

DERIVED CATEGORIES OF SCHEMES

08CU

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1. Introduction

08CV

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In this chapter we discuss derived categories of modules on schemes. Most of the material discussed here can be found in [TT90], [BN93], [BV03], and [LN07]. Of course there are many other references.

2. Conventions

08CW If \mathcal{A} is an abelian category and M is an object of \mathcal{A} then we also denote M the object of $K(\mathcal{A})$ and/or $D(\mathcal{A})$ corresponding to the complex which has M in degree 0 and is zero in all other degrees.

If we have a ring A , then $K(A)$ denotes the homotopy category of complexes of A -modules and $D(A)$ the associated derived category. Similarly, if we have a ringed space (X, \mathcal{O}_X) the symbol $K(\mathcal{O}_X)$ denotes the homotopy category of complexes of \mathcal{O}_X -modules and $D(\mathcal{O}_X)$ the associated derived category.

3. Derived category of quasi-coherent modules

06YZ In this section we discuss the relationship between quasi-coherent modules and all modules on a scheme X . A reference is [TT90, Appendix B]. By the discussion in Schemes, Section 24 the embedding $QCoh(\mathcal{O}_X) \subset Mod(\mathcal{O}_X)$ exhibits $QCoh(\mathcal{O}_X)$ as a weak Serre subcategory of the category of \mathcal{O}_X -modules. Denote

$$D_{QCoh}(\mathcal{O}_X) \subset D(\mathcal{O}_X)$$

the subcategory of complexes whose cohomology sheaves are quasi-coherent, see Derived Categories, Section 13. Thus we obtain a canonical functor

06VT (3.0.1)
$$D(QCoh(\mathcal{O}_X)) \longrightarrow D_{QCoh}(\mathcal{O}_X)$$

see Derived Categories, Equation (13.1.1).

08DT **Lemma 3.1.** *Let X be a scheme. Then $D_{QCoh}(\mathcal{O}_X)$ has direct sums.*

Proof. By Injectives, Lemma 13.4 the derived category $D(\mathcal{O}_X)$ has direct sums and they are computed by taking termwise direct sums of any representatives. Thus it is clear that the cohomology sheaf of a direct sum is the direct sum of the cohomology sheaves as taking direct sums is an exact functor (in any Grothendieck abelian category). The lemma follows as the direct sum of quasi-coherent sheaves is quasi-coherent, see Schemes, Section 24. \square

We will need some information on derived limits. We warn the reader that in the lemma below the derived limit will typically not be an object of D_{QCoh} .

0A0J **Lemma 3.2.** *Let X be a scheme. Let (K_n) be an inverse system of $D_{QCoh}(\mathcal{O}_X)$ with derived limit $K = R\lim K_n$ in $D(\mathcal{O}_X)$. Assume $H^q(K_{n+1}) \rightarrow H^q(K_n)$ is surjective for all $q \in \mathbf{Z}$ and $n \geq 1$. Then*

- (1) $H^q(K) = \lim H^q(K_n)$,
- (2) $R\lim H^q(K_n) = \lim H^q(K_n)$, and
- (3) for every affine open $U \subset X$ we have $H^p(U, \lim H^q(K_n)) = 0$ for $p > 0$.

Proof. Let \mathcal{B} be the set of affine opens of X . Since $H^q(K_n)$ is quasi-coherent we have $H^p(U, H^q(K_n)) = 0$ for $U \in \mathcal{B}$ by Cohomology of Schemes, Lemma 2.2. Moreover, the maps $H^0(U, H^q(K_{n+1})) \rightarrow H^0(U, H^q(K_n))$ are surjective for $U \in \mathcal{B}$ by Schemes, Lemma 7.5. Part (1) follows from Cohomology, Lemma 32.11 whose conditions we have just verified. Parts (2) and (3) follow from Cohomology, Lemma 32.4. \square

The following lemma will help us to “compute” a right derived functor on an object of $D_{QCoh}(\mathcal{O}_X)$.

08D3 **Lemma 3.3.** *Let X be a scheme. Let E be an object of $D_{QCoh}(\mathcal{O}_X)$. Then the canonical map $E \rightarrow R\lim \tau_{\geq -n}E$ is an isomorphism¹.*

Proof. Denote $\mathcal{H}^i = H^i(E)$ the i th cohomology sheaf of E . Let \mathcal{B} be the set of affine open subsets of X . Then $H^p(U, \mathcal{H}^i) = 0$ for all $p > 0$, all $i \in \mathbf{Z}$, and all $U \in \mathcal{B}$, see Cohomology of Schemes, Lemma 2.2. Thus the lemma follows from Cohomology, Lemma 32.9. \square

08D4 **Lemma 3.4.** *Let X be a scheme. Let $F : Mod(\mathcal{O}_X) \rightarrow Ab$ be an additive functor and $N \geq 0$ an integer. Assume that*

- (1) F commutes with countable direct products,
- (2) $R^p F(\mathcal{F}) = 0$ for all $p \geq N$ and \mathcal{F} quasi-coherent.

Then for $E \in D_{QCoh}(\mathcal{O}_X)$ the maps $R^p F(E) \rightarrow R^p F(\tau_{\geq p-N+1}E)$ are isomorphisms.

Proof. By shifting the complex we see it suffices to prove the assertion for $p = 0$. Write $E_n = \tau_{\geq -n}E$. We have $E = R\lim E_n$, see Lemma 3.3. Thus $RF(E) = R\lim RF(E_n)$ in $D(Ab)$ by Injectives, Lemma 13.6. Thus we have a short exact sequence

$$0 \rightarrow R^1 \lim R^{-1}F(E_n) \rightarrow R^0 F(E) \rightarrow \lim R^0 F(E_n) \rightarrow 0$$

see More on Algebra, Remark 75.9. To finish the proof we will show that the term on the left is zero and that the term on the right equals $R^0 F(E_{N-1})$.

We have a distinguished triangle

$$H^{-n}(E)[n] \rightarrow E_n \rightarrow E_{n-1} \rightarrow H^{-n}(E)[n+1]$$

(Derived Categories, Remark 12.4) in $D(\mathcal{O}_X)$. Since $H^{-n}(E)$ is quasi-coherent we have

$$R^p F(H^{-n}(E)[n]) = R^{p+n} F(H^{-n}(E)) = 0$$

for $p+n \geq N$ and

$$R^p F(H^{-n}(E)[n+1]) = R^{p+n+1} F(H^{-n}(E)) = 0$$

for $p+n+1 \geq N$. We conclude that

$$R^p F(E_n) \rightarrow R^p F(E_{n-1})$$

is an isomorphism for all $n \gg p$ and an isomorphism for $n \geq N$ for $p = 0$. Thus the systems $R^p F(E_n)$ all satisfy the ML condition and $R^1 \lim$ gives zero (see discussion in More on Algebra, Section 75). Moreover, the system $R^0 F(\tau_{\geq -n}E)$ is constant starting with $n = N - 1$ as desired. \square

The following lemma is the key ingredient to many of the results in this chapter.

06Z0 **Lemma 3.5.** *Let $X = \text{Spec}(A)$ be an affine scheme. All the functors in the diagram*

$$\begin{array}{ccc} D(QCoh(\mathcal{O}_X)) & \xrightarrow{(3.0.1)} & D_{QCoh}(\mathcal{O}_X) \\ & \searrow \sim & \swarrow R\Gamma(X, -) \\ & & D(A) \end{array}$$

¹In particular, E has a K-injective representative as in Cohomology, Lemma 33.1.

are equivalences of triangulated categories. Moreover, for E in $D_{QCoh}(\mathcal{O}_X)$ we have $H^0(X, E) = H^0(X, H^0(E))$.

Proof. The functor $R\Gamma(X, -)$ gives a functor $D(\mathcal{O}_X) \rightarrow D(A)$ and hence by restriction a functor

$$06VU \quad (3.5.1) \quad R\Gamma(X, -) : D_{QCoh}(\mathcal{O}_X) \longrightarrow D(A).$$

We will show this functor is quasi-inverse to (3.0.1) via the equivalence between quasi-coherent modules on X and the category of A -modules.

Elucidation. Denote (Y, \mathcal{O}_Y) the one point space with sheaf of rings given by A . Denote $\pi : (X, \mathcal{O}_X) \rightarrow (Y, \mathcal{O}_Y)$ the obvious morphism of ringed spaces. Then $R\Gamma(X, -)$ can be identified with $R\pi_*$ and the functor (3.0.1) via the equivalence $Mod(\mathcal{O}_Y) = Mod_A = QCoh(\mathcal{O}_X)$ can be identified with $L\pi^* = \pi^* = \sim$ (see Modules, Lemma 10.5 and Schemes, Lemmas 7.1 and 7.5). Thus the functors

$$D(A) \begin{array}{c} \xrightarrow{\quad} \\ \xleftarrow{\quad} \end{array} D_{QCoh}(\mathcal{O}_X)$$

are adjoint (by Cohomology, Lemma 29.1). In particular we obtain canonical adjunction mappings

$$a : R\Gamma(\widetilde{X}, E) \longrightarrow E$$

for E in $D(\mathcal{O}_X)$ and

$$b : M^\bullet \longrightarrow R\Gamma(X, \widetilde{M^\bullet})$$

for M^\bullet a complex of A -modules.

Let E be an object of $D_{QCoh}(\mathcal{O}_X)$. We may apply Lemma 3.4 to the functor $F(-) = \Gamma(X, -)$ with $N = 1$ by Cohomology of Schemes, Lemma 2.2. Hence

$$R^0\Gamma(X, E) = R^0\Gamma(X, \tau_{\geq 0}E) = \Gamma(X, H^0(E))$$

(the last equality by definition of the canonical truncation). Using this we will show that the adjunction mappings a and b induce isomorphisms $H^0(a)$ and $H^0(b)$. Thus a and b are quasi-isomorphisms (as the statement is invariant under shifts) and the lemma is proved.

In both cases we use that \sim is an exact functor (Schemes, Lemma 5.4). Namely, this implies that

$$H^0\left(R\Gamma(\widetilde{X}, E)\right) = R^0\Gamma(\widetilde{X}, E) = \Gamma(\widetilde{X}, \widetilde{H^0(E)})$$

which is equal to $H^0(E)$ because $H^0(E)$ is quasi-coherent. Thus $H^0(a)$ is an isomorphism. For the other direction we have

$$H^0(R\Gamma(X, \widetilde{M^\bullet})) = R^0\Gamma(X, \widetilde{M^\bullet}) = \Gamma(X, H^0(\widetilde{M^\bullet})) = \Gamma(X, \widetilde{H^0(M^\bullet)}) = H^0(M^\bullet)$$

which proves that $H^0(b)$ is an isomorphism. \square

08DV **Lemma 3.6.** *Let $X = \text{Spec}(A)$ be an affine scheme. If K^\bullet is a K -flat complex of A -modules, then $\widetilde{K^\bullet}$ is a K -flat complex of \mathcal{O}_X -modules.*

Proof. By More on Algebra, Lemma 55.5 we see that $K^\bullet \otimes_A A_{\mathfrak{p}}$ is a K -flat complex of $A_{\mathfrak{p}}$ -modules for every $\mathfrak{p} \in \text{Spec}(A)$. Hence we conclude from Cohomology, Lemma 27.4 (and Schemes, Lemma 5.4) that $\widetilde{K^\bullet}$ is K -flat. \square

0DJK **Lemma 3.7.** *If $f : X \rightarrow Y$ is a morphism of affine schemes given by the ring map $A \rightarrow B$, then the diagram*

$$\begin{array}{ccc} D(A) & \longrightarrow & D_{QCoh}(\mathcal{O}_X) \\ \downarrow & & \downarrow Rf_* \\ D(B) & \longrightarrow & D_{QCoh}(\mathcal{O}_Y) \end{array}$$

commutes.

Proof. Follows from Lemma 3.5 using that $R\Gamma(Y, Rf_*K) = R\Gamma(X, K)$ by Cohomology, Lemma 30.5. \square

08DW **Lemma 3.8.** *Let $f : Y \rightarrow X$ be a morphism of schemes.*

- (1) *The functor Lf^* sends $D_{QCoh}(\mathcal{O}_X)$ into $D_{QCoh}(\mathcal{O}_Y)$.*
- (2) *If X and Y are affine and f is given by the ring map $A \rightarrow B$, then the diagram*

$$\begin{array}{ccc} D(B) & \longrightarrow & D_{QCoh}(\mathcal{O}_Y) \\ \uparrow -\otimes_A^L B & & \uparrow Lf^* \\ D(A) & \longrightarrow & D_{QCoh}(\mathcal{O}_X) \end{array}$$

commutes.

Proof. We first prove the diagram

$$\begin{array}{ccc} D(B) & \longrightarrow & D(\mathcal{O}_Y) \\ \uparrow -\otimes_A^L B & & \uparrow Lf^* \\ D(A) & \longrightarrow & D(\mathcal{O}_X) \end{array}$$

commutes. This is clear from Lemma 3.6 and the constructions of the functors in question. To see (1) let E be an object of $D_{QCoh}(\mathcal{O}_X)$. To see that Lf^*E has quasi-coherent cohomology sheaves we may work locally on X . Note that Lf^* is compatible with restricting to open subschemes. Hence we can assume that f is a morphism of affine schemes as in (2). Then we can apply Lemma 3.5 to see that E comes from a complex of A -modules. By the commutativity of the first diagram of the proof the same holds for Lf^*E and we conclude (1) is true. \square

08DX **Lemma 3.9.** *Let X be a scheme.*

- (1) *For objects K, L of $D_{QCoh}(\mathcal{O}_X)$ the derived tensor product $K \otimes_{\mathcal{O}_X}^L L$ is in $D_{QCoh}(\mathcal{O}_X)$.*
- (2) *If $X = \text{Spec}(A)$ is affine then*

$$\widetilde{M^\bullet} \otimes_{\mathcal{O}_X}^L \widetilde{K^\bullet} = \widetilde{M^\bullet \otimes_A^L K^\bullet}$$

for any pair of complexes of A -modules K^\bullet, M^\bullet .

Proof. The equality of (2) follows immediately from Lemma 3.6 and the construction of the derived tensor product. To see (1) let K, L be objects of $D_{QCoh}(\mathcal{O}_X)$. To check that $K \otimes^L L$ is in $D_{QCoh}(\mathcal{O}_X)$ we may work locally on X , hence we may assume $X = \text{Spec}(A)$ is affine. By Lemma 3.5 we may represent K and L by complexes of A -modules. Then part (2) implies the result. \square

4. Total direct image

08DY The following lemma is the analogue of Cohomology of Schemes, Lemma 4.5.

08D5 **Lemma 4.1.** *Let $f : X \rightarrow S$ be a morphism of schemes. Assume that f is quasi-separated and quasi-compact.*

- (1) *The functor Rf_* sends $D_{QCoh}(\mathcal{O}_X)$ into $D_{QCoh}(\mathcal{O}_S)$.*
- (2) *If S is quasi-compact, there exists an integer $N = N(X, S, f)$ such that for an object E of $D_{QCoh}(\mathcal{O}_X)$ with $H^m(E) = 0$ for $m > 0$ we have $H^m(Rf_*E) = 0$ for $m \geq N$.*
- (3) *In fact, if S is quasi-compact we can find $N = N(X, S, f)$ such that for every morphism of schemes $S' \rightarrow S$ the same conclusion holds for the functor $R(f')_*$ where $f' : X' \rightarrow S'$ is the base change of f .*

Proof. Let E be an object of $D_{QCoh}(\mathcal{O}_X)$. To prove (1) we have to show that Rf_*E has quasi-coherent cohomology sheaves. The question is local on S , hence we may assume S is quasi-compact. Pick $N = N(X, S, f)$ as in Cohomology of Schemes, Lemma 4.5. Thus $R^p f_* \mathcal{F} = 0$ for all quasi-coherent \mathcal{O}_X -modules \mathcal{F} and all $p \geq N$ and the same remains true after base change.

First, assume E is bounded below. We will show (1) and (2) and (3) hold for such E with our choice of N . In this case we can for example use the spectral sequence

$$R^p f_* H^q(E) \Rightarrow R^{p+q} f_* E$$

(Derived Categories, Lemma 21.3), the quasi-coherence of $R^p f_* H^q(E)$, and the vanishing of $R^p f_* H^q(E)$ for $p \geq N$ to see that (1), (2), and (3) hold in this case.

Next we prove (2) and (3). Say $H^m(E) = 0$ for $m > 0$. Let $U \subset S$ be affine open. By Cohomology of Schemes, Lemma 4.6 and our choice of N we have $H^p(f^{-1}(U), \mathcal{F}) = 0$ for $p \geq N$ and any quasi-coherent \mathcal{O}_X -module \mathcal{F} . Hence we may apply Lemma 3.4 to the functor $\Gamma(f^{-1}(U), -)$ to see that

$$R\Gamma(U, Rf_*E) = R\Gamma(f^{-1}(U), E)$$

has vanishing cohomology in degrees $\geq N$. Since this holds for all $U \subset S$ affine open we conclude that $H^m(Rf_*E) = 0$ for $m \geq N$.

Next, we prove (1) in the general case. Recall that there is a distinguished triangle

$$\tau_{\leq -n-1}E \rightarrow E \rightarrow \tau_{\geq -n}E \rightarrow (\tau_{\leq -n-1}E)[1]$$

in $D(\mathcal{O}_X)$, see Derived Categories, Remark 12.4. By (2) we see that $Rf_*\tau_{\leq -n-1}E$ has vanishing cohomology sheaves in degrees $\geq -n+N$. Thus, given an integer q we see that $R^q f_*E$ is equal to $R^q f_*\tau_{\geq -n}E$ for some n and the result above applies. \square

08DZ **Lemma 4.2.** *Let $f : X \rightarrow S$ be a quasi-separated and quasi-compact morphism of schemes. Then $Rf_* : D_{QCoh}(\mathcal{O}_X) \rightarrow D_{QCoh}(\mathcal{O}_S)$ commutes with direct sums.*

Proof. Let E_i be a family of objects of $D_{QCoh}(\mathcal{O}_X)$ and set $E = \bigoplus E_i$. We want to show that the map

$$\bigoplus Rf_*E_i \longrightarrow Rf_*E$$

is an isomorphism. We will show it induces an isomorphism on cohomology sheaves in degree 0 which will imply the lemma. Choose an integer N as in Lemma 4.1. Then $R^0 f_*E = R^0 f_*\tau_{\geq -N}E$ and $R^0 f_*E_i = R^0 f_*\tau_{\geq -N}E_i$ by the lemma cited.

Observe that $\tau_{\geq -N}E = \bigoplus \tau_{\geq -N}E_i$. Thus we may assume all of the E_i have vanishing cohomology sheaves in degrees $< -N$. Next we use the spectral sequences

$$R^p f_* H^q(E) \Rightarrow R^{p+q} f_* E \quad \text{and} \quad R^p f_* H^q(E_i) \Rightarrow R^{p+q} f_* E_i$$

(Derived Categories, Lemma 21.3) to reduce to the case of a direct sum of quasi-coherent sheaves. This case is handled by Cohomology of Schemes, Lemma 6.1. \square

5. Affine morphisms

0AVV In this section we collect some information about pushforward along an affine morphism of schemes.

08I8 **Lemma 5.1.** *Let $f : X \rightarrow S$ be an affine morphism of schemes. Then $Rf_* : D_{QCoh}(\mathcal{O}_X) \rightarrow D_{QCoh}(\mathcal{O}_S)$ reflects isomorphisms.*

Proof. The statement means that a morphism $\alpha : E \rightarrow F$ of $D_{QCoh}(\mathcal{O}_X)$ is an isomorphism if $Rf_*\alpha$ is an isomorphism. We may check this on cohomology sheaves. In particular, the question is local on S . Hence we may assume S and therefore X is affine. In this case the statement is clear from the description of the derived categories $D_{QCoh}(\mathcal{O}_X)$ and $D_{QCoh}(\mathcal{O}_S)$ given in Lemma 3.5. Some details omitted. \square

08I9 **Lemma 5.2.** *Let $f : X \rightarrow S$ be an affine morphism of schemes. For E in $D_{QCoh}(\mathcal{O}_S)$ we have $Rf_* Lf^* E = E \otimes_{\mathcal{O}_S}^{\mathbf{L}} f_* \mathcal{O}_X$.*

Proof. Since f is affine the map $f_* \mathcal{O}_X \rightarrow Rf_* \mathcal{O}_X$ is an isomorphism (Cohomology of Schemes, Lemma 2.3). There is a canonical map $E \otimes_{\mathcal{O}_S}^{\mathbf{L}} f_* \mathcal{O}_X = E \otimes_{\mathcal{O}_S}^{\mathbf{L}} Rf_* \mathcal{O}_X \rightarrow Rf_* Lf^* E$ adjoint to the map

$$Lf^*(E \otimes_{\mathcal{O}_S}^{\mathbf{L}} Rf_* \mathcal{O}_X) = Lf^* E \otimes_{\mathcal{O}_X}^{\mathbf{L}} Lf^* Rf_* \mathcal{O}_X \longrightarrow Lf^* E \otimes_{\mathcal{O}_X}^{\mathbf{L}} \mathcal{O}_X = Lf^* E$$

coming from $1 : Lf^* E \rightarrow Lf^* E$ and the canonical map $Lf^* Rf_* \mathcal{O}_X \rightarrow \mathcal{O}_X$. To check the map so constructed is an isomorphism we may work locally on S . Hence we may assume S and therefore X is affine. In this case the statement is clear from the description of the derived categories $D_{QCoh}(\mathcal{O}_X)$ and $D_{QCoh}(\mathcal{O}_S)$ and the functor Lf^* given in Lemmas 3.5 and 3.8. Some details omitted. \square

Let Y be a scheme. Let \mathcal{A} be a sheaf of \mathcal{O}_Y -algebras. We will denote $D_{QCoh}(\mathcal{A})$ the inverse image of $D_{QCoh}(\mathcal{O}_X)$ under the restriction functor $D(\mathcal{A}) \rightarrow D(\mathcal{O}_X)$. In other words, $K \in D(\mathcal{A})$ is in $D_{QCoh}(\mathcal{A})$ if and only if its cohomology sheaves are quasi-coherent as \mathcal{O}_X -modules. If \mathcal{A} is quasi-coherent itself this is the same as asking the cohomology sheaves to be quasi-coherent as \mathcal{A} -modules, see Morphisms, Lemma 11.6.

0AVW **Lemma 5.3.** *Let $f : X \rightarrow Y$ be an affine morphism of schemes. Then f_* induces an equivalence*

$$\Phi : D_{QCoh}(\mathcal{O}_X) \longrightarrow D_{QCoh}(f_* \mathcal{O}_X)$$

whose composition with $D_{QCoh}(f_* \mathcal{O}_X) \rightarrow D_{QCoh}(\mathcal{O}_Y)$ is $Rf_* : D_{QCoh}(\mathcal{O}_X) \rightarrow D_{QCoh}(\mathcal{O}_Y)$.

Proof. Recall that Rf_* is computed on an object $K \in D_{QCoh}(\mathcal{O}_X)$ by choosing a K-injective complex \mathcal{I}^\bullet of \mathcal{O}_X -modules representing K and taking $f_* \mathcal{I}^\bullet$. Thus we let $\Phi(K)$ be the complex $f_* \mathcal{I}^\bullet$ viewed as a complex of $f_* \mathcal{O}_X$ -modules. Denote $g : (X, \mathcal{O}_X) \rightarrow (Y, f_* \mathcal{O}_X)$ the obvious morphism of ringed spaces. Then g is a

flat morphism of ringed spaces (see below for a description of the stalks) and Φ is the restriction of Rg_* to $D_{QCoh}(\mathcal{O}_X)$. We claim that Lg^* is a quasi-inverse. First, observe that Lg^* sends $D_{QCoh}(f_*\mathcal{O}_X)$ into $D_{QCoh}(\mathcal{O}_X)$ because g^* transforms quasi-coherent modules into quasi-coherent modules (Modules, Lemma 10.4). To finish the proof it suffices to show that the adjunction mappings

$$Lg^*\Phi(K) = Lg^*Rg_*K \rightarrow K \quad \text{and} \quad M \rightarrow Rg_*Lg^*M = \Phi(Lg^*M)$$

are isomorphisms for $K \in D_{QCoh}(\mathcal{O}_X)$ and $M \in D_{QCoh}(f_*\mathcal{O}_X)$. This is a local question, hence we may assume Y and therefore X are affine.

Assume $Y = \text{Spec}(B)$ and $X = \text{Spec}(A)$. Let $\mathfrak{p} = x \in \text{Spec}(A) = X$ be a point mapping to $\mathfrak{q} = y \in \text{Spec}(B) = Y$. Then $(f_*\mathcal{O}_X)_y = A_{\mathfrak{q}}$ and $\mathcal{O}_{X,x} = A_{\mathfrak{p}}$ hence g is flat. Hence g^* is exact and $H^i(Lg^*M) = g^*H^i(M)$ for any M in $D(f_*\mathcal{O}_X)$. For $K \in D_{QCoh}(\mathcal{O}_X)$ we see that

$$H^i(\Phi(K)) = H^i(Rf_*K) = f_*H^i(K)$$

by the vanishing of higher direct images (Cohomology of Schemes, Lemma 2.3) and Lemma 3.4. Thus it suffice to show that

$$g^*g_*\mathcal{F} \rightarrow \mathcal{F} \quad \text{and} \quad \mathcal{G} \rightarrow g_*g^*\mathcal{F}$$

are isomorphisms where \mathcal{F} is a quasi-coherent \mathcal{O}_X -module and \mathcal{G} is a quasi-coherent $f_*\mathcal{O}_X$ -module. This follows from Morphisms, Lemma 11.6. \square

6. The coherator

08D6 Let X be a scheme. The *coherator* is a functor

$$Q_X : \text{Mod}(\mathcal{O}_X) \longrightarrow \text{QCoh}(\mathcal{O}_X)$$

which is right adjoint to the inclusion functor $\text{QCoh}(\mathcal{O}_X) \rightarrow \text{Mod}(\mathcal{O}_X)$. It exists for any scheme X and moreover the adjunction mapping $Q_X(\mathcal{F}) \rightarrow \mathcal{F}$ is an isomorphism for every quasi-coherent module \mathcal{F} , see Properties, Proposition 23.4. Since Q_X is left exact (as a right adjoint) we can consider its right derived extension

$$RQ_X : D(\mathcal{O}_X) \longrightarrow D(\text{QCoh}(\mathcal{O}_X)).$$

Since Q_X is right adjoint to the inclusion functor $\text{QCoh}(\mathcal{O}_X) \rightarrow \text{Mod}(\mathcal{O}_X)$ we see that RQ_X is right adjoint to the canonical functor $D(\text{QCoh}(\mathcal{O}_X)) \rightarrow D(\mathcal{O}_X)$ by Derived Categories, Lemma 28.5.

In this section we will study the functor RQ_X . In Section 20 we will study the (closely related) right adjoint to the inclusion functor $D_{QCoh}(\mathcal{O}_X) \rightarrow D(\mathcal{O}_X)$ (when it exists).

08D7 **Lemma 6.1.** *Let $f : X \rightarrow Y$ be an affine morphism of schemes. Then f_* defines a derived functor $f_* : D(\text{QCoh}(\mathcal{O}_X)) \rightarrow D(\text{QCoh}(\mathcal{O}_Y))$. This functor has the property that*

$$\begin{array}{ccc} D(\text{QCoh}(\mathcal{O}_X)) & \longrightarrow & D_{QCoh}(\mathcal{O}_X) \\ f_* \downarrow & & \downarrow Rf_* \\ D(\text{QCoh}(\mathcal{O}_Y)) & \longrightarrow & D_{QCoh}(\mathcal{O}_Y) \end{array}$$

commutes.

Proof. The functor $f_* : QCoh(\mathcal{O}_X) \rightarrow QCoh(\mathcal{O}_Y)$ is exact, see Cohomology of Schemes, Lemma 2.3. Hence f_* defines a derived functor $f_* : D(QCoh(\mathcal{O}_X)) \rightarrow D(QCoh(\mathcal{O}_Y))$ by simply applying f_* to any representative complex, see Derived Categories, Lemma 17.9. For any complex of \mathcal{O}_X -modules \mathcal{F}^\bullet there is a canonical map $f_*\mathcal{F}^\bullet \rightarrow Rf_*\mathcal{F}^\bullet$. To finish the proof we show this is a quasi-isomorphism when \mathcal{F}^\bullet is a complex with each \mathcal{F}^n quasi-coherent. As the statement is invariant under shifts it suffices to show that $H^0(f_*(\mathcal{F}^\bullet)) \rightarrow R^0f_*\mathcal{F}^\bullet$ is an isomorphism. The statement is local on Y hence we may assume Y affine. By Lemma 4.1 we have $R^0f_*\mathcal{F}^\bullet = R^0f_*\tau_{\geq -n}\mathcal{F}^\bullet$ for all sufficiently large n . Thus we may assume \mathcal{F}^\bullet bounded below. As each \mathcal{F}^n is f_* -acyclic by Cohomology of Schemes, Lemma 2.3 we see that $f_*\mathcal{F}^\bullet \rightarrow Rf_*\mathcal{F}^\bullet$ is a quasi-isomorphism by Leray's acyclicity lemma (Derived Categories, Lemma 17.7). \square

08D8 **Lemma 6.2.** *Let $f : X \rightarrow Y$ be a morphism of schemes. Assume f is quasi-compact, quasi-separated, and flat. Then, denoting*

$$\Phi : D(QCoh(\mathcal{O}_X)) \rightarrow D(QCoh(\mathcal{O}_Y))$$

the right derived functor of $f_ : QCoh(\mathcal{O}_X) \rightarrow QCoh(\mathcal{O}_Y)$ we have $RQ_Y \circ Rf_* = \Phi \circ RQ_X$.*

Proof. We will prove this by showing that $RQ_Y \circ Rf_*$ and $\Phi \circ RQ_X$ are right adjoint to the same functor $D(QCoh(\mathcal{O}_Y)) \rightarrow D(\mathcal{O}_X)$.

Since f is quasi-compact and quasi-separated, we see that f_* preserves quasi-coherence, see Schemes, Lemma 24.1. Recall that $QCoh(\mathcal{O}_X)$ is a Grothendieck abelian category (Properties, Proposition 23.4). Hence any K in $D(QCoh(\mathcal{O}_X))$ can be represented by a K-injective complex \mathcal{I}^\bullet of $QCoh(\mathcal{O}_X)$, see Injectives, Theorem 12.6. Then we can define $\Phi(K) = f_*\mathcal{I}^\bullet$.

Since f is flat, the functor f^* is exact. Hence f^* defines $f^* : D(\mathcal{O}_Y) \rightarrow D(\mathcal{O}_X)$ and also $f^* : D(QCoh(\mathcal{O}_Y)) \rightarrow D(QCoh(\mathcal{O}_X))$. The functor $f^* = Lf^* : D(\mathcal{O}_Y) \rightarrow D(\mathcal{O}_X)$ is left adjoint to $Rf_* : D(\mathcal{O}_X) \rightarrow D(\mathcal{O}_Y)$, see Cohomology, Lemma 29.1. Similarly, the functor $f^* : D(QCoh(\mathcal{O}_Y)) \rightarrow D(QCoh(\mathcal{O}_X))$ is left adjoint to $\Phi : D(QCoh(\mathcal{O}_X)) \rightarrow D(QCoh(\mathcal{O}_Y))$ by Derived Categories, Lemma 28.5.

Let A be an object of $D(QCoh(\mathcal{O}_Y))$ and E an object of $D(\mathcal{O}_X)$. Then

$$\begin{aligned} \mathrm{Hom}_{D(QCoh(\mathcal{O}_Y))}(A, RQ_Y(Rf_*E)) &= \mathrm{Hom}_{D(\mathcal{O}_Y)}(A, Rf_*E) \\ &= \mathrm{Hom}_{D(\mathcal{O}_X)}(f^*A, E) \\ &= \mathrm{Hom}_{D(QCoh(\mathcal{O}_X))}(f^*A, RQ_X(E)) \\ &= \mathrm{Hom}_{D(QCoh(\mathcal{O}_Y))}(A, \Phi(RQ_X(E))) \end{aligned}$$

This implies what we want. \square

08D9 **Lemma 6.3.** *Let $X = \mathrm{Spec}(A)$ be an affine scheme. Then*

- (1) $Q_X : \mathrm{Mod}(\mathcal{O}_X) \rightarrow QCoh(\mathcal{O}_X)$ is the functor which sends \mathcal{F} to the quasi-coherent \mathcal{O}_X -module associated to the A -module $\Gamma(X, \mathcal{F})$,
- (2) $RQ_X : D(\mathcal{O}_X) \rightarrow D(QCoh(\mathcal{O}_X))$ is the functor which sends E to the complex of quasi-coherent \mathcal{O}_X -modules associated to the object $R\Gamma(X, E)$ of $D(A)$,
- (3) restricted to $D_{QCoh}(\mathcal{O}_X)$ the functor RQ_X defines a quasi-inverse to (3.0.1).

Proof. The functor Q_X is the functor

$$\mathcal{F} \mapsto \Gamma(\widetilde{X}, \mathcal{F})$$

by Schemes, Lemma 7.1. This immediately implies (1) and (2). The third assertion follows from (the proof of) Lemma 3.5. \square

08DA **Definition 6.4.** Let X be a scheme. Let E be an object of $D(\mathcal{O}_X)$. Let $T \subset X$ be a closed subset. We say E is *supported on T* if the cohomology sheaves $H^i(E)$ are supported on T .

08DB **Proposition 6.5.** *Let X be a quasi-compact scheme with affine diagonal. Then the functor (3.0.1)*

$$D(QCoh(\mathcal{O}_X)) \longrightarrow D_{QCoh}(\mathcal{O}_X)$$

is an equivalence with quasi-inverse given by RQ_X .

Proof. In this proof we will denote $i_X : D(QCoh(\mathcal{O}_X)) \rightarrow D_{QCoh}(\mathcal{O}_X)$ the functor of the lemma. Let E be an object of $D_{QCoh}(\mathcal{O}_X)$ and let A be an object of $D(QCoh(\mathcal{O}_X))$. We have to show that the adjunction maps

$$RQ_X(i_X(A)) \rightarrow A \quad \text{and} \quad E \rightarrow i_X(RQ_X(E))$$

are isomorphisms. We will prove this by induction on n : the smallest integer $n \geq 0$ such that E and $i_X(A)$ are supported on a closed subset of X which is contained in the union of n affine opens of X .

Base case: $n = 0$. In this case $E = 0$, hence the map $E \rightarrow i_X(RQ_X(E))$ is an isomorphism. Similarly $i_X(A) = 0$. Thus the cohomology sheaves of $i_X(A)$ are zero. Since the inclusion functor $QCoh(\mathcal{O}_X) \rightarrow Mod(\mathcal{O}_X)$ is fully faithful and exact, we conclude that the cohomology objects of A are zero, i.e., $A = 0$ and $RQ_X(i_X(A)) \rightarrow A$ is an isomorphism as well.

