LIMITS OF ALGEBRAIC SPACES

07SB

Contents

1. Introduction

07SC In this chapter we put material related to limits of algebraic spaces. A first topic is the characterization of algebraic spaces F locally of finite presentation over the base S as limit preserving functors. We continue with a study of limits of inverse systems over directed sets (Categories, Definition 21.1) with affine transition maps. We discuss absolute Noetherian approximation for quasi-compact and quasi-separated algebraic spaces following [CLO12]. Another approach is due to David Rydh (see [Ryd08]) whose results also cover absolute Noetherian approximation for certain algebraic stacks.

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2. Conventions

07SD The standing assumption is that all schemes are contained in a big fppf site Sch_{fppf}. And all rings A considered have the property that $\text{Spec}(A)$ is (isomorphic) to an object of this big site.

Let S be a scheme and let X be an algebraic space over S. In this chapter and the following we will write $X \times_S X$ for the product of X with itself (in the category of algebraic spaces over S), instead of $X \times X$.

3. Morphisms of finite presentation

049I In this section we generalize Limits, Proposition 6.1 to morphisms of algebraic spaces. The motivation for the following definition comes from the proposition just cited.

049J **Definition 3.1.** Let S be a scheme.

(1) A functor $F : (Sch/S)_{fppf}^{opp} \to Sets$ is said to be *limit preserving* or *locally of finite presentation* if for every affine scheme T over S which is a limit $T = \lim T_i$ of a directed inverse system of affine schemes T_i over S, we have
$$F(T) = \text{colim}\, F(T_i).$$
We sometimes say that F is *locally of finite presentation over S*.

(2) Let $F, G : (Sch/S)_{fppf}^{opp} \to Sets$. A transformation of functors $a : F \to G$ is *limit preserving* or *locally of finite presentation* if for every scheme T over S and every $y \in G(T)$ the functor
$$F_y : (Sch/T)_{fppf}^{opp} \longrightarrow Sets, \quad T'/T \longmapsto \{x \in F(T') \mid a(x) = y|_{T'}\}$$
is locally of finite presentation over T[1]. We sometimes say that F is *relatively limit preserving* over G.

The functor F_y is in some sense the fiber of $a : F \to G$ over y, except that it is a presheaf on the big fppf site of T. A formula for this functor is:

049K (3.1.1)
$$F_y = F|_{(Sch/T)_{fppf}} \times_{G|_{(Sch/T)_{fppf}}} *$$

Here $*$ is the final object in the category of (pre)sheaves on $(Sch/T)_{fppf}$ (see Sites, Example 10.2) and the map $* \to G|_{(Sch/T)_{fppf}}$ is given by y. Note that if $j : (Sch/T)_{fppf} \to (Sch/S)_{fppf}$ is the localization functor, then the formula above becomes $F_y = j^{-1}F \times_{j^{-1}G} *$ and $j_!F_y$ is just the fiber product $F \times_{G,y} T$. (See Sites, Section 24, for information on localization, and especially Sites, Remark 24.9 for information on $j_!$ for presheaves.)

At this point we temporarily have two definitions of what it means for a morphism $X \to Y$ of algebraic spaces over S to be locally of finite presentation. Namely, one by Morphisms of Spaces, Definition 28.1 and one using that $X \to Y$ is a transformation of functors so that Definition 3.1 applies (we will use the terminology "limit preserving" for this notion as much as possible). We will show in Proposition 3.9 that these two definitions agree.

06BC **Lemma 3.2.** *Let S be a scheme. Let $a : F \to G$ be a transformation of functors $(Sch/S)_{fppf}^{opp} \to Sets$. The following are equivalent*

[1] The characterization (2) in Lemma 3.2 may be easier to parse.

(1) *$a : F \to G$ is limit preserving, and*
(2) *for every every affine scheme T over S which is a limit $T = \lim T_i$ of a directed inverse system of affine schemes T_i over S the diagram of sets*

$$\begin{tikzcd} \mathrm{colim}_i F(T_i) \arrow[r] \arrow[d, "a"'] & F(T) \arrow[d, "a"] \\ \mathrm{colim}_i G(T_i) \arrow[r] & G(T) \end{tikzcd}$$

is a fibre product diagram.

Proof. Assume (1). Consider $T = \lim_{i \in I} T_i$ as in (2). Let (y, x_T) be an element of the fibre product $\mathrm{colim}_i G(T_i) \times_{G(T)} F(T)$. Then y comes from $y_i \in G(T_i)$ for some i. Consider the functor F_{y_i} on $(Sch/T_i)_{fppf}$ as in Definition 3.1. We see that $x_T \in F_{y_i}(T)$. Moreover $T = \lim_{i' \geq i} T_{i'}$ is a directed system of affine schemes over T_i. Hence (1) implies that x_T the image of a unique element x of $\mathrm{colim}_{i' \geq i} F_{y_i}(T_{i'})$. Thus x is the unique element of $\mathrm{colim}\, F(T_i)$ which maps to the pair (y, x_T). This proves that (2) holds.

Assume (2). Let T be a scheme and $y_T \in G(T)$. We have to show that F_{y_T} is limit preserving. Let $T' = \lim_{i \in I} T'_i$ be an affine scheme over T which is the directed limit of affine scheme T'_i over T. Let $x_{T'} \in F_{y_T}$. Pick $i \in I$ which is possible as I is a directed set. Denote $y_i \in F(T'_i)$ the image of $y_{T'}$. Then we see that $(y_i, x_{T'})$ is an element of the fibre product $\mathrm{colim}_i G(T'_i) \times_{G(T')} F(T')$. Hence by (2) we get a unique element x of $\mathrm{colim}_i F(T'_i)$ mapping to $(y_i, x_{T'})$. It is clear that x defines an element of $\mathrm{colim}_i F_y(T'_i)$ mapping to $x_{T'}$ and we win. □

049L **Lemma 3.3.** *Let S be a scheme contained in Sch_{fppf}. Let $F, G, H : (Sch/S)_{fppf}^{opp} \to Sets$. Let $a : F \to G$, $b : G \to H$ be transformations of functors. If a and b are limit preserving, then*

$$b \circ a : F \longrightarrow H$$

is limit preserving.

Proof. Let $T = \lim_{i \in I} T_i$ as in characterization (2) of Lemma 3.2. Consider the diagram

$$\begin{tikzcd} \mathrm{colim}_i F(T_i) \arrow[r] \arrow[d, "a"'] & F(T) \arrow[d, "a"] \\ \mathrm{colim}_i G(T_i) \arrow[r] \arrow[d, "b"'] & G(T) \arrow[d, "b"] \\ \mathrm{colim}_i H(T_i) \arrow[r] & H(T) \end{tikzcd}$$

By assumption the two squares are fibre product squares. Hence the outer rectangle is a fibre product diagram too which proves the lemma. □

049M **Lemma 3.4.** *Let S be a scheme contained in Sch_{fppf}. Let $F, G, H : (Sch/S)_{fppf}^{opp} \to Sets$. Let $a : F \to G$, $b : H \to G$ be transformations of functors. Consider the fibre* *product diagram*

$$\begin{tikzcd} H \times_{b,G,a} F \arrow[r, "b'"] \arrow[d, "a'"'] & F \arrow[d, "a"] \\ H \arrow[r, "b"] & G \end{tikzcd}$$

If a is limit preserving, then the base change a' is limit preserving.

Proof. Omitted. Hint: This is formal. □

049N **Lemma 3.5.** *Let T be an affine scheme which is written as a limit $T = \lim_{i \in I} T_i$ of a directed inverse system of affine schemes.*

(1) *Let $\mathcal{V} = \{V_j \to T\}_{j=1,\ldots,m}$ be a standard fppf covering of T, see Topologies, Definition 7.5. Then there exists an index i and a standard fppf covering $\mathcal{V}_i = \{V_{i,j} \to T_i\}_{j=1,\ldots,m}$ whose base change $T \times_{T_i} \mathcal{V}_i$ to T is isomorphic to $\mathcal{V}$.*
(2) *Let $\mathcal{V}_i$, $\mathcal{V}'_i$ be a pair of standard fppf coverings of T_i. If $f : T \times_{T_i} \mathcal{V} \to T \times_{T_i} \mathcal{V}'_i$ is a morphism of coverings of T, then there exists an index $i' \geq i$ and a morphism $f_{i'} : T_{i'} \times_{T_i} \mathcal{V} \to T_{i'} \times_{T_i} \mathcal{V}'_i$ whose base change to T is f.*
(3) *If $f, g : \mathcal{V} \to \mathcal{V}'_i$ are morphisms of standard fppf coverings of T_i whose base changes f_T, g_T to T are equal then there exists an index $i' \geq i$ such that $f_{T_{i'}} = g_{T_{i'}}$.*

In other words, the category of standard fppf coverings of T is the colimit over I of the categories of standard fppf coverings of T_i

Proof. By Limits, Lemma 10.1 the category of schemes of finite presentation over T is the colimit over I of the categories of finite presentation over T_i. By Limits, Lemmas 8.2 and 8.7 the same is true for category of schemes which are affine, flat and of finite presentation over T. To finish the proof of the lemma it suffices to show that if $\{V_{j,i} \to T_i\}_{j=1,\ldots,m}$ is a finite family of flat finitely presented morphisms with $V_{j,i}$ affine, and the base change $\coprod_j T \times_{T_i} V_{j,i} \to T$ is surjective, then for some $i' \geq i$ the morphism $\coprod T_{i'} \times_{T_i} V_{j,i} \to T_{i'}$ is surjective. Denote $W_{i'} \subset T_{i'}$, resp. $W \subset T$ the image. Of course $W = T$ by assumption. Since the morphisms are flat and of finite presentation we see that W_i is a quasi-compact open of T_i, see Morphisms, Lemma 24.9. Moreover, $W = T \times_{T_i} W_i$ (formation of image commutes with base change). Hence by Limits, Lemma 4.11 we conclude that $W_{i'} = T_{i'}$ for some large enough i' and we win. □

049O **Lemma 3.6.** *Let S be a scheme contained in Sch_{fppf}. Let $F : (Sch/S)_{fppf}^{opp} \to Sets$ be a functor. If F is limit preserving then its sheafification $F^\#$ is limit preserving.*

Proof. Assume F is limit preserving. It suffices to show that F^+ is limit preserving, since $F^\# = (F^+)^+$, see Sites, Theorem 10.10. Let T be an affine scheme over S, and let $T = \lim T_i$ be written as the directed limit of an inverse system of affine S schemes. Recall that $F^+(T)$ is the colimit of $\check{H}^0(\mathcal{V}, F)$ where the limit is over all coverings of T in $(Sch/S)_{fppf}$. Any fppf covering of an affine scheme can be refined by a standard fppf covering, see Topologies, Lemma 7.4. Hence we can write

$$F^+(T) = \mathrm{colim}_{\mathcal{V} \text{ standard covering } T}\, \check{H}^0(\mathcal{V}, F).$$

By Lemma 3.5 we may rewrite this as

$$\mathrm{colim}_{i \in I}\, \mathrm{colim}_{\mathcal{V}_i \text{ standard covering } T_i}\, \check{H}^0(T \times_{T_i} \mathcal{V}_i, F).$$

(The order of the colimits is irrelevant by Categories, Lemma 14.9.) Given a standard fppf covering $\mathcal{V}_i = \{V_j \to T_i\}_{j=1,\ldots,m}$ of T_i we see that

$$T \times_{T_i} V_j = \lim_{i' \geq i} T_{i'} \times_T V_j$$

by Limits, Lemma 2.3, and similarly

$$T \times_{T_i} (V_j \times_{T_i} V_{j'}) = \lim_{i' \geq i} T_{i'} \times_T (V_j \times_{T_i} V_{j'}).$$

As the presheaf F is limit preserving this means that

$$\check{H}^0(T \times_{T_i} \mathcal{V}_i, F) = \mathrm{colim}_{i' \geq i} \check{H}^0(T_{i'} \times_{T_i} \mathcal{V}_i, F).$$

Hence the colimit expression for $F^+(T)$ above collapses to

$$\mathrm{colim}_{i \in I} \mathrm{colim}_{\mathcal{V} \text{ standard covering } T_i} \check{H}^0(\mathcal{V}, F). = \mathrm{colim}_{i \in I} F^+(T_i).$$

In other words $F^+(T) = \mathrm{colim}_i F^+(T_i)$ and hence the lemma holds. □

049P **Lemma 3.7.** *Let S be a scheme. Let $F : (Sch/S)_{fppf}^{opp} \to Sets$ be a functor. Assume that*

1. *F is a sheaf, and*
2. *there exists an fppf covering $\{U_j \to S\}_{j \in J}$ such that $F|_{(Sch/U_j)_{fppf}}$ is limit preserving.*

Then F is limit preserving.

Proof. Let T be an affine scheme over S. Let I be a directed set, and let T_i be an inverse system of affine schemes over S such that $T = \lim T_i$. We have to show that the canonical map $\mathrm{colim}\, F(T_i) \to F(T)$ is bijective.

Choose some $0 \in I$ and choose a standard fppf covering $\{V_{0,k} \to T_0\}_{k=1,\ldots,m}$ which refines the pullback $\{U_j \times_S T_0 \to T_0\}$ of the given fppf covering of S. For each $i \geq 0$ we set $V_{i,k} = T_i \times_{T_0} V_{0,k}$, and we set $V_k = T \times_{T_0} V_{0,k}$. Note that $V_k = \lim_{i \geq 0} V_{i,k}$, see Limits, Lemma 2.3.

Suppose that $x, x' \in \mathrm{colim}\, F(T_i)$ map to the same element of $F(T)$. Say x, x' are given by elements $x_i, x'_i \in F(T_i)$ for some $i \in I$ (we may choose the same i for both as I is directed). By assumption (2) and the fact that x_i, x'_i map to the same element of $F(T)$ this implies that

$$x_i|_{V_{i',k}} = x'_i|_{V_{i',k}}$$

for some suitably large $i' \in I$. We can choose the same i' for each k as $k \in \{1, \ldots, m\}$ ranges over a finite set. Since $\{V_{i',k} \to T_{i'}\}$ is an fppf covering and F is a sheaf this implies that $x_i|_{T_{i'}} = x'_i|_{T_{i'}}$ as desired. This proves that the map $\mathrm{colim}\, F(T_i) \to F(T)$ is injective.

To show surjectivity we argue in a similar fashion. Let $x \in F(T)$. By assumption (2) for each k we can choose a i such that $x|_{V_k}$ comes from an element $x_{i,k} \in F(V_{i,k})$. As before we may choose a single i which works for all k. By the injectivity proved above we see that

$$x_{i,k}|_{V_{i',k} \times_{T_{i'}} V_{i',l}} = x_{i,l}|_{V_{i',k} \times_{T_{i'}} V_{i',l}}$$

for some large enough i'. Hence by the sheaf condition of F the elements $x_{i,k}|_{V_{i',k}}$ glue to an element $x_{i'} \in F(T_{i'})$ as desired. □

049Q **Lemma 3.8.** *Let S be a scheme contained in Sch_{fppf}. Let $F, G : (Sch/S)_{fppf}^{opp} \to Sets$ be functors. If $a : F \to G$ is a transformation which is limit preserving, then the induced transformation of sheaves $F^\# \to G^\#$ is limit preserving.*

Proof. Suppose that T is a scheme and $y \in G^\#(T)$. We have to show the functor $F^\#_y : (Sch/T)^{opp}_{fppf} \to Sets$ constructed from $F^\# \to G^\#$ and y as in Definition 3.1 is limit preserving. By Equation (3.1.1) we see that $F^\#_y$ is a sheaf. Choose an fppf covering $\{V_j \to T\}_{j \in J}$ such that $y|_{V_j}$ comes from an element $y_j \in F(V_j)$. Note that the restriction of $F^\#$ to $(Sch/V_j)_{fppf}$ is just $F^\#_{y_j}$. If we can show that $F^\#_{y_j}$ is limit preserving then Lemma 3.7 guarantees that $F^\#_y$ is limit preserving and we win. This reduces us to the case $y \in G(T)$.

Let $y \in G(T)$. In this case we claim that $F^\#_y = (F_y)^\#$. This follows from Equation (3.1.1). Thus this case follows from Lemma 3.6. $\square$

04AK **Proposition 3.9.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:*

1. *The morphism f is a morphism of algebraic spaces which is locally of finite presentation, see Morphisms of Spaces, Definition 28.1.*
2. *The morphism $f : X \to Y$ is limit preserving as a transformation of functors, see Definition 3.1.*

Proof. Assume (1). Let T be a scheme and let $y \in Y(T)$. We have to show that $T \times_Y X$ is limit preserving over T in the sense of Definition 3.1. Hence we are reduced to proving that if X is an algebraic space which is locally of finite presentation over S as an algebraic space, then it is limit preserving as a functor $X : (Sch/S)^{opp}_{fppf} \to Sets$. To see this choose a presentation $X = U/R$, see Spaces, Definition 9.3. It follows from Morphisms of Spaces, Definition 28.1 that both U and R are schemes which are locally of finite presentation over S. Hence by Limits, Proposition 6.1 we have

$$U(T) = \operatorname{colim} U(T_i), \quad R(T) = \operatorname{colim} R(T_i)$$

whenever $T = \lim_i T_i$ in $(Sch/S)_{fppf}$. It follows that the presheaf

$$(Sch/S)^{opp}_{fppf} \longrightarrow Sets, \quad W \longmapsto U(W)/R(W)$$

is limit preserving. Hence by Lemma 3.6 its sheafification $X = U/R$ is limit preserving too.

Assume (2). Choose a scheme V and a surjective étale morphism $V \to Y$. Next, choose a scheme U and a surjective étale morphism $U \to V \times_Y X$. By Lemma 3.4 the transformation of functors $V \times_Y X \to V$ is limit preserving. By Morphisms of Spaces, Lemma 38.8 the morphism of algebraic spaces $U \to V \times_Y X$ is locally of finite presentation, hence limit preserving as a transformation of functors by the first part of the proof. By Lemma 3.3 the composition $U \to V \times_Y X \to V$ is limit preserving as a transformation of functors. Hence the morphism of schemes $U \to V$ is locally of finite presentation by Limits, Proposition 6.1 (modulo a set theoretic remark, see last paragraph of the proof). This means, by definition, that (1) holds.

Set theoretic remark. Let $U \to V$ be a morphism of $(Sch/S)_{fppf}$. In the statement of Limits, Proposition 6.1 we characterize $U \to V$ as being locally of finite presentation if for *all* directed inverse systems $(T_i, f_{ii'})$ of affine schemes over V we have $U(T) = \operatorname{colim} V(T_i)$, but in the current setting we may only consider affine schemes T_i over V which are (isomorphic to) an object of $(Sch/S)_{fppf}$. So we have to make sure that there are enough affines in $(Sch/S)_{fppf}$ to make the proof work. Inspecting the proof of (2) $\Rightarrow$ (1) of Limits, Proposition 6.1 we see that the question reduces to the case that U and V are affine. Say $U = \mathrm{Spec}(A)$ and $V = \mathrm{Spec}(B)$. By construction of $(Sch/S)_{fppf}$ the spectrum of any ring of cardinality $\leq |B|$ is isomorphic to an object of $(Sch/S)_{fppf}$. Hence it suffices to observe that in the "only if" part of the proof of Algebra, Lemma 126.3 only A-algebras of cardinality $\leq |B|$ are used. $\square$

05N0 **Remark 3.10.** Here is an important special case of Proposition 3.9. Let S be a scheme. Let X be an algebraic space over S. Then X is locally of finite presentation over S if and only if X, as a functor $(Sch/S)^{opp} \to Sets$, is limit preserving. Compare with Limits, Remark 6.2. In fact, we will see in Lemma 3.11 below that it suffices if the map

$$\mathrm{colim}\, X(T_i) \longrightarrow X(T)$$

is surjective whenever $T = \lim T_i$ is a directed limit of affine schemes over S.

0CM6 **Lemma 3.11.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. If for every directed limit $T = \lim_{i \in I} T_i$ of affine schemes over S the map*

$$\mathrm{colim}\, X(T_i) \longrightarrow X(T) \times_{Y(T)} \mathrm{colim}\, Y(T_i)$$

is surjective, then f is locally of finite presentation. In other words, in Proposition 3.9 part (2) it suffices to check surjectivity in the criterion of Lemma 3.2.

Proof. Choose a scheme V and a surjective étale morphism $g : V \to Y$. Next, choose a scheme U and a surjective étale morphism $h : U \to V \times_Y X$. It suffices to show for $T = \lim T_i$ as in the lemma that the map

$$\mathrm{colim}\, U(T_i) \longrightarrow U(T) \times_{V(T)} \mathrm{colim}\, V(T_i)$$

is surjective, because then $U \to V$ will be locally of finite presentation by Limits, Lemma 6.3 (modulo a set theoretic remark exactly as in the proof of Proposition 3.9). Thus we take $a : T \to U$ and $b_i : T_i \to V$ which determine the same morphism $T \to V$. Picture

$$\begin{tikzcd}
T \arrow[rr, "p_i"'] \arrow[d, "a"'] & & T_i \arrow[d, "b_i"] \arrow[dl, dotted] \\
U \arrow[r, "h"] & X \times_Y V \arrow[r] \arrow[d] & V \arrow[d, "g"] \\
& X \arrow[r, "f"] & Y
\end{tikzcd}$$

By the assumption of the lemma after increasing i we can find a morphism $c_i : T_i \to X$ such that $h \circ a = (b_i, c_i) \circ p_i : T_i \to V \times_Y X$ and such that $f \circ c_i = g \circ b_i$. Since h is an étale morphism of algebraic spaces (and hence locally of finite presentation), we have the surjectivity of

$$\mathrm{colim}\, U(T_i) \longrightarrow U(T) \times_{(X \times_Y V)(T)} \mathrm{colim} (X \times_Y V)(T_i)$$

by Proposition 3.9. Hence after increasing i again we can find the desired morphism $a_i : T_i \to U$ with $a = a_i \circ p_i$ and $b_i = (U \to V) \circ a_i$. $\square$

4. Limits of algebraic spaces

07SE The following lemma explains how we think of limits of algebraic spaces in this chapter. We will use (without further mention) that the base change of an affine morphism of algebraic spaces is affine (see Morphisms of Spaces, Lemma 20.5).

