_{m}} \kappa(q)) = (A_{m})^\wedge \otimes_{A_{m}} \kappa(q)$$

Hence, as one of the components, the ring

$$(A_{m}^h)^\wedge \otimes_{A_{m}^h} \kappa(q^h)$$

is geometrically regular over $\kappa(q)$ by assumption on $A$. Since $\kappa(q^h)$ is separable algebraic over $\kappa(q)$ it follows from Algebra, Lemma 155.6 that

$$(A_{m}^h)^\wedge \otimes_{A_{m}^h} \kappa(q^h)$$

is geometrically regular over $\kappa(q^h)$ as desired.

\section{41. Excellent rings}

In this section we discuss Grothendieck’s notion of excellent rings. For the definitions of G-rings, J-2 rings, and universally catenary rings we refer to Definition 40.1, Definition 37.1, and Algebra, Definition 102.3.

\begin{definition}
Let $R$ be a ring.

(1) We say $R$ is quasi-excellent if $R$ is Noetherian, a G-ring, and J-2.

(2) We say $R$ is excellent if $R$ is quasi-excellent and universally catenary.
\end{definition}

Thus a Noetherian ring is quasi-excellent if it has geometrically regular formal fibres and if any finite type algebra over it has closed singular set. For such a ring to be excellent we require in addition that there exists (locally) a good dimension function.

\begin{lemma}
Any localization of a finite type ring over a (quasi-)excellent ring is (quasi-)excellent.
\end{lemma}

\begin{proof}
For finite type algebras this follows from the definitions for the properties J-2 and universally catenary. For G-rings, see Proposition 40.10. We omit the proof that localization preserves (quasi-)excellency.
\end{proof}

\begin{lemma}
A quasi-excellent ring is Nagata.
\end{lemma}

\begin{proof}
Let $R$ be quasi-excellent. Using that a finite type algebra over $R$ is quasi-excellent (Lemma 41.2) we see that it suffices to show that any quasi-excellent domain is N-1, see Algebra, Lemma 151.18. Applying Algebra, Lemma 151.30 (and using that a quasi-excellent ring is J-2) we reduce to showing that a quasi-excellent local domain $R$ is N-1. As $R \to R^\wedge$ is regular we see that $R^\wedge$ is reduced by Lemma 32.1. In other words, $R$ is analytically unramified. Hence $R$ is N-1 by Algebra, Lemma 151.26.
\end{proof}

\begin{proposition}
The following types of rings are excellent:
\end{proposition}
(1) fields,
(2) Noetherian complete local rings,
(3) \(\mathbb{Z}\),
(4) Dedekind domains with fraction field of characteristic zero,
(5) finite type ring extensions of any of the above.

**Proof.** See Propositions 40.12 and 38.6 to see that these rings are G-rings and have J-2. Any Cohen-Macaulay ring is universally catenary, see Algebra, Lemma 102.8. In particular fields, Dedekind rings, and more generally regular rings are universally catenary. Via the Cohen structure theorem we see that complete local rings are universally catenary. □

42. Abelian categories of modules

Let \(R\) be a ring. The category \(\text{Mod}_R\) of \(R\)-modules is an abelian category. Here are some examples of subcategories of \(\text{Mod}_R\) which are abelian (we use the terminology introduced in Homology, Definition 9.1 as well as Homology, Lemmas 9.2 and 9.3):

1. The category of coherent \(R\)-modules is a weak Serre subcategory of \(\text{Mod}_R\). This follows from Algebra, Lemma 88.2.
2. Let \(S \subset R\) be a multiplicative subset. The full subcategory consisting of \(R\)-modules \(M\) such that multiplication by \(s \in S\) is an isomorphism on \(M\) is a Serre subcategory of \(\text{Mod}_R\). This follows from Algebra, Lemma 9.5.
3. Let \(I \subset R\) be a finitely generated ideal. The full subcategory of \(f\)-power torsion modules is a Serre subcategory of \(\text{Mod}_R\). See Lemma 65.5.
4. In some texts a torsion module is defined as a module \(M\) such that for all \(x \in M\) there exists a nonzerodivisor \(f \in R\) such that \(fx = 0\). The full subcategory of torsion modules is a Serre subcategory of \(\text{Mod}_R\).
5. If \(R\) is not Noetherian, then the category \(\text{Mod}^\text{fg}_R\) of finitely generated \((i.e., finite) R\)-modules is not abelian. Namely, if \(I \subset R\) is a non-finitely generated ideal, then the map \(R \to R/I\) does not have a kernel in \(\text{Mod}^\text{fg}_R\).
6. If \(R\) is Noetherian, then coherent \((R=\text{finite}\) \(R\)-modules agree with finitely generated \((i.e., finite) R\)-modules, see Algebra, Lemmas 88.4, 88.3, and 30.4. Hence \(\text{Mod}^\text{fg}_R\) is abelian by (1) above, but in fact, in this case the category \(\text{Mod}^\text{fg}_R\) is a (strong) Serre subcategory of \(\text{Mod}_R\).

43. Injective abelian groups

In this section we show the category of abelian groups has enough injectives. Recall that an abelian group \(M\) is divisible if and only if for every \(x \in M\) and every \(n \in \mathbb{N}\) there exists a \(y \in M\) such that \(ny = x\).

**Lemma 43.1.** An abelian group \(J\) is an injective object in the category of abelian groups if and only if \(J\) is divisible.

**Proof.** Suppose that \(J\) is not divisible. Then there exists an \(x \in J\) and \(n \in \mathbb{N}\) such that there is no \(y \in J\) with \(ny = x\). Then the morphism \(\mathbb{Z} \to J\), \(m \mapsto mx\) does not extend to \(\frac{1}{n}\mathbb{Z} \supset \mathbb{Z}\). Hence \(J\) is not injective.

Let \(A \subset B\) be abelian groups. Assume that \(J\) is a divisible abelian group. Let \(\varphi : A \to J\) be a morphism. Consider the set of homomorphisms \(\varphi' : A' \to J\) with \(A \subset A' \subset B\) and \(\varphi'|_A = \varphi\). Define \((A', \varphi') \geq (A'', \varphi'')\) if and only if \(A' \supset A''\) and \(\varphi'|_{A''} = \varphi''\). If \((A_i, \varphi_i)_{i \in I}\) is a totally ordered collection of such pairs, then we
obtain a map $\bigcup_{i \in I} A_i \rightarrow J$ defined by $a \in A_i$ maps to $\varphi_i(a)$. Thus Zorn’s lemma applies. To conclude we have to show that if the pair $(A', \varphi')$ is maximal then $A' = B$. In other words, it suffices to show, given any subgroup $A \subset B$, $A \neq B$ and any $\varphi : A \rightarrow J$, then we can find $\varphi' : A' \rightarrow J$ with $A \subset A' \subset B$ such that (a) the inclusion $A \subset A'$ is strict, and (b) the morphism $\varphi'$ extends $\varphi$.

To prove this, pick $x \in B$, $x \notin A$. If there exists no $n \in \mathbb{N}$ such that $nx \in A$, then $A \oplus \mathbb{Z} \cong A + \mathbb{Z}x$. Hence we can extend $\varphi$ to $A' = A + \mathbb{Z}x$ by using $\varphi$ on $A$ and mapping $x$ to zero for example. If there does exist an $n \in \mathbb{N}$ such that $nx \in A$, then let $n$ be the minimal such integer. Let $z \in J$ be an element such that $nz = \varphi(nx)$.

Define a morphism $\tilde{\varphi} : A \oplus \mathbb{Z} \rightarrow J$ by $(a, m) \mapsto \varphi(a) + mz$. By our choice of $z$ the kernel of $\tilde{\varphi}$ contains the kernel of the map $A \oplus \mathbb{Z} \rightarrow B$, $(a, m) \mapsto a + mx$. Hence $\tilde{\varphi}$ factors through the image $A' = A + \mathbb{Z}x$, and this extends the morphism $\varphi$. □

We can use this lemma to show that every abelian group can be embedded in a injective abelian group. But this is a special case of the result of the following section.

### 44. Injective modules

Some lemmas on injective modules.

**Definition** 44.1. Let $R$ be a ring. An $R$-module $J$ is **injective** if and only if the functor $\text{Hom}_R(-, J) : \text{Mod}_R \rightarrow \text{Mod}_R$ is an exact functor.

The functor $\text{Hom}_R(-, M)$ is left exact for any $R$-module $M$, see Algebra, Lemma 10.1. Hence the condition for $J$ to be injective really signifies that given an injection of $R$-modules $M \rightarrow M'$ the map $\text{Hom}_R(M', J) \rightarrow \text{Hom}_R(M, J)$ is surjective.

Before we reformulate this in terms of $\text{Ext}$-modules we discuss the relationship between $\text{Ext}^1_R(M, N)$ and extensions as in Homology, Section 6.

**Lemma** 44.2. Let $R$ be a ring. Let $A$ be the abelian category of $R$-modules. There is a canonical isomorphism $\text{Ext}_A(M, N) = \text{Ext}^1_R(M, N)$ compatible with the long exact sequences of Algebra, Lemmas 69.6 and 69.7 and the 6-term exact sequences of Homology, Lemma 6.7.

Proof. Omitted. □

**Lemma** 44.3. Let $R$ be a ring. Let $J$ be an $R$-module. The following are equivalent

1. $J$ is injective,
2. $\text{Ext}^1_R(M, J) = 0$ for every $R$-module $M$.

Proof. Let $0 \rightarrow M'' \rightarrow M' \rightarrow M \rightarrow 0$ be a short exact sequence of $R$-modules. Consider the long exact sequence

$0 \rightarrow \text{Hom}_R(M, J) \rightarrow \text{Hom}_R(M', J) \rightarrow \text{Hom}_R(M'', J)$

$\rightarrow \text{Ext}^1_R(M, J) \rightarrow \text{Ext}^1_R(M', J) \rightarrow \text{Ext}^1_R(M'', J) \rightarrow \ldots$

of Algebra, Lemma 69.7. Thus we see that (2) implies (1). Conversely, if $J$ is injective then the Ext-group is zero by Homology, Lemma 23.2 and Lemma 44.2. □

**Lemma** 44.4. Let $R$ be a ring. Let $J$ be an $R$-module. The following are equivalent

1. $J$ is injective,
2. $\text{Ext}^1_R(R/I, J) = 0$ for every ideal $I \subset R$, and

Proof. □
(3) for an ideal \( I \subset R \) and module map \( I \to J \) there exists an extension \( R \to J \).

**Proof.** We have seen the implication (1) ⇒ (2) in Lemma 44.3. Given a module map \( I \to J \) as in (3) we obtain an extension of \( R/I \) by \( J \) by pushout

\[
\begin{array}{c@{\longrightarrow}c@{\longrightarrow}c}
0 & I & R \\
\downarrow & \downarrow & \downarrow \\
0 & J & E \\
& \downarrow & \downarrow \\
& R/I & 0
\end{array}
\]

If (2) holds, then the lower short exact sequence is split by Homology, Lemma 23.2. By choosing a splitting \( E \to J \), we obtain an extension \( R \to E \to J \) of the given map \( I \to J \). Thus (2) ⇒ (3).

Assume (3). Let \( M \subset N \) be an inclusion of \( R \)-modules. Let \( \varphi : M \to J \) be a homomorphism. We will show that \( \varphi \) extends to \( N \) which finishes the proof of the lemma. Consider the set of homomorphisms \( \{ \varphi' : M' \to J \mid M \subset M' \subset N \text{ and } \varphi'|_M = \varphi \} \). Define \( (M', \varphi') \geq (M'', \varphi'') \) if and only if \( M' \supset M'' \) and \( \varphi'|_M = \varphi'' \). If \( (M_i, \varphi_i)_{i \in I} \) is a totally ordered collection of such pairs, then we obtain a map \( \bigcup_{i \in I} M_i \to J \) defined by \( a \in M_i \) maps to \( \varphi_i(a) \). Thus Zorn's lemma applies. To conclude we have to show that if the pair \( (M', \varphi') \) is maximal then \( M' = N \). In other words, it suffices to show, given any subgroup \( M \subset N \) and any \( \varphi : M \to J \), then we can find \( \varphi' : M' \to J \) with \( M \subset M' \subset N \) such that (a) the inclusion \( M \subset M' \) is strict, and (b) the morphism \( \varphi' \) extends \( \varphi \).

To prove this, pick \( x \in N \), \( x \notin M \). Let \( I = \{ f \in R \mid fx \in M \} \). This is an ideal of \( R \). Define a homomorphism \( \psi : I \to J \) by \( f \mapsto \varphi(fx) \). Extend to a map \( \tilde{\psi} : R \to J \) which is possible by assumption (3). By our choice of \( I \) the kernel of \( M \oplus R \to J \), \( (y, f) \mapsto y - \tilde{\psi}(f) \) contains the kernel of the map \( M \oplus R \to N \), \( (y, f) \mapsto y + fx \). Hence this homomorphism factors through the image \( M' = M + Rx \) and this extends the given homomorphism as desired. \( \square \)

In the rest of this section we prove that there are enough injective modules over a ring \( R \). We start with the fact that \( \mathbb{Q}/\mathbb{Z} \) is an injective abelian group. This follows from Lemma 43.1.

**Definition 44.5.** Let \( R \) be a ring.

1. For any \( R \)-module \( M \) over \( R \) we denote \( M^\vee = \text{Hom}(M, \mathbb{Q}/\mathbb{Z}) \) with its natural \( R \)-module structure. We think of \( M \mapsto M^\vee \) as a contravariant functor from the category of \( R \)-modules to itself.

2. For any \( R \)-module \( M \) we denote

\[
F(M) = \bigoplus_{m \in M} R[m]
\]

the free module with basis given by the elements \([m]\) with \( m \in M \). We let \( F(M) \to M, \sum f_i[m_i] \mapsto \sum f_i m_i \) be the natural surjection of \( R \)-modules. We think of \( M \mapsto (F(M) \to M) \) as a functor from the category of \( R \)-modules to the category of arrows in \( R \)-modules.

**Lemma 44.6.** Let \( R \) be a ring. The functor \( M \mapsto M^\vee \) is exact.

**Proof.** This because \( \mathbb{Q}/\mathbb{Z} \) is an injective abelian group by Lemma 43.1. \( \square \)
There is a canonical map $ev : M \to (M^\vee)^\vee$ given by evaluation: given $x \in M$ we let $ev(x) \in (M^\vee)^\vee = \text{Hom}(M^\vee, \mathbb{Q}/\mathbb{Z})$ be the map $\varphi \mapsto \varphi(x)$.

**Lemma 44.7.** For any $R$-module $M$ the evaluation map $ev : M \to (M^\vee)^\vee$ is injective.

**Proof.** You can check this using that $\mathbb{Q}/\mathbb{Z}$ is an injective abelian group. Namely, if $x \in M$ is not zero, then let $M' \subset M$ be the cyclic group it generates. There exists a nonzero map $M' \to \mathbb{Q}/\mathbb{Z}$ which necessarily does not annihilate $x$. This extends to a map $\varphi : M \to \mathbb{Q}/\mathbb{Z}$ And then $ev(x)(\varphi) = \varphi(x) \neq 0$. □

The canonical surjection $F(M) \to M$ of $R$-modules turns into a a canonical injection, see above, of $R$-modules

$$(M^\vee)^\vee \longrightarrow (F(M^\vee))^\vee.$$ 

Set $J(M) = (F(M^\vee))^\vee$. The composition of $ev$ with this the displayed map gives $M \to J(M)$ functorially in $M$.

**Lemma 44.8.** Let $R$ be a ring. For every $R$-module $M$ the $R$-module $J(M)$ is injective.

**Proof.** Note that $J(M) \cong \prod_{\varphi \in M^\vee} R^\vee$ as an $R$-module. As the product of injective modules is injective, it suffices to show that $R^\vee$ is injective. For this we use that

$$\text{Hom}_R(N, R^\vee) = \text{Hom}_R(N, \text{Hom}_\mathbb{Z}(R, \mathbb{Q}/\mathbb{Z})) = N^\vee$$

and the fact that $(-)^\vee$ is an exact functor by Lemma 44.6 □

**Lemma 44.9.** Let $R$ be a ring. The construction above defines a covariant functor $M \mapsto (M \to J(M))$ from the category of $R$-modules to the category of arrows of $R$-modules such that for every module $M$ the output $M \to J(M)$ is an injective map of $M$ into an injective $R$-module $J(M)$.

**Proof.** Follows from the above. □

In particular, for any map of $R$-modules $M \to N$ there is an associated morphism $J(M) \to J(N)$ making the following diagram commute:

$$
\begin{array}{ccc}
M & \longrightarrow & N \\
\downarrow & & \downarrow \\
J(M) & \longrightarrow & J(N)
\end{array}
$$

This is the kind of construction we would like to have in general. In Homology, Section 23 we introduced terminology to express this. Namely, we say this means that the category of $R$-modules has functorial injective embeddings.

**45. Derived categories of modules**

In this section we put some generalities concerning the derived category of modules over a ring.

Let $A$ be a ring. The category of $A$-modules has products and products are exact. The category of $A$-modules has enough injectives by Lemma 44.9 Hence every complex of $A$-modules is quasi-isomorphic to a K-injective complex (Derived Categories, Lemma 29.9). It follows that $D(A)$ has countable products (Derived
Categories, Lemma \[32.2\] and in fact arbitrary products (Injectives, Lemma \[13.4\]). This implies that every inverse system of objects of \(D(A)\) has a derived limit (well defined up to isomorphism), see Derived Categories, Section \[32\].

**Lemma 45.1.** Let \(R \to S\) be a flat ring map. If \(I^\bullet\) is a \(K\)-injective complex of \(S\)-modules, then \(I^\bullet\) is \(K\)-injective as a complex of \(R\)-modules.

**Proof.** This is true because \(\Hom_{K(R)}(M^\bullet, I^\bullet) = \Hom_{K(S)}(M^\bullet \otimes_R S, I^\bullet)\) by Algebra, Lemma \[13.3\] and the fact that tensoring with \(S\) is exact. \(\square\)

**Lemma 45.2.** Let \(R \to S\) be an epimorphism of rings. Let \(I^\bullet\) be a complex of \(S\)-modules. If \(I^\bullet\) is \(K\)-injective as a complex of \(R\)-modules, then \(I^\bullet\) is a \(K\)-injective complex of \(S\)-modules.

**Proof.** This is true because \(\Hom_{K(R)}(N^\bullet, I^\bullet) = \Hom_{K(S)}(N^\bullet, I^\bullet)\) for any complex of \(S\)-modules \(N^\bullet\), see Algebra, Lemma \[104.14\]. \(\square\)

**Lemma 45.3.** Let \(A \to B\) be a ring map. If \(I^\bullet\) is a \(K\)-injective complex of \(A\)-modules, then \(\Hom_A(B, I^\bullet)\) is a \(K\)-injective complex of \(B\)-modules.

**Proof.** This is true because \(\Hom_{K(B)}(N^\bullet, \Hom_A(B, I^\bullet)) = \Hom_{K(A)}(N^\bullet, I^\bullet)\) by Algebra, Lemma \[13.4\]. \(\square\)

### 46. Computing Tor

Let \(R\) be a ring. We denote \(D(R)\) the derived category of the abelian category \(\text{Mod}_R\) of \(R\)-modules. Note that \(\text{Mod}_R\) has enough projectives as every free \(R\)-module is projective. Thus we can define the left derived functors of any additive functor from \(\text{Mod}_R\) to any abelian category.

This implies in particular to the functor \(- \otimes_R M : \text{Mod}_R \to \text{Mod}_R\) whose right derived functors are the Tor functors \(\text{Tor}_i^R(-, M)\), see Algebra, Section \[73\]. There is also a total right derived functor

\[
(46.0.1) \quad - \otimes_R^L M : D^-(-) \longrightarrow D^-(-)
\]

which is denoted \(- \otimes_R^L M\). Its satellites are the Tor modules, i.e., we have

\[
H^{-p}(N \otimes_R^L M) = \text{Tor}_p^R(N, M).
\]

A special situation occurs when we consider the tensor product with an \(R\)-algebra \(A\). In this case we think of \(- \otimes_R A\) as a functor from \(\text{Mod}_R\) to \(\text{Mod}_A\). Hence the total right derived functor

\[
(46.0.2) \quad - \otimes_R^L A : D^-(-) \longrightarrow D^-(-)
\]

which is denoted \(- \otimes_R^L A\). Its satellites are the tor groups, i.e., we have

\[
H^{-p}(N \otimes_R^L A) = \text{Tor}_p^R(N, A).
\]

In particular these Tor groups naturally have the structure of \(A\)-modules.
47. Derived tensor product

We can construct the derived tensor product in greater generality. In fact, it turns out that the boundedness assumptions are not necessary, provided we choose K-flat resolutions. In this section we use Homology, Example 22.2 and Homology, Definition 22.3 to turn a pair of complexes of modules into a double complex and its associated total complex.

**Lemma 47.1.** Let $R$ be a ring. Let $P^\bullet$ be a complex of $R$-modules. Let $\alpha, \beta : L^\bullet \to M^\bullet$ be homotopy equivalent maps of complexes. Then $\alpha$ and $\beta$ induce homotopy equivalent maps

$$\text{Tot}(\alpha \otimes \text{id}_P), \text{Tot}(\beta \otimes \text{id}_P) : \text{Tot}(L^\bullet \otimes_R P^\bullet) \to \text{Tot}(M^\bullet \otimes_R P^\bullet).$$

In particular the construction $L^\bullet \mapsto \text{Tot}(L^\bullet \otimes_R P^\bullet)$ defines an endo-functor of the homotopy category of complexes.

**Proof.** Say $\alpha = \beta + dh + hd$ for some homotopy $h$ defined by $h^n : L^n \to M^{n-1}$. Set

$$H^n = \bigoplus_{a+b=n} h^a \otimes \text{id}_P : \bigoplus_{a+b=n} L^a \otimes_R P^b \to \bigoplus_{a+b=n} M^{a-1} \otimes_R P^b$$

Then a straightforward computation shows that

$$\text{Tot}(\alpha \otimes \text{id}_P) = \text{Tot}(\beta \otimes \text{id}_P) + dH + Hd$$

as maps $\text{Tot}(L^\bullet \otimes_R P^\bullet) \to \text{Tot}(M^\bullet \otimes_R P^\bullet)$. \hfill $\square$

**Lemma 47.2.** Let $R$ be a ring. Let $P^\bullet$ be a complex of $R$-modules. The functor

$$K(\text{Mod}_R) \to K(\text{Mod}_R), \ L^\bullet \mapsto \text{Tot}(L^\bullet \otimes_R P^\bullet)$$

is an exact functor of triangulated categories.

**Proof.** By our definition of the triangulated structure on $K(\text{Mod}_R)$ we have to check that our functor maps a termwise split short exact sequence of complexes to a termwise split short exact sequence of complexes. As the terms of $\text{Tot}(L^\bullet \otimes_R P^\bullet)$ are direct sums of the tensor products $L^a \otimes_R P^b$ this is clear. \hfill $\square$

The following definition will allow us to think intelligently about derived tensor products of unbounded complexes.

**Definition 47.3.** Let $R$ be a ring. A complex $K^\bullet$ is called K-flat if for every acyclic complex $M^\bullet$ the total complex $\text{Tot}(M^\bullet \otimes_R K^\bullet)$ is acyclic.

**Lemma 47.4.** Let $R$ be a ring. Let $K^\bullet$ be a K-flat complex. Then the functor

$$K(\text{Mod}_R) \to K(\text{Mod}_R), \ L^\bullet \mapsto \text{Tot}(L^\bullet \otimes_R K^\bullet)$$

transforms quasi-isomorphisms into quasi-isomorphisms.

**Proof.** Follows from Lemma 47.2 and the fact that quasi-isomorphisms in $K(\text{Mod}_R)$ and $K(\text{Mod}_A)$ are characterized by having acyclic cones. \hfill $\square$

**Lemma 47.5.** Let $R \to R'$ be a ring map. If $K^\bullet$ is a K-flat complex of $R$-modules, then $K^\bullet \otimes_R R'$ is a K-flat complex of $R'$-modules.

**Proof.** Follows from the definitions and the fact that $(K^\bullet \otimes_R R') \otimes_{R'} L^\bullet = K^\bullet \otimes_R L^\bullet$ for any complex $L^\bullet$ of $R'$-modules. \hfill $\square$
Lemma 47.6. Let \( R \) be a ring. If \( K^\bullet, L^\bullet \) are \( K \)-flat complexes of \( R \)-modules, then \( \text{Tot}(K^\bullet \otimes_R L^\bullet) \) is a \( K \)-flat complex of \( R \)-modules.

Proof. Follows from the isomorphism
\[
\text{Tot}(M^\bullet \otimes_R \text{Tot}(K^\bullet \otimes_R L^\bullet)) = \text{Tot}(\text{Tot}(M^\bullet \otimes_R K^\bullet) \otimes_R L^\bullet)
\]
and the definition. \( \square \)

Lemma 47.7. Let \( R \) be a ring. Let \((K^1, K^2, K^3)\) be a distinguished triangle in \( K(\text{Mod}_R) \). If two out of three of \( K^\bullet \) are \( K \)-flat, so is the third.

Proof. Follows from Lemma 47.2 and the fact that in a distinguished triangle in \( K(\text{Mod}_A) \) if two out of three are acyclic, so is the third. \( \square \)

Lemma 47.8. Let \( R \) be a ring. Let \( P^\bullet \) be a bounded above complex of flat \( R \)-modules. Then \( P^\bullet \) is \( K \)-flat.

Proof. Let \( L^\bullet \) be an acyclic complex of \( R \)-modules. Let \( \xi \in H^n(\text{Tot}(L^\bullet \otimes_R P^\bullet)) \). We have to show that \( \xi = 0 \). Since \( \text{Tot}^n(L^\bullet \otimes_R P^\bullet) \) is a direct sum with terms \( L^n \otimes_R P^0 \) we see that \( \xi \) comes from an element in \( H^n(\text{Tot}(\tau_{\leq m} L^\bullet \otimes_R P^\bullet)) \) for some \( m \in \mathbb{Z} \). Since \( \tau_{\leq m} L^\bullet \) is also acyclic we may replace \( L^\bullet \) by \( \tau_{\leq m} L^\bullet \). Hence we may assume that \( L^\bullet \) is bounded above. In this case the spectral sequence of Homology, Lemma 22.6 has
\[
\varepsilon_{p,q}^{1} = H^p(L^\bullet \otimes_R P^q)
\]
which is zero as \( P^q \) is flat and \( L^\bullet \) acyclic. Hence \( H^*(\text{Tot}(L^\bullet \otimes_R P^\bullet)) = 0 \). \( \square \)

In the following lemma by a colimit of a system of complexes we mean the termwise colimit.

Lemma 47.9. Let \( R \) be a ring. Let \( K^1 \to K^2 \to \ldots \) be a system of \( K \)-flat complexes. Then \( \text{colim}_i K^i \) is \( K \)-flat.

Proof. Because we are taking termwise colimits it is clear that
\[
\text{colim}_i \text{Tot}(M^\bullet \otimes_R K^i) = \text{Tot}(M^\bullet \otimes_R \text{colim}_i K^i)
\]
Hence the lemma follows from the fact that filtered colimits are exact. \( \square \)

Lemma 47.10. Let \( R \) be a ring. For any complex \( M^\bullet \) there exists a \( K \)-flat complex \( K^\bullet \) and a quasi-isomorphism \( K^\bullet \to M^\bullet \). Moreover each \( K^n \) is a flat \( R \)-module.

Proof. Let \( P \subset \text{Ob}(\text{Mod}_R) \) be the class of flat \( R \)-modules. By Derived Categories, Lemma 28.1 there exists a system \( K^1 \to K^2 \to \ldots \) and a diagram
\[
\begin{array}{ccc}
K^1 & \to & K^2 \\
\downarrow & & \downarrow \\
\tau_{\leq 1} M^\bullet & \to & \tau_{\leq 2} M^\bullet \\
\end{array}
\]
with the properties (1), (2), (3) listed in that lemma. These properties imply each complex \( K^\bullet \) is a bounded above complex of flat modules. Hence \( K^\bullet \) is \( K \)-flat by Lemma 47.8. The induced map \( \text{colim}_i K^i \to M^\bullet \) is a quasi-isomorphism by construction. The complex \( \text{colim}_i K^i \) is \( K \)-flat by Lemma 47.9. The final assertion of the lemma is true because the colimit of a system of flat modules is flat, see Algebra, Lemma 38.2. \( \square \)
Remark 47.11. In fact, we can do better than Lemma 47.10. Namely, we can find a quasi-isomorphism $P^\bullet \to M^\bullet$ where $P^\bullet$ is a complex of $A$-modules endowed with a filtration

$$0 = F_{-1}P^\bullet \subset F_0P^\bullet \subset F_1P^\bullet \subset \ldots \subset P^\bullet$$

by subcomplexes such that

1. $P^\bullet = \bigcup F_pP^\bullet$,
2. the inclusions $F_iP^\bullet \to F_{i+1}P^\bullet$ are termwise split injections,
3. the quotients $F_{i+1}P^\bullet/F_iP^\bullet$ are isomorphic to direct sums of shifts $A[k]$ (as complexes, so differentials are zero).

This was shown in Differential Graded Algebra, Lemma 13.4. Moreover, given such a complex we obtain a distinguished triangle

$$\bigoplus F_iP^\bullet \to \bigoplus F_iP^\bullet \to M^\bullet \to \bigoplus F_iP^\bullet[1]$$

in $D(A)$. Using this we can sometimes reduce statements about general complexes to statements about $A[k]$ (this of course only works if the statement is preserved under taking direct sums). More precisely, let $T$ be a property of objects of $D(A)$.

Suppose that

1. if $K_i \in D(A)$, $i \in I$ is a family of objects with $T(K_i)$ for all $i \in I$, then $T(\bigoplus K_i)$,
2. if $K \to L \to M \to K[1]$ is a distinguished triangle and $T$ holds for two, then $T$ holds for the third object,
3. $T(A[k])$ holds for all $k$.

Then $T$ holds for all objects of $D(A)$.

Lemma 47.12. Let $R$ be a ring. Let $\alpha : P^\bullet \to Q^\bullet$ be a quasi-isomorphism of K-flat complexes of $R$-modules. For every complex $L^\bullet$ of $R$-modules the induced map

$$\text{Tot}(id_L \otimes \alpha) : \text{Tot}(L^\bullet \otimes_R P^\bullet) \to \text{Tot}(L^\bullet \otimes_R Q^\bullet)$$

is a quasi-isomorphism.

Proof. Choose a quasi-isomorphism $K^\bullet \to L^\bullet$ with $K^\bullet$ a K-flat complex, see Lemma 47.10. Consider the commutative diagram

$$\begin{array}{ccc}
\text{Tot}(K^\bullet \otimes_R P^\bullet) & \longrightarrow & \text{Tot}(K^\bullet \otimes_R Q^\bullet) \\
\downarrow & & \downarrow \\
\text{Tot}(L^\bullet \otimes_R P^\bullet) & \longrightarrow & \text{Tot}(L^\bullet \otimes_R Q^\bullet)
\end{array}$$

The result follows as by Lemma 47.4 the vertical arrows and the top horizontal arrow are quasi-isomorphisms. \qed

Let $R$ be a ring. Let $M^\bullet$ be an object of $D(R)$. Choose a K-flat resolution $K^\bullet \to M^\bullet$, see Lemma 47.10. By Lemmas 47.1 and 47.2 we obtain an exact functor of triangulated categories

$$K(\text{Mod}_R) \longrightarrow K(\text{Mod}_R), \quad L^\bullet \longmapsto \text{Tot}(L^\bullet \otimes_R K^\bullet)$$

By Lemma 47.4 this functor induces a functor $D(R) \to D(R)$ simply because $D(R)$ is the localization of $K(\text{Mod}_R)$ at quasi-isomorphism. By Lemma 47.12 the resulting functor (up to isomorphism) does not depend on the choice of the K-flat resolution.
Definition 47.13. Let $R$ be a ring. Let $M^\bullet$ be an object of $D(R)$. The derived tensor product
\[ - \otimes^L_R M^\bullet : D(R) \to D(R) \]
is the exact functor of triangulated categories described above. This functor extends the functor (46.0.1). It is clear from our explicit constructions that there is a canonical isomorphism
\[ M^\bullet \otimes^L_R L^\bullet \cong L^\bullet \otimes^L_R M^\bullet \]
whenever both $L^\bullet$ and $M^\bullet$ are in $D(R)$. Hence when we write $M^\bullet \otimes^L_R L^\bullet$ we will usually be agnostic about which variable we are using to define the derived tensor product with.

48. Derived change of rings

Let $R \to A$ be a ring map. Let $N$ be an $A$-module. We can also use K-flat resolutions to define a functor
\[ - \otimes^L_R N : D(R) \to D(A) \]
which is the left derived functor of the functor $\text{Mod}_R \to \text{Mod}_A$, $M \mapsto M \otimes_R N$. In particular, taking $N = A$ we obtain a derived base change functor
\[ - \otimes^L_R A : D(R) \to D(A) \]
esting the functor (46.0.2). Namely, for every complex of $R$-modules $M^\bullet$ we can choose a K-flat resolution $K^\bullet \to M^\bullet$ and set $M^\bullet \otimes^L_R N = K^\bullet \otimes^L_R N$. You can use Lemmas 47.10 and 47.12 to see that this is well defined. However, to cross all the t's and dot all the i's it is perhaps more convenient to use some general theory.

Lemma 48.1. The construction above is independent of choices and defines an exact functor of triangulated categories $- \otimes^L_R N : D(R) \to D(A)$. There is a functorial isomorphism
\[ E \otimes^L_R N = (E \otimes^L_R A) \otimes^L_A N \]
for $E$ in $D(R)$.

Proof. To prove the existence of the derived functor $- \otimes^L_R N$ we use the general theory developed in Derived Categories, Section 15. Set $\mathcal{D} = K(\text{Mod}_R)$ and $\mathcal{D}' = D(A)$. Let us write $F : \mathcal{D} \to \mathcal{D}'$ the exact functor of triangulated categories defined by the rule $F(M^\bullet) = M^\bullet \otimes_R N$. We let $S$ be the set of quasi-isomorphisms in $\mathcal{D} = K(\text{Mod}_R)$. This gives a situation as in Derived Categories, Situation 15.1 so that Derived Categories, Definition 15.2 applies. We claim that $LF$ is everywhere defined. This follows from Derived Categories, Lemma 15.15 with $\mathcal{P} \subset \text{Ob}(\mathcal{D})$ the collection of K-flat complexes: (1) follows from Lemma 47.10 and (2) follows from Lemma 47.12. Thus we obtain a derived functor
\[ LF : D(R) = S^{-1}\mathcal{D} \to \mathcal{D}' = D(A) \]
see Derived Categories, Equation (15.9.1). Finally, Derived Categories, Lemma 15.15 guarantees that $LF(K^\bullet) = F(K^\bullet) = K^\bullet \otimes_R N$ when $K^\bullet$ is K-flat, i.e., $LF$ is indeed computed in the way described above. Moreover, by Lemma 47.5 the complex $K^\bullet \otimes_R A$ is a K-flat complex of $A$-modules. Hence
\[ (K^\bullet \otimes^L_R A) \otimes^L_A N = (K^\bullet \otimes_R A) \otimes_A N = K^\bullet \otimes_A N = K^\bullet \otimes^L_A N \]
which proves the final statement of the lemma. □
Remark 48.2 (Warning). Let $R \to A$ be a ring map, and let $N$ and $N'$ be $A$-modules. Denote $N_R$ and $N'_R$ the restriction of $N$ and $N'$ to $R$-modules, see Algebra, Section 13. In this situation, the objects $N_R \otimes_R N'$ and $N \otimes_R N'_R$ of $D(A)$ are in general not isomorphic! In other words, one has to pay careful attention as to which of the two sides is being used to provide the $A$-module structure.

For a specific example, set $R = k[x,y]$, $A = R/(xy)$, $N = R/(x)$ and $N' = A = R/(xy)$. The resolution $0 \to R \xrightarrow{xy} R \to N_R' \to 0$ shows that $N \otimes_R N'_R = N[1] \oplus N$ in $D(A)$. The resolution $0 \to R \xrightarrow{x} R \to N_R \to 0$ shows that $N_R \otimes_R N'$ is represented by the complex $A \to A$. To see these two complexes are not isomorphic, one can show that the second complex is not isomorphic in $D(A)$ to the direct sum of its cohomology groups, or one can show that the first complex is not a perfect object of $D(A)$ whereas the second one is. Some details omitted.

Lemma 48.3. Let $A \to B \to C$ be ring maps. Let $M$ be an $A$-module, $N$ a $B$-module, and $K$ a $C$-module. Then

$$(M \otimes^L_A N) \otimes^L_B K = (M \otimes^L_A K) \otimes^L_B (N \otimes^L_B C) = (M \otimes^L_A C) \otimes^L_B (N \otimes^L_B K)$$

in $D(C)$.

Proof. Let $M^* \to M$ be a free resolution of $M$ as an $A$-module and let $N^* \to N$ be a free resolution of $N$ as a $B$-module. We have

$$M \otimes^L_A N = M^* \otimes_A N$$
$$= M^* \otimes_A B \otimes_B N$$
$$\leftarrow \text{Tot}((M^* \otimes_A B) \otimes_B N^*)$$
$$= \text{Tot}(M^* \otimes_A N^*)$$

Here the arrow is a quasi-isomorphism in $D(B)$ as $M^* \otimes_A B$ is a bounded above complex of free $B$-modules, hence K-flat (Lemma 47.8) and hence Lemma 47.4 applies. Now the complex $\text{Tot}(M^* \otimes_A N^*)$ is a complex of free $B$-modules hence we see that

$$(M \otimes^L_A N) \otimes^L_B K = \text{Tot}(M^* \otimes_A N^*) \otimes_B K = \text{Tot}(M^* \otimes_A N^* \otimes_B K)$$

On the other hand,

$$M \otimes^L_A K = M^* \otimes_A K \quad \text{and} \quad N \otimes^L_B C = N^* \otimes_B C$$

and the second is a bounded above complex of free $C$-modules hence we see that

$$(M \otimes^L_A K) \otimes^L_C (N \otimes^L_B C) = \text{Tot}((M^* \otimes_A K) \otimes_C (N^* \otimes_B C)) = \text{Tot}(M^* \otimes_A N^* \otimes_B K)$$

which proves the first equality of the statement of the lemma. To prove the second we use that

$$M \otimes^L_A C = M^* \otimes_A C \quad \text{and} \quad N \otimes^L_B K = N^* \otimes_B K$$

and the first is a bounded above complex of free $C$-modules so that

$$(M \otimes^L_A C) \otimes^L_C (N \otimes^L_B K) = \text{Tot}((M^* \otimes_A C) \otimes_C (N^* \otimes_B K)) = \text{Tot}(M^* \otimes_A N^* \otimes_B K)$$

as before. □
49. Tor independence

Consider a commutative diagram

\[
\begin{array}{ccc}
A & \\ & \downarrow \ & \\ R & \rightarrow & R'
\end{array}
\]

of rings. Given an object \( K \) of \( D(A) \) we can consider its restriction to an object of \( D(R) \). We can then consider take the derived change of rings of \( K \) to an object of \( D(A') \) and \( D(R') \). We claim there is a functorial comparison map

\[
(49.0.1) \quad K \otimes_{R}^{L} R' \rightarrow K \otimes_{A}^{L} A'
\]

in \( D(R') \). To construct this comparison map choose a \( K \)-flat complex \( K^\bullet \) of \( A \)-modules representing \( K \). Next, choose a quasi-isomorphism \( E^\bullet \rightarrow K^\bullet \) where \( E^\bullet \) is a \( K \)-flat complex of \( R \)-modules. The map above is the map

\[
K \otimes_{R}^{L} R' = E^\bullet \otimes_{R} R' \rightarrow K^\bullet \otimes_{A} A' = K \otimes_{A}^{L} A'
\]

In general there is no chance that this map is an isomorphism. However, we often encounter the situation where the diagram above is a “base change” diagram of rings, i.e., \( A' = A \otimes_{R} R' \). In this situation, for any \( A \)-module \( M \) we have \( M \otimes_{A} A' = M \otimes_{R} R' \). Thus \( - \otimes_{R} R' \) is equal to \( - \otimes_{A} A' \) as a functor \( \text{Mod}_A \rightarrow \text{Mod}_{A'} \). In general this equality does not extend to derived tensor products. In other words, the comparison map is not an isomorphism. A simple example is to take \( R = k[x] \), \( A = R' = A = k[x]/(x) = k \) and \( K^\bullet = A[0] \). Clearly, a necessary condition is that Tor\(_{p}^{R}(A, R') = 0 \) for all \( p > 0 \).

**Definition 49.1.** Let \( R \) be a ring. Let \( A, B \) be \( R \)-algebras. We say \( A \) and \( B \) are Tor independent over \( R \) if Tor\(_{p}^{R}(A, B) = 0 \) for all \( p > 0 \).

**Lemma 49.2.** The comparison map \((49.0.1)\) is an isomorphism if \( A' = A \otimes_{R} R' \) and \( A \) and \( R' \) are Tor independent over \( R \).

**Proof.** To prove this we choose a free resolution \( F^\bullet \rightarrow R' \) of \( R' \) as an \( R \)-module. Because \( A \) and \( R' \) are Tor independent over \( R \) we see that \( F^\bullet \otimes_{R} A \) is a free \( A \)-module resolution of \( A' \) over \( A \). By our general construction of the derived tensor product above we see that

\[
K^\bullet \otimes_{A} A' \cong \text{Tot}(K^\bullet \otimes_{A}(F^\bullet \otimes_{R} A)) = \text{Tot}(K^\bullet \otimes_{R} F^\bullet) \cong \text{Tot}(E^\bullet \otimes_{R} F^\bullet) \cong E^\bullet \otimes_{R} R'
\]

as desired. □

**Lemma 49.3.** Consider a commutative diagram of rings

\[
\begin{array}{ccc}
A' & \leftarrow & R' \rightarrow B' \\
& \uparrow \ & \\
A & \leftarrow & R \rightarrow B
\end{array}
\]

Assume that \( R' \) is flat over \( R \) and \( A' \) is flat over \( A \otimes_{R} R' \) and \( B' \) is flat over \( R' \otimes_{R} B \). Then

\[
\text{Tor}_{p}^{R}(A, B) \otimes_{(A \otimes_{R} B)} (A' \otimes_{R'} B') = \text{Tor}_{p}^{R}(A', B')
\]
Proof. By Algebra, Section 74 there are canonical maps

\[ \text{Tor}_i^R(A, B) \rightarrow \text{Tor}_i^R(A \otimes_R R', B \otimes_R R') \rightarrow \text{Tor}_i^R(A', B') \]

These induce a map from left to right in the formula of the lemma.

Take a free resolution \( F_* \rightarrow A \) of \( A \) as an \( R \)-module. Then we see that \( F_* \otimes_R R' \) is a resolution of \( A \otimes_R R' \). Hence \( \text{Tor}_i^R(A \otimes_R R', B \otimes_R R') \) is computed by \( F_* \otimes_R B \otimes_R R' \).

By our assumption that \( R' \) is flat over \( R \), this computes \( \text{Tor}_i^R(A, B) \otimes_R R' \). Thus \( \text{Tor}_i^R(A \otimes_R R', B \otimes_R R') = \text{Tor}_i^R(A, B) \otimes_R R' \) (uses only flatness of \( R' \) over \( R \)).

By Lazard’s theorem (Algebra, Theorem 79.4) we can write \( \text{Tor}_i^R(A, B) \otimes_R R' \) as one can see by writing everything out in terms of bases. Taking the colimit we get the result of the lemma. \( \square \)

Lemma 49.4. Let \( R \) be a ring. Let \( A, B \) be \( R \)-algebras. The following are equivalent:

1. \( A \) and \( B \) are Tor independent over \( R \).
2. For every pair of primes \( p \subset A \) and \( q \subset B \) lying over the same prime \( r \subset R \) the rings \( A_p \) and \( B_q \) are Tor independent over \( R_r \), and
3. For every prime \( s \) of \( A \otimes_R B \) the module

\[ \text{Tor}_i^R(A, B)_s = \text{Tor}_i^R(A_p, B_q)_s \]

(where \( p = A \cap s \), \( q = B \cap s \) and \( r = R \cap s \)) is zero.

Proof. Let \( s \) be a prime of \( A \otimes_R B \) as in (3). The equality

\[ \text{Tor}_i^R(A, B)_s = \text{Tor}_i^R(A_p, B_q)_s \]

where \( p = A \cap s \), \( q = B \cap s \) and \( r = R \cap s \) follows from Lemma 49.3. Hence (2) implies (3). Since we can test the vanishing of modules by localizing at primes (Algebra, Lemma 23.1) we conclude that (3) implies (1). For (1) \( \Rightarrow \) (2) we use that

\[ \text{Tor}_i^R(A_p, B_q) = \text{Tor}_i^R(A, B) \otimes_{(A \otimes_R B)} (A_p \otimes_R, B_q) \]

again by Lemma 49.3. \( \square \)

50. Spectral sequences for Tor

In this section we collect various spectral sequences that come up when considering the Tor functors.

Example 50.1. Let \( R \) be a ring. Let \( K_* \) be a bounded above chain complex of \( R \)-modules. Let \( M \) be an \( R \)-module. Then there is a spectral sequence with \( E_2 \)-page

\[ \text{Tor}_i^R(H_j(K_*), M) \Rightarrow H_{i+j}(K_* \otimes_R^L M) \]

and another spectral sequence with \( E_1 \)-page

\[ \text{Tor}_i^R(K_j, M) \Rightarrow H_{i+j}(K_* \otimes_R^L M) \]

This follows from the dual to Derived Categories, Lemma 21.3.
Example 50.2. Let \( R \rightarrow S \) be a ring map. Let \( M \) be an \( R \)-module and let \( N \) be an \( S \)-module. Then there is a spectral sequence
\[
\text{Tor}_n^S(\text{Tor}_m^R(M, S), N) \Rightarrow \text{Tor}_{n+m}^R(M, N).
\]
To construct it choose a \( R \)-free resolution \( P_\bullet \) of \( M \). Then we have
\[
M \otimes_R N = P_\bullet \otimes_R N = (P_\bullet \otimes_R S) \otimes_S N
\]
and then apply the first spectral sequence of Example 50.1.

Example 50.3. Consider a commutative diagram
\[
\begin{array}{ccc}
B & \rightarrow & B' = B \otimes_A A' \\
\uparrow & & \uparrow \\
A & \rightarrow & A'
\end{array}
\]
and \( B \)-modules \( M, N \). Set \( M' = M \otimes_A A' = M \otimes_B B' \) and \( N' = N \otimes_A A' = N \otimes_B B' \). Assume that \( A \rightarrow B \) is flat and that \( M' \) and \( N \) are \( A \)-flat. Then there is a spectral sequence
\[
\text{Tor}^A_i(\text{Tor}^R_{ij}(M, N), A') \Rightarrow \text{Tor}^{R'}_{i+j}(M', N')
\]
The reason is as follows. Choose free resolution \( F_\bullet \rightarrow M \) as a \( B \)-module. As \( B \) and \( M \) are \( A \)-flat we see that \( F_\bullet \otimes_A A' \) is a free \( B' \)-resolution of \( M' \). Hence we see that the groups \( \text{Tor}^A_i(M', N') \) are computed by the complex
\[
(F_\bullet \otimes_A A') \otimes_B N' = (F_\bullet \otimes_B N) \otimes_A A' = (F_\bullet \otimes_B N) \otimes_A L_A A'
\]
the last equality because \( F_\bullet \otimes_B N \) is a complex of flat \( A \)-modules as \( N \) is flat over \( A \). Hence we obtain the spectral sequence by applying the spectral sequence of Example 50.1.

Example 50.4. Let \( K^\bullet, L^\bullet \) be objects of \( D^-(R) \). Then there are spectral sequences
\[
E_2^{p,q} = H^p(K^\bullet \otimes_R^L H^q(L^\bullet)) \Rightarrow H^{p+q}(K^\bullet \otimes_R^L L^\bullet)
\]
with \( d_2^{p,q} : E_2^{p,q} \rightarrow E_2^{p+2,q-1} \) and
\[
H^q(H^p(K^\bullet) \otimes_R^L L^\bullet) \Rightarrow H^{p+q}(K^\bullet \otimes_R^L L^\bullet)
\]
After replacing \( K^\bullet \) and \( L^\bullet \) by bounded above complexes of projectives, these spectral sequences are simply the two spectral sequences for computing the cohomology of \( \text{Tot}(K^\bullet \otimes L^\bullet) \) discussed in Homology, Section 46.

51. Products and Tor

The simplest example of the product maps comes from the following situation. Suppose that \( K^\bullet, L^\bullet \in D(R) \) with one of them contained in \( D^-(R) \). Then there are maps
\[
H^i(K^\bullet) \otimes_R H^j(L^\bullet) \rightarrow H^{i+j}(K^\bullet \otimes_R^L L^\bullet)
\]
Namely, to define these maps we may assume that one of \( K^\bullet, L^\bullet \) is a bounded above complex of projective \( R \)-modules. In that case \( K^\bullet \otimes_R^L L^\bullet \) is represented by the complex \( \text{Tot}(K^\bullet \otimes_R L^\bullet) \), see Section 46. Next, suppose that \( \xi \in H^i(K^\bullet) \) and \( \zeta \in H^j(L^\bullet) \). Choose \( k \in \text{Ker}(K^i \rightarrow K^{i+1}) \) and \( l \in \text{Ker}(L^j \rightarrow L^{j+1}) \) representing \( \xi \) and \( \zeta \). Then we set
\[
\xi \cup \zeta = \text{class of } k \otimes l \text{ in } H^{i+j}(\text{Tot}(K^\bullet \otimes_R L^\bullet)).
\]
This make sense because the formula (see Homology, Definition \[22.3\]) for the differential \(d\) on the total complex shows that \(k \otimes l\) is a cocycle. Moreover, if \(k' = d_K(k'')\) for some \(k'' \in K^{i-1}\), then \(k' \otimes l = d(k'' \otimes l)\) because \(l\) is a cocycle. Similarly, altering the choice of \(l\) representing \(\zeta\) does not change the class of \(k \otimes l\). It is equally clear that \(\cup\) is bilinear, and hence to a general element of \(H^{i}(K^{\bullet}) \otimes_R H^{j}(L^{\bullet})\) we assign

\[
\sum \xi_i \otimes \zeta_i \mapsto \sum \xi_i \cup \zeta_i
\]
in \(H^{i+j}(\text{Tot}(K^{\bullet} \otimes_R L^{\bullet}))\).

Let \(R \to A\) be a ring map. Let \(K^{\bullet}, L^{\bullet} \in D^-(R)\). Then we have a canonical identification

\[(51.0.2) \quad (K^{\bullet} \otimes^L_R A) \otimes^L_A (L^{\bullet} \otimes^L_R A) = (K^{\bullet} \otimes^L_R L^{\bullet}) \otimes^L_R A\]
in \(D(A)\). It is constructed as follows. First, choose projective resolutions \(P^{\bullet} \to K^{\bullet}\) and \(Q^{\bullet} \to L^{\bullet}\) over \(R\). Then the left hand side is represented by the complex \(\text{Tot}((P^{\bullet} \otimes^L_R A) \otimes_A (Q^{\bullet} \otimes^L_R A))\) and the right hand side by the complex \(\text{Tot}(P^{\bullet} \otimes_R Q^{\bullet}) \otimes_R A\). These complexes are canonically isomorphic. Thus the construction above induces products

\[
\text{Tor}^R_n(K^{\bullet}, A) \otimes_A \text{Tor}^R_m(L^{\bullet}, A) \to \text{Tor}^R_{n+m}(K^{\bullet} \otimes_R L^{\bullet}, A)
\]
which are occasionally useful.

Let \(M, N\) be \(R\)-modules. Using the general construction above and functoriality of \(\text{Tor}\) we obtain canonical maps

\[(51.0.3) \quad \text{Tor}^R_n(M, A) \otimes_A \text{Tor}^R_m(N, A) \to \text{Tor}^R_{n+m}(M \otimes_R N, A)\]

Here is a direct construction using projective resolutions. First, choose projective resolutions

\[P_\bullet \to M, \quad Q_\bullet \to N, \quad T_\bullet \to M \otimes_R N\]
over \(R\). We have \(H_0(\text{Tot}(P_\bullet \otimes_R Q_\bullet)) = M \otimes_R N\) by right exactness of \(\otimes_R\). Hence Derived Categories, Lemmas \[19.6\] and \[19.7\] guarantee the existence and uniqueness of a map of complexes \(\mu : \text{Tot}(P_\bullet \otimes_R Q_\bullet) \to T_\bullet\) such that \(H_0(\mu) = \text{id}_{M \otimes_R N}\). This induces a canonical map

\[
(M \otimes^L_R A) \otimes^L_A (N \otimes^L_R A) = \text{Tot}((P_\bullet \otimes_R A) \otimes_A (Q_\bullet \otimes_R A))
\]

\[
= \text{Tot}(P_\bullet \otimes_R Q_\bullet) \otimes_R A
\]

\[
\to T_\bullet \otimes_R A
\]

\[
= (M \otimes_R N) \otimes^L_R A
\]
in \(D(A)\). Hence the products \((51.0.3)\) above are constructed using \((51.0.1)\) over \(A\) to construct

\[
\text{Tor}^R_n(M, A) \otimes_A \text{Tor}^R_m(N, A) \to H^{-n-m}((M \otimes^L_R A) \otimes^L_A (N \otimes^L_R A))
\]
and then composing by the displayed map above to end up in \(\text{Tor}^R_{n+m}(M \otimes_R N, A)\).

An interesting special case of the above occurs when \(M = N = B\) where \(B\) is an \(R\)-algebra. In this case we obtain maps

\[
\text{Tor}^R_n(B, A) \otimes_A \text{Tor}^R_m(B, A) \to \text{Tor}^R(B \otimes_R B, A) \to \text{Tor}^R(B, A)
\]
the second arrow being induced by the multiplication map \(B \otimes_R B \to B\) via functoriality for \(\text{Tor}\). In other words we obtain an \(A\)-algebra structure on \(\text{Tor}^R(B, A)\).
This algebra structure has many intriguing properties (associativity, graded commutative, \(B\)-algebra structure, divided powers in some case, etc) which we will discuss elsewhere (insert future reference here).

**Lemma 51.1.** Let \(R\) be a ring. Let \(A, B, C\) be \(R\)-algebras and let \(B \to C\) be an \(R\)-algebra map. Then the induced map

\[
\operatorname{Tor}_R^*(B, A) \longrightarrow \operatorname{Tor}_R^*(C, A)
\]

is an \(A\)-algebra homomorphism.

**Proof.** Omitted. Hint: You can prove this by working through the definitions, writing all the complexes explicitly. \(\square\)

**52. Pseudo-coherent modules**

Suppose that \(R\) is a ring. Recall that an \(R\)-module \(M\) is of finite type if there exists a surjection \(R^{\oplus a} \to M\) and of finite presentation if there exists a presentation \(R^{\oplus a_1} \to R^{\oplus a_0} \to M \to 0\). Similarly, we can consider those \(R\)-modules for which there exists a length \(n\) resolution

\[
R^{\oplus a_n} \to R^{\oplus a_{n-1}} \to \ldots \to R^{\oplus a_0} \to M \to 0
\]

by finite free \(R\)-modules. A module is called pseudo-coherent if we can find such a resolution for every \(n\). Here is the formal definition.

**Definition 52.1.** Let \(R\) be a ring. Denote \(D(R)\) its derived category. Let \(m \in \mathbb{Z}\).

1. An object \(K^*\) of \(D(R)\) is \(m\)-pseudo-coherent if there exists a bounded complex \(E^*\) of finite free \(R\)-modules and a morphism \(\alpha : E^* \to K^*\) such that \(H^i(\alpha)\) is an isomorphism for \(i > m\) and \(H^m(\alpha)\) is surjective.
2. An object \(K^*\) of \(D(R)\) is pseudo-coherent if it is quasi-isomorphic to a bounded above complex of finite free \(R\)-modules.
3. An \(R\)-module \(M\) is called \(m\)-pseudo-coherent if \(M[0]\) is an \(m\)-pseudo-coherent object of \(D(R)\).
4. An \(R\)-module \(M\) is called pseudo-coherent

\[\text{if } M[0] \text{ is a pseudo-coherent object of } D(R).\]

As usual we apply this terminology also to complexes of \(R\)-modules. Since any morphism \(E^* \to K^*\) in \(D(R)\) is represented by an actual map of complexes, see Derived Categories, Lemma \[19.8\] there is no ambiguity. It turns out that \(K^*\) is pseudo-coherent if and only if \(K^*\) is \(m\)-pseudo-coherent for all \(m \in \mathbb{Z}\), see Lemma \[52.5\]. Also, if the ring is Noetherian the condition can be understood as a finite generation condition on the cohomology, see Lemma \[52.16\]. Let us first relate this to the informal discussion above.

**Lemma 52.2.** Let \(R\) be a ring and \(m \in \mathbb{Z}\). Let \((K^*, L^*, M^*, f, g, h)\) be a distinguished triangle in \(D(R)\).

1. If \(K^*\) is \((m+1)\)-pseudo-coherent and \(L^*\) is \(m\)-pseudo-coherent then \(M^*\) is \(m\)-pseudo-coherent.
2. If \(K^*, M^*\) are \(m\)-pseudo-coherent, then \(L^*\) is \(m\)-pseudo-coherent.
3. If \(L^*\) is \((m+1)\)-pseudo-coherent and \(M^*\) is \(m\)-pseudo-coherent, then \(K^*\) is \((m+1)\)-pseudo-coherent.

\[6\]This clashes with what is meant by a pseudo-coherent module in \[Bou61\].
Proof. Proof of (1). Choose $\alpha : P^\bullet \to K^\bullet$ with $P^\bullet$ a bounded complex of finite free modules such that $H^i(\alpha)$ is an isomorphism for $i > m + 1$ and surjective for $i = m + 1$. We may replace $P^\bullet$ by $\sigma_{\geq m+1}P^\bullet$ and hence we may assume that $P^i = 0$ for $i < m + 1$. Choose $\beta : E^\bullet \to L^\bullet$ with $E^\bullet$ a bounded complex of finite free modules such that $H^i(\beta)$ is an isomorphism for $i > m$ and surjective for $i = m$. By Derived Categories, Lemma 19.11 we can find a map $\alpha : P^\bullet \to E^\bullet$ such that the diagram

$$
\begin{array}{ccc}
K^\bullet & \longrightarrow & L^\bullet \\
\downarrow & & \downarrow \\
P^\bullet & \xrightarrow{\alpha} & E^\bullet
\end{array}
$$

is commutative in $D(R)$. The cone $C(\alpha)^\bullet$ is a bounded complex of finite free $R$-modules, and the commutativity of the diagram implies that there exists a morphism of distinguished triangles

$$(P^\bullet, E^\bullet, C(\alpha)^\bullet) \longrightarrow (K^\bullet, L^\bullet, M^\bullet).$$

It follows from the induced map on long exact cohomology sequences and Homology, Lemmas 5.19 and 5.20 that $C(\alpha)^\bullet \to M^\bullet$ induces an isomorphism on cohomology in degrees $> m$ and a surjection in degree $m$. Hence $M^\bullet$ is $m$-pseudo-coherent.

Assertions (2) and (3) follow from (1) by rotating the distinguished triangle. □

Lemma 52.3. Let $R$ be a ring. Let $K^\bullet$ be a complex of $R$-modules. Let $m \in \mathbb{Z}$.

1. If $K^\bullet$ is $m$-pseudo-coherent and $H^i(K^\bullet) = 0$ for $i > m$, then $H^m(K^\bullet)$ is a finite type $R$-module.

2. If $K^\bullet$ is $m$-pseudo-coherent and $H^i(K^\bullet) = 0$ for $i > m + 1$, then $H^{m+1}(K^\bullet)$ is a finitely presented $R$-module.

