

# LIMITS OF ALGEBRAIC STACKS

0CMM

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## 1. Introduction

0CMN In this chapter we put material related to limits of algebraic stacks. Many results on limits of algebraic stacks and algebraic spaces have been obtained by David Rydh in [Ryd08].

## 2. Conventions

0CMP We continue to use the conventions and the abuse of language introduced in Properties of Stacks, Section 2.

## 3. Morphisms of finite presentation

0CMQ This section is the analogue of Limits of Spaces, Section 3. There we defined what it means for a transformation of functors on  $Sch$  to be limit preserving (we suggest looking at the characterization in Limits of Spaces, Lemma 3.2). In Criteria for Representability, Section 5 we defined the notion “limit preserving on objects”. Recall that in Artin’s Axioms, Section 11 we have defined what it means for a category fibred in groupoids over  $Sch$  to be limit preserving. Combining these we get the following notion.

0CMR **Definition 3.1.** Let  $S$  be a scheme. Let  $f : \mathcal{X} \rightarrow \mathcal{Y}$  be a 1-morphism of categories fibred in groupoids over  $(Sch/S)_{fppf}$ . We say  $f$  is *limit preserving* if for every directed limit  $U = \lim U_i$  of affine schemes over  $S$  the diagram

$$\begin{array}{ccc} \operatorname{colim} \mathcal{X}_{U_i} & \longrightarrow & \mathcal{X}_U \\ f \downarrow & & \downarrow f \\ \operatorname{colim} \mathcal{Y}_{U_i} & \longrightarrow & \mathcal{Y}_U \end{array}$$

of fibre categories is 2-cartesian.

0CMS **Lemma 3.2.** *Let  $S$  be a scheme. Let  $f : \mathcal{X} \rightarrow \mathcal{Y}$  be a 1-morphism of categories fibred in groupoids over  $(Sch/S)_{fppf}$ . If  $f$  is limit preserving (Definition 3.1), then  $f$  is limit preserving on objects (Criteria for Representability, Section 5).*

**Proof.** If for every directed limit  $U = \lim U_i$  of affine schemes over  $U$ , the functor

$$\operatorname{colim} \mathcal{X}_{U_i} \longrightarrow (\operatorname{colim} \mathcal{Y}_{U_i}) \times_{\mathcal{Y}_U} \mathcal{X}_U$$

is essentially surjective, then  $f$  is limit preserving on objects.  $\square$

0CMT **Lemma 3.3.** *Let  $p : \mathcal{X} \rightarrow \mathcal{Y}$  and  $q : \mathcal{Z} \rightarrow \mathcal{Y}$  be 1-morphisms of categories fibred in groupoids over  $(Sch/S)_{fppf}$ . If  $p : \mathcal{X} \rightarrow \mathcal{Y}$  is limit preserving, then so is the base change  $p' : \mathcal{X} \times_{\mathcal{Y}} \mathcal{Z} \rightarrow \mathcal{Z}$  of  $p$  by  $q$ .*

**Proof.** This is formal. Let  $U = \lim_{i \in I} U_i$  be the directed limit of affine schemes  $U_i$  over  $S$ . For each  $i$  we have

$$(\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z})_{U_i} = \mathcal{X}_{U_i} \times_{\mathcal{Y}_{U_i}} \mathcal{Z}_{U_i}$$

Filtered colimits commute with 2-fibre products of categories (details omitted) hence if  $p$  is limit preserving we get

$$\begin{aligned} \operatorname{colim}(\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z})_{U_i} &= \operatorname{colim} \mathcal{X}_{U_i} \times_{\operatorname{colim} \mathcal{Y}_{U_i}} \operatorname{colim} \mathcal{Z}_{U_i} \\ &= \mathcal{X}_U \times_{\mathcal{Y}_U} \operatorname{colim} \mathcal{Y}_{U_i} \times_{\operatorname{colim} \mathcal{Y}_{U_i}} \operatorname{colim} \mathcal{Z}_{U_i} \\ &= \mathcal{X}_U \times_{\mathcal{Y}_U} \operatorname{colim} \mathcal{Z}_{U_i} \\ &= \mathcal{X}_U \times_{\mathcal{Y}_U} \mathcal{Z}_U \times_{\mathcal{Z}_U} \operatorname{colim} \mathcal{Z}_{U_i} \\ &= (\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z})_U \times_{\mathcal{Z}_U} \operatorname{colim} \mathcal{Z}_{U_i} \end{aligned}$$

as desired.  $\square$

0CMU **Lemma 3.4.** *Let  $p : \mathcal{X} \rightarrow \mathcal{Y}$  and  $q : \mathcal{Y} \rightarrow \mathcal{Z}$  be 1-morphisms of categories fibred in groupoids over  $(Sch/S)_{fppf}$ . If  $p$  and  $q$  are limit preserving, then so is the composition  $q \circ p$ .*

