

ALGEBRAIC SPACES OVER FIELDS

06DR

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1. Introduction

06DS This chapter is the analogue of the chapter on varieties in the setting of algebraic spaces. A reference for algebraic spaces is [Knu71].

2. Conventions

06LX The standing assumption is that all schemes are contained in a big fppf site Sch_{fppf} . And all rings A considered have the property that $\mathrm{Spec}(A)$ is (isomorphic) to an object of this big site.

Let S be a scheme and let X be an algebraic space over S . In this chapter and the following we will write $X \times_S X$ for the product of X with itself (in the category of algebraic spaces over S), instead of $X \times X$.

3. Generically finite morphisms

0ACY This section continues the discussion in Decent Spaces, Section 20 and the analogue for morphisms of algebraic spaces of Varieties, Section 17.

0AD1 **Lemma 3.1.** *Let S be a scheme. Let $f : X \rightarrow Y$ be a morphism of algebraic spaces over S . Assume f is locally of finite type and Y is locally Noetherian. Let $y \in |Y|$ be a point of codimension ≤ 1 on Y . Let $X^0 \subset |X|$ be the set of points of codimension 0 on X . Assume in addition one of the following conditions is satisfied*

- (1) *for every $x \in X^0$ the transcendence degree of $x/f(x)$ is 0,*
- (2) *for every $x \in X^0$ with $f(x) \rightsquigarrow y$ the transcendence degree of $x/f(x)$ is 0,*
- (3) *f is quasi-finite at every $x \in X^0$,*
- (4) *f is quasi-finite at a dense set of points of $|X|$,*
- (5) *add more here.*

Then f is quasi-finite at every point of X lying over y .

Proof. We want to reduce the proof to the case of schemes. To do this we choose a commutative diagram

$$\begin{array}{ccc} U & \longrightarrow & X \\ g \downarrow & & \downarrow f \\ V & \longrightarrow & Y \end{array}$$

where U, V are schemes and where the horizontal arrows are étale and surjective. Pick $v \in V$ mapping to y . Observe that V is locally Noetherian and that $\dim(\mathcal{O}_{V,v}) \leq 1$ (see Properties of Spaces, Definitions 9.2 and Remark 7.3). The fibre U_v of $U \rightarrow V$ over v surjects onto $f^{-1}(\{y\}) \subset |X|$. The inverse image of X^0 in U is exactly the set of generic points of irreducible components of U (Properties of Spaces, Lemma 10.1). If $\eta \in U$ is such a point with image $x \in X^0$, then the transcendence degree of $x/f(x)$ is the transcendence degree of $\kappa(\eta)$ over $\kappa(g(\eta))$ (Morphisms of Spaces, Definition 32.1). Observe that $U \rightarrow V$ is quasi-finite at $u \in U$ if and only if f is quasi-finite at the image of u in X .

Case (1). Here case (1) of Varieties, Lemma 17.1 applies and we conclude that $U \rightarrow V$ is quasi-finite at all points of U_v . Hence f is quasi-finite at every point lying over y .

Case (2). Let $u \in U$ be a generic point of an irreducible component whose image in V specializes to v . Then the image $x \in X^0$ of u has the property that $f(x) \rightsquigarrow y$. Hence we see that case (2) of Varieties, Lemma 17.1 applies and we conclude as before.

Case (3) follows from case (3) of Varieties, Lemma 17.1.

In case (4), since $|U| \rightarrow |X|$ is open, we see that the set of points where $U \rightarrow V$ is quasi-finite is dense as well. Hence case (4) of Varieties, Lemma 17.1 applies. \square

0AD2 **Lemma 3.2.** *Let S be a scheme. Let $f : X \rightarrow Y$ be a morphism of algebraic spaces over S . Assume f is proper and Y is locally Noetherian. Let $y \in Y$ be a point of codimension ≤ 1 in Y . Let $X^0 \subset |X|$ be the set of points of codimension 0 on X . Assume in addition one of the following conditions is satisfied*

- (1) *for every $x \in X^0$ the transcendence degree of $x/f(x)$ is 0,*
- (2) *for every $x \in X^0$ with $f(x) \rightsquigarrow y$ the transcendence degree of $x/f(x)$ is 0,*
- (3) *f is quasi-finite at every $x \in X^0$,*
- (4) *f is quasi-finite at a dense set of points of $|X|$,*
- (5) *add more here.*

Then there exists an open subspace $Y' \subset Y$ containing y such that $Y' \times_Y X \rightarrow Y'$ is finite.

Proof. By Lemma 3.1 the morphism f is quasi-finite at every point lying over y . Let $\bar{y} : \text{Spec}(k) \rightarrow Y$ be a geometric point lying over y . Then $|X_{\bar{y}}|$ is a discrete space (Decent Spaces, Lemma 17.10). Since $X_{\bar{y}}$ is quasi-compact as f is proper we conclude that $|X_{\bar{y}}|$ is finite. Thus we can apply Cohomology of Spaces, Lemma 22.2 to conclude. \square

0BBQ **Lemma 3.3.** *Let S be a scheme. Let X be a Noetherian algebraic space over S . Let $f : Y \rightarrow X$ be a birational proper morphism of algebraic spaces with Y reduced. Let $U \subset X$ be the maximal open over which f is an isomorphism. Then U contains*

- (1) every point of codimension 0 in X ,
- (2) every $x \in |X|$ of codimension 1 on X such that the local ring of X at x is normal (Properties of Spaces, Remark 7.6), and
- (3) every $x \in |X|$ such that the fibre of $|Y| \rightarrow |X|$ over x is finite and such that the local ring of X at x is normal.

Proof. Part (1) follows from Decent Spaces, Lemma 21.5 (and the fact that the Noetherian algebraic spaces X and Y are quasi-separated and hence decent). Part (2) follows from part (3) and Lemma 3.2 (and the fact that finite morphisms have finite fibres). Let $x \in |X|$ be as in (3). By Cohomology of Spaces, Lemma 22.2 (which applies by Decent Spaces, Lemma 17.10) we may assume f is finite. Choose an affine scheme X' and an étale morphism $X' \rightarrow X$ and a point $x' \in X'$ mapping to x . It suffices to show there exists an open neighbourhood U' of $x' \in X'$ such that $Y \times_X X' \rightarrow X'$ is an isomorphism over U' (namely, then U contains the image of U' in X , see Spaces, Lemma 5.6). Then $Y \times_X X' \rightarrow X'$ is a finite birational (Decent Spaces, Lemma 21.6) morphism. Since a finite morphism is affine we reduce to the case of a finite birational morphism of Noetherian affine schemes $Y \rightarrow X'$ and $x \in X'$ such that $\mathcal{O}_{X',x}$ is a normal domain. This is treated in Varieties, Lemma 17.3. \square

4. Integral algebraic spaces

0AD3 We have not yet defined the notion of an integral algebraic space. The problem is that being integral is not an étale local property of schemes. We could use the property, that X is reduced and $|X|$ is irreducible, given in Properties, Lemma 3.4 to define integral algebraic spaces. In this case the algebraic space described in Spaces, Example 14.9 would be integral which does not seem right. To avoid this type of pathology we will in addition assume that X is a decent algebraic space, although perhaps a weaker alternative exists.

0AD4 **Definition 4.1.** Let S be a scheme. We say an algebraic space X over S is *integral* if it is reduced, decent, and $|X|$ is irreducible.

In this case the irreducible topological space $|X|$ is sober (Decent Spaces, Proposition 11.9). Hence it has a unique generic point x . Then x is contained in the schematic locus of X (Decent Spaces, Theorem 10.2) and we can look at its residue field as a substitute for the function field of X (not yet defined; insert future reference here). In Decent Spaces, Lemma 19.3 we characterized decent algebraic spaces

with finitely many irreducible components. Applying that lemma we see that an algebraic space X is integral if it is reduced, has an irreducible dense open subscheme X' with generic point x' and the morphism $x' \rightarrow X$ is quasi-compact.

0BH2 **Lemma 4.2.** *Let S be a scheme. Let X be an integral algebraic space over S . Then $\Gamma(X, \mathcal{O}_X)$ is a domain.*

Proof. Set $R = \Gamma(X, \mathcal{O}_X)$. If $f, g \in R$ are nonzero and $fg = 0$ then $X = V(f) \cup V(g)$ where $V(f)$ denotes the closed subspace of X cut out by f . Since X is irreducible, we see that either $V(f) = X$ or $V(g) = X$. Then either $f = 0$ or $g = 0$ by Properties of Spaces, Lemma 11.1. \square

The following lemma characterizes dominant morphisms of finite degree between integral algebraic spaces.