Proof. Proof of (1). Choose a bounded complex $E^\bullet$ of finite projective $R$-modules and a map $\alpha : E^\bullet \to K^\bullet$ which induces an isomorphism on cohomology in degrees $> m$ and a surjection in degree $m$. It is clear that it suffices to prove the result for $E^\bullet$. Let $n$ be the largest integer such that $E^n \neq 0$. If $n = m$, then the result is clear. If $n > m$, then $E^{n-1} \to E^n$ is surjective as $H^n(E^\bullet) = 0$. As $E^n$ is finite projective we see that $E^n = E' \oplus E^m$. Hence it suffices to prove the result for the complex $(E')^\bullet$ which is the same as $E^\bullet$ except has $E'$ in degree $n - 1$ and $0$ in degree $n$. We win by induction on $n$.

Proof of (2). Choose a bounded complex $E^\bullet$ of finite projective $R$-modules and a map $\alpha : E^\bullet \to K^\bullet$ which induces an isomorphism on cohomology in degrees $> m$ and a surjection in degree $m$. As in the proof of (1) we can reduce to the case that $E^i = 0$ for $i > m + 1$. Then we see that $H^{m+1}(K^\bullet) \cong H^{m+1}(E^\bullet) = \text{Coker}(E^m \to E^{m+1})$ which is of finite presentation. □

Lemma 52.4. Let $R$ be a ring. Let $M$ be an $R$-module. Then

1. $M$ is $0$-pseudo-coherent if and only if $M$ is a finite type $R$-module,

2. $M$ is $(-1)$-pseudo-coherent if and only if $M$ is a finitely presented $R$-module,

3. $M$ is $(-d)$-pseudo-coherent if and only if there exists a resolution

$$R^{\oplus a_d} \to R^{\oplus a_{d-1}} \to \ldots \to R^{\oplus a_0} \to M \to 0$$

of length $d$, and
(4) $M$ is pseudo-coherent if and only if there exists an infinite resolution
\[ \ldots \to R^a_{(2)} \to R^a_0 \to M \to 0 \]
by finite free $R$-modules.

**Proof.** If $M$ is of finite type (resp. of finite presentation), then $M$ is 0-pseudo-coherent (resp. ($-1$)-pseudo-coherent) as follows from the discussion preceding Definition 52.1. Conversely, if $M$ is 0-pseudo-coherent, then $M = H^0(M[0])$ is of finite type by Lemma 52.3. If $M$ is ($-1$)-pseudo-coherent, then it is 0-pseudo-coherent hence of finite type. Choose a surjection $R^a \to M$ and denote $K = \text{Ker}(R^a \to M)$. By Lemma 52.2 we see that $K$ is 0-pseudo-coherent, hence of finite type, whence $M$ is of finite presentation.

To prove the third and fourth statement use induction and an argument similar to the above (details omitted). $\square$

**Lemma 52.5.** Let $R$ be a ring. Let $K^\bullet$ be a complex of $R$-modules. The following are equivalent

1. $K^\bullet$ is pseudo-coherent,
2. $K^\bullet$ is $m$-pseudo-coherent for every $m \in \mathbb{Z}$, and
3. $K^\bullet$ is quasi-isomorphic to a bounded above complex of finite projective $R$-modules.

**Proof.** We see that (1) $\Rightarrow$ (3) as a finite free module is a finite projective $R$-module. Conversely, suppose $P^\bullet$ is a bounded above complex of finite projective $R$-modules. Say $P^i = 0$ for $i > n_0$. We choose a direct sum decompositions $F^{n_0} = P^{n_0} \oplus C^{n_0}$ with $F^{n_0}$ a finite free $R$-module, and inductively

\[ F^{n-1} = P^{n-1} \oplus C^n \oplus C^{n-1} \]

for $n \leq n_0$ with $F^{n_0}$ a finite free $R$-module. As a complex $F^\bullet$ has maps $F^{n-1} \to F^n$ which agree with $P^{n-1} \to P^n$, induce the identity $C^n \to C^n$, and are zero on $C^{n-1}$. The map $F^\bullet \to P^\bullet$ is a quasi-isomorphism (even a homotopy equivalence) and hence (3) implies (1).

Assume (1). Let $E^\bullet$ be a bounded above complex of finite free $R$-modules and let $E^\bullet \to K^\bullet$ be a quasi-isomorphism. Then the induced maps $\sigma_{\geq m}E^\bullet \to K^\bullet$ from the stupid truncation of $E^\bullet$ to $K^\bullet$ show that $K^\bullet$ is $m$-pseudo-coherent. Hence (1) implies (2).

Assume (2). We first apply (2) for $n = 0$ to obtain a map of complexes $\alpha : F^\bullet \to K^\bullet$ where $F^\bullet$ is bounded above, consists of finite free $R$-modules and such that $H^i(\alpha)$ is an isomorphism for $i > 0$ and surjective for $i = 0$. Note that these conditions remain satisfied after replacing $F^\bullet$ by $\sigma_{\geq 0}F^\bullet$. Picture

\[ F^0 \to F^1 \to \ldots \]
\[ K^{-1} \to K^0 \to K^1 \to \ldots \]

By induction on $n < 0$ we are going to extend $F^\bullet$ to a complex $F^n \to F^{n+1} \to \ldots \to F^{-1} \to F^0 \to \ldots$ of finite free $R$-modules and extend $\alpha$ such that $H^i(\alpha)$ is an isomorphism for $i > n$ and surjective for $i = n$. By shifting it suffices to prove the induction step for $n = -1$, i.e., it suffices to extend the diagram above by adding
Moreover, these maps define a morphism of distinguished triangles.

Let \( C^\bullet \) be the cone on \( \alpha \) (Derived Categories, Definition 9.1). The long exact sequence of cohomology shows that \( H^i(C^\bullet) = 0 \) for \( i \geq 0 \). By Lemma 52.2 we see that \( C^\bullet \) is \((-1\)-pseudo-coherent). By Lemma 52.3 we see that \( H^{-1}(C^\bullet) \) is a finite \( R \)-module. Choose a finite free \( R \)-module \( F^{-1} \) and a map \( \beta : F^{-1} \to C^{-1} \) such that the composition \( F^{-1} \to C^{-1} \to C^0 \) is zero and such that \( F^{-1} \) surjects onto \( H^{-1}(C^\bullet) \). Since \( C^{-1} = K^{-1} \oplus F^0 \) we can write \( \beta = (\alpha^{-1}, -d^{-1}) \). The vanishing of the composition \( F^{-1} \to C^{-1} \to C^0 \) implies these maps fit into a morphism of complexes,

\[
\begin{array}{ccc}
F^{-1} & \xrightarrow{\alpha^{-1}} & F^0 \\
\downarrow & & \downarrow \\
K^{-1} & \xrightarrow{\alpha} & K^0
\end{array}
\]

Moreover, these maps define a morphism of distinguished triangles

\[
(F^0 \to \ldots) \xrightarrow{(F^{-1} \to \ldots)} F^{-1} \xrightarrow{\beta} (F^0 \to \ldots)[1]
\]

Hence our choice of \( \beta \) implies that the map of complexes \( (F^{-1} \to \ldots) \to K^\bullet \) induces an isomorphism on cohomology in degrees \( \geq 0 \) and a surjection in degree \(-1\). This finishes the proof of the lemma.

**Lemma 52.6.** Let \( R \) be a ring. Let \((K^\bullet, L^\bullet, M^\bullet, f, g, h)\) be a distinguished triangle in \( D(R) \). If two out of three of \( K^\bullet, L^\bullet, M^\bullet \) are pseudo-coherent then the third is also pseudo-coherent.

**Proof.** Combine Lemmas 52.2 and 52.5.

**Lemma 52.7.** Let \( R \) be a ring. Let \( K^\bullet \) be a complex of \( R \)-modules. Let \( m \in \mathbb{Z} \).

1. If \( H^i(K^\bullet) = 0 \) for all \( i \geq m \), then \( K^\bullet \) is \( m \)-pseudo-coherent.
2. If \( H^i(K^\bullet) = 0 \) for \( i > m \) and \( H^m(K^\bullet) \) is a finite \( R \)-module, then \( K^\bullet \) is \( m \)-pseudo-coherent.
3. If \( H^i(K^\bullet) = 0 \) for \( i > m + 1 \), the module \( H^{m+1}(K^\bullet) \) is of finite presentation, and \( H^m(K^\bullet) \) is of finite type, then \( K^\bullet \) is \( m \)-pseudo-coherent.

**Proof.** It suffices to prove (3). Set \( M = H^{m+1}(K^\bullet) \). Note that \( \tau_{\geq m+1}K^\bullet \) is quasi-isomorphic to \( M[-m-1] \). By Lemma 52.4 we see that \( M[-m-1] \) is \( m \)-pseudo-coherent. Since we have the distinguished triangle

\[
(\tau_{\leq m}K^\bullet, K^\bullet, \tau_{\geq m+1}K^\bullet)
\]

(Derived Categories, Remark 12.4) by Lemma 52.2 it suffices to prove that \( \tau_{\leq m}K^\bullet \) is pseudo-coherent. By assumption \( H^m(\tau_{\leq m}K^\bullet) \) is a finite type \( R \)-module. Hence we can find a finite free \( R \)-module \( E \) and a map \( E \to \text{Ker}(d^m_R) \) such that the composition \( E \to \text{Ker}(d^m_R) \to H^m(\tau_{\leq m}K^\bullet) \) is surjective. Then \( E[-m] \to \tau_{\leq m}K^\bullet \) witnesses the fact that \( \tau_{\leq m}K^\bullet \) is \( m \)-pseudo-coherent.

**Lemma 52.8.** Let \( R \) be a ring. Let \( m \in \mathbb{Z} \). If \( K^\bullet \oplus L^\bullet \) is \( m \)-pseudo-coherent (resp. pseudo-coherent) so are \( K^\bullet \) and \( L^\bullet \).
Proof. In this proof we drop the superscript $\bullet$. Assume that $K \oplus L$ is $m$-pseudo-coherent. It is clear that $K, L \in D^{-}(R)$. Note that there is a distinguished triangle
\[(K \oplus L, K \oplus L, L \oplus L[1]) = (K, K, 0) \oplus (L, L, L \oplus L[1])\]
see Derived Categories, Lemma 4.9 By Lemma 52.2 we see that $L \oplus L[1]$ is $m$-pseudo-coherent. Hence also $L[1] \oplus L[2]$ is $m$-pseudo-coherent. By induction $L[n] \oplus L[n + 1]$ is $m$-pseudo-coherent. By Lemma 52.7 we see that $L[n]$ is $m$-pseudo-coherent for large $n$. Hence working backwards, using the distinguished triangles
\[(L[n], L[n] \oplus L[n - 1], L[n - 1])\]
we conclude that $L[n], L[n - 1], \ldots, L$ are $m$-pseudo-coherent as desired. The pseudo-coherent case follows from this and Lemma 52.5.

Lemma 52.9. Let $R$ be a ring. Let $m \in \mathbb{Z}$. Let $K^{\bullet}$ be a bounded above complex of $R$-modules such that $K^{i}$ is $(m - i)$-pseudo-coherent for all $i$. Then $K^{\bullet}$ is $m$-pseudo-coherent. In particular, if $K^{\bullet}$ is a bounded above complex of pseudo-coherent $R$-modules, then $K^{\bullet}$ is pseudo-coherent.

Proof. We may replace $K^{\bullet}$ by $\sigma_{\geq m-1}K^{\bullet}$ (for example) and hence assume that $K^{\bullet}$ is bounded. Then the complex $K^{\bullet}$ is $m$-pseudo-coherent as each $K^{i}[-i]$ is $m$-pseudo-coherent by induction on the length of the complex: use Lemma 52.2 and the stupid truncations. For the final statement, it suffices to prove that $K^{\bullet}$ is $m$-pseudo-coherent for all $m \in \mathbb{Z}$, see Lemma 52.5. This follows from the first part.

Lemma 52.10. Let $R$ be a ring. Let $m \in \mathbb{Z}$. Let $K^{\bullet} \in D^{-}(R)$ such that $H^{i}(K^{\bullet})$ is $(m - i)$-pseudo-coherent (resp. pseudo-coherent) for all $i$. Then $K^{\bullet}$ is $m$-pseudo-coherent (resp. pseudo-coherent).

Proof. Assume $K^{\bullet}$ is an object of $D^{-}(R)$ such that each $H^{i}(K^{\bullet})$ is $(m - i)$-pseudo-coherent. Let $n$ be the largest integer such that $H^{n}(K^{\bullet})$ is nonzero. We will prove the lemma by induction on $n$. If $n < m$, then $K^{\bullet}$ is $m$-pseudo-coherent by Lemma 52.7. If $n \geq m$, then we have the distinguished triangle
\[(\tau_{\leq n-1}K^{\bullet}, K^{\bullet}, H^{n}(K^{\bullet})[-n])\]
see Derived Categories, Remark 12.4 Since $H^{n}(K^{\bullet})[-n]$ is $m$-pseudo-coherent by assumption, we can use Lemma 52.2 to see that it suffices to prove that $\tau_{\leq n-1}K^{\bullet}$ is $m$-pseudo-coherent. By induction on $n$ we win. (The pseudo-coherent case follows from this and Lemma 52.5.)

Lemma 52.11. Let $A \rightarrow B$ be a ring map. Assume that $B$ is pseudo-coherent as an $A$-module. Let $K^{\bullet}$ be a complex of $B$-modules. The following are equivalent

1. $K^{\bullet}$ is $m$-pseudo-coherent as a complex of $B$-modules, and
2. $K^{\bullet}$ is pseudo-coherent as a complex of $A$-modules.

The same equivalence holds for pseudo-coherence.

Proof. Assume (1). Choose a bounded complex of finite free $B$-modules $E^{\bullet}$ and a map $\alpha : E^{\bullet} \rightarrow K^{\bullet}$ which is an isomorphism on cohomology in degrees $> m$ and a surjection in degree $m$. Consider the distinguished triangle $(E^{\bullet}, K^{\bullet}, C(\alpha)^{\bullet})$. By Lemma 52.7 $C(\alpha)^{\bullet}$ is $m$-pseudo-coherent as a complex of $A$-modules. Hence it suffices to prove that $E^{\bullet}$ is pseudo-coherent as a complex of $A$-modules, which
follows from Lemma \textsuperscript{52.9} The pseudo-coherent case of (1) ⇒ (2) follows from this and Lemma \textsuperscript{52.5}.

Assume (2). Let \( n \) be the largest integer such that \( H^n(K^\bullet) \neq 0 \). We will prove that \( K^\bullet \) is \( m \)-pseudo-coherent as a complex of \( B \)-modules by induction on \( n - m \). The case \( n < m \) follows from Lemma \textsuperscript{52.7} Choose a bounded complex of finite free \( A \)-modules \( E^\bullet \) and a map \( \alpha : E^\bullet \to K^\bullet \) which is an isomorphism on cohomology in degrees \( > m \) and a surjection in degree \( m \). Consider the induced map of complexes

\[
\alpha \otimes 1 : E^\bullet \otimes_A B \to K^\bullet.
\]

Note that \( C(n < m) \) is surjective by construction and since \( H^i(E^\bullet \otimes_A B) = 0 \) for \( i > n \) by the spectral sequence of Example \textsuperscript{50.4} On the other hand, \( C(n = m) \) is \( m \)-pseudo-coherent as a complex of \( A \)-modules because both \( K^\bullet \) and \( E^\bullet \otimes_A B \) (see Lemma \textsuperscript{52.9}) are so, see Lemma \textsuperscript{52.2} Hence by induction we see that \( C(n = m) \) is \( m \)-pseudo-coherent as a complex of \( B \)-modules. Finally another application of Lemma \textsuperscript{52.2} shows that \( K^\bullet \) is \( m \)-pseudo-coherent as a complex of \( B \)-modules (as clearly \( E^\bullet \otimes_A B \) is pseudo-coherent as a complex of \( B \)-modules). The pseudo-coherent case of (2) ⇒ (1) follows from this and Lemma \textsuperscript{52.5}.

**Lemma** \textsuperscript{52.12}. Let \( A \to B \) be a ring map. Let \( K^\bullet \) be an \( m \)-pseudo-coherent (resp. pseudo-coherent) complex of \( A \)-modules. Then \( K^\bullet \otimes^L_A B \) is an \( m \)-pseudo-coherent (resp. pseudo-coherent) complex of \( B \)-modules.

**Proof.** First we note that the statement of the lemma makes sense as \( K^\bullet \) is bounded above and hence \( K^\bullet \otimes^L_A B \) is defined by Equation \textsuperscript{46.0.2}. Having said this, choose a bounded complex \( E^\bullet \) of finite free \( A \)-modules and \( \bar{\alpha} : E^\bullet \to K^\bullet \) with \( H^i(\bar{\alpha}) \) an isomorphism for \( i > m \) and surjective for \( i = m \). Then the cone \( C(\alpha)^\bullet \) is acyclic in degrees \( \geq m \). Since \( - \otimes^L_A \) is an exact functor we get a distinguished triangle

\[
(E^\bullet \otimes^L_A B, K^\bullet \otimes^L_A B, C(\alpha)^\bullet \otimes^L_A B)
\]

of complexes of \( B \)-modules. By the dual to Derived Categories, Lemma \textsuperscript{17.1} we see that \( H^i(C(\alpha)^\bullet \otimes^L_A B) = 0 \) for \( i \geq m \). Since \( E^\bullet \) is a complex of projective \( R \)-modules we see that \( E^\bullet \otimes^L_A B = E^\bullet \otimes_A B \) and hence

\[
E^\bullet \otimes_A B \to K^\bullet \otimes^L_A B
\]

is a morphism of complexes of \( B \)-modules that witnesses the fact that \( K^\bullet \otimes^L_A B \) is \( m \)-pseudo-coherent. The case of pseudo-coherent complexes follows from the case of \( m \)-pseudo-coherent complexes via Lemma \textsuperscript{52.5}.

**Lemma** \textsuperscript{52.13}. Let \( A \to B \) be a flat ring map. Let \( M \) be an \( m \)-pseudo-coherent (resp. pseudo-coherent) \( A \)-module. Then \( M \otimes_A B \) is an \( m \)-pseudo-coherent (resp. pseudo-coherent) \( B \)-module.

**Proof.** Immediate consequence of Lemma \textsuperscript{52.12} and the fact that \( M \otimes^L_A B = M \otimes_A B \) because \( B \) is flat over \( A \).

The following lemma also follows from the stronger Lemma \textsuperscript{52.14}.

**Lemma** \textsuperscript{52.14}. Let \( R \) be a ring. Let \( f_1, \ldots, f_r \in R \) be elements which generate the unit ideal. Let \( m \in \mathbb{Z} \). Let \( K^\bullet \) be a complex of \( R \)-modules. If for each \( i \) the complex \( K^\bullet \otimes_R R f_i \) is \( m \)-pseudo-coherent (resp. pseudo-coherent), then \( K^\bullet \) is \( m \)-pseudo-coherent (resp. pseudo-coherent).
Proof. We will use without further mention that \(- \otimes_R R_f\) is an exact functor and that therefore
\[ H^i(K^*)_f = H^i(K^*) \otimes_R R_f = H^i(K^* \otimes_R R_f). \]
Assume \(K^* \otimes_R R_f\) is \(m\)-pseudo-coherent for \(i = 1, \ldots, r\). Let \(n \in \mathbb{Z}\) be the largest integer such that \(H^n(K^* \otimes_R R_f)\) is nonzero for some \(i\). This implies in particular that \(H^i(K^*) = 0\) for \(i > n\) (and that \(H^n(K^*) \neq 0\)) see Algebra, Lemma 23.2. We will prove the lemma by induction on \(n - m\). If \(n < m\), then the lemma is true by Lemma 52.7. If \(n \geq m\), then \(H^n(K^*)_f\) is a finite \(R_f\)-module for each \(i\), see Lemma 52.3. Hence \(H^n(K^*)\) is a finite \(R\)-module, see Algebra, Lemma 23.2. Choose a finite free \(R\)-module \(E\) and a surjection \(E \to H^n(K^*)\). As \(E\) is projective we can lift this to a map of complexes \(\alpha : E[-n] \to K^*\). Then the cone \(C(\alpha)^*\) has vanishing cohomology in degrees \(\geq n\). On the other hand, the complexes \(C(\alpha)^* \otimes_R R_f\) are \(m\)-pseudo-coherent for each \(i\), see Lemma 52.2. Hence by induction we see that \(C(\alpha)^*\) is \(m\)-pseudo-coherent as a complex of \(R\)-modules. Applying Lemma 52.2 once more we conclude.

Lemma 52.15. Let \(R\) be a ring. Let \(m \in \mathbb{Z}\). Let \(K^*\) be a complex of \(R\)-modules. Let \(R \to R'\) be a faithfully flat ring map. If the complex \(K^* \otimes_R R'\) is \(m\)-pseudo-coherent (resp. pseudo-coherent), then \(K^*\) is \(m\)-pseudo-coherent (resp. pseudo-coherent).

Proof. We will use without further mention that \(- \otimes_R R'\) is an exact functor and that therefore
\[ H^i(K^*) \otimes_R R' = H^i(K^* \otimes_R R'). \]
Assume \(K^* \otimes_R R'\) is \(m\)-pseudo-coherent. Let \(n \in \mathbb{Z}\) be the largest integer such that \(H^n(K^*)\) is nonzero; then \(n\) is also the largest integer such that \(H^n(K^* \otimes_R R')\) is nonzero. We will prove the lemma by induction on \(n - m\). If \(n < m\), then the lemma is true by Lemma 52.7. If \(n \geq m\), then \(H^n(K^*) \otimes_R R'\) is a finite \(R'\)-module, see Lemma 52.3. Hence \(H^n(K^*)\) is a finite \(R\)-module, see Algebra, Lemma 81.2. Choose a finite free \(R\)-module \(E\) and a surjection \(E \to H^n(K^*)\). As \(E\) is projective we can lift this to a map of complexes \(\alpha : E[-n] \to K^*\). Then the cone \(C(\alpha)^*\) has vanishing cohomology in degrees \(\geq n\). On the other hand, the complex \(C(\alpha)^* \otimes_R R'\) is \(m\)-pseudo-coherent, see Lemma 52.2. Hence by induction we see that \(C(\alpha)^*\) is \(m\)-pseudo-coherent as a complex of \(R\)-modules. Applying Lemma 52.2 once more we conclude.

Lemma 52.16. Let \(R\) be a Noetherian ring. Then

1. A complex of \(R\)-modules \(K^*\) is \(m\)-pseudo-coherent if and only if \(K^* \in D^+(R)\) and \(H^i(K^*)\) is a finite \(R\)-module for \(i \geq m\).
2. A complex of \(R\)-modules \(K^*\) is pseudo-coherent if and only if \(K^* \in D^-(R)\) and \(H^i(K^*)\) is a finite \(R\)-module for all \(i\).
3. An \(R\)-module is pseudo-coherent if and only if it is finite.

Proof. In Algebra, Lemma 69.1 we have seen that any finite \(R\)-module is pseudo-coherent. On the other hand, a pseudo-coherent module is finite, see Lemma 52.4. Hence (3) holds. Suppose that \(K^*\) is an \(m\)-pseudo-coherent complex. Then there exists a bounded complex of finite free \(R\)-modules \(E^*\) such that \(H^i(K^*)\) is isomorphic to \(H^i(E^*)\) for \(i > m\) and such that \(H^m(K^*)\) is a quotient of \(H^m(E^*)\). Thus it is clear that each \(H^i(K^*), i \geq m\) is a finite module. The converse implication in
(1) follows from Lemma 52.10 and part (3). Part (2) follows from (1) and Lemma 52.5.

**Remark 52.17.** Let \( R \) be ring map. Let \( L, M, N \) be \( R \)-modules. Consider the canonical map

\[
\text{Hom}_R(M, N) \otimes_R L \to \text{Hom}_R(M, N \otimes_R L)
\]

Choose a two term free resolution \( F_1 \to F_0 \to M \to 0 \). Assuming \( L \) flat over \( R \) we obtain a commutative diagram

\[
\begin{array}{cccc}
0 & \longrightarrow & \text{Hom}_R(M, N) \otimes_R L & \longrightarrow & \text{Hom}_R(F_0, N) \otimes_R L & \longrightarrow & \text{Hom}_R(F_1, N) \otimes_R L \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \text{Hom}_R(M, N \otimes_R L) & \longrightarrow & \text{Hom}_R(F_0, N \otimes_R L) & \longrightarrow & \text{Hom}_R(F_1, N \otimes_R L)
\end{array}
\]

with exact rows. We conclude that if \( F_0 \) and \( F_1 \) are finite free, i.e., if \( M \) is finitely presented, then the first displayed map is an isomorphism. Similarly, if \( M \) is \((-m)\)-pseudo-coherent and still assuming \( L \) is flat over \( R \), then the map

\[
\text{Ext}^i_R(M, N) \otimes_R L \to \text{Ext}^i_R(M, N \otimes_R L)
\]

is an isomorphism for \( i < m \).

**Remark 52.18.** Let \( R \) be ring map. Let \( M, N \) be \( R \)-modules. Let \( R \to R' \) be a flat ring map. By Algebra, Lemma 71.1 we have \( \text{Ext}^i_R(M \otimes_R R', N \otimes_R R') = \text{Ext}^i_R(M, N \otimes_R R') \). Combined with Remark 52.17 we conclude that

\[
\text{Hom}_R(M, N) \otimes_R R' = \text{Hom}_{R'}(M \otimes_R R', N \otimes_R R')
\]

if \( M \) is a finitely presented \( R \)-module and that

\[
\text{Ext}^i_R(M, N) \otimes_R R' = \text{Ext}^i_{R'}(M \otimes_R R', N \otimes_R R')
\]

is an isomorphism for \( i < m \) if \( M \) is \((-m)\)-pseudo-coherent. In particular if \( R \) is Noetherian and \( M \) is a finite module this holds for all \( i \).

### 53. Tor dimension

Instead of resolving by projective modules we can look at resolutions by flat modules. This leads to the following concept.

**Definition 53.1.** Let \( R \) be a ring. Denote \( D(R) \) its derived category. Let \( a, b \in \mathbb{Z} \).

1. An object \( K^\bullet \) of \( D(R) \) has tor-amplitude in \([a, b]\) if \( H^i(K^\bullet \otimes_R^L M) = 0 \) for all \( R \)-modules \( M \) and all \( i \not\in [a, b] \).
2. An object \( K^\bullet \) of \( D(R) \) has finite tor dimension if it has tor-amplitude in \([a, b]\) for some \( a, b \).
3. An \( R \)-module \( M \) has tor dimension \( \leq d \) if \( M[0] \) as an object of \( D(R) \) has tor-amplitude in \([-d, 0] \).
4. An \( R \)-module \( M \) has finite tor dimension if \( M[0] \) as an object of \( D(R) \) has finite tor dimension.

We observe that if \( K^\bullet \) has finite tor dimension, then \( K^\bullet \in D^b(R) \).

**Lemma 53.2.** Let \( R \) be a ring. Let \( K^\bullet \) be a bounded above complex of flat \( R \)-modules with tor-amplitude in \([a, b]\). Then \( \text{Coker}(d_K^{b-1}) \) is a flat \( R \)-module.
**Proof.** As $K^\bullet$ is a bounded above complex of flat modules we see that $K^\bullet \otimes_R M = K^\bullet \otimes_K^L M$. Hence for every $R$-module $M$ the sequence
$$K^{a-2} \otimes_R M \rightarrow K^{a-1} \otimes_R M \rightarrow K^a \otimes_R M$$
is exact in the middle. Since $K^{a-2} \rightarrow K^{a-1} \rightarrow K^a \rightarrow \mathrm{Coker}(d_K^{a-1}) \rightarrow 0$ is a flat resolution this implies that $\mathrm{Tor}_1^R(\mathrm{Coker}(d_K^{a-1}), M) = 0$ for all $R$-modules $M$. This means that $\mathrm{Coker}(d_K^{a-1})$ is flat, see Algebra, Lemma 73.8.

**Lemma 53.3.** Let $R$ be a ring. Let $K^\bullet$ be an object of $D(R)$. Let $a, b \in \mathbb{Z}$. The following are equivalent

1. $K^\bullet$ has tor-amplitude in $[a, b]$.
2. $K^\bullet$ is quasi-isomorphic to a complex $E^\bullet$ of flat $R$-modules with $E^i = 0$ for $i \not\in [a, b]$.

**Proof.** If (2) holds, then we may compute $K^\bullet \otimes_R^L M = E^\bullet \otimes_R M$ and it is clear that (1) holds. Assume that (1) holds. We may replace $K^\bullet$ by a projective resolution. Let $n$ be the largest integer such that $K^n \neq 0$. If $n > b$, then $K^{n-1} \rightarrow K^n$ is surjective as $H^n(K^\bullet) = 0$. As $K^n$ is projective we see that $K^{n-1} = K' \oplus K^n$. Hence it suffices to prove the result for the complex $(K')^\bullet$ which is the same as $K^\bullet$ except has $K'$ in degree $n-1$ and 0 in degree $n$. Thus, by induction on $n$, we reduce to the case that $K^\bullet$ is a complex of projective $R$-modules with $K^i = 0$ for $i > b$.

Set $E^\bullet = \tau_{\geq a}K^\bullet$. Everything is clear except that $E^a$ is flat which follows immediately from Lemma 53.2 and the definitions.

**Lemma 53.4.** Let $R$ be a ring. Let $(K^\bullet, L^\bullet, M^\bullet, f, g, h)$ be a distinguished triangle in $D(R)$. Let $a, b \in \mathbb{Z}$.

1. If $K^\bullet$ has tor-amplitude in $[a+1, b+1]$ and $L^\bullet$ has tor-amplitude in $[a, b]$ then $M^\bullet$ has tor-amplitude in $[a, b]$.
2. If $K^\bullet, M^\bullet$ have tor-amplitude in $[a, b]$, then $L^\bullet$ has tor-amplitude in $[a, b]$.
3. If $L^\bullet$ has tor-amplitude in $[a+1, b+1]$ and $M^\bullet$ has tor-amplitude in $[a, b]$, then $K^\bullet$ has tor-amplitude in $[a+1, b+1]$.

**Proof.** Omitted. Hint: This just follows from the long exact cohomology sequence associated to a distinguished triangle and the fact that $- \otimes_K^L M$ preserves distinguished triangles. The easiest one to prove is (2) and the others follow from it by translation.

**Lemma 53.5.** Let $R$ be a ring. Let $M$ be an $R$-module. Let $d \geq 0$. The following are equivalent.

1. $M$ has tor dimension $\leq d$, and
2. there exists a resolution
$$0 \rightarrow F_d \rightarrow \ldots \rightarrow F_1 \rightarrow F_0 \rightarrow M \rightarrow 0$$
with $F_i$ a flat $R$-module.

In particular an $R$-module has tor dimension 0 if and only if it is a flat $R$-module.

**Proof.** Assume (2). Then the complex $E^\bullet$ with $E^{-i} = F_i$ is quasi-isomorphic to $M$. Hence the Tor dimension of $M$ is at most $d$ by Lemma 53.3. Conversely, assume (1). Let $P^\bullet \rightarrow M$ be a projective resolution of $M$. By Lemma 53.2 we see that $\tau_{\geq -d}P^\bullet$ is a flat resolution of $M$ of length $d$, i.e., (2) holds.
Lemma 53.6. Let $R$ be a ring. Let $a, b \in \mathbb{Z}$. If $K^\bullet \oplus L^\bullet$ has tor amplitude in $[a, b]$ so do $K^\bullet$ and $L^\bullet$.

Proof. Clear from the fact that the Tor functors are additive.

Lemma 53.7. Let $R$ be a ring. Let $K^\bullet$ be a bounded complex of $R$-modules such that $K^i$ has tor amplitude in $[a - i, b - i]$ for all $i$. Then $K^\bullet$ has tor amplitude in $[a, b]$. In particular if $K^\bullet$ is a finite complex of $R$-modules of finite tor dimension, then $K^\bullet$ has finite tor dimension.

Proof. Follows by induction on the length of the finite complex: use Lemma 53.3 and the stupid truncations.

Lemma 53.8. Let $R$ be a ring. Let $a, b \in \mathbb{Z}$. Let $K^\bullet \in D^b(R)$ such that $H^i(K^\bullet)$ has tor amplitude in $[a - i, b - i]$ for all $i$. Then $K^\bullet$ has tor amplitude in $[a, b]$. In particular if $K^\bullet \in D^-(R)$ and all its cohomology groups have finite tor dimension then $K^\bullet$ has finite tor dimension.

Proof. Follows by induction on the length of the finite complex: use Lemma 53.3 and the canonical truncations.

Lemma 53.9. Let $A \to B$ be a ring map. Assume that $B$ is flat as an $A$-module. Let $K^\bullet$ be a complex of $B$-modules. Let $a, b \in \mathbb{Z}$. If $K^\bullet$ as a complex of $B$-modules has tor amplitude in $[a, b]$, then $K^\bullet$ as a complex of $A$-modules has tor amplitude in $[a, b]$.

Proof. This is true because $K^\bullet \otimes_A M = K^\bullet \otimes_B (M \otimes_A B)$ since any projective resolution of $K^\bullet$ as a complex of $B$-modules is a flat resolution of $K^\bullet$ as a complex of $A$-modules and can be used to compute $K^\bullet \otimes_A M$.

Lemma 53.10. Let $A \to B$ be a ring map. Assume that $B$ has tor dimension $\leq d$ as an $A$-module. Let $K^\bullet$ be a complex of $B$-modules. Let $a, b \in \mathbb{Z}$. If $K^\bullet$ as a complex of $B$-modules has tor amplitude in $[a, b]$, then $K^\bullet$ as a complex of $A$-modules has tor amplitude in $[a - d, b]$.

Proof. Let $M$ be an $A$-module. Choose a free resolution $F^\bullet \to M$. Then

$$K^\bullet \otimes_A M = \text{Tot}(K^\bullet \otimes_A F^\bullet) = \text{Tot}(K^\bullet \otimes_B (F^\bullet \otimes_A B)) = K^\bullet \otimes_B (M \otimes_A B).$$

By our assumption on $B$ as an $A$-module we see that $M \otimes_A B$ has cohomology only in degrees $-d, -d + 1, \ldots, 0$. Because $K^\bullet$ has tor amplitude in $[a, b]$ we see from the spectral sequence in Example 50.4 that $K^\bullet \otimes_B (M \otimes_A B)$ has cohomology only in degrees $[-d + a, b]$ as desired.

Lemma 53.11. Let $A \to B$ be a ring map. Let $a, b \in \mathbb{Z}$. Let $K^\bullet$ be a complex of $A$-modules with tor amplitude in $[a, b]$. Then $K^\bullet \otimes_A B$ as a complex of $B$-modules has tor amplitude in $[a, b]$.

Proof. By Lemma 53.3 we can find a quasi-isomorphism $E^\bullet \to K^\bullet$ where $E^\bullet$ is a complex of flat $A$-modules with $E^i = 0$ for $i \not\in [a, b]$. Then $E^\bullet \otimes_A B$ computes $K^\bullet \otimes_A B$ by construction and each $E^i \otimes_A B$ is a flat $B$-module by Algebra, Lemma 38.6. Hence we conclude by Lemma 53.3.

Lemma 53.12. Let $A \to B$ be a flat ring map. Let $d \geq 0$. Let $M$ be an $A$-module of tor dimension $\leq d$. Then $M \otimes_A B$ is a $B$-module of tor dimension $\leq d$. 

**Proof.** Immediate consequence of Lemma 53.11 and the fact that $M \otimes_A^L B = M \otimes_A B$ because $B$ is flat over $A$. □

**Lemma 53.13.** Let $R$ be a ring. Let $f_1, \ldots, f_r \in R$ be elements which generate the unit ideal. Let $a, b \in \mathbb{Z}$. Let $K^\bullet$ be a complex of $R$-modules. If for each $i$ the complex $K^\bullet \otimes_R f_i$ has tor amplitude in $[a, b]$, then $K^\bullet$ has tor amplitude in $[a, b]$.

**Proof.** Note that $- \otimes_R f_i$ is an exact functor and that therefore

$$H^i(K^\bullet f_i) = H^i(K^\bullet) \otimes_R f_i = H^i(K^\bullet \otimes_R f_i),$$

and similarly for every $R$-module $M$ we have

$$H^i(K^\bullet \otimes_R M f_i) = H^i(K^\bullet \otimes_R M) \otimes_R f_i = H^i(K^\bullet \otimes_R f_i \otimes_R M f_i).$$

Hence the result follows from the fact that an $R$-module $N$ is zero if and only if $N f_i$ is zero for each $i$, see Algebra, Lemma 23.2. □

**Lemma 53.14.** Let $R$ be a ring. Let $a, b \in \mathbb{Z}$. Let $K^\bullet$ be a complex of $R$-modules. Let $R \to R'$ be a faithfully flat ring map. If the complex $K^\bullet \otimes_R R'$ has tor amplitude in $[a, b]$, then $K^\bullet$ has tor amplitude in $[a, b]$.

**Proof.** Let $M$ be an $R$-module. Since $R \to R'$ is flat we see that

$$(M \otimes^L_R K^\bullet) \otimes_R R' = (M \otimes^L_R R') \otimes^L_R (K^\bullet \otimes_R R')$$

and taking cohomology commutes with tensoring with $R'$. Hence $\text{Tor}_i^R(M, K^\bullet) = \text{Tor}_i^{R'}(M \otimes_R R', K^\bullet \otimes_R R')$. Since $R \to R'$ is faithfully flat, the vanishing of $\text{Tor}_i^{R'}(M \otimes_R R', K^\bullet \otimes_R R')$ for $i \not\in [a, b]$ implies the same thing for $\text{Tor}_i^R(M, K^\bullet)$. □

**Lemma 53.15.** Let $R$ be a ring of finite global dimension $d$. Then

1. every module has finite tor dimension $\leq d$,
2. a complex of $R$-modules $K^\bullet$ with $H^i(K^\bullet) \neq 0$ only if $i \in [a, b]$ has tor amplitude in $[a - d, b]$, and
3. a complex of $R$-modules $K^\bullet$ has finite tor dimension if and only if $K^\bullet \in D^b(R)$.

**Proof.** The assumption on $R$ means that every module has a finite projective resolution of length at most $d$, in particular every module has finite tor dimension. The second statement follows from Lemma 53.5 and the definitions. The third statement is a rephrasing of the second. □

### 54. Spectral sequences for Ext

In this section we collect various spectral sequences that come up when considering the Ext functors. For any pair of objects $L$, $K$ of the derived category $D(R)$ of a ring $R$ we denote

$$\text{Ext}_R^j(L, K) = \text{Hom}_{D(R)}(L, K[n])$$

according to our general conventions in Derived Categories, Section 27.

For $M$ an $R$-module and $K \in D^+(R)$ there is a spectral sequence

$$(54.0.1) \quad \text{Ext}_R^j(M, H^i(K)) \Rightarrow \text{Ext}_R^{i+j}(M, K)$$

and if $K$ is represented by the bounded below complex $K^\bullet$ of $R$-modules there is a spectral sequence

$$(54.0.2) \quad \text{Ext}_R^j(M, K^i) \Rightarrow \text{Ext}_R^{i+j}(M, K)$$
55. Projective dimension

We defined the projective dimension of a module in Algebra, Definition 106.2.

**Definition 55.1.** Let $R$ be a ring. Let $K$ be an object of $D(R)$. We say $K$ has **finite projective dimension** if $K$ can be represented by a finite complex of projective modules. We say $K$ as **projective-amplitude in** $[a,b]$ if $K$ is quasi-isomorphic to a complex

$$
\cdots \to 0 \to P^a \to P^{a+1} \to \cdots \to P^{b-1} \to P^b \to 0 \to \cdots
$$

where $P^i$ is a projective $R$-module for all $i \in \mathbb{Z}$.

Clearly, $K$ has bounded projective dimension if and only if $K$ has projective-amplitude in $[a,b]$ for some $a,b \in \mathbb{Z}$. Furthermore, if $K$ has bounded projective dimension, then $K$ is bounded. Here is the obligatory lemma.

**Lemma 55.2.** Let $R$ be a ring. Let $K$ be an object of $D(R)$. Let $a,b \in \mathbb{Z}$. The following are equivalent

1. $K$ has projective-amplitude in $[a,b]$,
2. $\text{Ext}^i_R(K,N) = 0$ for all $R$-modules $N$ and all $i \notin [-b,-a]$.

**Proof.** Assume (1). We may assume $K$ is the complex

$$
\cdots \to 0 \to P^a \to P^{a+1} \to \cdots \to P^{b-1} \to P^b \to 0 \to \cdots
$$

where $P^i$ is a projective $R$-module for all $i \in \mathbb{Z}$. In this case we can compute the ext groups by the complex

$$
\cdots \to 0 \to \text{Hom}_R(P^b,N) \to \cdots \to \text{Hom}_R(P^a,N) \to 0 \to \cdots
$$

and we obtain (2).

Assume (2) holds. Choose an injection $H^n(K) \to I$ where $I$ is an injective $R$-module. Since $\text{Hom}_R(-,I)$ is an exact functor, we see that $\text{Ext}^{-n}(K,I) = \text{Hom}_R(H^n(K),I)$. We conclude that $H^n(K)$ is zero for $n \notin [a,b]$. In particular, $K$ is bounded above and we can choose a quasi-isomorphism

$$
P^\bullet \to K
$$

with $P^i$ projective (for example free) for all $i \in \mathbb{Z}$ and $P^i = 0$ for $i > b$. See Derived Categories, Lemma 16.5. Let $Q = \text{Coker}(P^{a-1} \to P^a)$. Then $K$ is quasi-isomorphic to the complex

$$
\cdots \to 0 \to Q \to P^{a+1} \to \cdots \to P^b \to 0 \to \cdots
$$

Denote $K' = (P^{a+1} \to \cdots \to P^b)$ the corresponding object of $D(R)$. We obtain a distinguished triangle

$$
K' \to K \to Q[-a] \to K'[1]
$$

in $D(R)$. Thus for every $R$-module $N$ an exact sequence

$$
\text{Ext}^{-a}(K',N) \to \text{Ext}^1(Q,N) \to \text{Ext}^{1-a}(K,N)
$$

By assumption the term on the right vanishes. By the implication $(1) \Rightarrow (2)$ the term on the left vanishes. Thus $Q$ is a projective $R$-module by Algebra, Lemma 75.2. 

□
**Example 55.3.** Let $k$ be a field and let $R$ be the ring of dual numbers over $k$, i.e., $R = k[x]/(x^2)$. Denote $\epsilon \in R$ the class of $x$. Let $M = R/(\epsilon)$. Then $M$ is quasi-isomorphic to the complex

$$R \xrightarrow{\epsilon} R \rightarrow \ldots$$

but $M$ does not have finite projective dimension as defined in Algebra, Definition 106.2. This explains why we consider bounded (in both directions) complexes of projective modules in our definition of bounded projective dimension of objects of $D(R)$.

**56. Injective dimension**

This section is the dual of the section on projective dimension.

**Definition 56.1.** Let $R$ be a ring. Let $K$ be an object of $D(R)$. We say $K$ has finite injective dimension if $K$ can be represented by a finite complex of injective $R$-modules. We say $K$ has injective-amplitude in $[a,b]$ if $K$ is isomorphic to a complex

$$\ldots \rightarrow 0 \rightarrow I^a \rightarrow I^{a+1} \rightarrow \ldots \rightarrow I^{b-1} \rightarrow I^b \rightarrow 0 \rightarrow \ldots$$

with $I^i$ an injective $R$-module for all $i \in \mathbb{Z}$.

Clearly, $K$ has bounded injective dimension if and only if $K$ has injective-amplitude in $[a,b]$ for some $a,b \in \mathbb{Z}$. Furthermore, if $K$ has bounded injective dimension, then $K$ is bounded. Here is the obligatory lemma.

**Lemma 56.2.** Let $R$ be a ring. Let $K$ be an object of $D(R)$. Let $a,b \in \mathbb{Z}$. The following are equivalent

1. $K$ has injective-amplitude in $[a,b]$,
2. $\text{Ext}^i_R(N,K) = 0$ for all $R$-modules $N$ and all $i \notin [a,b]$,
3. $\text{Ext}^i(R/I,K) = 0$ for all ideals $I \subset R$ and all $i \notin [a,b]$.

**Proof.** Assume (1). We may assume $K$ is the complex

$$\ldots \rightarrow 0 \rightarrow I^a \rightarrow I^{a+1} \rightarrow \ldots \rightarrow I^{b-1} \rightarrow I^b \rightarrow 0 \rightarrow \ldots$$

where $I^i$ is a injective $R$-module for all $i \in \mathbb{Z}$. In this case we can compute the ext groups by the complex

$$\ldots \rightarrow 0 \rightarrow \text{Hom}_R(N,I^a) \rightarrow \ldots \rightarrow \text{Hom}_R(N,I^b) \rightarrow 0 \rightarrow \ldots$$

and we obtain (2). It is clear that (2) implies (3).

Assume (3) holds. Choose a nonzero map $R \rightarrow H^n(K)$. Since $\text{Hom}_R(R,-)$ is an exact functor, we see that $\text{Ext}^n_R(R,K) = \text{Hom}_R(R,H^n(K)) = H^n(K)$. We conclude that $H^n(K)$ is zero for $n \notin [a,b]$. In particular, $K$ is bounded below and we can choose a quasi-isomorphism

$$K \rightarrow I^\bullet$$

with $I^i$ injetive for all $i \in \mathbb{Z}$ and $I^i = 0$ for $i < a$. See Derived Categories, Lemma 16.4

Let $J = \text{Ker}(I^b \rightarrow I^{b+1})$. Then $K$ is quasi-isomorphic to the complex

$$\ldots \rightarrow 0 \rightarrow I^a \rightarrow \ldots \rightarrow I^{b-1} \rightarrow J \rightarrow 0 \rightarrow \ldots$$

Denote $K' = (I^a \rightarrow \ldots \rightarrow I^{b-1})$ the corresponding object of $D(R)$. We obtain a distinguished triangle

$$J[-b] \rightarrow K \rightarrow K' \rightarrow J[1-b]$$
in \( D(R) \). Thus for every ideal \( I \subset R \) an exact sequence

\[
\text{Ext}^b(R/I, K') \to \text{Ext}^1(R/I, J) \to \text{Ext}^{1+b}(R/I, K)
\]

By assumption the term on the right vanishes. By the implication \((1) \Rightarrow (2)\) the term on the left vanishes. Thus \( J \) is a injective \( R \)-module by Lemma 44.4. \qedhere

**Example 56.3.** Let \( k \) be a field and let \( R \) be the ring of dual numbers over \( k \), i.e., \( R = k[x]/(x^2) \). Denote \( \epsilon \in R \) the class of \( x \). Let \( M = R/(\epsilon) \). Then \( M \) is quasi-isomorphic to the complex

\[
\cdots \to R \xrightarrow{\epsilon} R \to R \to R
\]

and \( R \) is an injective \( R \)-module. However one usually does not consider \( M \) to have finite injective dimension in this situation. This explains why we consider bounded (in both directions) complexes of injective modules in our definition of bounded injective dimension of objects of \( D(R) \).

**Lemma 56.4.** Let \( R \) be a ring. Let \( K \in D(R) \).

1. If \( K \) is in \( D^b(R) \) and \( H^i(K) \) has finite injective dimension for all \( i \), then \( K \) has finite injective dimension.
2. If \( K^\bullet \) represents \( K \), is a bounded complex of \( R \)-modules, and \( K^i \) has finite injective dimension for all \( i \), then \( K \) has finite injective dimension.

**Proof.** Omitted. Hint: Apply the spectral sequences of Derived Categories, Lemma 21.3 to the functor \( F = \text{Hom}_R(N, -) \) to get a computation of \( \text{Ext}^i_R(N, K) \) and use the criterion of Lemma 56.2. \qedhere

**Lemma 56.5.** Let \((R, m, \kappa)\) be a local Noetherian ring. Let \( K \in D^+(R) \) have finite cohomology modules. Then the following are equivalent

1. \( K \) has finite injective dimension, and
2. \( \text{Ext}^i_R(\kappa, K) = 0 \) for \( i \gg 0 \).

**Proof.** Say \( H^i(K) = 0 \) for \( i < a \). Then \( \text{Ext}^i(M, K) = 0 \) for \( i < a \) and all \( R \)-modules \( M \). Say \( \text{Ext}^i_R(\kappa, K) = 0 \) for \( i > b \). We will show by induction on \( \dim(M) \) that \( \text{Ext}^i(M, K) = 0 \) for all finite \( R \)-modules \( M \). This will prove the lemma by Lemma 56.2. We will use that the modules \( \text{Ext}^i(M, K) \) are finite by our assumption on \( K \) (bounded below with finite cohomology modules), the spectral sequence (54.0.1), and Algebra, Lemma 69.9.

The base case. If \( \dim(M) = 0 \) then we can use induction on the length of \( M \), see Algebra, Lemma 61.3. If the length is 1, then \( M = \kappa \) and the result holds. If \( \length(M) > 1 \), then we can find an exact sequence \( 0 \to M' \to M \to \kappa \to 0 \) with \( \length(M') < \length(M) \) and the result for \( M' \) follows from the result for \( M' \) and \( \kappa \) by the long exact sequence of Ext’s.

Assume \( \dim(M) > 0 \). Consider the exact sequence \( 0 \to C \to M \to M' \to 0 \) of Algebra, Lemma 66.2. Using the long exact sequence of Ext’s and the induction hypothesis for \( C \), we see that it suffices to prove the vanishing for \( M' \). Thus we may assume \( M \) has no embedded associated primes. Let \( f \in m \) be an element which is not contained in any associated primes of \( M \) (to find \( f \) use \( \dim(M) > 0 \), use \( M \) has no embedded associated primes, and use Algebra, Lemma 14.2). Then \( f \) is a
nonzerodivisor on $M$ (Algebra, Lemma \ref{AlgebraLemma29}) and we can consider the short exact sequence

$$0 \to M \to M \to M/fM \to 0$$

This produces the long exact sequence

$$\ldots \to \text{Ext}^b(M, K) \xrightarrow{f} \text{Ext}^b(M, K) \to \text{Ext}^{b+1}(M/fM, K) \to \ldots$$

By induction hypothesis for $M/fM$ we see that $\text{Ext}^{b+1}(M/fM, K)$ is zero. Since $f \in m$ and $\text{Ext}^b(M, K)$ is finite, we conclude by Nakayama’s lemma (Algebra, Lemma \ref{AlgebraLemma19}) that $\text{Ext}^b(M, K)$ is zero. □

57. Hom complexes

Let $R$ be a ring. Let $L^\bullet$ and $M^\bullet$ be two complexes of $R$-modules. We construct a complex $\text{Hom}^\bullet(L^\bullet, M^\bullet)$. Namely, for each $n$ we set

$$\text{Hom}^n(L^\bullet, M^\bullet) = \prod_{p+q=n} \text{Hom}_R(L^{-q}, M^p)$$

It is a good idea to think of $\text{Hom}^n$ as the $R$-module of all $R$-linear maps from $L^\bullet$ to $M^\bullet$ (viewed as graded modules) which are homogenous of degree $n$. In this terminology, we define the differential by the rule

$$d(f) = d_M \circ f - (-1)^n f \circ d_L$$

for $f \in \text{Hom}^n(L^\bullet, M^\bullet)$. We omit the verification that $d^2 = 0$. This construction is a special case of Differential Graded Algebra, Example \ref{Example19}. It follows immediately from the construction that we have

$$(57.0.1) \quad H^n(\text{Hom}^\bullet(L^\bullet, M^\bullet)) = \text{Hom}_{K(R)}(L^\bullet, M^\bullet[n])$$

for all $n \in \mathbb{Z}$.

**Lemma 57.1.** Let $R$ be a ring. Given complexes $K^\bullet, L^\bullet, M^\bullet$ of $R$-modules there is a canonical isomorphism

$$\text{Hom}^\bullet(K^\bullet, \text{Hom}^\bullet(L^\bullet, M^\bullet)) = \text{Hom}^\bullet(\text{Tot}(K^\bullet \otimes_R L^\bullet), M^\bullet)$$

of complexes of $R$-modules.

**Proof.** Let $\alpha$ be an element of degree $n$ on the left hand side. Thus

$$\alpha = (\alpha^{p,q}) \in \prod_{p+q=n} \text{Hom}_R(K^{-q}, \text{Hom}^p(L^\bullet, M^\bullet))$$

Each $\alpha^{p,q}$ is an element

$$\alpha^{p,q} = (\alpha^{r,s,q}) \in \prod_{r+s+q=n} \text{Hom}_R(K^{-q}, \text{Hom}_R(L^{-s}, M^r))$$

If we make the identifications

$$\text{Hom}_R(K^{-q}, \text{Hom}_R(L^{-s}, M^r)) = \text{Hom}_R(K^{-q} \otimes_R L^{-s}, M^r)$$

then by our sign rules we get

$$d(\alpha^{r,s,q}) = d_{\text{Hom}^\bullet}(L^\bullet, M^\bullet) \circ \alpha^{r,s,q} - (-1)^n \alpha^{r,s,q} \circ d_K$$

$$= d_M \circ \alpha^{r,s,q} - (-1)^{r+s} \alpha^{r,s,q} \circ d_L - (-1)^{r+s+q} \alpha^{r,s,q} \circ d_K$$

On the other hand, if $\beta$ is an element of degree $n$ of the right hand side, then

$$\beta = (\beta^{r,s,q}) \in \prod_{r+s+q=n} \text{Hom}_R(K^{-q} \otimes_R L^{-s}, M^r)$$
and by our sign rule (Homology, Definition 22.3) we get
\[ d(β^{r,s,q}) = d_M \circ β^{r,s,q} - (-1)^n β^{r,s,q} \circ d_{\text{Tot}(K^* \otimes L^*)} \]
\[ = d_M \circ β^{r,s,q} - (-1)^{r+s+q} (β^{r,s,q} \circ d_K + (-1)^{-q} β^{r,s,q} \circ d_L) \]
Thus we see that the map induced by the identifications (57.1.1) indeed is a morphism of complexes.

**Lemma 57.2.** Let \( R \) be a ring. Given complexes \( K^*, L^*, M^* \) of \( R \)-modules there is a canonical morphism
\[ \text{Tot}(\text{Hom}^*(L^*, M^*) \otimes_R \text{Hom}^*(K^*, L^*)) \longrightarrow \text{Hom}^*(K^*, M^*) \]
of complexes of \( R \)-modules.

**Proof.** An element \( α \) of degree \( n \) of the left hand side is
\[ α = (α^{p,q}) \in \bigoplus_{p+q=n} \text{Hom}^p(L^*, M^*) \otimes_R \text{Hom}^q(K^*, L^*) \]
The element \( α^{p,q} \) is a finite sum \( α^{p,q} = \sum \beta_i^p \otimes γ_i^q \) with
\[ β_i^p = (β_i^{r,s}) \in \prod_{r+s=p} \text{Hom}_R(L^{-s}, M^r) \]
and
\[ γ_i^q = (γ_i^{u,v}) \in \prod_{u+v=q} \text{Hom}_R(K^{-v}, L^u) \]
The map is given by sending \( α \) to \( δ = (δ^{r,v}) \) with
\[ δ^{r,v} = \sum_{i,s} β_i^{r,s} \circ γ_i^{-s,v} \in \text{Hom}_R(K^{-v}, M^r) \]
For given \( r + v = n \) this sum is finite as there are only finitely many nonzero \( α^{p,q} \), hence only finitely many nonzero \( β_i^p \) and \( γ_i^q \). By our sign rules we have
\[ d(α^{p,q}) = d_{\text{Hom}^*(L^*, M^*)}(α^{p,q}) + (-1)^p d_{\text{Hom}^*(K^*, L^*)}(α^{p,q}) \]
\[ = \sum \left( d_M \circ β_i^p \circ γ_i^q - (-1)^p β_i^p \circ d_L \circ γ_i^q \right) \]
\[ + (-1)^p \sum \left( β_i^p \circ d_L \circ γ_i^q - (-1)^q β_i^p \circ γ_i^q \circ d_K \right) \]
\[ = \sum \left( d_M \circ β_i^p \circ γ_i^q - (-1)^n β_i^p \circ γ_i^q \circ d_K \right) \]
It follows that the rules \( α \mapsto δ \) is compatible with differentials and the lemma is proved.

**Lemma 57.3.** Let \( R \) be a ring. Given complexes \( K^*, L^*, M^* \) of \( R \)-modules there is a canonical morphism
\[ \text{Tot}(\text{Hom}^*(L^*, M^*) \otimes_R K^*) \longrightarrow \text{Hom}^*(\text{Hom}^*(K^*, L^*), M^*) \]
of complexes of \( R \)-modules functorial in all three complexes.

**Proof.** Consider an element \( β \) of degree \( n \) of the right hand side. Then
\[ β = (β^{p,q}) \in \prod_{p+q=n} \text{Hom}_R(\text{Hom}^{-q}(K^*, L^*), M^p) \]
Each \( β^{p,q} \) is an element
\[ β^{p,q} = (β^{p,r,s}) \in \prod_{p+r+s=n} \text{Hom}_R(\text{Hom}_R(K^s, L^{-r}), M^p) \]
We can apply the differentials $d_M$ and $d_{\text{Hom}^*(K^*, L^*)}$ to the element $\beta^{p,q}$ and we can apply the differentials $d_K$, $d_L$, $d_M$ to the element $\beta^{p,r,s}$. We omit the precise definitions. The our sign rules tell us that

\[
d(\beta^{p,r,s}) = d_M(\beta^{p,r,s}) - (-1)^n d_{\text{Hom}^*(K^*, L^*)}(\beta^{p,r,s})
\]

\[
= d_M(\beta^{p,r,s}) - (-1)^n (d_L(\beta^{p,r,s}) - (-1)^{r+s} d_K(\beta^{p,r,s}))
\]

\[
= d_M(\beta^{p,r,s}) - (-1)^n d_L(\beta^{p,r,s}) + (-1)^p d_K(\beta^{p,r,s})
\]

On the other hand, an element $\alpha$ of degree $n$ of the left hand side looks like

\[
\alpha = (\alpha^{t,s}) \in \bigoplus_{t+s=n} \text{Hom}^t(L^*, M^*) \otimes K^s
\]

Each $\alpha^{t,s}$ maps to an element

\[
\alpha^{t,s} \mapsto (\alpha^{p,r,s}) \in \prod_{p+r+s=n} \text{Hom}_R(L^{-r}, M^p) \otimes_R K^s
\]

By our sign rules and with conventions as above we get

\[
d(\alpha^{p,r,s}) = d_{\text{Hom}^*(L^*, M^*)}(\alpha^{p,r,s}) + (-1)^{p+r} d_K(\alpha^{p,r,s})
\]

\[
= d_M(\alpha^{p,r,s}) - (-1)^{p+r} d_L(\alpha^{p,r,s}) + (-1)^{p+r} d_K(\alpha^{p,r,s})
\]

To define our map we will use the canonical maps

\[
\epsilon_{p,r,s} : \text{Hom}_R(L^{-r}, M^p) \otimes_R K^s \rightarrow \text{Hom}_R(\text{Hom}_R(K^s, L^{-r}), M^p)
\]

which sends $\varphi \otimes k$ to the map $\psi \mapsto \varphi(\psi(k))$. This is functorial in all three variables. However, since the signs above do not match we need to use instead some map

\[
\epsilon_{p,r,s} c_{p,r,s}
\]

for some sign $\epsilon_{p,r,s}$. Looking at the signs above we find that we need to find a solution for the equations

\[
\epsilon_{p,r,s} = \epsilon_{p+1,r,s}, \quad \epsilon_{p,r,s}(-1)^s = \epsilon_{p,r+1,s}, \quad \epsilon_{p,r,s}(-1)^r = \epsilon_{p,r,s+1}
\]

A good solution is to take $\epsilon_{p,r,s} = (-1)^{rs}$. The choice of this sign is explained in the remark following the proof.

