# Introduction

Basic topology will be explained in this document. A reference is [Eng77].

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2. Basic notions

The following notions are considered basic and will not be defined, and or proved. This does not mean they are all necessarily easy or well known.

1. $X$ is a topological space,
2. $x \in X$ is a point,
3. $E \subset X$ is a locally closed subset,
4. $x \in X$ is a closed point,
5. $E \subset X$ is a dense subset,
6. $f : X_1 \to X_2$ is continuous,
7. a continuous map of spaces $f : X \to Y$ is open if $f(U)$ is open in $Y$ for $U \subset X$ open,
8. a continuous map of spaces $f : X \to Y$ is closed if $f(Z)$ is closed in $Y$ for $Z \subset X$ closed,
9. a neighbourhood of $x \in X$ is any subset $E \subset X$ which contains an open subset that contains $x$,
10. the induced topology on a subset $E \subset X$,
11. $U : U = \bigcup_{i \in I} U_i$ is an open covering of $U$ (note: we allow any $U_i$ to be empty and we even allow, in case $U$ is empty, the empty set for $I$),
12. the open covering $V$ is a refinement of the open covering $U$ (if $V : V = \bigcup_{j \in J} V_j$ and $U : U = \bigcup_{i \in I} U_i$ this means each $V_j$ is completely contained in one of the $U_i$),
13. $\{E_i\}_{i \in I}$ is a fundamental system of neighbourhoods of $x$ in $X$,
14. a topological space $X$ is called Hausdorff or separated if and only if for every distinct pair of points $x, y \in X$ there exist disjoint opens $U, V \subset X$ such that $x \in U, y \in V$,
15. the product of two topological spaces,
16. the fibre product $X \times_Y Z$ of a pair of continuous maps $f : X \to Y$ and $g : Z \to Y$,
17. the discrete topology and the indiscrete topology on a set,
18. etc.

3. Hausdorff spaces

The category of topological spaces has finite products.

**Lemma 3.1.** Let $X$ be a topological space. The following are equivalent:

1. $X$ is Hausdorff,
2. the diagonal $\Delta(X) \subset X \times X$ is closed.

**Proof.** Omitted. □

**Lemma 3.2.** Let $f : X \to Y$ be a continuous map of topological spaces. If $Y$ is Hausdorff, then the graph of $f$ is closed in $X \times Y$.

**Proof.** The graph is the inverse image of the diagonal under the map $X \times Y \to Y \times Y$. Thus the lemma follows from Lemma 3.1. □

**Lemma 3.3.** Let $f : X \to Y$ be a continuous map of topological spaces. Let $s : Y \to X$ be a continuous map such that $f \circ s = id_Y$. If $X$ is Hausdorff, then $s(Y)$ is closed.
Proof. This follows from Lemma 3.1 as $s(Y) = \{x \in X \mid x = s(f(x))\}$. □

Lemma 3.4. Let $X \to Z$ and $Y \to Z$ be continuous maps of topological spaces. If $Z$ is Hausdorff, then $X \times_Z Y$ is closed in $X \times Y$.

Proof. This follows from Lemma 3.1 as $X \times_Z Y$ is the inverse image of $\Delta(Z)$ under $X \times Y \to Z \times Z$. □

4. Bases

Basic material on bases for topological spaces.

Definition 4.1. Let $X$ be a topological space. A collection of subsets $B$ of $X$ is called a base for the topology on $X$ or a basis for the topology on $X$ if the following conditions hold:

1. Every element $B \in B$ is open in $X$.
2. For every open $U \subset X$ and every $x \in U$, there exists an element $B \in B$ such that $x \in B \subset U$.

Let $X$ be a set and let $B$ be a collection of subsets. Assume that $X = \bigcup_{B \in B} B$ and that given $x \in B_1 \cap B_2$ with $B_1, B_2 \in B$ there is a $B_3 \in B$ with $x \in B_3 \subset B_1 \cap B_2$. Then there is a unique topology on $X$ such that $B$ is a basis for this topology. This remark is sometimes used to define a topology.

Lemma 4.2. Let $X$ be a topological space. Let $B$ be a basis for the topology on $X$. Let $\mathcal{U} : U = \bigcup_i U_i$ be an open covering of $U \subset X$. There exists an open covering $U = \bigcup V_j$ which is a refinement of $\mathcal{U}$ such that each $V_j$ is an element of the basis $B$.

Proof. Omitted. □

Definition 4.3. Let $X$ be a topological space. A collection of subsets $B$ of $X$ is called a subbase for the topology on $X$ or a subbasis for the topology on $X$ if the finite intersections of elements of $B$ form a basis for the topology on $X$.

In particular every element of $B$ is open.

Lemma 4.4. Let $X$ be a set. Given any collection $B$ of subsets of $X$ there is a unique topology on $X$ such that $B$ is a subbase for this topology.

Proof. Omitted. □

5. Submersive maps

If $X$ is a topological space and $E \subset X$ is a subset, then we usually endow $E$ with the induced topology.

Lemma 5.1. Let $X$ be a topological space. Let $Y$ be a set and let $f : Y \to X$ be an injective map of sets. The induced topology on $Y$ is the topology characterized by each of the following statements:

1. it is the weakest topology on $Y$ such that $f$ is continuous,
2. the open subsets of $Y$ are $f^{-1}(U)$ for $U \subset X$ open,
3. the closed subsets of $Y$ are the sets $f^{-1}(Z)$ for $Z \subset X$ closed.

Proof. Omitted. □
Dually, if $X$ is a topological space and $X \to Y$ is a surjection of sets, then $Y$ can be endowed with the quotient topology.

**Lemma 5.2.** Let $X$ be a topological space. Let $Y$ be a set and let $f : X \to Y$ be a surjective map of sets. The quotient topology on $Y$ is the topology characterized by each of the following statements:

1. it is the strongest topology on $Y$ such that $f$ is continuous,
2. a subset $V$ of $Y$ is open if and only if $f^{-1}(V)$ is open,
3. a subset $Z$ of $Y$ is closed if and only if $f^{-1}(Z)$ is closed.

**Proof.** Omitted. □

Let $f : X \to Y$ be a continuous map of topological spaces. In this case we obtain a factorization $X \to f(X) \to Y$ of maps of sets. We can endow $f(X)$ with the quotient topology coming from the surjection $X \to f(X)$ or with the induced topology coming from the injection $f(X) \to Y$. The map

$$(f(X), \text{quotient topology}) \to (f(X), \text{induced topology})$$

is continuous.

**Definition 5.3.** Let $f : X \to Y$ be a continuous map of topological spaces.

1. We say $f$ is a **strict map of topological spaces** if the induced topology and the quotient topology on $f(X)$ agree (see discussion above).
2. We say $f$ is **submersive** if $f$ is surjective and strict.

Thus a continuous map $f : X \to Y$ is submersive if $f$ is a surjection and for any $T \subset Y$ we have $T$ is open or closed if and only if $f^{-1}(T)$ is so. In other words, $Y$ has the quotient topology relative to the surjection $X \to Y$.

**Lemma 5.4.** Let $f : X \to Y$ be surjective, open, continuous map of topological spaces. Let $T \subset Y$ be a subset. Then

1. $f^{-1}(T) = \overline{f^{-1}(T)}$,
2. $T \subset Y$ is closed if and only $f^{-1}(T)$ is closed,
3. $T \subset Y$ is open if and only $f^{-1}(T)$ is open, and
4. $T \subset Y$ is locally closed if and only $f^{-1}(T)$ is locally closed.

In particular we see that $f$ is submersive.

**Proof.** It is clear that $\overline{f^{-1}(T)} \subset f^{-1}(T)$. If $x \in X$, and $x \notin f^{-1}(T)$, then there exists an open neighbourhood $x \in U \subset X$ with $U \cap f^{-1}(T) = \emptyset$. Since $f$ is open we see that $f(U)$ is an open neighbourhood of $f(x)$ not meeting $T$. Hence $x \notin f^{-1}(T)$. This proves (1). Part (2) is an easy consequence of (1). Part (3) is obvious from the fact that $f$ is open and surjective. For (4), if $f^{-1}(T)$ is locally closed, then $f^{-1}(T) \subset T = f^{-1}(\overline{T})$ is open, and hence by (3) applied to the map $f^{-1}(T) \to T$ we see that $T$ is open in $T$, i.e., $T$ is locally closed. □

**Lemma 5.5.** Let $f : X \to Y$ be surjective, closed, continuous map of topological spaces. Let $T \subset Y$ be a subset. Then

1. $\overline{T} = f(f^{-1}(T))$,
2. $T \subset Y$ is closed if and only $f^{-1}(T)$ is closed.

\[\text{This is very different from the notion of a submersion between differential manifolds! It is probably a good idea to use "strict and surjective" in stead of "submersive".}\]
(3) $T \subset Y$ is open if and only $f^{-1}(T)$ is open, and
(4) $T \subset Y$ is locally closed if and only $f^{-1}(T)$ is locally closed.

In particular we see that $f$ is submersive.

**Proof.** It is clear that $\overline{f^{-1}(T)} \subset f^{-1}(\overline{T})$. Then $T \subset f(\overline{f^{-1}(T)}) \subset \overline{T}$ is a closed subset, hence we get (1). Part (2) is obvious from the fact that $f$ is closed and surjective. Part (3) follows from (2) applied to the complement of $T$. For (4), if $f^{-1}(T)$ is locally closed, then $f^{-1}(T) \subset \overline{f^{-1}(T)}$ is open. Since the map $\overline{f^{-1}(T)} \to \overline{T}$ is surjective by (1) we can apply part (3) to the map $\overline{f^{-1}(T)} \to \overline{T}$ induced by $f$ to conclude that $T$ is open in $\overline{T}$, i.e., $T$ is locally closed. \(\square\)

6. Connected components

**Definition 6.1.** Let $X$ be a topological space.

(1) We say $X$ is connected if $X$ is not empty and whenever $X = T_1 \sqcup T_2$ with $T_i \subset X$ open and closed, then either $T_1 = \emptyset$ or $T_2 = \emptyset$.

(2) We say $T \subset X$ is a connected component of $X$ if $T$ is a maximal connected subset of $X$.

The empty space is not connected.

**Lemma 6.2.** Let $f : X \to Y$ be a continuous map of topological spaces. If $E \subset X$ is a connected subset, then $f(E) \subset Y$ is connected as well.

**Proof.** Omitted. \(\square\)

**Lemma 6.3.** Let $X$ be a topological space.

(1) If $T \subset X$ is connected, then so is its closure.

(2) Any connected component of $X$ is closed (but not necessarily open).

(3) Every connected subset of $X$ is contained in a connected component of $X$.

(4) Every point of $X$ is contained in a connected component, in other words, $X$ is the union of its connected components.

**Proof.** Let $\overline{T}$ be the closure of the connected subset $T$. Suppose $\overline{T} = T_1 \sqcup T_2$ with $T_i \subset \overline{T}$ open and closed. Then $T = (T \cap T_1) \sqcup (T \cap T_2)$. Hence $T$ equals one of the two, say $T = T_1 \cap T$. Thus clearly $\overline{T} \subset T$ as desired.

Pick a point $x \in X$. Consider the set $A$ of connected subsets $x \in T_0 \subset X$. Note that $A$ is nonempty since $\{x\} \in A$. There is a partial ordering on $A$ coming from inclusion: $\alpha \leq \alpha' \iff T_\alpha \subset T_{\alpha'}$. Choose a maximal totally ordered subset $A' \subset A$, and let $T = \bigcup_{\alpha \in A'} T_\alpha$. We claim that $T$ is connected. Namely, suppose that $T = T_1 \sqcup T_2$ is a disjoint union of two open and closed subsets of $T$. For each $\alpha \in A'$ we have either $T_\alpha \subset T_1$ or $T_\alpha \subset T_2$, by connectedness of $T_\alpha$. Suppose that for some $\alpha_0 \in A'$ we have $T_{\alpha_0} \not\subset T_1$ (say, if not we’re done anyway). Then, since $A'$ is totally ordered we see immediately that $T_\alpha \subset T_2$ for all $\alpha \in A'$. Hence $T = T_2$.

To get an example where connected components are not open, just take an infinite product $\prod_{n \in \mathbb{N}} \{0, 1\}$ with the product topology. Its connected components are singletons, which are not open. \(\square\)

**Lemma 6.4.** Let $f : X \to Y$ be a continuous map of topological spaces. Assume that

(1) all fibres of $f$ are connected, and
(2) a set $T \subset Y$ is closed if and only if $f^{-1}(T)$ is closed.

Then $f$ induces a bijection between the sets of connected components of $X$ and $Y$.

**Proof.** Let $T \subset Y$ be a connected component. Note that $T$ is closed, see Lemma 6.3. The lemma follows if we show that $f^{-1}(T)$ is connected because any connected subset of $X$ maps into a connected component of $Y$ by Lemma 6.2. Suppose that $f^{-1}(T) = Z_1 \cup Z_2$ with $Z_1$, $Z_2$ closed. For any $t \in T$ we see that $f^{-1}\{t\} = Z_1 \cap f^{-1}\{t\} \cup Z_2 \cap f^{-1}\{t\}$. By (1) we see $f^{-1}\{t\}$ is connected we conclude that either $f^{-1}\{t\} \subset Z_1$ or $f^{-1}\{t\} \subset Z_2$. In other words $T = T_1 \cup T_2$ with $f^{-1}(T_i) = Z_i$. By (2) we conclude that $T_i$ is closed in $Y$. Hence either $T_1 = \emptyset$ or $T_2 = \emptyset$ as desired.

**Lemma 6.5.** Let $f : X \to Y$ be a continuous map of topological spaces. Assume that (a) $f$ is open, (b) all fibres of $f$ are connected. Then $f$ induces a bijection between the sets of connected components of $X$ and $Y$.

**Proof.** This is a special case of Lemma 6.4.

**Lemma 6.6.** Let $f : X \to Y$ be a continuous map of nonempty topological spaces. Assume that (a) $Y$ is connected, (b) $f$ is open and closed, and (c) there is a point $y \in Y$ such that the fiber $f^{-1}(y)$ is a finite set. Then $X$ has at most $|f^{-1}(y)|$ connected components. Hence any connected component $T$ of $X$ is open and closed, and $f(T)$ is a nonempty open and closed subset of $Y$, which is therefore equal to $Y$.

**Proof.** If the topological space $X$ has at least $N$ connected components for some $N \in \mathbb{N}$, we find by induction a decomposition $X = X_1 \cup \ldots \cup X_N$ of $X$ as a disjoint union of $N$ nonempty open and closed subsets $X_1, \ldots, X_N$ of $X$. As $f$ is open and closed, each $f(X_1)$ is a nonempty open and closed subset of $Y$ and is hence equal to $Y$. In particular the intersection $X_i \cap f^{-1}(y)$ is nonempty for each $1 \leq i \leq N$. Hence $f^{-1}(y)$ has at least $N$ elements.

**Definition 6.7.** A topological space is totally disconnected if the connected components are all singletons.

A discrete space is totally disconnected. A totally disconnected space need not be discrete, for example $\mathbb{Q} \subset \mathbb{R}$ is totally disconnected but not discrete.

**Lemma 6.8.** Let $X$ be a topological space. Let $\pi_0(X)$ be the set of connected components of $X$. Let $X \to \pi_0(X)$ be the map which sends $x \in X$ to the connected component of $X$ passing through $x$. Endow $\pi_0(X)$ with the quotient topology. Then $\pi_0(X)$ is a totally disconnected space and any continuous map $X \to Y$ from $X$ to a totally disconnected space $Y$ factors through $\pi_0(X)$.

**Proof.** By Lemma 6.4 the connected components of $\pi_0(X)$ are the singletons. We omit the proof of the second statement.

**Definition 6.9.** A topological space $X$ is called locally connected if every point $x \in X$ has a fundamental system of connected neighbourhoods.

**Lemma 6.10.** Let $X$ be a topological space. If $X$ is locally connected, then

1. any open subset of $X$ is locally connected, and
2. the connected components of $X$ are open.

So also the connected components of open subsets of $X$ are open. In particular, every point has a fundamental system of open connected neighbourhoods.
7. Irreducible components

**Definition 7.1.** Let $X$ be a topological space.

1. We say $X$ is **irreducible**, if $X$ is not empty, and whenever $X = Z_1 \cup Z_2$ with $Z_i$ closed, we have $X = Z_1$ or $X = Z_2$.

2. We say $Z \subset X$ is an **irreducible component** of $X$ if $Z$ is a maximal irreducible subset of $X$.

An irreducible space is obviously connected.

**Lemma 7.2.** Let $f : X \to Y$ be a continuous map of topological spaces. If $E \subset X$ is an irreducible subset, then $f(E) \subset Y$ is irreducible as well.

