

RESTRICTED POWER SERIES

0AM7

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1. Introduction

0AM8 The title of this chapter is a bit misleading because the most basic material on restricted power series is in the chapter on formal algebraic spaces. For example Formal Spaces, Section 21 defines the restricted power series ring $A\{x_1, \dots, x_n\}$ given a linearly topologized ring A . In Formal Spaces, Section 22 we discuss the relationship between these restricted power series rings and morphisms of finite type between locally countably indexed formal algebraic spaces.

Let A be a Noetherian ring and let $I \subset A$ be an ideal. In the first part of this chapter (Sections 2 – 8) we discuss the category of I -adically complete algebras B topologically of finite type over a Noetherian ring A . It is shown that $B = A\{x_1, \dots, x_n\}/J$ for some (closed) ideal J in the restricted power series ring (where A is endowed with the I -adic topology). We show there is a good notion of a naive cotangent complex $NL_{B/A}^\wedge$. If some power of I annihilates $NL_{B/A}^\wedge$, then we think of $\mathrm{Spf}(B)$ as a rig-étale formal algebraic space over A . This leads to a definition of rig-étale morphisms of Noetherian formal algebraic spaces. After a certain amount of work we are able to prove the main result of the first part: if $\mathrm{Spf}(B)$ is rig-étale over A as above, then there exists a finite type A -algebra C such that B is isomorphic to the I -adic completion of C , see Lemma 7.4. One thing to note here is that we prove this without assuming the ring A is excellent or even a G-ring. In the last section of the first part we show that under the assumption that A is a G-ring there is a straightforward proof of the lemma based on Popescu's theorem.

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Many of the results discussed in the first part can be found in the paper [Elk73]. Other general references for this part are [DG67], [Abb10], and [FK].

In the second part of this chapter we use the main result of the first part to prove Artin's result on dilatations from [Art70]. The result on contractions will be the subject of a later chapter (insert future reference here). The main existence theorem is the equivalence of categories in Theorem 10.9. It is more general than Artin's result in that it shows that any rig-étale morphism $f : W \rightarrow \text{Spec}(A)_{/V(I)}$ is the completion of a morphism $Y \rightarrow \text{Spec}(A)$ of algebraic spaces X which is locally of finite type and isomorphism away from $V(I)$. In Artin's work the morphism f is assumed proper and rig-surjective. A special case of this is the main lemma mentioned above and the general case follows from this by a straightforward (somewhat lengthy) glueing procedure. There are several lemmas modifying the main theorem the final one of which is (almost) exactly the statement in Artin's paper. In the last section we apply the results to modifications of $\text{Spec}(A)$ before and after completion.

2. Two categories

0AL2 Let A be a ring and let $I \subset A$ be an ideal. In this section $\hat{}$ will mean I -adic completion. Set $A_n = A/I^n$ so that the I -adic completion of A is $A^\wedge = \lim A_n$. Let \mathcal{C} be the category

$$0AL3 \quad (2.0.1) \quad \mathcal{C} = \left\{ \begin{array}{l} \text{systems } (B_n, B_{n+1} \rightarrow B_n)_{n \in \mathbf{N}} \text{ where} \\ B_n \text{ is a finite type } A_n\text{-algebra,} \\ B_{n+1} \rightarrow B_n \text{ is an } A_{n+1}\text{-algebra map} \\ \text{which induces } B_{n+1}/I^n B_{n+1} \cong B_n \end{array} \right\}$$

Morphisms in \mathcal{C} are given by systems of homomorphisms. Let \mathcal{C}' be the category

$$0AL4 \quad (2.0.2) \quad \mathcal{C}' = \left\{ \begin{array}{l} A\text{-algebras } B \text{ which are } I\text{-adically complete} \\ \text{such that } B/IB \text{ is of finite type over } A/I \end{array} \right\}$$

Morphisms in \mathcal{C}' are A -algebra maps. There is a functor

$$0AJN \quad (2.0.3) \quad \mathcal{C}' \longrightarrow \mathcal{C}, \quad B \longmapsto (B/I^n B)$$

Indeed, since B/IB is of finite type over A/I the ring maps $A_n = A/I^n \rightarrow B/I^n B$ are of finite type (apply Algebra, Lemma 19.1 to a ring map $A/I^n[x_1, \dots, x_r] \rightarrow B/I^n B$ such that the images of x_1, \dots, x_r generate B/IB over A/I).

0AJP **Lemma 2.1.** *Let A be a ring and let $I \subset A$ be a finitely generated ideal. The functor*

$$\mathcal{C} \longrightarrow \mathcal{C}', \quad (B_n) \longmapsto B = \lim B_n$$

is a quasi-inverse to (2.0.3). The completions $A[x_1, \dots, x_r]^\wedge$ are in \mathcal{C}' and any object of \mathcal{C}' is of the form

$$B = A[x_1, \dots, x_r]^\wedge / J$$

for some ideal $J \subset A[x_1, \dots, x_r]^\wedge$.

Proof. Let (B_n) be an object of \mathcal{C} . By Algebra, Lemma 97.1 we see that $B = \lim B_n$ is I -adically complete and $B/I^n B = B_n$. Hence we see that B is an object of \mathcal{C}' and that we can recover the object (B_n) by taking the quotients. Conversely, if B is an object of \mathcal{C}' , then $B = \lim B/I^n B$ by assumption. Thus $B \mapsto (B/I^n B)$ is a quasi-inverse to the functor of the lemma.

Since $A[x_1, \dots, x_r]^\wedge = \lim A_n[x_1, \dots, x_r]$ it is an object of \mathcal{C}' by the first statement of the lemma. Finally, let B be an object of \mathcal{C}' . Choose $b_1, \dots, b_r \in B$ whose images in B/IB generate B/IB as an algebra over A/I . Since B is I -adically complete, the A -algebra map $A[x_1, \dots, x_r] \rightarrow B$, $x_i \mapsto b_i$ extends to an A -algebra map $A[x_1, \dots, x_r]^\wedge \rightarrow B$. To finish the proof we have to show this map is surjective which follows from Algebra, Lemma 95.1 as our map $A[x_1, \dots, x_r] \rightarrow B$ is surjective modulo I and as $B = B^\wedge$. \square

We warn the reader that, in case A is not Noetherian, the quotient of an object of \mathcal{C}' may not be an object of \mathcal{C}' . See Examples, Lemma 7.1. Next we show this does not happen when A is Noetherian.

- 0AJQ **Lemma 2.2.** *Let A be a Noetherian ring and let $I \subset A$ be an ideal. Then* [GD60, Proposition 7.5.5]
- (1) *every object of the category \mathcal{C}' , in particular the completion $A[x_1, \dots, x_r]^\wedge$, is Noetherian,*
 - (2) *if B is an object of \mathcal{C}' and $J \subset B$ is an ideal, then B/J is an object of \mathcal{C}' .*

Proof. To see (1) by Lemma 2.1 we reduce to the case of the completion of the polynomial ring. This case follows from Algebra, Lemma 96.6 as $A[x_1, \dots, x_r]$ is Noetherian (Algebra, Lemma 30.1). Part (2) follows from Algebra, Lemma 96.1 which tells us that every finite B -module is IB -adically complete. \square

- 0AL5 **Remark 2.3** (Base change). Let $\varphi : A_1 \rightarrow A_2$ be a ring map and let $I_i \subset A_i$ be ideals such that $\varphi(I_1^c) \subset I_2$ for some $c \geq 1$. This induces ring maps $A_{1,cn} = A_1/I_1^{cn} \rightarrow A_2/I_2^n = A_{2,n}$ for all $n \geq 1$. Let \mathcal{C}_i be the category (2.0.1) for (A_i, I_i) . There is a base change functor

$$0AJZ \quad (2.3.1) \quad \mathcal{C}_1 \longrightarrow \mathcal{C}_2, \quad (B_n) \longmapsto (B_{cn} \otimes_{A_{1,cn}} A_{2,n})$$

Let \mathcal{C}'_i be the category (2.0.2) for (A_i, I_i) . If I_2 is finitely generated, then there is a base change functor

$$0AK0 \quad (2.3.2) \quad \mathcal{C}'_1 \longrightarrow \mathcal{C}'_2, \quad B \longmapsto (B \otimes_{A_1} A_2)^\wedge$$

because in this case the completion is complete (Algebra, Lemma 95.3). If both I_1 and I_2 are finitely generated, then the two base change functors agree via the functors (2.0.3) which are equivalences by Lemma 2.1.

- 0AL6 **Remark 2.4** (Base change by closed immersion). Let A be a Noetherian ring and $I \subset A$ an ideal. Let $\mathfrak{a} \subset A$ be an ideal. Denote $\bar{A} = A/\mathfrak{a}$. Let $\bar{I} \subset \bar{A}$ be an ideal such that $I^c \bar{A} \subset \bar{I}$ and $\bar{I}^d \subset I \bar{A}$ for some $c, d \geq 1$. In this case the base change functor (2.3.2) for (A, I) to (\bar{A}, \bar{I}) is given by $B \mapsto \bar{B} = B/\mathfrak{a}B$. Namely, we have

$$0AK1 \quad (2.4.1) \quad \bar{B} = (B \otimes_A \bar{A})^\wedge = (B/\mathfrak{a}B)^\wedge = B/\mathfrak{a}B$$

the last equality because any finite B -module is I -adically complete by Algebra, Lemma 96.1 and if annihilated by \mathfrak{a} also \bar{I} -adically complete by Algebra, Lemma 95.9.

3. A naive cotangent complex

- 0AJL Let A be a Noetherian ring and let $I \subset A$ be an ideal. Let B be an A -algebra which is I -adically complete such that $A/I \rightarrow B/IB$ is of finite type, i.e., an object of (2.0.2). By Lemma 2.2 we can write

$$B = A[x_1, \dots, x_r]^\wedge / J$$

for some finitely generated ideal J . For a choice of presentation as above we define the naive cotangent complex in this setting by the formula

$$0AJR \quad (3.0.1) \quad NL_{B/A}^\wedge = (J/J^2 \longrightarrow \bigoplus Bdx_i)$$

with terms sitting in degrees -1 and 0 where the map sends the residue class of $g \in J$ to the differential $dg = \sum(\partial g/\partial x_i)dx_i$. Here the partial derivative is taken by thinking of g as a power series. The following lemma shows that $NL_{B/A}^\wedge$ is well defined in $D(B)$, i.e., independent of the chosen presentation, although this could be shown directly by comparing presentations as in Algebra, Section 132.

0AJS **Lemma 3.1.** *Let A be a Noetherian ring and let $I \subset A$ be an ideal. Let B be an object of (2.0.2). Then $NL_{B/A}^\wedge = R\lim NL_{B_n/A_n}$ in $D(B)$.*

Proof. In fact, the presentation $B = A[x_1, \dots, x_r]^\wedge/J$ defines presentations

$$B_n = B/I^n B = A_n[x_1, \dots, x_r]/J_n$$

where

$$J_n = JA_n[x_1, \dots, x_r] = J/(J \cap I^n A[x_1, \dots, x_r]^\wedge)$$

By Artin-Rees (Algebra, Lemma 50.2) in the Noetherian ring $A[x_1, \dots, x_r]^\wedge$ (Lemma 2.2) we see that we have canonical surjections

$$J/I^n J \rightarrow J_n \rightarrow J/I^{n-c} J, \quad n \geq c$$

for some $c \geq 0$. It follows that $\lim J_n/J_n^2 = J/J^2$ as any finite $A[x_1, \dots, x_r]^\wedge$ -module is I -adically complete (Algebra, Lemma 96.1). Thus

$$NL_{B/A}^\wedge = \lim(J_n/J_n^2 \longrightarrow \bigoplus B_n dx_i)$$

(termwise limit) and the transition maps in the system are termwise surjective. The two term complex $J_n/J_n^2 \longrightarrow \bigoplus B_n dx_i$ represents NL_{B_n/A_n} by Algebra, Section 132. It follows that $NL_{B/A}^\wedge$ represents $R\lim NL_{B_n/A_n}$ in the derived category by More on Algebra, Lemma 76.1. \square

0ALM **Lemma 3.2.** *Let A be a Noetherian ring and let $I \subset A$ be an ideal. Let $B \rightarrow C$ be a morphism of (2.0.2). Then there is an exact sequence*

$$\begin{array}{ccccccc} C \otimes_B H^0(NL_{B/A}^\wedge) & \longrightarrow & H^0(NL_{C/A}^\wedge) & \longrightarrow & H^0(NL_{C/B}^\wedge) & \longrightarrow & 0 \\ & & & & \swarrow & & \\ H^{-1}(NL_{B/A}^\wedge \otimes_B C) & \longrightarrow & H^{-1}(NL_{C/A}^\wedge) & \longrightarrow & H^{-1}(NL_{C/B}^\wedge) & & \end{array}$$

Proof. Choose a presentation $B = A[x_1, \dots, x_r]^\wedge/J$. Note that (B, IB) is a pair consisting of a Noetherian ring and an ideal, and C is in the corresponding category (2.0.2) for this pair. Hence we can choose a presentation $C = B[y_1, \dots, y_s]^\wedge/J'$. Combining these presentations gives a presentation

$$C = A[x_1, \dots, x_r, y_1, \dots, y_s]^\wedge/K$$

Then the reader verifies that we obtain a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \bigoplus Cdx_i & \longrightarrow & \bigoplus Cdx_i \oplus \bigoplus Cdy_j & \longrightarrow & \bigoplus Cdy_j \longrightarrow 0 \\ & & \uparrow & & \uparrow & & \uparrow \\ J/J^2 \otimes_B C & \longrightarrow & K/K^2 & \longrightarrow & J'/(J')^2 & \longrightarrow & 0 \end{array}$$

with exact rows. Note that the vertical arrow on the left hand side is the tensor product of the arrow defining $NL_{B/A}^\wedge$ with id_C . The lemma follows by applying the snake lemma (Algebra, Lemma 4.1). \square

0AQJ **Lemma 3.3.** *With assumptions as in Lemma 3.2 assume that $B/I^n B \rightarrow C/I^n C$ is a local complete intersection homomorphism for all n . Then $H^{-1}(NL_{B/A}^\wedge \otimes_B C) \rightarrow H^{-1}(NL_{C/A}^\wedge)$ is injective.*

Proof. By More on Algebra, Lemma 30.6 we see that this holds for the map between naive cotangent complexes of the situation modulo I^n for all n . In other words, we obtain a distinguished triangle in $D(C/I^n C)$ for every n . Using Lemma 3.1 this implies the lemma; details omitted. \square

Maps in the derived category out of a complex such as (3.0.1) are easy to understand by the result of the following lemma.

0ALN **Lemma 3.4.** *Let R be a ring. Let M^\bullet be a complex of modules over R with $M^i = 0$ for $i > 0$ and M^0 a projective R -module. Let K^\bullet be a second complex.*

- (1) *If $K^i = 0$ for $i \leq -2$, then $\text{Hom}_{D(R)}(M^\bullet, K^\bullet) = \text{Hom}_{K(R)}(M^\bullet, K^\bullet)$,*
- (2) *If $K^i = 0$ for $i \leq -3$ and $\alpha \in \text{Hom}_{D(R)}(M^\bullet, K^\bullet)$ composed with $K^\bullet \rightarrow K^{-2}[2]$ comes from an R -module map $a : M^{-2} \rightarrow K^{-2}$ with $a \circ d_M^{-3} = 0$, then α can be represented by a map of complexes $a^\bullet : M^\bullet \rightarrow K^\bullet$ with $a^{-2} = a$.*
- (3) *In (2) for any second map of complexes $(a')^\bullet : M^\bullet \rightarrow K^\bullet$ representing α with $a = (a')^{-2}$ there exist $h' : M^0 \rightarrow K^{-1}$ and $h : M^{-1} \rightarrow K^{-2}$ such that $h \circ d_M^{-2} = 0$, $(a')^{-1} = a^{-1} + d_K^{-2} \circ h + h' \circ d_M^{-1}$, $(a')^0 = a^0 + d_K^{-1} \circ h'$*

Proof. Set $F^0 = M^0$. Choose a free R -module F^{-1} and a surjection $F^{-1} \rightarrow M^{-1}$. Choose a free R -module F^{-2} and a surjection $F^{-2} \rightarrow M^{-2} \times_{M^{-1}} F^{-1}$. Continuing in this way we obtain a quasi-isomorphism $p^\bullet : F^\bullet \rightarrow M^\bullet$ which is termwise surjective and with F^i free for all i .