**Remark 57.4.** In the yoga of super vector spaces the sign used in the proof of Lemma 57.3 above can be explained as follows. A super vector space is just a vector space $V$ which comes with a direct sum decomposition $V = V^+ \oplus V^-$. Here we think of the elements of $V^+$ as the even elements and the elements of $V^-$ as the odd ones. Given two super vector spaces $V$ and $W$ we set

\[
(V \otimes W)^+ = (V^+ \otimes W^+) \oplus (V^- \otimes W^-)
\]

and similarly for the odd part. In the category of super vector spaces the isomorphism

\[
V \otimes W \rightarrow W \otimes V
\]

is defined to be the usual one, except that on the summand $V^+ \otimes W^-$ we use the negative of the usual identification. In this way we obtain a tensor category (where $\otimes$ is symmetric and associative with $1$). The category of super vector spaces has an internal hom which we denote $V^\vee$. One checks that the canonical isomorphisms $\text{Hom}(V, W) = W \otimes V^\vee$ and $\text{Hom}(V, W)^\vee = V \otimes W^\vee$ do not involve signs. Finally, given three super vector spaces $U, V, W$ we can consider the analogue

\[
c : \text{Hom}(V, W) \otimes U \rightarrow \text{Hom}(\text{Hom}(U, V), W)
\]
of the maps $c_{p,r,s}$ which occur in the lemma above. Using the formulae given above
(which do not involve signs) this becomes a map
\[ W \otimes V^\vee \otimes U \to W \otimes U \otimes V^\vee \]
which involves a $(-1)$ on elements $w \otimes v^\vee \otimes u$ if $v^\vee$ and $u$ are odd.

**Lemma 57.5.** Let $R$ be a ring. Given complexes $K^\bullet, L^\bullet$ of $R$-modules there is a
canonical morphism
\[ K^\bullet \to \text{Hom}^\bullet(L^\bullet, \text{Tot}(K^\bullet \otimes_R L^\bullet)) \]
of complexes of $R$-modules functorial in both complexes.

**Proof.** Let $\alpha$ be an element of degree $n$ of the right hand side. Thus
\[ \alpha = (\alpha^{p,q}) \in \prod_{p+q=n} \text{Hom}_R(L^{-q}, \text{Tot}^p(K^\bullet \otimes_R L^\bullet)) \]
Each $\alpha^{p,q}$ is an element $\alpha^{p,q} = (\alpha^{r,s,q}) \in \bigoplus_{r+s+q=n} \text{Hom}_R(L^{-q}, K^r \otimes_R L^s)$
By our sign rules we get
\[ d(\alpha^{r,s,q}) = d_{\text{Tot}^p(K^\bullet \otimes_R L^\bullet)} \circ \alpha^{r,s,q} - (-1)^n \alpha^{r,s,q} \circ d_L \]
\[ = d_K \circ \alpha^{r,s,q} + (-1)^r d_L \circ \alpha^{r,s,q} - (-1)^n \alpha^{r,s,q} \circ d_L \]
Now an element $\beta \in K^n$ we send to $\alpha$ with $\alpha^{n,n,q} = \beta \otimes \text{id}_L$ and $\alpha^{r,s,q} = 0$
if $r \neq n$. This is indeed an element as above, as for fixed $q$ there is only one
nonzero $\alpha^{r,s,q}$. The description of the differential shows this is compatible with
differentials. $\square$

**58. Derived hom**

The derived hom functor is an internal hom in the derived category of $R$-modules
in the sense that it is characterized by the formula
\[ \text{Hom}_{D(R)}(K, R \text{Hom}(L, M)) = \text{Hom}_{D(R)}(K \otimes_R L, M) \]
for objects $K, L, M$ of $D(R)$. Note that this formula characterizes the objects up to
unique isomorphism by the Yoneda lemma. A construction can be given as follows.
Choose a $K$-injective complex $I^\bullet$ of $R$-modules representing $M$, choose an complex
$L^\bullet$ representing $L$, and set
\[ R \text{Hom}(L, M) = \text{Hom}^\bullet(L^\bullet, I^\bullet) \]
with notation as in Section 57. A generalization of this construction is discussed in
Differential Graded Algebra, Section 21. From (57.0.1) and Derived Categories,
Lemma 29.2 that we have
\[ H^n(R \text{Hom}(L, M)) = \text{Hom}_{D(R)}(L, M[n]) \]
for all $n \in \mathbb{Z}$. In particular, the object $R \text{Hom}(L, M)$ of $D(R)$ is well defined, i.e.,
independent of the choice of the $K$-injective complex $I^\bullet$.

**Lemma 58.1.** Let $R$ be a ring. Let $K, L, M$ be objects of $D(R)$. There is a
canonical isomorphism
\[ R \text{Hom}(K, R \text{Hom}(L, M)) = R \text{Hom}(K \otimes_R L, M) \]
in $D(R)$ functorial in $K, L, M$ which recovers (58.0.1) by taking $H^0$. 


Proof. Choose a K-injective complex $I^*$ representing $M$ and a K-flat complex of $R$-modules $L^*$ representing $L$. For any complex of $R$-modules $K^*$ we have
\[
\text{Hom}^*(K^*, \text{Hom}^*(L^*, I^*)) = \text{Hom}^*(\text{Tot}(K^* \otimes_R L^*), I^*)
\]
by Lemma 57.1. The lemma follows by the definition of $R\text{Hom}$ and because $\text{Tot}(K^* \otimes_R L^*)$ represents the derived tensor product.

\[\square\]

Lemma 58.2. Let $R$ be a ring. Let $P^*$ be a bounded above complex of projective $R$-modules. Let $L^*$ be a complex of $R$-modules. Then $R\text{Hom}(P^*, L^*)$ is represented by the complex $\text{Hom}^*(P^*, L^*)$.

Proof. By (57.0.1) and Derived Categories, Lemma 19.8 the cohomology groups of the complex are “correct”. Hence if we choose a quasi-isomorphism $L^* \to I^*$ with $I^*$ a K-injective complex of $R$-modules then the induced map
\[
\text{Hom}^*(P^*, L^*) \to \text{Hom}^*(P^*, I^*)
\]
is a quasi-isomorphism. As the right hand side is our definition of $R\text{Hom}(P^*, L^*)$ we win.

\[\square\]

Lemma 58.3. Let $R$ be a ring. Given $K, L, M$ in $D(R)$ there is a canonical morphism
\[
R\text{Hom}(L, M) \otimes^L_R K \to R\text{Hom}(R\text{Hom}(K, L), M)
\]
in $D(R)$ functorial in $K, L, M$.

Proof. Choose a K-injective complex $I^*$ representing $M$, a K-injective complex $J^*$ representing $L$, and a K-flat complex $K^*$ representing $K$. The map is defined using the map
\[
\text{Tot}(\text{Hom}^*(J^*, I^*) \otimes_R K^*) \to \text{Hom}^*(\text{Hom}^*(K^*, J^*), I^*)
\]
of Lemma 57.3. We omit the proof that this is functorial in all three objects of $D(R)$.

\[\square\]

Lemma 58.4. Let $R$ be a ring. Given $K, L, M$ in $D(R)$ there is a canonical morphism
\[
R\text{Hom}(L, M) \otimes^L_R R\text{Hom}(K, L) \to R\text{Hom}(K, M)
\]
in $D(R)$.

Proof. In general (without suitable finiteness conditions) we do not see how to get this map from Lemma 57.2. Instead, we use the maps
\[
\begin{align*}
R\text{Hom}(L, M) \otimes^L_R R\text{Hom}(K, L) \otimes^L_R K \\
\downarrow \\
R\text{Hom}(R\text{Hom}(K, L), M) \otimes^L_R R\text{Hom}(K, L) \\
\downarrow \\
M
\end{align*}
\]
gotten by applying Lemma 58.3 twice. Finally, we use Lemma 58.1 to translate the composition
\[
R\text{Hom}(L, M) \otimes^L_R R\text{Hom}(K, L) \otimes^L_R K \to M
\]
into a map as in the statement of the lemma.

\[\square\]
**Lemma 58.5.** Let $R$ be a ring. Given complexes $K, L$ in $D(R)$ there is a canonical morphism

$$K \to R\text{Hom}(L, K \otimes_R L)$$

in $D(R)$ functorial in both $K$ and $L$.

**Proof.** Choose a $K$-flat complexes $K^\bullet$ and $L^\bullet$ representing $K$ and $L$. Choose a quasi-isomorphism $\text{Tot}(K^\bullet \otimes_R L^\bullet) \to I^\bullet$ where $I^\bullet$ is K-injective. Then we use the map

$$K^\bullet \to \text{Hom}^\bullet(L^\bullet, \text{Tot}(K^\bullet \otimes_R L^\bullet)) \to \text{Hom}^\bullet(L^\bullet, I^\bullet)$$

where the first map is the map from Lemma 57.5. □

### 59. Perfect complexes

A perfect complex is a pseudo-coherent complex of finite tor dimension. We will not use this as the definition, but define perfect complexes over a ring directly as follows.

**Definition 59.1.** Let $R$ be a ring. Denote $D(R)$ the derived category of the abelian category of $R$-modules.

1. An object $K$ of $D(R)$ is **perfect** if it is quasi-isomorphic to a bounded complex of finite projective $R$-modules.
2. An $R$-module $M$ is **perfect** if $M[0]$ is a perfect object in $D(R)$.

For example, over a Noetherian ring a finite module is perfect if and only if it has finite projective dimension, see Lemma 59.3 and Algebra, Definition 106.2.

**Lemma 59.2.** Let $K^\bullet$ be an object of $D(R)$. The following are equivalent

1. $K^\bullet$ is perfect, and
2. $K^\bullet$ is pseudo-coherent and has finite tor dimension.

**Proof.** It is clear that (1) implies (2), see Lemmas 52.5 and 53.3. Assume (2). Choose a bounded above complex $F^\bullet$ of finite free $R$-modules and a quasi-isomorphism $F^\bullet \to K^\bullet$. Assume that $K^\bullet$ has tor-amplitude in $[a, b]$. Set $E^\bullet = \tau_{\geq a}F^\bullet$. Note that $E_i$ is finite free except $E^a$ which is a finitely presented $R$-module. By Lemma 53.2 $E^a$ is flat. Hence by Algebra, Lemma 76.2 we see that $E^a$ is finite projective. □

**Lemma 59.3.** Let $M$ be a module over a ring $R$. The following are equivalent

1. $M$ is a perfect module, and
2. there exists a resolution

$$0 \to F_d \to \ldots \to F_1 \to F_0 \to M \to 0$$

with each $F_i$ a finite projective $R$-module.

**Proof.** Assume (2). Then the complex $E^\bullet$ with $E^{-i} = F_i$ is quasi-isomorphic to $M[0]$. Hence $M$ is perfect. Conversely, assume (1). By Lemmas 59.2 and 52.4 we can find resolution $E^\bullet \to M$ with $E^{-i}$ a finite free $R$-module. By Lemma 53.2 we see that $F_d = \text{Coker}(E^{d-1} \to E^d)$ is flat for some $d$ sufficiently large. By Algebra, Lemma 76.2 we see that $F_d$ is finite projective. Hence

$$0 \to F_d \to E^{-d+1} \to \ldots \to E^0 \to M \to 0$$

is the desired resolution. □
Lemma 59.4. Let $R$ be a ring. Let $(K^\bullet, L^\bullet, M^\bullet, f, g, h)$ be a distinguished triangle in $\mathcal{D}(R)$. If two out of three of $K^\bullet, L^\bullet, M^\bullet$ are perfect then the third is also perfect.

Proof. Combine Lemmas 59.2, 52.6, and 53.4.

Lemma 59.5. Let $R$ be a ring. If $K^\bullet \oplus L^\bullet$ is perfect, then so are $K^\bullet$ and $L^\bullet$.

Proof. Follows from Lemmas 59.2, 52.8, and 53.6.

Lemma 59.6. Let $R$ be a ring. Let $K^\bullet$ be a bounded complex of perfect $R$-modules. Then $K^\bullet$ is a perfect complex.

Proof. Follows by induction on the length of the finite complex: use Lemma 59.4 and the stupid truncations.

Lemma 59.7. Let $R$ be a ring. If $K^\bullet \in \mathcal{D}^b(R)$ and all its cohomology modules are perfect, then $K^\bullet$ is perfect.

Proof. Follows by induction on the length of the finite complex: use Lemma 59.4 and the canonical truncations.

Lemma 59.8. Let $A \to B$ be a ring map. Assume that $B$ is perfect as an $A$-module. Let $K^\bullet$ be a perfect complex of $B$-modules. Then $K^\bullet$ is perfect as a complex of $A$-modules.

Proof. Using Lemma 59.2, this translates into the corresponding results for pseudo-coherent modules and modules of finite tor dimension. See Lemma 53.10 and Lemma 52.11 for those results.

Lemma 59.9. Let $A \to B$ be a ring map. Let $K^\bullet$ be a perfect complex of $A$-modules. Then $K^\bullet \otimes_A B$ is a perfect complex of $B$-modules.

Proof. Using Lemma 59.2, this translates into the corresponding results for pseudo-coherent modules and modules of finite tor dimension. See Lemma 53.11 and Lemma 52.12 for those results.

Lemma 59.10. Let $A \to B$ be a flat ring map. Let $M$ be a perfect $A$-module. Then $M \otimes_A B$ is a perfect $B$-module.

Proof. By Lemma 59.3, the assumption implies that $M$ has a finite resolution $F_\bullet$ by finite projective $R$-modules. As $A \to B$ is flat the complex $F_\bullet \otimes_A B$ is a finite length resolution of $M \otimes_A B$ by finite projective modules over $B$. Hence $M \otimes_A B$ is perfect.

Lemma 59.11. Let $R$ be a ring. Let $f_1, \ldots, f_r \in R$ be elements which generate the unit ideal. Let $K^\bullet$ be a complex of $R$-modules. If for each $i$ the complex $K^\bullet \otimes_R R_{f_i}$ is perfect, then $K^\bullet$ is perfect.

Proof. Using Lemma 59.2, this translates into the corresponding results for pseudo-coherent modules and modules of finite tor dimension. See Lemma 53.13 and Lemma 52.14 for those results.

Lemma 59.12. Let $R$ be a ring. Let $a, b \in \mathbb{Z}$. Let $K^\bullet$ be a complex of $R$-modules. Let $R \to R'$ be a faithfully flat ring map. If the complex $K^\bullet \otimes_R R'$ has tor amplitude in $[a, b]$, then $K^\bullet$ has tor amplitude in $[a, b]$. 

Proof. Using Lemma 59.2 this translates into the corresponding results for pseudo-coherent modules and modules of finite tor dimension. See Lemma 53.14 and Lemma 52.15 for those results. □

Lemma 59.13. Let $R$ be a regular ring of finite dimension. Then

1. an $R$-module is perfect if and only if it is a finite $R$-module, and
2. a complex of $R$-modules $K^\bullet$ is perfect if and only if $K^\bullet \in D^b(R)$ and each $H^i(K^\bullet)$ is a finite $R$-module.

Proof. By Algebra, Lemma 107.8 the assumption on $R$ means that $R$ has finite global dimension. Hence every module has finite tor dimension, see Lemma 53.15.

On the other hand, as $R$ is Noetherian, a module is pseudo-coherent if and only if it is finite, see Lemma 52.16. This proves part (1).

Let $K^\bullet$ be a complex of $R$-modules. If $K^\bullet$ is perfect, then it is in $D^b(R)$ and it is quasi-isomorphic to a finite complex of finite projective $R$-modules so certainly each $H^i(K^\bullet)$ is a finite $R$-module (as $R$ is Noetherian). Conversely, suppose that $K^\bullet$ is in $D^b(R)$ and each $H^i(K^\bullet)$ is a finite $R$-module. Then by (1) each $H^i(K^\bullet)$ is a perfect $R$-module, whence $K^\bullet$ is perfect by Lemma 59.7. □

Lemma 59.14. Let $R$ be a ring. Let $p \subset R$ be a prime ideal. Let $K^\bullet$ be a pseudo-coherent complex of $R$-modules. Assume that for some $i \in \mathbb{Z}$ the map

$$H^i(K^\bullet) \otimes_R \kappa(p) \longrightarrow H^i(K^\bullet \otimes_R \kappa(p))$$

is surjective. Then there exists an $f \in R$, $f \not\in p$ such that $\tau_{2i+1}K^\bullet \otimes_R R_f$ is a perfect object of $D(R_f)$ with tor amplitude in $[i+1, \infty]$ and such that there exists an isomorphism

$$K^\bullet \otimes_R R_f \cong \tau_{\leq i}K^\bullet \otimes_R R_f \oplus \tau_{\geq i+1}K^\bullet \otimes_R R_f$$

in $D(R_f)$.

Proof. In this proof all tensor products are over $R$ and we write $\kappa = \kappa(p)$. We may assume that $K^\bullet$ is a bounded above complex of finite free $R$-modules. Let us inspect what is happening in degree $i$:

$$\ldots \rightarrow K^i \xrightarrow{d^i} K^{i+1} \rightarrow \ldots$$

Let $0 \subset V \subset W \subset K^i \otimes_R \kappa$ be defined by the formulas

$$V = \text{Im} \left( K^{i-1} \otimes_R \kappa \rightarrow K^i \otimes_R \kappa \right) \quad \text{and} \quad W = \text{Ker} \left( K^i \otimes_R \kappa \rightarrow K^{i+1} \otimes_R \kappa \right)$$

Set $\dim(V) = r$, $\dim(W/V) = s$, and $\dim(K^i \otimes_R \kappa/W) = t$. We can pick $x_1, \ldots, x_r \in K^{i-1}$ which map by $d^i_1$ to a basis of $V$. By our assumption we can pick $y_1, \ldots, y_s \in \text{Ker}(d^i)$ mapping to a basis of $W/V$. Finally, choose $z_1, \ldots, z_t \in K^i$ mapping to a basis of $K^i \otimes_R \kappa/W$. Then we see that the elements $d^i(z_1), \ldots, d^i(z_t) \in K^{i+1}$ are linearly independent in $K^{i+1} \otimes_R \kappa$. By Algebra, Lemma 77.3 we may after replacing $R$ by $R_f$ for some $f \in R$, $f \not\in p$ assume that

1. $d^i(x_a), y_b, z_c$ is an $R$-basis of $K^i$,
2. $d^i(z_1), \ldots, d^i(z_t)$ are $R$-linearly independent in $K^{i+1}$, and
3. the quotient $K^{i+1}/\sum Rd^i(z_c)$ is finite projective.

Since $d^i$ annihilates $d^{i-1}(x_a)$ and $y_b$, we deduced from condition (2) that $\text{Im}(d^{i-1}) \subset \sum Rd^{i-1}(x_a) + \sum Ry_b$. Set

$$E^\bullet = (\ldots \rightarrow 0 \rightarrow \bigoplus_{c=1,\ldots,t} Rz_c \rightarrow K^{i+1} \rightarrow \ldots)$$
We obtain a morphism of complexes $K^i \to E^i$ where in degree $i$ we take the projection of $K^i$ onto the summand generated by the basis vectors $z_{e_i}$ which is possible by (1). It is clear that this induces an isomorphism $\tau_{i+1}K^i \to \tau_{i+1}E^i$ and a quasi-isomorphism $E^i \to \tau_{i+1}E^i$. By condition (3) the complex $\tau_{i+1}E^i$ is a finite complex of finite projective modules supported in degrees $\geq i+1$, hence perfect of tor amplitude contained in $[i+1, \infty]$. Finally, the inclusion $E^i \subset K^i$ gives a section to the map $K^i \to \tau_{i+1}K^i$ in the derived category. Since we have the canonical distinguished triangle

$$\tau_{\leq i}K^i \to K^i \to \tau_{\geq i+1}K^i \to (\tau_{\leq i}K^i)[1]$$

in $D(R)$, see Derived Categories, Remark 12.4 we conclude by Derived Categories, Lemma 4.10

**Lemma 59.15.** Let $R$ be a ring. Let $p \subset R$ be a prime ideal. Let $K^i$ be a pseudo-coherent complex of $R$-modules. Assume that for some $i \in \mathbb{Z}$ the maps

$$H^i(K^i) \otimes_R \kappa(p) \to H^i(K^i \otimes_R \kappa(p))$$

are surjective. Then there exists an $f \in R$, $f \notin p$ such that

1. $\tau_{\geq i+1}K^i \otimes_R R_f$ is a perfect object of $D(R_f)$ with tor amplitude in $[i+1, \infty]$,
2. $H^i(K^i)_f$ is a finite projective $R_f$-module, and
3. $K^i \otimes_R R_f \cong \tau_{\leq i-1}K^i \otimes_R R_f \otimes H^i(K^i)_f \otimes \tau_{\geq i+1}K^i \otimes_R R_f$ in $D(R_f)$.

**Proof.** We get (1) from Lemma 59.14 as well as a splitting $K^i \otimes_R R_f = \tau_{\leq i}K^i \otimes_R R_f \oplus \tau_{\geq i+1}K^i \otimes_R R_f$ in $D(R_f)$. Applying Lemma 59.14 once more to $\tau_{\leq i}K^i \otimes_R R_f$ we obtain (after suitably choosing $f$) a splitting $\tau_{\leq i}K^i \otimes_R R_f = \tau_{\leq i-1}K^i \otimes_R R_f \oplus H^i(K^i)_f$ in $D(R_f)$ as well as the conclusion that $H^i(K^i)_f$ is a flat perfect module, i.e., finite projective.

**Lemma 59.16.** Let $R$ be a ring. Let $p \subset R$ be a prime ideal. Let $i \in \mathbb{Z}$. Let $K^i$ be a pseudo-coherent complex of $R$-modules such that $H^i(K^i \otimes_R \kappa(p)) = 0$. Then there exists an $f \in R$, $f \notin p$ such that

$$K^i \otimes_R R_f = \tau_{\geq i+1}K^i \otimes_R R_f \oplus \tau_{\leq i-1}K^i \otimes_R R_f$$

in $D(R_f)$ with $\tau_{\geq i+1}K^i \otimes_R R_f$ a perfect complex with tor amplitude in $[i+1, j]$ for some $j \in \mathbb{Z}$.

**Proof.** One can deduce this from the (more general) Lemma 59.14 but we will also prove it directly here. We may assume that $K^i$ is a bounded above complex of finite free $R$-modules. Let us inspect what is happening in degree $i$:

$$\ldots \to K^{i-2} \to R^{\oplus l} \to R^{\oplus m} \to R^{\oplus n} \to K^{i+2} \to \ldots$$

Let $A$ be the $m \times l$ matrix corresponding to $K^{i-1} \to K^i$ and let $B$ be the $n \times m$ matrix corresponding to $K^i \to K^{i+1}$. The assumption is that $A \mod p$ has rank $r$ and that $B \mod p$ has rank $m-r$. In other words, there is some $r \times r$ minor $a$ of $A$ which is not in $p$ and there is some $(m-r) \times (m-r)$-minor $b$ of $B$ which is not in $p$. Set $f = ab$. Then after inverting $f$ we can find direct sum decompositions $K^{i-1} = R^{\oplus l} \oplus R^{\oplus r}$, $K^i = R^{\oplus r} \oplus R^{\oplus m-r}$, $K^{i+1} = R^{\oplus m-r} \oplus R^{\oplus n-m+r}$ such that the module map $K^{i-1} \to K^i$ kills of $R^{\oplus r}$ and induces an isomorphism of $R^{\oplus r}$ onto the corresponding summand of $K^i$ and such that the module map $K^i \to K^{i+1}$ kills...
of $R^\oplus r$ and induces an isomorphism of $R^\oplus m - r$ onto the corresponding summand of $K^{i+1}$. Thus $K^\bullet$ becomes quasi-isomorphic to
\[ \ldots \to K^{i-2} \to R^\oplus l - r \to 0 \to R^\oplus n - m + r \to K^{i+2} \to \ldots \]
and everything is clear. \qed

Lemma 59.17. Let $R$ be a ring. Let $a, b \in \mathbb{Z}$. Let $K^\bullet$ be a pseudo-coherent complex of $R$-modules. The following are equivalent

1. $K^\bullet$ is perfect with tor amplitude in $[a, b]$,
2. for every prime $p$ we have $H^i(K^\bullet \otimes_R^L \kappa(p)) = 0$ for all $i \not\in [a, b]$, and
3. for every maximal ideal $m$ we have $H^i(K^\bullet \otimes_R^L \kappa(m)) = 0$ for all $i \not\in [a, b]$.

Proof. We omit the proof of the implications (1) \Rightarrow (2) \Rightarrow (3). Assume (3). Let $i \in \mathbb{Z}$ with $i \not\in [a, b]$. By Lemma 59.16 we see that the assumption implies that $H^i(K^\bullet)_m = 0$ for all maximal ideals of $R$. Hence $H^i(K^\bullet) = 0$, see Algebra, Lemma 23.1. Moreover, Lemma 59.16 now also implies that for every maximal ideal $m$ there exists an element $f \in R, f \not\in m$ such that $K^\bullet \otimes_R R_f$ is perfect with tor amplitude in $[a, b]$. Hence we conclude by appealing to Lemmas 59.11 and 53.13. \qed

Lemma 59.18. Let $R$ be a ring. Let $K^\bullet$ be a pseudo-coherent complex of $R$-modules. The following are equivalent

1. $K^\bullet$ is perfect,
2. for every prime ideal $p$ the complex $K^\bullet \otimes_R R_p$ is perfect,
3. for every prime $p$ we have $H^i(K^\bullet \otimes_R^L \kappa(p)) = 0$ for all $i \ll 0$,
4. for every maximal ideal $m$ the complex $K^\bullet \otimes_R^L \kappa(m)$ is perfect,
5. for every maximal ideal $m$ we have $H^i(K^\bullet \otimes_R^L \kappa(m)) = 0$ for all $i \ll 0$.

Proof. Assume (5). Pick a maximal ideal $m$ of $R$. By Lemma 59.16 we see that the assumption implies that $K^\bullet \otimes_R R_f$ is a perfect complex for some $f \in R, f \not\in m$. Since Spec($R$) is quasi-compact we conclude that $K^\bullet$ is perfect by Lemmas 59.11. The proof of the other implications is omitted. \qed

The following lemma useful in order to find perfect complexes over a polynomial ring $B = A[x_1, \ldots, x_d]$.

Lemma 59.19. Let $A \to B$ be a ring map. Let $a, b \in \mathbb{Z}$. Let $d \geq 0$. Let $K^\bullet$ be a complex of $B$-modules. Assume

1. the ring map $A \to B$ is flat,
2. for every prime $p \subset A$ the ring $B \otimes_A \kappa(p)$ has finite global dimension $\leq d$,
3. $K^\bullet$ is pseudo-coherent as a complex of $B$-modules, and
4. $K^\bullet$ has tor amplitude in $[a, b]$ as a complex of $A$-modules.

Then $K^\bullet$ is perfect as a complex of $B$-modules with tor amplitude in $[a - d, b]$.

Proof. We may assume that $K^\bullet$ is a bounded above complex of finite free $B$-modules. In particular, $K^\bullet$ is flat as a complex of $A$-modules and $K^\bullet \otimes_A M = K^\bullet \otimes_A^L M$ for any $A$-module $M$. For every prime $p$ of $A$ the complex

$K^\bullet \otimes_A \kappa(p)$

is a bounded above complex of finite free modules over $B \otimes_A \kappa(p)$ with vanishing $H^1$ except for $i \in [a, b]$. As $B \otimes_A \kappa(p)$ has global dimension $d$ we see from Lemma 53.15 that $K^\bullet \otimes_A \kappa(p)$ has tor amplitude in $[a - d, b]$. Let $q$ be a prime of $B$ lying
over \( p \). Since \( K^\bullet \otimes_A \kappa(p) \) is a bounded above complex of free \( B \otimes_A \kappa(q) \)-modules we see that

\[
K^\bullet \otimes_B^L \kappa(q) = K^\bullet \otimes_B \kappa(q) \\
= (K^\bullet \otimes_A \kappa(p)) \otimes_{B \otimes_A \kappa(q)} \kappa(q) \\
= (K^\bullet \otimes_A \kappa(p)) \otimes_B^L \kappa(q)
\]

Hence the arguments above imply that \( H^i(K^\bullet \otimes_B^L \kappa(q)) = 0 \) for \( i \not\in [a - d, b] \). We conclude by Lemma \[59.17\]

The following lemma is a local version of Lemma \[59.10\] It can be used to find perfect complexes over regular local rings.

**Lemma 59.20.** Let \( A \to B \) be a local ring homomorphism. Let \( a, b \in \mathbb{Z} \). Let \( d \geq 0 \). Let \( K^\bullet \) be a complex of \( B \)-modules. Assume

1. the ring map \( A \to B \) is flat,
2. the ring \( B/m_A B \) is regular of dimension \( d \),
3. \( K^\bullet \) is pseudo-coherent as a complex of \( B \)-modules, and
4. \( K^\bullet \) has tor amplitude in \( [a, b] \) as a complex of \( A \)-modules, in fact it suffices if \( H^i(K^\bullet \otimes_A^L \kappa(m_A)) \) is nonzero only for \( i \in [a, b] \).

Then \( K^\bullet \) is perfect as a complex of \( B \)-modules with tor amplitude in \( [a - d, b] \).

**Proof.** By (3) we may assume that \( K^\bullet \) is a bounded above complex of finite free \( B \)-modules. We compute

\[
K^\bullet \otimes_B^L \kappa(m_B) = K^\bullet \otimes_B \kappa(m_B) \\
= (K^\bullet \otimes_A \kappa(m_A)) \otimes_{B/m_A B} \kappa(m_B) \\
= (K^\bullet \otimes A \kappa(m_A)) \otimes_B^L \kappa(m_B)
\]

The first equality because \( K^\bullet \) is a bounded above complex of flat \( B \)-modules. The second equality follows from basic properties of the tensor product. The third equality holds because \( K^\bullet \otimes_A \kappa(m_A) = K^\bullet/m_A K^\bullet \) is a bounded above complex of flat \( B/m_A B \)-modules. Since \( K^\bullet \) is a bounded above complex of flat \( A \)-modules by (1), the cohomology modules \( H^i \) of the complex \( K^\bullet \otimes_A \kappa(m_A) \) are nonzero only for \( i \in [a, b] \) by assumption (4). Thus the spectral sequence of Example \[50.1\] and the fact that \( B/m_A B \) has finite global dimension \( d \) (by (2) and Algebra, Proposition \[107.1\]) shows that \( H^j(K^\bullet \otimes_B^L \kappa(m_B)) \) is zero for \( j \not\in [a - d, b] \). This finishes the proof by Lemma \[59.17\].

**Lemma 59.21.** Let \( K^\bullet \) be a perfect complex over a ring \( A \). There exists a perfect complex \( E^\bullet \) such that we have functorial isomorphisms

\[
H^0(K^\bullet \otimes_A^L L^\bullet) = \text{Ext}_A^0(E^\bullet, L^\bullet)
\]

for \( L^\bullet \in D(A) \).

**Proof.** We may assume that \( K^\bullet \) is a finite complex of finite projective \( A \)-modules. The cohomology group on the left is simply \( H^0(\text{Tot}(K^\bullet \otimes_A L^\bullet)) \). Set \( E^\bullet = \text{Hom}_A(K^\bullet, A) \), i.e., \( E^n = \text{Hom}_A(K^{-n}, A) \) with differentials the transpose of the differentials of \( K^\bullet \). Observe that \( E^\bullet \) is a finite complex of finite projective \( A \)-modules. The group on the right is \( \text{Mor}_{K(\text{Mod}_A)}(E^\bullet, L^\bullet) \) by Derived Categories,
Lemma 19.8 and the definition of Ext groups, see Derived Categories, Section 37

By definition this is the cohomology of

\[ \prod_n \text{Hom}_A(E^n, L^{n-1}) \rightarrow \prod_n \text{Hom}_A(E^n, L^n) \rightarrow \prod_n \text{Hom}_A(E^n, L^{n+1}) \]

Using \( \text{Hom}_A(E^n, L) = (K^n \otimes_A L) \) as \( K^{-n} \) is finite projective, we see that the cohomology groups are the same. \( \square \)

60. Characterizing perfect complexes

In this section we prove that the perfect complexes are exactly the compact objects of the derived category of a ring. First we show the following.

**Lemma 60.1.** Let \( R \) be a ring. The full subcategory \( D_{\text{perf}}(R) \subset D(R) \) of perfect objects is the smallest strictly full, saturated, triangulated subcategory containing \( R = R[0] \). In other words \( D_{\text{perf}}(R) = \langle R \rangle \). In particular, \( R \) is a classical generator for \( D_{\text{perf}}(R) \).

**Proof.** To see what the statement means, please look at Derived Categories, Definitions 6.1 and 33.2. It was shown in Lemmas 59.4 and 59.5 that \( D_{\text{perf}}(R) \subset D(R) \) is a strictly full, saturated, triangulated subcategory of \( D(R) \). Of course \( R \in D_{\text{perf}}(R) \).

Recall that \( \langle R \rangle = \bigcup \langle R \rangle_n \). To finish the proof we will show that if \( M \in D_{\text{perf}}(R) \) is represented by

\[ \ldots \rightarrow 0 \rightarrow M^a \rightarrow M^{a+1} \rightarrow \ldots \rightarrow M^b \rightarrow 0 \rightarrow \ldots \]

with \( M^i \) finite projective, then \( M \in \langle R \rangle_{b-a+1} \). The proof is by induction on \( b-a \).

By definition \( \langle R \rangle_1 \) contains any finite projective \( R \)-module placed in any degree; this deals with the base case \( b-a = 1 \) of the induction. In general, we consider the distinguished triangle

\[ M_b[-b] \rightarrow M^\bullet \rightarrow \sigma_{b-1}M^\bullet \rightarrow M_b[-b+1] \]

By induction the truncated complex \( \sigma_{b-1}M^\bullet \) is in \( \langle R \rangle_{b-a} \) and \( M_b[-b] \) is in \( \langle R \rangle_1 \).

Hence \( M^\bullet \in \langle R \rangle_{b-a+1} \) by definition. \( \square \)

Let \( R \) be a ring. Recall that \( D(R) \) has direct sums which are given simply by taking direct sums of complexes, see Derived Categories, Lemma 31.2. We will use this in the lemmas of this section without further mention.

**Lemma 60.2.** Let \( R \) be a ring. Let \( K \in D(R) \) be an object such that for every countable set of objects \( E_n \in D(R) \) the canonical map

\[ \bigoplus \text{Hom}_{D(R)}(K, E_n) \rightarrow \text{Hom}_{D(R)}(K, \bigoplus E_n) \]

is a bijection. Then, given any system \( L^\bullet \) of complexes over \( \mathbb{N} \) we have that

\[ \text{colim} \text{Hom}_{D(R)}(K, L^\bullet_n) \rightarrow \text{Hom}_{D(R)}(K, L^\bullet) \]

is a bijection, where \( L^\bullet \) is the termwise colimit, i.e., \( L^m = \text{colim} L^m_n \) for all \( m \in \mathbb{Z} \).

**Proof.** Consider the short exact sequence of complexes

\[ 0 \rightarrow \bigoplus L^\bullet_n \rightarrow \bigoplus L^\bullet_n \rightarrow L^\bullet \rightarrow 0 \]

where the first map is given by \( 1 - t_n \) in degree \( n \) where \( t_n : L^\bullet_n \rightarrow L^\bullet_{n+1} \) is the transition map. By Derived Categories, Lemma 12.1 this is a distinguished triangle
in $D(R)$. Apply the homological functor $\text{Hom}_{D(R)}(K, -)$, see Derived Categories, Lemma 34.2. Thus a long exact cohomology sequence

$$\cdots \to \text{Hom}_{D(R)}(K, \text{colim} L_n^•[-1]) \to \text{Hom}_{D(R)}(K, \bigoplus L_n^•) \to \text{Hom}_{D(R)}(K, \bigoplus L_n^•) \to \text{Hom}_{D(R)}(K, \text{colim} L_n^•) \to \cdots$$

Since we have assumed that $\text{Hom}_{D(R)}(K, \bigoplus L_n^•)$ is equal to $\bigoplus \text{Hom}_{D(R)}(K, L_n^•)$ we see that the first map on every row of the diagram is injective (by the explicit description of this map as the sum of the maps induced by $1 - t_n$). Hence we conclude that $\text{Hom}_{D(R)}(K, \text{colim} L_n^•)$ is the cokernel of the first map of the middle row in the diagram above which is what we had to show.

The following proposition, characterizing perfect complexes as the compact objects (Derived Categories, Definition 34.1) of the derived category, shows up in various places. See for example [Ric89, proof of Proposition 6.3] (this treats the bounded case), [TT90, Theorem 2.4.3] (the statement doesn’t match exactly), and [BN93, Proposition 6.4] (watch out for horrendous notational conventions).

**Proposition 60.3.** Let $R$ be a ring. For an object $K$ of $D(R)$ the following are equivalent

1. $K$ is perfect, and
2. $K$ is a compact object of $D(R)$.

**Proof.** Assume $K$ is perfect, i.e., $K$ is quasi-isomorphic to a bounded complex $P^•$ of finite projective modules, see Definition 59.1. If $E_i$ is represented by the complex $E_i^•$, then $\bigoplus E_i$ is represented by the complex whose degree $n$ term is $\bigoplus E_i^n$. On the other hand, as $P^n$ is projective for all $n$ we have $\text{Hom}_{D(R)}(P^•, K^•) = \text{Hom}_{K(R)}(P^•, K^•)$ for every complex of $R$-modules $K^•$, see Derived Categories, Lemma 19.8. Thus $\text{Hom}_{D(R)}(P^•, E^•)$ is the cohomology of the complex

$$\prod \text{Hom}_R(P^n, E^{n-1}) \to \prod \text{Hom}_R(P^n, E^n) \to \prod \text{Hom}_R(P^n, E^{n+1}).$$

Since $P^•$ is bounded we see that we may replace the $\prod$ signs by $\bigoplus$ signs in the complex above. Since each $P^n$ is a finite $R$-module we see that $\text{Hom}_R(P^n, \bigoplus E_i^m) = \bigoplus \text{Hom}_R(P^n, E_i^m)$ for all $n, m$. Combining these remarks we see that the map of Derived Categories, Definition 34.1 is a bijection.

Conversely, assume $K$ is compact. Represent $K$ by a complex $K^•$ and consider the map

$$K^• \to \bigoplus_{n \geq 0} \tau_{\geq n} K^•$$

where we have used the canonical truncations, see Homology, Section 13. This makes sense as in each degree the direct sum on the right is finite. By assumption this map factors through a finite direct sum. We conclude that $K \to \tau_{\geq n} K$ is zero for at least one $n$, i.e., $K$ is in $D^-(R)$.
Since $K \in D^-(R)$ and since every $R$-module is a quotient of a free module, we may represent $K$ by a bounded above complex $K^\bullet$ of free $R$-modules, see Derived Categories, Lemma 16.5. Note that we have

$$K^\bullet = \bigcup_{n \leq 0} \sigma_{\geq n} K^\bullet$$

where we have used the stupid truncations, see Homology, Section 13. Hence by Lemma 60.2 we see that $1 : K^\bullet \to K^\bullet$ factors through $\sigma_{\geq n} K^\bullet \to K^\bullet$ in $D(R)$. Thus we see that $1 : K^\bullet \to K^\bullet$ factors as

$$K^\bullet \xrightarrow{\varphi} L^\bullet \xrightarrow{\psi} K^\bullet$$

in $D(R)$ for some complex $L^\bullet$ which is bounded and whose terms are free $R$-modules. Say $L^i = 0$ for $i \not\in [a, b]$. Fix $a, b$ from now on. Let $c$ be the largest integer $\leq b + 1$ such that we can find a factorization of $1_{K^\bullet}$ as above with $L^i$ is finite free for $i < c$. We will show by induction that $c = b + 1$. Namely, write $L^c = \bigoplus_{\lambda \in \Lambda} R$. Since $L^{c-1}$ is finite free we can find a finite subset $\Lambda' \subset \Lambda$ such that $L^{c-1} \to L^c$ factors through $\bigoplus_{\lambda \in \Lambda'} R \subset L^c$. Consider the map of complexes

$$\pi : L^\bullet \longrightarrow \left( \bigoplus_{\lambda \in \Lambda \setminus \Lambda'} R \right)[-i]$$

given by the projection onto the factors corresponding to $\Lambda \setminus \Lambda'$ in degree $i$. By our assumption on $K$ we see that, after possibly replacing $\Lambda'$ by a larger finite subset, we may assume that $\pi \circ \varphi = 0$ in $D(R)$. Let $(L')^\bullet \subset L^\bullet$ be the kernel of $\pi$. Since $\pi$ is surjective we get a short exact sequence of complexes, which gives a distinguished triangle in $D(R)$ (see Derived Categories, Lemma 12.1). Since $\text{Hom}_{D(R)}(K, -)$ is homological (see Derived Categories, Lemma 4.2) and $\pi \circ \varphi = 0$, we can find a morphism $\varphi' : K^\bullet \to (L')^\bullet$ in $D(R)$ whose composition with $(L')^\bullet \to L^\bullet$ gives $\varphi$. Setting $\psi'$ equal to the composition of $\psi$ with $(L')^\bullet \to L^\bullet$ we obtain a new factorization. Since $(L')^\bullet$ agrees with $L^\bullet$ except in degree $c$ and since $(L')^c = \bigoplus_{\lambda \in \Lambda'} R$ the induction step is proved.

The conclusion of the discussion of the preceding paragraph is that $1_K : K \to K$ factors as

$$K \xrightarrow{\varphi} L \xrightarrow{\psi} K$$

in $D(R)$ where $L$ can be represented by a finite complex of free $R$-modules. In particular we see that $L$ is perfect. Note that $e = \varphi \circ \psi \in \text{End}_{D(R)}(L)$ is an idempotent. By Derived Categories, Lemma 4.12 we see that $L = \text{Ker}(e) \oplus \text{Ker}(1 - e)$. The map $\varphi : K \to L$ induces an isomorphism with $\text{Ker}(1 - e)$ in $D(R)$. Hence we finally conclude that $K$ is perfect by Lemma 59.5.

**Lemma 60.4.** Let $R$ be a ring. Let $I \subset R$ be an ideal. Let $\mathcal{P}^\bullet$ be a complex of $R/I$-modules. Let $K$ be an object of $D(R)$. Assume that

1. $\mathcal{P}^\bullet$ is a bounded above complex of projective $R/I$-modules,
2. $K \otimes_R R/I$ is represented by $\mathcal{P}^\bullet$ in $D(R/I)$, and
3. $I$ is a nilpotent ideal.

Then there exists a bounded above complex $P^\bullet$ of projective $R$-modules representing $K$ in $D(R)$ such that $P^\bullet \otimes_R R/I$ is isomorphic to $\mathcal{P}^\bullet$.

**Proof.** Assumption (3) means that $I^n = 0$ for some $n$. The result holds if $n = 1$. Below we will prove the result holds for $n = 2$. This implies for $n > 2$ that $\mathcal{P}^\bullet$ lifts
to a complex of projective \( R/I^2 \)-modules representing \( K \otimes_R^L R/I^2 \). Then the result will follow by induction on \( n \) as \((I^2)^{[n/2]} = 0\). Thus we may and do assume \( I^2 = 0 \).

Let us represent \( K \) by a \( K \)-flat complex \( K^\bullet \) with all \( K^a \) flat, see Lemma 47.10. Then we have a short exact sequence of complexes

\[
0 \to IK^\bullet \to K^\bullet \to K^\bullet/IK^\bullet \to 0
\]

and \( K^\bullet/IK^\bullet \) represents \( K \otimes_R^L R/I \) by construction of the derived tensor product.

By flatness we see that

\[
IK^\bullet = K^\bullet \otimes_R I = K^\bullet \otimes_R R/I \otimes_R I
\]

represents \( K \otimes_R^L R/I \otimes_R^L I \) because \( K^\bullet \otimes_R R/I \) is a \( K \)-flat complex over \( R/I \), see Lemma 47.5. By assumption (2) and Derived Categories, Lemma 19.8 there is a quasi-isomorphism \( \bar{\pi} : \bar{P}^\bullet \to K^\bullet/IK^\bullet \). Since \( \bar{P}^\bullet \) is \( K \)-flat we see that the induced map \( \bar{P}^\bullet \otimes_R/I I \to IK^\bullet \) is a quasi-isomorphism too.

Suppose that \( \bar{P}^i = 0 \) for \( i > b \). We will show by induction on \( a \) that we can find a complex \( P^a \to P^{a+1} \to \ldots \to P^b \) lifting \( \bar{P}^a \to \bar{P}^{a+1} \to \ldots \to \bar{P}^b \) and a map of complexes

\[
\alpha : (P^a \to P^{a+1} \to \ldots \to P^b) \to K
\]

lifting \( \bar{\alpha} \) such that \( H^i(\alpha) \) is an isomorphism for \( i > a \) and surjective for \( i = a \). If \( a > b \) then we just take \( P^i = 0 \) for all \( i \). Suppose we have a solution for some \( a \). Consider the cone \( C^\bullet \) of \( \alpha \), see Derived Categories, Definition 9.1. Note that \( C^\bullet/IC^\bullet \) and \( IC^\bullet \) are the cones of the induced maps \( \bar{\alpha} \) and \( \alpha \otimes \text{id}_I \). Then \( H^i(C^\bullet) \), \( H^i(C^\bullet/IC^\bullet) \), and \( H^i(IC^\bullet) \) are 0 for \( i \geq a \) (use long exact cohomology sequence associated to the cones; details omitted), in particular \( H^{a-1}(C^\bullet) \to H^{a-1}(C^\bullet/IC^\bullet) \) is surjective. The given maps \( \bar{P}^{a-1} \to K^{a-1}/IK^{a-1} \) and \( \bar{P}^a \to \bar{P}^b \) induce a map

\[
\bar{P}^{a-1}[a-1] \to C^\bullet/IC^\bullet
\]

in \( D(R/I) \). Choose a lift \( P^{a-1} \) of \( \bar{P}^{a-1}[a-1] \), see Algebra, Lemma 75.4. Because \( H^{a-1}(C^\bullet) \to H^{a-1}(C^\bullet/IC^\bullet) \) is surjective, see above, we can lift the displayed map to a map

\[
P^{a-1}[a-1] \to C^\bullet
\]

in \( D(R) \). The composition with the map \( C^\bullet \to (P^a \to \ldots \to P^b)[1] \) determines a map \( P^{a-1} \to P^a \) whose composition with \( P^a \to P^{a+1} \) is zero. On the other hand, we have \( C^{a-1} = K^{a-1} \oplus P^a \) and a calculation shows that the resulting map \( P^{a-1} \to K^{a-1} \) gives the desired extension of \( \alpha \).

□

Lemma 60.5. Let \( R \) be a ring. Let \( I \subset R \) be an ideal. Let \( K \) be an object of \( D(R) \).

Assume that

1. \( K \otimes_R^L R/I \) is perfect in \( D(R/I) \), and
2. \( I \) is a nilpotent ideal.

Then \( K \) is perfect in \( D(R) \).

Proof. Choose a finite complex \( P^\bullet \) of finite projective \( R/I \)-modules representing \( K \otimes_R^L R/I \), see Definition 39.1. By Lemma 60.4 there exists a complex \( P^\bullet \) of projective \( R \)-modules representing \( K \) such that \( P^\bullet = P^\bullet/IP^\bullet \). It follows from Nakayama’s lemma (Algebra, Lemma 19.1) that \( P^\bullet \) is a finite complex of finite projective \( R \)-modules.

□
Lemma 60.6. Let \( R \) be a ring. Let \( I, J \subseteq R \) be ideals. Let \( K \) be an object of \( D(R) \). Assume that

1. \( K \otimes_R R/I \) is perfect in \( D(R/I) \), and
2. \( K \otimes_R R/J \) is perfect in \( D(R/J) \).

Then \( K \otimes_R R/IJ \) is perfect in \( D(R/IJ) \).

Proof. It is clear that we may assume replace \( R \) by \( R/IJ \) and \( K \) by \( K \otimes_R R/IJ \). Then \( R \to R/(I \cap J) \) is a surjection whose kernel has square zero. Hence by Lemma 60.5 it suffices to prove that \( K \otimes_R R/(I \cap J) \) is perfect. Thus we may assume that \( I \cap J = 0 \).

We prove the lemma in case \( I \cap J = 0 \). First, we may represent \( K \) by a \( K \)-flat complex \( K \cdot \) with all \( K^n \) flat, see Lemma 47.10. Then we see that we have a short exact sequence of complexes

\[
0 \to K \to K/IK \oplus K/JK \to K/(I + J)K \to 0
\]

Note that \( K/IK \) represents \( K \otimes_R R/I \) by construction of the derived tensor product. Similarly for \( K/JK \) and \( K/(I + J)K \). Note that \( K/(I + J)K \) is a perfect complex of \( R/(I + J) \)-modules, see Lemma 59.9. Hence the complexes \( K/IK \), and \( K/JK \) and \( K/(I + J)K \) have finitely many nonzero cohomology groups (since a perfect complex has finite Tor-amplitude, see Lemma 59.2). We conclude that \( K \in D^b(R) \) by the long exact cohomology sequence associated to short exact sequence of complexes displayed above. In particular we assume \( K \) is a bounded above complex of free \( R \)-modules (see Derived Categories, Lemma 16.5).

We will now show that \( K \) is perfect using the criterion of Proposition 60.3. Thus we let \( E_j \in D(R) \) be a family of objects parametrized by a set \( J \). We choose complexes \( E_j \cdot \) with flat terms representing \( E_j \), see for example Lemma 47.10. It is clear that

\[
0 \to E_j \to E_j/IE_j \oplus E_j/JE_j \to E_j/(I + J)E_j \to 0
\]

is a short exact sequence of complexes. Taking direct sums we obtain a similar short exact sequence

\[
0 \to \bigoplus E_j \to \bigoplus E_j/IE_j \oplus \bigoplus E_j/JE_j \to \bigoplus E_j/(I + J)E_j \to 0
\]

(Note that \( - \otimes_R R/I \) commutes with direct sums.) This short exact sequence determines a distinguished triangle in \( D(R) \), see Derived Categories, Lemma 12.1. Apply the homological functor \( \text{Hom}_{D(R)}(K, -) \) (see Derived Categories, Lemma 16.5).
The following are equivalent

Lemma 61.1. Let 

In this case it is \textbf{not true} that

A counter example is \( R = k[x_1, x_2, x_3, \ldots], A = R, B = R/(x_i), \) and \( M = B. \) To “fix” this we introduce a relative notion of finite presentation.

Lemma 61.1. Let \( R \to A \) be a ring map of finite type. Let \( M \) be an \( A \)-module. The following are equivalent

(1) for some presentation \( \alpha : R[x_1, \ldots, x_n] \to A \) the module \( M \) is a finitely presented \( R[x_1, \ldots, x_n] \)-module,

(2) for all presentations \( \alpha : R[x_1, \ldots, x_n] \to A \) the module \( M \) is a finitely presented \( R[x_1, \ldots, x_n] \)-module, and

(3) for any surjection \( A' \to A \) where \( A' \) is a finitely presented \( R \)-algebra, the module \( M \) is finitely presented as \( A' \)-module.

In this case \( M \) is a finitely presented \( A \)-module.
Proof. If \( \alpha : R[x_1, \ldots, x_n] \to A \) and \( \beta : R[y_1, \ldots, y_m] \to A \) are presentations. Choose \( f_j \in R[x_1, \ldots, x_n] \) with \( \alpha(f_j) = \beta(y_j) \) and \( g_i \in R[y_1, \ldots, y_m] \) with \( \beta(g_i) = \alpha(x_i) \). Then we get a commutative diagram

\[
\begin{array}{ccc}
R[x_1, \ldots, x_n, y_1, \ldots, y_m] & \xrightarrow{y_j \mapsto f_j} & R[x_1, \ldots, x_n] \\
\downarrow^{x_i \mapsto g_i} & & \downarrow \\
R[y_1, \ldots, y_m] & \to & A
\end{array}
\]

Hence the equivalence of (1) and (2) follows by applying Algebra, Lemmas [6.4 and 7.4]. The equivalence of (2) and (3) follows by choosing a presentation \( A' = R[x_1, \ldots, x_n]/(f_1, \ldots, f_m) \) and using Algebra, Lemma [7.4] to show that \( M \) is finitely presented as \( A' \)-module if and only if \( M \) is finitely presented as a \( R[x_1, \ldots, x_n] \)-module.

**Definition 61.2.** Let \( R \to A \) be a finite type ring map. Let \( M \) be an \( A \)-module. We say \( M \) is an \( A \)-module finitely presented relative to \( R \) if the equivalent conditions of Lemma [61.1] hold.

Note that if \( R \to A \) is of finite presentation, then \( M \) is an \( A \)-module finitely presented relative to \( R \) if and only if \( M \) is a finitely presented \( A \)-module. It is equally clear that \( A \) as an \( A \)-module is finitely presented relative to \( R \) if and only if \( A \) is of finite presentation over \( R \). If \( R \) is Noetherian the notion is uninteresting. Now we can formulate the result we were looking for.

**Lemma 61.3.** Let \( R \) be a ring. Let \( A \to B \) be a finite map of finite type \( R \)-algebras. Let \( M \) be a \( B \)-module. Then \( M \) is an \( A \)-module finitely presented relative to \( R \) if and only if \( M \) is a \( B \)-module finitely presented relative to \( R \).

Proof. Choose a surjection \( R[x_1, \ldots, x_n] \to A \). Choose \( y_1, \ldots, y_m \in B \) which generate \( B \) over \( A \). As \( A \to B \) is finite each \( y_i \) satisfies a monic equation with coefficients in \( A \). Hence we can find monic polynomials \( P_j(T) \in R[x_1, \ldots, x_n][T] \) such that \( P_j(y_j) = 0 \) in \( B \). Then we get a commutative diagram

\[
\begin{array}{ccc}
R[x_1, \ldots, x_n] & \xrightarrow{y_j \mapsto P_j(T)} & R[x_1, \ldots, x_n, y_1, \ldots, y_m]/(P_j(y_j)) \\
\downarrow & & \downarrow \\
A & \to & B
\end{array}
\]

Since the top arrow is a finite and finitely presented ring map we conclude by Algebra, Lemma [7.4] and the definition.

With this result in hand we see that the relative notion makes sense and behaves well with regards to finite maps of rings of finite type over \( R \). It is also stable under localization, stable under base change, and "glues" well.

**Lemma 61.4.** Let \( R \) be a ring, \( f \in R \) an element, \( R_f \to A \) is a finite type ring map, \( g \in A \), and \( M \) an \( A \)-module. If \( M \) of finite presentation relative to \( R_f \), then \( M_g \) is an \( A_g \)-module of finite presentation relative to \( R \).

Proof. Choose a presentation \( R_f[x_1, \ldots, x_n] \to A \). We write \( R_f = R[x_0]/(fx_0 - 1) \). Consider the presentation \( R[x_0, x_1, \ldots, x_n, x_{n+1}] \to A_g \) which extends the
given map, maps $x_0$ to the image of $1/f$, and maps $x_{n+1}$ to $1/g$. Choose $g' \in R[x_0, x_1, \ldots, x_n]$ which maps to $g$ (this is possible). Suppose that

$$R_f[x_1, \ldots, x_n]^\otimes_s \to R_f[x_1, \ldots, x_n]^\otimes_t \to M \to 0$$

is a presentation of $M$ given by a matrix $(h_{ij})$. Pick $h'_{ij} \in R[x_0, x_1, \ldots, x_n]$ which map to $h_{ij}$. Then

$$R[x_0, x_1, \ldots, x_n, x_{n+1}]^\otimes_{s+2t} \to R[x_0, x_1, \ldots, x_n, x_{n+1}]^\otimes_t \to M_g \to 0$$

is a presentation of $M_f$. Here the $t \times (s + 2t)$ matrix defining the map has a first $t \times s$ block consisting of the matrix $h'_{ij}$, a second $t \times t$ block which is $(x_0f - 1)I_t$, and a third block which is $(x_{n+1}g' - 1)I_t$. \hfill $\square$

**Lemma 61.5.** Let $R \to A$ be a finite type ring map. Let $M$ be an $A$-module finitely presented relative to $R$. For any ring map $R \to R'$ the $A \otimes_R R'$-module

$$M \otimes_A A' = M \otimes_R R'$$

is finitely presented relative to $R'$.

**Proof.** Choose a surjection $R[x_1, \ldots, x_n] \to A$. Choose a presentation

$$R[x_1, \ldots, x_n]^\otimes_s \to R[x_1, \ldots, x_n]^\otimes_t \to M \to 0$$

Then

$$R'[x_1, \ldots, x_n]^\otimes_s \to R'[x_1, \ldots, x_n]^\otimes_t \to M \otimes_R R' \to 0$$

is a presentation of the base change and we win. \hfill $\square$

**Lemma 61.6.** Let $R \to A$ be a finite type ring map. Let $M$ be an $A$-module finitely presented relative to $R$. Let $A \to A'$ be a ring map of finite presentation. The $A'$-module $M \otimes_A A'$ is finitely presented relative to $R$.

**Proof.** Choose a surjection $R[x_1, \ldots, x_n] \to A$. Choose a presentation $A' = A[y_1, \ldots, y_m]/(g_1, \ldots, g_l)$. Pick $g'_i \in R[x_1, \ldots, x_n, y_1, \ldots, y_m]$ mapping to $g_i$. Say

$$R[x_1, \ldots, x_n]^\otimes_s \to R[x_1, \ldots, x_n]^\otimes_t \to M \to 0$$

is a presentation of $M$ given by a matrix $(h_{ij})$. Then

$$R[x_1, \ldots, x_n, y_1, \ldots, y_m]^\otimes_{s+t} \to R[x_0, x_1, \ldots, x_n, y_1, \ldots, y_m]^\otimes_t \to M \otimes_A A' \to 0$$

is a presentation of $M \otimes_A A'$. Here the $t \times (s + t)$ matrix defining the map has a first $t \times s$ block consisting of the matrix $h_{ij}$, followed by $l$ blocks of size $t \times t$ which are $g'_iI_t$. \hfill $\square$

**Lemma 61.7.** Let $R \to A \to B$ be finite type ring maps. Let $M$ be a $B$-module. If $M$ is finitely presented relative to $A$ and $A$ is of finite presentation over $R$, then $M$ is finitely presented relative to $R$.

**Proof.** Choose a surjection $A[x_1, \ldots, x_n] \to B$. Choose a presentation

$$A[x_1, \ldots, x_n]^\otimes_s \to A[x_1, \ldots, x_n]^\otimes_t \to M \to 0$$

given by a matrix $(h_{ij})$. Choose a presentation

$$A = R[y_1, \ldots, y_m]/(g_1, \ldots, g_l)$$

Choose $h'_{ij} \in R[y_1, \ldots, y_m, x_1, \ldots, x_n]$ mapping to $h_{ij}$. Then we obtain the presentation

$$R[y_1, \ldots, y_m, x_1, \ldots, x_n]^\otimes_{s+tu} \to R[y_1, \ldots, y_m, x_1, \ldots, x_n]^\otimes_t \to M \to 0$$
where the \( t \times (s + tu) \)-matrix is given by a first \( t \times s \) block consisting of \( h'_{ij} \) followed by \( u \) blocks of size \( t \times t \) given by \( g_i I_t \), \( i = 1, \ldots, u \).

**Lemma 61.8.** Let \( R \to A \) be a finite type ring map. Let \( M \) be an \( A \)-module. Let \( f_1, \ldots, f_r \in A \) generate the unit ideal. The following are equivalent

1. each \( M_{f_i} \) is finitely presented relative to \( R \), and
2. \( M \) is finitely presented relative to \( R \).

**Proof.** The implication (2) \( \Rightarrow \) (1) is in Lemma 61.4. Assume (1). Write \( 1 = \sum f_i g_i \) in \( A \). Choose a surjection \( R[x_1, \ldots, x_n, y_1, \ldots, y_r, z_1, \ldots, z_r] \to A \) such that \( y_i \) maps to \( f_i \) and \( z_i \) maps to \( g_i \). Then we see that there exists a surjection

\[
P = R[x_1, \ldots, x_n, y_1, \ldots, y_r, z_1, \ldots, z_r]/(\sum y_i z_i - 1) \to A.
\]

By Lemma 61.1 we see that \( M_{f_i} \) is a finitely presented \( A_{f_i} \)-module, hence by Algebra, Lemma 23.2 we see that \( M \) is a finitely presented \( A \)-module. Hence \( M \) is a finite \( P \)-module (with \( P \) as above). Choose a surjection \( P^{\oplus t} \to M \). We have to show that the kernel \( K \) of this map is a finite \( P \)-module. Since \( P_{y_i} \) surjects onto \( A_{f_i} \) we see by Lemma 61.1 and Algebra, Lemma 5.3 that the localization \( K_{y_i} \) is a finitely generated \( P_{y_i} \)-module. Choose elements \( k_{i,j} \in K \), \( i = 1, \ldots, r \), \( j = 1, \ldots, s_i \) such that the images of \( k_{i,j} \) in \( K_{y_i} \) generate. Set \( K' \subset K \) equal to the \( P \)-module generated by the elements \( k_{i,j} \). Then \( K/K' \) is a module whose localization at \( y_i \) is zero for all \( i \). Since \( (y_1, \ldots, y_r) = P \) we see that \( K/K' = 0 \) as desired. \( \square \)

**Lemma 61.9.** Let \( R \to A \) be a finite type ring map. Let \( 0 \to M' \to M \to M'' \to 0 \) be a short exact sequence of \( A \)-modules.