07SF **Lemma 4.1.** *Let S be a scheme. Let I be a directed set. Let $(X_i, f_{ii'})$ be an inverse system over I in the category of algebraic spaces over S. If the morphisms $f_{ii'} : X_i \to X_{i'}$ are affine, then the limit $X = \lim_i X_i$ (as an fppf sheaf) is an algebraic space. Moreover,*

(1) *each of the morphisms $f_i : X \to X_i$ is affine,*
(2) *for any $i \in I$ and any morphism of algebraic spaces $T \to X_i$ we have*

$$X \times_{X_i} T = \lim_{i' \geq i} X_{i'} \times_{X_i} T.$$

as algebraic spaces over S.

Proof. Part (2) is a formal consequence of the existence of the limit $X = \lim X_i$ as an algebraic space over S. Choose an element $0 \in I$ (this is possible as a directed set is nonempty). Choose a scheme U_0 and a surjective étale morphism $U_0 \to X_0$. Set $R_0 = U_0 \times_{X_0} U_0$ so that $X_0 = U_0/R_0$. For $i \geq 0$ set $U_i = X_i \times_{X_0} U_0$ and $R_i = X_i \times_{X_0} R_0 = U_i \times_{X_i} U_i$. By Limits, Lemma 2.2 we see that $U = \lim_{i \geq 0} U_i$ and $R = \lim_{i \geq 0} R_i$ are schemes. Moreover, the two morphisms $s, t : R \to U$ are the base change of the two projections $R_0 \to U_0$ by the morphism $U \to U_0$, in particular étale. The morphism $R \to U \times_S U$ defines an equivalence relation as directed a limit of equivalence relations is an equivalence relation. Hence the morphism $R \to U \times_S U$ is an étale equivalence relation. We claim that the natural map

07SG (4.1.1) $$U/R \longrightarrow \lim X_i$$

is an isomorphism of fppf sheaves on the category of schemes over S. The claim implies $X = \lim X_i$ is an algebraic space by Spaces, Theorem 10.5.

Let Z be a scheme and let $a : Z \to \lim X_i$ be a morphism. Then $a = (a_i)$ where $a_i : Z \to X_i$. Set $W_0 = Z \times_{a_0, X_0} U_0$. Note that $W_0 = Z \times_{a_i, X_i} U_i$ for all $i \geq 0$ by our choice of $U_i \to X_i$ above. Hence we obtain a morphism $W_0 \to \lim_{i \geq 0} U_i = U$. Since $W_0 \to Z$ is surjective and étale, we conclude that (4.1.1) is a surjective map of sheaves. Finally, suppose that Z is a scheme and that $a, b : Z \to U/R$ are two morphisms which are equalized by (4.1.1). We have to show that $a = b$. After replacing Z by the members of an fppf covering we may assume there exist morphisms $a', b' : Z \to U$ which give rise to a and b. The condition that a, b are equalized by (4.1.1) means that for each $i \geq 0$ the compositions $a'_i, b'_i : Z \to U \to U_i$ are equal as morphisms into $U_i/R_i = X_i$. Hence $(a'_i, b'_i) : Z \to U_i \times_S U_i$ factors through R_i, say by some morphism $c_i : Z \to R_i$. Since $R = \lim_{i \geq 0} R_i$ we see that $c = \lim c_i : Z \to R$ is a morphism which shows that a, b are equal as morphisms of Z into U/R.

Part (1) follows as we have seen above that $U_i \times_{X_i} X = U$ and $U \to U_i$ is affine by construction. $\square$

07SH **Lemma 4.2.** *Let S be a scheme. Let I be a directed set. Let $(X_i, f_{ii'})$ be an inverse system over I of algebraic spaces over S with affine transition maps. Let*

$X = \lim_i X_i$. Let $0 \in I$. Suppose that $T \to X_0$ is a morphism of algebraic spaces. Then

$$T \times_{X_0} X = \lim_{i \geq 0} T \times_{X_0} X_i$$

as algebraic spaces over S.

Proof. The limit X is an algebraic space by Lemma 4.1. The equality is formal, see Categories, Lemma 14.9. $\square$

0CUH **Lemma 4.3.** *Let S be a scheme. Let I be a directed set. Let $(X_i, f_{i'i}) \to (Y_i, g_{i'i})$ be a morphism of inverse systems over I of algebraic spaces over S. Assume*

1. *the morphisms $f_{i'i} : X_{i'} \to X_i$ are affine,*
2. *the morphisms $g_{i'i} : Y_{i'} \to Y_i$ are affine,*
3. *the morphisms $X_i \to Y_i$ are closed immersions.*

Then $\lim X_i \to \lim Y_i$ is a closed immersion.

Proof. Observe that $\lim X_i$ and $\lim Y_i$ exist by Lemma 4.1. Pick $0 \in I$ and choose an affine scheme V_0 and an étale morphism $V_0 \to Y_0$. Then the morphisms $V_i = Y_i \times_{Y_0} V_0 \to U_i = X_i \times_{Y_0} V_0$ are closed immersions of affine schemes. Hence the morphism $V = Y \times_{Y_0} V_0 \to U = X \times_{Y_0} V_0$ is a closed immersion because $V = \lim V_i$, $U = \lim U_i$ and because a limit of closed immersions of affine schemes is a closed immersion: a filtered colimit of surjective ring maps is surjective. Since the étale morphisms $V \to Y$ form an étale covering of Y as we vary our choice of $V_0 \to Y_0$ we see that the lemma is true. $\square$

0CUI **Lemma 4.4.** *Let S be a scheme. Let I be a directed set. Let $(X_i, f_{i'i})$ be an inverse systems over I of algebraic spaces over S. If X_i is reduced for all i, then X is reduced.*

Proof. Observe that $\lim X_i$ exists by Lemma 4.1. Pick $0 \in I$ and choose an affine scheme V_0 and an étale morphism $U_0 \to X_0$. Then the affine schemes $U_i = X_i \times_{X_0} U_0$ are reduced. Hence $U = X \times_{X_0} U_0$ is a reduced affine scheme as a limit of reduced affine schemes: a filtered colimit of reduced rings is reduced. Since the étale morphisms $U \to X$ form an étale covering of X as we vary our choice of $U_0 \to X_0$ we see that the lemma is true. $\square$

0CP4 **Lemma 4.5.** *Let S be a scheme. Let $X \to Y$ be a morphism of algebraic spaces over S. The equivalent conditions (1) and (2) of Proposition 3.9 are also equivalent to*

(3) *for every directed limit $T = \lim T_i$ of quasi-compact and quasi-separated algebraic spaces T_i over S with affine transition morphisms the diagram of sets*

$$\begin{tikzcd} \mathrm{colim}_i \mathrm{Mor}(T_i, X) \arrow[r] \arrow[d] & \mathrm{Mor}(T, X) \arrow[d] \\ \mathrm{colim}_i \mathrm{Mor}(T_i, Y) \arrow[r] & \mathrm{Mor}(T, Y) \end{tikzcd}$$

is a fibre product diagram.

Proof. It is clear that (3) implies (2). We will assume (2) and prove (3). The proof is rather formal and we encourage the reader to find their own proof.

Let us first prove that (3) holds when T_i is in addition assumed separated for all i. Choose $i \in I$ and choose a surjective étale morphism $U_i \to T_i$ where U_i is affine. Using Lemma 4.2 we see that with $U = U_i \times_{T_i} T$ and $U_{i'} = U_i \times_{T_i} T_{i'}$ we have $U = \lim_{i' \geq i} U_{i'}$. Of course U and $U_{i'}$ are affine (see Lemma 4.1). Since T_i is separated, the fibre product $V_i = U_i \times_{T_i} U_i$ is an affine scheme as well and we obtain affine schemes $V = V_i \times_{T_i} T$ and $V_{i'} = V_i \times_{T_i} T_{i'}$ with $V = \lim_{i' \geq i} V_{i'}$. Observe that $U \to T$ and $U_i \to T_i$ are surjective étale and that $V = U \times_T U$ and $V_{i'} = U_{i'} \times_{T_{i'}} U_{i'}$. Note that $\mathrm{Mor}(T, X)$ is the equalizer of the two maps $\mathrm{Mor}(U, X) \to \mathrm{Mor}(V, X)$; this is true for example because X as a sheaf on $(Sch/S)_{fppf}$ is the coequalizer of the two maps $h_V \to h_u$. Similarly $\mathrm{Mor}(T_{i'}, X)$ is the equalizer of the two maps $\mathrm{Mor}(U_{i'}, X) \to \mathrm{Mor}(V_{i'}, X)$. And of course the same thing is true with X replaced with Y. Condition (2) says that the diagrams of in (3) are fibre products in the case of $U = \lim U_i$ and $V = \lim V_i$. It follows formally that the same thing is true for $T = \lim T_i$.

In the general case, choose an affine scheme U, an $i \in I$, and a surjective étale morphism $U \to T_i$. Repeating the argument of the previous paragraph we still achieve the proof: the schemes $V_{i'}$, V are no longer affine, but they are still quasi-compact and separated and the result of the preceding paragraph applies. $\square$

5. Descending properties

0826 This section is the analogue of Limits, Section 4.

0CUJ **Lemma 5.1.** *Let S be a scheme. Let $X = \lim_{i \in I} X_i$ be the limit of a directed inverse system of algebraic spaces over S with affine transition morphisms (Lemma 4.1). If each X_i is decent (for example quasi-separated or locally separated) then $|X| = \lim_i |X_i|$ as sets.*

Proof. There is a canonical map $|X| \to \lim |X_i|$. Choose $0 \in I$. If $W_0 \subset X_0$ is an open subspace, then we have $f_0^{-1} W_0 = \lim_{i \geq 0} f_{i0}^{-1} W_0$, see Lemma 4.1. Hence, if we can prove the lemma for inverse systems where X_0 is quasi-compact, then the lemma follows in general. Thus we may and do assume X_0 is quasi-compact.

Choose an affine scheme U_0 and a surjective étale morphism $U_0 \to X_0$. Set $U_i = X_i \times_{X_0} U_0$ and $U = X \times_{X_0} U_0$. Set $R_i = U_i \times_{X_i} U_i$ and $R = U \times_X U$. Recall that $U = \lim U_i$ and $R = \lim R_i$, see proof of Lemma 4.1. Recall that $|X| = |U|/|R|$ and $|X_i| = |U_i|/|R_i|$. By Limits, Lemma 4.6 we have $|U| = \lim |U_i|$ and $|R| = \lim |R_i|$.

Surjectivity of $|X| \to \lim |X_i|$. Let $(x_i) \in \lim |X_i|$. Denote $S_i \subset |U_i|$ the inverse image of x_i. This is a finite nonempty set by the definition of decent spaces (Decent Spaces, Definition 6.1). Hence $\lim S_i$ is nonempty, see Categories, Lemma 21.7. Let $(u_i) \in \lim S_i \subset \lim |U_i|$. By the above this determines a point $u \in |U|$ which maps to an $x \in |X|$ mapping to the given element (x_i) of $\lim |X_i|$.

Injectivity of $|X| \to \lim |X_i|$. Suppose that $x, x' \in |X|$ map to the same point of $\lim |X_i|$. Choose lifts $u, u' \in |U|$ and denote $u_i, u'_i \in |U_i|$ the images. For each i let $T_i \subset |R_i|$ be the set of points mapping to $(u_i, u'_i) \in |U_i| \times |U_i|$. This is a finite set by the definition of decent spaces (Decent Spaces, Definition 6.1). Moreover T_i is nonempty as we've assumed that x and x' map to the same point of X_i. Hence $\lim T_i$ is nonempty, see Categories, Lemma 21.7. As before let $r \in |R| = \lim |R_i|$ be a point corresponding to an element of $\lim T_i$. Then r maps to (u, u') in $|U| \times |U|$ by construction and we see that $x = x'$ in $|X|$ as desired.

Parenthetical statement: A quasi-separated algebraic space is decent, see Decent Spaces, Section 6 (the key observation to this is Properties of Spaces, Lemma 6.7). A locally separated algebraic space is decent by Decent Spaces, Lemma 14.2. $\square$

086V **Lemma 5.2.** *With same notation and assumptions as in Lemma 5.1 we have* $|X| = \lim_i |X_i|$ *as topological spaces.*

Proof. We will use the criterion of Topology, Lemma 14.3. We have seen that $|X| = \lim_i |X_i|$ as sets in Lemma 5.1. The maps $f_i : X \to X_i$ are morphisms of algebraic spaces hence determine continuous maps $|X| \to |X_i|$. Thus $f_i^{-1}(U_i)$ is open for each open $U_i \subset |X_i|$. Finally, let $x \in |X|$ and let $x \in V \subset |X|$ be an open neighbourhood. We have to find an i and an open neighbourhood $W_i \subset |X_i|$ of the image x with $f_i^{-1}(W_i) \subset V$. Choose $0 \in I$. Choose a scheme U_0 and a surjective étale morphism $U_0 \to X_0$. Set $U = X \times_{X_0} U_0$ and $U_i = X_i \times_{X_0} U_0$ for $i \geq 0$. Then $U = \lim_{i \geq 0} U_i$ in the category of schemes by Lemma 4.1. Choose $u \in U$ mapping to x. By the result for schemes (Limits, Lemma 4.2) we can find an $i \geq 0$ and an open neighbourhood $E_i \subset U_i$ of the image of u whose inverse image in U is contained in the inverse image of V in U. Then we can set $W_i \subset |X_i|$ equal to the image of E_i. This works because $|U_i| \to |X_i|$ is open. $\square$

086W **Lemma 5.3.** *Let S be a scheme. Let $X = \lim_{i \in I} X_i$ be the limit of a directed inverse system of algebraic spaces over S with affine transition morphisms (Lemma 4.1). If each X_i is quasi-compact and nonempty, then $|X|$ is nonempty.*

Proof. Choose $0 \in I$. Choose an affine scheme U_0 and a surjective étale morphism $U_0 \to X_0$. Set $U_i = X_i \times_{X_0} U_0$ and $U = X \times_{X_0} U_0$. Then each U_i is a nonempty affine scheme. Hence $U = \lim U_i$ is nonempty (Limits, Lemma 4.3) and thus X is nonempty. $\square$

0CUK **Lemma 5.4.** *Let S be a scheme. Let $X = \lim_{i \in I} X_i$ be the limit of a directed inverse system of algebraic spaces over S with affine transition morphisms (Lemma 4.1). Let $x \in |X|$ with images $x_i \in |X_i|$. If each X_i is decent, then $\overline{\{x\}} = \lim_i \overline{\{x_i\}}$ as sets and as algebraic spaces if endowed with reduced induced scheme structure.*

Proof. Set $Z = \overline{\{x\}} \subset |X|$ and $Z_i = \overline{\{x_i\}} \subset |X_i|$. Since $|X| \to |X_i|$ is continuous we see that Z maps into Z_i for each i. Hence we obtain an injective map $Z \to \lim Z_i$ because $|X| = \lim |X_i|$ as sets (Lemma 5.1). Suppose that $x' \in |X|$ is not in Z. Then there is an open subset $U \subset |X|$ with $x' \in U$ and $x \notin U$. Since $|X| = \lim |X_i|$ as topological spaces (Lemma 5.2) we can write $U = \bigcup_{j \in J} f_j^{-1}(U_j)$ for some subset $J \subset I$ and opens $U_j \subset |X_j|$, see Topology, Lemma 14.2. Then we see that for some $j \in J$ we have $f_j(x') \in U_j$ and $f_j(x) \notin U_j$. In other words, we see that $f_j(x') \notin Z_j$. Thus $Z = \lim Z_i$ as sets.

Next, endow Z and Z_i with their reduced induced scheme structures, see Properties of Spaces, Definition 11.6. The transition morphisms $X_{i'} \to X_i$ induce affine morphisms $Z_{i'} \to Z_i$ and the projections $X \to X_i$ induce compatible morphisms $Z \to Z_i$. Hence we obtain morphisms $Z \to \lim Z_i \to X$ of algebraic spaces. By Lemma 4.3 we see that $\lim Z_i \to X$ is a closed immersion. By Lemma 4.4 the algebraic space $\lim Z_i$ is reduced. By the above $Z \to \lim Z_i$ is bijective on points. By uniqueness of the reduced induced closed subscheme structure we find that this morphism is an isomorphism of algebraic spaces. $\square$

084R **Situation 5.5.** Let S be a scheme. Let $X = \lim_{i \in I} X_i$ be the limit of a directed inverse system of algebraic spaces over S with affine transition morphisms (Lemma 4.1). We assume that X_i is quasi-compact and quasi-separated for all $i \in I$. We also choose an element $0 \in I$.

07SI **Lemma 5.6.** *Notation and assumptions as in Situation 5.5. Suppose that $\mathcal{F}_0$ is a quasi-coherent sheaf on X_0. Set $\mathcal{F}_i = f_{0i}^*\mathcal{F}_0$ for $i \geq 0$ and set $\mathcal{F} = f_0^*\mathcal{F}_0$. Then*

$$\Gamma(X, \mathcal{F}) = \text{colim}_{i \geq 0} \Gamma(X_i, \mathcal{F}_i)$$

Proof. Choose a surjective étale morphism $U_0 \to X_0$ where U_0 is an affine scheme (Properties of Spaces, Lemma 6.3). Set $U_i = X_i \times_{X_0} U_0$. Set $R_0 = U_0 \times_{X_0} U_0$ and $R_i = R_0 \times_{X_0} X_i$. In the proof of Lemma 4.1 we have seen that there exists a presentation $X = U/R$ with $U = \lim U_i$ and $R = \lim R_i$. Note that U_i and U are affine and that R_i and R are quasi-compact and separated (as X_i is quasi-separated). Hence Limits, Lemma 4.7 implies that

$$\mathcal{F}(U) = \text{colim}\, \mathcal{F}_i(U_i) \quad \text{and} \quad \mathcal{F}(R) = \text{colim}\, \mathcal{F}_i(R_i).$$

The lemma follows as $\Gamma(X, \mathcal{F}) = \text{Ker}(\mathcal{F}(U) \to \mathcal{F}(R))$ and similarly $\Gamma(X_i, \mathcal{F}_i) = \text{Ker}(\mathcal{F}_i(U_i) \to \mathcal{F}_i(R_i))$ □

0827 **Lemma 5.7.** *Notation and assumptions as in Situation 5.5. For any quasi-compact open subspace $U \subset X$ there exists an i and a quasi-compact open $U_i \subset X_i$ whose inverse image in X is U.*

Proof. Follows formally from the construction of limits in Lemma 4.1 and the corresponding result for schemes: Limits, Lemma 4.11. □

The following lemma will be superseded by the stronger Lemma 6.10.