Proof of (1). By Derived Categories, Lemma 19.8 we have

$$\text{Hom}_{D(R)}(M^\bullet, K^\bullet) = \text{Hom}_{K(R)}(F^\bullet, K^\bullet)$$

If $K^i = 0$ for $i \leq -2$, then any morphism of complexes $F^\bullet \rightarrow K^\bullet$ factors through p^\bullet . Similarly, any homotopy $\{h^i : F^i \rightarrow K^{i-1}\}$ factors through p^\bullet . Thus (1) holds.

Proof of (2). Choose $b^\bullet : F^\bullet \rightarrow K^\bullet$ representing α . The composition of α with $K^\bullet \rightarrow K^{-2}[2]$ is represented by $b^{-2} : F^{-2} \rightarrow K^{-2}$. As this is homotopic to $a \circ p^{-2} : F^{-2} \rightarrow M^{-2} \rightarrow K^{-2}$, there is a map $h : F^{-1} \rightarrow K^{-2}$ such that $b^{-2} = a \circ p^{-2} + h \circ d_F^{-2}$. Adjusting b^\bullet by h viewed as a homotopy from F^\bullet to K^\bullet , we find that $b^{-2} = a \circ p^{-2}$. Hence b^{-2} factors through p^{-2} . Since $F^0 = M^0$ the kernel of p^{-2} surjects onto the kernel of p^{-1} (for example because the kernel of p^\bullet is an acyclic complex or by a diagram chase). Hence b^{-1} necessarily factors through p^{-1} as well and we see that (2) holds for these factorizations and $a^0 = b^0$.

Proof of (3) is omitted. Hint: There is a homotopy between $a^\bullet \circ p^\bullet$ and $(a')^\bullet \circ p^\bullet$ and we argue as before that this homotopy factors through p^\bullet . \square

0AJT **Lemma 3.5.** *Let R be a ring. Let M^\bullet be a two term complex $M^{-1} \rightarrow M^0$ over R . If $\varphi, \psi \in \text{End}_{D(R)}(M^\bullet)$ are zero on $H^i(M^\bullet)$, then $\varphi \circ \psi = 0$.*

Proof. Apply Derived Categories, Lemma 12.5 to see that $\varphi \circ \psi$ factors through $\tau_{\leq -2}M^\bullet = 0$. \square

4. Rig-étale homomorphisms

0ALP In this and some of the later sections we will study ring maps as in Lemma 4.1. Condition (4) is one of the conditions used in [Art70] to define modifications. Ring maps like this are sometimes called rig-étale or rigid-étale ring maps in the literature. These and the analogously defined rig-smooth ring maps were studied in [Elk73]. A detailed exposition can also be found in [Abb10]. Our main goal will be to show that rig-étale ring maps are completions of finite type algebras, a result very similar to results found in Elkik's paper [Elk73].

0AJU **Lemma 4.1.** *Let A be a Noetherian ring and let $I \subset A$ be an ideal. Let B be an object of (2.0.2). The following are equivalent*

- 0AJV (1) *there exists a $c \geq 0$ such that multiplication by a on $NL_{B/A}^\wedge$ is zero in $D(B)$ for all $a \in I^c$,*
- 0AJW (2) *there exists a $c \geq 0$ such that $H^i(NL_{B/A}^\wedge)$, $i = -1, 0$ is annihilated by I^c ,*
- 0AJX (3) *there exists a $c \geq 0$ such that $H^i(NL_{B_n/A_n})$, $i = -1, 0$ is annihilated by I^c for all $n \geq 1$,*
- 0AJY (4) *$B = A[x_1, \dots, x_r]^\wedge/J$ and for every $a \in I$ there exists a $c \geq 0$ such that*
 - (a) *a^c annihilates $H^0(NL_{B/A}^\wedge)$, and*
 - (b) *there exist $f_1, \dots, f_r \in J$ such that $a^c J \subset (f_1, \dots, f_r) + J^2$.*

Proof. The equivalence of (1) and (2) follows from Lemma 3.5. The equivalence of (1) + (2) and (3) follows from Lemma 3.1. Some details omitted.

Assume the equivalent conditions (1), (2), (3) holds and let $B = A[x_1, \dots, x_r]^\wedge/J$ be a presentation (see Lemma 2.1). Let $a \in I$. Let c be such that multiplication by a^c is zero on $NL_{B/A}^\wedge$ which exists by (1). By Lemma 3.4 there exists a map $\alpha : \bigoplus Bdx_i \rightarrow J/J^2$ such that $d \circ \alpha$ and $\alpha \circ d$ are both multiplication by a^c . Let $f_i \in J$ be an element whose class modulo J^2 is equal to $\alpha(dx_i)$. Then we see that (4)(a), (b) hold.

Assume (4) holds. Say $I = (a_1, \dots, a_t)$. Let $c_i \geq 0$ be the integer such that (4)(a), (b) hold for $a_i^{c_i}$. Then we see that $I^{\sum c_i}$ annihilates $H^0(NL_{B/A}^\wedge)$. Let $f_{i,1}, \dots, f_{i,r} \in J$ be as in (4)(b) for a_i . Consider the composition

$$B^{\oplus r} \rightarrow J/J^2 \rightarrow \bigoplus Bdx_i$$

where the j th basis vector is mapped to the class of $f_{i,j}$ in J/J^2 . By (4)(a) and (b) the cokernel of the composition is annihilated by $a_i^{2c_i}$. Thus this map is surjective after inverting $a_i^{c_i}$, and hence an isomorphism (Algebra, Lemma 15.4). Thus the kernel of $B^{\oplus r} \rightarrow \bigoplus Bdx_i$ is a_i -power torsion, and hence $H^{-1}(NL_{B/A}^\wedge) = \text{Ker}(J/J^2 \rightarrow \bigoplus Bdx_i)$ is a_i -power torsion. Since B is Noetherian (Lemma 2.2), all modules including $H^{-1}(NL_{B/A}^\wedge)$ are finite. Thus $a_i^{d_i}$ annihilates $H^{-1}(NL_{B/A}^\wedge)$ for some $d_i \geq 0$. It follows that $I^{\sum d_i}$ annihilates $H^{-1}(NL_{B/A}^\wedge)$ and we see that (2) holds. \square

0ALQ **Lemma 4.2.** *Let A be a Noetherian ring and let I be an ideal. Let B be a finite type A -algebra.*

- (1) *If $\text{Spec}(B) \rightarrow \text{Spec}(A)$ is étale over $\text{Spec}(A) \setminus V(I)$, then B^\wedge satisfies the equivalent conditions of Lemma 4.1.*

(2) If B^\wedge satisfies the equivalent conditions of Lemma 4.1, then there exists $g \in 1 + IB$ such that $\text{Spec}(B_g)$ is étale over $\text{Spec}(A) \setminus V(I)$.

Proof. Assume B^\wedge satisfies the equivalent conditions of Lemma 4.1. The naive cotangent complex $NL_{B/A}$ is a complex of finite type B -modules and hence H^{-1} and H^0 are finite B -modules. Completion is an exact functor on finite B -modules (Algebra, Lemma 96.2) and $NL_{B^\wedge/A}$ is the completion of the complex $NL_{B/A}$ (this is easy to see by choosing presentations). Hence the assumption implies there exists a $c \geq 0$ such that $H^{-1}/I^n H^{-1}$ and $H^0/I^n H^0$ are annihilated by I^c for all n . By Nakayama's lemma (Algebra, Lemma 19.1) this means that $I^c H^{-1}$ and $I^c H^0$ are annihilated by an element of the form $g = 1 + x$ with $x \in IB$. After inverting g (which does not change the quotients $B/I^n B$) we see that $NL_{B/A}$ has cohomology annihilated by I^c . Thus $A \rightarrow B$ is étale at any prime of B not lying over $V(I)$ by the definition of étale ring maps, see Algebra, Definition 141.1.

Conversely, assume that $\text{Spec}(B) \rightarrow \text{Spec}(A)$ is étale over $\text{Spec}(A) \setminus V(I)$. Then for every $a \in I$ there exists a $c \geq 0$ such that multiplication by a^c is zero on $NL_{B/A}$. Since $NL_{B^\wedge/A}$ is the derived completion of $NL_{B/A}$ (see Lemma 3.1) it follows that B^\wedge satisfies the equivalent conditions of Lemma 4.1. \square

0AK2 **Lemma 4.3.** *Assume the map $(A_1, I_1) \rightarrow (A_2, I_2)$ is as in Remark 2.3 with A_1 and A_2 Noetherian. Let B_1 be in (2.0.2) for (A_1, I_1) . Let B_2 be the base change of B_1 . If multiplication by $f_1 \in B_1$ on NL_{B_1/A_1}^\wedge is zero in $D(B_1)$, then multiplication by the image $f_2 \in B_2$ on NL_{B_2/A_2}^\wedge is zero in $D(B_2)$.*

Proof. Choose a presentation $B_1 = A_1[x_1, \dots, x_r]^\wedge/J_1$. Since $A_2/I_2^n[x_1, \dots, x_r] = A_1/I_1^{cn}[x_1, \dots, x_r] \otimes_{A_1/I_1^{cn}} A_2/I_2^n$ we have

$$A_2[x_1, \dots, x_r]^\wedge = (A_1[x_1, \dots, x_r]^\wedge \otimes_{A_1} A_2)^\wedge$$

where we use I_2 -adic completion on both sides (but of course I_1 -adic completion for $A_1[x_1, \dots, x_r]^\wedge$). Set $J_2 = J_1 A_2[x_1, \dots, x_r]^\wedge$. Arguing similarly we get the presentation

$$\begin{aligned} B_2 &= (B_1 \otimes_{A_1} A_2)^\wedge \\ &= \lim \frac{A_1/I_1^{cn}[x_1, \dots, x_r]}{J_1(A_1/I_1^{cn}[x_1, \dots, x_r])} \otimes_{A_1/I_1^{cn}} A_2/I_2^n \\ &= \lim \frac{A_2/I_2^n[x_1, \dots, x_r]}{J_2(A_2/I_2^n[x_1, \dots, x_r])} \\ &= A_2[x_1, \dots, x_r]^\wedge/J_2 \end{aligned}$$

for B_2 over A_2 . Consider the commutative diagram

$$\begin{array}{ccc} NL_{B_1/A_1}^\wedge : & J_1/J_1^2 & \xrightarrow{\quad d \quad} \bigoplus B_1 dx_i \\ \downarrow & \downarrow & \downarrow \\ NL_{B_2/A_2}^\wedge : & J_2/J_2^2 & \longrightarrow \bigoplus B_2 dx_i \end{array}$$

The induced arrow $J_1/J_1^2 \otimes_{B_1} B_2 \rightarrow J_2/J_2^2$ is surjective because J_2 is generated by the image of J_1 . By Lemma 3.4 there is a map $\alpha_1 : \bigoplus B dx_i \rightarrow J_1/J_1^2$ such that $f_1 \text{id}_{\bigoplus B dx_i} = d \circ \alpha_1$ and $f_1 \text{id}_{J_1/J_1^2} = \alpha_1 \circ d$. We define $\alpha_2 : \bigoplus B_1 dx_i \rightarrow J_2/J_2^2$ by mapping dx_i to the image of $\alpha_1(dx_i)$ in J_2/J_2^2 . Because the image of the vertical

arrows contains generators of the modules J_2/J_2^2 and $\bigoplus B_2 dx_i$ it follows that α_2 also defines a homotopy between multiplication by f_2 and the zero map. \square

0AKJ **Lemma 4.4.** *Let A be a Noetherian ring. Let $I \subset A$ be an ideal. Let B be a finite type A -algebra such that $\text{Spec}(B) \rightarrow \text{Spec}(A)$ is étale over $\text{Spec}(A) \setminus V(I)$. Let C be a Noetherian A -algebra. Then any A -algebra map $B^\wedge \rightarrow C^\wedge$ of I -adic completions comes from a unique A -algebra map*

$$B \longrightarrow C^h$$

where C^h is the henselization of the pair (C, IC) as in More on Algebra, Lemma 10.14. Moreover, any A -algebra homomorphism $B \rightarrow C^h$ factors through some étale C -algebra C' such that $C'/IC \rightarrow C'/IC'$ is an isomorphism.

Proof. Uniqueness follows from the fact that C^h is a subring of C^\wedge , see for example More on Algebra, Lemma 10.17. The final assertion follows from the fact that C^h is the filtered colimit of these C -algebras C' , see proof of More on Algebra, Lemma 10.14. Having said this we now turn to the proof of existence.

Let $\varphi : B^\wedge \rightarrow C^\wedge$ be the given map. This defines a section

$$\sigma : (B \otimes_A C)^\wedge \longrightarrow C^\wedge$$

of the completion of the map $C \rightarrow B \otimes_A C$. We may replace (A, I, B, C, φ) by $(C, IC, B \otimes_A C, C, \sigma)$. In this way we see that we may assume that $A = C$.

Proof of existence in the case $A = C$. In this case the map $\varphi : B^\wedge \rightarrow A^\wedge$ is necessarily surjective. By Lemmas 4.2 and 3.2 we see that the cohomology groups of $NL_{A^\wedge/\varphi B^\wedge}^\wedge$ are annihilated by a power of I . Since φ is surjective, this implies that $\text{Ker}(\varphi)/\text{Ker}(\varphi)^2$ is annihilated by a power of I . Hence $\varphi : B^\wedge \rightarrow A^\wedge$ is the completion of a finite type B -algebra $B \rightarrow D$, see More on Algebra, Lemma 90.4. Hence $A \rightarrow D$ is a finite type algebra map which induces an isomorphism $A^\wedge \rightarrow D^\wedge$. By Lemma 4.2 we may replace D by a localization and assume that $A \rightarrow D$ is étale away from $V(I)$. Since $A^\wedge \rightarrow D^\wedge$ is an isomorphism, we see that $\text{Spec}(D) \rightarrow \text{Spec}(A)$ is also étale in a neighbourhood of $V(ID)$ (for example by More on Morphisms, Lemma 12.3). Thus $\text{Spec}(D) \rightarrow \text{Spec}(A)$ is étale. Therefore D maps to A^h and the lemma is proved. \square

5. Rig-étale morphisms

0AQK We can use the notion introduced in the previous section to define a new type of morphism of locally Noetherian formal algebraic spaces. Before we do so, we have to check it is a local property.

0AQL **Lemma 5.1.** *For morphisms $A \rightarrow B$ of the category $WAdm^{Noeth}$ (Formal Spaces, Section 16) consider the condition $P =$ “for some ideal of definition I of A the topology on B is the I -adic topology, the ring map $A/I \rightarrow B/IB$ is of finite type and $A \rightarrow B$ satisfies the equivalent conditions of Lemma 4.1”. Then P is a local property, see Formal Spaces, Remark 16.5.*

Proof. We have to show that Formal Spaces, Axioms (1), (2), and (3) hold for maps between Noetherian adic rings. For a Noetherian adic ring A with ideal of definition I we have $A\{x_1, \dots, x_r\} = A[x_1, \dots, x_r]^\wedge$ as topological A -algebras (see Formal Spaces, Remark 21.2). We will use without further mention that we know

the axioms hold for the property “ B is a quotient of $A[x_1, \dots, x_r]^\wedge$ ”, see Formal Spaces, Lemma 22.6.

Let a diagram as in Formal Spaces, Diagram (16.2.1) be given with A and B in the category $WAdm^{Noeth}$. Pick an ideal of definition $I \subset A$. By the remarks above the topology on each ring in the diagram is the I -adic topology. Since $A \rightarrow A'$ and $B \rightarrow B'$ are étale we see that $NL_{(A')^\wedge/A}^\wedge$ and $NL_{(B')^\wedge/B}^\wedge$ are zero. By Lemmas 3.2 and 3.3 we get

$$H^i(NL_{(B')^\wedge/(A')^\wedge}^\wedge) \cong H^i(NL_{(B')^\wedge/A}^\wedge) \quad \text{and} \quad H^i(NL_{B/A}^\wedge \otimes_B (B')^\wedge) \cong H^i(NL_{(B')^\wedge/A}^\wedge)$$

for $i = -1, 0$. Since B is Noetherian the ring map $B \rightarrow B' \rightarrow (B')^\wedge$ is flat (Algebra, Lemma 96.2) hence the tensor product comes out. Moreover, as B is I -adically complete, then if $B \rightarrow B'$ is faithfully flat, so is $B \rightarrow (B')^\wedge$. From these observations Formal Spaces, Axioms (1) and (2) follow immediately.

We omit the proof of Formal Spaces, Axiom (3). \square

0AQM **Definition 5.2.** Let S be a scheme. Let $f : X \rightarrow Y$ be a morphism of locally Noetherian formal algebraic spaces over S . We say f is *rig-étale* if f satisfies the equivalent conditions of Formal Spaces, Lemma 16.3 (in the setting of locally Noetherian formal algebraic spaces, see Formal Spaces, Remark 16.4) for the property P of Lemma 5.1.

