1. Introduction

This chapter discusses resolution of singularities of surfaces following Lipman [Lip78] and following the exposition in [Art86].

2. A trace map in positive characteristic

In this section \( p \) will be a prime number. Let \( R \) be an \( \mathbb{F}_p \)-algebra. Given an \( a \in R \) set \( S = R[x]/(x^p - a) \). Define an \( R \)-linear map

\[
\text{Tr}_x : \Omega_{S/R} \rightarrow \Omega_R
\]

by the rule

\[
x^i dx \mapsto \begin{cases} 0 & \text{if } 0 \leq i \leq p - 2, \\ da & \text{if } i = p - 1\end{cases}
\]

This makes sense as \( \Omega_{S/R} \) is a free \( R \)-module with basis \( x^i dx, 0 \leq i \leq p - 1 \). The following lemma implies that the trace map is well defined, i.e., independent of the choice of the coordinate \( x \).

**Lemma 2.1.** Let \( \varphi : R[x]/(x^p - a) \rightarrow R[y]/(y^p - b) \) be an \( R \)-algebra homomorphism. Then \( \text{Tr}_x = T_{y} \circ \varphi \).

**Proof.** Say \( \varphi(x) = \lambda_0 + \lambda_1 y + \ldots + \lambda_{p-1} y^{p-1} \) with \( \lambda_i \in R \). The condition that mapping \( x \) to \( \lambda_0 + \lambda_1 y + \ldots + \lambda_{p-1} y^{p-1} \) induces an \( R \)-algebra homomorphism \( R[x]/(x^p - a) \rightarrow R[y]/(y^p - b) \) is equivalent to the condition that

\[
a = \lambda_0^p + \lambda_1^p b + \ldots + \lambda_{p-1}^p b^{p-1}
\]

in the ring \( R \). Consider the polynomial ring

\[
R_{\text{univ}} = \mathbb{F}_p[b, \lambda_0, \ldots, \lambda_{p-1}]
\]
with the element \( a = \lambda_0^p + \lambda_1^p b + \ldots + \lambda_{p-1}^p b^{p-1} \) Consider the universal algebra map \( \varphi_{\text{univ}} : R_{\text{univ}}[x]/(x^p - a) \to R_{\text{univ}}[y]/(y^p - b) \) given by mapping \( x \) to \( \lambda_0 + \lambda_1 y + \ldots + \lambda_{p-1} y^{p-1} \). We obtain a canonical map

\[ R_{\text{univ}} \to R \]

sending \( b, \lambda \) to \( b, \lambda_i \). By construction we get a commutative diagram

\[
\begin{array}{ccc}
R_{\text{univ}}[x]/(x^p - a) & \longrightarrow & R[x]/(x^p - a) \\
\varphi_{\text{univ}} \downarrow & & \downarrow \varphi \\
R_{\text{univ}}[y]/(y^p - b) & \longrightarrow & R[y]/(y^p - b)
\end{array}
\]

and the horizontal arrows are compatible with the trace maps. Hence it suffices to prove the lemma for the map \( \varphi_{\text{univ}} \). Thus we may assume \( R = \mathbf{F}_p[b, \lambda_0, \ldots, \lambda_{p-1}] \) is a polynomial ring. We will check the lemma holds in this case by evaluating \( \text{Tr}_y(\varphi(x)^i d\varphi(x)) \) for \( i = 0, \ldots, p - 1 \).

The case \( 0 \leq i \leq p - 2 \). Expand

\[
(\lambda_0 + \lambda_1 y + \ldots + \lambda_{p-1} y^{p-1})^i(\lambda_1 + 2\lambda_2 y + \ldots + (p-1)\lambda_{p-1} y^{p-2})
\]

in the ring \( R[y]/(y^p - b) \). We have to show that the coefficient of \( y^{p-1} \) is zero. For this it suffices to show that the expression above as a polynomial in \( y \) has vanishing coefficients in front of the powers \( y^{p-1} \). Then we write our polynomial as

\[
\frac{d}{(i+1)dy} (\lambda_0 + \lambda_1 y + \ldots + \lambda_{p-1} y^{p-1})^{i+1}
\]

and indeed the coefficients of \( y^{kp-1} \) are all zero.

The case \( i = p - 1 \). Expand

\[
(\lambda_0 + \lambda_1 y + \ldots + \lambda_{p-1} y^{p-1})^{p-1}(\lambda_1 + 2\lambda_2 y + \ldots + (p-1)\lambda_{p-1} y^{p-2})
\]

in the ring \( R[y]/(y^p - b) \). To finish the proof we have to show that the coefficient of \( y^{p-1} \) times \( db \) is \( da \). Here we use that \( R \) is \( S/pS \) where \( S = \mathbf{Z}[b, \lambda_0, \ldots, \lambda_{p-1}] \). Then the above, as a polynomial in \( y \), is equal to

\[
\frac{d}{pdy} (\lambda_0 + \lambda_1 y + \ldots + \lambda_{p-1} y^{p-1})^p
\]

Since \( \frac{d}{dy}(y^k) = pk\lambda^k y^{k-1} \) it suffices to understand the coefficients of \( y^k \) in the polynomial \( (\lambda_0 + \lambda_1 y + \ldots + \lambda_{p-1} y^{p-1})^p \) modulo \( p \). The sum of these terms gives

\[
\lambda_0^p + \lambda_1^p y^p + \ldots + \lambda_{p-1}^p y^{(p-1)p} \mod p
\]

Whence we see that we obtain after applying the operator \( \frac{d}{pdy} \) and after reducing modulo \( y^p - b \) the value

\[
\lambda_0^p + 2\lambda_2^p b + \ldots + (p-1)\lambda_{p-1}^p b^{p-2}
\]

for the coefficient of \( y^{p-1} \) we wanted to compute. Now because \( a = \lambda_0^p + \lambda_1^p b + \ldots + \lambda_{p-1}^p b^{p-1} \) in \( R \) we obtain that

\[
da = (\lambda_0^p + 2\lambda_2^p b + \ldots + (p-1)\lambda_{p-1}^p b^{p-2})db
\]

in \( R \). This proves that the coefficient of \( y^{p-1} \) is as desired.
Lemma 2.2. Let \( F_p \subset \Lambda \subset R \subset S \) be ring extensions and assume that \( S \) is isomorphic to \( R[x]/(x^p - a) \) for some \( a \in R \). Then there are canonical \( R \)-linear maps

\[
Tr: \Omega_{S/\Lambda}^{t+1} \to \Omega_{R/\Lambda}^{t+1}
\]

for \( t \geq 0 \) such that

\[
\eta \land \ldots \land \eta \land x^t dx \mapsto \begin{cases} 
0 & \text{if } 0 \leq i \leq p - 2, \\
\eta_1 \land \ldots \land \eta_i \land da & \text{if } i = p - 1
\end{cases}
\]

for \( \eta \in \Omega_{R/\Lambda} \) and such that \( Tr \) annihilates the image of \( S \otimes_R \Omega_{R/\Lambda}^{t+1} \to \Omega_{S/\Lambda}^{t+1} \).

Proof. For \( t = 0 \) we use the composition

\[
\Omega_{S/\Lambda} \to \Omega_{S/R} \to \Omega_R \to \Omega_{R/\Lambda}
\]

where the second map is Lemma 2.1. There is an exact sequence

\[
H_1(L_{S/R}) \xrightarrow{\delta} \Omega_{R/\Lambda} \otimes_R S \to \Omega_{S/\Lambda} \to \Omega_{S/R} \to 0
\]

(Algebra, Lemma 130.4). The module \( \Omega_{S/R} \) is free over \( S \) with basis \( dx \) and the module \( H^1(L_{S/R}) \) is free over \( S \) with basis \( x^p - a \) which \( \delta \) maps to \( -da \otimes 1 \) in \( \Omega_{R/\Lambda} \otimes_R S \). In particular, if we set

\[
M = \text{Coker}(R \to \Omega_{R/\Lambda}, 1 \mapsto -da)
\]

then we see that \( \text{Coker}(\delta) = M \otimes_R S \). We obtain a canonical map

\[
\Omega_{S/\Lambda}^{t+1} \to \wedge^t_S(\text{Coker}(\delta)) \otimes_S \Omega_{S/R} = \wedge^t_R(M) \otimes_R \Omega_{S/R}
\]

Now, since the image of the map \( Tr: \Omega_{S/R} \to \Omega_{R/\Lambda} \) of Lemma 2.1 is contained in \( Rda \) we see that wedging with an element in the image annihilates \( da \). Hence there is a canonical map

\[
\wedge^t_R(M) \otimes_R \Omega_{S/R} \to \Omega_{R/\Lambda}^{t+1}
\]

mapping \( \eta_1 \land \ldots \land \eta_i \land \omega \) to \( \eta_1 \land \ldots \land \eta_i \land Tr(\omega) \).

Lemma 2.3. Let \( S \) be a scheme over \( F_p \). Let \( f: Y \to X \) be a finite morphism of Noetherian normal integral schemes over \( S \). Assume

1. the extension of function fields is purely inseparable of degree \( p \), and
2. \( \Omega_{X/S} \) is a coherent \( O_X \)-module (for example if \( X \) is of finite type over \( S \)).

For \( i \geq 1 \) there is a canonical map

\[
Tr: f_*\Omega_{Y/S}^i \to (\Omega_{X/S}^i)^{**}
\]

whose stalk in the generic point of \( X \) recovers the trace map of Lemma 2.2.

Proof. The exact sequence \( f^*\Omega_{X/S} \to \Omega_{Y/S} \to \Omega_{Y/X} \to 0 \) shows that \( \Omega_{Y/S} \) and hence \( f_*\Omega_{Y/S} \) are coherent modules as well. Thus it suffices to prove the trace map in the generic point extends to stalks at \( x \in X \) with \( \dim(O_{X,x}) = 1 \), see Divisors, Lemma 10.9. Thus we reduce to the case discussed in the next paragraph.

Assume \( X = \text{Spec}(A) \) and \( Y = \text{Spec}(B) \) with \( A \) a discrete valuation ring and \( B \) finite over \( A \). Since the induced extension \( K \subset L \) of fraction fields is purely inseparable, we see that \( B \) is local too. Hence \( B \) is a discrete valuation ring too. Then either

1. \( B/A \) has ramification index \( p \) and hence \( B = A[x]/(x^p - a) \) where \( a \in A \) is a uniformizer, or
Let $i$ can apply Lemma 2.2 to see that the trace map extends to $\Omega^i_{B/A} \to \Omega^i_{A/A}$ for all $i \geq 1$.

3. Modifications

Let $(A, m, \kappa)$ be a Noetherian local ring. We set $S = \text{Spec}(A)$ and $U = S \setminus \{m\}$. In this section we will consider the category

$$
\begin{align*}
\text{(3.0.1)} & \quad \left\{ f : X \to S \left| \begin{array}{c}
X \text{ is an algebraic space} \\
\text{f is a proper morphism} \\
f^{-1}(U) \to U \text{ is an isomorphism}
\end{array} \right. \right\}
\end{align*}
$$

A morphism from $X/S$ to $X'/S$ will be a morphism of algebraic spaces $X \to X'$ compatible with the structure morphisms over $S$. In Restricted Power Series, Section 13 we have seen that this category only depends on the completion of $A$ and we have proven some elementary properties of objects in this category. In this section we specifically study cases where $\dim(A) \leq 2$ or where the dimension of the closed fibre is at most 1.

**Lemma 3.1.** Let $(A, m, \kappa)$ be a 2-dimensional Noetherian local domain such that $U = \text{Spec}(A) \setminus \{m\}$ is a normal scheme. Then any modification $f : X \to S$ (as in Spaces over Fields, Definition 6.1) is a morphism as in (3.0.1).

**Proof.** Let $f : X \to S$ be a modification. We have to show that $f^{-1}(U) \to U$ is an isomorphism. By Spaces over Fields, Lemma 6.2 there exists a nonempty open $V \subset S$ such that $f^{-1}(V) \to V$ is an isomorphism. Since $X$ is integral we see that $f^{-1}(V)$ is dense in $X$. Note that every closed point $u$ of $U$ has codimension 1, i.e., that $\dim(O_{U,u}) = 1$. Thus we may apply Spaces over Fields, Lemma 4.4 to see that $f^{-1}(U) \to U$ is finite. In particular $f^{-1}(U)$ is a scheme. Then $f^{-1}(U) \to U$ is an isomorphism, see Morphisms, Lemma 48.19.