**Proof.** This is formal. Let  $U = \lim_{i \in I} U_i$  be the directed limit of affine schemes  $U_i$  over  $S$ . If  $p$  and  $q$  are limit preserving we get

$$\begin{aligned} \operatorname{colim} \mathcal{X}_{U_i} &= \mathcal{X}_U \times_{\mathcal{Y}_U} \operatorname{colim} \mathcal{Y}_{U_i} \\ &= \mathcal{X}_U \times_{\mathcal{Y}_U} \mathcal{Y}_U \times_{\mathcal{Z}_U} \operatorname{colim} \mathcal{Z}_{U_i} \\ &= \mathcal{X}_U \times_{\mathcal{Z}_U} \operatorname{colim} \mathcal{Z}_{U_i} \end{aligned}$$

as desired.  $\square$

0CMV **Lemma 3.5.** *Let  $p : \mathcal{X} \rightarrow \mathcal{Y}$  be a 1-morphism of categories fibred in groupoids over  $(Sch/S)_{fppf}$ . If  $p$  is representable by algebraic spaces, then the following are equivalent:*

- (1)  $p$  is limit preserving,
- (2)  $p$  is limit preserving on objects, and
- (3)  $p$  is locally of finite presentation (see Algebraic Stacks, Definition 10.1).

**Proof.** In Criteria for Representability, Lemma 5.3 we have seen that (2) and (3) are equivalent. Thus it suffices to show that (1) and (2) are equivalent. One direction we saw in Lemma 3.2. For the other direction, let  $U = \lim_{i \in I} U_i$  be the directed limit of affine schemes  $U_i$  over  $S$ . We have to show that

$$\operatorname{colim} \mathcal{X}_{U_i} \longrightarrow \mathcal{X}_U \times_{\mathcal{Y}_U} \operatorname{colim} \mathcal{Y}_{U_i}$$

is an equivalence. Since we are assuming (2) we know that it is essentially surjective. Hence we need to prove it is fully faithful. Since  $p$  is faithful on fibre categories (Algebraic Stacks, Lemma 9.2) we see that the functor is faithful. Let  $x_i$  and  $x'_i$  be objects in the fibre category of  $\mathcal{X}$  over  $U_i$ . The functor above sends  $x_i$  to  $(x_i|_U, p(x_i), can)$  where  $can$  is the canonical isomorphism  $p(x_i|_U) \rightarrow p(x_i)|_U$ . Thus we assume given a morphism

$$(\alpha, \beta_i) : (x_i|_U, p(x_i), can) \longrightarrow (x'_i|_U, p(x'_i), can)$$

in the category of the right hand side of the first displayed arrow of this proof. Our task is to produce an  $i' \geq i$  and a morphism  $x_i|_{U_{i'}} \rightarrow x'_i|_{U_{i'}}$  which maps to  $(\alpha, \beta_i|_{U_{i'}})$ .

Set  $y_i = p(x_i)$  and  $y'_i = p(x'_i)$ . By (Algebraic Stacks, Lemma 9.2) the functor

$$X_{y_i} : (Sch/U_i)^{opp} \rightarrow Sets, \quad V/U_i \mapsto \{(x, \phi) \mid x \in \text{Ob}(\mathcal{X}_V), \phi : f(x) \rightarrow y_i|_V\} / \cong$$

is an algebraic space over  $U_i$  and the same is true for the analogously defined functor  $X_{y'_i}$ . Since (2) is equivalent to (3) we see that  $X_{y'_i}$  is locally of finite presentation over  $U_i$ . Observe that  $(x_i, \text{id})$  and  $(x'_i, \text{id})$  define  $U_i$ -valued points of  $X_{y_i}$  and  $X_{y'_i}$ . There is a transformation of functors

$$\beta_i : X_{y_i} \rightarrow X_{y'_i}, \quad (x/V, \phi) \mapsto (x/V, \beta_i|_V \circ \phi)$$

in other words, this is a morphism of algebraic spaces over  $U_i$ . We claim that

$$\begin{array}{ccc} U & \longrightarrow & U_i \\ \downarrow & & \downarrow (x'_i, \text{id}) \\ U_i & \xrightarrow{(x_i, \text{id})} X_{y_i} \xrightarrow{\beta_i} & X_{y'_i} \end{array}$$

commutes. Namely, this is equivalent to the condition that the pairs  $(x_i|_U, \beta_i|_U)$  and  $(x'_i|_U, \text{id})$  as in the definition of the functor  $X_{y'_i}$  are isomorphic. And the morphism  $\alpha : x_i|_U \rightarrow x'_i|_U$  exactly produces such an isomorphism. Arguing backwards the reader sees that if we can find an  $i' \geq i$  such that the diagram

$$\begin{array}{ccc} U_{i'} & \longrightarrow & U_i \\ \downarrow & & \downarrow (x'_i, \text{id}) \\ U_i & \xrightarrow{(x_i, \text{id})} X_{y_i} \xrightarrow{\beta_i} & X_{y'_i} \end{array}$$

commutes, then we obtain an isomorphism  $x_i|_{U_{i'}} \rightarrow x'_i|_{U_{i'}}$  which is a solution to the problem posed in the preceding paragraph. However, the diagonal morphism

$$\Delta : X_{y'_i} \rightarrow X_{y'_i} \times_{U_i} X_{y'_i}$$

is locally of finite presentation (Morphisms of Spaces, Lemma 28.10) hence the fact that  $U \rightarrow U_i$  equalizes the two morphisms to  $X_{y'_i}$ , means that for some  $i' \geq i$  the morphism  $U_{i'} \rightarrow U_i$  equalizes the two morphisms, see Limits of Spaces, Proposition 3.9.  $\square$