084S **Lemma 5.8.** *Notation and assumptions as in Situation 5.5. Let $f_0 : Y_0 \to Z_0$ be a morphism of algebraic spaces over X_0. Assume (a) $Y_0 \to X_0$ and $Z_0 \to X_0$ are representable, (b) Y_0, Z_0 quasi-compact and quasi-separated, (c) f_0 locally of finite presentation, and (d) $Y_0 \times_{X_0} X \to Z_0 \times_{X_0} X$ an isomorphism. Then there exists an $i \geq 0$ such that $Y_0 \times_{X_0} X_i \to Z_0 \times_{X_0} X_i$ is an isomorphism.*

Proof. Choose an affine scheme U_0 and a surjective étale morphism $U_0 \to X_0$. Set $U_i = U_0 \times_{X_0} X_i$ and $U = U_0 \times_{X_0} X$. Apply Limits, Lemma 8.11 to see that $Y_0 \times_{X_0} U_i \to Z_0 \times_{X_0} U_i$ is an isomorphism of schemes for some $i \geq 0$ (details omitted). As $U_i \to X_i$ is surjective étale, it follows that $Y_0 \times_{X_0} X_i \to Z_0 \times_{X_0} X_i$ is an isomorphism (details omitted). □

084T **Lemma 5.9.** *Notation and assumptions as in Situation 5.5. If X is separated, then X_i is separated for some $i \in I$.*

Proof. Choose an affine scheme U_0 and a surjective étale morphism $U_0 \to X_0$. For $i \geq 0$ set $U_i = U_0 \times_{X_0} X_i$ and set $U = U_0 \times_{X_0} X$. Note that U_i and U are affine schemes which come equipped with surjective étale morphisms $U_i \to X_i$ and $U \to X$. Set $R_i = U_i \times_{X_i} U_i$ and $R = U \times_X U$ with projections $s_i, t_i : R_i \to U_i$ and $s, t : R \to U$. Note that R_i and R are quasi-compact separated schemes (as the algebraic spaces X_i and X are quasi-separated). The maps $s_i : R_i \to U_i$ and $s : R \to U$ are of finite type. By definition X_i is separated if and only if $(t_i, s_i) : R_i \to U_i \times U_i$ is a closed immersion, and since X is separated by assumption, the morphism $(t, s) : R \to U \times U$ is a closed immersion. Since $R \to U$

is of finite type, there exists an i such that the morphism $R \to U_i \times U$ is a closed immersion (Limits, Lemma 4.16). Fix such an $i \in I$. Apply Limits, Lemma 8.5 to the system of morphisms $R_{i'} \to U_i \times U_{i'}$ for $i' \geq i$ (this is permissible as indeed $R_{i'} = R_i \times_{U_i \times U_i} U_i \times U_{i'}$) to see that $R_{i'} \to U_i \times U_{i'}$ is a closed immersion for i' sufficiently large. This implies immediately that $R_{i'} \to U_{i'} \times U_{i'}$ is a closed immersion finishing the proof of the lemma. $\square$

07SQ **Lemma 5.10.** *Notation and assumptions as in Situation 5.5. If X is affine, then there exists an i such that X_i is affine.*

Proof. Choose $0 \in I$. Choose an affine scheme U_0 and a surjective étale morphism $U_0 \to X_0$. Set $U = U_0 \times_{X_0} X$ and $U_i = U_0 \times_{X_0} X_i$ for $i \geq 0$. Since the transition morphisms are affine, the algebraic spaces U_i and U are affine. Thus $U \to X$ is an étale morphism of affine schemes. Hence we can write $X = \mathrm{Spec}(A)$, $U = \mathrm{Spec}(B)$ and

$$B = A[x_1, \ldots, x_n]/(g_1, \ldots, g_n)$$

such that $\Delta = \det(\partial g_\lambda/\partial x_\mu)$ is invertible in B, see Algebra, Lemma 141.2. Set $A_i = \mathcal{O}_{X_i}(X_i)$. We have $A = \mathrm{colim}\, A_i$ by Lemma 5.6. After increasing 0 we may assume we have $g_{1,i}, \ldots, g_{n,i} \in A_i[x_1, \ldots, x_n]$ mapping to $g_1, \ldots, g_n$. Set

$$B_i = A_i[x_1, \ldots, x_n]/(g_{1,i}, \ldots, g_{n,i})$$

for all $i \geq 0$. Increasing 0 if necessary we may assume that $\Delta_i = \det(\partial g_{\lambda,i}/\partial x_\mu)$ is invertible in B_i for all $i \geq 0$. Thus $A_i \to B_i$ is an étale ring map. After increasing 0 we may assume also that $\mathrm{Spec}(B_i) \to \mathrm{Spec}(A_i)$ is surjective, see Limits, Lemma 8.13. Increasing 0 yet again we may choose elements $h_{1,i}, \ldots, h_{n,i} \in \mathcal{O}_{U_i}(U_i)$ which map to the classes of $x_1, \ldots, x_n$ in $B = \mathcal{O}_U(U)$ and such that $g_{\lambda,i}(h_{\nu,i}) = 0$ in $\mathcal{O}_{U_i}(U_i)$. Thus we obtain a commutative diagram

084U (5.10.1)
$$\begin{tikzcd} X_i \arrow[d] & U_i \arrow[l] \arrow[d] \\ \mathrm{Spec}(A_i) & \mathrm{Spec}(B_i) \arrow[l] \end{tikzcd}$$

By construction $B_i = B_0 \otimes_{A_0} A_i$ and $B = B_0 \otimes_{A_0} A$. Consider the morphism

$$f_0 : U_0 \longrightarrow X_0 \times_{\mathrm{Spec}(A_0)} \mathrm{Spec}(B_0)$$

This is a morphism of quasi-compact and quasi-separated algebraic spaces representable, separated and étale over X_0. The base change of f_0 to X is an isomorphism by our choices. Hence Lemma 5.8 guarantees that there exists an i such that the base change of f_0 to X_i is an isomorphism, in other words the diagram (5.10.1) is cartesian. Thus Descent, Lemma 36.1 applied to the fppf covering $\{\mathrm{Spec}(B_i) \to \mathrm{Spec}(A_i)\}$ combined with Descent, Lemma 34.1 give that $X_i \to \mathrm{Spec}(A_i)$ is representable by a scheme affine over $\mathrm{Spec}(A_i)$ as desired. (Of course it then also follows that $X_i = \mathrm{Spec}(A_i)$ but we don't need this.) $\square$

07SR **Lemma 5.11.** *Notation and assumptions as in Situation 5.5. If X is a scheme, then there exists an i such that X_i is a scheme.*

Proof. Choose a finite affine open covering $X = \bigcup W_j$. By Lemma 5.7 we can find an $i \in I$ and open subspaces $W_{j,i} \subset X_i$ whose base change to X is $W_j \to X$. By Lemma 5.10 we may assume that each $W_{j,i}$ is an affine scheme. This means that X_i is a scheme (see for example Properties of Spaces, Section 12). $\square$

0828 **Lemma 5.12.** *Let S be a scheme. Let B be an algebraic space over S. Let $X = \lim X_i$ be a directed limit of algebraic spaces over B with affine transition morphisms. Let $Y \to X$ be a morphism of algebraic spaces over B.*

1. *If $Y \to X$ is a closed immersion, X_i quasi-compact, and $Y \to B$ locally of finite type, then $Y \to X_i$ is a closed immersion for i large enough.*
2. *If $Y \to X$ is an immersion, X_i quasi-separated, $Y \to B$ locally of finite type, and Y quasi-compact, then $Y \to X_i$ is an immersion for i large enough.*
3. *If $Y \to X$ is an isomorphism, X_i quasi-compact, $X_i \to B$ locally of finite type, the transition morphisms $X_{i'} \to X_i$ are closed immersions, and $Y \to B$ is locally of finite presentation, then $Y \to X_i$ is an isomorphism for i large enough.*
4. *If $Y \to X$ is a monomorphism, X_i quasi-separated, $Y \to B$ locally of finite type, and Y quasi-compact, then $Y \to X_i$ is a monomorphism for i large enough.*

Proof. Proof of (1). Choose $0 \in I$. As X_0 is quasi-compact, we can choose an affine scheme W and an étale morphism $W \to B$ such that the image of $|X_0| \to |B|$ is contained in $|W| \to |B|$. Choose an affine scheme U_0 and an étale morphism $U_0 \to X_0 \times_B W$ such that $U_0 \to X_0$ is surjective. (This is possible by our choice of W and the fact that X_0 is quasi-compact; details omitted.) Let $V \to Y$, resp. $U \to X$, resp. $U_i \to X_i$ be the base change of $U_0 \to X_0$ (for $i \geq 0$). It suffices to prove that $V \to U_i$ is a closed immersion for i sufficiently large. Thus we reduce to proving the result for $V \to U = \lim U_i$ over W. This follows from the case of schemes, which is Limits, Lemma 4.16.

Proof of (2). Choose $0 \in I$. Choose a quasi-compact open subspace $X_0' \subset X_0$ such that $Y \to X_0$ factors through X_0'. After replacing X_i by the inverse image of X_0' for $i \geq 0$ we may assume all X_i' are quasi-compact and quasi-separated. Let $U \subset X$ be a quasi-compact open such that $Y \to X$ factors through a closed immersion $Y \to U$ (U exists as Y is quasi-compact). By Lemma 5.7 we may assume that $U = \lim U_i$ with $U_i \subset X_i$ quasi-compact open. By part (1) we see that $Y \to U_i$ is a closed immersion for some i. Thus (2) holds.

Proof of (3). Choose $0 \in I$. Choose an affine scheme U_0 and a surjective étale morphism $U_0 \to X_0$. Set $U_i = X_i \times_{X_0} U_0$, $U = X \times_{X_0} U_0 = Y \times_{X_0} U_0$. Then $U = \lim U_i$ is a limit of affine schemes, the transition maps of the system are closed immersions, and $U \to U_0$ is of finite presentation (because $U \to B$ is locally of finite presentation and $U_0 \to B$ is locally of finite type and Morphisms of Spaces, Lemma 28.9). Thus we've reduced to the following algebra fact: If $A = \lim A_i$ is a directed colimit of R-algebras with surjective transition maps and A of finite presentation over A_0, then $A = A_i$ for some i. Namely, write $A = A_0/(f_1, \ldots, f_n)$. Pick i such that $f_1, \ldots, f_n$ map to zero under the surjective map $A_0 \to A_i$.

Proof of (4). Set $Z_i = Y \times_{X_i} Y$. As the transition morphisms $X_{i'} \to X_i$ are affine hence separated, the transition morphisms $Z_{i'} \to Z_i$ are closed immersions, see Morphisms of Spaces, Lemma 4.5. We have $\lim Z_i = Y \times_X Y = Y$ as $Y \to X$ is a monomorphism. Choose $0 \in I$. Since $Y \to X_0$ is locally of finite type (Morphisms of Spaces, Lemma 23.6) the morphism $Y \to Z_0$ is locally of finite presentation (Morphisms of Spaces, Lemma 28.10). The morphisms $Z_i \to Z_0$ are locally of finite type (they are closed immersions). Finally, $Z_i = Y \times_{X_i} Y$ is quasi-compact as X_i

is quasi-separated and Y is quasi-compact. Thus part (3) applies to $Y = \lim_{i \geq 0} Z_i$ over Z_0 and we conclude $Y = Z_i$ for some i. This proves (4) and the lemma. □

086X **Lemma 5.13.** *Let S be a scheme. Let Y be an algebraic space over S. Let $X = \lim X_i$ be a directed limit of algebraic spaces over Y with affine transition morphisms. Assume*

1. *Y is quasi-separated,*
2. *X_i is quasi-compact and quasi-separated,*
3. *the morphism $X \to Y$ is separated.*

Then $X_i \to Y$ is separated for all i large enough.

Proof. Let $0 \in I$. Choose an affine scheme W and an étale morphism $W \to Y$ such that the image of $|W| \to |Y|$ contains the image of $|X_0| \to |Y|$. This is possible as X_0 is quasi-compact. It suffices to check that $W \times_Y X_i \to W$ is separated for some $i \geq 0$ because the diagonal of $W \times_Y X_i$ over W is the base change of $X_i \to X_i \times_Y X_i$ by the surjective étale morphism $(X_i \times_Y X_i) \times_Y W \to X_i \times_Y X_i$. Since Y is quasi-separated the algebraic spaces $W \times_Y X_i$ are quasi-compact (as well as quasi-separated). Thus we may base change to W and assume Y is an affine scheme. When Y is an affine scheme, we have to show that X_i is a separated algebraic space for i large enough and we are given that X is a separated algebraic space. Thus this case follows from Lemma 5.9. □

0A0R **Lemma 5.14.** *Let S be a scheme. Let Y be an algebraic space over S. Let $X = \lim X_i$ be a directed limit of algebraic spaces over Y with affine transition morphisms. Assume*

1. *Y quasi-compact and quasi-separated,*
2. *X_i quasi-compact and quasi-separated,*
3. *$X \to Y$ affine.*

Then $X_i \to Y$ is affine for i large enough.

Proof. Choose an affine scheme W and a surjective étale morphism $W \to Y$. Then $X \times_Y W$ is affine and it suffices to check that $X_i \times_Y W$ is affine for some i (Morphisms of Spaces, Lemma 20.3). This follows from Lemma 5.10. □

0A0S **Lemma 5.15.** *Let S be a scheme. Let Y be an algebraic space over S. Let $X = \lim X_i$ be a directed limit of algebraic spaces over Y with affine transition morphisms. Assume*

1. *Y quasi-compact and quasi-separated,*
2. *X_i quasi-compact and quasi-separated,*
3. *the transition morphisms $X_{i'} \to X_i$ are finite,*
4. *$X_i \to Y$ locally of finite type*
5. *$X \to Y$ integral.*

Then $X_i \to Y$ is finite for i large enough.

Proof. Choose an affine scheme W and a surjective étale morphism $W \to Y$. Then $X \times_Y W$ is finite over W and it suffices to check that $X_i \times_Y W$ is finite over W for some i (Morphisms of Spaces, Lemma 44.3). By Lemma 5.11 this reduces us to the case of schemes. In the case of schemes it follows from Limits, Lemma 4.19. □

0A0T **Lemma 5.16.** *Let S be a scheme. Let Y be an algebraic space over S. Let $X = \lim X_i$ be a directed limit of algebraic spaces over Y with affine transition morphisms. Assume*

1. *Y quasi-compact and quasi-separated,*
2. *X_i quasi-compact and quasi-separated,*
3. *the transition morphisms $X_{i'} \to X_i$ are closed immersions,*
4. *$X_i \to Y$ locally of finite type*
5. *$X \to Y$ is a closed immersion.*

Then $X_i \to Y$ is a closed immersion for i large enough.

Proof. Choose an affine scheme W and a surjective étale morphism $W \to Y$. Then $X \times_Y W$ is a closed subspace of W and it suffices to check that $X_i \times_Y W$ is a closed subspace W for some i (Morphisms of Spaces, Lemma 12.1). By Lemma 5.11 this reduces us to the case of schemes. In the case of schemes it follows from Limits, Lemma 4.20. □

6. Descending properties of morphisms

084V This section is the analogue of Section 5 for properties of morphisms. We will work in the following situation.

084W **Situation 6.1.** Let S be a scheme. Let $B = \lim B_i$ be a limit of a directed inverse system of algebraic spaces over S with affine transition morphisms (Lemma 4.1). Let $0 \in I$ and let $f_0 : X_0 \to Y_0$ be a morphism of algebraic spaces over B_0. Assume B_0, X_0, Y_0 are quasi-compact and quasi-separated. Let $f_i : X_i \to Y_i$ be the base change of f_0 to B_i and let $f : X \to Y$ be the base change of f_0 to B.

07SL **Lemma 6.2.** *With notation and assumptions as in Situation 6.1. If*

1. *f is étale,*
2. *f_0 is locally of finite presentation,*

then f_i is étale for some $i \geq 0$.

Proof. Choose an affine scheme V_0 and a surjective étale morphism $V_0 \to Y_0$. Choose an affine scheme U_0 and a surjective étale morphism $U_0 \to V_0 \times_{Y_0} X_0$. Diagram

$$\begin{tikzcd} U_0 \arrow[r] \arrow[d] & V_0 \arrow[d] \\ X_0 \arrow[r] & Y_0 \end{tikzcd}$$

The vertical arrows are surjective and étale by construction. We can base change this diagram to B_i or B to get

$$\begin{tikzcd} U_i \arrow[r] \arrow[d] & V_i \arrow[d] \\ X_i \arrow[r] & Y_i \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} U \arrow[r] \arrow[d] & V \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

Note that U_i, V_i, U, V are affine schemes, the vertical morphisms are surjective étale, and the limit of the morphisms $U_i \to V_i$ is $U \to V$. Recall that $X_i \to Y_i$ is étale if and only if $U_i \to V_i$ is étale and similarly $X \to Y$ is étale if and only if $U \to V$ is étale (Morphisms of Spaces, Lemma 38.2). Since f_0 is locally of finite presentation, so is the morphism $U_0 \to V_0$. Hence the lemma follows from Limits, Lemma 8.10. □

0CN2 **Lemma 6.3.** *With notation and assumptions as in Situation 6.1. If*

1. f *is smooth,*
2. f_0 *is locally of finite presentation,*

then f_i *is smooth for some* $i \geq 0$.

Proof. Choose an affine scheme V_0 and a surjective étale morphism $V_0 \to Y_0$. Choose an affine scheme U_0 and a surjective étale morphism $U_0 \to V_0 \times_{Y_0} X_0$. Diagram

$$\begin{CD} U_0 @>>> V_0 \\ @VVV @VVV \\ X_0 @>>> Y_0 \end{CD}$$

The vertical arrows are surjective and étale by construction. We can base change this diagram to B_i or B to get

$$\begin{CD} U_i @>>> V_i \\ @VVV @VVV \\ X_i @>>> Y_i \end{CD} \quad \text{and} \quad \begin{CD} U @>>> V \\ @VVV @VVV \\ X @>>> Y \end{CD}$$

Note that U_i, V_i, U, V are affine schemes, the vertical morphisms are surjective étale, and the limit of the morphisms $U_i \to V_i$ is $U \to V$. Recall that $X_i \to Y_i$ is smooth if and only if $U_i \to V_i$ is smooth and similarly $X \to Y$ is smooth if and only if $U \to V$ is smooth (Morphisms of Spaces, Definition 36.1). Since f_0 is locally of finite presentation, so is the morphism $U_0 \to V_0$. Hence the lemma follows from Limits, Lemma 8.9. $\square$

07SN **Lemma 6.4.** *With notation and assumptions as in Situation 6.1. If*

1. f *is surjective,*
2. f_0 *is locally of finite presentation,*

then f_i *is surjective for some* $i \geq 0$.

Proof. Choose an affine scheme V_0 and a surjective étale morphism $V_0 \to Y_0$. Choose an affine scheme U_0 and a surjective étale morphism $U_0 \to V_0 \times_{Y_0} X_0$. Diagram

$$\begin{CD} U_0 @>>> V_0 \\ @VVV @VVV \\ X_0 @>>> Y_0 \end{CD}$$

The vertical arrows are surjective and étale by construction. We can base change this diagram to B_i or B to get

$$\begin{CD} U_i @>>> V_i \\ @VVV @VVV \\ X_i @>>> Y_i \end{CD} \quad \text{and} \quad \begin{CD} U @>>> V \\ @VVV @VVV \\ X @>>> Y \end{CD}$$

Note that U_i, V_i, U, V are affine schemes, the vertical morphisms are surjective étale, the limit of the morphisms $U_i \to V_i$ is $U \to V$, and the morphisms $U_i \to X_i \times_{Y_i} V_i$ and $U \to X \times_Y V$ are surjective (as base changes of $U_0 \to X_0 \times_{Y_0} V_0$). In particular, we see that $X_i \to Y_i$ is surjective if and only if $U_i \to V_i$ is surjective and similarly $X \to Y$ is surjective if and only if $U \to V$ is surjective. Since f_0 is locally of finite presentation, so is the morphism $U_0 \to V_0$. Hence the lemma follows from the case of schemes (Limits, Lemma 8.13). □

084X **Lemma 6.5.** *Notation and assumptions as in Situation 6.1. If*

1. *f is universally injective,*
2. *f_0 is locally of finite type,*

then f_i is universally injective for some $i \geq 0$.

Proof. Recall that a morphism $X \to Y$ is universally injective if and only if the diagonal $X \to X \times_Y X$ is surjective (Morphisms of Spaces, Definition 19.3 and Lemma 19.2). Observe that $X_0 \to X_0 \times_{Y_0} X_0$ is of locally of finite presentation (Morphisms of Spaces, Lemma 28.10). Hence the lemma follows from Lemma 6.4 by considering the morphism $X_0 \to X_0 \times_{Y_0} X_0$. □

084Y **Lemma 6.6.** *Notation and assumptions as in Situation 6.1. If f is affine, then f_i is affine for some $i \geq 0$.*

Proof. Choose an affine scheme V_0 and a surjective étale morphism $V_0 \to Y_0$. Set $V_i = V_0 \times_{Y_0} Y_i$ and $V = V_0 \times_{Y_0} Y$. Since f is affine we see that $V \times_Y X = \lim V_i \times_{Y_i} X_i$ is affine. By Lemma 5.10 we see that $V_i \times_{Y_i} X_i$ is affine for some $i \geq 0$. For this i the morphism f_i is affine (Morphisms of Spaces, Lemma 20.3). □

084Z **Lemma 6.7.** *Notation and assumptions as in Situation 6.1. If*

1. *f is finite,*
2. *f_0 is locally of finite type,*

then f_i is finite for some $i \geq 0$.

Proof. Choose an affine scheme V_0 and a surjective étale morphism $V_0 \to Y_0$. Set $V_i = V_0 \times_{Y_0} Y_i$ and $V = V_0 \times_{Y_0} Y$. Since f is finite we see that $V \times_Y X = \lim V_i \times_{Y_i} X_i$ is a scheme finite over V. By Lemma 5.10 we see that $V_i \times_{Y_i} X_i$ is affine for some $i \geq 0$. Increasing i if necessary we find that $V_i \times_{Y_i} X_i \to V_i$ is finite by Limits, Lemma 8.3. For this i the morphism f_i is finite (Morphisms of Spaces, Lemma 44.3). □

0850 **Lemma 6.8.** *Notation and assumptions as in Situation 6.1. If*

1. *f is a closed immersion,*
2. *f_0 is locally of finite type,*

then f_i is a closed immersion for some $i \geq 0$.

Proof. Choose an affine scheme V_0 and a surjective étale morphism $V_0 \to Y_0$. Set $V_i = V_0 \times_{Y_0} Y_i$ and $V = V_0 \times_{Y_0} Y$. Since f is a closed immersion we see that $V \times_Y X = \lim V_i \times_{Y_i} X_i$ is a closed subscheme of the affine scheme V. By Lemma 5.10 we see that $V_i \times_{Y_i} X_i$ is affine for some $i \geq 0$. Increasing i if necessary we find that $V_i \times_{Y_i} X_i \to V_i$ is a closed immersion by Limits, Lemma 8.5. For this i the morphism f_i is a closed immersion (Morphisms of Spaces, Lemma 44.3). □

0851 **Lemma 6.9.** *Notation and assumptions as in Situation 6.1. If f is separated, then f_i is separated for some $i \geq 0$.*

Proof. Apply Lemma 6.8 to the diagonal morphism $\Delta_{X_0/Y_0} : X_0 \to X_0 \times_{Y_0} X_0$. (Diagonal morphisms are locally of finite type and the fibre product $X_0 \times_{Y_0} X_0$ is quasi-compact and quasi-separated. Some details omitted.) $\square$

0852 **Lemma 6.10.** *Notation and assumptions as in Situation 6.1. If*

1. *f is a isomorphism,*
2. *f_0 is locally of finite presentation,*

then f_i is a isomorphism for some $i \geq 0$.

Proof. Being an isomorphism is equivalent to being étale, universally injective, and surjective, see Morphisms of Spaces, Lemma 49.2. Thus the lemma follows from Lemmas 6.2, 6.4, and 6.5. $\square$

07SM **Lemma 6.11.** *Notation and assumptions as in Situation 6.1. If*

1. *f is a monomorphism,*
2. *f_0 is locally of finite type,*

then f_i is a monomorphism for some $i \geq 0$.

