1. Introduction

In this chapter we put some lemmas that have become “obsolete” (see [Mil17]).

2. Homological algebra

Remark 2.1. The following remarks are obsolete as they are subsumed in Homology, Lemmas [21.11] and [22.6]. Let $\mathcal{A}$ be an abelian category. Let $\mathcal{C} \subseteq \mathcal{A}$ be a weak Serre subcategory (see Homology, Definition [9.1]). Suppose that $K^{\bullet, \bullet}$ is a double complex to which Homology, Lemma [22.6] applies such that for some $r \geq 0$ all the objects $'E_{r+1}^{p,q}$ belong to $\mathcal{C}$. Then all the cohomology groups $H^n(sK^{\bullet})$ belong to $\mathcal{C}$. Namely, the assumptions imply that the kernels and images of $'d_r^{p,q}$ are in $\mathcal{C}$. Whereupon we see that each $'E_{r+1}^{p,q}$ is in $\mathcal{C}$. By induction we see that each $'E_{\infty}^{p,q}$ is in $\mathcal{C}$. Hence each $H^n(sK^{\bullet})$ has a finite filtration whose subquotients are in $\mathcal{C}$. Using that $\mathcal{C}$ is closed under extensions we conclude that $H^n(sK^{\bullet})$ is in $\mathcal{C}$ as claimed. The same result holds for the second spectral sequence associated to $K^{\bullet, \bullet}$. Similarly,
if \((K^\bullet, F)\) is a filtered complex to which Homology, Lemma \[21.11\] applies and for some \(r \geq 0\) all the objects \(E^p_r\) belong to \(C\), then each \(H^n(K^\bullet)\) is an object of \(C\).

3. Obsolete algebra lemmas

088X

Lemma 3.1. Let \(M\) be an \(R\)-module of finite presentation. For any surjection \(\alpha : R^{\oplus n} \to M\) the kernel of \(\alpha\) is a finite \(R\)-module.

Proof. This is a special case of Algebra, Lemma \[5.3\]. \qed

Lemma 3.2. Let \(\varphi : R \to S\) be a ring map. If

1. for any \(x \in S\) there exists \(n > 0\) such that \(x^n\) is in the image of \(\varphi\), and
2. for any \(x \in \ker(\varphi)\) there exists \(n > 0\) such that \(x^n = 0\),

then \(\varphi\) induces a homeomorphism on spectra. Given a prime number \(p\) such that

1. \(S\) is generated as an \(R\)-algebra by elements \(x\) such that there exists an \(n > 0\) with \(x^n \in \varphi(R)\) and \(p^n x \in \varphi(R)\), and
2. the kernel of \(\varphi\) is generated by nilpotent elements,

then (1) and (2) hold, and for any ring map \(R \to R'\) the ring map \(R' \to R' \otimes_R S\) also satisfies (a), (b), (1), and (2) and in particular induces a homeomorphism on spectra.

Proof. This is a combination of Algebra, Lemmas \[45.3\] and \[45.6\]. \qed

The following technical lemma says that you can lift any sequence of relations from a fibre to the whole space of a ring map which is essentially of finite type, in a suitable sense.

00SX Lemma 3.3. Let \(R \to S\) be a ring map. Let \(p \subset R\) be a prime. Let \(q \subset S\) be a prime lying over \(p\). Assume \(S_q\) is essentially of finite type over \(R_p\). Assume given

1. an integer \(n \geq 0\),
2. a prime \(a \subset \kappa(p)[x_1, \ldots, x_n] =: \psi : (\kappa(p)[x_1, \ldots, x_n]_a) \to S_q/pS_q,\)
3. a surjective \(\kappa(p)\)-homomorphism

\[
\psi : (\kappa(p)[x_1, \ldots, x_n])_a \to S_q/pS_q,
\]

and
4. elements \(f_1, \ldots, f_e\) in \(\ker(\psi)\).

Then there exist

1. an integer \(m \geq 0\),
2. and element \(g \in S\), \(g \notin q\),
3. a map

\[
\Psi : R[x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+m}] \to S_g
\]

and
4. elements \(f_1, \ldots, f_e, f_{e+1}, \ldots, f_{e+m}\) of \(\ker(\Psi)\)

such that
(1) the following diagram commutes

\[ \begin{array}{ccc}
R[x_1, \ldots, x_{n+m}] & \xrightarrow{\Psi} & (\kappa(p)[x_1, \ldots, x_n])_a \\
& \downarrow{x_{n+j} \rightarrow 0} & \\
S_y & \xrightarrow{\psi} & S_y/pS_y
\end{array} \]

(2) the element \( f_i, i \leq n \) maps to a unit times \( f_i \) in the local ring

\((\kappa(p)[x_1, \ldots, x_{n+m}])(a, x_{n+1}, \ldots, x_{n+m})\)

(3) the element \( f_{n+j} \) maps to a unit times \( x_{n+j} \) in the same local ring, and

(4) the induced map \( R[x_1, \ldots, x_{n+m}]_b \rightarrow S_q \) is surjective, where \( b = \Psi^{-1}(qS_q) \).

**Proof.** We claim that it suffices to prove the lemma in case \( R \) and \( S \) are local with maximal ideals \( p \) and \( q \). Namely, suppose we have constructed

\[ \Psi : R_p[x_1, \ldots, x_{n+m}] \rightarrow S_q \]

and \( f'_1, \ldots, f'_{n+m} \in R_p[x_1, \ldots, x_{n+m}] \) with all the required properties. Then there exists an element \( f \in R \), \( f \notin p \) such that each \( ff'_k \) comes from an element \( f_k \in R[x_1, \ldots, x_{n+m}] \). Moreover, for a suitable \( g \in S \), \( g \notin q \) the elements \( \Psi'(x_i) \) are the image of elements \( y_i \in S_q \). Let \( \Psi \) be the \( R \)-algebra map defined by the rule \( \Psi(x_i) = y_i \). Since \( \Psi(f_i) \) is zero in the localization \( S_q \) we may after possibly replacing \( g \) assume that \( \Psi(f_i) = 0 \). This proves the claim.

Thus we may assume \( R \) and \( S \) are local with maximal ideals \( p \) and \( q \). Pick \( y_1, \ldots, y_n \in S \) such that \( y_i \mod pS = \psi(x_i) \). Let \( y_{n+1}, \ldots, y_{n+m} \in S \) be elements which generate an \( R \)-subalgebra of which \( S \) is the localization. These exist by the assumption that \( S \) is essentially of finite type over \( R \). Since \( \psi \) is surjective we may write \( y_{n+j} \mod pS = \psi(h_j) \) for some \( h_j \in \kappa(p)[x_1, \ldots, x_n] \). Write \( h_j = g_j/d \), \( g_j \in \kappa(p)[x_1, \ldots, x_n] \) for some common denominator \( d \in \kappa(p)[x_1, \ldots, x_n] \), \( d \notin a \). Choose lifts \( G_j, D \in R[x_1, \ldots, x_n] \) of \( g_j \) and \( d \). Set \( y'_{n+j} = D(y_1, \ldots, y_n)y_{n+j} - G_j(y_1, \ldots, y_n) \). By construction \( y'_{n+j} \in pS \). It is clear that \( y_1, \ldots, y_n, y'_n, \ldots, y'_{n+m} \) generate an \( R \)-subalgebra of \( S \) whose localization is \( S \). We define

\[ \Psi : R[x_1, \ldots, x_{n+m}] \rightarrow S \]

to be the map that sends \( x_i \) to \( y_i \) for \( i = 1, \ldots, n \) and \( x_{n+j} \) to \( y'_{n+j} \) for \( j = 1, \ldots, m \).