To be sure, a rig-étale morphism is locally of finite type.

0AQN **Lemma 5.3.** *A rig-étale morphism of locally Noetherian formal algebraic spaces is locally of finite type.*

Proof. The property P in Lemma 5.1 implies the equivalent conditions (a), (b), (c), and (d) in Formal Spaces, Lemma 22.6. Hence this follows from Formal Spaces, Lemma 22.9. \square

6. Glueing rings along a principal ideal

0AK5 In this situation we prove some results about the categories \mathcal{C} and \mathcal{C}' of Section 2 in case A is a Noetherian ring and $I = (a)$ is a principal ideal.

0AK3 **Remark 6.1** (Linear approximation). Let A be a ring and $I \subset A$ be a finitely generated ideal. Let C be an I -adically complete A -algebra. Let $\psi : A[x_1, \dots, x_r]^\wedge \rightarrow C$ be a continuous A -algebra map. Suppose given $\delta_i \in C$, $i = 1, \dots, r$. Then we can consider

$$\psi' : A[x_1, \dots, x_r]^\wedge \rightarrow C, \quad x_i \mapsto \psi(x_i) + \delta_i$$

see Formal Spaces, Remark 21.1. Then we have

$$\psi'(g) = \psi(g) + \sum \psi(\partial g / \partial x_i) \delta_i + \xi$$

with error term $\xi \in (\delta_i \delta_j)$. This follows by writing g as a power series and working term by term. Convergence is automatic as the coefficients of g tend to zero. Details omitted.

0AK6 **Lemma 6.2.** *Let A be a Noetherian ring and $I = (a)$ a principal ideal. Let B be an object of (2.0.2). Assume given an integer $c \geq 0$ such that multiplication by a^c on $NL_{B/A}^\wedge$ is zero in $D(B)$. Let C be an I -adically complete A -algebra such that a is a nonzerodivisor on C . Let $n > 2c$. For any A_n -algebra map $\psi_n : B/a^n B \rightarrow C/a^n C$ there exists an A -algebra map $\varphi : B \rightarrow C$ such that $\psi_n \bmod a^{n-c} = \varphi \bmod a^{n-c}$.*

Proof. Choose a presentation $B = A[x_1, \dots, x_r]^\wedge / J$. Choose a lift

$$\psi : A[x_1, \dots, x_r]^\wedge \rightarrow C$$

of ψ_n . Then $\psi(J) \subset a^n C$ and $\psi(J^2) \subset a^{2n} C$ which determines a linear map

$$J/J^2 \rightarrow a^n C/a^{2n} C, \quad g \mapsto \psi(g)$$

By assumption and Lemma 3.4 there is a B -module map $\bigoplus B dx_i \rightarrow a^n C/a^{2n} C$, $dx_i \mapsto \delta_i$ such that $a^c \psi(g) = \sum \psi(\partial g / \partial x_i) \delta_i$ for all $g \in J$. Write $\delta_i = -a^c \delta'_i$ for some $\delta'_i \in a^{n-c} C$. Since a is a nonzerodivisor on C we see that $\psi(g) = -\sum \psi(\partial g / \partial x_i) \delta'_i$ in $C/a^{2n-c} C$. Then we look at the map

$$\psi' : A[x_1, \dots, x_r]^\wedge \rightarrow C, \quad x_i \mapsto \psi(x_i) + \delta'_i$$

A computation with power series (see Remark 6.1) shows that $\psi'(J) \subset a^{2n-2c} C$. Since $n > 2c$ we see that $n' = 2n - 2c = n + (n - 2c) > n$. Thus we obtain a morphism $\psi_{n'} : B/a^{n'} B \rightarrow C/a^{n'} C$ agreeing with ψ_n modulo a^{n-c} . Continuing in this fashion and taking the limit into $C = \lim C/a^t C$ we obtain the lemma. \square

0AK7 **Lemma 6.3.** *Let A be a Noetherian ring and $I = (a)$ a principal ideal. Let B be an object of (2.0.2). Assume given an integer $c \geq 0$ such that multiplication by a^c on $NL_{B/A}^\wedge$ is zero in $D(B)$. Let C be an I -adically complete A -algebra. Assume given an integer $d \geq 0$ such that $C[a^\infty] \cap a^d C = 0$. Let $n > \max(2c, c + d)$. For any A_n -algebra map $\psi_n : B/a^n B \rightarrow C/a^n C$ there exists an A -algebra map $\varphi : B \rightarrow C$ such that $\psi_n \bmod a^{n-c} = \varphi \bmod a^{n-c}$.*

If C is Noetherian we have $C[a^\infty] = C[a^e]$ for some $e \geq 0$. By Artin-Rees (Algebra, Lemma 50.2) there exists an integer f such that $a^n C \cap C[a^\infty] \subset a^{n-f} C[a^\infty]$ for all $n \geq f$. Then $d = e + f$ is an integer as in the lemma. This argument works in particular if C is an object of (2.0.2) by Lemma 2.2.

Proof. Let $C \rightarrow C'$ be the quotient of C by $C[a^\infty]$. The A -algebra C' is I -adically complete by Algebra, Lemma 95.10 and the fact that $\bigcap (C[a^\infty] + a^n C) = C[a^\infty]$ because for $n \geq d$ the sum $C[a^\infty] + a^n C$ is direct. For $m \geq d$ the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & C[a^\infty] & \longrightarrow & C & \longrightarrow & C' \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & C[a^\infty] & \longrightarrow & C/a^m C & \longrightarrow & C'/a^m C' \longrightarrow 0 \end{array}$$

has exact rows. Thus C is the fibre product of C' and $C/a^m C$ over $C'/a^m C'$. Thus the lemma now follows formally from the lifting result of Lemma 6.2. \square

0ALS **Lemma 6.4.** *Let A be a Noetherian ring and $I = (a)$ a principal ideal. Let B be an object of (2.0.2). Assume given an integer $c \geq 0$ such that multiplication by a^c on $NL_{B/A}^\wedge$ is zero in $D(B)$. Then there exists a finite type A -algebra C and an isomorphism $B \cong C^\wedge$.*

The rig-étale case of [Elk73, III Theorem 7] which handles the rig-smooth case.

Proof. Choose a presentation $B = A[x_1, \dots, x_r]^\wedge / J$. By Lemma 3.4 we can find a map $\alpha : \bigoplus B dx_i \rightarrow J/J^2$ such that $d \circ \alpha$ and $\alpha \circ d$ are both multiplication by a^c . Pick an element $f_i \in J$ whose class modulo J^2 is equal to $\alpha(dx_i)$. Then we see that $df_i = a^c dx_i$ in $\bigoplus dx_i$. In particular we have a ring map

$$A[x_1, \dots, x_r]^\wedge / (f_1, \dots, f_r, \Delta(f_1, \dots, f_r) - a^{rc}) \rightarrow B$$

where $\Delta(f_1, \dots, f_r) \in A[x_1, \dots, x_r]^\wedge$ is the determinant of the matrix of partial derivatives of the f_i .

Pick a large integer N . Pick $F_1, \dots, F_r \in A[x_1, \dots, x_r]$ such that $F_i - f_i \in I^N A[x_1, \dots, x_r]^\wedge$. Set

$$C = A[x_1, \dots, x_r, z]/(F_1, \dots, F_r, z\Delta(F_1, \dots, F_r) - a^{rc})$$

We claim that multiplication by a^{2rc} is zero on $NL_{C/A}$ in $D(C)$. Namely, the determinant of the matrix of the partial derivatives of the $r+1$ generators of the ideal of C with respect to the variables x_1, \dots, x_{r+1}, z is $\Delta(F_1, \dots, F_r)^2$. Since $\Delta(F_1, \dots, F_r)$ divides a^{rc} we in C the claim follows for example from Algebra, Lemma 14.4. Let C^\wedge be the I -adic completion of C . Since $NL_{C^\wedge/A}^\wedge$ is the I -adic completion of $NL_{C/A}$ we conclude that multiplication by a^{2rc} is zero on $NL_{C^\wedge/A}^\wedge$ as well.

By construction there is a (surjective) map $\psi_N : C/I^N C \rightarrow B/I^N B$ sending x_i to x_i and z to 1. By Lemma 6.3 (with the roles of B and C reversed) for N large enough we get a map $\varphi : C^\wedge \rightarrow B$ which agrees with ψ_N modulo I^{N-2rc} .

Since $\varphi : C^\wedge \rightarrow B$ is surjective modulo I we see that it is surjective (for example use Algebra, Lemma 95.1). By construction and assumption the naive cotangent complexes $NL_{C^\wedge/A}^\wedge$ and $NL_{B/A}^\wedge$ have cohomology annihilated by a fixed power of a . Thus the same thing is true for NL_{B/C^\wedge}^\wedge by Lemma 3.2. Since φ is surjective we conclude that $\text{Ker}(\varphi)/\text{Ker}(\varphi)^2$ is annihilated by a power of a . The result of the lemma now follows from More on Algebra, Lemma 90.4. \square

7. Glueing rings along an ideal

0AK8 Let A be a Noetherian ring. Let $I \subset A$ be an ideal. In this section we study I -adically complete A -algebras which are, in some vague sense, étale over the complement of $V(I)$ in $\text{Spec}(A)$.

0AK9 **Lemma 7.1.** *Let A be a Noetherian ring. Let $I \subset A$ be an ideal. Let t be the minimal number of generators for I . Let C be a Noetherian I -adically complete A -algebra. There exists an integer $d \geq 0$ depending only on $I \subset A \rightarrow C$ with the following property: given*

- (1) $c \geq 0$ and B in (2.0.2) such that for $a \in I^c$ multiplication by a on $NL_{B/A}^\wedge$ is zero in $D(B)$,
- (2) an integer $n > 2t \max(c, d)$,
- (3) an A/I^n -algebra map $\psi_n : B/I^n B \rightarrow C/I^n C$,

there exists a map $\varphi : B \rightarrow C$ of A -algebras such that $\psi_n \bmod I^{m-c} = \varphi \bmod I^{m-c}$ with $m = \lfloor \frac{n}{t} \rfloor$.

Proof. We prove this lemma by induction on the number of generators of I . Say $I = (a_1, \dots, a_t)$. If $t = 0$, then $I = 0$ and there is nothing to prove. If $t = 1$, then the lemma follows from Lemma 6.3 because $2 \max(c, d) \geq \max(2c, c + d)$. Assume $t > 1$.

Set $m = \lfloor \frac{n}{t} \rfloor$ as in the lemma. Set $\bar{A} = A/(a_t^m)$. Consider the ideal $\bar{I} = (\bar{a}_1, \dots, \bar{a}_{t-1})$ in \bar{A} . Set $\bar{C} = C/(a_t^m)$. Note that \bar{C} is a \bar{I} -adically complete Noetherian \bar{A} -algebra (use Algebra, Lemmas 96.1 and 95.9). Let \bar{d} be the integer for $\bar{I} \subset \bar{A} \rightarrow \bar{C}$ which exists by induction hypothesis.

Let $d_1 \geq 0$ be an integer such that $C[a_t^\infty] \cap a_t^{d_1} C = 0$ as in Lemma 6.3 (see discussion following the lemma and before the proof).

We claim the lemma holds with $d = \max(\bar{d}, d_1)$. To see this, let c, B, n, ψ_n be as in the lemma.

Note that $\bar{I} \subset I\bar{A}$. Hence by Lemma 4.3 multiplication by an element of \bar{I}^c on the cotangent complex of $\bar{B} = B/(a_t^m)$ is zero in $D(\bar{B})$. Also, we have

$$\bar{I}^{n-m+1} \supset I^n \bar{A}$$

Thus ψ_n gives rise to a map

$$\bar{\psi}_{n-m+1} : \bar{B}/\bar{I}^{n-m+1}\bar{B} \longrightarrow \bar{C}/\bar{I}^{n-m+1}\bar{C}$$

Since $n > 2t \max(c, d)$ and $d \geq \bar{d}$ we see that

$$n - m + 1 \geq (t - 1)n/t > 2(t - 1) \max(c, d) \geq 2(t - 1) \max(c, \bar{d})$$

Hence we can find a morphism $\varphi_m : \bar{B} \rightarrow \bar{C}$ agreeing with $\bar{\psi}_{n-m+1}$ modulo the ideal $\bar{I}^{m'-c}$ where $m' = \lfloor \frac{n-m+1}{t-1} \rfloor$.

Since $m \geq n/t > 2 \max(c, d) \geq 2 \max(c, d_1) \geq \max(2c, c+d_1)$, we can apply Lemma 6.3 for the ring map $A \rightarrow B$ and the ideal (a_t) to find a morphism $\varphi : B \rightarrow C$ agreeing modulo a_t^{m-c} with φ_m .

All in all we find $\varphi : B \rightarrow C$ which agrees with ψ_n modulo

$$(a_t^{m-c}) + (a_1, \dots, a_{t-1})^{m'-c} \subset I^{\min(m-c, m'-c)}$$

We leave it to the reader to see that $\min(m-c, m'-c) = m-c$. This concludes the proof. \square

0ALT Lemma 7.2. *Let A be a Noetherian ring and $I \subset A$ an ideal. Let $J \subset A$ be a nilpotent ideal. Consider a diagram*

$$\begin{array}{ccc} C & \longrightarrow & C/JC \\ \uparrow & & \uparrow \\ & & B_0 \\ & & \uparrow \\ A & \longrightarrow & A/J \end{array}$$

whose vertical arrows are of finite type such that

- (1) $\text{Spec}(C) \rightarrow \text{Spec}(A)$ is étale over $\text{Spec}(A) \setminus V(I)$,
- (2) $\text{Spec}(B_0) \rightarrow \text{Spec}(A/J)$ is étale over $\text{Spec}(A/J) \setminus V((I+J)/J)$, and
- (3) $B_0 \rightarrow C/JC$ is étale and induces an isomorphism $B_0/IB_0 = C/(I+J)C$.

Then we can fill in the diagram

$$\begin{array}{ccc}
 C & \longrightarrow & C/JC \\
 \uparrow & & \uparrow \\
 B & \longrightarrow & B_0 \\
 \uparrow & & \uparrow \\
 A & \longrightarrow & A/J
 \end{array}$$

with $A \rightarrow B$ of finite type, $B/JB = B_0$, $B \rightarrow C$ étale, and $\text{Spec}(B) \rightarrow \text{Spec}(A)$ étale over $\text{Spec}(A) \setminus V(I)$.

First proof. This proof uses algebraic spaces to construct B . Set $X = \text{Spec}(A)$, $X_0 = \text{Spec}(A/J)$, $Y_0 = \text{Spec}(B_0)$, $Z = \text{Spec}(C)$, $Z_0 = \text{Spec}(C/JC)$. Furthermore, denote $U \subset X$, $U_0 \subset X_0$, $V_0 \subset Y_0$, $W \subset Z$, $W_0 \subset Z_0$ the complement of the vanishing set of I . The conditions in the lemma guarantee that

$$\begin{array}{ccc}
 W_0 & \longrightarrow & Z_0 \\
 \downarrow & & \downarrow \\
 V_0 & \longrightarrow & Y_0
 \end{array}$$

is an elementary distinguished square. In addition we know that $W_0 \rightarrow U_0$ and $V_0 \rightarrow U_0$ are étale. The morphism $X_0 \subset X$ is a finite order thickening. By the topological invariance of the étale site we can find a unique étale morphism $V \rightarrow X$ with $V_0 = V \times_X X_0$ and we can lift the given morphism $W_0 \rightarrow V_0$ to a unique morphism $W \rightarrow V$. See More on Morphisms of Spaces, Theorem 8.1. By Pushouts of Spaces, Lemma 4.2 we can construct an elementary distinguished square

$$\begin{array}{ccc}
 W & \longrightarrow & Z \\
 \downarrow & & \downarrow \\
 V & \longrightarrow & Y
 \end{array}$$

in the category of algebraic spaces over X . Since the base change of an elementary distinguished square is an elementary distinguished square (Derived Categories of Spaces, Lemma 9.2) and since elementary distinguished squares are pushouts (Pushouts of Spaces, Lemma 4.1) we see that the base change of this diagram by $X_0 \rightarrow X$ gives the previous diagram. It follows that Y is affine by Limits of Spaces, Proposition 15.2. Write $Y = \text{Spec}(B)$. Then B fits into the desired diagram and satisfies all the properties required of it. \square

Second proof. This proof uses a little bit of deformation theory to construct B . By induction on the smallest n such that $J^n = 0$ we reduce to the case $J^2 = 0$. Denote by a subscript zero the base change of objects to $A_0 = A/J$. Since $J^2 = 0$ we see that JC is a C_0 -module.