Proof. Recall that a morphism is a monomorphism if and only if the diagonal is an isomorphism. The morphism $X_0 \to X_0 \times_{Y_0} X_0$ is locally of finite presentation by Morphisms of Spaces, Lemma 28.10. Since $X_0 \times_{Y_0} X_0$ is quasi-compact and quasi-separated we conclude from Lemma 6.10 that $\Delta_i : X_i \to X_i \times_{Y_i} X_i$ is an isomorphism for some $i \geq 0$. For this i the morphism f_i is a monomorphism. $\square$

08K0 **Lemma 6.12.** *Notation and assumptions as in Situation 6.1. Let $\mathcal{F}_0$ be a quasi-coherent $\mathcal{O}_{X_0}$-module and denote $\mathcal{F}_i$ the pullback to X_i and $\mathcal{F}$ the pullback to X. If*

1. *$\mathcal{F}$ is flat over Y,*
2. *$\mathcal{F}_0$ is of finite presentation, and*
3. *f_0 is locally of finite presentation,*

then $\mathcal{F}_i$ is flat over Y_i for some $i \geq 0$. In particular, if f_0 is locally of finite presentation and f is flat, then f_i is flat for some $i \geq 0$.

Proof. Choose an affine scheme V_0 and a surjective étale morphism $V_0 \to Y_0$. Choose an affine scheme U_0 and a surjective étale morphism $U_0 \to V_0 \times_{Y_0} X_0$. Diagram

$$\begin{tikzcd} U_0 \arrow[r] \arrow[d] & V_0 \arrow[d] \\ X_0 \arrow[r] & Y_0 \end{tikzcd}$$

The vertical arrows are surjective and étale by construction. We can base change this diagram to B_i or B to get

$$\begin{tikzcd} U_i \arrow[r] \arrow[d] & V_i \arrow[d] \\ X_i \arrow[r] & Y_i \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} U \arrow[r] \arrow[d] & V \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

Note that U_i, V_i, U, V are affine schemes, the vertical morphisms are surjective étale, and the limit of the morphisms $U_i \to V_i$ is $U \to V$. Recall that $\mathcal{F}_i$ is flat over Y_i if and only if $\mathcal{F}_i|_{U_i}$ is flat over V_i and similarly $\mathcal{F}$ is flat over Y if and only if $\mathcal{F}|_U$

is flat over V (Morphisms of Spaces, Definition 29.1). Since f_0 is locally of finite presentation, so is the morphism $U_0 \to V_0$. Hence the lemma follows from Limits, Lemma 10.4. □

08K1 **Lemma 6.13.** *Assumptions and notation as in Situation 6.1. If*

1. *f is proper, and*
2. *f_0 is locally of finite type,*

then there exists an i such that f_i is proper.

Proof. Choose an affine scheme V_0 and a surjective étale morphism $V_0 \to Y_0$. Set $V_i = Y_i \times_{Y_0} V_0$ and $V = Y \times_{Y_0} V_0$. It suffices to prove that the base change of f_i to V_i is proper, see Morphisms of Spaces, Lemma 39.2. Thus we may assume Y_0 is affine.

By Lemma 6.9 we see that f_i is separated for some $i \geq 0$. Replacing 0 by i we may assume that f_0 is separated. Observe that f_0 is quasi-compact. Thus f_0 is separated and of finite type. By Cohomology of Spaces, Lemma 18.1 we can choose a diagram

$$\begin{tikzcd} X_0 \arrow[dr] & X_0' \arrow[l, "\pi"'] \arrow[r] \arrow[d] & \mathbf{P}^n_{Y_0} \arrow[dl] \\ & Y_0 & \end{tikzcd}$$

where $X_0' \to \mathbf{P}^n_{Y_0}$ is an immersion, and $\pi : X_0' \to X_0$ is proper and surjective. Introduce $X' = X_0' \times_{Y_0} Y$ and $X_i' = X_0' \times_{Y_0} Y_i$. By Morphisms of Spaces, Lemmas 39.4 and 39.3 we see that $X' \to Y$ is proper. Hence $X' \to \mathbf{P}^n_Y$ is a closed immersion (Morphisms of Spaces, Lemma 39.6). By Morphisms of Spaces, Lemma 39.7 it suffices to prove that $X_i' \to Y_i$ is proper for some i. By Lemma 6.8 we find that $X_i' \to \mathbf{P}^n_{Y_i}$ is a closed immersion for i large enough. Then $X_i' \to Y_i$ is proper and we win. □

0D4K **Lemma 6.14.** *Assumptions and notation as in Situation 6.1. Let $d \geq 0$. If*

1. *f has relative dimension $\leq d$ (Morphisms of Spaces, Definition 32.2), and*
2. *f_0 is locally of finite type,*

then there exists an i such that f_i has relative dimension $\leq d$.

Proof. Choose an affine scheme V_0 and a surjective étale morphism $V_0 \to Y_0$. Choose an affine scheme U_0 and a surjective étale morphism $U_0 \to V_0 \times_{Y_0} X_0$. Diagram

$$\begin{tikzcd} U_0 \arrow[r] \arrow[d] & V_0 \arrow[d] \\ X_0 \arrow[r] & Y_0 \end{tikzcd}$$

The vertical arrows are surjective and étale by construction. We can base change this diagram to B_i or B to get

$$\begin{tikzcd} U_i \arrow[r] \arrow[d] & V_i \arrow[d] \\ X_i \arrow[r] & Y_i \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} U \arrow[r] \arrow[d] & V \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

Note that U_i, V_i, U, V are affine schemes, the vertical morphisms are surjective étale, and the limit of the morphisms $U_i \to V_i$ is $U \to V$. In this situation $X_i \to Y_i$ has relative dimension $\leq d$ if and only if $U_i \to V_i$ has relative dimension $\leq d$ (as defined in Morphisms, Definition 28.1). To see the equivalence, use that the definition for morphisms of algebraic spaces involves Morphisms of Spaces, Definition 32.1 which uses étale localization. The same is true for $X \to Y$ and $U \to V$. Since f_0 is locally of finite type, so is the morphism $U_0 \to V_0$. Hence the lemma follows from the more general Limits, Lemma 16.1. □

7. Descending relative objects

07SJ The following lemma is typical of the type of results in this section.

07SK **Lemma 7.1.** *Let S be a scheme. Let I be a directed set. Let $(X_i, f_{ii'})$ be an inverse system over I of algebraic spaces over S. Assume*

1. *the morphisms $f_{ii'} : X_i \to X_{i'}$ are affine,*
2. *the spaces X_i are quasi-compact and quasi-separated.*

Let $X = \lim_i X_i$. Then the category of algebraic spaces of finite presentation over X is the colimit over I of the categories of algebraic spaces of finite presentation over X_i.

Proof. Pick $0 \in I$. Choose a surjective étale morphism $U_0 \to X_0$ where U_0 is an affine scheme (Properties of Spaces, Lemma 6.3). Set $U_i = X_i \times_{X_0} U_0$. Set $R_0 = U_0 \times_{X_0} U_0$ and $R_i = R_0 \times_{X_0} X_i$. Denote $s_i, t_i : R_i \to U_i$ and $s, t : R \to U$ the two projections. In the proof of Lemma 4.1 we have seen that there exists a presentation $X = U/R$ with $U = \lim U_i$ and $R = \lim R_i$. Note that U_i and U are affine and that R_i and R are quasi-compact and separated (as X_i is quasi-separated). Let Y be an algebraic space over S and let $Y \to X$ be a morphism of finite presentation. Set $V = U \times_X Y$. This is an algebraic space of finite presentation over U. Choose an affine scheme W and a surjective étale morphism $W \to V$. Then $W \to Y$ is surjective étale as well. Set $R' = W \times_Y W$ so that $Y = W/R'$ (see Spaces, Section 9). Note that W is a scheme of finite presentation over U and that R' is a scheme of finite presentation over R (details omitted). By Limits, Lemma 10.1 we can find an index i and a morphism of schemes $W_i \to U_i$ of finite presentation whose base change to U gives $W \to U$. Similarly we can find, after possibly increasing i, a scheme R'_i of finite presentation over R_i whose base change to R is R'. The projection morphisms $s', t' : R' \to W$ are morphisms over the projection morphisms $s, t : R \to U$. Hence we can view s', resp. t' as a morphism between schemes of finite presentation over U (with structure morphism $R' \to U$ given by $R' \to R$ followed by s, resp. t). Hence we can apply Limits, Lemma 10.1 again to see that, after possibly increasing i, there exist morphisms $s'_i, t'_i : R'_i \to W_i$, whose base change to U is S', t'. By Limits, Lemmas 8.10 and 8.12 we may assume that s'_i, t'_i are étale and that $j'_i : R'_i \to W_i \times_{X_i} W_i$ is a monomorphism (here we view j'_i as a morphism of schemes of finite presentation over U_i via one of the projections – it doesn't matter which one). Setting $Y_i = W_i/R'_i$ (see Spaces, Theorem 10.5) we obtain an algebraic space of finite presentation over X_i whose base change to X is isomorphic to Y.

This shows that every algebraic space of finite presentation over X comes from an algebraic space of finite presentation over some X_i, i.e., it shows that the functor of the lemma is essentially surjective. To show that it is fully faithful, consider an index $0 \in I$ and two algebraic spaces Y_0, Z_0 of finite presentation over X_0. Set $Y_i = X_i \times_{X_0} Y_0$, $Y = X \times_{X_0} Y_0$, $Z_i = X_i \times_{X_0} Z_0$, and $Z = X \times_{X_0} Z_0$. Let $\alpha : Y \to Z$ be a morphism of algebraic spaces over X. Choose a surjective étale morphism $V_0 \to Y_0$ where V_0 is an affine scheme. Set $V_i = V_0 \times_{Y_0} Y_i$ and $V = V_0 \times_{Y_0} Y$ which are affine schemes endowed with surjective étale morphisms to Y_i and Y. The composition $V \to Y \to Z \to Z_0$ comes from a (essentially unique) morphism $V_i \to Z_0$ for some $i \geq 0$ by Proposition 3.9 (applied to $Z_0 \to X_0$ which is of finite presentation by assumption). After increasing i the two compositions

$$V_i \times_{Y_i} V_i \to V_i \to Z_0$$

are equal as this is true in the limit. Hence we obtain a (essentially unique) morphism $Y_i \to Z_0$. Since this is a morphism over X_0 it induces a morphism into $Z_i = Z_0 \times_{X_0} X_i$ as desired. □

07V7 **Lemma 7.2.** *With notation and assumptions as in Lemma 7.1. The category of $\mathcal{O}_X$-modules of finite presentation is the colimit over I of the categories $\mathcal{O}_{X_i}$-modules of finite presentation.*

Proof. Choose $0 \in I$. Choose an affine scheme U_0 and a surjective étale morphism $U_0 \to X_0$. Set $U_i = X_i \times_{X_0} U_0$. Set $R_0 = U_0 \times_{X_0} U_0$ and $R_i = R_0 \times_{X_0} X_i$. Denote $s_i, t_i : R_i \to U_i$ and $s, t : R \to U$ the two projections. In the proof of Lemma 4.1 we have seen that there exists a presentation $X = U/R$ with $U = \lim U_i$ and $R = \lim R_i$. Note that U_i and U are affine and that R_i and R are quasi-compact and separated (as X_i is quasi-separated). Moreover, it is also true that $R \times_{s,U,t} R = \text{colim}\, R_i \times_{s_i,U_i,t_i} R_i$. Thus we know that $QCoh(\mathcal{O}_U) = \text{colim}\, QCoh(\mathcal{O}_{U_i})$, $QCoh(\mathcal{O}_R) = \text{colim}\, QCoh(\mathcal{O}_{R_i})$, and $QCoh(\mathcal{O}_{R \times_{s,U,t} R}) = \text{colim}\, QCoh(\mathcal{O}_{R_i \times_{s_i,U_i,t_i} R_i})$ by Limits, Lemma 10.2. We have $QCoh(\mathcal{O}_X) = QCoh(U, R, s, t, c)$ and $QCoh(\mathcal{O}_{X_i}) = QCoh(U_i, R_i, s_i, t_i, c_i)$, see Properties of Spaces, Proposition 31.1. Thus the result follows formally. □

0D2X **Lemma 7.3.** *With notation and assumptions as in Lemma 7.1. Then any invertible $\mathcal{O}_X$-module is the pullback of an invertible $\mathcal{O}_{X_i}$-module for some i.*

Proof. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Since invertible modules are of finite presentation we can find an i and modules $\mathcal{L}_i$ and $\mathcal{N}_i$ of finite presentation over X_i such that $f_i^*\mathcal{L}_i \cong \mathcal{L}$ and $f_i^*\mathcal{N}_i \cong \mathcal{L}^{\otimes -1}$, see Lemma 7.2. Since pullback commutes with tensor product we see that $f_i^*(\mathcal{L}_i \otimes_{\mathcal{O}_{X_i}} \mathcal{N}_i)$ is isomorphic to $\mathcal{O}_X$. Since the tensor product of finitely presented modules is finitely presented, the same lemma implies that $f_{i'i}^*\mathcal{L}_i \otimes_{\mathcal{O}_{X_{i'}}} f_{i'i}^*\mathcal{N}_i$ is isomorphic to $\mathcal{O}_{X_{i'}}$ for some $i' \geq i$. It follows that $f_{i'i}^*\mathcal{L}_i$ is invertible (Modules on Sites, Lemma 31.2) and the proof is complete. □

8. Absolute Noetherian approximation

07SS The following result is [CLO12, Theorem 1.2.2]. A key ingredient in the proof is Decent Spaces, Lemma 8.6.

07SU **Proposition 8.1.** *Let X be a quasi-compact and quasi-separated algebraic space over $\text{Spec}(\mathbf{Z})$. There exist a directed set I and an inverse system of algebraic spaces $(X_i, f_{ii'})$ over I such that* Our proof follows closely the proof given in [CLO12, Theorem 1.2.2].

(1) *the transition morphisms $f_{ii'}$ are affine*
(2) *each X_i is quasi-separated and of finite type over $\mathbf{Z}$, and*
(3) *$X = \lim X_i$.*

Proof. We apply Decent Spaces, Lemma 8.6 to get open subspaces $U_p \subset X$, schemes V_p, and morphisms $f_p : V_p \to U_p$ with properties as stated. Note that $f_n : V_n \to U_n$ is an étale morphism of algebraic spaces whose restriction to the inverse image of $T_n = (V_n)_{red}$ is an isomorphism. Hence f_n is an isomorphism, for example by Morphisms of Spaces, Lemma 49.2. In particular U_n is a quasi-compact and separated scheme. Thus we can write $U_n = \lim U_{n,i}$ as a directed limit of schemes of finite type over $\mathbf{Z}$ with affine transition morphisms, see Limits, Proposition 5.4. Thus, applying descending induction on p, we see that we have reduced to the problem posed in the following paragraph.

Here we have $U \subset X$, $U = \lim U_i$, $Z \subset X$, and $f : V \to X$ with the following properties

(1) X is a quasi-compact and quasi-separated algebraic space,
(2) V is a quasi-compact and separated scheme,
(3) $U \subset X$ is a quasi-compact open subspace,
(4) $(U_i, g_{ii'})$ is a directed inverse system of quasi-separated algebraic spaces of finite type over $\mathbf{Z}$ with affine transition morphisms whose limit is U,
(5) $Z \subset X$ is a closed subspace such that $|X| = |U| \amalg |Z|$,
(6) $f : V \to X$ is a surjective étale morphism such that $f^{-1}(Z) \to Z$ is an isomorphism.

Problem: Show that the conclusion of the proposition holds for X.

Note that $W = f^{-1}(U) \subset V$ is a quasi-compact open subscheme étale over U. Hence we may apply Lemmas 7.1 and 6.2 to find an index $0 \in I$ and an étale morphism $W_0 \to U_0$ of finite presentation whose base change to U produces W. Setting $W_i = W_0 \times_{U_0} U_i$ we see that $W = \lim_{i \geq 0} W_i$. After increasing 0 we may assume the W_i are schemes, see Lemma 5.11. Moreover, W_i is of finite type over $\mathbf{Z}$.

Apply Limits, Lemma 5.3 to $W = \lim_{i \geq 0} W_i$ and the inclusion $W \subset V$. Replace I by the directed set J found in that lemma. This allows us to write V as a directed limit $V = \lim V_i$ of finite type schemes over $\mathbf{Z}$ with affine transition maps such that each V_i contains W_i as an open subscheme (compatible with transition morphisms). For each i we can form the push out

$$\begin{tikzcd} W_i \arrow[r] \arrow[d, "\Delta"'] & V_i \arrow[d] \\ W_i \times_{U_i} W_i \arrow[r] & R_i \end{tikzcd}$$

in the category of schemes. Namely, the left vertical and upper horizontal arrows are open immersions of schemes. In other words, we can construct R_i as the glueing of V_i and $W_i \times_{U_i} W_i$ along the common open W_i (see Schemes, Section 14). Note that the étale projection maps $W_i \times_{U_i} W_i \to W_i$ extend to étale morphisms $s_i, t_i : R_i \to V_i$. It is clear that the morphism $j_i = (t_i, s_i) : R_i \to V_i \times V_i$ is an étale equivalence relation on V_i. Note that $W_i \times_{U_i} W_i$ is quasi-compact (as U_i is quasi-separated and W_i quasi-compact) and V_i is quasi-compact, hence R_i is quasi-compact. For $i \geq i'$

the diagram

07SV (8.1.1)
$$\begin{tikzcd} R_i \arrow[r] \arrow[d, "s_i"'] & R_{i'} \arrow[d, "s_{i'}"] \\ V_i \arrow[r] & V_{i'} \end{tikzcd}$$

is cartesian because

$$(W_{i'} \times_{U_{i'}} W_{i'}) \times_{U_{i'}} U_i = W_{i'} \times_{U_{i'}} U_i \times_{U_i} U_i \times_{U_{i'}} W_{i'} = W_i \times_{U_i} W_i.$$

Consider the algebraic space $X_i = V_i/R_i$ (see Spaces, Theorem 10.5). As V_i is of finite type over $\mathbf{Z}$ and R_i is quasi-compact we see that X_i is quasi-separated and of finite type over $\mathbf{Z}$ (see Properties of Spaces, Lemma 6.5 and Morphisms of Spaces, Lemmas 8.5 and 23.4). As the construction of R_i above is compatible with transition morphisms, we obtain morphisms of algebraic spaces $X_i \to X_{i'}$ for $i \geq i'$. The commutative diagrams

$$\begin{tikzcd} V_i \arrow[r] \arrow[d] & V_{i'} \arrow[d] \\ X_i \arrow[r] & X_{i'} \end{tikzcd}$$

are cartesian as (8.1.1) is cartesian, see Groupoids, Lemma 20.7. Since $V_i \to V_{i'}$ is affine, this implies that $X_i \to X_{i'}$ is affine, see Morphisms of Spaces, Lemma 20.3. Thus we can form the limit $X' = \lim X_i$ by Lemma 4.1. We claim that $X \cong X'$ which finishes the proof of the proposition.

Proof of the claim. Set $R = \lim R_i$. By construction the algebraic space X' comes equipped with a surjective étale morphism $V \to X'$ such that

$$V \times_{X'} V \cong R$$

(use Lemma 4.1). By construction $\lim W_i \times_{U_i} W_i = W \times_U W$ and $V = \lim V_i$ so that R is the union of $W \times_U W$ and V glued along W. Property (6) implies the projections $V \times_X V \to V$ are isomorphisms over $f^{-1}(Z) \subset V$. Hence the scheme $V \times_X V$ is the union of the opens $\Delta_{V/X}(V)$ and $W \times_U W$ which intersect along $\Delta_{W/X}(W)$. We conclude that there exists a unique isomorphism $R \cong V \times_X V$ compatible with the projections to V. Since $V \to X$ and $V \to X'$ are surjective étale we see that

$$X = V/V \times_X V = V/R = V/V \times_{X'} V = X'$$

by Spaces, Lemma 9.1 and we win. $\square$

9. Applications

07V8 The following lemma can also be deduced directly from Decent Spaces, Lemma 8.6 without passing through absolute Noetherian approximation.

07V9 **Lemma 9.1.** *Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S. Every quasi-coherent $\mathcal{O}_X$-module is a filtered colimit of finitely presented $\mathcal{O}_X$-modules.*

Proof. We may view as an algebraic space over $\mathrm{Spec}(\mathbf{Z})$, see Spaces, Definition 16.2 and Properties of Spaces, Definition 3.1. Thus we may apply Proposition 8.1 and write $X = \lim X_i$ with X_i of finite presentation over $\mathbf{Z}$. Thus X_i is a Noetherian algebraic space, see Morphisms of Spaces, Lemma 28.6. The morphism $X \to X_i$ is affine, see Lemma 4.1. Conclusion by Cohomology of Spaces, Lemma 15.2. □

The rest of this section consists of straightforward applications of Lemma 9.1.