**Proof.** Suppose $f(E)$ is the union of $Z_1 \cap f(E)$ and $Z_2 \cap f(E)$, for two distinct closed subsets $Z_1$ and $Z_2$ of $Y$; this is equal to the intersection $(Z_1 \cup Z_2) \cap f(E)$, so $f(E)$ is then contained in the union $Z_1 \cup Z_2$. For the irreducibility of $f(E)$ it suffices to show that it is contained in either $Z_1$ or $Z_2$. The relation $f(E) \subset Z_1 \cup Z_2$ shows that $f^{-1}(f(E)) \subset f^{-1}(Z_1 \cup Z_2)$; as the right-hand side is clearly equal to $f^{-1}(Z_1) \cup f^{-1}(Z_2)$ and since $E \subset f^{-1}(f(E))$, it follows that $E \subset f^{-1}(Z_1) \cup f^{-1}(Z_2)$, from which one concludes by the irreducibility of $E$ that $E \subset f^{-1}(Z_1)$ or $E \subset f^{-1}(Z_2)$. Hence one sees that either $f(E) \subset f(f^{-1}(Z_1)) \subset Z_1$ or $f(E) \subset Z_2$. □

**Lemma 7.3.** Let $X$ be a topological space.

1. If $T \subset X$ is irreducible so is its closure in $X$.

2. Any irreducible component of $X$ is closed.

3. Any irreducible subset of $X$ is contained in an irreducible component of $X$.

4. Every point of $X$ is contained in some irreducible component of $X$, in other words, $X$ is the union of its irreducible components.

**Proof.** Let $\overline{T}$ be the closure of the irreducible subset $T$. If $\overline{T} = Z_1 \cup Z_2$ with $Z_i \subset \overline{T}$ closed, then $T = (T \cap Z_1) \cup (T \cap Z_2)$ and hence $T$ equals one of the two, say $T = Z_1 \cap T$. Thus clearly $\overline{T} \subset Z_1$. This proves (1). Part (2) follows immediately from (1) and the definition of irreducible components.

Let $T \subset X$ be irreducible. Consider the set $A$ of irreducible subsets $T \subset T_\alpha \subset X$. Note that $A$ is nonempty since $T \in A$. There is a partial ordering on $A$ coming from inclusion: $\alpha \leq \alpha' \iff T_\alpha \subset T_\alpha'$. Choose a maximal totally ordered subset $A' \subset A$, and let $T' = \bigcup_{\alpha \in A'} T_\alpha$. We claim that $T'$ is irreducible. Namely, suppose that $T' = Z_1 \cup Z_2$ is a union of two closed subsets of $T$. For each $\alpha \in A'$ we have either $T_\alpha \subset Z_1$ or $T_\alpha \subset Z_2$, by irreducibility of $T_\alpha$. Suppose that for some $\alpha_0 \in A'$ we have $T_{\alpha_0} \not\subset Z_1$ (say, if not we’re done anyway). Then, since $A'$ is totally ordered we see immediately that $T_\alpha \subset Z_2$ for all $\alpha \in A'$. Hence $T' = Z_2$. This proves (3). Part (4) is an immediate consequence of (3) as a singleton space is irreducible. □

A singleton is irreducible. Thus if $x \in X$ is a point then the closure $\overline{\{x\}}$ is an irreducible closed subset of $X$.

**Definition 7.4.** Let $X$ be a topological space.

1. Let $Z \subset X$ be an irreducible closed subset. A **generic point** of $Z$ is a point $\xi \in Z$ such that $Z = \overline{\{\xi\}}$. 

(2) The space $X$ is called Kolmogorov, if for every $x, x' \in X$, $x \neq x'$ there exists a closed subset of $X$ which contains exactly one of the two points.

(3) The space $X$ is called quasi-sober if every irreducible closed subset has a generic point.

(4) The space $X$ is called sober if every irreducible closed subset has a unique generic point.

A topological space $X$ is Kolmogorov, quasi-sober or sober, resp., if and only if the map $x \mapsto \{x\}$ from $X$ to the set of irreducible closed subsets of $X$ is injective, surjective or bijective, resp. Hence we see that a topological space is sober if and only if it is quasi-sober and Kolmogorov.

**Lemma 7.5.** Let $X$ be a topological space and let $Y \subset X$.

1. If $X$ is Kolmogorov then so is $Y$.
2. Suppose $Y$ is locally closed in $X$. If $X$ is quasi-sober then so is $Y$.
3. Suppose $Y$ is locally closed in $X$. If $X$ is sober then so is $Y$.

**Proof.** Proof of (1). Suppose $X$ is Kolmogorov. Let $x, y \in X$ with $x \neq y$. Then \( \{x\} \cap Y = \{x\} \neq \{y\} = \{y\} \cap Y \). Hence \( \{x\} \cap Y \neq \{y\} \cap Y \). This shows that $Y$ is Kolmogorov.

Proof of (2). Suppose $X$ is quasi-sober. It suffices to consider the cases $Y$ is closed and $Y$ is open. First, suppose $Y$ is closed. Let $Z$ be an irreducible closed subset of $Y$. Then $Z$ is an irreducible closed subset of $X$. Hence there exists $x \in Y$ with $\{x\} = Y$. It follows $\{x\} \cap Y = Y$. This shows $Y$ is quasi-sober. Second, suppose $Y$ is open. Let $Z$ be an irreducible closed subset of $Y$. Then $Z$ is an irreducible closed subset of $X$. Hence there exists $x \in Z$ with $\{x\} = Z$. If $x \notin Y$ we get the contradiction $Z = Z \cap Y \subset Z \cap Y = \{x\} \cap Y = \emptyset$. Therefore $x \in Y$. It follows $Z = Z \cap Y = \{x\} \cap Y$. This shows $Y$ is quasi-sober.

Proof of (3). Immediately from (1) and (2). \qed

**Lemma 7.6.** Let $X$ be a topological space and let $\{X_i\}_{i \in I}$ be a covering of $X$.

1. Suppose $X_i$ is locally closed in $X$ for every $i \in I$. Then, $X$ is Kolmogorov if and only if $X_i$ is Kolmogorov for every $i \in I$.
2. Suppose $X_i$ is open in $X$ for every $i \in I$. Then, $X$ is quasi-sober if and only if $X_i$ is quasi-sober for every $i \in I$.
3. Suppose $X_i$ is open in $X$ for every $i \in I$. Then, $X$ is sober if and only if $X_i$ is sober for every $i \in I$.

**Proof.** Proof of (1). If $X$ is Kolmogorov then so is $X_i$ for every $i \in I$ by Lemma 7.5. Suppose $X_i$ is Kolmogorov for every $i \in I$. Let $x, y \in X$ with $\overline{\{x\}} = \overline{\{y\}}$. There exists $i \in I$ with $x \in X_i$. There exists an open subset $U \subset X$ such that $X_i$ is a closed subset of $U$. If $y \notin U$ we get the contradiction $x \in \overline{\{x\}} \cap U = \overline{\{y\}} \cap U = \emptyset$. Hence $y \in U$. It follows $y \in \overline{\{y\}} \cap U = \overline{\{x\}} \cap U \subset X_i$. This shows $y \in X_i$. It follows $\overline{\{x\}} \cap X_i = \overline{\{y\}} \cap X_i$. Since $X_i$ is Kolmogorov we get $x = y$. This shows $X$ is Kolmogorov.

Proof of (2). If $X$ is quasi-sober then so is $X_i$ for every $i \in I$ by Lemma 7.5. Suppose $X_i$ is quasi-sober for every $i \in I$. Let $Y$ be an irreducible closed subset of $X$. As $Y \neq \emptyset$ there exists $i \in I$ with $X_i \cap Y \neq \emptyset$. As $X_i$ is open in $X$ it follows $X_i \cap Y$ is non-empty and open in $Y$, hence irreducible and dense in $Y$. Thus $X_i \cap Y = X_i \cap X_i \cap Y = X_i \cap \overline{\{x\}} \cap U \subset X_i$. This shows $X$ is quasi-sober.
is an irreducible closed subset of $X_i$. As $X_i$ is quasi-sober there exists $x \in X_i \cap Y$ with $X_i \cap Y = \{x\} \cap X_i \subset \{x\}$. Since $X_i \cap Y$ is dense in $Y$ and $Y$ is closed in $X$ it follows $Y = X_i \cap Y \subset X_i \cap Y \subset \{x\} \subset Y$. Therefore $Y = \{x\}$. This shows $X$ is quasi-sober.

Proof of (3). Immediately from (1) and (2). \qed

**Example 7.7.** Let $X$ be an indiscrete space of cardinality at least 2. Then $X$ is quasi-sober but not Kolmogorov. Moreover, the family of its singletons is a covering of $X$ by discrete and hence Kolmogorov spaces.

**Example 7.8.** Let $Y$ be an infinite set, furnished with the topology whose closed sets are $Y$ and the finite subsets of $Y$. Then $Y$ is Kolmogorov but not quasi-sober. However, the family of its singletons (which are its irreducible components) is a covering by discrete and hence sober spaces.

**Example 7.9.** Let $X$ and $Y$ be as in Example 7.7 and Example 7.8. Then, $X \amalg Y$ is neither Kolmogorov nor quasi-sober.

**Example 7.10.** Let $Z$ be an infinite set and let $z \in Z$. We furnish $Z$ with the topology whose closed sets are $Z$ and the finite subsets of $Z \setminus \{z\}$. Then $Z$ is sober but its subspace $Z \setminus \{z\}$ is not quasi-sober.

**Example 7.11.** Recall that a topological space $X$ is Hausdorff iff for every distinct pair of points $x, y \in X$ there exist disjoint opens $U, V \subset X$ such that $x \in U$, $y \in V$. In this case $X$ is irreducible if and only if $X$ is a singleton. Similarly, any subset of $X$ is irreducible if and only if it is a singleton. Hence a Hausdorff space is sober.

**Lemma 7.12.** Let $f : X \to Y$ be a continuous map of topological spaces. Assume that (a) $Y$ is irreducible, (b) $f$ is open, and (c) there exists a dense collection of points $y \in Y$ such that $f^{-1}(y)$ is irreducible. Then $X$ is irreducible.

**Proof.** Suppose $X = Z_1 \cup Z_2$ with $Z_i$ closed. Consider the open sets $U_1 = Z_1 \setminus Z_2 = X \setminus Z_2$ and $U_2 = Z_2 \setminus Z_1 = X \setminus Z_1$. To get a contradiction assume that $U_1$ and $U_2$ are both nonempty. By (b) we see that $f(U_1)$ is open. By (a) we have $Y$ irreducible and hence $f(U_1) \cap f(U_2) \neq \emptyset$. By (c) there is a point $y$ which corresponds to a point of this intersection such that the fibre $X_y = f^{-1}(y)$ is irreducible. Then $X_y \cap U_1$ and $X_y \cap U_2$ are nonempty disjoint open subsets of $X_y$ which is a contradiction. \qed

**Lemma 7.13.** Let $f : X \to Y$ be a continuous map of topological spaces. Assume that (a) $f$ is open, and (b) for every $y \in Y$ the fibre $f^{-1}(y)$ is irreducible. Then $f$ induces a bijection between irreducible components.

**Proof.** We point out that assumption (b) implies that $f$ is surjective (see Definition 7.1). Let $T \subset Y$ be an irreducible component. Note that $T$ is closed, see Lemma 7.3. The lemma follows if we show that $f^{-1}(T)$ is irreducible because any irreducible subset of $X$ maps into an irreducible component of $Y$ by Lemma 7.2. Note that $f^{-1}(T) \to T$ satisfies the assumptions of Lemma 7.12. Hence we win. \qed

The construction of the following lemma is sometimes called the “soberification”.

**Lemma 7.14.** Let $X$ be a topological space. There is a canonical continuous map $c : X \to X'$.
from $X$ to a sober topological space $X'$ which is universal among continuous maps from $X$ to sober topological spaces. Moreover, the assignment $U' \mapsto c^{-1}(U')$ is a bijection between opens of $X'$ and $X$ which commutes with finite intersections and arbitrary unions. The image $c(X)$ is a Kolmogorov topological space and the map $c : X \to c(X)$ is universal for maps of $X$ into Kolmogorov spaces.

**Proof.** Let $X'$ be the set of irreducible closed subsets of $X$ and let $$c : X \to X', \ x \mapsto \{x\}.$$ For $U \subset X$ open, let $U' \subset X'$ denote the set of irreducible closed subsets of $X$ which meet $U$. Then $c^{-1}(U') = U$. In particular, if $U_1 \neq U_2$ are open in $X$, then $U_1' \neq U_2'$. Hence $c$ induces a bijection between the subsets of $X'$ of the form $U'$ and the opens of $X$.

Let $U_1, U_2$ be open in $X$. Suppose that $Z \in U_1'$ and $Z \in U_2'$. Then $Z \cap U_1$ and $Z \cap U_2$ are nonempty open subsets of the irreducible space $Z$ and hence $Z \cap U_1 \cap U_2$ is nonempty. Thus $(U_1 \cap U_2)' = U_1' \cap U_2'$. The rule $U \mapsto U'$ is also compatible with arbitrary unions (details omitted). Thus it is clear that the collection of $U'$ form a topology on $X'$ and that we have a bijection as stated in the lemma.

Next we show that $X'$ is sober. Let $T \subset X'$ be an irreducible closed subset. Let $U \subset X$ be the open such that $X' \setminus T = U'$. Then $Z = X \setminus U$ is irreducible because of the properties of the bijection of the lemma. We claim that $Z \in T$ is the unique generic point. Namely, any open of the form $V' \subset X'$ which does not contain $Z$ must come from an open $V \subset X$ which misses $Z$, i.e., is contained in $U$.

Finally, we check the universal property. Let $f : X \to Y$ be a continuous map to a sober topological space. Then we let $f' : X' \to Y$ be the map which sends the irreducible closed $Z \subset X$ to the unique generic point of $f(Z)$. It follows immediately that $f' \circ c = f$ as maps of sets, and the properties of $c$ imply that $f'$ is continuous. We omit the verification that the continuous map $f'$ is unique. We also omit the proof of the statements on Kolmogorov spaces. 

\[\square\]

8. Noetherian topological spaces

**Definition** 8.1. A topological space is called **Noetherian** if the descending chain condition holds for closed subsets of $X$. A topological space is called **locally Noetherian** if every point has a neighbourhood which is Noetherian.

**Lemma** 8.2. Let $X$ be a Noetherian topological space.

1. Any subset of $X$ with the induced topology is Noetherian.
2. The space $X$ has finitely many irreducible components.
3. Each irreducible component of $X$ contains a nonempty open of $X$.

**Proof.** Let $T \subset X$ be a subset of $X$. Let $T_1 \supset T_2 \supset \ldots$ be a descending chain of closed subsets of $T$. Write $T_i = T \cap Z_i$ with $Z_i \subset X$ closed. Consider the descending chain of closed subsets $Z_1 \supset Z_1 \cap Z_2 \supset Z_1 \cap Z_2 \cap Z_3 \ldots$ This stabilizes by assumption and hence the original sequence of $T_i$ stabilizes. Thus $T$ is Noetherian.

Let $A$ be the set of closed subsets of $X$ which do not have finitely many irreducible components. Assume that $A$ is not empty to arrive at a contradiction. The set $A$ is partially ordered by inclusion: $\alpha \leq \alpha' \iff Z_\alpha \subset Z_{\alpha'}$. By the descending chain condition we may find a smallest element of $A$, say $Z$. As $Z$ is not a finite union of
irreducible components, it is not irreducible. Hence we can write \( Z = Z' \cup Z'' \) and both are strictly smaller closed subsets. By construction \( Z' = \bigcup Z'_i \) and \( Z'' = \bigcup Z''_i \) are finite unions of their irreducible components. Hence \( Z = \bigcup Z'_i \cup \bigcup Z''_i \) is a finite union of irreducible closed subsets. After removing redundant members of this expression, this will be the decomposition of \( Z \) into its irreducible components, a contradiction.

Let \( Z \subset X \) be an irreducible component of \( X \). Let \( Z_1, \ldots, Z_n \) be the other irreducible components of \( X \). Consider \( U = Z \setminus (Z_1 \cup \ldots \cup Z_n) \). This is not empty since otherwise the irreducible space \( Z \) would be contained in one of the other \( Z_i \). Because \( X = Z \cup Z_1 \cup \ldots \cup Z_n \) (see Lemma 7.3), also \( U = X \setminus (Z_1 \cup \ldots \cup Z_n) \) and hence open in \( X \). Thus \( Z \) contains a nonempty open of \( X \).

**Lemma 8.3.** Let \( f : X \to Y \) be a continuous map of topological spaces.

1. If \( X \) is Noetherian, then \( f(X) \) is Noetherian.
2. If \( X \) is locally Noetherian and \( f \) open, then \( f(X) \) is locally Noetherian.

**Proof.** In case (1), suppose that \( Z_1 \supset Z_2 \supset Z_3 \supset \ldots \) is a descending chain of closed subsets of \( f(X) \) (as usual with the induced topology as a subset of \( Y \)). Then \( f^{-1}(Z_1) \supset f^{-1}(Z_2) \supset f^{-1}(Z_3) \supset \ldots \) is a descending chain of closed subsets of \( X \). Hence this chain stabilizes. Since \( f(f^{-1}(Z_i)) = Z_i \) we conclude that \( Z_1 \supset Z_2 \supset Z_3 \supset \ldots \) stabilizes also. In case (2), let \( y \in f(X) \). Choose \( x \in X \) with \( f(x) = y \). By assumption there exists a neighbourhood \( E \subset X \) of \( x \) which is Noetherian. Then \( f(E) \subset f(X) \) is a neighbourhood which is Noetherian by part (1).

**Lemma 8.4.** Let \( X \) be a topological space. Let \( X_i \subset X \), \( i = 1, \ldots, n \) be a finite collection of subsets. If each \( X_i \) is Noetherian (with the induced topology), then \( \bigcup_{i=1,\ldots,n} X_i \) is Noetherian (with the induced topology).