1. If \( M', M'' \) are finitely presented relative to \( R \), then so is \( M \).
2. If \( M' \) is a finite type \( A \)-module and \( M \) is finitely presented relative to \( R \), then \( M'' \) is finitely presented relative to \( R \).

**Proof.** Follows immediately from Algebra, Lemma 5.3. \( \square \)

**Lemma 61.10.** Let \( R \to A \) be a finite type ring map. Let \( M, M' \) be \( A \)-modules. If \( M \oplus M' \) is finitely presented relative to \( R \), then so are \( M \) and \( M' \).

**Proof.** Omitted. \( \square \)

### 62. Relatively pseudo-coherent modules

This section is the analogue of Section 61 for pseudo-coherence.

**Lemma 62.1.** Let \( R \) be a ring. Let \( K^\bullet \) be an object of \( D^{-}(R) \). Consider the \( R \)-algebra map \( R[x] \to R \) which maps \( x \) to zero. Then

\[
K^\bullet \otimes^L_{R[x]} R \cong K^\bullet \oplus K^\bullet[1]
\]

in \( D(R) \).

**Proof.** Choose a projective resolution \( P^\bullet \to K^\bullet \) over \( R \). Then

\[
P^\bullet \otimes_R R[x] \xrightarrow{\sim} P^\bullet \otimes_R R[x]
\]

is a double complex of projective \( R[x] \)-modules whose associated total complex is quasi-isomorphic to \( P^\bullet \). Hence

\[
K^\bullet \otimes^L_{R[x]} R \cong \text{Tot}(P^\bullet \otimes_R R[x] \xrightarrow{\sim} P^\bullet \otimes_R R[x]) \otimes_R R = \text{Tot}(P^\bullet \xrightarrow{0} P^\bullet)
\]

\[
= P^\bullet \oplus P^\bullet[1] \cong K^\bullet \oplus K^\bullet[1]
\]
as desired.

\[\text{Lemma 62.2.} \text{ Let } R \text{ be a ring and } K^\bullet \text{ a complex of } R\text{-modules. Let } m \in \mathbb{Z}. \text{ Consider the } R\text{-algebra map } R[x] \to R \text{ which maps } x \text{ to zero. Then } K^\bullet \text{ is } m\text{-pseudo-coherent as a complex of } R\text{-modules if and only if } K^\bullet \text{ is } m\text{-pseudo-coherent as a complex of } R[x]\text{-modules.} \]

\[\text{Proof.} \text{ This is a special case of Lemma } 52.11 \text{ We also prove it in another way as follows.} \]

Note that \(0 \to R[x] \to R[x] \to R \to 0\) is exact. Hence \(R\) is pseudo-coherent as an \(R[x]\)-module. Thus one implication of the lemma follows from Lemma 52.11. To prove the other implication, assume that \(K^\bullet\) is \(m\)-pseudo-coherent as a complex of \(R[x]\)-modules. By Lemma 52.12 we see that \(K^\bullet \otimes_{R[x]}^L R\) is \(m\)-pseudo-coherent as a complex of \(R\)-modules. By Lemma 62.1 we see that \(K^\bullet \oplus K^\bullet[1]\) is \(m\)-pseudo-coherent as a complex of \(R\)-modules. Finally, we conclude that \(K^\bullet\) is \(m\)-pseudo-coherent as a complex of \(R\)-modules from Lemma 52.8. 

\[\text{Lemma 62.3.} \text{ Let } R \to A \text{ be a ring map of finite type. Let } K^\bullet \text{ be a complex of } A\text{-modules. Let } m \in \mathbb{Z}. \text{ The following are equivalent} \]

\[\begin{enumerate}
\item \text{for some presentation } \alpha : R[x_1, \ldots, x_n] \to A \text{ the complex } K^\bullet \text{ is an } m\text{-pseudo-coherent complex of } R[x_1, \ldots, x_n]\text{-modules,} 
\item \text{for all presentations } \alpha : R[x_1, \ldots, x_n] \to A \text{ the complex } K^\bullet \text{ is an } m\text{-pseudo-coherent complex of } R[x_1, \ldots, x_n]\text{-modules.}
\end{enumerate} \]

\[\text{In particular the same equivalence holds for pseudo-coherence.} \]

\[\text{Proof.} \text{ If } \alpha : R[x_1, \ldots, x_n] \to A \text{ and } \beta : R[y_1, \ldots, y_m] \to A \text{ are presentations. Choose } f_j \in R[x_1, \ldots, x_n] \text{ with } \alpha(f_j) = \beta(y_j) \text{ and } g_i \in R[y_1, \ldots, y_m] \text{ with } \beta(g_i) = \alpha(x_i). \text{ Then we get a commutative diagram} \]

\[
\begin{array}{ccc}
R[x_1, \ldots, x_n, y_1, \ldots, y_m] & \xrightarrow{y_i \mapsto f_j} & R[x_1, \ldots, x_n] \\
x_i \mapsto g_i & & \\
R[y_1, \ldots, y_m] & \xrightarrow{f_j} & A
\end{array}
\]

\[\text{After a change of coordinates the ring homomorphism } R[x_1, \ldots, x_n, y_1, \ldots, y_m] \to R[x_1, \ldots, x_n] \text{ is isomorphic to the ring homomorphism which maps each } y_i \text{ to zero. Similarly for the left vertical map in the diagram. Hence, by induction on the number of variables this lemma follows from Lemma 62.2. The pseudo-coherent case follows from this and Lemma 52.3.} \]

\[\text{Definition 62.4.} \text{ Let } R \to A \text{ be a finite type ring map. Let } K^\bullet \text{ be a complex of } A\text{-modules. Let } m \in \mathbb{Z}. \]

\[\begin{enumerate}
\item \text{We say } K^\bullet \text{ is } m\text{-pseudo-coherent relative to } R \text{ if the equivalent conditions of Lemma 62.3 hold.} 
\item \text{We say } K^\bullet \text{ is pseudo-coherent relative to } R \text{ if } K^\bullet \text{ is } m\text{-pseudo-coherent relative to } R \text{ for all } m \in \mathbb{Z}. 
\item \text{We say } M \text{ is } m\text{-pseudo-coherent relative to } R \text{ if } M[0] \text{ is } m\text{-pseudo-coherent.} 
\item \text{We say } M \text{ is pseudo-coherent relative to } R \text{ if } M[0] \text{ is pseudo-coherent relative to } R.
\end{enumerate} \]
Part (2) means that $K^\bullet$ is pseudo-coherent as a complex of $R[x_1, \ldots, x_n]$-modules for any surjection $R[y_1, \ldots, y_m] \to A$, see Lemma 52.5. This definition has the following pleasing property.

**Lemma 62.5.** Let $R$ be a ring. Let $A \to B$ be a finite map of finite type $R$-algebras. Let $m \in \mathbb{Z}$. Let $K^\bullet$ be a complex of $B$-modules. Then $K^\bullet$ is $m$-pseudo-coherent (resp. pseudo-coherent) relative to $R$ if and only if $K^\bullet$ seen as a complex of $A$-modules is $m$-pseudo-coherent (pseudo-coherent) relative to $R$.

**Proof.** Choose a surjection $R[x_1, \ldots, x_n] \to A$. Choose $y_1, \ldots, y_m \in B$ which generate $B$ over $A$. As $A \to B$ is finite each $y_i$ satisfies a monic equation with coefficients in $A$. Hence we can find monic polynomials $P_j(T) \in R[x_1, \ldots, x_n][T]$ such that $P_j(y_j) = 0$ in $B$. Then we get a commutative diagram

$$
\begin{array}{ccc}
R[x_1, \ldots, x_n, y_1, \ldots, y_m] & \longrightarrow & R[x_1, \ldots, x_n, y_1, \ldots, y_m]/(P_j(y_j)) \\
\downarrow & & \downarrow \\
A & \longrightarrow & B \\
\end{array}
$$

The top horizontal arrow and the top right vertical arrow satisfy the assumptions of Lemma 52.11. Hence $K^\bullet$ is $m$-pseudo-coherent (resp. pseudo-coherent) as a complex of $R[x_1, \ldots, x_n]$-modules if and only if $K^\bullet$ is $m$-pseudo-coherent (resp. pseudo-coherent) as a complex of $R[x_1, \ldots, x_n, y_1, \ldots, y_m]$-modules. \qed

**Lemma 62.6.** Let $R$ be a ring. Let $R \to A$ be a finite type ring map. Let $m \in \mathbb{Z}$. Let $(K^\bullet, L^\bullet, M^\bullet, f, g, h)$ be a distinguished triangle in $D(A)$.

1. If $K^\bullet$ is $(m+1)$-pseudo-coherent relative to $R$ and $L^\bullet$ is $m$-pseudo-coherent relative to $R$ then $M^\bullet$ is $m$-pseudo-coherent relative to $R$.
2. If $K^\bullet, M^\bullet$ are $m$-pseudo-coherent relative to $R$, then $L^\bullet$ is $m$-pseudo-coherent relative to $R$.
3. If $L^\bullet$ is $(m+1)$-pseudo-coherent relative to $R$ and $M^\bullet$ is $m$-pseudo-coherent relative to $R$, then $K^\bullet$ is $(m+1)$-pseudo-coherent relative to $R$.

Moreover, if two out of three of $K^\bullet, L^\bullet, M^\bullet$ are pseudo-coherent relative to $R$, the so is the third.

**Proof.** Follows immediately from Lemma 52.2 and the definitions. \qed

**Lemma 62.7.** Let $R \to A$ be a finite type ring map. Let $M$ be an $A$-module. Then

1. $M$ is 0-pseudo-coherent relative to $R$ if and only if $M$ is a finite type $A$-module,
2. $M$ is $(-1)$-pseudo-coherent relative to $R$ if and only if $M$ is a finitely presented relative to $R$,
3. $M$ is $(-d)$-pseudo-coherent relative to $R$ if and only if for every surjection $R[x_1, \ldots, x_n] \to A$ there exists a resolution $R[x_1, \ldots, x_n]^{\oplus d} \to R[x_1, \ldots, x_n]^{\oplus d-1} \to \cdots \to R[x_1, \ldots, x_n]^{\oplus 0} \to M \to 0$ of length $d$, and
(4) $M$ is pseudo-coherent relative to $R$ if and only if for every presentation $R[x_1, \ldots, x_n] \to A$ there exists an infinite resolution
$$\ldots \to R[x_1, \ldots, x_n]^{\oplus a_1} \to R[x_1, \ldots, x_n]^{\oplus a_0} \to M \to 0$$
by finite free $R[x_1, \ldots, x_n]$-modules.

**Proof.** Follows immediately from Lemma 52.3 and the definitions. \(\square\)

**Lemma 62.8.** Let $R \to A$ be a finite type ring map. Let $m \in \mathbb{Z}$. Let $K^\bullet, L^\bullet \in D(A)$. If $K^\bullet \oplus L^\bullet$ is $m$-pseudo-coherent (resp. pseudo-coherent) relative to $R$ so are $K^\bullet$ and $L^\bullet$.

**Proof.** Immediate from Lemma 52.8 and the definitions. \(\square\)

**Lemma 62.9.** Let $R \to A$ be a finite type ring map. Let $m \in \mathbb{Z}$. Let $K^\bullet$ be a bounded above complex of $A$-modules such that $K^i$ is $(m-i)$-pseudo-coherent relative to $R$ for all $i$. Then $K^\bullet$ is $m$-pseudo-coherent relative to $R$. In particular, if $K^\bullet$ is a bounded above complex of $A$-modules pseudo-coherent relative to $R$, then $K^\bullet$ is pseudo-coherent relative to $R$.

**Proof.** Immediate from Lemma 52.9 and the definitions. \(\square\)

**Lemma 62.10.** Let $R \to A$ be a finite type ring map. Let $m \in \mathbb{Z}$. Let $K^\bullet \in D^-(A)$ such that $H^i(K^\bullet)$ is $(m-i)$-pseudo-coherent (resp. pseudo-coherent) relative to $R$ for all $i$. Then $K^\bullet$ is $m$-pseudo-coherent (resp. pseudo-coherent) relative to $R$.

**Proof.** Immediate from Lemma 52.10 and the definitions. \(\square\)

**Lemma 62.11.** Let $R$ be a ring, $f \in R$ an element, $R_f \to A$ is a finite type ring map, $g \in A$, and $K^\bullet$ a complex of $A$-modules. If $K^\bullet$ is $m$-pseudo-coherent (resp. pseudo-coherent) relative to $R_f$, then $K^\bullet \otimes_A A_g$ is $m$-pseudo-coherent (resp. pseudo-coherent) relative to $R$.

**Proof.** First we show that $K^\bullet$ is $m$-pseudo-coherent relative to $R$. Namely, suppose $R_f[x_1, \ldots, x_n] \to A$ is surjective. Write $R_f = R[x_0]/(fx_0 - 1)$. Then $R[x_0, x_1, \ldots, x_n] \to A$ is surjective, and $R_f[x_1, \ldots, x_n]$ is pseudo-coherent as an $R[x_0, \ldots, x_n]$-module. Hence by Lemma 52.11 we see that $K^\bullet$ is $m$-pseudo-coherent as a complex of $R[x_0, x_1, \ldots, x_n]$-modules.

Choose an element $g' \in R[x_0, x_1, \ldots, x_n]$ which maps to $g \in A$. By Lemma 52.12 we see that
$$K^\bullet \otimes_{R[x_0, x_1, \ldots, x_n]} R[x_0, x_1, \ldots, x_n, \frac{1}{g}] = K^\bullet \otimes_{R[x_0, x_1, \ldots, x_n]} R[x_0, x_1, \ldots, x_n, \frac{1}{g}] = K^\bullet \otimes_A A_f$$
is $m$-pseudo-coherent as a complex of $R[x_0, x_1, \ldots, x_n, \frac{1}{g}]$-modules. write
$$R[x_0, x_1, \ldots, x_n, \frac{1}{g}] = R[x_0, \ldots, x_n, x_{n+1}]/(x_{n+1}g' - 1).$$

As $R[x_0, x_1, \ldots, x_n, \frac{1}{g}]$ is pseudo-coherent as a $R[x_0, \ldots, x_n, x_{n+1}]$-module we conclude (see Lemma 52.11) that $K^\bullet \otimes_A A_g$ is $m$-pseudo-coherent as a complex of $R[x_0, \ldots, x_n, x_{n+1}]$-modules as desired. \(\square\)
Lemma 62.12. Let $R \to A$ be a finite type ring map. Let $m \in \mathbb{Z}$. Let $K^\bullet$ be a complex of $A$-modules which is $m$-pseudo-coherent (resp. pseudo-coherent) relative to $R$. Let $R \to R'$ be a ring map such that $A$ and $R'$ are Tor independent over $R$. Set $A' = A \otimes_R R'$. Then $K^\bullet \otimes^L_A A'$ is $m$-pseudo-coherent (resp. pseudo-coherent) relative to $R'$.

Proof. Choose a surjection $R[x_1, \ldots, x_n] \to A$. Note that

$$K^\bullet \otimes^L_A A' = K^\bullet \otimes^L_R R' = K^\bullet \otimes^L_{R[x_1, \ldots, x_n]} R'[x_1, \ldots, x_n]$$

by Lemma 49.2 applied twice. Hence we win by Lemma 62.12.

Lemma 62.13. Let $R \to A \to B$ be finite type ring maps. Let $m \in \mathbb{Z}$. Let $K^\bullet$ be a complex of $A$-modules. Assume $B$ as a $B$-module is pseudo-coherent relative to $A$. If $K^\bullet$ is $m$-pseudo-coherent (resp. pseudo-coherent) relative to $R$, then $K^\bullet \otimes^L_A B$ is $m$-pseudo-coherent (resp. pseudo-coherent) relative to $R$.

Proof. Choose a surjection $A[y_1, \ldots, y_m] \to B$. Choose a surjection $R[x_1, \ldots, x_n] \to A$. Combined we get a surjection $R[x_1, \ldots, x_n, y_1, \ldots, y_m] \to B$. Choose a resolution $E^\bullet \to B$ of $B$ by a complex of finite free $A[y_1, \ldots, y_m]$-modules (which is possible by our assumption on the ring map $A \to B$). We may assume that $K^\bullet$ is a bounded above complex of flat $A$-modules. Then

$$K^\bullet \otimes^L_A B = \text{Tot}(K^\bullet \otimes_A B[0])$$

$$= \text{Tot}(K^\bullet \otimes_A A[y_1, \ldots, y_m] \otimes_A[y_1, \ldots, y_m] B[0])$$

$$\cong \text{Tot}((K^\bullet \otimes_A A[y_1, \ldots, y_m]) \otimes_A[y_1, \ldots, y_m] E^\bullet)$$

$$= \text{Tot}(K^\bullet \otimes_A E^\bullet)$$

in $D(A[y_1, \ldots, y_m])$. The quasi-isomorphism $\cong$ comes from an application of Lemma 47.8. Thus we have to show that $\text{Tot}(K^\bullet \otimes_A E^\bullet)$ is $m$-pseudo-coherent as a complex of $R[x_1, \ldots, x_n, y_1, \ldots, y_m]$-modules. Note that $\text{Tot}(K^\bullet \otimes_A E^\bullet)$ has a filtration by subcomplexes with successive quotients the complexes $K^\bullet \otimes_A E^\bullet[-i]$. Note that for $i \leq 0$ the complexes $K^\bullet \otimes_A E^\bullet[-i]$ have zero cohomology in degrees $\leq m$ and hence are $m$-pseudo-coherent (over any ring). Hence, applying Lemma 62.14 and induction, it suffices to show that $K^\bullet \otimes_A E^\bullet[-i]$ is pseudo-coherent relative to $R$ for all $i$. Note that $E^i = 0$ for $i > 0$. Since also $E^i$ is finite free this reduces to proving that $K^\bullet \otimes_A A[y_1, \ldots, y_m]$ is $m$-pseudo-coherent relative to $R$ which follows from Lemma 62.12 for instance.

Lemma 62.14. Let $R \to A \to B$ be finite type ring maps. Let $m \in \mathbb{Z}$. Let $M$ be an $A$-module. Assume $B$ is flat over $A$ and $B$ as a $B$-module is pseudo-coherent relative to $A$. If $M$ is $m$-pseudo-coherent (resp. pseudo-coherent) relative to $R$, then $M \otimes_A B$ is $m$-pseudo-coherent (resp. pseudo-coherent) relative to $R$.


Lemma 62.15. Let $R$ be a ring. Let $A \to B$ be a map of finite type $R$-algebras. Let $m \in \mathbb{Z}$. Let $K^\bullet$ be a complex of $B$-modules. Assume $A$ is pseudo-coherent relative to $R$. Then the following are equivalent

(1) $K^\bullet$ is $m$-pseudo-coherent (resp. pseudo-coherent) relative to $A$, and
(2) $K^\bullet$ is $m$-pseudo-coherent (resp. pseudo-coherent) relative to $R$. 
Proof. Choose a surjection $R[x_1, \ldots, x_n] \to A$. Choose a surjection $A[y_1, \ldots, y_m] \to B$. Then we get a surjection

$$R[x_1, \ldots, x_n, y_1, \ldots, y_m] \to A[y_1, \ldots, y_m]$$

which is a flat base change of $R[x_1, \ldots, x_n] \to A$. By assumption $A$ is a pseudo-coherent module over $R[x_1, \ldots, x_n]$ hence by Lemma 52.13 we see that $A[y_1, \ldots, y_m]$ is pseudo-coherent over $R[x_1, \ldots, x_n, y_1, \ldots, y_m]$. Thus the lemma follows from Lemma 52.11 and the definitions. □

Lemma 62.16. Let $R \to A$ be a finite type ring map. Let $K^\bullet$ be a complex of $A$-modules. Let $m \in \mathbb{Z}$. Let $f_1, \ldots, f_r \in A$ generate the unit ideal. The following are equivalent

1. each $K^\bullet \otimes_A A_{f_i}$ is $m$-pseudo-coherent relative to $R$, and
2. $K^\bullet$ is $m$-pseudo-coherent relative to $R$.

The same equivalence holds for pseudo-coherence.

Proof. The implication (2) ⇒ (1) is in Lemma 62.11. Assume (1). Write $1 = \sum f_i g_i$ in $A$. Choose a surjection $R[x_1, \ldots, x_n, y_1, \ldots, y_r, z_1, \ldots, z_r] \to A$ such that $y_i$ maps to $f_i$ and $z_i$ maps to $g_i$. Then we see that there exists a surjection

$$P = R[x_1, \ldots, x_n, y_1, \ldots, y_r, z_1, \ldots, z_r]/(\sum y_i z_i - 1) \to A.$$

Note that $P$ is pseudo-coherent as an $R[x_1, \ldots, x_n, y_1, \ldots, y_r, z_1, \ldots, z_r]$-module and that $P[1/y_i]$ is pseudo-coherent as an $R[x_1, \ldots, x_n, y_1, \ldots, y_r, z_1, \ldots, z_r, 1/y_i]$-module. Hence by Lemma 52.11 we see that $K^\bullet \otimes_A A_{f_i}$ is an $m$-pseudo-coherent complex of $P[1/y_i]$-modules for each $i$. Thus by Lemma 52.14 we see that $K^\bullet$ is pseudo-coherent as a complex of $P$-modules, and Lemma 52.11 shows that $K^\bullet$ is pseudo-coherent as a complex of $R[x_1, \ldots, x_n, y_1, \ldots, y_r, z_1, \ldots, z_r]$-modules. □

Lemma 62.17. Let $R$ be a Noetherian ring. Let $R \to A$ be a finite type ring map. Then

1. A complex of $A$-modules $K^\bullet$ is $m$-pseudo-coherent relative to $R$ if and only if $K^i \in D^-(A)$ and $H^i(K^\bullet)$ is a finite $A$-module for $i \geq m$.
2. A complex of $A$-modules $K^\bullet$ is pseudo-coherent relative to $R$ if and only if $K^i \in D^-(A)$ and $H^i(K^\bullet)$ is a finite $A$-module for all $i$.
3. An $A$-module is pseudo-coherent relative to $R$ if and only if it is finite.

Proof. Immediate consequence of Lemma 52.16 and the definitions. □

63. Pseudo-coherent and perfect ring maps

We can define these types of ring maps as follows.

Definition 63.1. Let $A \to B$ be a ring map.

1. We say $A \to B$ is a pseudo-coherent ring map if it is of finite type and $B$, as a $B$-module, is pseudo-coherent relative to $A$.
2. We say $A \to B$ is a perfect ring map if it is a pseudo-coherent ring map such that $B$ as an $A$-module has finite tor dimension.

This terminology may be nonstandard. Using Lemma 62.7 we see that $A \to B$ is pseudo-coherent if and only if $B = A[x_1, \ldots, x_n]/I$ and $B$ as an $A[x_1, \ldots, x_n]$-module has a resolution by finite free $A[x_1, \ldots, x_n]$-modules. The motivation for
the definition of a perfect ring map is Lemma 59.2. The following lemmas gives a more useful and intuitive characterization of a perfect ring map.

**Lemma 63.2.** A ring map $A \to B$ is perfect if and only if $B = A[x_1, \ldots, x_n]/I$ and $B$ as an $A[x_1, \ldots, x_n]$-module has a finite resolution by finite projective $A[x_1, \ldots, x_n]$-modules.

**Proof.** If $A \to B$ is perfect, then $B = A[x_1, \ldots, x_n]/I$ and $B$ is pseudo-coherent as an $A[x_1, \ldots, x_n]$-module and has finite tor dimension as an $A$-module. Hence Lemma 59.19 implies that $B$ is perfect as an $A[x_1, \ldots, x_n]$-module, i.e., it has a finite resolution by finite projective $A[x_1, \ldots, x_n]$-modules (Lemma 59.3). Conversely, if $B = A[x_1, \ldots, x_n]/I$ and $B$ as an $A[x_1, \ldots, x_n]$-module has a finite resolution by finite projective $A[x_1, \ldots, x_n]$-modules then $B$ is pseudo-coherent as an $A[x_1, \ldots, x_n]$-module, hence $A \to B$ is pseudo-coherent. Moreover, the given resolution over $A[x_1, \ldots, x_n]$ is a finite resolution by flat $A$-modules and hence $B$ has finite tor dimension as an $A$-module. □

Lots of the results of the preceding sections can be reformulated in terms of this terminology. We also refer to More on Morphisms, Sections 40 and 41 for the corresponding discussion concerning morphisms of schemes.

**Lemma 63.3.** A finite type ring map of Noetherian rings is pseudo-coherent.

**Proof.** See Lemma 62.17. □

**Lemma 63.4.** A ring map which is flat and of finite presentation is perfect.

**Proof.** Let $A \to B$ be a ring map which is flat and of finite presentation. It is clear that $B$ has finite tor dimension. By Algebra, Lemma 157.1 there exists a finite type $\mathbf{Z}$-algebra $A_0 \subset A$ and a flat finite type ring map $A_0 \to B_0$ such that $B = B_0 \otimes_{A_0} A$. By Lemma 62.17 we see that $A_0 \to B_0$ is pseudo-coherent. As $A_0 \to B_0$ is flat we see that $B_0$ and $A$ are tor independent over $A_0$, hence we may use Lemma 62.12 to conclude that $A \to B$ is pseudo-coherent. □

**Lemma 63.5.** Let $A \to B$ be a finite type ring map with $A$ a regular ring of finite dimension. Then $A \to B$ is perfect.

**Proof.** By Algebra, Lemma 107.8 the assumption on $A$ means that $A$ has finite global dimension. Hence every module has finite tor dimension, see Lemma 53.15 in particular $B$ does. By Lemma 63.3 the map is pseudo-coherent. □

**Lemma 63.6.** A local complete intersection homomorphism is perfect.

**Proof.** Let $A \to B$ be a local complete intersection homomorphism. By Definition 24.2 this means that $I = A[x_1, \ldots, x_n]/I$ where $I$ is a Koszul ideal in $A[x_1, \ldots, x_n]$. By Lemmas 63.2 and 59.3 it suffices to show that $I$ is a perfect module over $A[x_1, \ldots, x_n]$. By Lemma 59.11 this is a local question. Hence we may assume that $I$ is generated by a Koszul-regular sequence (by Definition 23.1). Of course this means that $I$ has a finite free resolution and we win. □

### 64. Rlim of abelian groups and modules

We briefly discuss $R$lim on abelian groups and modules. In this section we will denote $A\mathcal{B}(\mathbf{N})$ the abelian category of inverse systems of abelian groups. This makes sense as an inverse system of abelian groups is the same thing as a sheaf of
groups on the category $\mathbf{N}$ (with a unique morphism $i \to j$ if $i \leq j$), see Remark \ref{remark-cohomology-limit}. Many of the arguments in this section duplicate the arguments used to construct the cohomological machinery for modules on ringed sites.

**Lemma 64.1.** The functor $\text{lim} : \text{Ab}(\mathbf{N}) \to \text{Ab}$ has a right derived functor

\[
R\text{lim} : D(\text{Ab}(\mathbf{N})) \to D(\text{Ab})
\]

As usual we set $R^p\text{lim}(K) = H^p(R\text{lim}(K))$. Moreover, we have

1. for any $(A_n)$ in $\text{Ab}(\mathbf{N})$ we have $R^p\text{lim} A_n = 0$ for $p > 1$,
2. the object $R\text{lim} A_n$ of $D(\text{Ab})$ is represented by the complex
   \[
   \prod A_n \to \prod A_n, \quad (x_n) \mapsto (x_n - f_{n+1}(x_{n+1}))
   \]
   sitting in degrees 0 and 1,
3. if $(A_n)$ is ML, then $R^1\text{lim} A_n = 0$, i.e., $(A_n)$ is right acyclic for $\text{lim}$,
4. every $K^* \in D(\text{Ab}(\mathbf{N}))$ is quasi-isomorphic to a complex whose terms are right acyclic for $\text{lim}$, and
5. if each $K^p = (K^p_n)$ is right acyclic for $\text{lim}$, i.e., of $R^1\text{lim} K^p_n = 0$, then $R\text{lim} K$ is represented by the complex whose term in degree $p$ is $\text{lim}_n K^p_n$.

**Proof.** Let $(A_n)$ be an arbitrary inverse system. Let $(B_n)$ be the inverse system with

\[
B_n = A_n \oplus A_{n-1} \oplus \ldots \oplus A_1
\]

and transition maps given by projections. Let $A_n \to B_n$ be given by $(1, f_n, f_{n-1} \circ f_n, \ldots, f_2 \circ \ldots \circ f_n)$ where $f_i : A_i \to A_{i-1}$ are the transition maps. In this way we see that every inverse system is a subobject of a ML system (Homology, Section \ref{section-algebra}). It follows from Derived Categories, Lemma \ref{lemma-derived-functor-limit} using Homology, Lemma \ref{lemma-limit-acyclic} that every ML system is right acyclic for $\text{lim}$, i.e., (3) holds. This already implies that $RF$ is defined on $D^+(\text{Ab}(\mathbf{N}))$, see Derived Categories, Proposition \ref{proposition-derived-limit}. Set $C_n = A_{n-1} \oplus \ldots \oplus A_1$ for $n > 1$ and $C_1 = 0$ with transition maps given by projections as well. Then there is a short exact sequence of inverse systems

\[
0 \to (A_n) \to (B_n) \to (C_n) \to 0
\]

where $B_n \to C_n$ is given by $(x_i) \mapsto (x_i - f_{i+1}(x_{i+1}))$. Since $(C_n)$ is ML as well, we conclude that (2) holds (by proposition reference above) which also implies (1). Finally, this implies by Derived Categories, Lemma \ref{lemma-lim-limit} that $R\text{lim}$ is in fact defined on all of $D(\text{Ab}(\mathbf{N}))$. In fact, the proof of Derived Categories, Lemma \ref{lemma-lim-limit} proceeds by proving assertions (4) and (5). \hfill $\square$

We give two simple applications. The first is the “correct” formulation of Homology, Lemma \ref{lemma-homology-limit}.

**Lemma 64.2.** Let

\[
(A_n^{-2} \to A_n^{-1} \to A_n^0 \to A_n^1)
\]

be an inverse system of complexes of abelian groups and denote $A^{-2} \to A^{-1} \to A^0 \to A^1$ its limit. Denote $(H_n^1)$, $(H_n^0)$ the inverse systems of cohomologies, and denote $H^{-1}$, $H^0$ the cohomologies of $A^{-2} \to A^{-1} \to A^0 \to A^1$. If

1. $(A_n^{-2})$ and $(A_n^{-1})$ have vanishing $R^1\text{lim}$,
2. $(H_n^{-1})$ has vanishing $R^1\text{lim}$,

then $H^0 = \text{lim} H^0_n$. 

**Proof.** Let \( K \in D(Ab(N)) \) be the object represented by the system of complexes whose \( n \)th constituent is the complex \( A_n^{-2} \to A_n^{-1} \to A_n^0 \to A_n^1 \). We will compute \( H^0(R\lim K) \) using both spectral sequences\(^7\) of Derived Categories, Lemma \( \text{[21.3]} \). The first has \( E_1 \)-page

\[
\begin{array}{cccc}
0 & 0 & R^1 \lim A_n^0 & R^1 \lim A_n^1 \\
A_n^{-2} & A_n^{-1} & A_n^0 & A_n^1
\end{array}
\]

with horizontal differentials and all higher differentials are zero. The second has \( E_2 \) page

\[
\begin{array}{cccc}
R^1 \lim H_n^{-2} & 0 & R^1 \lim H_n^0 & R^1 \lim H_n^1 \\
\lim H_n^{-2} & \lim H_n^{-1} & \lim H_n^0 & \lim H_n^1
\end{array}
\]

and degenerates at this point. The result follows. \( \square \)

**Lemma 64.3.** Let \( \mathcal{D} \) be a triangulated category. Let \((K_n)\) be an inverse system of objects of \( \mathcal{D} \). Let \( K \) be a derived limit of the system \((K_n)\). Then for every \( L \) in \( \mathcal{D} \) we have short exact sequences

\[
0 \to R^1 \lim \text{Hom}_{\mathcal{D}}(L,K_n[1]) \to \text{Hom}_{\mathcal{D}}(L,K) \to \lim \text{Hom}_{\mathcal{D}}(L,K_n) \to 0
\]

**Proof.** This follows from Derived Categories, Definition \( \text{[32.1]} \) and Lemma \( \text{[4.2]} \) and the description of \( \lim \) and \( R^1 \lim \) in Lemma \( \text{[64.1]} \) above. \( \square \)

**Remark 64.4** (\( R\lim \) as cohomology). Consider the category \( N \) whose objects are natural numbers and whose morphisms are unique arrows \( i \to j \) if \( j \geq i \). Endow \( N \) with the chaotic topology (Sites, Example \( \text{[6.6]} \)) so that a sheaf \( \mathcal{F} \) is the same thing as an inverse system of sets over \( N \). Note that \( \Gamma(N,\mathcal{F}) = \lim \mathcal{F}_n \). For an inverse system of abelian groups \( \mathcal{F}_n \) we have

\[
R^p \lim \mathcal{F}_n = H^p(N,\mathcal{F})
\]

because both sides are the higher right derived functors of \( \mathcal{F} \mapsto \lim \mathcal{F}_n = H^0(N,\mathcal{F}) \). Thus the existence of \( R\lim \) also follows from the general material in Cohomology on Sites, Sections \( \text{[3]} \) and \( \text{[19]} \).

**Warning.** An object of \( D(Ab(N)) \) is a complex of inverse systems of abelian groups. You can also think of this as an inverse system \((K_n^\bullet)\) of complexes. However, this is not the same thing as an inverse system of objects of \( D(Ab) \); we will come back and explain the difference later.

The products in the following lemma can be seen as termwise products of complexes or as products in the derived category \( D(Ab) \), see Derived Categories, Lemma \( \text{[32.2]} \). This lemma in particular shows the notation in this section is compatible with the notation introduced in Derived Categories, Section \( \text{[32]} \). See Remark \( \text{[64.16]} \) for more explanation.

**Lemma 64.5.** Let \( K = (K_n^\bullet) \) be an object of \( D(Ab(N)) \). There exists a canonical distinguished triangle

\[
R\lim K \to \prod K_n^\bullet \to \prod K_n^\bullet \to R\lim K[1]
\]

\(^7\)To use these spectral sequences we have to show that \( Ab(N) \) has enough injectives. A inverse system \((I_n)\) of abelian groups is injective if and only if each \( I_n \) is an injective abelian group and the transition maps are split surjections. Every system embeds in one of these. Details omitted.
in $D(\text{Ab})$ where the middle map fits into the commutative diagrams

\[
\begin{array}{ccc}
\prod_n K_n^\bullet & \longrightarrow & \prod_n K_n^\bullet \\
\downarrow & & \downarrow \\
K_n^\bullet \oplus K_{n+1}^\bullet & \xrightarrow{1-\pi} & K_n^\bullet
\end{array}
\]

whose vertical maps are projections and where $\pi : K_{n+1}^\bullet \to K_n^\bullet$ is the transition map of the system.

**Proof.** Suppose that for each $p$ the inverse system $(K_p^\bullet)$ is right acyclic for $\lim$. By Lemma 64.1 this gives a short exact sequence

\[
0 \to \lim_n K_n^p \to \prod_n K_n^p \to \prod_n K_n^p \to 0
\]

for each $p$. Since the complex consisting of $\lim_n K_n^p$ computes $\text{R lim} K$ by Lemma 64.1 we see that the lemma holds in this case.

Next, assume $K = (K_n^\bullet)$ is general. By Lemma 64.1 there is a quasi-isomorphism $K \to L$ in $D(\text{Ab}(\mathbb{N}))$ such that $(L_n^p)$ is acyclic for each $p$. Then $\prod_n K_n^\bullet$ is quasi-isomorphic to $\prod_n L_n^\bullet$ as products are exact in $\text{Ab}$, whence the result for $L$ (proved above) implies the result for $K$.

**Lemma 64.6.** With notation as in Lemma 64.5 the long exact cohomology sequence associated to the distinguished triangle breaks up into short exact sequences

\[
0 \to R^1 \lim_n H^{p-1}(K_n^\bullet) \to H^p(\text{R lim} K) \to \lim_n H^p(K_n^\bullet) \to 0
\]

**Proof.** The long exact sequence of the distinguished triangle is

\[
\ldots \to H^p(\text{R lim} K) \to \prod_n H^p(K_n^\bullet) \to \prod_n H^p(K_n^\bullet) \to H^{p+1}(\text{R lim} K) \to \ldots
\]

The map in the middle has kernel $\lim_n H^p(K_n^\bullet)$ by its explicit description given in the lemma. The cokernel of this map is $R^1 \lim_n H^p(K_n^\bullet)$ by Lemma 64.1.

**Lemma 64.7.** Let $E \to D$ be a morphism of $D(\text{Ab}(\mathbb{N}))$. Let $(E_n)$, resp. $(D_n)$ be the system of objects of $D(\text{Ab})$ associated to $E$, resp. $D$. If $(E_n) \to (D_n)$ is an isomorphism of pro-objects, then $R \text{lim} E \to R \text{lim} D$ is an isomorphism in $D(\text{Ab})$.

**Proof.** The assumption in particular implies that the pro-objects $H^p(E_n)$ and $H^p(D_n)$ are isomorphic. By the short exact sequences of Lemma 64.6 it suffices to show that given a map $(A_n) \to (B_n)$ of inverse systems of abelian groupsc which induces an isomorphism of pro-objects, then $\lim A_n \cong \lim B_n$ and $R^1 \lim A_n \cong R^1 \lim B_n$.

The assumption implies there are $1 \leq m_1 < m_2 < m_3 < \ldots$ and maps $\varphi_n : B_{m_n} \to A_n$ such that $(\varphi_n) : (B_{m_n}) \to (A_n)$ is a map of systems which is inverse to the given map $\psi = (\psi_n) : (A_n) \to (B_n)$ as a morphism of pro-objects. What this means is that (after possibly replacing $m_n$ by larger integers) we may assume that the compositions $A_{m_n} \to B_{m_n} \to A_n$ and $B_{m_n} \to A_n \to B_n$ are equal to the transition maps of the inverse systems. Now, if $(b_n) \in \lim B_n$ we can set $a_n = \varphi_{m_n}(b_{m_n})$. This defines an inverse $\lim B_n \to \lim A_n$ (computation omitted). Let us use the cokernel of the map

\[
\prod B_n \longrightarrow \prod B_n
\]
as an avatar of $R^1\lim B_n$ (Lemma 64.1). Any element in this cokernel can be represented by an element $(b_i)$ with $b_i = 0$ if $i \neq m_n$ for some $n$ (computation omitted). We can define a map $R^1\lim B_n \to R^1\lim A_n$ by mapping the class of such a special element $(b_n)$ to the class of $(\varphi_n(b_{m_n}))$. We omit the verification this map is inverse to the map $R^1\lim A_n \to R^1\lim B_n$.

**Lemma 64.8.** Let $(A_n)$ be an inverse system of abelian groups. The following are equivalent:

1. $(A_n)$ is zero as a pro-object,
2. $\lim_n A_n = 0$ and $R^1\lim A_n = 0$ and the same holds for $\bigoplus_{i \in \mathbb{N}}(A_n)$.

**Proof.** It follows from Lemma 64.7 that (1) implies (2). For $m \geq n$ let $\Delta_{n,m} = \text{Im}(A_n \to A_m)$ so that $\Delta_n = \bigcup_{n=1}^{\infty} \Delta_{n,n+1}$. Note that $(A_n)$ is zero as a pro-object if and only if for every $n$ there is an $m \geq n$ such that $\Delta_{n,m} = 0$. Note that $(A_n)$ is ML if and only if for every $n$ there is an $m_n \geq n$ such that $\Delta_{n,m_n} = A_{n,m_n} = 0$. In the ML case it is clear that $\lim_n A_n = 0$ implies that $\lim_n \Delta_{n,m_n} = 0$ because the maps $\Delta_{n+1,m_n+1} \to \Delta_{n,m_n}$ are surjective.

Assume $(A_n)$ is not zero as a pro-object and not ML. Then we can pick an $n$ and a sequence of integers $n < m_1 < m_2 < \ldots$ and elements $x_i \in A_{m_i}$ whose image $y_i \in A_n$ is not in $\Delta_{n,m_i+1}$. Set $B_n = \bigoplus_{i \in \mathbb{N}} A_{m_i}$. Let $\xi = (\xi_n) = \prod B_n$ be the element with $\xi_n = 0$ unless $n = m_i$ and $\xi_{m_i} = (0, \ldots, 0, x_i, 0, \ldots)$ with $x_i$ placed in the $i$th summand. We claim that $\xi$ is not in the image of the map $\prod B_n \to \prod B_n$ of Lemma 64.1. This shows that $R^1\lim B_n$ is nonzero and finishes the proof. Namely, suppose that $\xi$ is the image of $\eta = (z_1, z_2, \ldots)$ with $z_n = \sum z_{n,i} \in \bigoplus_i A_{m_i}$, where $z_{n,i} \in \bigoplus_i A_{m_i}$. Let $x_i = z_{m_i,i} \bmod A_{m_i,m_i+1}$. Then $z_{m_i-1,i}$ is the image of $z_{m_i,i}$ under $A_{m_i} \to A_{m_i-1}$, and so on, and we conclude that $\xi$ is the image of $z_{n,i}$ under $A_{m_i} \to A_n$. We conclude that $z_{n,i}$ is congruent to $y_i$ modulo $A_{n,m_i+1}$. In particular $z_{n,i} \neq 0$. This is impossible as $\sum z_{n,i} \in \bigoplus_i A_{m_i}$, hence only a finite number of $z_{n,i}$ can be nonzero.

Let $(A_n)$ be an inverse system of rings. We will denote $\text{Mod}(\mathbb{N}, (A_n))$ the category of inverse systems $(M_n)$ of abelian groups such that each $M_n$ is given the structure of an $A_n$-module and the transition maps $M_{n+1} \to M_n$ are $A_{n+1}$-module maps. This is an abelian category. Set $A = \lim_{\rightarrow} A_n$. Given an object $(M_n)$ of $\text{Mod}(\mathbb{N}, (A_n))$ the limit $M_n$ is an $A$-module.

**Lemma 64.9.** In the situation above. The functor $\lim : \text{Mod}(\mathbb{N}, (A_n)) \to \text{Mod}_A$ has a right derived functor

$$R\lim : D(\text{Mod}(\mathbb{N}, (A_n))) \to D(A)$$

As usual we set $R^p\lim(K) = H^p(R\lim(K))$. Moreover, we have

1. for any $(A_n)$ in $\text{Mod}(\mathbb{N}, (A_n))$ we have $R^p\lim A_n = 0$ for $p > 1$,
2. the object $R\lim A_n$ of $D(\text{Mod}_A)$ is represented by the complex

$$\prod A_n \to \prod A_n, \quad (x_n) \mapsto (x_n - f_{n+1}(x_{n+1}))$$

sitting in degrees 0 and 1,
3. if $(A_n)$ is ML, then $R^1\lim A_n = 0$, i.e., $(A_n)$ is right acyclic for $\lim$,
4. every $K^p \in D(\text{Mod}(\mathbb{N}, (A_n)))$ is quasi-isomorphic to a complex whose terms are right acyclic for $\lim$, and
5. if each $K^p = (K^p_n)$ is right acyclic for $\lim$, i.e., of $R^1\lim_n K_n^p = 0$, then $R\lim K$ is represented by the complex whose term in degree $p$ is $\lim_n K_n^p$. 

**Proof.** The proof of this is word for word the same as the proof of Lemma 64.1. □

**Remark 64.10.** This remark is a continuation of Remark 64.4. A sheaf of rings on $\mathbf{N}$ is just an inverse system of rings $(A_n)$. A sheaf of modules over $(A_n)$ is exactly the same thing as an object of the category $\text{Mod}(\mathbf{N}, (A_n))$ defined above. The derived functor $R\lim$ of Lemma 64.9 is simply $R\Gamma(\mathbf{N}, -)$ from the derived category of modules to the derived category of modules over the global sections of the structure sheaf. is true in general that cohomology of groups and modules agree, see Cohomology on Sites, Lemma 12.4.

**Lemma 64.11.** Let $(A_n)$ be an inverse system of rings. Every $K \in D(\text{Mod}(\mathbf{N}, (A_n)))$ can be represented by a system of complexes $(M_n^\bullet)$ such that all the transition maps $M_{n+1}^i \to M_n^i$ are surjective.

**Proof.** Let $K$ be represented by the system $(K_n^\bullet)$. Set $M_1^i = K_1^i$. Suppose we have constructed surjective maps of complexes $M_n^i \to M_{n-1}^i \to \cdots \to M_1^i$ and homotopy equivalences $\psi_e : K_e^i \to M_e^i$ such that the diagrams

$$\begin{array}{ccc}
K_{n+1}^i & \longrightarrow & K_n^i \\
\downarrow & & \downarrow \\
M_{n+1}^i & \longrightarrow & M_n^i
\end{array}$$

commute for all $e < n$. Then we consider the diagram

$$\begin{array}{ccc}
K_{n+1}^i & \longrightarrow & K_n^i \\
\downarrow & & \downarrow \\
M_{n+1}^i & \longrightarrow & M_n^i
\end{array}$$

By Derived Categories, Lemma 9.8 we can factor the composition $K_{n+1}^i \to M_n^i$ as $K_{n+1}^i \to M_{n+1}^i \to M_n^i$ such that the first arrow is a homotopy equivalence and the second a termwise split surjection. The lemma follows from this and induction. □

**Lemma 64.12.** Let $(A_n)$ be an inverse system of rings. Every $K \in D(\text{Mod}(\mathbf{N}, (A_n)))$ can be represented by a system of complexes $(K_n^\bullet)$ such that each $K_n^\bullet$ is K-flat.

**Proof.** First use Lemma 64.11 to represent $K$ by a system of complexes $(M_n^\bullet)$ such that all the transition maps $M_{n+1}^i \to M_n^i$ are surjective. Next, let $K_1^\bullet \to M_1^\bullet$ be a quasi-isomorphism with $K_1^\bullet$ a K-flat complex of $A_1$-modules (Lemma 47.10). Suppose we have constructed $K_n^\bullet \to K_{n-1}^\bullet \to \cdots \to K_1^\bullet$ and maps of complexes $\psi_e : K_e^i \to M_e^i$ such that

$$\begin{array}{ccc}
K_{e+1}^i & \longrightarrow & K_e^i \\
\downarrow & & \downarrow \\
M_{e+1}^i & \longrightarrow & M_e^i
\end{array}$$

commutes for all $e < n$. Then we consider the diagram

$$\begin{array}{ccc}
C^\bullet & \longrightarrow & K_n^\bullet \\
\downarrow & & \downarrow \\
M_{n+1}^i & \varphi_n & M_n^i
\end{array}$$
in \( D(A_{n+1}) \). As \( M_{n+1}^\bullet \to M_n^\bullet \) is termwise surjective, the complex \( C^\bullet \) fitting into the left upper corner with terms

\[
C^p = M^p_{n+1} \times_{M^p_n} K^p_n
\]

is quasi-isomorphic to \( M_{n+1}^\bullet \) (details omitted). Choose a quasi-isomorphism \( K_{n+1}^\bullet \to C^\bullet \) with \( K_{n+1}^\bullet \) K-flat. Thus the lemma holds by induction. \( \square \)

**Lemma 64.13.** Let \((A_n)\) be an inverse system of rings. Given \( K, L \in D(\text{Mod}(N, (A_n)))\) there is a canonical derived tensor product \( K \otimes^L L \) in \( D(\text{Mod}(N, (A_n))) \) compatible with the maps to \( D(A_n) \). The construction is symmetric in \( K \) and \( L \) and an exact functor of triangulated categories in each variable.

**Proof.** Choose a representative \((K_n^\bullet)\) for \( K \) such that each \( K_n^\bullet \) is a K-flat complex (Lemma 64.12). Then you can define \( K \otimes^L L \) as the object represented by the system of complexes

\[
(\text{Tot}(K_n^\bullet \otimes_{A_n} L_n^\bullet))
\]

for any choice of representative \((L_n^\bullet)\) for \( L \). This is well defined in both variables by Lemmas 47.4 and 47.12. Compatibility with the map to \( D(A_n) \) is clear. Exactness follows exactly as in Lemma 47.2. \( \square \)

As in the case of abelian groups an object \( M = (M_n^\bullet) \in D(\text{Mod}(N, (A_n))) \) is an inverse system of complexes of modules, which is not the same thing as an inverse system of objects in the derived categories. In the following lemma we show how an inverse system of objects in derived categories always lifts to an object of \( D(\text{Mod}(N, (A_n))) \).

**Lemma 64.14.** Let \((A_n)\) be an inverse system of rings. Suppose that we are given

1. for every \( n \) an object \( K_n^\bullet \) of \( D(A_n) \), and
2. for every \( n \) a map \( \varphi_n : K_{n+1}^\bullet \to K_n^\bullet \) of \( D(A_{n+1}) \) by restriction via the restriction map \( A_{n+1} \to A_n \).

There exists an object \( M = (M_n^\bullet) \in D(\text{Mod}(N, (A_n))) \) and isomorphisms \( \psi_n : M_n^\bullet \to K_n^\bullet \) in \( D(A_n) \) such that the diagrams

\[
\begin{array}{ccc}
M_{n+1}^\bullet & \to & M_n^\bullet \\
\downarrow \psi_{n+1} & & \downarrow \psi_n \\
K_{n+1}^\bullet & \xrightarrow{\varphi_n} & K_n^\bullet
\end{array}
\]

commute in \( D(A_{n+1}) \).

**Proof.** Namely, set \( M_1^\bullet = K_1^\bullet \). Suppose we have constructed \( M_n^\bullet \to M_{n-1}^\bullet \to \ldots \to M_1^\bullet \) and maps of complexes \( \psi_e : M_e^\bullet \to K_e^\bullet \) such that the diagrams above commute for all \( e < n \). Then we consider the diagram

\[
\begin{array}{ccc}
M_n^\bullet & \to & M_{n+1}^\bullet \\
\downarrow \psi_n & & \downarrow \psi_{n+1} \\
K_n^\bullet & \xrightarrow{\varphi_n} & K_{n+1}^\bullet
\end{array}
\]

in \( D(A_{n+1}) \). By the definition of morphisms in \( D(A_{n+1}) \) we can find a quasi-isomorphism \( \psi_{n+1} : M_{n+1}^\bullet \to K_{n+1}^\bullet \) of complexes of \( A_{n+1} \)-modules such that there
exists a morphism of complexes $M_{n+1}^* \to M_n^*$ of $A_{n+1}$-modules representing the composition $\psi_n^{-1} \circ \varphi_n \circ \psi_{n+1}$ in $D(A_{n+1})$. Thus the lemma holds by induction. □

**Remark 64.15.** With assumptions as in Lemma 64.14. A priori there are many isomorphism classes of objects $M$ of $D(\text{Mod}(\mathbf{N}, (A_n)))$ which give rise to the system $(K_n^*, \varphi_n)$ as above. For each such $M$ we can consider the complex $R\lim M \in D(A)$ where $A = \text{lim} A_n$. By Lemma 64.5 there exists a canonical distinguished triangle

$$R\lim M \to \prod_n K_n^* \to \prod_n K_n^* \to R\lim M[1]$$

in $D(A)$. Hence we see that the isomorphism class of $R\lim M$ in $D(A)$ is independent of the choices made in constructing $M$, by axiom TR3 of triangulated categories and Derived Categories, Lemma 4.3.

**Remark 64.16.** Let $(K_n)$ be an inverse system of objects of $D(\text{Ab})$. Let $K = R\lim K_n$ be a derived limit of this system (see Derived Categories, Section 32). Such a derived limit exists because $D(\text{Ab})$ has countable products (Derived Categories, Lemma 32.2). By Lemma 64.14 we can also lift $(K_n)$ to an object $M$ of $D(\mathbf{N})$. Then $K \cong R\lim M$ where $R\lim$ is the functor 64.1.1 because $R\lim M$ is also a derived limit of the system $(K_n)$ (by Lemma 64.5) and derived limits are unique up to isomorphism. In particular for every $p \in \mathbf{Z}$ there is a canonical short exact sequence

$$0 \to R^1 \lim H^{p-1}(K_n) \to H^p(K) \to \lim H^p(K_n) \to 0$$

as follows from Lemma 64.5 for $M$. This can also been seen directly, without invoking the existence of $M$, by applying the argument of the proof of Lemma 64.5 to the (defining) distinguished triangle $K \to \prod K_n \to \prod K_n \to K[1]$.

**Remark 64.17.** Let $A$ be a ring. Let $(E_n)$ be an inverse system of objects of $D(A)$. We’ve seen above that a derived limit $R\lim E_n$ exists. Thus for every object $K$ of $D(A)$ also the derived limit $R\lim (K \otimes_A E_n)$ exists. It turns out that we can construct these derived limits functorially in $K$ and obtain an exact functor

$$R\lim(- \otimes_A E_n) : D(A) \to D(A)$$

of triangulated categories. Namely, we first lift $(E_n)$ to an object $E$ of $D(\mathbf{N}, A)$, see Lemma 64.14 (The functor will depend on the choice of this lift.) Next, observe that there is a “diagonal” or “constant” functor

$$\Delta : D(A) \to D(\mathbf{N}, A)$$

mapping the complex $K^*$ to the constant inverse system of complexes with value $K^*$. Then we simply define

$$R\lim(K \otimes_A E_n) = R\lim(\Delta(K) \otimes E)$$

where on the right hand side we use the functor $R\lim$ of Lemma 64.9 and the functor $- \otimes_A -$ of Lemma 64.13.

**Lemma 64.18.** Let $A$ be a ring. Let $E \to D \to F \to E[1]$ be a distinguished triangle of $D(\mathbf{N}, A)$. Let $(E_n)$, resp. $(D_n)$, resp. $(F_n)$ be the system of objects of $D(A)$ associated to $E$, resp. $D$, resp. $F$. Then for every $K \in D(A)$ there is a canonical distinguished triangle

$$R\lim(K \otimes_A E_n) \to R\lim(K \otimes_A D_n) \to R\lim(K \otimes_A F_n) \to R\lim(K \otimes_A E_n)[1]$$

in $D(A)$ with notation as in Remark 64.17.
Proof. This is clear from the construction in Remark 64.17 and the fact that $\Delta : D(A) \to D(N, A)$, $- \otimes^L -$, and $R\lim$ are exact functors of triangulated categories.

Lemma 64.19. Let $A$ be a ring. Let $E \to D$ be a morphism of $D(N, A)$. Let $(E_n)$, resp. $(D_n)$ be the system of objects of $D(A)$ associated to $E$, resp. $D$. If $(E_n) \to (D_n)$ is an isomorphism of pro-objects, then for every $K \in D(A)$ the corresponding map

$$R\lim(K \otimes^L A E_n) \to R\lim(K \otimes^L A D_n)$$

in $D(A)$ is an isomorphism (notation as in Remark 64.17).

Proof. Follows from the definitions and Lemma 64.7.

65. Torsion modules

In this section “torsion modules” will refer to modules supported on a given closed subset $V(I)$ of an affine scheme $\text{Spec}(R)$. This is different, but analogous to, the notion of a torsion module over a domain (Definition 15.1).

Definition 65.1. Let $R$ be a ring. Let $M$ be an $R$-module.

1. Let $I \subset R$ be an ideal. We say $M$ is an $I$-power torsion module if for every $m \in M$ there exists an $n > 0$ such that $I^n m = 0$.
2. Let $f \in R$. We say $M$ is an $f$-power torsion module if for each $m \in M$, there exists an $n > 0$ such that $f^n m = 0$.

Thus an $f$-power torsion module is the same thing as an $I$-power torsion module for $I = (f)$. We will use the notation

$$M[I^n] = \{m \in M \mid I^n m = 0\}$$

and

$$M[I^\infty] = \bigcup M[I^n]$$

for an $R$-module $M$. Thus $M$ is $I$-power torsion if and only if $M = M[I^\infty]$ if and only if $M = \bigcup M[I^n]$.

Lemma 65.2. Let $R$ be a ring. Let $I$ be an ideal of $R$. Let $M$ be an $I$-power torsion module. Then $M$ admits a resolution

$$\ldots \to K_2 \to K_1 \to K_0 \to M \to 0$$

with each $K_i$ a direct sum of copies of $R/I^n$ for $n$ variable.

Proof. There is a canonical surjection

$$\bigoplus_{m \in M} R/I^{n_m} \to M \to 0$$

where $n_m$ is the smallest positive integer such that $I^{n_m} m = 0$. The kernel of the preceding surjection is also an $I$-power torsion module. Proceeding inductively, we construct the desired resolution of $M$.

Lemma 65.3. Let $R$ be a ring. Let $I$ be an ideal of $R$. For any $R$-module $M$ set $M[I^n] = \{m \in M \mid I^n m = 0\}$. If $I$ is finitely generated then the following are equivalent

1. $M[I] = 0$,
2. $M[I^n] = 0$ for all $n \geq 1$, and
Lemma 65.4. Let $R$ be a ring. Let $I$ be a finitely generated ideal of $R$.

(1) For any $R$-module $M$ we have $(M/M[I^\infty])[I] = 0$.

(2) An extension of $I$-power torsion modules is $I$-power torsion.

Proof. Let $m \in M$. If $m$ maps to an element of $(M/M[I^\infty])[I]$ then $Im \subset M[I^\infty]$. Write $I = (f_1, \ldots, f_t)$. Then we see that $f_im \in M[I^\infty]$, i.e., $I^i f_im = 0$ for some $n_i > 0$. Thus we see that $I^N m = 0$ with $N = \sum n_i + 2$. Hence $m$ maps to zero in $(M/M[I^\infty])$ which proves the first statement of the lemma.

For the second, suppose that $0 \to M' \to M \to M'' \to 0$ is a short exact sequence of modules with $M'$ and $M''$ both $I$-power torsion modules. Then $M[I^\infty] \supset M'$ and hence $M/M[I^\infty]$ is a quotient of $M''$ and therefore $I$-power torsion. Combined with the first statement and Lemma 65.3 this implies that it is zero.

Lemma 65.5. Let $I$ be a finitely generated ideal of a ring $R$. The $I$-power torsion modules form a Serre subcategory of the abelian category $\text{Mod}_R$, see Homology, Definition [9.1]

Proof. It is clear that a submodule and a quotient module of an $I$-power torsion module is $I$-power torsion. Moreover, the extension of two $I$-power torsion modules is $I$-power torsion by Lemma 65.4. Hence the statement of the lemma by Homology, Lemma 9.2.

Lemma 65.6. Let $R$ be a ring and let $I \subset R$ be a finitely generated ideal. The subcategory $I^\infty$-torsion $\subset \text{Mod}_R$ depends only on the closed subset $Z = V(I) \subset \text{Spec}(R)$. In fact, an $R$-module $M$ is $I$-power torsion if and only if its support is contained in $Z$.

Proof. Let $M$ be an $R$-module. Let $x \in M$. If $x \in M[I^\infty]$, then $x$ maps to zero in $M_f$ for all $f \in I$. Hence $x$ maps to zero in $M_p$ for all $p \not\ni I$. Conversely, if $x$ maps to zero in $M_p$ for all $p \not\ni I$, then $x$ maps to zero in $M_f$ for all $f \in I$. Hence if $I = (f_1, \ldots, f_t)$, then $f_1^n x = 0$ for some $n_i \geq 1$. It follows that $x \in M[I^{\sum n_i}]$. Thus $M[I^\infty]$ is the kernel of $M \to \prod_{p \ni Z} M_p$. The second statement of the lemma follows and it implies the first.

66. Formal glueing of module categories

Fix a noetherian scheme $X$, and a closed subscheme $Z$ with complement $U$. Our goal is to explain how coherent sheaves on $X$ can be constructed (uniquely) from coherent sheaves on the formal completion of $X$ along $Z$, and those on $U$ with a suitable compatibility on the overlap. We first do this using only commutative algebra (this section) and later we explain this in the setting of algebraic spaces (Pushouts of Spaces, Section 3).