0829 **Lemma 9.2.** *Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Then $\mathcal{F}$ is the directed colimit of its finite type quasi-coherent submodules.*

Proof. If $\mathcal{G}, \mathcal{H} \subset \mathcal{F}$ are finite type quasi-coherent $\mathcal{O}_X$-submodules then the image of $\mathcal{G} \oplus \mathcal{H} \to \mathcal{F}$ is another finite type quasi-coherent $\mathcal{O}_X$-submodule which contains both of them. In this way we see that the system is directed. To show that $\mathcal{F}$ is the colimit of this system, write $\mathcal{F} = \operatorname{colim}_i \mathcal{F}_i$ as a directed colimit of finitely presented quasi-coherent sheaves as in Lemma 9.1. Then the images $\mathcal{G}_i = \operatorname{Im}(\mathcal{F}_i \to \mathcal{F})$ are finite type quasi-coherent subsheaves of $\mathcal{F}$. Since $\mathcal{F}$ is the colimit of these the result follows. □

086Y **Lemma 9.3.** *Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S. Let $\mathcal{F}$ be a finite type quasi-coherent $\mathcal{O}_X$-module. Then we can write $\mathcal{F} = \lim \mathcal{F}_i$ where each $\mathcal{F}_i$ is an $\mathcal{O}_X$-module of finite presentation and all transition maps $\mathcal{F}_i \to \mathcal{F}_{i'}$ surjective.*

Proof. Write $\mathcal{F} = \operatorname{colim} \mathcal{G}_i$ as a filtered colimit of finitely presented $\mathcal{O}_X$-modules (Lemma 9.1). We claim that $\mathcal{G}_i \to \mathcal{F}$ is surjective for some i. Namely, choose an étale surjection $U \to X$ where U is an affine scheme. Choose finitely many sections $s_k \in \mathcal{F}(U)$ generating $\mathcal{F}|_U$. Since U is affine we see that s_k is in the image of $\mathcal{G}_i \to \mathcal{F}$ for i large enough. Hence $\mathcal{G}_i \to \mathcal{F}$ is surjective for i large enough. Choose such an i and let $\mathcal{K} \subset \mathcal{G}_i$ be the kernel of the map $\mathcal{G}_i \to \mathcal{F}$. Write $\mathcal{K} = \operatorname{colim} \mathcal{K}_a$ as the filtered colimit of its finite type quasi-coherent submodules (Lemma 9.2). Then $\mathcal{F} = \operatorname{colim} \mathcal{G}_i/\mathcal{K}_a$ is a solution to the problem posed by the lemma. □

Let X be an algebraic space. In the following lemma we use the notion of a *finitely presented quasi-coherent $\mathcal{O}_X$-algebra $\mathcal{A}$*. This means that for every affine $U = \operatorname{Spec}(R)$ étale over X we have $\mathcal{A}|_U = \widetilde{A}$ where A is a (commutative) R-algebra which is of finite presentation as an R-algebra.

082A **Lemma 9.4.** *Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S. Let $\mathcal{A}$ be a quasi-coherent $\mathcal{O}_X$-algebra. Then $\mathcal{A}$ is a directed colimit of finitely presented quasi-coherent $\mathcal{O}_X$-algebras.*

Proof. First we write $\mathcal{A} = \operatorname{colim}_i \mathcal{F}_i$ as a directed colimit of finitely presented quasi-coherent sheaves as in Lemma 9.1. For each i let $\mathcal{B}_i = \operatorname{Sym}(\mathcal{F}_i)$ be the symmetric algebra on $\mathcal{F}_i$ over $\mathcal{O}_X$. Write $\mathcal{I}_i = \operatorname{Ker}(\mathcal{B}_i \to \mathcal{A})$. Write $\mathcal{I}_i = \operatorname{colim}_j \mathcal{F}_{i,j}$ where $\mathcal{F}_{i,j}$ is a finite type quasi-coherent submodule of $\mathcal{I}_i$, see Lemma 9.2. Set $\mathcal{I}_{i,j} \subset \mathcal{I}_i$ equal to the $\mathcal{B}_i$-ideal generated by $\mathcal{F}_{i,j}$. Set $\mathcal{A}_{i,j} = \mathcal{B}_i/\mathcal{I}_{i,j}$. Then $\mathcal{A}_{i,j}$ is a quasi-coherent finitely presented $\mathcal{O}_X$-algebra. Define $(i,j) \leq (i',j')$ if $i \leq i'$ and the map $\mathcal{B}_i \to \mathcal{B}_{i'}$ maps the ideal $\mathcal{I}_{i,j}$ into the ideal $\mathcal{I}_{i',j'}$. Then it is clear that $\mathcal{A} = \operatorname{colim}_{i,j} \mathcal{A}_{i,j}$. □

Let X be an algebraic space. In the following lemma we use the notion of a *quasi-coherent $\mathcal{O}_X$-algebra $\mathcal{A}$ of finite type*. This means that for every affine $U = \operatorname{Spec}(R)$ étale over X we have $\mathcal{A}|_U = \widetilde{A}$ where A is a (commutative) R-algebra which is of finite type as an R-algebra.

082B **Lemma 9.5.** *Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S. Let $\mathcal{A}$ be a quasi-coherent $\mathcal{O}_X$-algebra. Then $\mathcal{A}$ is the directed colimit of its finite type quasi-coherent $\mathcal{O}_X$-subalgebras.*

Proof. Omitted. Hint: Compare with the proof of Lemma 9.2. □

Let X be an algebraic space. In the following lemma we use the notion of a *finite (resp. integral) quasi-coherent $\mathcal{O}_X$-algebra $\mathcal{A}$*. This means that for every affine $U = \text{Spec}(R)$ étale over X we have $\mathcal{A}|_U = \widetilde{A}$ where A is a (commutative) R-algebra which is finite (resp. integral) as an R-algebra.

086Z **Lemma 9.6.** *Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S. Let $\mathcal{A}$ be a finite quasi-coherent $\mathcal{O}_X$-algebra. Then $\mathcal{A} = \text{colim}\, \mathcal{A}_i$ is a directed colimit of finite and finitely presented quasi-coherent $\mathcal{O}_X$-algebras with surjective transition maps.*

Proof. By Lemma 9.3 there exists a finitely presented $\mathcal{O}_X$-module $\mathcal{F}$ and a surjection $\mathcal{F} \to \mathcal{A}$. Using the algebra structure we obtain a surjection

$$\text{Sym}^*_{\mathcal{O}_X}(\mathcal{F}) \longrightarrow \mathcal{A}$$

Denote $\mathcal{J}$ the kernel. Write $\mathcal{J} = \text{colim}\, \mathcal{E}_i$ as a filtered colimit of finite type $\mathcal{O}_X$-submodules $\mathcal{E}_i$ (Lemma 9.2). Set

$$\mathcal{A}_i = \text{Sym}^*_{\mathcal{O}_X}(\mathcal{F})/(\mathcal{E}_i)$$

where $(\mathcal{E}_i)$ indicates the ideal sheaf generated by the image of $\mathcal{E}_i \to \text{Sym}^*_{\mathcal{O}_X}(\mathcal{F})$. Then each $\mathcal{A}_i$ is a finitely presented $\mathcal{O}_X$-algebra, the transition maps are surjective, and $\mathcal{A} = \text{colim}\, \mathcal{A}_i$. To finish the proof we still have to show that $\mathcal{A}_i$ is a finite $\mathcal{O}_X$-algebra for i sufficiently large. To do this we choose an étale surjective map $U \to X$ where U is an affine scheme. Take generators $f_1, \ldots, f_m \in \Gamma(U, \mathcal{F})$. As $\mathcal{A}(U)$ is a finite $\mathcal{O}_X(U)$-algebra we see that for each j there exists a monic polynomial $P_j \in \mathcal{O}(U)[T]$ such that $P_j(f_j)$ is zero in $\mathcal{A}(U)$. Since $\mathcal{A} = \text{colim}\, \mathcal{A}_i$ by construction, we have $P_j(f_j) = 0$ in $\mathcal{A}_i(U)$ for all sufficiently large i. For such i the algebras $\mathcal{A}_i$ are finite. □

082C **Lemma 9.7.** *Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S. Let $\mathcal{A}$ be an integral quasi-coherent $\mathcal{O}_X$-algebra. Then*

1. *$\mathcal{A}$ is the directed colimit of its finite quasi-coherent $\mathcal{O}_X$-subalgebras, and*
2. *$\mathcal{A}$ is a directed colimit of finite and finitely presented $\mathcal{O}_X$-algebras.*

Proof. By Lemma 9.5 we have $\mathcal{A} = \text{colim}\, \mathcal{A}_i$ where $\mathcal{A}_i \subset \mathcal{A}$ runs through the quasi-coherent $\mathcal{O}_X$-sub algebras of finite type. Any finite type quasi-coherent $\mathcal{O}_X$-subalgebra of $\mathcal{A}$ is finite (use Algebra, Lemma 35.5 on affine schemes étale over X). This proves (1).

To prove (2), write $\mathcal{A} = \text{colim}\, \mathcal{F}_i$ as a colimit of finitely presented $\mathcal{O}_X$-modules using Lemma 9.1. For each i, let $\mathcal{J}_i$ be the kernel of the map

$$\text{Sym}^*_{\mathcal{O}_X}(\mathcal{F}_i) \longrightarrow \mathcal{A}$$

For $i' \geq i$ there is an induced map $\mathcal{J}_i \to \mathcal{J}_{i'}$ and we have $\mathcal{A} = \text{colim}\, \text{Sym}^*_{\mathcal{O}_X}(\mathcal{F}_i)/\mathcal{J}_i$. Moreover, the quasi-coherent $\mathcal{O}_X$-algebras $\text{Sym}^*_{\mathcal{O}_X}(\mathcal{F}_i)/\mathcal{J}_i$ are finite (see above).

Write $\mathcal{J}_i = \operatorname{colim} \mathcal{E}_{ik}$ as a colimit of finitely presented $\mathcal{O}_X$-modules. Given $i' \geq i$ and k there exists a k' such that we have a map $\mathcal{E}_{ik} \to \mathcal{E}_{i'k'}$ making

$$\begin{array}{ccc} \mathcal{J}_i & \longrightarrow & \mathcal{J}_{i'} \\ \uparrow & & \uparrow \\ \mathcal{E}_{ik} & \longrightarrow & \mathcal{E}_{i'k'} \end{array}$$

commute. This follows from Cohomology of Spaces, Lemma 5.3. This induces a map

$$\mathcal{A}_{ik} = \text{Sym}^*_{\mathcal{O}_X}(\mathcal{F}_i)/(\mathcal{E}_{ik}) \longrightarrow \text{Sym}^*_{\mathcal{O}_X}(\mathcal{F}_{i'})/(\mathcal{E}_{i'k'}) = \mathcal{A}_{i'k'}$$

where $(\mathcal{E}_{ik})$ denotes the ideal generated by $\mathcal{E}_{ik}$. The quasi-coherent $\mathcal{O}_X$-algebras $\mathcal{A}_{ki}$ are of finite presentation and finite for k large enough (see proof of Lemma 9.6). Finally, we have

$$\operatorname{colim} \mathcal{A}_{ik} = \operatorname{colim} \mathcal{A}_i = \mathcal{A}$$

Namely, the first equality was shown in the proof of Lemma 9.6 and the second equality because $\mathcal{A}$ is the colimit of the modules $\mathcal{F}_i$. $\square$

0853 **Lemma 9.8.** *Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S. Let $U \subset X$ be a quasi-compact open. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $\mathcal{G} \subset \mathcal{F}|_U$ be a quasi-coherent $\mathcal{O}_U$-submodule which is of finite type. Then there exists a quasi-coherent submodule $\mathcal{G}' \subset \mathcal{F}$ which is of finite type such that $\mathcal{G}'|_U = \mathcal{G}$.*

Proof. Denote $j : U \to X$ the inclusion morphism. As X is quasi-separated and U quasi-compact, the morphism j is quasi-compact. Hence $j_*\mathcal{G} \subset j_*\mathcal{F}|_U$ are quasi-coherent modules on X (Morphisms of Spaces, Lemma 11.2). Let $\mathcal{H} = \text{Ker}(j_*\mathcal{G} \oplus \mathcal{F} \to j_*\mathcal{F}|_U)$. Then $\mathcal{H}|_U = \mathcal{G}$. By Lemma 9.2 we can find a finite type quasi-coherent submodule $\mathcal{H}' \subset \mathcal{H}$ such that $\mathcal{H}'|_U = \mathcal{H}|_U = \mathcal{G}$. Set $\mathcal{G}' = \text{Im}(\mathcal{H}' \to \mathcal{F})$ to conclude. $\square$

10. Relative approximation

09NR The title of this section refers to the following result.

09NS **Lemma 10.1.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume that*

1. *X is quasi-compact and quasi-separated, and*
2. *Y is quasi-separated.*

Then $X = \lim X_i$ is a limit of a directed inverse system of algebraic spaces X_i of finite presentation over Y with affine transition morphisms over Y.

Proof. Since $|f|(|X|)$ is quasi-compact we may replace Y by a quasi-compact open subspace whose set of points contains $|f|(|X|)$. Hence we may assume Y is quasi-compact as well. Write $X = \lim X_a$ and $Y = \lim Y_b$ as in Proposition 8.1, i.e., with X_a and Y_b of finite type over $\mathbf{Z}$ and with affine transition morphisms. By Proposition 3.9 we find that for each b there exists an a and a morphism $f_{a,b} : X_a \to Y_b$ making the diagram

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ X_a & \longrightarrow & Y_b \end{array}$$

commute. Moreover the same proposition implies that, given a second triple $(a', b', f_{a',b'})$, there exists an $a'' \geq a'$ such that the compositions $X_{a''} \to X_a \to X_b$ and $X_{a''} \to X_{a'} \to X_{b'} \to X_b$ are equal. Consider the set of triples $(a, b, f_{a,b})$ endowed with the preordering

$$(a, b, f_{a,b}) \geq (a', b', f_{a',b'}) \Leftrightarrow a \geq a',\ b' \geq b, \text{ and } f_{a',b'} \circ h_{a,a'} = g_{b',b} \circ f_{a,b}$$

where $h_{a,a'} : X_a \to X_{a'}$ and $g_{b',b} : Y_{b'} \to Y_b$ are the transition morphisms. The remarks above show that this system is directed. It follows formally from the equalities $X = \lim X_a$ and $Y = \lim Y_b$ that

$$X = \lim_{(a,b,f_{a,b})} X_a \times_{f_{a,b}, Y_b} Y.$$

where the limit is over our directed system above. The transition morphisms $X_a \times_{Y_b} Y \to X_{a'} \times_{Y_{b'}} Y$ are affine as the composition

$$X_a \times_{Y_b} Y \to X_a \times_{Y_{b'}} Y \to X_{a'} \times_{Y_{b'}} Y$$

where the first morphism is a closed immersion (by Morphisms of Spaces, Lemma 4.5) and the second is a base change of an affine morphism (Morphisms of Spaces, Lemma 20.5) and the composition of affine morphisms is affine (Morphisms of Spaces, Lemma 20.4). The morphisms $f_{a,b}$ are of finite presentation (Morphisms of Spaces, Lemmas 28.7 and 28.9) and hence the base changes $X_a \times_{f_{a,b}, S_b} S \to S$ are of finite presentation (Morphisms of Spaces, Lemma 28.3). □

11. Finite type closed in finite presentation

07SP This section is the analogue of Limits, Section 9.

0870 **Lemma 11.1.** *Let S be a scheme. Let $f : X \to Y$ be an affine morphism of algebraic spaces over S. If Y quasi-compact and quasi-separated, then X is a directed limit $X = \lim X_i$ with each X_i affine and of finite presentation over Y.*

Proof. Consider the quasi-coherent $\mathcal{O}_Y$-module $\mathcal{A} = f_*\mathcal{O}_X$. By Lemma 9.4 we can write $\mathcal{A} = \operatorname{colim} \mathcal{A}_i$ as a directed colimit of finitely presented $\mathcal{O}_Y$-algebras $\mathcal{A}_i$. Set $X_i = \underline{\mathrm{Spec}}_Y(\mathcal{A}_i)$, see Morphisms of Spaces, Definition 20.8. By construction $X_i \to Y$ is affine and of finite presentation and $X = \lim X_i$. □

09YA **Lemma 11.2.** *Let S be a scheme. Let $f : X \to Y$ be an integral morphism of algebraic spaces over S. Assume Y quasi-compact and quasi-separated. Then X can be written as a directed limit $X = \lim X_i$ where X_i are finite and of finite presentation over Y.*

Proof. Consider the finite quasi-coherent $\mathcal{O}_Y$-module $\mathcal{A} = f_*\mathcal{O}_X$. By Lemma 9.7 we can write $\mathcal{A} = \operatorname{colim} \mathcal{A}_i$ as a directed colimit of finite and finitely presented $\mathcal{O}_Y$-algebras $\mathcal{A}_i$. Set $X_i = \underline{\mathrm{Spec}}_Y(\mathcal{A}_i)$, see Morphisms of Spaces, Definition 20.8. By construction $X_i \to Y$ is finite and of finite presentation and $X = \lim X_i$. □

07VR **Lemma 11.3.** *Let S be a scheme. Let $f : X \to Y$ be a finite morphism of algebraic spaces over S. Assume Y quasi-compact and quasi-separated. Then X can be written as a directed limit $X = \lim X_i$ where the transition maps are closed immersions and the objects X_i are finite and of finite presentation over Y.*

Proof. Consider the finite quasi-coherent $\mathcal{O}_Y$-module $\mathcal{A} = f_*\mathcal{O}_X$. By Lemma 9.6 we can write $\mathcal{A} = \operatorname{colim} \mathcal{A}_i$ as a directed colimit of finite and finitely presented $\mathcal{O}_Y$-algebras $\mathcal{A}_i$ with surjective transition maps. Set $X_i = \underline{\mathrm{Spec}}_Y(\mathcal{A}_i)$, see Morphisms of

Spaces, Definition 20.8. By construction $X_i \to Y$ is finite and of finite presentation, the transition maps are closed immersions, and $X = \lim X_i$. $\square$

0A0U **Lemma 11.4.** *Let S be a scheme. Let $f : X \to Y$ be a closed immersion of algebraic spaces over S. Assume Y quasi-compact and quasi-separated. Then X can be written as a directed limit $X = \lim X_i$ where the transition maps are closed immersions and the morphisms $X_i \to Y$ are closed immersions of finite presentation.*

Proof. Let $\mathcal{I} \subset \mathcal{O}_Y$ be the quasi-coherent sheaf of ideals defining X as a closed subspace of Y. By Lemma 9.2 we can write $\mathcal{I} = \operatorname{colim} \mathcal{I}_i$ as the filtered colimit of its finite type quasi-coherent submodules. Let X_i be the closed subspace of X cut out by $\mathcal{I}_i$. Then $X_i \to Y$ is a closed immersion of finite presentation, and $X = \lim X_i$. Some details omitted. $\square$

0871 **Lemma 11.5.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume*

(1) *f is locally of finite type and quasi-affine, and*
(2) *Y is quasi-compact and quasi-separated.*

Then there exists a morphism of finite presentation $f' : X' \to Y$ and a closed immersion $X \to X'$ over Y.

Proof. By Morphisms of Spaces, Lemma 21.6 we can find a factorization $X \to Z \to Y$ where $X \to Z$ is a quasi-compact open immersion and $Z \to Y$ is affine. Write $Z = \lim Z_i$ with Z_i affine and of finite presentation over Y (Lemma 11.1). For some $0 \in I$ we can find a quasi-compact open $U_0 \subset Z_0$ such that X is isomorphic to the inverse image of U_0 in Z (Lemma 5.7). Let U_i be the inverse image of U_0 in Z_i, so $U = \lim U_i$. By Lemma 5.12 we see that $X \to U_i$ is a closed immersion for some i large enough. Setting $X' = U_i$ finishes the proof. $\square$

0872 **Lemma 11.6.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume:*

(1) *f is of locally of finite type.*
(2) *X is quasi-compact and quasi-separated, and*
(3) *Y is quasi-compact and quasi-separated.*

Then there exists a morphism of finite presentation $f' : X' \to Y$ and a closed immersion $X \to X'$ of algebraic spaces over Y.

Proof. By Proposition 8.1 we can write $X = \lim_i X_i$ with X_i quasi-separated of finite type over $\mathbf{Z}$ and with transition morphisms $f_{ii'} : X_i \to X_{i'}$ affine. Consider the commutative diagram

$$\begin{tikzcd} X \arrow[r] \arrow[rd] & X_{i,Y} \arrow[r] \arrow[d] & X_i \arrow[d] \\ & Y \arrow[r] & \operatorname{Spec}(\mathbf{Z}) \end{tikzcd}$$

Note that X_i is of finite presentation over $\operatorname{Spec}(\mathbf{Z})$, see Morphisms of Spaces, Lemma 28.7. Hence the base change $X_{i,Y} \to Y$ is of finite presentation by Morphisms of Spaces, Lemma 28.3. Observe that $\lim X_{i,Y} = X \times Y$ and that $X \to X \times Y$ is a monomorphism. By Lemma 5.12 we see that $X \to X_{i,Y}$ is a monomorphism for i large enough. Fix such an i. Note that $X \to X_{i,Y}$ is locally of finite type

(Morphisms of Spaces, Lemma 23.6) and a monomorphism, hence separated and locally quasi-finite (Morphisms of Spaces, Lemma 27.10). Hence $X \to X_{i,Y}$ is representable. Hence $X \to X_{i,Y}$ is quasi-affine because we can use the principle Spaces, Lemma 5.8 and the result for morphisms of schemes More on Morphisms, Lemma 38.2. Thus Lemma 11.5 gives a factorization $X \to X' \to X_{i,Y}$ with $X \to X'$ a closed immersion and $X' \to X_{i,Y}$ of finite presentation. Finally, $X' \to Y$ is of finite presentation as a composition of morphisms of finite presentation (Morphisms of Spaces, Lemma 28.2). □

0873 **Proposition 11.7.** *Let S be a scheme. $f : X \to Y$ be a morphism of algebraic spaces over S. Assume*

(1) *f is of finite type and separated, and*
(2) *Y is quasi-compact and quasi-separated.*

Then there exists a separated morphism of finite presentation $f' : X' \to Y$ and a closed immersion $X \to X'$ over Y.