Properties (1) and (4) are clear by construction. Moreover the ideal \( b \) maps onto the ideal \( (a, x_{n+1}, \ldots, x_{n+m}) \) in the polynomial ring \( \kappa(p)[x_1, \ldots, x_{n+m}] \).

Denote \( J = \text{Ker}(\Psi) \). We have a short exact sequence

\[ 0 \rightarrow J_b \rightarrow R[x_1, \ldots, x_{n+m}]_b \rightarrow S_q \rightarrow 0. \]

The surjectivity comes from our choice of \( y_1, \ldots, y_n, y'_n, \ldots, y'_{n+m} \) above. This implies that

\[ J_b/pJ_b \rightarrow \kappa(p)[x_1, \ldots, x_{n+m}](a, x_{n+1}, \ldots, x_{n+m}) \rightarrow S_q/pS_q \rightarrow 0. \]

is exact. By construction \( x_i \) maps to \( \psi(x_i) \) and \( x_{n+j} \) maps to zero under the last map. Thus it is easy to choose \( f_i \) as in (2) and (3) of the lemma. \( \square \)

**Remark 3.4** (Projective resolutions). Let \( R \) be a ring. For any set \( S \) we let \( F(S) \) denote the free \( R \)-module on \( S \). Then any left \( R \)-module has the following two step resolution

\[ F(M \times M) \oplus F(R \times M) \rightarrow F(M) \rightarrow M \rightarrow 0. \]
The first map is given by the rule
\[ [m_1, m_2] \oplus [r, m] \mapsto [m_1 + m_2] - [m_1] - [m_2] + [rm] - r[m]. \]

**Lemma 3.5.** Let \( S \) be a multiplicative set of \( A \). Then the map
\[ f : \text{Spec}(S^{-1}A) \longrightarrow \text{Spec}(A) \]
induced by the canonical ring map \( A \to S^{-1}A \) is a homeomorphism onto its image and \( \text{Im}(f) = \{ p \in \text{Spec}(A) : p \cap S = \emptyset \} \).

**Proof.** This is a duplicate of Algebra, Lemma 16.5 \( \square \)

**Lemma 3.6.** Let \( A \to B \) be a finite type, flat ring map with \( A \) an integral domain. Then \( B \) is a finitely presented \( A \)-algebra.

**Proof.** Special case of More on Flatness, Proposition 13.9 \( \square \)

**Lemma 3.7.** Let \( R \) be a domain with fraction field \( K \). Let \( S = R[x_1, \ldots, x_n] \) be a polynomial ring over \( R \). Let \( M \) be a finite \( S \)-module. Assume that \( M \) is flat over \( R \). If for every subring \( R \subset R' \subset K \), \( R \neq R' \) the module \( M \otimes_R R' \) is finitely presented over \( S \otimes_R R' \), then \( M \) is finitely presented over \( S \).

**Proof.** This lemma is true because \( M \) is finitely presented even without the assumption that \( M \otimes_R R' \) is finitely presented for every \( R' \) as in the statement of the lemma. This follows from More on Flatness, Proposition 13.9. Originally this lemma had an erroneous proof (thanks to Ofer Gabber for finding the gap) and was used in an alternative proof of the proposition cited. To reinstate this lemma, we need a correct argument in case \( R \) is a local normal domain using only results from the chapters on commutative algebra; please email stacks.project@gmail.com if you have an argument. \( \square \)

**Lemma 3.8.** Let \( A \to B \) be a ring map. Let \( f \in B \). Assume that
\begin{enumerate}
\item \( A \to B \) is flat,
\item \( f \) is a nonzerodivisor, and
\item \( A \to B/fB \) is flat.
\end{enumerate}
Then for every ideal \( I \subset A \) the map \( f : B/IB \to B/IB \) is injective.

**Proof.** Note that \( IB = I \otimes_A B \) and \( I(B/fB) = I \otimes_A B/fB \) by the flatness of \( B \) and \( B/fB \) over \( A \). In particular \( IB/f1B \cong I \otimes_A B/fB \) maps injectively into \( B/fB \). Hence the result follows from the snake lemma applied to the diagram
\[
\begin{array}{ccc}
0 & \rightarrow & I \otimes_A B & \rightarrow & B & \rightarrow & B/IB & \rightarrow & 0 \\
& & \downarrow f & & \downarrow f & & \downarrow f & & \\
0 & \rightarrow & I \otimes_A B & \rightarrow & B & \rightarrow & B/IB & \rightarrow & 0
\end{array}
\]
with exact rows. \( \square \)

**Lemma 3.9.** If \( R \to S \) is a faithfully flat ring map then for every \( R \)-module \( M \) the map \( M \to S \otimes_R M, x \mapsto 1 \otimes x \) is injective.

**Proof.** This lemma is a duplicate of Algebra, Lemma 81.11 \( \square \)
4. Lemmas related to ZMT

073V The lemmas in this section were originally used in the proof of the (algebraic version of) Zariski’s Main Theorem, Algebra, Theorem 122.13.

00PR Lemma 4.1. Let \( \varphi : R \to S \) be a ring map. Suppose \( t \in S \) satisfies the relation \( \varphi(a_0) + \varphi(a_1)t + \ldots + \varphi(a_n)t^n = 0 \). Set \( u_n = \varphi(a_n) \), \( u_{n-1} = u_nt + \varphi(a_{n-1}) \), and so on till \( u_1 = u_2t + \varphi(a_1) \). Then all of \( u_n, u_{n-1}, \ldots, u_1 \) and \( u_nt, u_{n-1}t, \ldots, u_1t \) are integral over \( R \), and the ideals \((\varphi(a_0), \ldots, \varphi(a_n))\) and \((u_n, \ldots, u_1)\) of \( S \) are equal.

Proof. We prove this by induction on \( n \). As \( u_n = \varphi(a_n) \) we conclude from Algebra, Lemma 122.1 that \( u_nt \) is integral over \( R \). Of course \( u_n = \varphi(a_n) \) is integral over \( R \). Then \( u_{n-1} = u_nt + \varphi(a_{n-1}) \) is integral over \( R \) (see Algebra, Lemma 55.7) and we have

\[ \varphi(a_0) + \varphi(a_1)t + \ldots + \varphi(a_{n-1})t^{n-1} + u_{n-1}t^{n-1} = 0. \]

Hence by the induction hypothesis applied to the map \( S' \to S \) where \( S' \) is the integral closure of \( R \) in \( S \) and the displayed equation we see that \( u_{n-1}, \ldots, u_1 \) and \( u_{n-1}t, \ldots, u_1t \) are all in \( S' \) too. The statement on the ideals is immediate from the shape of the elements and the fact that \( u_1t + \varphi(a_0) = 0 \).

00PS Lemma 4.2. Let \( \varphi : R \to S \) be a ring map. Suppose \( t \in S \) satisfies the relation \( \varphi(a_0) + \varphi(a_1)t + \ldots + \varphi(a_i)t^i = 0 \). Let \( J \subset S \) be an ideal such that for at least one \( i \) we have \( \varphi(a_i) \notin J \). Then there exists \( a \in S, u \notin J \) such that both \( a \) and \( ut \) are integral over \( R \).