**Lemma 3.2.** Let $(A, m, \kappa)$ be a Noetherian local ring. Let $g : X \to Y$ be a morphism in the category (3.0.1). If the induced morphism $X_\kappa \to Y_\kappa$ of special fibres is a closed immersion, then $g$ is a closed immersion.

**Proof.** This is a special case of More on Morphisms of Spaces, Lemma 37.3.

**Lemma 3.3.** Let $(A, m, \kappa)$ be a complete Noetherian local ring. Let $X$ be an algebraic space over $\text{Spec}(A)$. If $X \to \text{Spec}(A)$ is proper and $\dim(X_\kappa) \leq 1$, then $X$ is a scheme projective over $A$.

**Proof.** By Spaces over Fields, Lemma 7.5 the algebraic space $X_\kappa$ is a scheme. Hence $X_\kappa$ is a proper scheme of dimension $\leq 1$ over $\kappa$. By Varieties, Lemma 27.4 we see that $X_\kappa$ is H-projective over $\kappa$. Let $\mathcal{L}$ be an ample invertible sheaf on $X_\kappa$.

We are going to show that $\mathcal{L}$ lifts to a compatible system $\{\mathcal{L}_n\}$ of invertible sheaves on the $n$th infinitesimal neighbourhoods

$$X_n = X \times_{\text{Spec}(A)} \text{Spec}(A/m^n)$$

of $X_\kappa = X_1$. Recall that the étale sites of $X_\kappa$ and all $X_n$ are canonically equivalent, see More on Morphisms of Spaces, Lemma 8.6. In the rest of the proof we do not
distinguish between sheaves on $X_n$ and sheaves on $X_m$ or $X_\kappa$. Suppose, given a lift $L_n$ to $X_n$. We consider the exact sequence

$$1 \to (1 + m^n \mathcal{O}_X/m^{n+1} \mathcal{O}_X)^* \to \mathcal{O}_{X_{n+1}}^* \to \mathcal{O}_{X_n}^* \to 1$$

of sheaves on $X_{n+1}$. We have $(1 + m^n \mathcal{O}_X/m^{n+1} \mathcal{O}_X)^* \cong m^n \mathcal{O}_X/m^{n+1} \mathcal{O}_X$ as abelian sheaves on $X_{n+1}$. The class of $L_n$ in $H^1(X_n, \mathcal{O}_{X_n}^*)$ (see Cohomology on Sites, Lemma 7.1) can be lifted to an element of $H^1(X_{n+1}, \mathcal{O}_{X_{n+1}}^*)$ if and only if the obstruction in $H^2(X_{n+1}, m^n \mathcal{O}_X/m^{n+1} \mathcal{O}_X)$ is zero. Note that $m^n \mathcal{O}_X/m^{n+1} \mathcal{O}_X$ is a quasi-coherent $\mathcal{O}_{X_\kappa}$-module on $X_\kappa$. Hence its étale cohomology agrees with its cohomology on the scheme $X_\kappa$, see Descent, Proposition 7.10. However, as $X_\kappa$ is a Noetherian scheme of dimension $\leq 1$ this cohomology group vanishes (Cohomology, Proposition 21.16).

By Grothendieck’s algebraization theorem (Cohomology of Schemes, Theorem 23.4) we find a projective morphism of schemes $Y \to \text{Spec}(A)$ and a compatible system of isomorphisms $X_n \to Y_n$. (Here we use the assumption that $A$ is complete.) By More on Morphisms of Spaces, Lemma 32.3 we see that $X \cong Y$ and the proof is complete.

**Lemma 3.4.** Let $(A, m, \kappa)$ be a Noetherian local domain of dimension $\geq 1$. Let $f: X \to \text{Spec}(A)$ be a morphism of algebraic spaces. Assume one of the following conditions is satisfied

1. $f$ is a modification (Spaces over Fields, Definition 6.7).
2. $f$ is an alteration (Spaces over Fields, Definition 6.3).
3. $f$ is locally of finite type, quasi-separated, $X$ is integral, and there is exactly one point of $|X|$ mapping to the generic point of $\text{Spec}(A)$.
4. $f$ is locally of finite type, $X$ is decent, and the points of $|X|$ mapping to the generic point of $\text{Spec}(A)$ are the generic points of irreducible components of $|X|$.
5. add more here.

Then $\dim(X_\kappa) \leq \dim(A) - 1$.

**Proof.** Cases (1), (2), and (3) are special cases of (4). Choose an affine scheme $U = \text{Spec}(B)$ and an étale morphism $U \to X$. The ring map $A \to B$ is of finite type. We have to show that $\dim(U_\kappa) \leq \dim(A) - 1$. Since $X$ is decent, the generic points of irreducible components of $U$ are the points lying over generic points of irreducible components of $|X|$, see Decent Spaces, Lemma 10.8. Hence the fibre of $\text{Spec}(B) \to \text{Spec}(A)$ over $(0)$ is the (finite) set of minimal primes $q_1, \ldots, q_r$ of $B$. Thus $A_f \to B_f$ is finite for some nonzero $f \in A$ (Algebra, Lemma 119.9). We conclude the field extensions $f.f.(A) \subset \kappa(q_i)$ are finite. Let $q \subset B$ be a prime lying over $m$. Then

$$\dim(B_q) = \max \dim((B/q_i)_q) \leq \dim(A)$$

the inequality by the dimension formula for $A \subset B/q_i$, see Algebra, Lemma 110.1. However, the dimension of $B_q/mB_q$ (which is the local ring of $U_\kappa$ at the corresponding point) is at least one less because the minimal primes $q_i$ are not in $V(m)$. We conclude by Properties, Lemma 10.2.

**Lemma 3.5.** If $(A, m, \kappa)$ is a complete Noetherian local domain of dimension 2, then every modification of $\text{Spec}(A)$ is projective over $A$. 

Proof. By Lemma 3.3 it suffices to show that the special fibre of any modification $X$ of $\text{Spec}(A)$ has dimension $\leq 1$. This follows from Lemma 3.4.

4. Quadratic transformations

In this section we study what happens when we blow up a nonsingular point on a surface. We hesitate the formally define such a morphism as a quadratic transformation as on the one hand often other names are used and on the other hand the phrase “quadratic transformation” is sometimes used with a different meaning.

**Lemma 4.1.** Let $(A, \mathfrak{m}, \kappa)$ be a regular local ring of dimension 2. Let $f : X \to S = \text{Spec}(A)$ be the blowing up of $A$ in $\mathfrak{m}$. There is a closed immersion

$$r : X \to \mathbb{P}^1_S$$

over $S$ such that $\mathcal{O}_X(1) = r^*\mathcal{O}_{\mathbb{P}^1_S}(1)$ and such that $r|_E : E \to \mathbb{P}^1_{\kappa}$ is an isomorphism.

**Proof.** As $A$ is regular of dimension 2 we can write $\mathfrak{m} = (x, y)$. Then $x$ and $y$ placed in degree 1 generate the Rees algebra $\bigoplus_{n \geq 0} \mathfrak{m}^n$ over $A$. Recall that $X = \text{Proj}(\bigoplus_{n \geq 0} \mathfrak{m}^n)$, see Divisors, Lemma 21.2. Thus the surjection

$$A[T_0, T_1] \to \bigoplus_{n \geq 0} \mathfrak{m}^n, \quad T_0 \mapsto x, \quad T_1 \mapsto y$$

of graded $A$-algebras induces a closed immersion $r : X \to \mathbb{P}^1_{\kappa} = \text{Proj}(A[T_0, T_1])$ such that $\mathcal{O}_X(1) = r^*\mathcal{O}_{\mathbb{P}^1_{\kappa}}(1)$, see Constructions, Lemma 11.5. To prove the final statement note that

$$(\bigoplus_{n \geq 0} \mathfrak{m}^n) \otimes_A \kappa = \bigoplus_{n \geq 0} \mathfrak{m}^n/\mathfrak{m}^{n+1} \cong \kappa[\overline{x}, \overline{y}]$$

a polynomial algebra, see Algebra, Lemma 103.1. This proves that the fibre of $X \to S$ over $\text{Spec}(\kappa)$ is equal to $\text{Proj}(\kappa[\overline{x}, \overline{y}]) = \mathbb{P}^1_{\kappa}$, see Constructions, Lemma 11.6. Recall that $E$ is the closed subscheme of $X$ defined by $\mathfrak{m}\mathcal{O}_X$, i.e., $E = X_\kappa$. By our choice of the morphism $r$ we see that $r|_E$ in fact produces the identification of $E = X_\kappa$ with the special fibre of $\mathbb{P}^1_{\kappa} \to S$.

**Lemma 4.2.** Let $(A, \mathfrak{m}, \kappa)$ be a regular local ring of dimension 2. Let $f : X \to S = \text{Spec}(A)$ be the blowing up of $A$ in $\mathfrak{m}$. Then $X$ is an irreducible regular scheme.

**Proof.** Observe that $X$ is integral by Divisors, Lemma 21.7 and Algebra, Lemma 103.2. To see $X$ is regular it suffices to check that $\mathcal{O}_{X, x}$ is regular for closed points $x \in X$, see Properties, Lemma 9.2. Let $x \in X$ be a closed point. Since $f$ is proper $x$ maps to $\mathfrak{m}$, i.e., $x$ is a point of the exceptional divisor $E$. Then $E$ is an effective Cartier divisor and $E \cong \mathbb{P}^1_{\kappa}$. Thus if $f \in \mathfrak{m}_x \subset \mathcal{O}_{X, x}$ is a local equation for $E$, then $\mathcal{O}_{X, x}/(f) \cong \mathcal{O}_{\mathbb{P}^1_{\kappa}, x}$. Since $\mathbb{P}^1_{\kappa}$ is covered by two affine opens which are the spectrum of a polynomial ring over $\kappa$, we see that $\mathcal{O}_{\mathbb{P}^1_{\kappa}, x}$ is regular by Algebra, Lemma 111.1. We conclude by Algebra, Lemma 103.7.

**Lemma 4.3.** Let $(A, \mathfrak{m}, \kappa)$ be a regular local ring of dimension 2. Let $f : X \to S = \text{Spec}(A)$ be the blowing up of $A$ in $\mathfrak{m}$. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module.

1. $H^p(X, \mathcal{F}) = 0$ for $p \notin \{0, 1\}$,
2. $H^1(X, \mathcal{O}_X(n)) = 0$ for $n \geq -1$,
3. $H^1(X, \mathcal{F}) = 0$ if $\mathcal{F}$ or $\mathcal{F}(1)$ is globally generated,
4. $H^0(X, \mathcal{O}_X(n)) = \mathfrak{m}^{\max(0, n)}$, 


(5) \( \text{length}_A H^1(X, \mathcal{O}_X(n)) = -n(n - 1)/2 \) if \( n < 0 \).

**Proof.** If \( m = (x, y) \), then \( X \) is covered by the spectra of the affine blowup algebras \( A[\frac{m}{2}] \) and \( A[\frac{m}{2}] \) because \( x \) and \( y \) placed in degree 1 generate the Rees algebra \( \bigoplus m^n \) over \( A \). See Divisors, Lemma \ref{divisor-chinese} and Constructions, Lemma \ref{construction-lem}. Since \( X \) is separated by Constructions, Lemma \ref{construction-lemma} we see that cohomology of quasi-coherent sheaves vanishes in degrees \( \geq 2 \) by Cohomology of Schemes, Lemma \ref{cohomology-lem}.