0AD5 **Lemma 4.3.** *Let S be a scheme. Let X, Y be integral algebraic spaces over S . Let $x \in |X|$ and $y \in |Y|$ be the generic points. Let $f : X \rightarrow Y$ be locally of finite type. Assume f is dominant (Morphisms of Spaces, Definition 18.1). The following are equivalent:*

- (1) *the transcendence degree of x/y is 0,*
- (2) *the extension $\kappa(x) \supset \kappa(y)$ (see proof) is finite,*
- (3) *there exist nonempty affine opens $U \subset X$ and $V \subset Y$ such that $f(U) \subset V$ and $f|_U : U \rightarrow V$ is finite,*
- (4) *f is quasi-finite at x , and*
- (5) *x is the only point of $|X|$ mapping to y .*

If f is separated or if f is quasi-compact, then these are also equivalent to

- (6) *there exists a nonempty affine open $V \subset Y$ such that $f^{-1}(V) \rightarrow V$ is finite.*

Proof. By elementary topology, we see that $f(x) = y$ as f is dominant. Let $Y' \subset Y$ be the schematic locus of Y and let $X' \subset f^{-1}(Y')$ be the schematic locus of $f^{-1}(Y')$. By the discussion above, using Decent Spaces, Proposition 11.9 and Theorem 10.2, we see that $x \in |X'|$ and $y \in |Y'|$. Then $f|_{X'} : X' \rightarrow Y'$ is a morphism of integral schemes which is locally of finite type. Thus we see that (1), (2), (3) are equivalent by Morphisms, Lemma 48.7.

Condition (4) implies condition (1) by Morphisms of Spaces, Lemma 32.3 applied to $X \rightarrow Y \rightarrow Y$. On the other hand, condition (3) implies condition (4) as a finite morphism is quasi-finite and as $x \in U$ because x is the generic point. Thus (1) – (4) are equivalent.

Assume the equivalent conditions (1) – (4). Suppose that $x' \mapsto y$. Then $x \rightsquigarrow x'$ is a specialization in the fibre of $|X| \rightarrow |Y|$ over y . If $x' \neq x$, then f is not quasi-finite at x by Decent Spaces, Lemma 17.9. Hence $x = x'$ and (5) holds. Conversely, if (5) holds, then (5) holds for the morphism of schemes $X' \rightarrow Y'$ (see above) and we can use Morphisms, Lemma 48.7 to see that (1) holds.

Observe that (6) implies the equivalent conditions (1) – (5) without any further assumptions on f . To finish the proof we have to show the equivalent conditions (1) – (5) imply (6). This follows from Decent Spaces, Lemma 20.4. \square

0AD6 **Definition 4.4.** *Let S be a scheme. Let X and Y be integral algebraic spaces over S . Let $f : X \rightarrow Y$ be locally of finite type and dominant. Assume any of*

the equivalent conditions (1) – (5) of Lemma 4.3. Let $x \in |X|$ and $y \in |Y|$ be the generic points. Then the positive integer

$$\deg(X/Y) = [\kappa(x) : \kappa(y)]$$

is called the *degree of X over Y* .

Here is a lemma about normal integral algebraic spaces.

0AYH **Lemma 4.5.** *Let S be a scheme. Let X be a normal integral algebraic space over S . For every $x \in |X|$ there exists a normal integral affine scheme U and an étale morphism $U \rightarrow X$ such that x is in the image.*

Proof. Choose an affine scheme U and an étale morphism $U \rightarrow X$ such that x is in the image. Let u_i , $i \in I$ be the generic points of irreducible components of U . Then each u_i maps to the generic point of X (Decent Spaces, Lemma 19.1). By our definition of a decent space (Decent Spaces, Definition 6.1), we see that I is finite. Hence $U = \text{Spec}(A)$ where A is a normal ring with finitely many minimal primes. Thus $A = \prod_{i \in I} A_i$ is a product of normal domains by Algebra, Lemma 36.16. Then $U = \coprod U_i$ with $U_i = \text{Spec}(A_i)$ and x is in the image of $U_i \rightarrow X$ for some i . This proves the lemma. \square

0BH3 **Lemma 4.6.** *Let S be a scheme. Let X be a normal integral algebraic space over S . Then $\Gamma(X, \mathcal{O}_X)$ is a normal domain.*

Proof. Set $R = \Gamma(X, \mathcal{O}_X)$. Then R is a domain by Lemma 4.2. Let $f = a/b$ be an element of the fraction field of R which is integral over R . For any $U \rightarrow X$ étale with U a scheme there is at most one $f_U \in \Gamma(U, \mathcal{O}_U)$ with $b|_U f_U = a|_U$. Namely, U is reduced and the generic points of U map to the generic point of X which implies that $b|_U$ is a nonzerodivisor. For every $x \in |X|$ we choose $U \rightarrow X$ as in Lemma 4.5. Then there is a unique $f_U \in \Gamma(U, \mathcal{O}_U)$ with $b|_U f_U = a|_U$ because $\Gamma(U, \mathcal{O}_U)$ is a normal domain by Properties, Lemma 7.9. By the uniqueness mentioned above these f_U glue and define a global section f of the structure sheaf, i.e., of R . \square

5. Modifications and alterations

0AD7 Using our notion of an integral algebraic space we can define a modification as follows.

0AD8 **Definition 5.1.** Let S be a scheme. Let X be an integral algebraic space over S . A *modification of X* is a birational proper morphism $f : X' \rightarrow X$ of algebraic spaces over S with X' integral.

For birational morphisms of algebraic spaces, see Decent Spaces, Definition 21.1.

0AD9 **Lemma 5.2.** *Let $f : X' \rightarrow X$ be a modification as in Definition 5.1. There exists a nonempty open $U \subset X$ such that $f^{-1}(U) \rightarrow U$ is an isomorphism.*

Proof. By Lemma 4.3 there exists a nonempty $U \subset X$ such that $f^{-1}(U) \rightarrow U$ is finite. By generic flatness (Morphisms of Spaces, Proposition 31.1) we may assume $f^{-1}(U) \rightarrow U$ is flat and of finite presentation. So $f^{-1}(U) \rightarrow U$ is finite locally free (Morphisms of Spaces, Lemma 45.6). Since f is birational, the degree of X' over X is 1. Hence $f^{-1}(U) \rightarrow U$ is finite locally free of degree 1, in other words it is an isomorphism. \square

0ADA **Definition 5.3.** Let S be a scheme. Let X be an integral algebraic space over S . An *alteration* of X is a proper dominant morphism $f : Y \rightarrow X$ of algebraic spaces over S with Y integral such that $f^{-1}(U) \rightarrow U$ is finite for some nonempty open $U \subset X$.

If $f : Y \rightarrow X$ is a dominant and proper morphism between integral algebraic spaces, then it is an alteration as soon as the induced extension of residue fields in generic points is finite. Here is the precise statement.

0ADB **Lemma 5.4.** *Let S be a scheme. Let $f : X \rightarrow Y$ be a proper dominant morphism of integral algebraic spaces over S . Then f is an alteration if and only if any of the equivalent conditions (1) – (6) of Lemma 4.3 hold.*

Proof. Immediate consequence of the lemma referenced in the statement. \square

0DMN **Lemma 5.5.** *Let S be a scheme. Let $f : X \rightarrow Y$ be a proper surjective morphism of algebraic spaces over S . Assume Y is integral. Then there exists an integral closed subspace $X' \subset X$ such that $f' = f|_{X'} : X' \rightarrow Y$ is an alteration.*

Proof. Let $V \subset Y$ be a nonempty open affine (Decent Spaces, Theorem 10.2). Let $\eta \in V$ be the generic point. Then X_η is a nonempty proper algebraic space over η . Choose a closed point $x \in |X_\eta|$ (exists because $|X_\eta|$ is a quasi-compact, sober topological space, see Decent Spaces, Proposition 11.9 and Topology, Lemma 12.8.) Let X' be the reduced induced closed subspace structure on $\overline{\{x\}} \subset |X|$ (Properties of Spaces, Definition 11.6. Then $f' : X' \rightarrow Y$ is surjective as the image contains η . Also f' is proper as a composition of a closed immersion and a proper morphism. Finally, the fibre X'_η has a single point; to see this use Decent Spaces, Lemma 17.6 for both $X \rightarrow Y$ and $X' \rightarrow Y$ and the point η . Since Y is decent and $X' \rightarrow Y$ is separated we see that X' is decent (Decent Spaces, Lemmas 16.2 and 16.5). Thus f' is an alteration by Lemma 5.4. \square

6. Schematic locus

06LY We have already proven a number of results on the schematic locus of an algebraic space. Here is a list of references:

- (1) Properties of Spaces, Sections 12 and 13,
- (2) Decent Spaces, Section 10,
- (3) Properties of Spaces, Lemma 14.3 \Leftarrow Decent Spaces, Lemma 11.12 \Leftarrow Decent Spaces, Lemma 13.2,
- (4) Limits of Spaces, Section 15, and
- (5) Limits of Spaces, Section 17.