Induction step. Suppose that E and $i_X(A)$ are supported on a closed subset T of X contained in $U_1 \cup \dots \cup U_n$ with $U_i \subset X$ affine open. Set $U = U_n$. The inclusion morphism $j : U \rightarrow X$ is flat and affine (Morphisms, Lemma 11.11). Consider the distinguished triangles

$$A \rightarrow j_*(A|_U) \rightarrow A' \rightarrow A[1] \quad \text{and} \quad E \rightarrow Rj_*(E|_U) \rightarrow E' \rightarrow E[1]$$

where j_* is as in Lemma 6.1. Note that $E \rightarrow Rj_*(E|_U)$ is a quasi-isomorphism over $U = U_n$. Since $i_X \circ j_* = Rj_* \circ i_U$ by Lemma 6.1 and since $i_X(A)|_U = i_U(A|_U)$ we see that $i_X(A) \rightarrow i_X(j_*(A|_U))$ is a quasi-isomorphism over U . Hence $i_X(A')$ and E' are supported on the closed subset $T \setminus U$ of X which is contained in $U_1 \cup \dots \cup U_{n-1}$. By induction hypothesis the statement is true for A' and E' . By Derived Categories, Lemma 4.3 it suffices to prove the maps

$$RQ_X(i_X(j_*(A|_U))) \rightarrow j_*(A|_U) \quad \text{and} \quad Rj_*(E|_U) \rightarrow i_X(RQ_X(Rj_*E|_U))$$

are isomorphisms. By Lemmas 6.1 and 6.2 we have

$$RQ_X(i_X(j_*(A|_U))) = RQ_X(Rj_*(i_U(A|_U))) = j_*RQ_U(i_U(A|_U))$$

and

$$i_X(RQ_X(Rj_*(E|_U))) = i_X(j_*RQ_U(E|_U)) = Rj_*(i_U(RQ_U(E|_U)))$$

Finally, the maps

$$RQ_U(i_U(A|_U)) \rightarrow A|_U \quad \text{and} \quad E|_U \rightarrow i_U(RQ_U(E|_U))$$

are isomorphisms by Lemma 6.3. The result follows. \square

09T6 **Remark 6.6.** Analyzing the proof of Proposition 6.5 we see that we have shown the following. Let X be a quasi-compact and quasi-separated scheme. Suppose that for every affine open $U \subset X$ the right derived functor

$$\Phi : D(QCoh(\mathcal{O}_U)) \rightarrow D(QCoh(\mathcal{O}_X))$$

of the left exact functor $j_* : QCoh(\mathcal{O}_U) \rightarrow QCoh(\mathcal{O}_X)$ fits into a commutative diagram

$$\begin{array}{ccc} D(QCoh(\mathcal{O}_U)) & \xrightarrow{i_U} & D_{QCoh}(\mathcal{O}_U) \\ \Phi \downarrow & & \downarrow Rj_* \\ D(QCoh(\mathcal{O}_X)) & \xrightarrow{i_X} & D_{QCoh}(\mathcal{O}_X) \end{array}$$

Then the functor (3.0.1)

$$D(QCoh(\mathcal{O}_X)) \longrightarrow D_{QCoh}(\mathcal{O}_X)$$

is an equivalence with quasi-inverse given by RQ_X .

0CRX **Lemma 6.7.** *Let $f : X \rightarrow Y$ be a morphism of schemes. Assume X and Y are quasi-compact and have affine diagonal. Then, denoting*

$$\Phi : D(QCoh(\mathcal{O}_X)) \rightarrow D(QCoh(\mathcal{O}_Y))$$

the right derived functor of $f_ : QCoh(\mathcal{O}_X) \rightarrow QCoh(\mathcal{O}_Y)$ the diagram*

$$\begin{array}{ccc} D(QCoh(\mathcal{O}_X)) & \longrightarrow & D_{QCoh}(\mathcal{O}_X) \\ \Phi \downarrow & & \downarrow Rf_* \\ D(QCoh(\mathcal{O}_Y)) & \longrightarrow & D_{QCoh}(\mathcal{O}_Y) \end{array}$$

is commutative.

Proof. Observe that the horizontal arrows in the diagram are equivalences of categories by Proposition 6.5. Hence we can identify these categories (and similarly for other quasi-compact schemes with affine diagonal). The statement of the lemma is that the canonical map $\Phi(K) \rightarrow Rf_*(K)$ is an isomorphism for all K in $D(QCoh(\mathcal{O}_X))$. Note that if $K_1 \rightarrow K_2 \rightarrow K_3 \rightarrow K_1[1]$ is a distinguished triangle in $D(QCoh(\mathcal{O}_X))$ and the statement is true for two-out-of-three, then it is true for the third.

Let $U \subset X$ be an affine open. Since the diagonal of X is affine, the inclusion morphism $j : U \rightarrow X$ is affine (Morphisms, Lemma 11.11). Similarly, the composition $g = f \circ j : U \rightarrow Y$ is affine. Let \mathcal{I}^\bullet be a K-injective complex in $QCoh(\mathcal{O}_U)$. Since $j_* : QCoh(\mathcal{O}_U) \rightarrow QCoh(\mathcal{O}_X)$ has an exact left adjoint $j^* : QCoh(\mathcal{O}_X) \rightarrow QCoh(\mathcal{O}_U)$ we see that $j_*\mathcal{I}^\bullet$ is a K-injective complex in $QCoh(\mathcal{O}_X)$, see Derived Categories, Lemma 29.9. It follows that

$$\Phi(j_*\mathcal{I}^\bullet) = f_*j_*\mathcal{I}^\bullet = g_*\mathcal{I}^\bullet$$

By Lemma 6.1 we see that $j_*\mathcal{I}^\bullet$ represents $Rj_*\mathcal{I}^\bullet$ and $g_*\mathcal{I}^\bullet$ represents $Rg_*\mathcal{I}^\bullet$. On the other hand, we have $Rf_* \circ Rj_* = Rg_*$. Hence $f_*j_*\mathcal{I}^\bullet$ represents $Rf_*(j_*\mathcal{I}^\bullet)$. We conclude that the lemma is true for any complex of the form $j_*\mathcal{G}^\bullet$ with \mathcal{G}^\bullet a complex of quasi-coherent modules on U . (Note that if $\mathcal{G}^\bullet \rightarrow \mathcal{I}^\bullet$ is a quasi-isomorphism, then $j_*\mathcal{G}^\bullet \rightarrow j_*\mathcal{I}^\bullet$ is a quasi-isomorphism as well since j_* is an exact functor on quasi-coherent modules.)

Let \mathcal{F}^\bullet be a complex of quasi-coherent \mathcal{O}_X -modules. Let $T \subset X$ be a closed subset such that the support of \mathcal{F}^p is contained in T for all p . We will use induction on the minimal number n of affine opens U_1, \dots, U_n such that $T \subset U_1 \cup \dots \cup U_n$. The base case $n = 0$ is trivial. If $n \geq 1$, then set $U = U_1$ and denote $j : U \rightarrow X$ the open immersion as above. We consider the map of complexes $c : \mathcal{F}^\bullet \rightarrow j_* j^* \mathcal{F}^\bullet$. We obtain two short exact sequences of complexes:

$$0 \rightarrow \text{Ker}(c) \rightarrow \mathcal{F}^\bullet \rightarrow \text{Im}(c) \rightarrow 0$$

and

$$0 \rightarrow \text{Im}(c) \rightarrow j_* j^* \mathcal{F}^\bullet \rightarrow \text{Coker}(c) \rightarrow 0$$

The complexes $\text{Ker}(c)$ and $\text{Coker}(c)$ are supported on $T \setminus U \subset U_2 \cup \dots \cup U_n$ and the result holds for them by induction. The result holds for $j_* j^* \mathcal{F}^\bullet$ by the discussion in the preceding paragraph. We conclude by looking at the distinguished triangles associated to the short exact sequences and using the initial remark of the proof. \square

0CRY **Remark 6.8** (Warning). Let X be a quasi-compact scheme with affine diagonal. Even though we know that $D(QCoh(\mathcal{O}_X)) = D_{QCoh}(\mathcal{O}_X)$ by Proposition 6.5 strange things can happen and it is easy to make mistakes with this material. One pitfall is to carelessly assume that this equality means derived functors are the same. For example, suppose we have a quasi-compact open $U \subset X$. Then we can consider the higher right derived functors

$$R^i(QCoh)\Gamma(U, -) : QCoh(\mathcal{O}_X) \rightarrow Ab$$

of the left exact functor $\Gamma(U, -)$. Since this is a universal δ -functor, and since the functors $H^i(U, -)$ (defined for all abelian sheaves on X) restricted to $QCoh(\mathcal{O}_X)$ form a δ -functor, we obtain canonical transformations

$$t^i : R^i(QCoh)\Gamma(U, -) \rightarrow H^i(U, -).$$

These transformations aren't in general isomorphisms even if $X = \text{Spec}(A)$ is affine! Namely, we have $R^1(QCoh)\Gamma(U, \tilde{I}) = 0$ if I an injective A -module by construction of right derived functors and the equivalence of $QCoh(\mathcal{O}_X)$ and Mod_A . But Examples, Lemma 39.2 shows there exists A, I , and U such that $H^1(U, \tilde{I}) \neq 0$.

7. The coherator for Noetherian schemes

09T1 In the case of Noetherian schemes we can use the following lemma.

09T2 **Lemma 7.1.** *Let X be a Noetherian scheme. Let \mathcal{J} be an injective object of $QCoh(\mathcal{O}_X)$. Then \mathcal{J} is a flasque sheaf of \mathcal{O}_X -modules.*

Proof. Let $U \subset X$ be an open subset and let $s \in \mathcal{J}(U)$ be a section. Let $\mathcal{I} \subset X$ be the quasi-coherent sheaf of ideals defining the reduced induced scheme structure on $X \setminus U$ (see Schemes, Definition 12.5). By Cohomology of Schemes, Lemma 10.4 the section s corresponds to a map $\sigma : \mathcal{I}^n \rightarrow \mathcal{J}$ for some n . As \mathcal{J} is an injective object of $QCoh(\mathcal{O}_X)$ we can extend σ to a map $\tilde{s} : \mathcal{O}_X \rightarrow \mathcal{J}$. Then \tilde{s} corresponds to a global section of \mathcal{J} restricting to s . \square

09T3 **Lemma 7.2.** *Let $f : X \rightarrow Y$ be a morphism of Noetherian schemes. Then f_* on quasi-coherent sheaves has a right derived extension $\Phi : D(QCoh(\mathcal{O}_X)) \rightarrow$*

$D(QCoh(\mathcal{O}_Y))$ such that the diagram

$$\begin{array}{ccc} D(QCoh(\mathcal{O}_X)) & \longrightarrow & D_{QCoh}(\mathcal{O}_X) \\ \Phi \downarrow & & \downarrow Rf_* \\ D(QCoh(\mathcal{O}_Y)) & \longrightarrow & D_{QCoh}(\mathcal{O}_Y) \end{array}$$

commutes.

Proof. Since X and Y are Noetherian schemes the morphism is quasi-compact and quasi-separated (see Properties, Lemma 5.4 and Schemes, Remark 21.18). Thus f_* preserve quasi-coherence, see Schemes, Lemma 24.1. Next, Let K be an object of $D(QCoh(\mathcal{O}_X))$. Since $QCoh(\mathcal{O}_X)$ is a Grothendieck abelian category (Properties, Proposition 23.4), we can represent K by a K-injective complex \mathcal{I}^\bullet such that each \mathcal{I}^n is an injective object of $QCoh(\mathcal{O}_X)$, see Injectives, Theorem 12.6. Thus we see that the functor Φ is defined by setting

$$\Phi(K) = f_*\mathcal{I}^\bullet$$

where the right hand side is viewed as an object of $D(QCoh(\mathcal{O}_Y))$. To finish the proof of the lemma it suffices to show that the canonical map

$$f_*\mathcal{I}^\bullet \longrightarrow Rf_*\mathcal{I}^\bullet$$

is an isomorphism in $D(\mathcal{O}_Y)$. To see this it suffices to prove the map induces an isomorphism on cohomology sheaves. Pick any $m \in \mathbf{Z}$. Let $N = N(X, Y, f)$ be as in Lemma 4.1. Consider the short exact sequence

$$0 \rightarrow \sigma_{\geq m-N-1}\mathcal{I}^\bullet \rightarrow \mathcal{I}^\bullet \rightarrow \sigma_{\leq m-N-2}\mathcal{I}^\bullet \rightarrow 0$$

of complexes of quasi-coherent sheaves on X . By Lemma 4.1 we see that the cohomology sheaves of $Rf_*\sigma_{\leq m-N-2}\mathcal{I}^\bullet$ are zero in degrees $\geq m-1$. Thus we see that $R^m f_*\mathcal{I}^\bullet$ is isomorphic to $R^m f_*\sigma_{\geq m-N-1}\mathcal{I}^\bullet$. In other words, we may assume that \mathcal{I}^\bullet is a bounded below complex of injective objects of $QCoh(\mathcal{O}_X)$. This follows from Leray's acyclicity lemma (Derived Categories, Lemma 17.7) via Cohomology, Lemma 13.5 and Lemma 7.1. \square

09T4 **Proposition 7.3.** *Let X be a Noetherian scheme. Then the functor (3.0.1)*

$$D(QCoh(\mathcal{O}_X)) \longrightarrow D_{QCoh}(\mathcal{O}_X)$$

is an equivalence with quasi-inverse given by RQ_X .

Proof. This follows using the exact same argument as in the proof of Proposition 6.5 using Lemma 7.2. See discussion in Remark 6.6. \square

8. Koszul complexes

08CX Let A be a ring and let f_1, \dots, f_r be a sequence of elements of A . We have defined the Koszul complex $K_\bullet(f_1, \dots, f_r)$ in More on Algebra, Definition 26.2. It is a chain complex sitting in degrees $r, \dots, 0$. We turn this into a cochain complex $K^\bullet(f_1, \dots, f_r)$ by setting $K^{-n}(f_1, \dots, f_r) = K_n(f_1, \dots, f_r)$ and using the same differentials. In the rest of this section all the complexes will be cochain complexes.

We define a complex $I^\bullet(f_1, \dots, f_r)$ such that we have a distinguished triangle

$$I^\bullet(f_1, \dots, f_r) \rightarrow A \rightarrow K^\bullet(f_1, \dots, f_r) \rightarrow I^\bullet(f_1, \dots, f_r)[1]$$

in $K(A)$. In other words, we set

$$I^i(f_1, \dots, f_r) = \begin{cases} K^{i-1}(f_1, \dots, f_r) & \text{if } i \leq 0 \\ 0 & \text{else} \end{cases}$$

and we use the negative of the differential on $K^\bullet(f_1, \dots, f_r)$. The maps in the distinguished triangle are the obvious ones. Note that $I^0(f_1, \dots, f_r) = A^{\oplus r} \rightarrow A$ is given by multiplication by f_i on the i th factor. Hence $I^\bullet(f_1, \dots, f_r) \rightarrow A$ factors as

$$I^\bullet(f_1, \dots, f_r) \rightarrow I \rightarrow A$$

where $I = (f_1, \dots, f_r)$. In fact, there is a short exact sequence

$$0 \rightarrow H^{-1}(K^\bullet(f_1, \dots, f_r)) \rightarrow H^0(I^\bullet(f_1, \dots, f_r)) \rightarrow I \rightarrow 0$$

and for every $i < 0$ we have $H^i(I^\bullet(f_1, \dots, f_r)) = H^{i-1}(K^\bullet(f_1, \dots, f_r))$. Observe that given a second sequence g_1, \dots, g_r of elements of A there are canonical maps

$$I^\bullet(f_1 g_1, \dots, f_r g_r) \rightarrow I^\bullet(f_1, \dots, f_r) \quad \text{and} \quad K^\bullet(f_1 g_1, \dots, f_r g_r) \rightarrow K^\bullet(f_1, \dots, f_r)$$

compatible with the maps described above. The first of these maps is given by multiplication by g_i on the i th summand of $I^0(f_1 g_1, \dots, f_r g_r) = A^{\oplus r}$. In particular, given f_1, \dots, f_r we obtain an inverse system of complexes

$$08CY \quad (8.0.1) \quad I^\bullet(f_1, \dots, f_r) \leftarrow I^\bullet(f_1^2, \dots, f_r^2) \leftarrow I^\bullet(f_1^3, \dots, f_r^3) \leftarrow \dots$$

which will play an important role in that which is to follow. To easily formulate the following lemmas we fix some notation.

08CZ **Situation 8.1.** Here A is a ring and f_1, \dots, f_r is a sequence of elements of A . We set $X = \text{Spec}(A)$ and $U = D(f_1) \cup \dots \cup D(f_r) \subset X$. We denote $\mathcal{U} : U = \bigcup_{i=1, \dots, r} D(f_i)$ the given open covering of U .

Our first lemma is that the complexes above can be used to compute the cohomology of quasi-coherent sheaves on U . Suppose given a complex I^\bullet of A -modules and an A -module M . Then we define $\text{Hom}_A(I^\bullet, M)$ to be the complex with n th term $\text{Hom}_A(I^{-n}, M)$ and differentials given as the contragredients of the differentials on I^\bullet .

08D0 **Lemma 8.2.** *In Situation 8.1. Let M be an A -module and denote \mathcal{F} the associated \mathcal{O}_X -module. Then there is a canonical isomorphism of complexes*

$$\text{colim}_e \text{Hom}_A(I^\bullet(f_1^e, \dots, f_r^e), M) \longrightarrow \check{\mathcal{C}}_{alt}^\bullet(\mathcal{U}, \mathcal{F})$$

functorial in M .

Proof. Recall that the alternating Čech complex is the subcomplex of the usual Čech complex given by alternating cochains, see Cohomology, Section 24. As usual we view a p -cochain in $\check{\mathcal{C}}_{alt}^\bullet(\mathcal{U}, \mathcal{F})$ as an alternating function s on $\{1, \dots, r\}^{p+1}$ whose value $s_{i_0 \dots i_p}$ at (i_0, \dots, i_p) lies in $M_{f_{i_0} \dots f_{i_p}} = \mathcal{F}(U_{i_0 \dots i_p})$. On the other hand, a p -cochain t in $\text{Hom}_A(I^\bullet(f_1^e, \dots, f_r^e), M)$ is given by a map $t : \wedge^{p+1}(A^{\oplus r}) \rightarrow M$. Write $[i] \in A^{\oplus r}$ for the i th basis element and write

$$[i_0, \dots, i_p] = [i_0] \wedge \dots \wedge [i_p] \in \wedge^{p+1}(A^{\oplus r})$$

Then we send t as above to s with

$$s_{i_0 \dots i_p} = \frac{t([i_0, \dots, i_p])}{f_{i_0}^e \dots f_{i_p}^e}$$

It is clear that s so defined is an alternating cochain. The construction of this map is compatible with the transition maps of the system as the transition map

$$I^\bullet(f_1^e, \dots, f_r^e) \leftarrow I^\bullet(f_1^{e+1}, \dots, f_r^{e+1}),$$

of the (8.0.1) sends $[i_0, \dots, i_p]$ to $f_{i_0} \dots f_{i_p}[i_0, \dots, i_p]$. It is clear from the description of the localizations $M_{f_{i_0} \dots f_{i_p}}$ in Algebra, Lemma 9.9 that these maps define an isomorphism of cochain modules in degree p in the limit. To finish the proof we have to show that the map is compatible with differentials. To see this recall that

$$\begin{aligned} d(s)_{i_0 \dots i_{p+1}} &= \sum_{j=0}^{p+1} (-1)^j s_{i_0 \dots \hat{i}_j \dots i_p} \\ &= \sum_{j=0}^{p+1} (-1)^j \frac{t([i_0, \dots, \hat{i}_j, \dots, i_{p+1}])}{f_{i_0}^e \dots \hat{f}_{i_j}^e \dots f_{i_{p+1}}^e} \end{aligned}$$

On the other hand, we have

$$\begin{aligned} \frac{d(t)([i_0, \dots, i_{p+1}])}{f_{i_0}^e \dots f_{i_{p+1}}^e} &= \frac{t(d[i_0, \dots, i_{p+1}])}{f_{i_0}^e \dots f_{i_{p+1}}^e} \\ &= \frac{\sum_j (-1)^j f_{i_j}^e t([i_0, \dots, \hat{i}_j, \dots, i_{p+1}])}{f_{i_0}^e \dots f_{i_{p+1}}^e} \end{aligned}$$

The two formulas agree by inspection. \square

Suppose given a finite complex I^\bullet of A -modules and a complex of A -modules M^\bullet . We obtain a double complex $H^{\bullet, \bullet} = \text{Hom}_A(I^\bullet, M^\bullet)$ where $H^{p,q} = \text{Hom}_A(I^p, M^q)$. The first differential comes from the differential on $\text{Hom}_A(I^\bullet, M^q)$ and the second from the differential on M^\bullet . Associated to this double complex is the total complex with degree n term given by

$$\bigoplus_{p+q=n} \text{Hom}_A(I^p, M^q)$$

and differential as in Homology, Definition 22.3. As our complex I^\bullet has only finitely many nonzero terms, the direct sum displayed above is finite. The conventions for taking the total complex associated to a Čech complex of a complex are as in Cohomology, Section 26.

08D1 **Lemma 8.3.** *In Situation 8.1. Let M^\bullet be a complex of A -modules and denote \mathcal{F}^\bullet the associated complex of \mathcal{O}_X -modules. Then there is a canonical isomorphism of complexes*

$$\text{colim}_e \text{Tot}(\text{Hom}_A(I^\bullet(f_1^e, \dots, f_r^e), M^\bullet)) \longrightarrow \text{Tot}(\check{\mathcal{C}}_{\text{alt}}^\bullet(\mathcal{U}, \mathcal{F}^\bullet))$$

functorial in M^\bullet .

Proof. Immediate from Lemma 8.2 and our conventions for taking associated total complexes. \square

08D2 **Lemma 8.4.** *In Situation 8.1. Let \mathcal{F}^\bullet be a complex of quasi-coherent \mathcal{O}_X -modules. Then there is a canonical isomorphism*

$$\text{Tot}(\check{\mathcal{C}}_{\text{alt}}^\bullet(\mathcal{U}, \mathcal{F}^\bullet)) \longrightarrow R\Gamma(U, \mathcal{F}^\bullet)$$

in $D(A)$ functorial in \mathcal{F}^\bullet .

Proof. Let \mathcal{B} be the set of affine opens of U . Since the higher cohomology groups of a quasi-coherent module on an affine scheme are zero (Cohomology of Schemes, Lemma 2.2) this is a special case of Cohomology, Lemma 34.2. \square

In Situation 8.1 denote I_e the object of $D(\mathcal{O}_X)$ corresponding to the complex of A -modules $I^\bullet(f_1^e, \dots, f_r^e)$ via the equivalence of Lemma 3.5. The maps (8.0.1) give a system

$$I_1 \leftarrow I_2 \leftarrow I_3 \leftarrow \dots$$

Moreover, there is a compatible system of maps $I_e \rightarrow \mathcal{O}_X$ which become isomorphisms when restricted to U . Thus we see that for every object E of $D(\mathcal{O}_X)$ there is a canonical map

$$08DC \quad (8.4.1) \quad \operatorname{colim}_e \operatorname{Hom}_{D(\mathcal{O}_X)}(I_e, E) \longrightarrow H^0(U, E)$$

constructed by sending a map $I_e \rightarrow E$ to its restriction to U and using that $\operatorname{Hom}_{D(\mathcal{O}_U)}(\mathcal{O}_U, E|_U) = H^0(U, E)$.

08DD **Proposition 8.5.** *In Situation 8.1. For every object E of $D_{Q\text{Coh}}(\mathcal{O}_X)$ the map (8.4.1) is an isomorphism.*

Proof. By Lemma 3.5 we may assume that E is given by a complex of quasi-coherent sheaves \mathcal{F}^\bullet . Let $M^\bullet = \Gamma(X, \mathcal{F}^\bullet)$ be the corresponding complex of A -modules. By Lemmas 8.3 and 8.4 we have quasi-isomorphisms

$$\operatorname{colim}_e \operatorname{Tot}(\operatorname{Hom}_A(I^\bullet(f_1^e, \dots, f_r^e), M^\bullet)) \longrightarrow \operatorname{Tot}(\check{C}_{alt}^\bullet(\mathcal{U}, \mathcal{F}^\bullet)) \longrightarrow R\Gamma(U, \mathcal{F}^\bullet)$$

Taking H^0 on both sides we obtain

$$\operatorname{colim}_e \operatorname{Hom}_{D(A)}(I^\bullet(f_1^e, \dots, f_r^e), M^\bullet) = H^0(U, E)$$

Since $\operatorname{Hom}_{D(A)}(I^\bullet(f_1^e, \dots, f_r^e), M^\bullet) = \operatorname{Hom}_{D(\mathcal{O}_X)}(I_e, E)$ by Lemma 3.5 the lemma follows. \square

In Situation 8.1 denote K_e the object of $D(\mathcal{O}_X)$ corresponding to the complex of A -modules $K^\bullet(f_1^e, \dots, f_r^e)$ via the equivalence of Lemma 3.5. Thus we have distinguished triangles

$$I_e \rightarrow \mathcal{O}_X \rightarrow K_e \rightarrow I_e[1]$$

and a system

$$K_1 \leftarrow K_2 \leftarrow K_3 \leftarrow \dots$$

compatible with the system (I_e) . Moreover, there is a compatible system of maps

$$K_e \rightarrow H^0(K_e) = \mathcal{O}_X/(f_1^e, \dots, f_r^e)$$

08E3 **Lemma 8.6.** *In Situation 8.1. Let E be an object of $D_{Q\text{Coh}}(\mathcal{O}_X)$. Assume that $H^i(E)|_U = 0$ for $i = -r+1, \dots, 0$. Then given $s \in H^0(X, E)$ there exists an $e \geq 0$ and a morphism $K_e \rightarrow E$ such that s is in the image of $H^0(X, K_e) \rightarrow H^0(X, E)$.*

Proof. Since U is covered by r affine opens we have $H^j(U, \mathcal{F}) = 0$ for $j \geq r$ and any quasi-coherent module (Cohomology of Schemes, Lemma 4.2). By Lemma 3.4 we see that $H^0(U, E)$ is equal to $H^0(U, \tau_{\geq -r+1}E)$. There is a spectral sequence

$$H^j(U, H^i(\tau_{\geq -r+1}E)) \Rightarrow H^{i+j}(U, \tau_{\geq -N}E)$$

see Derived Categories, Lemma 21.3. Hence $H^0(U, E) = 0$ by our assumed vanishing of cohomology sheaves of E . We conclude that $s|_U = 0$. Think of s as a morphism $\mathcal{O}_X \rightarrow E$ in $D(\mathcal{O}_X)$. By Proposition 8.5 the composition $I_e \rightarrow \mathcal{O}_X \rightarrow E$

is zero for some e . By the distinguished triangle $I_e \rightarrow \mathcal{O}_X \rightarrow K_e \rightarrow I_e[1]$ we obtain a morphism $K_e \rightarrow E$ such that s is the composition $\mathcal{O}_X \rightarrow K_e \rightarrow E$. \square

9. Pseudo-coherent and perfect complexes

08E4 In this section we make the connection between the general notions defined in Cohomology, Sections 40, 41, 42, and 43 and the corresponding notions for complexes of modules in More on Algebra, Sections 60, 61, and 67.

08E5 **Lemma 9.1.** *Let X be a scheme. If E is an m -pseudo-coherent object of $D(\mathcal{O}_X)$, then $H^i(E)$ is a quasi-coherent \mathcal{O}_X -module for $i > m$ and $H^m(E)$ is a quotient of a quasi-coherent \mathcal{O}_X -module. If E is pseudo-coherent, then E is an object of $D_{QCoh}(\mathcal{O}_X)$.*

Proof. Locally on X there exists a strictly perfect complex \mathcal{E}^\bullet such that $H^i(E)$ is isomorphic to $H^i(\mathcal{E}^\bullet)$ for $i > m$ and $H^m(E)$ is a quotient of $H^m(\mathcal{E}^\bullet)$. The sheaves \mathcal{E}^i are direct summands of finite free modules, hence quasi-coherent. The lemma follows. \square

08E7 **Lemma 9.2.** *Let $X = \text{Spec}(A)$ be an affine scheme. Let M^\bullet be a complex of A -modules and let E be the corresponding object of $D(\mathcal{O}_X)$. Then E is an m -pseudo-coherent (resp. pseudo-coherent) as an object of $D(\mathcal{O}_X)$ if and only if M^\bullet is m -pseudo-coherent (resp. pseudo-coherent) as a complex of A -modules.*

Proof. It is immediate from the definitions that if M^\bullet is m -pseudo-coherent, so is E . To prove the converse, assume E is m -pseudo-coherent. As $X = \text{Spec}(A)$ is quasi-compact with a basis for the topology given by standard opens, we can find a standard open covering $X = D(f_1) \cup \dots \cup D(f_n)$ and strictly perfect complexes \mathcal{E}_i^\bullet on $D(f_i)$ and maps $\alpha_i : \mathcal{E}_i^\bullet \rightarrow E|_{U_i}$ inducing isomorphisms on H^j for $j > m$ and surjections on H^m . By Cohomology, Lemma 40.8 after refining the open covering we may assume α_i is given by a map of complexes $\mathcal{E}_i^\bullet \rightarrow \widehat{M}^\bullet|_{U_i}$ for each i . By Modules, Lemma 14.6 the terms \mathcal{E}_i^n are finite locally free modules. Hence after refining the open covering we may assume each \mathcal{E}_i^n is a finite free \mathcal{O}_{U_i} -module. From the definition it follows that $M_{f_i}^\bullet$ is an m -pseudo-coherent complex of A_{f_i} -modules. We conclude by applying More on Algebra, Lemma 60.15.

The case “pseudo-coherent” follows from the fact that E is pseudo-coherent if and only if E is m -pseudo-coherent for all m (by definition) and the same is true for M^\bullet by More on Algebra, Lemma 60.5. \square

08E8 **Lemma 9.3.** *Let X be a Noetherian scheme. Let E be an object of $D_{QCoh}(\mathcal{O}_X)$. For $m \in \mathbf{Z}$ the following are equivalent*

- (1) $H^i(E)$ is coherent for $i \geq m$ and zero for $i \gg 0$, and
- (2) E is m -pseudo-coherent.

In particular, E is pseudo-coherent if and only if E is an object of $D_{Coh}^-(\mathcal{O}_X)$.

Proof. As X is quasi-compact we see that in both (1) and (2) the object E is bounded above. Thus the question is local on X and we may assume X is affine. Say $X = \text{Spec}(A)$ for some Noetherian ring A . In this case E corresponds to a complex of A -modules M^\bullet by Lemma 3.5. By Lemma 9.2 we see that E is m -pseudo-coherent if and only if M^\bullet is m -pseudo-coherent. On the other hand, $H^i(E)$ is coherent if and only if $H^i(M^\bullet)$ is a finite A -module (Properties, Lemma 16.1). Thus the result follows from More on Algebra, Lemma 60.18. \square

08E9 **Lemma 9.4.** *Let $X = \text{Spec}(A)$ be an affine scheme. Let M^\bullet be a complex of A -modules and let E be the corresponding object of $D(\mathcal{O}_X)$. Then*

- (1) *E has tor amplitude in $[a, b]$ if and only if M^\bullet has tor amplitude in $[a, b]$.*
- (2) *E has finite tor dimension if and only if M^\bullet has finite tor dimension.*

Proof. Part (2) follows trivially from part (1). In the proof of (1) we will use the equivalence $D(A) = D_{QCoh}(X)$ of Lemma 3.5 without further mention. Assume M^\bullet has tor amplitude in $[a, b]$. Then K^\bullet is isomorphic in $D(A)$ to a complex \widetilde{K}^\bullet of flat A -modules with $\widetilde{K}^i = 0$ for $i \notin [a, b]$, see More on Algebra, Lemma 61.3. Then E is isomorphic to \widetilde{K}^\bullet . Since each \widetilde{K}^i is a flat \mathcal{O}_X -module, we see that E has tor amplitude in $[a, b]$ by Cohomology, Lemma 42.3.

Assume that E has tor amplitude in $[a, b]$. Then E is bounded whence M^\bullet is in $K^-(A)$. Thus we may replace M^\bullet by a bounded above complex of A -modules. We may even choose a projective resolution and assume that M^\bullet is a bounded above complex of free A -modules. Then for any A -module N we have

$$E \otimes_{\mathcal{O}_X}^{\mathbf{L}} \widetilde{N} \cong \widetilde{M^\bullet} \otimes_{\mathcal{O}_X}^{\mathbf{L}} \widetilde{N} \cong \widetilde{M^\bullet \otimes_A N}$$

in $D(\mathcal{O}_X)$. Thus the vanishing of cohomology sheaves of the left hand side implies M^\bullet has tor amplitude in $[a, b]$. \square

0DHY **Lemma 9.5.** *Let $f : X \rightarrow S$ be a morphism of affine schemes corresponding to the ring map $R \rightarrow A$. Let M^\bullet be a complex of A -modules and let E be the corresponding object of $D(\mathcal{O}_X)$. Then*

- (1) *E as an object of $D(f^{-1}\mathcal{O}_S)$ has tor amplitude in $[a, b]$ if and only if M^\bullet has tor amplitude in $[a, b]$ as an object of $D(R)$.*
- (2) *E locally has finite tor dimension as an object of $D(f^{-1}\mathcal{O}_S)$ if and only if M^\bullet has finite tor dimension as an object of $D(R)$.*

Proof. Consider a prime $\mathfrak{q} \subset A$ lying over $\mathfrak{p} \subset R$. Let $x \in X$ and $s = f(x) \in S$ be the corresponding points. Then $(f^{-1}\mathcal{O}_S)_x = \mathcal{O}_{S,s} = R_{\mathfrak{p}}$ and $E_x = M_{\mathfrak{q}}^\bullet$. Keeping this in mind we can see the equivalence as follows.

If M^\bullet has tor amplitude in $[a, b]$ as a complex of R -modules, then the same is true for the localization of M^\bullet at any prime of A . Then we conclude by Cohomology, Lemma 42.5 that E has tor amplitude in $[a, b]$ as a complex of sheaves of $f^{-1}\mathcal{O}_S$ -modules. Conversely, assume that E has tor amplitude in $[a, b]$ as an object of $D(f^{-1}\mathcal{O}_S)$. We conclude (using the last cited lemma) that $M_{\mathfrak{q}}^\bullet$ has tor amplitude in $[a, b]$ as a complex of $R_{\mathfrak{p}}$ -modules for every prime $\mathfrak{q} \subset A$ lying over $\mathfrak{p} \subset R$. By More on Algebra, Lemma 61.15 we find that M^\bullet has tor amplitude in $[a, b]$ as a complex of R -modules. This finishes the proof of (1).

Since X is quasi-compact, if E locally has finite tor dimension as a complex of $f^{-1}\mathcal{O}_S$ -modules, then actually E has tor amplitude in $[a, b]$ for some a, b as a complex of $f^{-1}\mathcal{O}_S$ -modules. Thus (2) follows from (1). \square

08EA **Lemma 9.6.** *Let X be a quasi-separated scheme. Let E be an object of $D_{QCoh}(\mathcal{O}_X)$. Let $a \leq b$. The following are equivalent*

- (1) *E has tor amplitude in $[a, b]$, and*
- (2) *for all \mathcal{F} in $QCoh(\mathcal{O}_X)$ we have $H^i(E \otimes_{\mathcal{O}_X}^{\mathbf{L}} \mathcal{F}) = 0$ for $i \notin [a, b]$.*

Proof. It is clear that (1) implies (2). Assume (2). Let $U \subset X$ be an affine open. As X is quasi-separated the morphism $j : U \rightarrow X$ is quasi-compact and separated, hence j_* transforms quasi-coherent modules into quasi-coherent modules (Schemes, Lemma 24.1). Thus the functor $QCoh(\mathcal{O}_X) \rightarrow QCoh(\mathcal{O}_U)$ is essentially surjective. It follows that condition (2) implies the vanishing of $H^i(E|_U \otimes_{\mathcal{O}_U}^{\mathbf{L}} \mathcal{G})$ for $i \notin [a, b]$ for all quasi-coherent \mathcal{O}_U -modules \mathcal{G} . Write $U = \text{Spec}(A)$ and let M^\bullet be the complex of A -modules corresponding to $E|_U$ by Lemma 3.5. We have just shown that $M^\bullet \otimes_A^{\mathbf{L}} N$ has vanishing cohomology groups outside the range $[a, b]$, in other words M^\bullet has tor amplitude in $[a, b]$. By Lemma 9.4 we conclude that $E|_U$ has tor amplitude in $[a, b]$. This proves the lemma. \square

08EB **Lemma 9.7.** *Let $X = \text{Spec}(A)$ be an affine scheme. Let M^\bullet be a complex of A -modules and let E be the corresponding object of $D(\mathcal{O}_X)$. Then E is a perfect object of $D(\mathcal{O}_X)$ if and only if M^\bullet is perfect as an object of $D(A)$.*

Proof. This is a logical consequence of Lemmas 9.2 and 9.4, Cohomology, Lemma 43.5, and More on Algebra, Lemma 67.2. \square

As a consequence of our description of pseudo-coherent complexes on schemes we can prove certain internal homs are quasi-coherent.