Let \( i : E \to X \) be the exceptional divisor, see Divisors, Definition \ref{divisor-def}. Recall that \( \mathcal{O}_X(-E) = \mathcal{O}_X(1) \) is \( f \)-relatively ample, see Divisors, Lemma \ref{divisor-lem}. Hence we know that \( H^1(X, \mathcal{O}_X(-nE)) = 0 \) for some \( n > 0 \), see Cohomology of Schemes, Lemma \ref{cohomology-lem}. Consider the filtration

\[
\mathcal{O}_X(-nE) \subset \mathcal{O}_X(-(n-1)E) \subset \ldots \subset \mathcal{O}_X(-E) \subset \mathcal{O}_X \subset \mathcal{O}_X(E)
\]

The successive quotients are the sheaves

\[
\mathcal{O}_X(-(tE))/\mathcal{O}_X(-(t+1)E) = \mathcal{O}_X(t)/\mathcal{I}(t) = i_* \mathcal{O}_E(t)
\]

where \( \mathcal{I} = \mathcal{O}_X(-E) \) is the ideal sheaf of \( E \). By Lemma \ref{lem}, we have \( E = \mathbf{P}_k^1 \) and \( \mathcal{O}_E(1) \) indeed corresponds to the usual Serre twist of the structure sheaf on \( \mathbf{P}_k^1 \). Hence the cohomology of \( \mathcal{O}_E(t) \) vanishes in degree 1 for \( t \geq -1 \), see Cohomology of Schemes, Lemma \ref{cohomology-lem}. Since this is equal to \( H^1(X, i_* \mathcal{O}_E(t)) \) (by Cohomology of Schemes, Lemma \ref{cohomology-lem}) we find that \( H^1(X, \mathcal{O}_X(-(t+1)E)) \to H^1(X, \mathcal{O}_X(-(tE))) \) is surjective for \( t \geq -1 \). Hence

\[
0 = H^1(X, \mathcal{O}_X(-nE)) \to H^1(X, \mathcal{O}_X(-tE)) = H^1(X, \mathcal{O}_X(t))
\]

is surjective for \( t \geq -1 \) which proves (2).

Let \( \mathcal{F} \) be globally generated. This means there exists a short exact sequence

\[
0 \to \mathcal{G} \to \bigoplus_{i \in I} \mathcal{O}_X \to \mathcal{F} \to 0
\]

Note that \( H^1(X, \bigoplus_{i \in I} \mathcal{O}_X) = \bigoplus_{i \in I} H^1(X, \mathcal{O}_X) \) by Cohomology, Lemma \ref{cohomology-lem}. By part (2) we have \( H^1(X, \mathcal{O}_X) = 0 \). If \( \mathcal{F}(1) \) is globally generated, then we can find a surjection \( \bigoplus_{i \in I} \mathcal{O}_X(-1) \to \mathcal{F} \) and argue in a similar fashion. In other words, part (3) follows from part (2).

For part (4) we note that for all \( n \) large enough we have \( \Gamma(X, \mathcal{O}_X(n)) = \mathcal{M}^n \), see Cohomology of Schemes, Lemma \ref{cohomology-lem}. If \( n \geq 0 \), then we can use the short exact sequence

\[
0 \to \mathcal{O}_X(n) \to \mathcal{O}_X(n-1) \to i_* \mathcal{O}_E(n-1) \to 0
\]

and the vanishing of \( H^1 \) for the sheaf on the left to get a commutative diagram

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & m^{\max(0,n)} & \longrightarrow & m^{\max(0,n-1)} & \longrightarrow & m^{\max(0,n)}/m^{\max(0,n-1)} & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Gamma(X, \mathcal{O}_X(n)) & \longrightarrow & \Gamma(X, \mathcal{O}_X(n-1)) & \longrightarrow & \Gamma(E, \mathcal{O}_E(n-1)) & \longrightarrow & 0
\end{array}
\]

with exact rows. In fact, the rows are exact also for \( n < 0 \) because in this case the groups on the right are zero. In the proof of Lemma \ref{lem} we have seen that the right vertical arrow is an isomorphism (details omitted). Hence if the left vertical arrow is an isomorphism, so is the middle one. In this way we see that (4) holds by descending induction on \( n \).
Finally, we prove (5) by descending induction on $n$ and the sequences

$$0 \to \mathcal{O}_X(n) \to \mathcal{O}_X(n-1) \to i_*\mathcal{O}_E(n-1) \to 0$$

Namely, for $n \geq -1$ we already know $H^1(X, \mathcal{O}_X(n)) = 0$. Since

$$H^1(X, i_*\mathcal{O}_E(-2)) = H^1(E, \mathcal{O}_E(-2)) = H^1(\mathbb{P}^1, \mathcal{O}(-2)) \cong \kappa$$

by Cohomology of Schemes, Lemma 8.1 which has length 1 as an $A$-module, we conclude from the long exact cohomology sequence that (5) holds for $n = -2$. And so on and so forth.

**Lemma 4.4.** Let $(A, \mathfrak{m})$ be a regular local ring of dimension 2. Let $f : X \to S = \text{Spec}(A)$ be the blowing up of $A$ in $\mathfrak{m}$. Let $\mathfrak{m}^n \subset I \subset \mathfrak{m}$ be an ideal. Let $d \geq 0$ be the largest integer such that

$$I\mathcal{O}_X \subset \mathcal{O}_X(-dE)$$

where $E$ is the exceptional divisor. Set $I' = I\mathcal{O}_X(dE) \subset \mathcal{O}_X$. Then $d > 0$, the sheaf $\mathcal{O}_X/I'$ is supported in finitely many closed points $x_1, \ldots, x_r$ of $X$, and

$$\text{length}_A(A/I) > \text{length}_A(\Gamma(X, \mathcal{O}_X/I'))$$

$$\geq \sum_{i=1, \ldots, r} \text{length}_{\mathcal{O}_{X,x_i}}(\mathcal{O}_{X,x_i}/I'_{x_i})$$

**Proof.** Since $I \subset \mathfrak{m}$ we see that every element of $I$ vanishes on $E$. Thus we see that $d \geq 1$. On the other hand, since $\mathfrak{m}^n \subset I$ we see that $d \leq n$. Consider the short exact sequence

$$0 \to I\mathcal{O}_X \to \mathcal{O}_X \to \mathcal{O}_X/I\mathcal{O}_X \to 0$$

Since $I\mathcal{O}_X$ is globally generated, we see that $H^1(X, I\mathcal{O}_X) = 0$ by Lemma 4.3. Hence we obtain a surjection $A/I \to \Gamma(X, \mathcal{O}_X/I\mathcal{O}_X)$. Consider the short exact sequence

$$0 \to \mathcal{O}_X(-dE)/I\mathcal{O}_X \to \mathcal{O}_X/I\mathcal{O}_X \to \mathcal{O}_X/\mathcal{O}_X(-dE) \to 0$$

By Divisors, Lemma 12.7 we see that $\mathcal{O}_X(-dE)/I\mathcal{O}_X$ is supported in finitely many closed points of $X$. In particular, this coherent sheaf has vanishing higher cohomology groups (detail omitted). Thus in the following diagram

$$\begin{array}{c}
A/I \\
\downarrow \\
\Gamma(X, \mathcal{O}_X(-dE)/I\mathcal{O}_X) \longrightarrow \Gamma(X, \mathcal{O}_X/I\mathcal{O}_X) \longrightarrow \Gamma(X, \mathcal{O}_X/\mathcal{O}_X(-dE)) \longrightarrow 0
\end{array}$$

the bottom row is exact and the vertical arrow surjective. We have

$$\text{length}_A(\Gamma(X, \mathcal{O}_X(-dE)/I\mathcal{O}_X)) < \text{length}_A(A/I)$$

since $\Gamma(X, \mathcal{O}_X/\mathcal{O}_X(-dE))$ is nonzero. Namely, the image of $1 \in \Gamma(X, \mathcal{O}_X)$ is nonzero as $d > 0$.

To finish the proof we translate the results above into the statements of the lemma. Since $\mathcal{O}_X(dE)$ is invertible we have

$$\mathcal{O}_X/I' = \mathcal{O}_X(-dE)/I\mathcal{O}_X \otimes_{\mathcal{O}_X} \mathcal{O}_X(dE).$$

Thus $\mathcal{O}_X/I'$ and $\mathcal{O}_X(-dE)/I\mathcal{O}_X$ are supported in the same set of finitely many closed points, say $x_1, \ldots, x_r \in E \subset X$. Moreover we obtain

$$\Gamma(X, \mathcal{O}_X(-dE)/I\mathcal{O}_X) = \bigoplus \mathcal{O}_X(-dE)_{x_i}/I\mathcal{O}_{X,x_i} \cong \bigoplus \mathcal{O}_{X,x_i}/I'_{x_i} = \Gamma(X, \mathcal{O}_X/I')$$
Lemma 4.5. Let $(A, m, \kappa)$ be a regular local ring of dimension 2. Let $f : X \to S = \text{Spec}(A)$ be the blowing up of $A$ in $m$. Then $\Omega_{X/S} = i_* \Omega_{E/\kappa}$, where $i : E \to X$ is the immersion of the exceptional divisor.

Proof. Writing $P^1 = P^1_S$, let $r : X \to P^1$ be as in Lemma 4.1. Then we have an exact sequence

\[ C_{X/P^1} \to r^* \Omega_{P^1/S} \to \Omega_{X/S} \to 0 \]

see Morphisms, Lemma 34.15. Since $\Omega_{P^1/S}|E = \Omega_{E/\kappa}$ by Morphisms, Lemma 34.10 it suffices to see that the first arrow defines a surjection onto the kernel of the canonical map $r^* \Omega_{P^1/S} \to i_* \Omega_{E/\kappa}$. This we can do locally. With notation as in the proof of Lemma 4.1 on an affine open of $X$ the morphism $f$ corresponds to the ring map

\[ A \to A[t]/(xt - y) \]

where $x, y \in m$ are generators. Thus $d(xt - y) = xdt$ and $ydt = t \cdot xdt$ which proves what we want. \qed

5. Quadratic transformations of spaces

Using the result above we can prove that blowups in points dominate any modification of a regular 2 dimensional algebraic space.

Let $X$ be a decent algebraic space over some base scheme $S$. Let $x \in |X|$ be a closed point. By Decent Spaces, Lemma 12.5 we can represent $x$ by a closed immersion $i : \text{Spec}(k) \to X$. Then the blowing up of $X$ at $x$ means the blowing up of $X$ in the closed subspace $Z = i(\text{Spec}(k)) \subset X$.

Lemma 5.1. Let $S$ be a scheme. Let $X$ be a Noetherian algebraic space over $S$. Let $T \subset |X|$ be a finite set of closed points $x$ such that (1) $X$ is regular at $x$ and (2) the local ring of $X$ at $x$ has dimension 2. Let $I \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals such that $\mathcal{O}_X/I$ is supported on $T$. Then there exists a sequence

\[ X_n \to X_{n-1} \to \ldots \to X_1 \to X_0 = X \]

where $X_{i+1} \to X_i$ is the blowing up of $X_i$ at a closed point $x_i$ lying above a point of $T$ such that $I\mathcal{O}_{X_n}$ is an invertible ideal sheaf.