Proof. This is immediate from Lemma 4.1 since one of the elements \( u_i \) will not be in \( J \).

The following two lemmas are a way of describing closed subschemes of \( \mathbf{P}^d_k \) cut out by one (nondegenerate) equation.

00Q4 Lemma 4.3. Let \( R \) be a ring. Let \( F(X,Y) \in R[X,Y] \) be homogeneous of degree \( d \). Assume that for every prime \( p \) of \( R \) at least one coefficient of \( F \) is not in \( p \). Let \( S = R[X,Y]/(F) \) as a graded ring. Then for all \( n \geq d \) the \( R \)-module \( S_n \) is finite locally free of rank \( d \).

Proof. The \( R \)-module \( S_n \) has a presentation

\[ R[X,Y]_{n-d} \to R[X,Y]_n \to S_n \to 0. \]

Thus by Algebra, Lemma 78.3 it is enough to show that multiplication by \( F \) induces an injective map \( \kappa(p)[X,Y] \to \kappa(p)[X,Y] \) for all primes \( p \). This is clear from the assumption that \( F \) does not map to the zero polynomial mod \( p \). The assertion on ranks is clear from this as well.

00Q5 Lemma 4.4. Let \( k \) be a field. Let \( F, G \in k[X,Y] \) be homogeneous of degrees \( d, e \). Assume \( F, G \) relatively prime. Then multiplication by \( G \) is injective on \( S = k[X,Y]/(F) \).

Proof. This is one way to define “relatively prime”. If you have another definition, then you can show it is equivalent to this one.

00Q6 Lemma 4.5. Let \( R \) be a ring. Let \( F(X,Y) \in R[X,Y] \) be homogeneous of degree \( d \). Let \( S = R[X,Y]/(F) \) as a graded ring. Let \( p \subset R \) be a prime such that some coefficient of \( F \) is not in \( p \). There exists an \( f \in R, f \notin p \), an integer \( e \), and a
G \in R[X,Y]_e \text{ such that multiplication by } G \text{ induces isomorphisms } (S_n)_f \to (S_{n+e})_f \text{ for all } n \geq d.

**Proof.** During the course of the proof we may replace } R \text{ by } R_f \text{ for } f \in R, \ f \not\in p \text{ (finitely often). As a first step we do such a replacement such that some coefficient of } F \text{ is invertible in } R. \text{ In particular the modules } S_n \text{ are now locally free of rank } d \text{ for } n \geq d \text{ by Lemma 4.3. Pick any } G \in R[X,Y]_e \text{ such that the image of } G \text{ in } \kappa(p)[X,Y] \text{ is relatively prime to the image of } F(X,Y) \text{ (this is possible for some } e). \text{ Apply Algebra, Lemma 78.3 to the map induced by multiplication by } G \text{ from } S_d \to S_{d+e}. \text{ By our choice of } G \text{ and Lemma 4.4 we see } S_d \otimes \kappa(p) \to S_{d+e} \otimes \kappa(p) \text{ is bijective. Thus, after replacing } R \text{ by } R_f \text{ for a suitable } f \text{ we may assume that } G : S_d \to S_{d+e} \text{ is bijective. This in turn implies that the image of } G \text{ in } \kappa(p')[X,Y] \text{ is relatively prime to the image of } F \text{ for all primes } p' \text{ of } R. \text{ And then by Algebra, Lemma 78.3 again we see that all the maps } G : S_d \to S_{d+e}, \ n \geq d \text{ are isomorphisms.} \] □

**Remark 4.6.** Let } R \text{ be a ring. Suppose that we have } F \in R[X,Y]_d \text{ and } G \in R[X,Y]_e \text{ such that, setting } S = R[X,Y]/(F) \text{ we have (1) } S_n \text{ is finite locally free of rank } d \text{ for all } n \geq d, \text{ and (2) multiplication by } G \text{ defines isomorphisms } S_n \to S_{n+e} \text{ for all } n \geq d. \text{ In this case we may define a finite, locally free } R\text{-algebra } A \text{ as follows:}

1. as an } R\text{-module } A = S_{cd}, \text{ and
2. multiplication } A \times A \to A \text{ is given by the rule that } H_1H_2 = H_3 \text{ if and only if } G^dH_3 = H_1H_2 \text{ in } S_{2cd}.

This makes sense because multiplication by } G^d \text{ induces a bijective map } S_{cd} \to S_{2cd}. \text{ It is easy to see that this defines a ring structure. Note the confusing fact that the element } G^d \text{ defines the unit element of the ring } A.

**Lemma 4.7.** Let } R \text{ be a ring, let } f \in R. \text{ Suppose we have } S, S' \text{ and the solid arrows forming the following commutative diagram of rings}

\[ \begin{array}{ccc}
S'' & \rightarrow & S' \\
\downarrow & & \downarrow \\
R & \rightarrow & S \\
\downarrow & & \downarrow \\
R_f & \rightarrow & S_f
\end{array} \]

Assume that } R_f \to S' \text{ is finite. Then we can find a finite ring map } R \to S'' \text{ and dotted arrows as in the diagram such that } S' = (S'')_f.

**Proof.** Namely, suppose that } S' \text{ is generated by } x_i \text{ over } R_f, \ i = 1, \ldots, w. \text{ Let } P_i(t) \in R_f[t] \text{ be a monic polynomial such that } P_i(x_i) = 0. \text{ Say } P_i \text{ has degree } d_i > 0. \text{ Write } P_i(t) = t^{d_i} + \sum_{j<d_i}(a_{ij}/f^n)t^j \text{ for some uniform } n. \text{ Also write the image of } x_i \text{ in } S_f \text{ as } g_i/f^n \text{ for suitable } g_i \in S. \text{ Then we know that the element } \xi_i = f^{nd_i}g_i^{d_i} + \sum_{j<d_i}f^{n(d_i-j)}a_{ij}g_i^j \text{ of } S \text{ is killed by a power of } f. \text{ Hence upon increasing } n \text{ to } n', \text{ which replaces } g_i \text{ by } f^{n'-n}g_i \text{ we may assume } \xi_i = 0. \text{ Then } S' \text{ is generated by the elements } f^n x_i, \text{ each of which is a zero of the monic polynomial } Q_i(t) = t^{d_i} + \sum_{j<d_i}f^{n(d_i-j)}a_{ij}t^j \text{ with coefficients in } R. \text{ Also, by construction } Q_i(f^n g_i) = 0 \text{ in } S. \text{ Thus we get a finite } R\text{-algebra } S'' = R[z_1, \ldots, z_w]/(Q_1(z_1), \ldots, Q_w(z_w)) \text{ which fits into a commutative diagram as above. The map } \alpha : S'' \to S \text{ maps } z_i \text{ to } f^n g_i \text{ and}
the map $\beta : S'' \to S'$ maps $z_i$ to $f^n x_i$. It may not yet be the case that $\beta$ induces an isomorphism $(S'')_f \cong S'$. For the moment we only know that this map is surjective. The problem is that there could be elements $h/f^n \in (S'')_f$ which map to zero in $S'$ but are not zero. In this case $\beta(h)$ is an element of $S$ such that $f^N \beta(h) = 0$ for some $N$. Thus $f^N h$ is an element of the ideal $J = \{ h \in S'' \mid \alpha(h) = 0 \text{ and } \beta(h) = 0 \}$ of $S''$. OK, and it is easy to see that $S''/J$ does the job. 

\[ \square \]

5. Formally smooth ring maps

07GD

Lemma 5.1. Let $R$ be a ring. Let $S$ be a $R$-algebra. If $S$ is of finite presentation and formally smooth over $R$ then $S$ is smooth over $R$.