There are some cases where certain types of morphisms of algebraic spaces are automatically representable, for example separated, locally quasi-finite morphisms (Morphisms of Spaces, Lemma 49.1), and flat monomorphisms (More on Morphisms of Spaces, Lemma 4.1) In Section 7 we will study what happens with the schematic locus under extension of base field.

06LZ **Lemma 6.1.** *Let S be a scheme. Let X be an algebraic space over S . In each of the following cases X is a scheme:*

- (1) X is quasi-compact and quasi-separated and $\dim(X) = 0$,
- (2) X is locally of finite type over a field k and $\dim(X) = 0$,
- (3) X is Noetherian and $\dim(X) = 0$, and

(4) *add more here.*

Proof. Cases (2) and (3) follow immediately from case (1) but we will give a separate proofs of (2) and (3) as these proofs use significantly less theory.

Proof of (3). Let U be an affine scheme and let $U \rightarrow X$ be an étale morphism. Set $R = U \times_X U$. The two projection morphisms $s, t : R \rightarrow U$ are étale morphisms of schemes. By Properties of Spaces, Definition 8.2 we see that $\dim(U) = 0$ and $\dim(R) = 0$. Since R is a locally Noetherian scheme of dimension 0, we see that R is a disjoint union of spectra of Artinian local rings (Properties, Lemma 10.5). Since we assumed that X is Noetherian (so quasi-separated) we conclude that R is quasi-compact. Hence R is an affine scheme (use Schemes, Lemma 6.8). The étale morphisms $s, t : R \rightarrow U$ induce finite residue field extensions. Hence s and t are finite by Algebra, Lemma 53.4 (small detail omitted). Thus Groupoids, Proposition 23.9 shows that $X = U/R$ is an affine scheme.

Proof of (2) – almost identical to the proof of (4). Let U be an affine scheme and let $U \rightarrow X$ be an étale morphism. Set $R = U \times_X U$. The two projection morphisms $s, t : R \rightarrow U$ are étale morphisms of schemes. By Properties of Spaces, Definition 8.2 we see that $\dim(U) = 0$ and similarly $\dim(R) = 0$. On the other hand, the morphism $U \rightarrow \text{Spec}(k)$ is locally of finite type as the composition of the étale morphism $U \rightarrow X$ and $X \rightarrow \text{Spec}(k)$, see Morphisms of Spaces, Lemmas 23.2 and 38.9. Similarly, $R \rightarrow \text{Spec}(k)$ is locally of finite type. Hence by Varieties, Lemma 20.2 we see that U and R are disjoint unions of spectra of local Artinian k -algebras finite over k . The same thing is therefore true of $U \times_{\text{Spec}(k)} U$. As

$$R = U \times_X U \longrightarrow U \times_{\text{Spec}(k)} U$$

is a monomorphism, we see that R is a finite(!) union of spectra of finite k -algebras. It follows that R is affine, see Schemes, Lemma 6.8. Applying Varieties, Lemma 20.2 once more we see that R is finite over k . Hence s, t are finite, see Morphisms, Lemma 42.13. Thus Groupoids, Proposition 23.9 shows that the open subspace U/R of X is an affine scheme. Since the schematic locus of X is an open subspace (see Properties of Spaces, Lemma 12.1), and since $U \rightarrow X$ was an arbitrary étale morphism from an affine scheme we conclude that X is a scheme.

Proof of (1). By Cohomology of Spaces, Lemma 10.1 we have vanishing of higher cohomology groups for all quasi-coherent sheaves \mathcal{F} on X . Hence X is affine (in particular a scheme) by Cohomology of Spaces, Proposition 16.7. \square

The following lemma tells us that a quasi-separated algebraic space is a scheme away from codimension 1.

0ADC **Lemma 6.2.** *Let S be a scheme. Let X be a quasi-separated algebraic space over S . Let $x \in |X|$. The following are equivalent*

- (1) *x is a point of codimension 0 on X ,*
- (2) *the local ring of X at x has dimension 0, and*
- (3) *x is a generic point of an irreducible component of $|X|$.*

If true, then there exists an open subspace of X containing x which is a scheme.

Proof. The equivalence of (1), (2), and (3) follows from Decent Spaces, Lemma 19.1 and the fact that a quasi-separated algebraic space is decent (Decent Spaces, Section 6). However in the next paragraph we will give a more elementary proof of the equivalence.

Note that (1) and (2) are equivalent by definition (Properties of Spaces, Definition 9.2). To prove the equivalence of (1) and (3) we may assume X is quasi-compact. Choose

$$\emptyset = U_{n+1} \subset U_n \subset U_{n-1} \subset \dots \subset U_1 = X$$

and $f_i : V_i \rightarrow U_i$ as in Decent Spaces, Lemma 8.6. Say $x \in U_i$, $x \notin U_{i+1}$. Then $x = f_i(y)$ for a unique $y \in V_i$. If (1) holds, then y is a generic point of an irreducible component of V_i (Properties of Spaces, Lemma 10.1). Since $f_i^{-1}(U_{i+1})$ is a quasi-compact open of V_i not containing y , there is an open neighbourhood $W \subset V_i$ of y disjoint from $f_i^{-1}(U_{i+1})$ (see Properties, Lemma 2.2 or more simply Algebra, Lemma 25.4). Then $f_i|_W : W \rightarrow X$ is an isomorphism onto its image and hence $x = f_i(y)$ is a generic point of $|X|$. Conversely, assume (3) holds. Then f_i maps $\overline{\{y\}}$ onto the irreducible component $\overline{\{x\}}$ of $|U_i|$. Since $|f_i|$ is bijective over $\overline{\{x\}}$, it follows that $\overline{\{y\}}$ is an irreducible component of U_i . Thus x is a point of codimension 0.

The final statement of the lemma is Properties of Spaces, Proposition 12.3. \square

The following lemma says that a separated locally Noetherian algebraic space is a scheme in codimension 1, i.e., away from codimension 2.

0ADD **Lemma 6.3.** *Let S be a scheme. Let X be an algebraic space over S . Let $x \in |X|$. If X is separated, locally Noetherian, and the dimension of the local ring of X at x is ≤ 1 (Properties of Spaces, Definition 9.2), then there exists an open subspace of X containing x which is a scheme.*

Proof. (Please see the remark below for a different approach avoiding the material on finite groupoids.) We can replace X by an quasi-compact neighbourhood of x , hence we may assume X is quasi-compact, separated, and Noetherian. There exists a scheme U and a finite surjective morphism $U \rightarrow X$, see Limits of Spaces, Proposition 16.1. Let $R = U \times_X U$. Then $j : R \rightarrow U \times_S U$ is an equivalence relation and we obtain a groupoid scheme (U, R, s, t, c) over S with s, t finite and U Noetherian and separated. Let $\{u_1, \dots, u_n\} \subset U$ be the set of points mapping to x . Then $\dim(\mathcal{O}_{U, u_i}) \leq 1$ by Decent Spaces, Lemma 11.11.

By More on Groupoids, Lemma 14.10 there exists an R -invariant affine open $W \subset U$ containing the orbit $\{u_1, \dots, u_n\}$. Since $U \rightarrow X$ is finite surjective the continuous map $|U| \rightarrow |X|$ is closed surjective, hence submersive by Topology, Lemma 6.5. Thus $f(W)$ is open and there is an open subspace $X' \subset X$ with $f : W \rightarrow X'$ a surjective finite morphism. Then X' is an affine scheme by Cohomology of Spaces, Lemma 17.1 and the proof is finished. \square

0ADE **Remark 6.4.** Here is a sketch of a proof of Lemma 6.3 which avoids using More on Groupoids, Lemma 14.10.

Step 1. We may assume X is a reduced Noetherian separated algebraic space (for example by Cohomology of Spaces, Lemma 17.1 or by Limits of Spaces, Lemma 15.3) and we may choose a finite surjective morphism $Y \rightarrow X$ where Y is a Noetherian scheme (by Limits of Spaces, Proposition 16.1).