0A6H **Lemma 9.8.** *Let X be a scheme.*

- (1) *If L is in $D_{QCoh}^+(\mathcal{O}_X)$ and K in $D(\mathcal{O}_X)$ is pseudo-coherent, then $R\mathcal{H}om(K, L)$ is in $D_{QCoh}(\mathcal{O}_X)$ and locally bounded below.*
- (2) *If L is in $D_{QCoh}(\mathcal{O}_X)$ and K in $D(\mathcal{O}_X)$ is perfect, then $R\mathcal{H}om(K, L)$ is in $D_{QCoh}(\mathcal{O}_X)$.*
- (3) *If $X = \text{Spec}(A)$ is affine and $K, L \in D(A)$ then*

$$R\mathcal{H}om(\widetilde{K}, \widetilde{L}) = R\widetilde{\text{Hom}}(K, L)$$

in the following two cases

- (a) *K is pseudo-coherent and L is bounded below,*
- (b) *K is perfect and L arbitrary.*
- (4) *If $X = \text{Spec}(A)$ and K, L are in $D(A)$, then the n th cohomology sheaf of $R\mathcal{H}om(\widetilde{K}, \widetilde{L})$ is the sheaf associated to the presheaf*

$$X \supset D(f) \longmapsto \text{Ext}_{A_f}^n(K \otimes_A A_f, L \otimes_A A_f)$$

for $f \in A$.

Proof. The construction of the internal hom in the derived category of \mathcal{O}_X commutes with localization (see Cohomology, Section 36). Hence to prove (1) and (2) we may replace X by an affine open. By Lemmas 3.5, 9.2, and 9.7 in order to prove (1) and (2) it suffices to prove (3).

Part (3) follows from the computation of the internal hom of Cohomology, Lemma 40.10 by representing K by a bounded above (resp. finite) complex of finite projective A -modules and L by a bounded below (resp. arbitrary) complex of A -modules.

To prove (4) recall that on any ringed space the n th cohomology sheaf of $R\mathcal{H}om(A, B)$ is the sheaf associated to the presheaf

$$U \mapsto \text{Hom}_{D(U)}(A|_U, B|_U[n]) = \text{Ext}_{D(\mathcal{O}_U)}^n(A|_U, B|_U)$$

See Cohomology, Section 36. On the other hand, the restriction of \widetilde{K} to a principal open $D(f)$ is the image of $K \otimes_A A_f$ and similarly for L . Hence (4) follows from the equivalence of categories of Lemma 3.5. \square

0ATN **Lemma 9.9.** *Let X be a scheme. Let K, L, M be objects of $D_{QCoh}(\mathcal{O}_X)$. The map*

$$K \otimes_{\mathcal{O}_X}^L R\mathcal{H}om(M, L) \longrightarrow R\mathcal{H}om(M, K \otimes_{\mathcal{O}_X}^L L)$$

of Cohomology, Lemma 36.7 is an isomorphism in the following cases

- (1) M perfect, or
- (2) K is perfect, or
- (3) M is pseudo-coherent, $L \in D^+(\mathcal{O}_X)$, and K has finite tor dimension.

Proof. Lemma 9.8 reduces cases (1) and (3) to the affine case which is treated in More on Algebra, Lemma 84.4. (You also have to use Lemmas 9.2, 9.7, and 9.4 to do the translation into algebra.) If K is perfect but no other assumptions are made, then we do not know that either side of the arrow is in $D_{QCoh}(\mathcal{O}_X)$ but the result is still true because we can work locally and reduce to the case that K is a finite complex of finite free modules in which case it is clear. \square

10. Derived category of coherent modules

08E0 Let X be a locally Noetherian scheme. In this case the category $Coh(\mathcal{O}_X) \subset Mod(\mathcal{O}_X)$ of coherent \mathcal{O}_X -modules is a weak Serre subcategory, see Homology, Section 9 and Cohomology of Schemes, Lemma 9.2. Denote

$$D_{Coh}(\mathcal{O}_X) \subset D(\mathcal{O}_X)$$

the subcategory of complexes whose cohomology sheaves are coherent, see Derived Categories, Section 13. Thus we obtain a canonical functor

08E1 (10.0.1)
$$D(Coh(\mathcal{O}_X)) \longrightarrow D_{Coh}(\mathcal{O}_X)$$

see Derived Categories, Equation (13.1.1).

08E2 **Lemma 10.1.** *Let S be a Noetherian scheme. Let $f : X \rightarrow S$ be a morphism of schemes which is locally of finite type. Let E be an object of $D_{Coh}^b(\mathcal{O}_X)$ such that the support of $H^i(E)$ is proper over S for all i . Then Rf_*E is an object of $D_{Coh}^b(\mathcal{O}_S)$.*

Proof. Consider the spectral sequence

$$R^p f_* H^q(E) \Rightarrow R^{p+q} f_* E$$

see Derived Categories, Lemma 21.3. By assumption and Cohomology of Schemes, Lemma 25.10 the sheaves $R^p f_* H^q(E)$ are coherent. Hence $R^{p+q} f_* E$ is coherent, i.e., $E \in D_{Coh}(\mathcal{O}_S)$. Boundedness from below is trivial. Boundedness from above follows from Cohomology of Schemes, Lemma 4.5 or from Lemma 4.1. \square

0D0B **Lemma 10.2.** *Let S be a Noetherian scheme. Let $f : X \rightarrow S$ be a morphism of schemes which is locally of finite type. Let E be an object of $D_{Coh}^+(\mathcal{O}_X)$ such that the support of $H^i(E)$ is proper over S for all i . Then Rf_*E is an object of $D_{Coh}^+(\mathcal{O}_S)$.*

Proof. The proof is the same as the proof of Lemma 10.1. You can also deduce it from Lemma 10.1 by considering what the exact functor Rf_* does to the distinguished triangles $\tau_{\leq a} E \rightarrow E \rightarrow \tau_{\geq a+1} E \rightarrow \tau_{\leq a} E[1]$. \square

0D0C **Lemma 10.3.** *Let X be a locally Noetherian scheme. If L is in $D_{\text{Coh}}^+(\mathcal{O}_X)$ and K in $D_{\text{Coh}}^-(\mathcal{O}_X)$, then $R\mathcal{H}om(K, L)$ is in $D_{\text{Coh}}^+(\mathcal{O}_X)$.*

Proof. It suffices to prove this when X is the spectrum of a Noetherian ring A . By Lemma 9.3 we see that K is pseudo-coherent. Then we can use Lemma 9.8 to translate the problem into the following algebra problem: for $L \in D_{\text{Coh}}^+(A)$ and K in $D_{\text{Coh}}^-(A)$, then $R\mathcal{H}om(K, L)$ is in $D_{\text{Coh}}^+(A)$. Since L is bounded below and K is bounded below there is a convergent spectral sequence

$$\text{Ext}_A^p(K, H^q(L)) \Rightarrow \text{Ext}_A^{p+q}(K, L)$$

and there are convergent spectral sequences

$$\text{Ext}_A^i(H^{-j}(K), H^q(L)) \Rightarrow \text{Ext}_A^{i+j}(K, H^q(L))$$

This finishes the proof as the modules $\text{Ext}_A^p(M, N)$ are finite for finite A -modules M, N by Algebra, Lemma 70.9. \square

0D0D **Lemma 10.4.** *Let A be a Noetherian ring. Let X be a proper scheme over A . For L in $D_{\text{Coh}}^+(\mathcal{O}_X)$ and K in $D_{\text{Coh}}^-(\mathcal{O}_X)$, the A -modules $\text{Ext}_{\mathcal{O}_X}^n(K, L)$ are finite.*

Proof. Recall that

$$\text{Ext}_{\mathcal{O}_X}^n(K, L) = H^n(X, R\mathcal{H}om_{\mathcal{O}_X}(K, L)) = H^n(\text{Spec}(A), Rf_*R\mathcal{H}om_{\mathcal{O}_X}(K, L))$$

see Cohomology, Lemma 36.1 and Cohomology, Section 14. Thus the result follows from Lemmas 10.3 and 10.2. \square

11. Descent finiteness properties of complexes

09UC This section is the analogue of Descent, Section 7 for objects of the derived category of a scheme. The easiest such result is probably the following.

09UD **Lemma 11.1.** *Let $f : X \rightarrow Y$ be a surjective flat morphism of schemes (or more generally locally ringed spaces). Let $E \in D(\mathcal{O}_Y)$. Let $a, b \in \mathbf{Z}$. Then E has tor-amplitude in $[a, b]$ if and only if Lf^*E has tor-amplitude in $[a, b]$.*

Proof. Pullback always preserves tor-amplitude, see Cohomology, Lemma 42.4. We may check tor-amplitude in $[a, b]$ on stalks, see Cohomology, Lemma 42.5. A flat local ring homomorphism is faithfully flat by Algebra, Lemma 38.17. Thus the result follows from More on Algebra, Lemma 61.17. \square

09UE **Lemma 11.2.** *Let $\{f_i : X_i \rightarrow X\}$ be an fpqc covering of schemes. Let $E \in D_{\text{QCoh}}(\mathcal{O}_X)$. Let $m \in \mathbf{Z}$. Then E is m -pseudo-coherent if and only if each Lf_i^*E is m -pseudo-coherent.*

Proof. Pullback always preserves m -pseudo-coherence, see Cohomology, Lemma 41.3. Conversely, assume that Lf_i^*E is m -pseudo-coherent for all i . Let $U \subset X$ be an affine open. It suffices to prove that $E|_U$ is m -pseudo-coherent. Since $\{f_i : X_i \rightarrow X\}$ is an fpqc covering, we can find finitely many affine open $V_j \subset X_{a(j)}$ such that $f_{a(j)}(V_j) \subset U$ and $U = \bigcup f_{a(j)}(V_j)$. Set $V = \coprod V_j$. Thus we may replace X by U and $\{f_i : X_i \rightarrow X\}$ by $\{V \rightarrow U\}$ and assume that X is affine and our covering is given by a single surjective flat morphism $\{f : Y \rightarrow X\}$ of affine schemes. In this case the result follows from More on Algebra, Lemma 60.16 via Lemmas 3.5 and 9.2. \square

09UF **Lemma 11.3.** *Let $\{f_i : X_i \rightarrow X\}$ be an fppf covering of schemes. Let $E \in D(\mathcal{O}_X)$. Let $m \in \mathbf{Z}$. Then E is m -pseudo-coherent if and only if each Lf_i^*E is m -pseudo-coherent.*

Proof. Pullback always preserves m -pseudo-coherence, see Cohomology, Lemma 41.3. Conversely, assume that Lf_i^*E is m -pseudo-coherent for all i . Let $U \subset X$ be an affine open. It suffices to prove that $E|_U$ is m -pseudo-coherent. Since $\{f_i : X_i \rightarrow X\}$ is an fppf covering, we can find finitely many affine open $V_j \subset X_{a(j)}$ such that $f_{a(j)}(V_j) \subset U$ and $U = \bigcup f_{a(j)}(V_j)$. Set $V = \coprod V_i$. Thus we may replace X by U and $\{f_i : X_i \rightarrow X\}$ by $\{V \rightarrow U\}$ and assume that X is affine and our covering is given by a single surjective flat morphism $\{f : Y \rightarrow X\}$ of finite presentation.

Since f is flat the derived functor Lf^* is just given by f^* and f^* is exact. Hence $H^i(Lf^*E) = f^*H^i(E)$. Since Lf^*E is m -pseudo-coherent, we see that $Lf^*E \in D^-(\mathcal{O}_Y)$. Since f is surjective and flat, we see that $E \in D^-(\mathcal{O}_X)$. Let $i \in \mathbf{Z}$ be the largest integer such that $H^i(E)$ is nonzero. If $i < m$, then we are done. Otherwise, $f^*H^i(E)$ is a finite type \mathcal{O}_Y -module by Cohomology, Lemma 41.9. Then by Descent, Lemma 7.2 the \mathcal{O}_X -module $H^i(E)$ is of finite type. Thus, after replacing X by the members of a finite affine open covering, we may assume there exists a map

$$\alpha : \mathcal{O}_X^{\oplus n}[-i] \longrightarrow E$$

such that $H^i(\alpha)$ is a surjection. Let C be the cone of α in $D(\mathcal{O}_X)$. Pulling back to Y and using Cohomology, Lemma 41.4 we find that Lf^*C is m -pseudo-coherent. Moreover $H^j(C) = 0$ for $j \geq i$. Thus by induction on i we see that C is m -pseudo-coherent. Using Cohomology, Lemma 41.4 again we conclude. \square

09UG **Lemma 11.4.** *Let $\{f_i : X_i \rightarrow X\}$ be an fpqc covering of schemes. Let $E \in D(\mathcal{O}_X)$. Then E is perfect if and only if each Lf_i^*E is perfect.*

Proof. Pullback always preserves perfect complexes, see Cohomology, Lemma 43.6. Conversely, assume that Lf_i^*E is perfect for all i . Then the cohomology sheaves of each Lf_i^*E are quasi-coherent, see Lemma 9.1 and Cohomology, Lemma 43.5. Since the morphisms f_i is flat we see that $H^p(Lf_i^*E) = f_i^*H^p(E)$. Thus the cohomology sheaves of E are quasi-coherent by Descent, Proposition 5.2. Having said this the lemma follows formally from Cohomology, Lemma 43.5 and Lemmas 11.1 and 11.2. \square

09VA **Lemma 11.5.** *Let $i : Z \rightarrow X$ be a morphism of ringed spaces such that i is a closed immersion of underlying topological spaces and such that $i_*\mathcal{O}_Z$ is pseudo-coherent as an \mathcal{O}_X -module. Let $E \in D(\mathcal{O}_Z)$. Then E is m -pseudo-coherent if and only if Ri_*E is m -pseudo-coherent.*

Proof. Throughout this proof we will use that i_* is an exact functor, and hence that $Ri_* = i_*$, see Modules, Lemma 6.1.

Assume E is m -pseudo-coherent. Let $x \in X$. We will find a neighbourhood of x such that i_*E is m -pseudo-coherent on it. If $x \notin Z$ then this is clear. Thus we may assume $x \in Z$. We will use that $U \cap Z$ for $x \in U \subset X$ open form a fundamental system of neighbourhoods of x in Z . After shrinking X we may assume E is bounded above. We will argue by induction on the largest integer p such that $H^p(E)$ is nonzero. If $p < m$, then there is nothing to prove. If $p \geq m$, then $H^p(E)$ is an \mathcal{O}_Z -module of finite type, see Cohomology, Lemma 41.9. Thus

we may choose, after shrinking X , a map $\mathcal{O}_Z^{\oplus n}[-p] \rightarrow E$ which induces a surjection $\mathcal{O}_Z^{\oplus n} \rightarrow H^p(E)$. Choose a distinguished triangle

$$\mathcal{O}_Z^{\oplus n}[-p] \rightarrow E \rightarrow C \rightarrow \mathcal{O}_Z^{\oplus n}[-p+1]$$

We see that $H^j(C) = 0$ for $j \geq p$ and that C is m -pseudo-coherent by Cohomology, Lemma 41.4. By induction we see that i_*C is m -pseudo-coherent on X . Since $i_*\mathcal{O}_Z$ is m -pseudo-coherent on X as well, we conclude from the distinguished triangle

$$i_*\mathcal{O}_Z^{\oplus n}[-p] \rightarrow i_*E \rightarrow i_*C \rightarrow i_*\mathcal{O}_Z^{\oplus n}[-p+1]$$

and Cohomology, Lemma 41.4 that i_*E is m -pseudo-coherent.

Assume that i_*E is m -pseudo-coherent. Let $z \in Z$. We will find a neighbourhood of z such that E is m -pseudo-coherent on it. We will use that $U \cap Z$ for $z \in U \subset X$ open form a fundamental system of neighbourhoods of z in Z . After shrinking X we may assume i_*E and hence E is bounded above. We will argue by induction on the largest integer p such that $H^p(E)$ is nonzero. If $p < m$, then there is nothing to prove. If $p \geq m$, then $H^p(i_*E) = i_*H^p(E)$ is an \mathcal{O}_X -module of finite type, see Cohomology, Lemma 41.9. Choose a complex \mathcal{E}^\bullet of \mathcal{O}_Z -modules representing E . We may choose, after shrinking X , a map $\alpha : \mathcal{O}_X^{\oplus n}[-p] \rightarrow i_*\mathcal{E}^\bullet$ which induces a surjection $\mathcal{O}_X^{\oplus n} \rightarrow i_*H^p(\mathcal{E}^\bullet)$. By adjunction we find a map $\alpha : \mathcal{O}_Z^{\oplus n}[-p] \rightarrow \mathcal{E}^\bullet$ which induces a surjection $\mathcal{O}_Z^{\oplus n} \rightarrow H^p(\mathcal{E}^\bullet)$. Choose a distinguished triangle

$$\mathcal{O}_Z^{\oplus n}[-p] \rightarrow E \rightarrow C \rightarrow \mathcal{O}_Z^{\oplus n}[-p+1]$$

We see that $H^j(C) = 0$ for $j \geq p$. From the distinguished triangle

$$i_*\mathcal{O}_Z^{\oplus n}[-p] \rightarrow i_*E \rightarrow i_*C \rightarrow i_*\mathcal{O}_Z^{\oplus n}[-p+1]$$

the fact that $i_*\mathcal{O}_Z$ is pseudo-coherent and Cohomology, Lemma 41.4 we conclude that i_*C is m -pseudo-coherent. By induction we conclude that C is m -pseudo-coherent. By Cohomology, Lemma 41.4 again we conclude that E is m -pseudo-coherent. \square

09VB **Lemma 11.6.** *Let $f : X \rightarrow Y$ be a finite morphism of schemes such that $f_*\mathcal{O}_X$ is pseudo-coherent as an \mathcal{O}_Y -module². Let $E \in D_{QCoh}(\mathcal{O}_X)$. Then E is m -pseudo-coherent if and only if Rf_*E is m -pseudo-coherent.*

Proof. This is a translation of More on Algebra, Lemma 60.12 into the language of schemes. To do the translation, use Lemmas 3.5 and 9.2. \square

12. Lifting complexes

08EC Let $U \subset X$ be an open subspace of a ringed space and denote $j : U \rightarrow X$ the inclusion morphism. The functor $D(\mathcal{O}_X) \rightarrow D(\mathcal{O}_U)$ is essentially surjective as Rj_* is a right inverse to restriction. In this section we extend this to complexes with quasi-coherent cohomology sheaves, etc.

08ED **Lemma 12.1.** *Let X be a scheme and let $j : U \rightarrow X$ be a quasi-compact open immersion. The functors*

$$D_{QCoh}(\mathcal{O}_X) \rightarrow D_{QCoh}(\mathcal{O}_U) \quad \text{and} \quad D_{QCoh}^+(\mathcal{O}_X) \rightarrow D_{QCoh}^+(\mathcal{O}_U)$$

are essentially surjective. If X is quasi-compact, then the functors

$$D_{QCoh}^-(\mathcal{O}_X) \rightarrow D_{QCoh}^-(\mathcal{O}_U) \quad \text{and} \quad D_{QCoh}^b(\mathcal{O}_X) \rightarrow D_{QCoh}^b(\mathcal{O}_U)$$

²This means that f is pseudo-coherent, see More on Morphisms, Lemma 49.8.

are essentially surjective.

Proof. The argument preceding the lemma applies for the first case because Rj_* maps $D_{QCoh}(\mathcal{O}_U)$ into $D_{QCoh}(\mathcal{O}_X)$ by Lemma 4.1. It is clear that Rj_* maps $D_{QCoh}^+(\mathcal{O}_U)$ into $D_{QCoh}^+(\mathcal{O}_X)$ which implies the statement on bounded below complexes. Finally, Lemma 4.1 guarantees that Rj_* maps $D_{QCoh}^-(\mathcal{O}_U)$ into $D_{QCoh}^-(\mathcal{O}_X)$ if X is quasi-compact. Combining these two we obtain the last statement. \square

08EE **Lemma 12.2.** *Let X be an affine scheme and let $U \subset X$ be a quasi-compact open subscheme. For any pseudo-coherent object E of $D(\mathcal{O}_U)$ there exists a bounded above complex of finite free \mathcal{O}_X -modules whose restriction to U is isomorphic to E .*

Proof. By Lemma 9.1 we see that E is an object of $D_{QCoh}(\mathcal{O}_U)$. By Lemma 12.1 we may assume $E = E'|_U$ for some object E' of $D_{QCoh}(\mathcal{O}_X)$. Write $X = \text{Spec}(A)$. By Lemma 3.5 we can find a complex M^\bullet of A -modules whose associated complex of \mathcal{O}_X -modules is a representative of E' .

Choose $f_1, \dots, f_r \in A$ such that $U = D(f_1) \cup \dots \cup D(f_r)$. By Lemma 9.2 the complexes $M_{f_j}^\bullet$ are pseudo-coherent complexes of A_{f_j} -modules. Let n be an integer. Assume we have a map of complexes $\alpha : F^\bullet \rightarrow M^\bullet$ where F^\bullet is bounded above, $F^i = 0$ for $i < n$, each F^i is a finite free R -module, such that

$$H^i(\alpha_{f_j}) : H^i(F_{f_j}^\bullet) \rightarrow H^i(M_{f_j}^\bullet)$$

is an isomorphism for $i > n$ and surjective for $i = n$. Picture

$$\begin{array}{ccccccc} & & F^n & \longrightarrow & F^{n+1} & \longrightarrow & \dots \\ & & \downarrow \alpha & & \downarrow \alpha & & \\ M^{n-1} & \longrightarrow & M^n & \longrightarrow & M^{n+1} & \longrightarrow & \dots \end{array}$$

Since each $M_{f_j}^\bullet$ has vanishing cohomology in large degrees we can find such a map for $n \gg 0$. By induction on n we are going to extend this to a map of complexes $F^\bullet \rightarrow M^\bullet$ such that $H^i(\alpha_{f_j})$ is an isomorphism for all i . The lemma will follow by taking \widetilde{F}^\bullet .

The induction step will be to extend the diagram above by adding F^{n-1} . Let C^\bullet be the cone on α (Derived Categories, Definition 9.1). The long exact sequence of cohomology shows that $H^i(C_{f_j}^\bullet) = 0$ for $i \geq n$. By More on Algebra, Lemma 60.2 we see that $C_{f_j}^\bullet$ is $(n-1)$ -pseudo-coherent. By More on Algebra, Lemma 60.3 we see that $H^{-1}(C_{f_j}^\bullet)$ is a finite A_{f_j} -module. Choose a finite free A -module F^{n-1} and an A -module $\beta : F^{n-1} \rightarrow C^{n-1}$ such that the composition $F^{n-1} \rightarrow C^{n-1} \rightarrow C^n$ is zero and such that $F_{f_j}^{n-1}$ surjects onto $H^{n-1}(C_{f_j}^\bullet)$. (Some details omitted; hint: clear denominators.) Since $C^{n-1} = M^{n-1} \oplus F^n$ we can write $\beta = (\alpha^{n-1}, -d^{n-1})$. The vanishing of the composition $F^{n-1} \rightarrow C^{n-1} \rightarrow C^n$ implies these maps fit into a morphism of complexes

$$\begin{array}{ccccccc} & & F^{n-1} & \longrightarrow & F^n & \longrightarrow & F^{n+1} & \longrightarrow & \dots \\ & & \downarrow \alpha^{n-1} & & \downarrow \alpha & & \downarrow \alpha & & \\ \dots & \longrightarrow & M^{n-1} & \longrightarrow & M^n & \longrightarrow & M^{n+1} & \longrightarrow & \dots \end{array}$$

Moreover, these maps define a morphism of distinguished triangles

$$\begin{array}{ccccccc}
 (F^n \rightarrow \dots) & \longrightarrow & (F^{n-1} \rightarrow \dots) & \longrightarrow & F^{n-1} & \longrightarrow & (F^n \rightarrow \dots)[1] \\
 \downarrow & & \downarrow & & \downarrow \beta & & \downarrow \\
 (F^n \rightarrow \dots) & \longrightarrow & M^\bullet & \longrightarrow & C^\bullet & \longrightarrow & (F^n \rightarrow \dots)[1]
 \end{array}$$

Hence our choice of β implies that the map of complexes $(F^{-1} \rightarrow \dots) \rightarrow M^\bullet$ induces an isomorphism on cohomology localized at f_j in degrees $\geq n$ and a surjection in degree -1 . This finishes the proof of the lemma. \square

08EF **Lemma 12.3.** *Let X be a quasi-compact and quasi-separated scheme. Let $E \in D_{QCoh}^b(\mathcal{O}_X)$. There exists an integer $n_0 > 0$ such that $\text{Ext}_{D(\mathcal{O}_X)}^n(\mathcal{E}, E) = 0$ for every finite locally free \mathcal{O}_X -module \mathcal{E} and every $n \geq n_0$.*

Proof. Recall that $\text{Ext}_{D(\mathcal{O}_X)}^n(\mathcal{E}, E) = \text{Hom}_{D(\mathcal{O}_X)}(\mathcal{E}, E[n])$. We have Mayer-Vietoris for morphisms in the derived category, see Cohomology, Lemma 31.3. Thus if $X = U \cup V$ and the result of the lemma holds for $E|_U$, $E|_V$, and $E|_{U \cap V}$ for some bound n_0 , then the result holds for E with bound $n_0 + 1$. Thus it suffices to prove the lemma when X is affine, see Cohomology of Schemes, Lemma 4.1.

Assume $X = \text{Spec}(A)$ is affine. Choose a complex of A -modules M^\bullet whose associated complex of quasi-coherent modules represents E , see Lemma 3.5. Write $\mathcal{E} = \tilde{P}$ for some A -module P . Since \mathcal{E} is finite locally free, we see that P is a finite projective A -module. We have

$$\begin{aligned}
 \text{Hom}_{D(\mathcal{O}_X)}(\mathcal{E}, E[n]) &= \text{Hom}_{D(A)}(P, M^\bullet[n]) \\
 &= \text{Hom}_{K(A)}(P, M^\bullet[n]) \\
 &= \text{Hom}_A(P, H^n(M^\bullet))
 \end{aligned}$$

The first equality by Lemma 3.5, the second equality by Derived Categories, Lemma 19.8, and the final equality because $\text{Hom}_A(P, -)$ is an exact functor. As E and hence M^\bullet is bounded we get zero for all sufficiently large n . \square

08EG **Lemma 12.4.** *Let X be an affine scheme. Let $U \subset X$ be a quasi-compact open. For every perfect object E of $D(\mathcal{O}_U)$ there exists an integer r and a finite locally free sheaf \mathcal{F} on U such that $\mathcal{F}[-r] \oplus E$ is the restriction of a perfect object of $D(\mathcal{O}_X)$.*

Proof. Say $X = \text{Spec}(A)$. Recall that a perfect complex is pseudo-coherent, see Cohomology, Lemma 43.5. By Lemma 12.2 we can find a bounded above complex \mathcal{F}^\bullet of finite free A -modules such that E is isomorphic to $\mathcal{F}^\bullet|_U$ in $D(\mathcal{O}_U)$. By Cohomology, Lemma 43.5 and since U is quasi-compact, we see that E has finite tor dimension, say E has tor amplitude in $[a, b]$. Pick $r < a$ and set

$$\mathcal{F} = \text{Ker}(\mathcal{F}^r \rightarrow \mathcal{F}^{r+1}) = \text{Im}(\mathcal{F}^{r-1} \rightarrow \mathcal{F}^r).$$

Since E has tor amplitude in $[a, b]$ we see that $\mathcal{F}|_U$ is flat (Cohomology, Lemma 42.2). Hence $\mathcal{F}|_U$ is flat and of finite presentation, thus finite locally free (Properties, Lemma 20.2). It follows that

$$(\mathcal{F} \rightarrow \mathcal{F}^r \rightarrow \mathcal{F}^{r+1} \rightarrow \dots)|_U$$

is a strictly perfect complex on U representing E . We obtain a distinguished triangle

$$\mathcal{F}|_U[-r-1] \rightarrow E \rightarrow (\mathcal{F}^r \rightarrow \mathcal{F}^{r+1} \rightarrow \dots)|_U \rightarrow \mathcal{F}|_U[-r]$$

Note that $(\mathcal{F}^r \rightarrow \mathcal{F}^{r+1} \rightarrow \dots)$ is a perfect complex on X . To finish the proof it suffices to pick r such that the map $\mathcal{F}|_U[-r-1] \rightarrow E$ is zero in $D(\mathcal{O}_U)$, see Derived Categories, Lemma 4.10. By Lemma 12.3 this holds if $r \ll 0$. \square

08EH **Lemma 12.5.** *Let X be an affine scheme. Let $U \subset X$ be a quasi-compact open. Let E, E' be objects of $D_{Q\text{Coh}}(\mathcal{O}_X)$ with E perfect. For every map $\alpha : E|_U \rightarrow E'|_U$ there exist maps*

$$E \xleftarrow{\beta} E_1 \xrightarrow{\gamma} E'$$

of perfect complexes on X such that $\beta : E_1 \rightarrow E$ restricts to an isomorphism on U and such that $\alpha = \gamma|_U \circ \beta|_U^{-1}$. Moreover we can assume $E_1 = E \otimes_{\mathcal{O}_X}^L I$ for some perfect complex I on X .

Proof. Write $X = \text{Spec}(A)$. Write $U = D(f_1) \cup \dots \cup D(f_r)$. Choose finite complex of finite projective A -modules M^\bullet representing E (Lemma 9.7). Choose a complex of A -modules $(M')^\bullet$ representing E' (Lemma 3.5). In this case the complex $H^\bullet = \text{Hom}_A(M^\bullet, (M')^\bullet)$ is a complex of A -modules whose associated complex of quasi-coherent \mathcal{O}_X -modules represents $R\mathcal{H}om(E, E')$, see Cohomology, Lemma 40.9. Then α determines an element s of $H^0(U, R\mathcal{H}om(E, E'))$, see Cohomology, Lemma 36.1. There exists an e and a map

$$\xi : I^\bullet(f_1^e, \dots, f_r^e) \rightarrow \text{Hom}_A(M^\bullet, (M')^\bullet)$$

corresponding to s , see Proposition 8.5. Letting E_1 be the object corresponding to complex of quasi-coherent \mathcal{O}_X -modules associated to

$$\text{Tot}(I^\bullet(f_1^e, \dots, f_r^e) \otimes_A M^\bullet)$$

we obtain $E_1 \rightarrow E$ using the canonical map $I^\bullet(f_1^e, \dots, f_r^e) \rightarrow A$ and $E_1 \rightarrow E'$ using ξ and Cohomology, Lemma 36.1. \square

08EI **Lemma 12.6.** *Let X be an affine scheme. Let $U \subset X$ be a quasi-compact open. For every perfect object F of $D(\mathcal{O}_U)$ the object $F \oplus F[1]$ is the restriction of a perfect object of $D(\mathcal{O}_X)$.*

Proof. By Lemma 12.4 we can find a perfect object E of $D(\mathcal{O}_X)$ such that $E|_U = \mathcal{F}[r] \oplus F$ for some finite locally free \mathcal{O}_U -module \mathcal{F} . By Lemma 12.5 we can find a morphism of perfect complexes $\alpha : E_1 \rightarrow E$ such that $(E_1)|_U \cong E|_U$ and such that $\alpha|_U$ is the map

$$\begin{pmatrix} \text{id}_{\mathcal{F}[r]} & 0 \\ 0 & 0 \end{pmatrix} : \mathcal{F}[r] \oplus F \rightarrow \mathcal{F}[r] \oplus F$$

Then the cone on α is a solution. \square

08EJ **Lemma 12.7.** *Let X be a quasi-compact and quasi-separated scheme. Let $f \in \Gamma(X, \mathcal{O}_X)$. For any morphism $\alpha : E \rightarrow E'$ in $D_{Q\text{Coh}}(\mathcal{O}_X)$ such that*

- (1) E is perfect, and
- (2) E' is supported on $T = V(f)$

there exists an $n \geq 0$ such that $f^n \alpha = 0$.

Proof. We have Mayer-Vietoris for morphisms in the derived category, see Cohomology, Lemma 31.3. Thus if $X = U \cup V$ and the result of the lemma holds for $f|_U$, $f|_V$, and $f|_{U \cap V}$, then the result holds for f . Thus it suffices to prove the lemma when X is affine, see Cohomology of Schemes, Lemma 4.1.

Let $X = \text{Spec}(A)$. Then $f \in A$. We will use the equivalence $D(A) = D_{QCoh}(X)$ of Lemma 3.5 without further mention. Represent E by a finite complex of finite projective A -modules P^\bullet . This is possible by Lemma 9.7. Let t be the largest integer such that P^t is nonzero. The distinguished triangle

$$P^t[-t] \rightarrow P^\bullet \rightarrow \sigma_{\leq t-1}P^\bullet \rightarrow P^t[-t+1]$$

shows that by induction on the length of the complex P^\bullet we can reduce to the case where P^\bullet has a single nonzero term. This and the shift functor reduces us to the case where P^\bullet consists of a single finite projective A -module P in degree 0. Represent E' by a complex M^\bullet of A -modules. Then α corresponds to a map $P \rightarrow H^0(M^\bullet)$. Since the module $H^0(M^\bullet)$ is supported on $V(f)$ by assumption (2) we see that every element of $H^0(M^\bullet)$ is annihilated by a power of f . Since P is a finite A -module the map $f^n \alpha : P \rightarrow H^0(M^\bullet)$ is zero for some n as desired. \square

08EK **Lemma 12.8.** *Let X be an affine scheme. Let $T \subset X$ be a closed subset such that $X \setminus T$ is quasi-compact. Let $U \subset X$ be a quasi-compact open. For every perfect object F of $D(\mathcal{O}_U)$ supported on $T \cap U$ the object $F \oplus F[1]$ is the restriction of a perfect object E of $D(\mathcal{O}_X)$ supported in T .*

Proof. Say $T = V(g_1, \dots, g_s)$. After replacing g_j by a power we may assume multiplication by g_j is zero on F , see Lemma 12.7. Choose E as in Lemma 12.6. Note that $g_j : E \rightarrow E$ restricts to zero on U . Choose a distinguished triangle

$$E \xrightarrow{g_1} E \rightarrow C_1 \rightarrow E[1]$$

By Derived Categories, Lemma 4.10 the object C_1 restricts to $F \oplus F[1] \oplus F[1] \oplus F[2]$ on U . Moreover, $g_1 : C_1 \rightarrow C_1$ has square zero by Derived Categories, Lemma 4.5. Namely, the diagram

$$\begin{array}{ccccc} E & \longrightarrow & C_1 & \longrightarrow & E[1] \\ 0 \downarrow & & g_1 \downarrow & & 0 \downarrow \\ E & \longrightarrow & C_1 & \longrightarrow & E[1] \end{array}$$

is commutative since the compositions $E \xrightarrow{g_1} E \rightarrow C_1$ and $C_1 \rightarrow E[1] \xrightarrow{g_1} E[1]$ are zero. Continuing, setting C_{i+1} equal to the cone of the map $g_i : C_i \rightarrow C_i$ we obtain a perfect complex C_s on X supported on T whose restriction to U gives

$$F \oplus F[1]^{\oplus s} \oplus F[2]^{\oplus \binom{s}{2}} \oplus \dots \oplus F[s]$$

Choose morphisms of perfect complexes $\beta : C' \rightarrow C_s$ and $\gamma : C' \rightarrow C_s$ as in Lemma 12.5 such that $\beta|_U$ is an isomorphism and such that $\gamma|_U \circ \beta|_U^{-1}$ is the morphism

$$F \oplus F[1]^{\oplus s} \oplus F[2]^{\oplus \binom{s}{2}} \oplus \dots \oplus F[s] \rightarrow F \oplus F[1]^{\oplus s} \oplus F[2]^{\oplus \binom{s}{2}} \oplus \dots \oplus F[s]$$

which is the identity on all summands except for F where it is zero. By Lemma 12.5 we also have $C' = C_s \otimes^{\mathbf{L}} I$ for some perfect complex I on X . Hence the nullity of $g_j^2 \text{id}_{C_s}$ implies the same thing for C' . Thus C' is supported on T as well. Then $\text{Cone}(\gamma)$ is a solution. \square

A special case of the following lemma can be found in [Nee96].