Proof. See Algebra, Proposition 136.13

6. Cohomology

0BM0 The following lemma computes the cohomology sheaves of the derived limit in a special case.

0A08 Lemma 6.1. Let $(C, \mathcal{O})$ be a ringed site. Let $(K_n)$ be an inverse system of objects of $D(\mathcal{O})$. Let $\mathcal{B} \subset \text{Ob}(\mathcal{C})$ be a subset. Let $d \in \mathbb{N}$. Assume

- $(1)$ $K_n$ is an object of $D^+(\mathcal{O})$ for all $n$,
- $(2)$ for $q \in \mathbb{Z}$ there exists $n(q)$ such that $H^q(K_{n+1}) \to H^q(K_n)$ is an isomorphism for $n \geq n(q)$,
- $(3)$ every object of $\mathcal{C}$ has a covering whose members are elements of $\mathcal{B}$,
- $(4)$ for every $U \in \mathcal{B}$ we have $H^p(U, H^q(K_n)) = 0$ for $p > d$ and all $q$.

Then we have $H^m(R \lim K_n) = \lim H^m(K_n)$ for all $m \in \mathbb{Z}$.

Proof. Set $K = R \lim K_n$. Let $U \in \mathcal{B}$. For each $n$ there is a spectral sequence

$$H^p(U, H^q(K_n)) \Rightarrow H^{p+q}(U, K_n)$$

which converges as $K_n$ is bounded below, see Derived Categories, Lemma 21.3. If we fix $m \in \mathbb{Z}$, then we see from our assumption $(4)$ that only $H^p(U, H^q(K_n))$ contribute to $H^m(U, K_n)$ for $0 \leq p \leq d$ and $m - d \leq q \leq m$. By assumption $(2)$ this implies that $H^m(U, K_{n+1}) \to H^m(U, K_n)$ is an isomorphism as soon as $n \geq \max n(m), \ldots, n(m - d)$. The functor $R\Gamma(U, -)$ commutes with derived limits by Injectives, Lemma 13.6. Thus we have

$$H^m(U, K) = H^m(R \lim R\Gamma(U, K_n))$$

On the other hand we have just seen that the complexes $R\Gamma(U, K_n)$ have eventually constant cohomology groups. Thus by More on Algebra, Remark 68.16 we find that $H^m(U, K)$ is equal to $H^m(U, K_n)$ for all $n \gg 0$ for some bound independent of $U \in \mathcal{B}$. Pick such an $n$. Finally, recall that $H^m(K)$ is the sheafification of the presheaf $U \mapsto H^m(U, K)$ and $H^m(K_n)$ is the sheafification of the presheaf $U \mapsto H^m(U, K_n)$. On the elements of $\mathcal{B}$ these presheaves have the same values. Therefore assumption $(3)$ guarantees that the sheafifications are the same too. The lemma follows. \[ \square \]
7. Simplicial methods

Lemma 7.1. Assumptions and notation as in Simplicial, Lemma 32.1. There exists a section \( g : U \to V \) to the morphism \( f \) and the composition \( g \circ f \) is homotopy equivalent to the identity on \( V \). In particular, the morphism \( f \) is a homotopy equivalence.

Proof. Immediate from Simplicial, Lemmas 32.1 and 30.8

Lemma 7.2. Let \( C \) be a category with finite coproducts and finite limits. Let \( X \) be an object of \( C \). Let \( k \geq 0 \). The canonical map
\[
\text{Hom}(\Delta[k], X) \to \text{cosk}_1 \text{sk}_1 \text{Hom}(\Delta[k], X)
\]
is an isomorphism.

Proof. For any simplicial object \( V \) we have
\[
\text{Mor}(V, \text{cosk}_1 \text{sk}_1 \text{Hom}(\Delta[k], X)) = \text{Mor}(\text{sk}_1 V, \text{sk}_1 \text{Hom}(\Delta[k], X)) = \text{Mor}(i_1! \text{sk}_1 V, \text{Hom}(\Delta[k], X)) = \text{Mor}(i_1! \text{sk}_1 V \times \Delta[k], X)
\]
The first equality by the adjointness of \( \text{sk} \) and \( \text{cosk} \), the second equality by the adjointness of \( i_1! \) and \( \text{sk}_1 \), and the first equality by Simplicial, Definition 17.1 where the last \( X \) denotes the constant simplicial object with value \( X \). By Simplicial, Lemma 20.2 an element in this set depends only on the terms of degree 0 and 1 of \( i_1! \text{sk}_1 V \times \Delta[k] \). These agree with the degree 0 and 1 terms of \( V \times \Delta[k] \), see Simplicial, Lemma 21.3. Thus the set above is equal to \( \text{Mor}(V \times \Delta[k], X) = \text{Mor}(V, \text{Hom}(\Delta[k], X)) \).

Lemma 7.3. Let \( C \) be a category. Let \( X \) be an object of \( C \) such that the self products \( X \times \ldots \times X \) exist. Let \( k \geq 0 \) and let \( C[k] \) be as in Simplicial, Example 7.6. With notation as in Simplicial, Lemma 15.2 the canonical map
\[
\text{Hom}(C[k], X)_1 \to (\text{cosk}_0 \text{sk}_0 \text{Hom}(C[k], X))_1
\]
is identified with the map
\[
\prod_{\alpha : [k] \to [1]} X \to X \times X
\]
which is the projection onto the factors where \( \alpha \) is a constant map.

Proof. This is shown in the proof of Hypercoverings, Lemma 6.3

8. Obsolete lemmas on schemes

Lemma 8.1. Let \( (R, m, \kappa) \) be a local ring. Let \( X \subset \mathbb{P}_R^n \) be a closed subscheme. Assume that \( R = \Gamma(X, \mathcal{O}_X) \). Then the special fibre \( X_k \) is geometrically connected.

Proof. This is a special case of More on Morphisms, Theorem 38.4

Lemma 8.2. Let \( X \) be a Noetherian scheme. Let \( Z_0 \subset X \) be an irreducible closed subset with generic point \( \xi \). Let \( \mathcal{P} \) be a property of coherent sheaves on \( X \) such that
\begin{enumerate}
\item For any short exact sequence of coherent sheaves if two out of three of them have property \( \mathcal{P} \) then so does the third.
\end{enumerate}
(2) If \( P \) holds for a direct sum of coherent sheaves then it holds for both.

(3) For every integral closed subscheme \( Z \subset Z_0 \subset X \), \( Z \neq Z_0 \) and every quasi-coherent sheaf of ideals \( \mathcal{I} \subset \mathcal{O}_Z \) we have \( P \) for \( (Z \to X)_{\mathcal{I}} \).

(4) There exists some coherent sheaf \( \mathcal{G} \) on \( X \) such that
   (a) \( \text{Supp}(\mathcal{G}) = Z_0 \),
   (b) \( \mathcal{G}_\xi \) is annihilated by \( m_\xi \), and
   (c) property \( P \) holds for \( \mathcal{G} \).

Then property \( P \) holds for every coherent sheaf \( \mathcal{F} \) on \( X \) whose support is contained in \( Z_0 \).