Step 2. After replacing X by an open neighbourhood of x , there exists a birational finite morphism $X' \rightarrow X$ and a closed subscheme $Y' \subset X' \times_X Y$ such that $Y' \rightarrow X'$ is surjective finite locally free. Namely, because X is reduced there is a dense open subspace $U \subset X$ over which Y is flat (Morphisms of Spaces, Proposition 31.1). Then we can choose a U -admissible blow up $b : \tilde{X} \rightarrow X$ such that the strict

transform \tilde{Y} of Y is flat over \tilde{X} , see More on Morphisms of Spaces, Lemma 33.1. (An alternative is to use Hilbert schemes if one wants to avoid using the result on blow ups). Then we let $X' \subset \tilde{X}$ be the scheme theoretic closure of $b^{-1}(U)$ and $Y' = X' \times_{\tilde{X}} \tilde{Y}$. Since x is a codimension 1 point, we see that $X' \rightarrow X$ is finite over a neighbourhood of x (Lemma 3.2).

Step 3. After shrinking X to a smaller neighbourhood of x we get that X' is a scheme. This holds because Y' is a scheme and $Y' \rightarrow X'$ being finite locally free and because every finite set of codimension 1 points of Y' is contained in an affine open. Use Properties of Spaces, Proposition 13.1 and Varieties, Proposition 40.7.

Step 4. There exists an affine open $W' \subset X'$ containing all points lying over x which is the inverse image of an open subspace of X . To prove this let $Z \subset X$ be the closure of the set of points where $X' \rightarrow X$ is not an isomorphism. We may assume $x \in Z$ otherwise we are already done. Then x is a generic point of an irreducible component of Z and after shrinking X we may assume Z is an affine scheme (Lemma 6.2). Then the inverse image $Z' \subset X'$ is an affine scheme as well. Say $x_1, \dots, x_n \in Z'$ are the points mapping to x . Then we can find an affine open W' in X' whose intersection with Z' is the inverse image of a principal open of Z containing x . Namely, we first pick an affine open $W' \subset X'$ containing x_1, \dots, x_n using Varieties, Proposition 40.7. Then we pick a principal open $D(f) \subset Z$ containing x whose inverse image $D(f|_{Z'})$ is contained in $W' \cap Z'$. Then we pick $f' \in \Gamma(W', \mathcal{O}_{W'})$ restricting to $f|_{Z'}$ and we replace W' by $D(f') \subset W'$. Since $X' \rightarrow X$ is an isomorphism away from $Z' \rightarrow Z$ the choice of W' guarantees that the image $W \subset X$ of W' is open with inverse image W' in X' .

Step 5. Then $W' \rightarrow W$ is a finite surjective morphism and W is a scheme by Cohomology of Spaces, Lemma 17.1 and the proof is complete.

7. Schematic locus and field extension

0B82 It can happen that a nonrepresentable algebraic space over a field k becomes representable (i.e., a scheme) after base change to an extension of k . See Spaces, Example 14.2. In this section we address this issue.

0B83 **Lemma 7.1.** *Let k be a field. Let X be an algebraic space over k . If there exists a purely inseparable field extension $k \subset k'$ such that $X_{k'}$ is a scheme, then X is a scheme.*

Proof. The morphism $X_{k'} \rightarrow X$ is integral, surjective, and universally injective. Hence this lemma follows from Limits of Spaces, Lemma 15.4. \square

0B84 **Lemma 7.2.** *Let k be a field with algebraic closure \bar{k} . Let X be a quasi-separated algebraic space over k .*

- (1) *If there exists a field extension $k \subset K$ such that X_K is a scheme, then $X_{\bar{k}}$ is a scheme.*
- (2) *If X is quasi-compact and there exists a field extension $k \subset K$ such that X_K is a scheme, then $X_{k'}$ is a scheme for some finite separable extension k' of k .*

Proof. Since every algebraic space is the union of its quasi-compact open subspaces, we see that the first part of the lemma follows from the second part (some details omitted). Thus we assume X is quasi-compact and we assume given an

extension $k \subset K$ with K_K representable. Write $K = \bigcup A$ as the colimit of finitely generated k -subalgebras A . By Limits of Spaces, Lemma 5.11 we see that X_A is a scheme for some A . Choose a maximal ideal $\mathfrak{m} \subset A$. By the Hilbert Nullstellensatz (Algebra, Theorem 33.1) the residue field $k' = A/\mathfrak{m}$ is a finite extension of k . Thus we see that $X_{k'}$ is a scheme. If $k' \supset k$ is not separable, let $k' \supset k'' \supset k$ be the subextension found in Fields, Lemma 14.6. Since k'/k'' is purely inseparable, by Lemma 7.1 the algebraic space $X_{k''}$ is a scheme. Since k''/k is separable the proof is complete. \square

0B86 **Lemma 7.3.** *Let $k \subset k'$ be a finite Galois extension with Galois group G . Let X be an algebraic space over k . Then G acts freely on the algebraic space $X_{k'}$ and $X = X_{k'}/G$ in the sense of Properties of Spaces, Lemma 33.1.*

Proof. Omitted. Hints: First show that $\text{Spec}(k) = \text{Spec}(k')/G$. Then use compatibility of taking quotients with base change. \square

0B87 **Lemma 7.4.** *Let S be a scheme. Let X be an algebraic space over S and let G be a finite group acting freely on X . Set $Y = X/G$ as in Properties of Spaces, Lemma 33.1. For $y \in |Y|$ the following are equivalent*

- (1) y is in the schematic locus of Y , and
- (2) there exists an affine open $U \subset X$ containing the preimage of y .

Proof. It follows from the construction of $Y = X/G$ in Properties of Spaces, Lemma 33.1 that the morphism $X \rightarrow Y$ is surjective and étale. Of course we have $X \times_Y X = X \times G$ hence the morphism $X \rightarrow Y$ is even finite étale. It is also surjective. Thus the lemma follows from Decent Spaces, Lemma 10.3. \square

0B85 **Lemma 7.5.** *Let k be a field. Let X be a quasi-separated algebraic space over k . If there exists a purely transcendental field extension $k \subset K$ such that X_K is a scheme, then X is a scheme.*

Proof. Since every algebraic space is the union of its quasi-compact open subspaces, we may assume X is quasi-compact (some details omitted). Recall (Fields, Definition 26.1) that the assumption on the extension K/k signifies that K is the fraction field of a polynomial ring (in possibly infinitely many variables) over k . Thus $K = \bigcup A$ is the union of subalgebras each of which is a localization of a finite polynomial algebra over k . By Limits of Spaces, Lemma 5.11 we see that X_A is a scheme for some A . Write

$$A = k[x_1, \dots, x_n][1/f]$$

for some nonzero $f \in k[x_1, \dots, x_n]$.

If k is infinite then we can finish the proof as follows: choose $a_1, \dots, a_n \in k$ with $f(a_1, \dots, a_n) \neq 0$. Then (a_1, \dots, a_n) define a k -algebra map $A \rightarrow k$ mapping x_i to a_i and $1/f$ to $1/f(a_1, \dots, a_n)$. Thus the base change $X_A \times_{\text{Spec}(A)} \text{Spec}(k) \cong X$ is a scheme as desired.

In this paragraph we finish the proof in case k is finite. In this case we write $X = \lim X_i$ with X_i of finite presentation over k and with affine transition morphisms (Limits of Spaces, Lemma 10.1). Using Limits of Spaces, Lemma 5.11 we see that $X_{i,A}$ is a scheme for some i . Thus we may assume $X \rightarrow \text{Spec}(k)$ is of finite presentation. Let $x \in |X|$ be a closed point. We may represent x by a closed immersion $\text{Spec}(\kappa) \rightarrow X$ (Decent Spaces, Lemma 13.6). Then $\text{Spec}(\kappa) \rightarrow \text{Spec}(k)$

is of finite type, hence κ is a finite extension of k (by the Hilbert Nullstellensatz, see Algebra, Theorem 33.1; some details omitted). Say $[\kappa : k] = d$. Choose an integer $n \gg 0$ prime to d and let $k \subset k'$ be the extension of degree n . Then k'/k is Galois with $G = \text{Aut}(k'/k)$ cyclic of order n . If n is large enough there will be k -algebra homomorphism $A \rightarrow k'$ by the same reason as above. Then $X_{k'}$ is a scheme and $X = X_{k'}/G$ (Lemma 7.3). On the other hand, since n and d are relatively prime we see that

$$\text{Spec}(\kappa) \times_X X_{k'} = \text{Spec}(\kappa) \times_{\text{Spec}(k)} \text{Spec}(k') = \text{Spec}(\kappa \otimes_k k')$$

is the spectrum of a field. In other words, the fibre of $X_{k'} \rightarrow X$ over x consists of a single point. Thus by Lemma 7.4 we see that x is in the schematic locus of X as desired. \square

0BA7 **Remark 7.6.** Let k be finite field. Let $K \supset k$ be a geometrically irreducible field extension. Then K is the limit of geometrically irreducible finite type k -algebras A . Given A the estimates of Lang and Weil [LW54], show that for $n \gg 0$ there exists an k -algebra homomorphism $A \rightarrow k'$ with k'/k of degree n . Analyzing the argument given in the proof of Lemma 7.5 we see that if X is a quasi-separated algebraic space over k and X_K is a scheme, then X is a scheme. If we ever need this result we will precisely formulate it and prove it here.