Here are some references treating some of the material in this section: [Art70 Section 2], [FR70 Appendix], [BL95], [MB96], and [LJ95 Section 4.6].

Lemma 66.1. Let $\varphi : R \to S$ be a ring map. Let $I \subset R$ be an ideal. The following are equivalent

(1) $\varphi$ is flat and $R/I \to S/IS$ is faithfully flat,
\( \varphi \) is flat, and the map \( \text{Spec}(S/IS) \to \text{Spec}(R/I) \) is surjective.

(3) \( \varphi \) is flat, and the base change functor \( M \mapsto M \otimes_R S \) is faithful on modules annihilated by \( I \), and 

(4) \( \varphi \) is flat, and the base change functor \( M \mapsto M \otimes_R S \) is faithful on \( I \)-power torsion modules.

**Proof.** If \( R \to S \) is flat, then \( R/I^n \to S/I^nS \) is flat for every \( n \), see Algebra, Lemma 38.6. Hence (1) and (2) are equivalent by Algebra, Lemma 38.15. The equivalence of (1) with (3) follows by identifying \( I \)-torsion \( R \)-modules with \( R/I \)-modules, using that 

\[
M \otimes_R S = M \otimes_{R/I} S/IS
\]

for \( R \)-modules \( M \) annihilated by \( I \), and Algebra, Lemma 38.13. The implication (4) \( \Rightarrow \) (3) is immediate. Assume (3). We have seen above that \( R/I^n \to S/I^nS \) is flat, and by assumption it induces a surjection on spectra, as \( \text{Spec}(R/I^n) = \text{Spec}(R/I) \) and similarly for \( S \). Hence the base change functor is faithful on modules annihilated by \( I^n \). Since any \( I \)-power torsion module \( M \) is the union \( M = \bigcup M_n \) where \( M_n \) is annihilated by \( I^n \) we see that the base change functor is faithful on the category of all \( I \)-power torsion modules (as tensor product commutes with colimits). \( \square \)

**Lemma 66.2.** Assume \( (\varphi : R \to S, I) \) satisfies the equivalent conditions of Lemma 66.1. The following are equivalent

(1) for any \( I \)-power torsion module \( M \), the natural map \( M \to M \otimes_R S \) is an isomorphism, and 

(2) \( R/I \to S/IS \) is an isomorphism.

**Proof.** The implication (1) \( \Rightarrow \) (2) is immediate. Assume (2). First assume that \( M \) is annihilated by \( I \). In this case, \( M \) is an \( R/I \)-module. Hence, we have an isomorphism

\[
M \otimes_R S = M \otimes_{R/I} S/IS = M \otimes_{R/I} R/I = M
\]

proving the claim. Next we prove by induction that \( M \to M \otimes_R S \) is an isomorphism for any module \( M \) is annihilated by \( I^n \). Assume the induction hypothesis holds for \( n \) and assume \( M \) is annihilated by \( I^{n+1} \). Then we have a short exact sequence

\[
0 \to I^n M \to M \to M/I^nM \to 0
\]

and as \( R \to S \) is flat this gives rise to a short exact sequence

\[
0 \to I^n M \otimes_R S \to M \otimes_R S \to M/I^nM \otimes_R S \to 0
\]

Using that the canonical map is an isomorphism for \( M' = I^n M \) and \( M'' = M/I^nM \) (by induction hypothesis) we conclude the same thing is true for \( M \). Finally, suppose that \( M \) is a general \( I \)-power torsion module. Then \( M = \bigcup M_n \) where \( M_n \) is annihilated by \( I^n \) and we conclude using that tensor products commute with colimits. \( \square \)

**Lemma 66.3.** Assume \( \varphi : R \to S \) is a flat ring map and \( I \subset R \) is a finitely generated ideal such that \( R/I \to S/IS \) is an isomorphism. Then

(1) for any \( R \)-module \( M \) the map \( M \to M \otimes_R S \) induces an isomorphism \( M[I^\infty] \to (M \otimes_R S)[(IS)^\infty] \) of \( I \)-power torsion submodules, 

(2) the natural map

\[
\text{Hom}_R(M, N) \to \text{Hom}_S(M \otimes_R S, N \otimes_R S)
\]

is an isomorphism if either \( M \) or \( N \) is \( I \)-power torsion, and
(3) the base change functor $M \mapsto M \otimes_R S$ defines an equivalence of categories between $I$-power torsion modules and $IS$-power torsion modules.

**Proof.** Note that the equivalent conditions of both Lemma 66.1 and Lemma 66.2 are satisfied. We will use these without further mention. We first prove (1). Let $M$ be any $R$-module. Set $M' = M/M[I^\infty]$ and consider the exact sequence

$$0 \to M[I^\infty] \to M \to M' \to 0$$

As $M[I^\infty] = M[I^\infty] \otimes_R S$ we see that it suffices to show that $(M' \otimes_R S)[(IS)^\infty] = 0$. Write $I = (f_1, \ldots, f_t)$. By Lemma 66.4 we see that $M'[I^\infty] = 0$. Hence for every $n > 0$ the map

$$M' \to \bigoplus_{i=1}^t M', \quad x \mapsto (f_1^nx, \ldots, f_t^nx)$$

is injective. As $S$ is flat over $R$ also the corresponding map $M' \otimes_R S \to \bigoplus_{i=1}^t M' \otimes_R S$ is injective. This means that $(M' \otimes_R S)[F^n] = 0$ as desired.

Next we prove (2). If $N$ is $I$-power torsion, then $N \otimes_R S = N$ and the displayed map of (2) is an isomorphism by Algebra, Lemma [13.3]. If $M$ is $I$-power torsion, then the image of any map $M \to N$ factors through $M[I^\infty]$ and the image of any map $M \otimes_R S \to N \otimes_R S$ factors through $(N \otimes_R S)[(IS)^\infty]$. Hence in this case part (1) guarantees that we may replace $N$ by $N[I^\infty]$ and the result follows from the case where $N$ is $I$-power torsion we just discussed.

Next we prove (3). The functor is fully faithful by (2). For essential surjectivity, we simply note that for any $IS$-power torsion $S$-module $N$, the natural map $N \otimes_R S \to N$ is an isomorphism.

**Lemma 66.4.** Assume $\varphi : R \to S$ is a flat ring map and $I \subset R$ is a finitely generated ideal such that $R/I \to S/IS$ is an isomorphism. For any $f_1, \ldots, f_r \in R$ such that $V(f_1, \ldots, f_r) = V(I)$

1. the map of Koszul complexes $K(R, f_1, \ldots, f_r) \to K(S, f_1, \ldots, f_r)$ is an isomorphism, and

2. The map of extended alternating Čech complexes

$$R \to \prod_{i_0} R_{f_{i_0}} \to \prod_{i_0 < i_1} R_{f_{i_0}f_{i_1}} \to \cdots \to R_{f_1 \cdots f_r}$$

$$S \to \prod_{i_0} S_{f_{i_0}} \to \prod_{i_0 < i_1} S_{f_{i_0}f_{i_1}} \to \cdots \to S_{f_1 \cdots f_r}$$

is a quasi-isomorphism.

**Proof.** In both cases we have a complex $K_\bullet$ of $R$ modules and we want to show that $K_\bullet \to K_\bullet \otimes_R S$ is a quasi-isomorphism. By Lemma 66.2 and the flatness of $R \to S$ this will hold as soon as all homology groups of $K$ are $I$-power torsion. This is true for the Koszul complex by Lemma 21.6. Since the alternating Čech complex is a colimit of Koszul complexes (Lemma 21.13) the case of the Koszul complex implies the second statement too.

**Lemma 66.5.** Let $R$ be a ring. Let $I = (f_1, \ldots, f_n)$ be a finitely generated ideal of $R$. Let $M$ be the $R$-module generated by elements $e_1, \ldots, e_n$ subject to the relations $f_ie_j - f_je_i = 0$. There exists a short exact sequence

$$0 \to K \to M \to I \to 0$$
such that $K$ is annihilated by $I$.

**Proof.** This is just a truncation of the Koszul complex. The map $M \to I$ is determined by the rule $e_i \mapsto f_i$. If $m = \sum a_i e_i$ is in the kernel of $M \to I$, i.e., $\sum a_i f_i = 0$, then $f_i m = \sum f_i a_i e_i = (\sum f_i a_i) e_i = 0$. □

**Lemma 66.6.** Let $R$ be a ring. Let $I = (f_1, \ldots, f_n)$ be a finitely generated ideal of $R$. For any $R$-module $N$ set

$$H_1(N, f_\bullet) = \{ (x_1, \ldots, x_n) \in N^{\oplus n} \mid f_i x_j = f_j x_i \}$$

For any $R$-module $N$ there exists a canonical short exact sequence

$$0 \to \text{Ext}_R(R/I, N) \to H_1(N, f_\bullet) \to \text{Hom}_R(K, N)$$

where $K$ is as in Lemma 66.5.

**Proof.** The notation above indicates the Ext-groups in $\text{Mod}_R$ as defined in Homology, Section 6. These are denoted $\text{Ext}_R(M, N)$. Using the long exact sequence of Homology, Lemma 6.4 associated to the short exact sequence $0 \to I \to R \to R/I \to 0$ and the fact that $\text{Ext}_R(R, N) = 0$ we see that

$$\text{Ext}_R(R/I, N) = \text{Coker}(N \to \text{Hom}(I, N))$$

Using the short exact sequence of Lemma 66.3 we see that we get a complex

$$\cdots \to N \to \text{Hom}(M, N) \to \text{Hom}_R(K, N)$$

whose homology in the middle is canonically isomorphic to $\text{Ext}_R(R/I, N)$. The proof of the lemma is now complete as the cokernel of the first map is canonically isomorphic to $H_1(N, f_\bullet)$. □

**Lemma 66.7.** Let $R$ be a ring. Let $I = (f_1, \ldots, f_n)$ be a finitely generated ideal of $R$. For any $R$-module $N$ the Koszul homology group $H_1(N, f_\bullet)$ defined in Lemma 66.6 is annihilated by $I$.

**Proof.** Let $(x_1, \ldots, x_n) \in N^{\oplus n}$ with $f_i x_j = f_j x_i$. Then we have $f_i (x_1, \ldots, x_n) = (f_i x_1, \ldots, f_i x_n)$. In other words $f_i$ annihilates $H_1(N, f_\bullet)$. □

We can improve on the full faithfulness of Lemma 66.3 by showing that Ext-groups whose source is $I$-power torsion are insensitive to passing to $S$ as well. See Dualizing Complexes, Lemma 8.14 for a derived version of the following lemma.

**Lemma 66.8.** Assume $\varphi : R \to S$ is a flat ring map and $I \subset R$ is a finitely generated ideal such that $R/I \to S/IS$ is an isomorphism. Let $M, N$ be $R$-modules. Assume $M$ is $I$-power torsion. Given a short exact sequence

$$0 \to N \otimes_R S \to \tilde{E} \to M \otimes_R S \to 0$$

there exists a commutative diagram

$$
\begin{array}{c}
\begin{array}{ccc}
0 & \to & N \\
& | & | \\
& \downarrow & \downarrow \\
0 & \to & N \otimes_R S
\end{array} \\
\begin{array}{ccc}
& & \\
\downarrow & & \\
& & \\
\begin{array}{ccc}
& & \\
0 & \to & \tilde{E} \\
& | & | \\
& \downarrow & \downarrow \\
0 & \to & M \otimes_R S
\end{array}
\end{array}
\end{array}
$$

with exact rows.
Proof. As $M$ is $I$-power torsion we see that $M \otimes_R S = M$, see Lemma \[66.2\]. We will use this identification without further mention. As $R \to S$ is flat, the base change functor is exact and we obtain a functorial map of Ext-groups

$$\text{Ext}_R(M, N) \to \text{Ext}_S(M \otimes_R S, N \otimes_R S),$$

see Homology, Lemma \[7.2\]. The claim of the lemma is that this map is surjective when $M$ is $I$-power torsion. In fact we will show that it is an isomorphism. By Lemma \[65.2\] we can find a surjection $M' \to M$ with $M'$ a direct sum of modules of the form $R/I^n$. Using the long exact sequence of Homology, Lemma \[6.4\] we see that it suffices to prove the lemma for $M'$. Using compatibility of Ext with direct sums (details omitted) we reduce to the case where $M = R/I^n$ for some $n$.

Let $f_1, \ldots, f_t$ be generators for $I^n$. By Lemma \[66.6\] we have a commutative diagram

$$
\begin{array}{ccc}
0 & \to & \text{Ext}_R(R/I^n, N) \to H_1(N, f_\bullet) \to \text{Hom}_R(K, N) \\
\downarrow & & \downarrow \\
0 & \to & \text{Ext}_S(S/I^n S, N \otimes S) \to H_1(N \otimes S, f_\bullet) \to \text{Hom}_S(K \otimes S, N \otimes S)
\end{array}
$$

with exact rows where $K$ is as in Lemma \[66.5\]. Hence it suffices to prove that the two right vertical arrows are isomorphisms. Since $K$ is annihilated by $I^n$ we see that $\text{Hom}_R(K, N) = \text{Hom}_S(K \otimes_R S, N \otimes_R S)$ by Lemma \[66.3\]. As $R \to S$ is flat we have $H_1(N, f_\bullet) \otimes_R S = H_1(N \otimes_R S, f_\bullet)$. As $H_1(N, f_\bullet)$ is annihilated by $I^n$, see Lemma \[66.7\] we have $H_1(N, f_\bullet) \otimes_R S = H_1(N, f_\bullet)$ by Lemma \[66.2\].

Let $R \to S$ be a ring map. Let $f_1, \ldots, f_t \in R$ and $I = (f_1, \ldots, f_t)$. Then for any $R$-module $M$ we can define a complex

$$\text{(66.8.1) } 0 \to M \xrightarrow{\alpha} M \otimes_R S \times \prod M_{f_i} \xrightarrow{\beta} \prod (M \otimes_R S)_{f_i} \times \prod M_{f_i f_j}$$

where $\alpha(m) = (m \otimes 1, m/1, \ldots, m/1)$ and $\beta(m', m_1, \ldots, m_t) = ((m'/1-m_1 \otimes 1, \ldots, m'/1-m_t \otimes 1), (m_1-m_2, \ldots, m_{t-1}-m_t)$.

We would like to know when this complex is exact.

Lemma \[66.9\]. Assume $\varphi: R \to S$ is a flat ring map and $I = (f_1, \ldots, f_t) \subset R$ is an ideal such that $R/I \to S/IS$ is an isomorphism. Let $M$ be an $R$-module. Then the complex \text{(66.8.1)} is exact.

Proof. First proof. Denote $\hat{\mathcal{C}}_R \to \hat{\mathcal{C}}_S$ the quasi-isomorphism of extended alternating Čech complexes of Lemma \[66.4\]. Since these complexes are bounded with flat terms, we see that $M \otimes_R \hat{\mathcal{C}}_R \to M \otimes_R \hat{\mathcal{C}}_S$ is a quasi-isomorphism too (Lemmas \[47.8\] and \[47.12\]). Now the complex \text{(66.8.1)} is a truncation of the cone of the map $M \otimes_R \hat{\mathcal{C}}_R \to M \otimes_R \hat{\mathcal{C}}_S$ and we win.

Second computational proof. Let $m \in M$. If $\alpha(m) = 0$, then $m \in M[I^{\infty}]$, see Lemma \[65.3\]. Pick $n$ such that $I^nm = 0$ and consider the map $\varphi: R/I^n \to M$. If $m \otimes 1 = 0$, then $\varphi \otimes 1_S = 0$, hence $\varphi = 0$ (see Lemma \[66.3\]), hence $m = 0$. In this way we see that $\alpha$ is injective.

Let $(m'_1, m'_2, \ldots, m'_t) \in \text{Ker}(\beta)$. Write $m'_i = m_i/f_i^n$ for some $n > 0$ and $m_i \in M$. We may, after possibly enlarging $n$ assume that $f_i^n m' = m_i \otimes 1$ in $M \otimes_R S$ and $f_i^n m_i - f_i^n m_j = 0$ in $M$. In particular we see that $(m_1, \ldots, m_t)$ defines an element
ξ of $H_1(M, (f^1_i, \ldots, f^n_i))$. Since $H_1(M, (f^1_i, \ldots, f^n_i))$ is annihilated by $I^{n+1}$ (see Lemma 66.7) and since $R \to S$ is flat we see that

$$H_1(M, (f^1_i, \ldots, f^n_i)) = H_1(M, (f^1_i, \ldots, f^n_i)) \otimes_R S = H_1(M \otimes_R S, (f^1_i, \ldots, f^n_i))$$


by Lemma 66.2. The existence of $m'$ implies that $ξ$ maps to zero in the last group, i.e., the element $ξ$ is zero. Thus there exists an $m \in M$ such that $m_i = f^m_i m$. Then

$$(m', m'_1, \ldots, m'_n) - m(m''_1, 0, \ldots, 0)$$

for some $m'' \in (M \otimes_R S)[(IS)\infty]$. By Lemma 66.3 we conclude that $m'' \in M[I^n]$ and we win. □

**Remark 66.10.** In this remark we define a category of glueing data. Let $R \to S$ be a ring map. Let $f_1, \ldots, f_t \in R$ and $I = (f_1, \ldots, f_t)$. Consider the category $\text{Glue}(R \to S, f_1, \ldots, f_t)$ as the category whose objects are systems $(M', M_i, \alpha_i, \alpha_{ij})$, where $M'$ is an $S$-module, $M_i$ is an $R_{f_i}$-module, $\alpha_i : (M')_{f_i} \to M_i \otimes_R S$ is an isomorphism, and $\alpha_{ij} : (M_i)_{f_i} \to (M_j)_{f_j}$ are isomorphisms such that

(a) $\alpha_{ij} \circ \alpha_i = \alpha_j$ as maps $(M')_{f_i f_j} \to (M_j)_{f_j}$, and

(b) $\alpha_{ik} \circ \alpha_{ij} = \alpha_{jk}$ as maps $(M_j)_{f_j f_k} \to (M_k)_{f_k f_j}$ (cocycle condition).

There are maps $\varphi' : M' \to M'$, $\varphi_i : M_i \to M_i$ compatible with the given maps $\alpha_i, \beta_i, \alpha_{ij}, \beta_{ij}$.

There is a canonical functor

$$\text{Can} : \text{Mod}_R \to \text{Glue}(R \to S, f_1, \ldots, f_t), \quad M \mapsto (M \otimes_R S, M_i, \text{can}_i, \text{can}_{ij})$$

where $\text{can}_i : (M \otimes_R S)_{f_i} \to M_i \otimes_R S$ and $\text{can}_{ij} : (M_i)_{f_i} \to (M_j)_{f_j}$ are the canonical isomorphisms. For any object $M = (M', M_i, \alpha_i, \alpha_{ij})$ of the category $\text{Glue}(R \to S, f_1, \ldots, f_t)$ we define

$$H^0(M) = \{(m', m_i) \mid \alpha_i(m') = m_i \otimes 1, \alpha_{ij}(m_i) = m_j\}$$

in other words defined by the exact sequence

$$0 \to H^0(M) \to M' \times \prod M_i \to \prod M_{f'_i} \times \prod (M_i)_{f_j}$$

similar to 66.8.1. We think of $H^0(M)$ as an $R$-module. Thus we also get a functor

$$H^0 : \text{Glue}(R \to S, f_1, \ldots, f_t) \to \text{Mod}_R$$

Our next goal is to show that the functors $\text{Can}$ and $H^0$ are sometimes quasi-inverse to each other.

**Lemma 66.11.** Assume $\varphi : R \to S$ is a flat ring map and $I = (f_1, \ldots, f_t) \subset R$ is an ideal such that $R/I \to S/IS$ is an isomorphism. Then the functor $H^0$ is a left quasi-inverse to the functor $\text{Can}$ of Remark 66.10.

**Proof.** This is a reformulation of Lemma 66.9.

**Lemma 66.12.** Assume $\varphi : R \to S$ is a flat ring map and let $I = (f_1, \ldots, f_t) \subset R$ be an ideal. Then $\text{Glue}(R \to S, f_1, \ldots, f_t)$ is an abelian category, and the functor $\text{Can}$ is exact and commutes with arbitrary colimits.

**Proof.** Given a morphism $(\varphi', \varphi_i) : (M', M_i, \alpha_i, \alpha_{ij}) \to (N', N_i, \beta_i, \beta_{ij})$ of the category $\text{Glue}(R \to S, f_1, \ldots, f_t)$ we see that its kernel exists and is equal to the object $(\text{Ker}(\varphi'), \text{Ker}(\varphi_i), \alpha_i, \alpha_{ij})$ and its cokernel exists and is equal to the object $(\text{Coker}(\varphi'), \text{Coker}(\varphi_i), \beta_i, \beta_{ij})$. This works because $R \to S$ is flat, hence taking kernels/cokernels commutes with $- \otimes_R S$. Details omitted. The exactness follows
from the $R$-flatness of $R_{f_i}$ and $S$, while commuting with colimits follows as tensor products commute with colimits.

**Lemma 66.13.** Let $\varphi : R \to S$ be a flat ring map and $(f_1, \ldots, f_i) = R$. Then $\text{Can}$ and $H^0$ are quasi-inverse equivalences of categories

$$
\text{Mod}_R = \text{Glue}(R \to S, f_1, \ldots, f_i)
$$

**Proof.** Consider an object $M = (M', M_i, \alpha_i, \alpha_{ij})$ of $\text{Glue}(R \to S, f_1, \ldots, f_i)$. By Algebra, Lemma 23.4 there exists a unique module $M$ and isomorphisms $M_f \to M_i$ which recover the glueing data $\alpha_{ij}$. Then both $M'$ and $M \otimes_R S$ are $S$-modules which recover the modules $M_i \otimes_R S$ upon localizing at $f_i$. Whence there is a canonical isomorphism $M \otimes_R S \to M'$. This shows that $M$ is in the essential image of $\text{Can}$. Combined with Lemma 66.11 the lemma follows. □

**Lemma 66.14.** Let $\varphi : R \to S$ be a flat ring map and $I = (f_1, \ldots, f_i)$ and ideal. Let $R \to R'$ be a flat ring map, and set $S' = S \otimes_R R'$. Then we obtain a commutative diagram of categories and functors

$$
\begin{array}{ccc}
\text{Mod}_R & \xrightarrow{\text{Can}} & \text{Glue}(R \to S, f_1, \ldots, f_i) \\
\downarrow{\otimes_R R'} & & \downarrow{\otimes_R R'} \\
\text{Mod}_{R'} & \xrightarrow{\text{Can}} & \text{Glue}(R' \to S', f_1, \ldots, f_i)
\end{array}
$$

**Proof.** Omitted. □

**Proposition 66.15.** Assume $\varphi : R \to S$ is a flat ring map and $I = (f_1, \ldots, f_i) \subset R$ is an ideal such that $R/I \to S/IS$ is an isomorphism. Then $\text{Can}$ and $H^0$ are quasi-inverse equivalences of categories

$$
\text{Mod}_R = \text{Glue}(R \to S, f_1, \ldots, f_i)
$$

**Proof.** We have already seen that $H^0 \circ \text{Can}$ is isomorphic to the identity functor, see Lemma 66.11. Consider an object $M = (M', M_i, \alpha_i, \alpha_{ij})$ of $\text{Glue}(R \to S, f_1, \ldots, f_i)$. We get a natural morphism

$$
\Psi : (H^0(M) \otimes_R S, H^0(M)_{f_i}, \text{can}_i, \text{can}_{ij}) \to (M', M_i, \alpha_i, \alpha_{ij}).
$$

Namely, by definition $H^0(M)$ comes equipped with compatible $R$-module maps $H^0(M) \to M'$ and $H^0(M) \to M_i$. We have to show that this map is an isomorphism.

Pick an index $i$ and set $R' = R_{f_i}$. Combining Lemmas 66.14 and 66.13 we see that $\Psi \otimes_R R'$ is an isomorphism. Hence the kernel, resp. cokernel of $\Psi$ is a system of the form $(K, 0, 0, 0)$, resp. $(Q, 0, 0, 0)$. Note that $H^0((K, 0, 0, 0)) = K$, that $H^0$ is left exact, and that by construction $H^0(\Psi)$ is bijective. Hence we see $K = 0$, i.e., the kernel of $\Psi$ is zero.

The conclusion of the above is that we obtain a short exact sequence

$$
0 \to H^0(M) \otimes_R S \to M' \to Q \to 0
$$

and that $M_i = H^0(M)_{f_i}$. Note that we may think of $Q$ as an $R$-module which is $I$-power torsion so that $Q = Q \otimes_R S$. By Lemma 66.8 we see that there exists a
Lemma 66.16. Let \( \varphi : R \to S \) be a flat ring map and let \( I \subset R \) be a finitely generated ideal such that \( R/I \) is an isomorphism.

1. Given an \( R \)-module \( N \), an \( S \)-module \( M' \) and an \( S \)-module map \( \varphi : M' \to N \otimes_R S \) whose kernel and cokernel are \( I \)-power torsion, there exists an \( R \)-module map \( \psi : M \to N \) and an isomorphism \( M \otimes_R S = M' \) compatible with \( \varphi \) and \( \psi \).

2. Given an \( R \)-module \( M \), an \( S \)-module \( N' \) and an \( S \)-module map \( \varphi : M \otimes_R S \to N' \) whose kernel and cokernel are \( I \)-power torsion, there exists an \( R \)-module map \( \psi : M \to N \) and an isomorphism \( N \otimes_R S = N' \) compatible with \( \varphi \) and \( \psi \).

In both cases we have \( \text{Ker}(\varphi) \cong \text{Ker}(\psi) \) and \( \text{Coker}(\varphi) \cong \text{Coker}(\psi) \).

Proof. Proof of (1). Say \( I = (f_1, \ldots, f_l) \). It is clear that the localization \( \varphi_{f_i} \) is an isomorphism. Thus we see that \( (M', N_{f_i}, \varphi_{f_i}, \text{can}_{ij}) \) is an object of \( \text{Glue}(R \to S, f_1, \ldots, f_l) \), see Remark 66.10. By Proposition 66.15 we conclude that there exists an \( R \)-module \( M \) such that \( M' = M \otimes_R S \) and \( N_{f_i} = M_{f_i} \) compatibly with the isomorphisms \( \varphi_{f_i} \) and \( \text{can}_{ij} \). There is a morphism

\[
(M \otimes_R S, M_{f_i}, \text{can}_{i,}, \text{can}_{ij}) \to (N \otimes_R S, N_{f_i}, \text{can}_{i,}, \text{can}_{ij})
\]

of \( \text{Glue}(R \to S, f_1, \ldots, f_l) \) which uses \( \varphi \) in the first component. This corresponds to an \( R \)-module map \( \psi : M \to N \) (by the equivalence of categories of Proposition 66.15). The composition of the base change of \( M \to N \) with the isomorphism \( M' \cong M \otimes_R S \) is \( \varphi \), in other words \( M \to N \) is compatible with \( \varphi \).

Proof of (2). This is just the dual of the argument above. Namely, the localization \( \varphi_{f_i} \) is an isomorphism. Thus we see that \( (N', M_{f_i}, \varphi_{f_i}, \text{can}_{ij}) \) is an object of \( \text{Glue}(R \to S, f_1, \ldots, f_l) \), see Remark 66.10. By Proposition 66.15 we conclude that there exists an \( R \)-module \( N \) such that \( N' = N \otimes_R S \) and \( N_{f_i} = M_{f_i} \) compatibly with the isomorphisms \( \varphi_{f_i}^{-1} \) and \( \text{can}_{ij} \). There is a morphism

\[
(M \otimes_R S, M_{f_i}, \text{can}_{i,}, \text{can}_{ij}) \to (N' \otimes_R S, N_{f_i}, \varphi_{f_i}, \text{can}_{ij})
\]

of \( \text{Glue}(R \to S, f_1, \ldots, f_l) \) which uses \( \varphi \) in the first component. This corresponds to an \( R \)-module map \( \psi : M \to N \) (by the equivalence of categories of Proposition 66.15). The composition of the base change of \( M \to N \) with the isomorphism \( N' \cong N \otimes_R S \) is \( \varphi \), in other words \( M \to N \) is compatible with \( \varphi \).

The final statement follows for example from Lemma 66.3. □

Next, we specialize Proposition 66.15 to get something more usable. Namely, if \( I = (f) \) is a principal ideal then the objects of \( \text{Glue}(R \to S, f) \) are simply triples \( (M', M_1, \alpha_1) \) and there is no cocycle condition to check!
Theorem 66.17. Let $R$ be a ring, and let $f \in R$. Let $\varphi : R \to S$ be a flat ring map inducing an isomorphism $R/fR \to S/fS$. Then the functor 
\[ \text{Mod}_R \to \text{Mod}_S \times_{\text{Mod}_{fR}} \text{Mod}_{fR}, \quad M \mapsto (M \otimes_R S, M_f, \text{can}) \]
is an equivalence.

Proof. The category appearing on the right side of the arrow is the category of triples $(M', M_1, \alpha_1)$ where $M'$ is an $S$-module, $M_1$ is a $R_f$-module, and $\alpha_1 : M'_1 \to M'_1 \otimes_R S$ is a $S_f$-isomorphism, see Categories, Example 30.3. Hence this theorem is a special case of Proposition 66.15. \hfill \Box

A useful special case of Theorem 66.17 is when $R$ is noetherian, and $S$ is a completion of $R$ at an element $f$. The completion $R \to S$ is flat, and the functor $M \mapsto (M \otimes_R S, M_f, \text{can})$ can be identified with the $f$-adic completion functor when $M$ is finitely generated. To state this more precisely, let $\text{Mod}^g_R$ denote the category of finitely generated $R$-modules.

Proposition 66.18. Let $R$ be a noetherian ring. Let $f \in R$ be an element. Let $R^\wedge$ be the $f$-adic completion of $R$. Then the functor $M \mapsto (M^\wedge, M_f, \text{can})$ defines an equivalence
\[ \text{Mod}^g_R \to \text{Mod}^g_{R^\wedge} \times_{\text{Mod}^g_{R_f}} \text{Mod}^g_{R_f} \]

Proof. The ring map $R \to R^\wedge$ is flat by Algebra, Lemma 94.3. It is clear that $R/fR = R^\wedge/fR^\wedge$. By Algebra, Lemma 94.2 the completion of a finite $R$-module $M$ is equal to $M \otimes_R R^\wedge$. Hence the displayed functor of the proposition is equal to the functor occurring in Theorem 66.17. In particular it is fully faithful. Let $(M_1, M_2, \psi)$ be an object of the right hand side. By Theorem 66.17 there exists an $R$-module $M$ such that $M_1 = M \otimes_R R^\wedge$ and $M_2 = M_f$. As $R \to R^\wedge \times R_f$ is faithfully flat we conclude from Algebra, Lemma 23.2 that $M$ is finitely generated, i.e., $M \in \text{Mod}^g_R$. This proves the proposition. \hfill \Box

Remark 66.19. The equivalences of Proposition 66.15, Theorem 66.17, and Proposition 66.18 preserve properties of modules. For example if $M$ corresponds to $M = (M', M_i, \alpha_i, \alpha_{ij})$ then $M$ is finite, or finitely presented, or flat, or projective over $R$ if and only if $M'$ and $M_i$ have the corresponding property over $S$ and $R_f$. This follows from the fact that $R \to S \times \coprod R_f$ is faithfully flat and descend and ascent of these properties along faithfully flat maps, see Algebra, Lemma 81.2 and Theorem 93.5. These functors also preserve the $\otimes$-structures on either side. Thus, it defines equivalences of various categories built out of the pair $(\text{Mod}_R, \otimes)$, such as the category of algebras.

Remark 66.20. Given a differential manifold $X$ with a compact closed submanifold $Z$ having complement $U$, specifying a sheaf on $X$ is the same as specifying a sheaf on $U$, a sheaf on an unspecified tubular neighbourhood $T$ of $Z$ in $X$, and an isomorphism between the two resulting sheaves along $T \cap U$. Tubular neighbourhoods do not exist in algebraic geometry as such, but results such as Proposition 66.15, Theorem 66.17, and Proposition 66.18 allow us to work with formal neighbourhoods instead.
67. Derived Completion

Some references for the material in this section are [DG02], [GM92], [Lur11] (especially Chapter 4). Our exposition follows [BS13]. The analogue (or “dual”) of this section for torsion modules is Dualizing Complexes, Section 8. The relationship between the derived category of complexes with torsion cohomology and derived complete complexes can be found in Dualizing Complexes, Section 12.

Let $K \in D(A)$. Let $f \in A$. We denote $T(K, f)$ a derived limit of the system

$$\ldots \to K \xrightarrow{f} K \xrightarrow{f} K$$

in $D(A)$.

**Lemma 67.1.** Let $A$ be a ring. Let $f \in A$. Let $K \in D(A)$. The following are equivalent:

1. $\Ext^n_A(A_f, K) = 0$ for all $n$,
2. $\Hom_{D(A)}(E, K) = 0$ for all $E$ in $D(A_f)$,
3. $T(K, f) = 0$,
4. for every $p \in \mathbb{Z}$ we have $T(H^p(K), f) = 0$,
5. for every $p \in \mathbb{Z}$ we have $\Hom_A(A_f, H^p(K)) = 0$ and $\Ext_A(A_f, H^p(K)) = 0$,
6. $R\Hom_A(A_f, K) = 0$,
7. add more here.

**Proof.** It is clear that (2) implies (1) and that (1) is equivalent to (6). Assume (1). Let $I^\bullet$ be a $K$-injective complex of $A$-modules representing $K$. Condition (1) signifies that $\Hom_A(A_f, I^\bullet)$ is acyclic. Let $M^\bullet$ be a complex of $A_f$-modules representing $E$. Then

$$\Hom_{D(A)}(E, K) = \Hom_{K(A)}(M^\bullet, I^\bullet) = \Hom_{K(A_f)}(M^\bullet, \Hom_A(A_f, I^\bullet))$$

by Algebra, Lemma [13.4]. As $\Hom_A(A_f, I^\bullet)$ is a $K$-injective complex of $A_f$-modules by Lemma [45.3] the fact that it is acyclic implies that it is homotopy equivalent to zero (Derived Categories, Lemma [29.2]). Thus we get (2).

A free resolution of the $A$-module $A_f$ is given by

$$0 \to \bigoplus_{n \in \mathbb{N}} A \to \bigoplus_{n \in \mathbb{N}} A \to A_f \to 0$$

where the first map sends the $(x_0, x_1, \ldots)$ to $(fx_0 - x_1, fx_1 - x_2, \ldots)$ and the second map sends $(x_0, x_1, \ldots)$ to $x_0 + x_1/f + x_2/f^2 + \ldots$. Applying $\Hom_A(\_, I^\bullet)$ we get

$$0 \to \Hom_A(A_f, I^\bullet) \to \prod I^\bullet \to \prod I^\bullet \to 0$$

This means that the object $T(K, f)$ is a representative of $R\Hom_A(A_f, K)$ in $D(A)$. Thus the equivalence of (1) and (3).

There is a spectral sequence

$$E_2^{p,q} = \Ext^p_A(A_f, H^q(K)) \Rightarrow \Ext^{p+q}_A(A_f, K)$$

(details omitted). This spectral sequence degenerates at $E_2$ because $A_f$ has a length 1 resolution by projective $A$-modules (see above) hence the $E_2$-page has only 2 nonzero rows. Thus we obtain short exact sequences

$$0 \to \Ext^1_A(A_f, H^{p-1}(K)) \to \Ext^p_A(A_f, K) \to \Hom_A(A_f, H^p(K)) \to 0$$

This proves (4) and (5) are equivalent to (1). $\square$
Lemma 67.2. Let $A$ be a ring. Let $K \in D(A)$. The set $I$ of $f \in A$ such that $T(K, f) = 0$ is an ideal of $A$.

Proof. We will use the results of Lemma 67.1 without further mention. If $f \in I$, and $g \in A$, then $A_{gf}$ is an $A_f$-module hence $\text{Ext}^{n}_{A}(A_{gf}, K) = 0$ for all $n$, hence $gf \in I$. Suppose $f, g \in I$. Then there is a short exact sequence

$$0 \to A_{f+g} \to A_{f(f+g)} \oplus A_{g(f+g)} \to A_{gf(f+g)} \to 0$$

because $f, g$ generate the unit ideal in $A_{f+g}$. This follows from Algebra, Lemma 22.1 and the easy fact that the last arrow is surjective. By the long exact sequence of $\text{Ext}$ and the vanishing of $\text{Ext}^{n}_{A}(A_{f+g}, K)$, $\text{Ext}^{n}_{A}(A_{g(f+g)}, K)$, and $\text{Ext}^{n}_{A}(A_{gf(f+g)}, K)$ we get the vanishing of $\text{Ext}^{n}_{A}(A_{f+g}, K)$.

□

Lemma 67.3. Let $A$ be a ring. Let $I \subset A$ be an ideal. Let $M$ be an $A$-module.

1. If $M$ is $I$-adically complete, then $T(M, f) = 0$ for all $f \in I$.
2. Conversely, if $T(M, f) = 0$ for all $f \in I$ and $I$ is finitely generated, then $M \rightarrow \text{lim} M/f^n M$ is surjective.

Proof. Proof of (1). Assume $M$ is $I$-adically complete. By Lemma 67.1 it suffices to prove $\text{Ext}^{1}_{A}(A_{f}, M) = 0$ and $\text{Hom}_{A}(A_{f}, M) = 0$. Since $M = \text{lim} M/f^n M$ and since $\text{Hom}_{A}(A_{f}, M/f^n M) = 0$ it follows that $\text{Hom}_{A}(A_{f}, M) = 0$. Suppose we have an extension

$$0 \rightarrow M \rightarrow E \rightarrow A_{f} \rightarrow 0$$

For $n \geq 0$ pick $e_n \in E$ mapping to $1/f^n$. Set $\delta_n = fe_{n+1} - e_n \in M$ for $n \geq 0$. Replace $e_n$ by

$$e'_n = e_n + \delta_n + f\delta_{n+1} + f^2\delta_{n+2} + \ldots$$

The infinite sum exists as $M$ is complete with respect to $I$ and $f \in I$. A simple calculation shows that $f e'_{n+1} = e'_n$. Thus we get a splitting of the extension by mapping $1/f^n$ to $e'_n$.

Proof of (2). Assume that $I = (f_1, \ldots, f_r)$ and that $T(M, f_i) = 0$ for $i = 1, \ldots, r$. By Algebra, Lemma 94.12 we may assume $I = (f)$ and $T(M, f) = 0$. Let $x_n \in M$ for $n \geq 0$. Consider the extension

$$0 \rightarrow M \rightarrow E \rightarrow A_{f} \rightarrow 0$$

given by

$$E = M \oplus \bigoplus A e_n \langle x_n - fe_{n+1} + e_n \rangle$$

mapping $e_n$ to $1/f^n$ in $A_f$ (see above). By assumption and Lemma 67.1 this extension is split, hence we obtain an element $x + e_0$ which generates a copy of $A_f$ in $E$. Then

$$x + e_0 = x - x_0 + fe_1 = x - x_0 - x_1 + f^2 e_2 = \ldots$$

Since $M/f^n M = E/f^n E$ by the snake lemma, we see that $x = x_0 + f x_1 + \ldots + f^{n-1} x_{n-1}$ modulo $f^n M$. In other words, the map $M \rightarrow \text{lim} M/f^n M$ is surjective as desired.

Motivated by the results above we make the following definition.

Definition 67.4. Let $A$ be a ring. Let $K \in D(A)$. Let $I \subset A$ be an ideal. We say $K$ is derived complete with respect to $I$ if for every $f \in I$ we have $T(K, f) = 0$. If $M$ is an $A$-module, then we say $M$ is derived complete with respect to $I$ if $M[0] \in D(A)$ is derived complete with respect to $I$.
The full subcategory $D_{\text{comp}}(A) = D_{\text{comp}}(A, I) \subset D(A)$ consisting of derived complete objects is a strictly full, saturated triangulated subcategory, see Derived Categories, Definitions [3.4 and 6.1]. This subcategory is preserved under products and homotopy limits in $D(A)$. But it is not preserved under countable direct sums in general. We will often simply say $M$ is a derived complete module if the choice of the ideal $I$ is clear from the context.

**Proposition 67.5.** Let $I \subset A$ be a finitely generated ideal of a ring $A$. Let $M$ be an $A$-module. The following are equivalent

1. $M$ is $I$-adically complete, and
2. $M$ is derived complete with respect to $I$ and $\bigcap I^n M = 0$.

**Proof.** This is clear from the results of Lemma 67.3. □

The next lemma shows that the category $C$ of derived complete modules is abelian. It turns out that $C$ is not a Grothendieck abelian category, see Examples, Section 10.

**Lemma 67.6.** Let $I$ be an ideal of a ring $A$.

1. The derived complete $A$-modules form a weak Serre subcategory $C$ of $\text{Mod}_A$.
2. $D_C(A) \subset D(A)$ is the full subcategory of derived complete objects.

**Proof.** Part (2) is immediate from Lemma 67.1 and the definitions. For part (1), suppose that $M \rightarrow N$ is a map of derived complete modules. Denote $K = (M \rightarrow N)$ the corresponding object of $D(A)$. Pick $f \in I$. Then $\text{Ext}_A^n(A_f, K)$ is zero for all $n$ because $\text{Ext}_A^n(A_f, M)$ and $\text{Ext}_A^n(A_f, N)$ are zero for all $n$. Hence $K$ is derived complete. By (2) we see that $\text{Ker}(M \rightarrow N)$ and $\text{Coker}(M \rightarrow N)$ are objects of $C$. Finally, suppose that $0 \rightarrow M_1 \rightarrow M_2 \rightarrow M_3 \rightarrow 0$ is a short exact sequence of $A$-modules and $M_1, M_3$ are derived complete. Then it follows from the long exact sequence of Ext’s that $M_2$ is derived complete. Thus $C$ is a weak Serre subcategory by Homology, Lemma 9.3. □

If the ring is $I$-adically complete, then one obtains an ample supply of derived complete complexes.

**Lemma 67.7.** Let $A$ be a ring and $I \subset A$ an ideal. If $A$ is $I$-adically complete then any pseudo-coherent object of $D(A)$ is derived complete.

**Proof.** Let $K$ be a pseudo-coherent object of $D(A)$. By definition this means $K$ is represented by a bounded above complex $K^\bullet$ of finite free $A$-modules. Since $A$ is $I$-adically complete, hence derived complete (Lemma 67.3). It follows that $H^n(K)$ is derived complete for all $n$, by part (1) of Lemma 67.6. This in turn implies that $K$ is derived complete by part (2) of the same lemma. □

**Lemma 67.8.** Let $A$ be a ring. Let $f, g \in A$. Then for $K \in D(A)$ we have $R\text{Hom}(A_f, R\text{Hom}(A_g, K)) = R\text{Hom}(A_{fg}, K)$.

**Proof.** This follows from Lemma 58.1. □

**Lemma 67.9.** Let $I$ be a finitely generated ideal of a ring $A$. The inclusion functor $D_{\text{comp}}(A, I) \rightarrow D(A)$ has a left adjoint, i.e., given any object $K$ of $D(A)$ there exists a map $K \rightarrow K^\wedge$ of $K$ into a derived complete object of $D(A)$ such that the map

$$\text{Hom}_{D(A)}(K^\wedge, E) \rightarrow \text{Hom}_{D(A)}(K, E)$$
is bijective whenever \( E \) is a derived complete object of \( D(A) \). In fact, if \( I \) is generated by \( f_1, \ldots, f_r \in A \), then we have

\[
K^\wedge = \operatorname{RHom}\left((A \to \prod_{i_0} A_{f_{i_0}} \to \prod_{i_0 < i_1} A_{f_{i_0} f_{i_1}} \to \cdots \to A_{f_1 \cdots f_r}), K\right)
\]

functorially in \( K \).

**Proof.** Define \( K^\wedge \) by the last displayed formula of the lemma. There is a map of complexes

\[
(A \to \prod_{i_0} A_{f_{i_0}} \to \prod_{i_0 < i_1} A_{f_{i_0} f_{i_1}} \to \cdots \to A_{f_1 \cdots f_r}) \to A
\]

which induces a map \( K \to K^\wedge \). It suffices to prove that \( K^\wedge \) is derived complete and that \( K \to K^\wedge \) is an isomorphism if \( K \) is derived complete.

Let \( f \in A \). By Lemma \( \ref{lem:67.10} \) the object \( \operatorname{RHom}(A_f, K^\wedge) \) is equal to

\[
\operatorname{RHom}\left((A_f \to \prod_{i_0} A_{f f_{i_0}} \to \prod_{i_0 < i_1} A_{f f_{i_0} f_{i_1}} \to \cdots \to A_{f f_1 \cdots f_r}), K\right)
\]

If \( f \in I \), then \( f_1, \ldots, f_r \) generate the unit ideal in \( A_f \), hence the extended alternating Čech complex

\[
A_f \to \prod_{i_0} A_{f f_{i_0}} \to \prod_{i_0 < i_1} A_{f f_{i_0} f_{i_1}} \to \cdots \to A_{f f_1 \cdots f_r}
\]

is zero in \( D(A) \) by Lemma \( \ref{lem:21.13} \) (In fact, if \( f = f_i \) for some \( i \), then this complex is homotopic to zero; this is the only case we need.) Hence \( \operatorname{RHom}(A_f, K^\wedge) = 0 \) and we conclude that \( K^\wedge \) is derived complete by Lemma \( \ref{lem:67.1} \).

Conversely, if \( K \) is derived complete, then \( \operatorname{RHom}(A_f, K) \) is zero for all \( f = f_{i_0} \cdots f_{i_p}, p \geq 0 \). Thus \( K \to K^\wedge \) is an isomorphism in \( D(A) \).

**Alternative proof existence completion.** For each \( i \in \{1, \ldots, r\} \) let \( D_i(A) \) denote the full subcategory of objects which are derived complete with respect to \( (f_i) \).

Then \( D_{\text{comp}}(A) = D_1(A) \cap \cdots \cap D_r(A) \). A formal argument shows it suffices to construct a left adjoint for \( D_i(A) \to D(A) \). Thus we may and do assume \( I = (f) \) for some \( f \in A \). Any object of \( D(A) \) can be represented by a \( K \)-injective complex of \( A \)-modules \( J^\bullet \). Then \( \operatorname{Hom}_A(A_f, J^\bullet) \) is a \( K \)-injective complex of \( A_f \)-modules and \( K \)-injective as a complex of \( A \)-modules, by Lemmas \( \ref{lem:45.3} \) and \( \ref{lem:45.1} \). We claim that

\[
C^\bullet = \text{Cone}(\operatorname{Hom}_A(A_f, J^\bullet) \to J^\bullet)
\]

endowed with the canonical from \( J^\bullet \) is the derived completion. Namely, we have a distinguished triangle

\[
\operatorname{Hom}_A(A_f, J^\bullet) \to J^\bullet \to C^\bullet \to \operatorname{Hom}_A(A_f, J^\bullet)[1]
\]

and for any derived complete complex \( N^\bullet \) we have

\[
\operatorname{Hom}_{D(A)}(\operatorname{Hom}_A(A_f, J^\bullet), N^\bullet) = 0
\]

by Lemma \( \ref{lem:67.1} \). The long exact sequence of Hom’s easily shows that \( C^\bullet \) has the desired universal property.

**Lemma 67.10.** Let \( A \) be a ring and let \( I \subset A \) be a finitely generated ideal. Let \( K^\bullet \) be a complex of \( A \)-modules such that \( f : K^\bullet \to K^\bullet \) is an isomorphism for some \( f \in I \), i.e., \( K^\bullet \) is a complex of \( A_f \)-modules. Then the derived completion of \( K^\bullet \) is zero.
Proof. Indeed, in this case the $R\text{Hom}(K,L)$ is zero for any derived complete complex $L$, see Lemma 67.1. Hence $K^\wedge$ is zero by the universal property in Lemma 67.9.

Lemma 67.11. Let $A$ be a ring and let $I \subset A$ be a finitely generated ideal. Let $K, L \in D(A)$. Then

$$R\text{Hom}(K,L)^\wedge = R\text{Hom}(K,L^\wedge) = R\text{Hom}(K^\wedge,L^\wedge)$$

Proof. By Lemma 67.9 we know that derived completion is given by $R\text{Hom}(C,-)$ for some $C \in D(A)$. Then

$$R\text{Hom}(C,R\text{Hom}(K,L)) = R\text{Hom}(C \otimes K,L) = R\text{Hom}(K,R\text{Hom}(C,L))$$

by Lemma 58.1. This proves the first equation. The map $K \to K^\wedge$ induces a map $R\text{Hom}(K^\wedge,L^\wedge) \to R\text{Hom}(K,L^\wedge)$ which is an isomorphism in $D(A)$ by definition of the derived completion as the left adjoint to the inclusion functor.

Let $A$ be a ring and let $I \subset A$ be an ideal. For any $K \in D(A)$ we can consider the derived limit $K' = R\text{lim}(K \otimes_A A/I^n)$. This is a functor in $K$, see Remark 64.17. The system of maps $A \to A/I^n$ induces a map $K \to K'$ which is an isomorphism in $D(A)$ by definition of the derived completion as the left adjoint to the inclusion functor.

Lemma 67.12. Let $A$ be a ring and let $I \subset A$ be an ideal. For $K \in D(A)$ the naive derived completion $K' = R\text{lim}(K \otimes_A A/I^n)$ is derived complete with respect to $I$.

Proof. Let $f \in I$. The groups $\text{Ext}_A^p(A_f,K')$ sit in short exact sequences

$$0 \to R^1 \text{lim}\text{Ext}_A^{p-1}(A_f, K \otimes A/I^n) \to \text{Ext}_A^p(A_f,K') \to \text{lim}\text{Ext}_A^p(A_f, K \otimes A/I^n) \to 0$$

by Lemma 64.3. We conclude since $f$ acts both as an isomorphism and nilpotently on the outer terms.

Lemma 67.13. Let $A$ be a ring. Let $f \in A$. If there exists an integer $c \geq 1$ such that $A[f^c] = A[f^{c+1}] = A[f^{c+2}] = \ldots$ (for example if $A$ is Noetherian), then for all $n \geq 1$ there exist maps

$$(A \xrightarrow{f^n} A) \to A/(f^n), \quad A/(f^{n+c}) \to (A \xrightarrow{f^n} A)$$

in $D(A)$ inducing an isomorphism of the pro-objects $\{A/(f^n)\}$ and $\{(f^n : A \to A)\}$ in $D(A)$.
Proof. The first displayed arrow is obvious. We can define the second arrow of the lemma by the diagram

\[
\begin{array}{ccc}
A/A[f^n] & \longrightarrow & A \\
f^n & \longrightarrow & 1 \\
A & \longrightarrow & A
\end{array}
\]

Since the top horizontal arrow is injective the complex in the top row is quasi-isomorphic to \(A/f^n A\). We omit the calculation of compositions needed to show the statement on pro objects. \(\square\)

Let \(A\) be a ring. Let \(f_1, \ldots, f_r \in A\). We are going to consider the sequence of Koszul complexes \(K_n^* = K^*(A, f_1^n, \ldots, f_r^n)\) placed in cohomological degrees \(-r, -r + 1, \ldots, 0\). Using the functoriality of Lemma 67.14, we get maps

\[
\cdots \rightarrow K_4^* \rightarrow K_3^* \rightarrow K_2^* \rightarrow K_1^*
\]

compatible with \(H^0(K_n^*) = A/(f_1^n, \ldots, f_r^n)\) and the natural maps between these quotients. A key feature of the discussion below will use that for \(m > n\) the map

\[
K_m^{-p} = \wedge^p(A^{[m]}) \rightarrow \wedge^p(A^{[r]}) = K_n^{-p}
\]

is given by multiplication by \(f_1^{m-n} \cdots f_r^{m-n}\) on the basis element \(e_{i_1} \wedge \ldots \wedge e_{i_p}\). Finally, note that there is a compatible system of maps \(A \rightarrow K_n^*\).

**Lemma 67.14.** With notation as above, let \(I = (f_1, \ldots, f_r) \subset A\). For \(K \in D(A)\) the object \(K' = R\lim(K \otimes_A^L K_n^*)\) is derived complete with respect to \(I\).

**Proof.** Let \(f \in I\). We have short exact sequences

\[
0 \rightarrow R^1 \lim Ext^p_A(A_f, K \otimes_A^L K_n^*) \rightarrow Ext^p_A(A_f, K') \rightarrow \lim Ext^p_A(A_f, K \otimes_A^L K_n^*) \rightarrow 0
\]

by Lemma 64.3. We conclude the middle term is zero as \(f\) acts both as an isomorphism and nilpotently on the outer terms (recall that \(f_i^n\) acts by an endomorphism of \(K_n^*\) which is homotopic to zero). Thus \(K\) is derived complete with respect to \(I\). \(\square\)

**Lemma 67.15.** With notation as above. Let \(I = (f_1, \ldots, f_r)\). Let \(K \in D(A)\). The following are equivalent

1. \(K\) is derived complete with respect to \(I\), and
2. the canonical map \(K \rightarrow R\lim(K \otimes_A^L K_n^*)\) is an isomorphism of \(D(A)\).

**Proof.** If (2) holds, then \(K\) is derived complete with respect to \(I\) by Lemma 67.14. Conversely, assume that \(K\) is derived complete with respect to \(I\). Consider the filtrations

\[
K_n^* \supset \sigma_{r-1} K_n^* \supset \sigma_{r-2} K_n^* \supset \ldots \supset \sigma_{2-r} K_n^* \supset \sigma_0 K_n^* = A
\]

by stupid truncations (Homology, Section 13). Because the construction \(R\lim(K \otimes E)\) is exact in the second variable (Lemma 64.18) we see that it suffices to show

\[
R\lim \left( K \otimes_A^L (\sigma_{i} K_n^*/\sigma_{i+1} K_n^*) \right) = 0
\]

for \(p < 0\). The explicit description of the Koszul complexes above shows that

\[
R\lim \left( K \otimes_A^L (\sigma_{i} K_n^*/\sigma_{i+1} K_n^*) \right) = \bigoplus_{i_1, \ldots, i_{-p}} T(K, f_{i_1} \cdots f_{i_{-p}})
\]

which is zero for \(p < 0\) by assumption on \(K\). \(\square\)
Lemma 67.16. With notation as above. Let $I = (f_1, \ldots, f_r) \subset A$. The functor which sends $K \in D(A)$ to the derived limit $K' = R\lim (K \otimes_A^L K_n^\bullet)$ is the left adjoint to the inclusion functor $D_{comp}(A) \to D(A)$ constructed in Lemma 67.9.

Proof. The assignment $K \mapsto K'$ is a functor and $K'$ is derived complete with respect to $I$ by Lemma 67.14. By a formal argument (omitted) we see that it suffices to show $K \to K'$ is an isomorphism if $K$ is derived complete with respect to $I$. This is Lemma 67.15. $\square$

Lemma 67.17. Let $A$ be a ring and let $I \subset A$ be an ideal which can be generated by $r$ elements. Then derived completion has finite cohomological dimension:

1. If $K \to L$ is a morphism of $D(A)$ which induces an isomorphism on $H^i(K) \to H^i(L)$ for $i \geq 0$ then $H^i(K^\wedge) \to H^i(L^\wedge)$ is an isomorphism for $i \geq 1$.

2. If $K \to L$ is a morphism of $D(A)$ which induces an isomorphism on $H^i(K) \to H^i(L)$ for $i \leq 0$ then $H^i(K^\wedge) \to H^i(L^\wedge)$ is an isomorphism for $i \leq -r - 1$.

Proof. Say $I$ is generated by $f_1, \ldots, f_r$. By Lemma 67.16 we have

$$H^i(K^\wedge) = H^i(R\lim K \otimes_A^L K_n^\bullet)$$

and hence this fits into a short exact sequence

$$0 \to R^1 \lim H^{i-1}(K \otimes_A^L K_n^\bullet) \to H^i(K^\wedge) \to \lim H^i(K \otimes_A^L K_n^\bullet) \to 0$$

by Lemma 64.6. Thus it suffices to prove that $H^i(K \otimes_A^L K_n^\bullet)$ only depends on $H^j(K)$ for $j \in \{i, \ldots, i + r\}$. As $K_n^\bullet$ is a complex of finite free modules sitting in degrees $-r, \ldots, 0$ this follows from a straightforward argument which we omit. $\square$

Lemma 67.18. With notation as above. If $A$ is Noetherian, then for every $n$ there exists an $m \geq n$ such that $K_m^\bullet \to K_n^\bullet$ factors through the map $K_m^\bullet \to A/(f_1^n, \ldots, f_r^n)$. In other words, the pro-objects $\{K_n^\bullet\}$ and $\{A/(f_1^n, \ldots, f_r^n)\}$ of $D(A)$ are isomorphic.

Proof. Note that the Koszul complexes have length $r$. Thus the dual of Derived Categories, Lemma 12.5 implies it suffices to show that for every $p < 0$ and $n \in \mathbf{N}$ there exists an $m \geq n$ such that $H^p(K_m^\bullet) \to H^p(K_n^\bullet)$ is zero. Since $A$ is Noetherian, we see that

$$H^p(K_n^\bullet) = \frac{\text{Ker}(K_n^p \to K_n^{p+1})}{\text{Im}(K_n^{p-1} \to K_n^p)}$$

is a finite $A$-module. Moreover, the map $K_n^p \to K_n^p$ is given by a diagonal matrix whose entries are in the ideal $(f_1^{m-n}, \ldots, f_r^{m-n})$ if $p < 0$ (in fact they are in the $|p|$th power of that ideal). Note that $H^p(K_n^\bullet)$ is annihilated by $I = (f_1^n, \ldots, f_r^n)$, see Lemma 21.6. Now $I^t \subset (f_1^{m-n}, \ldots, f_r^{m-n})$ for $m = n + tr$. Thus by Artin-Rees (Algebra, Lemma 49.2) for some $m$ large enough we see that the image of $K_n^p \to K_n^p$ intersected with $\text{Ker}(K_n^p \to K_n^{p+1})$ is contained in $I\text{Ker}(K_n^p \to K_n^{p+1})$. For this $m$ we get the zero map. $\square$

Proposition 67.19. Let $A$ be a Noetherian ring. Let $I \subset A$ be an ideal. The functor which sends $K \in D(A)$ to the derived limit $K' = R\lim (K \otimes_A^L A/I^n)$ is the left adjoint to the inclusion functor $D_{comp}(A) \to D(A)$ constructed in Lemma 67.9.
Proof. Say \((f_1, \ldots, f_r) = I\) and let \(K^*_n\) be the Koszul complex with respect to \(f_1^n, \ldots, f_r^n\). By Lemma 67.16 it suffices to prove that

\[ R\lim(K \otimes_{A} L K^*_n) = R\lim(K \otimes_{A} A/(f_1^n, \ldots, f_r^n)) = R\lim(K \otimes_{A} A/I^n). \]

By Lemma 67.15 the pro-objects \(\{K^*_n\}\) and \(\{A/(f_1^n, \ldots, f_r^n)\}\) of \(D(A)\) are isomorphic. It is clear that the pro-objects \(\{A/(f_1^n, \ldots, f_r^n)\}\) and \(\{A/I^n\}\) are isomorphic. Thus the map from left to right is an isomorphism by Lemma 67.19.

As an application of the proposition above we identify the derived completion in the Noetherian case for pseudo-coherent complexes.

**Lemma 67.20.** Let \(A\) be a Noetherian ring and \(I \subset A\) an ideal. Let \(K\) be an object of \(D(A)\) such that \(H^n(A)\) a finite \(A\)-module for all \(n \in \mathbb{Z}\). Then the cohomology modules \(H^n(K^\wedge)\) of the derived completion are the \(I\)-adic completions of the cohomology modules \(H^n(K)\).

**Proof.** The complex \(\tau_{\leq m}K\) is pseudo-coherent for all \(m\) by Lemma 52.16. Thus \(\tau_{\leq m}K\) is represented by a bounded above complex \(F^*\) of finite free \(A\)-modules. Then \(\tau_{\leq m}K \otimes_{A} A/I^n = F^*/I^nF^*\). Hence \((\tau_{\leq m}K)^\wedge = R\lim P^*/I^nP^*\) (Proposition 67.19) and since the \(R\lim\) is just given by termwise limit (Lemma 64.9) and since \(I\)-adic completion is an exact functor on finite \(A\)-modules (Algebra, Lemma 94.3) we conclude the result holds for \(\tau_{\leq m}K\). Hence the result holds for \(K\) as derived completion has finite cohomological dimension, see Lemma 67.17.