**Proof.** The proof is a variant on the proof of Cohomology of Schemes, Lemma \( 12.5 \). In exactly the same manner as in that proof we see that any coherent sheaf whose support is strictly contained in \( Z_0 \) has property \( P \).

Consider a coherent sheaf \( \mathcal{G} \) as in (3). By Cohomology of Schemes, Lemma \( 12.2 \) there exists a sheaf of ideals \( \mathcal{I} \) on \( Z_0 \) and a short exact sequence

\[
0 \to ((Z_0 \to X)_{\mathcal{I}})^{\oplus r} \to \mathcal{G} \to \mathcal{Q} \to 0
\]

where the support of \( \mathcal{Q} \) is strictly contained in \( Z_0 \). In particular \( r > 0 \) and \( \mathcal{I} \) is nonzero because the support of \( \mathcal{G} \) is equal to \( Z \). Since \( \mathcal{Q} \) has property \( P \) we conclude that also \( ((Z_0 \to X)_{\mathcal{I}})^{\oplus r} \) has property \( P \). By (2) we deduce property \( P \) for \( (Z_0 \to X)_{\mathcal{I}} \). Slotting this into the proof of Cohomology of Schemes, Lemma \( 12.5 \) at the appropriate point gives the lemma. Some details omitted. \( \square \)

**Lemma 8.3.** Let \( X \) be a Noetherian scheme. Let \( P \) be a property of coherent sheaves on \( X \) such that

(1) For any short exact sequence of coherent sheaves if two out of three of them have property \( P \) then so does the third.

(2) If \( P \) holds for a direct sum of coherent sheaves then it holds for both.

(3) For every integral closed subscheme \( Z \subset X \) with generic point \( \xi \) there exists some coherent sheaf \( \mathcal{G} \) such that
   (a) \( \text{Supp}(\mathcal{G}) = Z \),
   (b) \( \mathcal{G}_\xi \) is annihilated by \( m_\xi \), and
   (c) property \( P \) holds for \( \mathcal{G} \).

Then property \( P \) holds for every coherent sheaf \( \mathcal{F} \) on \( X \) whose support is contained in \( Z_0 \).

**Proof.** This follows from Lemma 8.2 in exactly the same way that Cohomology of Schemes, Lemma \( 12.6 \) follows from Cohomology of Schemes, Lemma \( 12.5 \). \( \square \)

**Lemma 8.4.** Let \( X \) be a scheme. Let \( \mathcal{L} \) be an invertible \( \mathcal{O}_X \)-module. Let \( s \in \Gamma(X, \mathcal{L}) \) be a section. Let \( \mathcal{F}' \subset \mathcal{F} \) be quasi-coherent \( \mathcal{O}_X \)-modules. Assume that

(1) \( X \) is quasi-compact,
(2) \( \mathcal{F} \) is of finite type, and
(3) \( \mathcal{F}'|_{X_s} = \mathcal{F}|_{X_s} \).

Then there exists an \( n \geq 0 \) such that multiplication by \( s^n \) on \( \mathcal{F} \) factors through \( \mathcal{F}' \).

**Proof.** In other words we claim that \( s^n \mathcal{F} \subset \mathcal{F}' \otimes_{\mathcal{O}_X} \mathcal{L}^\otimes n \) for some \( n \geq 0 \). In other words, we claim that the quotient map \( \mathcal{F} \to \mathcal{F}/\mathcal{F}' \) becomes zero after multiplying by a power of \( s \). This follows from Properties, Lemma \( 17.3 \). \( \square \)
9. Functor of quotients

082R Lemma 9.1. Let $S = \text{Spec}(R)$ be an affine scheme. Let $X$ be an algebraic space over $S$. Let $q_i : F \to Q_i$, $i = 1, 2$ be surjective maps of quasi-coherent $\mathcal{O}_X$-modules. Assume $Q_1$ flat over $S$. Let $T \to S$ be a quasi-compact morphism of schemes such that there exists a factorization

\[
\begin{array}{ccc}
F_T & \xrightarrow{q_1_T} & Q_1_T \\
\downarrow & & \downarrow \\
Q_2_T & \xleftarrow{q_2_T} & Q_2_T
\end{array}
\]

Then exists a closed subscheme $Z \subset S$ such that (a) $T \to S$ factors through $Z$ and (b) $q_1, Z$ factors through $q_2, Z$. If $\text{Ker}(q_2)$ is a finite type $\mathcal{O}_X$-module and $X$ quasi-compact, then we can take $Z \to S$ of finite presentation.

Proof. Apply Quot, Lemma 7.5 to the map $\text{Ker}(q_2) \to Q_1$.

10. Spaces and fpqc coverings

0ARG The material here was made obsolete by Gabber’s argument showing that algebraic spaces satisfy the sheaf condition with respect to fpqc coverings. Please visit Properties of Spaces, Section 16.

03WA Lemma 10.1. Let $S$ be a scheme. Let $X$ be an algebraic space over $S$. Let \( \{T_i : T_i \to T\}_{i \in I} \) be a fpqc covering of schemes over $S$. Then the map

\[ \text{Mor}_S(T, X) \to \prod_{i \in I} \text{Mor}_S(T_i, X) \]

is injective.


03WB Lemma 10.2. Let $S$ be a scheme. Let $X$ be an algebraic space over $S$. Let $X = \bigcup_{j \in J} X_j$ be a Zariski covering, see Spaces, Definition 12.5. If each $X_j$ satisfies the sheaf property for the fpqc topology then $X$ satisfies the sheaf property for the fpqc topology.

Proof. This is true because all algebraic spaces satisfy the sheaf property for the fpqc topology, see Properties of Spaces, Proposition 16.1.

03WC Remark 10.3. This remark used to discuss to what extent the original proof of Lemma 10.3 (of December 18, 2009) generalizes.
11. Very reasonable algebraic spaces

Lemma 11.1. Let $S$ be a scheme. Let $X$ be a reasonable algebraic space over $S$. Then $|X|$ is Kolmogorov (see Topology, Definition 7.4).

Proof. Follows from the definitions and Decent Spaces, Lemma 10.8

In the rest of this section we make some remarks about very reasonable algebraic spaces. If there exists a scheme $U$ and a surjective, étale, quasi-compact morphism $U \to X$, then $X$ is very reasonable, see Decent Spaces, Lemma 4.7

Lemma 11.2. A scheme is very reasonable.

Proof. This is true because the identity map is a quasi-compact, surjective étale morphism.

Lemma 11.3. Let $S$ be a scheme. Let $X$ be an algebraic space over $S$. If there exists a Zariski open covering $X = \bigcup X_i$ such that each $X_i$ is very reasonable, then $X$ is very reasonable.

Proof. This is case (e) of Decent Spaces, Lemma 5.2

Lemma 11.4. An algebraic space which is Zariski locally quasi-separated is very reasonable. In particular any quasi-separated algebraic space is very reasonable.

Proof. This is one of the implications of Decent Spaces, Lemma 5.1

Lemma 11.5. Let $S$ be a scheme. Let $X$, $Y$ be algebraic spaces over $S$. Let $Y \to X$ be a representable morphism. If $X$ is very reasonable, so is $Y$.