Consider the canonical map

$$\gamma : J \otimes_{A_0} C_0 \longrightarrow JC$$

Since $\text{Spec}(C) \rightarrow \text{Spec}(A)$ is étale over the complement of $V(I)$ (and hence flat) we see that γ is an isomorphism away from $V(IC_0)$, see More on Morphisms, Lemma

10.1. In particular, the kernel and cokernel of γ are annihilated by a power of I (use that C_0 is Noetherian and that the modules in question are finite). Observe that $J \otimes_{A_0} C_0 = (J \otimes_{A_0} B_0) \otimes_{B_0} C_0$. Hence by More on Algebra, Lemma 78.16 there exists a unique B_0 -module homomorphism

$$c : J \otimes_{A_0} B_0 \rightarrow N$$

with $c \otimes \text{id}_{C_0} = \gamma$ and $\text{Ker}(\gamma) = \text{Ker}(c)$ and $\text{Coker}(\gamma) = \text{Coker}(c)$. Moreover, N is a finite B_0 -module, see More on Algebra, Remark 78.19.

Choose a presentation $B_0 = A[x_1, \dots, x_r]/K$. To construct B we try to find the dotted arrow m fitting into the following pushout diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & N & \xrightarrow{\dots\dots\dots} & B & \xrightarrow{\dots\dots\dots} & B_0 \longrightarrow 0 \\ & & \uparrow \text{ } m & & \uparrow & & \parallel \\ 0 & \longrightarrow & K/K^2 & \longrightarrow & A[x_1, \dots, x_r]/K^2 & \longrightarrow & A[x_1, \dots, x_r]/K \longrightarrow 0 \\ & & \uparrow & & & & \\ & & J \otimes_{A_0} B_0 & & & & \end{array}$$

where the curved arrow is the map c constructed above and the map $J \otimes_{A_0} B_0 \rightarrow K/K^2$ is the obvious one.

As $B_0 \rightarrow C_0$ is étale we can write $C_0 = B_0[y_1, \dots, y_r]/(g_{0,1}, \dots, g_{0,r})$ such that the determinant of the partial derivatives of the $g_{0,j}$ is invertible in C_0 , see Algebra, Lemma 141.2. We combine this with the chosen presentation of B_0 to get a presentation $C_0 = A[x_1, \dots, x_r, y_1, \dots, y_s]/L$. Choose a lift $\psi : A[x_i, y_j] \rightarrow C$ of the map to C_0 . Then it is the case that C fits into the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & JC & \longrightarrow & C & \longrightarrow & C_0 \longrightarrow 0 \\ & & \uparrow \mu & & \uparrow & & \parallel \\ 0 & \longrightarrow & L/L^2 & \longrightarrow & A[x_i, y_j]/L^2 & \longrightarrow & A[x_i, y_j]/L \longrightarrow 0 \\ & & \uparrow & & & & \\ & & J \otimes_{A_0} C_0 & & & & \end{array}$$

where the curved arrow is the map γ constructed above and the map $J \otimes_{A_0} C_0 \rightarrow L/L^2$ is the obvious one. By our choice of presentations and the fact that C_0 is a complete intersection over B_0 we have

$$L/L^2 = K/K^2 \otimes_{B_0} C_0 \oplus \bigoplus C_0 g_j$$

where $g_j \in L$ is any lift of $g_{0,j}$, see More on Algebra, Lemma 30.6.

Consider the three term complex

$$K^\bullet : J \otimes_{A_0} B_0 \rightarrow K/K^2 \rightarrow \bigoplus B_0 dx_i$$

where the second arrow is the differential in the naive cotangent complex of B_0 over A for the given presentation and the last term is placed in degree 0. Since $\text{Spec}(B_0) \rightarrow \text{Spec}(A_0)$ is étale away from $V(I)$ the cohomology modules of this complex are supported on $V(IB_0)$. Namely, for $a \in I$ after inverting a we can apply More on Algebra, Lemma 30.6 for the ring maps $A_a \rightarrow A_{0,a} \rightarrow B_{0,a}$ and

use that $NL_{A_0,a/A_a} = J_a$ and $NL_{B_0,a/A_0,a} = 0$ (some details omitted). Hence these cohomology groups are annihilated by a power of I .

Similarly, consider the three term complex

$$L^\bullet : J \otimes_{A_0} C_0 \rightarrow L/L^2 \rightarrow \bigoplus C_0 dx_i \oplus \bigoplus C_0 dy_j$$

By our direct sum decomposition of L/L^2 above and the fact that the the determinant of the partial derivatives of the $g_{0,j}$ is invertible in C_0 we see that the natural map $K^\bullet \rightarrow L^\bullet$ induces a quasi-isomorphism

$$K^\bullet \otimes_{B_0} C_0 \longrightarrow L^\bullet$$

Applying Dualizing Complexes, Lemma 9.8 we find that

$$0AM9 \quad (7.2.1) \quad \text{Hom}_{D(B_0)}(K^\bullet, E) = \text{Hom}_{D(C_0)}(L^\bullet, E \otimes_{B_0} C_0)$$

for any object $E \in D(B_0)$.

The maps $\text{id}_{J \otimes_{A_0} C_0}$ and μ define an element in

$$\text{Hom}_{D(C_0)}(L^\bullet, (J \otimes_{A_0} C_0 \rightarrow JC))$$

(the target two term complex is placed in degree -2 and -1) such that the composition with the map to $J \otimes_{A_0} C_0[2]$ is the element in $\text{Hom}_{D(C_0)}(L^\bullet, J \otimes_{A_0} C_0[2])$ corresponding to $\text{id}_{J \otimes_{A_0} C_0}$. Picture

$$\begin{array}{ccccc} J \otimes_{A_0} C_0 & \longrightarrow & L/L^2 & \longrightarrow & \bigoplus C_0 dx_i \oplus \bigoplus C_0 dy_j \\ \text{id}_{J \otimes_{A_0} C_0} \downarrow & & \downarrow \mu & & \\ J \otimes_{A_0} C_0 & \xrightarrow{\gamma} & JC & & \end{array}$$

Applying (7.2.1) we obtain a unique element

$$\xi \in \text{Hom}_{D(B_0)}(K^\bullet, (J \otimes_{A_0} B_0 \rightarrow N))$$

Its composition with the map to $J \otimes_{A_0} B_0[2]$ is the element in $\text{Hom}_{D(C_0)}(K^\bullet, J \otimes_{A_0} B_0[2])$ corresponding to $\text{id}_{J \otimes_{A_0} B_0}$. By Lemma 3.4 we can find a map of complexes $K^\bullet \rightarrow (J \otimes_{A_0} B_0 \rightarrow N)$ representing ξ and equal to $\text{id}_{J \otimes_{A_0} B_0}$ in degree -2 . Denote $m : K/K^2 \rightarrow N$ the degree -1 part of this map. Picture

$$\begin{array}{ccccc} J \otimes_{A_0} B_0 & \longrightarrow & K/K^2 & \longrightarrow & \bigoplus B_0 dx_i \\ \text{id}_{J \otimes_{A_0} B_0} \downarrow & & \downarrow m & & \\ J \otimes_{A_0} B_0 & \xrightarrow{c} & N & & \end{array}$$

Thus we can use m to create an algebra B by push out as explained above. However, we may still have to change m a bit to make sure that B maps to C in the correct manner.

Denote $m \otimes \text{id}_{C_0} \oplus 0 : L/L^2 \rightarrow JC$ the map coming from the direct sum decomposition of L/L^2 (see above), using that $N \otimes_{B_0} C_0 = JC$, and using 0 on the second factor. By our choice of m above the maps of complexes $(\text{id}_{J \otimes_{A_0} C_0}, \mu, 0)$ and $(\text{id}_{J \otimes_{A_0} C_0}, m \otimes \text{id}_{C_0} \oplus 0, 0)$ define the same element of $\text{Hom}_{D(C_0)}(L^\bullet, (J \otimes_{A_0} C_0 \rightarrow JC))$. By Lemma 3.4 there exist maps $h : L^{-1} \rightarrow J \otimes_{A_0} C_0$ and $h' : L^0 \rightarrow JC$

which define a homotopy between $(\text{id}_{J \otimes_{A_0} C_0}, \mu, 0)$ and $(\text{id}_{J \otimes_{A_0} C_0}, m \otimes \text{id}_{C_0} \oplus 0, 0)$.
Picture

$$\begin{array}{ccccc}
 J \otimes_{A_0} C_0 & \longrightarrow & K/K^2 \otimes_{B_0} C_0 \oplus \bigoplus C_0 g_j & \longrightarrow & \bigoplus C_0 dx_i \oplus \bigoplus C_0 dy_j \\
 \text{id}_{J \otimes_{A_0} C_0} \downarrow & & \swarrow h & & \downarrow \mu \\
 J \otimes_{A_0} C_0 & \xrightarrow{\gamma} & JC & & \swarrow h' \\
 & & \downarrow m \otimes \text{id}_{C_0} \oplus 0 & & \\
 & & JC & &
 \end{array}$$

Since h precomposed with d_L^{-2} is zero it defines an element in $\text{Hom}_{D(C_0)}(L^\bullet, J \otimes_{A_0} C_0[1])$ which comes from a unique element χ of $\text{Hom}_{D(B_0)}(K^\bullet, J \otimes_{A_0} B_0[1])$ by (7.2.1). Applying Lemma 3.4 again we represent χ by a map $g : K/K^2 \rightarrow J \otimes_{A_0} B_0$. Then the base change $g \otimes \text{id}_{C_0}$ and h differ by a homotopy $h'' : L^0 \rightarrow J \otimes_{A_0} C_0$. Hence if we modify m into $m + c \circ g$, then we find that $m \otimes \text{id}_{C_0} \oplus 0$ and μ just differ by a map $h' : L^0 \rightarrow JC$.

Changing our choice of the map $\psi : A[x_i, y_j] \rightarrow C$ by sending x_i to $\psi(x_i) + h'(dx_i)$ and sending y_j to $\psi(y_j) + h'(dy_j)$, we find a commutative diagram

$$\begin{array}{ccc}
 N & \longrightarrow & JC \\
 \uparrow m & & \uparrow \mu \\
 K/K^2 & \longrightarrow & L/L^2 \\
 \uparrow c & & \uparrow \gamma \\
 J \otimes_{A_0} B_0 & \longrightarrow & J \otimes_{A_0} C_0
 \end{array}$$

At this point we can define B as the pushout in the first commutative diagram of the proof. The commutativity of the diagram just displayed, shows that there is an A -algebra map $B \rightarrow C$ compatible with the given map $N = JB \rightarrow JC$. As $N \otimes_{B_0} C_0 = JC$ it follows from More on Morphisms, Lemma 10.1 that $B \rightarrow C$ is flat. From this it easily follows that it is étale. We omit the proof of the other properties as they are mostly self evident at this point. \square

0AKA **Lemma 7.3.** *Let A be a Noetherian ring. Let $I \subset A$ be an ideal. Let B be an object of (2.0.2). Assume there is an integer $c \geq 0$ such that for $a \in I^c$ multiplication by a on $NL_{B/A}^\wedge$ is zero in $D(B)$. Then there exists a finite type A -algebra C and an isomorphism $B \cong C^\wedge$.*

In Section 8 we will give a simpler proof of this result in case A is a G-ring.

Proof. We prove this lemma by induction on the number of generators of I . Say $I = (a_1, \dots, a_t)$. If $t = 0$, then $I = 0$ and there is nothing to prove. If $t = 1$, then the lemma follows from Lemma 6.4. Assume $t > 1$.

For any $m \geq 1$ set $\bar{A}_m = A/(a_t^m)$. Consider the ideal $\bar{I}_m = (\bar{a}_1, \dots, \bar{a}_{t-1})$ in \bar{A}_m . Let $B_m = B/(a_t^m)$ be the base change of B for the map $(A, I) \rightarrow (\bar{A}_m, \bar{I}_m)$, see (2.4.1). By Lemma 4.3 the assumption of the lemma holds for $\bar{I}_m \subset \bar{A}_m \rightarrow B_m$.

By induction hypothesis (on t) we can find a finite type \bar{A}_m -algebra C_m and a map $C_m \rightarrow B_m$ which induces an isomorphism $C_m^\wedge \cong B_m$ where the completion is with respect to \bar{I}_m . By Lemma 4.2 we may assume that $\text{Spec}(C_m) \rightarrow \text{Spec}(\bar{A}_m)$ is étale over $\text{Spec}(\bar{A}_m) \setminus V(\bar{I}_m)$.

We claim that we may choose $A_m \rightarrow C_m \rightarrow B_m$ as in the previous paragraph such that moreover there are isomorphisms $C_m/(a_t^{m-1}) \rightarrow C_{m-1}$ compatible with the given A -algebra structure and the maps to $B_{m-1} = B_m/(a_t^{m-1})$. Namely, first fix a choice of $A_1 \rightarrow C_1 \rightarrow B_1$. Suppose we have found $C_{m-1} \rightarrow C_{m-2} \rightarrow \dots \rightarrow C_1$ with the desired properties. Note that $C_m/(a_t^{m-1})$ is étale over $\text{Spec}(\bar{A}_{m-1}) \setminus V(\bar{I}_{m-1})$. Hence by Lemma 4.4 there exists an étale extension $C_{m-1} \rightarrow C'_{m-1}$ which induces an isomorphism modulo \bar{I}_{m-1} and an \bar{A}_{m-1} -algebra map $C_m/(a_t^{m-1}) \rightarrow C'_{m-1}$ inducing the isomorphism $B_m/(a_t^{m-1}) \rightarrow B_{m-1}$ on completions. Note that $C_m/(a_t^{m-1}) \rightarrow C'_{m-1}$ is étale over the complement of $V(\bar{I}_{m-1})$ by Morphisms, Lemma 34.18 and over $V(\bar{I}_{m-1})$ induces an isomorphism on completions hence is étale there too (for example by More on Morphisms, Lemma 12.3). Thus $C_m/(a_t^{m-1}) \rightarrow C'_{m-1}$ is étale. By the topological invariance of étale morphisms (Étale Morphisms, Theorem 15.2) there exists an étale ring map $C_m \rightarrow C'_m$ such that $C_m/(a_t^{m-1}) \rightarrow C'_{m-1}$ is isomorphic to $C_m/(a_t^{m-1}) \rightarrow C'_m/(a_t^{m-1})$. Observe that the \bar{I}_m -adic completion of C'_m is equal to the \bar{I}_m -adic completion of C_m , i.e., to B_m (details omitted). We apply Lemma 7.2 to the diagram

$$\begin{array}{ccc}
 & C'_m & \longrightarrow & C'_m/(a_t^{m-1}) \\
 & \nearrow & & \uparrow \\
 C''_m & \xrightarrow{\quad} & C_{m-1} & \\
 & \nwarrow & & \uparrow \\
 & \bar{A}_m & \longrightarrow & \bar{A}_{m-1}
 \end{array}$$

to see that there exists a “lift” of C''_m of C_{m-1} to an algebra over \bar{A}_m with all the desired properties.

By construction (C_m) is an object of the category (2.0.1) for the principal ideal (a_t) . Thus the inverse limit $B' = \varprojlim C_m$ is an (a_t) -adically complete A -algebra such that $B'/a_t B'$ is of finite type over $A/(a_t)$, see Lemma 2.1. By construction the I -adic completion of B' is isomorphic to B (details omitted). Consider the complex $NL_{B'/A}^\wedge$ constructed using the (a_t) -adic topology. Choosing a presentation for B' (which induces a similar presentation for B) the reader immediately sees that $NL_{B'/A}^\wedge \otimes_{B'} B = NL_{B/A}^\wedge$. Since $a_t \in I$ and since the cohomology modules of $NL_{B'/A}^\wedge$ are finite B' -modules (hence complete for the a_t -adic topology), we conclude that a_t^c acts as zero on these cohomologies as the same thing is true by assumption for $NL_{B/A}^\wedge$. Thus multiplication by a_t^{2c} is zero on $NL_{B'/A}^\wedge$ by Lemma 3.5. Hence finally, we may apply Lemma 6.4 to $(a_t) \subset A \rightarrow B'$ to finish the proof. \square

0AKG **Lemma 7.4.** *Let A be a Noetherian ring. Let $I \subset A$ be an ideal. Let B be an I -adically complete A -algebra with $A/I \rightarrow B/IB$ of finite type. The equivalent conditions of Lemma 4.1 are also equivalent to*

0AKH (5) *there exists a finite type A -algebra C with $\text{Spec}(C) \rightarrow \text{Spec}(A)$ is étale over $\text{Spec}(A) \setminus V(I)$ such that $B \cong C^\wedge$.*

Proof. First, assume conditions (1) – (4) hold. Then there exists a finite type A -algebra C with such that $B \cong C^\wedge$ by Lemma 7.3. In other words, $B_n = C/I^n C$. The naive cotangent complex $NL_{C/A}$ is a complex of finite type C -modules and

hence H^{-1} and H^0 are finite C -modules. By assumption there exists a $c \geq 0$ such that $H^{-1}/I^n H^{-1}$ and $H^0/I^n H^0$ are annihilated by I^c for some n . By Nakayama's lemma this means that $I^c H^{-1}$ and $I^c H^0$ are annihilated by an element of the form $f = 1 + x$ with $x \in IC$. After inverting f (which does not change the quotients $B_n = C/I^n C$) we see that $NL_{C/A}$ has cohomology annihilated by I^c . Thus $A \rightarrow C$ is étale at any prime of C not lying over $V(I)$ by the definition of étale ring maps, see Algebra, Definition 141.1.