09IM **Lemma 12.9.** *Let X be a quasi-compact and quasi-separated scheme. Let $U \subset X$ be a quasi-compact open. Let $T \subset X$ be a closed subset with $X \setminus T$ retro-compact in X . Let E be an object of $D_{QCoh}(\mathcal{O}_X)$. Let $\alpha : P \rightarrow E|_U$ be a map where*

P is a perfect object of $D(\mathcal{O}_U)$ supported on $T \cap U$. Then there exists a map $\beta : R \rightarrow E$ where R is a perfect object of $D(\mathcal{O}_X)$ supported on T such that P is a direct summand of $R|_U$ in $D(\mathcal{O}_U)$ compatible α and $\beta|_U$.

Proof. Since X is quasi-compact there exists an integer m such that $X = U \cup V_1 \cup \dots \cup V_m$ for some affine opens V_j of X . Arguing by induction on m we see that we may assume $m = 1$. In other words, we may assume that $X = U \cup V$ with V affine. By Lemma 12.8 we can choose a perfect object Q in $D(\mathcal{O}_V)$ supported on $T \cap V$ and an isomorphism $Q|_{U \cap V} \rightarrow (P \oplus P[1])|_{U \cap V}$. By Lemma 12.5 we can replace Q by $Q \otimes^{\mathbf{L}} I$ (still supported on $T \cap V$) and assume that the map

$$Q|_{U \cap V} \rightarrow (P \oplus P[1])|_{U \cap V} \rightarrow P|_{U \cap V} \rightarrow E|_{U \cap V}$$

lifts to $Q \rightarrow E|_V$. By Cohomology, Lemma 39.1 we find an morphism $a : R \rightarrow E$ of $D(\mathcal{O}_X)$ such that $a|_U$ is isomorphic to $P \oplus P[1] \rightarrow E|_U$ and $a|_V$ isomorphic to $Q \rightarrow E|_V$. Thus R is perfect and supported on T as desired. \square

09IN **Remark 12.10.** The proof of Lemma 12.9 shows that

$$R|_U = P \oplus P^{\oplus n_1}[1] \oplus \dots \oplus P^{\oplus n_m}[m]$$

for some $m \geq 0$ and $n_j \geq 0$. Thus the highest degree cohomology sheaf of $R|_U$ equals that of P . By repeating the construction for the map $P^{\oplus n_1}[1] \oplus \dots \oplus P^{\oplus n_m}[m] \rightarrow R|_U$, taking cones, and using induction we can achieve equality of cohomology sheaves of $R|_U$ and P above any given degree.

13. Approximation by perfect complexes

08EL In this section we discuss the observation, due to Neeman and Lipman, that a pseudo-coherent complex can be “approximated” by perfect complexes.

08EM **Definition 13.1.** Let X be a scheme. Consider triples (T, E, m) where

- (1) $T \subset X$ is a closed subset,
- (2) E is an object of $D_{QCoh}(\mathcal{O}_X)$, and
- (3) $m \in \mathbf{Z}$.

We say *approximation holds for the triple* (T, E, m) if there exists a perfect object P of $D(\mathcal{O}_X)$ supported on T and a map $\alpha : P \rightarrow E$ which induces isomorphisms $H^i(P) \rightarrow H^i(E)$ for $i > m$ and a surjection $H^m(P) \rightarrow H^m(E)$.

Approximation cannot hold for every triple. Namely, it is clear that if approximation holds for the triple (T, E, m) , then

- (1) E is m -pseudo-coherent, see Cohomology, Definition 41.1, and
- (2) the cohomology sheaves $H^i(E)$ are supported on T for $i \geq m$.

Moreover, the “support” of a perfect complex is a closed subscheme whose complement is retrocompact in X (details omitted). Hence we cannot expect approximation to hold without this assumption on T . This partly explains the conditions in the following definition.

08EN **Definition 13.2.** Let X be a scheme. We say *approximation by perfect complexes holds* on X if for any closed subset $T \subset X$ with $X \setminus T$ retro-compact in X there exists an integer r such that for every triple (T, E, m) as in Definition 13.1 with

- (1) E is $(m - r)$ -pseudo-coherent, and
- (2) $H^i(E)$ is supported on T for $i \geq m - r$

approximation holds.

We will prove that approximation by perfect complexes holds for quasi-compact and quasi-separated schemes. It seems that the second condition is necessary for our method of proof. It is possible that the first condition may be weakened to “ E is m -pseudo-coherent” by carefully analyzing the arguments below.

08EP **Lemma 13.3.** *Let X be a scheme. Let $U \subset X$ be an open subscheme. Let (T, E, m) be a triple as in Definition 13.1. If*

- (1) $T \subset U$,
- (2) approximation holds for $(T, E|_U, m)$, and
- (3) the sheaves $H^i(E)$ for $i \geq m$ are supported on T ,

then approximation holds for (T, E, m) .

Proof. Let $j : U \rightarrow X$ be the inclusion morphism. If $P \rightarrow E|_U$ is an approximation of the triple $(T, E|_U, m)$ over U , then $j_!P = Rj_*P \rightarrow j_!(E|_U) \rightarrow E$ is an approximation of (T, E, m) over X . See Cohomology, Lemmas 31.6 and 43.10. \square

08EQ **Lemma 13.4.** *Let X be an affine scheme. Then approximation holds for every triple (T, E, m) as in Definition 13.1 such that there exists an integer $r \geq 0$ with*

- (1) E is m -pseudo-coherent,
- (2) $H^i(E)$ is supported on T for $i \geq m - r + 1$,
- (3) $X \setminus T$ is the union of r affine opens.

In particular, approximation by perfect complexes holds for affine schemes.

Proof. Say $X = \text{Spec}(A)$. Write $T = V(f_1, \dots, f_r)$. (The case $r = 0$, i.e., $T = X$ follows immediately from Lemma 9.2 and the definitions.) Let (T, E, m) be a triple as in the lemma. Let t be the largest integer such that $H^t(E)$ is nonzero. We will proceed by induction on t . The base case is $t < m$; in this case the result is trivial. Now suppose that $t \geq m$. By Cohomology, Lemma 41.9 the sheaf $H^t(E)$ is of finite type. Since it is quasi-coherent it is generated by finitely many sections (Properties, Lemma 16.1). For every $s \in \Gamma(X, H^t(E)) = H^t(X, E)$ (see proof of Lemma 3.5) we can find an $e > 0$ and a morphism $K_e[-t] \rightarrow E$ such that s is in the image of $H^0(K_e) = H^t(K_e[-t]) \rightarrow H^t(E)$, see Lemma 8.6. Taking a finite direct sum of these maps we obtain a map $P \rightarrow E$ where P is a perfect complex supported on T , where $H^i(P) = 0$ for $i > t$, and where $H^t(P) \rightarrow E$ is surjective. Choose a distinguished triangle

$$P \rightarrow E \rightarrow E' \rightarrow P[1]$$

Then E' is m -pseudo-coherent (Cohomology, Lemma 41.4), $H^i(E') = 0$ for $i \geq t$, and $H^i(E')$ is supported on T for $i \geq m - r + 1$. By induction we find an approximation $P' \rightarrow E'$ of (T, E', m) . Fit the composition $P' \rightarrow E' \rightarrow P[1]$ into a distinguished triangle $P \rightarrow P'' \rightarrow P' \rightarrow P[1]$ and extend the morphisms $P' \rightarrow E'$ and $P[1] \rightarrow P[1]$ into a morphism of distinguished triangles

$$\begin{array}{ccccccc} P & \longrightarrow & P'' & \longrightarrow & P' & \longrightarrow & P[1] \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ P & \longrightarrow & E & \longrightarrow & E' & \longrightarrow & P[1] \end{array}$$

using TR3. Then P'' is a perfect complex (Cohomology, Lemma 43.7) supported on T . An easy diagram chase shows that $P'' \rightarrow E$ is the desired approximation. \square

08ER **Lemma 13.5.** *Let X be a scheme. Let $X = U \cup V$ be an open covering with U quasi-compact, V affine, and $U \cap V$ quasi-compact. If approximation by perfect complexes holds on U , then approximation holds on X .*

Proof. Let $T \subset X$ be a closed subset with $X \setminus T$ retro-compact in X . Let r_U be the integer of Definition 13.2 adapted to the pair $(U, T \cap U)$. Set $T' = T \setminus U$. Note that $T' \subset V$ and that $V \setminus T' = (X \setminus T) \cap U \cap V$ is quasi-compact by our assumption on T . Let r' be the number of affines needed to cover $V \setminus T'$. We claim that $r = \max(r_U, r')$ works for the pair (X, T) .

To see this choose a triple (T, E, m) such that E is $(m - r)$ -pseudo-coherent and $H^i(E)$ is supported on T for $i \geq m - r$. Let t be the largest integer such that $H^t(E)|_U$ is nonzero. (Such an integer exists as U is quasi-compact and $E|_U$ is $(m - r)$ -pseudo-coherent.) We will prove that E can be approximated by induction on t .

Base case: $t \leq m - r'$. This means that $H^i(E)$ is supported on T' for $i \geq m - r'$. Hence Lemma 13.4 guarantees the existence of an approximation $P \rightarrow E|_V$ of $(T', E|_V, m)$ on V . Applying Lemma 13.3 we see that (T', E, m) can be approximated. Such an approximation is also an approximation of (T, E, m) .

Induction step. Choose an approximation $P \rightarrow E|_U$ of $(T \cap U, E|_U, m)$. This in particular gives a surjection $H^t(P) \rightarrow H^t(E|_U)$. By Lemma 12.8 we can choose a perfect object Q in $D(\mathcal{O}_V)$ supported on $T \cap V$ and an isomorphism $Q|_{U \cap V} \rightarrow (P \oplus P[1])|_{U \cap V}$. By Lemma 12.5 we can replace Q by $Q \otimes^{\mathbf{L}} I$ and assume that the map

$$Q|_{U \cap V} \rightarrow (P \oplus P[1])|_{U \cap V} \rightarrow P|_{U \cap V} \rightarrow E|_{U \cap V}$$

lifts to $Q \rightarrow E|_V$. By Cohomology, Lemma 39.1 we find a morphism $a : R \rightarrow E$ of $D(\mathcal{O}_X)$ such that $a|_U$ is isomorphic to $P \oplus P[1] \rightarrow E|_U$ and $a|_V$ is isomorphic to $Q \rightarrow E|_V$. Thus R is perfect and supported on T and the map $H^t(R) \rightarrow H^t(E)$ is surjective on restriction to U . Choose a distinguished triangle

$$R \rightarrow E \rightarrow E' \rightarrow R[1]$$

Then E' is $(m - r)$ -pseudo-coherent (Cohomology, Lemma 41.4), $H^i(E')|_U = 0$ for $i \geq t$, and $H^i(E')$ is supported on T for $i \geq m - r$. By induction we find an approximation $R' \rightarrow E'$ of (T, E', m) . Fit the composition $R' \rightarrow E' \rightarrow R[1]$ into a distinguished triangle $R \rightarrow R'' \rightarrow R' \rightarrow R[1]$ and extend the morphisms $R' \rightarrow E'$ and $R[1] \rightarrow R[1]$ into a morphism of distinguished triangles

$$\begin{array}{ccccccc} R & \longrightarrow & R'' & \longrightarrow & R' & \longrightarrow & R[1] \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ R & \longrightarrow & E & \longrightarrow & E' & \longrightarrow & R[1] \end{array}$$

using TR3. Then R'' is a perfect complex (Cohomology, Lemma 43.7) supported on T . An easy diagram chase shows that $R'' \rightarrow E$ is the desired approximation. \square

08ES **Theorem 13.6.** *Let X be a quasi-compact and quasi-separated scheme. Then approximation by perfect complexes holds on X .*

Proof. This follows from the induction principle of Cohomology of Schemes, Lemma 4.1 and Lemmas 13.5 and 13.4. \square

14. Generating derived categories

09IP In this section we prove that the derived category $D_{Qcoh}(\mathcal{O}_X)$ of a quasi-compact and quasi-separated scheme can be generated by a single perfect object. We urge the reader to read the proof of this result in the wonderful paper by Bondal and van den Bergh, see [BV03].

09IQ **Lemma 14.1.** *Let X be a quasi-compact and quasi-separated scheme. Let U be a quasi-compact open subscheme. Let P be a perfect object of $D(\mathcal{O}_U)$. Then P is a direct summand of the restriction of a perfect object of $D(\mathcal{O}_X)$.*

Proof. Special case of Lemma 12.9. \square

09IR **Lemma 14.2.** *In Situation 8.1 denote $j : U \rightarrow X$ the open immersion and let K be the perfect object of $D(\mathcal{O}_X)$ corresponding to the Koszul complex on f_1, \dots, f_r over A . For $E \in D_{Qcoh}(\mathcal{O}_X)$ the following are equivalent* [BN93, Proposition 6.1]

- (1) $E = Rj_*(E|_U)$, and
- (2) $\mathrm{Hom}_{D(\mathcal{O}_X)}(K[n], E) = 0$ for all $n \in \mathbf{Z}$.

Proof. Choose a distinguished triangle $E \rightarrow Rj_*(E|_U) \rightarrow N \rightarrow E[1]$. Observe that

$$\mathrm{Hom}_{D(\mathcal{O}_X)}(K[n], Rj_*(E|_U)) = \mathrm{Hom}_{D(\mathcal{O}_U)}(K|_U[n], E) = 0$$

for all n as $K|_U = 0$. Thus it suffices to prove the result for N . In other words, we may assume that E restricts to zero on U . Observe that there are distinguished triangles

$$K^\bullet(f_1^{e_1}, \dots, f_i^{e'_i}, \dots, f_r^{e_r}) \rightarrow K^\bullet(f_1^{e_1}, \dots, f_i^{e'_i + e''_i}, \dots, f_r^{e_r}) \rightarrow K^\bullet(f_1^{e_1}, \dots, f_i^{e''_i}, \dots, f_r^{e_r}) \rightarrow \dots$$

of Koszul complexes, see More on Algebra, Lemma 26.11. Hence if $\mathrm{Hom}_{D(\mathcal{O}_X)}(K[n], E) = 0$ for all $n \in \mathbf{Z}$ then the same thing is true for the K replaced by K_e as in Lemma 8.6. Thus our lemma follows immediately from that one and the fact that E is determined by the complex of A -modules $R\Gamma(X, E)$, see Lemma 3.5. \square

09IS **Theorem 14.3.** *Let X be a quasi-compact and quasi-separated scheme. The category $D_{Qcoh}(\mathcal{O}_X)$ can be generated by a single perfect object. More precisely, there exists a perfect object P of $D(\mathcal{O}_X)$ such that for $E \in D_{Qcoh}(\mathcal{O}_X)$ the following are equivalent*

- (1) $E = 0$, and
- (2) $\mathrm{Hom}_{D(\mathcal{O}_X)}(P[n], E) = 0$ for all $n \in \mathbf{Z}$.

Proof. We will prove this using the induction principle of Cohomology of Schemes, Lemma 4.1.

If X is affine, then \mathcal{O}_X is a perfect generator. This follows from Lemma 3.5.

Assume that $X = U \cup V$ is an open covering with U quasi-compact such that the theorem holds for U and V is an affine open. Let P be a perfect object of $D(\mathcal{O}_U)$ which is a generator for $D_{Qcoh}(\mathcal{O}_U)$. Using Lemma 14.1 we may choose a perfect object Q of $D(\mathcal{O}_X)$ whose restriction to U is a direct sum one of whose summands is P . Say $V = \mathrm{Spec}(A)$. Let $Z = X \setminus U$. This is a closed subset of V with $V \setminus Z$ quasi-compact. Choose $f_1, \dots, f_r \in A$ such that $Z = V(f_1, \dots, f_r)$. Let $K \in D(\mathcal{O}_V)$ be the perfect object corresponding to the Koszul complex on f_1, \dots, f_r over A . Note that since K is supported on $Z \subset V$ closed, the pushforward $K' = R(V \rightarrow X)_* K$

is a perfect object of $D(\mathcal{O}_X)$ whose restriction to V is K (see Cohomology, Lemma 43.10). We claim that $Q \oplus K'$ is a generator for $D_{QCoh}(\mathcal{O}_X)$.

Let E be an object of $D_{QCoh}(\mathcal{O}_X)$ such that there are no nontrivial maps from any shift of $Q \oplus K'$ into E . By Cohomology, Lemma 31.6 we have $K' = R(V \rightarrow X)_!K$ and hence

$$\mathrm{Hom}_{D(\mathcal{O}_X)}(K'[n], E) = \mathrm{Hom}_{D(\mathcal{O}_V)}(K[n], E|_V)$$

Thus by Lemma 14.2 the vanishing of these groups implies that $E|_V$ is isomorphic to $R(U \cap V \rightarrow V)_*E|_{U \cap V}$. This implies that $E = R(U \rightarrow X)_*E|_U$ (small detail omitted). If this is the case then

$$\mathrm{Hom}_{D(\mathcal{O}_X)}(Q[n], E) = \mathrm{Hom}_{D(\mathcal{O}_U)}(Q|_U[n], E|_U)$$

which contains $\mathrm{Hom}_{D(\mathcal{O}_U)}(P[n], E|_U)$ as a direct summand. Thus by our choice of P the vanishing of these groups implies that $E|_U$ is zero. Whence E is zero. \square

The following result is a strengthening of Theorem 14.3 proved using exactly the same methods. Let $T \subset X$ be a closed subset of a scheme X . Let's denote $D_T(\mathcal{O}_X)$ the strictly full, saturated, triangulated subcategory consisting of complexes whose cohomology sheaves are supported on T .

0A9A **Lemma 14.4.** *Let X be a quasi-compact and quasi-separated scheme. Let $T \subset X$ be a closed subset such that $X \setminus T$ is quasi-compact. With notation as above, the category $D_{QCoh,T}(\mathcal{O}_X)$ is generated by a single perfect object.* [Rou08, Theorem 6.8]

Proof. We will prove this using the induction principle of Cohomology of Schemes, Lemma 4.1.

Assume $X = \mathrm{Spec}(A)$ is affine. In this case there exist $f_1, \dots, f_r \in A$ such that $T = V(f_1, \dots, f_r)$. Let K be the Koszul complex on f_1, \dots, f_r as in Lemma 14.2. Then K is a perfect object with cohomology supported on T and hence a perfect object of $D_{QCoh,T}(\mathcal{O}_X)$. On the other hand, if $E \in D_{QCoh,T}(\mathcal{O}_X)$ and $\mathrm{Hom}(K, E[n]) = 0$ for all n , then Lemma 14.2 tells us that $E = Rj_*(E|_{X \setminus T}) = 0$. Hence K generates $D_{QCoh,T}(\mathcal{O}_X)$, (by our definition of generators of triangulated categories in Derived Categories, Definition 33.2).

Assume that $X = U \cup V$ is an open covering with V affine and U quasi-compact such that the lemma holds for U . Let P be a perfect object of $D(\mathcal{O}_U)$ supported on $T \cap U$ which is a generator for $D_{QCoh,T \cap U}(\mathcal{O}_U)$. Using Lemma 12.9 we may choose a perfect object Q of $D(\mathcal{O}_X)$ supported on T whose restriction to U is a direct sum one of whose summands is P . Write $V = \mathrm{Spec}(B)$. Let $Z = X \setminus U$. Then Z is a closed subset of V such that $V \setminus Z$ is quasi-compact. As X is quasi-separated, it follows that $Z \cap T$ is a closed subset of V such that $W = V \setminus (Z \cap T)$ is quasi-compact. Thus we can choose $g_1, \dots, g_s \in B$ such that $Z \cap T = V(g_1, \dots, g_s)$. Let $K \in D(\mathcal{O}_V)$ be the perfect object corresponding to the Koszul complex on g_1, \dots, g_s over B . Note that since K is supported on $(Z \cap T) \subset V$ closed, the pushforward $K' = R(V \rightarrow X)_*K$ is a perfect object of $D(\mathcal{O}_X)$ whose restriction to V is K (see Cohomology, Lemma 43.10). We claim that $Q \oplus K'$ is a generator for $D_{QCoh,T}(\mathcal{O}_X)$.

Let E be an object of $D_{QCoh,T}(\mathcal{O}_X)$ such that there are no nontrivial maps from any shift of $Q \oplus K'$ into E . By Cohomology, Lemma 31.6 we have $K' = R(V \rightarrow X)_!K$ and hence

$$\mathrm{Hom}_{D(\mathcal{O}_X)}(K'[n], E) = \mathrm{Hom}_{D(\mathcal{O}_V)}(K[n], E|_V)$$

Thus by Lemma 14.2 we have $E|_V = Rj_*E|_W$ where $j : W \rightarrow V$ is the inclusion. Picture

$$\begin{array}{ccccc} W & \xrightarrow{\quad} & V & \xleftarrow{\quad} & Z \cap T \\ j' \uparrow & & \nearrow j & & \downarrow \\ U \cap V & & & & Z \\ & & \nwarrow j'' & & \end{array}$$

Since E is supported on T we see that $E|_W$ is supported on $T \cap W = T \cap U \cap V$ which is closed in W . We conclude that

$$E|_V = Rj_*(E|_W) = Rj_*(Rj'_*(E|_{U \cap V})) = Rj''_*(E|_{U \cap V})$$

where the second equality is part (1) of Cohomology, Lemma 31.6. This implies that $E = R(U \rightarrow X)_*E|_U$ (small detail omitted). If this is the case then

$$\mathrm{Hom}_{D(\mathcal{O}_X)}(Q[n], E) = \mathrm{Hom}_{D(\mathcal{O}_U)}(Q|_U[n], E|_U)$$

which contains $\mathrm{Hom}_{D(\mathcal{O}_U)}(P[n], E|_U)$ as a direct summand. Thus by our choice of P the vanishing of these groups implies that $E|_U$ is zero. Whence E is zero. \square

15. An example generator

0BQQ In this section we prove that the derived category of projective space over a ring is generated by a vector bundle, in fact a direct sum of shifts of the structure sheaf.

The following lemma says that $\bigoplus_{n \geq 0} \mathcal{L}^{\otimes -n}$ is a generator if \mathcal{L} is ample.

0BQR **Lemma 15.1.** *Let X be a scheme and \mathcal{L} an ample invertible \mathcal{O}_X -module. If K is a nonzero object of $D_{QCoh}(\mathcal{O}_X)$, then for some $n \geq 0$ and $p \in \mathbf{Z}$ the cohomology group $H^p(X, K \otimes_{\mathcal{O}_X}^{\mathbf{L}} \mathcal{L}^{\otimes n})$ is nonzero.*

Proof. Recall that as X has an ample invertible sheaf, it is quasi-compact and separated (Properties, Definition 26.1 and Lemma 26.7). Thus we may apply Proposition 6.5 and represent K by a complex \mathcal{F}^\bullet of quasi-coherent modules. Pick any p such that $\mathcal{H}^p = \mathrm{Ker}(\mathcal{F}^p \rightarrow \mathcal{F}^{p+1}) / \mathrm{Im}(\mathcal{F}^{p-1} \rightarrow \mathcal{F}^p)$ is nonzero. Choose a point $x \in X$ such that the stalk \mathcal{H}_x^p is nonzero. Choose an $n \geq 0$ and $s \in \Gamma(X, \mathcal{L}^{\otimes n})$ such that X_s is an affine open neighbourhood of x . Choose $\tau \in \mathcal{H}^p(X_s)$ which maps to a nonzero element of the stalk \mathcal{H}_x^p ; this is possible as \mathcal{H}^p is quasi-coherent and X_s is affine. Since taking sections over X_s is an exact functor on quasi-coherent modules, we can find a section $\tau' \in \mathcal{F}^p(X_s)$ mapping to zero in $\mathcal{F}^{p+1}(X_s)$ and mapping to τ in $\mathcal{H}^p(X_s)$. By Properties, Lemma 17.2 there exists an m such that $\tau' \otimes s^{\otimes m}$ is the image of a section $\tau'' \in \Gamma(X, \mathcal{F}^p \otimes \mathcal{L}^{\otimes mn})$. Applying the same lemma once more, we find $l \geq 0$ such that $\tau'' \otimes s^{\otimes l}$ maps to zero in $\mathcal{F}^{p+1} \otimes \mathcal{L}^{\otimes (m+l)n}$. Then τ'' gives a nonzero class in $H^p(X, K \otimes_{\mathcal{O}_X}^{\mathbf{L}} \mathcal{L}^{\otimes (m+l)n})$ as desired. \square

0BQS **Lemma 15.2.** *Let A be a ring. Let $X = \mathbf{P}_A^n$. For every $a \in \mathbf{Z}$ there exists an exact complex*

$$0 \rightarrow \mathcal{O}_X(a) \rightarrow \dots \rightarrow \mathcal{O}_X(a+i)^{\oplus \binom{n+1}{i}} \rightarrow \dots \rightarrow \mathcal{O}_X(a+n+1) \rightarrow 0$$

of vector bundles on X .

Proof. Recall that \mathbf{P}_A^n is $\mathrm{Proj}(A[X_0, \dots, X_n])$, see Constructions, Definition 13.2. Consider the Koszul complex

$$K_\bullet = K_\bullet(A[X_0, \dots, X_n], X_0, \dots, X_n)$$

over $S = A[X_0, \dots, X_n]$ on X_0, \dots, X_n . Since X_0, \dots, X_n is clearly a regular sequence in the polynomial ring S , we see that (More on Algebra, Lemma 27.2) that the Koszul complex K_\bullet is exact, except in degree 0 where the cohomology is $S/(X_0, \dots, X_n)$. Note that K_\bullet becomes a complex of graded modules if we put the generators of K_i in degree $+i$. In other words an exact complex

$$0 \rightarrow S(-n-1) \rightarrow \dots \rightarrow S(-n-1+i)^{\oplus \binom{n}{i}} \rightarrow \dots \rightarrow S \rightarrow S/(X_0, \dots, X_n) \rightarrow 0$$

Applying the exact functor \sim functor of Constructions, Lemma 8.4 and using that the last term is in the kernel of this functor, we obtain the exact complex

$$0 \rightarrow \mathcal{O}_X(-n-1) \rightarrow \dots \rightarrow \mathcal{O}_X(-n-1+i)^{\oplus \binom{n+1}{i}} \rightarrow \dots \rightarrow \mathcal{O}_X \rightarrow 0$$

Twisting by the invertible sheaves $\mathcal{O}_X(n+a)$ we get the exact complexes of the lemma. \square

0A9V **Lemma 15.3.** *Let A be a ring. Let $X = \mathbf{P}_A^n$. Then*

$$E = \mathcal{O}_X \oplus \mathcal{O}_X(-1) \oplus \dots \oplus \mathcal{O}_X(-n)$$

is a generator (Derived Categories, Definition 33.2) of $D_{QCoh}(X)$.

Proof. Let $K \in D_{QCoh}(\mathcal{O}_X)$. Assume $\text{Hom}(E, K[p]) = 0$ for all $p \in \mathbf{Z}$. We have to show that $K = 0$. By Derived Categories, Lemma 33.3 we see that $\text{Hom}(E', K[p])$ is zero for all $E' \in \langle E \rangle$ and $p \in \mathbf{Z}$. By Lemma 15.2 applied with $a = -n-1$ we see that $\mathcal{O}_X(-n-1) \in \langle E \rangle$ because it is quasi-isomorphic to a finite complex whose terms are finite direct sums of summands of E . Repeating the argument with $a = -n-2$ we see that $\mathcal{O}_X(-n-2) \in \langle E \rangle$. Arguing by induction we find that $\mathcal{O}_X(-m) \in \langle E \rangle$ for all $m \geq 0$. Since

$$\text{Hom}(\mathcal{O}_X(-m), K[p]) = H^p(X, K \otimes_{\mathcal{O}_X}^{\mathbf{L}} \mathcal{O}_X(m)) = H^p(X, K \otimes_{\mathcal{O}_X}^{\mathbf{L}} \mathcal{O}_X(1)^{\otimes m})$$

we conclude that $K = 0$ by Lemma 15.1. (This also uses that $\mathcal{O}_X(1)$ is an ample invertible sheaf on X which follows from Properties, Lemma 26.12.) \square

0BQT **Remark 15.4.** Let $f : X \rightarrow Y$ be a morphism of quasi-compact and quasi-separated schemes. Let $E \in D_{QCoh}(\mathcal{O}_Y)$ be a generator (see Theorem 14.3). Then the following are equivalent

- (1) for $K \in D_{QCoh}(\mathcal{O}_X)$ we have $Rf_*K = 0$ if and only if $K = 0$,
- (2) $Rf_* : D_{QCoh}(\mathcal{O}_X) \rightarrow D_{QCoh}(\mathcal{O}_Y)$ reflects isomorphisms, and
- (3) Lf^*E is a generator for $D_{QCoh}(\mathcal{O}_X)$.

The equivalence between (1) and (2) is a formal consequence of the fact that $Rf_* : D_{QCoh}(\mathcal{O}_X) \rightarrow D_{QCoh}(\mathcal{O}_Y)$ is an exact functor of triangulated categories. Similarly, the equivalence between (1) and (3) follows formally from the fact that Lf^* is the left adjoint to Rf_* . These conditions hold if f is affine (Lemma 5.1) or if f is an open immersion, or if f is a composition of such. We conclude that

- (1) if X is a quasi-affine scheme then \mathcal{O}_X is a generator for $D_{QCoh}(\mathcal{O}_X)$,
- (2) if $X \subset \mathbf{P}_A^n$ is a quasi-compact locally closed subscheme, then $\mathcal{O}_X \oplus \mathcal{O}_X(-1) \oplus \dots \oplus \mathcal{O}_X(-n)$ is a generator for $D_{QCoh}(\mathcal{O}_X)$ by Lemma 15.3.

16. Compact and perfect objects

09M0 Let X be a Noetherian scheme of finite dimension. By Cohomology, Proposition 21.7 and Cohomology on Sites, Lemma 42.4 the sheaves of modules $j_!\mathcal{O}_U$ are compact objects of $D(\mathcal{O}_X)$ for all opens $U \subset X$. These sheaves are typically not quasi-coherent, hence these do not give perfect object of the derived category $D(\mathcal{O}_X)$. However, if we restrict ourselves to complexes with quasi-coherent cohomology sheaves, then this does not happen. Here is the precise statement.

09M1 **Proposition 16.1.** *Let X be a quasi-compact and quasi-separated scheme. An object of $D_{QCoh}(\mathcal{O}_X)$ is compact if and only if it is perfect.*

Proof. By Cohomology, Lemma 44.1 the perfect objects define compact objects of $D(\mathcal{O}_X)$. Conversely, let K be a compact object of $D_{QCoh}(\mathcal{O}_X)$. To show that K is perfect, it suffices to show that $K|_U$ is perfect for every affine open $U \subset X$, see Cohomology, Lemma 43.2. Observe that $j : U \rightarrow X$ is a quasi-compact and separated morphism. Hence $Rj_* : D_{QCoh}(\mathcal{O}_U) \rightarrow D_{QCoh}(\mathcal{O}_X)$ commutes with direct sums, see Lemma 4.2. Thus the adjointness of restriction to U and Rj_* implies that $K|_U$ is a compact object of $D_{QCoh}(\mathcal{O}_U)$. Hence we reduce to the case that X is affine.

Assume $X = \text{Spec}(A)$ is affine. By Lemma 3.5 the problem is translated into the same problem for $D(A)$. For $D(A)$ the result is More on Algebra, Proposition 70.3. \square

The following result is a strengthening of Proposition 16.1. Let $T \subset X$ be a closed subset of a scheme X . As before $D_T(\mathcal{O}_X)$ denotes the strictly full, saturated, triangulated subcategory consisting of complexes whose cohomology sheaves are supported on T . Since taking direct sums commutes with taking cohomology sheaves, it follows that $D_T(\mathcal{O}_X)$ has direct sums and that they are equal to direct sums in $D(\mathcal{O}_X)$.

0A9B **Lemma 16.2.** *Let X be a quasi-compact and quasi-separated scheme. Let $T \subset X$ be a closed subset such that $X \setminus T$ is quasi-compact. An object of $D_{QCoh,T}(\mathcal{O}_X)$ is compact if and only if it is perfect as an object of $D(\mathcal{O}_X)$.*

Proof. We observe that $D_{QCoh,T}(\mathcal{O}_X)$ is a triangulated category with direct sums by the remark preceding the lemma. By Cohomology, Lemma 44.1 the perfect objects define compact objects of $D(\mathcal{O}_X)$ hence a fortiori of any subcategory preserved under taking direct sums. For the converse we will use there exists a generator $E \in D_{QCoh,T}(\mathcal{O}_X)$ which is a perfect complex of \mathcal{O}_X -modules, see Lemma 14.4. Hence by the above, E is compact. Then it follows from Derived Categories, Proposition 34.6 that E is a classical generator of the full subcategory of compact objects of $D_{QCoh,T}(\mathcal{O}_X)$. Thus any compact object can be constructed out of E by a finite sequence of operations consisting of (a) taking shifts, (b) taking finite direct sums, (c) taking cones, and (d) taking direct summands. Each of these operations preserves the property of being perfect and the result follows. \square

The following lemma is an application of the ideas that go into the proof of the preceding lemma.

0A9C **Lemma 16.3.** *Let X be a quasi-compact and quasi-separated scheme. Let $T \subset X$ be a closed subset such that $U = X \setminus T$ is quasi-compact. Let $\alpha : P \rightarrow E$ be a morphism of $D_{QCoh}(\mathcal{O}_X)$ with either*

- (1) P is perfect and E supported on T , or
 (2) P pseudo-coherent, E supported on T , and E bounded below.

Then there exists a perfect complex of \mathcal{O}_X -modules I and a map $I \rightarrow \mathcal{O}_X[0]$ such that $I \otimes^{\mathbf{L}} P \rightarrow E$ is zero and such that $I|_U \rightarrow \mathcal{O}_U[0]$ is an isomorphism.

Proof. Set $\mathcal{D} = D_{QCoh,T}(\mathcal{O}_X)$. In both cases the complex $K = R\mathcal{H}om(P, E)$ is an object of \mathcal{D} . See Lemma 9.8 for quasi-coherence. It is clear that K is supported on T as formation of $R\mathcal{H}om$ commutes with restriction to opens. The map α defines an element of $H^0(K) = \text{Hom}_{D(\mathcal{O}_X)}(\mathcal{O}_X[0], K)$. Then it suffices to prove the result for the map $\alpha : \mathcal{O}_X[0] \rightarrow K$.

Let $E \in \mathcal{D}$ be a perfect generator, see Lemma 14.4. Write

$$K = \text{hocolim} K_n$$

as in Derived Categories, Lemma 34.3 using the generator E . Since the functor $\mathcal{D} \rightarrow D(\mathcal{O}_X)$ commutes with direct sums, we see that $K = \text{hocolim} K_n$ holds in $D(\mathcal{O}_X)$. Since \mathcal{O}_X is a compact object of $D(\mathcal{O}_X)$ we find an n and a morphism $\alpha_n : \mathcal{O}_X \rightarrow K_n$ which gives rise to α , see Derived Categories, Lemma 31.9. By Derived Categories, Lemma 34.4 applied to the morphism $\mathcal{O}_X[0] \rightarrow K_n$ in the ambient category $D(\mathcal{O}_X)$ we see that α_n factors as $\mathcal{O}_X[0] \rightarrow Q \rightarrow K_n$ where Q is an object of $\langle E \rangle$. We conclude that Q is a perfect complex supported on T .

Choose a distinguished triangle

$$I \rightarrow \mathcal{O}_X[0] \rightarrow Q \rightarrow I[1]$$

By construction I is perfect, the map $I \rightarrow \mathcal{O}_X[0]$ restricts to an isomorphism over U , and the composition $I \rightarrow K$ is zero as α factors through Q . This proves the lemma. \square

17. Derived categories as module categories

09M2 In this section we draw some conclusions of what has gone before. Before we do so we need a couple more lemmas.

09M3 **Lemma 17.1.** *Let X be a scheme. Let K^\bullet be a complex of \mathcal{O}_X -modules whose cohomology sheaves are quasi-coherent. Let $(E, d) = \text{Hom}_{\text{Comp}^{dg}(\mathcal{O}_X)}(K^\bullet, K^\bullet)$ be the endomorphism differential graded algebra. Then the functor*

$$- \otimes_E^{\mathbf{L}} K^\bullet : D(E, d) \longrightarrow D(\mathcal{O}_X)$$

of Differential Graded Algebra, Lemma 26.3 has image contained in $D_{QCoh}(\mathcal{O}_X)$.