OCMW **Lemma 3.6.** *Let  $p : \mathcal{X} \rightarrow \mathcal{Y}$  be a 1-morphism of categories fibred in groupoids over  $(Sch/S)_{fppf}$ . The following are equivalent*

- (1) *the diagonal  $\Delta : \mathcal{X} \rightarrow \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$  is limit preserving, and*

(2) for every directed limit  $U = \lim U_i$  of affine schemes over  $S$  the functor

$$\operatorname{colim} \mathcal{X}_{U_i} \longrightarrow \mathcal{X}_U \times_{\mathcal{Y}_U} \operatorname{colim} \mathcal{Y}_{U_i}$$

is fully faithful.

In particular, if  $p$  is limit preserving, then  $\Delta$  is too.

**Proof.** Let  $U = \lim U_i$  be a directed limit of affine schemes over  $S$ . We claim that the functor

$$\operatorname{colim} \mathcal{X}_{U_i} \longrightarrow \mathcal{X}_U \times_{\mathcal{Y}_U} \operatorname{colim} \mathcal{Y}_{U_i}$$

is fully faithful if and only if the functor

$$\operatorname{colim} \mathcal{X}_{U_i} \longrightarrow \mathcal{X}_U \times_{(\mathcal{X} \times_{\mathcal{Y}} \mathcal{X})_U} \operatorname{colim} (\mathcal{X} \times_{\mathcal{Y}} \mathcal{X})_{U_i}$$

is an equivalence. This will prove the lemma. Since  $(\mathcal{X} \times_{\mathcal{Y}} \mathcal{X})_U = \mathcal{X}_U \times_{\mathcal{Y}_U} \mathcal{X}_U$  and  $(\mathcal{X} \times_{\mathcal{Y}} \mathcal{X})_{U_i} = \mathcal{X}_{U_i} \times_{\mathcal{Y}_{U_i}} \mathcal{X}_{U_i}$  this is a purely category theoretic assertion which we discuss in the next paragraph.

Let  $\mathcal{I}$  be a filtered index category. Let  $(\mathcal{C}_i)$  and  $(\mathcal{D}_i)$  be systems of groupoids over  $\mathcal{I}$ . Let  $p : (\mathcal{C}_i) \rightarrow (\mathcal{D}_i)$  be a map of systems of groupoids over  $\mathcal{I}$ . Suppose we have a functor  $p : \mathcal{C} \rightarrow \mathcal{D}$  of groupoids and functors  $f : \operatorname{colim} \mathcal{C}_i \rightarrow \mathcal{C}$  and  $g : \operatorname{colim} \mathcal{D}_i \rightarrow \mathcal{D}$  fitting into a commutative diagram

$$\begin{array}{ccc} \operatorname{colim} \mathcal{C}_i & \xrightarrow{f} & \mathcal{C} \\ p \downarrow & & \downarrow p \\ \operatorname{colim} \mathcal{D}_i & \xrightarrow{g} & \mathcal{D} \end{array}$$

Then we claim that

$$A : \operatorname{colim} \mathcal{C}_i \longrightarrow \mathcal{C} \times_{\mathcal{D}} \operatorname{colim} \mathcal{D}_i$$

is fully faithful if and only if the functor

$$B : \operatorname{colim} \mathcal{C}_i \longrightarrow \mathcal{C} \times_{\Delta, \mathcal{C} \times_{\mathcal{D}} \mathcal{C}, f \times_{g,f}} \operatorname{colim} (\mathcal{C}_i \times_{\mathcal{D}_i} \mathcal{C}_i)$$

is an equivalence. Set  $\mathcal{C}' = \operatorname{colim} \mathcal{C}_i$  and  $\mathcal{D}' = \operatorname{colim} \mathcal{D}_i$ . Since 2-fibre products commute with filtered colimits we see that  $A$  and  $B$  become the functors

$$A' : \mathcal{C}' \rightarrow \mathcal{C} \times_{\mathcal{D}} \mathcal{D}' \quad \text{and} \quad B' : \mathcal{C}' \longrightarrow \mathcal{C} \times_{\Delta, \mathcal{C} \times_{\mathcal{D}} \mathcal{C}, f \times_{g,f}} (\mathcal{C}' \times_{\mathcal{D}'} \mathcal{C}')$$