Proof. By Lemma 11.6 there is a closed immersion $X \to Z$ with Z/Y of finite presentation. Let $\mathcal{I} \subset \mathcal{O}_Z$ be the quasi-coherent sheaf of ideals defining X as a closed subscheme of Y. By Lemma 9.2 we can write $\mathcal{I}$ as a directed colimit $\mathcal{I} = \text{colim}_{a \in A} \mathcal{I}_a$ of its quasi-coherent sheaves of ideals of finite type. Let $X_a \subset Z$ be the closed subspace defined by $\mathcal{I}_a$. These form an inverse system indexed by A. The transition morphisms $X_a \to X_{a'}$ are affine because they are closed immersions. Each X_a is quasi-compact and quasi-separated since it is a closed subspace of Z and Z is quasi-compact and quasi-separated by our assumptions. We have $X = \lim_a X_a$ as follows directly from the fact that $\mathcal{I} = \text{colim}_{a \in A} \mathcal{I}_a$. Each of the morphisms $X_a \to Z$ is of finite presentation, see Morphisms, Lemma 20.7. Hence the morphisms $X_a \to Y$ are of finite presentation. Thus it suffices to show that $X_a \to Y$ is separated for some $a \in A$. This follows from Lemma 5.13 as we have assumed that $X \to Y$ is separated. □

12. Approximating proper morphisms

0A0V

0A0W **Lemma 12.1.** *Let S be a scheme. Let $f : X \to Y$ be a proper morphism of algebraic spaces over S with Y quasi-compact and quasi-separated. Then $X = \lim X_i$ is a directed limit of algebraic spaces X_i proper and of finite presentation over Y and with transition morphisms and morphisms $X \to X_i$ closed immersions.*

Proof. By Proposition 11.7 we can find a closed immersion $X \to X'$ with X' separated and of finite presentation over Y. By Lemma 11.4 we can write $X = \lim X_i$ with $X_i \to X'$ a closed immersion of finite presentation. We claim that for all i large enough the morphism $X_i \to Y$ is proper which finishes the proof.

To prove this we may assume that Y is an affine scheme, see Morphisms of Spaces, Lemma 39.2. Next, we use the weak version of Chow's lemma, see Cohomology of Spaces, Lemma 18.1, to find a diagram

$$\begin{tikzcd} X' \arrow[dr] & X'' \arrow[l, "\pi"'] \arrow[r] \arrow[d] & \mathbf{P}^n_Y \arrow[dl] \\ & Y & \end{tikzcd}$$

where $X'' \to \mathbf{P}^n_Y$ is an immersion, and $\pi : X'' \to X'$ is proper and surjective. Denote $X_i' \subset X''$, resp. $\pi^{-1}(X)$ the scheme theoretic inverse image of $X_i \subset X'$, resp. $X \subset X'$. Then $\lim X_i' = \pi^{-1}(X)$. Since $\pi^{-1}(X) \to Y$ is proper (Morphisms of Spaces, Lemmas 39.4), we see that $\pi^{-1}(X) \to \mathbf{P}^n_Y$ is a closed immersion (Morphisms of Spaces, Lemmas 39.6 and 12.3). Hence for i large enough we find that $X_i' \to \mathbf{P}^n_Y$ is a closed immersion by Lemma 5.16. Thus X_i' is proper over Y. For such i the morphism $X_i \to Y$ is proper by Morphisms of Spaces, Lemma 39.7. □

0A0X **Lemma 12.2.** *Let $f : X \to Y$ be a proper morphism of algebraic spaces over $\mathbf{Z}$ with Y quasi-compact and quasi-separated. Then there exists a directed set I, an inverse system $(f_i : X_i \to Y_i)$ of morphisms of algebraic spaces over I, such that the transition morphisms $X_i \to X_{i'}$ and $Y_i \to Y_{i'}$ are affine, such that f_i is proper and of finite presentation, such that Y_i is of finite presentation over $\mathbf{Z}$, and such that $(X \to Y) = \lim(X_i \to Y_i)$.*

Proof. By Lemma 12.1 we can write $X = \lim_{k \in K} X_k$ with $X_k \to Y$ proper and of finite presentation. Next, by absolute Noetherian approximation (Proposition 8.1) we can write $Y = \lim_{j \in J} Y_j$ with Y_j of finite presentation over $\mathbf{Z}$. For each k there exists a j and a morphism $X_{k,j} \to Y_j$ of finite presentation with $X_k \cong Y \times_{Y_j} X_{k,j}$ as algebraic spaces over Y, see Lemma 7.1. After increasing j we may assume $X_{k,j} \to Y_j$ is proper, see Lemma 6.13. The set I will be consist of these pairs (k, j) and the corresponding morphism is $X_{k,j} \to Y_j$. For every $k' \geq k$ we can find a $j' \geq j$ and a morphism $X_{j',k'} \to X_{j,k}$ over $Y_{j'} \to Y_j$ whose base change to Y gives the morphism $X_{k'} \to X_k$ (follows again from Lemma 7.1). These morphisms form the transition morphisms of the system. Some details omitted. □

Recall the scheme theoretic support of a finite type quasi-coherent module, see Morphisms of Spaces, Definition 15.4.

08K2 **Lemma 12.3.** *Assumptions and notation as in Situation 6.1. Let $\mathcal{F}_0$ be a quasi-coherent $\mathcal{O}_{X_0}$-module. Denote $\mathcal{F}$ and $\mathcal{F}_i$ the pullbacks of $\mathcal{F}_0$ to X and X_i. Assume*

1. *f_0 is locally of finite type,*
2. *$\mathcal{F}_0$ is of finite type,*
3. *the scheme theoretic support of $\mathcal{F}$ is proper over Y.*

Then the scheme theoretic support of $\mathcal{F}_i$ is proper over Y_i for some i.

Proof. We may replace X_0 by the scheme theoretic support of $\mathcal{F}_0$. By Morphisms of Spaces, Lemma 15.2 this guarantees that X_i is the support of $\mathcal{F}_i$ and X is the support of $\mathcal{F}$. Then, if $Z \subset X$ denotes the scheme theoretic support of $\mathcal{F}$, we see that $Z \to X$ is a universal homeomorphism. We conclude that $X \to Y$ is proper as this is true for $Z \to Y$ by assumption, see Morphisms, Lemma 39.8. By Lemma 6.13 we see that $X_i \to Y$ is proper for some i. Then it follows that the scheme theoretic support Z_i of $\mathcal{F}_i$ is proper over Y by Morphisms of Spaces, Lemmas 39.5 and 39.4. □

13. Embedding into affine space

088K Some technical lemmas to be used in the proof of Chow's lemma later.

088L **Lemma 13.1.** *Let S be a scheme. Let $f : U \to X$ be a morphism of algebraic spaces over S. Assume U is an affine scheme, f is locally of finite type, and X*

quasi-separated and locally separated. Then there exists an immersion $U \to \mathbf{A}^n_X$ over X.

Proof. Say $U = \mathrm{Spec}(A)$. Write $A = \mathrm{colim}\, A_i$ as a filtered colimit of finite type $\mathbf{Z}$-subalgebras. For each i the morphism $U \to U_i = \mathrm{Spec}(A_i)$ induces a morphism

$$U \longrightarrow X \times U_i$$

over X. In the limit the morphism $U \to X \times U$ is an immersion as X is locally separated, see Morphisms of Spaces, Lemma 4.6. By Lemma 5.12 we see that $U \to X \times U_i$ is an immersion for some i. Since U_i is isomorphic to a closed subscheme of $\mathbf{A}^n_{\mathbf{Z}}$ the lemma follows. $\square$

088M **Remark 13.2.** We have seen in Examples, Section 22 that Lemma 13.1 does not hold if we drop the assumption that X be locally separated. This raises the question: Does Lemma 13.1 hold if we drop the assumption that X be quasi-separated? If you know the answer, please email stacks.project@gmail.com.

088N **Lemma 13.3.** *Let S be a scheme. Let $f : Y \to X$ be a morphism of algebraic spaces over S. Assume X Noetherian and f of finite presentation. Then there exists a dense open $V \subset Y$ and an immersion $V \to \mathbf{A}^n_X$.*

Proof. The assumptions imply that Y is Noetherian (Morphisms of Spaces, Lemma 28.6). Then Y is quasi-separated, hence has a dense open subscheme (Properties of Spaces, Proposition 12.3). Thus we may assume that Y is a Noetherian scheme. By removing intersections of irreducible components of Y (use Topology, Lemma 9.2 and Properties, Lemma 5.5) we may assume that Y is a disjoint union of irreducible Noetherian schemes. Since there is an immersion

$$\mathbf{A}^n_X \amalg \mathbf{A}^m_X \longrightarrow \mathbf{A}^{\max(n,m)+1}_X$$

(details omitted) we see that it suffices to prove the result in case Y is irreducible.

Assume Y is an irreducible scheme. Let $T \subset |X|$ be the closure of the image of $f : Y \to X$. Note that since $|Y|$ and $|X|$ are sober topological spaces (Properties of Spaces, Lemma 14.1) T is irreducible with a unique generic point ξ which is the image of the generic point η of Y. Let $\mathcal{I} \subset X$ be a quasi-coherent sheaf of ideals cutting out the reduced induced space structure on T (Properties of Spaces, Definition 11.6). Since $\mathcal{O}_{Y,\eta}$ is an Artinian local ring we see that for some $n > 0$ we have $f^{-1}\mathcal{I}^n\mathcal{O}_{Y,\eta} = 0$. As $f^{-1}\mathcal{I}\mathcal{O}_Y$ is a finite type quasi-coherent ideal we conclude that $f^{-1}\mathcal{I}^n\mathcal{O}_V = 0$ for some nonempty open $V \subset Y$. Let $Z \subset X$ be the closed subspace cut out by $\mathcal{I}^n$. By construction $V \to Y \to X$ factors through Z. Because $\mathbf{A}^n_Z \to \mathbf{A}^n_X$ is an immersion, we may replace X by Z and Y by V. Hence we reach the situation where Y and X are irreducible and $Y \to X$ maps the generic point of Y onto the generic point of X.

Assume Y and X are irreducible, Y is a scheme, and $Y \to X$ maps the generic point of Y onto the generic point of X. By Properties of Spaces, Proposition 12.3 X has a dense open subscheme $U \subset X$. Choose a nonempty affine open $V \subset Y$ whose image in X is contained in U. By Morphisms, Lemma 37.2 we may factor $V \to U$ as $V \to \mathbf{A}^n_U \to U$. Composing with $\mathbf{A}^n_U \to \mathbf{A}^n_X$ we obtain the desired immersion. $\square$

14. Sections with support in a closed subset

0854 This section is the analogue of Properties, Section 24.

0855 **Lemma 14.1.** *Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space. Let $U \subset X$ be an open subspace. The following are equivalent:*

1. *$U \to X$ is quasi-compact,*
2. *U is quasi-compact, and*
3. *there exists a finite type quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_X$ such that $|X| \setminus |U| = |V(\mathcal{I})|$.*

Proof. Let W be an affine scheme and let $\varphi : W \to X$ be a surjective étale morphism, see Properties of Spaces, Lemma 6.3. If (1) holds, then $\varphi^{-1}(U) \to W$ is quasi-compact, hence $\varphi^{-1}(U)$ is quasi-compact, hence U is quasi-compact (as $|\varphi^{-1}(U)| \to |U|$ is surjective). If (2) holds, then $\varphi^{-1}(U)$ is quasi-compact because φ is quasi-compact since X is quasi-separated (Morphisms of Spaces, Lemma 8.9). Hence $\varphi^{-1}(U) \to W$ is a quasi-compact morphism of schemes by Properties, Lemma 24.1. It follows that $U \to X$ is quasi-compact by Morphisms of Spaces, Lemma 8.7. Thus (1) and (2) are equivalent.

Assume (1) and (2). By Properties of Spaces, Lemma 11.4 there exists a unique quasi-coherent sheaf of ideals $\mathcal{J}$ cutting out the reduced induced closed subspace structure on $|X| \setminus |U|$. Note that $\mathcal{J}|_U = \mathcal{O}_U$ which is an $\mathcal{O}_U$-modules of finite type. As U is quasi-compact it follows from Lemma 9.2 that there exists a quasi-coherent subsheaf $\mathcal{I} \subset \mathcal{J}$ which is of finite type and has the property that $\mathcal{I}|_U = \mathcal{J}|_U$. Then $|X| \setminus |U| = |V(\mathcal{I})|$ and we obtain (3). Conversely, if $\mathcal{I}$ is as in (3), then $\varphi^{-1}(U) \subset W$ is a quasi-compact open by the lemma for schemes (Properties, Lemma 24.1) applied to $\varphi^{-1}\mathcal{I}$ on W. Thus (2) holds. □

0856 **Lemma 14.2.** *Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Consider the sheaf of $\mathcal{O}_X$-modules $\mathcal{F}'$ which associates to every object U of $X_{étale}$ the module*

$$\mathcal{F}'(U) = \{s \in \mathcal{F}(U) \mid \mathcal{I}s = 0\}$$

Assume $\mathcal{I}$ is of finite type. Then

1. *$\mathcal{F}'$ is a quasi-coherent sheaf of $\mathcal{O}_X$-modules,*
2. *for affine U in $X_{étale}$ we have $\mathcal{F}'(U) = \{s \in \mathcal{F}(U) \mid \mathcal{I}(U)s = 0\}$, and*
3. *$\mathcal{F}'_x = \{s \in \mathcal{F}_x \mid \mathcal{I}_x s = 0\}$.*

Proof. It is clear that the rule defining $\mathcal{F}'$ gives a subsheaf of $\mathcal{F}$. Hence we may work étale locally on X to verify the other statements. Thus the lemma reduces to the case of schemes which is Properties, Lemma 24.2. □

0857 **Definition 14.3.** Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals of finite type. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. The subsheaf $\mathcal{F}' \subset \mathcal{F}$ defined in Lemma 14.2 above is called the *subsheaf of sections annihilated by $\mathcal{I}$*.

0858 **Lemma 14.4.** *Let S be a scheme. Let $f : X \to Y$ be a quasi-compact and quasi-separated morphism of algebraic spaces over S. Let $\mathcal{I} \subset \mathcal{O}_Y$ be a quasi-coherent sheaf of ideals of finite type. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $\mathcal{F}' \subset \mathcal{F}$ be the subsheaf of sections annihilated by $f^{-1}\mathcal{I}\mathcal{O}_X$. Then $f_*\mathcal{F}' \subset f_*\mathcal{F}$ is the subsheaf of sections annihilated by $\mathcal{I}$.*

Proof. Omitted. Hint: The assumption that f is quasi-compact and quasi-separated implies that $f_*\mathcal{F}$ is quasi-coherent (Morphisms of Spaces, Lemma 11.2) so that Lemma 14.2 applies to $\mathcal{I}$ and $f_*\mathcal{F}$. $\square$

Next we come to the sheaf of sections supported in a closed subset. Again this isn't always a quasi-coherent sheaf, but if the complement of the closed is "retrocompact" in the given algebraic space, then it is.

0859 **Lemma 14.5.** *Let S be a scheme. Let X be an algebraic space over S. Let $T \subset |X|$ be a closed subset and let $U \subset X$ be the open subspace such that $T \amalg |U| = |X|$. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Consider the sheaf of $\mathcal{O}_X$-modules $\mathcal{F}'$ which associates to every object $\varphi : W \to X$ of $X_{étale}$ the module*

$$\mathcal{F}'(W) = \{s \in \mathcal{F}(W) \mid \textit{the support of } s \textit{ is contained in } |\varphi|^{-1}(T)\}$$

If $U \to X$ is quasi-compact, then

1. *for W affine there exist a finitely generated ideal $I \subset \mathcal{O}_X(W)$ such that $|\varphi|^{-1}(T) = V(I)$,*
2. *for W and I as in (1) we have $\mathcal{F}'(W) = \{x \in \mathcal{F}(W) \mid I^n x = 0 \text{ for some } n\}$,*
3. *$\mathcal{F}'$ is a quasi-coherent sheaf of $\mathcal{O}_X$-modules.*

Proof. It is clear that the rule defining $\mathcal{F}'$ gives a subsheaf of $\mathcal{F}$. Hence we may work étale locally on X to verify the other statements. Thus the lemma reduces to the case of schemes which is Properties, Lemma 24.5. $\square$

085A **Definition 14.6.** Let S be a scheme. Let X be an algebraic space over S. Let $T \subset |X|$ be a closed subset whose complement corresponds to an open subspace $U \subset X$ with quasi-compact inclusion morphism $U \to X$. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. The quasi-coherent subsheaf $\mathcal{F}' \subset \mathcal{F}$ defined in Lemma 14.5 above is called the *subsheaf of sections supported on T*.

085B **Lemma 14.7.** *Let S be a scheme. Let $f : X \to Y$ be a quasi-compact and quasi-separated morphism of algebraic spaces over S. Let $T \subset |Y|$ be a closed subset. Assume $|Y| \setminus T$ corresponds to an open subspace $V \subset Y$ such that $V \to Y$ is quasi-compact. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $\mathcal{F}' \subset \mathcal{F}$ be the subsheaf of sections supported on $|f|^{-1}T$. Then $f_*\mathcal{F}' \subset f_*\mathcal{F}$ is the subsheaf of sections supported on T.*

Proof. Omitted. Hints: $|X| \setminus |f|^{-1}T$ is the support of the open subspace $U = f^{-1}V \subset X$. Since $V \to Y$ is quasi-compact, so is $U \to X$ (by base change). The assumption that f is quasi-compact and quasi-separated implies that $f_*\mathcal{F}$ is quasi-coherent. Hence Lemma 14.5 applies to T and $f_*\mathcal{F}$ as well as to $|f|^{-1}T$ and $\mathcal{F}$. The equality of the given quasi-coherent modules is immediate from the definitions. $\square$

15. Characterizing affine spaces

07VQ This section is the analogue of Limits, Section 11.

07VS **Lemma 15.1.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume that f is surjective and finite, and assume that X is affine. Then Y is affine.*

Proof. We may and do view $f : X \to Y$ as a morphism of algebraic space over $\mathrm{Spec}(\mathbf{Z})$ (see Spaces, Definition 16.2). Note that a finite morphism is affine and universally closed, see Morphisms of Spaces, Lemma 44.7. By Morphisms of Spaces, Lemma 9.8 we see that Y is a separated algebraic space. As f is surjective and X is quasi-compact we see that Y is quasi-compact.

By Lemma 11.3 we can write $X = \lim X_a$ with each $X_a \to Y$ finite and of finite presentation. By Lemma 5.10 we see that X_a is affine for a large enough. Hence we may and do assume that $f : X \to Y$ is finite, surjective, and of finite presentation.

By Proposition 8.1 we may write $Y = \lim Y_i$ as a directed limit of algebraic spaces of finite presentation over $\mathbf{Z}$. By Lemma 7.1 we can find $0 \in I$ and a morphism $X_0 \to Y_0$ of finite presentation such that $X_i = X_0 \times_{Y_0} Y_i$ for $i \geq 0$ and such that $X = \lim_i X_i$. By Lemma 6.7 we see that $X_i \to Y_i$ is finite for i large enough. By Lemma 6.4 we see that $X_i \to Y_i$ is surjective for i large enough. By Lemma 5.10 we see that X_i is affine for i large enough. Hence for i large enough we can apply Cohomology of Spaces, Lemma 17.1 to conclude that Y_i is affine. This implies that Y is affine and we conclude. $\square$

07VT **Proposition 15.2.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume that f is surjective and integral, and assume that X is affine. Then Y is affine.*

Proof. We may and do view $f : X \to Y$ as a morphism of algebraic spaces over $\mathrm{Spec}(\mathbf{Z})$ (see Spaces, Definition 16.2). Note that integral morphisms are affine and universally closed, see Morphisms of Spaces, Lemma 44.7. By Morphisms of Spaces, Lemma 9.8 we see that Y is a separated algebraic space. As f is surjective and X is quasi-compact we see that Y is quasi-compact.

Consider the sheaf $\mathcal{A} = f_*\mathcal{O}_X$. This is a quasi-coherent sheaf of $\mathcal{O}_Y$-algebras, see Morphisms of Spaces, Lemma 11.2. By Lemma 9.1 we can write $\mathcal{A} = \mathrm{colim}_i \mathcal{F}_i$ as a filtered colimit of finite type $\mathcal{O}_Y$-modules. Let $\mathcal{A}_i \subset \mathcal{A}$ be the $\mathcal{O}_Y$-subalgebra generated by $\mathcal{F}_i$. Since the map of algebras $\mathcal{O}_Y \to \mathcal{A}$ is integral, we see that each $\mathcal{A}_i$ is a finite quasi-coherent $\mathcal{O}_Y$-algebra. Hence

$$X_i = \underline{\mathrm{Spec}}_Y(\mathcal{A}_i) \longrightarrow Y$$

is a finite morphism of algebraic spaces. Here $\underline{\mathrm{Spec}}$ is the construction of Morphisms of Spaces, Lemma 20.7. It is clear that $X = \lim_i X_i$. Hence by Lemma 5.10 we see that for i sufficiently large the scheme X_i is affine. Moreover, since $X \to Y$ factors through each X_i we see that $X_i \to Y$ is surjective. Hence we conclude that Y is affine by Lemma 15.1. $\square$

The following corollary of the result above can be found in [CLO12].