Proof. Let P be a differential graded E -module with property (P) and let F_\bullet be a filtration on P as in Differential Graded Algebra, Section 13. Then we have

$$P \otimes_E K^\bullet = \text{hocolim} F_i P \otimes_E K^\bullet$$

Each of the $F_i P$ has a finite filtration whose graded pieces are direct sums of $E[k]$. The result follows easily. \square

The following lemma can be strengthened (there is a uniformity in the vanishing over all L with nonzero cohomology sheaves only in a fixed range).

09M4 **Lemma 17.2.** *Let X be a quasi-compact and quasi-separated scheme. Let K be a perfect object of $D(\mathcal{O}_X)$. Then*

- (1) *there exist integers $a \leq b$ such that $\mathrm{Hom}_{D(\mathcal{O}_X)}(K, L) = 0$ for $L \in D_{Qcoh}(\mathcal{O}_X)$ with $H^i(L) = 0$ for $i \in [a, b]$, and*
 (2) *if L is bounded, then $\mathrm{Ext}_{D(\mathcal{O}_X)}^n(K, L)$ is zero for all but finitely many n .*

Proof. Part (2) follows from (1) as $\mathrm{Ext}_{D(\mathcal{O}_X)}^n(K, L) = \mathrm{Hom}_{D(\mathcal{O}_X)}(K, L[n])$. We prove (1). Since K is perfect we have

$$\mathrm{Hom}_{D(\mathcal{O}_X)}(K, L) = H^0(X, K^\vee \otimes_{\mathcal{O}_X}^{\mathbf{L}} L)$$

where K^\vee is the “dual” perfect complex to K , see Cohomology, Lemma 43.11. Note that $K^\vee \otimes_{\mathcal{O}_X}^{\mathbf{L}} L$ is in $D_{Qcoh}(X)$ by Lemmas 3.9 and 9.1 (to see that a perfect complex has quasi-coherent cohomology sheaves). Say K^\vee has tor amplitude in $[a, b]$. Then the spectral sequence

$$E_1^{p,q} = H^p(K^\vee \otimes_{\mathcal{O}_X}^{\mathbf{L}} H^q(L)) \Rightarrow H^{p+q}(K^\vee \otimes_{\mathcal{O}_X}^{\mathbf{L}} L)$$

shows that $H^j(K^\vee \otimes_{\mathcal{O}_X}^{\mathbf{L}} L)$ is zero if $H^q(L) = 0$ for $q \in [j - b, j - a]$. Let N be the integer d of Cohomology of Schemes, Lemma 4.4. Then $H^0(X, K^\vee \otimes_{\mathcal{O}_X}^{\mathbf{L}} L)$ vanishes if the cohomology sheaves

$$H^{-N}(K^\vee \otimes_{\mathcal{O}_X}^{\mathbf{L}} L), H^{-N+1}(K^\vee \otimes_{\mathcal{O}_X}^{\mathbf{L}} L), \dots, H^0(K^\vee \otimes_{\mathcal{O}_X}^{\mathbf{L}} L)$$

are zero. Namely, by the lemma cited and Lemma 3.4, we have

$$H^0(X, K^\vee \otimes_{\mathcal{O}_X}^{\mathbf{L}} L) = H^0(X, \tau_{\geq -N}(K^\vee \otimes_{\mathcal{O}_X}^{\mathbf{L}} L))$$

and by the vanishing of cohomology sheaves, this is equal to $H^0(X, \tau_{\geq 1}(K^\vee \otimes_{\mathcal{O}_X}^{\mathbf{L}} L))$ which is zero by Derived Categories, Lemma 17.1. It follows that $\mathrm{Hom}_{D(\mathcal{O}_X)}(K, L)$ is zero if $H^i(L) = 0$ for $i \in [-b - N, -a]$. \square

The following result is taken from [BV03].

09M5 **Theorem 17.3.** *Let X be a quasi-compact and quasi-separated scheme. Then there exist a differential graded algebra (E, d) with only a finite number of nonzero cohomology groups $H^i(E)$ such that $D_{Qcoh}(\mathcal{O}_X)$ is equivalent to $D(E, d)$.*

Proof. Let K^\bullet be a K-injective complex of \mathcal{O} -modules which is perfect and generates $D_{Qcoh}(\mathcal{O}_X)$. Such a thing exists by Theorem 14.3 and the existence of K-injective resolutions. We will show the theorem holds with

$$(E, d) = \mathrm{Hom}_{\mathrm{Comp}^{dg}(\mathcal{O}_X)}(K^\bullet, K^\bullet)$$

where $\mathrm{Comp}^{dg}(\mathcal{O}_X)$ is the differential graded category of complexes of \mathcal{O} -modules. Please see Differential Graded Algebra, Section 26. Since K^\bullet is K-injective we have

09M6 (17.3.1)
$$H^n(E) = \mathrm{Ext}_{D(\mathcal{O}_X)}^n(K^\bullet, K^\bullet)$$

for all $n \in \mathbf{Z}$. Only a finite number of these Exts are nonzero by Lemma 17.2. Consider the functor

$$- \otimes_E^{\mathbf{L}} K^\bullet : D(E, d) \longrightarrow D(\mathcal{O}_X)$$

of Differential Graded Algebra, Lemma 26.3. Since K^\bullet is perfect, it defines a compact object of $D(\mathcal{O}_X)$, see Proposition 16.1. Combined with (17.3.1) the functor above is fully faithful as follows from Differential Graded Algebra, Lemmas 26.6. It has a right adjoint

$$R\mathrm{Hom}(K^\bullet, -) : D(\mathcal{O}_X) \longrightarrow D(E, d)$$

by Differential Graded Algebra, Lemmas 26.5 which is a left quasi-inverse functor by generalities on adjoint functors. On the other hand, it follows from Lemma 17.1 that we obtain

$$- \otimes_E^{\mathbf{L}} K^\bullet : D(E, d) \longrightarrow D_{QCoh}(\mathcal{O}_X)$$

and by our choice of K^\bullet as a generator of $D_{QCoh}(\mathcal{O}_X)$ the kernel of the adjoint restricted to $D_{QCoh}(\mathcal{O}_X)$ is zero. A formal argument shows that we obtain the desired equivalence, see Derived Categories, Lemma 7.2. \square

0DJL **Remark 17.4** (Variant with support). Let X be a quasi-compact and quasi-separated scheme. Let $T \subset X$ be a closed subset such that $X \setminus T$ is quasi-compact. The analogue of Theorem 17.3 holds for $D_{QCoh, T}(\mathcal{O}_X)$. This follows from the exact same argument as in the proof of the theorem, using Lemmas 14.4 and 16.2 and a variant of Lemma 17.1 with supports. If we ever need this, we will precisely state the result here and give a detailed proof.

09SU **Remark 17.5** (Uniqueness of dga). Let X be a quasi-compact and quasi-separated scheme over a ring R . By the construction of the proof of Theorem 17.3 there exists a differential graded algebra (A, d) over R such that $D_{QCoh}(X)$ is R -linearly equivalent to $D(A, d)$ as a triangulated category. One may ask: how unique is (A, d) ? The answer is (only) slightly better than just saying that (A, d) is well defined up to derived equivalence. Namely, suppose that (B, d) is a second such pair. Then we have

$$(A, d) = \mathrm{Hom}_{\mathrm{Comp}^{dg}(\mathcal{O}_X)}(K^\bullet, K^\bullet)$$

and

$$(B, d) = \mathrm{Hom}_{\mathrm{Comp}^{dg}(\mathcal{O}_X)}(L^\bullet, L^\bullet)$$

for some K -injective complexes K^\bullet and L^\bullet of \mathcal{O}_X -modules corresponding to perfect generators of $D_{QCoh}(\mathcal{O}_X)$. Set

$$\Omega = \mathrm{Hom}_{\mathrm{Comp}^{dg}(\mathcal{O}_X)}(K^\bullet, L^\bullet) \quad \Omega' = \mathrm{Hom}_{\mathrm{Comp}^{dg}(\mathcal{O}_X)}(L^\bullet, K^\bullet)$$

Then Ω is a differential graded $B^{opp} \otimes_R A$ -module and Ω' is a differential graded $A^{opp} \otimes_R B$ -module. Moreover, the equivalence

$$D(A, d) \rightarrow D_{QCoh}(\mathcal{O}_X) \rightarrow D(B, d)$$

is given by the functor $- \otimes_A^{\mathbf{L}} \Omega'$ and similarly for the quasi-inverse. Thus we are in the situation of Differential Graded Algebra, Remark 28.10. If we ever need this remark we will provide a precise statement with a detailed proof here.

18. Characterizing pseudo-coherent complexes, I

0DJM We can use the methods above to characterize pseudo-coherent objects as derived homotopy limits of approximations by perfect objects.

0DJN **Lemma 18.1.** *Let X be a quasi-compact and quasi-separated scheme. Let $K \in D(\mathcal{O}_X)$. The following are equivalent*

- (1) K is pseudo-coherent, and
- (2) $K = \mathrm{hocolim} K_n$ where K_n is perfect and $\tau_{\geq -n} K_n \rightarrow \tau_{\geq -n} K$ is an isomorphism for all n .

Proof. The implication (2) \Rightarrow (1) is true on any ringed space. Namely, assume (2) holds. Recall that a perfect object of the derived category is pseudo-coherent, see Cohomology, Lemma 43.5. Then it follows from the definitions that $\tau_{\geq -n}K_n$ is $(-n+1)$ -pseudo-coherent and hence $\tau_{\geq -n}K$ is $(-n+1)$ -pseudo-coherent, hence K is $(-n+1)$ -pseudo-coherent. This is true for all n , hence K is pseudo-coherent, see Cohomology, Definition 41.1.

Assume (1). We start by choosing an approximation $K_1 \rightarrow K$ of $(X, K, -2)$ by a perfect complex K_1 , see Definitions 13.1 and 13.2 and Theorem 13.6. Suppose by induction we have

$$K_1 \rightarrow K_2 \rightarrow \dots \rightarrow K_n \rightarrow K$$

with K_i perfect such that $\tau_{\geq -i}K_i \rightarrow \tau_{\geq -i}K$ is an isomorphism for all $1 \leq i \leq n$. Then we pick $a \leq b$ as in Lemma 17.2 for the perfect object K_n . Choose an approximation $K_{n+1} \rightarrow K$ of $(X, K, \min(a-1, -n-1))$. Choose a distinguished triangle

$$K_{n+1} \rightarrow K \rightarrow C \rightarrow K_{n+1}[1]$$

Then we see that $C \in D_{QCoh}(\mathcal{O}_X)$ has $H^i(C) = 0$ for $i \geq a$. Thus by our choice of a, b we see that $\text{Hom}_{D(\mathcal{O}_X)}(K_n, C) = 0$. Hence the composition $K_n \rightarrow K \rightarrow C$ is zero. Hence by Derived Categories, Lemma 4.2 we can factor $K_n \rightarrow K$ through K_{n+1} proving the induction step.

We still have to prove that $K = \text{hocolim}K_n$. This follows by an application of Derived Categories, Lemma 31.8 to the functors $H^i(-) : D(\mathcal{O}_X) \rightarrow \text{Mod}(\mathcal{O}_X)$ and our choice of K_n . \square

0DJP **Lemma 18.2.** *Let X be a quasi-compact and quasi-separated scheme. Let $T \subset X$ be a closed subset such that $X \setminus T$ is quasi-compact. Let $K \in D(\mathcal{O}_X)$ supported on T . The following are equivalent*

- (1) K is pseudo-coherent, and
- (2) $K = \text{hocolim}K_n$ where K_n is perfect, supported on T , and $\tau_{\geq -n}K_n \rightarrow \tau_{\geq -n}K$ is an isomorphism for all n .

Proof. The proof of this lemma is exactly the same as the proof of Lemma 18.1 except that in the choice of the approximations we use the triples (T, K, m) . \square

19. An example equivalence

0CS7 In Section 15 we proved that the derived category of projective space \mathbf{P}_A^n over a ring A is generated by a vector bundle, in fact a direct sum of shifts of the structure sheaf. In this section we prove this determines an equivalence of $D_{QCoh}(\mathcal{O}_{\mathbf{P}_A^n})$ with the derived category of an A -algebra.

Before we can state the result we need some notation. Let A be a ring. Let $X = \mathbf{P}_A^n = \text{Proj}(S)$ where $S = A[X_0, \dots, X_n]$. By Lemma 15.3 we know that

$$0CS8 \quad (19.0.1) \quad P = \mathcal{O}_X \oplus \mathcal{O}_X(-1) \oplus \dots \oplus \mathcal{O}_X(-n)$$

is a perfect generator of $D_{QCoh}(\mathcal{O}_X)$. Consider the (noncommutative) A -algebra

$$0CS9 \quad (19.0.2) \quad R = \text{Hom}_{\mathcal{O}_X}(P, P) = \begin{pmatrix} S_0 & S_1 & S_2 & \dots & \dots \\ 0 & S_0 & S_1 & \dots & \dots \\ 0 & 0 & S_0 & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & \dots & S_0 \end{pmatrix}$$

with obvious multiplication and addition. If we view P as a complex of \mathcal{O}_X -modules in the usual way (i.e., with P in degree 0 and zero in every other degree), then we have

$$R = \mathrm{Hom}_{\mathrm{Comp}^{dg}(\mathcal{O}_X)}(P, P)$$

where on the right hand side we view R as a differential graded algebra over A with zero differential (i.e., with R in degree 0 and zero in every other degree). According to the discussion in Differential Graded Algebra, Section 26 we obtain a derived functor

$$- \otimes_R^{\mathbf{L}} P : D(R) \longrightarrow D(\mathcal{O}_X),$$

see especially Differential Graded Algebra, Lemma 26.3. By Lemma 17.1 we see that the essential image of this functor is contained in $D_{QCoh}(\mathcal{O}_X)$.

0BQU Lemma 19.1. *Let A be a ring. Let $X = \mathbf{P}_A^n = \mathrm{Proj}(S)$ where $S = A[X_0, \dots, X_n]$. With P as in (19.0.1) and R as in (19.0.2) the functor*

$$- \otimes_R^{\mathbf{L}} P : D(R) \longrightarrow D_{QCoh}(\mathcal{O}_X)$$

is an A -linear equivalence of triangulated categories sending R to P .

In words: the derived category of quasi-coherent modules on projective space is equivalent to the derived category of modules over a (noncommutative) algebra. This property of projective space appears to be quite unusual among all projective schemes over A .

Proof. To prove that our functor is fully faithful it suffices to prove that $\mathrm{Ext}_X^i(P, P)$ is zero for $i \neq 0$ and equal to R for $i = 0$, see Differential Graded Algebra, Lemma 26.6. As in the proof of Lemma 17.2 we see that

$$\mathrm{Ext}_X^i(P, P) = H^i(X, P^\wedge \otimes P) = \bigoplus_{0 \leq a, b \leq n} H^i(X, \mathcal{O}_X(a-b))$$

By the computation of cohomology of projective space (Cohomology of Schemes, Lemma 8.1) we find that these Ext-groups are zero unless $i = 0$. For $i = 0$ we recover R because this is how we defined R in (19.0.2). By Differential Graded Algebra, Lemma 26.5 our functor has a right adjoint, namely $R\mathrm{Hom}(P, -) : D_{QCoh}(\mathcal{O}_X) \rightarrow D(R)$. Since P is a generator for $D_{QCoh}(\mathcal{O}_X)$ by Lemma 15.3 we see that the kernel of $R\mathrm{Hom}(P, -)$ is zero. Hence our functor is an equivalence of triangulated categories by Derived Categories, Lemma 7.2. \square

20. The coherator revisited

0CQZ In Section 6 we constructed and studied the right adjoint RQ_X to the canonical functor $D(QCoh(\mathcal{O}_X)) \rightarrow D(\mathcal{O}_X)$. It was constructed as the right derived extension of the coherator $Q_X : \mathrm{Mod}(\mathcal{O}_X) \rightarrow QCoh(\mathcal{O}_X)$. In this section, we study when the inclusion functor

$$D_{QCoh}(\mathcal{O}_X) \longrightarrow D(\mathcal{O}_X)$$

has a right adjoint. If this right adjoint exists, we will denote it ³ by

$$DQ_X : D(\mathcal{O}_X) \longrightarrow D_{QCoh}(\mathcal{O}_X)$$

It turns out that quasi-compact and quasi-separated schemes have such a right adjoint.

³This is probably nonstandard notation. However, we have already used Q_X for the coherator and RQ_X for its derived extension.

0CR0 **Lemma 20.1.** *Let X be a quasi-compact and quasi-separated scheme. The inclusion functor $D_{QCoh}(\mathcal{O}_X) \rightarrow D(\mathcal{O}_X)$ has a right adjoint DQ_X .*

First proof. We will use the induction principle as in Cohomology of Schemes, Lemma 4.1 to prove this. If $D(QCoh(\mathcal{O}_X)) \rightarrow D_{QCoh}(\mathcal{O}_X)$ is an equivalence, then the lemma is true because the functor RQ_X of Section 6 is a right adjoint to the functor $D(QCoh(\mathcal{O}_X)) \rightarrow D(\mathcal{O}_X)$. In particular, our lemma is true for affine schemes, see Lemma 6.3. Thus we see that it suffices to show: if $X = U \cup V$ is a union of two quasi-compact opens and the lemma holds for U , V , and $U \cap V$, then the lemma holds for X .

The adjoint exists if and only if for every object K of $D(\mathcal{O}_X)$ we can find a distinguished triangle

$$E' \rightarrow E \rightarrow K \rightarrow E'[1]$$

in $D(\mathcal{O}_X)$ such that E' is in $D_{QCoh}(\mathcal{O}_X)$ and such that $\text{Hom}(M, K) = 0$ for all M in $D_{QCoh}(\mathcal{O}_X)$. See Derived Categories, Lemma 36.3. Consider the distinguished triangle

$$E \rightarrow Rj_{U,*}E|_U \oplus Rj_{V,*}E|_V \rightarrow Rj_{U \cap V,*}E|_{U \cap V} \rightarrow E[1]$$

in $D(\mathcal{O}_X)$ of Cohomology, Lemma 31.2. By Derived Categories, Lemma 36.2 it suffices to construct the desired distinguished triangles for $Rj_{U,*}E|_U$, $Rj_{V,*}E|_V$, and $Rj_{U \cap V,*}E|_{U \cap V}$. This reduces us to the statement discussed in the next paragraph.

Let $j : U \rightarrow X$ be an open immersion corresponding with U a quasi-compact open for which the lemma is true. Let L be an object of $D(\mathcal{O}_U)$. Then there exists a distinguished triangle

$$E' \rightarrow Rj_*L \rightarrow K \rightarrow E'[1]$$

in $D(\mathcal{O}_X)$ such that E' is in $D_{QCoh}(\mathcal{O}_X)$ and such that $\text{Hom}(M, K) = 0$ for all M in $D_{QCoh}(\mathcal{O}_X)$. To see this we choose a distinguished triangle

$$L' \rightarrow L \rightarrow Q \rightarrow L'[1]$$

in $D(\mathcal{O}_U)$ such that L' is in $D_{QCoh}(\mathcal{O}_U)$ and such that $\text{Hom}(N, Q) = 0$ for all N in $D_{QCoh}(\mathcal{O}_U)$. This is possible because the statement in Derived Categories, Lemma 36.3 is an if and only if. We obtain a distinguished triangle

$$Rj_*L' \rightarrow Rj_*L \rightarrow Rj_*Q \rightarrow Rj_*L'[1]$$

in $D(\mathcal{O}_X)$. Observe that Rj_*L' is in $D_{QCoh}(\mathcal{O}_X)$ by Lemma 4.1. On the other hand, if M in $D_{QCoh}(\mathcal{O}_X)$, then

$$\text{Hom}(M, Rj_*Q) = \text{Hom}(Lj^*M, Q) = 0$$

because Lj^*M is in $D_{QCoh}(\mathcal{O}_U)$ by Lemma 3.8. This finishes the proof. \square

Second proof. The adjoint exists by Derived Categories, Proposition 35.2. The hypotheses are satisfied: First, note that $D_{QCoh}(\mathcal{O}_X)$ has direct sums and direct sums commute with the inclusion functor (Lemma 3.1). On the other hand, $D_{QCoh}(\mathcal{O}_X)$ is compactly generated because it has a perfect generator Theorem 14.3 and because perfect objects are compact by Proposition 16.1. \square

0CR1 **Lemma 20.2.** *Let $f : X \rightarrow Y$ be a quasi-compact and quasi-separated morphism of schemes. If the right adjoints DQ_X and DQ_Y of the inclusion functors $D_{QCoh} \rightarrow D$ exist for X and Y , then*

$$Rf_* \circ DQ_X = DQ_Y \circ Rf_*$$

Proof. The statement makes sense because Rf_* sends $D_{QCoh}(\mathcal{O}_X)$ into $D_{QCoh}(\mathcal{O}_Y)$ by Lemma 4.1. The statement is true because Lf^* similarly maps $D_{QCoh}(\mathcal{O}_Y)$ into $D_{QCoh}(\mathcal{O}_X)$ (Lemma 3.8) and hence both $Rf_* \circ DQ_X$ and $DQ_Y \circ Rf_*$ are right adjoint to $Lf^* : D_{QCoh}(\mathcal{O}_Y) \rightarrow D(\mathcal{O}_X)$. \square

0CR2 **Remark 20.3.** Let X be a quasi-compact and quasi-separated scheme. Let $X = U \cup V$ with U and V quasi-compact open. By Lemma 20.1 the functors DQ_X , DQ_U , DQ_V , $DQ_{U \cap V}$ exist. Moreover, there is a canonical distinguished triangle

$$DQ_X(K) \rightarrow Rj_{U,*}DQ_U(K|_U) \oplus Rj_{V,*}DQ_V(K|_V) \rightarrow Rj_{U \cap V,*}DQ_{U \cap V}(K|_{U \cap V}) \rightarrow$$

for any $K \in D(\mathcal{O}_X)$. This follows by applying the exact functor DQ_X to the distinguished triangle of Cohomology, Lemma 31.2 and using Lemma 20.2 three times.

0CSA **Lemma 20.4.** *Let X be a quasi-compact and quasi-separated scheme. The functor DQ_X of Lemma 20.1 has the following boundedness property: there exists an integer $N = N(X)$ such that, if K in $D(\mathcal{O}_X)$ with $H^i(U, K) = 0$ for U affine open in X and $i \notin [a, b]$, then the cohomology sheaves $H^i(DQ_X(K))$ are zero for $i \notin [a, b + N]$.*

Proof. We will prove this using the induction principle of Cohomology of Schemes, Lemma 4.1.

If X is affine, then the lemma is true with $N = 0$ because then $RQ_X = DQ_X$ is given by taking the complex of quasi-coherent sheaves associated to $R\Gamma(X, K)$. See Lemmas 3.5 and 6.3.

Assume U, V are quasi-compact open in X and the lemma holds for U , V , and $U \cap V$. Say with integers $N(U)$, $N(V)$, and $N(U \cap V)$. Now suppose K is in $D(\mathcal{O}_X)$ with $H^i(W, K) = 0$ for all affine open $W \subset X$ and all $i \notin [a, b]$. Then $K|_U$, $K|_V$, $K|_{U \cap V}$ have the same property. Hence we see that $RQ_U(K|_U)$ and $RQ_V(K|_V)$ and $RQ_{U \cap V}(K|_{U \cap V})$ have vanishing cohomology sheaves outside the interval $[a, b + \max(N(U), N(V), N(U \cap V))]$. Since the functors $Rj_{U,*}$, $Rj_{V,*}$, $Rj_{U \cap V,*}$ have finite cohomological dimension on D_{QCoh} by Lemma 4.1 we see that there exists an N such that $Rj_{U,*}DQ_U(K|_U)$, $Rj_{V,*}DQ_V(K|_V)$, and $Rj_{U \cap V,*}DQ_{U \cap V}(K|_{U \cap V})$ have vanishing cohomology sheaves outside the interval $[a, b + N]$. Then finally we conclude by the distinguished triangle of Remark 20.3. \square

0CSB **Example 20.5.** Let X be a quasi-compact and quasi-separated scheme. Let (\mathcal{F}_n) be an inverse system of quasi-coherent sheaves. Since DQ_X is a right adjoint it commutes with products and therefore with derived limits. Hence we see that

$$DQ_X(R \lim \mathcal{F}_n) = (R \lim \text{ in } D_{QCoh}(\mathcal{O}_X))(\mathcal{F}_n)$$

where the first $R \lim$ is taken in $D(\mathcal{O}_X)$. In fact, let's write $K = R \lim \mathcal{F}_n$ for this. For any affine open $U \subset X$ we have

$$H^i(U, K) = H^i(R\Gamma(U, R \lim \mathcal{F}_n)) = H^i(R \lim R\Gamma(U, \mathcal{F}_n)) = H^i(R \lim \Gamma(U, \mathcal{F}_n))$$

since cohomology commutes with derived limits and since the quasi-coherent sheaves \mathcal{F}_n have no higher cohomology on affines. By the computation of $R \lim$ in the category of abelian groups, we see that $H^i(U, K) = 0$ unless $i \in [0, 1]$. Then finally we conclude that the $R \lim$ in $D_{QCoh}(\mathcal{O}_X)$, which is $DQ_X(K)$ by the above, is in $D_{QCoh}^b(\mathcal{O}_X)$ by Lemma 20.4.

21. Cohomology and base change, IV

08ET This section continues the discussion of Cohomology of Schemes, Section 22. First, we have a very general version of the projection formula for quasi-compact and quasi-separated morphisms of schemes and complexes with quasi-coherent cohomology sheaves.

08EU **Lemma 21.1.** *Let $f : X \rightarrow Y$ be a quasi-compact and quasi-separated morphism of schemes. For E in $D_{QCoh}(\mathcal{O}_X)$ and K in $D_{QCoh}(\mathcal{O}_Y)$ the map*

$$Rf_*(E) \otimes_{\mathcal{O}_Y}^L K \longrightarrow Rf_*(E \otimes_{\mathcal{O}_X}^L Lf^*K)$$

defined in Cohomology, Equation (45.2.1) is an isomorphism.

Proof. To check the map is an isomorphism we may work locally on Y . Hence we reduce to the case that Y is affine.

Suppose that $K = \bigoplus \tilde{K}_i$ is a direct sum of some complexes $\tilde{K}_i \in D_{QCoh}(\mathcal{O}_Y)$. If the statement holds for each \tilde{K}_i , then it holds for K . Namely, the functors Lf^* and \otimes^L preserve direct sums by construction and Rf_* commutes with direct sums (for complexes with quasi-coherent cohomology sheaves) by Lemma 4.2. Moreover, suppose that $K \rightarrow L \rightarrow M \rightarrow K[1]$ is a distinguished triangle in $D_{QCoh}(Y)$. Then if the statement of the lemma holds for two of K, L, M , then it holds for the third (as the functors involved are exact functors of triangulated categories).

Assume Y affine, say $Y = \text{Spec}(A)$. The functor $\tilde{} : D(A) \rightarrow D_{QCoh}(\mathcal{O}_Y)$ is an equivalence (Lemma 3.5). Let T be the property for $K \in D(A)$ that the statement of the lemma holds for \tilde{K} . The discussion above and More on Algebra, Remark 55.12 shows that it suffices to prove T holds for $A[k]$. This finishes the proof, as the statement of the lemma is clear for shifts of the structure sheaf. \square

08IA **Definition 21.2.** Let S be a scheme. Let X, Y be schemes over S . We say X and Y are *Tor independent over S* if for every $x \in X$ and $y \in Y$ mapping to the same point $s \in S$ the rings $\mathcal{O}_{X,x}$ and $\mathcal{O}_{Y,y}$ are Tor independent over $\mathcal{O}_{S,s}$ (see More on Algebra, Definition 57.1).

08IB **Lemma 21.3.** *Let $g : S' \rightarrow S$ be a morphism of schemes. Let $f : X \rightarrow S$ be quasi-compact and quasi-separated. Consider the base change diagram*

$$\begin{array}{ccc} X' & \xrightarrow{\quad} & X \\ f' \downarrow & & \downarrow f \\ S' & \xrightarrow{\quad g \quad} & S \end{array}$$

*If X and S' are Tor independent over S , then for all $E \in D_{QCoh}(\mathcal{O}_X)$ we have $Rf'_*L(g')^*E = Lg^*Rf_*E$.*

Proof. For any object E of $D(\mathcal{O}_X)$ we can use Cohomology, Remark 29.3 to get a canonical base change map $Lg^*Rf_*E \rightarrow Rf'_*L(g')^*E$. To check this is an isomorphism we may work locally on S' . Hence we may assume $g : S' \rightarrow S$ is a morphism of affine schemes. In particular, g is affine and it suffices to show that

$$Rg_*Lg^*Rf_*E \rightarrow Rg_*Rf'_*L(g')^*E = Rf_*(Rh_*L(g')^*E)$$

is an isomorphism, see Lemma 5.1 (and use Lemmas 3.8, 3.9, and 4.1 to see that the objects $Rf'_*L(g')^*E$ and Lg^*Rf_*E have quasi-coherent cohomology sheaves).

Note that g' is affine as well (Morphisms, Lemma 11.8). By Lemma 5.2 the map becomes a map

$$Rf_*E \otimes_{\mathcal{O}_S}^{\mathbf{L}} g_*\mathcal{O}_{S'} \longrightarrow Rf_*(E \otimes_{\mathcal{O}_X}^{\mathbf{L}} g'_*\mathcal{O}_{X'})$$

Observe that $g'_*\mathcal{O}_{X'} = f^*g_*\mathcal{O}_{S'}$. Thus by Lemma 21.1 it suffices to prove that $Lf^*g_*\mathcal{O}_{S'} = f^*g_*\mathcal{O}_{S'}$. This follows from our assumption that X and S' are Tor independent over S . Namely, to check it we may work locally on X , hence we may also assume X is affine. Say $X = \text{Spec}(A)$, $S = \text{Spec}(R)$ and $S' = \text{Spec}(R')$. Our assumption implies that A and R' are Tor independent over R (More on Algebra, Lemma 57.5), i.e., $\text{Tor}_i^R(A, R') = 0$ for $i > 0$. In other words $A \otimes_R^{\mathbf{L}} R' = A \otimes_R R'$ which exactly means that $Lf^*g_*\mathcal{O}_{S'} = f^*g_*\mathcal{O}_{S'}$ (use Lemma 3.8). \square

The following lemma will be used in the chapter on dualizing complexes.

0AA7 **Lemma 21.4.** *Consider a cartesian square*

$$\begin{array}{ccc} X' & \longrightarrow & X \\ f' \downarrow & & \downarrow f \\ S' & \xrightarrow{g} & S \end{array}$$

of quasi-compact and quasi-separated schemes. Assume g and f Tor independent and $S = \text{Spec}(R)$, $S' = \text{Spec}(R')$ affine. For $M, K \in D(\mathcal{O}_X)$ the canonical map

$$R\text{Hom}_X(M, K) \otimes_R^{\mathbf{L}} R' \longrightarrow R\text{Hom}_{X'}(L(g')^*M, L(g')^*K)$$

in $D(R')$ is an isomorphism in the following two cases

- (1) $M \in D(\mathcal{O}_X)$ is perfect and $K \in D_{Q\text{Coh}}(X)$, or
- (2) $M \in D(\mathcal{O}_X)$ is pseudo-coherent, $K \in D_{Q\text{Coh}}^+(X)$, and R' has finite tor dimension over R .

Proof. There is a canonical map $R\text{Hom}_X(M, K) \rightarrow R\text{Hom}_{X'}(L(g')^*M, L(g')^*K)$ in $D(\Gamma(X, \mathcal{O}_X))$ of global hom complexes, see Cohomology, Section 38. Restricting scalars we can view this as a map in $D(R)$. Then we can use the adjointness of restriction and $-\otimes_R^{\mathbf{L}} R'$ to get the displayed map of the lemma. Having defined the map it suffices to prove it is an isomorphism in the derived category of abelian groups.

The right hand side is equal to

$$R\text{Hom}_X(M, R(g')_*L(g')^*K) = R\text{Hom}_X(M, K \otimes_{\mathcal{O}_X}^{\mathbf{L}} g'_*\mathcal{O}_{X'})$$

by Lemma 5.2. In both cases the complex $R\mathcal{H}om(M, K)$ is an object of $D_{Q\text{Coh}}(\mathcal{O}_X)$ by Lemma 9.8. There is a natural map

$$R\mathcal{H}om(M, K) \otimes_{\mathcal{O}_X}^{\mathbf{L}} g'_*\mathcal{O}_{X'} \longrightarrow R\mathcal{H}om(M, K \otimes_{\mathcal{O}_X}^{\mathbf{L}} g'_*\mathcal{O}_{X'})$$

which is an isomorphism in both cases Lemma 9.9. To see that this lemma applies in case (2) we note that $g'_*\mathcal{O}_{X'} = Rg'_*\mathcal{O}_{X'} = Lf^*g_*\mathcal{O}_X$ the second equality by Lemma 21.3. Using Lemma 9.4 and Cohomology, Lemma 42.4 we conclude that $g'_*\mathcal{O}_{X'}$ has finite Tor dimension. Hence, in both cases by replacing K by $R\mathcal{H}om(M, K)$ we reduce to proving

$$R\Gamma(X, K) \otimes_A^{\mathbf{L}} A' \longrightarrow R\Gamma(X, K \otimes_{\mathcal{O}_X}^{\mathbf{L}} g'_*\mathcal{O}_{X'})$$

is an isomorphism. Note that the left hand side is equal to $R\Gamma(X', L(g')^*K)$ by Lemma 5.2. Hence the result follows from Lemma 21.3. \square

0BZA **Remark 21.5.** With notation as in Lemma 21.4. The diagram

$$\begin{array}{ccc} R\mathrm{Hom}_X(M, Rg'_*L) \otimes_R^{\mathbf{L}} R' & \longrightarrow & R\mathrm{Hom}_{X'}(L(g')^*M, L(g')^*Rg'_*L) \\ \mu \downarrow & & \downarrow a \\ R\mathrm{Hom}_X(M, R(g')_*L) & \xlongequal{\quad} & R\mathrm{Hom}_{X'}(L(g')^*M, L) \end{array}$$

is commutative where the top horizontal arrow is the map from the lemma, μ is the multiplication map, and a comes from the adjunction map $L(g')^*Rg'_*L \rightarrow L$. The multiplication map is the adjunction map $K' \otimes_R^{\mathbf{L}} R' \rightarrow K'$ for any $K' \in D(R')$.

0C0V **Lemma 21.6.** *Consider a cartesian square of schemes*

$$\begin{array}{ccc} X' & \longrightarrow & X \\ f' \downarrow & & \downarrow f \\ S' & \xrightarrow{g} & S \end{array}$$

Assume g and f Tor independent.

- (1) *If $E \in D(\mathcal{O}_X)$ has tor amplitude in $[a, b]$ as a complex of $f^{-1}\mathcal{O}_S$ -modules, then $L(g')^*E$ has tor amplitude in $[a, b]$ as a complex of $f'^{-1}\mathcal{O}_{S'}$ -modules.*
- (2) *If \mathcal{G} is an \mathcal{O}_X -module flat over S , then $L(g')^*\mathcal{G} = (g')^*\mathcal{G}$.*

Proof. We can compute tor dimension at stalks, see Cohomology, Lemma 42.5. If $x' \in X'$ with image $x \in X$, then

$$(L(g')^*E)_{x'} = E_x \otimes_{\mathcal{O}_{X,x}}^{\mathbf{L}} \mathcal{O}_{X',x'}$$

Let $s' \in S'$ and $s \in S$ be the image of x' and x . Since X and S' are tor independent over S , we can apply More on Algebra, Lemma 57.2 to see that the right hand side of the displayed formula is equal to $E_x \otimes_{\mathcal{O}_{S,s}}^{\mathbf{L}} \mathcal{O}_{S',s'}$ in $D(\mathcal{O}_{S',s'})$. Thus (1) follows from More on Algebra, Lemma 61.13. To see (2) observe that flatness of \mathcal{G} is equivalent to the condition that $\mathcal{G}[0]$ has tor amplitude in $[0, 0]$. Applying (1) we conclude. \square

22. Cohomology and base change, V

0DJ6 In Section 21 we saw a base change theorem holds when the morphisms are tor independent. Even in the affine case there cannot be a base change theorem without such a condition, see More on Algebra, Section 57. In this section we analyze when one can get a base change result “one complex at a time”.