Thus it suffices to prove that if

$$\begin{array}{ccc} \mathcal{C}' & \xrightarrow{f} & \mathcal{C} \\ p \downarrow & & \downarrow p \\ \mathcal{D}' & \xrightarrow{g} & \mathcal{D} \end{array}$$

is a commutative diagram of groupoids, then  $A'$  is fully faithful if and only if  $B'$  is an equivalence. This follows from Categories, Lemma 34.9 (with trivial, i.e., punctual, base category) because

$$\mathcal{C} \times_{\Delta, \mathcal{C} \times_{\mathcal{D}} \mathcal{C}, f \times_{g,f}} (\mathcal{C}' \times_{\mathcal{D}'} \mathcal{C}') = \mathcal{C}' \times_{A', \mathcal{C} \times_{\mathcal{D}} \mathcal{D}', A'} \mathcal{C}'$$

This finishes the proof.  $\square$

OCMX **Lemma 3.7.** *Let  $S$  be a scheme. Let  $\mathcal{X}$  be an algebraic stack over  $S$ . If  $\mathcal{X} \rightarrow S$  is locally of finite presentation, then  $\mathcal{X}$  is limit preserving in the sense of Artin's Axioms, Definition 11.1 (equivalently: the morphism  $\mathcal{X} \rightarrow S$  is limit preserving).*

**Proof.** Choose a surjective smooth morphism  $U \rightarrow \mathcal{X}$  for some scheme  $U$ . Then  $U \rightarrow S$  is locally of finite presentation, see Morphisms of Stacks, Section 26. We can write  $\mathcal{X} = [U/R]$  for some smooth groupoid in algebraic spaces  $(U, R, s, t, c)$ , see Algebraic Stacks, Lemma 16.2. Since  $U$  is locally of finite presentation over  $S$  it follows that the algebraic space  $R$  is locally of finite presentation over  $S$ . Recall that  $[U/R]$  is the stack in groupoids over  $(Sch/S)_{fppf}$  obtained by stacky-fying the category fibred in groupoids whose fibre category over  $T$  is the groupoid  $(U(T), R(T), s, t, c)$ . Since  $U$  and  $R$  are limit preserving as functors (Limits of Spaces, Proposition 3.9) this category fibred in groupoids is limit preserving. Thus it suffices to show that fppf stackyfication preserves the property of being limit preserving. This is true (hint: use Limits of Spaces, Lemma 3.5). However, we give a direct proof below using that in this case we know what the stackyfication amounts to.

Let  $T = \lim T_\lambda$  be a directed limit of affine schemes over  $S$ . We have to show that the functor

$$\operatorname{colim}[U/R]_{T_\lambda} \longrightarrow [U/R]_T$$

is an equivalence of categories. Let us show this functor is essentially surjective. Let  $x \in \operatorname{Ob}([U/R]_T)$ . In Groupoids in Spaces, Lemma 23.1 the reader finds a description of the category  $[U/R]_T$ . In particular  $x$  corresponds to an fppf covering  $\{T_i \rightarrow T\}_{i \in I}$  and a  $[U/R]$ -descent datum  $(u_i, r_{ij})$  relative to this covering. After refining this covering we may assume it is a standard fppf covering of the affine scheme  $T$ . By Limits of Spaces, Lemma 3.5 we may choose a  $\lambda$  and a standard fppf covering  $\{T_{\lambda,i} \rightarrow T_\lambda\}_{i \in I}$  whose base change to  $T$  is equal to  $\{T_i \rightarrow T\}_{i \in I}$ . For each  $i$ , after increasing  $\lambda$ , we can find a  $u_{\lambda,i} : T_{\lambda,i} \rightarrow U$  whose composition with  $T_i \rightarrow T_{\lambda,i}$  is the given morphism  $u_i$  (this is where we use that  $U$  is limit preserving). Similarly, for each  $i, j$ , after increasing  $\lambda$ , we can find a  $r_{\lambda,ij} : T_{\lambda,i} \times_{T_\lambda} T_{\lambda,j} \rightarrow R$  whose composition with  $T_{ij} \rightarrow T_{\lambda,ij}$  is the given morphism  $r_{ij}$  (this is where we use that  $R$  is limit preserving). After increasing  $\lambda$  we can further assume that

$$s \circ r_{\lambda,ij} = u_{\lambda,i} \circ \operatorname{pr}_0 \quad \text{and} \quad t \circ r_{\lambda,ij} = u_{\lambda,j} \circ \operatorname{pr}_1,$$

and

$$c \circ (r_{\lambda,jk} \circ \operatorname{pr}_{12}, r_{\lambda,ij} \circ \operatorname{pr}_{01}) = r_{\lambda,ik} \circ \operatorname{pr}_{02}.$$

In other words, we may assume that  $(u_{\lambda,i}, r_{\lambda,ij})$  is a  $[U/R]$ -descent datum relative to the covering  $\{T_{\lambda,i} \rightarrow T_\lambda\}_{i \in I}$ . Then we obtain a corresponding object of  $[U/R]$  over  $T_\lambda$  whose pullback to  $T$  is isomorphic to  $x$  as desired. The proof of fully faithfulness works in exactly the same way using the description of morphisms in the fibre categories of  $[U/T]$  given in Groupoids in Spaces, Lemma 23.1.  $\square$

OCMY **Proposition 3.8.** *Let  $f : \mathcal{X} \rightarrow \mathcal{Y}$  be a morphism of algebraic stacks. The following are equivalent*

- (1)  $f$  is limit preserving,
- (2)  $f$  is limit preserving on objects, and
- (3)  $f$  is locally of finite presentation.