Proof. Say $T = \{x_1, \ldots, x_r\}$. Pick an étale morphism $U \to X$ where $U$ is a scheme with points $u_i \in U$ lying over $x_i$. By Decent Spaces, Lemma 10.3 the points $u_i$ are closed points. After shrinking $U$ we may assume these are the only points of $U$ mapping to $T$. The local rings $\mathcal{O}_{U, u_i}$ are regular local of dimension 2, see Properties of Spaces, Definitions 23.2 and 20.2. Let $I_i \subset \mathcal{O}_{U, u_i}$ be the stalk of $I|U$ at $u_i$. Set

\[ n_i = \text{length}_{\mathcal{O}_{U, u_i}}(\mathcal{O}_{U, u_i}/I_i) \]

This is finite as $\mathcal{O}_X/I$ is supported on $T$ and hence $\mathcal{O}_{U, u_i}/I_i$ has support equal to $\{u_i\}$ (see Algebra, Lemma 61.3). We are going to use induction on $\sum n_i$. If $n_i = 0$ for all $i$, then $I = \mathcal{O}_X$ and we are done.
Suppose $n_i > 0$. Let $X' \to X$ be the blowing up of $X$ in $x_i$ (see discussion above the lemma). Since $U \to X$ is étale and $u_i$ is the unique point of $U$ lying over $x$ we see that $U' = U \times_X X'$ is the blowup of $U$ in $u_i$, see Divisors on Spaces, Lemma 6.3. Since Spec($O_{U,u_i}$) $\to U$ is flat we see that $U' \times_U$ Spec($O_{U,u_i}$) is the blowup of the ring $O_{U,u_i}$ in the maximal ideal. Hence both squares in the commutative diagram

\[
\begin{array}{ccc}
\text{Proj}(\bigoplus_{d \geq 0} m_{u_i}^d) & \longrightarrow & U' \longrightarrow X' \\
\downarrow & & \downarrow \\
\text{Spec}(O_{U,u_i}) & \longrightarrow & U \longrightarrow X
\end{array}
\]

are cartesian. Let $E \subset X'$, $E' \subset U'$, $E'' \subset \text{Proj}(\bigoplus_{d \geq 0} m_{u_i}^d)$ be the exceptional divisors. Let $d \geq 1$ be the integer found in Lemma 4.4 for the ideal $I_i \subset O_{U,u_i}$. Since the horizontal arrows in the diagram are flat, since $E'' \to E$ is surjective, and since $E''$ is the pullback of $E$, we see that

\[I \mathcal{O}_{X'} \subset O_{X'}(-dE)\]

(some details omitted). Set $T' = T \mathcal{O}_{X'}(dE) \subset O_{X'}$. Then we see that $O_{X'}/T'$ is supported in finitely many closed points $T' \subset |X'|$ because this holds over $X \setminus \{x_i\}$ and for the pullback to Proj($\bigoplus_{d \geq 0} m_{u_i}^d$). The final assertion of Lemma 4.4 tells us that the sum of the lengths of the stalks $O_{U',u_i}/T' \otimes O_{U',u_i}$ for $u_i$ lying over $u_i$ is $< n_i$. Hence the sum of the lengths has decreased.

By induction hypothesis, there exists a sequence

\[X_n' \to \ldots \to X_1' \to X'\]

of blowups at closed points lying over $T'$ such that $T' \mathcal{O}_{X_1'}$ is invertible. Since $T' \mathcal{O}_{X'}(-dE) = T \mathcal{O}_{X'}$, we see that $T \mathcal{O}_{X_1'} = T' \mathcal{O}_{X_1'}(-d(f')^{-1}E)$ where $f' : X_n' \to X'$ is the composition. Note that $(f')^{-1}E$ is an effective Cartier divisor by Divisors on Spaces, Lemma 6.8. Thus we are done by Divisors on Spaces, Lemma 2.7. □

**Lemma 5.2.** Let $S$ be a scheme. Let $X$ be a Noetherian algebraic space over $S$. Let $T \subset |X|$ be a finite set of closed points $x$ such that (1) $X$ is regular at $x$ and (2) the local ring of $X$ at $x$ has dimension 2. Let $f : Y \to X$ be a proper morphism of algebraic spaces which is an isomorphism over $U = X \setminus T$. Then there exists a sequence

\[X_n \to X_{n-1} \to \ldots \to X_1 \to X_0 = X\]

where $X_{i+1} \to X_i$ is the blowing up of $X_i$ at a closed point $x_i$ lying above a point of $T$ and a factorization $X_n \to Y \to X$ of the composition.

**Proof.** By More on Morphisms of Spaces, Lemma 28.4 there exists a $U$-admissible blowup $X' \to X$ which dominates $Y \to X$. Hence we may assume there exists an ideal sheaf $\mathcal{I} \subset O_X$ such that $O_X/\mathcal{I}$ is supported on $T$ and such that $Y$ is the blowing up of $X$ in $\mathcal{I}$. By Lemma 5.1 there exists a sequence

\[X_n \to X_{n-1} \to \ldots \to X_1 \to X_0 = X\]

where $X_{i+1} \to X_i$ is the blowing up of $X_i$ at a closed point $x_i$ lying above a point of $T$ such that $\mathcal{I} \mathcal{O}_{X_i}$ is an invertible ideal sheaf. By the universal property of blowing up (Divisors on Spaces, Lemma 6.5) we find the desired factorization. □
6. Vanishing

In this section we will often work in the following setting.

**Situation 6.1.** Here \((A, m, \kappa)\) be a local Noetherian normal domain of dimension 2. Let \(s\) be the closed point of \(S = \text{Spec}(A)\) and \(U = S \setminus \{s\}\). Let \(f : X \to \text{Spec}(A)\) be a modification (as in Spaces over Fields, Definition 6.1). We denote \(C_1, \ldots, C_r\) the irreducible components of the special fibre \(X_s\) of \(f\).

By Lemma 3.1 the morphism \(f\) defines an isomorphism \(f^{-1}(U) \to U\). The special fibre \(X_s\) is proper over \(\text{Spec}(\kappa)\), has dimension at most 1 (Lemma 3.4), and therefore is a scheme (Spaces over Fields, Lemma 7.5). By Stein factorization (more precisely, More on Morphisms of Spaces, Lemma 25.5) we have \(f_\ast \mathcal{O}_X = \mathcal{O}_S\) and the special fibre \(X_s\) is geometrically connected over \(\kappa\). If \(X_s\) has dimension 0, then \(f\) is finite (More on Morphisms of Spaces, Lemma 24.6) and hence an isomorphism (Morphisms, Lemma 48.19). We will discard this uninteresting case and we conclude that \(\dim(C_i) = 1\) for \(i = 1, \ldots, r\). The schematic locus of \(X\) contains every point of codimension 1 of \(X\) (Spaces over Fields, Lemma 7.5), in particular the generic point of \(C_i\).

The following lemma allows one to reduce to the case where \(X\) is a scheme in many of the following lemmas.

**Lemma 6.2.** In Situation 6.1 there exists a morphism \(g : X' \to X\) such that \(X'\) is an integral scheme, projective over \(A\), and \(X' \to \text{Spec}(A)\) is a modification (an isomorphism over \(U\)).

**Proof.** We can find a morphism \(X' \to X\) where \(X' \to S\) is a \(U\)-admissible blow up, see Restricted Power Series, Lemma 13.3. Then \(X'\) satisfies all the conditions of the lemma.

**Lemma 6.3.** In Situation 6.1 there exists a nonzero \(f \in m\) such that for every \(i = 1, \ldots, r\) there exist

1. a closed point \(x_i \in C_i\) in the schematic locus of \(X\) with \(x_i \notin C_j\) for \(j \neq i\),
2. a factorization \(f = g_i f_i\) of \(f\) in \(O_{X,x_i}\) such that \(g_i \in m_{x_i}\) maps to a nonzero element of \(O_{C_i,x_i}\).

**Proof.** We will use the observations made following Situation 6.1 without further mention. Pick a closed point \(x_i \in C_i\) contained in the schematic locus of \(X\) which is not in \(C_j\) for \(j \neq i\). Pick \(g_i \in m_{x_i}\) which maps to a nonzero element of \(O_{C_i,x_i}\). Since the fraction field of \(A\) is the fraction field of \(O_{X,x_i}\), we can write \(g_i = a_i/b_i\) for some \(a_i, b_i \in A\). Take \(f = \prod a_i\).

**Lemma 6.4.** In Situation 6.1 assume \(X\) is normal. If \(Z \subset X\) is a nonempty effective Cartier divisor such that \(|Z| \subset |X_s|\), then the conormal sheaf of \(Z\) is not trivial. More precisely, there exists an \(i\) such that \(C_i \subset Z\) and \(\deg(C_{Z/X}|C_i) > 0\).

**Proof.** We will use the observations made following Situation 6.1 without further mention. Let \(f\) be a function as in Lemma 3.1. Let \(\xi_i \in C_i\) be the generic point. Let \(O_i\) be the local ring of \(X\) at \(\xi_i\). Then \(O_i\) is a discrete valuation ring. Let \(e_i\) be the valuation of \(f\) in \(O_i\), so \(e_i > 0\). Let \(h_i \in O_i\) be a local equation for \(Z\) and let \(d_i\) be its valuation. Then \(d_i \geq 0\). Choose and fix \(i\) with \(d_i/e_i\) maximal (then \(d_i > 0\) as \(Z\) is not empty). Replace \(f\) by \(f^{d_i}\) and \(Z\) by \(e_i Z\). This is permissible, by the relation \(O_X(e_i Z) = O_X(Z)^{\otimes e_i}\), the relation between the conormal sheaf and
$\mathcal{O}_X(Z)$ (see Divisors on Spaces, Lemmas 2.15 and 2.14) and since the degree gets multiplied by $e_i$, see Varieties, Lemma 28.8. Let $I$ be the ideal sheaf of $Z$ so that $\mathcal{C}_{Z/X} = I|_Z$. Consider the image $\overline{f}$ of $f$ in $\Gamma(Z, \mathcal{O}_Z)$. By our choices above we see that $\overline{f}$ vanishes in the generic points of irreducible components of $Z$ (these are all generic points of $C_j$ as $Z$ is contained in the special fibre). On the other hand, $Z$ is $(S_1)$ by Divisors on Spaces, Lemma 2.22. Thus the scheme $Z$ has no embedded associated points and we conclude that $\overline{f} = 0$ (Divisors, Lemmas 4.3 and 5.6). Hence $f$ is a global section of $I$ which generates $I|_{C_i}$ by construction. Thus the image $s_i$ of $f$ in $\Gamma(C_i, I|_{C_i})$ is nonzero. However, our choice of $f$ guarantees that $s_i$ has a zero at $x_i$. Hence the degree of $I|_{C_i}$ is $> 0$ by Varieties, Lemma 28.11 \(\Box\)

**Lemma 6.5.** In Situation 6.1 assume $X$ is normal and A Nagata. The map

$$H^1(X, \mathcal{O}_X) \to H^1(f^{-1}(U), \mathcal{O}_X)$$

is injective.

**Proof.** Let $0 \to \mathcal{O}_X \to \mathcal{E} \to \mathcal{O}_X \to 0$ be the extension corresponding to a non-trivial element $\xi$ of $H^1(X, \mathcal{O}_X)$ (Cohomology on Sites, Lemma 6.1). Let $\pi : P = \mathbb{P}(\mathcal{E}) \to X$ be the projective bundle associated to $\mathcal{E}$. The surjection $\mathcal{E} \to \mathcal{O}_X$ defines a section $\sigma : X \to P$ whose conormal sheaf is isomorphic to $\mathcal{O}_X$ (Divisors on Spaces, Lemma 5.4). If the restriction of $\xi$ to $f^{-1}(U)$ is trivial, then we get a map $\mathcal{E}|_{f^{-1}(U)} \to \mathcal{O}_{f^{-1}(U)}$ splitting the injection $\mathcal{O}_X \to \mathcal{E}$. This defines a second section $\sigma' : f^{-1}(U) \to P$ disjoint from $\sigma$. Since $\xi$ is nontrivial we conclude that $\sigma'$ cannot extend to all of $X$ and be disjoint from $\sigma$. Let $X' \subset P$ be the scheme theoretic image of $\sigma'$ (Morphisms of Spaces, Definition 16.2). Picture

```
X' \quad \quad \quad \quad X
\downarrow \quad \quad \quad \quad \downarrow \pi
f^{-1}(U) \quad \quad \quad \quad P
\downarrow \quad \quad \quad \quad \downarrow \sigma
f^{-1}(U) \quad \quad \quad \quad X
```

The morphism $P \setminus \sigma(X) \to X$ is affine. If $X' \cap \sigma(X) = \emptyset$, then $X' \to X$ is both affine and proper, hence finite (Morphisms of Spaces, Lemma 41.9), hence an isomorphism (as $X$ is normal, see Decent Spaces, Lemma 18.4). This is impossible as mentioned above.