Conversely, assume that $A \rightarrow C$ of finite type is given such that $\text{Spec}(C) \rightarrow \text{Spec}(A)$ is étale over $\text{Spec}(A) \setminus V(I)$. Then for every $a \in I$ there exists an $c \geq 0$ such that multiplication by a^c is zero $NL_{C/A}$. Since $NL_{C^\wedge/A}$ is the derived completion of $NL_{C/A}$ (see Lemma 3.1) it follows that $B = C^\wedge$ satisfies the equivalent conditions of Lemma 4.1. \square

8. In case the base ring is a G-ring

0ALU If the base ring A is a Noetherian G-ring, then some of the material above simplifies somewhat and we obtain some additional results.

Proof of Lemma 7.3 in case A is a G-ring. This proof is easier in that it does not depend on the somewhat delicate deformation theory argument given in the proof of Lemma 7.2, but of course it requires a very strong assumption on the Noetherian ring A .

Choose a presentation $B = A[x_1, \dots, x_r]^\wedge/J$. Choose generators $g_1, \dots, g_m \in J$. Choose generators k_1, \dots, k_t of the module of relations between g_1, \dots, g_m , i.e., such that

$$(A[x_1, \dots, x_r]^\wedge)^{\oplus t} \xrightarrow{k_1, \dots, k_t} (A[x_1, \dots, x_r]^\wedge)^{\oplus m} \xrightarrow{g_1, \dots, g_m} A[x_1, \dots, x_r]^\wedge$$

is exact in the middle. Write $k_i = (k_{i1}, \dots, k_{im})$ so that we have

$$0AKB \quad (8.0.1) \quad \sum k_{ij} g_j = 0$$

for $i = 1, \dots, t$. Let $I^c = (a_1, \dots, a_s)$. For each $l \in \{1, \dots, s\}$ we know that multiplication by a_l on $NL_{B/A}^\wedge$ is zero in $D(B)$. By Lemma 3.4 we can find a map $\alpha_l : \bigoplus B dx_i \rightarrow J/J^2$ such that $d \circ \alpha_l$ and $\alpha_l \circ d$ are both multiplication by a_l . Pick an element $f_{l,i} \in J$ whose class modulo J^2 is equal to $\alpha_l(dx_i)$. Then we have for all $l = 1, \dots, s$ and $i = 1, \dots, r$ that

$$0AKC \quad (8.0.2) \quad \sum_{i'} (\partial f_{l,i} / \partial x_{i'}) dx_{i'} = a_l dx_i + \sum h_{l,i}^{j',i'} g_{j'} dx_{i'}$$

for some $h_{l,i}^{j',i'} \in A[x_1, \dots, x_r]^\wedge$. We also have for $j = 1, \dots, m$ and $l = 1, \dots, s$ that

$$0AKD \quad (8.0.3) \quad a_l g_j = \sum h_{l,j}^i f_{l,i} + \sum h_{l,j}^{j'',j''} g_{j'} g_{j''}$$

for some $h_{l,j}^i$ and $h_{l,j}^{j'',j''}$ in $A[x_1, \dots, x_r]^\wedge$. Of course, since $f_{l,i} \in J$ we can write for $l = 1, \dots, s$ and $i = 1, \dots, r$

$$0AKE \quad (8.0.4) \quad f_{l,i} = \sum h_{l,i}^j g_j$$

for some $h_{l,i}^j$ in $A[x_1, \dots, x_r]^\wedge$.

Let $A[x_1, \dots, x_r]^h$ be the henselization of the pair $(A[x_1, \dots, x_r], IA[x_1, \dots, x_r])$, see More on Algebra, Lemma 10.14. Since A is a Noetherian G-ring, so is $A[x_1, \dots, x_r]$,

see More on Algebra, Proposition 47.10. Hence we have approximation for the map $A[x_1, \dots, x_r]^h \rightarrow A[x_1, \dots, x_r]^\wedge$ with respect to the ideal generated by I , see Smoothing Ring Maps, Lemma 14.1. Choose a large integer M . Choose

$$G_j, K_{ij}, F_{l,i}, H_{l,j}^i, H_{l,j}^{j',j''}, H_{l,i}^j \in A[x_1, \dots, x_r]^h$$

such that analogues of equations (8.0.1), (8.0.3), and (8.0.4) hold for these elements in $A[x_1, \dots, x_r]^h$, i.e.,

$$\sum K_{ij} G_j = 0, \quad a_l G_j = \sum H_{l,j}^i F_{l,i} + \sum H_{l,j}^{j',j''} G_{j'} G_{j''}, \quad F_{l,i} = \sum H_{l,i}^j G_j$$

and such that we have

$$G_j - g_j, K_{ij} - k_{ij}, F_{l,i} - f_{l,i}, H_{l,j}^i - h_{l,j}^i, H_{l,j}^{j',j''} - h_{l,j}^{j',j''}, H_{l,i}^j - h_{l,i}^j \in I^M A[x_1, \dots, x_r]^h$$

where we take liberty of thinking of $A[x_1, \dots, x_r]^h$ as a subring of $A[x_1, \dots, x_r]^\wedge$. Note that we cannot guarantee that the analogue of (8.0.2) holds in $A[x_1, \dots, x_r]^h$, because it is not a polynomial equation. But since taking partial derivatives is A -linear, we do get the analogue modulo I^M . More precisely, we see that

$$\text{OAKF} \quad (8.0.5) \quad \sum_{i'} (\partial F_{l,i} / \partial x_{i'}) dx_{i'} - a_l dx_i - \sum h_{l,i}^{j',i'} G_{j'} dx_{i'} \in I^M A[x_1, \dots, x_r]^\wedge$$

for $l = 1, \dots, s$ and $i = 1, \dots, r$.

With these choices, consider the ring

$$C^h = A[x_1, \dots, x_r]^h / (G_1, \dots, G_r)$$

and denote C^\wedge its I -adic completion, namely

$$C^\wedge = A[x_1, \dots, x_r]^\wedge / J', \quad J' = (G_1, \dots, G_r) A[x_1, \dots, x_r]^\wedge$$

In the following paragraphs we establish the fact that C^\wedge is isomorphic to B . Then in the final paragraph we deal with show that C^h comes from a finite type algebra over A as in the statement of the lemma.

First consider the cokernel

$$\Omega = \text{Coker}(J' / (J')^2 \rightarrow \bigoplus C^\wedge dx_i)$$

This C^\wedge module is generated by the images of the elements dx_i . Since $F_{l,i} \in J'$ by the analogue of (8.0.4) we see from (8.0.5) we see that $a_l dx_i \in I^M \Omega$. As $I^c = (a_l)$ we see that $I^c \Omega \subset I^M \Omega$. Since $M > c$ we conclude that $I^c \Omega = 0$ by Algebra, Lemma 19.1.

Next, consider the kernel

$$H_1 = \text{Ker}(J' / (J')^2 \rightarrow \bigoplus C^\wedge dx_i)$$

By the analogue of (8.0.3) we see that $a_l J' \subset (F_{l,i}) + (J')^2$. On the other hand, the determinant Δ_l of the matrix $(\partial F_{l,i} / \partial x_{i'})$ satisfies $\Delta_l = a_l^r \text{ mod } I^M C^\wedge$ by (8.0.5). It follows that $a_l^{r+1} H_1 \subset I^M H_1$ (some details omitted; use Algebra, Lemma 14.4). Now $(a_l^{r+1}, \dots, a_s^{r+1}) \supset I^{(sr+1)c}$. Hence $I^{(sr+1)c} H_1 \subset I^M H_1$ and since $M > (sr+1)c$ we conclude that $I^{(sr+1)c} H_1 = 0$.

By Lemma 3.5 we conclude that multiplication by an element of $I^{2(sr+1)c}$ on $NL_{C^\wedge/A}^\wedge$ is zero (note that the bound does not depend on M or the choice of

the approximation, as long as M is large enough). Since $G_j - g_j$ is in the ideal generated by I^M we see that there is an isomorphism

$$\psi_M : C^\wedge / I^M C^\wedge \rightarrow B / I^M B$$

As M is large enough we can use Lemma 7.1 with $d = d(I \subset A \rightarrow B)$, with C^\wedge playing the role of B , with $2(rs+1)c$ instead of c , to find a morphism

$$\psi : C^\wedge \rightarrow B$$

which agrees with ψ_M modulo $I^{q-2(rs+1)c}$ where q is the quotient of M by the number of generators of I . We claim ψ is an isomorphism. Since C^\wedge and B are I -adically complete the map ψ is surjective because it is surjective modulo I (see Algebra, Lemma 95.1). On the other hand, as M is large enough we see that

$$\mathrm{Gr}_I(C^\wedge) \cong \mathrm{Gr}_I(B)$$

as graded $\mathrm{Gr}_I(A[x_1, \dots, x_r]^\wedge)$ -modules by More on Algebra, Lemma 4.2. Since ψ is compatible with this isomorphism as it agrees with ψ_M modulo I , this means that $\mathrm{Gr}_I(\psi)$ is an isomorphism. As C^\wedge and B are I -adically complete, it follows that ψ is an isomorphism.

This paragraph serves to deal with the issue that C^h is not of finite type over A . Namely, the ring $A[x_1, \dots, x_r]^h$ is a filtered colimit of étale $A[x_1, \dots, x_r]$ algebras A' such that $A/I[x_1, \dots, x_r] \rightarrow A'/IA'$ is an isomorphism (see proof of More on Algebra, Lemma 10.14). Pick an A' such that G_1, \dots, G_m are the images of $G'_1, \dots, G'_m \in A'$. Setting $C = A'/(G'_1, \dots, G'_m)$ we get the finite type algebra we were looking for. \square

The following lemma isn't true in general if A is not a G-ring but just Noetherian. Namely, if (A, \mathfrak{m}) is local and $I = \mathfrak{m}$, then the lemma is equivalent to Artin approximation for A^h (as in Smoothing Ring Maps, Theorem 13.1) which does not hold for every Noetherian local ring.

0AK4 **Lemma 8.1.** *Let A be a Noetherian G-ring. Let $I \subset A$ be an ideal. Let B, C be finite type A -algebras. For any A -algebra map $\varphi : B^\wedge \rightarrow C^\wedge$ of I -adic completions and any $N \geq 1$ there exist*

- (1) *an étale ring map $C \rightarrow C'$ which induces an isomorphism $C/IC \rightarrow C'/IC'$,*
- (2) *an A -algebra map $\varphi : B \rightarrow C'$*

such that φ and ψ agree modulo I^N into $C^\wedge = (C')^\wedge$.

Proof. The statement of the lemma makes sense as $C \rightarrow C'$ is flat (Algebra, Lemma 141.3) hence induces an isomorphism $C/I^n C \rightarrow C'/I^n C'$ for all n (More on Algebra, Lemma 78.2) and hence an isomorphism on completions. Let C^h be the henselization of the pair (C, IC) , see More on Algebra, Lemma 10.14. Then C^h is the filtered colimit of the algebras C' and the maps $C \rightarrow C' \rightarrow C^h$ induce isomorphism on completions (More on Algebra, Lemma 10.17). Thus it suffices to prove there exists an A -algebra map $B \rightarrow C^h$ which is congruent to ψ modulo I^N . Write $B = A[x_1, \dots, x_n]/(f_1, \dots, f_m)$. The ring map ψ corresponds to elements $\hat{c}_1, \dots, \hat{c}_n \in C^\wedge$ with $f_j(\hat{c}_1, \dots, \hat{c}_n) = 0$ for $j = 1, \dots, m$. Namely, as A is a Noetherian G-ring, so is C , see More on Algebra, Proposition 47.10. Thus Smoothing Ring Maps, Lemma 14.1 applies to give elements $c_1, \dots, c_n \in C^h$ such that $f_j(c_1, \dots, c_n) = 0$ for $j = 1, \dots, m$ and such that $\hat{c}_i - c_i \in I^N C^h$. This determines the map $B \rightarrow C^h$ as desired. \square

9. Rig-surjective morphisms

0AQP For morphisms locally of finite type between locally Noetherian formal algebraic spaces a definition borrowed from [Art70] can be used. See Remark 9.10 for a discussion of what to do in more general cases.

0AQQ **Definition 9.1.** Let S be a scheme. Let $f : X \rightarrow Y$ be a morphism of formal algebraic spaces over S . Assume that X and Y are locally Noetherian and that f is locally of finite type. We say f is *rig-surjective* if for every solid diagram

$$\begin{array}{ccc} \mathrm{Spf}(R') & \cdots\cdots\cdots & X \\ \downarrow & & \downarrow f \\ \mathrm{Spf}(R) & \xrightarrow{p} & Y \end{array}$$

where R is a complete discrete valuation ring and where p is an adic morphism there exists an extension of complete discrete valuation rings $R \subset R'$ and a morphism $\mathrm{Spf}(R') \rightarrow X$ making the displayed diagram commute.

We prove a few lemmas to explain what this means.

0AQR **Lemma 9.2.** *Let S be a scheme. Let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ be morphisms of formal algebraic spaces over S . Assume X, Y, Z are locally Noetherian and f and g locally of finite type. Then if f and g are rig-surjective, so is $g \circ f$.*

Proof. Follows in a straightforward manner from the definitions (and Formal Spaces, Lemma 18.3). \square

0AQS **Lemma 9.3.** *Let S be a scheme. Let $f : X \rightarrow Y$ and $Z \rightarrow Y$ be morphisms of formal algebraic spaces over S . Assume X, Y, Z are locally Noetherian and f and g locally of finite type. If f is rig-surjective, then the base change $Z \times_Y X \rightarrow Z$ is too.*

Proof. Follows in a straightforward manner from the definitions (and Formal Spaces, Lemmas 18.9 and 18.4). \square

0AQT **Lemma 9.4.** *Let S be a scheme. Let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ be morphisms of formal algebraic spaces over S . Assume X, Y, Z locally Noetherian and f and g locally of finite type. If $g \circ f : X \rightarrow Z$ is rig-surjective, so is $g : Y \rightarrow Z$.*

Proof. Immediate from the definition. \square

0AQU **Lemma 9.5.** *Let S be a scheme. Let $f : X \rightarrow Y$ be a morphism of formal algebraic spaces which is representable by algebraic spaces, étale, and surjective. Assume X and Y locally Noetherian. Then f is rig-surjective.*

Proof. Let $p : \mathrm{Spf}(R) \rightarrow Y$ be an adic morphism where R is a complete discrete valuation ring. Let $Z = \mathrm{Spf}(R) \times_Y X$. Then $Z \rightarrow \mathrm{Spf}(R)$ is representable by algebraic spaces, étale, and surjective. Hence Z is nonempty. Pick a nonempty affine formal algebraic space V and an étale morphism $V \rightarrow Z$ (possible by our definitions). Then $V \rightarrow \mathrm{Spf}(R)$ corresponds to $R \rightarrow A^\wedge$ where $R \rightarrow A$ is an étale ring map, see Formal Spaces, Lemma 14.13. Since $A^\wedge \neq 0$ (as $V \neq \emptyset$) we can find a maximal ideal \mathfrak{m} of A lying over \mathfrak{m}_R . Then $A_\mathfrak{m}$ is a discrete valuation ring (More on Algebra, Lemma 41.4). Then $R' = A_\mathfrak{m}^\wedge$ is a complete discrete valuation ring (More on Algebra, Lemma 40.5). Applying Formal Spaces, Lemma 5.10. we find the desired morphism $\mathrm{Spf}(R') \rightarrow V \rightarrow Z \rightarrow X$. \square

0AQV **Remark 9.6.** Let S be a scheme. Let $f : X \rightarrow Y$ be a morphism of locally Noetherian formal algebraic spaces which is locally of finite type. The upshot of the lemmas above is that we may check whether $f : X \rightarrow Y$ is rig-surjective, étale locally on Y . For example, suppose that $\{Y_i \rightarrow Y\}$ is a covering as in Formal Spaces, Definition 7.1. Then f is rig-surjective if and only if $f_i : X \times_Y Y_i \rightarrow Y_i$ is rig-surjective. Namely, if f is rig-surjective, so is any base change (Lemma 9.3). Conversely, if all f_i are rig-surjective, so is $\coprod f_i : \coprod X \times_Y Y_i \rightarrow \coprod Y_i$. By Lemma 9.5 the morphism $\coprod Y_i \rightarrow Y$ is rig-surjective. Hence $\coprod X \times_Y Y_i \rightarrow Y$ is rig-surjective (Lemma 9.2). Since this morphism factors through $X \rightarrow Y$ we see that $X \rightarrow Y$ is rig-surjective by Lemma 9.4.