**Proof.** Assume (3). Let  $T = \lim T_i$  be a directed limit of affine schemes. Consider the functor

$$\operatorname{colim} \mathcal{X}_{T_i} \longrightarrow \mathcal{X}_T \times_{\mathcal{Y}_T} \operatorname{colim} \mathcal{Y}_{T_i}$$

This is a special case of [EG15, Lemma 2.3.15]

Let  $(x, y_i, \beta)$  be an object on the right hand side, i.e.,  $x \in \text{Ob}(\mathcal{X}_T)$ ,  $y_i \in \text{Ob}(\mathcal{Y}_{T_i})$ , and  $\beta : f(x) \rightarrow y_i|_T$  in  $\mathcal{Y}_T$ . Then we can consider  $(x, y_i, \beta)$  as an object of the algebraic stack  $\mathcal{X}_{y_i} = \mathcal{X} \times_{\mathcal{Y}, y_i} T_i$  over  $T$ . Since  $\mathcal{X}_{y_i} \rightarrow T_i$  is locally of finite presentation (as a base change of  $f$ ) we see that it is limit preserving by Lemma 3.7. This means that  $(x, y_i, \beta)$  comes from an object over  $T_{i'}$  for some  $i' \geq i$  and unwinding the definitions we find that  $(x, y_i, \beta)$  is in the essential image of the displayed functor. In other words, the displayed functor is essentially surjective. Another formulation is that this means  $f$  is limit preserving on objects. Now we apply this to the diagonal  $\Delta$  of  $f$ . Namely, by Morphisms of Stacks, Lemma 26.7 the morphism  $\Delta$  is locally of finite presentation. Thus the argument above shows that  $\Delta$  is limit preserving on objects. By Lemma 3.5 this implies that  $\Delta$  is limit preserving. By Lemma 3.6 we conclude that the displayed functor above is fully faithful. Thus it is an equivalence (as we already proved essential surjectivity) and we conclude that (1) holds.

The implication (1)  $\Rightarrow$  (2) is trivial. Assume (2). Choose a scheme  $V$  and a surjective smooth morphism  $V \rightarrow \mathcal{Y}$ . By Criteria for Representability, Lemma 5.1 the base change  $\mathcal{X} \times_{\mathcal{Y}} V \rightarrow V$  is limit preserving on objects. Choose a scheme  $U$  and a surjective smooth morphism  $U \rightarrow \mathcal{X} \times_{\mathcal{Y}} V$ . Since a smooth morphism is locally of finite presentation, we see that  $U \rightarrow \mathcal{X} \times_{\mathcal{Y}} V$  is limit preserving (first part of the proof). By Criteria for Representability, Lemma 5.2 we find that the composition  $U \rightarrow V$  is limit preserving on objects. We conclude that  $U \rightarrow V$  is locally of finite presentation, see Criteria for Representability, Lemma 5.3. This is exactly the condition that  $f$  is locally of finite presentation, see Morphisms of Stacks, Definition 26.1.  $\square$

#### 4. Descending properties

0CPX This section is the analogue of Limits, Section 4.

0CPY **Situation 4.1.** Let  $Y = \lim_{i \in I} Y_i$  be a limit of a directed system of algebraic spaces with affine transition morphisms. We assume that  $X_i$  is quasi-compact and quasi-separated for all  $i \in I$ . We also choose an element  $0 \in I$ .

0CPZ **Lemma 4.2.** *In Situation 4.1 assume that  $\mathcal{X}_0 \rightarrow Y_0$  is a morphism from algebraic stack to  $Y_0$ . Assume  $\mathcal{X}_0$  is quasi-compact and quasi-separated. If  $Y \times_{Y_0} \mathcal{X}_0 \rightarrow Y$  is separated, then  $Y_i \times_{Y_0} \mathcal{X}_0 \rightarrow Y_i$  is separated for all sufficiently large  $i \in I$ .*

**Proof.** Write  $\mathcal{X} = Y \times_{Y_0} \mathcal{X}_0$  and  $\mathcal{X}_i = Y_i \times_{Y_0} \mathcal{X}_0$ . Choose an affine scheme  $U_0$  and a surjective smooth morphism  $U_0 \rightarrow \mathcal{X}_0$ . Set  $U = Y \times_{Y_0} U_0$  and  $U_i = Y_i \times_{Y_0} U_0$ . Then  $U$  and  $U_i$  are affine and  $U \rightarrow \mathcal{X}$  and  $U_i \rightarrow \mathcal{X}_i$  are smooth and surjective. Set  $R_0 = U_0 \times_{\mathcal{X}_0} U_0$ . Set  $R = Y \times_{Y_0} R_0$  and  $R_i = Y_i \times_{Y_0} R_0$ . Then  $R = U \times_{\mathcal{X}} U$  and  $R_i = U_i \times_{\mathcal{X}_i} U_i$ .