Let $X''$ be the normalization of $X'$ (as constructed in Morphisms of Spaces, Lemma 43.11 using a surjective étale morphism $U'' \to X'$ with $U'$ affine). Since $A$ is Nagata, we see that $X'' \to X'$ is finite (because the corresponding morphism $U'' \to U'$ is finite by Morphisms, Lemmas 48.21 and 19.2). Let $Z \subset X''$ be the pullback of the effective Cartier divisor $\sigma(X) \subset P$. By the above we see that $Z$ is not empty and is contained in the closed fibre of $X'' \to S$. Since $P \to X$ is smooth, we see that $\sigma(X)$ is an effective Cartier divisor (check étale locally on $X$ and use Divisors, Lemma 17.7). Hence $Z \subset X''$ is an effective Cartier divisor too. Since the conormal sheaf of $\sigma(X)$ in $P$ is $\mathcal{O}_X$, the conormal sheaf of $Z$ in $X''$ (which is a priori invertible) is $\mathcal{O}_Z$ by More on Morphisms of Spaces, Lemma 4.1. This is impossible by Lemma 6.4 and the proof is complete. \(\Box\)

**Lemma 6.6.** In Situation 6.1 assume $X$ is normal and A Nagata. Then

$$\text{Hom}_{D(A)}(\kappa[-1], Rf_*\mathcal{O}_X)$$

is zero. This uses $D(A) = D_{QCoh}(\mathcal{O}_S)$ to think of $Rf_*\mathcal{O}_X$ as an object of $D(A)$.
Proof. By adjointness of $Rf_*$ and $Lf^*$ such a map is the same thing as a map $\alpha : Lf^*\kappa[-1] \to \mathcal{O}_X$. Note that
\[
H^i(Lf^*\kappa[-1]) = \begin{cases} 
0 & \text{if } i > 1 \\
\mathcal{O}_{X_s} & \text{if } i = 1 \\
some \mathcal{O}_{X_s}\text{-module} & \text{if } i \leq 0
\end{cases}
\]
Since $\text{Hom}(H^0(Lf^*\kappa[-1]), \mathcal{O}_X) = 0$ as $\mathcal{O}_X$ is torsion free, the spectral sequence for $\text{Ext}$ (Cohomology on Sites, Example 24.1) implies that $\text{Hom}_{\mathcal{D}(\mathcal{O}_X)}(Lf^*\kappa[-1], \mathcal{O}_X)$ is equal to $\text{Ext}^1_{\mathcal{O}_X}(\mathcal{O}_{X_s}, \mathcal{O}_X)$. We conclude that $\alpha : Lf^*\kappa[-1] \to \mathcal{O}_X$ is given by an extension
\[
0 \to \mathcal{O}_X \to \mathcal{E} \to \mathcal{O}_{X_s} \to 0
\]
By Lemma 6.6, the pullback of this extension via the surjection $\mathcal{O}_X \to \mathcal{O}_{X_s}$ is zero (since this pullback is clearly split over $f^{-1}(U)$). Thus $1 \in \mathcal{O}_{X_s}$ lifts to a global section $s$ of $\mathcal{E}$. Multiplying $s$ by the ideal sheaf $\mathcal{I}$ of $X_s$ we obtain an $\mathcal{O}_{X_s}$-module map $c_s : \mathcal{I} \to \mathcal{O}_{X_s}$. Applying $f_*$ we obtain an $A$-linear map $f_*c_s : \mathfrak{m} \to A$. Since $A$ is a Noetherian normal local domain this map is given by multiplication by an element $a \in A$. Changing $s$ into $s - a$ we find that $s$ is annihilated by $\mathcal{I}$ and the extension is trivial as desired. 

The Grauert-Riemenschneider vanishing of the next proposition is a formal consequence of Lemma 6.6 and the general theory of duality. However, since we have sofar only developed this theory for morphisms of schemes, we restrict ourselves to this case.

Proposition 6.7 (Grauert-Riemenschneider). In Situation 6.1 assume
1. $X$ is a normal scheme,
2. $A$ is Nagata and has a dualizing complex $\omega_A^*$.

Let $\omega_X$ be the dualizing module of $X$ (Dualizing Complexes, Example 29.1). Then $R^1f_*\omega_X = 0$.

Proof. In this proof we will use the identification $D(A) = D_{\text{QCoh}}(\mathcal{O}_S)$ to identify quasi-coherent $\mathcal{O}_S$-modules with $A$-modules. Moreover, we may assume that $\omega_A^*$ is normalized, see Dualizing Complexes, Section 16. Since $X$ is a Noetherian normal $2$-dimensional scheme it is Cohen-Macaulay (Properties, Lemma 12.6). Thus $\omega_X^* = \omega_X[2]$ (Dualizing Complexes, Lemma 30.3 and the normalization in Dualizing Complexes, Example 29.1). If the proposition is false, then we can find a nonzero map $R^1f_*\omega_X \to \kappa$. In other words we obtain a nonzero map $\alpha : Rf_*\omega_X^* \to \kappa[1]$. Applying $R\text{Hom}_A(-, \omega_A^*)$ we get a nonzero map
\[
\beta : \kappa[-1] \to Rf_*\mathcal{O}_X
\]
which is impossible by Lemma 6.6. To see that $R\text{Hom}_A(-, \omega_A^*)$ does what we said, first note that
\[
R\text{Hom}_A(\kappa[1], \omega_A^*) = R\text{Hom}_A(\kappa, \omega_A^*)[-1] = \kappa[-1]
\]
as $\omega_A^*$ is normalized and we have
\[
R\text{Hom}_A(Rf_*\omega_X^*, \omega_A^*) = Rf_*R\text{Hom}_{\mathcal{O}_X}(\omega_X^*, \omega_X^*) = Rf_*\mathcal{O}_X
\]
The first equality by Dualizing Complexes, Lemma 20.10 and the fact that $\omega_X^* = f^*\omega_A^*$ by construction, and the second equality because $\omega_X^*$ is a dualizing complex for $X$ (which goes back to Dualizing Complexes, Lemma 25.5). 

□
7. Boundedness

In this section we begin the discussion which will lead to a reduction to the case of rational singularities for 2-dimensional schemes.

**Lemma 7.1.** Let \((A, m, \kappa)\) be a Noetherian normal local domain of dimension 2. Consider a commutative diagram

\[
\begin{array}{ccc}
S' & \xrightarrow{g} & S \\
\downarrow f' & & \downarrow f \\
\text{Spec}(A) & & 
\end{array}
\]

where \(f\) and \(f'\) are modifications as in Situation 6.1 and \(S\) normal. Then we have a short exact sequence

\[
0 \to H^1(X, \mathcal{O}_S) \to H^1(S', \mathcal{O}_{S'}) \to H^0(S, R^1 g_* \mathcal{O}_{S'}) \to 0
\]

Also \(\dim(\text{Supp}(R^1 g_* \mathcal{O}_{S'})) = 0\) and \(R^1 g_* \mathcal{O}_{S'}\) is generated by global sections.

**Proof.** We will use the observations made following Situation 6.1 without further mention. As \(S\) is normal and \(g\) is dominant and birational, we have \(g_* \mathcal{O}_{S'} = \mathcal{O}_S\), see for example More on Morphisms of Spaces, Lemma 25.5. Since the fibres of \(g\) have dimension \(\leq 1\), we have \(R^p g_* \mathcal{O}_{S'} = 0\) for \(p > 1\), see for example Cohomology of Spaces, Lemma 20.9. The support of \(R^1 g_* \mathcal{O}_{S'}\) is contained in the set of points of \(|X|\) where the fibres of \(g'\) have dimension \(\geq 1\). Thus it is contained in the set of images of those irreducible components \(C' \subset S'\) which map to points of \(S\) which is a finite set of closed points (recall that \(S' \to S\) is a morphism of proper 1-dimensional schemes over \(\kappa\)). Then \(R^1 g_* \mathcal{O}_{S'}\) is globally generated by Cohomology of Schemes, Lemma 9.10. Using the morphism \(f : S \to S\) and the references above we find that \(H^p(S, F) = 0\) for \(p > 1\) for any coherent \(S\)-module \(F\). Hence the short exact sequence of the lemma is a consequence of the Leray spectral sequence for \(g\) and \(\mathcal{O}_{S'}\), see Cohomology on Sites, Lemma 14.5. \(\square\)

**Lemma 7.2.** Let \(A\) be a Noetherian local normal domain of dimension 2. For \(f \in m\) nonzero denote \(\text{div}(f) = \sum n_i(p_i)\) the divisor associated to \(f\) on the punctured spectrum of \(A\). We set \(|f| = \sum n_i\). There exist integers \(N\) and \(M\) such that \(|f + g| \leq M\) for all \(g \in m^N\).

**Proof.** Pick \(h \in m\) such that \(f, h\) is a regular sequence in \(A\) (this follows from Algebra, Lemmas 147.4 and 70.7). We will prove the lemma with \(M = \text{length}_A(A/(f, h))\) and with \(N\) any integer such that \(m^N \subset (f, h)\). Such an integer \(N\) exists because \(|f, h| = m\). Note that \(M = \text{length}_A(A/(f + g, h))\) for all \(g \in m^N\) because \((f, h) = (f + g, h)\). This moreover implies that \(f + g, h\) is a regular sequence in \(A\) too, see Algebra, Lemma 101.2. Now suppose that \(\text{div}(f + g) = \sum m_j(q_j)\). Then consider the map

\[
c : A/(f + g) \to \prod A/q_j^{(m_j)}
\]

where \(q_j^{(m_j)}\) is the symbolic power, see Algebra, Section 63. Since \(A\) is normal, we see that \(A_{q_j}\) is a discrete valuation ring and hence

\[
A_{q_i}/(f + g) = A_{q_i}/q_j^{m_i} A_{q_i} = (A/q_j^{(m_i)})_{q_i}
\]

Since \(V(f + g, h) = \{m\}\) this implies that \(c\) becomes an isomorphism on inverting \(h\) (small detail omitted). Since \(h\) is a nonzerodivisor on \(A/(f + g)\) we see that
the length of $A/(f + g, h)$ equals the Herbrand quotient $e_A(A/(f + g), 0, h)$ as defined in Chow Homology, Section 3. Similarly the length of $A/(h, q_j^{(m_j)})$ equals $e_A(A/q_j^{(m_j)}, 0, h)$. Then we have

$$M = \text{length}_A(A/(f + g, h)) = e_A(A/(f + g), 0, h) = \sum q_j e_A(A/q_j^{(m_j)}, 0, h) = \sum \sum m=0,\ldots,m_j-1 e_A(q_j^{(m)}/q_j^{(m+1)}, 0, h)$$

The equalities follow from Chow Homology, Lemma 3.3 using in particular that the cokernel of $c$ has finite length as discussed above. It is straightforward to prove that $e_A(q^{(m)}/q^{(m+1)}, 0, h)$ is at least 1 by Nakayama’s lemma. This finishes the proof of the lemma.

**Lemma 7.3.** Let $A$ be a Noetherian local normal domain of dimension 2. Let $p_1, \ldots, p_r$ be pairwise distinct primes of height 1. There exists an element $f \in p_1 \cap \ldots \cap p_r$ such that $A/\mathfrak{p}A$ is reduced.

**Proof.** As a first approximation pick any nonzero $f \in p_1 \cap \ldots \cap p_r$. Pick integers $N$ and $M$ as in Lemma 7.2 adapted to $f$. Write

$$\text{div}(f) = \sum_{i=1,\ldots,s} (q_i) + \sum_{j=1,\ldots,t} m_j (r_j)$$

with $m_j > 1$ and with no equalities among the primes $q_i$ and $r_j$ (in other words the set $\{q_i, r_j\}$ has $r + s$ elements). We have $r + \sum m_j \leq M$ is bounded among all $f$ in $f + m^N$ hence we may assume $f \in p_1 \cap \ldots \cap p_r$ is chosen with $s$ maximal. We claim that $t = 0$. If not, then we choose

$$g \in m^N \cap q_1^2 \cap \ldots \cap q_s^2 \cap r_1 \cap \ldots \cap r_t \text{ and } g \notin \{r_1 \cup \ldots \cup r_t\}$$

First choose $g_0 \in m^N$, $g_i \in q_i$ and $g'_i \in r_i$ and each not contained in any other of the primes (using prime avoidance Algebra, Lemma 14.2) and then take $g = g_0 g_1^2 \ldots g_s^2 g'_1 \ldots g'_t$. Observe that $g \in p_1 \cap \ldots \cap p_r$ as $\{p_i\} \subset \{q_i, r_j\}$. Now we note that

$$\text{div}(f + g) = \sum_{i=1,\ldots,s} (q_i) + \sum_{j=1,\ldots,t} (r_j) + \sum e_k(s_k)$$

for some height one primes $s_k \notin \{p_i, q_j, r_t\}$. This is a contradiction with maximality of $s$ unless $t = 0$ which is what we wanted to show.