**Proof.** Omitted.

**Example 8.5.** Any nonempty, Kolmogorov Noetherian topological space has a closed point (combine Lemmas 11.8 and 11.13). Let \( X = \{1, 2, 3, \ldots\} \). Define a topology on \( X \) with opens \( \emptyset, \{1, 2, \ldots, n\}, \ n \geq 1 \) and \( X \). Thus \( X \) is a locally Noetherian topological space, without any closed points. This space cannot be the underlying topological space of a locally Noetherian scheme, see Properties, Lemma 5.8.

**Lemma 8.6.** Let \( X \) be a locally Noetherian topological space. Then \( X \) is locally connected.

**Proof.** Let \( x \in X \). Let \( E \) be a neighbourhood of \( x \). We have to find a connected neighbourhood of \( x \) contained in \( E \). By assumption there exists a neighbourhood \( E' \) of \( x \) which is Noetherian. Then \( E \cap E' \) is Noetherian, see Lemma 8.2. Let \( E \cap E' = Y_1 \cup \ldots \cup Y_n \) be the decomposition into irreducible components, see Lemma 8.2. Let \( E'' = \bigcup_{Y_i} Y_i \). This is a connected subset of \( E \cap E' \) containing \( x \). It contains the open \( E \cap E' \setminus \left( \bigcup_{Y_i} Y_i \right) \) of \( E \cap E' \) and hence it is a neighbourhood of \( x \) in \( X \). This proves the lemma.

9. Krull dimension

**Definition 9.1.** Let \( X \) be a topological space.
(1) A chain of irreducible closed subsets of $X$ is a sequence $Z_0 \subset Z_1 \subset \ldots \subset Z_n \subset X$ with $Z_i$ closed irreducible and $Z_i \neq Z_{i+1}$ for $i = 0, \ldots, n - 1$.

(2) The length of a chain $Z_0 \subset Z_1 \subset \ldots \subset Z_n \subset X$ of irreducible closed subsets of $X$ is the integer $n$.

(3) The dimension or more precisely the Krull dimension $\dim(X)$ of $X$ is the element of $\{-\infty, 0, 1, 2, 3, \ldots, \infty\}$ defined by the formula:

$$\dim(X) = \sup\{\text{lengths of chains of irreducible closed subsets}\}$$

Thus $\dim(X) = -\infty$ if and only if $X$ is the empty space.

(4) Let $x \in X$. The Krull dimension of $X$ at $x$ is defined as

$$\dim_x(X) = \min\{\dim(U), x \in U \subset X \text{ open}\}$$

the minimum of $\dim(U)$ where $U$ runs over the open neighbourhoods of $x$ in $X$.

Note that if $U' \subset U \subset X$ are open then $\dim(U') \leq \dim(U)$. Hence if $\dim_x(X) = d$ then $x$ has a fundamental system of open neighbourhoods $U$ with $\dim(U) = \dim_x(X)$.

**Example 9.2.** The Krull dimension of the usual Euclidean space $\mathbb{R}^n$ is 0.

**Example 9.3.** Let $X = \{s, \eta\}$ with open sets given by $\{\emptyset, \{\eta\}, \{s, \eta\}\}$. In this case a maximal chain of irreducible closed subsets is $\{s\} \subset \{s, \eta\}$. Hence $\dim(X) = 1$. It is easy to generalize this example to get a $(n + 1)$-element topological space of Krull dimension $n$.

**Definition 9.4.** Let $X$ be a topological space. We say that $X$ is equidimensional if every irreducible component of $X$ has the same dimension.

### 10. Codimension and catenary spaces

We only define the codimension of irreducible closed subsets.

**Definition 10.1.** Let $X$ be a topological space. Let $Y \subset X$ be an irreducible closed subset. The codimension of $Y$ in $X$ is the supremum of the lengths $e$ of chains

$$Y = Y_0 \subset Y_1 \subset \ldots \subset Y_e \subset X$$

of irreducible closed subsets in $X$ starting with $Y$. We will denote this $\text{codim}(Y, X)$.

The codimension is an element of $\{0, 1, 2, \ldots\} \cup \{\infty\}$. If $\text{codim}(Y, X) < \infty$, then every chain can be extended to a maximal chain (but these do not all have to have the same length).

**Lemma 10.2.** Let $X$ be a topological space. Let $Y \subset X$ be an irreducible closed subset. Let $U \subset X$ be an open subset such that $Y \cap U$ is nonempty. Then

$$\text{codim}(Y, X) = \text{codim}(Y \cap U, U)$$

**Proof.** The rule $T \mapsto T$ defines a bijective inclusion preserving map between the closed irreducible subsets of $U$ and the closed irreducible subsets of $X$ which meet $U$. Using this the lemma easily follows. Details omitted. □
Example 10.3. Let $X = [0, 1]$ be the unit interval with the following topology: The sets $[0, 1], (1 - 1/n, 1]$ for $n \in \mathbb{N}$, and $\emptyset$ are open. So the closed sets are $\emptyset, \{0\}, [0, 1 - 1/n]$ for $n > 1$ and $[0, 1]$. This is clearly a Noetherian topological space. But the irreducible closed subset $Y = \{0\}$ has infinite codimension $\text{codim}(Y, X) = \infty$. To see this we just remark that all the closed sets $[0, 1 - 1/n]$ are irreducible.

Definition 10.4. Let $X$ be a topological space. We say $X$ is catenary if for every pair of irreducible closed subsets $T \subset T'$ we have $\text{codim}(T, T') < \infty$ and every maximal chain of irreducible closed subsets

$$T = T_0 \subset T_1 \subset \ldots \subset T_e = T'$$

has the same length (equal to the codimension).

Lemma 10.5. Let $X$ be a topological space. The following are equivalent:

1. $X$ is catenary,
2. $X$ has an open covering by catenary spaces.

Moreover, in this case any locally closed subspace of $X$ is catenary.

Proof. Suppose that $X$ is catenary and that $U \subset X$ is an open subset. The rule $T \mapsto T$ defines a bijective inclusion preserving map between the closed irreducible subsets of $U$ and the closed irreducible subsets of $X$ which meet $U$. Using this the lemma easily follows. Details omitted. □

Lemma 10.6. Let $X$ be a topological space. The following are equivalent:

1. $X$ is catenary,
2. for every pair of irreducible closed subsets $Y \subset Y'$ we have $\text{codim}(Y, Y') < \infty$ and for every triple $Y \subset Y' \subset Y''$ of irreducible closed subsets we have $\text{codim}(Y, Y'') = \text{codim}(Y, Y') + \text{codim}(Y', Y'')$.

Proof. Omitted. □

11. Quasi-compact spaces and maps

The phrase “compact” will be reserved for Hausdorff topological spaces. And many spaces occurring in algebraic geometry are not Hausdorff.

Definition 11.1. Quasi-compactness.

1. We say that a topological space $X$ is quasi-compact if every open covering of $X$ has a finite refinement.
2. We say that a continuous map $f : X \to Y$ is quasi-compact if the inverse image $f^{-1}(V)$ of every quasi-compact open $V \subset Y$ is quasi-compact.
3. We say a subset $Z \subset X$ is retrocompact if the inclusion map $Z \to X$ is quasi-compact.

In many texts on topology a space is called compact if it is quasi-compact and Hausdorff; and in other texts the Hausdorff condition is omitted. To avoid confusion in algebraic geometry we use the term quasi-compact. Note that the notion of quasi-compactness of a map is very different from the notion of a “proper map” in topology, since there one requires the inverse image of any (quasi-)compact subset of the target to be (quasi-)compact, whereas in the definition above we only consider quasi-compact open sets.

Lemma 11.2. A composition of quasi-compact maps is quasi-compact.
Lemma 11.3. A closed subset of a quasi-compact topological space is quasi-compact.

Proof. This is immediate from the definition. □

Lemma 11.4. Let \( E \subset X \) be a closed subset of the quasi-compact space \( X \). Let \( E = \bigcup V_j \) be an open covering. Choose \( U_j \subset X \) open such that \( V_j = E \cap U_j \). Then \( X = (X \setminus E) \cup \bigcup U_j \) is an open covering of \( X \). Hence \( X = (X \setminus E) \cup U_{j_1} \cup \ldots \cup U_{j_n} \) for some \( n \) and indices \( j_i \). Thus \( E = V_{j_1} \cup \ldots \cup V_{j_n} \) as desired. □

Lemma 11.5. Let \( X \) be a Hausdorff topological space. Assume that \( E \) is closed. Then

1. If \( E \subset X \) is quasi-compact, then it is closed.
2. If \( E_1, E_2 \subset X \) are disjoint quasi-compact subsets then there exists opens \( E_i \subset U_i \) with \( U_1 \cap U_2 = \emptyset \).

Proof. Proof of (1). Let \( x \in X, x \notin E \). For every \( e \in E \) we can find disjoint opens \( V_e \) and \( U_e \) with \( e \in V_e \) and \( x \in U_e \). Since \( E \subset \bigcup V_e \) we can find finitely many \( e_1, \ldots, e_n \) such that \( E \subset V_{e_1} \cup \ldots \cup V_{e_n} \). Then \( U = U_{e_1} \cap \ldots \cap U_{e_n} \) is an open neighbourhood of \( x \) which avoids \( V_{e_1} \cup \ldots \cup V_{e_n} \). In particular it avoids \( E \). Thus \( E \) is closed.

Proof of (2). In the proof of (1) we have seen that given \( x \in E_1 \) we can find an open neighbourhood \( x \in U_x \) and an open \( E_2 \subset V_x \) such that \( U_x \cap V_x = \emptyset \). Because \( E_1 \) is quasi-compact we can find a finite number \( x_i \in E_1 \) such that \( E_1 \subset U = U_{x_1} \cup \ldots \cup U_{x_n} \). We take \( V = V_{x_1} \cap \ldots \cap V_{x_n} \) to finish the proof. □

Lemma 11.6. Let \( X \) be a Hausdorff topological space. Let \( E \subset X \). The following are equivalent: (a) \( E \) is closed in \( X \), (b) \( E \) is quasi-compact.

Proof. The implication (a) \( \Rightarrow \) (b) is Lemma 11.3. The implication (b) \( \Rightarrow \) (a) is Lemma 11.5. □

The following is really a reformulation of the quasi-compact property.

Lemma 11.7. Let \( f : X \to Y \) be a continuous map of topological spaces.

1. If \( X \) is quasi-compact, then \( f(X) \) is quasi-compact.
2. If \( f \) is quasi-compact, then \( f(X) \) is retrocompact.

Proof. If \( f(X) = \bigcup V_i \) is an open covering, then \( X = \bigcup f^{-1}(V_i) \) is an open covering. Hence if \( X \) is quasi-compact then \( X = f^{-1}(V_{i_1}) \cup \ldots \cup f^{-1}(V_{i_n}) \) for some \( i_1, \ldots, i_n \in I \) and hence \( f(X) = V_{i_1} \cup \ldots \cup V_{i_n} \). This proves (1). Assume \( f \) is quasi-compact, and let \( V \subset Y \) be quasi-compact open. Then \( f^{-1}(V) \) is quasi-compact, hence by (1) we see that \( f(f^{-1}(V)) = f(X) \cap V \) is quasi-compact. Hence \( f(X) \) is retrocompact. □

Lemma 11.8. Let \( X \) be a topological space. Assume that

1. \( X \) is nonempty,
2. \( X \) is quasi-compact, and
3. \( X \) is Kolmogorov.

Then \( X \) has a closed point.
Proof. Consider the set
\[ T = \{ Z \subset X \mid Z = \{x\} \text{ for some } x \in X \} \]
of all closures of singletons in \( X \). It is nonempty since \( X \) is nonempty. Make \( T \) into a partially ordered set using the relation of inclusion. Suppose \( Z_\alpha, \alpha \in A \) is a totally ordered subset of \( T \). By Lemma \[11.6\] we see that \( \bigcap_{\alpha \in A} Z_\alpha \neq \emptyset \). Hence there exists some \( x \in \bigcap_{\alpha \in A} Z_\alpha \) and we see that \( Z = \{x\} \in T \) is a lower bound for the family. By Zorn’s lemma there exists a minimal element \( Z \in T \). As \( X \) is Kolmogorov we conclude that \( Z = \{x\} \) for some \( x \in X \) is a closed point. □

Lemma \[11.9\]. Let \( X \) be a quasi-compact Kolmogorov space. Then the set \( X_0 \) of closed points of \( X \) is quasi-compact.

Proof. Let \( X_0 = \bigcup U_{1,0} \) be an open covering. Write \( U_{1,0} = X_0 \cap U_i \) for some open \( U_i \subset X \). Consider the complement \( Z \) of \( \bigcup U_i \). This is a closed subset of \( X \), hence quasi-compact (Lemma \[11.3\]) and Kolmogorov. By Lemma \[11.8\] if \( Z \) is nonempty it would have a closed point which contradicts the fact that \( X_0 \subset \bigcup U_i \). Hence \( Z = \emptyset \) and \( X = \bigcup U_i \). Since \( X \) is quasi-compact this covering has a finite subcover and we conclude. □

Lemma \[11.10\]. Let \( X \) be a topological space. Assume

1. \( X \) is quasi-compact,
2. \( X \) has a basis for the topology consisting of quasi-compact opens, and
3. the intersection of two quasi-compact opens is quasi-compact.

For any \( x \in X \) the connected component of \( X \) containing \( x \) is the intersection of all open and closed subsets of \( X \) containing \( x \).

Proof. Let \( T \) be the connected component containing \( x \). Let \( S = \bigcap_{\alpha \in A} Z_\alpha \) be the intersection of all open and closed subsets \( Z_\alpha \) of \( X \) containing \( x \). Note that \( S \) is closed in \( X \). Note that any finite intersection of \( Z_\alpha \)'s is a \( Z_\alpha \). Because \( T \) is connected and \( x \in T \) we have \( T \subset S \). It suffices to show that \( S \) is connected. If not, then there exists a disjoint union decomposition \( S = B \sqcup C \) with \( B \) and \( C \) open and closed in \( S \). In particular, \( B \) and \( C \) are closed in \( X \), and so quasi-compact by Lemma \[11.3\] and assumption (1). By assumption (2) there exist quasi-compact opens \( U, V \subset X \) with \( B = S \cap U \) and \( C = S \cap V \) (details omitted). Then \( U \cap V \cap S = \emptyset \). Hence \( \bigcap_{\alpha \in A} U \cap V \cap Z_\alpha = \emptyset \). By assumption (3) the intersection \( U \cap V \) is quasi-compact. By Lemma \[11.6\] for some \( \alpha' \in A \) we have \( U \cap V \cap Z_{\alpha'} = \emptyset \). Since \( X \setminus (U \cup V) \) is disjoint from \( S \) and closed in \( X \) hence quasi-compact, we can use the same lemma to see that \( Z_{\alpha''} \subset U \cup V \) for some \( \alpha'' \in A \). Then \( Z_\alpha = Z_{\alpha'} \cap Z_{\alpha''} \) is contained in \( U \cup V \) and disjoint from \( U \cap V \). Hence \( Z_\alpha = U \cap Z_\alpha \sqcup V \cap Z_\alpha \) is a decomposition into two open pieces, hence \( U \cap Z_\alpha \) and \( V \cap Z_\alpha \) are open and closed in \( X \). Thus, if \( x \in B \) say, then we see that \( S \subset U \cap Z_\alpha \) and we conclude that \( C = \emptyset \). □

Lemma \[11.11\]. Let \( X \) be a topological space. Assume \( X \) is quasi-compact and Hausdorff. For any \( x \in X \) the connected component of \( X \) containing \( x \) is the intersection of all open and closed subsets of \( X \) containing \( x \).

Proof. Let \( T \) be the connected component containing \( x \). Let \( S = \bigcap_{\alpha \in A} Z_\alpha \) be the intersection of all open and closed subsets \( Z_\alpha \) of \( X \) containing \( x \). Note that \( S \) is closed in \( X \). Note that any finite intersection of \( Z_\alpha \)'s is a \( Z_\alpha \). Because \( T \) is connected and \( x \in T \) we have \( T \subset S \). It suffices to show that \( S \) is connected. If not,
then there exists a disjoint union decomposition $S = B \sqcup C$ with $B$ and $C$ open and closed in $S$. In particular, $B$ and $C$ are closed in $X$, and so quasi-compact by Lemma 11.3. By Lemma 11.4 there exist disjoint opens $U, V \subset X$ with $B \subset U$ and $C \subset V$. Then $X \setminus U \cup V$ is closed in $X$ hence quasi-compact (Lemma 11.3). It follows that $(X \setminus U \cup V) \cap Z_\alpha = \emptyset$ for some $\alpha$ by Lemma 11.6. In other words, $Z_\alpha \subset U \cup V$. Thus $Z_\alpha = Z_\alpha \cap V \sqcup Z_\alpha \cap U$ is a decomposition into two open pieces, hence $U \cap Z_\alpha$ and $V \cap Z_\alpha$ are open and closed in $X$. Thus, if $x \in B$ say, then we see that $S \subset U \cap Z_\alpha$ and we conclude that $C = \emptyset$.

Lemma 11.12. Let $X$ be a topological space. Assume

1. $X$ is quasi-compact,
2. $X$ has a basis for the topology consisting of quasi-compact opens, and
3. the intersection of two quasi-compact opens is quasi-compact.