With this notation note that  $\mathcal{X} \rightarrow Y$  is separated implies that  $R \rightarrow U \times_Y U$  is proper as the base change of  $\mathcal{X} \rightarrow \mathcal{X} \times_Y \mathcal{X}$  by  $U \times_Y U \rightarrow \mathcal{X} \times_Y \mathcal{X}$ . Conversely, we see that  $\mathcal{X}_i \rightarrow Y_i$  is separated if  $R_i \rightarrow U_i \times_{Y_i} U_i$  is proper because  $U_i \times_{Y_i} U_i \rightarrow \mathcal{X}_i \times_{Y_i} \mathcal{X}_i$  is surjective and smooth, see Properties of Stacks, Lemma 3.3. Observe that  $R_0 \rightarrow U_0 \times_{Y_0} U_0$  is locally of finite type and that  $R_0$  is quasi-compact and quasi-separated. By Limits of Spaces, Lemma 6.13 we see that  $R_i \rightarrow U_i \times_{Y_i} U_i$  is proper for large enough  $i$  which finishes the proof.  $\square$

## 5. Descending relative objects

0CN3 This section is the analogue of Limits of Spaces, Section 7.

0CN4 **Lemma 5.1.** *Let  $I$  be a directed set. Let  $(X_i, f_{ii'})$  be an inverse system of algebraic spaces over  $I$ . Assume*

- (1) *the morphisms  $f_{ii'} : X_i \rightarrow X_{i'}$  are affine,*
- (2) *the spaces  $X_i$  are quasi-compact and quasi-separated.*

*Let  $X = \lim X_i$ . If  $\mathcal{X}$  is an algebraic stack of finite presentation over  $X$ , then there exists an  $i \in I$  and an algebraic stack  $\mathcal{X}_i$  of finite presentation over  $X_i$  with  $\mathcal{X} \cong \mathcal{X}_i \times_{X_i} X$  as algebraic stacks over  $X$ .*

**Proof.** By Morphisms of Stacks, Definition 26.1 the morphism  $\mathcal{X} \rightarrow X$  is quasi-compact, locally of finite presentation, and quasi-separated. Since  $X$  is quasi-compact and  $\mathcal{X} \rightarrow X$  is quasi-compact, we see that  $\mathcal{X}$  is quasi-compact (Morphisms of Stacks, Definition 7.2). Hence we can find an affine scheme  $U$  and a surjective smooth morphism  $U \rightarrow \mathcal{X}$  (Properties of Stacks, Lemma 6.2). Set  $R = U \times_{\mathcal{X}} U$ . We obtain a smooth groupoid in algebraic spaces  $(U, R, s, t, c)$  over  $X$  such that  $\mathcal{X} = [U/R]$ , see Algebraic Stacks, Lemma 16.2. Since  $\mathcal{X} \rightarrow X$  is quasi-separated and  $X$  is quasi-separated we see that  $\mathcal{X}$  is quasi-separated (Morphisms of Stacks, Lemma 4.10). Thus  $R \rightarrow U \times U$  is quasi-compact and quasi-separated (Morphisms of Stacks, Lemma 4.7) and hence  $R$  is a quasi-separated and quasi-compact algebraic space. On the other hand  $U \rightarrow X$  is locally of finite presentation and hence also  $R \rightarrow X$  is locally of finite presentation (because  $s : R \rightarrow U$  is smooth hence locally of finite presentation). Thus  $(U, R, s, t, c)$  is a groupoid object in the category of algebraic spaces which are of finite presentation over  $X$ . By Limits of Spaces, Lemma 7.1 there exists an  $i$  and a groupoid in algebraic spaces  $(U_i, R_i, s_i, t_i, c_i)$  over  $X_i$  whose pullback to  $X$  is isomorphic to  $(U, R, s, t, c)$ . After increasing  $i$  we may assume that  $s_i$  and  $t_i$  are smooth, see Limits of Spaces, Lemma 6.3. The quotient stack  $\mathcal{X}_i = [U_i/R_i]$  is an algebraic stack (Algebraic Stacks, Theorem 17.3).

There is a morphism  $[U/R] \rightarrow [U_i/R_i]$ , see Groupoids in Spaces, Lemma 20.1. We claim that combined with the morphisms  $[U/R] \rightarrow X$  and  $[U_i/R_i] \rightarrow X_i$  (Groupoids in Spaces, Lemma 19.2) we obtain an isomorphism (i.e., equivalence)

$$[U/R] \longrightarrow [U_i/R_i] \times_{X_i} X$$

The corresponding map

$$[U/pR] \longrightarrow [U_i/pR_i] \times_{X_i} X$$

on the level of “presheaves of groupoids” as in Groupoids in Spaces, Equation (19.0.1) is an isomorphism. Thus the claim follows from the fact that stackification commutes with fibre products, see Stacks, Lemma 8.4.  $\square$

## 6. Finite type closed in finite presentation

0CQ0 This section is the analogue of Limits of Spaces, Section 11.