0AQW **Lemma 9.7.** *Let S be a scheme. Let $f : X \rightarrow Y$ be a proper surjective morphism of locally Noetherian algebraic spaces over S . Let $T \subset |Y|$ be a closed subset and let $T' = |f|^{-1}(T) \subset |X|$. Then $X_{/T'} \rightarrow Y_{/T}$ is rig-surjective.*

Proof. The statement makes sense by Formal Spaces, Lemmas 15.6 and 18.10. Let $Y_j \rightarrow Y$ be a jointly surjective family of étale morphism where Y_j is an affine scheme for each j . Denote $T_j \subset Y_j$ the inverse image of T . Then $\{(Y_j)_{/T_j} \rightarrow Y_{/T}\}$ is a covering as in Formal Spaces, Definition 7.1. Moreover, setting $X_j = Y_j \times_Y X$ and $T'_j \subset |X_j|$ the inverse image of T , we have

$$(X_j)_{/T'_j} = (Y_j)_{/T_j} \times_{(Y_{/T})} X_{/T'}$$

By the discussion in Remark 9.6 we reduce to the case where Y is an affine Noetherian scheme treated in the next paragraph.

Assume $Y = \text{Spec}(A)$ where A is a Noetherian ring. This implies that $Y_{/T} = \text{Spf}(A^\wedge)$ where A^\wedge is the I -adic completion of A for some ideal $I \subset A$. Let $p : \text{Spf}(R) \rightarrow \text{Spf}(A^\wedge)$ be an adic morphism where R is a complete discrete valuation ring. Let K be the field of fractions of R . Consider the composition $A \rightarrow A^\wedge \rightarrow R$. Since $X \rightarrow Y$ is surjective, the fibre $X_K = \text{Spec}(K) \times_Y X$ is nonempty. Thus we may choose an affine scheme U and an étale morphism $U \rightarrow X$ such that U_K is nonempty. Let $u \in U_K$ be a closed point (possible as U_K is affine). By Morphisms, Lemma 19.3 the residue field $L = \kappa(u)$ is a finite extension of K . Let $R' \subset L$ be the integral closure of R in L . By More on Algebra, Remark 93.6 we see that R' is a discrete valuation ring. Because $X \rightarrow Y$ is proper we see that the given morphism $\text{Spec}(L) = u \rightarrow U_K \rightarrow X_K \rightarrow X$ extends to a morphism $\text{Spec}(R') \rightarrow X$ over the given morphism $\text{Spec}(R) \rightarrow Y$ (Morphisms of Spaces, Lemma 43.1). By commutativity of the diagram the induced morphisms $\text{Spec}(R'/\mathfrak{m}_{R'}^n) \rightarrow X$ are points of $X_{/T'}$ and we find

$$\text{Spf}((R')^\wedge) = \text{colim } \text{Spec}(R'/\mathfrak{m}_{R'}^n) \longrightarrow X_{/T'}$$

as desired (note that $(R')^\wedge$ is a complete discrete valuation ring by More on Algebra, Lemma 40.5; in fact in the current situation $R' = (R')^\wedge$ but we do not need this). \square

0AQX **Lemma 9.8.** *Let A be a Noetherian ring complete with respect to an ideal I . Let B be an I -adically complete A -algebra. If $A/I^n \rightarrow B/I^n B$ is of finite type and flat for all n and faithfully flat for $n = 1$, then $\text{Spf}(B) \rightarrow \text{Spf}(A)$ is rig-surjective.*

Proof. We will use without further mention that morphisms between formal spectra are given by continuous maps between the corresponding topological rings, see Formal Spaces, Lemma 5.10. Let $\varphi : A \rightarrow R$ be a continuous map into a complete

discrete valuation ring A . This implies that $\varphi(I) \subset \mathfrak{m}_R$. On the other hand, since we only need to produce the lift $\varphi' : B' \rightarrow R'$ in the case that φ corresponds to an adic morphism, we may assume that $\varphi(I) \neq 0$. Thus we may consider the base change $C = B \widehat{\otimes}_A R$, see Remark 2.3 for example. Then C is an \mathfrak{m}_R -adically complete R -algebra such that $C/\mathfrak{m}_R^n C$ is of finite type and flat over R/\mathfrak{m}_R^n and such that $C/\mathfrak{m}_R C$ is nonzero. Pick any maximal ideal $\mathfrak{m} \subset C$ lying over \mathfrak{m}_R . By flatness (which implies going down) we see that $\text{Spec}(C_{\mathfrak{m}}) \setminus V(\mathfrak{m}_R C_{\mathfrak{m}})$ is a nonempty open. Hence We can pick a prime $\mathfrak{q} \subset \mathfrak{m}$ such that \mathfrak{q} defines a closed point of $\text{Spec}(C_{\mathfrak{m}}) \setminus \{\mathfrak{m}\}$ and such that $\mathfrak{q} \not\subset V(IC_{\mathfrak{m}})$, see Properties, Lemma 6.4. Then C/\mathfrak{q} is a dimension 1-local domain and we can find $C/\mathfrak{q} \subset R'$ with R' a discrete valuation ring (Algebra, Lemma 118.13). By construction $\mathfrak{m}_R R' \subset \mathfrak{m}_{R'}$ and we see that $C \rightarrow R'$ extends to a continuous map $C \rightarrow (R')^\wedge$ (in fact we can pick R' such that $R' = (R')^\wedge$ in our current situation but we do not need this). Since the completion of a discrete valuation ring is a discrete valuation ring, we see that the assumption gives a commutative diagram of rings

$$\begin{array}{ccccc} (R')^\wedge & \longleftarrow & C & \longleftarrow & B \\ \uparrow & & \uparrow & & \uparrow \\ R & \longleftarrow & R & \longleftarrow & A \end{array}$$

which gives the desired lift. □

0AQY **Lemma 9.9.** *Let A be a Noetherian ring complete with respect to an ideal I . Let B be an I -adically complete A -algebra. Assume that*

- (1) *the I -torsion in A is 0,*
- (2) *$A/I^n \rightarrow B/I^n B$ is flat and of finite type for all n .*

Then $\text{Spf}(B) \rightarrow \text{Spf}(A)$ is rig-surjective if and only if $A/I \rightarrow B/IB$ is faithfully flat.

Proof. Faithful flatness implies rig-surjectivity by Lemma 9.8. To prove the converse we will use without further mention that the vanishing of I -torsion is equivalent to the vanishing of I -power torsion (More on Algebra, Lemma 77.3). We will also use without further mention that morphisms between formal spectra are given by continuous maps between the corresponding topological rings, see Formal Spaces, Lemma 5.10.

Assume $\text{Spf}(B) \rightarrow \text{Spf}(A)$ is rig-surjective. Choose a maximal ideal $I \subset \mathfrak{m} \subset A$. The open $U = \text{Spec}(A_{\mathfrak{m}}) \setminus V(I_{\mathfrak{m}})$ of $\text{Spec}(A_{\mathfrak{m}})$ is nonempty as the $I_{\mathfrak{m}}$ -torsion of $A_{\mathfrak{m}}$ is zero (use Algebra, Lemma 61.4). Thus we can find a prime $\mathfrak{q} \subset A_{\mathfrak{m}}$ which defines a point of U (i.e., $IA_{\mathfrak{m}} \not\subset \mathfrak{q}$) and which corresponds to a closed point of $\text{Spec}(A_{\mathfrak{m}}) \setminus \{\mathfrak{m}\}$, see Properties, Lemma 6.4. Then $A_{\mathfrak{m}}/\mathfrak{q}$ is a dimension 1 local domain. Thus we can find an injective local homomorphism of local rings $A_{\mathfrak{m}}/\mathfrak{q} \subset R$ where R is a discrete valuation ring (Algebra, Lemma 118.13). By construction $IR \subset \mathfrak{m}_R$ and we see that $A \rightarrow R$ extends to a continuous map $A \rightarrow R^\wedge$. Since the completion of a discrete valuation ring is a discrete valuation ring, we see that the assumption

gives a commutative diagram of rings

$$\begin{array}{ccc} R' & \longleftarrow & B \\ \uparrow & & \uparrow \\ R^\wedge & \longleftarrow & A \end{array}$$

Thus we find a prime ideal of B lying over \mathfrak{m} . It follows that $\mathrm{Spec}(B/IB) \rightarrow \mathrm{Spec}(A/I)$ is surjective, whence $A/I \rightarrow B/IB$ is faithfully flat (Algebra, Lemma 38.16). \square

0AQZ **Remark 9.10.** The condition as formulated in Definition 9.1 is not right for morphisms of locally adic* formal algebraic spaces. For example, if $A = (\bigcup_{n \geq 1} k[t^{1/n}])^\wedge$ where the completion is the t -adic completion, then there are no adic morphisms $\mathrm{Spf}(R) \rightarrow \mathrm{Spf}(A)$ where R is a complete discrete valuation ring. Thus any morphism $X \rightarrow \mathrm{Spf}(A)$ would be rig-surjective, but since A is a domain and $t \in A$ is not zero, we want to think of A as having at least one “rig-point”, and we do not want to allow $X = \emptyset$. To cover this particular case, one can consider adic morphisms

$$\mathrm{Spf}(R) \longrightarrow Y$$

where R is a valuation ring complete with respect to a principal ideal J whose radical is $\mathfrak{m}_R = \sqrt{J}$. In this case the value group of R can be embedded into $(\mathbf{R}, +)$ and one obtains the point of view used by Berkovich in defining an analytic space associated to Y , see [Ber90]. Another approach is championed by Huber. In his theory, one drops the hypothesis that $\mathrm{Spec}(R/J)$ is a singleton, see [Hub93].

0AR0 **Lemma 9.11.** *Let S be a scheme. Let $f : X \rightarrow Y$ be a morphism of formal algebraic spaces. Assume X and Y are locally Noetherian, f locally of finite type, and f a monomorphism. Then f is rig surjective if and only if every adic morphism $\mathrm{Spf}(R) \rightarrow Y$ where R is a complete discrete valuation ring factors through X .*

Proof. One direction is trivial. For the other, suppose that $\mathrm{Spf}(R) \rightarrow Y$ is an adic morphism such that there exists an extension of complete discrete valuation rings $R \subset R'$ with $\mathrm{Spf}(R') \rightarrow \mathrm{Spf}(R) \rightarrow X$ factoring through Y . Then $\mathrm{Spec}(R'/\mathfrak{m}_R^n R') \rightarrow \mathrm{Spec}(R/\mathfrak{m}_R^n)$ is surjective and flat, hence the morphisms $\mathrm{Spec}(R/\mathfrak{m}_R^n) \rightarrow X$ factor through X as X satisfies the sheaf condition for fpqc coverings, see Formal Spaces, Lemma 25.1. In other words, $\mathrm{Spf}(R) \rightarrow Y$ factors through X . \square

10. Algebraization

0AR1 In this section we prove a generalization of the result on dilatations from the paper of Artin [Art70]. We first reformulate the algebra results proved above into the language of formal algebraic spaces.

Let S be a scheme. Let V be a locally Noetherian formal algebraic space over S . We denote \mathcal{C}_V the category of formal algebraic spaces W over V such that the structure morphism $W \rightarrow V$ is rig-étale.

Let S be a scheme. Let X be an algebraic space over S . Let $T \subset |X|$ be a closed subset. Recall that $X_{/T}$ denotes the formal completion of X along T , see Formal Spaces, Section 9. More generally, for any algebraic space Y over X we denote $Y_{/T}$ the completion of Y along the inverse image of T in $|Y|$, so that $Y_{/T}$ is a formal algebraic space over $X_{/T}$.

0AR2 **Lemma 10.1.** *Let S be a scheme. Let X be a locally Noetherian algebraic space over S . Let $T \subset |X|$ be a closed subset. If $Y \rightarrow X$ is morphism of algebraic spaces which is locally of finite type and étale over $X \setminus T$, then $Y_{/T} \rightarrow X_{/T}$ is rig-étale, i.e., $Y_{/T}$ is an object of $\mathcal{C}_{X_{/T}}$ defined above.*

Proof. Choose a surjective étale morphism $U \rightarrow X$ with $U = \coprod U_i$ a disjoint union of affine schemes, see Properties of Spaces, Lemma 6.1. For each i choose a surjective étale morphism $V_i \rightarrow Y \times_X U_i$ where $V_i = \coprod V_{ij}$ is a disjoint union of affines. Write $U_i = \text{Spec}(A_i)$ and $V_{ij} = \text{Spec}(B_{ij})$. Let $I_i \subset A_i$ be an ideal cutting out the inverse image of T in U_i . Then we may apply Lemma 4.2 to see that the map of I_i -adic completions $A_i^\wedge \rightarrow B_{ij}^\wedge$ has the property P of Lemma 5.1. Since $\{\text{Spf}(A_i^\wedge) \rightarrow X_{/T}\}$ and $\{\text{Spf}(B_{ij}^\wedge) \rightarrow Y_{/T}\}$ are coverings as in Formal Spaces, Definition 7.1 we see that $Y_{/T} \rightarrow X_{/T}$ is rig-étale by definition. \square

0AR3 **Lemma 10.2.** *Let X be a Noetherian affine scheme. Let $T \subset X$ be a closed subset. Let U be an affine scheme and let $U \rightarrow X$ a finite type morphism étale over $X \setminus T$. Let V be a Noetherian affine scheme over X . For any morphism $c' : V_{/T} \rightarrow U_{/T}$ over $X_{/T}$ there exists an étale morphism $b : V' \rightarrow V$ of affine schemes which induces an isomorphism $b_{/T} : V'_{/T} \rightarrow V_{/T}$ and a morphism $a : V' \rightarrow U$ such that $c' = a_{/T} \circ b_{/T}^{-1}$.*

Proof. This is a reformulation of Lemma 4.4. \square

0AR4 **Lemma 10.3.** *Let X be a Noetherian affine scheme. Let $T \subset X$ be a closed subset. Let $W \rightarrow X_{/T}$ be a rig-étale morphism of formal algebraic spaces with W an affine formal algebraic space. Then there exists an affine scheme U , a finite type morphism $U \rightarrow X$ étale over $X \setminus T$ such that $W \cong U_{/T}$. Moreover, if $W \rightarrow X_{/T}$ is étale, then $U \rightarrow X$ is étale.*

Proof. The existence of U is a restatement of Lemma 7.4. The final statement follows from More on Morphisms, Lemma 12.3. \square

Let S be a scheme. Let X be a locally Noetherian algebraic space over S and let $T \subset |X|$ be a closed subset. Let us denote $\mathcal{C}_{X,T}$ the category of algebraic spaces Y over X such that the structure morphism $f : Y \rightarrow X$ is locally of finite type and an isomorphism over the complement of T . Formal completion defines a functor

$$0AR5 \quad (10.3.1) \quad F_{X,T} : \mathcal{C}_{X,T} \longrightarrow \mathcal{C}_{X_{/T}}, \quad (f : Y \rightarrow X) \longmapsto (f_{/T} : Y_{/T} \rightarrow X_{/T})$$

see Lemma 10.1.

0AR6 **Lemma 10.4.** *Let S be a scheme. Let $f : X \rightarrow Y$ and $g : Z \rightarrow Y$ be morphisms of algebraic spaces. Let $T \subset |X|$ be closed. Assume that*

- (1) X is locally Noetherian,
- (2) g is a monomorphism and locally of finite type,
- (3) $f|_{X \setminus T} : X \setminus T \rightarrow Y$ factors through g , and
- (4) $f_{/T} : X_{/T} \rightarrow Y$ factors through g ,

then f factors through g .