For a subset $T \subset X$ the following are equivalent:

(a) $T$ is an intersection of open and closed subsets of $X$, and
(b) $T$ is closed in $X$ and is a union of connected components of $X$.

Proof. It is clear that (a) implies (b). Assume (b). Let $x \in X$, $x \notin T$. Let $x \in C \subset X$ be the connected component of $X$ containing $x$. By Lemma 11.10 we see that $C = \bigcap V_\alpha$ is the intersection of all open and closed subsets $V_\alpha$ of $X$ which contain $C$. In particular, any pairwise intersection $V_\alpha \cap V_\beta$ occurs as a $V_\alpha$. As $T$ is a union of connected components of $X$ we see that $C \cap T = \emptyset$. Hence $T \cap \bigcap V_\alpha = \emptyset$. Since $T$ is quasi-compact as a closed subset of a quasi-compact space (see Lemma 11.3) we deduce that $T \cap V_\alpha = \emptyset$ for some $\alpha$, see Lemma 11.6. For this $\alpha$ we see that $U_\alpha = X \setminus V_\alpha$ is an open and closed subset of $X$ which contains $T$ and not $x$. The lemma follows.

Lemma 11.13. Let $X$ be a Noetherian topological space.

1. The space $X$ is quasi-compact.
2. Any subset of $X$ is retrocompact.

Proof. Suppose $X = \bigcup U_i$ is an open covering of $X$ indexed by the set $I$ which does not have a refinement by a finite open covering. Choose $i_1, i_2, \ldots$ elements of $I$ inductively in the following way: Choose $i_{n+1}$ such that $U_{i_{n+1}}$ is not contained in $U_{i_1} \cup \ldots \cup U_{i_n}$. Thus we see that $X \supset (X \setminus U_{i_1}) \supset (X \setminus U_{i_1} \cup U_{i_2}) \supset \ldots$ is a strictly decreasing infinite sequence of closed subsets. This contradicts the fact that $X$ is Noetherian. This proves the first assertion. The second assertion is now clear since every subset of $X$ is Noetherian by Lemma 8.2.


Proof. The conditions imply immediately that $X$ has a finite covering by Noetherian subsets, and hence is Noetherian by Lemma 8.4.

Lemma 11.15 (Alexander subbase theorem). Let $X$ be a topological space. Let $\mathcal{B}$ be a subbase for $X$. If every covering of $X$ by elements of $\mathcal{B}$ has a finite refinement, then $X$ is quasi-compact.

Proof. Assume there is an open covering of $X$ which does not have a finite refinement. Using Zorn’s lemma we can choose a maximal open covering $X = \bigcup_{i \in I} U_i$ which does not have a finite refinement (details omitted). In other words, if $U \subset X$ is any open which does not occur as one of the $U_i$, then the covering $X = U \cup \bigcup_{i \in I} U_i$
does have a finite refinement. Let $I' \subset I$ be the set of indices such that $U_i \in \mathcal{B}$. Then $\bigcup_{i \in I'} U_i \neq X$, since otherwise we would get a finite refinement covering $X$ by our assumption on $\mathcal{B}$. Pick $x \in X$, $x \notin \bigcup_{i \in I'} U_i$. Pick $i \in I$ with $x \in U_i$. Pick \( V_1, \ldots, V_n \in \mathcal{B} \) such that $x \in V_1 \cap \ldots \cap V_n \subset U_i$. This is possible as $\mathcal{B}$ is a subbasis for $X$. Note that $V_j$ does not occur as a $U_i$. By maximality of the chosen covering we see that for each $j$ there exist $i_{j,1}, \ldots, i_{j,n_j} \in I$ such that $X = V_j \cup U_{i_{j,1}} \cup \ldots \cup U_{i_{j,n_j}}$. Since $V_1 \cap \ldots \cap V_n \subset U_i$ we conclude that $X = U_i \cup \bigcup U_{i,j}$ a contradiction. \( \Box \)

12. Locally quasi-compact spaces

Recall that a neighbourhood of a point need not be open.

**Definition** 12.1. A topological space $X$ is called **locally quasi-compact** if every point has a fundamental system of quasi-compact neighbourhoods.

The term *locally compact space* in the literature often refers to a space as in the following lemma.

**Lemma 12.2.** A Hausdorff space is locally quasi-compact if and only if every point has a quasi-compact neighbourhood.

**Proof.** Let $X$ be a Hausdorff space. Let $x \in X$ and let $x \in E \subset X$ be a quasi-compact neighbourhood. Then $\overline{E}$ is closed by Lemma 11.4. Suppose that $x \in U \subset X$ is an open neighbourhood of $x$. Then $Z = E \setminus U$ is a closed subset of $E$ not containing $x$. Hence we can find a pair of disjoint open subsets $W, V \subset E$ of $E$ such that $x \in V$ and $Z \subset W$, see Lemma 11.4. It follows that $\overline{V} \subset E$ is a closed neighbourhood of $x$ contained in $E \cap U$. Also $\overline{V}$ is quasi-compact as a closed subset of $E$ (Lemma 11.3). In this way we obtain a fundamental system of quasi-compact neighbourhoods of $x$. \( \Box \)

**Lemma 12.3.** Let $X$ be a Hausdorff and quasi-compact space. Let $X = \bigcup_{i \in I} U_i$ be an open covering. Then there exists an open covering $X = \bigcup_{i \in I} V_i$ such that $V_i \subset U_i$ for all $i$.

**Proof.** Let $x \in X$. Choose an $i(x) \in I$ such that $x \in U_{i(x)}$. Since $X \setminus U_{i(x)}$ and $\{x\}$ are disjoint closed subsets of $X$, by Lemmas 11.3 and 11.4 there exists an open neighbourhood $U_x$ of $x$ whose closure is disjoint from $X \setminus U_{i(x)}$. Thus $U_x \subset U_{i(x)}$. Since $X$ is quasi-compact, there is a finite list of points $x_1, \ldots, x_m$ such that $X = U_{x_1} \cup \ldots \cup U_{x_m}$. Setting $V_i = \bigcup_{i=i(x_j)} U_{x_j}$ the proof is finished. \( \Box \)

**Lemma 12.4.** Let $X$ be a Hausdorff and quasi-compact space. Let $X = \bigcup_{i \in I} U_i$ be an open covering. Suppose given an integer $p \geq 0$ and for every $(p + 1)$-tuple $i_0, \ldots, i_p$ of $I$ an open covering $U_{i_0} \cap \ldots \cap U_{i_p} = \bigcup W_{i_0 \ldots i_p,k}$. Then there exists an open covering $X = \bigcup_{j \in J} V_j$ and a map $\alpha : J \to I$ such that $V_j \subset U_{\alpha(j)}$ and such that each $V_{j_0} \cap \ldots \cap V_{j_p}$ is contained in $W_{\alpha(j_0)\ldots \alpha(j_p),k}$ for some $k$.

**Proof.** Since $X$ is quasi-compact, there is a reduction to the case where $I$ is finite (details omitted). We prove the result for $I$ finite by induction on $p$. The base

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2This may not be standard notation. Alternative notions used in the literature are: (1) Every point has some quasi-compact neighbourhood, and (2) Every point has a closed quasi-compact neighbourhood. A scheme has the property that every point has a fundamental system of open quasi-compact neighbourhoods.
case $p = 0$ is immediate by taking a covering as in Lemma 12.3 refining the open covering $X = \bigcup W_{i_0, k}$.

Induction step. Assume the lemma proven for $p - 1$. For all $p$-tuples $i_0', \ldots, i_{p-1}'$ of $I$ let $U_{i_0'} \cap \ldots \cap U_{i_{p-1}'} = \bigcup W_{i_0' \ldots i_{p-1}'}$, $k$ be a common refinement of the coverings $U_{i_0} \cap \ldots \cap U_{i_p} = \bigcup W_{i_0 \ldots i_p}$ for those $(p + 1)$-tuples such that $\{i_0', \ldots, i_{p-1}'\} = \{i_0, \ldots, i_p\}$ (equality of sets). (There are finitely many of these as $I$ is finite.) By induction there exists a solution for these opens, say $X = \bigcup V_j$ and $\alpha : J \to I$. At this point the covering $X = \bigcup_{j \in J} V_j$ and $\alpha$ satisfy $V_j \subseteq U_{\alpha(j)}$ and each $V_{j_0} \cap \ldots \cap V_{j_p}$ is contained in $W_{\alpha(j_0) \ldots \alpha(j_p), k}$ for some $k$ if there is a repetition in $\alpha(j_0), \ldots, \alpha(j_p)$. Of course, we may and do assume that $J$ is finite.

Fix $i_0, \ldots, i_p \in I$ pairwise distinct. Consider $(p + 1)$-tuples $j_0, \ldots, j_p \in J$ with $i_0 = \alpha(j_0), \ldots, i_p = \alpha(j_p)$ such that $V_{j_0} \cap \ldots \cap V_{j_p}$ is not contained in $W_{\alpha(j_0) \ldots \alpha(j_p), k}$ for any $k$. Let $N$ be the number of such $(p + 1)$-tuples. We will show how to decrease $N$. Since

$$V_{j_0} = (V_{j_0} \setminus (V_{j_1} \cap \ldots \cap V_{j_p})) \cup \bigcup_{k \in K} V_{j_0} \cap W_{i_0 \ldots i_p, k}$$

we find a finite set $K = \{k_1, \ldots, k_t\}$ such that the LHS is contained in $\bigcup_{k \in K} W_{i_0 \ldots i_p, k}$. Then we consider the open covering

$$V_{j_0} \cap \ldots \cap V_{j_p} \subseteq U_{i_0} \cap \ldots \cap U_{i_p} = \bigcup W_{i_0 \ldots i_p, k}$$

The first open on the RHS intersects $V_{j_0} \cap \ldots \cap V_{j_p}$ in the empty set and the other opens $U_{i_k}$ of the RHS satisfy $V_{j_0, k} \cap V_{j_1} \cap \ldots \cap V_{j_p} \subseteq W_{\alpha(j_0) \ldots \alpha(j_p), k}$. Set $J' = J \cup K$. For $j \in J$ set $V'_j = V_j$ if $j \neq j_0$ and set $V'_{j_0} = V_{j_0} \setminus (V_{j_1} \cap \ldots \cap V_{j_p})$. For $k \in K$ set $V'_k = V_{j_0, k}$. Finally, the map $\alpha' : J' \to I$ is given by $\alpha$ on $J$ and maps every element of $K$ to $i_0$. A simple check shows that $N$ has decreased by one under this replacement. Repeating this procedure $N$ times we arrive at the situation where $N = 0$.

To finish the proof we argue by induction on the number $M$ of $(p + 1)$-tuples $i_0, \ldots, i_p \in I$ with pairwise distinct entries for which there exists a $(p + 1)$-tuple $j_0, \ldots, j_p \in J$ with $i_0 = \alpha(j_0), \ldots, i_p = \alpha(j_p)$ such that $V_{j_0} \cap \ldots \cap V_{j_p}$ is not contained in $W_{\alpha(j_0) \ldots \alpha(j_p), k}$ for any $k$. To do this, we claim that the operation performed in the previous paragraph does not increase $M$. This follows formally from the fact that the map $\alpha' : J' \to I$ factors through a map $\beta : J' \to J$ such that $V'_{\beta(j')} \subseteq V_{\beta(j')}$.

**Lemma 12.5.** Let $X$ be a Hausdorff and locally quasi-compact space. Let $Z \subseteq X$ be a quasi-compact (hence closed) subset. Suppose given an integer $p \geq 0$, a set $I$, for every $i \in I$ an open $U_i \subseteq X$, and for every $(p + 1)$-tuple $i_0, \ldots, i_p$ of $I$ an open $W_{i_0 \ldots i_p} \subseteq U_{i_0} \cap \ldots \cap U_{i_p}$ such that

1. $Z \subseteq \bigcup U_i$, and
2. for every $i_0, \ldots, i_p$ we have $W_{i_0 \ldots i_p} \cap Z = U_{i_0} \cap \ldots \cap U_{i_p} \cap Z$.

Then there exists opens $V_i$ of $X$ such that we have $Z \subseteq \bigcup V_i$, for all $i$ we have $V_i \subseteq U_i$, and we have $V_{i_0} \cap \ldots \cap V_{i_p} \subseteq W_{i_0 \ldots i_p}$ for all $(p + 1)$-tuples $i_0, \ldots, i_p$.

**Proof.** Since $Z$ is quasi-compact, there is a reduction to the case where $I$ is finite (details omitted). Because $X$ is locally quasi-compact and $Z$ is quasi-compact, we can find a neighbourhood $Z \subseteq E$ which is quasi-compact, i.e., $E$ is quasi-compact. 
and contains an open neighbourhood of \( Z \) in \( X \). If we prove the result after replacing \( X \) by \( E \), then the result follows. Hence we may assume \( X \) is quasi-compact.

We prove the result in case \( I \) is finite and \( X \) is quasi-compact by induction on \( p \). The base case is \( p = 0 \). In this case we have \( X = (X \setminus Z) \cup \bigcup W_i \). By Lemma 12.3 we can find a covering \( X = V \cup \bigcup V_i \) by opens \( V_i \subset W_i \) and \( V \subset X \setminus Z \) with \( V_i \subset W_i \) for all \( i \). Then we see that we obtain a solution of the problem posed by the lemma.

Induction step. Assume the lemma proven for \( p - 1 \). Set \( W_{j_0, \ldots, j_{p-1}} \) equal to the intersection of all \( W_{i_0, \ldots, i_p} \) with \( \{ j_0, \ldots, j_{p-1} \} = \{ i_0, \ldots, i_p \} \) (equality of sets). By induction there exists a solution for these opens, say \( V_i \subset U_i \). It follows from our choice of \( W_{j_0, \ldots, j_{p-1}} \) that we have \( V_{i_0} \cap \ldots \cap V_{i_p} \subset W_{i_0, \ldots, i_p} \) for all \((p + 1)\)-tuples \( i_0, \ldots, i_p \) where \( i_a = i_b \) for some \( 0 \leq a < b \leq p \). Thus we only need to modify our choice of \( V_i \) if \( V_{i_0} \cap \ldots \cap V_{i_p} \not\subset W_{i_0, \ldots, i_p} \) for some \((p + 1)\)-tuple \( i_0, \ldots, i_p \) with pairwise distinct elements. In this case we have

\[
T = V_{i_0} \cap \ldots \cap V_{i_p} \setminus W_{i_0, \ldots, i_p} \subset V_{i_0} \cap \ldots \cap V_{i_p} \setminus W_{i_0, \ldots, i_p}
\]

is a closed subset of \( X \) contained in \( U_{i_0} \cap \ldots \cap U_{i_p} \) not meeting \( Z \). Hence we can replace \( V_{i_0} \) by \( V_{i_0} \setminus T \) to “fix” the problem. After repeating this finitely many times for each of the problem tuples, the lemma is proven.

13. Limits of spaces

The category of topological spaces has products. Namely, if \( I \) is a set and for \( i \in I \) we are given a topological space \( X_i \) then we endow \( \prod_{i \in I} X_i \) with the product topology. As a basis for the topology we use sets of the form \( \prod U_i \) where \( U_i \subset X_i \) is open and \( U_i = X_i \) for almost all \( i \).

The category of topological spaces has equalizers. Namely, if \( a, b : X \to Y \) are morphisms of topological spaces, then the equalizer of \( a \) and \( b \) is the subset \( \{ x \in X \mid a(x) = b(x) \} \subset X \) endowed with the induced topology.

**Lemma 13.1.** The category of topological spaces has limits and the forgetful functor to sets commutes with them.

**Proof.** This follows from the discussion above and Categories, Lemma [14.10]. It follows from the description above that the forgetful functor commutes with limits. Another way to see this is to use Categories, Lemma [24.4] and use that the forgetful functor has a left adjoint, namely the functor which assigns to a set the corresponding discrete topological space.

**Lemma 13.2.** Let \( I \) be a cofiltered category. Let \( i \mapsto X_i \) be a diagram of topological spaces over \( I \). Let \( X = \lim X_i \) be the limit with projection maps \( f_i : X \to X_i \).

1. Any open of \( X \) is of the form \( \bigcup_{j \in J} f_j^{-1}(U_j) \) for some subset \( J \subset I \) and opens \( U_j \subset X_j \).
2. Any quasi-compact open of \( X \) is of the form \( f_i^{-1}(U_i) \) for some \( i \) and some \( U_i \subset X_i \) open.

**Proof.** The construction of the limit given above shows that \( X \subset \prod X_i \) with the induced topology. A basis for the topology of \( \prod X_i \) are the opens \( \prod U_i \) where
$U_i \subset X_i$ is open and $U_i = X_i$ for almost all $i$. Say $i_1, \ldots, i_n \in \text{Ob}(\mathcal{I})$ are the objects such that $U_{i_j} \neq X_{i_j}$. Then

$$X \cap \prod U_i = f_{i_1}^{-1}(U_{i_1}) \cap \ldots \cap f_{i_n}^{-1}(U_{i_n})$$

For a general limit of topological spaces these form a basis for the topology on $X$. However, if $\mathcal{I}$ is cofiltered as in the statement of the lemma, then we can pick a $j \in \text{Ob}(\mathcal{I})$ and morphisms $j \to i_l, l = 1, \ldots, n$. Let

$$U_j = (X_j \to X_{i_1})^{-1}(U_{i_1}) \cap \ldots \cap (X_j \to X_{i_n})^{-1}(U_{i_n})$$

Then it is clear that $X \cap \prod U_i = f_{i_j}^{-1}(U_j)$. Thus for any open $W$ of $X$ there is a set $A$ and a map $\alpha : A \to \text{Ob}(\mathcal{I})$ and opens $U_a \subset X_{\alpha(a)}$ such that $W = \bigcup f_{\alpha(a)}^{-1}(U_a)$. Set $J = \text{Im}(\alpha)$ and for $j \in J$ set $U_j = \bigcup_{\alpha(a)=j} U_a$ to see that $W = \bigcup_{j \in J} f_j^{-1}(U_j)$. This proves (1).