0CQ1 **Lemma 6.1.** *Let  $f : \mathcal{X} \rightarrow Y$  be a morphism from an algebraic stack to an algebraic space. Assume:*

- (1)  *$f$  is of finite type and quasi-separated,*
- (2)  *$Y$  is quasi-compact and quasi-separated.*

Then there exists a morphism of finite presentation  $f' : \mathcal{X}' \rightarrow Y$  and a closed immersion  $\mathcal{X} \rightarrow \mathcal{X}'$  of algebraic stacks over  $Y$ .

**Proof.** Write  $Y = \lim_{i \in I} Y_i$  as a limit of algebraic spaces over a directed set  $I$  with affine transition morphisms and with  $Y_i$  Noetherian, see Limits of Spaces, Proposition 8.1. We will use the material from Limits of Spaces, Section 21.

Choose a presentation  $\mathcal{X} = [U/R]$ . Denote  $(U, R, s, t, c, e, i)$  the corresponding groupoid in algebraic spaces over  $Y$ . We may and do assume  $U$  is affine. Then  $U, R, R \times_{s,U,t} R$  are quasi-separated algebraic spaces of finite type over  $Y$ . We have two morphisms  $s, t : R \rightarrow U$ , three morphisms  $c : R \times_{s,U,t} R \rightarrow R$ ,  $\text{pr}_1 : R \times_{s,U,t} R \rightarrow R$ ,  $\text{pr}_2 : R \times_{s,U,t} R \rightarrow R$ , a morphism  $e : U \rightarrow R$ , and finally a morphism  $i : R \rightarrow R$ . These morphisms satisfy a list of axioms which are detailed in Groupoids, Section 13.

According to Limits of Spaces, Remark 21.5 we can find an  $i_0 \in I$  and inverse systems

- (1)  $(U_i)_{i \geq i_0}$ ,
- (2)  $(R_i)_{i \geq i_0}$ ,
- (3)  $(T_i)_{i \geq i_0}$

over  $(Y_i)_{i \geq i_0}$  such that  $U = \lim_{i \geq i_0} U_i$ ,  $R = \lim_{i \geq i_0} R_i$ , and  $R \times_{s,U,t} R = \lim_{i \geq i_0} T_i$  and such that there exist morphisms of systems

- (1)  $(s_i)_{i \geq i_0} : (R_i)_{i \geq i_0} \rightarrow (U_i)_{i \geq i_0}$ ,
- (2)  $(t_i)_{i \geq i_0} : (R_i)_{i \geq i_0} \rightarrow (U_i)_{i \geq i_0}$ ,
- (3)  $(c_i)_{i \geq i_0} : (T_i)_{i \geq i_0} \rightarrow (R_i)_{i \geq i_0}$ ,
- (4)  $(p_i)_{i \geq i_0} : (T_i)_{i \geq i_0} \rightarrow (R_i)_{i \geq i_0}$ ,
- (5)  $(q_i)_{i \geq i_0} : (T_i)_{i \geq i_0} \rightarrow (R_i)_{i \geq i_0}$ ,
- (6)  $(e_i)_{i \geq i_0} : (U_i)_{i \geq i_0} \rightarrow (R_i)_{i \geq i_0}$ ,
- (7)  $(i_i)_{i \geq i_0} : (R_i)_{i \geq i_0} \rightarrow (R_i)_{i \geq i_0}$

with  $s = \lim_{i \geq i_0} s_i$ ,  $t = \lim_{i \geq i_0} t_i$ ,  $c = \lim_{i \geq i_0} c_i$ ,  $\text{pr}_1 = \lim_{i \geq i_0} p_i$ ,  $\text{pr}_2 = \lim_{i \geq i_0} q_i$ ,  $e = \lim_{i \geq i_0} e_i$ , and  $i = \lim_{i \geq i_0} i_i$ . By Limits of Spaces, Lemma 21.7 we see that we may assume that  $s_i$  and  $t_i$  are smooth (this may require increasing  $i_0$ ). By Limits of Spaces, Lemma 21.6 we may assume that the maps  $R \rightarrow U \times_{U_i, s_i} R_i$  given by  $s$  and  $R \rightarrow R_i$  and  $R \rightarrow U \times_{U_i, t_i} R_i$  given by  $t$  and  $R \rightarrow R_i$  are isomorphisms for all  $i \geq i_0$ . By Limits of Spaces, Lemma 21.9 we see that we may assume that the diagrams

$$\begin{array}{ccc} T_i & \xrightarrow{q_i} & R_i \\ p_i \downarrow & & \downarrow t_i \\ R_i & \xrightarrow{s_i} & U_i \end{array}$$

are cartesian. The uniqueness of Limits of Spaces, Lemma 21.4 then guarantees that for a sufficiently large  $i$  the relations between the morphisms  $s, t, c, e, i$  mentioned above are satisfied by  $s_i, t_i, c_i, e_i, i_i$ . Fix such an  $i$ .