**Lemma 7.4.** Let $(A, \mathfrak{m}, \kappa)$ be a Noetherian normal local domain of dimension 2. If $a \in \mathfrak{m}^2$ is nonzero, then there exists an element $c \in A$ such that $A/cA$ is reduced and such that $c$ divides $c^n$ for some $n$.

**Proof.** Let $\text{div}(a) = \sum_{i=1,\ldots,r} n_i(p_i)$. Choose $c \in p_1 \cap \ldots \cap p_r$ with $A/cA$ reduced, see Lemma 7.3. For $n \geq \max(n_i)$ we see that $-\text{div}(a) + \text{div}(c^n)$ is an effective divisor (all coefficients nonnegative). Thus $c^n/a \in A$ by Algebra, Lemma 147.6.

**Lemma 7.5.** Let $(A, \mathfrak{m}, \kappa)$ be a Noetherian local normal Nagata domain of dimension 2. Let $a \in A$ be nonzero. There exists an integer $N$ such that for every modification $f : X \rightarrow \text{Spec}(A)$ with $X$ normal the $A$-module

$$M_{X,a} = \text{Coker}(A \rightarrow H^0(Z, \mathcal{O}_Z))$$

where $Z \subset X$ is cut out by $a$ has length bounded by $N$. 

```
Proof. By the short exact sequence $0 \to \mathcal{O}_X \xrightarrow{a} \mathcal{O}_X \to \mathcal{O}_Z \to 0$ we see that
\[(7.5.1)\] 
$M_{X,a} = H^1(X, \mathcal{O}_X)[a]$ \hspace{2cm}

Here $N[a] = \{ n \in N \mid an = 0 \}$ for an $A$-module $N$. Thus if $a$ divides $b$, then $M_{X,a} \subset M_{X,b}$. Suppose that for some $c \in A$ the modules $M_{X,c}$ have bounded length. Then for every $X$ we have an exact sequence
\[0 \to M_{X,c} \to M_{X,c^2} \to M_{X,c}\]
where the second arrow is given by multiplication by $c$. Hence we see that $M_{X,c^2}$ has bounded length as well. Thus it suffices to find a $c \in A$ for which the lemma is true such that $a$ divides $c^n$ for some $n > 0$. By Lemma 7.4 we may assume $A/(a)$ is a reduced ring.

Assume that $A/(a)$ is reduced. Let $A/(a) \subset B$ be the normalization of $A/(a)$ in its quotient ring. Because $A$ is Nagata, we see that $\text{Coker}(A \to B)$ is finite. We claim the length of this finite module is a bound. To see this, consider $f : X \to \text{Spec}(A)$ as in the lemma and let $Z' \subset Z$ be the scheme theoretic closure of $Z \cap f^{-1}(U)$. Then $Z' \to \text{Spec}(A/(a))$ is finite because $\dim(A/(a)) = 1$, so $Z'' \to \text{Spec}(A/(a))$ has finite fibres by Lemma 3.4, so it is finite by More on Morphisms of Spaces, Lemma 24.6. Hence $Z' = \text{Spec}(B')$ with $A/(a) \subset B' \subset B$. On the other hand, we claim the map
\[H^0(Z, \mathcal{O}_Z) \to H^0(Z', \mathcal{O}_{Z'})\]
is injective. Namely, if $s \in H^0(Z, \mathcal{O}_Z)$ is in the kernel, then the restriction of $s$ to $f^{-1}(U) \cap Z$ is zero. Hence the image of $s$ in $H^1(X, \mathcal{O}_X)$ vanishes in $H^1(f^{-1}(U), \mathcal{O}_X)$. By Lemma 6.5 we see that $s$ comes from an element $\tilde{s}$ of $A$. But by assumption $\tilde{s}$ maps to zero in $B'$ which implies that $s = 0$. Putting everything together we see that $M_{X,a}$ is a subquotient of $B'/A$, namely not every element of $B'$ extends to a global section of $\mathcal{O}_Z$, but in any case the length of $M_{X,a}$ is bounded by the length of $B/A$. \hfill $\square$

In some cases, resolution of singularities reduces to the case of rational singularities.

**Definition 7.6.** Let $(A, m, \kappa)$ be a Noetherian local normal Nagata domain of dimension 2.

1. We say $A$ defines a rational singularity if for every normal modification $X \to \text{Spec}(A)$ we have $H^1(X, \mathcal{O}_X) = 0$.
2. We say that reduction to rational singularities is possible for $A$ if the length of the $A$-modules
\[H^1(X, \mathcal{O}_X)\]
is bounded for all modifications $X \to \text{Spec}(A)$ with $X$ normal.

The reason for this terminology is the following lemma.

**Lemma 7.7.** Let $(A, m, \kappa)$ be a Noetherian local normal Nagata domain of dimension 2. If reduction to rational singularities is possible for $A$, then there exists a modification $X \to \text{Spec}(A)$ where $X$ is a normal scheme projective over $A$, such that for any closed point $x \in X$ the local ring $\mathcal{O}_{X,x}$ defines a rational singularity.

**Proof.** We choose an $X \to \text{Spec}(A)$ which maximizes the length of $H^1(X, \mathcal{O}_X)$. By Lemma 7.4 for any further modification $g : X' \to X$ with $X'$ normal we have $R^1g_*\mathcal{O}_{X'} = 0$ and $H^1(X, \mathcal{O}_X) = H^1(X', \mathcal{O}_{X'})$. 
We first massage $X$ to turn it into a scheme projective over $A$. Let $X' \to X$ be as in Lemma 6.2 and let $X''$ be the normalization of $X'$. As $A$ is Nagata, the morphism $X'' \to X'$ is finite (Morphisms, Lemma 48.21). Then $X'' \to S$ is projective (More on Morphisms, Lemma 36.2). Thus we may replace $X$ by $X''$ and assume $X$ is a normal scheme projective over $A$ with $H^1(X, \mathcal{O}_X)$ maximal.

Let $X \to \text{Spec}(A)$ be a modification with $X$ projective over $A$ and $H^1(X, \mathcal{O}_X)$ maximal (among all normal modifications). Let $x \in X$ be a closed point. Let $Y \to \text{Spec}(\mathcal{O}_{X,x})$ be a modification with $Y$ normal. We want to show that $H^1(Y, \mathcal{O}_Y) = 0$. Arguing as in the second paragraph we may assume $Y$ is a scheme. By Limits, Lemma 15.1 we can find a morphism of schemes $g : X' \to X$ which is an isomorphism over $X \setminus \{x\}$ such that $X' \times_X \text{Spec}(\mathcal{O}_{X,x})$ is isomorphic to $Y$. Then $g$ is a modification. By maximality we have $R^1g_*\mathcal{O}_{X'} = 0$ (see first paragraph). Clearly this means that $H^1(Y, \mathcal{O}_Y) = 0$ as desired.

\begin{lemma}
Let $A \to B$ be a finite injective local ring map of Noetherian local normal Nagata domains of dimension 2. Assume that the induced extension of fraction fields is separable. If reduction to rational singularities is possible for $A$ then it is possible for $B$.
\end{lemma}

\begin{proof}
Let $n$ be the degree of the fraction field extension $K \subset L$. Let $\text{Tr} : L \to K$ be the trace. Since the extension is finite separable the trace pairing $(h,g) \mapsto \text{Tr}(fg)$ is a nondegenerate quadratic form on $L$ over $K$. Pick $b_1, \ldots, b_n \in B$ which form a basis of $L$ over $K$. By the above $d = \det(\text{Tr}(b_ib_j)) \in A$ is nonzero.

Let $Y \to \text{Spec}(B)$ be a modification with $Y$ normal. We can find a $U$-admissible blow up $X'$ of $\text{Spec}(A)$ such that the strict transform $Y'$ of $Y$ is finite over $X'$, see More on Morphisms of Spaces, Lemma 28.2. Picture

\[
\begin{array}{ccc}
Y' & \longrightarrow & Y \longrightarrow \text{Spec}(B) \\
\downarrow & & \downarrow \\
X' & \longrightarrow & \text{Spec}(A)
\end{array}
\]

After replacing $X'$ and $Y'$ by their normalizations we may assume that $X'$ and $Y'$ are normal modifications of $\text{Spec}(A)$ and $\text{Spec}(B)$. In this way we reduce to the case where there exists a commutative diagram

\[
\begin{array}{ccc}
Y & \longrightarrow & \text{Spec}(B) \\
\downarrow & & \downarrow \\
X & \longrightarrow & \text{Spec}(A)
\end{array}
\]

with $X$ and $Y$ normal modifications of $\text{Spec}(A)$ and $\text{Spec}(B)$ and $\pi$ finite.

The trace map on $L$ over $K$ extends to a map of $\mathcal{O}_X$-modules $\text{Tr} : \pi_*\mathcal{O}_Y \to \mathcal{O}_X$. Consider the map

$$
\Phi : \pi_*\mathcal{O}_Y \longrightarrow \mathcal{O}_X^{\oplus n}, \quad s \mapsto (\text{Tr}(b_1s), \ldots, \text{Tr}(b_ns))
$$

This map is injective (because it is injective in the generic point) and there is a map

$$
\mathcal{O}_X^{\oplus n} \longrightarrow \pi_*\mathcal{O}_Y, \quad (s_1, \ldots, s_n) \mapsto \sum b_is_i
$$

whose composition with $\Phi$ has matrix $\text{Tr}(b_i b_j)$. Hence the cokernel of $\Phi$ is annihilated by $d$. Thus we see that we have an exact sequence

$$H^0(X, \text{Coker}(\Phi)) \to H^1(Y, \mathcal{O}_Y) \to H^1(X, \mathcal{O}_X)^{\otimes n}$$

Since the right hand side is bounded by assumption, it suffices to show that the $d$-torsion in $H^1(Y, \mathcal{O}_Y)$ is bounded. This is the content of Lemma 7.5 and (7.5.1).

**Lemma 7.9.** Let $A$ be a Nagata regular local ring of dimension 2. Then $A$ defines a rational singularity.

**Proof.** (The assumption that $A$ be Nagata is not necessary for this proof, but we’ve only defined the notion of rational singularity in the case of Nagata 2-dimensional normal local domains.) Let $X \to \text{Spec}(A)$ be a modification. By Lemma 5.2 we can dominate $X$ by a scheme $X_n$ which is the last in a sequence

$$X_n \to X_{n-1} \to \ldots \to X_1 \to X_0 = \text{Spec}(A)$$

of blowing ups in closed points. By Lemma 4.2 the schemes $X_i$ are regular, in particular normal (Algebra, Lemma 147.5). By Lemma 7.1 we have $H^1(X, \mathcal{O}_X) \subset H^1(X_n, \mathcal{O}_{X_n})$. Thus it suffices to prove $H^1(X_n, \mathcal{O}_{X_n}) = 0$. Using Lemma 7.1 again, we see that it suffices to prove $H^1(X_i, \mathcal{O}_{X_i}) = 0$ for $i = 1, \ldots, n$.

This follows from Lemma 4.3.

**Remark 7.10.** Let $X$ be an integral Noetherian normal scheme of dimension 2. In this case the following are equivalent

1. $X$ has a dualizing complex $\omega_X^\bullet$,
2. there is a coherent $\mathcal{O}_X$-module $\omega_X$ such that $\omega_X[n]$ is a dualizing complex, where $n$ can be any integer.

This follows from the fact that $X$ is Cohen-Macaulay (Properties, Lemma 12.6) and Dualizing Complexes, Lemma 30.3. In this situation we will say that $\omega_X$ is a dualizing module in accordance with Dualizing Complexes, Section 29. In particular, when $A$ is a Noetherian normal local domain of dimension 2, then we say $A$ has a dualizing module $\omega_A$ if the above is true. In this case, if $X \to \text{Spec}(A)$ is a normal modification and $X$ is a scheme, then $X$ has a dualizing module too, see Dualizing Complexes, Example 29.1. In this situation we always denote $\omega_X$ the dualizing module normalized with respect to $\omega_A$, i.e., such that $\omega_X[2]$ is the dualizing complex normalized relative to $\omega_A[2]$.