**Lemma 67.21.** Let \(I\) be a finitely generated ideal of a ring \(A\). Let \(M\) be a derived complete \(A\)-module. If \(M/IM = 0\), then \(M = 0\).

**Proof.** Assume that \(M/IM\) is zero. Let \(I = (f_1, \ldots, f_r)\). Let \(i < r\) be the largest integer such that \(N = M/(f_1, \ldots, f_i)M\) is nonzero. If \(i\) does not exist, then \(M = 0\) which is what we want to show. Then \(N\) is derived complete as a cokernel of a map between derived complete modules, see Lemma 67.6. By our choice of \(i\) we have that \(f_{i+1} : N \to N\) is surjective. Hence

\[ \lim(\ldots \to N \xrightarrow{f_{i+1}} N \xrightarrow{f_{i+1}} N) \]

is nonzero, contradicting the derived completeness of \(N\).

**Lemma 67.22.** Let \(I\) be an ideal of a Noetherian ring \(A\). Let \(M\) be a derived complete \(A\)-module. If \(M/IM\) is a finite \(A/I\)-module, then \(M = \lim M/IM\) and \(M\) is a finite \(A\wedge\)-module.

**Proof.** Assume \(M/IM\) is finite. Pick \(x_1, \ldots, x_t \in M\) which map to generators of \(M/IM\). We obtain a map \(A^{\oplus t} \to M\) mapping the \(i\)th basis vector to \(x_i\). By Proposition 67.19 the derived completion of \(A\) is \(A^\wedge = \lim A/I^n\). As \(M\) is derived complete, we see that our map factors through a map \(q : (A^\wedge)^{\oplus t} \to M\). The module \(\text{Coker}(q)\) is zero by Lemma 67.21. Thus \(M\) is a finite \(A^\wedge\)-module. Since \(A^\wedge\) is Noetherian and complete with respect to \(IA^\wedge\), it follows that \(M\) is \(I\)-adically complete (use Algebra, Lemmas 94.9, 94.16 and 49.2).

**Remark 67.23.** Let \(A\) be a ring and \(f \in A\). Set \(I = (f)\). In this situation we have the naive derived \(I\)-adic completion functor \(K \mapsto K^\wedge = R\lim(K \otimes_{A} L A/f^nA)\) and the functor

\[ K \mapsto K^\wedge = R\lim(K \otimes_{A} L (A \xrightarrow{f^n} A))\]
of Lemmas \[67.16\] and \[67.9\] There is a natural transformation of functors \( K^\wedge \to K' \). Thus we ask: When is this transformation an isomorphism of functors? We have seen in Proposition \[67.19\] that this is true if \( A \) is Noetherian. More generally, the same argument shows this is true if the pro-objects \( \{ (f^n : A \to A) \} \) and \( \{ A/f^nA \} \) are equal, for example if \( f \)-torsion is bounded (Lemma \[67.13\]). Conversely, we see from Lemma \[64.18\] that the condition is exactly that

\[
R \text{lim}(K \otimes^L_A A[f^n])
\]

is zero for all \( K \in D(A) \). Here the maps of the system \( (A[f^n]) \) are given by multiplication by \( f \). Taking \( K = A \) and \( K = \bigoplus_{n \in \mathbb{N}} A \) we see from Lemma \[64.8\] this implies \( (A[f^n]) \) is zero as a pro-object, i.e., \( f^{n-1}A[f^n] = 0 \) for some \( n \), i.e., \( A[f^{n-1}] = A[f^n] \), i.e., the \( f \)-torsion is bounded.

**Example 67.24.** Let \( A \) be a ring. Let \( f \in A \) be a nonzerodivisor. An example to keep in mind is \( A = \mathbb{Z}_p \) and \( f = p \). Let \( M \) be an \( A \)-module. Claim: \( M \) is derived complete with respect to \( f \) if and only if there exists a short exact sequence

\[
0 \to K \to L \to M \to 0
\]

where \( K, L \) are \( f \)-adically complete modules whose \( f \)-torsion is zero. Namely, if there is such a short exact sequence, then

\[
M \otimes^L_A (A \xrightarrow{f} A) = (K/f^nK \to L/f^nL)
\]

because \( f \) is a nonzerodivisor on \( K \) and \( L \) and we conclude that \( R \text{lim}(M \otimes^L_A (A \xrightarrow{f} A)) \) is quasi-isomorphic to \( K \to L \), i.e., \( M \). This shows that \( M \) is derived complete by Lemma \[67.15\]. Conversely, suppose that \( M \) is derived complete. Choose a surjection \( F \to M \) where \( F \) is a free \( A \)-module. Since \( f \) is a nonzerodivisor on \( F \) the derived completion of \( F \) is \( L = \lim F/f^nF \). Note that \( L \) is \( f \)-torsion free: if \((x_n)\) with \( x_n \in F \) represents an element \( \xi \) of \( L \) and \( f \xi = 0 \), then \( x_n = x_{n+1} + f^n z_n \) and \( f x_n = f^n y_n \) for some \( z_n, y_n \in F \). Then \( f^n y_n = f x_n = f x_{n+1} + f^{n+1} z_n = f^{n+1} y_{n+1} + f^{n+1} z_n \) and since \( f \) is a nonzerodivisor on \( F \) we see that \( y_n \in f F \) which implies that \( x_n \in f^n F \), i.e., \( \xi = 0 \). Since \( L \) is the derived completion, the universal property gives a map \( L \to M \) factoring \( F \to M \). Let \( K = \text{Ker}(L \to M) \) be the kernel. Again \( K \) is \( f \)-torsion free, hence the derived completion of \( K \) is \( \text{lim} K/f^n K \).

On the other hand, both \( K \) and \( L \) are derived complete, hence \( K \) is too by Lemma \[67.6\]. It follows that \( K = \text{lim} K/f^n K \) and the claim is proved.

**Lemma 67.25.** Let \( A \to B \) be a ring map. Let \( I \subset A \) be an ideal. The inverse image of \( D_{\text{comp}}(A, I) \) under the restriction functor \( D(B) \to D(A) \) is \( D_{\text{comp}}(B, IB) \).

**Proof.** Using Lemma \[67.2\] we see that \( L \in D(B) \) is in \( D_{\text{comp}}(B, IB) \) if and only if \( T(L, f) \) is zero for every local section \( f \in I \). Observe that the cohomology of \( T(L, f) \) is computed in the category of abelian groups, so it doesn’t matter whether we think of \( f \) as an element of \( A \) or take the image of \( f \) in \( B \). The lemma follows immediately from this and the definition of derived complete objects.

**Lemma 67.26.** Let \( A \to B \) be a ring map. Let \( I \subset A \) be a finitely generated ideal. If \( A \to B \) is flat and \( A/I \cong B/IB \), then the restriction functor \( D(B) \to D(A) \) induces an equivalence \( D_{\text{comp}}(B, IB) \to D_{\text{comp}}(A, I) \).

**Proof.** Choose generators \( f_1, \ldots, f_r \) of \( I \). Denote \( \check{C}_{A}^\bullet \to \check{C}_{B}^\bullet \) the quasi-isomorphism of extended alternating Čech complexes of Lemma \[66.4\] Let \( K \in D_{\text{comp}}(A, I) \). Let
\( I^* \) be a K-injective complex of \( A \)-modules representing \( K \). Since \( \text{Ext}^n_A(A/I, K) \) and \( \text{Ext}_A^n(B, K) \) are zero for all \( f \in I \) and \( n \in \mathbb{Z} \) (Lemma 67.1) we conclude that \( \mathcal{C}_A \rightarrow A \) and \( \mathcal{C}_B \rightarrow B \) induce quasi-isomorphisms
\[
I^* = \text{Hom}_A(A, I^*) \rightarrow \text{Tot}(\text{Hom}_A(\mathcal{C}_A^*, I^*))
\]
and
\[
\text{Hom}_A(B, I^*) \rightarrow \text{Tot}(\text{Hom}_A(\mathcal{C}_B^*, I^*))
\]
Some details omitted. Since \( \mathcal{C}_A \rightarrow \mathcal{C}_B \) is a quasi-isomorphism and \( I^* \) is K-injective we conclude that \( \text{Hom}_A(B, I^*) \rightarrow I^* \) is a quasi-isomorphism. As the complex \( \text{Hom}_A(B, I^*) \) is a complex of \( B \)-modules we conclude that \( K \) is in the image of the restriction map, i.e., the functor is essentially surjective.

In fact, the argument shows that \( F: D_{\text{comp}}(A, I) \rightarrow D_{\text{comp}}(B, IB), K \mapsto \text{Hom}_A(B, I^*) \) is a left inverse to restriction. Finally, suppose that \( L \in D_{\text{comp}}(B, IB) \). Represent \( L \) by a K-injective complex \( J^* \) of \( B \)-modules. Then \( J^* \) is also K-injective as a complex of \( A \)-modules (Lemma 45.1) hence \( F(\text{restriction of } L) = \text{Hom}_A(B, J^*) \). There is a map \( J^* \rightarrow \text{Hom}_A(B, J^*) \) of complexes of \( B \)-modules, whose composition with \( \text{Hom}_A(B, J^*) \rightarrow J^* \) is the identity. We conclude that \( F \) is also a right inverse to restriction and the proof is finished. \( \square \)

### 68. Taking limits of complexes

In this section we discuss what happens when we have a “formal deformation” of a complex and we take its limit. More precisely, we have a ring \( A \) an ideal \( I \) and objects \( K_n \in D(A/I^n) \) which fit together in the sense that
\[
K_n = K_{n+1} \otimes_{A/I^{n+1}} A/I^n.
\]
Under some additional hypotheses we can show that \( K = R\lim K_n \) reproduces the system in the sense that \( K_n = K \otimes_A^n A/I^n \). We do not know if the following lemma holds for unbounded complexes.

**Lemma 68.1.** Let \( A \) be a ring and \( I \subset A \) an ideal. Suppose given \( K_n \in D(A/I^n) \) and maps \( K_{n+1} \rightarrow K_n \) in \( D(A/I^{n+1}) \). If

1. \( A \) is Noetherian,
2. \( K_1 \) is bounded above, and
3. the maps induce isomorphisms \( K_{n+1} \otimes_{A/I^{n+1}} A/I^n \rightarrow K_n \),

then \( K = R\lim K_n \) is a derived complete object of \( D^-(A) \) and \( K \otimes_A^n A/I^n \rightarrow K_n \) is an isomorphism for all \( n \).

**Proof.** Suppose that \( H^i(K_1) = 0 \) for \( i > b \). Then we can find a complex of free \( A/I \)-modules \( P^*_i \) representing \( K_1 \) with \( P^*_i = 0 \) for \( i > b \). By Lemma 60.4 we can, by induction on \( n > 1 \), find complexes \( P^*_n \) of free \( A/I^n \)-modules representing \( K_n \) and maps \( P^*_n \rightarrow P^*_{n-1} \) representing the maps \( K_n \rightarrow K_{n-1} \) inducing isomorphisms of complexes \( P^*_n/I^n-1 P^*_n \rightarrow P^*_{n-1} \).

Thus we have arrived at the situation where \( R\lim K_n \) is represented by \( P^* = \lim P^*_n \), see Lemma 64.9 and Remark 64.15. The complexes \( P^*_n \) are uniformly bounded above complexes of flat \( A/I^n \)-modules and the transition maps are termwise surjective. Then \( P^* \) is a bounded above complex of flat \( A \)-modules by Lemma 20.4. It follows that \( K \otimes_{A/I}^A/I \) is represented by \( P^* \otimes_A A/I \). We have \( P^* \otimes_A A/I \lim P^*_n \otimes_A A/I^n \) termwise by Lemma 20.4. The transition maps \( P^*_{n+1} \otimes_A A/I^{n+1} \rightarrow P^*_n \otimes_A A/I^n \) are
isomorphisms for \( n \geq t \). Hence we have \( \lim P_n^* \otimes_A A/I^t = R \lim P_n^* \otimes_A A/I^t \).

By assumption and our choice of \( P_n^* \) the complex \( P_n^* \otimes_A A/I^t = P_n^* \otimes_{A/I^n} A/I^t \) represents \( K_n \otimes_{A/I^n} A/I^t = K_t \) for all \( n \geq t \). We conclude

\[
P^* \otimes_A A/I^t = R \lim P_n^* \otimes_A A/I^t = R \lim K_t = K_t
\]

In other words, we have \( K \otimes_A^L A/I^t = K_t \). This proves the lemma as it follows that \( K \) is derived complete by Proposition \[67.19\]

**Lemma 68.2.** Let \( A \) be a ring and \( I \subset A \) an ideal. Suppose given \( K_n \in D(A/I^n) \) and maps \( K_{n+1} \to K_n \) in \( D(A/I^{n+1}) \). Assume

1. \( A \) is \( I \)-adically complete,
2. \( K_1 \) is pseudo-coherent, and
3. the maps induce isomorphisms \( K_{n+1} \otimes_{A/I^{n+1}}^L A/I^n \to K_n \).

Then \( K = R \lim K_n \) is a pseudo-coherent, derived complete object of \( D(A) \) and \( K \otimes_A^L A/I^n \to K_n \) is an isomorphism for all \( n \).

**Proof.** By assumption we can find a bounded above complex of finite free \( A/I \)-modules \( P^*_n \) representing \( K_1 \), see Definition \[62.1\] By Lemma \[60.4\] we can, by induction on \( n > 1 \), find complexes \( P^*_n \) of finite free \( A/I^n \)-modules representing \( K_n \) and maps \( P^*_n \to P^*_{n-1} \) representing the maps \( K_n \to K_{n-1} \) inducing isomorphisms (1) of complexes \( P^*_n/I^n P^*_n \to P^*_{n-1} \).

Thus \( R \lim K_n \) is represented by \( P^* = \lim P^*_n \), see Lemma \[64.9\] and Remark \[64.15\]

Since \( A \) is \( I \)-adically complete the modules \( P^n \) are finite free \( A \)-modules. Thus \( K \) is pseudo-coherent. Moreover, \( P^* \) is a bounded above complex of flat \( A \)-modules. It follows that \( K \otimes_A^L A/I^t \) is represented by \( P^* \otimes_A A/I^t \). We have \( P^* \otimes_A A/I^t = \lim P_n^* \otimes_A A/I^t \) termwise. The transition maps \( P^*_{n+1} \otimes_A A/I^t \to P^*_n \otimes_A A/I^t \) are isomorphisms for \( n \geq t \). Hence we have \( \lim P_n^* \otimes_A A/I^t = R \lim P_n^* \otimes_A A/I^t \).

By assumption and our choice of \( P_n^* \) the complex \( P_n^* \otimes_A A/I^t = P_n^* \otimes_{A/I^n} A/I^t \) represents \( K_n \otimes_{A/I^n} A/I^t = K_t \) for all \( n \geq t \). We conclude

\[
P^* \otimes_A A/I^t = R \lim P_n^* \otimes_A A/I^t = R \lim K_t = K_t
\]

In other words, we have \( K \otimes_A^L A/I^t = K_t \). Finally, \( K_n \) is a derived complete object of \( D(A) \) as it is annihilated by \( I^n \). Since the category of derived objects is preserved under homotopy limits we see that \( K \) is derived complete. This proves the lemma. \qed

**Lemma 68.3.** Let \( A \) be a ring and \( I \subset A \) an ideal. Suppose given \( K_n \in D(A/I^n) \) and maps \( K_{n+1} \to K_n \) in \( D(A/I^{n+1}) \). Assume

1. \( A \) is \( I \)-adically complete,
2. \( K_1 \) is a perfect object, and
3. the maps induce isomorphisms \( K_{n+1} \otimes_{A/I^{n+1}}^L A/I^n \to K_n \).

Then \( K = R \lim K_n \) is a perfect, derived complete object of \( D(A) \) and \( K \otimes_A^L A/I^n \to K_n \) is an isomorphism for all \( n \).

**Proof.** By Lemma \[68.2\] we see that \( K \) is bounded above, pseudo-coherent, and that \( K \otimes_A^L A/I^n \to K_n \) is an isomorphism for all \( n \). Thus it suffices to show that \( H^i(K \otimes_A^L \kappa) = 0 \) for \( i \ll 0 \) and every surjective map \( A \to \kappa \) whose kernel is a
maximal ideal \( m \), see Lemma 59.18. Since \( A \) is \( I \)-adically complete we have \( I \subset m \), see Algebra, Lemma 94.11. Hence
\[
K \otimes_A L \kappa = K \otimes_A A/I \otimes_A L \kappa = K_1 \otimes_{A/I} \kappa
\]
and we get what we want as \( K_1 \) has finite tor dimension by Lemma 59.2. \( \square \)

69. Some evaluation maps

In this section we prove that certain canonical maps of \( R \) Hom’s are isomorphisms for suitable types of complexes.

**Lemma 69.1.** Let \( R \) be a ring. Let \( K, L, M \) be objects of \( D(R) \). the map
\[
R \text{Hom}(L, M) \otimes_R L K \rightarrow R \text{Hom}(R \text{Hom}(K, L), M)
\]
of Lemma 58.3 is an isomorphism in the following two cases
(1) \( K \) perfect, or
(2) \( K \) is pseudo-coherent, \( L \in D^+(R) \), and \( M \) finite injective dimension.

**Proof.** Choose a \( K \)-injective complex \( I^\bullet \) representing \( M \), a \( K \)-injective complex \( J^\bullet \) representing \( L \), and a bounded above complex of finite projective modules \( K^\bullet \) representing \( K \). Consider the map of complexes
\[
\text{Tot}(\text{Hom}^\bullet(J^\bullet, I^\bullet)) \otimes_R K^\bullet \rightarrow \text{Hom}^\bullet(\text{Hom}^\bullet(K^\bullet, J^\bullet), I^\bullet)
\]
of Lemma 57.3. Note that
\[
\left( \prod_{p+r=t} \text{Hom}_R(J^{-r}, I^p) \right) \otimes_R K^s = \prod_{p+r=t} \text{Hom}_R(J^{-r}, I^p) \otimes_R K^s
\]
because \( K^s \) is finite projective. The map is given by the maps
\[
e_{p,r,s} : \text{Hom}_R(J^{-r}, I^p) \otimes_R K^s \rightarrow \text{Hom}_R(\text{Hom}_R(K^s, J^{-r}), I^p)
\]
which are isomorphisms as \( K^s \) is finite projective. For every element \( \alpha = (\alpha^{p,r,s}) \) of degree \( n \) of the left hand side, there are only finitely many values of \( s \) such that \( \alpha^{p,r,s} \) is nonzero (for some \( p, r \) with \( n = p+r+s \)). Hence our map is an isomorphism if the same vanishing condition is forced on the elements \( \beta = (\beta^{p,r,s}) \) of the right hand side. If \( K^\bullet \) is a bounded complex of finite projective modules, this is clear. On the other hand, if we can choose \( J^\bullet \) bounded and \( J^\bullet \) bounded below, then \( \beta^{p,r,s} \) is zero for \( p \) outside a fixed range, for \( s \gg 0 \), and for \( r \gg 0 \). Hence among solutions of \( n = p + r + s \) with \( \beta^{p,r,s} \) nonzero only a finite number of \( s \) values occur. \( \square \)

**Lemma 69.2.** Let \( R \) be a ring. Let \( K, L, M \) be objects of \( D(R) \). the map
\[
R \text{Hom}(L, M) \otimes_R L K \rightarrow R \text{Hom}(R \text{Hom}(K, L), M)
\]
of Lemma 58.3 is an isomorphism if the following three conditions are satisfied
(1) \( L, M \) have finite injective dimension,
(2) \( R \text{Hom}(L, M) \) has finite tor dimension,
(3) for every \( n \in \mathbb{Z} \) the truncation \( \tau_{\leq n} K \) is pseudo-coherent

**Proof.** Pick an integer \( n \) and consider the distinguished triangle
\[
\tau_{\leq n} K \rightarrow K \rightarrow \tau_{\geq n+1} K \rightarrow \tau_{\leq n} K[1]
\]
see Derived Categories, Remark 12.4. By assumption (3) and Lemma 69.1 the map is an isomorphism for \( \tau_{\leq n} K \). Hence it suffices to show that both
\[
R \text{Hom}(L, M) \otimes_R \tau_{\geq n+1} K \quad \text{and} \quad R \text{Hom}(R \text{Hom}(\tau_{\geq n+1} K, L), M)
\]
have vanishing cohomology in degrees $\leq n - c$ for some $c$. This follows immediately from assumptions (2) and (1).

**Lemma 69.3.** Let $R \to R'$ be a flat ring map. Let $K, L \in D(R)$. If $K$ is pseudo-coherent and $L \in D^+(R)$, then there is a canonical isomorphism

$$R\text{Hom}(K, L) \otimes_R R' \to R\text{Hom}(K \otimes_R R', L \otimes_R R')$$

in $D(R')$.

**Proof.** We represent $K$ by a bounded above complex $K^\bullet$ of finite free $R$-modules. We represent $L$ by a bounded below complex $L^\bullet$ of $R$-modules. Then we see that $R\text{Hom}(K, L)$ is represented by $\text{Hom}^\bullet(K^\bullet, L^\bullet)$ and that $R\text{Hom}(K \otimes_R R', L \otimes_R R')$ is represented by $\text{Hom}^\bullet(K^\bullet \otimes_R R', L^\bullet \otimes_R R')$. See Lemma 58.2. Thus it suffices to observe that the canonical map

$$\text{Hom}^\bullet(K^\bullet, L^\bullet) \otimes_R R' \to \text{Hom}^\bullet(K^\bullet \otimes_R R', L^\bullet \otimes_R R')$$

coming from the maps on components

$$\text{Hom}_R(K^{-q}, L^p) \otimes_R R' \to \text{Hom}_R(K^{-q} \otimes_R R', L^p \otimes_R R')$$

is an isomorphism. Each of the component maps is an isomorphism as $K^{-q}$ is finite free and the map in total is an isomorphism as the products in the definition of $\text{Hom}^\bullet(K^\bullet, L^\bullet)$ are finite (whence commute with tensor products) by the boundedness properties of the complexes $K^\bullet$ and $L^\bullet$.

**Lemma 69.4.** Let $R$ be a ring. Let $K, L, M$ be objects of $D(R)$. There is a canonical map

$$K \otimes^L_R R\text{Hom}(M, L) \to R\text{Hom}(M, K \otimes^L_R L)$$

which is an isomorphism in the following cases

1. $M$ perfect, or
2. $K$ is perfect, or
3. $M$ is pseudo-coherent, $L \in D^+(R)$, and $K$ has finite tor dimension.

**Proof.** The map is obtained as the composition

$$K \otimes^L_R R\text{Hom}(M, L) \to R\text{Hom}(K, K \otimes^L_R L) \otimes^L_R R\text{Hom}(M, L) \to R\text{Hom}(M, K \otimes^L_R L)$$

where we have used the maps of Lemmas 58.5 and 58.4.

Proof in case $M$ is perfect. Note that both sides of the arrow transform distinguished triangles in $M$ into distinguished triangles and commute with direct sums. Hence it suffices to check it holds when $M = R[n]$, see Derived Categories, Remark 33.3 and Lemma 60.1. In this case the result is obvious.

Proof in case $K$ is perfect. Same argument as in the previous case.

Proof in case (3). We may represent $K$ by a finite complex $K^\bullet$ of flat $R$-modules, see Lemma 53.3. We represent $M$ by a bounded above complex $M^\bullet$ of finite free $R$-modules, see Definition 52.1. We represent $L$ by a bounded below complex $L^\bullet$ of injectives. Then the object on the LHS is represented by

$$\text{Tot}(K^\bullet \otimes_R \text{Hom}^\bullet(M^\bullet, L^\bullet))$$

and the object on the RHS by

$$\text{Hom}^\bullet(M^\bullet, \text{Tot}(K^\bullet \otimes_R L^\bullet))$$
Both complexes have in degree $n$ the module
\[ \bigoplus_{p,q+r=n} K^p \otimes \text{Hom}_R(M^{-r}, L^n) = \bigoplus_{p,q+r=n} \text{Hom}_R(M^{-r}, K^p \otimes_R L^n) \]
because $M^{-r}$ is finite free (as well these are finite direct sums). We omit the verification that the map defined above induces the canonical isomorphism between these modules. \qed

## 70. Miscellany

Some results which do not fit anywhere else.

**Lemma 70.1.** Let $A$ be a Noetherian ring. Let $I \subset A$ be an ideal. Let $M$, $N$ be finite $A$-modules. Set $M_n = M/I^n M$ and $N_n = N/I^n N$. Then the systems $(\text{Hom}_A(M_n, N_n))$ and $(\text{Isom}_A(M_n, N_n))$ are Mittag-Leffler.

**Proof.** Note that $\text{Hom}_A(M_n, N_n) = \text{Hom}_A(M, N)$. Choose a presentation
\[ A^{\oplus t} \xrightarrow{T} A^{\oplus s} \rightarrow M \rightarrow 0 \]
The transpose of $T$ induces a map $\varphi : N^{\oplus s} \rightarrow N^{\oplus t}$ such that
\[ \text{Hom}_A(M_n, N_n) = \varphi^{-1}(I^n N^{\oplus t})/I^n N^{\oplus s}. \]
By Artin-Rees there exists an integer $c$ such that
\[ \varphi^{-1}(I^n N^{\oplus t}) = \text{Ker}(\varphi) + I^{n-c} \varphi^{-1}(I^n N^{\oplus t}) \]
for all $n \geq c$, see Algebra, Lemma [49.3]. Thus it is clear that the images of $\text{Hom}_A(M, N_n) \rightarrow \text{Hom}_A(M, N_m)$ stabilize for $n \geq m + c$.

The result for isomorphisms follows from the case of homomorphisms applied to both $(\text{Hom}(M_n, N_n))$ and $(\text{Hom}(N_n, M_n))$ and the following fact: for $n > m > 0$, if we have maps $\alpha : M_n \rightarrow N_n$ and $\beta : N_n \rightarrow M_n$ which induce an isomorphisms $M_m \rightarrow N_m$ and $N_m \rightarrow M_m$, then $\alpha$ and $\beta$ are isomorphisms. Namely, then $\alpha \circ \beta$ is surjective by Nakayama’s lemma (Algebra, Lemma [19.1]) hence $\alpha \circ \beta$ is an isomorphism by Algebra, Lemma [15.4]. \qed

**Lemma 70.2.** Let $A$ be a Noetherian ring. Let $I \subset A$ be an ideal. Let $M$, $N$ be finite $A$-modules. Set $M_n = M/I^n M$ and $N_n = N/I^n N$. If $M_n \cong N_n$ for all $n$, then $M^\wedge \cong N^\wedge$ as $A^\wedge$-modules.

**Proof.** By Lemma [70.1] the system $(\text{Isom}_A(M_n, N_n))$ is Mittag-Leffler. By assumption each of the sets $\text{Isom}_A(M_n, N_n)$ is nonempty. Hence $\lim \text{Isom}_A(M_n, N_n)$ is nonempty. Since $\lim \text{Isom}_A(M_n, N_n) = \text{Isom}(M^\wedge, N^\wedge)$ (use Algebra, Lemma 95.1) we obtain an isomorphism. \qed

**Lemma 70.3.** Let $A$ be a Noetherian ring. Let $I \subset A$ be an ideal. Let $M$, $N$ be finite $A$-modules with $N$ annihilated by $I$. For each $p > 0$ there exists an $n$ such that the map $\text{Ext}_A^p(M, N) \rightarrow \text{Ext}_A^p(I^n M, N)$ is zero.

**Proof.** The result is clear for $p = 0$ (with $n = 1$). Choose a short exact sequence $0 \rightarrow K \rightarrow A^{\oplus t} \rightarrow M \rightarrow 0$. For $n$ pick a short exact sequence $0 \rightarrow L \rightarrow A^{\oplus s} \rightarrow
Lemma 70.4. Let \( p > 0 \) we see that \( L \to K \) has image contained in \( I^{n-c}K \) if \( n \geq c \). At this point the exact sequence

\[
\text{Hom}_A(A^t, N) \to \text{Hom}_A(K, N) \to \text{Ext}^1_A(M, N) \to 0
\]

and the corresponding sequence for \( \text{Ext}^1_A(I^n M, N) \) show that the lemma holds for \( p = 1 \) with \( n = c + 1 \). Moreover, we see that the result for \( p - 1 \) and the module \( K \) implies the result for \( p \) and the module \( M \) by the commutativity of the diagram

\[
\begin{array}{c}
\text{Ext}^{p-1}_A(L, N) \\
\downarrow \\
\text{Ext}^{p-1}_A(I^{n-c}K, N)
\end{array} \overset{\cong}{\longrightarrow} \begin{array}{c}
\text{Ext}^{p-1}_A(I^n M, N) \\
\downarrow \\
\text{Ext}^p_A(M, N)
\end{array}
\]

for \( p > 1 \). Some details omitted.

\[\square\]

Lemma 70.4. Let \( A \) be a Noetherian ring. Let \( I \subset A \) be an ideal. Let \( M \) be a finite \( A \)-module. There exists an integer \( n > 0 \) such that \( I^n M \to M \) factors through the map \( I \otimes_A^L M \to M \) in \( D(A) \).

Proof. Consider the distinguished triangle

\( I \otimes_A^L M \to M \to A/I \otimes_A^L M \to I \otimes_A^L M[1] \)

By the axioms of a triangulated category it suffices to prove that \( I^n M \to A/I \otimes_A^L M \) is zero in \( D(A) \) for some \( n \). Choose generators \( f_1, \ldots, f_r \) of \( I \) and let \( K = K_\bullet(A, f_1, \ldots, f_r) \) be the Koszul complex. Consider the factorization \( A \to K \to A/I \) of the quotient map. Then we see that it suffices to show that \( I^n M \to K \otimes_A M \) is zero in \( D(A) \) for some \( n > 0 \). Suppose that we have found an \( n > 0 \) such that \( I^n M \to K \otimes_A M \) factors through \( \tau_{\geq 1}(K \otimes_A M) \) in \( D(A) \). Then the obstruction to factoring through \( \tau_{\geq 1+1}(K \otimes_A M) \) is an element in \( \text{Ext}^t(I^n M, H_t(K \otimes_A M)) \). The finite \( A \)-module \( H_t(K \otimes_A M) \) is annihilated by \( I \). Then by Lemma 70.3 we can after increasing \( n \) assume this obstruction element is zero. Repeating this a finite number of times we find \( n \) such that \( I^n M \to K \otimes_A M \) factors through \( 0 = \tau_{\geq r+1}(K \otimes_A M) \) in \( D(A) \) and we win.

\[\square\]

Lemma 70.5. Let \( R \) be a Noetherian local ring. Let \( I \subset R \) be an ideal and let \( E \) be a nonzero module over \( R/I \). If \( R/I \) has finite projective dimension and \( E \) has finite projective dimension over \( R/I \), then \( E \) has finite projective dimension over \( R \) and

\[\text{pd}_R(E) = \text{pd}_R(R/I) + \text{pd}_{R/I}(E)\]

Proof. We will use that, for a finite module, having finite projective dimension over \( R \), resp. \( R/I \) is the same as being a perfect module, see discussion following Definition 59.1. We see that \( E \) has finite projective dimension over \( R \) by Lemma
Thus we can apply Auslander-Buchsbaum (Algebra, Proposition 108.1) to see that
\[
\text{pd}_R(E) + \text{depth}(E) = \text{depth}(R), \quad \text{pd}_{R/I}(E) + \text{depth}(E) = \text{depth}(R/I),
\]
and
\[
\text{pd}_R(R/I) + \text{depth}(R/I) = \text{depth}(R).
\]
Note that in the first equation we take the depth of $E$ as an $R$-module and in the second as an $R/I$-module. However these depths are the same (this is trivial but also follows from Algebra, Lemma 70.8). This concludes the proof. 

71. Weakly étale ring maps

Most of the results in this section are from the paper [Oli83] by Olivier. See also the related paper [Fer67].

**Definition 71.1.** A ring $A$ is called absolutely flat if every $A$-module is flat over $A$. A ring map $A \to B$ is weakly étale or absolutely flat if both $A \to B$ and $B \otimes_A B \to B$ are flat.

For example a localization is weakly étale. An étale ring map is weakly étale. Here is a simple, yet key property.

**Lemma 71.2.** Let $A \to B$ be a ring map such that $B \otimes_A B \to B$ is flat. Let $N$ be a $B$-module. If $N$ is flat as an $A$-module, then $N$ is flat as a $B$-module.

**Proof.** Assume $N$ is a flat as an $A$-module. Then the functor
\[
\text{Mod}_B \to \text{Mod}_{B\otimes_A B}, \quad N' \mapsto N \otimes_A N'
\]
is exact. As $B \otimes_A B \to B$ is flat we conclude that the functor
\[
\text{Mod}_B \to \text{Mod}_B, \quad N' \mapsto (N \otimes_A N') \otimes_{B\otimes_A B} B = N \otimes_B N'
\]
is exact, hence $N$ is flat over $B$. 

**Definition 71.3.** Let $A$ be a ring. Let $d \geq 0$ be an integer. We say that $A$ has weak dimension $\leq d$ if every $A$-module has tor dimension $\leq d$.

**Lemma 71.4.** Let $A \to B$ be a weakly étale ring map. If $A$ has weak dimension at most $d$, then so does $B$.

**Proof.** Let $N$ be a $B$-module. If $d = 0$, then $N$ is flat as an $A$-module, hence flat as a $B$-module by Lemma 71.2. Assume $d > 0$. Choose a resolution $F_\bullet \to N$ by free $B$-modules. Our assumption implies that $K = \text{Im}(F_d \to F_{d-1})$ is $A$-flat, see Lemma 53.2. Hence it is $B$-flat by Lemma 71.2. Thus $0 \to K \to F_{d-1} \to \ldots \to F_0 \to N \to 0$ is a flat resolution of length $d$ and we see that $N$ has tor dimension at most $d$.

**Lemma 71.5.** Let $A$ be a ring. The following are equivalent

1. $A$ has weak dimension $\leq 0$,
2. $A$ is absolutely flat, and
3. $A$ is reduced and every prime is maximal.

In this case every local ring of $A$ is a field.
Proof. The equivalence of (1) and (2) is immediate. Assume $A$ is absolutely flat. This implies every ideal of $A$ is pure, see Algebra, Definition [105.1]. Hence every finitely generated ideal is generated by an idempotent by Algebra, Lemma [105.5]. If $f \in A$, then $(f) = (e)$ for some idempotent $e \in A$ and $D(f) = D(e)$ is open and closed (Algebra, Lemma [20.1]). This already implies every ideal of $A$ is maximal for example by Algebra, Lemma [25.5]. Moreover, if $f$ is nilpotent, then $e = 0$ hence $f = 0$. Thus $A$ is reduced.

Assume $A$ is reduced and every prime of $A$ is maximal. Let $M$ be an $A$-module. Our goal is to show that $M$ is flat. We may write $M$ as a filtered colimit of finite $A$-modules, hence we may assume $M$ is finite (Algebra, Lemma [38.2]). There is a finite filtration of $M$ by modules of the form $A/I$ (Algebra, Lemma [5.4]), hence we may assume that $M = A/I$ (Algebra, Lemma [38.12]). Thus it suffices to show every ideal of $A$ is pure. Since $A$ every local ring of $\mathop{	ext{Spec}}(A)$ is a field (by Algebra, Lemma [24.1] and the fact that every prime of $A$ is minimal), we see that every ideal $I \subset A$ is radical. Note that every closed subset of $\mathop{	ext{Spec}}(A)$ is closed under specialization. Thus every (radical) ideal of $A$ is pure by Algebra, Lemma [105.4].

Lemma 71.6. A product of fields is an absolutely flat ring.

Proof. Let $K_i$ be a family of fields. If $f = (f_i) \in \prod K_i$, then the ideal generated by $f$ is the same as the ideal generated by the idempotent $e = (e_i)$ with $e_i = 0, 1$ according to whether $f_i$ is 0, 1. Thus $D(f) = D(e)$ is open and closed and we conclude by Lemma [71.5] and Algebra, Lemma [25.5].

Lemma 71.7. Let $A \to B$ and $A \to A'$ be ring maps. Let $B' = B \otimes_A A'$ be the base change of $B$.

1. If $B \otimes_A B \to B$ is flat, then $B' \otimes_{A'} B' \to B'$ is flat.
2. If $A \to B$ is weakly étale, then $A' \to B'$ is weakly étale.

Proof. Assume $B \otimes_A B \to B$ is flat. The ring map $B' \otimes_{A'} B' \to B'$ is the base change of $B \otimes_A B \to B$ by $A \to A'$. Hence it is flat by Algebra, Lemma [38.6]. This proves (1). Part (2) follows from (1) and the fact (just used) that the base change of a flat ring map is flat.

Lemma 71.8. Let $A \to B$ be a ring map such that $B \otimes_A B \to B$ is flat.

1. If $A$ is an absolutely flat ring, then so is $B$.
2. If $A$ is reduced and $A \to B$ is weakly étale, then $B$ is reduced.

Proof. Part (1) follows immediately from Lemma [71.2] and the definitions. If $A$ is reduced, then there exists an injection $A \to A' = \prod_{\text{fp-min}} A_p$ of $A$ into an absolutely flat ring (Algebra, Lemma [24.2] and Lemma [71.6]). If $A \to B$ is flat, then the induced map $B \to B' = B \otimes_A A'$ is injective too. By Lemma [71.7] the ring map $A' \to B'$ is weakly étale. By part (1) we see that $B'$ is absolutely flat. By Lemma [71.5] the ring $B'$ is reduced. Hence $B$ is reduced.

Lemma 71.9. Let $A \to B$ and $B \to C$ be ring maps.

1. If $B \otimes_A B \to B$ and $C \otimes_B C \to C$ are flat, then $C \otimes_A C \to C$ is flat.
2. If $A \to B$ and $B \to C$ are weakly étale, then $A \to C$ is weakly étale.

Proof. Part (1) follows from the factorization

$$C \otimes_A C \to C \otimes_B C \to C$$
of the multiplication map, the fact that
\[ C \otimes_B C = (C \otimes_A C) \otimes_{B \otimes A} B, \]
the fact that a base change of a flat map is flat, and the fact that the composition of flat ring maps is flat. See Algebra, Lemmas \(38.6\) and \(38.3\). Part (2) follows from (1) and the fact (just used) that the composition of flat ring maps is flat. □

**Lemma 71.10.** Let \( A \to B \to C \) be ring maps.

1. If \( B \to C \) is faithfully flat and \( C \otimes_A C \to C \) is flat, then \( B \otimes_A B \to B \) is flat.
2. If \( B \to C \) is faithfully flat and \( A \to C \) is weakly étale, then \( A \to B \) is weakly étale.

**Proof.** Assume \( B \to C \) is faithfully flat and \( C \otimes_A C \to C \) is flat. Consider the commutative diagram

\[
\begin{array}{ccc}
C \otimes_A C & \longrightarrow & C \\
\uparrow & & \uparrow \\
B \otimes_A B & \longrightarrow & B
\end{array}
\]

The vertical arrows are flat, the top horizontal arrow is flat. Hence \( C \) is flat as a \( B \otimes_A B \)-module. The map \( B \to C \) is faithfully flat and \( C = B \otimes_B C \). Hence \( B \) is flat as a \( B \otimes_A B \)-module by Algebra, Lemma \(38.8\). This proves (1). Part (2) follows from (1) and the fact that \( A \to B \) is flat if \( A \to C \) is flat and \( B \to C \) is faithfully flat (Algebra, Lemma \(38.8\)). □

**Lemma 71.11.** Let \( A \) be a ring. Let \( B \to C \) be an \( A \)-algebra map of weakly étale \( A \)-algebras. Then \( B \to C \) is weakly étale.

**Proof.** Write \( B \to C \) as the composition \( B \to B \otimes_A C \to C \). The first map is flat as the base change of the flat ring map \( A \to C \). The second is the base change of the flat ring map \( B \otimes_A B \to B \) by the ring map \( B \otimes_A B \to B \otimes_A C \), hence flat. Thus \( B \to C \) is flat. The ring map \( C \otimes_A C \to C \otimes_B C \) is surjective, hence an epimorphism. Thus Lemma \(71.2\) implies, that since \( C \) is flat over \( C \otimes_A C \) it follows that \( C \) is flat over \( C \otimes_B C \). □

**Lemma 71.12.** Let \( A \to B \) be a ring map such that \( B \otimes_A B \to B \) is flat. Then \( \Omega_{B/A} = 0 \), i.e., \( B \) is formally unramified over \( A \).

**Proof.** Let \( I \subset B \otimes_A B \) be the kernel of the flat surjective map \( B \otimes_A B \to B \). Then \( I \) is a pure ideal (Algebra, Definition \(105.1\)), so \( I^2 = I \) (Algebra, Lemma \(105.2\)). Since \( \Omega_{B/A} = I/I^2 \) (Algebra, Lemma \(128.13\)) we obtain the vanishing. This means \( B \) is formally unramified over \( A \) by Algebra, Lemma \(142.2\). □

**Lemma 71.13.** Let \( A \to B \) be a ring map. Then \( A \to B \) is weakly étale in each of the following cases

1. \( B = S^{-1}A \) is a localization of \( A \),
2. \( A \to B \) is étale,
3. \( B \) is a filtered colimit of weakly étale \( A \)-algebras.

**Proof.** An étale ring map is flat and the map \( B \otimes_A B \to B \) is also étale as a map between étale \( A \)-algebras (Algebra, Lemma \(139.9\)). This proves (2).
Let $B_i$ be a directed system of weakly étale $A$-algebras. Then $B = \text{colim} B_i$ is flat over $A$ by Algebra, Lemma \[38.2\]. Note that the transition maps $B_i \to B_i'$ are flat by Lemma \[71.11\]. Hence $B$ is flat over $B_i$ for each $i$, and we see that $B$ is flat over $B \otimes_A B_i$ by Algebra, Lemma \[38.3\]. Thus $B$ is flat over $B \otimes_A B = \text{colim} B_i \otimes_A B_i$ by Algebra, Lemma \[38.5\].

Part (1) can be proved directly, but also follows by combining (2) and (3).

**Lemma 71.14.** Let $K \subset L$ be an extension of fields. If $L \otimes_K L \to L$ is flat, then $L$ is an algebraic separable extension of $K$.

**Proof.** By Lemma \[71.10\] we see that any subfield $K \subset L' \subset L$ the map $L' \otimes_K L' \to L'$ is flat. Thus we may assume $L$ is a finitely generated field extension of $K$. In this case the fact that $L/K$ is formally unramified (Lemma \[71.12\]) implies that $L/K$ is finite separable, see Algebra, Lemma \[148.1\].

**Lemma 71.15.** Let $K$ be a field. Let $K \to B$ be a ring map such that $B \otimes_K B \to B$ is flat. Then $B$ is a filtered colimit of étale $K$-algebras.

**Proof.** A field is absolutely flat ring, hence $B$ is a absolutely flat ring by Lemma \[71.8\]. Hence $B$ is reduced and every local ring is a field, see Lemma \[71.3\]. Let $q \subset B$ be a prime. The ring map $B \to B_q$ is weakly étale, hence $B_q$ is weakly étale over $K$ (Lemma \[71.9\]). Thus $B_q$ is a separable algebraic extension of $K$ by Lemma \[71.14\].

Let $K \subset A \subset B$ be a finitely generated $K$-sub algebra. Then every minimal prime $p \subset A$ is the image of a prime $q$ of $B$, see Algebra, Lemma \[29.5\]. Thus $\kappa(p)$ as a subfield of $B_q = \kappa(q)$ is is separable algebraic over $K$. Hence every generic point of $\text{Spec}(A)$ is closed (Algebra, Lemma \[34.9\]). Thus $\dim(A) = 0$. Then $A$ is the product of its local rings, e.g., by Algebra, Proposition \[59.6\]. Moreover, since $A$ is reduced, all local rings are equal to their residue fields which are finite separable over $K$. This means that $A$ is étale over $K$ by Algebra, Lemma \[139.4\] and finishes the proof.

**Lemma 71.16.** Let $A \to B$ be a ring map. If $A \to B$ is weakly étale, then $A \to B$ induces separable algebraic residue field extensions.

**Proof.** Let $p$ be a prime of $A$. Then $\kappa(p) \to B \otimes_A \kappa(p)$ is weakly étale by Lemma \[71.7\]. Hence $B \otimes_A \kappa(p)$ is a filtered colimit of étale $\kappa(p)$-algebras by Lemma \[71.15\]. Hence for $q \subset B$ lying over $p$ the extension $\kappa(p) \subset \kappa(q)$ is a filtered colimit of finite separable extensions by Algebra, Lemma \[139.4\].

**Lemma 71.17.** Let $A$ be a ring. The following are equivalent

1. $A$ has weak dimension $\leq 1$,
2. every ideal of $A$ is flat,
3. every finitely generated ideal of $A$ is flat,
4. every submodule of a flat $A$-module is flat, and
5. every local ring of $A$ is a valuation ring.

**Proof.** If $A$ has weak dimension $\leq 1$, then the resolution $0 \to I \to A \to A/I \to 0$ shows that every ideal $I$ is is flat by Lemma \[53.2\]. Hence (1) $\Rightarrow$ (2).
Assume (4). Let $M$ be an $A$-module. Choose a surjection $F \to M$ where $F$ is a free $A$-module. Then $\text{Ker}(F \to M)$ is flat by assumption, and we see that $M$ has tor dimension $\leq 1$ by Lemma 53.5. Hence (4) $\Rightarrow$ (1).

Every ideal is the union of the finitely generated ideals contained in it. Hence (3) implies (2) by Algebra, Lemma 38.2. Thus (3) $\iff$ (2).

Assume (2). Suppose that $N \subset M$ with $M$ a flat $A$-module. We will prove that $N$ is flat. We can write $M = \operatorname{colim} M_i$ with each $M_i$ finite free, see Algebra, Theorem 79.4. Setting $N_i \subset M_i$ the inverse image of $N$ we see that $N = \operatorname{colim} N_i$. By Algebra, Lemma 38.2 it suffices to prove $N_i$ is flat and we reduce to the case $M = R^{[n]}$. In this case the module $N$ has a finite filtration by the submodules $R^{[j]} \cap N$ whose subquotients are ideals. By (2) these ideals are flat and hence $N$ is flat by Algebra, Lemma 38.12. Thus (2) $\Rightarrow$ (4).

Assume $A$ satisfies (1) and let $p \subset A$ be a prime ideal. By Lemmas 71.13 and 71.4 we see that $A_p$ satisfies (1). We will show $A$ is a valuation ring if $A$ is a local ring satisfying (3). Let $f \in m$ be a nonzero element. Then $(f)$ is a flat nonzero module generated by one element. Hence it is a free $A$-module by Algebra, Lemma 76.4. It follows that $f$ is a nonzerodivisor and $A$ is a domain. If $I \subset A$ is a finitely generated ideal, then we similarly see that $I$ is a finite free $A$-module, hence (by considering the rank) free of rank 1 and $I$ is a principal ideal. Thus $A$ is a valuation ring by Algebra, Lemma 48.15. Thus (1) $\Rightarrow$ (5).

Assume (5). Let $I \subset A$ be a finitely generated ideal. Then $I_p \subset A_p$ is a finitely generated ideal in a valuation ring, hence principal (Algebra, Lemma 48.15). We reduce to the case $I$ is flat by Algebra, Lemma 38.19. Thus (5) $\Rightarrow$ (3). This finishes the proof of the lemma.

**Lemma 71.18.** Let $J$ be a set. For each $j \in J$ let $A_j$ be a valuation ring with fraction field $K_j$. Set $A = \prod A_j$ and $K = \prod K_j$. Then $A$ has weak dimension at most 1 and $A \to K$ is a localization.

**Proof.** Let $I \subset A$ be a finitely generated ideal. By Lemma 71.17 it suffices to show that $I$ is a flat $A$-module. Let $I_j \subset A_j$ be the image of $I$. Observe that $I_j = I \otimes_A A_j$, hence $I \to \prod I_j$ is surjective by Algebra, Proposition 87.2. Thus $I = \prod I_j$. Since $A_j$ is a valuation ring, the ideal $I_j$ is generated by a single element (Algebra, Lemma 48.15). Let $I_j = (f_j)$. Then $I$ is generated by the element $f = (f_j)$. Let $e \in A$ be the idempotent which has a 0 or 1 in $A_j$ depending on whether $f_j$ is 0 or 1. Then $f = ge$ for some nonzerodivisor $g \in A$: take $g = (g_j)$ with $g_j = 1$ if $f_j = 0$ and $g_j = f_j$ else. Thus $I \cong (e)$ as a module. We conclude $I$ is flat as $(e)$ is a direct summand of $A$. The final statement is true because $K = S^{-1}A$ where $S = \prod (A_j \setminus \{0\})$.

**Lemma 71.19.** Let $A$ be a normal domain with fraction field $K$. There exists a cartesian diagram

$$
\begin{array}{ccc}
A & \longrightarrow & K \\
\downarrow & & \downarrow \\
V & \longrightarrow & L
\end{array}
$$

of rings where $V$ has weak dimension at most 1 and $V \to L$ is a flat, injective, epimorphism of rings.
Proof. For every \( x \in K \), \( x \notin A \) pick \( V_x \subset K \) as in Algebra, Lemma \[48.11\] Set \( V = \prod_{x \in K \setminus A} V_x \) and \( L = \prod_{x \in K \setminus A} K \). The ring \( V \) has weak dimension at most 1 by Lemma \[71.18\] which also shows that \( V \to K \) is a localization. A localization is flat and an epimorphism, see Algebra, Lemmas \[38.19\] and \[104.3\]. □

Lemma 71.20. Let \( A \) be a ring of weak dimension at most 1. If \( A \to B \) is a flat, injective, epimorphism of rings, then \( A \) is integrally closed in \( B \).

Proof. Let \( x \in B \) be integral over \( A \). Let \( A' = A[x] \subset B \). Then \( A' \) is a finite ring extension of \( A \) by Algebra, Lemma \[35.5\]. To show \( A = A' \) it suffices to show \( A \to A' \) is an epimorphism by Algebra, Lemma \[104.6\]. Note that \( A' \) is flat over \( A \) by assumption on \( A \) and the fact that \( B \) is flat over \( A \) (Lemma \[71.17\]). Hence the composition

\[
A' \otimes_A A' \to B \otimes_A A' \to B \otimes_A B \to B
\]

is injective, i.e., \( A' \otimes_A A' \cong A' \) and the lemma is proved. □

Lemma 71.21. Let \( A \) be a normal domain with fraction field \( K \). Let \( A \to B \) be weakly étale. Then \( B \) is integrally closed in \( B \otimes_A K \).

Proof. Choose a diagram as in Lemma \[71.19\] As \( A \to B \) is flat, the base change gives a cartesian diagram

\[
\begin{array}{ccc}
B & \to & B \otimes_A K \\
\downarrow & & \downarrow \\
B \otimes_A V & \to & B \otimes_A L
\end{array}
\]

of rings. Note that \( V \to B \otimes_A V \) is weakly étale (Lemma \[71.7\]), hence \( B \otimes_A V \) has weak dimension at most 1 by Lemma \[71.4\]. Note that \( B \otimes_A V \to B \otimes_A L \) is a flat, injective, epimorphism of rings as a flat base change of such (Algebra, Lemmas \[38.6\] and \[104.3\]). By Lemma \[71.20\] we see that \( B \otimes_A V \) is integrally closed in \( B \otimes_A L \). It follows from the cartesian property of the diagram that \( B \) is integrally closed in \( B \otimes_A K \). □

Lemma 71.22. Let \( A \to B \) be a ring homomorphism. Assume

1. \( A \) is a henselian local ring,
2. \( A \to B \) is integral,
3. \( B \) is a domain.

Then \( B \) is a henselian local ring and \( A \to B \) is a local homomorphism. If \( A \) is strictly henselian, then \( B \) is a strictly henselian local ring and the extension \( \kappa(m_A) \subset \kappa(m_B) \) of residue fields is purely inseparable.

Proof. Write \( B \) as a filtered colimit \( B = \text{colim} \ B_i \) of finite \( A \)-sub algebras. If we prove the results for each \( B_i \), then the result follows for \( B \). See Algebra, Lemma \[146.3\]. If \( A \to B \) is finite, then \( B \) is a product of local henselian rings by Algebra, Lemma \[146.4\]. Since \( B \) is a domain we see that \( B \) is a local ring. The maximal ideal of \( B \) lies over the maximal ideal of \( A \) by going up for \( A \to B \) (Algebra, Lemma \[35.20\]). If \( A \) is strictly henselian, then the field extension \( \kappa(m_A) \subset \kappa(m_B) \) being algebraic, has to be purely inseparable. Of course, then \( \kappa(m_B) \) is separably algebraically closed and \( B \) is strictly henselian. □
Lemma 71.23. Let $A \to B$ and $A \to C$ be local homomorphisms of local rings. If $A \to C$ is integral and $\kappa(m_A) \subset \kappa(m_C)$ is purely inseparable, then $D = B \otimes_A C$ is a local ring and $B \to D$ and $C \to D$ are local.

Proof. Any maximal ideal of $D$ lies over the maximal ideal of $B$ by going up for the integral ring map $B \to D$ (Algebra, Lemma 35.20). Now $D/m_B D = \kappa(m_B) \otimes_A C = \kappa(m_B) \otimes_{\kappa(m_A)} C/m_A C$. The spectrum of $C/m_A C$ consists of a single point, namely $m_C$. Thus the spectrum of $D/m_B D$ is the same as the spectrum of $\kappa(m_B) \otimes_{\kappa(m_A)} \kappa(m_C)$ which is a single point by our assumption that $\kappa(m_A) \subset \kappa(m_C)$ is purely inseparable. This proves that $D$ is local and that the ring maps $B \to D$ and $C \to D$ are local. □

Theorem 71.24 (Olivier). Let $A \to B$ be a local homomorphism of local rings. If $A$ is strictly henselian and $A \to B$ is weakly étale, then $A = B$.

Proof. We will show that for all $p \subset A$ there is a unique prime $q \subset B$ lying over $p$ and $\kappa(p) = \kappa(q)$. This implies that $B \otimes_A B \to B$ is bijective on spectra as well as surjective and flat. Hence it is an isomorphism for example by the description of pure ideals in Algebra, Lemma 105.4. Hence $A \to B$ is a faithfully flat epimorphism of rings. We get $A = B$ by Algebra, Lemma 104.7.

Note that the fibre ring $B \otimes_A \kappa(p)$ is a colimit of étale extensions of $\kappa(p)$ by Lemmas 71.7 and 71.13. Hence, if there exists more than one prime lying over $p$ or if $\kappa(p) \neq \kappa(q)$ for some $q$, then $B \otimes_A L$ has a nontrivial idempotent for some (separable) algebraic field extension $L \supset \kappa(p)$.

Let $\kappa(p) \subset L$ be an algebraic field extension. Let $A' \subset L$ be the integral closure of $A/p$ in $L$. By Lemma 71.22 we see that $A'$ is a strictly henselian local ring whose residue field is a purely inseparable extension of the residue field of $A$. Thus $B \otimes_A A'$ is a local ring by Lemma 71.23. On the other hand, $B \otimes_A A'$ is integrally closed in $B \otimes_A L$ by Lemma 71.21. Since $B \otimes_A A'$ is local, it follows that the ring $B \otimes_A L$ does not have nontrivial idempotents which is what we wanted to prove. □

72. Ramification theory

In this section and the next we use the following definitions.

Definition 72.1. We say that $A \to B$ or $A \subset B$ is an extension of discrete valuation rings if $A$ and $B$ are discrete valuation rings and $A \to B$ is injective and local. In particular, if $\pi_A$ and $\pi_B$ are uniformizers of $A$ and $B$, then $\pi_A = u \pi_B^e$ for some $e \geq 1$ and unit $u$ of $B$. The integer $e$ does not depend on the choice of the uniformizers as it is also the unique integer $\geq 1$ such that $m_A B = m_B^e$.

The integer $e$ is called the ramification index of $B$ over $A$. We say that $B$ is weakly unramified over $A$ if $e = 1$. If the extension of residue fields $\kappa_A = A/m_A \subset \kappa_B = B/m_B$ is finite, then we set $f = [\kappa_B : \kappa_A]$ and we call it the residual degree or residue degree of the extension $A \subset B$.

Note that we do not require the extension of fraction fields to be finite.

Lemma 72.2. Let $A \subset B$ be an extension of discrete valuation rings with fraction fields $K \subset L$. If the extension $K \subset L$ is finite, then the residue field extension is finite and we have $ef \leq [L : K]$. 

Proof. Finiteness of the residue field extension is Algebra, Lemma \[116.9\] The inequality follows from Algebra, Lemmas \[116.8\] and \[50.12\].

Lemma 72.3. Let $A \subset B$ be an extension of discrete valuation rings inducing the field extension $K \subset L$. If the characteristic of $K$ is $p > 0$ and $L$ is purely inseparable over $K$, then the ramification index $e$ is a power of $p$.

Proof. Write $\pi_A = u\pi_B^p$ for some $u \in B^\times$. On the other hand, we have $\pi_B^q \in K$ for some $p$-power $q$. Write $\pi_B^q = v\pi_A^k$ for some $v \in A^\times$ and $k \in \mathbb{Z}$. Then $\pi_A^q = u^q v^e \pi_A^ke$. Taking valuations in $B$ we conclude that $ke = q$. □

Lemma 72.4. Let $A \subset B$ be an extension of discrete valuation rings. The following are equivalent

1. $A \to B$ is formally smooth in the $m_B$-adic topology, and
2. $A \to B$ is weakly unramified and $\kappa_A \subset \kappa_B$ is a separable field extension.

Proof. This follows from Proposition \[30.4\] and Algebra, Proposition \[148.9\]. □

Remark 72.5. Let $A$ be a discrete valuation ring with fraction field $K$. Let $K \subset L$ be a finite separable field extension. Let $B \subset L$ be the integral closure of $A$ in $L$. Picture:

\[
\begin{array}{ccc}
B & \longrightarrow & L \\
\uparrow & & \uparrow \\
A & \longrightarrow & K
\end{array}
\]

By Algebra, Lemma \[151.8\] the ring extension $A \subset B$ is finite, hence $B$ is Noetherian. By Algebra, Lemma \[109.4\] the dimension of $B$ is 1, hence $B$ is a Dedekind domain, see Algebra, Lemma \[117.14\]. Let $m_1, \ldots, m_n$ be the maximal ideals of $B$ (i.e., the primes lying over $m_A$). We obtain extensions of discrete valuation rings

\[ A \subset B_{m_i}, \]

and hence ramification indices $e_i$ and residue degrees $f_i$. We have

\[ [L : K] = \sum e_i f_i, \]

by Algebra, Lemma \[121.1\]. We observe that $n = 1$ if $A$ is henselian (by Algebra, Lemma \[146.4\]), e.g. if $A$ is complete.

Definition 72.6. Let $A$ be a discrete valuation ring with fraction field $K$. Let $L \supset K$ be a finite separable extension. With $B$ and $m_i$, $i = 1, \ldots, n$ as in Remark 72.5 we say the extension $L/K$ is

1. unramified if $e_i = 1$ and the extension $\kappa_A \subset \kappa(m_i)$ is separable for all $i$,
2. totally ramified if $n = 1$ and the residue field extension $\kappa_A \subset \kappa(m_1)$ is trivial,
3. tamely ramified if either the characteristic of $\kappa_A$ is 0 or the characteristic of $\kappa_A$ is $p > 0$ and the field extensions $\kappa_A \subset \kappa(m_i)$ are separable and the ramification indices $e_i$ are prime to $p$.