Proof. Consider the fibre product $E = X \times_Y Z \rightarrow X$. By assumption the the open immersion $X \setminus T \rightarrow X$ factors through E and any morphism $\varphi : X' \rightarrow X$ with $|\varphi|(|X'|) \subset T$ factors through E as well, see Formal Spaces, Section 9. By More on Morphisms of Spaces, Lemma 20.3 this implies that $E \rightarrow X$ is étale at every point

of E mapping to a point of T . Hence $E \rightarrow X$ is an étale monomorphism, hence an open immersion (Morphisms of Spaces, Lemma 49.2). Then it follows that $E = X$ since our assumptions imply that $|X| = |E|$. \square

0AR7 **Lemma 10.5.** *Let S be a scheme. Let X, Y be locally Noetherian algebraic spaces over S . Let $T \subset |X|$ and $T' \subset |Y|$ be closed subsets. Let $a, b : X \rightarrow Y$ be morphisms of algebraic spaces over S such that $a|_{X \setminus T} = b|_{X \setminus T}$, such that $|a|(T) \subset T'$ and $|b|(T) \subset T'$, and such that $a|_T = b|_T$ as morphisms $X|_T \rightarrow Y|_{T'}$. Then $a = b$.*

Proof. Let E be the equalizer of a and b . Then E is an algebraic space and $E \rightarrow X$ is locally of finite type and a monomorphism, see Morphisms of Spaces, Lemma 4.1. Our assumptions imply we can apply Lemma 10.4 to the two morphisms $f = \text{id} : X \rightarrow X$ and $g : E \rightarrow X$ and the closed subset T of $|X|$. \square

0AR8 **Lemma 10.6.** *Let S be a scheme. Let X be a locally Noetherian algebraic space over S . Let $T \subset |X|$ be a closed subset. Let $s, t : R \rightarrow U$ be two morphisms of algebraic spaces over X . Assume*

- (1) R, U are locally of finite type over X ,
- (2) the base change of s and t to $X \setminus T$ is an étale equivalence relation, and
- (3) the formal completion $(t|_T, s|_T) : R|_T \rightarrow U|_T \times_{X|_T} U|_T$ is an equivalence relation too.

Then $(t, s) : R \rightarrow U \times_X U$ is an étale equivalence relation.

Proof. The morphisms $s, t : R \rightarrow U$ are étale over $X \setminus T$ by assumption. Since the formal completions of the maps $s, t : R \rightarrow U$ are étale, we see that s and t are étale for example by More on Morphisms, Lemma 12.3. Applying Lemma 10.4 to the morphisms $\text{id} : R \times_{U \times_X U} R \rightarrow R \times_{U \times_X U} R$ and $\Delta : R \rightarrow R \times_{U \times_X U} R$ we conclude that (t, s) is a monomorphism. Applying it again to $(t \circ \text{pr}_0, s \circ \text{pr}_1) : R \times_{s, U, t} R \rightarrow U \times_X U$ and $(t, s) : R \rightarrow U \times_X U$ we find that “transitivity” holds. We omit the proof of the other two axioms of an equivalence relation. \square

0AR9 **Remark 10.7.** Let S, X , and $T \subset |X|$ be as in (10.3.1). Let $U \rightarrow X$ be an algebraic space over X such that $U \rightarrow X$ is locally of finite type and étale outside of T . We will construct a factorization

$$U \longrightarrow Y \longrightarrow X$$

with Y in $\mathcal{C}_{X, T}$ such that $U|_T \rightarrow Y|_T$ is an isomorphism. We may assume the image of $U \rightarrow X$ contains $X \setminus T$, otherwise we replace U by $U \amalg (X \setminus T)$. For an algebraic space Z over X , let us denote Z° the open subspace which is the inverse image of $X \setminus T$. Let

$$R = U \amalg_{U^\circ} (U \times_X U)^\circ$$

be the pushout of $U^\circ \rightarrow U$ and the diagonal morphism $U^\circ \rightarrow U^\circ \times_X U^\circ = (U \times_X U)^\circ$. Since $U^\circ \rightarrow X$ is étale, the diagonal is an open immersion and we see that R is an algebraic space (this follows for example from Spaces, Lemma 8.4). The two projections $(U \times_X U)^\circ \rightarrow U$ extend to R and we obtain two étale morphisms $s, t : R \rightarrow U$. Checking on each piece separately we find that R is an étale equivalence relation on U . Set $Y = U/R$ which is an algebraic space by Bootstrap, Theorem 10.1. Since $U^\circ \rightarrow X \setminus T$ is a surjective étale morphism and since $R^\circ = U^\circ \times_{X \setminus T} U^\circ$ we see that $Y^\circ \rightarrow X \setminus T$ is an isomorphism. In other words, $Y \rightarrow X$ is an object of $\mathcal{C}_{X, T}$. On the other hand, the morphism $U \rightarrow Y$ induces an isomorphism $U|_T \rightarrow Y|_T$. Namely, the formal completion of R along the inverse image of T is

equal to the formal completion of U along the inverse image of T by our choice of R . By our construction of the formal completion in Formal Spaces, Section 9 we conclude that $U_{/T} = Y_{/T}$.

0ARA Lemma 10.8. *Let S be a scheme. Let X be a Noetherian affine algebraic space over S . Let $T \subset |X|$ be a closed subset. Then the functor $F_{X,T}$ is an equivalence.*

Before we prove this lemma let us discuss an example. Suppose that $S = \text{Spec}(k)$, $X = \mathbf{A}_k^1$, and $T = \{0\}$. Then $X_{/T} = \text{Spf}(k[[x]])$. Let $W = \text{Spf}(k[[x]] \times k[[x]])$. Then the corresponding Y is the affine line with zero doubled (Schemes, Example 14.3). Moreover, this is the output of the construction in Remark 10.7 starting with $U = X \amalg X$.

Proof. For any scheme or algebraic space Z over X , let us denote $Z_0 \subset Z$ the inverse image of T with the induced reduced closed subscheme or subspace structure. Note that $Z_0 = (Z_{/T})_{red}$ is the reduction of the formal completion.

The functor $F_{X,T}$ is faithful by Lemma 10.5.

Let Y, Y' be objects of $\mathcal{C}_{X,T}$ and let $a' : Y_{/T} \rightarrow Y'_{/T}$ be a morphism in $\mathcal{C}_{X_{/T}}$. To prove $F_{X,T}$ is fully faithful, we will construct a morphism $a : Y \rightarrow Y'$ in $\mathcal{C}_{X,T}$ such that $a' = a_{/T}$.

Let U be an affine scheme and let $U \rightarrow Y$ be an étale morphism. Because U is affine, U_0 is affine and the image of $U_0 \rightarrow Y_0 \rightarrow Y'_0$ is a quasi-compact subspace of $|Y'_0|$. Thus we can choose an affine scheme V and an étale morphism $V \rightarrow Y'$ such that the image of $|V_0| \rightarrow |Y'_0|$ contains this quasi-compact subset. Consider the formal algebraic space

$$W = U_{/T} \times_{Y'_{/T}} V_{/T}$$

By our choice of V the above, the map $W \rightarrow U_{/T}$ is surjective. Thus there exists an affine formal algebraic space W' and an étale morphism $W' \rightarrow W$ such that $W' \rightarrow W \rightarrow U_{/T}$ is surjective. Then $W' \rightarrow U_{/T}$ is étale. By Lemma 10.3 $W' = U'_{/T}$ for $U' \rightarrow U$ étale and U' affine. Write $V = \text{Spec}(C)$. By Lemma 10.2 there exists an étale morphism $U'' \rightarrow U'$ of affines which is an isomorphism on completions and a morphism $U'' \rightarrow V$ whose completion is the composition $U''_{/T} \rightarrow U'_{/T} \rightarrow W \rightarrow V_{/T}$. Thus we get

$$Y \longleftarrow U'' \longrightarrow Y'$$

over X agreeing with the given map on formal completions such that the image of $U''_0 \rightarrow Y_0$ is the same as the image of $U_0 \rightarrow Y_0$.

Taking a disjoint union of U'' as constructed in the previous paragraph, we find a scheme U , an étale morphism $U \rightarrow Y$, and a morphism $b : U \rightarrow Y'$ over X , such that the diagram

$$\begin{array}{ccc} U_{/T} & & \\ \downarrow & \searrow^{b_{/T}} & \\ Y_{/T} & \xrightarrow{a'} & Y'_{/T} \end{array}$$

is commutative and such that $U_0 \rightarrow Y_0$ is surjective. Taking a disjoint union with the open $X \setminus T$ (which is also open in Y and Y'), we find that we may even assume that $U \rightarrow Y$ is a surjective étale morphism. Let $R = U \times_Y U$. Then the two compositions $R \rightarrow U \rightarrow Y'$ agree both over $X \setminus T$ and after formal completion

along T , whence are equal by Lemma 10.5. This means exactly that b factors as $U \rightarrow Y \rightarrow Y'$ to give us our desired morphism $a : Y \rightarrow Y'$.

Essential surjectivity. Let W be an object of $\mathcal{C}_{X/T}$. We prove W is in the essential image in a number of steps.

Step 1: W is an affine formal algebraic space. Then we can find $U \rightarrow X$ of finite type and étale over $X \setminus T$ such that $U_{/T}$ is isomorphic to W , see Lemma 10.3. Thus we see that W is in the essential image by the construction in Remark 10.7.

Step 2: W is separated. Choose $\{W_i \rightarrow W\}$ as in Formal Spaces, Definition 7.1. By Step 1 the formal algebraic spaces W_i and $W_i \times_W W_j$ are in the essential image. Say $W_i = (Y_i)_{/T}$ and $W_i \times_W W_j = (Y_{ij})_{/T}$. By fully faithfulness we obtain morphisms $t_{ij} : Y_{ij} \rightarrow Y_i$ and $s_{ij} : Y_{ij} \rightarrow Y_j$ matching the projections $W_i \times_W W_j \rightarrow W_i$ and $W_i \times_W W_j \rightarrow W_j$. Set $R = \coprod Y_{ij}$ and $U = \coprod Y_i$ and denote $s = \coprod s_{ij} : R \rightarrow U$ and $t = \coprod t_{ij} : R \rightarrow U$. Applying Lemma 10.6 we find that $(t, s) : R \rightarrow U \times_X U$ is an étale equivalence relation. Thus we can take the quotient $Y = U/R$ and it is an algebraic space, see Bootstrap, Theorem 10.1. Since completion commutes with fibre products and taking quotient sheaves, we find that $Y_{/T} \cong W$ in $\mathcal{C}_{X/T}$.

Step 3: W is general. Choose $\{W_i \rightarrow W\}$ as in Formal Spaces, Definition 7.1. The formal algebraic spaces W_i and $W_i \times_W W_j$ are separated. Hence by Step 2 the formal algebraic spaces W_i and $W_i \times_W W_j$ are in the essential image. Then we argue exactly as in the previous paragraph to see that W is in the essential image as well. This concludes the proof. \square

0ARB **Theorem 10.9.** *Let S be a scheme. Let X be a locally Noetherian algebraic space over S . Let $T \subset |X|$ be a closed subset. The functor $F_{X,T}$ (10.3.1)*

$$\left\{ \begin{array}{l} \text{algebraic spaces } Y \text{ locally of finite} \\ \text{type over } X \text{ such that } Y \rightarrow X \\ \text{is an isomorphism over } X \setminus T \end{array} \right\} \longrightarrow \left\{ \begin{array}{l} \text{formal algebraic spaces } W \text{ endowed} \\ \text{with a rig-étale morphism } W \rightarrow X_{/T} \end{array} \right\}$$

given by formal completion is an equivalence.

Proof. The theorem is essentially a formal consequence of Lemma 10.8. We give the details but we encourage the reader to think it through for themselves. Let $g : U \rightarrow X$ be a surjective étale morphism with $U = \coprod U_i$ and each U_i affine. Denote $F_{U,T}$ the functor for U and the inverse image of T in $|U|$.

Since $U = \coprod U_i$ both the category $\mathcal{C}_{U,T}$ and the category $\mathcal{C}_{U_{/T}}$ decompose as a product of categories, one for each i . Since the functors $F_{U_i,T}$ are equivalences for all i by the lemma we find that the same is true for $F_{U,T}$.

Since $F_{U,T}$ is faithful, it follows that $F_{X,T}$ is faithful too. Namely, if $a, b : Y \rightarrow Y'$ are morphisms in $\mathcal{C}_{X,T}$ such that $a_{/T} = b_{/T}$, then we find on pulling back that the base changes $a_U, b_U : U \times_X Y \rightarrow U \times_X Y'$ are equal. Since $U \times_X Y \rightarrow Y$ is surjective étale, this implies that $a = b$.

At this point we know that $F_{X,T}$ is faithful for every situation as in the theorem. Let $R = U \times_X U$ where U is as above. Let $t, s : R \rightarrow U$ be the projections. Since X is Noetherian, so is R . Thus the functor $F_{R,T}$ (defined in the obvious manner) is faithful. Let $Y \rightarrow X$ and $Y' \rightarrow X$ be objects of $\mathcal{C}_{X,T}$. Let $a' : Y_{/T} \rightarrow Y'_{/T}$ be a morphism in the category $\mathcal{C}_{X_{/T}}$. Taking the base change to U we obtain a morphism $a'_U : (U \times_X Y)_{/T} \rightarrow (U \times_X Y')_{/T}$ in the category $\mathcal{C}_{U_{/T}}$. Since the functor $F_{U,T}$ is

fully faithful we obtain a morphism $a_U : U \times_X Y \rightarrow U \times_X Y'$ with $F_{U,T}(a_U) = a'_U$. Since $s^*(a'_U) = t^*(a'_U)$ and since $F_{R,T}$ is faithful, we find that $s^*(a_U) = t^*(a_U)$. Since

$$R \times_X Y \begin{array}{c} \xrightarrow{\quad} \\ \xrightarrow{\quad} \end{array} U \times_X Y \longrightarrow Y$$

is an equalizer diagram of sheaves, we find that a_U descends to a morphism $a : Y \rightarrow Y'$. We omit the proof that $F_{X,T}(a) = a'$.

At this point we know that $F_{X,T}$ is faithful for every situation as in the theorem. To finish the proof we show that $F_{X,T}$ is essentially surjective. Let $W \rightarrow X_{/T}$ be an object of $\mathcal{C}_{X_{/T}}$. Then $U \times_X W$ is an object of $\mathcal{C}_{U_{/T}}$. By the affine case we find an object $V \rightarrow U$ of $\mathcal{C}_{U,T}$ and an isomorphism $\alpha : F_{U,T}(V) \rightarrow U \times_X W$ in $\mathcal{C}_{U_{/T}}$. By fully faithfulness of $F_{R,T}$ we find a unique morphism $h : s^*V \rightarrow t^*V$ in the category $\mathcal{C}_{R,T}$ such that $F_{R,T}(h)$ corresponds, via the isomorphism α , to the canonical descent datum on $U \times_X W$ in the category $\mathcal{C}_{R_{/T}}$. Using faithfulness of our functor on $R \times_{s,U,t} R$ we see that h satisfies the cocycle condition. We conclude, for example by the much more general Bootstrap, Lemma 11.2, that there exists an object $Y \rightarrow X$ of $\mathcal{C}_{X,T}$ and an isomorphism $\beta : U \times_X Y \rightarrow V$ such that the descent datum h corresponds, via β , to the canonical descent datum on $U \times_X Y$. We omit the verification that $F_{X,T}(Y)$ is isomorphic to W ; hint: in the category of formal algebraic spaces there is descent for morphisms along étale coverings. \square

We are often interested as to whether the output of the construction of Theorem 10.9 is a separated algebraic space. In the next few lemmas we match properties of $Y \rightarrow X$ and the corresponding completion $Y_{/T} \rightarrow X_{/T}$.

0ARU Lemma 10.10. *Let S be a scheme. Let X be a locally Noetherian algebraic space over S . Let $T \subset |X|$ be a closed subset. Let $W \rightarrow X_{/T}$ be an object of the category $\mathcal{C}_{X_{/T}}$ and let $Y \rightarrow X$ be the object corresponding to W via Theorem 10.9. Then $Y \rightarrow X$ is quasi-compact if and only if $W \rightarrow X_{/T}$ is so.*

Proof. These conditions may be checked after base change to an affine scheme étale over X , resp. a formal affine algebraic space étale over $X_{/T}$, see Morphisms of Spaces, Lemma 8.7 as well as Formal Spaces, Lemma 12.3. If $U \rightarrow X$ ranges over étale morphisms with U affine, then the formal completions $U_{/T} \rightarrow X_{/T}$ give a family of formal affine coverings as in Formal Spaces, Definition 7.1. Thus we may and do assume X is affine.