**Lemma 7.11.** Let $A$ be a Nagata Noetherian local normal domain of dimension 2 which has a dualizing complex $\omega_A^\bullet$. If there exists a nonzero $d \in A$ such that for all normal modifications $X \to \text{Spec}(A)$ with $X$ a scheme the cokernel of the trace map

$$\Gamma(X, \omega_X) \to \omega_A$$

is annihilated by $d$, then reduction to rational singularities is possible for $A$.

**Proof.** Any modification of $\text{Spec}(A)$ can be dominated by a modification which is a scheme, see Lemma 6.2 which we can normalize to obtain a normal modification (using that $A$ is Nagata, see Morphisms, Lemma 48.21). Thus by Lemma 7.1 it suffices to bound the length of $H^1(X, \mathcal{O}_X)$ for normal modifications which are schemes.

Let $X \to \text{Spec}(A)$ be as in the statement. Let $\omega_X$ be the dualizing module of $X$ as in the statement of Grauert-Riemenschneider (Proposition 6.7). The trace map is
the map $R_f \omega_X \to \omega_A$ described in Dualizing Complexes, Section 2. By Grauert-Riemenschneider we have $R_f \omega_X = f_* \omega_X$ thus the trace map indeed produces a map $\Gamma(X, \omega_X) \to \omega_A$. By duality we have $R_f \omega_X = R \text{Hom}_A(R_f \mathcal{O}_X, \omega_A)$ (this uses that $\omega_X[2]$ is the dualizing complex on $X$ normalized relative to $\omega_A[2]$, see Dualizing Complexes, Lemma 27.8 or more directly Section 20 or even more directly Lemma 20.10). The distinguished triangle

$$A \to Rf_* \mathcal{O}_X \to R^1f_* \mathcal{O}_X[-1] \to A[1]$$

is transformed by $R \text{Hom}_A(-, \omega_A)$ into the short exact sequence

$$0 \to f_* \omega_X \to \omega_A \to \text{Ext}^2_A(R^1f_* \mathcal{O}_X, \omega_A) \to 0$$

(and $\text{Ext}^i_A(R^1f_* \mathcal{O}_X, \omega_A) = 0$ for $i \neq 2$; this will follow from the discussion below as well). Since $R^1f_* \mathcal{O}_X$ is supported in $\{m\}$, the local duality theorem tells us that

$$\text{Ext}^2_A(R^1f_* \mathcal{O}_X, \omega_A) = \text{Ext}^0_A(R^1f_* \mathcal{O}_X, \omega_A[2]) = \text{Hom}_A(R^1f_* \mathcal{O}_X, E)$$

is the Matlis dual of $R^1f_* \mathcal{O}_X$ (and the other ext groups are zero), see Dualizing Complexes, Lemma 18.4. By the equivalence of categories inherent in Matlis duality (Dualizing Complexes, Proposition 7.8), if $R^1f_* \mathcal{O}_X$ is not annihilated by $d$, then neither is the $\text{Ext}^2$ above. Hence we see that $H^1(X, \mathcal{O}_X)$ is annihilated by $d$. Thus the required boundedness follows from Lemma 7.5 and (7.5.1).

**Lemma 7.12.** Let $p$ be a prime number. Let $A$ be a regular local ring of dimension 2 and characteristic $p$. Let $A_0 \subset A$ be a subring such that $\Omega_{A/A_0}$ is free of rank $r < \infty$. Set $\omega_A = \Omega_{A/A_0}$. If $X \to \text{Spec}(A)$ is the result of a sequence of blowings up in closed points, then there exists a map

$$\varphi_X : (\Omega_X^{r}/\text{Spec}(A_0))^{**} \to \omega_X$$

extending the given identification in the generic point.

**Proof.** Suppose we have constructed the map $\varphi_X : (\Omega_X^{r}/\text{Spec}(A_0))^{**} \to \omega_X$ and suppose that $b : X' \to X$ is a blow up in a closed point. Set $\Omega_{X'} = (\Omega_{X/A_0})^{**}$ and $\Omega_{X'} = (\Omega_{X'/A_0})^{**}$. By the universal property of the dualizing module, a map $\Omega_{X'} \to \omega_{X'}$ is the same thing as a map $b_* \Omega_{X'} \to \omega_X$, see Dualizing Complexes, Lemma 29.7. Thus in turn it suffices to produce a map

$$b_* \Omega_{X'} \to \Omega_X$$

The sheaves $\Omega_{X'}$ and $\Omega_X$ are invertible, see Divisors, Lemma 10.10. Consider the exact sequence

$$b^* \Omega_{X/A_0} \to \Omega_{X'/A_0} \to \Omega_{X'/X} \to 0$$

A local calculation shows that $\Omega_{X'/X}$ is isomorphic to an invertible module on the exceptional divisor $E$, see Lemma 4.5. It follows that either

$$\Omega_{X'} \cong (b^* \Omega_X)(E) \quad \text{or} \quad \Omega_{X'} \cong b^* \Omega_X$$

see Divisors, Lemma 12.9. The second possibility never happens in characteristic zero, but can happen in characteristic $p$. In both cases we see that $b_* \Omega_{X'} = \Omega_X$ by Lemma 4.3.

**Lemma 7.13.** Let $p$ be a prime number. Let $A$ be a complete regular local ring of dimension 2 and characteristic $p$. Let $K = f.f.(A) \subset L$ be a degree $p$ inseparable extension and let $B \subset L$ be the integral closure of $A$. Then reduction to rational singularities is possible for $B$. 

Proof. We have $A = k[[x, y]]$. Write $L = K[x]/(x^p - f)$ for some $f \in A$ and denote $g \in B$ the congruence class of $x$, i.e., the element such that $g^p = f$. By More on Algebra, Lemma 36.5 there exists a subfield $k' \subseteq k$ with $p^e = [k : k'] < \infty$ such that $f$ is not contained in the fraction field $K_0$ of $A_0 = k'[\langle x, y^p \rangle] \subseteq A$. Then

$$
\Omega_{A/A_0} = A \otimes_k \Omega_{k'/k} \oplus Adx \oplus Ady
$$

is finite free of rank $e + 2$. Set $\omega_A = \Omega^{e+2}_{A/A_0}$. Consider the canonical map

$$
\text{Tr} : \Omega^{e+2}_{B/A_0} \to \Omega^{e+2}_{A/A_0} = \omega_A
$$

of Lemma 2.3. By duality this determines a map

$$
c : \Omega^{e+2}_{B/A_0} \to \omega_B = \text{Hom}_A(B, \omega_A)
$$

Claim: the cokernel of $c$ is annihilated by a nonzero element of $B$.

Since $df$ is nonzero in $\Omega_{A/A_0}$ (Algebra, Lemma 148.2) we can find $\eta_1, \ldots, \eta_{e+1} \in \Omega_{A/A_0}$ such that $\theta = \eta_1 \wedge \ldots \wedge \eta_{e+1} \wedge df$ is nonzero in $\omega_A = \Omega^{e+2}_{A/A_0}$. To prove the claim we will construct elements $\omega_i$ of $\Omega^{e+2}_{B/A_0}$, $i = 0, \ldots, p-1$ which are mapped to $\varphi_i \in \omega_B = \text{Hom}_A(B, \omega_A)$ with $\varphi_i(g^j) = \delta_{ij}\theta$ for $j = 0, \ldots, p-1$. Since $\{1, g, \ldots, g^{p-1}\}$ is a basis for $L/K$ this proves the claim. We set $\eta = \eta_1 \wedge \ldots \wedge \eta_{e+1}$ so that $\theta = \eta \wedge df$. Set $\omega_i = \eta \wedge g^{p-1-i}dg$. Then by construction we have

$$
\varphi_i = \text{Tr}(g^i \eta \wedge g^{p-1-i}dg) = \text{Tr}(g^i \eta \wedge g^{p-1-i}dg) = \delta_{ij}\theta
$$

by the explicit description of the trace map in Lemma 2.2.

Let $Y \to \text{Spec}(B)$ be a normal modification. Exactly as in the proof of Lemma 7.8 we can reduce to the case where $Y$ is finite over a modification $X$ of $\text{Spec}(A)$. Arguing as in the proof of Lemma 7.9 we may even assume that $X = X_n$ where

$$
X_n \to X_{n-1} \to \ldots \to X_1 \to X_0 = X
$$

is a sequence of blowing ups in closed points. By Lemma 2.3 we obtain the first arrow in

$$
\pi_* : (\Omega^{e+2}_{Y/A_0})^* \to (\Omega^{e+2}_{X/A_0})^* \to \omega_X
$$

and the second arrow is from Lemma 7.12. By duality this corresponds to a map

$$
c_Y : \Omega^{e+2}_{Y/A_0} \to \omega_Y
$$

extending the map $c$ above. Hence we see that the image of $\Gamma(Y, \omega_Y) \to \omega_B$ contains the image of $c$. By our claim we see that the cokernel of $c$ is annihilated by a fixed nonzero element of $B$. We conclude by Lemma 7.11. $\square$

8. Rational singularities

In this section we reduce from rational singular points to Gorenstein rational singular points. See [Lip69] and [Mat70].

**Situation 8.1.** Here $(A, \mathfrak{m}, k)$ be a local Noetherian normal domain of dimension 2 which defines a rational singularity. Let $s$ be the closed point of $S = \text{Spec}(A)$ and $U = S \setminus \{s\}$. Let $f : X \to \text{Spec}(A)$ be a normal proper birational morphism of schemes. We denote $C_1, \ldots, C_r$ the irreducible components of the special fibre $X_s$ of $f$.

**Lemma 8.2.** In Situation 8.1. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Then

- (1) $H^p(X, \mathcal{F}) = 0$ for $p \notin \{0, 1\}$, and
Proof. Part (1) follows from Cohomology of Schemes, Lemma 18.9. If $\mathcal{F}$ is globally generated, then there is a surjection $\bigoplus_{i \in I} \mathcal{O}_X \to \mathcal{F}$. By part (1) and the long exact sequence of cohomology this induces a surjection on $H^1$. Since $H^1(X, \mathcal{O}_X) = 0$ as $S$ has a rational singularity, and since $H^1(X, -)$ commutes with direct sums (Cohomology, Lemma 20.1) we conclude.

Lemma 8.3. In Situation 8.1 assume $E = X_*$ is an effective Cartier divisor. Let $\mathcal{I}$ be the ideal sheaf of $E$. Then $H^0(X, \mathcal{I}^n) = m^n$ and $H^1(X, \mathcal{I}^n) = 0$.

Proof. We have $H^0(X, \mathcal{O}_X) = A$, see discussion following Situation 6.1. Then $m \subset H^0(X, \mathcal{I}) \subset H^0(X, \mathcal{O}_X)$. The second inclusion is not an equality as $X_* \neq \emptyset$. Thus $H^0(X, \mathcal{I}) = m$. As $\mathcal{I}^n = m^n \mathcal{O}_X$ our Lemma 8.2 shows that $H^1(X, \mathcal{I}^n) = 0$.

Choose generators $x_1, \ldots, x_{\mu+1}$ of $m$. These define global sections of $\mathcal{I}$ which generate it. Hence a short exact sequence

$$0 \to \mathcal{F} \to \mathcal{O}_{X}^{\oplus \mu+1} \to \mathcal{I} \to 0$$

Then $\mathcal{F}$ is a finite locally free $\mathcal{O}_{X}$-module of rank $\mu$ and $\mathcal{F} \otimes \mathcal{I}$ is globally generated by Constructions, Lemma 13.8. Hence $\mathcal{F} \otimes \mathcal{I}^n$ is globally generated for all $n \geq 1$. Thus for $n \geq 2$ we can consider the exact sequence

$$0 \to \mathcal{F} \otimes \mathcal{I}^{n-1} \to (\mathcal{I}^{n-1})^{\oplus \mu+1} \to \mathcal{I}^n \to 0$$

Applying the long exact sequence of cohomology using that $H^1(X, \mathcal{F}) = 0$ by Lemma 8.2 we obtain that every element of $H^0(X, \mathcal{I}^n)$ is of the form $\sum x_i a_i$ for some $a_i \in H^0(X, \mathcal{I}^{n-1})$. This shows that $H^0(X, \mathcal{I}^n) = m^n$ by induction.

Lemma 8.4. In Situation 8.1 assume $A$ is Nagata. Then the blow up of $\text{Spec}(A)$ in $m$ is normal.

Proof. Let $X' \to \text{Spec}(A)$ be the blow up, in other words

$$X' = \text{Proj}(A \oplus m \oplus m^2 \oplus \ldots)$$