0B88 **Lemma 7.7.** Let k be a field with algebraic closure \bar{k} . Let X be an algebraic space over k such that

- (1) X is decent and locally of finite type over k ,
- (2) $X_{\bar{k}}$ is a scheme, and
- (3) any finite set of \bar{k} -rational points of $X_{\bar{k}}$ are contained in an affine.

Then X is a scheme.

Proof. If $k \subset K$ is an extension, then the base change X_K is decent (Decent Spaces, Lemma 6.5) and locally of finite type over K (Morphisms of Spaces, Lemma 23.3). By Lemma 7.1 it suffices to prove that X becomes a scheme after base change to the perfection of k , hence we may assume k is a perfect field (this step isn't strictly necessary, but makes the other arguments easier to think about). By covering X by quasi-compact opens we see that it suffices to prove the lemma in case X is quasi-compact (small detail omitted). In this case $|X|$ is a sober topological space (Decent Spaces, Proposition 11.9). Hence it suffices to show that every closed point in $|X|$ is contained in the schematic locus of X (use Properties of Spaces, Lemma 12.1 and Topology, Lemma 12.8).

Let $x \in |X|$ be a closed point. By Decent Spaces, Lemma 13.6 we can find a closed immersion $\text{Spec}(l) \rightarrow X$ representing x . Then $\text{Spec}(l) \rightarrow \text{Spec}(k)$ is of finite type (Morphisms of Spaces, Lemma 23.2) and we conclude that l is a finite extension of k by the Hilbert Nullstellensatz (Algebra, Theorem 33.1). It is separable because k is perfect. Thus the scheme

$$\text{Spec}(l) \times_X X_{\bar{k}} = \text{Spec}(l) \times_{\text{Spec}(k)} \text{Spec}(\bar{k}) = \text{Spec}(l \otimes_k \bar{k})$$

is the disjoint union of a finite number of \bar{k} -rational points. By assumption (3) we can find an affine open $W \subset X_{\bar{k}}$ containing these points.

By Lemma 7.2 we see that $X_{k'}$ is a scheme for some finite extension k'/k . After enlarging k' we may assume that there exists an affine open $U' \subset X_{k'}$ whose

base change to \bar{k} recovers W (use that $X_{\bar{k}}$ is the limit of the schemes $X_{k''}$ for $k' \subset k'' \subset \bar{k}$ finite and use Limits, Lemmas 4.11 and 4.13). We may assume that k'/k is a Galois extension (take the normal closure Fields, Lemma 16.3 and use that k is perfect). Set $G = \text{Gal}(k'/k)$. By construction the G -invariant closed subscheme $\text{Spec}(l) \times_X X_{k'}$ is contained in U' . Thus x is in the schematic locus by Lemmas 7.3 and 7.4. \square

The following two lemmas should go somewhere else. Please compare the next lemma to Decent Spaces, Lemma 17.8.

06S0 **Lemma 7.8.** *Let k be a field. Let X be an algebraic space over k . The following are equivalent*

- (1) X is locally quasi-finite over k ,
- (2) X is locally of finite type over k and has dimension 0,
- (3) X is a scheme and is locally quasi-finite over k ,
- (4) X is a scheme and is locally of finite type over k and has dimension 0, and
- (5) X is a disjoint union of spectra of Artinian local k -algebras A over k with $\dim_k(A) < \infty$.

Proof. Because we are over a field relative dimension of X/k is the same as the dimension of X . Hence by Morphisms of Spaces, Lemma 33.6 we see that (1) and (2) are equivalent. Hence it follows from Lemma 6.1 (and trivial implications) that (1) – (4) are equivalent. Finally, Varieties, Lemma 20.2 shows that (1) – (4) are equivalent with (5). \square

06S1 **Lemma 7.9.** *Let k be a field. Let $f : X \rightarrow Y$ be a monomorphism of algebraic spaces over k . If Y is locally quasi-finite over k so is X .*

Proof. Assume Y is locally quasi-finite over k . By Lemma 7.8 we see that $Y = \coprod \text{Spec}(A_i)$ where each A_i is an Artinian local ring finite over k . By Decent Spaces, Lemma 18.1 we see that X is a scheme. Consider $X_i = f^{-1}(\text{Spec}(A_i))$. Then X_i has either one or zero points. If X_i has zero points there is nothing to prove. If X_i has one point, then $X_i = \text{Spec}(B_i)$ with B_i a zero dimensional local ring and $A_i \rightarrow B_i$ is an epimorphism of rings. In particular $A_i/\mathfrak{m}_{A_i} = B_i/\mathfrak{m}_{A_i}B_i$ and we see that $A_i \rightarrow B_i$ is surjective by Nakayama's lemma, Algebra, Lemma 19.1 (because \mathfrak{m}_{A_i} is a nilpotent ideal!). Thus B_i is a finite local k -algebra, and we conclude by Lemma 7.8 that $X \rightarrow \text{Spec}(k)$ is locally quasi-finite. \square

8. Geometrically reduced algebraic spaces

0DMP If X is a reduced algebraic space over a field, then it can happen that X becomes nonreduced after extending the ground field. This does not happen for geometrically reduced algebraic spaces.

0DMQ **Definition 8.1.** Let k be a field. Let X be an algebraic space over k .

- (1) Let $x \in |X|$ be a point. We say X is *geometrically reduced at x* if $\mathcal{O}_{X,\bar{x}}$ is geometrically reduced over k .
- (2) We say X is *geometrically reduced over k* if X is geometrically reduced at every point of X .

Here are the obligatory lemmas connecting this with the corresponding property for schemes over a field.

ODMR **Lemma 8.2.** *Let k be a field. Let X be an algebraic space over k . Let $x \in |X|$. The following are equivalent*

- (1) X is geometrically reduced at x ,
- (2) for some étale neighbourhood $(U, u) \rightarrow (X, x)$ where U is a scheme, U is geometrically reduced at u ,
- (3) for any étale neighbourhood $(U, u) \rightarrow (X, x)$ where U is a scheme, U is geometrically reduced at u .

Proof. Recall that the local ring $\mathcal{O}_{X, \bar{x}}$ is the strict henselization of $\mathcal{O}_{U, u}$, see Properties of Spaces, Lemma 21.1. By Varieties, Lemma 6.2 we find that U is geometrically reduced at u if and only if $\mathcal{O}_{U, u}$ is geometrically reduced over k . Thus we have to show: if A is a local k -algebra, then A is geometrically reduced over k if and only if A^{sh} is geometrically reduced over k . We check this using the definition of geometrically reduced algebras (Algebra, Definition 42.1). Let K/k be a field extension. Since $A \rightarrow A^{sh}$ is faithfully flat (More on Algebra, Lemma 41.1) we see that $A \otimes_k K \rightarrow A^{sh} \otimes_k K$ is faithfully flat (Algebra, Lemma 38.7). Hence if $A^{sh} \otimes_k K$ is reduced, so is $A \otimes_k K$ by Algebra, Lemma 158.2. Conversely, recall that A^{sh} is a colimit of étale A -algebras, see Algebra, Lemma 150.2. Thus $A^{sh} \otimes_k K$ is a filtered colimit of étale $A \otimes_k K$ -algebras. We conclude by Algebra, Lemma 157.7. \square

ODMS **Lemma 8.3.** *Let k be a field. Let X be an algebraic space over k . The following are equivalent*

- (1) X is geometrically reduced,
- (2) for some surjective étale morphism $U \rightarrow X$ where U is a scheme, U is geometrically reduced,
- (3) for any étale morphism $U \rightarrow X$ where U is a scheme, U is geometrically reduced.

Proof. Immediate from the definitions and Lemma 8.2. \square

9. Geometrically connected algebraic spaces

0A0Y If X is a connected algebraic space over a field, then it can happen that X becomes disconnected after extending the ground field. This does not happen for geometrically connected schemes.

0A0Z **Definition 9.1.** Let X be an algebraic space over the field k . We say X is *geometrically connected* over k if the base change $X_{k'}$ is connected for every field extension k' of k .

By convention a connected topological space is nonempty; hence a fortiori geometrically connected algebraic spaces are nonempty.