07VU **Lemma 15.3.** *Let S be a scheme. Let X be an algebraic space over S. If X_{red} is a scheme, then X is a scheme.* [CLO12, 3.1.12]

Proof. Let $U' \subset X_{red}$ be an open affine subscheme. Let $U \subset X$ be the open subspace corresponding to the open $|U'| \subset |X_{red}| = |X|$. Then $U' \to U$ is surjective and integral. Hence U is affine by Proposition 15.2. Thus every point is contained in an open subscheme of X, i.e., X is a scheme. $\square$

07VV **Lemma 15.4.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume f is integral and induces a bijection $|X| \to |Y|$. Then X is a scheme if and only if Y is a scheme.*

Proof. An integral morphism is representable by definition, hence if Y is a scheme, so is X. Conversely, assume that X is a scheme. Let $U \subset X$ be an affine open. An integral morphism is closed and $|f|$ is bijective, hence $|f|(|U|) \subset |Y|$ is open as the complement of $|f|(|X| \setminus |U|)$. Let $V \subset Y$ be the open subspace with $|V| = |f|(|U|)$, see Properties of Spaces, Lemma 4.8. Then $U \to V$ is integral and surjective, hence V is an affine scheme by Proposition 15.2. This concludes the proof. □

08B2 **Lemma 15.5.** *Let S be a scheme. Let $f : X \to B$ and $B' \to B$ be morphisms of algebraic spaces over S. Assume*

1. *$B' \to B$ is a closed immersion,*
2. *$|B'| \to |B|$ is bijective,*
3. *$X \times_B B' \to B'$ is a closed immersion, and*
4. *$X \to B$ is of finite type or $B' \to B$ is of finite presentation.*

Then $f : X \to B$ is a closed immersion.

Proof. Assumptions (1) and (2) imply that $B_{red} = B'_{red}$. Set $X' = X \times_B B'$. Then $X' \to X$ is closed immersion and $X'_{red} = X_{red}$. Let $U \to B$ be an étale morphism with U affine. Then $X' \times_B U \to X \times_B U$ is a closed immersion of algebraic spaces inducing an isomorphism on underlying reduced spaces. Since $X' \times_B U$ is a scheme (as $B' \to B$ and $X' \to B'$ are representable) so is $X \times_B U$ by Lemma 15.3. Hence $X \to B$ is representable too. Thus we reduce to the case of schemes, see Morphisms, Lemma 43.7. □

16. Finite cover by a scheme

0ACX As an application of the limit results of this chapter, we prove that given any quasi-compact and quasi-separated algebraic space X, there is a scheme Y and a surjective, finite morphism $Y \to X$. We will rely on the already proven result that we can find a finite integral cover by a scheme, which was proved in Decent Spaces, Section 9.

09YC **Proposition 16.1.** *Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S.*

1. *There exists a surjective finite morphism $Y \to X$ of finite presentation where Y is a scheme,*
2. *given a surjective étale morphism $U \to X$ we may choose $Y \to X$ such that for every $y \in Y$ there is an open neighbourhood $V \subset Y$ such that $V \to X$ factors through U.*

Proof. Part (1) is the special case of (2) with $U = X$. Let $Y \to X$ be as in Decent Spaces, Lemma 9.1. Choose a finite affine open covering $Y = \bigcup V_j$ such that $V_j \to X$ factors through U. We can write $Y = \lim Y_i$ with $Y_i \to X$ finite and of finite presentation, see Lemma 11.2. For large enough i the algebraic space Y_i is a scheme, see Lemma 5.11. For large enough i we can find affine opens $V_{i,j} \subset Y_i$ whose inverse image in Y recovers V_j, see Lemma 5.7. For even larger i the morphisms $V_j \to U$ over X come from morphisms $V_{i,j} \to U$ over X, see Proposition 3.9. This finishes the proof. □

17. Obtaining schemes

0B7X A few more techniques to show an algebraic space is a scheme. The first is that we can show there is a minimal closed subspace which is not a scheme.

0B7Y **Lemma 17.1.** *Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S. If X is not a scheme, then there exists a closed subspace $Z \subset X$ such that Z is not a scheme, but every proper closed subspace $Z' \subset Z$ is a scheme.*

Proof. We prove this by Zorn's lemma. Let $\mathcal{Z}$ be the set of closed subspaces Z which are not schemes ordered by inclusion. By assumption $\mathcal{Z}$ contains X, hence is nonempty. If Z_α is a totally ordered subset of $\mathcal{Z}$, then $Z = \bigcap Z_\alpha$ is in $\mathcal{Z}$. Namely,

$$Z = \lim Z_\alpha$$

and the transition morphisms are affine. Thus we may apply Lemma 5.11 to see that if Z were a scheme, then so would one of the Z_α. (This works even if $Z = \emptyset$, but note that by Lemma 5.3 this cannot happen.) Thus $\mathcal{Z}$ has minimal elements by Zorn's lemma. $\square$

Now we can prove a little bit about these minimal non-schemes.

0B7Z **Lemma 17.2.** *Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S. Assume that every proper closed subspace $Z \subset X$ is a scheme, but X is not a scheme. Then X is reduced and irreducible.*

Proof. We see that X is reduced by Lemma 15.3. Choose closed subsets $T_1 \subset |X|$ and $T_2 \subset |X|$ such that $|X| = T_1 \cup T_2$. If T_1 and T_2 are proper closed subsets, then the corresponding reduced induced closed subspaces $Z_1, Z_2 \subset X$ (Properties of Spaces, Definition 11.6) are schemes and so is $Z = Z_1 \times_X Z_2 = Z_1 \cap Z_2$ as a closed subscheme of either Z_1 or Z_2. Observe that the coproduct $Z_1 \amalg_Z Z_2$ exists in the category of schemes, see More on Morphisms, Lemma 14.5. One way to proceed, is to show that $Z_1 \amalg_Z Z_2$ is isomorphic to X, but we cannot use this here as the material on pushouts of algebraic spaces comes later in the theory. Instead we will use Lemma 15.1 to find an affine neighbourhood of every point. Namely, let $x \in |X|$. If $x \notin Z_1$, then x has a neighbourhood which is a scheme, namely, $X \setminus Z_1$. Similarly if $x \notin Z_2$. If $x \in Z = Z_1 \cap Z_2$, then we choose an affine open $U \subset Z_1 \amalg_Z Z_2$ containing z. Then $U_1 = Z_1 \cap U$ and $U_2 = Z_2 \cap U$ are affine opens whose intersections with Z agree. Since $|Z_1| = T_1$ and $|Z_2| = T_2$ are closed subsets of $|X|$ which intersect in $|Z|$, we find an open $W \subset |X|$ with $W \cap T_1 = |U_1|$ and $W \cap T_2 = |U_2|$. Let W denote the corresponding open subspace of X. Then $x \in |W|$ and the morphism $U_1 \amalg U_2 \to W$ is a surjective finite morphism whose source is an affine scheme. Thus W is an affine scheme by Lemma 15.1. $\square$

A key point in the following lemma is that we only need to check the condition in the images of points of X.

0B80 **Lemma 17.3.** *Let $f : X \to S$ be a quasi-compact and quasi-separated morphism from an algebraic space to a scheme S. If for every $x \in |X|$ with image $s = f(x) \in S$ the algebraic space $X \times_S \mathrm{Spec}(\mathcal{O}_{S,s})$ is a scheme, then X is a scheme.*

Proof. Let $x \in |X|$. It suffices to find an open neighbourhood U of $s = f(x)$ such that $X \times_S U$ is a scheme. As $X \times_S \mathrm{Spec}(\mathcal{O}_{S,s})$ is a scheme, then, since $\mathcal{O}_{S,s} = \mathrm{colim}\, \mathcal{O}_S(U)$ where the colimit is over affine open neighbourhoods of s in S we see that

$$X \times_S \mathrm{Spec}(\mathcal{O}_{S,s}) = \lim X \times_S U$$

By Lemma 5.11 we see that $X \times_S U$ is a scheme for some U. $\square$

Instead of restricting to local rings as in Lemma 17.3, we can restrict to closed subschemes of the base.

0B81 **Lemma 17.4.** *Let $\varphi : X \to \mathrm{Spec}(A)$ be a quasi-compact and quasi-separated morphism from an algebraic space to an affine scheme. If X is not a scheme, then there exists an ideal $I \subset A$ such that the base change $X_{A/I}$ is not a scheme, but for every $I \subset I'$, $I \neq I'$ the base change $X_{A/I'}$ is a scheme.*

Proof. We prove this by Zorn's lemma. Let $\mathcal{I}$ be the set of ideals I such that $X_{A/I}$ is not a scheme. By assumption $\mathcal{I}$ contains (0). If I_α is a chain of ideals in $\mathcal{I}$, then $I = \bigcup I_\alpha$ is in $\mathcal{I}$. Namely, $A/I = \mathrm{colim}\, A/I_\alpha$, hence

$$X_{A/I} = \lim X_{A/I_\alpha}$$

Thus we may apply Lemma 5.11 to see that if $X_{A/I}$ were a scheme, then so would be one of the X_{A/I_α}. Thus $\mathcal{I}$ has maximal elements by Zorn's lemma. $\square$

18. Application to modifications

0BGX Using limits we can describe the category of modifications of a decent algebraic space over a closed point in terms of the henselian local ring.

0BGY **Lemma 18.1.** *Let S be a scheme. Consider a separated étale morphism $f : V \to W$ of algebraic spaces over S. Assume there exists a closed subspace $T \subset W$ such that $f^{-1}T \to T$ is an isomorphism. Then, with $W^0 = W \setminus T$ and $V^0 = f^{-1}W^0$ the base change functor*

$$\left\{\begin{matrix} g : X \to W \text{ morphism of algebraic spaces} \\ g^{-1}(W^0) \to W^0 \text{ is an isomorphism} \end{matrix}\right\} \longrightarrow \left\{\begin{matrix} h : Y \to V \text{ morphism of algebraic spaces} \\ h^{-1}(V^0) \to V^0 \text{ is an isomorphism} \end{matrix}\right\}$$

is an equivalence of categories.

Proof. Since $V \to W$ is separated we see that $V \times_W V = \Delta(V) \amalg U$ for some open and closed subspace U of $V \times_W V$. By the assumption that $f^{-1}T \to T$ is an isomorphism we see that $U \times_W T = \emptyset$, i.e., the two projections $U \to V$ maps into V^0.

Given $h : Y \to V$ in the right hand category, consider the contravariant functor X on $(Sch/S)_{fppf}$ defined by the rule

$$X(T) = \{(w, y) \mid w : T \to W,\ y : T \times_{w,W} V \to Y \text{ morphism over } V\}$$

Denote $g : X \to W$ the map sending $(w, y) \in X(T)$ to $w \in W(T)$. Since $h^{-1}V^0 \to V^0$ is an isomorphism, we see that if $w : T \to W$ maps into W^0, then there is a unique choice for h. In other words $X \times_{g,W} W^0 = W^0$. On the other hand, consider a T-valued point (w, y, v) of $X \times_{g,W,f} V$. Then $w = f \circ v$ and

$$y : T \times_{f\circ v,W} V \longrightarrow V$$

is a morphism over V. Consider the morphism

$$T \times_{f\circ v,W} V \xrightarrow{(v,\mathrm{id}_V)} V \times_W V = V \amalg U$$

The inverse image of V is T embedded via $(\mathrm{id}_T, v) : T \to T \times_{f\circ v,W} V$. The composition $y' = y \circ (\mathrm{id}_T, v) : T \to Y$ is a morphism with $v = h \circ y'$ which determines y because the restriction of y to the other part is uniquely determined as U maps into V^0 by the second projection. It follows that $X \times_{g,W,f} V \to Y$, $(w, y, v) \mapsto y'$ is an isomorphism.

Thus if we can show that X is an algebraic space, then we are done. Since $V \to W$ is separated and étale it is representable by Morphisms of Spaces, Lemma 49.1 (and Morphisms of Spaces, Lemma 38.5). Of course $W^0 \to W$ is representable and étale as it is an open immersion. Thus

$$W^0 \amalg Y = X \times_{g,W} W^0 \amalg X \times_{g,W,f} V = X \times_{g,W} (W^0 \amalg V) \longrightarrow X$$

is representable, surjective, and étale by Spaces, Lemmas 3.3 and 5.5. Thus X is an algebraic space by Spaces, Lemma 11.2. $\square$

0BGZ **Lemma 18.2.** *Notation and assumptions as in Lemma 18.1. Let $g : X \to W$ correspond to $h : Y \to V$ via the equivalence. Then g is quasi-compact, quasi-separated, separated, locally of finite presentation, of finite presentation, locally of finite type, of finite type, proper, integral, finite, and add more here if and only if h is so.*

Proof. If g is quasi-compact, quasi-separated, separated, locally of finite presentation, of finite presentation, locally of finite type, of finite type, proper, finite, so is h as a base change of g by Morphisms of Spaces, Lemmas 8.3, 4.4, 28.3, 23.3, 39.3, 44.5. Conversely, let P be a property of morphisms of algebraic spaces which is étale local on the base and which holds for the identity morphism of any algebraic space. Since $\{W^0 \to W, V \to W\}$ is an étale covering, to prove that g has P it suffices to show that h has P. Thus we conclude using Morphisms of Spaces, Lemmas 8.7, 4.12, 28.4, 23.4, 39.2, 44.3. $\square$

0BH0 **Lemma 18.3.** *Let S be a scheme. Let X be a decent algebraic space over S. Let $x \in |X|$ be a closed point such that $U = X \setminus \{x\} \to X$ is quasi-compact. With $V = \mathrm{Spec}(\mathcal{O}^h_{X,x}) \setminus \{\mathfrak{m}^h_x\}$ the base change functor*

$$\left\{\begin{matrix} f : Y \to X \text{ of finite presentation} \\ f^{-1}(U) \to U \text{ is an isomorphism} \end{matrix}\right\} \longrightarrow \left\{\begin{matrix} g : Y \to \mathrm{Spec}(\mathcal{O}^h_{X,x}) \text{ of finite presentation} \\ g^{-1}(V) \to V \text{ is an isomorphism} \end{matrix}\right\}$$

is an equivalence of categories.

Proof. Let $a : (W, w) \to (X, x)$ be an elementary étale neighbourhood of x with W affine as in Decent Spaces, Lemma 11.2. Since x is a closed point of X and w is the unique point of W lying over x, we see that w is a closed point of W. Since a is étale and identifies residue fields at x and w, it follows that a induces an isomorphism $a^{-1}x \to x$ (as closed subspaces of X and W). Thus we may apply Lemma 18.1 and 18.2 to reduce the problem to the case where X is an affine scheme.

Assume X is an affine scheme. Recall that $\mathcal{O}^h_{X,x}$ is the colimit of $\Gamma(U, \mathcal{O}_U)$ over affine elementary étale neighbourhoods $(U, u) \to (X, x)$. Recall that the category of these neighbourhoods is cofiltered, see Decent Spaces, Lemma 11.4 or More on Morphisms, Lemma 31.4. Then $\mathrm{Spec}(\mathcal{O}^h_{X,x}) = \lim U$ and $V = \lim U \setminus \{u\}$ (Lemma 4.1) where the limits are taken over the same category. Thus by Lemma 7.1 The category on the right is the colimit of the categories for the pairs (U, u). And by the material in the first paragraph, each of these categories is equivalent to the category for the pair (X, x). This finishes the proof. $\square$

19. Universally closed morphisms

0CM7

In this section we discuss when a quasi-compact (but not necessarily separated) morphism is universally closed. We first prove a lemma which will allow us to check universal closedness after a base change which is locally of finite presentation.

0CM8 **Lemma 19.1.** *Let S be a scheme. Let $f : X \to Y$ and $g : Z \to Y$ be morphisms of algebraic spaces over S. Let $z \in |Z|$ and let $T \subset |X \times_Y Z|$ be a closed subset with $z \notin \mathrm{Im}(T \to |Z|)$. If f is quasi-compact, then there exists an étale neighbourhood $(V, v) \to (Z, z)$, a commutative diagram*

$$\begin{tikzcd} V \arrow[r, "a"'] \arrow[d] & Z' \arrow[d, "b"] \\ Z \arrow[r, "g"] & Y, \end{tikzcd}$$

and a closed subset $T' \subset |X \times_Y Z'|$ such that

1. *the morphism $b : Z' \to Y$ is locally of finite presentation,*
2. *with $z' = a(v)$ we have $z' \notin \mathrm{Im}(T' \to |Z'|)$, and*
3. *the inverse image of T in $|X \times_Y V|$ maps into T' via $|X \times_Y V| \to |X \times_Y Z'|$.*

Moreover, we may assume V and Z' are affine schemes and if Z is a scheme we may assume V is an affine open neighbourhood of z.

Proof. We will deduce this from the corresponding result for morphisms of schemes. Let $y \in |Y|$ be the image of z. First we choose an affine étale neighbourhood $(U, u) \to (Y, y)$ and then we choose an affine étale neighbourhood $(V, v) \to (Z, z)$ such that the morphism $V \to Y$ factors through U. Then we may replace

1. $X \to Y$ by $X \times_Y U \to U$,
2. $Z \to Y$ by $V \to U$,
3. z by v, and
4. T by its inverse image in $|(X \times_Y U) \times_U V| = |X \times_Y V|$.

In fact, below we will show that after replacing V by an affine open neighbourhood of v there will be a morphism $a : V \to Z'$ for some $Z' \to U$ of finite presentation and a closed subset T' of $|(X \times_Y U) \times_U Z'| = |X \times_Y Z'|$ such that T maps into T' and $a(v) \notin \mathrm{Im}(T' \to |Z'|)$. Thus we may and do assume that Z and Y are affine schemes with the proviso that we need to find a solution where V is an open neighbourhood of z.

Since f is quasi-compact and Y is affine, the algebraic space X is quasi-compact. Choose an affine scheme W and a surjective étale morphism $W \to X$. Let $T_W \subset |W \times_Y Z|$ be the inverse image of T. Then z is not in the image of T_W. By the schemes case (Limits, Lemma 14.1) we can find an open neighbourhood $V \subset Z$ of z a commutative diagram of schemes

$$\begin{tikzcd} V \arrow[r, "a"'] \arrow[d] & Z' \arrow[d, "b"] \\ Z \arrow[r, "g"] & Y, \end{tikzcd}$$

and a closed subset $T' \subset |W \times_Y Z'|$ such that

1. the morphism $b : Z' \to Y$ is locally of finite presentation,
2. with $z' = a(z)$ we have $z' \notin \mathrm{Im}(T' \to Z')$, and
3. $T_W \cap |W \times_Y V|$ maps into T' via $|W \times_Y V| \to |W \times_Y Z'|$.

The commutative diagram

$$\begin{tikzcd} W \times_Y V \arrow[r, "b"'] \arrow[d, "c"'] & W \times_Y Z' \arrow[d, "q"'] \\ X \times_Y V \arrow[r, "a"] & X \times_Y Z' \end{tikzcd}$$

is cartesian. The vertical maps are surjective étale hence surjective and open. Also $T_1 = T_W \cap |W \times_Y V|$ is the inverse image of $T_2 = T \cap |X \times_Y V|$ by c. By Properties of Spaces, Lemma 4.3 we get $b(T_1) = q^{-1}(a(T_2))$. By Topology, Lemma 6.4 we get

$$q^{-1}(\overline{a(T_1)}) = \overline{q^{-1}(a(T_1))} = \overline{b(T_2)} \subset T'$$

As q is surjective the image of $\overline{a(T_1)} \to |Z'|$ does not contain z' since the same is true for T'. This concludes the proof. $\square$

0CM9 **Lemma 19.2.** *Let S be a scheme. Let $f : X \to Y$ be a quasi-compact morphism of algebraic spaces over S. The following are equivalent*

1. *f is universally closed,*
2. *for every morphism $Z \to Y$ which is locally of finite presentation the map $|X \times_Y Z| \to |Z|$ is closed, and*
3. *there exists a scheme V and a surjective étale morphism $V \to Y$ such that $|\mathbf{A}^n \times (X \times_Y V)| \to |\mathbf{A}^n \times V|$ is closed for all $n \geq 0$.*

Proof. It is clear that (1) implies (2). Suppose that $|X \times_Y Z| \to |Z|$ is not closed for some morphism of algebraic spaces $Z \to Y$ over S. This means that there exists some closed subset $T \subset |X \times_Y Z|$ such that $\text{Im}(T \to |Z|)$ is not closed. Pick $z \in |Z|$ in the closure of the image of T but not in the image. Apply Lemma 19.1. We find an étale neighbourhood $(V, v) \to (Z, z)$, a commutative diagram

$$\begin{tikzcd} V \arrow[r, "a"'] \arrow[d] & Z' \arrow[d, "b"] \\ Z \arrow[r, "g"] & Y, \end{tikzcd}$$

and a closed subset $T' \subset |X \times_Y Z'|$ such that

1. the morphism $b : Z' \to Y$ is locally of finite presentation,
2. with $z' = a(v)$ we have $z' \notin \text{Im}(T' \to |Z'|)$, and
3. the inverse image of T in $|X \times_Y V|$ maps into T' via $|X \times_Y V| \to |X \times_Y Z'|$.

We claim that z' is in the closure of $\text{Im}(T' \to |Z'|)$ which implies that $|X \times_Y Z'| \to |Z'|$ is not closed. The claim shows that (2) implies (1). To see the claim is true we suggest the reader contemplate the following commutative diagram

$$\begin{tikzcd} X \times_Y Z \arrow[d] & X \times_Y V \arrow[l] \arrow[r] \arrow[d] & X \times_Y Z' \arrow[d] \\ Z & V \arrow[l] \arrow[r, "a"] & Z' \end{tikzcd}$$

Let $T_V \subset |X \times_Y V|$ be the inverse image of T. By Properties of Spaces, Lemma 4.3 the image of T_V in $|V|$ is the inverse image of the image of T in $|Z|$. Then since z is in the closure of the image of $T \to |Z|$ and since $|V| \to |Z|$ is open, we see that v is in the closure of the image of $T_V \to |V|$. Since the image of T_V in $|X \times_Y Z'|$

is contained in $|T'|$ it follows immediately that $z' = a(v)$ is in the closure of the image of T'.