To see (2) suppose that $\bigcup_{j \in J} f_j^{-1}(U_j)$ is quasi-compact. Then it is equal to $f_{i_j}^{-1}(U_{i_j}) \cup \ldots \cup f_{i_m}^{-1}(U_{i_m})$ for some $j_1, \ldots, j_m \in J$. Since $\mathcal{I}$ is cofiltered, we can pick a $i \in \text{Ob}(\mathcal{I})$ and morphisms $i \to j_l, l = 1, \ldots, m$. Let

$$U_i = (X_i \to X_{i_1})^{-1}(U_{i_1}) \cup \ldots \cup (X_i \to X_{i_m})^{-1}(U_{i_m})$$

Then our open equals $f_i^{-1}(U_i)$ as desired. \hfill

**Lemma 13.3.** Let $\mathcal{I}$ be a cofiltered category. Let $i \mapsto X_i$ be a diagram of topological spaces over $\mathcal{I}$. Let $X$ be a topological space such that

1. $X = \lim_i X_i$ as a set (denote $f_i$ the projection maps),
2. the sets $f_i^{-1}(U_i)$ for $i \in \text{Ob}(\mathcal{I})$ and $U_i \subset X_i$ open form a basis for the topology of $X$.

Then $X$ is the limit of the $X_i$ as a topological space.

**Proof.** Follows from the description of the limit topology in Lemma 13.2. \hfill

**Theorem 13.4** (Tychonov). A product of quasi-compact spaces is quasi-compact.

**Proof.** Let $I$ be a set and for $i \in I$ let $X_i$ be a quasi-compact topological space. Set $X = \prod X_i$. Let $\mathcal{B}$ be the set of subsets of $X$ of the form $U_i \times \prod_{i' \neq i} X_{i'}$ where $U_i \subset X_i$ is open. By construction this family is a subbasis for the topology on $X$. By Lemma 11.15 it suffices to show that any covering $X = \bigcup_{j \in J} B_j$ by elements $B_j$ of $\mathcal{B}$ has a finite refinement. We can decompose $J = \prod J_i$ so that if $j \in J_i$, then $B_j = U_j \times \prod_{i' \neq i} X_{i'}$ with $U_j \subset X_j$ open. If $X_i = \bigcup_{j \in J_i} U_j$, then there is a finite refinement and we conclude that $X = \bigcup_{j \in J} B_j$ has a finite refinement. If this is not the case, then for every $i$ we can choose an point $x_i \in X_i$ which is not in $\bigcup_{j \in J_i} U_j$. But then the point $x = (x_i)_{i \in I}$ is an element of $X$ not contained in $\bigcup_{j \in J} B_j$, a contradiction. \hfill

The following lemma does not hold if one drops the assumption that the spaces $X_i$ are Hausdorff, see Examples, Section 4.

**Lemma 13.5.** Let $\mathcal{I}$ be a category and let $i \mapsto X_i$ be a diagram over $\mathcal{I}$ in the category of topological spaces. If each $X_i$ is quasi-compact and Hausdorff, then $\lim X_i$ is quasi-compact.
Proof. Recall that \( \lim X_i \) is a subspace of \( \prod X_i \). By Theorem \([13.4]\) this product is quasi-compact. Hence it suffices to show that \( \lim X_i \) is a closed subspace of \( \prod X_i \) (Lemma \([11.3]\)). If \( \varphi : j \to k \) is a morphism of \( \mathcal{I} \), then let \( \Gamma_{\varphi} \subset X_j \times X_k \) denote the graph of the corresponding continuous map \( X_j \to X_k \). By Lemma \([5.2]\) this graph is closed. It is clear that \( \lim X_i \) is the intersection of the closed subsets
\[
\Gamma_{\varphi} \times \prod_{i \neq j,k} X_i \subset \prod X_i
\]
Thus the result follows. \( \square \)

The following lemma generalizes Categories, Lemma \([21.5]\) and partially generalizes Lemma \([11.6]\).

Lemma 13.6. Let \( \mathcal{I} \) be a cofiltered category and let \( i \to X_i \) be a diagram over \( \mathcal{I} \) in the category of topological spaces. If each \( X_i \) is quasi-compact, Hausdorff, and nonempty, then \( \lim X_i \) is nonempty.

Proof. In the proof of Lemma \([13.5]\) we have seen that \( X = \lim X_i \) is the intersection of the closed subsets
\[
Z_{\varphi} = \Gamma_{\varphi} \times \prod_{i \neq j,k} X_i
\]
inside the quasi-compact space \( \prod X_i \) where \( \varphi : j \to k \) is a morphism of \( \mathcal{I} \) and \( \Gamma_{\varphi} \subset X_j \times X_k \) is the graph of the corresponding morphism \( X_j \to X_k \). Hence by Lemma \([11.6]\) it suffices to show any finite intersection of these subsets is nonempty. Assume \( \varphi_t : j_t \to k_t \), \( t = 1, \ldots, n \) is a finite collection of morphisms of \( \mathcal{I} \). Since \( \mathcal{I} \) is cofiltered, we can pick an object \( j \) and a morphism \( \psi_t : j \to j_t \) for each \( t \). For each pair \( t, t' \) such that either (a) \( j_t = j_{t'} \), or (b) \( j_t = k_{t'} \), or (c) \( k_t = k_{t'} \), we obtain two morphisms \( j \to l \) with \( l = j_t \) in case (a), (b) or \( l = k_t \) in case (c). Because \( \mathcal{I} \) is cofiltered and since there are finitely many pairs \( (t, t') \) we may choose a map \( j' \to j \) which equalizes these two morphisms for all such pairs \( (t, t') \). Pick an element \( x \in X_{j'} \) and for each \( t \) let \( x_{j_t} \), resp. \( x_{k_t} \) be the image of \( x \) under the morphism \( X_{j'} \to X_j \to X_{j_t} \), resp. \( X_{j'} \to X_j \to X_{k_t} \). For any index \( i \in \text{Ob}(\mathcal{I}) \) which is not equal to \( j_t \) or \( k_t \) for some \( t \) we pick an arbitrary element \( x_i \in X_i \) (using the axiom of choice). Then \( (x_i)_{i \in \text{Ob}(\mathcal{I})} \) is in the intersection
\[
Z_{\varphi_1} \cap \ldots \cap Z_{\varphi_n}
\]
by construction and the proof is complete. \( \square \)

14. Constructible sets

Definition 14.1. Let \( X \) be a topological space. Let \( E \subset X \) be a subset of \( X \).

(1) We say \( E \) is constructible\(^3\) in \( X \) if \( E \) is a finite union of subsets of the form \( U \cap V^c \) where \( U, V \subset X \) are open and retrocompact in \( X \).

(2) We say \( E \) is locally constructible in \( X \) if there exists an open covering \( X = \bigcup V_i \) such that each \( E \cap V_i \) is constructible in \( V_i \).

Lemma 14.2. The collection of constructible sets is closed under finite intersections, finite unions and complements.

\(^3\)In the second edition of EGA I \([GD71]\) this was called a “globally constructible” set and the terminology “constructible” was used for what we call a locally constructible set.
Proof. Note that if \( U_1, U_2 \) are open and retrocompact in \( X \) then so is \( U_1 \cup U_2 \) because the union of two quasi-compact subsets of \( X \) is quasi-compact. It is also true that \( U_1 \cap U_2 \) is retrocompact. Namely, suppose \( U \subset X \) is quasi-compact open, then \( U_2 \cap U \) is quasi-compact because \( U_2 \) is retrocompact in \( X \), and then we conclude \( U_1 \cap (U_2 \cap U) \) is quasi-compact because \( U_1 \) is retrocompact in \( X \). From this it is formal to show that the complement of a constructible set is constructible, that finite unions of constructibles are constructible, and that finite intersections of constructibles are constructible. \( \square \)

**Lemma 14.3.** Let \( f : X \to Y \) be a continuous map of topological spaces. If the inverse image of every retrocompact open subset of \( Y \) is retrocompact in \( X \), then inverse images of constructible sets are constructible.

**Proof.** This is true because \( f^{-1}(U \cap V^c) = f^{-1}(U) \cap f^{-1}(V)^c \), combined with the definition of constructible sets. \( \square \)

**Lemma 14.4.** Let \( U \subset X \) be open. For a constructible set \( E \subset X \) the intersection \( E \cap U \) is constructible in \( U \).

**Proof.** Suppose that \( V \subset X \) is retrocompact open in \( X \). It suffices to show that \( V \cap U \) is retrocompact in \( U \) by Lemma 14.3. To show this let \( W \subset U \) be open and quasi-compact. Then \( W \) is open and quasi-compact in \( X \). Hence \( V \cap W = V \cap U \cap W \) is quasi-compact as \( V \) is retrocompact in \( X \). \( \square \)

**Lemma 14.5.** Let \( U \subset X \) be a retrocompact open. Let \( E \subset U \). If \( E \) is constructible in \( U \), then \( E \) is constructible in \( X \).

**Proof.** Suppose that \( V,W \subset U \) are retrocompact open in \( U \). Then \( V,W \) are retrocompact open in \( X \) (Lemma 11.2). Hence \( V \cap (U \setminus W) = V \cap (X \setminus W) \) is constructible in \( X \). We conclude since every constructible subset of \( U \) is a finite union of subsets of the form \( V \cap (U \setminus W) \). \( \square \)

**Lemma 14.6.** Let \( X \) be a topological space. Let \( E \subset X \) be a subset. Let \( X = V_1 \cup \ldots \cup V_m \) be a finite covering by retrocompact opens. Then \( E \) is constructible in \( X \) if and only if \( E \cap V_j \) is constructible in \( V_j \) for each \( j = 1,\ldots,m \).

**Proof.** If \( E \) is constructible in \( X \), then by Lemma 14.4 we see that \( E \cap V_j \) is constructible in \( V_j \) for all \( j \). Conversely, suppose that \( E \cap V_j \) is constructible in \( V_j \) for each \( j = 1,\ldots,m \). Then \( E = \bigcup E \cap V_j \) is a finite union of constructible sets by Lemma 14.5 and hence constructible. \( \square \)

**Lemma 14.7.** Let \( X \) be a topological space. Let \( Z \subset X \) be a closed subset such that \( X \setminus Z \) is quasi-compact. Then for a constructible set \( E \subset X \) the intersection \( E \cap Z \) is constructible in \( Z \).

**Proof.** Suppose that \( V \subset X \) is retrocompact open in \( X \). It suffices to show that \( V \cap Z \) is retrocompact in \( Z \) by Lemma 14.3. To show this let \( W \subset Z \) be open and quasi-compact. The subset \( W' = W \cup (X \setminus Z) \) is quasi-compact, open, and \( W = Z \cap W' \). Hence \( V \cap Z \cap W = V \cap Z \cap W' \) is a closed subset of the quasi-compact open \( V \cap W' \) as \( V \) is retrocompact in \( X \). Thus \( V \cap Z \cap W \) is quasi-compact by Lemma 11.3. \( \square \)

**Lemma 14.8.** Let \( X \) be a topological space. Let \( T \subset X \) be a subset. Suppose

1. \( T \) is retrocompact in \( X \),

...
(2) quasi-compact opens form a basis for the topology on $X$.

Then for a constructible set $E \subset X$ the intersection $E \cap T$ is constructible in $T$.

**Proof.** Suppose that $V \subset X$ is retrocompact open in $X$. It suffices to show that $V \cap T$ is retrocompact in $T$ by Lemma [14.3]. To show this let $W \subset T$ be open and quasi-compact. By assumption (2) we can find a quasi-compact open $W' \subset X$ such that $W = T \cap W'$ (details omitted). Hence $V \cap T \cap W = V \cap T \cap W'$ is the intersection of $T$ with the quasi-compact open $V \cap W'$ as $V$ is retrocompact in $X$. Thus $V \cap T \cap W$ is quasi-compact. □

**Lemma 14.9.** Let $Z \subset X$ be a closed subset whose complement is retrocompact open. Let $E \subset Z$. If $E$ is constructible in $Z$, then $E$ is constructible in $X$.

**Proof.** Suppose that $V \subset Z$ is retrocompact open in $Z$. Consider the open subset $\tilde{V} = V \cup (X \setminus Z)$ of $X$. Let $W \subset X$ be quasi-compact open. Then

$$W \cap \tilde{V} = (V \cap W) \cup ((X \setminus Z) \cap W).$$

The first part is quasi-compact as $V \cap W = V \cap (Z \cap W)$ and $(Z \cap W)$ is quasi-compact open in $Z$ (Lemma [14.3]) and $V$ is retrocompact in $Z$. The second part is quasi-compact as $(X \setminus Z)$ is retrocompact in $X$. In this way we see that $\tilde{V}$ is retrocompact in $X$. Thus if $V_1, V_2 \subset Z$ are retrocompact open, then

$$V_1 \cap (Z \setminus V_2) = \tilde{V}_1 \cap (X \setminus \tilde{V}_2)$$

is constructible in $X$. We conclude since every constructible subset of $Z$ is a finite union of subsets of the form $V_1 \cap (Z \setminus V_2)$. □

**Lemma 14.10.** Let $X$ be a topological space. Every constructible subset of $X$ is retrocompact.

**Proof.** Let $E = \bigcup_{i=1,\ldots,n} U_i \cap V_i^c$ with $U_i, V_i$ retrocompact open in $X$. Let $W \subset X$ be quasi-compact open. Then $E \cap W = \bigcup_{i=1,\ldots,n} U_i \cap V_i^c \cap W$. Thus it suffices to show that $U \cap V_i^c \cap W$ is quasi-compact if $U, V_i$ are retrocompact open and $W$ is quasi-compact open. This is true because $U \cap V_i^c \cap W$ is a closed subset of the quasi-compact $U \cap W$ so Lemma [14.3] applies. □

Question: Does the following lemma also hold if we assume $X$ is a quasi-compact topological space? Compare with Lemma [14.7]

**Lemma 14.11.** Let $X$ be a topological space. Assume $X$ has a basis consisting of quasi-compact opens. For $E, E'$ constructible in $X$, the intersection $E \cap E'$ is constructible in $E$.

**Proof.** Combine Lemmas [14.8] and [14.10] □

**Lemma 14.12.** Let $X$ be a topological space. Assume $X$ has a basis consisting of quasi-compact opens. Let $E$ be constructible in $X$ and $F \subset E$ constructible in $E$. Then $F$ is constructible in $X$.

**Proof.** Observe that any retrocompact subset $T$ of $X$ has a basis for the induced topology consisting of quasi-compact opens. In particular this holds for any constructible subset (Lemma [14.10]). Write $E = E_1 \cup \ldots \cup E_n$ with $E_i = U_i \cap V_i^c$ where $U_i, V_i \subset X$ are retrocompact open. Note that $E_i = E \cap E_i$ is constructible in $E$ by Lemma [14.11]. Hence $F \cap E_i$ is constructible in $E_i$ by Lemma [14.11]. Thus it
suffices to prove the lemma in case $E = U \cap V^c$ where $U, V \subset X$ are retrocompact open. In this case the inclusion $E \subset X$ is a composition

$$E = U \cap V^c \to U \to X$$

Then we can apply Lemma 14.9 to the first inclusion and Lemma 14.5 to the second.

Lemma 14.13. Let $X$ be a topological space which has a basis for the topology consisting of quasi-compact opens. Let $E \subset X$ be a subset. Let $X = E_1 \cup \ldots \cup E_m$ be a finite covering by constructible subsets. Then $E$ is constructible in $X$ if and only if $E \cap E_j$ is constructible in $E_j$ for each $j = 1, \ldots, m$.


Lemma 14.14. Let $X$ be a topological space. Suppose that $Z \subset X$ is irreducible. Let $E \subset X$ be a finite union of locally closed subsets (e.g. $E$ is constructible). The following are equivalent:

1. The intersection $E \cap Z$ contains an open dense subset of $Z$.
2. The intersection $E \cap Z$ is dense in $Z$.

If $Z$ has a generic point $\xi$, then this is also equivalent to

3. We have $\xi \in E$.

Proof. Write $E = \bigcup U_i \cap Z_i$ as the finite union of intersections of open sets $U_i$ and closed sets $Z_i$. Suppose that $E \cap Z$ is dense in $Z$. Note that the closure of $E \cap Z$ is the union of the closures of the intersections $U_i \cap Z_i \cap Z$. As $Z$ is irreducible we conclude that the closure of $U_i \cap Z_i \cap Z$ is $Z$ for some $i$. Fix such an $i$. It follows that $Z \subset Z_i$ since otherwise the closed subset $Z \cap Z_i$ of $Z$ would not be dense in $Z$. Then $U_i \cap Z_i \cap Z = U_i \cap Z$ is an open nonempty subset of $Z$. Because $Z$ is irreducible, it is open dense. Hence $E \cap Z$ contains an open dense subset of $Z$. The converse is obvious.