0A10 **Lemma 9.2.** *Let X be an algebraic space over the field k . Let $k \subset k'$ be a field extension. Then X is geometrically connected over k if and only if $X_{k'}$ is geometrically connected over k' .*

Proof. If X is geometrically connected over k , then it is clear that $X_{k'}$ is geometrically connected over k' . For the converse, note that for any field extension $k \subset k''$ there exists a common field extension $k' \subset k'''$ and $k'' \subset k'''$. As the morphism $X_{k'''} \rightarrow X_{k''}$ is surjective (as a base change of a surjective morphism between spectra of fields) we see that the connectedness of $X_{k'''}$ implies the connectedness

of $X_{k''}$. Thus if $X_{k'}$ is geometrically connected over k' then X is geometrically connected over k . \square

0A11 **Lemma 9.3.** *Let k be a field. Let X, Y be algebraic spaces over k . Assume X is geometrically connected over k . Then the projection morphism*

$$p : X \times_k Y \longrightarrow Y$$

induces a bijection between connected components.

Proof. Let $y \in |Y|$ be represented by a morphism $\text{Spec}(K) \rightarrow Y$ where K is a field. The fibre of $|X \times_k Y| \rightarrow |Y|$ over y is the image of $|Y_K| \rightarrow |X \times_k Y|$ by Properties of Spaces, Lemma 4.3. Thus these fibres are connected by our assumption that Y is geometrically connected. By Morphisms of Spaces, Lemma 6.6 the map $|p|$ is open. Thus we may apply Topology, Lemma 7.5 to conclude. \square

0A12 **Lemma 9.4.** *Let $k \subset k'$ be an extension of fields. Let X be an algebraic space over k . Assume k separably algebraically closed. Then the morphism $X_{k'} \rightarrow X$ induces a bijection of connected components. In particular, X is geometrically connected over k if and only if X is connected.*

Proof. Since k is separably algebraically closed we see that k' is geometrically connected over k , see Algebra, Lemma 47.4. Hence $Z = \text{Spec}(k')$ is geometrically connected over k by Varieties, Lemma 7.5. Since $X_{k'} = Z \times_k X$ the result is a special case of Lemma 9.3. \square

0A13 **Lemma 9.5.** *Let k be a field. Let X be an algebraic space over k . Let \bar{k} be a separable algebraic closure of k . Then X is geometrically connected if and only if the base change $X_{\bar{k}}$ is connected.*

Proof. Assume $X_{\bar{k}}$ is connected. Let $k \subset k'$ be a field extension. There exists a field extension $\bar{k} \subset \bar{k}'$ such that k' embeds into \bar{k}' as an extension of k . By Lemma 9.4 we see that $X_{\bar{k}'}$ is connected. Since $X_{\bar{k}'} \rightarrow X_{k'}$ is surjective we conclude that $X_{k'}$ is connected as desired. \square

Let k be a field. Let $k \subset \bar{k}$ be a (possibly infinite) Galois extension. For example \bar{k} could be the separable algebraic closure of k . For any $\sigma \in \text{Gal}(\bar{k}/k)$ we get a corresponding automorphism $\text{Spec}(\sigma) : \text{Spec}(\bar{k}) \rightarrow \text{Spec}(\bar{k})$. Note that $\text{Spec}(\sigma) \circ \text{Spec}(\tau) = \text{Spec}(\tau \circ \sigma)$. Hence we get an action

$$\text{Gal}(\bar{k}/k)^{opp} \times \text{Spec}(\bar{k}) \longrightarrow \text{Spec}(\bar{k})$$

of the opposite group on the scheme $\text{Spec}(\bar{k})$. Let X be an algebraic space over k . Since $X_{\bar{k}} = \text{Spec}(\bar{k}) \times_{\text{Spec}(k)} X$ by definition we see that the action above induces a canonical action

0A14 (9.5.1)
$$\text{Gal}(\bar{k}/k)^{opp} \times X_{\bar{k}} \longrightarrow X_{\bar{k}}.$$

0A15 **Lemma 9.6.** *Let k be a field. Let X be an algebraic space over k . Let \bar{k} be a (possibly infinite) Galois extension of k . Let $V \subset X_{\bar{k}}$ be a quasi-compact open. Then*

- (1) *there exists a finite subextension $k \subset k' \subset \bar{k}$ and a quasi-compact open $V' \subset X_{k'}$ such that $V = (V')_{\bar{k}}$,*

- (2) *there exists an open subgroup $H \subset \text{Gal}(\bar{k}/k)$ such that $\sigma(V) = V$ for all $\sigma \in H$.*

Proof. Choose a scheme U and a surjective étale morphism $U \rightarrow X$. Choose a quasi-compact open $W \subset U_{\bar{k}}$ whose image in $X_{\bar{k}}$ is V . This is possible because $|U_{\bar{k}}| \rightarrow |X_{\bar{k}}|$ is continuous and because $|U_{\bar{k}}|$ has a basis of quasi-compact opens. We can apply Varieties, Lemma 7.9 to $W \subset U_{\bar{k}}$ to obtain the lemma. \square

0A16 **Lemma 9.7.** *Let k be a field. Let $k \subset \bar{k}$ be a (possibly infinite) Galois extension. Let X be an algebraic space over k . Let $\bar{T} \subset |X_{\bar{k}}|$ have the following properties*

- (1) \bar{T} is a closed subset of $|X_{\bar{k}}|$,
- (2) for every $\sigma \in \text{Gal}(\bar{k}/k)$ we have $\sigma(\bar{T}) = \bar{T}$.

Then there exists a closed subset $T \subset |X|$ whose inverse image in $|X_{k'}|$ is \bar{T} .

Proof. Let $T \subset |X|$ be the image of \bar{T} . Since $|X_{\bar{k}}| \rightarrow |X|$ is surjective, the statement means that T is closed and that its inverse image is \bar{T} . Choose a scheme U and a surjective étale morphism $U \rightarrow X$. By the case of schemes (see Varieties, Lemma 7.10) there exists a closed subset $T' \subset |U|$ whose inverse image in $|U_{\bar{k}}|$ is the inverse image of \bar{T} . Since $|U_{\bar{k}}| \rightarrow |X_{\bar{k}}|$ is surjective, we see that T' is the inverse image of T via $|U| \rightarrow |X|$. By our construction of the topology on $|X|$ this means that T is closed. In the same manner one sees that \bar{T} is the inverse image of T . \square

0A17 **Lemma 9.8.** *Let k be a field. Let X be an algebraic space over k . The following are equivalent*

- (1) X is geometrically connected,
- (2) for every finite separable field extension $k \subset k'$ the scheme $X_{k'}$ is connected.

Proof. This proof is identical to the proof of Varieties, Lemma 7.11 except that we replace Varieties, Lemma 7.7 by Lemma 9.5, we replace Varieties, Lemma 7.9 by Lemma 9.6, and we replace Varieties, Lemma 7.10 by Lemma 9.7. We urge the reader to read that proof in stead of this one.

It follows immediately from the definition that (1) implies (2). Assume that X is not geometrically connected. Let $k \subset \bar{k}$ be a separable algebraic closure of k . By Lemma 9.5 it follows that $X_{\bar{k}}$ is disconnected. Say $X_{\bar{k}} = \bar{U} \amalg \bar{V}$ with \bar{U} and \bar{V} open, closed, and nonempty algebraic subspaces of $X_{\bar{k}}$.

Suppose that $W \subset X$ is any quasi-compact open subspace. Then $W_{\bar{k}} \cap \bar{U}$ and $W_{\bar{k}} \cap \bar{V}$ are open and closed subspaces of $W_{\bar{k}}$. In particular $W_{\bar{k}} \cap \bar{U}$ and $W_{\bar{k}} \cap \bar{V}$ are quasi-compact, and by Lemma 9.6 both $W_{\bar{k}} \cap \bar{U}$ and $W_{\bar{k}} \cap \bar{V}$ are defined over a finite subextension and invariant under an open subgroup of $\text{Gal}(\bar{k}/k)$. We will use this without further mention in the following.

Pick $W_0 \subset X$ quasi-compact open subspace such that both $W_{0,\bar{k}} \cap \bar{U}$ and $W_{0,\bar{k}} \cap \bar{V}$ are nonempty. Choose a finite subextension $k \subset k' \subset \bar{k}$ and a decomposition $W_{0,k'} = U'_0 \amalg V'_0$ into open and closed subsets such that $W_{0,\bar{k}} \cap \bar{U} = (U'_0)_{\bar{k}}$ and $W_{0,\bar{k}} \cap \bar{V} = (V'_0)_{\bar{k}}$. Let $H = \text{Gal}(\bar{k}/k') \subset \text{Gal}(\bar{k}/k)$. In particular $\sigma(W_{0,\bar{k}} \cap \bar{U}) = W_{0,\bar{k}} \cap \bar{U}$ and similarly for \bar{V} .