To make this work, suppose we have a commutative diagram

$$\begin{array}{ccc} X' & \longrightarrow & X \\ f' \downarrow & & \downarrow f \\ S' & \xrightarrow{g} & S \end{array}$$

of schemes (usually we will assume it is cartesian). Let $K \in D_{Qcoh}(\mathcal{O}_X)$ and let $L(g')^*K \rightarrow K'$ be a map in $D_{Qcoh}(\mathcal{O}_{X'})$. For a point $x' \in X'$ set $x = g'(x') \in X$, $s' = f'(x') \in S'$ and $s = f(x) = g(s')$. Then we can consider the maps

$$K_x \otimes_{\mathcal{O}_{S,s}}^{\mathbf{L}} \mathcal{O}_{S',s'} \rightarrow K_x \otimes_{\mathcal{O}_{X,x}}^{\mathbf{L}} \mathcal{O}_{X',x'} \rightarrow K'_{x'}$$

where the first arrow is More on Algebra, Equation (57.0.1) and the second comes from $(L(g')^*K)_{x'} = K_x \otimes_{\mathcal{O}_{X,x}}^{\mathbf{L}} \mathcal{O}_{X',x'}$ and the given map $L(g')^*K \rightarrow K'$. For each $i \in \mathbf{Z}$ we obtain a $\mathcal{O}_{X,x} \otimes_{\mathcal{O}_{S,s}} \mathcal{O}_{S',s'}$ -module structure on $H^i(K_x \otimes_{\mathcal{O}_{S,s}}^{\mathbf{L}} \mathcal{O}_{S',s'})$. Putting everything together we obtain canonical maps

$$0DJ7 \quad (22.0.1) \quad H^i(K_x \otimes_{\mathcal{O}_{S,s}}^{\mathbf{L}} \mathcal{O}_{S',s'}) \otimes_{(\mathcal{O}_{X,x} \otimes_{\mathcal{O}_{S,s}} \mathcal{O}_{S',s'})} \mathcal{O}_{X',x'} \longrightarrow H^i(K'_{x'})$$

of $\mathcal{O}_{X',x'}$ -modules.

0DJ8 **Lemma 22.1.** *Let*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ f' \downarrow & & \downarrow f \\ S' & \xrightarrow{g} & S \end{array}$$

be a cartesian diagram of schemes. Let $K \in D_{QCoh}(\mathcal{O}_X)$ and let $L(g')^*K \rightarrow K'$ be a map in $D_{QCoh}(\mathcal{O}_{X'})$. The following are equivalent

- (1) for any $x' \in X'$ and $i \in \mathbf{Z}$ the map (22.0.1) is an isomorphism,
- (2) for $U \subset X$, $V' \subset S'$ affine open both mapping into the affine open $V \subset S$ with $U' = V' \times_V U$ the composition

$$R\Gamma(U, K) \otimes_{\mathcal{O}_S(U)}^{\mathbf{L}} \mathcal{O}_{S'}(V') \rightarrow R\Gamma(U, K) \otimes_{\mathcal{O}_X(U)}^{\mathbf{L}} \mathcal{O}_{X'}(U') \rightarrow R\Gamma(U', K')$$

is an isomorphism in $D(\mathcal{O}_{S'}(V'))$, and

- (3) there is a set I of quadruples U_i, V'_i, V_i, U'_i , $i \in I$ as in (2) with $X' = \bigcup U'_i$.

Proof. The second arrow in (2) comes from the equality

$$R\Gamma(U, K) \otimes_{\mathcal{O}_X(U)}^{\mathbf{L}} \mathcal{O}_{X'}(U') = R\Gamma(U', L(g')^*K)$$

of Lemma 3.8 and the given arrow $L(g')^*K \rightarrow K'$. The first arrow of (2) is More on Algebra, Equation (57.0.1). It is clear that (2) implies (3). Observe that (1) is local on X' . Therefore it suffices to show that if X, S, S', X' are affine, then (1) is equivalent to the condition that

$$R\Gamma(X, K) \otimes_{\mathcal{O}_S(S)}^{\mathbf{L}} \mathcal{O}_{S'}(S') \rightarrow R\Gamma(X, K) \otimes_{\mathcal{O}_X(X)}^{\mathbf{L}} \mathcal{O}_{X'}(X') \rightarrow R\Gamma(X', K')$$

is an isomorphism in $D(\mathcal{O}_{S'}(S'))$. Say $S = \text{Spec}(R)$, $X = \text{Spec}(A)$, $S' = \text{Spec}(R')$, $X' = \text{Spec}(A')$, K corresponds to the complex M^\bullet of A -modules, and K' corresponds to the complex N^\bullet of A' -modules. Note that $A' = A \otimes_R R'$. The condition above is that the composition

$$M^\bullet \otimes_R^{\mathbf{L}} R' \rightarrow M^\bullet \otimes_A^{\mathbf{L}} A' \rightarrow N^\bullet$$

is an isomorphism in $D(R')$. Equivalently, it is that for all $i \in \mathbf{Z}$ the map

$$H^i(M^\bullet \otimes_R^{\mathbf{L}} R') \rightarrow H^i(M^\bullet \otimes_A^{\mathbf{L}} A') \rightarrow H^i(N^\bullet)$$

is an isomorphism. Observe that this is a map of $A \otimes_R R'$ -modules, i.e., of A' -modules. On the other hand, (1) is the requirement that for compatible primes $\mathfrak{q}' \subset A'$, $\mathfrak{q} \subset A$, $\mathfrak{p}' \subset R'$, $\mathfrak{p} \subset R$ the composition

$$H^i(M_{\mathfrak{q}}^\bullet \otimes_{R_{\mathfrak{p}}}^{\mathbf{L}} R'_{\mathfrak{p}'}) \otimes_{(A_{\mathfrak{q}} \otimes_{R_{\mathfrak{p}}} R'_{\mathfrak{p}'})} A'_{\mathfrak{q}'} \rightarrow H^i(M_{\mathfrak{q}}^\bullet \otimes_{A_{\mathfrak{q}}}^{\mathbf{L}} A'_{\mathfrak{q}'}) \rightarrow H^i(N_{\mathfrak{q}'}^\bullet)$$

is an isomorphism. Since

$$H^i(M_{\mathfrak{q}}^\bullet \otimes_{R_{\mathfrak{p}}}^{\mathbf{L}} R'_{\mathfrak{p}'}) \otimes_{(A_{\mathfrak{q}} \otimes_{R_{\mathfrak{p}}} R'_{\mathfrak{p}'})} A'_{\mathfrak{q}'} = H^i(M^\bullet \otimes_R^{\mathbf{L}} R') \otimes_{A'} A'_{\mathfrak{q}'}$$

is the localization at \mathfrak{q}' , we see that these two conditions are equivalent by Algebra, Lemma 23.1. \square

0DJ9 **Lemma 22.2.** *Let*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ f' \downarrow & & \downarrow f \\ S' & \xrightarrow{g} & S \end{array}$$

be a cartesian diagram of schemes. Let $K \in D_{QCoh}(\mathcal{O}_X)$ and let $L(g')^*K \rightarrow K'$ be a map in $D_{QCoh}(\mathcal{O}_{X'})$. If

- (1) the equivalent conditions of Lemma 22.1 hold, and
- (2) f is quasi-compact and quasi-separated,

then the composition $Lg^*Rf_*K \rightarrow Rf'_*L(g')^*K \rightarrow Rf'_*K'$ is an isomorphism.

Proof. We could prove this using the same method as in the proof of Lemma 21.3 but instead we will prove it using the induction principle and relative Mayer-Vietoris.

To check the map is an isomorphism we may work locally on S' . Hence we may assume $g : S' \rightarrow S$ is a morphism of affine schemes. In particular X is a quasi-compact and quasi-separated scheme. We will use the induction principle of Cohomology of Schemes, Lemma 4.1 to prove that for any quasi-compact open $U \subset X$ the similarly constructed map $Lg^*R(U \rightarrow S)_*K|_U \rightarrow R(U' \rightarrow S')_*K'|_{U'}$ is an isomorphism. Here $U' = (g')^{-1}(U)$.

If $U \subset X$ is an affine open, then we find that the result is true by assumption, see Lemma 22.1 part (2) and the translation into algebra afforded to us by Lemmas 3.5 and 3.8.

The induction step. Suppose that $X = U \cup V$ is an open covering with $U, V, U \cap V$ quasi-compact such that the result holds for U, V , and $U \cap V$. Denote $a = f|_U$, $b = f|_V$ and $c = f|_{U \cap V}$. Let $a' : U' \rightarrow S'$, $b' : V' \rightarrow S'$ and $c' : U' \cap V' \rightarrow S'$ be the base changes of a, b , and c . Using the distinguished triangles from relative Mayer-Vietoris (Cohomology, Lemma 31.5) we obtain a commutative diagram

$$\begin{array}{ccc} Lg^*Rf_*K & \longrightarrow & Rf'_*K' \\ \downarrow & & \downarrow \\ Lg^*Ra_*K|_U \oplus Lg^*Rb_*K|_V & \longrightarrow & Ra'_*K'|_{U'} \oplus Rb'_*K'|_{V'} \\ \downarrow & & \downarrow \\ Lg^*Rc_*K|_{U \cap V} & \longrightarrow & Rc'_*K'|_{U' \cap V'} \\ \downarrow & & \downarrow \\ Lg^*Rf_*K[1] & \longrightarrow & Rf'_*K'[1] \end{array}$$

Since the 2nd and 3rd horizontal arrows are isomorphisms so is the first (Derived Categories, Lemma 4.3) and the proof of the lemma is finished. \square

0DJA **Lemma 22.3.** *Let*

$$\begin{array}{ccc} X' & \xrightarrow{\quad} & X \\ f' \downarrow & & \downarrow f \\ S' & \xrightarrow{\quad g \quad} & S \end{array}$$

be a cartesian diagram of schemes. Let $K \in D_{QCoh}(\mathcal{O}_X)$ and let $L(g')^*K \rightarrow K'$ be a map in $D_{QCoh}(\mathcal{O}_{X'})$. If the equivalent conditions of Lemma 22.1 hold, then

- (1) for $E \in D_{QCoh}(\mathcal{O}_X)$ the equivalent conditions of Lemma 22.1 hold for $L(g')^*(E \otimes^{\mathbf{L}} K) \rightarrow L(g')^*E \otimes^{\mathbf{L}} K'$,
- (2) if E in $D(\mathcal{O}_X)$ is perfect the equivalent conditions of Lemma 22.1 hold for $L(g')^*R\mathcal{H}om(E, K) \rightarrow R\mathcal{H}om(L(g')^*E, K')$, and
- (3) if K is bounded below and E in $D(\mathcal{O}_X)$ pseudo-coherent the equivalent conditions of Lemma 22.1 hold for $L(g')^*R\mathcal{H}om(E, K) \rightarrow R\mathcal{H}om(L(g')^*E, K')$.

Proof. The statement makes sense as the complexes involved have quasi-coherent cohomology sheaves by Lemmas 3.8, 3.9, and 9.8 and Cohomology, Lemmas 41.3 and 43.6. Having said this, we can check the maps (22.0.1) are isomorphisms in case (1) by computing the source and target of (22.0.1) using the transitive property of tensor product, see More on Algebra, Lemma 55.16. The map in (2) and (3) is the composition

$$L(g')^*R\mathcal{H}om(E, K) \rightarrow R\mathcal{H}om(L(g')^*E, L(g')^*K) \rightarrow R\mathcal{H}om(L(g')^*E, K')$$

where the first arrow is Cohomology, Remark 36.11 and the second arrow comes from the given map $L(g')^*K \rightarrow K'$. To prove the maps (22.0.1) are isomorphisms one represents E_x by a bounded complex of finite projective $\mathcal{O}_{X,x}$ -modules in case (2) or by a bounded above complex of finite free modules in case (3) and computes the source and target of the arrow. Some details omitted. \square

0A1D **Lemma 22.4.** *Let $f : X \rightarrow S$ be a quasi-compact and quasi-separated morphism of schemes. Let $E \in D_{QCoh}(\mathcal{O}_X)$. Let \mathcal{G}^\bullet be a bounded above complex of quasi-coherent \mathcal{O}_X -modules flat over S . Then formation of*

$$Rf_*(E \otimes_{\mathcal{O}_X}^{\mathbf{L}} \mathcal{G}^\bullet)$$

commutes with arbitrary base change (see proof for precise statement).

Proof. The statement means the following. Let $g : S' \rightarrow S$ be a morphism of schemes and consider the base change diagram

$$\begin{array}{ccc} X' & \xrightarrow{\quad} & X \\ f' \downarrow & & \downarrow f \\ S' & \xrightarrow{\quad g \quad} & S \end{array}$$

in other words $X' = S' \times_S X$. The lemma asserts that

$$Lg^*Rf_*(E \otimes_{\mathcal{O}_X}^{\mathbf{L}} \mathcal{G}^\bullet) \longrightarrow Rf'_*(L(g')^*E \otimes_{\mathcal{O}_{X'}}^{\mathbf{L}} (g')^*\mathcal{G}^\bullet)$$

is an isomorphism. Observe that on the right hand side we do **not** use the derived pullback on \mathcal{G}^\bullet . To prove this, we apply Lemmas 22.2 and 22.3 to see that it suffices to prove the canonical map

$$L(g')^*\mathcal{G}^\bullet \rightarrow (g')^*\mathcal{G}^\bullet$$

satisfies the equivalent conditions of Lemma 22.1. This follows by checking the condition on stalks, where it immediately follows from the fact that $\mathcal{G}_x^\bullet \otimes_{\mathcal{O}_{S,s}} \mathcal{O}_{S',s'}$ computes the derived tensor product by our assumptions on the complex \mathcal{G}^\bullet . \square

08IE **Lemma 22.5.** *Let $f : X \rightarrow S$ be a quasi-compact and quasi-separated morphism of schemes. Let E be an object of $D(\mathcal{O}_X)$. Let \mathcal{G}^\bullet be a complex of quasi-coherent \mathcal{O}_X -modules. If*

- (1) *E is perfect, \mathcal{G}^\bullet is a bounded above, and \mathcal{G}^n is flat over S , or*
- (2) *E is pseudo-coherent, \mathcal{G}^\bullet is bounded, and \mathcal{G}^n is flat over S ,*

then formation of

$$Rf_* R\mathcal{H}om(E, \mathcal{G}^\bullet)$$

commutes with arbitrary base change (see proof for precise statement).

Proof. The statement means the following. Let $g : S' \rightarrow S$ be a morphism of schemes and consider the base change diagram

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ f' \downarrow & & \downarrow f \\ S' & \xrightarrow{g} & S \end{array}$$

in other words $X' = S' \times_S X$. The lemma asserts that

$$Lg^* Rf_* R\mathcal{H}om(E, \mathcal{G}^\bullet) \longrightarrow R(f')_* R\mathcal{H}om(L(g')^* E, (g')^* \mathcal{G}^\bullet)$$

is an isomorphism. Observe that on the right hand side we do **not** use the derived pullback on \mathcal{G}^\bullet . To prove this, we apply Lemmas 22.2 and 22.3 to see that it suffices to prove the canonical map

$$L(g')^* \mathcal{G}^\bullet \rightarrow (g')^* \mathcal{G}^\bullet$$

satisfies the equivalent conditions of Lemma 22.1. This was shown in the proof of Lemma 22.4. \square

23. Producing perfect complexes

0A1E The following lemma is our main technical tool for producing perfect complexes. Later versions of this result will reduce to this by Noetherian approximation, see Section 26.

08EV **Lemma 23.1.** *Let S be a Noetherian scheme. Let $f : X \rightarrow S$ be a morphism of schemes which is locally of finite type. Let $E \in D(\mathcal{O}_X)$ such that*

- (1) *$E \in D_{Coh}^b(\mathcal{O}_X)$,*
- (2) *the support of $H^i(E)$ is proper over S for all i , and*
- (3) *E has finite tor dimension as an object of $D(f^{-1}\mathcal{O}_S)$.*

Then $Rf_ E$ is a perfect object of $D(\mathcal{O}_S)$.*

Proof. By Lemma 10.1 we see that $Rf_* E$ is an object of $D_{Coh}^b(\mathcal{O}_S)$. Hence $Rf_* E$ is pseudo-coherent (Lemma 9.3). Hence it suffices to show that $Rf_* E$ has finite tor dimension, see Cohomology, Lemma 43.5. By Lemma 9.6 it suffices to check that $Rf_*(E) \otimes_{\mathcal{O}_S}^L \mathcal{F}$ has universally bounded cohomology for all quasi-coherent sheaves \mathcal{F} on S . Bounded from above is clear as $Rf_*(E)$ is bounded from above. Let $T \subset X$ be the union of the supports of $H^i(E)$ for all i . Then T is proper over S by assumptions (1) and (2), see Cohomology of Schemes, Lemma 25.6. In particular

there exists a quasi-compact open $X' \subset X$ containing T . Setting $f' = f|_{X'}$ we have $Rf_*(E) = Rf'_*(E|_{X'})$ because E restricts to zero on $X \setminus T$. Thus we may replace X by X' and assume f is quasi-compact. Moreover, f is quasi-separated by Morphisms, Lemma 14.7. Now

$$Rf_*(E) \otimes_{\mathcal{O}_S}^{\mathbf{L}} \mathcal{F} = Rf_* \left(E \otimes_{\mathcal{O}_X}^{\mathbf{L}} Lf^* \mathcal{F} \right) = Rf_* \left(E \otimes_{f^{-1}\mathcal{O}_S}^{\mathbf{L}} f^{-1} \mathcal{F} \right)$$

by Lemma 21.1 and Cohomology, Lemma 28.4. By assumption (3) the complex $E \otimes_{f^{-1}\mathcal{O}_S}^{\mathbf{L}} f^{-1} \mathcal{F}$ has cohomology sheaves in a given finite range, say $[a, b]$. Then Rf_* of it has cohomology in the range $[a, \infty)$ and we win. \square

0DJQ **Lemma 23.2.** *Let S be a Noetherian scheme. Let $f : X \rightarrow S$ be a morphism of schemes which is locally of finite type. Let $E \in D(\mathcal{O}_X)$ be perfect. Let \mathcal{G}^\bullet be a bounded complex of coherent \mathcal{O}_X -modules flat over S with support proper over S . Then $K = Rf_*(E \otimes_{\mathcal{O}_X}^{\mathbf{L}} \mathcal{G}^\bullet)$ is a perfect object of $D(\mathcal{O}_S)$.*

Proof. The object K is perfect by Lemma 23.1. We check the lemma applies: Locally E is isomorphic to a finite complex of finite free \mathcal{O}_X -modules. Hence locally $E \otimes_{\mathcal{O}_X}^{\mathbf{L}} \mathcal{G}^\bullet$ is isomorphic to a finite complex whose terms are of the form

$$\bigoplus_{i=a, \dots, b} (\mathcal{G}^i)^{\oplus r_i}$$

for some integers a, b, r_a, \dots, r_b . This immediately implies the cohomology sheaves $H^i(E \otimes_{\mathcal{O}_X}^{\mathbf{L}} \mathcal{G}^\bullet)$ are coherent. The hypothesis on the tor dimension also follows as \mathcal{G}^i is flat over $f^{-1}\mathcal{O}_S$. \square

0DJR **Lemma 23.3.** *Let S be a Noetherian scheme. Let $f : X \rightarrow S$ be a morphism of schemes which is locally of finite type. Let $E \in D(\mathcal{O}_X)$ be perfect. Let \mathcal{G}^\bullet be a bounded complex of coherent \mathcal{O}_X -modules flat over S with support proper over S . Then $K = Rf_* R\mathcal{H}om(E, \mathcal{G}^\bullet)$ is a perfect object of $D(\mathcal{O}_S)$.*

Proof. Since E is a perfect complex there exists a dual perfect complex E^\vee , see Cohomology, Lemma 43.11. Observe that $R\mathcal{H}om(E, \mathcal{G}^\bullet) = E^\vee \otimes_{\mathcal{O}_X}^{\mathbf{L}} \mathcal{G}^\bullet$. Thus the perfectness of K follows from Lemma 23.2. \square

We will generalize the following lemma to flat and proper morphisms over general bases in Lemma 26.4 and to perfect proper morphisms in More on Morphisms, Lemma 50.12.

0B6F **Lemma 23.4.** *Let S be a Noetherian scheme. Let $f : X \rightarrow S$ be a flat proper morphism of schemes. Let $E \in D(\mathcal{O}_X)$ be perfect. Then $Rf_* E$ is a perfect object of $D(\mathcal{O}_S)$.*

Proof. We claim that Lemma 23.1 applies. Conditions (1) and (2) are immediate. Condition (3) is local on X . Thus we may assume X and S affine and E represented by a strictly perfect complex of \mathcal{O}_X -modules. Since \mathcal{O}_X is flat as a sheaf of $f^{-1}\mathcal{O}_S$ -modules we find that condition (3) is satisfied. \square

24. A projection formula for Ext

08IC Lemma 24.3 (or similar results in the literature) is sometimes used to verify one of Artin's criteria for Quot functors, Hilbert schemes, and other moduli problems.

Suppose that $f : X \rightarrow S$ is a proper, flat, finitely presented morphism of schemes and $E \in D(\mathcal{O}_X)$ is perfect. Here the lemma says

$$\mathrm{Ext}_X^i(E, f^*\mathcal{F}) = \mathrm{Ext}_S^i((Rf_*E^\vee)^\vee, \mathcal{F})$$

for \mathcal{F} quasi-coherent on S . Writing it this way makes it look like a projection formula for Ext and indeed the result follows rather easily from Lemma 21.1.

0A1F **Lemma 24.1.** *Assumptions and notation as in Lemma 23.2. Then there are functorial isomorphisms*

$$H^i(S, K \otimes_{\mathcal{O}_S}^{\mathbf{L}} \mathcal{F}) \longrightarrow H^i(X, E \otimes_{\mathcal{O}_X}^{\mathbf{L}} (\mathcal{G}^\bullet \otimes_{\mathcal{O}_X} f^*\mathcal{F}))$$

for \mathcal{F} quasi-coherent on S compatible with boundary maps (see proof).

Proof. We have

$$\mathcal{G}^\bullet \otimes_{\mathcal{O}_X}^{\mathbf{L}} Lf^*\mathcal{F} = \mathcal{G}^\bullet \otimes_{f^{-1}\mathcal{O}_S}^{\mathbf{L}} f^{-1}\mathcal{F} = \mathcal{G}^\bullet \otimes_{f^{-1}\mathcal{O}_S} f^{-1}\mathcal{F} = \mathcal{G}^\bullet \otimes_{\mathcal{O}_X} f^*\mathcal{F}$$

the first equality by Cohomology, Lemma 28.4, the second as \mathcal{G}^n is a flat $f^{-1}\mathcal{O}_S$ -module, and the third by definition of pullbacks. Hence we obtain

$$\begin{aligned} H^i(X, E \otimes_{\mathcal{O}_X}^{\mathbf{L}} (\mathcal{G}^\bullet \otimes_{\mathcal{O}_X} f^*\mathcal{F})) &= H^i(X, E \otimes_{\mathcal{O}_X}^{\mathbf{L}} \mathcal{G}^\bullet \otimes_{\mathcal{O}_X}^{\mathbf{L}} Lf^*\mathcal{F}) \\ &= H^i(S, Rf_*(E \otimes_{\mathcal{O}_X}^{\mathbf{L}} \mathcal{G}^\bullet \otimes_{\mathcal{O}_X}^{\mathbf{L}} Lf^*\mathcal{F})) \\ &= H^i(S, Rf_*(E \otimes_{\mathcal{O}_X}^{\mathbf{L}} \mathcal{G}^\bullet) \otimes_{\mathcal{O}_S}^{\mathbf{L}} \mathcal{F}) \\ &= H^i(S, K \otimes_{\mathcal{O}_S}^{\mathbf{L}} \mathcal{F}) \end{aligned}$$

The first equality by the above, the second by Leray (Cohomology, Lemma 14.1), and the third equality by Lemma 21.1. The statement on boundary maps means the following: Given a short exact sequence $0 \rightarrow \mathcal{F}_1 \rightarrow \mathcal{F}_2 \rightarrow \mathcal{F}_3 \rightarrow 0$ of quasi-coherent \mathcal{O}_S -modules, the isomorphisms fit into commutative diagrams

$$\begin{array}{ccc} H^i(S, K \otimes_{\mathcal{O}_S}^{\mathbf{L}} \mathcal{F}_3) & \longrightarrow & H^i(X, E \otimes_{\mathcal{O}_X}^{\mathbf{L}} (\mathcal{G}^\bullet \otimes_{\mathcal{O}_X} f^*\mathcal{F}_3)) \\ \delta \downarrow & & \downarrow \delta \\ H^{i+1}(S, K \otimes_{\mathcal{O}_S}^{\mathbf{L}} \mathcal{F}_1) & \longrightarrow & H^{i+1}(X, E \otimes_{\mathcal{O}_X}^{\mathbf{L}} (\mathcal{G}^\bullet \otimes_{\mathcal{O}_X} f^*\mathcal{F}_1)) \end{array}$$

where the boundary maps come from the distinguished triangle

$$K \otimes_{\mathcal{O}_S}^{\mathbf{L}} \mathcal{F}_1 \rightarrow K \otimes_{\mathcal{O}_S}^{\mathbf{L}} \mathcal{F}_2 \rightarrow K \otimes_{\mathcal{O}_S}^{\mathbf{L}} \mathcal{F}_3 \rightarrow K \otimes_{\mathcal{O}_S}^{\mathbf{L}} \mathcal{F}_1[1]$$

and the distinguished triangle in $D(\mathcal{O}_X)$ associated to the short exact sequence

$$0 \rightarrow \mathcal{G}^\bullet \otimes_{\mathcal{O}_X} f^*\mathcal{F}_1 \rightarrow \mathcal{G}^\bullet \otimes_{\mathcal{O}_X} f^*\mathcal{F}_2 \rightarrow \mathcal{G}^\bullet \otimes_{\mathcal{O}_X} f^*\mathcal{F}_3 \rightarrow 0$$

of complexes of \mathcal{O}_X -modules. This sequence is exact because \mathcal{G}^n is flat over S . We omit the verification of the commutativity of the displayed diagram. \square

08ID **Lemma 24.2.** *Assumptions and notation as in Lemma 23.3. Then there are functorial isomorphisms*

$$H^i(S, K \otimes_{\mathcal{O}_S}^{\mathbf{L}} \mathcal{F}) \longrightarrow \mathrm{Ext}_{\mathcal{O}_X}^i(E, \mathcal{G}^\bullet \otimes_{\mathcal{O}_X} f^*\mathcal{F})$$

for \mathcal{F} quasi-coherent on S compatible with boundary maps (see proof).

Proof. As in the proof of Lemma 23.3 let E^\vee be the dual perfect complex and recall that $K = Rf_*(E^\vee \otimes_{\mathcal{O}_X}^{\mathbf{L}} \mathcal{G}^\bullet)$. Since we also have

$$\mathrm{Ext}_{\mathcal{O}_X}^i(E, \mathcal{G}^\bullet \otimes_{\mathcal{O}_X} f^* \mathcal{F}) = H^i(X, E^\vee \otimes_{\mathcal{O}_X}^{\mathbf{L}} (\mathcal{G}^\bullet \otimes_{\mathcal{O}_X} f^* \mathcal{F}))$$

by construction of E^\vee , the existence of the isomorphisms follows from Lemma 24.1 applied to E^\vee and \mathcal{G}^\bullet . The statement on boundary maps means the following: Given a short exact sequence $0 \rightarrow \mathcal{F}_1 \rightarrow \mathcal{F}_2 \rightarrow \mathcal{F}_3 \rightarrow 0$ then the isomorphisms fit into commutative diagrams

$$\begin{array}{ccc} H^i(S, K \otimes_{\mathcal{O}_S}^{\mathbf{L}} \mathcal{F}_3) & \longrightarrow & \mathrm{Ext}_{\mathcal{O}_X}^i(E, \mathcal{G}^\bullet \otimes_{\mathcal{O}_X} f^* \mathcal{F}_3) \\ \delta \downarrow & & \downarrow \delta \\ H^{i+1}(S, K \otimes_{\mathcal{O}_S}^{\mathbf{L}} \mathcal{F}_1) & \longrightarrow & \mathrm{Ext}_{\mathcal{O}_X}^{i+1}(E, \mathcal{G}^\bullet \otimes_{\mathcal{O}_X} f^* \mathcal{F}_1) \end{array}$$

where the boundary maps come from the distinguished triangle

$$K \otimes_{\mathcal{O}_S}^{\mathbf{L}} \mathcal{F}_1 \rightarrow K \otimes_{\mathcal{O}_S}^{\mathbf{L}} \mathcal{F}_2 \rightarrow K \otimes_{\mathcal{O}_S}^{\mathbf{L}} \mathcal{F}_3 \rightarrow K \otimes_{\mathcal{O}_S}^{\mathbf{L}} \mathcal{F}_1[1]$$

and the distinguished triangle in $D(\mathcal{O}_X)$ associated to the short exact sequence

$$0 \rightarrow \mathcal{G}^\bullet \otimes_{\mathcal{O}_X} f^* \mathcal{F}_1 \rightarrow \mathcal{G}^\bullet \otimes_{\mathcal{O}_X} f^* \mathcal{F}_2 \rightarrow \mathcal{G}^\bullet \otimes_{\mathcal{O}_X} f^* \mathcal{F}_3 \rightarrow 0$$

of complexes. This sequence is exact because \mathcal{G} is flat over S . We omit the verification of the commutativity of the displayed diagram. \square

08IF **Lemma 24.3.** *Let $f : X \rightarrow S$ be a morphism of schemes, $E \in D(\mathcal{O}_X)$ and \mathcal{G}^\bullet a complex of \mathcal{O}_X -modules. Assume*

- (1) S is Noetherian,
- (2) f is locally of finite type,
- (3) $E \in D_{\mathrm{Coh}}^-(\mathcal{O}_X)$,
- (4) \mathcal{G}^\bullet is a bounded complex of coherent \mathcal{O}_X -modules flat over S with support proper over S .

Then the following two statements are true

- (A) for every $m \in \mathbf{Z}$ there exists a perfect object K of $D(\mathcal{O}_S)$ and functorial maps

$$\alpha_{\mathcal{F}}^i : \mathrm{Ext}_{\mathcal{O}_X}^i(E, \mathcal{G}^\bullet \otimes_{\mathcal{O}_X} f^* \mathcal{F}) \longrightarrow H^i(S, K \otimes_{\mathcal{O}_S}^{\mathbf{L}} \mathcal{F})$$

for \mathcal{F} quasi-coherent on S compatible with boundary maps (see proof) such that $\alpha_{\mathcal{F}}^i$ is an isomorphism for $i \leq m$

- (B) there exists a pseudo-coherent $L \in D(\mathcal{O}_S)$ and functorial isomorphisms

$$\mathrm{Ext}_{\mathcal{O}_S}^i(L, \mathcal{F}) \longrightarrow \mathrm{Ext}_{\mathcal{O}_X}^i(E, \mathcal{G}^\bullet \otimes_{\mathcal{O}_X} f^* \mathcal{F})$$

for \mathcal{F} quasi-coherent on S compatible with boundary maps.

Proof. Proof of (A). Suppose \mathcal{G}^i is nonzero only for $i \in [a, b]$. We may replace X by a quasi-compact open neighbourhood of the union of the supports of \mathcal{G}^i . Hence we may assume X is Noetherian. In this case X and f are quasi-compact and quasi-separated. Choose an approximation $P \rightarrow E$ by a perfect complex P of $(X, E, -m - 1 + a)$ (possible by Theorem 13.6). Then the induced map

$$\mathrm{Ext}_{\mathcal{O}_X}^i(E, \mathcal{G}^\bullet \otimes_{\mathcal{O}_X} f^* \mathcal{F}) \longrightarrow \mathrm{Ext}_{\mathcal{O}_X}^i(P, \mathcal{G}^\bullet \otimes_{\mathcal{O}_X} f^* \mathcal{F})$$

is an isomorphism for $i \leq m$. Namely, the kernel, resp. cokernel of this map is a quotient, resp. submodule of

$$\mathrm{Ext}_{\mathcal{O}_X}^i(C, \mathcal{G}^\bullet \otimes_{\mathcal{O}_X} f^* \mathcal{F}) \quad \text{resp.} \quad \mathrm{Ext}_{\mathcal{O}_X}^{i+1}(C, \mathcal{G}^\bullet \otimes_{\mathcal{O}_X} f^* \mathcal{F})$$

where C is the cone of $P \rightarrow E$. Since C has vanishing cohomology sheaves in degrees $\geq -m-1+a$ these Ext-groups are zero for $i \leq m+1$ by Derived Categories, Lemma 27.3. This reduces us to the case that E is a perfect complex which is Lemma 24.2. The statement on boundaries is explained in the proof of Lemma 24.2.

Proof of (B). As in the proof of (A) we may assume X is Noetherian. Observe that E is pseudo-coherent by Lemma 9.3. By Lemma 18.1 we can write $E = \mathrm{hocolim} E_n$ with E_n perfect and $E_n \rightarrow E$ inducing an isomorphism on truncations $\tau_{\geq -n}$. Let E_n^\vee be the dual perfect complex (Cohomology, Lemma 43.11). We obtain an inverse system $\dots \rightarrow E_3^\vee \rightarrow E_2^\vee \rightarrow E_1^\vee$ of perfect objects. This in turn gives rise to an inverse system

$$\dots \rightarrow K_3 \rightarrow K_2 \rightarrow K_1 \quad \text{with} \quad K_n = Rf_*(E_n^\vee \otimes_{\mathcal{O}_X}^{\mathbf{L}} \mathcal{G}^\bullet)$$

perfect on S , see Lemma 23.2. By Lemma 24.2 and its proof and by the arguments in the previous paragraph (with $P = E_n$) for any quasi-coherent \mathcal{F} on S we have functorial canonical maps

$$\begin{array}{ccc} & \mathrm{Ext}_{\mathcal{O}_X}^i(E, \mathcal{G}^\bullet \otimes_{\mathcal{O}_X} f^* \mathcal{F}) & \\ & \swarrow \quad \searrow & \\ H^i(S, K_{n+1} \otimes_{\mathcal{O}_S}^{\mathbf{L}} \mathcal{F}) & \xrightarrow{\quad} & H^i(S, K_n \otimes_{\mathcal{O}_S}^{\mathbf{L}} \mathcal{F}) \end{array}$$

which are isomorphisms for $i \leq n+a$. Let $L_n = K_n^\vee$ be the dual perfect complex. Then we see that $L_1 \rightarrow L_2 \rightarrow L_3 \rightarrow \dots$ is a system of perfect objects in $D(\mathcal{O}_S)$ such that for any quasi-coherent \mathcal{F} on S the maps

$$\mathrm{Ext}_{\mathcal{O}_S}^i(L_{n+1}, \mathcal{F}) \longrightarrow \mathrm{Ext}_{\mathcal{O}_S}^i(L_n, \mathcal{F})$$

are isomorphisms for $i \leq n+a-1$. This implies that $L_n \rightarrow L_{n+1}$ induces an isomorphism on truncations $\tau_{\geq -n-a+2}$ (hint: take cone of $L_n \rightarrow L_{n+1}$ and look at its last nonvanishing cohomology sheaf). Thus $L = \mathrm{hocolim} L_n$ is pseudo-coherent, see Lemma 18.1. The mapping property of homotopy colimits gives that $\mathrm{Ext}_{\mathcal{O}_S}^i(L, \mathcal{F}) = \mathrm{Ext}_{\mathcal{O}_S}^i(L_n, \mathcal{F})$ for $i \leq n+a-3$ which finishes the proof. \square

0DJS **Remark 24.4.** The pseudo-coherent complex L of part (B) of Lemma 24.3 is canonically associated to the situation. For example, formation of L as in (B) is compatible with base change. In other words, given a cartesian diagram

$$\begin{array}{ccc} X' & \xrightarrow{\quad} & X \\ f' \downarrow & & \downarrow f \\ S' & \xrightarrow{\quad g \quad} & S \end{array}$$

of schemes we have canonical functorial isomorphisms

$$\mathrm{Ext}_{\mathcal{O}_{S'}}^i(Lg^*L, \mathcal{F}') \longrightarrow \mathrm{Ext}_{\mathcal{O}_X}^i(L(g')^*E, (g')^*\mathcal{G}^\bullet \otimes_{\mathcal{O}_{X'}} (f')^*\mathcal{F}')$$

for \mathcal{F}' quasi-coherent on S' . Observe that we do **not** use derived pullback on \mathcal{G}^\bullet on the right hand side. If we ever need this, we will formulate a precise result here and give a detailed proof.