It is clear that (1) implies (3). Let $V \to Y$ be as in (3). If we can show that $X \times_Y V \to V$ is universally closed, then f is universally closed by Morphisms of Spaces, Lemma 9.5. Thus it suffices to show that $f : X \to Y$ satisfies (2) if f is a quasi-compact morphism of algebraic spaces, Y is a scheme, and $|\mathbf{A}^n \times X| \to |\mathbf{A}^n \times Y|$ is closed for all n. Let $Z \to Y$ be locally of finite presentation. We have to show the map $|X \times_Y Z| \to |Z|$ is closed. This question is étale local on Z hence we may assume Z is affine (some details omitted). Since Y is a scheme, Z is affine, and $Z \to Y$ is locally of finite presentation we can find an immersion $Z \to \mathbf{A}^n \times Y$, see Morphisms, Lemma 37.2. Consider the cartesian diagram

$$\begin{tikzcd} X \times_Y Z \arrow[r] \arrow[d] & \mathbf{A}^n \times X \arrow[d] \\ Z \arrow[r] & \mathbf{A}^n \times Y \end{tikzcd} \quad \text{inducing the cartesian square} \quad \begin{tikzcd} {|X \times_Y Z|} \arrow[r] \arrow[d] & {|\mathbf{A}^n \times X|} \arrow[d] \\ {|Z|} \arrow[r] & {|\mathbf{A}^n \times Y|} \end{tikzcd}$$

of topological spaces whose horizontal arrows are homeomorphisms onto locally closed subsets (Properties of Spaces, Lemma 11.2). Thus every closed subset T of $|X \times_Y Z|$ is the pullback of a closed subset T' of $|\mathbf{A}^n \times Y|$. Since the assumption is that the image of T' in $|\mathbf{A}^n \times X|$ is closed we conclude that the image of T in $|Z|$ is closed as desired. ☐

0CMA **Lemma 19.3.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume f separated and of finite type. The following are equivalent*

(1) *The morphism f is proper.*
(2) *For any morphism $Y \to Z$ which is locally of finite presentation the map $|X \times_Y Z| \to |Z|$ is closed, and*
(3) *there exists a scheme V and a surjective étale morphism $V \to Y$ such that $|\mathbf{A}^n \times (X \times_Y V)| \to |\mathbf{A}^n \times V|$ is closed for all $n \geq 0$.*

Proof. In view of the fact that a proper morphism is the same thing as a separated, finite type, and universally closed morphism, this lemma is a special case of Lemma 19.2. ☐

20. Noetherian valuative criterion

0CMB We have already proved some results in Cohomology of Spaces, Section 19. The corresponding section for schemes is Limits, Section 15. Currently we are missing the analogues of Limits, Lemmas 15.2, 15.3, and 15.4.

Many of the results in this section can (and perhaps should) be proved by appealing to the following lemma, although we have not always done so.

0CMC **Lemma 20.1.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume f finite type and Y locally Noetherian. Let $y \in |Y|$ be a point in the closure of the image of $|f|$. Then there exists a commutative diagram*

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \mathrm{Spec}(A) \arrow[r] & Y \end{tikzcd}$$

where A is a discrete valuation ring and K is its field of fractions mapping the closed point of $\mathrm{Spec}(A)$ *to* y*. Moreover, we can assume that the point* $x \in |X|$ *corresponding to* $\mathrm{Spec}(K) \to X$ *is a codimension* 0 *point*[2] *and that* K *is the residue field of a point on a scheme étale over* X*.*

Proof. Choose an affine scheme V, a point $v \in V$ and an étale morphism $V \to Y$ mapping v to y. The map $|V| \to |Y|$ is open and by Properties of Spaces, Lemma 4.3 the image of $|X \times_Y V| \to |V|$ is the inverse image of the image of $|f|$. We conclude that the point v is in the closure of the image of $|X \times_Y V| \to |V|$. If we prove the lemma for $X \times_Y V \to V$ and the point v, then the lemma follows for f and y. In this way we reduce to the situation described in the next paragraph.

Assume we have $f : X \to Y$ and $y \in |Y|$ as in the lemma where Y is an affine scheme. Since f is quasi-compact, we conclude that X is quasi-compact. Hence we can choose an affine scheme W and a surjective étale morphism $W \to X$. Then the image of $|f|$ is the same as the image of $W \to Y$. In this way we reduce to the case of schemes which is Limits, Lemma 15.1. $\square$

0CMD **Lemma 20.2.** *Let S be a scheme. Let $f : X \to Y$ and $h : U \to X$ be morphisms of algebraic spaces over S. Assume that Y is locally Noetherian, that f and h are of finite type, that f is separated, and that the image of $|h| : |U| \to |X|$ is dense in $|X|$. If given any commutative solid diagram*

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r] \arrow[d] & U \arrow[r, "h"] & X \arrow[d, "f"] \\ \mathrm{Spec}(A) \arrow[rr] \arrow[urr, dashed] & & Y \end{tikzcd}$$

where A is a discrete valuation ring with field of fractions K, there exists a dotted arrow making the diagram commute, then f is proper.

Proof. It suffices to prove that f is universally closed. Let $V \to Y$ be an étale morphism where V is an affine scheme. By Morphisms of Spaces, Lemma 9.5 it suffices to prove that the base change $X \times_Y V \to V$ is universally closed. By Properties of Spaces, Lemma 4.3 the image I of $|U \times_Y V| \to |X \times_Y V|$ is the inverse image of the image of $|h|$. Since $|X \times_Y V| \to |X|$ is open (Properties of Spaces, Lemma 15.7) we conclude that I is dense in $|X \times_Y V|$. Therefore the assumptions of the lemma are satisfied for the morphisms $U \times_Y V \to X \times_Y V \to V$. Hence we may assume Y is an affine scheme.

Assume Y is an affine scheme. Then U is quasi-compact. Choose an affine scheme and a surjective étale morphism $W \to U$. Then we may and do replace U by W and assume that U is affine. By the weak version of Chow's lemma (Cohomology of Spaces, Lemma 18.1) we can choose a surjective proper morphism $X' \to X$ where X' is a scheme. Then $U' = X' \times_X U$ is a scheme and $U' \to X'$ is of finite type. We may replace X' by the scheme theoretic image of $h' : U' \to X'$ and hence $h'(U')$ is

[2] See discussion in Properties of Spaces, Section 10.

dense in X'. We claim that for every diagram

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r] \arrow[d] & U' \arrow[r, "h"] & X' \arrow[d, "f'"] \\ \mathrm{Spec}(A) \arrow[rr] \arrow[rru, dashed] & & Y \end{tikzcd}$$

where A is a discrete valuation ring with field of fractions K, there exists a dotted arrow making the diagram commute. Namely, we first get an arrow $\mathrm{Spec}(A) \to X$ by the assumption of the lemma and then we lift this to an arrow $\mathrm{Spec}(A) \to X'$ using the valuative criterion for properness (Morphisms of Spaces, Lemma 43.1). The morphism $X' \to Y$ is separated as a composition of a proper and a separated morphism. Thus by the case of schemes the morphism $X' \to Y$ is proper (Limits, Lemma 15.5). By Morphisms of Spaces, Lemma 39.7 we conclude that $X \to Y$ is proper. □

0CME **Lemma 20.3.** *Let S be a scheme. Let $f : X \to Y$ and $h : U \to X$ be morphisms of algebraic spaces over S. Assume that Y is locally Noetherian, that f is locally of finite type and quasi-separated, that h is of finite type, and that the image of $|h| : |U| \to |X|$ is dense in $|X|$. If given any commutative solid diagram*

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r] \arrow[d] & U \arrow[r, "h"] & X \arrow[d, "f"] \\ \mathrm{Spec}(A) \arrow[rr] \arrow[rru, dashed] & & Y \end{tikzcd}$$

where A is a discrete valuation ring with field of fractions K, there exists at most one dotted arrow making the diagram commute, then f is separated.

Proof. We will apply Lemma 20.2 to the morphisms $U \to X$ and $\Delta : X \to X \times_Y X$. We check the conditions. Observe that Δ is quasi-compact because f is quasi-separated. Of course Δ is locally of finite type and separated (true for any diagonal morphism). Finally, suppose given a commutative solid diagram

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r] \arrow[d] & U \arrow[r, "h"] & X \arrow[d, "\Delta"] \\ \mathrm{Spec}(A) \arrow[rr, "{(a,b)}"] \arrow[rru, dashed] & & X \times_Y X \end{tikzcd}$$

where A is a discrete valuation ring with field of fractions K. Then a and b give two dotted arrows in the diagram of the lemma and have to be equal. Hence as dotted arrow we can use $a = b$ which gives existence. This finishes the proof. □

0CMF **Lemma 20.4.** *Let S be a scheme. Let $f : X \to Y$ and $h : U \to X$ be morphisms of algebraic spaces over S. Assume that Y is locally Noetherian, that f and h are of finite type, and that $h(U)$ is dense in X. If given any commutative solid diagram*

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r] \arrow[d] & U \arrow[r, "h"] & X \arrow[d, "f"] \\ \mathrm{Spec}(A) \arrow[rr] \arrow[rru, dashed] & & Y \end{tikzcd}$$

where A is a discrete valuation ring with field of fractions K, there exists a unique dotted arrow making the diagram commute, then f is proper.

Proof. Combine Lemmas 20.3 and 20.2. ☐

21. Descending finite type spaces

0CP5 This section continues the theme of Section 11 in the spirit of the results discussed in Section 7. It is also the analogue of Limits, Section 18 for algebraic spaces.

0CP6 **Situation 21.1.** Let S be a scheme, for example $\mathrm{Spec}(\mathbf{Z})$. Let $B = \lim_{i \in I} B_i$ be the limit of a directed inverse system of Noetherian spaces over S with affine transition morphisms $B_{i'} \to B_i$ for $i' \geq i$.

0CP7 **Lemma 21.2.** *In Situation 21.1. Let $X \to B$ be a quasi-separated and finite type morphism of algebraic spaces. Then there exists an $i \in I$ and a diagram*

0CP8 (21.2.1)
$$\begin{tikzcd} X \arrow[r] \arrow[d] & W \arrow[d] \\ B \arrow[r] & B_i \end{tikzcd}$$

such that $W \to B_i$ is of finite type and such that the induced morphism $X \to B \times_{B_i} W$ is a closed immersion.

Proof. By Lemma 11.6 we can find a closed immersion $X \to X'$ over B where X' is an algebraic space of finite presentation over B. By Lemma 7.1 we can find an i and a morphism of finite presentation $X'_i \to B_i$ whose pull back is X'. Set $W = X'_i$. ☐

0CP9 **Lemma 21.3.** *In Situation 21.1. Let $X \to B$ be a quasi-separated and finite type morphism of algebraic spaces. Given $i \in I$ and a diagram*

$$\begin{tikzcd} X \arrow[r] \arrow[d] & W \arrow[d] \\ B \arrow[r] & B_i \end{tikzcd}$$

as in (21.2.1) for $i' \geq i$ let $X_{i'}$ be the scheme theoretic image of $X \to B_{i'} \times_{B_i} W$. Then $X = \lim_{i' \geq i} X_{i'}$.

Proof. Since X is quasi-compact and quasi-separated formation of the scheme theoretic image of $X \to B_{i'} \times_{B_i} W$ commutes with étale localization (Morphisms of Spaces, Lemma 16.3). Hence we may and do assume W is affine and maps into an affine U_i étale over B_i. Then

$$B_{i'} \times_{B_i} W = B_{i'} \times_{B_i} U_i \times_{U_i} W = U_{i'} \times_{U_i} W$$

where $U_{i'} = B_{i'} \times_{B_i} U_i$ is affine as the transition morphisms are affine. Thus the lemma follows from the case of schemes which is Limits, Lemma 18.3. ☐

0CPA **Lemma 21.4.** *In Situation 21.1. Let $f : X \to Y$ be a morphism of algebraic spaces quasi-separated and of finite type over B. Let*

$$\begin{tikzcd} X \arrow[r] \arrow[d] & W \arrow[d] \\ B \arrow[r] & B_{i_1} \end{tikzcd} \quad \textit{and} \quad \begin{tikzcd} Y \arrow[r] \arrow[d] & V \arrow[d] \\ B \arrow[r] & B_{i_2} \end{tikzcd}$$

be diagrams as in (21.2.1). Let $X = \lim_{i \geq i_1} X_i$ and $Y = \lim_{i \geq i_2} Y_i$ be the corresponding limit descriptions as in Lemma 21.3. Then there exists an $i_0 \geq \max(i_1, i_2)$ and a morphism

$$(f_i)_{i \geq i_0} : (X_i)_{i \geq i_0} \to (Y_i)_{i \geq i_0}$$

of inverse systems over $(B_i)_{i \geq i_0}$ such that such that $f = \lim_{i \geq i_0} f_i$. If $(g_i)_{i \geq i_0} : (X_i)_{i \geq i_0} \to (Y_i)_{i \geq i_0}$ is a second morphism of inverse systems over $(B_i)_{i \geq i_0}$ such that such that $f = \lim_{i \geq i_0} g_i$ then $f_i = g_i$ for all $i \gg i_0$.

Proof. Since $V \to B_{i_2}$ is of finite presentation and $X = \lim_{i \geq i_1} X_i$ we can appeal to Proposition 3.9 as improved by Lemma 4.5 to find an $i_0 \geq \max(i_1, i_2)$ and a morphism $h : X_{i_0} \to V$ over B_{i_2} such that $X \to X_{i_0} \to V$ is equal to $X \to Y \to V$. For $i \geq i_0$ we get a commutative solid diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & X_i \arrow[r] \arrow[d, dotted] \arrow[dd, bend right=60] & X_{i_0} \arrow[d, "h"] \\ Y \arrow[r] \arrow[d] & Y_i \arrow[r] \arrow[d] & V \arrow[d] \\ B \arrow[r] & B_i \arrow[r] & B_{i_0} \end{tikzcd}$$

Since $X \to X_i$ has scheme theoretically dense image and since Y_i is the scheme theoretic image of $Y \to B_i \times_{B_{i_2}} V$ we find that the morphism $X_i \to B_i \times_{B_{i_2}} V$ induced by the diagram factors through Y_i (Morphisms of Spaces, Lemma 16.6). This proves existence.

Uniqueness. Let $E_i \to X_i$ be the equalizer of f_i and g_i for $i \geq i_0$. We have $E_i = Y_i \times_{\Delta, Y_i \times_{B_i} Y_i, (f_i, g_i)} X_i$. Hence $E_i \to X_i$ is a monomorphism of finite presentation as a base change of the diagonal of Y_i over B_i, see Morphisms of Spaces, Lemmas 4.1 and 28.10. Since X_i is a closed subspace of $B_i \times_{B_{i_0}} X_{i_0}$ and similarly for Y_i we see that

$$E_i = X_i \times_{(B_i \times_{B_{i_0}} X_{i_0})} (B_i \times_{B_{i_0}} E_{i_0}) = X_i \times_{X_{i_0}} E_{i_0}$$

Similarly, we have $X = X \times_{X_{i_0}} E_{i_0}$. Hence we conclude that $E_i = X_i$ for i large enough by Lemma 6.10. $\square$

0CPB **Remark 21.5.** In Situation 21.1 Lemmas 21.2, 21.3, and 21.4 tell us that the category of algebraic spaces quasi-separated and of finite type over B is equivalent to certain types of inverse systems of algebraic spaces over $(B_i)_{i \in I}$, namely the ones produced by applying Lemma 21.3 to a diagram of the form (21.2.1). For example, given $X \to B$ finite type and quasi-separated if we choose two different diagrams $X \to V_1 \to B_{i_1}$ and $X \to V_2 \to B_{i_2}$ as in (21.2.1), then applying Lemma 21.4 to id_X (in two directions) we see that the corresponding limit descriptions of X are canonically isomorphic (up to shrinking the directed set I). And so on and so forth.

0CPC **Lemma 21.6.** *Notation and assumptions as in Lemma 21.4. If f is flat and of finite presentation, then there exists an $i_3 > i_0$ such that for $i \geq i_3$ we have f_i is flat, $X_i = Y_i \times_{Y_{i_3}} X_{i_3}$, and $X = Y \times_{Y_{i_3}} X_{i_3}$.*

Proof. By Lemma 7.1 we can choose an $i \geq i_2$ and a morphism $U \to Y_i$ of finite presentation such that $X = Y \times_{Y_i} U$ (this is where we use that f is of finite presentation). After increasing i we may assume that $U \to Y_i$ is flat, see Lemma 6.12. As discussed in Remark 21.5 we may and do replace the initial diagram used to define the system $(X_i)_{i \geq i_1}$ by the system corresponding to $X \to U \to B_i$. Thus $X_{i'}$ for $i' \geq i$ is defined as the scheme theoretic image of $X \to B_{i'} \times_{B_i} U$.

Because $U \to Y_i$ is flat (this is where we use that f is flat), because $X = Y \times_{Y_i} U$, and because the scheme theoretic image of $Y \to Y_i$ is Y_i, we see that the scheme theoretic image of $X \to U$ is U (Morphisms of Spaces, Lemma 29.12). Observe that $Y_{i'} \to B_{i'} \times_{B_i} Y_i$ is a closed immersion for $i' \geq i$ by construction of the system of Y_j. Then the same argument as above shows that the scheme theoretic image of $X \to B_{i'} \times_{B_i} U$ is equal to the closed subspace $Y_{i'} \times_{Y_i} U$. Thus we see that $X_{i'} = Y_{i'} \times_{Y_i} U$ for all $i' \geq i$ and hence the lemma holds with $i_3 = i$. □

0CPD **Lemma 21.7.** *Notation and assumptions as in Lemma 21.4. If f is smooth, then there exists an $i_3 > i_0$ such that for $i \geq i_3$ we have f_i is smooth.*

Proof. Combine Lemmas 21.6 and 6.3. □

0CPE **Lemma 21.8.** *Notation and assumptions as in Lemma 21.4. If f is proper, then there exists an $i_3 \geq i_0$ such that for $i \geq i_3$ we have f_i is proper.*

Proof. By the discussion in Remark 21.5 the choice of i_1 and W fitting into a diagram as in (21.2.1) is immaterial for the truth of the lemma. Thus we choose W as follows. First we choose a closed immersion $X \to X'$ with $X' \to Y$ proper and of finite presentation, see Lemma 12.1. Then we choose an $i_3 \geq i_2$ and a proper morphism $W \to Y_{i_3}$ such that $X' = Y \times_{Y_{i_3}} W$. This is possible because $Y = \lim_{i \geq i_2} Y_i$ and Lemmas 10.1 and 6.13. With this choice of W it is immediate from the construction that for $i \geq i_3$ the algebraic space X_i is a closed subspace of $Y_i \times_{Y_{i_3}} W \subset B_i \times_{B_{i_3}} W$ and hence proper over Y_i. □

0CPF **Lemma 21.9.** *In Situation 21.1 suppose that we have a cartesian diagram*

$$\begin{tikzcd} X^1 \arrow[r, "p"] \arrow[d, "q"'] & X^3 \arrow[d, "a"] \\ X^2 \arrow[r, "b"] & X^4 \end{tikzcd}$$

of algebraic spaces quasi-separated and of finite type over B. For each $j = 1, 2, 3, 4$ choose $i_j \in I$ and a diagram

$$\begin{tikzcd} X^j \arrow[r] \arrow[d] & W^j \arrow[d] \\ B \arrow[r] & B_{i_j} \end{tikzcd}$$

as in (21.2.1). Let $X^j = \lim_{i \geq i_j} X_i^j$ be the corresponding limit descriptions as in Lemma 21.4. Let $(a_i)_{i \geq i_5}$, $(b_i)_{i \geq i_6}$, $(p_i)_{i \geq i_7}$, and $(q_i)_{i \geq i_8}$ be the corresponding *morphisms of inverse systems contructed in Lemma 21.4. Then there exists an* $i_9 \geq \max(i_5, i_6, i_7, i_8)$ *such that for* $i \geq i_9$ *we have* $a_i \circ p_i = b_i \circ q_i$ *and such that*

$$(q_i, p_i) : X_i^1 \longrightarrow X_i^2 \times_{b_i, X_i^4, a_i} X_i^3$$

is a closed immersion. If a *and* b *are flat and of finite presentation, then there exists an* $i_{10} \geq \max(i_5, i_6, i_7, i_8, i_9)$ *such that for* $i \geq i_{10}$ *the last displayed morphism is an isomorphism.*

Proof. According to the discussion in Remark 21.5 the choice of W^1 fitting into a diagram as in (21.2.1) is immaterial for the truth of the lemma. Thus we may choose $W^1 = W^2 \times_{W^4} W^3$. Then it is immediate from the construction of X_i^1 that $a_i \circ p_i = b_i \circ q_i$ and that

$$(q_i, p_i) : X_i^1 \longrightarrow X_i^2 \times_{b_i, X_i^4, a_i} X_i^3$$

is a closed immersion.

If a and b are flat and of finite presentation, then so are p and q as base changes of a and b. Thus we can apply Lemma 21.6 to each of a, b, p, q, and $a \circ p = b \circ q$. It follows that there exists an $i_9 \in I$ such that

$$(q_i, p_i) : X_i^1 \to X_i^2 \times_{X_i^4} X_i^3$$

is the base change of (q_{i_9}, p_{i_9}) by the morphism by the morphism $X_i^4 \to X_{i_9}^4$ for all $i \geq i_9$. We conclude that (q_i, p_i) is an isomorphism for all sufficiently large i by Lemma 6.10. □

22. Other chapters

References

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