Proof. This is case (e) of Decent Spaces, Lemma 5.3

Remark 11.6. Very reasonable algebraic spaces form a strictly larger collection than Zariski locally quasi-separated algebraic spaces. Consider an algebraic space of the form $X = [U/G]$ (see Spaces, Definition 14.4) where $G$ is a finite group acting without fixed points on a non-quasi-separated scheme $U$. Namely, in this case $U \times_X U = U \times G$ and clearly both projections to $U$ are quasi-compact, hence $X$ is very reasonable. On the other hand, the diagonal $U \times_X U \to U \times U$ is not quasi-compact, hence this algebraic space is not quasi-separated. Now, take $U$ the infinite affine space over a field $k$ of characteristic $\neq 2$ with zero doubled, see Schemes, Example 21.4. Let $0_1, 0_2$ be the two zeros of $U$. Let $G = \{+1, -1\}$, and let $-1$ act by $-1$ on all coordinates, and by switching $0_1$ and $0_2$. Then $[U/G]$ is very reasonable but not Zariski locally quasi-separated (details omitted).

Warning: The following lemma should be used with caution, as the schemes $U_i$ in it are not necessarily separated or even quasi-separated.

Lemma 11.7. Let $S$ be a scheme. Let $X$ be a very reasonable algebraic space over $S$. There exists a set of schemes $U_i$ and morphisms $U_i \to X$ such that

1. each $U_i$ is a quasi-compact scheme,
2. each $U_i \to X$ is étale,
3. both projections $U_i \times_X U_i \to U_i$ are quasi-compact, and
4. the morphism $\coprod U_i \to X$ is surjective (and étale).
This section gives an alternative construction of the cotangent complex of a morphism over $Y$ is the image of $A$ where it is understood that (a) $A$ is a morphism of schemes where

(1) $U$ is an open subscheme of $X$,

(2) $V$ is an open subscheme of $Y$, and

(3) there exists an isomorphism $A = V \times \text{Spec}(P)$ over $V$ where $P$ is a polynomial algebra over $\mathbb{Z}$ (on some set of variables).

In other words, $A$ is an (infinite dimensional) affine space over $V$. Morphisms are given by commutative diagrams.

**Notation.** An object of $\mathcal{C}_{X/Y}$, i.e., a diagram (12.0.1), is often denoted $U \to A$ where it is understood that (a) $U$ is an open subscheme of $X$, (b) $U \to A$ is a morphism over $Y$, (c) the image of the structure morphism $A \to Y$ is an open $V \subset Y$, and (d) $A \to V$ is an affine space. We'll write $U \to A/V$ to indicate $V \subset Y$ is the image of $A \to Y$. Recall that $X_{\text{Zar}}$ denotes the small Zariski site $X$. There are forgetful functors

$$\mathcal{C}_{X/Y} \to X_{\text{Zar}}, (U \to A) \mapsto U \quad \text{and} \quad \mathcal{C}_{X/Y} \to Y_{\text{Zar}}, (U \to A/V) \mapsto V.$$  

**Lemma 12.1.** Let $X \to Y$ be a morphism of schemes.

(1) The category $\mathcal{C}_{X/Y}$ is fibred over $X_{\text{Zar}}$.

(2) The category $\mathcal{C}_{X/Y}$ is fibred over $Y_{\text{Zar}}$.

(3) The category $\mathcal{C}_{X/Y}$ is fibred over the category of pairs $(U, V)$ where $U \subset X$, $V \subset Y$ are open and $f(U) \subset V$.

**Proof.** Ad (1). Given an object $U \to A$ of $\mathcal{C}_{X/Y}$ and a morphism $U' \to U$ of $X_{\text{Zar}}$ consider the object $i' : U' \to A$ of $\mathcal{C}_{X/Y}$ where $i'$ is the composition of $i$ and $U' \to U$. The morphism $(U' \to A) \to (U \to A)$ of $\mathcal{C}_{X/Y}$ is strongly cartesian over $X_{\text{Zar}}$.

Ad (2). Given an object $U \to A/V$ and $V' \to V$ we can set $U' = U \cap f^{-1}(V')$ and $A' = V' \times_Y A$ to obtain a strongly cartesian morphism $(U' \to A') \to (U \to A)$ over $V' \to V$.
Ad (3). Denote \((X/Y)_{\text{Zar}}\) the category in (3). Given \(U \to A/V\) and a morphism 
\((U', V') \to (U, V)\) in \((X/Y)_{\text{Zar}}\) we can consider \(A' = V' \times_V A\). Then the morphism 
\((U' \to A'/V') \to (U \to A/V)\) is strongly cartesian in \(\mathcal{C}_{X/Y}\) over \((X/Y)_{\text{Zar}}\). \(\square\)

We obtain a topology \(\tau_X\) on \(\mathcal{C}_{X/Y}\) by using the topology inherited from \(X_{\text{Zar}}\) (see Stacks, Section \[\text{10}\]). If not otherwise stated this is the topology on \(\mathcal{C}_{X/Y}\) we will consider. To be precise, a family of morphisms \(\{(U_i \to A_i) \to (U \to A)\}\) is a covering of \(\mathcal{C}_{X/Y}\) if and only if

1. \(U = \bigcup U_i\), and
2. \(A_i = A\) for all \(i\).

We obtain the same collection of sheaves if we allow \(A_i \cong A\) in (2). The functor \(u\) defines a morphism of topoi \(\pi : \mathcal{S}h(\mathcal{C}_{X/Y}) \to \mathcal{S}h(X_{\text{Zar}})\).

The site \(\mathcal{C}_{X/Y}\) comes with several sheaves of rings.

1. The sheaf \(\mathcal{O}\) given by the rule \((U \to A) \mapsto \mathcal{O}(A)\).
2. The sheaf \(\mathcal{O}_X = \pi^{-1}\mathcal{O}_X\) given by the rule \((U \to A) \mapsto \mathcal{O}(U)\).
3. The sheaf \(\mathcal{O}_Y\) given by the rule \((U \to A/V) \mapsto \mathcal{O}(V)\).

We obtain morphisms of ringed topoi

\[
\begin{array}{ccc}
\mathcal{S}h(\mathcal{C}_{X/Y}), \mathcal{O}_X & \xrightarrow{i} & \mathcal{S}h(\mathcal{C}_{X/Y}), \mathcal{O} \\
\downarrow \pi & & \downarrow \pi \\
\mathcal{S}h(X_{\text{Zar}}), \mathcal{O}_X & & \mathcal{S}h(X_{\text{Zar}}), \mathcal{O}
\end{array}
\]

(08T8 \[12.1.1\])

The morphism \(i\) is the identity on underlying topos and \(i^\sharp : \mathcal{O} \to \mathcal{O}_X\) is the obvious map. The map \(\pi\) is a special case of Cohomology on Sites, Situation \[29.1\]. An important role will be played in the following by the derived functors \(\mathcal{L}i_* : D(\mathcal{O}) \to D(\mathcal{O}_X)\) left adjoint to \(Ri_* = i_* : D(\mathcal{O}_X) \to D(\mathcal{O})\) and \(\mathcal{L}\pi_* : D(\mathcal{O}_X) \to D(\mathcal{O}_X)\) left adjoint to \(\pi^* = \pi^{-1} : D(\mathcal{O}_X) \to D(\mathcal{O}_X)\).