25. Limits and derived categories

09RC In this section we collect some results about the derived category of a scheme which is the limit of an inverse system of schemes. More precisely, we will work in the following setting.

09RD **Situation 25.1.** Let $S = \lim_{i \in I} S_i$ be a limit of a directed system of schemes with affine transition morphisms $f_{i'i} : S_{i'} \rightarrow S_i$. We assume that S_i is quasi-compact and quasi-separated for all $i \in I$. We denote $f_i : S \rightarrow S_i$ the projection. We also fix an element $0 \in I$.

09RE **Lemma 25.2.** *In Situation 25.1. Let E_0 and K_0 be objects of $D(\mathcal{O}_{S_0})$. Set $E_i = Lf_{i0}^* E_0$ and $K_i = Lf_{i0}^* K_0$ for $i \geq 0$ and set $E = Lf_0^* E_0$ and $K = Lf_0^* K_0$. Then the map*

$$\operatorname{colim}_{i \geq 0} \operatorname{Hom}_{D(\mathcal{O}_{S_i})}(E_i, K_i) \longrightarrow \operatorname{Hom}_{D(\mathcal{O}_S)}(E, K)$$

is an isomorphism if either

- (1) E_0 is perfect and $K_0 \in D_{QCoh}(\mathcal{O}_{S_0})$, or
- (2) E_0 is pseudo-coherent and $K_0 \in D_{QCoh}(\mathcal{O}_{S_0})$ has finite tor dimension.

Proof. For every open $U_0 \subset S_0$ consider the condition P that the canonical map

$$\operatorname{colim}_{i \geq 0} \operatorname{Hom}_{D(\mathcal{O}_{U_i})}(E_i|_{U_i}, K_i|_{U_i}) \longrightarrow \operatorname{Hom}_{D(\mathcal{O}_U)}(E|_U, K|_U)$$

is an isomorphism, where $U = f_0^{-1}(U_0)$ and $U_i = f_{i0}^{-1}(U_0)$. We will prove P holds for all quasi-compact opens U_0 by the induction principle of Cohomology of Schemes, Lemma 4.1. Condition (2) of this lemma follows immediately from Mayer-Vietoris for hom in the derived category, see Cohomology, Lemma 31.3. Thus it suffices to prove the lemma when S_0 is affine.

Assume S_0 is affine. Say $S_0 = \operatorname{Spec}(A_0)$, $S_i = \operatorname{Spec}(A_i)$, and $S = \operatorname{Spec}(A)$. We will use Lemma 3.5 without further mention.

In case (1) the object E_0^\bullet corresponds to a finite complex of finite projective A_0 -modules, see Lemma 9.7. We may represent the object K_0 by a K -flat complex K_0^\bullet of A_0 -modules. In this situation we are trying to prove

$$\operatorname{colim}_{i \geq 0} \operatorname{Hom}_{D(A_i)}(E_0^\bullet \otimes_{A_0} A_i, K_0^\bullet \otimes_{A_0} A_i) \longrightarrow \operatorname{Hom}_{D(A)}(E_0^\bullet \otimes_{A_0} A, K_0^\bullet \otimes_{A_0} A)$$

Because E_0^\bullet is a bounded above complex of projective modules we can rewrite this as

$$\operatorname{colim}_{i \geq 0} \operatorname{Hom}_{K(A_0)}(E_0^\bullet, K_0^\bullet \otimes_{A_0} A_i) \longrightarrow \operatorname{Hom}_{K(A_0)}(E_0^\bullet, K_0^\bullet \otimes_{A_0} A)$$

Since there are only a finite number of nonzero modules E_0^n and since these are all finitely presented modules, this map is an isomorphism.

In case (2) the object E_0 corresponds to a bounded above complex E_0^\bullet of finite free A_0 -modules, see Lemma 9.2. We may represent K_0 by a finite complex K_0^\bullet of flat A_0 -modules, see Lemma 9.4 and More on Algebra, Lemma 61.3. In particular K_0^\bullet is K -flat and we can argue as before to arrive at the map

$$\operatorname{colim}_{i \geq 0} \operatorname{Hom}_{K(A_0)}(E_0^\bullet, K_0^\bullet \otimes_{A_0} A_i) \longrightarrow \operatorname{Hom}_{K(A_0)}(E_0^\bullet, K_0^\bullet \otimes_{A_0} A)$$

It is clear that this map is an isomorphism (only a finite number of terms are involved since K_0^\bullet is bounded). \square

09RF **Lemma 25.3.** *In Situation 25.1 the category of perfect objects of $D(\mathcal{O}_S)$ is the colimit of the categories of perfect objects of $D(\mathcal{O}_{S_i})$.*

Proof. For every open $U_0 \subset S_0$ consider the condition P that the functor

$$\operatorname{colim}_{i \geq 0} D_{\text{perf}}(\mathcal{O}_{U_i}) \longrightarrow D_{\text{perf}}(\mathcal{O}_U)$$

is an equivalence where $_{\text{perf}}$ indicates the full subcategory of perfect objects and where $U = f_0^{-1}(U_0)$ and $U_i = f_{i0}^{-1}(U_0)$. We will prove P holds for all quasi-compact opens U_0 by the induction principle of Cohomology of Schemes, Lemma 4.1. First, we observe that we already know the functor is fully faithful by Lemma 25.2. Thus it suffices to prove essential surjectivity.

We first check condition (2) of the induction principle. Thus suppose that we have $S_0 = U_0 \cup V_0$ and that P holds for U_0 , V_0 , and $U_0 \cap V_0$. Let E be a perfect object of $D(\mathcal{O}_S)$. We can find $i \geq 0$ and $E_{U,i}$ perfect on U_i and $E_{V,i}$ perfect on V_i whose pullback to U and V are isomorphic to $E|_U$ and $E|_V$. Denote

$$a : E_{U,i} \rightarrow (Rf_{i,*}E)|_{U_i} \quad \text{and} \quad b : E_{V,i} \rightarrow (Rf_{i,*}E)|_{V_i}$$

the maps adjoint to the isomorphisms $Lf_i^*E_{U,i} \rightarrow E|_U$ and $Lf_i^*E_{V,i} \rightarrow E|_V$. By fully faithfulness, after increasing i , we can find an isomorphism $c : E_{U,i}|_{U_i \cap V_i} \rightarrow E_{V,i}|_{U_i \cap V_i}$ which pulls back to the identifications

$$Lf_i^*E_{U,i}|_{U \cap V} \rightarrow E|_{U \cap V} \rightarrow Lf_i^*E_{V,i}|_{U \cap V}.$$

Apply Cohomology, Lemma 39.1 to get an object E_i on S_i and a map $d : E_i \rightarrow Rf_{i,*}E$ which restricts to the maps a and b over U_i and V_i . Then it is clear that E_i is perfect and that d is adjoint to an isomorphism $Lf_i^*E_i \rightarrow E$.

Finally, we check condition (1) of the induction principle, in other words, we check the lemma holds when S_0 is affine. Say $S_0 = \operatorname{Spec}(A_0)$, $S_i = \operatorname{Spec}(A_i)$, and $S = \operatorname{Spec}(A)$. Using Lemmas 3.5 and 9.7 we see that we have to show that

$$D_{\text{perf}}(A) = \operatorname{colim} D_{\text{perf}}(A_i)$$

This is clear from the fact that perfect complexes over rings are given by finite complexes of finite projective (hence finitely presented) modules. See More on Algebra, Lemma 67.16 for details. \square

26. Cohomology and base change, VI

0A1G A final section on cohomology and base change continuing the discussion of Sections 21, 22, and 23. An easy to grok special case is given in Remark 26.2.

0A1H **Lemma 26.1.** *Let $f : X \rightarrow S$ be a morphism of finite presentation. Let $E \in D(\mathcal{O}_X)$ be a perfect object. Let \mathcal{G}^\bullet be a bounded complex of finitely presented \mathcal{O}_X -modules, flat over S , with support proper over S . Then*

$$K = Rf_*(E \otimes_{\mathcal{O}_X}^{\mathbf{L}} \mathcal{G}^\bullet)$$

is a perfect object of $D(\mathcal{O}_S)$ and its formation commutes with arbitrary base change.

Proof. The statement on base change is Lemma 22.4. Thus it suffices to show that K is a perfect object. If S is Noetherian, then this follows from Lemma 23.2. We will reduce to this case by Noetherian approximation. We encourage the reader to skip the rest of this proof.

The question is local on S , hence we may assume S is affine. Say $S = \operatorname{Spec}(R)$. We write $R = \operatorname{colim} R_i$ as a filtered colimit of Noetherian rings R_i . By Limits, Lemma 10.1 there exists an i and a scheme X_i of finite presentation over R_i whose base change to R is X . By Limits, Lemma 10.2 we may assume after increasing i ,

that there exists a bounded complex of finitely presented \mathcal{O}_{X_i} -modules \mathcal{G}_i^\bullet whose pullback to X is \mathcal{G}^\bullet . After increasing i we may assume \mathcal{G}_i^n is flat over R_i , see Limits, Lemma 10.4. After increasing i we may assume the support of \mathcal{G}_i^n is proper over R_i , see Limits, Lemma 13.4 and Cohomology of Schemes, Lemma 25.7. Finally, by Lemma 25.3 we may, after increasing i , assume there exists a perfect object E_i of $D(\mathcal{O}_{X_i})$ whose pullback to X is E . Applying Lemma 23.2 to $X_i \rightarrow \text{Spec}(R_i)$, E_i , \mathcal{G}_i^\bullet and using the base change property already shown we obtain the result. \square

0A11 **Remark 26.2.** Let R be a ring. Let X be a scheme of finite presentation over R . Let \mathcal{G} be a finitely presented \mathcal{O}_X -module flat over R with support proper over R . By Lemma 26.1 there exists a finite complex of finite projective R -modules M^\bullet such that we have

$$R\Gamma(X_{R'}, \mathcal{G}_{R'}) = M^\bullet \otimes_R R'$$

functorially in the R -algebra R' .

0CSC **Lemma 26.3.** *Let $f : X \rightarrow S$ be a morphism of finite presentation. Let $E \in D(\mathcal{O}_X)$ be a pseudo-coherent object. Let \mathcal{G}^\bullet be a bounded above complex of finitely presented \mathcal{O}_X -modules, flat over S , with support proper over S . Then*

$$K = Rf_*(E \otimes_{\mathcal{O}_X}^{\mathbf{L}} \mathcal{G}^\bullet)$$

is a pseudo-coherent object of $D(\mathcal{O}_S)$ and its formation commutes with arbitrary base change.

Proof. The statement on base change is Lemma 22.4. Thus it suffices to show that K is a pseudo-coherent object. This will follow from Lemma 26.1 by approximation by perfect complexes. We encourage the reader to skip the rest of the proof.

The question is local on S , hence we may assume S is affine. Then X is quasi-compact and quasi-separated. Moreover, there exists an integer N such that total direct image $Rf_* : D_{QCoh}(\mathcal{O}_X) \rightarrow D_{QCoh}(\mathcal{O}_S)$ has cohomological dimension N as explained in Lemma 4.1. Choose an integer b such that $\mathcal{G}^i = 0$ for $i > b$. It suffices to show that K is m -pseudo-coherent for every m . Choose an approximation $P \rightarrow E$ by a perfect complex P of $(X, E, m - N - 1 - b)$. This is possible by Theorem 13.6. Choose a distinguished triangle

$$P \rightarrow E \rightarrow C \rightarrow P[1]$$

in $D_{QCoh}(\mathcal{O}_X)$. The cohomology sheaves of C are zero in degrees $\geq m - N - 1 - b$. Hence the cohomology sheaves of $C \otimes^{\mathbf{L}} \mathcal{G}^\bullet$ are zero in degrees $\geq m - N - 1$. Thus the cohomology sheaves of $Rf_*(C \otimes^{\mathbf{L}} \mathcal{G}^\bullet)$ are zero in degrees $\geq m - 1$. Hence

$$Rf_*(P \otimes^{\mathbf{L}} \mathcal{G}^\bullet) \rightarrow Rf_*(E \otimes^{\mathbf{L}} \mathcal{G}^\bullet)$$

is an isomorphism on cohomology sheaves in degrees $\geq m$. Next, suppose that $H^i(P) = 0$ for $i > a$. Then $P \otimes^{\mathbf{L}} \sigma_{\geq m - N - 1 - a} \mathcal{G}^\bullet \rightarrow P \otimes^{\mathbf{L}} \mathcal{G}^\bullet$ is an isomorphism on cohomology sheaves in degrees $\geq m - N - 1$. Thus again we find that

$$Rf_*(P \otimes^{\mathbf{L}} \sigma_{\geq m - N - 1 - a} \mathcal{G}^\bullet) \rightarrow Rf_*(P \otimes^{\mathbf{L}} \mathcal{G}^\bullet)$$

is an isomorphism on cohomology sheaves in degrees $\geq m$. By Lemma 26.1 the source is a perfect complex. We conclude that K is m -pseudo-coherent as desired. \square

0B91 **Lemma 26.4.** *Let S be a scheme. Let $f : X \rightarrow S$ be a flat proper morphism of finite presentation.*

- (1) Let $E \in D(\mathcal{O}_X)$ be perfect. Then Rf_*E is a perfect object of $D(\mathcal{O}_S)$ and its formation commutes with arbitrary base change.
- (2) Let \mathcal{G} be an \mathcal{O}_X -module of finite presentation, flat over S . Then $Rf_*\mathcal{G}$ is a perfect object of $D(\mathcal{O}_S)$ and its formation commutes with arbitrary base change.

Proof. Special cases of Lemma 26.1 applied with (1) \mathcal{G}^\bullet equal to \mathcal{O}_X in degree 0 and (2) $E = \mathcal{O}_X$ and \mathcal{G}^\bullet consisting of \mathcal{G} sitting in degree 0. \square

0CSD **Lemma 26.5.** *Let S be a scheme. Let $f : X \rightarrow S$ be a flat proper morphism of finite presentation. Let $E \in D(\mathcal{O}_X)$ be pseudo-coherent. Then Rf_*E is a pseudo-coherent object of $D(\mathcal{O}_S)$ and its formation commutes with arbitrary base change.*

More generally, if $f : X \rightarrow S$ is proper and E on X is pseudo-coherent relative to S (More on Morphisms, Definition 48.2), then Rf_*E is pseudo-coherent (but formation does not commute with base change in this generality). See [Kie72].

Proof. Special case of Lemma 26.3 applied with \mathcal{G}^\bullet equal to \mathcal{O}_X in degree 0. \square

0D2Q **Lemma 26.6.** *Let R be a ring. Let X be a scheme and let $f : X \rightarrow \text{Spec}(R)$ be proper, flat, and of finite presentation. Let (M_n) be an inverse system of R -modules with surjective transition maps. Then the canonical map*

$$\mathcal{O}_X \otimes_R (\lim M_n) \longrightarrow \lim \mathcal{O}_X \otimes_R M_n$$

induces an isomorphism from the source to DQ_X applied to the target.

Proof. The statement means that for any object E of $D_{QCoh}(\mathcal{O}_X)$ the induced map

$$\text{Hom}(E, \mathcal{O}_X \otimes_R (\lim M_n)) \longrightarrow \text{Hom}(E, \lim \mathcal{O}_X \otimes_R M_n)$$

is an isomorphism. Since $D_{QCoh}(\mathcal{O}_X)$ has a perfect generator (Theorem 14.3) it suffices to check this for perfect E . By Lemma 3.2 we have $\lim \mathcal{O}_X \otimes_R M_n = R \lim \mathcal{O}_X \otimes_R M_n$. The exact functor $R\text{Hom}_X(E, -) : D_{QCoh}(\mathcal{O}_X) \rightarrow D(R)$ of Cohomology, Section 38 commutes with products and hence with derived limits, whence

$$R\text{Hom}_X(E, \lim \mathcal{O}_X \otimes_R M_n) = R \lim R\text{Hom}_X(E, \mathcal{O}_X \otimes_R M_n)$$

Let E^\vee be the dual perfect complex, see Cohomology, Lemma 43.11. We have

$$R\text{Hom}_X(E, \mathcal{O}_X \otimes_R M_n) = R\Gamma(X, E^\vee \otimes_{\mathcal{O}_X}^L Lf^* M_n) = R\Gamma(X, E^\vee) \otimes_R^L M_n$$

by Lemma 21.1. From Lemma 26.4 we see $R\Gamma(X, E^\vee)$ is a perfect complex of R -modules. In particular it is a pseudo-coherent complex and by More on Algebra, Lemma 85.7 we obtain

$$R \lim R\Gamma(X, E^\vee) \otimes_R^L M_n = R\Gamma(X, E^\vee) \otimes_R^L \lim M_n$$

as desired. \square

0A1J **Lemma 26.7.** *Let $f : X \rightarrow S$ be a morphism of finite presentation. Let $E \in D(\mathcal{O}_X)$ be a perfect object. Let \mathcal{G}^\bullet be a bounded complex of finitely presented \mathcal{O}_X -modules, flat over S , with support proper over S . Then*

$$K = Rf_* R\mathcal{H}om(E, \mathcal{G}^\bullet)$$

is a perfect object of $D(\mathcal{O}_S)$ and its formation commutes with arbitrary base change.

Proof. The statement on base change is Lemma 22.5. Thus it suffices to show that K is a perfect object. If S is Noetherian, then this follows from Lemma 23.3. We will reduce to this case by Noetherian approximation. We encourage the reader to skip the rest of this proof.

The question is local on S , hence we may assume S is affine. Say $S = \text{Spec}(R)$. We write $R = \text{colim } R_i$ as a filtered colimit of Noetherian rings R_i . By Limits, Lemma 10.1 there exists an i and a scheme X_i of finite presentation over R_i whose base change to R is X . By Limits, Lemma 10.2 we may assume after increasing i , that there exists a bounded complex of finitely presented \mathcal{O}_{X_i} -modules \mathcal{G}_i^\bullet whose pullback to X is \mathcal{G}^\bullet . After increasing i we may assume \mathcal{G}_i^n is flat over R_i , see Limits, Lemma 10.4. After increasing i we may assume the support of \mathcal{G}_i^n is proper over R_i , see Limits, Lemma 13.4 and Cohomology of Schemes, Lemma 25.7. Finally, by Lemma 25.3 we may, after increasing i , assume there exists a perfect object E_i of $D(\mathcal{O}_{X_i})$ whose pullback to X is E . Applying Lemma 23.3 to $X_i \rightarrow \text{Spec}(R_i)$, E_i , \mathcal{G}_i^\bullet and using the base change property already shown we obtain the result. \square

27. Perfect complexes

0BDH We first talk about jumping loci for betti numbers of perfect complexes. Given a complex E on a scheme X and a point x of X we often write $E \otimes_{\mathcal{O}_X}^{\mathbf{L}} \kappa(x)$ instead of the more correct Li_x^*E , where $i_x : x \rightarrow X$ is the canonical morphism.

0BDI **Lemma 27.1.** *Let X be a scheme. Let $E \in D(\mathcal{O}_X)$ be pseudo-coherent (for example perfect). For any $i \in \mathbf{Z}$ consider the function*

$$\beta_i : X \longrightarrow \{0, 1, 2, \dots\}, \quad x \longmapsto \dim_{\kappa(x)} H^i(E \otimes_{\mathcal{O}_X}^{\mathbf{L}} \kappa(x))$$

Then we have

- (1) formation of β_i commutes with arbitrary base change,
- (2) the functions β_i are upper semi-continuous, and
- (3) the level sets of β_i are locally constructible in X .

Proof. Consider a morphism of schemes $f : Y \rightarrow X$ and a point $y \in Y$. Let x be the image of y and consider the commutative diagram

$$\begin{array}{ccc} y & \xrightarrow{\quad j \quad} & Y \\ g \downarrow & & \downarrow f \\ x & \xrightarrow{\quad i \quad} & X \end{array}$$

Then we see that $Lg^* \circ Li^* = Lj^* \circ Lf^*$. This implies that the function β'_i associated to the perfect complex Lf^*K is the pullback of the function β_i , in a formula: $\beta'_i = \beta_i \circ f$. This is the meaning of (1).

Fix i and let $x \in X$. It is enough to prove (2) and (3) holds in an open neighbourhood of x , hence we may assume X affine. Then we can represent E by a bounded above complex \mathcal{F}^\bullet of of finite free modules (Lemma 12.2). Then $P = \sigma_{\geq i-1} \mathcal{F}^\bullet$ is a perfect object and $P \rightarrow E$ induces an isomorphism

$$H^i(P \otimes_{\mathcal{O}_X}^{\mathbf{L}} \kappa(x')) \rightarrow H^i(E \otimes_{\mathcal{O}_X}^{\mathbf{L}} \kappa(x'))$$

for all $x' \in X$. Thus we may assume E is perfect. In this case by More on Algebra, Lemma 68.6 there exists an affine open neighbourhood U of x and $a \leq b$ such that

$K|_U$ is represented by a complex

$$\dots \rightarrow 0 \rightarrow \mathcal{O}_U^{\oplus \beta_a(x)} \rightarrow \mathcal{O}_U^{\oplus \beta_{a+1}(x)} \rightarrow \dots \rightarrow \mathcal{O}_U^{\oplus \beta_{b-1}(x)} \rightarrow \mathcal{O}_U^{\oplus \beta_b(x)} \rightarrow 0 \rightarrow \dots$$

(This also uses earlier results to turn the problem into algebra, for example Lemmas 3.5 and 9.7.) It follows immediately that $\beta_i(x') \leq \beta_i(x)$ for all $x' \in U$. This proves that β_i is upper semi-continuous.

To prove (3) we may assume that X is affine and K is given by a complex of finite free \mathcal{O}_X -modules (for example by arguing as in the previous paragraph, or by using Cohomology, Lemma 43.3). Thus we have to show that given a complex

$$\mathcal{O}_X^{\oplus a} \rightarrow \mathcal{O}_X^{\oplus b} \rightarrow \mathcal{O}_X^{\oplus c}$$

the function associated to a point $x \in X$ the dimension of the cohomology of $\kappa_x^{\oplus a} \rightarrow \kappa_x^{\oplus b} \rightarrow \kappa_x^{\oplus c}$ in the middle has constructible level sets. Let $A \in \text{Mat}(a \times b, \Gamma(X, \mathcal{O}_X))$ be the matrix of the first arrow. The rank of the image of A in $\text{Mat}(a \times b, \kappa(x))$ is equal to r if all $(r+1) \times (r+1)$ -minors of A vanish at x and there is some $r \times r$ -minor of A which does not vanish at x . Thus the set of points where the rank is r is a constructible locally closed set. Arguing similarly for the second arrow and putting everything together we obtain the desired result. \square

0BDJ **Lemma 27.2.** *Let X be a scheme. Let $E \in D(\mathcal{O}_X)$ be perfect. The function*

$$\chi_E : X \rightarrow \mathbf{Z}, \quad x \mapsto \sum (-1)^i \dim_{\kappa(x)} H^i(E \otimes_{\mathcal{O}_X}^{\mathbf{L}} \kappa(x))$$

is locally constant on X .

Proof. By Cohomology, Lemma 43.3 we see that we can, locally on X , represent E by a finite complex \mathcal{E}^\bullet of finite free \mathcal{O}_X -modules. On such an open the function χ_E is constant with value $\sum (-1)^i \text{rank}(\mathcal{E}^i)$. \square

0BDK **Lemma 27.3.** *Let X be a scheme. Let $E \in D(\mathcal{O}_X)$ be perfect. Given $i, r \in \mathbf{Z}$, there exists an open subscheme $U \subset X$ characterized by the following*

- (1) $E|_U \cong H^i(E|_U)[-i]$ and $H^i(E|_U)$ is a locally free \mathcal{O}_U -module of rank r ,
- (2) a morphism $f : Y \rightarrow X$ factors through U if and only if Lf^*E is isomorphic to a locally free module of rank r placed in degree i .

Proof. Let $\beta_j : X \rightarrow \{0, 1, 2, \dots\}$ for $j \in \mathbf{Z}$ be the functions of Lemma 27.1. Then the set

$$W = \{x \in X \mid \beta_j(x) \leq 0 \text{ for all } j \neq i\}$$

is open in X and its formation commutes with pullback to any Y over X . This follows from the lemma using that a priori in a neighbourhood of any point only a finite number of the β_j are nonzero. Thus we may replace X by W and assume that $\beta_j(x) = 0$ for all $x \in X$ and all $j \neq i$. In this case $H^i(E)$ is a finite locally free module and $E \cong H^i(E)[-i]$, see for example More on Algebra, Lemma 68.6. Thus X is the disjoint union of the open subschemes where the rank of $H^i(E)$ is fixed and we win. \square

0BDL **Lemma 27.4.** *Let X be a scheme. Let $E \in D(\mathcal{O}_X)$ be perfect of tor-amplitude in $[a, b]$ for some $a, b \in \mathbf{Z}$. Then there exists a locally closed subscheme $j : Z \rightarrow X$ characterized by the following*

- (1) $H^a(Lj^*E)$ is an invertible \mathcal{O}_Z -module, and

- (2) a morphism $f : Y \rightarrow X$ factors through Z if and only if for all morphisms $g : Y' \rightarrow Y$ the $\mathcal{O}_{Y'}$ -module $H^a(L(f \circ g)^*E)$ is invertible.

Moreover, we have

- (3) if $f : Y \rightarrow X$ factors as $Y \xrightarrow{g} Z \rightarrow X$, then $H^a(Lf^*E) = g^*H^a(Lj^*E)$,
(4) if $\beta_a(x) \leq 1$ for all $x \in X$, then j is a closed immersion and given $f : Y \rightarrow X$ the following are equivalent
(a) $f : Y \rightarrow X$ factors through Z ,
(b) $H^0(Lf^*E)$ is an invertible \mathcal{O}_Y -module,
(c) $\mathcal{O}_Y \rightarrow \mathcal{H}om_{\mathcal{O}_Y}(H^0(Lf^*E), H^0(Lf^*E))$ is injective.

Proof. First, let $U \subset X$ be the open subscheme where the function β_a of Lemma 27.1 has values ≤ 1 . Let $f : Y \rightarrow X$ be as in (2). Then for any $y \in Y$ we have $\beta_a(Lf^*E) = 1$ hence y maps into U by Lemma 27.1. Hence f as in (2) factors through U . Thus we may replace X by U and assume that $\beta_a(x) \in \{0, 1\}$ for all $x \in X$. We will show that in this case there is a closed subscheme $Z \subset X$ characterized by the equivalence of (4) (a), (b), (c) and that (3) holds. This will finish the proof because it will a fortiori show that morphisms as in (2) factor through Z .

If $x \in X$ and $\beta_a(x) = 0$, then there is an open neighbourhood of x where $\beta_a = 0$ (Lemma 27.1). In this way we see that set theoretically at least Z is a closed subset.

To get a scheme theoretic structure, consider a point $x \in X$ with $\beta_a(x) = 1$. Set $\beta = \beta_{a+1}(x)$. By More on Algebra, Lemma 68.6 there exists an affine open neighbourhood U of x such that $K|_U$ is represented by a complex

$$\dots \rightarrow 0 \rightarrow \mathcal{O}_U \xrightarrow{f_1, \dots, f_\beta} \mathcal{O}_U^{\oplus \beta} \rightarrow \dots \rightarrow \mathcal{O}_U^{\oplus \beta_{b-1}(x)} \rightarrow \mathcal{O}_U^{\oplus \beta_b(x)} \rightarrow 0 \rightarrow \dots$$

(This also uses earlier results to turn the problem into algebra, for example Lemmas 3.5 and 9.7.) Now, if $g : Y \rightarrow U$ is any morphism of schemes such that $g^\sharp(f_j)$ is nonzero for some j , then $H^0(Lg^*E)$ is not an invertible \mathcal{O}_Y -module and $H^0(Lg^*E)$ does not satisfy (4)(c) since it is annihilated by a nonzero function. And trivially it is an invertible \mathcal{O}_Y -module if $g^\sharp(f_j) = 0$ for all j . Thus we see that over U the closed subscheme cut out by f_1, \dots, f_β satisfies (3) and we have the equivalence of (4)(a), (b), and (c). This finishes the proof as the characterization of Z shows that the locally constructed patches glue (details omitted). \square

28. Applications

0BDM Mostly applications of cohomology and base change. In the future we may generalize these results to the situation discussed in Lemma 26.1.

0BDN **Lemma 28.1.** *Let $f : X \rightarrow S$ be a flat, proper morphism of finite presentation. Let \mathcal{F} be an \mathcal{O}_X -module of finite presentation, flat over S . For fixed $i \in \mathbf{Z}$ consider the function*

$$\beta_i : X \rightarrow \{0, 1, 2, \dots\}, \quad s \mapsto \dim_{\kappa(s)} H^i(X_s, \mathcal{F}_s)$$

Then we have

- (1) formation of β_i commutes with arbitrary base change,
- (2) the functions β_i are upper semi-continuous, and
- (3) the level sets of β_i are locally constructible in S .

Proof. By cohomology and base change (more precisely by Lemma 26.4) the object $K = Rf_*\mathcal{F}$ is a perfect object of the derived category of S whose formation commutes with arbitrary base change. In particular we have

$$H^i(X_s, \mathcal{F}_s) = H^i(K \otimes_{\mathcal{O}_S}^{\mathbf{L}} \kappa(s))$$

Thus the lemma follows from Lemma 27.1. \square

0B9T **Lemma 28.2.** *Let $f : X \rightarrow S$ be a flat, proper morphism of finite presentation. Let \mathcal{F} be an \mathcal{O}_X -module of finite presentation, flat over S . The function*

$$s \mapsto \chi(X_s, \mathcal{F}_s)$$

is locally constant on S . Formation of this function commutes with base change.

Proof. By cohomology and base change (more precisely by Lemma 26.4) the object $K = Rf_*\mathcal{F}$ is a perfect object of the derived category of S whose formation commutes with arbitrary base change. Thus we have to show the map

$$s \mapsto \sum (-1)^i \dim_{\kappa(s)} H^i(K \otimes_{\mathcal{O}_S}^{\mathbf{L}} \kappa(s))$$

is locally constant on S . This is Lemma 27.2. \square

0B9S **Lemma 28.3.** *Let $f : X \rightarrow S$ be a flat, proper morphism of finite presentation. Let \mathcal{F} be an \mathcal{O}_X -module of finite presentation, flat over S . Fix $i, r \in \mathbf{Z}$. Then there exists an open subscheme $U \subset S$ with the following property: A morphism $T \rightarrow S$ factors through U if and only if $Rf_{T,*}\mathcal{F}_T$ is isomorphic to a finite locally free module of rank r placed in degree i .*

Proof. By cohomology and base change (more precisely by Lemma 26.4) the object $K = Rf_*\mathcal{F}$ is a perfect object of the derived category of S whose formation commutes with arbitrary base change. Thus this lemma follows immediately from Lemma 27.3. \square

0D4E **Lemma 28.4.** *Let $f : X \rightarrow S$ be a morphism of finite presentation. Let \mathcal{F} be an \mathcal{O}_X -module of finite presentation, flat over S with support proper over S . If $R^i f_*\mathcal{F} = 0$ for $i > 0$, then $f_*\mathcal{F}$ is locally free and its formation commutes with arbitrary base change (see proof for explanation).*

Proof. By Lemma 26.1 the object $E = Rf_*\mathcal{F}$ of $D(\mathcal{O}_S)$ is perfect and its formation commutes with arbitrary base change, in the sense that $Rf'_*(g')^*\mathcal{F} = Lg^*E$ for any cartesian diagram

$$\begin{array}{ccc} X' & \xrightarrow{\quad} & X \\ f' \downarrow & & \downarrow f \\ S' & \xrightarrow{\quad g \quad} & S \end{array}$$

of schemes. Since there is never any cohomology in degrees < 0 , we see that E (locally) has tor-amplitude in $[0, b]$ for some b . If $H^i(E) = R^i f_*\mathcal{F} = 0$ for $i > 0$, then E has tor amplitude in $[0, 0]$. Whence $E = H^0(E)[0]$. We conclude $H^0(E) = f_*\mathcal{F}$ is finite locally free by More on Algebra, Lemma 67.2 (and the characterization of finite projective modules in Algebra, Lemma 77.2). Commutation with base change means that $g^*f_*\mathcal{F} = f'_*(g')^*\mathcal{F}$ for a diagram as above and it follows from the already established commutation of base change for E . \square

0E0L **Lemma 28.5.** *Let $f : X \rightarrow S$ be a morphism of schemes. Assume*

- (1) f is proper, flat, and of finite presentation, and
- (2) the geometric fibres of f are reduced and connected.

Then $f_*\mathcal{O}_X = \mathcal{O}_S$ and this holds after any base change.

Proof. Since the assumptions are preserved by base change it suffices to show that $f_*\mathcal{O}_X = \mathcal{O}_S$. By cohomology and base change (more precisely Lemma 26.4) the complex $E = Rf_*\mathcal{O}_X$ is perfect and its formation commutes with arbitrary base change. In particular, for $s \in S$ we see that $H^0(E \otimes^{\mathbf{L}} \kappa(s)) = H^0(X_s, \mathcal{O}_{X_s}) = \kappa(s)$. The last equality follows from Varieties, Lemma 9.3 and the fact that X_s is geometrically connected and geometrically reduced. Thus $\beta_0(s) \leq 1$ for all $s \in S$ with notation as in Lemma 27.1. Apply Lemma 27.4 with $a = 0$. We obtain a universal closed subscheme $j : Z \rightarrow S$ with $H^0(Lj^*E)$ invertible characterized by the equivalence of (4)(a), (b), and (c) of the lemma. Since formation of E commutes with base change, we have

$$Lf^*E = R\mathrm{pr}_{1,*}\mathcal{O}_{X \times_S X}$$

The morphism $\mathrm{pr}_1 : X \times_S X$ has a section namely the diagonal morphism Δ for X over S . We obtain maps

$$\mathcal{O}_X \longrightarrow R\mathrm{pr}_{1,*}\mathcal{O}_{X \times_S X} \longrightarrow \mathcal{O}_X$$

in $D(\mathcal{O}_X)$ whose composition is the identity. Thus $R\mathrm{pr}_{1,*}\mathcal{O}_{X \times_S X} = \mathcal{O}_X \oplus E'$ in $D(\mathcal{O}_X)$. Thus \mathcal{O}_X is a direct summand of $H^0(Lf^*E)$ and we conclude that $X \rightarrow S$ factors through Z by the equivalence of (4)(c) and (4)(a) of the lemma cited above. Since $\{X \rightarrow S\}$ is an fppf covering, we have $Z = S$. Thus $f_*\mathcal{O}_X$ is an invertible \mathcal{O}_S -module. We conclude because a ring map $A \rightarrow B$ such that B is invertible as an A -module is an isomorphism. \square

29. Other applications

0CRN In this section we state and prove some results that can be deduced from the theory worked out above.

0CRP **Lemma 29.1.** *Let X be a quasi-compact and quasi-separated scheme. Let K be an object of $D_{\mathrm{QCoh}}(\mathcal{O}_X)$ such that the cohomology sheaves $H^i(K)$ have countable sets of sections over affine opens. Then for any quasi-compact open $U \subset X$ and any perfect object E in $D(\mathcal{O}_X)$ the sets*

$$H^i(U, K \otimes^{\mathbf{L}} E), \quad \mathrm{Ext}^i(E|_U, K|_U)$$

are countable.