Suppose that $\xi \in Z$ is a generic point. Of course if (1) $\Leftrightarrow$ (2) holds, then $\xi \in E$. Conversely, if $\xi \in E$, then $\xi \in U_i \cap Z_i$ for some $i = i_0$. Clearly this implies $Z \subset Z_{i_0}$ and hence $U_{i_0} \cap Z_{i_0} \cap Z = U_{i_0} \cap Z$ is an open not empty subset of $Z$. We conclude as before.

15. Constructible sets and Noetherian spaces

Lemma 15.1. Let $X$ be a Noetherian topological space. The constructible sets in $X$ are precisely the finite unions of locally closed subsets of $X$.

Proof. This follows immediately from Lemma 11.13.

Lemma 15.2. Let $f : X \to Y$ be a continuous map of Noetherian topological spaces. If $E \subset Y$ is constructible in $Y$, then $f^{-1}(E)$ is constructible in $X$.

Proof. Follows immediately from Lemma 15.1 and the definition of a continuous map.

Lemma 15.3. Let $X$ be a Noetherian topological space. Let $E \subset X$ be a subset. The following are equivalent:

1. $E$ is constructible in $X$, and
2. for every irreducible closed $Z \subset X$ the intersection $E \cap Z$ either contains a nonempty open of $Z$ or is not dense in $Z$. 
**Proof.** Assume $E$ is constructible and $Z \subset X$ irreducible closed. Then $E \cap Z$ is constructible in $Z$ by Lemma 15.2. Hence $E \cap Z$ is a finite union of nonempty locally closed subsets $T_i$ of $Z$. Clearly if none of the $T_i$ is open in $Z$, then $E \cap Z$ is not dense in $Z$. In this way we see that (1) implies (2).

Conversely, assume (2) holds. Consider the set $S$ of closed subsets $Y$ of $X$ such that $E \cap Y$ is not constructible in $Y$. If $S \neq \emptyset$, then it has a smallest element $Y$ as $X$ is Noetherian. Let $Y = Y_1 \cup \ldots \cup Y_r$ be the decomposition of $Y$ into its irreducible components, see Lemma 8.2. If $r > 1$, then each $Y_i \cap E$ is constructible in $Y_i$ and hence a finite union of locally closed subsets of $Y_i$. Thus $E \cap Y$ is a finite union of locally closed subsets of $Y$ too and we conclude that $E \cap Y$ is constructible in $Y$ by Lemma 15.1. This is a contradiction and so $r = 1$. If $r = 1$, then $Y$ is irreducible, and by assumption (2) we see that $E \cap Y$ either (a) contains an open $V$ of $Y$ or (b) is not dense in $Y$. In case (a) we see, by minimality of $Y$, that $E \cap (Y \setminus V)$ is a finite union of locally closed subsets of $Y \setminus V$. Thus $E \cap Y$ is a finite union of locally closed subsets of $Y$ and is constructible by Lemma 15.1. This is a contradiction and so we must be in case (b). In case (b) we see that $E \cap Y = E \cap Y'$ for some proper closed subset $Y' \subset Y$. By minimality of $Y$ we see that $E \cap Y'$ is a finite union of locally closed subsets of $Y'$ and we see that $E \cap Y' = E \cap Y$ is a finite union of locally closed subsets of $Y$ and is constructible by Lemma 15.1. This contradiction finishes the proof of the lemma. □

**Lemma 15.4.** Let $X$ be a Noetherian topological space. Let $x \in X$. Let $E \subset X$ be constructible in $X$. The following are equivalent:

1. $E$ is a neighbourhood of $x$, and
2. for every irreducible closed subset $Y$ of $X$ which contains $x$ the intersection $E \cap Y$ is dense in $Y$.

**Proof.** It is clear that (1) implies (2). Assume (2). Consider the set $S$ of closed subsets $Y$ of $X$ containing $x$ such that $E \cap Y$ is not a neighbourhood of $x$ in $Y$. If $S \neq \emptyset$, then it has a minimal element $Y$ as $X$ is Noetherian. Suppose $Y = Y_1 \cup Y_2$ with two smaller nonempty closed subsets $Y_1$, $Y_2$. If $x \in Y_i$ for $i = 1, 2$, then $Y_i \cap E$ is a neighbourhood of $x$ in $Y_i$ and we conclude $Y \cap E$ is a neighbourhood of $x$ in $Y$ which is a contradiction. If $x \in Y_1$ but $x \notin Y_2$ (say), then $Y_1 \cap E$ is a neighbourhood of $x$ in $Y_1$ and hence also in $Y$, which is a contradiction as well. We conclude that $Y$ is irreducible closed. By assumption (2) we see that $E \cap Y$ is dense in $Y$. Thus $E \cap Y$ contains an open $V$ of $Y$, see Lemma 15.3. If $x \in V$ then $E \cap Y$ is a neighbourhood of $x$ in $Y$ which is a contradiction. If $x \notin V$, then $Y' = Y \setminus V$ is a proper closed subset of $Y$ containing $x$. By minimality of $Y$ we see that $E \cap Y'$ contains an open neighbourhood $V' \subset Y'$ of $x$ in $Y'$. But then $V' \cup V$ is an open neighbourhood of $x$ in $Y$ contained in $E$, a contradiction. This contradiction finishes the proof of the lemma. □

**Lemma 15.5.** Let $X$ be a Noetherian topological space. Let $E \subset X$ be a subset. The following are equivalent:

1. $E$ is open in $X$, and
2. for every irreducible closed subset $Y$ of $X$ the intersection $E \cap Y$ is either empty or contains a nonempty open of $Y$.

**Proof.** This follows formally from Lemmas 15.3 and 15.4. □
16. Characterizing proper maps

We include a section discussing the notion of a proper map in usual topology. It turns out that in topology, the notion of being proper is the same as the notion of being universally closed, in the sense that any base change is a closed morphism (not just taking products with spaces). The reason for doing this is that in algebraic geometry we use this notion of universal closedness as the basis for our definition of properness.

Lemma 16.1 (Tube lemma). Let $X$ and $Y$ be topological spaces. Let $A \subset X$ and $B \subset Y$ be quasi-compact subsets. Let $A \times B \subset W \subset X \times Y$ with $W$ open in $X \times Y$. Then there exists opens $A \subset U \subset X$ and $B \subset V \subset Y$ such that $U \times V \subset W$.

Proof. For every $a \in A$ and $b \in B$ there exist opens $U_{(a,b)}$ of $X$ and $V_{(a,b)}$ of $Y$ such that $(a,b) \in U_{(a,b)} \times V_{(a,b)} \subset W$. Fix $b$ and we see there exist a finite number $a_1, \ldots, a_n$ such that $A \subset \bigcup_{a \in I} U_{(a,b)}$. Hence

$A \times \{b\} \subset \bigcup_{a \in I} U_{(a,b)} \times (V_{(a,b)} \cap \cap V_{(a,b)}) \subset W.$

Thus for every $b \in B$ there exists opens $U_b \subset X$ and $V_b \subset Y$ such that $A \times \{b\} \subset U_b \times V_b$. As above there exist a finite number $b_1, \ldots, b_m$ such that $B \subset V_{b_1} \cup \ldots \cup V_{b_m}$. Then we win because $A \times B \subset (U_{b_1} \cap \cap U_{b_m}) \times (V_{b_1} \cup \ldots \cup V_{b_m})$. □

The notation in the following definition may be slightly different from what you are used to.

Definition 16.2. Let $f : X \rightarrow Y$ be a continuous map between topological spaces.

1. We say that the map $f$ is closed iff the image of every closed subset is closed.
2. We say that the map $f$ is proper iff the map $Z \times X \rightarrow Z \times Y$ is closed for any topological space $Z$.
3. We say that the map $f$ is quasi-proper iff the inverse image $f^{-1}(V)$ of every quasi-compact subset $V \subset Y$ is quasi-compact.
4. We say that $f$ is universally closed iff the map $f^\prime : Z \times Y \rightarrow Z$ is closed for any map $g : Z \rightarrow Y$.

The following lemma is useful later.

Lemma 16.3. A topological space $X$ is quasi-compact if and only if the projection map $Z \times X \rightarrow Z$ is closed for any topological space $Z$.

Proof. (See also remark below.) If $X$ is not quasi-compact, there exists an open covering $X = \bigcup_{i \in I} U_i$ such that no finite number of $U_i$ cover $X$. Let $Z$ be the subset of the power set $\mathcal{P}(I)$ of $I$ consisting of $I$ and all nonempty finite subsets of $I$. Define a topology on $Z$ with as a basis for the topology the following sets:

1. All subsets of $Z \setminus \{I\}$.
2. For every finite subset $K$ of $I$ the set $U_K := \{ J \subset I \mid J \in Z, \ K \subset J \}$.

It is left to the reader to verify this is the basis for a topology. Consider the subset of $Z \times X$ defined by the formula

$M = \{(J,x) \mid J \in Z, \ x \in \bigcap_{i \in J} U_i \}$

This is the terminology used in [Bour]. Usually this is what is called “universally closed” in the literature. Thus our notion of proper does not involve any separation conditions.
If \((J, x) \notin M\), then \(x \in U_i\) for some \(i \in J\). Hence \(U_{(i)} \times U_i \subset Z \times X\) is an open subset containing \((J, x)\) and not intersecting \(M\). Hence \(M\) is closed. The projection of \(M\) to \(Z\) is \(Z \setminus \{I\}\) which is not closed. Hence \(Z \times X \to Z\) is not closed.

Assume \(X\) is quasi-compact. Let \(Z\) be a topological space. Let \(M \subset Z \times X\) be closed. Let \(z \in Z\) be a point which is not in \(\text{pr}_1(M)\). By the Tube Lemma \([16.1]\) there exists an open \(U \subset Z\) such that \(U \times X\) is contained in the complement of \(M\). Hence \(\text{pr}_1(M)\) is closed.

\[\text{Remark 16.4.}\] Lemma \([16.3]\) is a combination of \([\text{Bon71, I, p. 75, Lemme 1}]\) and \([\text{Bon71, I, p. 76, Corrolaire 1}]\).

**Theorem 16.5.** Let \(f : X \to Y\) be a continuous map between topological spaces. The following conditions are equivalent:

1. The map \(f\) is quasi-proper and closed.
2. The map \(f\) is proper.
3. The map \(f\) is universally closed.
4. The map \(f\) is closed and \(f^{-1}(y)\) is quasi-compact for any \(y \in Y\).

\[\text{Proof.}\] (See also the remark below.) If the map \(f\) satisfies (1), it automatically satisfies (4) because any single point is quasi-compact.

Assume map \(f\) satisfies (4). We will prove it is universally closed, i.e., (3) holds. Let \(g : Z \to Y\) be a continuous map of topological spaces and consider the diagram

\[
\begin{array}{ccc}
Z \times_Y X & \xrightarrow{f'} & X \\
\downarrow g' & & \downarrow f \\
Z & \xrightarrow{g} & Y
\end{array}
\]

During the proof we will use that \(Z \times_Y X \to Z \times X\) is a homeomorphism onto its image, i.e., that we may identify \(Z \times_Y X\) with the corresponding subset of \(Z \times X\) with the induced topology. The image of \(f' : Z \times_Y X \to Z\) is \(\text{Im}(f') = \{z : g(z) \in f(X)\}\).

Because \(f(X)\) is closed, we see that \(\text{Im}(f')\) is a closed subspace of \(Z\). Consider a closed subset \(P \subset Z \times_Y X\). Let \(z \in Z\), \(z \notin f'(P)\). If \(z \notin \text{Im}(f')\), then \(Z \setminus \text{Im}(f')\) is an open neighbourhood which avoids \(f'(P)\). If \(z\) is in \(\text{Im}(f')\) then \((f')^{-1}\{z\} = \{z\} \times f^{-1}\{g(z)\}\) and \(f^{-1}\{g(z)\}\) is quasi-compact by assumption. Because \(P\) is a closed subset of \(Z \times_Y X\), we have a closed \(P'\) of \(Z \times X\) such that \(P = P' \cap Z \times_Y X\). Since \((f')^{-1}\{z\}\) is a subset of \(P^c = P^c \cup (Z \times_Y X)^c\), and since \((f')^{-1}\{z\}\) is disjoint from \((Z \times_Y X)^c\) we see that \((f')^{-1}\{z\}\) is contained in \(P^c\). We may apply the Tube Lemma \([16.1]\) to \((f')^{-1}\{z\} = \{z\} \times f^{-1}\{g(z)\} \subset (P')^c \subset Z \times X\). This gives \(V \times U\) containing \((f')^{-1}\{z\}\) where \(U\) and \(V\) are open sets in \(X\) and \(Z\) respectively and \(V \times U\) has empty intersection with \(P^c\). Then the set \(V \cap g^{-1}(Y - f(U^c))\) is open in \(Z\) since \(f\) is closed, contains \(z\), and has empty intersection with the image of \(P\). Thus \(f'(P)\) is closed. In other words, the map \(f\) is universally closed.

The implication (3) \(\Rightarrow\) (2) is trivial. Namely, given any topological space \(Z\) consider the projection morphism \(g : Z \times Y \to Y\). Then it is easy to see that \(f'\) is the map \(Z \times X \to Z \times Y\), in other words that \((Z \times Y) \times_Y X = Z \times X\). (This identification is a purely categorical property having nothing to do with topological spaces per se.)

Assume \(f\) satisfies (2). We will prove it satisfies (1). Note that \(f\) is closed as \(f\) can be identified with the map \(\{pt\} \times X \to \{pt\} \times Y\) which is assumed closed.
Choose any quasi-compact subset \( K \subset Y \). Let \( Z \) be any topological space. Because \( Z \times X \rightarrow Z \times Y \) is closed we see the map \( Z \times f^{-1}(K) \rightarrow Z \times K \) is closed (if \( T \) is closed in \( Z \times f^{-1}(K) \), write \( T = Z \times f^{-1}(K) \cap T' \) for some closed \( T' \subset Z \times X \)).

Because \( K \) is quasi-compact, \( K \times Z \rightarrow Z \) is closed by Lemma 16.3. Hence the composition \( Z \times f^{-1}(K) \rightarrow Z \times K \rightarrow Z \) is closed and therefore \( f^{-1}(K) \) must be quasi-compact by Lemma 16.3 again. □

Remark 16.6. Here are some references to the literature. In [Bou71, I, p. 75, Theorem 1] you can find: (2) ⇔ (4). In [Bou71, I, p. 77, Proposition 6] you can find: (2) ⇒ (1). Of course, trivially we have (1) ⇒ (4). Thus (1), (2) and (4) are equivalent. Fan Zhou claimed and proved that (3) and (4) are equivalent; let me know if you find a reference in the literature.

Lemma 16.7. Let \( f : X \rightarrow Y \) be a continuous map of topological spaces. If \( X \) is quasi-compact and \( Y \) is Hausdorff, then \( f \) is proper.

Proof. Since every point of \( Y \) is closed, we see from Lemma 11.3 that the closed subset \( f^{-1}(y) \) of \( X \) is quasi-compact for all \( y \in Y \). Thus, by Theorem 16.5 it suffices to show that \( f \) is closed. If \( E \subset X \) is closed, then it is quasi-compact (Lemma 11.3), hence \( f(E) \subset Y \) is quasi-compact (Lemma 11.7), hence \( f(E) \) is closed in \( Y \) (Lemma 11.4). □

Lemma 16.8. Let \( f : X \rightarrow Y \) be a continuous map of topological spaces. If \( f \) is bijective, \( X \) is quasi-compact, and \( Y \) is Hausdorff, then \( f \) is a homeomorphism.

Proof. This follows immediately from Lemma 16.7 which tells us that \( f \) is closed, i.e., \( f^{-1} \) is continuous. □

17. Jacobson spaces

Definition 17.1. Let \( X \) be a topological space. Let \( X_0 \) be the set of closed points of \( X \). We say that \( X \) is Jacobson if every closed subset \( Z \subset X \) is the closure of \( Z \cap X_0 \).

Note that a topological space \( X \) is Jacobson if and only if every nonempty locally closed subset of \( X \) has a point closed in \( X \).

Let \( X \) be a Jacobson space and let \( X_0 \) be the set of closed points of \( X \) with the induced topology. Clearly, the definition implies that the morphism \( X_0 \rightarrow X \) induces a bijection between the closed subsets of \( X_0 \) and the closed subsets of \( X \). Thus many properties of \( X \) are inherited by \( X_0 \). For example, the Krull dimensions of \( X \) and \( X_0 \) are the same.

Lemma 17.2. Let \( X \) be a topological space. Let \( X_0 \) be the set of closed points of \( X \). Suppose that for every point \( x \in X \) the intersection \( X_0 \cap \{x\} \) is dense in \( \{x\} \). Then \( X \) is Jacobson.