Remark 12.2. We obtain a second topology \(\tau_Y\) on \(\mathcal{C}_{X/Y}\) by taking the topology inherited from \(Y_{\text{Zar}}\). There is a third topology \(\tau_{X \to Y}\) where a family of morphisms \(\{(U_i \to A_i) \to (U \to A)\}\) is a covering if and only if \(U = \bigcup U_i, V = \bigcup V_i\) and \(A_i \cong V_i \times_V A\). This is the topology inherited from the topology on the site \((X/Y)_{\text{Zar}}\) whose underlying category is the category of pairs \((U, V)\) as in Lemma \[12.1\] part (3). The coverings of \((X/Y)_{\text{Zar}}\) are families \(\{(U_i, V_i) \to (U, V)\}\) such that \(U = \bigcup U_i\) and \(V = \bigcup V_i\). There are morphisms of topos

\[
\mathcal{S}h(\mathcal{C}_{X/Y}) = \mathcal{S}h(\mathcal{C}_{X/Y}, \tau_X) \leftarrow \mathcal{S}h(\mathcal{C}_{X/Y}, \tau_{X \to Y}) \to \mathcal{S}h(\mathcal{C}_{X/Y}, \tau_Y)
\]

(recall that \(\tau_X\) is our “default” topology). The pullback functors for these arrows are sheafification and pushforward is the identity on underlying presheaves. The diagram of topos

\[
\begin{array}{ccc}
\mathcal{S}h(X_{\text{Zar}}) & \xleftarrow{\pi} & \mathcal{S}h(\mathcal{C}_{X/Y}) \\
\downarrow f & & \downarrow \pi \\
\mathcal{S}h(Y_{\text{Zar}}) & \xleftarrow{\pi} & \mathcal{S}h(\mathcal{C}_{X/Y}, \tau_Y)
\end{array}
\]

is not commutative. Namely, the pullback of a nonzero abelian sheaf on \(Y\) is a nonzero abelian sheaf on \((\mathcal{C}_{X/Y}, \tau_{X \to Y})\), but we can certainly find examples where
such a sheaf pulls back to zero on $X$. Note that any presheaf $\mathcal{F}$ on $Y_{zar}$ gives a sheaf $\mathcal{E}$ on $\mathcal{C}_{Y/X}$ by the rule which assigns to $(U \to A/V)$ the set $\mathcal{F}(V)$. Even if $\mathcal{F}$ happens to be a sheaf it isn’t true in general that $\mathcal{E} = \pi^{-1} f^{-1} \mathcal{F}$. This is related to the noncommutativity of the diagram above, as we can describe $\mathcal{E}$ as the pushforward of the pullback of $\mathcal{F}$ to $\text{Sh}(\mathcal{C}_{X/Y}, \tau_{X \to Y})$ via the lower horizontal and right vertical arrows. An example is the sheaf $\mathcal{O}_Y$. But what is true is that there is a map $\mathcal{E} \to \pi^{-1} f^{-1} \mathcal{F}$ which is transformed (as we shall see later) into an isomorphism after applying $\pi$.

13. Deformations and obstructions of flat modules

In this section we sketch a construction of a deformation theory for the stack of coherent sheaves for any algebraic space $X$ over a ring $\Lambda$. This material is obsolete due to the improved discussion in Quot, Section \ref{section:deformation-theory}.

Our setup will be the following. We assume given

1. a ring $\Lambda$,
2. an algebraic space $X$ over $\Lambda$,
3. a $\Lambda$-algebra $A$, set $X_A = X \times_{\text{Spec}(\Lambda)} \text{Spec}(A)$, and
4. a finitely presented $\mathcal{O}_{X_A}$-module $\mathcal{F}$ flat over $A$.

In this situation we will consider all possible surjections

$$0 \to I \to A' \to A \to 0$$

where $A'$ is a $\Lambda$-algebra whose kernel $I$ is an ideal of square zero in $A'$. Given $A'$ we obtain a first order thickening $X_A \to X_{A'}$ of algebraic spaces over $\text{Spec}(\Lambda)$. For each of these we consider the problem of lifting $\mathcal{F}$ to a finitely presented module $\mathcal{F}'$ on $X_{A'}$ flat over $A'$. We would like to replicate the results of Deformation Theory, Lemma \ref{lemma:deformation-theory} in this setting.

To be more precise let $\text{Lift}(\mathcal{F}, A')$ denote the category of pairs $(\mathcal{F}', \alpha)$ where $\mathcal{F}'$ is a finitely presented module on $X_{A'}$ flat over $A'$ and $\alpha : \mathcal{F}'|_{X_A} \to \mathcal{F}$ is an isomorphism. Morphisms $(\mathcal{F}', \alpha_1) \to (\mathcal{F}', \alpha_2)$ are isomorphisms $\mathcal{F}'_1 \to \mathcal{F}'_2$ which are compatible with $\alpha_1$ and $\alpha_2$. The set of isomorphism classes of $\text{Lift}(\mathcal{F}, A')$ is denoted $\text{Lift}(\mathcal{F}, A')$.

Let $\mathcal{G}$ be a sheaf of $\mathcal{O}_X \otimes_A A$-modules on $X_{etale}$ flat over $A$. We introduce the category $\text{Lift}(\mathcal{G}, A')$ of pairs $(\mathcal{G}', \beta)$ where $\mathcal{G}'$ is a sheaf of $\mathcal{O}_X \otimes_A A'$-modules flat over $A'$ and $\beta$ is an isomorphism $\mathcal{G}' \otimes_{A'} A \to \mathcal{G}$.

Lemma \ref{lemma:deformation-theory}. Notation and assumptions as above. Let $p : X_A \to X$ denote the projection. Given $A'$ denote $p' : X_{A'} \to X$ the projection. The functor $p'_* \otimes_A A'$ induces an equivalence of categories between

1. the category $\text{Lift}(\mathcal{F}, A')$, and
2. the category $\text{Lift}(p_* \mathcal{F}, A')$.

Proof. \text{FIXME}. \qed

Let $\mathcal{H}$ be a sheaf of $\mathcal{O} \otimes_A A$-modules on $\mathcal{C}_{X/A}$ flat over $A$. We introduce the category $\text{Lift}_\mathcal{O}(\mathcal{H}, A')$ whose objects are pairs $(\mathcal{H}', \gamma)$ where $\mathcal{H}'$ is a sheaf of $\mathcal{O} \otimes_A A'$-modules flat over $A'$ and $\gamma : \mathcal{H}' \otimes_A A' \to \mathcal{H}$ is an isomorphism of $\mathcal{O} \otimes_A A$-modules.

Let $\mathcal{G}$ be a sheaf of $\mathcal{O}_X \otimes_A A$-modules on $X_{etale}$ flat over $A$. Consider the morphisms $i$ and $\pi$ of Cotangent, Equation \eqref{equation:co tangent}. Denote $\mathcal{G} = \pi^{-1}(\mathcal{G})$. It is simply given
by the rule \( (U \rightarrow A) \mapsto \mathcal{G}(U) \) hence it is a sheaf of \( \mathcal{O}_X \otimes_A A \)-modules. Denote \( i_* \mathcal{G} \) the same sheaf but viewed as a sheaf of \( \mathcal{O} \otimes_A A \)-modules.

**Lemma 13.2.** Notation and assumptions as above. The functor \( \pi_! \) induces an equivalence of categories between

1. the category \( \text{Lift}_{\mathcal{O}}(i_* \mathcal{G}, A') \), and
2. the category \( \text{Lift}(\mathcal{G}, A') \).