Proof. Using Cohomology, Lemma 43.11 we see that it suffices to prove the result for the groups $H^i(U, K \otimes^{\mathbf{L}} E)$. We will use the induction principle to prove the lemma, see Cohomology of Schemes, Lemma 4.1.

First we show that it holds when $U = \mathrm{Spec}(A)$ is affine. Namely, we can represent K by a complex of A -modules K^\bullet and E by a finite complex of finite projective A -modules P^\bullet . See Lemmas 3.5 and 9.7 and our definition of perfect complexes of A -modules (More on Algebra, Definition 67.1). Then $(E \otimes^{\mathbf{L}} K)|_U$ is represented by the total complex associated to the double complex $P^\bullet \otimes_A K^\bullet$ (Lemma 3.9). Using induction on the length of the complex P^\bullet (or using a suitable spectral sequence) we see that it suffices to show that $H^i(P^a \otimes_A K^\bullet)$ is countable for each a . Since

P^a is a direct summand of $A^{\oplus n}$ for some n this follows from the assumption that the cohomology group $H^i(K^\bullet)$ is countable.

To finish the proof it suffices to show: if $U = V \cup W$ and the result holds for V , W , and $V \cap W$, then the result holds for U . This is an immediate consequence of the Mayer-Vietoris sequence, see Cohomology, Lemma 31.4. \square

0CRQ **Lemma 29.2.** *Let X be a quasi-compact and quasi-separated scheme such that the sets of sections of \mathcal{O}_X over affine opens are countable. Let K be an object of $D_{QCoh}(\mathcal{O}_X)$. The following are equivalent*

- (1) $K = \text{hocolim} E_n$ with E_n a perfect object of $D(\mathcal{O}_X)$, and
- (2) the cohomology sheaves $H^i(K)$ have countable sets of sections over affine opens.

Proof. If (1) is true, then (2) is true because homotopy colimits commutes with taking cohomology sheaves (by Derived Categories, Lemma 31.8) and because a perfect complex is locally isomorphic to a finite complex of finite free \mathcal{O}_X -modules and therefore satisfies (2) by assumption on X .

Assume (2). Choose a K-injective complex \mathcal{K}^\bullet representing K . Choose a perfect generator E of $D_{QCoh}(\mathcal{O}_X)$ and represent it by a K-injective complex \mathcal{I}^\bullet . According to Theorem 17.3 and its proof there is an equivalence of triangulated categories $F : D_{QCoh}(\mathcal{O}_X) \rightarrow D(A, d)$ where (A, d) is the differential graded algebra

$$(A, d) = \text{Hom}_{\text{Comp}^{dg}(\mathcal{O}_X)}(\mathcal{I}^\bullet, \mathcal{I}^\bullet)$$

which maps K to the differential graded module

$$M = \text{Hom}_{\text{Comp}^{dg}(\mathcal{O}_X)}(\mathcal{I}^\bullet, \mathcal{K}^\bullet)$$

Note that $H^i(A) = \text{Ext}^i(E, E)$ and $H^i(M) = \text{Ext}^i(E, K)$. Moreover, since F is an equivalence it and its quasi-inverse commute with homotopy colimits. Therefore, it suffices to write M as a homotopy colimit of compact objects of $D(A, d)$. By Differential Graded Algebra, Lemma 29.3 it suffices show that $\text{Ext}^i(E, E)$ and $\text{Ext}^i(E, K)$ are countable for each i . This follows from Lemma 29.1. \square

0CRR **Lemma 29.3.** *Let A be a ring. Let X be a scheme of finite presentation over A . Let $f : U \rightarrow X$ be a flat morphism of finite presentation. Then*

- (1) there exists an inverse system of perfect objects L_n of $D(\mathcal{O}_X)$ such that

$$R\Gamma(U, Lf^*K) = \text{hocolim} R\text{Hom}(L_n, K)$$

in $D(A)$ functorially in K in $D_{QCoh}(\mathcal{O}_X)$, and

- (2) there exists a system of perfect objects E_n of $D(\mathcal{O}_X)$ such that

$$R\Gamma(U, Lf^*K) = \text{hocolim} R\Gamma(X, E_n \otimes^{\mathbf{L}} K)$$

in $D(A)$ functorially in K in $D_{QCoh}(\mathcal{O}_X)$.

Proof. By Lemma 21.1 we have

$$R\Gamma(U, Lf^*K) = R\Gamma(X, Rf_*\mathcal{O}_U \otimes^{\mathbf{L}} K)$$

functorially in K . Observe that $R\Gamma(X, -)$ commutes with homotopy colimits because it commutes with direct sums by Lemma 4.2. Similarly, $-\otimes^{\mathbf{L}} K$ commutes with derived colimits because $-\otimes^{\mathbf{L}} K$ commutes with direct sums (because direct sums in $D(\mathcal{O}_X)$ are given by direct sums of representing complexes). Hence to prove (2) it suffices to write $Rf_*\mathcal{O}_U = \text{hocolim} E_n$ for a system of perfect objects E_n of

$D(\mathcal{O}_X)$. Once this is done we obtain (1) by setting $L_n = E_n^\vee$, see Cohomology, Lemma 43.11.

Write $A = \text{colim } A_i$ with A_i of finite type over \mathbf{Z} . By Limits, Lemma 10.1 we can find an i and morphisms $U_i \rightarrow X_i \rightarrow \text{Spec}(A_i)$ of finite presentation whose base change to $\text{Spec}(A)$ recovers $U \rightarrow X \rightarrow \text{Spec}(A)$. After increasing i we may assume that $f_i : U_i \rightarrow X_i$ is flat, see Limits, Lemma 8.7. By Lemma 21.3 the derived pullback of $Rf_{i,*}\mathcal{O}_{U_i}$ by $g : X \rightarrow X_i$ is equal to $Rf_*\mathcal{O}_U$. Since Lg^* commutes with derived colimits, it suffices to prove what we want for f_i . Hence we may assume that U and X are of finite type over \mathbf{Z} .

Assume $f : U \rightarrow X$ is a morphism of schemes of finite type over \mathbf{Z} . To finish the proof we will show that $Rf_*\mathcal{O}_U$ is a homotopy colimit of perfect complexes. To see this we apply Lemma 29.2. Thus it suffices to show that $R^i f_*\mathcal{O}_U$ has countable sets of sections over affine opens. This follows from Lemma 29.1 applied to the structure sheaf. \square

30. Characterizing pseudo-coherent complexes, II

0CSE This section is a continuation of Section 18. In this section we discuss characterizations of pseudo-coherent complexes in terms of cohomology. More results of this nature can be found in More on Morphisms, Section 57.

0CSF **Lemma 30.1.** *Let A be a ring. Let R be a (possibly noncommutative) A -algebra which is finite free as an A -module. Then any object M of $D(R)$ which is pseudo-coherent in $D(A)$ can be represented by a bounded above complex of finite free (right) R -modules.*

Proof. Choose a complex M^\bullet of right R -modules representing M . Since M is pseudo-coherent we have $H^i(M) = 0$ for large enough i . Let m be the smallest index such that $H^m(M)$ is nonzero. Then $H^m(M)$ is a finite A -module by More on Algebra, Lemma 60.3. Thus we can choose a finite free R -module F^m and a map $F^m \rightarrow M^m$ such that $F^m \rightarrow M^m \rightarrow M^{m+1}$ is zero and such that $F^m \rightarrow H^m(M)$ is surjective. Picture:

$$\begin{array}{ccccccc} & & F^i & \longrightarrow & 0 & \longrightarrow & \dots \\ & & \downarrow \alpha & & \downarrow & & \\ M^{i-1} & \longrightarrow & M^i & \longrightarrow & M^{i+1} & \longrightarrow & \dots \end{array}$$

By descending induction on $n \leq m$ we are going to construct finite free R -modules F^i for $i \geq n$, differentials $d^i : F^i \rightarrow F^{i+1}$ for $i \geq n$, maps $\alpha : F^i \rightarrow M^i$ compatible with differentials, such that (1) $H^i(\alpha)$ is an isomorphism for $i > n$ and surjective for $i = n$, and (2) $F^i = 0$ for $i > b$. Picture

$$\begin{array}{ccccccccccc} & & F^n & \longrightarrow & F^{n+1} & \longrightarrow & \dots & \longrightarrow & F^i & \longrightarrow & 0 & \longrightarrow & \dots \\ & & \downarrow \alpha & & \downarrow \alpha & & & & \downarrow \alpha & & \downarrow & & \\ M^{n-1} & \longrightarrow & M^n & \longrightarrow & M^{n+1} & \longrightarrow & \dots & \longrightarrow & M^i & \longrightarrow & M^{i+1} & \longrightarrow & \dots \end{array}$$

The base case is $n = m$ which we've done above. Induction step. Let C^\bullet be the cone on α (Derived Categories, Definition 9.1). The long exact sequence of cohomology shows that $H^i(C^\bullet) = 0$ for $i \geq n$. Observe that F^\bullet is pseudo-coherent as a complex of A -modules because R is finite free as an A -module. Hence by More

on Algebra, Lemma 60.2 we see that C^\bullet is $(n-1)$ -pseudo-coherent as a complex of A -modules. By More on Algebra, Lemma 60.3 we see that $H^{n-1}(C^\bullet)$ is a finite A -module. Choose a finite free R -module F^{n-1} and a map $\beta : F^{n-1} \rightarrow C^{n-1}$ such that the composition $F^{n-1} \rightarrow C^{n-1} \rightarrow C^n$ is zero and such that F^{n-1} surjects onto $H^{n-1}(C^\bullet)$. Since $C^{n-1} = M^{n-1} \oplus F^n$ we can write $\beta = (\alpha^{n-1}, -d^{n-1})$. The vanishing of the composition $F^{n-1} \rightarrow C^{n-1} \rightarrow C^n$ implies these maps fit into a morphism of complexes

$$\begin{array}{ccccccc} F^{n-1} & \xrightarrow{\quad} & F^n & \longrightarrow & F^{n+1} & \longrightarrow & \dots \\ \downarrow \alpha^{n-1} & & \downarrow \alpha & & \downarrow \alpha & & \\ \dots & \longrightarrow & M^{n-1} & \longrightarrow & M^n & \longrightarrow & M^{n+1} \longrightarrow \dots \end{array}$$

Moreover, these maps define a morphism of distinguished triangles

$$\begin{array}{ccccccc} (F^n \rightarrow \dots) & \longrightarrow & (F^{n-1} \rightarrow \dots) & \longrightarrow & F^{n-1} & \longrightarrow & (F^n \rightarrow \dots)[1] \\ \downarrow & & \downarrow & & \downarrow \beta & & \downarrow \\ (F^n \rightarrow \dots) & \longrightarrow & M^\bullet & \longrightarrow & C^\bullet & \longrightarrow & (F^n \rightarrow \dots)[1] \end{array}$$

Hence our choice of β implies that the map of complexes $(F^{n-1} \rightarrow \dots) \rightarrow M^\bullet$ induces an isomorphism on cohomology in degrees $\geq n$ and a surjection in degree $n-1$. This finishes the proof of the lemma. \square

0CSG Lemma 30.2. *Let A be a ring. Let $n \geq 0$. Let $K \in D_{QCoh}(\mathcal{O}_{\mathbf{P}_A^n})$. The following are equivalent*

- (1) K is pseudo-coherent,
- (2) $R\Gamma(\mathbf{P}_A^n, E \otimes^{\mathbf{L}} K)$ is a pseudo-coherent object of $D(A)$ for each pseudo-coherent object E of $D(\mathcal{O}_{\mathbf{P}_A^n})$,
- (3) $R\Gamma(\mathbf{P}_A^n, E \otimes^{\mathbf{L}} K)$ is a pseudo-coherent object of $D(A)$ for each perfect object E of $D(\mathcal{O}_{\mathbf{P}_A^n})$,
- (4) $R\mathrm{Hom}_{\mathbf{P}_A^n}(E, K)$ is a pseudo-coherent object of $D(A)$ for each perfect object E of $D(\mathcal{O}_{\mathbf{P}_A^n})$,
- (5) $R\Gamma(\mathbf{P}_A^n, K \otimes^{\mathbf{L}} \mathcal{O}_{\mathbf{P}_A^n}(d))$ is pseudo-coherent object of $D(A)$ for $d = 0, 1, \dots, n$.

Proof. Recall that

$$R\mathrm{Hom}_{\mathbf{P}_A^n}(E, K) = R\Gamma(\mathbf{P}_A^n, R\mathcal{H}om_{\mathcal{O}_{\mathbf{P}_A^n}}(E, K))$$

by definition, see Cohomology, Section 38. Thus parts (4) and (3) are equivalent by Cohomology, Lemma 43.11.

Since every perfect complex is pseudo-coherent, it is clear that (2) implies (3).

Assume (1) holds. Then $E \otimes^{\mathbf{L}} K$ is pseudo-coherent for every pseudo-coherent E , see Cohomology, Lemma 41.5. By Lemma 26.5 the direct image of such a pseudo-coherent complex is pseudo-coherent and we see that (2) is true.

Part (3) implies (5) because we can take $E = \mathcal{O}_{\mathbf{P}_A^n}(d)$ for $d = 0, 1, \dots, n$.

To finish the proof we have to show that (5) implies (1). Let P be as in (19.0.1) and R as in (19.0.2). By Lemma 19.1 we have an equivalence

$$- \otimes_R^{\mathbf{L}} P : D(R) \longrightarrow D_{QCoh}(\mathcal{O}_{\mathbf{P}_A^n})$$

Let $M \in D(R)$ be an object such that $M \otimes^{\mathbf{L}} P = K$. By Differential Graded Algebra, Lemma 26.4 there is an isomorphism

$$R\mathrm{Hom}(R, M) = R\mathrm{Hom}_{\mathbf{P}_A^n}(P, K)$$

in $D(A)$. Arguing as above we obtain

$$R\mathrm{Hom}_{\mathbf{P}_A^n}(P, K) = R\Gamma(\mathbf{P}_A^n, R\mathrm{Hom}_{\mathcal{O}_{\mathbf{P}_A^n}}(E, K)) = R\Gamma(\mathbf{P}_A^n, P^\vee \otimes_{\mathcal{O}_{\mathbf{P}_A^n}}^{\mathbf{L}} K).$$

Using that P^\vee is the direct sum of $\mathcal{O}_{\mathbf{P}_A^n}(d)$ for $d = 0, 1, \dots, n$ and (5) we conclude $R\mathrm{Hom}(R, M)$ is pseudo-coherent as a complex of A -modules. Of course $M = R\mathrm{Hom}(R, M)$ in $D(A)$. Thus M is pseudo-coherent as a complex of A -modules. By Lemma 30.1 we may represent M by a bounded above complex F^\bullet of finite free R -modules. Then $F^\bullet = \bigcup_{p \geq 0} \sigma_{\geq p} F^\bullet$ is a filtration which shows that F^\bullet is a differential graded R -module with property (P), see Differential Graded Algebra, Section 13. Hence $K = M \otimes_R^{\mathbf{L}} P$ is represented by $F^\bullet \otimes_R P$ (follows from the construction of the derived tensor functor, see for example the proof of Differential Graded Algebra, Lemma 26.3). Since $F^\bullet \otimes_R P$ is a bounded above complex whose terms are direct sums of copies of P we conclude that the lemma is true. \square

0CSH **Lemma 30.3.** *Let A be a ring. Let X be a scheme over A which is quasi-compact and quasi-separated. Let $K \in D_{Q\mathrm{Coh}}^-(\mathcal{O}_X)$. If $R\Gamma(X, E \otimes^{\mathbf{L}} K)$ is pseudo-coherent in $D(A)$ for every perfect E in $D(\mathcal{O}_X)$, then $R\Gamma(X, E \otimes^{\mathbf{L}} K)$ is pseudo-coherent in $D(A)$ for every pseudo-coherent E in $D(\mathcal{O}_X)$.*

This lemma is false if one drops the assumption that K is bounded above.

Proof. There exists an integer N such that $R\Gamma(X, -) : D_{Q\mathrm{Coh}}(\mathcal{O}_X) \rightarrow D(A)$ has cohomological dimension N as explained in Lemma 4.1. Let $b \in \mathbf{Z}$ be such that $H^i(K) = 0$ for $i > b$. Let E be pseudo-coherent on X . It suffices to show that $R\Gamma(X, E \otimes^{\mathbf{L}} K)$ is m -pseudo-coherent for every m . Choose an approximation $P \rightarrow E$ by a perfect complex P of $(X, E, m - N - 1 - b)$. This is possible by Theorem 13.6. Choose a distinguished triangle

$$P \rightarrow E \rightarrow C \rightarrow P[1]$$

in $D_{Q\mathrm{Coh}}(\mathcal{O}_X)$. The cohomology sheaves of C are zero in degrees $\geq m - N - 1 - b$. Hence the cohomology sheaves of $C \otimes^{\mathbf{L}} K$ are zero in degrees $\geq m - N - 1$. Thus the cohomology of $R\Gamma(X, C \otimes^{\mathbf{L}} K)$ are zero in degrees $\geq m - 1$. Hence

$$R\Gamma(X, P \otimes^{\mathbf{L}} K) \rightarrow R\Gamma(X, E \otimes^{\mathbf{L}} K)$$

is an isomorphism on cohomology in degrees $\geq m$. By assumption the source is pseudo-coherent. We conclude that $R\Gamma(X, E \otimes^{\mathbf{L}} K)$ is m -pseudo-coherent as desired. \square

31. Relatively perfect objects

0DHZ In this section we introduce a notion from [Lie06].

0DIO **Definition 31.1.** Let $f : X \rightarrow S$ be a morphism of schemes which is flat and locally of finite presentation. An object E of $D(\mathcal{O}_X)$ is *perfect relative to S* or *S -perfect* if E is pseudo-coherent (Cohomology, Definition 41.1) and E locally has finite tor dimension as an object of $D(f^{-1}\mathcal{O}_S)$ (Cohomology, Definition 42.1).

Please see Remark 31.13 for a discussion.

0DI1 **Example 31.2.** Let k be a field. Let X be a scheme of finite presentation over k (in particular X is quasi-compact). Then an object E of $D(\mathcal{O}_X)$ is k -perfect if and only if it is bounded and pseudo-coherent (by definition), i.e., if and only if it is in $D_{Coh}^b(X)$ (by Lemma 9.3). Thus being relatively perfect does **not** mean “perfect on the fibres”.

The corresponding algebra concept is studied in More on Algebra, Section 74. We can link the notion for schemes with the algebraic notion as follows.

0DI2 **Lemma 31.3.** *Let $f : X \rightarrow S$ be a morphism of schemes which is flat and locally of finite presentation. Let E be an object of $D_{QCoh}(\mathcal{O}_X)$. The following are equivalent*

- (1) E is S -perfect,
- (2) for any affine open $U \subset X$ mapping into an affine open $V \subset S$ the complex $R\Gamma(U, E)$ is $\mathcal{O}_S(V)$ -perfect.
- (3) there exists an affine open covering $S = \bigcup V_i$ and for each i an affine open covering $f^{-1}(V_i) = \bigcup U_{ij}$ such that the complex $R\Gamma(U_{ij}, E)$ is $\mathcal{O}_S(V_i)$ -perfect.

Proof. Being pseudo-coherent is a local property and “locally having finite tor dimension” is a local property. Hence this lemma immediately reduces to the statement: if X and S are affine, then E is S -perfect if and only if $K = R\Gamma(X, E)$ is $\mathcal{O}_S(S)$ -perfect. Say $X = \text{Spec}(A)$, $S = \text{Spec}(R)$ and E corresponds to $K \in D(A)$, i.e., $K = R\Gamma(X, E)$, see Lemma 3.5.

Observe that K is R -perfect if and only if K is pseudo-coherent and has finite tor dimension as a complex of R -modules (More on Algebra, Definition 74.1). By Lemma 9.2 we see that E is pseudo-coherent if and only if K is pseudo-coherent. By Lemma 9.5 we see that E has finite tor dimension over $f^{-1}\mathcal{O}_S$ if and only if K has finite tor dimension as a complex of R -modules. \square

0DI3 **Lemma 31.4.** *Let $f : X \rightarrow S$ be a morphism of schemes which is flat and locally of finite presentation. The full subcategory of $D(\mathcal{O}_X)$ consisting of S -perfect objects is a saturated⁴ triangulated subcategory.*

Proof. This follows from Cohomology, Lemmas 41.4, 41.6, 42.6, and 42.8. \square

0DI4 **Lemma 31.5.** *Let $f : X \rightarrow S$ be a morphism of schemes which is flat and locally of finite presentation. A perfect object of $D(\mathcal{O}_X)$ is S -perfect. If $K, M \in D(\mathcal{O}_X)$, then $K \otimes_{\mathcal{O}_X}^L M$ is S -perfect if K is perfect and M is S -perfect.*

Proof. First proof: reduce to the affine case using Lemma 31.3 and then apply More on Algebra, Lemma 74.3. \square

0DI5 **Lemma 31.6.** *Let $f : X \rightarrow S$ be a morphism of schemes which is flat and locally of finite presentation. Let $g : S' \rightarrow S$ be a morphism of schemes. Set $X' = S' \times_S X$ and denote $g' : X' \rightarrow X$ the projection. If $K \in D(\mathcal{O}_X)$ is S -perfect, then $L(g')^*K$ is S' -perfect.*

Proof. First proof: reduce to the affine case using Lemma 31.3 and then apply More on Algebra, Lemma 74.5.

Second proof: $L(g')^*K$ is pseudo-coherent by Cohomology, Lemma 41.3 and the bounded tor dimension property follows from Lemma 21.6. \square

⁴Derived Categories, Definition 6.1.

0DI6 **Situation 31.7.** Let $S = \lim_{i \in I} S_i$ be a limit of a directed system of schemes with affine transition morphisms $g_{i' i} : S_{i'} \rightarrow S_i$. We assume that S_i is quasi-compact and quasi-separated for all $i \in I$. We denote $g_i : S \rightarrow S_i$ the projection. We fix an element $0 \in I$ and a flat morphism of finite presentation $X_0 \rightarrow S_0$. We set $X_i = S_i \times_{S_0} X_0$ and $X = S \times_{S_0} X_0$ and we denote the transition morphisms $f_{i' i} : X_{i'} \rightarrow X_i$ and $f_i : X \rightarrow X_i$ the projections.

0DI7 **Lemma 31.8.** *In Situation 31.7. Let K_0 and L_0 be objects of $D(\mathcal{O}_{X_0})$. Set $K_i = Lf_{i0}^* K_0$ and $L_i = Lf_{i0}^* L_0$ for $i \geq 0$ and set $K = Lf_0^* K_0$ and $L = Lf_0^* L_0$. Then the map*

$$\operatorname{colim}_{i \geq 0} \operatorname{Hom}_{D(\mathcal{O}_{X_i})}(K_i, L_i) \longrightarrow \operatorname{Hom}_{D(\mathcal{O}_X)}(K, L)$$

is an isomorphism if K_0 is pseudo-coherent and $L_0 \in D_{QCoh}(\mathcal{O}_{X_0})$ has (locally) finite tor dimension as an object of $D((X_0 \rightarrow S_0)^{-1} \mathcal{O}_{S_0})$

Proof. For every quasi-compact open $U_0 \subset X_0$ consider the condition P that

$$\operatorname{colim}_{i \geq 0} \operatorname{Hom}_{D(\mathcal{O}_{U_i})}(K_i|_{U_i}, L_i|_{U_i}) \longrightarrow \operatorname{Hom}_{D(\mathcal{O}_U)}(K|_U, L|_U)$$

is an isomorphism where $U = f_0^{-1}(U_0)$ and $U_i = f_{i0}^{-1}(U_0)$. If P holds for U_0 , V_0 and $U_0 \cap V_0$, then it holds for $U_0 \cup V_0$ by Mayer-Vietoris for hom in the derived category, see Cohomology, Lemma 31.3.

Denote $\pi_0 : X_0 \rightarrow S_0$ the given morphism. Then we can first consider $U_0 = \pi_0^{-1}(W_0)$ with $W_0 \subset S_0$ quasi-compact open. By the induction principle of Cohomology of Schemes, Lemma 4.1 applied to quasi-compact opens of S_0 and the remark above, we find that it is enough to prove P for $U_0 = \pi_0^{-1}(W_0)$ with W_0 affine. In other words, we have reduced to the case where S_0 is affine. Next, we apply the induction principle again, this time to all quasi-compact and quasi-separated opens of X_0 , to reduce to the case where X_0 is affine as well.

If X_0 and S_0 are affine, the result follows from More on Algebra, Lemma 74.6. Namely, by Lemmas 9.1 and 3.5 the statement is translated into computations of homs in the derived categories of modules. Then Lemma 9.2 shows that the complex of modules corresponding to K_0 is pseudo-coherent. And Lemma 9.5 shows that the complex of modules corresponding to L_0 has finite tor dimension over $\mathcal{O}_{S_0}(S_0)$. Thus the assumptions of More on Algebra, Lemma 74.6 are satisfied and we win. \square

0DI8 **Lemma 31.9.** *In Situation 31.7 the category of S -perfect objects of $D(\mathcal{O}_X)$ is the colimit of the categories of S_i -perfect objects of $D(\mathcal{O}_{X_i})$.*

Proof. For every quasi-compact open $U_0 \subset X_0$ consider the condition P that the functor

$$\operatorname{colim}_{i \geq 0} D_{S_i\text{-perfect}}(\mathcal{O}_{U_i}) \longrightarrow D_{S\text{-perfect}}(\mathcal{O}_U)$$

is an equivalence where $U = f_0^{-1}(U_0)$ and $U_i = f_{i0}^{-1}(U_0)$. We observe that we already know this functor is fully faithful by Lemma 31.8. Thus it suffices to prove essential surjectivity.

Suppose that P holds for quasi-compact opens U_0, V_0 of X_0 . We claim that P holds for $U_0 \cup V_0$. We will use the notation $U_i = f_{i0}^{-1}U_0$, $U = f_0^{-1}U_0$, $V_i = f_{i0}^{-1}V_0$, and $V = f_0^{-1}V_0$ and we will abusively use the symbol f_i for all the morphisms $U \rightarrow U_i$, $V \rightarrow V_i$, $U \cap V \rightarrow U_i \cap V_i$, and $U \cup V \rightarrow U_i \cup V_i$. Suppose E is an S -perfect object of $D(\mathcal{O}_{U \cup V})$. Goal: show E is in the essential image of the functor. By assumption,

we can find $i \geq 0$, an S_i -perfect object $E_{U,i}$ on U_i , an S_i -perfect object $E_{V,i}$ on V_i , and isomorphisms $Lf_i^* E_{U,i} \rightarrow E|_U$ and $Lf_i^* E_{V,i} \rightarrow E|_V$. Let

$$a : E_{U,i} \rightarrow (Rf_{i,*} E)|_{U_i} \quad \text{and} \quad b : E_{V,i} \rightarrow (Rf_{i,*} E)|_{V_i}$$

the maps adjoint to the isomorphisms $Lf_i^* E_{U,i} \rightarrow E|_U$ and $Lf_i^* E_{V,i} \rightarrow E|_V$. By fully faithfulness, after increasing i , we can find an isomorphism $c : E_{U,i}|_{U_i \cap V_i} \rightarrow E_{V,i}|_{U_i \cap V_i}$ which pulls back to the identifications

$$Lf_i^* E_{U,i}|_{U \cap V} \rightarrow E|_{U \cap V} \rightarrow Lf_i^* E_{V,i}|_{U \cap V}.$$

Apply Cohomology, Lemma 39.1 to get an object E_i on $U_i \cup V_i$ and a map $d : E_i \rightarrow Rf_{i,*} E$ which restricts to the maps a and b over U_i and V_i . Then it is clear that E_i is S_i -perfect (because being relatively perfect is a local property) and that d is adjoint to an isomorphism $Lf_i^* E_i \rightarrow E$.

By exactly the same argument as used in the proof of Lemma 31.8 using the induction principle (Cohomology of Schemes, Lemma 4.1) we reduce to the case where both X_0 and S_0 are affine. (First work with opens in S_0 to reduce to S_0 affine, then work with opens in X_0 to reduce to X_0 affine.) In the affine case the result follows from More on Algebra, Lemma 74.6. The translation into algebra is done by Lemma 31.3. \square

0DJT **Lemma 31.10.** *Let $f : X \rightarrow S$ be a morphism of schemes which is flat, proper, and of finite presentation. Let $E \in D(\mathcal{O}_X)$ be S -perfect. Then $Rf_* E$ is a perfect object of $D(\mathcal{O}_S)$ and its formation commutes with arbitrary base change.*

Proof. The statement on base change is Lemma 22.4 (with \mathcal{G}^\bullet equal to \mathcal{O}_X in degree 0). Thus it suffices to show that $Rf_* E$ is a perfect object. We will reduce to the case where S is Noetherian affine by a limit argument.

The question is local on S , hence we may assume S is affine. Say $S = \text{Spec}(R)$. We write $R = \text{colim } R_i$ as a filtered colimit of Noetherian rings R_i . By Limits, Lemma 10.1 there exists an i and a scheme X_i of finite presentation over R_i whose base change to R is X . By Limits, Lemmas 13.1 and 8.7 we may assume X_i is proper and flat over R_i . By Lemma 31.9 we may assume there exists a R_i -perfect object E_i of $D(\mathcal{O}_{X_i})$ whose pullback to X is E . Applying Lemma 23.1 to $X_i \rightarrow \text{Spec}(R_i)$ and E_i and using the base change property already shown we obtain the result. \square

0DJU **Lemma 31.11.** *Let $f : X \rightarrow S$ be a morphism of schemes. Let $E, K \in D(\mathcal{O}_X)$. Assume*

- (1) S is quasi-compact and quasi-separated,
- (2) f is proper, flat, and of finite presentation,
- (3) E is S -perfect,
- (4) K is pseudo-coherent.

Then there exists a pseudo-coherent $L \in D(\mathcal{O}_S)$ such that

$$Rf_* R\mathcal{H}om(K, E) = R\mathcal{H}om(L, \mathcal{O}_S)$$

and the same is true after arbitrary base change: given

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ f' \downarrow & & \downarrow f \\ S' & \xrightarrow{g} & S \end{array} \quad \begin{array}{l} \text{cartesian, then we have} \\ Rf'_* R\mathcal{H}om(L(g')^* K, L(g')^* E) \\ = R\mathcal{H}om(Lg^* L, \mathcal{O}_{S'}) \end{array}$$

Proof. Since S is quasi-compact and quasi-separated, the same is true for X . By Lemma 18.1 we can write $K = \text{hocolim} K_n$ with K_n perfect and $K_n \rightarrow K$ inducing an isomorphism on truncations $\tau_{\geq -n}$. Let K_n^\vee be the dual perfect complex (Cohomology, Lemma 43.11). We obtain an inverse system $\dots \rightarrow K_3^\vee \rightarrow K_2^\vee \rightarrow K_1^\vee$ of perfect objects. By Lemma 31.5 we see that $K_n^\vee \otimes_{\mathcal{O}_X} E$ is S -perfect. Thus we may apply Lemma 31.10 to $K_n^\vee \otimes_{\mathcal{O}_X} E$ and we obtain an inverse system

$$\dots \rightarrow M_3 \rightarrow M_2 \rightarrow M_1$$

of perfect complexes on S with

$$M_n = Rf_*(K_n^\vee \otimes_{\mathcal{O}_X}^{\mathbf{L}} E) = Rf_* R\mathcal{H}om(K_n, E)$$

Moreover, the formation of these complexes commutes with any base change, namely $Lg^* M_n = Rf'_*((L(g')^* K_n)^\vee \otimes_{\mathcal{O}_{X'}}^{\mathbf{L}} L(g')^* E) = Rf'_* R\mathcal{H}om(L(g')^* K_n, L(g')^* E)$.

As $K_n \rightarrow K$ induces an isomorphism on $\tau_{\geq -n}$, we see that $K_n \rightarrow K_{n+1}$ induces an isomorphism on $\tau_{\geq -n}$. It follows that $K_{n+1}^\vee \rightarrow K_n^\vee$ induces an isomorphism on $\tau_{\leq n}$ as $K_n^\vee = R\mathcal{H}om(K_n, \mathcal{O}_X)$. Suppose that E has tor amplitude in $[a, b]$ as a complex of $f^{-1}\mathcal{O}_Y$ -modules. Then the same is true after any base change, see Lemma 21.6. We find that $K_{n+1}^\vee \otimes_{\mathcal{O}_X} E \rightarrow K_n^\vee \otimes_{\mathcal{O}_X} E$ induces an isomorphism on $\tau_{\leq n+a}$ and the same is true after any base change. Applying the right derived functor Rf_* we conclude the maps $M_{n+1} \rightarrow M_n$ induce isomorphisms on $\tau_{\leq n+a}$ and the same is true after any base change. Choose a distinguished triangle

$$M_{n+1} \rightarrow M_n \rightarrow C_n \rightarrow M_{n+1}[1]$$

Take S' equal to the spectrum of the residue field at a point $s \in S$ and pull back to see that $C_n \otimes_{\mathcal{O}_S}^{\mathbf{L}} \kappa(s)$ has nonzero cohomology only in degrees $\geq n+a$. By More on Algebra, Lemma 68.6 we see that the perfect complex C_n has tor amplitude in $[n+a, m_n]$ for some integer m_n . In particular, the dual perfect complex C_n^\vee has tor amplitude in $[-m_n, -n-a]$.

Let $L_n = M_n^\vee$ be the dual perfect complex. The conclusion from the discussion in the previous paragraph is that $L_n \rightarrow L_{n+1}$ induces isomorphisms on $\tau_{\geq -n-a}$. Thus $L = \text{hocolim} L_n$ is pseudo-coherent, see Lemma 18.1. Since we have

$$R\mathcal{H}om(K, E) = R\mathcal{H}om(\text{hocolim} K_n, E) = R\lim R\mathcal{H}om(K_n, E) = R\lim K_n^\vee \otimes_{\mathcal{O}_X} E$$

(Cohomology, Lemma 43.12) and since $R\lim$ commutes with Rf_* we find that

$$Rf_* R\mathcal{H}om(K, E) = R\lim M_n = R\lim R\mathcal{H}om(L_n, \mathcal{O}_S) = R\mathcal{H}om(L, \mathcal{O}_S)$$

This proves the formula over S . Since the construction of M_n is compatible with base change, the formula continues to hold after any base change. \square

0DJV **Remark 31.12.** The reader may have noticed the similarity between Lemma 31.11 and Lemma 24.3. Indeed, the pseudo-coherent complex L of Lemma 31.11 may be characterized as the unique pseudo-coherent complex on S such that there are functorial isomorphisms

$$\text{Ext}_{\mathcal{O}_S}^i(L, \mathcal{F}) \longrightarrow \text{Ext}_{\mathcal{O}_X}^i(K, E \otimes_{\mathcal{O}_X}^{\mathbf{L}} Lf^* \mathcal{F})$$

compatible with boundary maps for \mathcal{F} ranging over $QCoh(\mathcal{O}_S)$. If we ever need this we will formulate a precise result here and give a detailed proof.

0DI9 **Remark 31.13.** Our Definition 31.1 of a relatively perfect complex is equivalent to the one given in [Lie06] whenever our definition applies⁵. Next, suppose that $f : X \rightarrow S$ is only assumed to be locally of finite type (not necessarily flat, nor locally of finite presentation). The definition in the paper cited above is that $E \in D(\mathcal{O}_X)$ is relatively perfect if

- (A) locally on X the object E should be quasi-isomorphic to a finite complex of S -flat, finitely presented \mathcal{O}_X -modules.

On the other hand, the natural generalization of our Definition 31.1 is

- (B) E is pseudo-coherent relative to S (More on Morphisms, Definition 48.2) and E locally has finite tor dimension as an object of $D(f^{-1}\mathcal{O}_S)$ (Cohomology, Definition 42.1).

The advantage of condition (B) is that it clearly defines a triangulated subcategory of $D(\mathcal{O}_X)$, whereas we suspect this is not the case for condition (A). The advantage of condition (A) is that it is easier to work with in particular in regards to limits.

32. Other chapters

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⁵To see this, use Lemma 31.3 and More on Algebra, Lemma 74.4.

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