Having chosen W_0, k' as above, for every quasi-compact open subspace $W \subset X$ we set

$$U_W = \bigcap_{\sigma \in H} \sigma(W_{\bar{k}} \cap \bar{U}), \quad V_W = \bigcup_{\sigma \in H} \sigma(W_{\bar{k}} \cap \bar{V}).$$

Now, since $W_{\bar{k}} \cap \bar{U}$ and $W_{\bar{k}} \cap \bar{V}$ are fixed by an open subgroup of $\text{Gal}(\bar{k}/k)$ we see that the union and intersection above are finite. Hence U_W and V_W are both open and closed subspaces. Also, by construction $W_{\bar{k}} = U_W \amalg V_W$.

We claim that if $W \subset W' \subset X$ are quasi-compact open subspaces, then $W_{\bar{k}} \cap U_{W'} = U_W$ and $W_{\bar{k}} \cap V_{W'} = V_W$. Verification omitted. Hence we see that upon defining $U = \bigcup_{W \subset X} U_W$ and $V = \bigcup_{W \subset X} V_W$ we obtain $X_{\bar{k}} = U \amalg V$ is a disjoint union of open and closed subsets. It is clear that V is nonempty as it is constructed by taking unions (locally). On the other hand, U is nonempty since it contains $W_0 \cap \bar{U}$ by construction. Finally, $U, V \subset X_{\bar{k}}$ are closed and H -invariant by construction. Hence by Lemma 9.7 we have $U = (U')_{\bar{k}}$, and $V = (V')_{\bar{k}}$ for some closed $U', V' \subset X_{k'}$. Clearly $X_{k'} = U' \amalg V'$ and we see that $X_{k'}$ is disconnected as desired. \square

10. Geometrically irreducible algebraic spaces

ODMT Spaces, Example 14.9 shows that it is best not to think about irreducible algebraic spaces in complete generality¹. For decent (for example quasi-separated) algebraic spaces this kind of disaster doesn't happen. Thus we make the following definition only under the assumption that our algebraic space is decent.

ODMU **Definition 10.1.** Let k be a field. Let X be a decent algebraic space over k . We say X is *geometrically irreducible* if the topological space $|X_{k'}|$ is irreducible² for any field extension k' of k .

Observe that $X_{k'}$ is a decent algebraic space (Decent Spaces, Lemma 6.5). Hence the topological space $|X_{k'}|$ is sober. Decent Spaces, Proposition 11.9.

11. Geometrically integral algebraic spaces

ODMV Recall that integral algebraic spaces are by definition decent, see Section 4.

ODMW **Definition 11.1.** Let X be an algebraic space over the field k . We say X is *geometrically integral* over k if the algebraic space $X_{k'}$ is integral (Definition 4.1) for every field extension k' of k .

In particular X is a decent algebraic space. We can relate this to being geometrically reduced and geometrically irreducible as follows.

ODMX **Lemma 11.2.** *Let k be a field. Let X be a decent algebraic space over k . Then X is geometrically integral over k if and only if X is both geometrically reduced and geometrically irreducible over k .*

Proof. This is an immediate consequence of the definitions because our notion of integral (in the presence of decency) is equivalent to reduced and irreducible. \square

ODMY **Lemma 11.3.** *Let k be a field. Let X be a proper algebraic space over k .*

- (1) $A = H^0(X, \mathcal{O}_X)$ is a finite dimensional k -algebra,
- (2) $A = \prod_{i=1, \dots, n} A_i$ is a product of Artinian local k -algebras, one factor for each connected component of $|X|$,
- (3) if X is reduced, then $A = \prod_{i=1, \dots, n} k_i$ is a product of fields, each a finite extension of k ,

¹To be sure, if we say “the algebraic space X is irreducible”, we probably mean to say “the topological space $|X|$ is irreducible”.

²An irreducible space is nonempty.

- (4) if X is geometrically reduced, then k_i is finite separable over k ,
- (5) if X is geometrically irreducible, then A is geometrically irreducible over k ,
- (6) if X is geometrically integral, then $A = k$.

Proof. By Cohomology of Spaces, Lemma 20.3 we see that $A = H^0(X, \mathcal{O}_X)$ is a finite dimensional k -algebra. This proves (1).

Then A is a product of local rings by Algebra, Lemma 52.2 and Algebra, Proposition 59.6. If $X = Y \amalg Z$ with Y and Z open subspaces of X , then we obtain an idempotent $e \in A$ by taking the section of \mathcal{O}_X which is 1 on Y and 0 on Z . Conversely, if $e \in A$ is an idempotent, then we get a corresponding decomposition of $|X|$. Finally, as $|X|$ is a Noetherian topological space (by Morphisms of Spaces, Lemma 28.6 and Properties of Spaces, Lemma 23.2) its connected components are open. Hence the connected components of $|X|$ correspond 1-to-1 with primitive idempotents of A . This proves (2).

If X is reduced, then A is reduced (Properties of Spaces, Lemma 11.1). Hence the local rings $A_i = k_i$ are reduced and therefore fields (for example by Algebra, Lemma 24.1). This proves (3).

If X is geometrically reduced, then same thing is true for $A \otimes_k \bar{k} = H^0(X_{\bar{k}}, \mathcal{O}_{X_{\bar{k}}})$ (see Cohomology of Spaces, Lemma 11.2 for equality). This implies that $k_i \otimes_k \bar{k}$ is a product of fields and hence k_i/k is separable for example by Algebra, Lemmas 43.1 and 43.3. This proves (4).

If X is geometrically irreducible, then $A \otimes_k \bar{k} = H^0(X_{\bar{k}}, \mathcal{O}_{X_{\bar{k}}})$ is a zero dimensional local ring by part (2) and hence its spectrum has one point, in particular it is irreducible. Thus A is geometrically irreducible. This proves (5).

If X is geometrically irreducible and geometrically reduced (or equivalently geometrically integral by Lemma 11.2), then $A = k_1$ is a field and the extension k_1/k is finite separable and geometrically irreducible. However, then $k_1 \otimes_k \bar{k}$ is a product of $[k_1 : k]$ copies of \bar{k} and we conclude that $k_1 = k$. \square

0DMZ **Lemma 11.4.** *Let k be a field. Let X be a proper integral algebraic space over k . Let \mathcal{L} be an invertible \mathcal{O}_X -module. If $H^0(X, \mathcal{L})$ and $H^0(X, \mathcal{L}^{\otimes -1})$ are both nonzero, then $\mathcal{L} \cong \mathcal{O}_X$.*

Proof. Let $s \in H^0(X, \mathcal{L})$ and $t \in H^0(X, \mathcal{L}^{\otimes -1})$ be nonzero sections. Let $x \in |X|$ be a point in the support of s . Choose an affine étale neighbourhood $(U, u) \rightarrow (X, x)$ such that $\mathcal{L}|_U \cong \mathcal{O}_U$. Then $s|_U$ corresponds to a nonzero regular function on the reduced (because X is reduced) scheme U and hence is nonvanishing in a generic point of an irreducible component of U . By Decent Spaces, Lemma 19.1 we conclude that the generic point η of $|X|$ is in the support of s . The same is true for t . Then of course st must be nonzero because the local ring of X at η is a field (by aforementioned lemma the local ring has dimension zero, as X is reduced the local ring is reduced, and Algebra, Lemma 24.1). However, we have seen that $K = H^0(X, \mathcal{O}_X)$ is a field in Lemma 11.3. Thus st is everywhere nonzero and we see that $s : \mathcal{O}_X \rightarrow \mathcal{L}$ is an isomorphism. \square

12. Spaces smooth over fields

06M0 This section is the analogue of Varieties, Section 24.