It follows that  $(U_i, R_i, s_i, t_i, c_i, e_i, i_i)$  is a smooth groupoid in algebraic spaces over  $Y_i$ . Hence  $\mathcal{X}_i = [U_i/R_i]$  is an algebraic stack (Algebraic Stacks, Theorem 17.3). The morphism of groupoids

$$(U, R, s, t, c, e, i) \rightarrow (U_i, R_i, s_i, t_i, c_i, e_i, i_i)$$



over  $Y \rightarrow Y_i$  determines a commutative diagram

$$\begin{array}{ccc} \mathcal{X} & \longrightarrow & \mathcal{X}_i \\ \downarrow & & \downarrow \\ Y & \longrightarrow & Y_i \end{array}$$

(Groupoids in Spaces, Lemma 20.1). We claim that the morphism  $\mathcal{X} \rightarrow Y \times_{Y_i} \mathcal{X}_i$  is a closed immersion. The claim finishes the proof because the algebraic stack  $\mathcal{X}_i \rightarrow Y_i$  is of finite presentation by construction. To prove the claim, note that the left diagram

$$\begin{array}{ccc} U & \longrightarrow & U_i \\ \downarrow & & \downarrow \\ \mathcal{X} & \longrightarrow & \mathcal{X}_i \end{array} \quad \begin{array}{ccc} U & \longrightarrow & Y \times_{Y_i} U_i \\ \downarrow & & \downarrow \\ \mathcal{X} & \longrightarrow & Y \times_{Y_i} \mathcal{X}_i \end{array}$$

is cartesian by Groupoids in Spaces, Lemma 24.3 and the results mentioned above. Hence the right commutative diagram is cartesian too. Then the desired result follows from the fact that  $U \rightarrow Y \times_{Y_i} U_i$  is a closed immersion by construction of the inverse system  $(U_i)$  in Limits of Spaces, Lemma 21.3, the fact that  $Y \times_{Y_i} U_i \rightarrow Y \times_{Y_i} \mathcal{X}_i$  is smooth and surjective, and Properties of Stacks, Lemma 9.4.  $\square$

There is a version for separated algebraic stacks.

0CQ2 **Lemma 6.2.** *Let  $f : \mathcal{X} \rightarrow Y$  be a morphism from an algebraic stack to an algebraic space. Assume:*

- (1)  $f$  is of finite type and separated,
- (2)  $Y$  is quasi-compact and quasi-separated.

*Then there exists a separated morphism of finite presentation  $f' : \mathcal{X}' \rightarrow Y$  and a closed immersion  $\mathcal{X} \rightarrow \mathcal{X}'$  of algebraic stacks over  $Y$ .*

**Proof.** First we use exactly the same procedure as in the proof of Lemma 6.1 (and we borrow its notation) to construct the embedding  $\mathcal{X} \rightarrow \mathcal{X}'$  as a morphism  $\mathcal{X} \rightarrow \mathcal{X}' = Y \times_{Y_i} \mathcal{X}_i$  with  $\mathcal{X}_i = [U_i/R_i]$ . Thus it is enough to show that  $\mathcal{X}_i \rightarrow Y_i$  is separated for sufficiently large  $i$ . In other words, it is enough to show that  $\mathcal{X}_i \rightarrow \mathcal{X}_i \times_{Y_i} \mathcal{X}_i$  is proper for  $i$  sufficiently large. Since the morphism  $U_i \times_{Y_i} U_i \rightarrow \mathcal{X}_i \times_{Y_i} \mathcal{X}_i$  is surjective and smooth and since  $R_i = \mathcal{X}_i \times_{\mathcal{X}_i \times_{Y_i} \mathcal{X}_i} U_i \times_{Y_i} U_i$  it is enough to show that the morphism  $(s_i, t_i) : R_i \rightarrow U_i \times_{Y_i} U_i$  is proper for  $i$  sufficiently large, see Properties of Stacks, Lemma 3.3. We prove this in the next paragraph.

Observe that  $U \times_Y U \rightarrow Y$  is quasi-separated and of finite type. Hence we can use the construction of Limits of Spaces, Remark 21.5 to find an  $i_1 \in I$  and an inverse system  $(V_i)_{i \geq i_1}$  with  $U \times_Y U = \lim_{i \geq i_1} V_i$ . By Limits of Spaces, Lemma 21.9 for  $i$  sufficiently large the functoriality of the construction applied to the projections  $U \times_Y U \rightarrow U$  gives closed immersions

$$V_i \rightarrow U_i \times_{Y_i} U_i$$

(There is a small mismatch here because in truth we should replace  $Y_i$  by the scheme theoretic image of  $Y \rightarrow Y_i$ , but clearly this does not change the fibre product.) On the other hand, by Limits of Spaces, Lemma 21.8 the functoriality applied to the proper morphism  $(s, t) : R \rightarrow U \times_Y U$  (here we use that  $\mathcal{X}$  is separated) leads to morphisms  $R_i \rightarrow V_i$  which are proper for large enough  $i$ . Composing these

morphisms we obtain a proper morphisms  $R_i \rightarrow U_i \times_{Y_i} U_i$  for all  $i$  large enough. The functoriality of the construction of Limits of Spaces, Remark 21.5 shows that this is the morphism is the same as  $(s_i, t_i)$  for large enough  $i$  and the proof is complete.  $\square$

## 7. Other chapters

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