Proof. Let \( Z \) be closed subset of \( X \) and \( U \) be and open subset of \( X \) such that \( U \cap Z \) is nonempty. Then for \( x \in U \cap Z \) we have that \( \{x\} \cap U \) is a nonempty subset of \( Z \cap U \), and by hypothesis it contains a point closed in \( X \) as required. □

Lemma 17.3. Let \( X \) be a Kolmogorov topological space with a basis of quasi-compact open sets. If \( X \) is not Jacobson, then there exists a non-closed point \( x \in X \) such that \( \{x\} \) is locally closed.
Proof. As $X$ is not Jacobson there exists a closed set $Z$ and an open set $U$ in $X$ such that $Z \cap U$ is nonempty and does not contain points closed in $X$. As $X$ has a basis of quasi-compact open sets we may replace $U$ by an open quasi-compact neighborhood of a point in $Z \cap U$ and so we may assume that $U$ is quasi-compact open. By Lemma 11.8 there exists a point $x \in Z \cap U$ closed in $Z \cap U$, and so $\{x\}$ is locally closed but not closed in $X$. □

Lemma 17.4. Let $X$ be a topological space. Let $X = \bigcup U_i$ be an open covering. Then $X$ is Jacobson if and only if each $U_i$ is Jacobson. Moreover, in this case $X_0 = \bigcup U_{i,0}$.

Proof. Let $X$ be a topological space. Let $X_0$ be the set of closed points of $X$. Let $U_{i,0}$ be the set of closed points of $U_i$. Then $X_0 \cap U_i \subset U_{i,0}$ but equality may not hold in general.

First, assume that each $U_i$ is Jacobson. We claim that in this case $X_0 \cap U_i = U_{i,0}$. Namely, suppose that $x \in U_{i,0}$, i.e., $x$ is closed in $U_i$. Let $\{x\}$ be the closure in $X$. Consider $\{x\} \cap U_j$. If $x \notin U_j$, then $\{x\} \cap U_j = \emptyset$. If $x \in U_j$, then $U_i \cap U_j \subset U_j$ is an open subset of $U_j$ containing $x$. Let $T' = U_j \setminus U_i \cap U_j$ and $T = \{x\} \cup T'$. Then $T, T'$ are closed subsets of $U_j$ and $T$ contains $x$. As $U_j$ is Jacobson we see that the closed points of $U_j$ are dense in $T$. Because $T = \{x\} \cup T'$ this can only be the case if $x$ is closed in $U_j$. Hence $\{x\} \cap U_j = \{x\}$. We conclude that $\{x\} = \{x\}$ as desired.

Let $Z \subset X$ be a closed subset (still assuming each $U_i$ is Jacobson). Since now we know that $X_0 \cap Z \cap U_i = U_{i,0} \cap Z$ are dense in $Z \cap U_i$ it follows immediately that $X_0 \cap Z$ is dense in $Z$.

Conversely, assume that $X$ is Jacobson. Let $Z \subset U_i$ be closed. Then $X_0 \cap Z$ is dense in $Z$. Hence also $X_0 \cap Z$ is dense in $Z$, because $Z \setminus Z$ is closed. As $X_0 \cap U_i \subset U_{i,0}$ we see that $U_{i,0} \cap Z$ is dense in $Z$. Thus $U_i$ is Jacobson as desired. □

Lemma 17.5. Let $X$ be Jacobson. The following types of subsets $T \subset X$ are Jacobson:

1. Open subspaces.
2. Closed subspaces.
3. Locally closed subspaces.
4. Unions of locally closed subspaces.
5. Constructible sets.
6. Any subset $T \subset X$ which locally on $X$ is a union of locally closed subsets.

In each of these cases closed points of $T$ are closed in $X$.

Proof. Let $X_0$ be the set of closed points of $X$. For any subset $T \subset X$ we let $(\ast)$ denote the property:

$(\ast)$ Every nonempty locally closed subset of $T$ has a point closed in $X$.

Note that always $X_0 \cap T \subset T_0$. Hence property $(\ast)$ implies that $T$ is Jacobson. In addition it clearly implies that every closed point of $T$ is closed in $X$.

Suppose that $T = \bigcup T_i$ with $T_i$ locally closed in $X$. Take $A \subset T$ a locally closed nonempty subset in $T$, then there exists a $T_i$ such that $A \cap T_i$ is nonempty, it is locally closed in $T_i$ and so in $X$. As $X$ is Jacobson $A$ has a point closed in $X$. □

Lemma 17.6. A finite Jacobson space is discrete.
Proof. If $X$ is finite Jacobson, $X_0 \subset X$ the subset of closed points, then, on the one hand, $\overline{X_0} = X$. On the other hand, $X$, and hence $X_0$ is finite, so $X_0 = \{x_1, \ldots, x_n\} = \bigcup_{i=1,\ldots,n} \{x_i\}$ is a finite union of closed sets, hence closed, so $X = \overline{X_0} = X_0$. Every point is closed, and by finiteness, every point is open. □

Lemma 17.7. Suppose $X$ is a Jacobson topological space. Let $X_0$ be the set of closed points of $X$. There is a bijective, inclusion preserving correspondence

\[ \{\text{finite unions loc. closed subsets of } X\} \leftrightarrow \{\text{finite unions loc. closed subsets of } X_0\}\]

given by $E \mapsto E \cap X_0$. This correspondence preserves the subsets of locally closed, of open and of closed subsets.

Proof. We just prove that the correspondence $E \mapsto E \cap X_0$ is injective. Indeed if $E \neq E'$ then without loss of generality $E \setminus E'$ is nonempty, and it is a finite union of locally closed sets (details omitted). As $X$ is Jacobson, we see that $(E \setminus E') \cap X_0 = E \cap X_0 \setminus E' \cap X_0$ is not empty. □

Lemma 17.8. Suppose $X$ is a Jacobson topological space. Let $X_0$ be the set of closed points of $X$. There is a bijective, inclusion preserving correspondence

\[ \{\text{constructible subsets of } X\} \leftrightarrow \{\text{constructible subsets of } X_0\}\]

given by $E \mapsto E \cap X_0$. This correspondence preserves the subset of retrocompact open subsets, as well as complements of these.

Proof. From Lemma 17.7 above, we just have to see that if $U$ is open in $X$ then $U \cap X_0$ is retrocompact in $X_0$ if and only if $U$ is retrocompact in $X$. This follows if we prove that for $U$ open in $X$ then $U \cap X_0$ is quasi-compact if and only if $U$ is quasi-compact. From Lemma 17.5 it follows that we may replace $X$ by $U$ and assume that $U = X$. Finally notice that any collection of opens $U$ of $X$ cover $X$ if and only if they cover $X_0$, using the Jacobson property of $X$ in the closed $X \setminus \bigcup U$ to find a point in $X_0$ if it were nonempty. □

18. Specialization

Definition 18.1. Let $X$ be a topological space.

1. If $x, x' \in X$ then we say $x$ is a specialization of $x'$, or $x'$ is a generalization of $x$ if $x \in \overline{\{x'\}}$. Notation: $x' \leadsto x$.
2. A subset $T \subset X$ is stable under specialization if for all $x' \in T$ and every specialization $x' \leadsto x$ we have $x \in T$.
3. A subset $T \subset X$ is stable under generalization if for all $x \in T$ and every generalization $x' \leadsto x$ we have $x' \in T$.

Lemma 18.2. Let $X$ be a topological space.

1. Any closed subset of $X$ is stable under specialization.
2. Any open subset of $X$ is stable under generalization.
3. A subset $T \subset X$ is stable under specialization if and only if the complement $T^c$ is stable under generalization.

Proof. Omitted. □

Definition 18.3. Let $f : X \rightarrow Y$ be a continuous map of topological spaces.
We say that specializations lift along \( f \) or that \( f \) is specializing if given \( y' \leadsto y \) in \( Y \) and any \( x' \in X \) with \( f(x') = y' \) there exists a specialization \( x' \leadsto x \) in \( X \) such that \( f(x) = y \).

We say that generalizations lift along \( f \) or that \( f \) is generalizing if given \( y' \leadsto y \) in \( Y \) and any \( x \in X \) with \( f(x) = y \) there exists a generalization \( x' \leadsto x \) in \( X \) such that \( f(x') = y' \).

**Lemma 18.4.** Suppose \( f : X \to Y \) and \( g : Y \to Z \) are continuous maps of topological spaces. If specializations lift along both \( f \) and \( g \) then specializations lift along \( g \circ f \). Similarly for “generalizations lift along”.

**Proof.** Omitted. \( \square \)

**Lemma 18.5.** Let \( f : X \to Y \) be a continuous map of topological spaces.

1. If specializations lift along \( f \), and if \( T \subseteq X \) is stable under specialization, then \( f(T) \subseteq Y \) is stable under specialization.
2. If generalizations lift along \( f \), and if \( T \subseteq X \) is stable under generalization, then \( f(T) \subseteq Y \) is stable under generalization.

**Proof.** Omitted. \( \square \)

**Lemma 18.6.** Let \( f : X \to Y \) be a continuous map of topological spaces.

1. If \( f \) is closed then specializations lift along \( f \).
2. If \( f \) is open, \( X \) is a Noetherian topological space, each irreducible closed subset of \( X \) has a generic point, and \( Y \) is Kolmogorov then generalizations lift along \( f \).

**Proof.** Assume \( f \) is closed. Let \( y' \leadsto y \) in \( Y \) and any \( x' \in X \) with \( f(x') = y' \) be given. Consider the closed subset \( T = \{ x' \} \subseteq X \). Then \( f(T) \subseteq Y \) is a closed subset, and \( y' \in f(T) \). Hence also \( y \in f(T) \). Hence \( y = f(x) \) with \( x \in T \), i.e., \( x' \leadsto x \).

Assume \( f \) is open, \( X \) Noetherian, every irreducible closed subset of \( X \) has a generic point, and \( Y \) is Kolmogorov. Let \( y' \leadsto y \) in \( Y \) and any \( x \in X \) with \( f(x) = y \) be given. Consider \( T = f^{-1}(\{ y' \}) \subseteq X \). Take an open neighbourhood \( x \in U \subseteq X \) of \( x \). Then \( f(U) \subseteq Y \) is open and \( y \in f(U) \). Hence also \( y' \in f(U) \). In other words, \( T \cap U \neq \emptyset \). This proves that \( x \in T \). Since \( X \) is Noetherian, \( T \) is Noetherian (Lemma 8.2). Hence it has a decomposition \( T = T_1 \cup \ldots \cup T_n \) into irreducible components. Then correspondingly \( T = T_1 \cup \ldots \cup T_n \). By the above \( x \in T_i \) for some \( i \). By assumption there exists a generic point \( x' \in T_i \), and we see that \( x' \leadsto x \). As \( x' \in T \) we see that \( f(x') \in \{ y' \} \). Note that \( f(T_i) = f(\{ x' \}) \subseteq f(\{ x' \}) \). If \( f(x') \neq y' \), then since \( Y \) is Kolmogorov \( f(x') \) is not a generic point of the irreducible closed subset \( \{ y' \} \) and the inclusion \( f(x') \in \{ y' \} \) is strict, i.e., \( y' \notin f(T_i) \). This contradicts the fact that \( f(T_i) = \{ y' \} \). Hence \( f(x') = y' \) and we win. \( \square \)

**Lemma 18.7.** Suppose that \( s, t : R \to U \) and \( \pi : U \to X \) are continuous maps of topological spaces such that

1. \( \pi \) is open,
2. \( U \) is sober,
3. \( s, t \) have finite fibres,
4. generalizations lift along \( s, t \),
(5) \((t, s)(R) \subset U \times U\) is an equivalence relation on \(U\) and \(X\) is the quotient of \(U\) by this equivalence relation (as a set).

Then \(X\) is Kolmogorov.

**Proof.** Properties (3) and (5) imply that a point \(x\) corresponds to an finite equivalence class \(\{u_1, \ldots, u_n\} \subset U\) of the equivalence relation. Suppose that \(x' \in X\) is a second point corresponding to the equivalence class \(\{u'_1, \ldots, u'_m\} \subset U\). Suppose that \(u_i \leadsto u'_j\) for some \(i, j\). Then for any \(r' \in R\) with \(s(r') = u_i\) by (4) we can find \(r \leadsto r'\) with \(s(r) = u_i\). Hence \(t(r) \leadsto t(r')\). Since \(\{u'_1, \ldots, u'_m\} = t(s^{-1}(\{u'_j\}))\) we conclude that every element of \(\{u'_1, \ldots, u'_m\}\) is the specialization of an element of \(\{u_1, \ldots, u_n\}\). Thus \(\{u_1\} \cup \ldots \cup \{u_n\}\) is a union of equivalence classes, hence of the form \(\pi^{-1}(Z)\) for some subset \(Z \subset X\). By (1) we see that \(Z\) is closed in \(X\) and in fact \(Z = \{x\}\) because \(\pi(\{u_i\}) \subset \{x\}\) for each \(i\). In other words, \(x \leadsto x'\) if and only if some lift of \(x\) in \(U\) specializes to some lift of \(x'\) in \(U\), if and only if every lift of \(x'\) in \(U\) is a specialization of some lift of \(x\) in \(U\).

Suppose that both \(x \leadsto x'\) and \(x' \leadsto x\). Say \(x\) corresponds to \(\{u_1, \ldots, u_n\}\) and \(x'\) corresponds to \(\{u'_1, \ldots, u'_m\}\) as above. Then, by the results of the preceding paragraph, we can find a sequence

\[
\ldots \leadsto u'_{j_3} \leadsto u_{i_3} \leadsto u'_{j_2} \leadsto u_{i_2} \leadsto u'_{j_1} \leadsto u_{i_1}
\]

which must repeat, hence by (2) we conclude that \(\{u_1, \ldots, u_n\} = \{u'_1, \ldots, u'_m\}\), i.e., \(x = x'\). Thus \(X\) is Kolmogorov. \(\square\)

**Lemma 18.8.** Let \(f : X \rightarrow Y\) be a morphism of topological spaces. Suppose that \(Y\) is a sober topological space, and \(f\) is surjective. If either specializations or generalizations lift along \(f\), then \(\dim(X) \geq \dim(Y)\).

**Proof.** Assume specializations lift along \(f\). Let \(Z_0 \subset Z_1 \subset \ldots \subset Z_e \subset Y\) be a chain of irreducible closed subsets of \(X\). Let \(\xi_e \in X\) be a point mapping to the generic point of \(Z_e\). By assumption there exists a specialization \(\xi_e \leadsto \xi_{e-1}\) in \(X\) such that \(\xi_{e-1}\) maps to the generic point of \(Z_{e-1}\). Continuing in this manner we find a sequence of specializations

\(\xi_e \leadsto \xi_{e-1} \leadsto \ldots \leadsto \xi_0\)

with \(\xi_i\) mapping to the generic point of \(Z_i\). This clearly implies the sequence of irreducible closed subsets

\(\{\xi_0\} \subset \{\xi_1\} \subset \ldots \subset \{\xi_e\}\)

is a chain of length \(e\) in \(X\). The case when generalizations lift along \(f\) is similar. \(\square\)

**Lemma 18.9.** Let \(X\) be a Noetherian sober topological space. Let \(E \subset X\) be a subset of \(X\).

1. If \(E\) is constructible and stable under specialization, then \(E\) is closed.
2. If \(E\) is constructible and stable under generalization, then \(E\) is open.

**Proof.** Let \(E\) be constructible and stable under generalization. Let \(Y \subset X\) be an irreducible closed subset with generic point \(\xi \in Y\). If \(E \cap Y\) is nonempty, then it contains \(\xi\) (by stability under generalization) and hence is dense in \(Y\), hence it contains a nonempty open of \(Y\), see Lemma \([15.3]\) Thus \(E\) is open by Lemma \([15.5]\). This proves (2). To prove (1) apply (2) to the complement of \(E\) in \(X\). \(\square\)
19. Dimension functions

It scarcely makes sense to consider dimension functions unless the space considered is sober (Definition 7.2). Thus the definition below can be improved by considering the sober topological space associated to \( X \). Since the underlying topological space of a scheme is sober we do not bother with this improvement.

**Definition 19.1.** Let \( X \) be a topological space.

1. Let \( x, y \in X \), \( x \neq y \). Suppose \( x \mathbin{\rightsquigarrow} y \), that is \( y \) is a specialization of \( x \). We say \( y \) is an immediate specialization of \( x \) if there is no \( z \in X \setminus \{x, y\} \) with \( x \mathbin{\rightsquigarrow} z \) and \( z \mathbin{\rightsquigarrow} y \).

2. A map \( \delta : X \rightarrow \mathbb{Z} \) is called a dimension function\footnote{This is likely nonstandard notation. This notion is usually introduced only for (locally) Noetherian schemes, in which case condition (a) is implied by (b).} if
   
   (a) whenever \( x \mathbin{\rightsquigarrow} y \) and \( x \neq y \) we have \( \delta(x) > \delta(y) \), and
   
   (b) for every immediate specialization \( x \mathbin{\rightsquigarrow} y \) in \( X \) we have \( \delta(x) = \delta(y) + 1 \).

It is clear that if \( \delta \) is a dimension function, then so is \( \delta + t \) for any \( t \in \mathbb{Z} \). Here is an fun lemma.