06M1 **Lemma 12.1.** *Let k be a field. Let X be an algebraic space smooth over k . Then X is a regular algebraic space.*

Proof. Choose a scheme U and a surjective étale morphism $U \rightarrow X$. The morphism $U \rightarrow \text{Spec}(k)$ is smooth as a composition of an étale (hence smooth) morphism and a smooth morphism (see Morphisms of Spaces, Lemmas 38.6 and 36.2). Hence U is regular by Varieties, Lemma 24.3. By Properties of Spaces, Definition 7.2 this means that X is regular. \square

07W4 **Lemma 12.2.** *Let k be a field. Let X be an algebraic space smooth over $\text{Spec}(k)$. The set of $x \in |X|$ which are image of morphisms $\text{Spec}(k') \rightarrow X$ with $k' \supset k$ finite separable is dense in $|X|$.*

Proof. Choose a scheme U and a surjective étale morphism $U \rightarrow X$. The morphism $U \rightarrow \text{Spec}(k)$ is smooth as a composition of an étale (hence smooth) morphism and a smooth morphism (see Morphisms of Spaces, Lemmas 38.6 and 36.2). Hence we can apply Varieties, Lemma 24.6 to see that the closed points of U whose residue fields are finite separable over k are dense. This implies the lemma by our definition of the topology on $|X|$. \square

13. Euler characteristics

0DN0 In this section we prove some elementary properties of Euler characteristics of coherent sheaves on algebraic spaces proper over fields.

0DN1 **Definition 13.1.** Let k be a field. Let X be a proper algebraic over k . Let \mathcal{F} be a coherent \mathcal{O}_X -module. In this situation the *Euler characteristic of \mathcal{F}* is the integer

$$\chi(X, \mathcal{F}) = \sum_i (-1)^i \dim_k H^i(X, \mathcal{F}).$$

For justification of the formula see below.

In the situation of the definition only a finite number of the vector spaces $H^i(X, \mathcal{F})$ are nonzero (Cohomology of Spaces, Lemma 7.3) and each of these spaces is finite dimensional (Cohomology of Spaces, Lemma 20.3). Thus $\chi(X, \mathcal{F}) \in \mathbf{Z}$ is well defined. Observe that this definition depends on the field k and not just on the pair (X, \mathcal{F}) .

0DN2 **Lemma 13.2.** *Let k be a field. Let X be a proper algebraic space over k . Let $0 \rightarrow \mathcal{F}_1 \rightarrow \mathcal{F}_2 \rightarrow \mathcal{F}_3 \rightarrow 0$ be a short exact sequence of coherent modules on X . Then*

$$\chi(X, \mathcal{F}_2) = \chi(X, \mathcal{F}_1) + \chi(X, \mathcal{F}_3)$$

Proof. Consider the long exact sequence of cohomology

$$0 \rightarrow H^0(X, \mathcal{F}_1) \rightarrow H^0(X, \mathcal{F}_2) \rightarrow H^0(X, \mathcal{F}_3) \rightarrow H^1(X, \mathcal{F}_1) \rightarrow \dots$$

associated to the short exact sequence of the lemma. The rank-nullity theorem in linear algebra shows that

$$0 = \dim H^0(X, \mathcal{F}_1) - \dim H^0(X, \mathcal{F}_2) + \dim H^0(X, \mathcal{F}_3) - \dim H^1(X, \mathcal{F}_1) + \dots$$

This immediately implies the lemma. \square

14. Numerical intersections

0DN3 In this section we play around with the Euler characteristic of coherent sheaves on proper algebraic spaces to obtain numerical intersection numbers for invertible modules. Our main tool will be the following lemma.

0DN4 **Lemma 14.1.** *Let k be a field. Let X be a proper algebraic space over k . Let \mathcal{F} be a coherent \mathcal{O}_X -module. Let $\mathcal{L}_1, \dots, \mathcal{L}_r$ be invertible \mathcal{O}_X -modules. The map*

$$(n_1, \dots, n_r) \mapsto \chi(X, \mathcal{F} \otimes \mathcal{L}_1^{\otimes n_1} \otimes \dots \otimes \mathcal{L}_r^{\otimes n_r})$$

is a numerical polynomial in n_1, \dots, n_r of total degree at most the dimension of the scheme theoretic support of \mathcal{F} .

Proof. Let $Z \subset X$ be the scheme theoretic support of \mathcal{F} . Then $\mathcal{F} = i_*\mathcal{G}$ for some coherent \mathcal{O}_Z -module \mathcal{G} (Cohomology of Spaces, Lemma 12.7) and we have

$$\chi(X, \mathcal{F} \otimes \mathcal{L}_1^{\otimes n_1} \otimes \dots \otimes \mathcal{L}_r^{\otimes n_r}) = \chi(Z, \mathcal{G} \otimes i^*\mathcal{L}_1^{\otimes n_1} \otimes \dots \otimes i^*\mathcal{L}_r^{\otimes n_r})$$

by the projection formula (Cohomology on Sites, Lemma 40.1) and Cohomology of Spaces, Lemma 8.3. Since $|Z| = \text{Supp}(\mathcal{F})$ we see that it suffices to show

$$P_{\mathcal{F}}(n_1, \dots, n_r) : (n_1, \dots, n_r) \mapsto \chi(X, \mathcal{F} \otimes \mathcal{L}_1^{\otimes n_1} \otimes \dots \otimes \mathcal{L}_r^{\otimes n_r})$$

is a numerical polynomial in n_1, \dots, n_r of total degree at most $\dim(X)$. Let us say property \mathcal{P} holds for the coherent \mathcal{O}_X -module \mathcal{F} if the above is true.

We will prove this statement by devissage, more precisely we will check conditions (1), (2), and (3) of Cohomology of Spaces, Lemma 14.6 are satisfied.

Verification of condition (1). Let

$$0 \rightarrow \mathcal{F}_1 \rightarrow \mathcal{F}_2 \rightarrow \mathcal{F}_3 \rightarrow 0$$

be a short exact sequence of coherent sheaves on X . By Lemma 13.2 we have

$$P_{\mathcal{F}_2}(n_1, \dots, n_r) = P_{\mathcal{F}_1}(n_1, \dots, n_r) + P_{\mathcal{F}_3}(n_1, \dots, n_r)$$

Then it is clear that if 2-out-of-3 of the sheaves \mathcal{F}_i have property \mathcal{P} , then so does the third.

Condition (2) follows because $P_{\mathcal{F}^{\oplus m}}(n_1, \dots, n_r) = mP_{\mathcal{F}}(n_1, \dots, n_r)$.

Proof of (3). Let $Z \subset X$ be a reduced closed subspace with $|Z|$ irreducible. We apply Cohomology of Spaces, Lemma 18.1 to the morphism $Z \rightarrow \text{Spec}(k)$. Thus we get a surjective proper morphism $f : Y \rightarrow Z$ over $\text{Spec}(k)$ where Y is a closed subscheme of \mathbf{P}_k^m for some m . After replacing Y by a closed subscheme we may assume that Y is integral and $f : Y \rightarrow Z$ is an alteration, see Lemma 5.5. Denote $\mathcal{O}_Y(n)$ the pullback of $\mathcal{O}_{\mathbf{P}_k^m}(n)$. Pick $n > 0$ such that $R^p f_* \mathcal{O}_Y(n) = 0$ for $p > 0$, see Cohomology of Spaces, Lemma 20.1. We claim that $\mathcal{G} = i_* f_* \mathcal{O}_Y(n)$ satisfies \mathcal{P} . Namely, by the case of schemes (Varieties, Lemma 43.1) we know that

$$(n_1, \dots, n_r) \mapsto \chi(Y, \mathcal{O}_Y(n) \otimes f^* i^*(\mathcal{L}_1^{\otimes n_1} \otimes \dots \otimes \mathcal{L}_r^{\otimes n_r}))$$

is a numerical polynomial in n_1, \dots, n_r of total degree at most $\dim(Y)$. On the other hand, by the projection formula (Cohomology on Sites, Lemma 40.1)

$$\begin{aligned} i_* Rf_* (\mathcal{O}_Y(n) \otimes f^* i^*(\mathcal{L}_1^{\otimes n_1} \otimes \dots \otimes \mathcal{L}_r^{\otimes n_r})) &= i_* Rf_* \mathcal{O}_Y(n) \otimes \mathcal{L}_1^{\otimes n_1} \otimes \dots \otimes \mathcal{L}_r^{\otimes n_r} \\ &= \mathcal{G} \otimes \mathcal{L}_1^{\otimes n_1} \otimes \dots \otimes \mathcal{L}_r^{\otimes n_r} \end{aligned}$$

the last equality by our choice of n . By Leray (Cohomology on Sites, Lemma 15.6) we get

$$\chi(Y, \mathcal{O}_Y(n) \otimes f^* i^* (\mathcal{L}_1^{\otimes n_1} \otimes \dots \otimes \mathcal{L}_r^{\otimes n_r})) = P_G(n_1, \dots, n_r)$$

and we conclude because $\dim(Y) \leq \dim(Z) \leq \dim(X)$. The first inequality hold by Morphisms of Spaces, Lemma 34.2 and the fact that $Y \rightarrow Z$ is an alteration (and hence the induced extension of residue fields in generic points is finite). \square

15. Other chapters

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