Lemma 72.7. Let $A$ be a discrete valuation ring with fraction field $K$. Let $K \subset L$ be a Galois extension with Galois group $G$. Then $G$ acts on the ring $B$ of Remark 72.3 and acts transitively on the set of maximal ideals of $B$. 
Proof. If there are two or more orbits of the action, then we can find an element \( b \in B \) which vanishes at all the maximal ideals of one orbit and has residue 1 at all the maximal ideals in another orbit. Then \( b' = \prod_{\sigma \in G} \sigma(b) \) is a \( G \)-invariant element of \( B \subset L \) which is in some maximal ideals of \( B \) but not in all maximal ideals of \( B \). Since \( K = L^G \) we conclude that \( b' \in K \). Since \( b' \) maps to an element of \( B \) we see that \( b' \in A \). Then on the one hand it must be true that \( b' \in m_A \) as \( b' \) is in some maximal ideal of \( B \) and on the other hand it must be true that \( b' \notin m_A \) as \( b' \) is not in all maximal ideals of \( B \). This contradiction finishes the proof of the lemma. \( \square \)

Lemma 72.8. Let \( A \) be a discrete valuation ring with fraction field \( K \). Let \( K \subset L \) be a Galois extension. Then there are \( e \geq 1 \) and \( f \geq 1 \) such that \( e_i = e \) and \( f_i = f \) for all \( i \) (notation as in Remark [72.5]). In particular \( [L : K] = e \).

Proof. Immediate consequence of Lemma 72.7 and the definitions. \( \square \)

Lemma 72.10. Let \( A \) be a discrete valuation ring with fraction field \( K \) and residue field \( \kappa \). Let \( K \subset L \) be a Galois extension with Galois group \( G \). Let \( B \) be the integral closure of \( A \) in \( L \).

(1) For a maximal ideal \( m \subset B \) the decomposition group associated to \( m \) is the subgroup \( D = \{ \sigma \in G | \sigma(m) = m \} \) of \( G \).

(2) The kernel \( I \) of the map \( D \rightarrow \text{Aut}(\kappa(m)/\kappa_A) \) is called the inertia group.

Note that the field \( \kappa(m) \) may be inseparable over \( \kappa_A \). In particular the field extension \( \kappa_A \subset \kappa(m) \) need not be Galois. If \( \kappa_A \) is perfect, then it is.

Proof. Pick an element \( \theta \in \kappa(m) \). Pick \( b \in B \) mapping to \( \theta \) and with zero residue modulo all other maximal ideals of \( B \). Let \( P(t) \in K[t] \) be the minimal polynomial of \( b \) over \( K \). By Algebra, Lemma 57.7 the polynomial \( P \) has coefficients in \( A \). Thus \( \theta \) is a root of the image \( \overline{P} \in \kappa[t] \). Thus the minimal polynomial of \( \theta \) divides \( \overline{P} \). Since \( P \) splits completely as \( P(t) = \prod (t - b_i) \) over \( B \) where the \( b_i \) are the conjugates of \( b \) in \( L \), we conclude that the minimal polynomial of \( \theta \) over \( \kappa \) splits completely in \( \kappa(m) \). This shows \( \kappa(m) \) is a normal extension of \( \kappa \).

Since \( \kappa(m)/\kappa_A \) is normal we may assume \( \kappa(m) = \kappa_1 \otimes_{\kappa} \kappa_2 \) with \( \kappa \subset \kappa_1 \) purely inseparable and \( \kappa \subset \kappa_2 \) Galois, see Fields, Lemma 24.3. Pick \( \theta \in \kappa_2 \) which generates \( \kappa_2 \) over \( \kappa \). If \( \theta' \in \kappa_2 \) is a conjugate of \( \theta \), then the above shows there exists a \( \sigma \in G \) such that \( \sigma(b) \) maps to \( \theta' \). By our choice of \( b \) (vanishing at other maximal ideals) this implies \( \sigma \in D \) and that the image of \( \sigma \) in \( \text{Aut}(\kappa(m)/\kappa_A) \) maps \( \theta \) to \( \theta' \). Hence the surjectivity.
The "equivalently all" part of the lemma follows from Lemma 72.7. Assume \( \kappa_A \subset \kappa(m) \) is separable. Parts (3) and (4) follow immediately from (1) and (2).

**Lemma 72.11.** Let \( A \) be a discrete valuation ring with fraction field \( K \). Let \( K \subset L \) be a Galois extension with Galois group \( G \). Let \( B \) be the integral closure of \( A \) in \( L \). Let \( m \subset B \) be a maximal ideal. The inertia group \( I \) of \( m \) has the following structure:

1. If the characteristic of \( \kappa_A \) is 0, then \( I \) is finite cyclic of order \( e \).
2. If the characteristic of \( \kappa_A \) is \( p > 0 \), then there is a short exact sequence of groups \( 1 \to \mathcal{P} \to I \to I_1 \to 0 \) where \( \mathcal{P} \) is a \( p \)-group and \( I_1 \) is cyclic of order prime to \( p \). In fact, the order of \( I_1 \) is the prime to \( p \) part of the integer \( e \).

Here \( e \) is the integer of Lemma 72.8.

**Proof.** Recall that \( |G| = [L : K] = nef \), see Lemma 72.8. Since \( G \) acts transitively on the set \( \{m_1, \ldots, m_n\} \) of maximal ideals of \( B \) (Lemma 72.7) and since \( D \) is the stabilizer of an element we see that \( |D| = ef \). By Lemma 72.10 we have

\[
ef = |D| = |I| \cdot |\text{Aut}(\kappa(m)/\kappa_A)|
\]

As \( \kappa(m) \) is normal over \( \kappa_A \) the order of \( \text{Aut}(\kappa(m)/\kappa_A) \) differs from \( f \) by a power of \( p \) (small detail omitted). Hence the prime to \( p \) part of \( |I| \) is equal to the prime to \( p \) part of \( e \).

Set \( C = B_m \). Then \( I \) acts on \( C \) over \( A \) and trivially on the residue field of \( C \). Let \( \pi_A \in A \) and \( \pi_C \in C \) be uniformizers. Write \( \pi_A = u \pi_C^e \) for some unit \( u \) in \( B \). For \( \sigma \in I \) write \( \sigma(\pi_C) = u_\sigma \pi_C \). Then we have

\[
\pi_A = \sigma(\pi_A) = \sigma(u_\sigma \pi_C)^e = \sigma(u)u_\sigma^e \pi_C^e = \frac{\sigma(u)}{u} u_\sigma^e \pi_A
\]

Since \( \sigma(u) \equiv u \mod \kappa_C \) we see that \( u_\sigma \) maps to an \( e \)-th root of unity in \( \kappa_C \). We obtain a homomorphism

\[
\chi : I \to \mu_e(\kappa_C)
\]

Since \( \kappa_C \) has characteristic \( p \), the group \( \mu_e(\kappa_C) \) is cyclic of order at most the prime to \( p \) part of \( e \) (some facts about roots of unity in fields omitted). Thus it suffices to prove that the kernel of \( \chi \) is a \( p \)-group. Let \( \sigma \) be a nontrivial element of the kernel. Then \( \sigma(m_C^i) \subset m_C^{i+1} \) for all \( i \). Let \( m \) be the order of \( \sigma \). Pick \( c \in C \) such that \( \sigma(c) \neq c \). Then \( \sigma(c) - c \in m_C^i, \sigma(c) - c \notin m_C^{i+1} \) for some \( i \) and we have

\[
0 = \sigma^m(c) - c = \sigma^m(c) - \sigma^{m-1}(c) + \ldots + \sigma(c) - c = \sum_{j=0}^{m-1} \sigma^j(\sigma(c) - c) \equiv m(\sigma(c) - c) \mod m_C^{i+1}
\]

It follows that \( p|m \) (or \( m = 0 \) if \( p = 1 \)). Thus every element of the kernel of \( \chi \) has order divisible by \( p \), i.e., \( \text{Ker}(\chi) \) is a \( p \)-group.

**Lemma 72.12.** Let \( \varphi : R \to S \) be a ring map such that

1. the kernel of \( \varphi \) is locally nilpotent, and
2. \( S \) is generated as an \( R \)-algebra by elements \( s \in S \) for which there exists a polynomial \( P(T) \in R[T] \) whose image in \( S[T] \) is \( (T - s)^n \) for some \( n = n(s) > 0 \).
Then Spec(S) → Spec(R) is a homeomorphism and R → S induces purely inseparable extensions of residue fields. Moreover, conditions (1) and (2) remain true on arbitrary base change.

**Proof.** We may replace R by R/Ker(ϕ), see Algebra, Lemma 45.2. Assumption (2) implies S is generated over R by elements which are integral over R. Hence R ⊂ S is integral (Algebra, Lemma 35.7). In particular Spec(S) → Spec(R) is surjective and closed (Algebra, Lemmas 35.15 and 35.20). Let p ⊂ R be a prime and let k ⊃ κ(p) be the perfection. Set A = S ⊗_R k. By the surjectivity the ring A' = S ⊗_R κ(p) is not zero (Algebra, Lemma 16.9) whence A = A' ⊗_κ(p) k is nonzero. Assumption (2) implies that A is generated over k by elements a ∈ A for which there exist a P(T) ∈ k[T] with P(T) → (T − a)^n(α) in A[T]. Since A ≠ 0 and k is perfect, this implies that P = (T − λ_a)^n(α) for some λ_a ∈ k (some details omitted). Hence k = A_red because a and λ_a map to the same element of A_red for all of these generators. It follows that Spec(A) is a singleton. Hence Spec(S) → Spec(R) is bijective and all residue field extensions are purely inseparable. Since a bijective, continuous, closed map of topological spaces is a homeomorphism the proof is finished.

**Lemma 72.13.** Let R be a ring. Let G be a finite group acting on R. Let I ⊂ R be an ideal such that σ(I) ⊂ I for all σ ∈ G. The ring maps

\[ R^G/I^G \rightarrow R^G/(R^G \cap I) \rightarrow (R/I)^G \]

induce homeomorphisms on spectra and purely inseparable extensions of residue fields.

**Proof.** Let n be the order of G. Let x ∈ R^G \cap I. Then x^n = \prod_{σ ∈ G} σ(x) is in I^G. Hence the kernel of the first map is locally nilpotent. Since the first map is also surjective, we see that the result of the lemma is true for the first map.

The second map is injective. Let x ∈ R be an element which maps to an element of (R/I)^G. Consider the polynomial P(T) = \prod(T − σ(x)) ∈ R^G[T]. The image of P in (R/I)^G[T] is (T − x)^n. Thus Lemma 72.12 applies to both the second map and the composition.

**Lemma 72.14.** Let A be a discrete valuation ring with fraction field K. Let L be a Galois extension of K. Let m ⊂ B be a maximal ideal of the integral closure of A in L. Let I ⊂ G be the corresponding inertia subgroup. Then B^I is the integral closure of A in L^I and A → (B^I)_B^I/m is étale.

**Proof.** It follows from the definitions that B^I is the integral closure of A in L^I.

We first prove the final statement in case B is a discrete valuation ring, i.e., when G is the decomposition group of m. As I acts trivially on κ_B it follows from Lemma 72.13 that the extension κ_{B^I} = B^I/(B^I \cap I) ⊂ κ_B is purely inseparable. Since G/I acts faithfully on κ_B, we conclude that G/I acts faithfully on κ_{B^I} over κ_A. By Galois theory we see that [κ_{B^I} : κ_A] ≥ |G/I|. On the other hand, we have [L^I : K] = |G/I| by Galois theory. By Lemma 72.2 we see that A ⊂ B^I is weakly unramified and that [κ_{B^I} : κ_A] = |G/I|. Thus κ_{B^I} is Galois over κ_A (with group G/I) and we conclude that L^I is unramified over K as desired.

In general we reduce to the case discussed in the previous paragraph by splitting B using Algebra, Lemma 139.23. (An alternative is to use completion to do this.) We omit the details.
Lemma 72.15 (Krasner’s lemma). Let $A$ be a complete local domain of dimension 1. Let $P(t) \in A[t]$ be a polynomial with coefficients in $A$. Let $\alpha \in A$ be a root of $P$ but not a root of the derivative $P' = dP/dt$. For every $c \geq 0$ there exists an integer $n$ such that for any $Q \in A[t]$ whose coefficients are in $m_A^n$ the polynomial $P + Q$ has a root $\beta \in A$ with $\beta - \alpha \in m_A^n$.

Proof. Choose a nonzero $\pi \in m$. Since the dimension of $A$ is 1 we have $m = \sqrt{(\pi)}$. By assumption we may write $P'(\alpha)^{-1} = \pi^{-m}a$ for some $m \geq 0$ and $\alpha \in A$. We may and do assume that $c \geq m + 1$. Pick $n$ such that $m_A^n \subset (\pi^{c+m})$. Pick any $Q$ as in the statement. For later use we observe that we can write

$$P(x + y) = P(x) + P'(x)y + R(x,y)y^2$$

for some $R(x,y) \in A[x,y]$. We will show by induction that we can find a sequence $\alpha_m, \alpha_{m+1}, \alpha_{m+2}, \ldots$ such that

1. $\alpha_k \equiv \alpha \mod \pi^c$,
2. $\alpha_{k+1} - \alpha_k \in (\pi^k)$, and
3. $(P + Q)(\alpha_k) \in (\pi^{m+k})$.

Setting $\beta = \lim \alpha_k$ will finish the proof.

Base case. Since the coefficients of $Q$ are in $(\pi^{c+m})$ we have $(P + Q)(\alpha) \in (\pi^{c+m})$. Hence $\alpha_m = \alpha$ works. This choice guarantees that $\alpha_k \equiv \alpha \mod \pi^c$ for all $k \geq m$.

Induction step. Given $\alpha_k$ we write $\alpha_{k+1} = \alpha_k + \delta$ for some $\delta \in (\pi^k)$. Then we have

$$(P + Q)(\alpha_{k+1}) = P(\alpha_k + \delta) + Q(\alpha_k + \delta)$$

Because the coefficients of $Q$ are in $(\pi^{c+m})$ we see that $Q(\alpha_k + \delta) \equiv Q(\alpha_k) \mod \pi^{c+m+k}$. On the other hand we have

$$P(\alpha_k + \delta) = P(\alpha_k) + P'(\alpha_k)\delta + R(\alpha_k, \delta)\delta^2$$

Note that $P'(\alpha_k) \equiv P'(\alpha) \mod (\pi^{m+1})$ as $\alpha_k \equiv \alpha \mod \pi^{m+1}$. Hence we obtain

$$P(\alpha_k + \delta) \equiv P(\alpha_k) + P'(\alpha)\delta \mod \pi^{k+m+1}$$

Recombining the two terms we see that

$$(P + Q)(\alpha_{k+1}) \equiv (P + Q)(\alpha_k) + P'(\alpha)\delta \mod \pi^{k+m+1}$$

Thus a solution is to take $\delta = -P'(\alpha)^{-1}(P + Q)(\alpha_k) = -\pi^{-m}a(P + Q)(\alpha_k)$ which is contained in $(\pi^k)$ by induction assumption. \qed

Lemma 72.16. Let $A$ be a discrete valuation ring with field of fractions $K$. Let $A^\wedge$ be the completion of $A$ with fraction field $K^\wedge$. If $K^\wedge \subset M$ is a finite separable extension, then there exists a finite separable extension $K \subset L$ such that $M = K^\wedge \otimes_K L$.

Proof. Note that $A^\wedge$ is a discrete valuation ring too (by Lemmas 33.4 and 33.1). In particular $A^\wedge$ is a domain. The proof will work more generally for Noetherian local rings $A$ such that $A^\wedge$ is a local domain of dimension 1.

Let $\theta \in M$ be an element that generates $M$ over $K^\wedge$. (Theorem of the primitive element.) Let $P(t) \in K^\wedge[t]$ be the minimal polynomial of $\theta$ over $K^\wedge$. Let $\pi \in m_A$ be a nonzero element. After replacing $\theta$ by $\pi^n \theta$ we may assume that the coefficients of $P(t)$ are in $A^\wedge$. Let $B = A^\wedge[\theta] = A^\wedge[t]/(P(t))$. Note that $B$ is a complete local domain of dimension 1 because it is finite over $A$ and contained in $M$. Since $M$ is separable over $K$ the element $\theta$ is not a root of the derivative of $P$. For any integer
n we can find a monic polynomial $P_1 \in A[t]$ such that $P - P_1$ has coefficients in $\pi^n A^\wedge[t]$. By Krasner’s lemma (Lemma 72.15) we see that $P_1$ has a root $\beta$ in $B$ for $n$ sufficiently large. Moreover, we may assume (if $n$ is chosen large enough) that $\theta - \beta \in \pi B$. Consider the map $\Phi : A^\wedge[t]/(P_1) \to B$ of $A^\wedge$-algebras which maps $t$ to $\beta$. Since $B = \pi B + \sum_{i < \deg(P_1)} A^\wedge \theta^i$, the map $\Phi$ is surjective by Nakayama’s lemma. As $\deg(P_1) = \deg(P)$ it follows that $\Phi$ is an isomorphism. We conclude that the ring extension $L = K[t]/(P_1(t))$ satisfies $K^\wedge \otimes_K L \cong M$. This implies that $L$ is a field and the proof is complete. \hfill \Box

**Definition 72.17.** Let $A$ be a discrete valuation ring. We say $A$ has *mixed characteristic* if the characteristic of the residue field of $A$ is $p > 0$ and the characteristic of the fraction field of $A$ is $0$. In this case we obtain an extension of discrete valuation rings $\mathbb{Z}_{(p)} \subset A$ and the *absolute ramification index* of $A$ is the ramification index of this extension.

**73. Eliminating ramification**

In this section we discuss a result of Helmut Epp, see [Epp73]. We strongly encourage the reader to read the original. Our approach is slightly different as we try to handle the mixed and equicharacteristic cases by the same method. For related results, see also [Pon98], [Pon99], [Kuhl03], and [ZK99].

**Remark 73.1.** Let $A \to B$ be an extension of discrete valuation rings with fraction fields $K \subset L$. Let $K \subset K_1$ be a finite extension of fields. Let $A_1 \subset K_1$ be the integral closure of $A$ in $K_1$. On the other hand, let $L_1 = (L \otimes_K K_1)_{red}$. Then $L_1$ is a nonempty finite product of finite field extensions of $L$. Let $B_1$ be the integral closure of $B$ in $L_1$. We obtain compatible commutative diagrams

\[
\begin{array}{cccc}
L & \longrightarrow & L_1 & \quad B & \longrightarrow & B_1 \\
\uparrow & & \uparrow & \quad \uparrow & & \uparrow \\
K & \longrightarrow & K_1 & \quad A & \longrightarrow & A_1
\end{array}
\]

In this situation we have the following:

1. By Algebra, Lemma 117.15 the ring $A_1$ is a Dedekind domain and $B_1$ is a finite product of Dedekind domains.
2. Note that $L \otimes_K K_1 = (B \otimes_A A_1)_\pi$ where $\pi \in A$ is a uniformizer and that $\pi$ is a nonzerodivisor on $B \otimes_A A_1$. Thus the ring map $B \otimes_A A_1 \to B_1$ is integral with kernel consisting of nilpotent elements. Hence $\text{Spec}(B_1) \to \text{Spec}(B \otimes_A A_1)$ is surjective on spectra (Algebra, Lemma 35.15). The map $\text{Spec}(B \otimes_A A_1) \to \text{Spec}(A_1)$ is surjective as $A_1/m_A A_1 \to B/m_A B \otimes_{k_A} A_1/m_A A_1$ is an injective ring map with $A_1/m_A A_1$ Artinian. We conclude that $\text{Spec}(B_1) \to \text{Spec}(A_1)$ is surjective.
3. Let $m_i$, $i = 1, \ldots, n$ with $n \geq 1$ be the maximal ideals of $A_1$. For each $i = 1, \ldots, n$ let $m_{i,j}$, $j = 1, \ldots, m_i$ with $m_j \geq 1$ be the maximal ideals of $B_1$ lying over $m_i$. We observe diagrams

\[
\begin{array}{cccc}
B & \longrightarrow & (B_1)_{m_{i,j}} \\
\uparrow & & \uparrow & \quad \uparrow \\
A & \longrightarrow & (A_1)_{m_i}
\end{array}
\]
of extensions of discrete valuation rings.

1. If \( A \) is henselian (for example complete), then \( A_1 \) is a discrete valuation ring, i.e., \( n = 1 \). Namely, \( A_1 \) is a union of finite extensions of \( A \) which are domains, hence local by Algebra, Lemma \[146.4\].

2. If \( B \) is henselian (for example complete), then \( B_1 \) is a product of discrete valuation rings, i.e., \( m_i = 1 \) for \( i = 1, \ldots, n \).

3. If \( K \subset K_1 \) is purely inseparable, then \( A_1 \) and \( B_1 \) are both discrete valuation rings, i.e., \( n = 1 \) and \( m_1 = 1 \). This is true because for every \( b \in B_1 \) a \( p \)-power power of \( b \) is in \( B \), hence \( B_1 \) can only have one maximal ideal.

4. If \( K \subset K_1 \) is finite separable, then \( L_1 = L \otimes_K K_1 \) and is a finite product of finite separable extensions too. Hence \( A \subset A_1 \) and \( B \subset B_1 \) are finite by Algebra, Lemma \[151.8\].

5. If \( A \) is Nagata, then \( A \subset A_1 \) is finite.

6. If \( B \) is Nagata, then \( B \subset B_1 \) is finite.

The goal in this section is to find extensions \( K \subset K_1 \) as in Remark \[73.1\] such that the extensions \( (A_1)_{m_i} \subset (B_1)_{m_{ij}} \) are all weakly unramified or even formally smooth.

**Definition 73.2.** Let \( A \to B \) be an extension of discrete valuation rings with fraction fields \( K \subset L \).

1. We say a finite field extension \( K \subset K_1 \) is a weak solution for \( A \subset B \) if all the extensions \( (A_1)_{m_i} \subset (B_1)_{m_{ij}} \) of Remark \[73.1\] are weakly unramified.

2. We say a finite field extension \( K \subset K_1 \) is a solution for \( A \subset B \) if each extension \( (A_1)_{m_i} \subset (B_1)_{m_{ij}} \) of Remark \[73.1\] is formally smooth in the \( m_{ij} \)-adic topology.

We say a solution \( K \subset K_1 \) is a separable solution if \( K \subset K_1 \) is separable.

In general (weak) solutions do not exist. The following example shows that in general one needs inseparable extensions to get a solution.

**Example 73.3.** Let \( k \) be a perfect field of characteristic \( p > 0 \). Let \( A = k[[x]] \) and \( K = k((x)) \). Then \( B = A[x^{1/p}] \). Any weak solution \( K \subset K_1 \) for \( A \to B \) is inseparable (and any finite inseparable extension of \( K \) is a solution). We omit the proof.

Solutions are stable under further extensions, but this may not be true for weak solutions. Weak solutions are stable under totally ramified solutions.

**Lemma 73.4.** Let \( A \to B \) be an extension of discrete valuation rings with fraction fields \( K \subset L \). Assume that \( A \to B \) is formally smooth in the \( m_B \)-adic topology. Then for any finite extension \( K \subset K_1 \) we have \( L_1 = L \otimes_K K_1 \), \( B_1 = B \otimes_A A_1 \), and each extension \( (A_1)_{m_i} \subset (B_1)_{m_{ij}} \) (see Remark \[73.1\]) is formally smooth in the \( m_{ij} \)-adic topology.

**Proof.** We will use the equivalence of Lemma \[72.4\] without further mention. Let \( \pi \in A \) and \( \pi_i \in (A_1)_{m_i} \) be uniformizers. As \( \kappa_A \subset \kappa_B \) is separable, the ring

\[
\frac{(B \otimes_A (A_1)_{m_i})/\pi_i(B \otimes_A (A_1)_{m_i})}{B/\pi B \otimes_A (A_1)_{m_i}/\pi_i(A_1)_{m_i}}
\]

is a product of fields each separable over \( \kappa_m \). Hence the element \( \pi_i \) in \( B \otimes_A (A_1)_{m_i} \) is a nonzerodivisor and the quotient by this element is a product of fields. It follows that \( B \otimes_A A_1 \) is a Dedekind domain in particular reduced. Thus \( B \otimes_A A_1 \subset B_1 \) is an equality. \( \square \)
**Lemma 73.5.** Let \( A \to B \) be an extension of discrete valuation rings with fraction fields \( K \subset L \). Assume that \( A \to B \) is weakly unramified. Then for any totally ramified separable extension \( K \subset K_1 \) we have that \( L_1 = L \otimes_K K_1 \) is a field, \( A_1 \) and \( B_1 = B \otimes_A A_1 \) are discrete valuation rings, and the extension \( A_1 \subset B_1 \) (see Remark [73.4]) is weakly unramified.

**Proof.** Let \( \pi \in A \) and \( \pi_1 \in A_1 \) be uniformizers. As \( K \subset K_1 \) is totally ramified we have \( \pi_1^e = u \pi \) for some unit \( u \) in \( A \). Hence \( A_1 \) is generated by \( \pi_1 \) over \( A \) and the minimal polynomial \( P(t) \) of \( \pi_1 \) over \( K \) has the form

\[
P(t) = t^e + a_{e-1} t^{e-1} + \ldots + a_0
\]

with \( a_i \in (\pi) \) and \( a_0 = u \pi \) for some unit \( u \) of \( A \). Note that \( e = [K_1 : K] \) as well. Since \( A \to B \) is weakly unramified we see that \( \pi \) is a uniformizer of \( B \) and hence \( B_1 = B[t]/(P(t)) \) is a discrete valuation ring with uniformizer the class of \( t \). Thus the lemma is clear. \( \square \)

**Lemma 73.6.** Let \( A \to B \to C \) be extensions of discrete valuation rings with fraction fields \( K \subset L \subset M \). Let \( K \subset K_1 \) be a finite extension.

1. If \( K_1 \) is a (weak) solution for \( A \to C \), then \( K_1 \) is a (weak) solution for \( A \to B \).
2. If \( K_1 \) is a (weak) solution for \( A \to B \) and \( L_1 = (L \otimes_K K_1)_{\text{red}} \) is a product of fields which are (weak) solutions for \( B \to C \), then \( K_1 \) is a weak solution for \( A \to C \).

**Proof.** Let \( L_1 = (L \otimes_K K_1)_{\text{red}} \) and \( M_1 = (M \otimes_K K_1)_{\text{red}} \) and let \( B_1 \subset L_1 \) and \( C_1 \subset M_1 \) be the integral closure of \( B \) and \( C \). Note that \( M_1 = (M \otimes_L L_1)_{\text{red}} \) and that \( L_1 \) is a (nonempty) finite product of finite extensions of \( L \). Hence the ring map \( \text{Spec}(C_1) \to \text{Spec}(B_1) \) is surjective. Choose a maximal ideal \( m \subset C_1 \) and consider the extensions of discrete valuation rings

\[
(A_1)_{A_1 \cap m} \to (B_1)_{B_1 \cap m} \to (C_1)_m
\]

If the composition is weakly unramified, so is the map \( (A_1)_{A_1 \cap m} \to (B_1)_{B_1 \cap m} \). If the residue field extension \( \kappa_{A_1 \cap m} \to \kappa_m \) is separable, so is the subextension \( \kappa_{A_1 \cap m} \to \kappa_{B_1 \cap m} \). Taking into account Lemma [72.4] this proves (1). A similar argument works for (2). \( \square \)

**Lemma 73.7.** Let \( A \to B \) be an extension of discrete valuation rings. There exists a commutative diagram

\[
\begin{array}{ccc}
B & \longrightarrow & B' \\
\uparrow & & \uparrow \\
A & \longrightarrow & A'
\end{array}
\]

of extensions of discrete valuation rings such that

1. the extensions \( K \subset K' \) and \( L \subset L' \) of fraction fields are separable algebraic,
2. the residue fields of \( A' \) and \( B' \) are separable algebraic closures of the residue fields of \( A \) and \( B \), and
3. if a solution, weak solution, or separable solution exists for \( A' \to B' \), then a solution, weak solution, or separable solution exists for \( A \to B \).
Proof. By Algebra, Lemma 149.2 there exists an extension $A \subset A'$ which is a filtered colimit of finite étale extensions such that the residue field of $A'$ is a separable algebraic closure of the residue field of $A$. Then $A \subset A'$ is an extension of discrete valuation rings such that the induced extension $K \subset K'$ of fraction fields is separable algebraic.

Let $B \subset B'$ be a strict henselization of $B$. Then $B \subset B'$ is an extension of discrete valuation rings whose fraction field extension is separable algebraic. By Algebra, Lemma 146.25 there exists a commutative diagram as in the statement of the lemma. Parts (1) and (2) of the lemma are clear.

Let $K' \subset K'_1$ be a (weak) solution for $A' \to B'$. Since $A'$ is a colimit, we can find a finite étale extension $A \subset A'_1$ and a finite extension $f.f.(A'_1) \subset K_1$ such that $K'_1 = K' \otimes_{f.f.(A'_1)} K_1$. As $A \subset A'_1$ is finite étale and $B'$ strictly henselian, it follows that $B' \otimes_A A'_1$ is a finite product of rings isomorphic to $B'$. Hence

$$L' \otimes_K K_1 = L' \otimes_K f.f.(A'_1) \otimes_{f.f.(A'_1)} K_1$$

is a finite product of rings isomorphic to $L' \otimes_K K'_1$. Thus we see that $K \subset K_1$ is a (weak) solution for $A \to B$. Hence it is also a (weak) solution for $A \to B$ by Lemma 73.6. □

Lemma 73.8. Let $A \to B$ be an extension of discrete valuation rings with fraction fields $K \subset L$. Let $K \subset K_1$ be a normal extension. Say $G = \text{Aut}(K_1/K)$. Then $G$ acts on the rings $K_1, L_1, A_1$ and $B_1$ of Remark 73.1 and acts transitively on the set of maximal ideals of $B_1$.

Proof. Everything is clear apart from the last assertion. If there are two or more orbits of the action, then we can find an element $b \in B_1$ which vanishes at all the maximal ideals of one orbit and has residue 1 at all the maximal ideals in another orbit. Then $b' = \prod_{\sigma \in G} \sigma(b)$ is a $G$-invariant element of $B_1 \subset L_1 = (L \otimes_K K_1)_{\text{red}}$ which is in some maximal ideals of $B_1$ but not in all maximal ideals of $B_1$. Lifting it to an element of $L \otimes_K K_1$ and raising to a high power we obtain a $G$-invariant element $b''$ of $L \otimes_K K_1$ mapping to $(b')^N$ for some $N > 0$; in fact, we only need to do this in case the characteristic is $p > 0$ and in this case raising to a suitably large $p$-power $q$ defines a canonical map $(L \otimes_K K_1)_{\text{red}} \to L \otimes_K K_1$. Since $K = (K_1)^G$ we conclude that $b'' \in L$. Since $b''$ maps to an element of $B_1$ we see that $b'' \in B$ (as $B$ is normal). Then on the one hand it must be true that $b'' \in m_B$ as $b'$ is in some maximal ideal of $B_1$ and on the other hand it must be true that $b'' \notin m_B$ as $b'$ is not in all maximal ideals of $B_1$. This contradiction finishes the proof of the lemma. □

Lemma 73.9. Let $A$ be a discrete valuation ring with uniformizer $\pi$. Let $n \geq 2$. Then $K_1 = K[\pi^{1/n}]$ is a degree $n$ extension of $K$ and the integral closure $A_1$ of $A$ in $K_1$ is the ring $A[\pi^{1/n}]$ which is a discrete valuation ring with ramification index $n$ over $A$.

Proof. This lemma proves itself. □

Lemma 73.10. Let $A$ be a discrete valuation ring with uniformizer $\pi$. If the residue characteristic of $A$ is $p > 0$, then for every $n > 1$ and $p$-power $q$ there exists a degree $q$ separable totally ramified extension $K \subset L$ such that the integral closure $B$ of $A$ in $L$ has ramification index $q$ and a uniformizer $\pi_B$ such that $\pi_B^q = \pi + \pi^b$ and $\pi_B^q = \pi + (\pi_B)^n b'$ for some $b, b' \in B$. □
Proof. If the characteristic of $K$ is zero, then we can take the extension given by $\pi_B^n = \pi$, see Lemma 73.9. If the characteristic of $K$ is $p > 0$, then we can take the extension of $K$ given by $y^q - \pi^{n+q-1}y = \pi$ where $y = \pi z$. Taking $\pi_B = y$ we obtain the desired result. \hfill \blacksquare

Lemma 73.11. Let $A$ be a discrete valuation ring. Assume the residue field $\kappa_A$ has characteristic $p > 0$ and that $a \in A$ is an element whose residue class in $\kappa_A$ is not a $p$th power. Then $a$ is not a $p$th power in $K$ and the integral closure of $A$ in $K[a^{1/p}]$ is the ring $A[a^{1/p}]$ which is a discrete valuation ring weakly unramified over $A$.

Proof. This lemma proves itself. \hfill \blacksquare

Lemma 73.12. Let $A \subset B \subset C$ be extensions of discrete valuation rings with fractions fields $K \subset L \subset M$. Let $\pi \in A$ be a uniformizer. Assume

(1) $B$ is a Nagata ring,
(2) $A \subset B$ is weakly unramified,
(3) $M$ is a degree $p$ purely inseparable extension of $L$.

Then either

(1) $A \to C$ is weakly unramified, or
(2) $C = B[\pi^{1/p}]$, or
(3) there exists a degree $p$ separable, totally ramified extension $K \subset K_1$ such that $L_1 = L \otimes_K K_1$ and $M_1 = M \otimes_K K_1$ are fields and the maps of integral closures $A_1 \to B_1 \to C_1$ are weakly unramified extensions of discrete valuation rings.

Proof. Let $e$ be the ramification index of $C$ over $B$. If $e = 1$, then we are done. If not, then $e = p$ by Lemmas 72.2 and 72.3. This in turn implies that the residue fields of $B$ and $C$ agree. Choose a uniformizer $\pi_C$ of $C$. Write $\pi_C^n = u\pi$ for some unit $u$ of $C$. Since $\pi_C^n \in L$, we see that $u \in B^*$. Also $M = L[\pi_C]$.

Suppose there exists an integer $m \geq 0$ such that

$$u = \sum_{0 \leq i < m} b_i^p \pi^i + b^m$$

with $b_i \in B$ and with $b \in B$ an element whose image in $\kappa_B$ is not a $p$th power. Choose an extension $K \subset K_1$ as in Lemma 73.10 with $n = m + 2$ and denote $\pi'$ the uniformizer of the integral closure $A_1$ of $A$ in $K_1$ such that $\pi = (\pi')^p + (\pi')^{np}a$ for some $a \in A_1$. Let $B_1$ be the integral closure of $B$ in $L \otimes_K K_1$. Observe that $A_1 \to B_1$ is weakly unramified by Lemma 73.5. In $B_1$ we have

$$u\pi = \left(\sum_{0 \leq i < m} b_i (\pi')^{i+1}\right)^p + b(\pi')^{(m+1)p} + (\pi')^{np}b_1$$

for some $b_1 \in B_1$ (computation omitted). We conclude that $M_1$ is obtained from $L_1$ by adjoining a $p$th root of

$$b + (\pi')^{n-m-1}b_1$$

Since the residue field of $B_1$ equals the residue field of $B$ we see from Lemma 73.11 that $M_1/L_1$ has degree $p$ and the integral closure $C_1$ of $B_1$ is weakly unramified over $B_1$. Thus we conclude in this case.

If there does not exist an integer $m$ as in the preceding paragraph, then $u$ is a $p$th power in the $\pi$-adic completion of $B_1$. Since $B$ is Nagata, this means that $u$ is a $p$th...
more on algebra

power in \( B_1 \) by Algebra, Lemma 151.34. Whence the second case of the statement of the lemma holds. □

Lemma 73.13. Let \( A \) be a local ring annihilated by a prime \( p \) whose maximal ideal is nilpotent. There exists a ring map \( \sigma : \kappa_A \to A \) which is a section to the residue map \( A \to \kappa_A \). If \( A \to A' \) is a local homomorphism of local rings, then we can choose a similar ring map \( \sigma' : \kappa_{A'} \to A' \) compatible with \( \sigma \) provided that the extension \( \kappa_A \subset \kappa_{A'} \) is separable.

Proof. Separable extensions are formally smooth by Algebra, Proposition 148.9. Thus the existence of \( \sigma \) follows from the fact that \( \mathbf{F}_p \to \kappa_A \) is separable. Similarly for the existence of \( \sigma' \) compatible with \( \sigma \). □

Lemma 73.14. Let \( A \) be a discrete valuation ring with fraction field \( K \) of characteristic \( p > 0 \). Let \( \xi \in K \). Let \( L \) be an extension of \( K \) obtained by adjoining a root of \( z^p - z = \xi \). Then \( L/K \) is Galois and one of the following happens

1. \( L = K \),
2. \( L/K \) is unramified of degree \( p \),
3. \( L/K \) is totally ramified with ramification index \( p \), and
4. \( L/K \) is weakly unramified, the integral closure \( B \) of \( A \) in \( L \) is a discrete valuation ring and \( A \to B \) induces a purely inseparable residue field extension of degree \( p \).

Let \( \pi \) be a uniformizer of \( A \). We have the following implications:

(A) If \( \xi \in A \), then we are in case (1) or (2).
(B) If \( \xi = \pi^{-n}a \) where \( n > 0 \) is not divisible by \( p \) and \( a \) is a unit in \( A \), then we are in case (3)
(C) If \( \xi = \pi^{-n}a \) where \( n > 0 \) is divisible by \( p \) and the image of \( a \) in \( \kappa_A \) is not a \( p \)th power, then we are in case (4).

Proof. The extension is Galois of order dividing \( p \) by the discussion in Fields, Section 22. It immediately follows from the discussion in Section 72 that we are in one of the cases (1) – (4) listed in the lemma.

Case (A). Here we see that \( A \to A[x]/(x^p - x - \xi) \) is a finite \( \acute{e} \)tale ring extension. Hence we are in cases (1) or (2).

Case (B). Write \( \xi = \pi^{-n}a \) where \( p \) does not divide \( n \). Let \( B \subset L \) be the integral closure of \( A \) in \( L \). If \( C = B_m \) for some maximal ideal \( m \), then it is clear that \( \text{pord}_C(z) = -\text{nord}_C(\pi) \). In particular \( A \subset C \) has ramification index divisible by \( p \). It follows that it is \( p \) and that \( B = C \).

Case (C). Set \( k = n/p \). Then we can rewrite the equation as 

\[
(\pi^k z)^p - \pi^{n-k}(\pi^k z) = a
\]

Since \( A[y]/(y^p - \pi^{n-k}y - a) \) is a discrete valuation ring weakly unramified over \( A \), the lemma follows. □

Lemma 73.15. Let \( A \subset B \subset C \) be extensions of discrete valuation rings with fractions fields \( K \subset L \subset M \). Assume

1. \( A \subset B \) weakly unramified,
2. the characteristic of \( K \) is \( p \),
3. \( M \) is a degree \( p \) Galois extension of \( L \), and
(4) \( \kappa_A = \bigcap_{n \geq 1} \kappa^n_B \).

Then there exists a totally ramified Galois extension \( K_1 \) of \( K \) which is a weak solution for \( A \to C \).

**Proof.** Since the characteristic of \( L \) is \( p \) we know that \( M \) is an Artin-Schreier extension of \( L \) (Fields, Lemma 22.1). Thus we may pick \( z \in M \), \( z \notin L \) such that \( \xi = z^p - z \in L \). Choose \( n \geq 0 \) such that \( \pi^n \xi \in B \). We pick \( z \) such that \( n \) is minimal. If \( n = 0 \), then \( M \) is unramified over \( L \) (Lemma 73.14) and we are done. Thus we have \( n > 0 \).

Assumption (4) implies that \( \kappa_A \) is perfect. Thus we may choose compatible ring maps \( \tilde{\sigma} : \kappa_A \to A/\pi^n A \) and \( \sigma : \kappa_B \to B/\pi^n B \) as in Lemma 73.13. We lift the second of these to a map of sets \( \sigma : \kappa_B \to B \). Then we can write

\[
\xi = \sum_{i=n, \ldots, 1} \sigma(\lambda_i) \pi^{-i} + b
\]

for some \( \lambda_i \in \kappa_B \) and \( b \in B \). Let

\[
I = \{ i \in \{ n, \ldots, 1 \} \mid \lambda_i \in \kappa_A \}
\]

and

\[
J = \{ j \in \{ n, \ldots, 1 \} \mid \lambda_j \notin \kappa_A \}
\]

We will argue by induction on the size of the finite set \( J \).

The case \( J = \emptyset \). Here for all \( i \in \{ n, \ldots, 1 \} \) we have \( \sigma(\lambda_i) = a_i + \pi^n b_i \) for some \( a_i \in A \) and \( b_i \in B \) by our choice of \( \sigma \). Thus \( \xi = \pi^{-n} a + b \) for some \( a \in A \) and \( b \in B \). If \( p \mid n \), then we write \( a = a_0^p + \pi a_1 \) for some \( a_0, a_1 \in A \) (as the residue field of \( A \) is perfect). We compute

\[
(z - \pi^{-n/p} a_0)^p - (z - \pi^{-n/p} a_0) = \pi^{-(n-1)}(a_1 + \pi^{-n-1-n/p} a_0) + b'
\]

for some \( b' \in B \). This would contradict the minimality of \( n \). Thus \( p \) does not divide \( n \).

Consider the degree \( p \) extension \( K_1 \) of \( K \) given by \( w^p - w = \pi^{-n} a \). By Lemma 73.14 this extension is a totally ramified Galois extension. Thus \( L_1 = L \otimes_K K_1 \) is a field and \( A_1 \subset B_1 \) is weakly unramified (Lemma 73.5). By Lemma 73.14 the ring \( M_1 = M \otimes_K K_1 \) is either a product of \( p \) copies of \( L_1 \) (in which case we are done) or a field extension of \( L_1 \) of degree \( p \). Moreover, in the second case, \( M_1 \) is either weakly unramified over \( L_1 \) (in which case we are done) or a degree \( p \) totally ramified Galois extension. In this last case the extension \( M_1/L_1 \) is generated by the element \( z - w \) and

\[
(z - w)^p - (z - w) = z^p - z - (w^p - w) = b
\]

with \( b \in B \) (see above). Thus by Lemma 73.14 once more the extension \( M_1/L_1 \) is unramified and we conclude that \( K_1 \) is a weak solution for \( A \to C \). From now on we assume \( J \neq \emptyset \).

Suppose that \( j', j \in J \) such that \( j' = p^r j \) for some \( r > 0 \). Then we change our choice of \( z \) into

\[
z' = z - (\sigma(\lambda_j) \pi^{-j} + \sigma(\lambda_{j'}^p) \pi^{-p j} + \ldots + \sigma(\lambda_{j'}^{p^{r-1}}) \pi^{-p^{r-1} j})
\]

Then \( \xi \) changes into \( \xi' = (z')^p - (z') \) as follows

\[
\xi' = \xi - \sigma(\lambda_j) \pi^{-j} + \sigma(\lambda_{j'}^p) \pi^{-j' + 1} + \text{something in } B
\]

8If \( B \) is complete, then we can choose \( \sigma \) to be a ring map. If \( A \) is also complete and \( \sigma \) is a ring map, then \( \sigma \) maps \( \kappa_A \) into \( A \).
Writing $\xi' = \sum_{i=n-1}^{i=0} \sigma(\lambda'_i)\pi^{-i} + b'$ as before we find that $\lambda'_i = \lambda_i$ for $i \neq j, j'$ and $\lambda'_j = 0$. Thus the set $J$ has gotten smaller. By induction on the size of $J$ we may assume no such pair $j, j'$ exists. (Please observe that in this procedure we may get thrown back into the case that $J = \emptyset$ we treated above.)

For $j \in J$ write $\lambda_j = \mu_j^{p^{r_j}}$ for some $r_j \geq 0$ and $\mu_j \in \mathbb{Z}_B$ which is not a $p$th power. This is possible by our assumption (4). Let $j \in J$ be the unique index such that $jP^{r_j - 1}$ is maximal. (The index is unique by the result of the preceding paragraph.) Choose $r > \max(r_j + 1)$ and such that $jp^{r - r_j} > n$ for $j \in J$. Choose a separable totally ramified extension $K \subset K_1$ of degree $p^r$ such that the corresponding discrete valuation ring $A_1 \subset K_1$ has uniformizer $\pi'$ with $(\pi')^{p^r} = \pi + \pi^{n+1}a$ for some $a \in A_1$ (Lemma [73.10]). Observe that $L_1 = L \otimes_K K_1$ is a field and $B_1 \subset L_1$ a discrete valuation ring totally ramified over $B$ (Lemma [73.5]). Computing in $B_1$ we get

$$\xi = \sum_{i \in I} \sigma(\lambda_i)(\pi')^{-ip^r} + \sum_{j \in J} \sigma(\mu_j)^{p^{r_j}}(\pi')^{-jp^r} + b_1$$

for some $b_1 \in B_1$. Note that $\sigma(\lambda_i)$ for $i \in I$ is a $q$th power modulo $\pi^n$, i.e., modulo $(\pi')^{np^r}$. Hence we can rewrite the above as

$$\xi = \sum_{i \in I} x_i^{p^r}(\pi')^{-ip^r} + \sum_{j \in J} \sigma(\mu_j)^{p^{r_j}}(\pi')^{-jp^r} + b_1$$

As in the previous paragraph we change our choice of $z$ into

$$z' = z - \sum_{i \in I} \left( x_i(\pi')^{-i} + \ldots + x_i^{p^{r_i-1}}(\pi')^{-ip^{r_i-1}} \right) - \sum_{j \in J} \left( \sigma(\mu_j)(\pi')^{-jp^{r_j-1}} + \ldots + \sigma(\mu_j)^{p^{r_j}}(\pi')^{-jp^{r_j-1}} \right)$$

to obtain

$$(z')^p - z' = \sum_{i \in I} x_i(\pi')^{-i} + \sum_{j \in J} \sigma(\mu_j)(\pi')^{-jp^{r_j-1}} + b'_1$$

for some $b'_1 \in B_1$. Since there is a unique $j$ such that $jP^{r_j}$ is maximal and since $jP^{r_j-1}$ is bigger than $i \in I$ and divisible by $p$, we see that $M_1/L_1$ falls into case (C) of Lemma [73.14]. This finishes the proof. \[ \square \]

**Lemma 73.16.** Let $A$ be a ring which contains a primitive $p$th root of unity $\zeta$. Set $w = 1 - \zeta$. Then

$$P(z) = \frac{1 + wz - 1}{wp} = z^p - z + \sum_{0 < i < p} a_i z^i$$

is an element of $A[z]$ and in fact $a_i \in (w)$. Moreover, we have

$$P(z_1 + z_2 + wz_1z_2) = P(z_1) + P(z_2) + w^p P(z_1) P(z_2)$$

in the polynomial ring $A[z_1, z_2]$.

**Proof.** It suffices to prove this when

$$A = \mathbb{Z}[\zeta] = \mathbb{Z}[x]/(x^{p-1} + \ldots + x + 1)$$

is the ring of integers of the cyclotomic field. The polynomial identity $t^p - 1 = (t-1)(t-\zeta)\ldots(t-\zeta^{p-1})$ (which is proved by looking at the roots on both sides) shows that $t^p - 1 + \ldots + t + 1 = (t-\zeta)\ldots(t-\zeta^{p-1})$. Substituting $t = 1$ we obtain $p = (1-\zeta)(1-\zeta^2)\ldots(1-\zeta^{p-1})$. The maximal ideal $(p, w) = (w)$ is the unique
prime ideal of $A$ lying over $p$ (as fields of characteristic $p$ do not have nontrivial $p$th roots of 1). It follows that $p = uu^{p-1}$ for some unit $u$. This implies that

$$a_i = \frac{1}{p} \binom{p}{i} uu^{i-1}$$

for $p \geq 1$ and $-1 + a_1 = pw/w^p = u$. Since $P(-1) = 0$ we see that $0 = (-1)^p - u$ modulo $(w)$. Hence $a_1 \in (w)$ and the proof if the first part is done. The second part follows from a direct computation we omit.

\[\square\]

**Lemma 73.17.** Let $A$ be a discrete valuation ring of mixed characteristic $(0, p)$ which contains a primitive $p$th root of 1. Let $P(t) \in A[t]$ be the polynomial of Lemma 73.16. Let $\xi \in K$. Let $L$ be an extension of $K$ obtained by adjoining a root of $P(z) = \xi$. Then $L/K$ is Galois and one of the following happens

1. $L = K$,
2. $L/K$ is unramified of degree $p$,
3. $L/K$ is totally ramified with ramification index $p$, and
4. $L/K$ is weakly unramified, the integral closure $B$ of $A$ in $L$ is a discrete valuation ring and $A \rightarrow B$ induces a purely inseparable residue field extension of degree $p$.

Let $\pi$ be a uniformizer of $A$. We have the following implications:

(A) If $\xi \in A$, then we are in case (1) or (2).

(B) If $\xi = \pi^{-n}a$ where $n > 0$ is not divisible by $p$ and $a$ is a unit in $A$, then we are in case (3).

(C) If $\xi = \pi^{-n}a$ where $n > 0$ is divisible by $p$ and the image of $a$ in $\kappa_A$ is not a $p$th power, then we are in case (4).

**Proof.** Adjoining a root of $P(z) = \xi$ is the same thing as adjoining a root of $y^p = w^p(1 + \xi)$. Since $K$ contains a primitive $p$th root of 1 the extension is Galois of order dividing $p$ by the discussion in Fields, Section 21. It immediately follows from the discussion in Section 12 that we are in one of the cases (1) – (4) listed in the lemma.

Case (A). Here we see that $A \rightarrow A[x]/(P(x) - \xi)$ is a finite étale ring extension. Hence we are in cases (1) or (2).

Case (B). Write $\xi = \pi^{-n}a$ where $p$ does not divide $n$. Let $B \subset L$ be the integral closure of $A$ in $L$. If $C = B_m$ for some maximal ideal $m$, then it is clear that $\text{ord}_C(\pi) = -n \text{ord}_C(\pi)$. In particular $A \subset C$ has ramification index divisible by $p$. It follows that it is $p$ and that $B = C$.

Case (C). Set $k = n/p$. Then we can rewrite the equation as

$$(\pi^k z)^p - \pi^{n-k}(\pi^k z) + \sum a_i \pi^{n-ik}(\pi^k z)^i = a$$

Since $A[y]/(y^p - \pi^{n-k}y - \sum a_i \pi^{n-ik}y^i - a)$ is a discrete valuation ring weakly unramified over $A$, the lemma follows.

Let $A$ be a discrete valuation ring of mixed characteristic $(0, p)$ containing a primitive $p$th root of 1. Let $w \in A$ and $P(t) \in A[t]$ be as in Lemma 73.16. Let $L$ be a finite extension of $K$. We say $L/K$ is a degree $p$ extension of finite level if $L$ is a degree $p$ extension of $K$ obtained by adjoining a root of the equation $P(z) = \xi$ where $\xi \in K$ is an element with $w^p \xi \in m_A$. 
This definition is relevant to the discussion in this section due to the following straightforward lemma.

**Lemma 73.18.** Let $A \subset B \subset C$ be extensions of discrete valuation rings with fractions fields $K \subset L \subset M$. Assume that

1. $A$ has mixed characteristic $(0, p)$,
2. $A \subset B$ is weakly unramified,
3. $B$ contains a primitive $p$th root of 1, and
4. $M/L$ is Galois of degree $p$.

Then there exists a totally ramified Galois extension $K \subset K_1$ which is either a weak solution for $A \to C$ or is such that $M_1/L_1$ is a degree $p$ extension of finite level.

**Proof.** Let $\pi \in A$ be a uniformizer. By Kummer theory (Fields, Lemma 21.1) $M$ is obtained from $L$ by adjoining the root of $y^p = b$ for some $b \in L$.

If $\text{ord}_B(b)$ is prime to $p$, then we choose a degree $p$ separable, totally ramified extension $K \subset K_1$ (for example using Lemma 73.10). Let $A_1$ be the integral closure of $A$ in $K_1$. By Lemma 73.5 the integral closure $B_1$ of $B$ in $L_1 = L \otimes_K K_1$ is a discrete valuation ring weakly unramified over $A_1$. If $K \subset K_1$ is not a weak solution for $A \to C$, then the integral closure $C_1$ of $C$ in $M_1 = M \otimes_K K_1$ is a discrete valuation ring and $B_1 \to C_1$ has ramification index $p$. In this case, the field $M_1$ is obtained from $L_1$ by adjoining the $p$th root of $b$ with $\text{ord}_{B_1}(b)$ divisible by $p$. Replacing $A$ by $A_1$, etc we may assume that $b = \pi^n u$ where $u \in B$ is a unit and $n$ is divisible by $p$. Of course, in this case the extension $M$ is obtained from $L$ by adjoining the $p$th root of a unit.

Suppose $M$ is obtained from $L$ by adjoining the root of $y^p = u$ for some unit $u$ of $B$. If the residue class of $u$ in $\kappa_B$ is not a $p$th power, then $B \subset C$ is weakly unramified (Lemma 73.11) and we are done. Otherwise, we can replace our choice of $y$ by $y/v$ where $v^p$ and $u$ have the same image in $\kappa_B$. After such a replacement we have

$$y^p = 1 + \pi b$$

for some $b \in B$. Then we see that $P(z) = \pi b/w^p$ where $z = (y - 1)/w$. Thus we see that the extension is a degree $p$ extension of finite level with $\xi = \pi b/w^p$. \qed

Let $A$ be a discrete valuation ring of mixed characteristic $(0, p)$ containing a primitive $p$th root of 1. Let $w \in A$ and $P(t) \in A[t]$ be as in Lemma 73.16. Let $L$ be a degree $p$ extension of $K$ of finite level. Choose $z \in L$ generating $L$ over $K$ with $\xi = P(z) \in K$. Choose a uniformizer $\pi$ for $A$ and write $w = u\pi^{e_1}$ for some integer $e_1 = \text{ord}_A(w)$ and unit $u \in A$. Finally, pick $n \geq 0$ such that

$$\pi^n \xi \in A$$

The *level of $L/K$* is the smallest value of the quantity $n/e_1$ taking over all $z$ generating $L/K$ with $\xi = P(z) \in K$.

We make a couple of remarks. Since the extension is of finite level we know that we can choose $z$ such that $n < pe_1$. Thus the level is a rational number contained in $[0, p)$. If the level is zero then $L/K$ is unramified by Lemma 73.17. Our next goal is to lower the level.

**Lemma 73.19.** Let $A \subset B \subset C$ be extensions of discrete valuation rings with fractions fields $K \subset L \subset M$. Assume

1. $A$ has mixed characteristic $(0, p)$,
Consider the degree \( p \) the shape of the polynomial \( \xi \) \( (73.19.2) \). These to a map of sets \( \text{for some} \ \lambda \) and by equations \( (73.19.1) \) and \( (73.19.2) \) we have 

\[ z \text{ and } 1_{n \geq p} \]

Thus we may choose compatible ring maps \( \sigma : \kappa_A \rightarrow A/\pi^m A \) and \( \sigma : \kappa_B \rightarrow B/\pi^m B \) as in Lemma \( 73.13 \). We lift the second of these to a map of sets \( \sigma : \kappa_B \rightarrow B \). Then we can write 

\[ \xi = \sum_{i=n}^{n-m} \sigma(\lambda_i)\pi^{-i + \pi^{-n+m}}b \]

for some \( \lambda_i \in \kappa_B \) and \( b \in B \). Let 

\[ I = \{ i \in \{ n, \ldots, n - m + 1 \} \mid \lambda_i \in \kappa_A \} \]

and 

\[ J = \{ j \in \{ n, \ldots, n - m + 1 \} \mid \lambda_i \notin \kappa_A \} \]

We will argue by induction on the size of the finite set \( J \).

The case \( J = \emptyset \). Here for all \( i \in \{ n, \ldots, n - m + 1 \} \) we have \( \sigma(\lambda_i) = a_i + \pi^{-n+m}b_i \) for some \( a_i \in A \) and \( b_i \in B \) by our choice of \( \sigma \). Thus \( \xi = \pi^{-n}a + \pi^{-n+m}b \) for some \( a \in A \) and \( b \in B \). If \( p | n \), then we write \( a = a_0^p + \pi a_1 \) for some \( a_0, a_1 \in A \) (as the residue field of \( A \) is perfect). Set \( z_1 = -\pi^{-n/p}a_0 \). Note that \( P(z_1) \in \pi^{-n}B \) and that \( z + z_1 + wz \) \( z_1 \) is an element generating \( M \) over \( L \) (note that \( wz_1 \neq -1 \) as \( n < p \)). Moreover, by Lemma \( 73.16 \) we have 

\[ P(z + z_1 + wz) = P(z) + P(z_1) + w^pP(z_1) \in K \]

and by equations \( (73.19.1) \) and \( (73.19.2) \) we have 

\[ P(z) + P(z_1) + w^pP(z_1) = \xi + z_1^p - z_1 \equiv \pi^{-n+m}B \]

for some \( b' \in B \). This contradict the minimality of \( n \). Thus \( p \) does not divide \( n \). Consider the degree \( p \) extension \( K_1 \) of \( K \) given by \( P(y) = -\pi^{-n}a \). By Lemma
this extension is separable, totally ramified. Thus \( L_1 = L \otimes_K K_1 \) is a field and \( A_1 \subset B_1 \) is weakly unramified (Lemma \( \ref{lem:23.5} \)). By Lemma \( \ref{lem:73.17} \), the ring \( M_1 = M \otimes_K K_1 \) is either a product of \( p \) copies of \( L_1 \) (in which case we are done) or a field extension of \( L_1 \) of degree \( p \). Moreover, in the second case, \( M_1 \) is either weakly unramified over \( L_1 \) (in which case we are done) or a degree \( p \) totally ramified Galois extension. In this last case the extension \( M_1/L_1 \) is generated by the element \( z + y + wz \) and we see that \( P(z + y + wz) \in L_1 \) and

\[
P(z + y + wz) = P(z) + P(y) + w^p P(z) P(y) + \xi - \pi^{-n} a \text{ mod } \pi^{-n+m} B_1
\]

\[
\equiv 0 \text{ mod } \pi^{-n+m} B_1
\]

in exactly the same manner as above. By our choice of \( m \) this means exactly that \( M_1/L_1 \) has level at most \( \max(0, l - 1, 2l - p) \). From now on we assume that \( J \neq \emptyset \).

Suppose that \( j', j \in J \) such that \( j' = p^r j \) for some \( r > 0 \). Then we set

\[
z_1 = -\sigma(\lambda_j) \pi^{-j} - \sigma(\lambda_j^p) \pi^{-p j} - \ldots - \sigma(\lambda_j^{p^{r-1}}) \pi^{-p^{r-1} j}
\]

and we change \( z \) into \( z' = z + z_1 + wz \).

Observe that \( z' \in M \) generates \( M \) over \( L \) and that we have \( \xi' = P(z') = P(z) + P(z_1) + wP(z)P(z_1) \in L \) with

\[
\xi' \equiv \xi - \sigma(\lambda_j) \pi^{-j} + \sigma(\lambda_j^{p^r}) \pi^{-j'} \text{ mod } \pi^{-n+m} B
\]

by using equations \( \ref{eq:73.19.1} \) and \( \ref{eq:73.19.2} \) as above. Writing

\[
\xi' = \sum_{i=n, \ldots, 2n+1} \sigma(\lambda_i) \pi^{-i} + \pi^{-n+m} b'
\]

as before we find that \( \lambda_i' = \lambda_i \) for \( i \neq j, j' \) and \( \lambda_j' = 0 \). Thus the set \( J \) has gotten smaller. By induction on the size of \( J \) we may assume there is no pair \( j, j' \) of \( J \) such that \( j'/j \) is a power of \( p \). (Please observe that in this procedure we may get thrown back into the case that \( J = \emptyset \) we treated above.)