**Proof.** Fixme. \( \square \)

**Lemma 13.3.** Notation and assumptions as in Lemma 13.2. Consider the object

\[
L = L(A, X, A, \mathcal{G}) = L_{\pi_!}(L_i^*(i_* \mathcal{G}))
\]

of \( D(\mathcal{O}_X \otimes A) \). Given a surjection \( A' \rightarrow A \) of \( \Lambda \)-algebras with square zero kernel \( I \) we have

1. The category \( \text{Lift}(\mathcal{G}, A') \) is nonempty if and only if a certain class \( \xi \in \text{Ext}^2_{\mathcal{O}_X \otimes A}(L, \mathcal{G} \otimes A I) \) is zero.
2. If \( \text{Lift}(\mathcal{G}, A') \) is nonempty, then \( \text{Lift}(\mathcal{G}, A') \) is principal homogeneous under \( \text{Ext}^1_{\mathcal{O}_X \otimes A}(L, \mathcal{G} \otimes A I) \).
3. Given a lift \( \mathcal{G}' \), the set of automorphisms of \( \mathcal{G}' \) which pull back to \( \text{id}_\mathcal{G} \) is canonically isomorphic to \( \text{Ext}^0_{\mathcal{O}_X \otimes A}(L, \mathcal{G} \otimes A I) \).

**Proof.** Fixme. \( \square \)

Finally, we put everything together as follows.

**Proposition 13.4.** With \( \Lambda, X, A, F \) as above. There exists a canonical object

\[
L = L(A, X, A, F) = L_{\pi_!}(L_i^*(i_* (F)))
\]

of \( D(\mathcal{O}_X \otimes A) \). Given a surjection \( A' \rightarrow A \) of \( \Lambda \)-algebras with square zero kernel \( I \) we have

1. The category \( \text{Lift}(F, A') \) is nonempty if and only if a certain class \( \xi \in \text{Ext}^2_{\mathcal{O}_X \otimes A}(L, F \otimes A I) \) is zero.
2. If \( \text{Lift}(F, A') \) is nonempty, then \( \text{Lift}(F, A') \) is principal homogeneous under \( \text{Ext}^1_{\mathcal{O}_X \otimes A}(L, F \otimes A I) \).
3. Given a lift \( F' \), the set of automorphisms of \( F' \) which pull back to \( \text{id}_F \) is canonically isomorphic to \( \text{Ext}^0_{\mathcal{O}_X \otimes A}(L, F \otimes A I) \).

**Proof.** Fixme. \( \square \)

**Lemma 13.5.** In the situation of Proposition 13.4, if \( X \rightarrow \text{Spec}(\Lambda) \) is locally of finite type and \( \Lambda \) is Noetherian, then \( L \) is pseudo-coherent.

**Proof.** Fixme. \( \square \)

### 14. Modifications

Here is a obsolete result on the category of Restricted Power Series, Equation 13.0.1. Please visit Restricted Power Series, Section 13 for the current material.

**Lemma 14.1.** Let \( (A, m, \kappa) \) be a Noetherian local ring. The category of Restricted Power Series, Equation 13.0.1 for \( A \) is equivalent to the category Restricted Power Series, Equation 13.0.1 for the henselization \( A^h \) of \( A \).

**Proof.** This is a special case of Restricted Power Series, Lemma 13.3. \( \square \)
The following lemma on rational singularities is no longer needed in the chapter on resolving surface singularities.

**Lemma 14.2.** In Resolution of Surfaces, Situation 9.1. Let $M$ be a finite reflexive $A$-module. Let $M \otimes_A O_X$ denote the pullback of the associated $O_S$-module. Then $M \otimes_A O_X$ maps onto its double dual.

**Proof.** Let $F = (M \otimes_A O_X)^{**}$ be the double dual and let $F' \subset F$ be the image of the evaluation map $M \otimes_A O_X \to F$. Then we have a short exact sequence

$$0 \to F' \to F \to Q \to 0$$

Since $X$ is normal, the local rings $O_{X,x}$ are discrete valuation rings for points of codimension 1 (see Properties, Lemma 12.5). Hence $Q_x = 0$ for such points by More on Algebra, Lemma 17.3. Thus $Q$ is supported in finitely many closed points and is globally generated by Cohomology of Schemes, Lemma 9.10. We obtain the exact sequence

$$0 \to H^0(X, F') \to H^0(X, F) \to H^0(X, Q) \to 0$$

because $F'$ is generated by global sections (Resolution of Surfaces, Lemma 9.2). Since $X \to \text{Spec}(A)$ is an isomorphism over the complement of the closed point, and since $M$ is reflexive, we see that the maps

$$M \to H^0(X, F') \to H^0(X, F)$$

induce isomorphisms after localization at any nonmaximal prime of $A$. Hence these maps are isomorphisms by More on Algebra, Lemma 17.11 and the fact that reflexive modules over normal rings have property $(S_2)$ (More on Algebra, Lemma 17.14). Thus we conclude that $Q = 0$ as desired. \hfill $\square$

### 15. Intersection theory

**Lemma 15.1.** Let $(S, \delta)$ be as in Chow Homology, Situation 8.7. Let $X$ be locally of finite type over $S$. Let $X$ be integral and $n = \dim(X)$. Let $a \in \Gamma(X, O_X)$ be a nonzero function. Let $i : D = Z(a) \to X$ be the closed immersion of the zero scheme of $a$. Let $f \in R(X)^*$. In this case $i^* \text{div}_X(f) = 0$ in $A^n - 2(D)$.

**Proof.** Special case of Chow Homology, Lemma 29.1

### 16. Duplicate references

This section is a place where we collect duplicates.

**Lemma 16.1.** Let $S$ be a scheme. Let $X$ be an algebraic space over $S$. The map \{Spec($k$) \to $X$ monomorphism\} \to $|X|$ is injective.

**Proof.** This is a duplicate of Properties of Spaces, Lemma 4.11

**Theorem 16.2.** Let $S = \text{Spec}(K)$ with $K$ a field. Let $\overline{s}$ be a geometric point of $S$. Let $G = \text{Gal}_{\text{alg}(s)}$ denote the absolute Galois group. Then there is an equivalence of categories $\text{Sh}(S_{\text{etale}}) \to G\text{-Sets}$, $\mathcal{F} \mapsto \mathcal{F}_{\overline{s}}$.

**Proof.** This is a duplicate of Étale Cohomology, Theorem 56.3

**Remark 16.3.** You got here because of a duplicate tag. Please see Formal Deformation Theory, Section 11 for the actual content.
# 17. Other chapters

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## Schemes

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39. More on Groupoid Schemes
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43. Picard Schemes of Curves
44. Adequate Modules
45. Dualizing Complexes
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## Algebraic Spaces

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53. Properties of Algebraic Spaces
54. Morphisms of Algebraic Spaces
55. Decent Algebraic Spaces
56. Cohomology of Algebraic Spaces
57. Limits of Algebraic Spaces
58. Divisors on Algebraic Spaces
59. Algebraic Spaces over Fields
60. Topologies on Algebraic Spaces
61. Descent and Algebraic Spaces
62. Derived Categories of Spaces
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65. Groupoids in Algebraic Spaces
66. More on Groupoids in Spaces
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## Deformation Theory

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## Algebraic Stacks

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78. Sheaves on Algebraic Stacks
79. Criteria for Representability
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83. Morphisms of Algebraic Stacks
References