Let $V \rightarrow Y$ be a surjective étale morphism where $V = \coprod_{j \in J} V_j$ is a disjoint union of affines. Then $V_{/T} \rightarrow Y_{/T} = W$ is a surjective étale morphism. Thus if Y is quasi-compact, we can choose J is finite, and we conclude that W is quasi-compact. Conversely, if W is quasi-compact, then we can find a finite subset $J' \subset J$ such that $\coprod_{j \in J'} (V_j)_{/T} \rightarrow W$ is surjective. Then it follows that

$$(X \setminus T) \amalg \coprod_{j \in J'} V_j \longrightarrow Y$$

is surjective. This either follows from the construction of Y in the proof of Lemma 10.8 or it follows since we have

$$|Y| = |X \setminus T| \amalg |W_{red}|$$

as $Y_{/T} = W$. \square

0ARV **Lemma 10.11.** *Let S be a scheme. Let X be a locally Noetherian algebraic space over S . Let $T \subset |X|$ be a closed subset. Let $W \rightarrow X_{/T}$ be an object of the category $\mathcal{C}_{X_{/T}}$ and let $Y \rightarrow X$ be the object corresponding to W via Theorem 10.9. Then $Y \rightarrow X$ is quasi-separated if and only if $W \rightarrow X_{/T}$ is so.*

Proof. These conditions may be checked after base change to an affine scheme étale over X , resp. a formal affine algebraic space étale over $X_{/T}$, see Morphisms of Spaces, Lemma 4.12 as well as Formal Spaces, Lemma 23.5. If $U \rightarrow X$ ranges over étale morphisms with U affine, then the formal completions $U_{/T} \rightarrow X_{/T}$ give a family of formal affine coverings as in Formal Spaces, Definition 7.1. Thus we may and do assume X is affine.

Let $V \rightarrow Y$ be a surjective étale morphism where $V = \coprod_{j \in J} V_j$ is a disjoint union of affines. Then Y is quasi-separated if and only if $V_j \times_Y V_{j'}$ is quasi-compact for all $j, j' \in J$. Similarly, W is quasi-separated if and only if $(V_j \times_Y V_{j'})_{/T} = (V_j)_{/T} \times_{Y_{/T}} (V_{j'})_{/T}$ is quasi-compact for all $j, j' \in J$. Since X is Noetherian affine, we see that

$$(V_j \times_Y V_{j'}) \times_X (X \setminus T)$$

is quasi-compact. Hence we conclude the equivalence holds by the equality

$$|V_j \times_Y V_{j'}| = |(V_j \times_Y V_{j'}) \times_X (X \setminus T)| \amalg |(V_j \times_Y V_{j'})_{/T}|$$

and the fact that the second summand is closed in the left hand side. \square

0ARW **Lemma 10.12.** *Let S be a scheme. Let X be a locally Noetherian algebraic space over S . Let $T \subset |X|$ be a closed subset. Let $W \rightarrow X_{/T}$ be an object of the category $\mathcal{C}_{X_{/T}}$ and let $Y \rightarrow X$ be the object corresponding to W via Theorem 10.9. Then $Y \rightarrow X$ is separated if and only if $W \rightarrow X_{/T}$ is separated and $\Delta : W \rightarrow W \times_{X_{/T}} W$ is rig-surjective.*

Proof. These conditions may be checked after base change to an affine scheme étale over X , resp. a formal affine algebraic space étale over $X_{/T}$, see Morphisms of Spaces, Lemma 4.12 as well as Formal Spaces, Lemma 23.5. If $U \rightarrow X$ ranges over étale morphisms with U affine, then the formal completions $U_{/T} \rightarrow X_{/T}$ give a family of formal affine coverings as in Formal Spaces, Definition 7.1. Thus we may and do assume X is affine. In the proof of both directions we may assume that $Y \rightarrow X$ and $W \rightarrow X_{/T}$ are quasi-separated by Lemma 10.11.

Proof of easy direction. Assume $Y \rightarrow X$ is separated. Then $Y \rightarrow Y \times_X Y$ is a closed immersion and it follows that $W \rightarrow W \times_{X_{/T}} W$ is a closed immersion too, i.e., we see that $W \rightarrow X_{/T}$ is separated. Let

$$p : \mathrm{Spf}(R) \longrightarrow W \times_{X_{/T}} W = (Y \times_X Y)_{/T}$$

be an adic morphism where R is a complete discrete valuation ring with fraction field K . The composition into $Y \times_X Y$ corresponds to a morphism $g : \mathrm{Spec}(R) \rightarrow Y \times_X Y$, see Formal Spaces, Lemma 26.3. Since p is an adic morphism, so is the composition $\mathrm{Spf}(R) \rightarrow X$. Thus we see that $g(\mathrm{Spec}(K))$ is a point of

$$(Y \times_X Y) \times_X (X \setminus T) \cong X \setminus T \cong Y \times_X (X \setminus T)$$

(small detail omitted). Hence this lifts to a K -point of Y and we obtain a commutative diagram

$$\begin{array}{ccc} \mathrm{Spec}(K) & \longrightarrow & Y \\ \downarrow & \nearrow \text{dotted} & \downarrow \\ \mathrm{Spec}(R) & \longrightarrow & Y \times_X Y \end{array}$$

Since $Y \rightarrow X$ was assumed separated we find the dotted arrow exists (Cohomology of Spaces, Lemma 19.1). Applying the functor completion along T we find that p can be lifted to a morphism into W , i.e., $W \rightarrow W \times_{X/T} W$ is rig-surjective.

Proof of hard direction. Assume $W \rightarrow X/T$ separated and $W \rightarrow W \times_{X/T} W$ rig-surjective. By Cohomology of Spaces, Lemma 19.1 and Remark 19.3 it suffices to show that given any commutative diagram

$$\begin{array}{ccc} \mathrm{Spec}(K) & \longrightarrow & Y \\ \downarrow & \nearrow \text{dotted} & \downarrow \\ \mathrm{Spec}(R) & \xrightarrow{g} & Y \times_X Y \end{array}$$

where R is a complete discrete valuation ring with fraction field K , there is at most one dotted arrow making the diagram commute. Let $h : \mathrm{Spec}(R) \rightarrow X$ be the composition of g with the morphism $Y \times_X Y \rightarrow X$. There are three cases: Case I: $h(\mathrm{Spec}(R)) \subset (X \setminus T)$. This case is trivial because $Y \times_X (X \setminus T) = X \setminus T$. Case II: h maps $\mathrm{Spec}(R)$ into T . This case follows from our assumption that $W \rightarrow X/T$ is separated. Namely, if T denotes the reduced induced closed subspace structure on T , then h factors through T and

$$W \times_{X/T} T = Y \times_X T \longrightarrow T$$

is separated by assumption (and for example Formal Spaces, Lemma 23.5) which implies we get the lifting property by Cohomology of Spaces, Lemma 19.1 applied to the displayed arrow. Case III: $h(\mathrm{Spec}(K))$ is not in T but h maps the closed point of $\mathrm{Spec}(R)$ into T . In this case the corresponding morphism

$$g_{/T} : \mathrm{Spf}(R) \longrightarrow (Y \times_X Y)_{/T} = W \times_{X/T} W$$

is an adic morphism (detail omitted). Hence our assumption that $W \rightarrow W \times_{X/T} W$ be rig-surjective implies we can lift $g_{/T}$ to a morphism $e : \mathrm{Spf}(R) \rightarrow W = Y_{/T}$ (see Lemma 9.11 for why we do not need to extend R). Algebraizing the composition $\mathrm{Spf}(R) \rightarrow Y$ using Formal Spaces, Lemma 26.3 we find a morphism $\mathrm{Spec}(R) \rightarrow Y$ lifting g as desired. \square

0ARX **Lemma 10.13.** *Let S be a scheme. Let X be a locally Noetherian algebraic space over S . Let $T \subset |X|$ be a closed subset. Let $W \rightarrow X_{/T}$ be an object of the category $\mathcal{C}_{X/T}$ and let $Y \rightarrow X$ be the object corresponding to W via Theorem 10.9. Then $Y \rightarrow X$ is proper if and only if the following conditions hold*

- (1) $W \rightarrow X_{/T}$ is proper,
- (2) $W \rightarrow X_{/T}$ is rig-surjective, and
- (3) $\Delta : W \rightarrow W \times_{X/T} W$ is rig-surjective.

Proof. These conditions may be checked after base change to an affine scheme étale over X , resp. a formal affine algebraic space étale over $X_{/T}$, see Morphisms

of Spaces, Lemma 39.2 as well as Formal Spaces, Lemma 24.2. If $U \rightarrow X$ ranges over étale morphisms with U affine, then the formal completions $U_{/T} \rightarrow X_{/T}$ give a family of formal affine coverings as in Formal Spaces, Definition 7.1. Thus we may and do assume X is affine. In the proof of both directions we may assume that $Y \rightarrow X$ and $W \rightarrow X_{/T}$ are separated and quasi-compact and that $W \rightarrow W \times_{X_{/T}} W$ is rig-surjective by Lemmas 10.10 and 10.12.

Proof of the easy direction. Assume $Y \rightarrow X$ is proper. Then $Y_{/T} = Y \times_X X_{/T} \rightarrow X_{/T}$ is proper too. Let

$$p : \mathrm{Spf}(R) \longrightarrow X_{/T}$$

be an adic morphism where R is a complete discrete valuation ring with fraction field K . Then p corresponds to a morphism $g : \mathrm{Spec}(R) \rightarrow X$, see Formal Spaces, Lemma 26.3. Since p is an adic morphism, we have $p(\mathrm{Spec}(K)) \notin T$. Since $Y \rightarrow X$ is an isomorphism over $X \setminus T$ we can lift to X and obtain a commutative diagram

$$\begin{array}{ccc} \mathrm{Spec}(K) & \longrightarrow & Y \\ \downarrow & \nearrow & \downarrow \\ \mathrm{Spec}(R) & \longrightarrow & X \end{array}$$

Since $Y \rightarrow X$ was assumed proper we find the dotted arrow exists. (Cohomology of Spaces, Lemma 19.2). Applying the functor completion along T we find that p can be lifted to a morphism into W , i.e., $W \rightarrow X_{/T}$ is rig-surjective.

Proof of hard direction. Assume $W \rightarrow X_{/T}$ proper, $W \rightarrow W \times_{X_{/T}} W$ rig-surjective, and $W \rightarrow X_{/T}$ rig-surjective. By Cohomology of Spaces, Lemma 19.2 and Remark 19.3 it suffices to show that given any commutative diagram

$$\begin{array}{ccc} \mathrm{Spec}(K) & \longrightarrow & Y \\ \downarrow & \nearrow & \downarrow \\ \mathrm{Spec}(R) & \xrightarrow{g} & X \end{array}$$

where R is a complete discrete valuation ring with fraction field K , there is a dotted arrow making the diagram commute. Let $h : \mathrm{Spec}(R) \rightarrow X$ be the composition of g with the morphism $Y \times_X Y \rightarrow X$. There are three cases: Case I: $h(\mathrm{Spec}(R)) \subset (X \setminus T)$. This case is trivial because $Y \times_X (X \setminus T) = X \setminus T$. Case II: h maps $\mathrm{Spec}(R)$ into T . This case follows from our assumption that $W \rightarrow X_{/T}$ is proper. Namely, if T denotes the reduced induced closed subspace structure on T , then h factors through T and

$$W \times_{X_{/T}} T = Y \times_X T \longrightarrow T$$

is proper by assumption which implies we get the lifting property by Cohomology of Spaces, Lemma 19.2 applied to the displayed arrow. Case III: $h(\mathrm{Spec}(K))$ is not in T but h maps the closed point of $\mathrm{Spec}(R)$ into T . In this case the corresponding morphism

$$g_{/T} : \mathrm{Spf}(R) \longrightarrow Y_{/T} = W$$

is an adic morphism (detail omitted). Hence our assumption that $W \rightarrow X_{/T}$ be rig-surjective implies we can lift $g_{/T}$ to a morphism $e : \mathrm{Spf}(R') \rightarrow W = Y_{/T}$ for some extension of complete discrete valuation rings $R \subset R'$. Algebraizing the composition $\mathrm{Spf}(R') \rightarrow Y$ using Formal Spaces, Lemma 26.3 we find a morphism

$\text{Spec}(R') \rightarrow Y$ lifting g . By the discussion in Cohomology of Spaces, Remark 19.3 this is sufficient to conclude that $Y \rightarrow X$ is proper. \square

11. Application to modifications

0AS1 Let A be a Noetherian ring and let $I \subset A$ be an ideal. We set $S = \text{Spec}(A)$ and $U = S \setminus V(I)$. In this section we will consider the category

$$0AS2 \quad (11.0.1) \quad \left\{ f : X \longrightarrow S \quad \left| \quad \begin{array}{l} X \text{ is an algebraic space} \\ f \text{ is locally of finite type} \\ f^{-1}(U) \rightarrow U \text{ is an isomorphism} \end{array} \right. \right\}$$

A morphism from X/S to X'/S will be a morphism of algebraic spaces $X \rightarrow X'$ compatible with the structure morphisms over S .

Let $A \rightarrow B$ be a homomorphism of Noetherian rings and let $J \subset B$ be an ideal such that $J = \sqrt{IB}$. Then base change along the morphism $\text{Spec}(B) \rightarrow \text{Spec}(A)$ gives a functor from the category (11.0.1) for A to the category (11.0.1) for B .

0AE5 **Lemma 11.1.** *Let (A, I) be a pair consisting of a Noetherian ring and an ideal I . Let A^\wedge be the I -adic completion of A . Then base change defines an equivalence of categories between the category (11.0.1) for A with the category (11.0.1) for the completion A^\wedge .*

Proof. Set $S = \text{Spec}(A)$ as in (11.0.1) and $T = V(I)$. Similarly, write $S' = \text{Spec}(A^\wedge)$ and $T' = V(IA^\wedge)$. The morphism $S' \rightarrow S$ defines an isomorphism $S'_{/T'} \rightarrow S_{/T}$ of formal completions. Let $\mathcal{C}_{S,T}$, $\mathcal{C}_{S_{/T}}$, $\mathcal{C}_{S'_{/T'}}$, and $\mathcal{C}_{S',T'}$ be the corresponding categories as used in (10.3.1). By Theorem 10.9 (in fact we only need the affine case treated in Lemma 10.8) we see that

$$\mathcal{C}_{S,T} = \mathcal{C}_{S_{/T}} = \mathcal{C}_{S'_{/T'}} = \mathcal{C}_{S',T'}$$

Since $\mathcal{C}_{S,T}$ is the category (11.0.1) for A and $\mathcal{C}_{S',T'}$ the category (11.0.1) for A^\wedge this proves the lemma. \square

0BH5 **Lemma 11.2.** *Notation and assumptions as in Lemma 11.1. Let $f : X \rightarrow \text{Spec}(A)$ correspond to $g : Y \rightarrow \text{Spec}(A^\wedge)$ via the equivalence. Then f is quasi-compact, quasi-separated, separated, proper, finite, and add more here if and only if g is so.*

Proof. You can deduce this for the statements quasi-compact, quasi-separated, separated, and proper by using Lemmas 10.10, 10.11, 10.12, 10.11, and 10.13 to translate the corresponding property into a property of the formal completion and using the argument of the proof of Lemma 11.1. However, there is a direct argument using fpqc descent as follows. First, note that $\{U \rightarrow \text{Spec}(A), \text{Spec}(A^\wedge) \rightarrow \text{Spec}(A)\}$ is an fpqc covering with $U = \text{Spec}(A) \setminus V(I)$ as before. The base change of f by $U \rightarrow \text{Spec}(A)$ is id_U by definition of our category (11.0.1). Let P be a property of morphisms of algebraic spaces which is fpqc local on the base (Descent on Spaces, Definition 9.1) such that P holds for identity morphisms. Then we see that P holds for f if and only if P holds for g . This applies to P equal to quasi-compact, quasi-separated, separated, proper, and finite by Descent on Spaces, Lemmas 10.1, 10.2, 10.18, 10.19, and 10.23. \square

0AF7 **Lemma 11.3.** *Let $A \rightarrow B$ be a local map of local Noetherian rings such that*

- (1) $A \rightarrow B$ is flat,

- (2) $\mathfrak{m}_B = \mathfrak{m}_A B$, and
- (3) $\kappa(\mathfrak{m}_A) = \kappa(\mathfrak{m}_B)$

(equivalently, $A \rightarrow B$ induces an isomorphism on completions, see *More on Algebra*, Lemma 40.8). Then the base change functor from the category (11.0.1) for (A, \mathfrak{m}_A) to the category (11.0.1) for (B, \mathfrak{m}_B) is an equivalence.

Proof. This follows immediately from Lemma 11.1. □

0AE6 **Lemma 11.4.** *Let $(A, \mathfrak{m}, \kappa)$ be a Noetherian local ring. Let $f : X \rightarrow S$ be an object of (11.0.1). Then there exists a U -admissible blowup $S' \rightarrow S$ which dominates X .*

Proof. Special case of *More on Morphisms of Spaces*, Lemma 37.4. □

12. Other chapters

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