For \( j \in J \) write \( \lambda_j = \mu_j \pi^r \) for some \( r_j \geq 0 \) and \( \mu_j \in \kappa_B \) which is not a \( p \)-th power. This is possible by our assumption (4). Let \( j \in J \) be the unique index such that \( j \pi^r \) is maximal. (The index is unique by the result of the preceding paragraph.) Choose \( r > \max(r_j + 1) \) and such that \( j \pi^r > n \) for \( j \in J \). Let \( K \subset K_1 \) be the totally ramified extension of degree \( p^r \) defined by \( (\pi')^{p^r} = \pi \). Observe that \( \pi' \) is the uniformizer of the corresponding discrete valuation ring \( A_1 \subset K_1 \). Observe that \( L_1 = L \otimes_K K_1 \) is a field and \( B_1 \subset L_1 \) a discrete valuation ring totally ramified over \( B \) (Lemma \( \ref{lem:73.5} \)). Computing in \( B_1 \) we get

\[
\xi = \sum_{i \in I} \sigma(\lambda_i) (\pi')^{-i} \pi^r + \sum_{j \in J} \sigma(\mu_j) (\pi')^{-j} \pi^r + \pi^{-n+m} b_1
\]

for some \( b_1 \in B_1 \). Note that \( \sigma(\lambda_i) \) for \( i \in I \) is a \( q \)-th power modulo \( \pi^m \), i.e., modulo \( (\pi')^{mp^r} \). Hence we can rewrite the above as

\[
\xi = \sum_{i \in I} x_i (\pi')^{-i} \pi^r + \sum_{j \in J} \sigma(\mu_j) (\pi')^{-j} \pi^r + \pi^{-n+m} b_1
\]

Similar to our choice in the previous paragraph we set

\[
z_1 - \sum_{i \in I} \left( x_i (\pi')^{-i} + \ldots + x_i^{p^{r-1}} (\pi')^{-ip^{r-1}} \right)
\]

\[
- \sum_{j \in J} \left( \sigma(\mu_j) (\pi')^{-j} \pi^r + \ldots + \sigma(\mu_j) (\pi')^{-j} \pi^r \right)
\]
and we change our choice of \( z \) into \( z' = z + z_1 + wz_1 \). Then \( z' \) generates \( M_1 \) over \( L_1 \) and \( \xi' = P(z') = P(z) + P(z_1) + wP(z_1)P(z_1) \in L_1 \) and a calculation shows that
\[
\xi' = \sum_{i \in I} x_i(\pi')^{-i} + \sum_{j \in J} \sigma(\mu_j)(\pi')^{-jp^{rj}} + (\pi')^{-n + m}p^{rj}b_1'
\]
for some \( b_1' \in B_1 \). There is a unique \( j \) such that \( jp^{rj} \) is maximal and \( jp^{rj} \) is bigger than \( i \in I \). If \( jp^{rj} \leq (n - m)p^{r} \) then the level of the extension \( M_1/L_1 \) is less than \( \max(0, l - 1, 2l - p) \). If not, then, as \( p \) divides \( jp^{rj} \), we see that \( M_1/L_1 \) falls into case (C) of Lemma 73.17. This finishes the proof. □

**Lemma 73.20.** Let \( A \subset B \subset C \) be extensions of discrete valuation rings with fraction fields \( K \subset L \subset M \). Assume

1. the residue field \( k \) of \( A \) is algebraically closed of characteristic \( p > 0 \),
2. \( A \) and \( B \) are complete,
3. \( A \to B \) is weakly unramified,
4. \( M \) is a finite extension of \( L \),
5. \( k = \bigcap_{n \geq 1} \kappa_B^n \)

Then there exists a finite extension \( K \subset K_1 \) which is a weak solution for \( A \to C \).

**Proof.** Let \( M' \) be any finite extension of \( L \) and consider the integral closure \( C' \) of \( B \) in \( M' \). Then \( C' \) is finite over \( B \) as \( B \) is Nagata by Algebra, Lemma 151.23 Moreover, \( C' \) is a discrete valuation ring, see discussion in Remark 73.1 Moreover \( C' \) is complete as a \( B \)-module, hence complete as a discrete valuation ring, see Algebra, Section 91. It follows in particular that \( C \) is the integral closure of \( B \) in \( M \) (by definition of valuation rings as maximal for the relation of domination).

Let \( M \subset M' \) be a finite extension and let \( C' \subset M' \) be the integral closure of \( B \) as above. By Lemma 73.6 it suffices to prove the result for \( A \to B \to C' \). Hence we may assume that \( M/L \) is normal, see Fields, Lemma 15.3

If \( M/L \) is normal, we can find a chain of finite extensions
\[
L = L^0 \subset L^1 \subset L^2 \subset \ldots \subset L^r = M
\]
such that each extension \( L^{j+1}/L^j \) is either: (a) purely inseparable of degree \( p \), (b) totally ramified Galois of degree \( p \), (c) totally ramified Galois cyclic of order prime to \( p \), or (d) Galois and unramified. Namely, since \( M/L \) is normal we can write it as a compositum of a Galois extension and a purely inseparable extension (Fields, Lemma 24.3). For the purely inseparable extension the existence of the filtration is clear. In the Galois case, note that \( G \) is “the” decomposition group and let \( I \subset G \) be the inertia group. Then on the one hand \( I \) is solvable by Lemma 72.11 and on the other hand the extension \( M^I/L \) is unramified by Lemma 72.14 This proves we have a filtration as stated.

We are going to argue by induction on the integer \( r \). Suppose that we can find a finite extension \( K \subset K_1 \) which is a weak solution for \( A \to B^1 \) where \( B^1 \) is the integral closure of \( B \) in \( L^1 \). Let \( K'_1 \) be the normal closure of \( K_1/K \) (Fields, Lemma 15.3). Since \( A \) is complete and the residue field of \( A \) is algebraically closed we see that \( K_1 \subset K'_1 \) is separable and totally ramified (some details omitted). Hence \( K \subset K'_1 \) is a weak solution for \( A \to B^1 \) as well by Lemma 73.5 In other words,
we may and do assume that \( K \) is a normal extension of \( K \). Having done so we consider the sequence
\[
L_0^0 = (L_0^0 \otimes_K K_1)^{red} \subset L_1^1 = (L_1^1 \otimes_K K_1)^{red} \subset \ldots \subset L_r^r = (L_r^r \otimes_K K_1)^{red}
\]
and the corresponding integral closures \( B_i^r \). Note that \( C_1 = B_1^r \) is a product of discrete valuation rings which are transitively permuted by \( G = \text{Aut}(K_1/K) \) by Lemma \[73.8\]. In particular all the extensions of discrete valuation rings \( A_1 \to (C_1)_m \) are isomorphic and a solution for one will be a solution for all of them. We can apply the induction hypothesis to the sequence
\[
A_1 \to (B_1^1)_{B_1^1 \cap m} \to (B_2^2)_{B_2^2 \cap m} \to \ldots \to (B_r^r)_{B_r^r \cap m} = (C_1)_m
\]
to get a solution \( K_1 \subset K_2 \) for \( A_1 \to (C_1)_m \). The extension \( K \subset K_1 \) will then be a solution for \( A \to C \) by what we said before. Note that the induction hypothesis applies: the ring map \( A_1 \to (B_1^1)_{B_1^1 \cap m} \) is weakly unramified by our choice of \( K_1 \) and the sequence of fraction field extensions each still have one of the properties (a), (b), (c), or (d) listed above. Moreover, observe that for any finite extension \( \kappa_B \subset \kappa \) we still have \( k = \bigcap \kappa_B^{p^n} \).

Thus everything boils down to finding a weak solution when the field extension \( L \subset M \) satisfies one of the properties (a), (b), (c), or (d).

Case (d). This case is trivial as here \( B \to C \) is unramified already.

Case (c). This is the tamely ramified case. Say \( M/L \) is cyclic of order \( n \) prime to \( p \). Choose uniformizers \( \pi \in A \) and \( \pi_C \in C \). Then \( \pi_C^e = u\pi \) for some unit \( u \) of \( C \). Let \( K_1 = K[\pi^{1/n}] \). Then the ring \( M_1 = M \otimes_K K_1 \) is a product of fields each of which is obtained from \( M \) by adjoining a root of \( z^n - u = 0 \). Since \( C[z]/(z^n - u) \) is finite étale over \( C \) we conclude that each of these extensions is unramified over \( M \). Considering the commutative diagram
\[
\begin{array}{ccc}
M & \longrightarrow & M_1 \\
\downarrow & & \downarrow \\
L & \longrightarrow & L_1
\end{array}
\]
and using multiplicativity of ramification indices, it follows that each factor of \( M_1 \) is unramified over \( L_1 \) as well.

Case (b). We divide this case into the mixed characteristic case and the equicharacteristic case. In the equicharacteristic case this is Lemma \[73.15\]. In the mixed characteristic case, we first replace \( K \) by a finite extension to get to the situation where \( M/L \) is a degree \( p \) extension of finite level using Lemma \[73.18\]. Then the level is a rational number \( l \in [0, p) \), see discussion preceding Lemma \[73.19\]. If the level is 0, then \( B \to C \) is weakly unramified and we’re done. If not, then we can replacing the field \( K \) by a finite extension to obtain a new situation with level \( l' \leq \max(0, l - 1, 2l - p) \) by Lemma \[73.19\]. If \( l = p - \epsilon \) for \( \epsilon < 1 \) then we see that \( l' \leq p - 2\epsilon \). Hence after a finite number of replacements we obtain a case with level \( \leq p - 1 \). Then after at most \( p - 1 \) more such replacements we reach the situation where the level is zero.

Case (a) is Lemma \[73.12\]. This is the only case where we possibly need a purely inseparable extension of \( K \), namely, in case (2) of the statement of the lemma we
At this point we have collected all the lemmas we need to prove the main result of this section.

**Theorem 73.21** (Epp). Let $A \subset B$ be an extension of discrete valuation rings with fraction fields $K \subset L$. If the characteristic of $\kappa_A$ is $p > 0$, assume that every element of

$$\bigcap_{n \geq 1} \kappa_p^n$$

is separable algebraic over $\kappa_A$. Then there exists a finite extension $K \subset K_1$ which is a weak solution for $A \to B$ as defined in Definition 73.2.

**Proof.** We first prove the result in case the characteristic of $\kappa_A$ is zero as the result is easy in that case. Namely, suppose the ramification index is $e$. Choose a uniformizer $\pi_B \in B$ and a uniformizer $\pi \in A$. Write $\pi_B^e = u\pi$ for some unit $u \in B$.

Let $K \subset K_1$ be the totally ramified extension obtained by adjoining an $e$th root of $\pi$ (see Lemma 73.9). Then $L_1 = L \otimes_K K_1$ is a product of fields, each obtained by adjoining a root of $z^e = u$ to $L$. However, since $B[x]/(x^e - u)$ is finite étale over $B$ (since the characteristic is zero) we conclude that each of these extensions is unramified over $L$. Considering the commutative diagram

$$
\begin{array}{c}
L & \longrightarrow & L_1 \\
\uparrow & & \uparrow \\
K & \longrightarrow & K_1
\end{array}
$$

and using multiplicativity of ramification indices, it follows that each factor of $L_1$ is unramified over $K_1$ as well. This finishes the proof in residue characteristic 0.

From now on we let $p$ be a prime number and we assume that $\kappa_A$ has characteristic $p$. We first apply Lemma 73.7 to reduce to the case that $A$ and $B$ have separably closed residue fields. Since $\kappa_A$ and $\kappa_B$ are replaced by their separable algebraic closures by this procedure we see that we obtain

$$\kappa_A \supset \bigcap_{n \geq 1} \kappa_p^n$$

from the condition of the theorem.

Let $\pi \in A$ be a uniformizer. Let $A^\wedge$ and $B^\wedge$ be the completions of $A$ and $B$. We have a commutative diagram

$$
\begin{array}{c}
B & \longrightarrow & B^\wedge \\
\uparrow & & \uparrow \\
A & \longrightarrow & A^\wedge
\end{array}
$$

of extensions of discrete valuation rings. Let $K^\wedge$ be the fraction field of $A^\wedge$. Suppose that we can find a finite extension $K^\wedge \subset M$ which is (a) a weak solution for $A^\wedge \to B^\wedge$ and (b) a compositum of a separable extension and an extension obtained by adjoining a $p$-power root of $\pi$. Then by Lemma 72.16 we can find a finite extension $K \subset K_1$ such that $K^\wedge \otimes_K K_1 = M$. Let $A_1$, resp. $A_1^\wedge$ be the integral closure of $A$, resp. $A^\wedge$ in $K_1$, resp. $M$. Since $A \to A^\wedge$ is formally smooth (Lemma 72.4) we see that $A_1 \to A_1^\wedge$ is formally smooth (Lemma 73.4 and $A_1$ and $A_1^\wedge$ are
discrete valuation rings by discussion in Remark 73.1. We conclude from Lemma 73.6 part (2) that $K \subset K_1$ is a weak solution for $A \to B^\wedge$. Applying Lemma 73.6 part (1) we see that $K \subset K_1$ is a weak solution for $A \to B$.

Thus we may assume $A$ and $B$ are complete discrete valuation rings with separably closed residue fields of characteristic $p$ and with $\kappa_A \supseteq \bigcap_{n \geq 1} \kappa_B^{p^n}$. We are also given a uniformizer $\pi \in A$ and we have to find a weak solution for $A \to B$ which is a compositum of a separable extension and a field obtained by taking $p$-power roots of $\pi$. Note that the second condition is automatic if $A$ has mixed characteristic.

Set $k = \bigcap_{n \geq 1} \kappa_B^{p^n}$. Observe that $k$ is an algebraically closed field of characteristic $p$. If $A$ has mixed characteristic let $\Lambda$ be a Cohen ring for $k$ and in the equicharacteristic case set $\Lambda = k[[t]]$. We can choose a ring map $\Lambda \to A$ which maps $t$ to $\pi$ in the equicharacteristic case. In the equicharacteristic case this follows from the Cohen structure theorem (Algebra, Theorem 150.8) and in the mixed characteristic case this follows as $\mathbb{Z}_p \to \Lambda$ is formally smooth in the adic topology (Lemmas 72.4 and 28.5). Applying Lemma 73.6 we see that it suffices to prove the existence of a weak solution for $\Lambda \to B$ which in the equicharacteristic $p$ case is a compositum of a separable extension and a field obtained by taking $p$-power roots of $t$. However, since $\Lambda = k[[t]]$ in the equicharacteristic case and any extension of $k((t))$ is such a compositum, we can now drop this requirement!

Thus we arrive at the situation where $A$ and $B$ are complete, the residue field $k$ of $A$ is algebraically closed of characteristic $p > 0$, we have $k = \bigcap_{n \geq 1} \kappa_B^{p^n}$, and in the mixed characteristic case $p$ is a uniformizer of $A$ (i.e., $A$ is a Cohen ring for $k$). If $A$ has mixed characteristic choose a Cohen ring $\Lambda$ for $\kappa_B$ and in the equicharacteristic case set $\Lambda = \kappa_B[[t]]$. Arguing as above we may choose a ring map $\Lambda \to A$ lifting $k \to \kappa_B$ and mapping a uniformizer to a uniformizer. Since $k \subset \kappa_B$ is separable the ring map $A \to \Lambda$ is formally smooth in the adic topology (Lemma 72.4). Hence we can find a ring map $\Lambda \to B$ such that the composition $A \to \Lambda \to B$ is the given ring map $A \to B$ (see Lemma 28.5). Since $A$ and $B$ are complete discrete valuation rings with the same residue field, $\bar{B}$ is finite over $\Lambda$ (Algebra, Lemma 94.17). This reduces us to the special case discussed in Lemma 73.20.

Lemma 73.22. Let $A \to B$ be an extension of discrete valuation rings with fraction fields $K \subset L$. Assume $B$ is essentially of finite type over $A$. Let $K \subset K'$ be an algebraic extension of fields such that the integral closure $A'$ of $A$ in $K'$ is Noetherian. Then the integral closure $B'$ of $B$ in $L' = (L \otimes_K K')_{\text{red}}$ is Noetherian as well. Moreover, the map $\text{Spec}(B') \to \text{Spec}(A')$ is surjective and the corresponding residue field extensions are finitely generated field extensions.

Proof. Let $A \to C$ be a finite type ring map such that $B$ is a localization of $C$ at a prime $\mathfrak{p}$. Then $C' = C \otimes_A A'$ is a finite type $A'$-algebra, in particular Noetherian. Since $A \to A'$ is integral, so is $C \to C'$. Thus $B = B' \subset C'$ is integral too. It follows that the dimension of $C'$ is 1 (Algebra, Lemma 109.4). Of course $C_p'$ is Noetherian. Let $q_1, \ldots, q_n$ be the minimal primes of $C_p'$. Let $B_i'$ be the integral closure of $B = B_p$, or equivalently by the above of $C_p'$ in the field of fractions of $C_p'/q_i$. It follows from Krull-Akizuki (Algebra, Lemma 116.11 applied to the finitely many localizations of $C_p'$ at its maximal ideals) that each $B_i'$ is Noetherian. Moreover the residue field extensions in $C_p' \to B_i'$ are finite by Algebra, Lemma 116.9. Finally, we observe that $B' = \prod B_i'$ is the integral closure of $B$ in $L' = (L \otimes_K K')_{\text{red}}$. □
Proposition 73.23. Let $A \to B$ be an extension of discrete valuation rings with fraction fields $K \subset L$. If $B$ is essentially of finite type over $A$, then there exists a finite extension $K \subset K_1$ which is a solution for $A \to B$ as defined in Definition 73.2.

Proof. Observe that a weak solution is a solution if the residue field of $A$ is perfect, see Lemma 72.4. Thus the proposition follows immediately from Theorem 73.21 if the residue characteristic of $A$ is 0 (and in fact we do not need the assumption that $A \to B$ is essentially of finite type). If the residue characteristic of $A$ is $p > 0$ we will also deduce it from Epp’s theorem.

Let $x_i \in A$, $i \in I$ be a set of elements mapping to a $p$-base of the residue field $\kappa$ of $A$. Set

$$A' = \bigcup_{n \geq 1} A[t_{i,n}]/(t_{i,n}^p - x_i)$$

where the transition maps send $t_{i,n+1}$ to $t_{i,n}^p$. Observe that $A'$ is a filtered colimit of weakly unramified finite extensions of discrete valuation rings over $A$. Thus $A'$ is a discrete valuation ring and $A \to A'$ is weakly unramified. By construction the residue field $\kappa' = A'/m_A A'$ is the perfection of $\kappa$.

Let $K' = f.f.(A')$. We may apply Lemma 73.22 to the extension $K \subset K'$. Thus $B'$ is a finite product of Dedekind domains. Let $m_1, \ldots, m_n$ be the maximal ideals of $B'$. Using Epp’s theorem (Theorem 73.21) we find a weak solution $K' = f.f.(A') \subset K_1'$ for each of the extensions $A' \subset B'_{m_i}$. Since the residue field of $A'$ is perfect, these are actually solutions. Let $K' \subset K_1'$ be a finite extension which contains each $K_1'$. Then $K' \subset K_1'$ is still a solution for each $A' \subset B'_{m_i}$, by Lemma 73.4.

Let $A_1'$ be the integral closure of $A$ in $K_1'$. Note that $A_1'$ is a Dedekind domain by the discussion in Remark 73.1 applied to $K' \subset K_1'$. Thus Lemma 73.22 applies to $K \subset K_1'$. Therefore the integral closure $B_1'$ of $B$ in $L_1' = (L \otimes_K K_1')_{red}$ is a Dedekind domain and because $K' \subset K_1'$ is a solution for each $A' \subset B'_{m_i}$ we see that $(A_1')_{A_1' \cap m} \to (B_1')_m$ is formally smooth for each maximal ideal $m \subset B_1'$.

By construction, the field $K_1'$ is a filtered colimit of finite extensions of $K$. Say $K_1' = \colim_{i \in I} K_i'$. For each $i$ let $A_i$, resp. $B_i$ be the integral closure of $A$, resp. $B$ in $K_i$, resp. $L_i = (L \otimes_K K_i)_{red}$. Then it is clear that

$$A_1' = \colim A_i \quad \text{and} \quad B_1' = \colim B_i$$

Since the ring maps $A_i \to A_1'$ and $B_i \to B_1'$ are injective integral ring maps and since $A_1'$ and $B_1'$ have finite spectra, we see that for all $i$ large enough the ring maps $A_i \to A_1'$ and $B_i \to B_1'$ are bijective on spectra. Once this is true, for all $i$ large enough the maps $A_i \to A_1'$ and $B_i \to B_1'$ will be weakly unramified (once the uniformizer is in the image). It follows from multiplicativity of ramification indices that $A_i \to B_i$ induces weakly unramified maps on all localizations at maximal ideals of $B_i$ for such $i$. Increasing $i$ a bit more we see that

$$B_i \otimes_{A_i} A_1' \longrightarrow B_1'$$

induces surjective maps on residue fields (because the residue fields of $B_1'$ are finitely generated over those of $A_1'$ by Lemma 73.22). Picture of residue fields at maximal
ideals lying under a chosen maximal ideal of $B'_i$:

\[
\begin{array}{c c c}
\kappa_{B_1} & \rightarrow & \kappa_{B_1}' \\
\uparrow & & \uparrow \\
\kappa_{A_1} & \rightarrow & \kappa_{A_1}' \\
\kappa_{B_1} & \rightarrow & \kappa_{B_1}' \\
\vdots & & \vdots \\
\kappa_{A_1} & \rightarrow & \kappa_{A_1}' \\
\kappa_{B_1} & \rightarrow & \kappa_{B_1}' \\
\uparrow & & \uparrow \\
\kappa_{A_1} & \rightarrow & \kappa_{A_1}' \\
\kappa_{B_1} & \rightarrow & \kappa_{B_1}' \\
\vdots & & \vdots \\
\kappa_{A_1} & \rightarrow & \kappa_{A_1}' \\
\kappa_{B_1} & \rightarrow & \kappa_{B_1}' \\
\end{array}
\]

Thus $\kappa_{B_1}$ is a finitely generated extension of $\kappa_{A_1}$ such that the compositum of $\kappa_{B_1}$ and $\kappa_{A_1}'$ in $\kappa_{B_1}'$ is separable over $\kappa_{A_1}'$. Then that happens already at a finite stage: for example, say $\kappa_{B_1}'$ is finite separable over $\kappa_{A_1}'(x_1, \ldots, x_n)$, then just increase $i$ such that $x_1, \ldots, x_n$ are in $\kappa_{B_1}$, and such that all generators satisfy separable polynomial equations over $\kappa_{A_1}(x_1, \ldots, x_n)$. This means that $A_1 \rightarrow B_i$ is formally smooth at all maximal ideals of $B_i$ and the proof is complete. □

74. Picard groups of rings

Let $R$ be a ring. An invertible $R$-module is a finite locally free module of rank 1. The set of isomorphism classes of these modules is often called the class group or Picard group of $R$. The group structure is determined by assigning to the isomorphism classes of the invertible modules $L$ and $L'$ the isomorphism class of $L \otimes_R L'$. The inverse of an invertible module $L$ is the module

\[ L^{-1} = \text{Hom}_R(L, R), \]

namely, the evaluation map $L \otimes_R L^{-1} \rightarrow R$ is an isomorphism. Let us denote the Picard group of $R$ by $\text{Pic}(R)$.

Recall that we have defined in Algebra, Section 53 a group $K_0(R)$ as the free group on isomorphism classes of finite projective $R$-modules modulo the relations $[M'] + [M''] = [M' \oplus M'']$.

**Lemma 74.1.** Let $R$ be a ring. There is a map

\[ \det : K_0(R) \rightarrow \text{Pic}(R) \]

which maps $[M]$ to the class of the invertible module $\wedge^n(M)$ if $M$ is a finite locally free module of rank $n$.

**Proof.** Let $M$ be a finite projective $R$-module. There exists a product decomposition $R = R_0 \times \ldots \times R_i$ such that in the corresponding decomposition $M = M_0 \times \ldots \times M_i$ of $M$ we have that $M_i$ is finite locally free of rank $i$ over $R_i$. This follows from Algebra, Lemma 76.2 (to see that the rank is locally constant) and Algebra, Lemmas 20.3 and 22.3 (to decompose $R$ into a product). In this situation we define

\[ \det(M) = \wedge_{R_0}^0(M_0) \times \ldots \times \wedge_{R_i}^i(M_i) \]

as an $R$-module. This is a finite locally free module of rank 1 as each term is finite locally free of rank 1. To finish the proof we have to show that

\[ \det(M' \oplus M'') \cong \det(M') \otimes \det(M'') \]

whenever $M'$ and $M''$ are finite projective $R$-modules. Decompose $R$ into a product of rings $R_{ij}$ such that $M' = \prod M'_{ij}$ and $M'' = \prod M''_{ij}$ where $M'_{ij}$ has rank $i$ and $M''_{ij}$ has rank $j$. This reduces us to the case where $M'$ and $M''$ have constant rank say $i$ and $j$. In this case we have to prove that

\[ \wedge^{i+j}(M' \oplus M'') \cong \wedge^i(M') \otimes \wedge^j(M'') \]
Lemma 74.2. Let $R$ be a ring. There is a map
\[ c : \text{perfect complexes over } R \to K_0(R) \]
with the following properties

1. $c(K[n]) = (-1)^n c(K)$ for a perfect complex $K$,
2. if $K \to L \to M \to K[1]$ is a distinguished triangle of perfect complexes,
   then $c(L) = c(K) + c(M)$,
3. if $K$ is represented by a finite complex $M^\bullet$ consisting of finite projective modules,
   then $c(K) = \sum (-1)^i [M_i]$.

Proof. Let $K$ be a perfect object of $D(R)$. By definition we can represent $K$ by a
finite complex $M^\bullet$ of finite projective $R$-modules. We define $c$ by setting
\[ c(K) = \sum (-1)^n [M^n] \]
in $K_0(R)$. Of course we have to show that this is well defined, but once it is well
defined, then (1) and (3) are immediate. For the moment we view the map $c$ as
defined on complexes of finite projective $R$-modules.

Suppose that $L^\bullet \to M^\bullet$ is a surjective map of finite complexes of finite projective
$R$-modules. Let $K^\bullet$ be the kernel. Then we obtain short exact sequences of $R$-
modules
\[ 0 \to K^n \to L^n \to M^n \to 0 \]
which are split because $M^n$ is projective. Hence $K^\bullet$ is also a finite complex of finite
projective $R$-modules and $c(L^\bullet) = c(K^\bullet) + c(M^\bullet)$ in $K_0(R)$.

Suppose given finite complex $M^\bullet$ of finite projective $R$-modules which is acyclic.
Say $M^n = 0$ for $n \notin [a,b]$. Then we can break $M^\bullet$ into short exact sequences
\begin{align*}
0 & \to M^a \to M^{a+1} \to N^{a+1} \to 0, \\
0 & \to N^{a+1} \to M^{a+2} \to N^{a+3} \to 0, \\
& \quad \vdots \\
0 & \to N^{b-3} \to M^{b-2} \to N^{b-2} \to 0, \\
0 & \to N^{b-2} \to M^{b-1} \to M^b \to 0
\end{align*}
Arguing by descending induction we see that $N^{b-2}, \ldots, N^{a+1}$ are finite projective
$R$-modules, the sequences are split exact, and
\[ c(M^\bullet) = \sum (-1)[M^n] = \sum (-1)^n ([N^{a-1}] + [N^n]) = 0 \]
Thus our construction gives zero on acyclic complexes.

It follows formally from the results of the preceding two paragraphs that $c$ is well
defined and satisfies (2). Namely, suppose the finite complexes $M^\bullet$ and $L^\bullet$ of finite
projective $R$-modules represent the same object of $D(R)$. Then we can represent
the isomorphism by a map $f : M^\bullet \to L^\bullet$ of complexes, see Derived Categories,
Lemma \[19.8\] We obtain a short exact sequence of complexes
\[ 0 \to L^\bullet \to C(f)^\bullet \to K^\bullet[1] \to 0 \]
see Derived Categories, Definition \[3.1\] Since $f$ is a quasi-isomorphism, the cone
$C(f)^\bullet$ is acyclic (this follows for example from the discussion in Derived Categories,
Section \[12\]). Hence
\[ 0 = c(C(f)^\bullet) = c(L^\bullet) + c(K^\bullet[1]) = c(L^\bullet) - c(K^\bullet) \]
as desired. We omit the proof of (2) which is similar. □

**Lemma 74.3.** Let $R$ be a regular local ring. Let $f \in R$. Then $\text{Pic}(R_f) = 0$.

**Proof.** Let $L$ be an invertible $R_f$-module. In particular $L$ is a finite $R_f$-module. There exists a finite $R$-module $M$ such that $M_f \cong L$, see Algebra, Lemma 123.3. By Algebra, Proposition 107.1 we see that $M$ has a finite free resolution $F_*$ over $R$. It follows that $L$ is quasi-isomorphic to a finite complex of free $R_f$-modules. Hence by Lemma 74.2 we see that $[L] = n[R_f]$ in $K_0(R)$ for some $n \in \mathbb{Z}$. Applying the map of Lemma 74.1 we see that $L$ is trivial. □

**Lemma 74.4.** A regular local ring is a UFD.

**Proof.** Recall that a regular local ring is a domain, see Algebra, Lemma 103.2. We will prove the unique factorization property by induction on the dimension of the regular local ring $R$. If $\dim(R) = 0$, then $R$ is a field and in particular a UFD. Assume $\dim(R) > 0$. Let $x \in m$, $x \notin m^2$. Then $R/(x)$ is regular by Algebra, Lemma 103.3 hence a domain by Algebra, Lemma 103.2 hence $x$ is a prime element. Let $p \subset R$ be a height 1 prime. We have to show that $p$ is principal, see Algebra, Lemma 117.6. We may assume $x \not\in p$, since if $x \in p$, then $p = (x)$ and we are done. For every nonmaximal prime $q \subset R$ the local ring $R_q$ is a regular local ring, see Algebra, Lemma 107.6. By induction we see that $pR_q$ is principal. In particular, the $R_x$-module $p_x = pR_x \subset R_x$ is a finitely presented $R_x$-module whose localization at any prime is free of rank 1. By Algebra, Lemma 76.2 we see that $p_x$ is an invertible $R_x$-module. By Lemma 74.3 we see that $p_x = (y)$ for some $y \in R_x$. We can write $y = x^ef$ for some $f \in p$ and $e \in \mathbb{Z}$. Factor $f = a_1 \ldots a_r$ into irreducible elements of $R$ (Algebra, Lemma 117.3). Since $p$ is prime, we see that $a_i \in p$ for some $i$. Since $p_x = (y)$ is prime and $a_i[y]$ in $R_x$, it follows that $p_x$ is generated by $a_i$ in $R_x$, i.e., the image of $a_i$ in $R_x$ is prime. As $x$ is a prime element, we find that $a_i$ is prime in $R$ by Algebra, Lemma 117.7. Since $(a_i) \subset p$ and $p$ has height 1 we conclude that $(a_i) = p$ as desired. □

**75. Extensions of valuation rings**

This section is the analogue of Section 72 for general valuation rings.

**Definition 75.1.** We say that $A \to B$ or $A \subset B$ is an extension of valuation rings if $A$ and $B$ are valuation rings and $A \to B$ is injective and local. Such an extension induces a commutative diagram

$$
\begin{array}{ccc}
A \setminus \{0\} & \longrightarrow & B \setminus \{0\} \\
\downarrow v & & \downarrow v \\
\Gamma_A & \longrightarrow & \Gamma_B \\
\end{array}
$$

where $\Gamma_A$ and $\Gamma_B$ are the value groups. We say that $B$ is weakly unramified over $A$ if the lower horizontal arrow is a bijection. If the extension of residue fields $\kappa_A = A/m_A \subset \kappa_B = B/m_B$ is finite, then we set $f = [\kappa_B : \kappa_A]$ and we call it the residual degree or residue degree of the extension $A \subset B$.

Note that $\Gamma_A \to \Gamma_B$ is injective, because the units of $A$ are the inverse of the units of $B$ under the map $A \to B$. Note also, that we do not require the extension of fraction fields to be finite.
**Lemma 75.2.** Let $A \subset B$ be an extension of valuation rings with fraction fields $K \subset L$. If the extension $K \subset L$ is finite, then the residue field extension is finite, the index of $\Gamma_A$ in $\Gamma_B$ is finite, and

$$[\Gamma_B : \Gamma_A][\kappa_B : \kappa_A] \leq [L : K].$$

**Proof.** Let $b_1, \ldots, b_n \in B$ be units whose images in $\kappa_B$ are linearly independent over $\kappa_A$. Let $c_1, \ldots, c_m \in B$ be nonzero elements whose images in $\Gamma_B/\Gamma_A$ are pairwise distinct. We claim that $b_ic_j$ are $K$-linearly independent in $L$. Namely, we claim a sum

$$\sum a_{ij}b_ic_j$$

with $a_{ij} \in K$ not all zero cannot be zero. Choose $(i_0, j_0)$ with $v(a_{i_0,j_0}b_{i_0}c_{j_0})$ minimal. Replace $a_{ij}$ by $a_{ij}/a_{i_0,j_0}$, so that $a_{i_0,j_0} = 1$. Let

$$P = \{ (i, j) \mid v(a_{ij}b_ic_j) = v(a_{i_0,j_0}b_{i_0}c_{j_0}) \}$$

By our choice of $c_1, \ldots, c_m$ we see that $(i, j) \in P$ implies $j = j_0$. Hence if $(i, j) \in P$, then $v(a_{ij}) = v(a_{i_0,j_0}) = 0$, i.e., $a_{ij}$ is a unit. By our choice of $b_1, \ldots, b_n$ we see that

$$\sum_{(i, j) \in P} a_{ij}b_i$$

is a unit in $B$. Thus the valuation of $\sum_{(i, j) \in P} a_{ij}b_ic_j$ is $v(c_{j_0}) = v(a_{i_0,j_0}b_{i_0}c_{j_0})$. Since the terms with $(i, j) \not\in P$ in the first displayed sum have strictly bigger valuation, we conclude that this sum cannot be zero, thereby proving the lemma. $\square$

**Lemma 75.3.** Let $A \to B$ be a flat local homomorphism of Noetherian local normal domains. Let $f \in A$ and $h \in B$ such that $f = uh^n$ for some $n > 1$ and some unit $u$ of $B$. Assume that for every height 1 prime $p \subset A$ there is a height 1 prime $q \subset B$ lying over $p$ such that the extension $A_p \subset B_q$ is weakly unramified. Then $f = ug^n$ for some $g \in A$ and unit $u$ of $A$.

**Proof.** The local rings of $A$ and $B$ at height 1 primes are discrete valuation rings (Algebra, Lemma 116.6). Thus the assumption makes sense (via Definition 72.1). Let $p_1, \ldots, p_r$ be the primes of $A$ minimal over $f$. These have height 1 by Algebra, Lemma 59.10. For each $i$ let $q_{i,j} \subset B$, $j = 1, \ldots, r_i$ be the height 1 primes of $B$ lying over $p_i$. Say we number them so that $A_{p_i} \to B_{q_{i,j}}$ is weakly unramified. Since $f$ maps to an $n$th power times a unit in $B_{q_{i,j}}$ we see that the valuation $v_i$ of $f$ in $A_{p_i}$ is divisible by $n$. Consider the exact sequence

$$0 \to I \to A \to \prod_{i=1,\ldots,r} A_{p_i}/p_i^{v_i/n}A_{p_i}$$

Applying the exact functor $- \otimes_A B$ we obtain

$$0 \to I \otimes_A B \to B \to \prod_{i=1,\ldots,r} \prod_{j=1,\ldots,r_i} B_{q_{i,j}}/q_{i,j}^{e_{i,j}v_i/n}A_{p_i}$$

where $e_{i,j}$ is the ramification index of $A_{p_i} \to B_{q_{i,j}}$. It follows that $I \otimes_A B$ is the set of elements $h'$ of $B$ which have valuation $\geq e_{i,j}v_i/n$ at $q_{i,j}$. Since $f = uh^n$ in $B$ we see that $h$ has valuation $e_{i,j}v_i/n$ at $q_{i,j}$. Thus $h' = h \in B$ by Algebra, Lemma 147.6. It follows that $I \otimes_A B$ is a free $B$-module of rank 1. Therefore $I$ is a free $A$-module of rank 1, see Algebra, Lemma 76.5. Let $g \in I$ be a generator. Then we see that $g$ and $h$ differ by a unit in $B$. Working backwards we conclude that the valuation of $g$ in $A_{p_i}$ is $v_i/n$. Hence $g^n$ and $f$ differ by a unit in $A$ (by Algebra, Lemma 147.6) as desired. $\square$
Lemma 75.4. Let $A$ be a valuation ring. Let $A \to B$ be an étale ring map and let $\mathfrak{m} \subset B$ be a prime lying over the maximal ideal of $A$. Then $A \subset B_\mathfrak{m}$ is an extension of valuation rings which is weakly unramified.

Proof. The ring $A$ has weak dimension $\leq 1$ by Lemma 71.17. Then $B$ has weak dimension $\leq 1$ by Lemmas 71.4 and 71.13 hence the local ring $B_\mathfrak{m}$ is a valuation ring by Lemma 71.17. Since the extension $f.f.(A) \subset f.f.(B_\mathfrak{m})$ is finite, we see that the $\Gamma_A$ has finite index in the value group of $B_\mathfrak{m}$. Thus for every $h \in B_\mathfrak{m}$ there exists an $n > 0$, an element $f \in A$, and a unit $w \in B_\mathfrak{m}$ such that $f = wh^n$ in $B_\mathfrak{m}$. We will show that this implies $f = ug^n$ for some $g \in A$ and unit $u \in A$; this will show that the value groups of $A$ and $B_\mathfrak{m}$ agree, as claimed in the lemma.

Write $A = \text{colim } A_i$ as the colimit of its local subrings which are essentially of finite type over $\mathbb{Z}$. Since $A$ is a normal domain (Algebra, Lemma 48.10), we may assume that each $A_i$ is normal (here we use that taking normalizations the local rings remain essentially of finite type over $\mathbb{Z}$ by Algebra, Proposition 151.32). For some $i$ we can find an étale extension $A_i \to B_i$ such that $B = A \otimes_{A_i} B_i$, see Algebra, Lemma 139.3. Let $\mathfrak{m}$ be the intersection of $B_i$ with $\mathfrak{m}$. Then we may apply Lemma 75.3 to the ring map $A_i \to (B_i)_\mathfrak{m}$ to conclude. The hypotheses of the lemma are satisfied because:

1. $A_i$ and $(B_i)_\mathfrak{m}$ are Noetherian as they are essentially of finite type over $\mathbb{Z}$,
2. $A_i \to (B_i)_\mathfrak{m}$ is flat as $A_i \to B_i$ is étale,
3. $B_i$ is normal as $A_i \to B_i$ is étale, see Algebra, Lemma 152.7,
4. for every height 1 prime of $A_i$ there exists a height 1 prime of $(B_i)_\mathfrak{m}$ lying over it by Algebra, Lemma 110.2 and the fact that $\text{Spec}((B_i)_\mathfrak{m}) \to \text{Spec}(A_i)$ is surjective,
5. the induced extensions $(A_i)_p \to (B_i)_q$ are unramified for every prime $q$ lying over a prime $p$ as $A_i \to B_i$ is étale.

This concludes the proof of the lemma. \(\square\)

Lemma 75.5. Let $A$ be a valuation ring. Let $A^h$, resp. $A^{sh}$ be its henselization, resp. strict henselization. Then

$$A \subset A^h \subset A^{sh}$$

are extensions of valuation rings which induce bijections on value groups, i.e., which are weakly unramified.

Proof. Write $A^h = \text{colim}(B_i)_q$, where $A \to B_i$ is étale and $q_i \subset B_i$ is a prime ideal lying over $\mathfrak{m}_A$, see Algebra, Lemma 146.21. Then Lemma 75.4 tells us that $(B_i)_q$, is a valuation ring and that the induced map

$$(A \setminus \{0\})/A^* \to ((B_i)_q \setminus \{0\})/(B_i)_q^*,$$

is bijective. By Algebra, Lemma 48.5 we conclude that $A^h$ is a valuation ring. It also follows that $(A \setminus \{0\})/A^* \to (A^h \setminus \{0\})/(A^h)^*$ is bijective. This proves the lemma for the inclusion $A \subset A^h$. To prove it for $A \subset A^{sh}$ we can use exactly the same argument except we replace $A$, see Algebra, Lemma 146.21 by Algebra, Lemma 146.27. Since $A^{sh} = (A^h)^{sh}$ we see that this also proves the assertions of the lemma for the inclusion $A^h \subset A^{sh}$. \(\square\)
76. Structure of modules over a PID

We work a little bit more generally (following the papers [War69] and [War70] by Warfield) so that the proofs work over valuation rings.

**Lemma 76.1.** Let $P$ be a module over a ring $R$. The following are equivalent

1. $P$ is a direct summand of a direct sum of modules of the form $R/fR$, for $f \in R$ varying.
2. for every short exact sequence $0 \to A \to B \to C \to 0$ of $R$-modules such that $fA = A \cap fB$ for all $f \in R$ the map $\text{Hom}_R(P,B) \to \text{Hom}_R(P,C)$ is surjective.

**Proof.** Let $0 \to A \to B \to C \to 0$ be an exact sequence as in (2). To prove that (1) implies (2) it suffices to prove that $\text{Hom}_R(R/fR,B) \to \text{Hom}_R(R/fR,C)$ is surjective for every $f \in R$. Let $\psi : R/fR \to C$ be a map. Say $\psi(1)$ is the image of $b \in B$. Then $fb \in A$. Hence there exists an $a \in A$ such that $fa = fb$. Then $f(b - a) = 0$ hence we get a morphism $\varphi : R/fR \to B$ mapping 1 to $b - a$ which lifts $\psi$.

Conversely, assume that (2) holds. Let $I$ be the set of pairs $(f, \varphi)$ where $f \in R$ and $\varphi : R/fR \to P$. For $i \in I$ denote $(f_i, \varphi_i)$ the corresponding pair. Consider the map

$$B = \bigoplus_{i \in I} R/f_iR \to P$$

which sends the element $r$ in the summand $R/f_iR$ to $\varphi_i(r)$ in $P$. Let $A = \text{Ker}(F \to P)$. Then we see that (1) is true if the sequence

$$0 \to A \to B \to P \to 0$$

is an exact sequence as in (2). To see this suppose $f \in R$ and $a \in A$ maps to $fb$ in $B$. Write $b = (r_i)_{i \in I}$ with almost all $r_i = 0$. Then we see that

$$f \sum \varphi_i(r_i) = 0$$

in $P$. Hence there is an $i_0 \in I$ such that $f_{i_0} = f$ and $\varphi_{i_0}(1) = \sum \varphi_i(r_i)$. Let $x_{i_0} \in R/f_iR$ be the class of 1. Then we see that

$$a = (r_i)_{i \in I} - (0, \ldots, 0, x_{i_0}, 0, \ldots)$$

is an element of $A$ and $fa = b$ as desired. \qed

**Lemma 76.2 (Generalized valuation rings).** Let $R$ be a ring. The following are equivalent

1. For $a, b \in R$ either $a$ divides $b$ or $b$ divides $a$.
2. Every finitely generated ideal is principal and $R$ is local.
3. The set of ideals of $R$ are linearly ordered by inclusion.

This holds in particular if $R$ is a valuation ring.

**Proof.** Assume (2) and let $a, b \in R$. Then $(a, b) = (c)$. If $c = 0$, then $a = b = 0$ and $a$ divides $b$. Assume $c \neq 0$. Write $c = ua + vb$ and $a = wc$ and $b = zc$. Then $c(1 - uw - vz) = 0$. Since $R$ is local, this implies that $1 - uw - vz \in \mathfrak{m}$. Hence either $w$ or $z$ is a unit, so either $a$ divides $b$ or $b$ divides $a$. Thus (2) implies (1).

Assume (1). If $R$ has two maximal ideals $\mathfrak{m}_1$, we can choose $a \in \mathfrak{m}_1$ with $a \not\in \mathfrak{m}_2$ and $b \in \mathfrak{m}_2$ with $b \not\in \mathfrak{m}_1$. Then $a$ does not divide $b$ and $b$ does not divide $a$. Hence
\(R\) has a unique maximal ideal and is local. It follows easily from condition (1) and induction that every finitely generated ideal is principal. Thus (1) implies (2).

It is straightforward to prove that (1) and (3) are equivalent. The final statement is Algebra, Lemma \[18.3\].

**Lemma 76.3.** Let \(R\) be a ring satisfying the equivalent conditions of Lemma \[76.2\]. Then every finitely presented \(R\)-module is isomorphic to a finite direct sum of modules of the form \(R/fR\).

**Proof.** Let \(M\) be a finitely presented \(R\)-module. Let \(x_1, \ldots, x_n \in M\) be a minimal set of generators. Let \(I \subset R\) be the annihilator of \(M\). For some \(i\) the annihilator \(I_i\) of \(x_i\) is \(I\): we have \(I = \bigcap I_i\) and the set of ideals are linearly ordered. After renumbering we may assume \(I_1 = I\). We set \(A = Rx_1 \subset M\). Consider the exact sequence \(0 \to A \to M \to M/A \to 0\). Since \(A\) is finite, we see that \(M/A\) is a finitely presented \(R\)-module (Algebra, Lemma \[5.3\]) with fewer generators. Hence \(M/A \cong \bigoplus_{i=1}^n R/f_j R\) by induction. On the other hand, we claim that \(A \to M\) satisfies the property: if \(f \in R\), then \(fA = A \cap fM\). Namely, if \(x \in A \cap fM\), then \(x = \sum f_i x_i\), and \(x = gx_1\). Hence \(g = f_{x_1}\) and we see that \(x \in fA\). By Lemma \[76.1\] the sequence is split and we find \(M \cong A \oplus \bigoplus_{j=1}^m R/f_j R\). Then \(A = R/I\) is finitely presented (as a summand of \(M\)) and hence \(I\) is finitely generated, hence principal. This finishes the proof. \(\square\)

**Lemma 76.4.** Let \(R\) be a ring such that every local ring of \(R\) at a maximal ideal satisfies the equivalent conditions of Lemma \[76.4\]. Then every finitely presented \(R\)-module is a summand of a finite direct sum of modules of the form \(R/fR\) for \(f\) in \(R\) varying.

**Proof.** Let \(M\) be a finitely presented \(R\)-module. We first show that \(M\) is a summand of a direct sum of modules of the form \(R/fR\) and at the end we argue the direct sum can be taken to be finite. Let

\[0 \to A \to B \to C \to 0\]

be a short exact sequence of \(R\)-modules such that \(fA = A \cap fB\) for all \(f \in R\). By Lemma \[76.7\] we have to show that \(\text{Hom}_R(M, B) \to \text{Hom}_R(M, C)\) is surjective. It suffices to prove this after localization at maximal ideals \(m\), see Algebra, Lemma \[23.1\]. Note that the localized sequences \(0 \to A_m \to B_m \to C_m \to 0\) satisfy the condition that \(fA_m = A_m \cap fB_m\) for all \(f \in R_m\) (because we can write \(f = uf'\) with \(u \in R_m\) a unit and \(f' \in R\) and because localization is exact). Since \(M\) is finitely presented, we see that

\[\text{Hom}_R(M, B)_m = \text{Hom}_{R_m}(M_m, B_m)\quad \text{and} \quad \text{Hom}_R(M, C)_m = \text{Hom}_{R_m}(M_m, C_m)\]

by Algebra, Lemma \[10.2\]. The module \(M_m\) is a finitely presented \(R_m\)-module. By Lemma \[76.3\] we see that \(M_m\) is a direct sum of modules of the form \(R_m/fR_m\). Thus we conclude by Lemma \[76.1\] that the map on localizations is surjective.

At this point we know that \(M\) is a summand of \(\bigoplus_{i \in I} R/f_i R\). Consider the map \(M \to \bigoplus_{i \in I'} R/f_i R\). Since \(M\) is a finite \(R\)-module, the image is contained in \(\bigoplus_{i \in I'} R/f_i R\) for some finite subset \(I' \subset I\). This finishes the proof. \(\square\)

**Definition 76.5.** Let \(R\) be a domain.

1. We say \(R\) is a Bezout domain if every finitely generated ideal of \(R\) is principal.
We say $R$ is an \textit{elementary divisor domain} if for all $n, m \geq 1$ and every $n \times m$ matrix $A$, there exist invertible matrices $U, V$ of size $n \times n, m \times m$ such that

$$UAV = \begin{pmatrix}
  f_1 & 0 & 0 & \ldots \\
  0 & f_2 & 0 & \ldots \\
  0 & 0 & f_3 & \ldots \\
  \vdots & \vdots & \vdots & \ddots & \ldots
\end{pmatrix}$$

with $f_1, \ldots, f_{\min(n,m)} \in R$ and $f_1 | f_2 | \ldots$.

It is apparently still an open question as to whether every Bézout domain $R$ is an elementary divisor domain (or not). This is equivalent to the question of whether every finitely presented module over $R$ is a direct sum of cyclic modules. The converse implication is true.

\textbf{Lemma 76.6.} An elementary divisor domain is Bézout.

\textbf{Proof.} Let $a, b \in R$ be nonzero. Consider the $1 \times 2$ matrix $A = (a \ b)$. Then we see that $u(a \ b)V = (f \ 0)$ with $u \in R$ invertible and $V = (g_{ij})$ an invertible $2 \times 2$ matrix. Then $f = uag_{11} + ubg_{21}$ and $(g_{11}, g_{21}) = R$. An induction argument (omitted) then shows any finitely generated ideal in $R$ is generated by one element. \hfill $\square$

\textbf{Lemma 76.7.} The localization of a Bézout domain is Bézout. Every local ring of a Bézout domain is a valuation ring. A local domain is Bézout if and only if it is a valuation ring.

\textbf{Proof.} We omit the proof of the statement on localizations. The final statement is Algebra, Lemma 48.15. The second statement follows from the other two. \hfill $\square$

\textbf{Lemma 76.8.} Let $R$ be a Bézout domain.

1. Every finite submodule of a free module is finite free.
2. Every finitely presented $R$-module $M$ is a direct sum of a finite free module and a torsion module $M_{\text{tors}}$ which is a summand of a module of the form $\bigoplus_{i=1,\ldots,n} R/f_i R$ with $f_1, \ldots, f_n \in R$ nonzero.

\textbf{Proof.} Proof of (1). Let $M \subset F$ be a finite submodule of a free module $F$. Since $M$ is finite, we may assume $F$ is a finite free module (details omitted). Say $F = R^{\oplus n}$. We argue by induction on $n$. If $n = 1$, then $M$ is a finitely generated ideal, hence principal by our assumption that $R$ is Bézout. If $n > 1$, then we consider the image $I$ of $M$ under the projection $R^{\oplus n} \to R$ onto the last summand. If $I = (0)$, then $M \subset R^{\oplus n-1}$ and we are done by induction. If $I \neq 0$, then $I = (f) \cong R$. Hence $M \cong R \oplus \ker(M \to I)$ and we are done by induction as well.

Let $M$ be a finitely presented $R$-module. Since the localizations of $R$ are maximal ideals are valuation rings (Lemma 76.7) we may apply Lemma 76.4. Thus $M$ is a summand of a module of the form $R^{\oplus r} \oplus \bigoplus_{i=1,\ldots,n} R/f_i R$ with $f_i \neq 0$. Since taking the torsion submodule is a functor we see that $M_{\text{tors}}$ is a summand of the module $\bigoplus_{i=1,\ldots,n} R/f_i R$ and $M/M_{\text{tors}}$ is a summand of $R^{\oplus r}$. By the first part of the proof we see that $M/M_{\text{tors}}$ is finite free. Hence $M \cong M_{\text{tors}} \oplus M/M_{\text{tors}}$ as desired. \hfill $\square$

\textbf{Lemma 76.9.} Let $R$ be a PID. Every finite $R$-module $M$ is of isomorphic to a module of the form $R^{\oplus r} \oplus \bigoplus_{i=1,\ldots,n} R/f_i R$.
for some \( r, n \geq 0 \) and \( f_1, \ldots, f_n \in R \) nonzero.

**Proof.** A PID is a Noetherian Bézout ring. By Lemma 76.8 it suffices to prove the result if \( M \) is nonzero. Since \( M \) is finite, this means that the annihilator of \( M \) is nonzero. Say \( fM = 0 \) for some \( f \in R \) nonzero. Then we can think of \( M \) as a module over \( R/fR \). Since \( R/fR \) is Noetherian of dimension 0 (small detail omitted) we see that \( R/fR = \prod R_i \) is a finite product of Artinian local rings \( R_i \) (Algebra, Proposition 59.6). Each \( R_i \), being a local ring and a quotient of a PID, is a generalized valuation ring in the sense of Lemma 76.2 (small detail omitted). Write \( M = \prod M_j \) with \( M_j = e_jM \) where \( e_j \in R/fR \) is the idempotent corresponding to the factor \( R_j \). By Lemma 76.3 we see that \( M_j = \sum_{i=1}^n R_j/R_j \) for some \( f^i_j \in R_j \). Choose lifts \( f^i_j \in R \) and choose \( g^i_j \in R \) with \( (g^i_j) = (f^i_j, f_j) \). Then we conclude that

\[
M \cong \bigoplus R/g^i_j R
\]
as an \( R \)-module which finishes the proof. □

One can also prove that a PID is a elementary divisor domain (insert future reference here), by proving lemmas similar to the following.

**Lemma 76.10.** Let \( R \) be a Bézout domain. Let \( n \geq 1 \) and \( f_1, \ldots, f_n \in R \) generate the unit ideal. There exists an invertible \( n \times n \) matrix in \( R \) whose first row is \( f_1 \cdots f_n \).

**Proof.** This follows from Lemma 76.8 but we can also prove it directly as follows. By induction on \( n \). The result holds for \( n = 1 \). Assume \( n > 1 \). We may assume \( f_1 \neq 0 \) after renumbering. Choose \( f \in R \) such that \( (f) = (f_1, \ldots, f_{n-1}) \). Let \( A \) be an \( (n-1) \times (n-1) \) matrix whose first row is \( f_1/f, \ldots, f_{n-1}/f \). Choose \( a, b \in R \) such that \( af - bf_n = 1 \) which is possible because \( 1 \in (f_1, \ldots, f_n) = (f, f_n) \). Then a solution is the matrix

\[
\begin{pmatrix}
  f & 0 & \ldots & 0 & f_n \\
  0 & 1 & \ldots & 0 & 0 \\
  \vdots & & & & \vdots \\
  0 & 0 & \ldots & 1 & 0 \\
  b & 0 & \ldots & 0 & a
\end{pmatrix}
\begin{pmatrix}
  A & 0 \\
  0 & \ldots & 0 & 0 \\
  0 & \ldots & 0 & 1
\end{pmatrix}
\]

Observe that the left matrix is invertible because it has determinant 1. □

77. Other chapters
<table>
<thead>
<tr>
<th>Topic</th>
<th>Subtopics</th>
</tr>
</thead>
<tbody>
<tr>
<td>(23)</td>
<td>Divided Power Algebra</td>
</tr>
<tr>
<td>(24)</td>
<td>Hypercoverings</td>
</tr>
<tr>
<td>(25)</td>
<td>Schemes</td>
</tr>
<tr>
<td>(26)</td>
<td>Constructions of Schemes</td>
</tr>
<tr>
<td>(27)</td>
<td>Properties of Schemes</td>
</tr>
<tr>
<td>(28)</td>
<td>Morphisms of Schemes</td>
</tr>
<tr>
<td>(29)</td>
<td>Cohomology of Schemes</td>
</tr>
<tr>
<td>(30)</td>
<td>Divisors</td>
</tr>
<tr>
<td>(31)</td>
<td>Limits of Schemes</td>
</tr>
<tr>
<td>(32)</td>
<td>Varieties</td>
</tr>
<tr>
<td>(33)</td>
<td>Topologies on Schemes</td>
</tr>
<tr>
<td>(34)</td>
<td>Descent</td>
</tr>
<tr>
<td>(35)</td>
<td>Derived Categories of Schemes</td>
</tr>
<tr>
<td>(36)</td>
<td>More on Morphisms</td>
</tr>
<tr>
<td>(37)</td>
<td>More on Flatness</td>
</tr>
<tr>
<td>(38)</td>
<td>Groupoid Schemes</td>
</tr>
<tr>
<td>(39)</td>
<td>More on Groupoid Schemes</td>
</tr>
<tr>
<td>(40)</td>
<td>Étale Morphisms of Schemes</td>
</tr>
<tr>
<td>(41)</td>
<td>Topics in Scheme Theory</td>
</tr>
<tr>
<td>(42)</td>
<td>Chow Homology</td>
</tr>
<tr>
<td>(43)</td>
<td>Intersection Theory</td>
</tr>
<tr>
<td>(44)</td>
<td>Adequate Modules</td>
</tr>
<tr>
<td>(45)</td>
<td>Dualizing Complexes</td>
</tr>
<tr>
<td>(46)</td>
<td>Étale Cohomology</td>
</tr>
<tr>
<td>(47)</td>
<td>Crystalline Cohomology</td>
</tr>
<tr>
<td>(48)</td>
<td>Pro-étale Cohomology</td>
</tr>
<tr>
<td>(49)</td>
<td>Algebraic Spaces</td>
</tr>
<tr>
<td>(50)</td>
<td>Properties of Algebraic Spaces</td>
</tr>
<tr>
<td>(51)</td>
<td>Morphisms of Algebraic Spaces</td>
</tr>
<tr>
<td>(52)</td>
<td>Decent Algebraic Spaces</td>
</tr>
<tr>
<td>(53)</td>
<td>Cohomology of Algebraic Spaces</td>
</tr>
<tr>
<td>(54)</td>
<td>Limits of Algebraic Spaces</td>
</tr>
<tr>
<td>(55)</td>
<td>Divisors on Algebraic Spaces</td>
</tr>
<tr>
<td>(56)</td>
<td>Algebraic Spaces over Fields</td>
</tr>
<tr>
<td>(57)</td>
<td>Topologies on Algebraic Spaces</td>
</tr>
<tr>
<td>(58)</td>
<td>Derived Categories of Spaces</td>
</tr>
<tr>
<td>(59)</td>
<td>More on Morphisms of Spaces</td>
</tr>
<tr>
<td>(60)</td>
<td>Pushouts of Algebraic Spaces</td>
</tr>
<tr>
<td>(61)</td>
<td>Groupoids in Algebraic Spaces</td>
</tr>
<tr>
<td>(62)</td>
<td>More on Groupoids in Spaces</td>
</tr>
<tr>
<td>(63)</td>
<td>Bootstrap</td>
</tr>
<tr>
<td>(64)</td>
<td>Topics in Geometry</td>
</tr>
<tr>
<td>(65)</td>
<td>Quotients of Groupoids</td>
</tr>
<tr>
<td>(66)</td>
<td>Simplicial Spaces</td>
</tr>
<tr>
<td>(67)</td>
<td>Formal Algebraic Spaces</td>
</tr>
<tr>
<td>(68)</td>
<td>Restricted Power Series</td>
</tr>
<tr>
<td>(69)</td>
<td>Resolution of Surfaces</td>
</tr>
<tr>
<td>(70)</td>
<td>Deformation Theory</td>
</tr>
<tr>
<td>(71)</td>
<td>Formal Deformation Theory</td>
</tr>
<tr>
<td>(72)</td>
<td>Algebraic Stacks</td>
</tr>
<tr>
<td>(73)</td>
<td>Examples of Stacks</td>
</tr>
<tr>
<td>(74)</td>
<td>Sheaves on Algebraic Stacks</td>
</tr>
<tr>
<td>(75)</td>
<td>Criteria for Representability</td>
</tr>
<tr>
<td>(76)</td>
<td>Artin’s Axioms</td>
</tr>
<tr>
<td>(77)</td>
<td>Quot and Hilbert Spaces</td>
</tr>
<tr>
<td>(78)</td>
<td>Properties of Algebraic Stacks</td>
</tr>
<tr>
<td>(79)</td>
<td>Morphisms of Algebraic Stacks</td>
</tr>
<tr>
<td>(80)</td>
<td>Cohomology of Algebraic Stacks</td>
</tr>
<tr>
<td>(81)</td>
<td>Derived Categories of Stacks</td>
</tr>
<tr>
<td>(82)</td>
<td>Introducing Algebraic Stacks</td>
</tr>
<tr>
<td>(83)</td>
<td>Miscellany</td>
</tr>
<tr>
<td>(84)</td>
<td>Examples</td>
</tr>
<tr>
<td>(85)</td>
<td>Exercises</td>
</tr>
<tr>
<td>(86)</td>
<td>Guide to Literature</td>
</tr>
<tr>
<td>(87)</td>
<td>Desirables</td>
</tr>
<tr>
<td>(88)</td>
<td>Coding Style</td>
</tr>
<tr>
<td>(89)</td>
<td>Obsolete</td>
</tr>
<tr>
<td>(90)</td>
<td>GNU Free Documentation License</td>
</tr>
<tr>
<td>(91)</td>
<td>Auto Generated Index</td>
</tr>
</tbody>
</table>

References


More on Algebra