is the Proj of the Rees algebra. This in particular shows that $X'$ is integral and that $X' \to \text{Spec}(A)$ is a projective modification. Let $X$ be the normalization of $X'$. Since $A$ is Nagata, we see that $\nu : X \to X'$ is finite (Morphisms, Lemma 88.21). Let $E' \subset X'$ be the exceptional divisor and let $E \subset X$ be the inverse image. Let $\mathcal{I}' \subset \mathcal{O}_{X'}$ and $\mathcal{I} \subset \mathcal{O}_X$ be their ideal sheaves. Recall that $\mathcal{I}' = \mathcal{O}_{X'}(1)$ (Divisors, Lemma 21.11). Observe that $\mathcal{I} = \nu^* \mathcal{I}'$ and that $E$ is an effective Cartier divisor (Divisors, Lemma 11.12). We are trying to show that $\nu$ is an isomorphism. As $\nu$ is finite, it suffices to show that $\mathcal{O}_{X'} \to \nu_* \mathcal{O}_X$ is an isomorphism. If not, then we can find an $n \geq 0$ such that

$$H^0(X', (\mathcal{I}')^n) \neq H^0(X', (\nu_* \mathcal{O}_X) \otimes (\mathcal{I}')^n)$$

for example because we can recover quasi-coherent $\mathcal{O}_{X'}$-modules from their associated graded modules, see Properties, Lemma 26.3. By the projection formula we have

$$H^0(X', (\nu_* \mathcal{O}_X) \otimes (\mathcal{I}')^n) = H^0(X, \nu^* (\mathcal{I})^n) = H^0(X, \mathcal{I}^n) = m^n$$

the last equality by Lemma 8.3. On the other hand, there is clearly an injection $m^n \to H^0(X', (\mathcal{I}')^n)$. Since $H^0(X', (\mathcal{I}')^n)$ is torsion free we conclude equality holds for all $n$, hence $X = X'$. 

Lemma 8.5. In Situation \ref{situation}. Let $M$ be a finite reflexive $A$-module. Let $M \otimes_A O_X$ denote the pullback of the associated $O_S$-module. Then $M \otimes_A O_X$ maps onto its double dual.

Proof. Let $\mathcal{F} = (M \otimes_A O_X)^{**}$ be the double dual and let $\mathcal{F}' \subset \mathcal{F}$ be the image of the evaluation map $M \otimes_A O_X \to \mathcal{F}$. Then we have a short exact sequence

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{Q} \to 0$$

Since $X$ is normal, the local rings $O_{X,x}$ are discrete valuation rings for points of codimension 1 (see Properties, Lemma \ref{properties}). Hence $Q_x = 0$ for such points by More on Algebra, Lemma \ref{more-on-algebra}. Thus $\mathcal{Q}$ is supported in finitely many closed points and is globally generated by Cohomology of Schemes, Lemma \ref{cohomology-of-schemes}. We obtain the exact sequence

$$0 \to H^0(X, \mathcal{F}') \to H^0(X, \mathcal{F}) \to H^0(X, \mathcal{Q}) \to 0$$

because $\mathcal{F}'$ is generated by global sections (Lemma \ref{global-sections}). Since $X \to \text{Spec}(A)$ is an isomorphism over the complement of the closed point, and since $M$ is reflexive, we see that the maps

$$M \to H^0(X, \mathcal{F}') \to H^0(X, \mathcal{F})$$

induce isomorphisms after localization at any nonmaximal prime of $A$. Hence these maps are isomorphisms by More on Algebra, Lemma \ref{more-on-algebra} and the fact that reflexive modules over normal rings have property $(S_2)$ (More on Algebra, Lemma \ref{more-on-algebra}). Thus we conclude that $\mathcal{Q} = 0$ as desired. \qed

9. Examples

Some examples related to the results earlier in this chapter.

Example 9.1. Let $k$ be a field. The ring $A = k[x, y, z]/(x^r + y^s + z^t)$ is a UFD for $r, s, t$ pairwise coprime integers. Namely, since $x^r + y^s + z^t$ is irreducible $A$ is a domain. The element $z$ is a prime element, i.e., generates a prime ideal in $A$. On the other hand, if $r = 1 + ers$ for some $e$, then $A[1/z] \cong k[x', y', 1/z]$ where $x' = x/z^e$, $y' = y/z^e$ and $z = (x')^r + (y')^s$. Thus $A[1/z]$ is a localization of a polynomial ring and hence a UFD. It follows from an argument of Nagata that $A$ is a UFD. See Algebra, Lemma \ref{algebra}. A similar argument can be given if $r$ is not congruent to 1 modulo $rs$.

Example 9.2. The ring $A = \mathbb{C}[[x, y, z]]/(x^r + y^s + z^t)$ is not a UFD when $r < s < t$ are pairwise coprime integers and not equal to 2, 3, 5. For example consider the special case $A = \mathbb{C}[[x, y, z]]/(x^2 + y^3 + z^7)$. Consider the maps

$$\psi_\zeta : \mathbb{C}[[x, y, z]]/(x^2 + y^3 + z^7) \to \mathbb{C}[t]$$

given by

$$x \mapsto t^\zeta, \quad y \mapsto t^3, \quad z \mapsto -\zeta t^2(1 + t)^{1/7}$$

where $\zeta$ is a 7th root of unity. The kernel $p_\zeta$ of $\psi_\zeta$ is a height one prime, hence if $A$ is a UFD, then it is principal, say given by $f_\zeta \in \mathbb{C}[[x, y, z]]$. Note that $V(x^3 - y^7) = \bigcup V(p_\zeta)$ and $A/(x^3 - y^7)$ is reduced away from the closed point. Hence, still assuming $A$ is a UFD, we would obtain

$$\prod_\zeta f_\zeta = u(x^3 - y^7) + a(x^2 + y^3 + z^7) \quad \text{in} \quad \mathbb{C}[[x, y, z]]$$
for some unit $u \in \mathbb{C}[x, y, z]$ and some element $a \in \mathbb{C}[x, y, z]$. After scaling by a constant we may assume $u(0, 0, 0) = 1$. Note that the left hand side vanishes to order 7. Hence $a = -x \ mod \ m^2$. But then we get a term $xy^3$ on the right hand side which does not occur on the left hand side. A contradiction.

**Example 9.3.** There exists an excellent 2-dimensional Noetherian local ring and a modification $X \to S = \text{Spec}(A)$ which is not a scheme. We sketch a construction. Let $X$ be a normal surface over $\mathbb{C}$ with a unique singular point $x \in X$. Assume that there exists a resolution $\pi : X' \to X$ such that the exceptional fibre $C = \pi^{-1}(x)_{\text{red}}$ is a smooth projective curve. Furthermore, assume there exists a point $c \in C$ such that if $\mathcal{O}_C(nc)$ is in the image of $\text{Pic}(X') \to \text{Pic}(C)$, then $n = 0$. Then we let $X'' \to X'$ be the blowing up in the nonsingular point $c$. Let $C' \subset X''$ be the strict transform of $C$ and let $E' \subset X''$ be the exceptional fibre. By Artin’s results ([Art70]; use for example [Mum61] to see that the normal bundle of $C'$ is negative) we can blow down the curve $C'$ in $X''$ to obtain an algebraic space $X'''$. Picture

We claim that $X'''$ is not a scheme. This provides us with our example because $X'''$ is a scheme if and only if the base change of $X''$ to $A = \mathcal{O}_{X,x}$ is a scheme (details omitted). If $X''$ where a scheme, then the image of $C'$ in $X'''$ would have an affine neighbourhood. The complement of this neighbourhood would be an effective Cartier divisor on $X'''$ (because $X'''$ is nonsingular apart from 1 point). This effective Cartier divisor would correspond to an effective Cartier divisor on $X''$ meeting $E$ and avoiding $C'$. Taking the image in $X'$ we obtain an effective Cartier divisor meeting $C$ (set theoretically) in $c$. This is impossible as no multiple of $c$ is the restriction of a Cartier divisor by assumption.

To finish we have to find such a singular surface $X$. We can just take $X$ to be the affine surface given by

$$x^3 + y^3 + z^3 + x^4 + y^4 + z^4 = 0$$

in $\mathbb{A}^3_\mathbb{C} = \text{Spec}(\mathbb{C}[x, y, z])$ and singular point $(0, 0, 0)$. Then $(0, 0, 0)$ is the only singular point. Blowing up $X$ in the maximal ideal corresponding to $(0, 0, 0)$ we find three charts each isomorphic to the smooth affine surface

$$1 + s^3 + t^3 + x(1 + s^4 + t^4) = 0$$

which is nonsingular with exceptional divisor $C$ given by $x = 0$. The reader will recognize $C$ as an elliptic curve. Finally, the surface $X$ is rational as projection from $(0, 0, 0)$ shows, or because in the equation for the blow up we can solve for $x$. Finally, the Picard group of a nonsingular rational surface is countable, whereas the Picard group of an elliptic curve over the complex numbers is uncountable. Hence we can find a closed point $c$ as indicated.
10. Other chapters

Preliminaries

- (1) Introduction
- (2) Conventions
- (3) Set Theory
- (4) Categories
- (5) Topology
- (6) Sheaves on Spaces
- (7) Sites and Sheaves
- (8) Stacks
- (9) Fields
- (10) Commutative Algebra
- (11) Brauer Groups
- (12) Homological Algebra
- (13) Derived Categories
- (14) Simplicial Methods
- (15) More on Algebra
- (16) Smoothing Ring Maps
- (17) Sheaves of Modules
- (18) Modules on Sites
- (19) Injectives
- (20) Cohomology of Sheaves
- (21) Cohomology on Sites
- (22) Differential Graded Algebra
- (23) Divided Power Algebra
- (24) Hypercoverings

Schemes

- (25) Schemes
- (26) Constructions of Schemes
- (27) Properties of Schemes
- (28) Morphisms of Schemes
- (29) Cohomology of Schemes
- (30) Divisors
- (31) Limits of Schemes
- (32) Varieties
- (33) Topologies on Schemes
- (34) Descent
- (35) Derived Categories of Schemes
- (36) More on Morphisms
- (37) More on Flatness
- (38) Groupoid Schemes
- (39) More on Groupoid Schemes
- (40) Étale Morphisms of Schemes

Topics in Scheme Theory

- (41) Chow Homology
- (42) Intersection Theory

Algebraic Spaces

- (43) Adequate Modules
- (44) Dualizing Complexes
- (45) Étale Cohomology
- (46) Crystalline Cohomology
- (47) Pro-étale Cohomology

- (48) Algebraic Spaces
- (49) Properties of Algebraic Spaces
- (50) Morphisms of Algebraic Spaces
- (51) Decent Algebraic Spaces
- (52) Cohomology of Algebraic Spaces
- (53) Limits of Algebraic Spaces
- (54) Divisors on Algebraic Spaces
- (55) Algebraic Spaces over Fields
- (56) Topologies on Algebraic Spaces
- (57) Descent and Algebraic Spaces
- (58) Derived Categories of Spaces
- (59) More on Morphisms of Spaces
- (60) Pushouts of Algebraic Spaces
- (61) Groupoids in Algebraic Spaces
- (62) More on Groupoids in Spaces
- (63) Bootstrap

Topics in Geometry

- (64) Quotients of Groupoids
- (65) Simplicial Spaces
- (66) Formal Algebraic Spaces
- (67) Restricted Power Series
- (68) Resolution of Surfaces

Deformation Theory

- (69) Formal Deformation Theory
- (70) Deformation Theory
- (71) The Cotangent Complex

Algebraic Stacks

- (72) Algebraic Stacks
- (73) Examples of Stacks
- (74) Sheaves on Algebraic Stacks
- (75) Criteria for Representability
- (76) Artin’s Axioms
- (77) Quot and Hilbert Spaces
- (78) Properties of Algebraic Stacks
- (79) Morphisms of Algebraic Stacks
- (80) Cohomology of Algebraic Stacks
- (81) Derived Categories of Stacks
- (82) Introducing Algebraic Stacks

Miscellany
References