**Lemma 19.2.** Let \( X \) be a topological space. If \( X \) is sober and has a dimension function, then \( X \) is catenary. Moreover, for any \( x \mathbin{\rightsquigarrow} y \) we have

\[
\delta(x) - \delta(y) = \text{codim} \left( \left\{ y \right\}, \left\{ x \right\} \right).
\]

**Proof.** Suppose \( Y \subset Y' \subset X \) are irreducible closed subsets. Let \( \xi \in Y \), \( \xi' \in Y' \) be their generic points. Then we see immediately from the definitions that

\[
\text{codim}(Y, Y') \leq \delta(\xi) - \delta(\xi') < \infty.
\]

In fact the first inequality is an equality. Namely, suppose

\[
Y = Y_0 \subset Y_1 \subset \ldots \subset Y_e = Y'
\]

is any maximal chain of irreducible closed subsets. Let \( \xi_i \in Y_i \) denote the generic point. Then we see that \( \xi_i \mathbin{\rightsquigarrow} \xi_{i+1} \) is an immediate specialization. Hence we see that \( e = \delta(\xi) - \delta(\xi') \) as desired. This also proves the last statement of the lemma. \( \square \)

**Lemma 19.3.** Let \( X \) be a topological space. Let \( \delta, \delta' \) be two dimension functions on \( X \). If \( X \) is locally Noetherian and sober then \( \delta - \delta' \) is locally constant on \( X \).

**Proof.** Let \( x \in X \) be a point. We will show that \( \delta - \delta' \) is constant in a neighbourhood of \( x \). We may replace \( X \) by an open neighbourhood of \( x \) in \( X \) which is Noetherian. Hence we may assume \( X \) is Noetherian and sober. Let \( Z_1, \ldots, Z_r \) be the irreducible components of \( X \) passing through \( x \). (There are finitely many as \( X \) is Noetherian, see Lemma 8.2.) Let \( \xi_i \in Z_i \) be the generic point. Note \( Z_1 \cup \ldots \cup Z_r \) is a neighbourhood of \( x \) in \( X \) (not necessarily closed). We claim that \( \delta - \delta' \) is constant on \( Z_1 \cup \ldots \cup Z_r \). Namely, if \( y \in Z_i \), then

\[
\delta(x) - \delta(y) = \delta(x) - \delta(\xi_i) + \delta(\xi_i) - \delta(y) = -\text{codim}(\{x\}, Z_i) + \text{codim}(\{y\}, Z_i)
\]

by Lemma 19.2. Similarly for \( \delta' \). Whence the result. \( \square \)

**Lemma 19.4.** Let \( X \) be locally Noetherian, sober and catenary. Then any point has an open neighbourhood \( U \subset X \) which has a dimension function.
Proof. We will use repeatedly that an open subspace of a catenary space is catenary, see Lemma 10.5 and that a Noetherian topological space has finitely many irreducible components, see Lemma 8.2. In the proof of Lemma 19.3 we saw how to construct such a function. Namely, we first replace $X$ by a Noetherian open neighbourhood of $x$. Next, we let $Z_1, \ldots, Z_r \subset X$ be the irreducible components of $X$. Let

$$Z_i \cap Z_j = \bigcup Z_{ijk}$$

be the decomposition into irreducible components. We replace $X$ by

$$X \setminus \left( \bigcup_{x \notin Z_i} Z_i \cup \bigcup_{x \notin Z_{ijk}} Z_{ijk} \right)$$

so that we may assume $x \in Z_i$ for all $i$ and $x \in Z_{ijk}$ for all $i, j, k$. For $y \in X$ choose any $i$ such that $y \in Z_i$ and set

$$\delta(y) = -\text{codim}(\{x\}, Z_i) + \text{codim}(\{y\}, Z_i).$$

We claim this is a dimension function. First we show that it is well defined, i.e., independent of the choice of $i$. Namely, suppose that $y \in Z_{ijk}$ for some $i, j, k$. Then we have (using Lemma 10.6)

$$\delta(y) = -\text{codim}(\{x\}, Z_i) + \text{codim}(\{y\}, Z_i)$$

$$= -\text{codim}(\{x\}, Z_{ijk}) - \text{codim}(Z_{ijk}, Z_i) + \text{codim}(\{y\}, Z_{ijk}) + \text{codim}(Z_{ijk}, Z_i)$$

$$= -\text{codim}(\{x\}, Z_{ijk}) + \text{codim}(\{y\}, Z_{ijk})$$

which is symmetric in $i$ and $j$. We omit the proof that it is a dimension function. □

Remark 19.5. Combining Lemmas 19.3 and 19.4 we see that on a catenary, locally Noetherian, sober topological space the obstruction to having a dimension function is an element of $H^1(X, Z)$.

20. Nowhere dense sets

Definition 20.1. Let $X$ be a topological space.

1. Given a subset $T \subset X$ the interior of $T$ is the largest open subset of $X$ contained in $T$.

2. A subset $T \subset X$ is called nowhere dense if the closure of $T$ has empty interior.

Lemma 20.2. Let $X$ be a topological space. The union of a finite number of nowhere dense sets is a nowhere dense set.

Proof. Omitted. □

Lemma 20.3. Let $X$ be a topological space. Let $U \subset X$ be an open. Let $T \subset U$ be a subset. If $T$ is nowhere dense in $U$, then $T$ is nowhere dense in $X$.

Proof. Assume $T$ is nowhere dense in $U$. Suppose that $x \in X$ is an interior point of the closure $\overline{T}$ of $T$ in $X$. Say $x \in V \subset T$ with $V \subset X$ open in $X$. Note that $T \cap U$ is the closure of $T$ in $U$. Hence the interior of $T \cap U$ being empty implies $V \cap U = \emptyset$. Thus $x$ cannot be in the closure of $U$, a fortiori cannot be in the closure of $T$, a contradiction. □
Lemma 20.4. Let $X$ be a topological space. Let $X = \bigcup U_i$ be an open covering. Let $T \subset X$ be a subset. If $T \cap U_i$ is nowhere dense in $U_i$ for all $i$, then $T$ is nowhere dense in $X$.

Proof. Omitted. (Hint: closure commutes with intersecting with opens.)

Lemma 20.5. Let $f : X \to Y$ be a continuous map of topological spaces. Let $T \subset X$ be a subset. If $f$ is a homeomorphism of $X$ onto a closed subset of $Y$ and $T$ is nowhere dense in $X$, then also $f(T)$ is nowhere dense in $Y$.

Proof. Omitted.

Lemma 20.6. Let $f : X \to Y$ be a continuous map of topological spaces. Let $T \subset Y$ be a subset. If $f$ is open and $T$ is a closed nowhere dense subset of $Y$, then also $f^{-1}(T)$ is a closed nowhere dense subset of $X$. If $f$ is surjective and open, then $T$ is closed nowhere dense if and only if $f^{-1}(T)$ is closed nowhere dense.

Proof. Omitted. (Hint: In the first case the interior of $f^{-1}(T)$ maps into the interior of $T$, and in the second case the interior of $f^{-1}(T)$ maps onto the interior of $T$.)

21. Profinite spaces

Here is the definition.

Definition 21.1. A topological space is profinite if it is homeomorphic to a limit of a diagram of finite discrete spaces.

This is not the most convenient characterization of a profinite space.

Lemma 21.2. Let $X$ be a topological space. The following are equivalent

1. $X$ is a profinite space, and
2. $X$ is Hausdorff, quasi-compact, and totally disconnected.

If this is true, then $X$ is a cofiltered limit of finite discrete spaces.

Proof. Assume (1). Choose a diagram $i \mapsto X_i$ of finite discrete spaces such that $X = \lim X_i$. As each $X_i$ is Hausdorff and quasi-compact we find that $X$ is quasi-compact by Lemma [13.5]. If $x, x' \in X$ are distinct points, then $x$ and $x'$ map to distinct points in some $X_i$. Hence $x$ and $x'$ have disjoint open neighbourhoods, i.e., $X$ is Hausdorff. In exactly the same way we see that $X$ is totally disconnected.

Assume (2). Let $\mathcal{I}$ be the set of finite disjoint union decompositions $X = \bigsqcup_{i \in \mathcal{I}} U_i$ with $U_i$ open (and closed). For each $I \in \mathcal{I}$ there is a continuous map $X \to I$ sending a point of $U_i$ to $i$. We define a partial ordering: $I \leq I'$ for $I, I' \in \mathcal{I}$ if and only if the covering corresponding to $I'$ refines the covering corresponding to $I$. In this case we obtain a canonical map $I' \to I$. In other words we obtain an inverse system of finite discrete spaces over $\mathcal{I}$. The maps $X \to I$ fit together and we obtain a continuous map

$$X \longrightarrow \lim_{I \in \mathcal{I}} I$$

We claim this map is a homeomorphism, which finishes the proof. (The final assertion follows too as the partially ordered set $\mathcal{I}$ is directed: given two disjoint union decompositions of $X$ we can find a third refining either.) Namely, the map is injective as $X$ is totally disconnected and hence $\{x\}$ is the intersection of all open and closed subsets of $X$ containing $x$ (Lemma [11.11]) and the map is surjective by Lemma [11.6]. By Lemma [16.8] the map is a homeomorphism. □
Lemma 21.3. Let $X$ be a profinite space. Every open covering of $X$ has a refinement by a finite covering $X = \coprod U_i$ with $U_i$ open and closed.

Proof. Write $X = \lim X_i$ as a limit of an inverse system of finite discrete spaces over a directed partially ordered set $I$ (Lemma 21.2). Denote $f_i : X \to X_i$ the projection. For every point $x = (x_i) \in X$ a fundamental system of open neighbourhoods is the collection $f_i^{-1}(\{x_i\})$. Thus, as $X$ is quasi-compact, we may assume we have an open covering $X = f_i^{-1}(\{x_i\}) \cup \ldots \cup f_n^{-1}(\{x_n\})$ Choose $i \in I$ with $i \geq i_j$ for $j = 1, \ldots, n$ (this is possible as $I$ is a directed partially ordered set). Then we see that the covering $X = \coprod_{t \in X_i} f_i^{-1}(\{t\})$ refines the given covering and is of the desired form. □

Lemma 21.4. Let $X$ be a topological space. If $X$ is quasi-compact and every connected component of $X$ is the intersection of the open and closed subsets containing it, then $\pi_0(X)$ is a profinite space.

Proof. We will use Lemma 21.2 to prove this. Since $\pi_0(X)$ is the image of a quasi-compact space it is quasi-compact (Lemma 11.7). It is totally disconnected by construction (Lemma 6.8). Let $C, D \subset X$ be distinct connected components of $X$. Write $C = \bigcap U_\alpha$ as the intersection of the open and closed subsets of $X$ containing $C$. Any finite intersection of $U_\alpha$’s is another. Since $\bigcap U_\alpha \cap D = \emptyset$ we conclude that $U_\alpha \cap D = \emptyset$ for some $\alpha$ (use Lemmas 6.3 [11.3] and 11.6). Since $U_\alpha$ is open and closed, it is the union of the connected components it contains, i.e., $U_\alpha$ is the inverse image of some open and closed subset $V_\alpha \subset \pi_0(X)$. This proves that the points corresponding to $C$ and $D$ are contained in disjoint open subsets, i.e., $\pi_0(X)$ is Hausdorff. □

22. Spectral spaces

The material in this section is taken from [Hoc69] and [Hoc67]. In his thesis Hochster proves (among other things) that the spectral spaces are exactly the topological spaces that occur as the spectrum of a ring.

Definition 22.1. A topological space $X$ is called spectral if it is sober, quasi-compact, the intersection of two quasi-compact opens is quasi-compact, and the collection of quasi-compact opens forms a basis for the topology. A continuous map $f : X \to Y$ of spectral spaces is called spectral if the inverse image of a quasi-compact open is quasi-compact.

In other words a continuous map of spectral spaces is spectral if and only if it is quasi-compact (Definition 11.1).

Let $X$ be a spectral space. The constructible topology on $X$ is the topology which has as a subbase of opens the sets $U$ and $U^c$ where $U$ is a quasi-compact open of $X$. Note that since $X$ is spectral an open $U \subset X$ is retrocompact if and only if $U$ is quasi-compact. Hence the constructible topology can also be characterized as the coarsest topology such that every constructible subset of $X$ is both open and closed. Since the collection of quasi-compact opens is a basis for the topology on $X$ we see that the constructible topology is stronger than the given topology on $X$. 
Lemma 22.2. Let $X$ be a spectral space. The constructible topology is Hausdorff and quasi-compact.

**Proof.** Since the collection of all quasi-compact opens forms a basis for the topology on $X$ and $X$ is sober, it is clear that $X$ is Hausdorff in the constructible topology.

Let $B$ be the collection of subsets $B \subset X$ with $B$ either quasi-compact open or closed with quasi-compact complement. If $B \in B$ then $B' \in B$. It suffices to show every covering $X = \bigcup_{i \in I} B_i$ with $B_i \in B$ has a finite refinement, see Lemma 11.15. Taking complements we see that we have to show that any family $\{B_i\}_{i \in I}$ of elements of $B$ such that $B_i \cap \ldots \cap B_{i_n} \neq \emptyset$ for all $n$ and all $i_1, \ldots, i_n \in I$ has a common point of intersection. We may and do assume $B_i \neq B_i'$ for $i \neq i'$.

To get a contradiction assume $\{B_i\}_{i \in I}$ is a family of elements of $B$ having the finite intersection property but empty intersection. An application of Zorn's lemma shows that we may assume our family is maximal (details omitted). Let $I' \subset I$ be those indices such that $B_i$ is closed and set $Z = \bigcap_{i \in I'} B_i$. This is a closed subset of $X$. If $Z$ is reducible, then we can write $Z = Z' \cup Z''$ as a union of two closed subsets, neither equal to $Z$. This means in particular that we can find a quasi-compact open $U' \subset X$ meeting $Z'$ but not $Z''$. Similarly, we can find a quasi-compact open $U'' \subset X$ meeting $Z''$ but not $Z'$. Set $B' = X \setminus U'$ and $B'' = X \setminus U''$. Note that $Z'' \subset B'$ and $Z' \subset B''$. If there exist a finite number of indices $i_1, \ldots, i_n \in I$ such that $B_i' \cap \ldots \cap B_{i_n}' = \emptyset$ as well as a finite number of indices $j_1, \ldots, j_m \in I$ such that $B_i'' \cap B_{j_1}'' \cap \ldots \cap B_{j_m}'' = \emptyset$ then we find that $Z \cap B_{i_1} \cap \ldots \cap B_{i_n} \cap B_{j_1} \cap \ldots \cap B_{j_m} = \emptyset$. However, the set $B_{i_1} \cap \ldots \cap B_{i_n} \cap B_{j_1} \cap \ldots \cap B_{j_m}$ is quasi-compact hence we would find a finite number of indices $i'_1, \ldots, i'_l \in I'$ with $B_{i'_1} \cap \ldots \cap B_{i'_l} \cap B_{j'_1} \cap \ldots \cap B_{j'_l} = \emptyset$, a contradiction. Thus we see that we may add either $B'$ or $B''$ to the given family contradicting maximality. We conclude that $Z$ is irreducible. However, this leads to a contradiction as well, as now every nonempty (by the same argument as above) open $Z \cap B_i$ for $i \in I \setminus I'$ contains the unique generic point of $Z$. This contradiction proves the lemma.

Lemma 22.3. Let $f : X \to Y$ be a spectral map of spectral spaces. Then

1. $f$ is continuous in the constructible topology,
2. the fibres of $f$ are quasi-compact, and
3. the image is closed in the constructible topology.

**Proof.** Let $X'$ and $Y'$ denote $X$ and $Y$ endowed with the constructible topology which are quasi-compact Hausdorff spaces by Lemma 22.2. Part (1) says $X' \to Y'$ is continuous and follows immediately from the definitions. Part (3) follows as $f(X')$ is a quasi-compact subset of the Hausdorff space $Y'$, see Lemma 11.4. We have a commutative diagram

$$
\begin{array}{ccc}
X' & \longrightarrow & X \\
\downarrow & & \downarrow \\
Y' & \longrightarrow & Y
\end{array}
$$

of continuous maps of topological spaces. Since $Y'$ is Hausdorff we see that the fibres $X'_y$ are closed in $X'$. As $X'$ is quasi-compact we see that $X'_y$ is quasi-compact (Lemma 11.3). As $X'_y \to X_y$ is a surjective continuous map we conclude that $X_y$ is quasi-compact (Lemma 11.7).
**Lemma 22.4.** Let $X$ be a spectral space. Let $E \subset X$ be closed in the constructible topology (for example constructible or closed). Then $E$ with the induced topology is a spectral space.

**Proof.** Let $Z \subset E$ be a closed irreducible subset. Let $\eta$ be the generic point of the closure $\overline{Z}$ of $Z$ in $X$. To prove that $E$ is sober, we show that $\eta \in E$. If not, then since $E$ is closed in the constructible topology, there exists a constructible subset $F \subset X$ such that $\eta \in F$ and $F \cap E = \emptyset$. By Lemma 14.14 this implies $F \cap \overline{Z}$ contains a nonempty open subset of $\overline{Z}$. But this is impossible as $\overline{Z}$ is the closure of $Z$ and $Z \cap F = \emptyset$.

Since $E$ is closed in the constructible topology, it is quasi-compact in the constructible topology (Lemmas 11.3 and 22.2). Hence a fortiori it is quasi-compact in the topology coming from $X$. If $U \subset X$ is a quasi-compact open, then $E \cap U$ is closed in the constructible topology, hence quasi-compact (as seen above). It follows that the quasi-compact open subsets of $E$ are the intersections $E \cap U$ with $U$ quasi-compact open in $X$. These form a basis for the topology. Finally, given two $U, U' \subset X$ quasi-compact opens, the intersection $(E \cap U) \cap (E \cap U') = E \cap (U \cap U')$ and $U \cap U'$ is quasi-compact as $X$ is spectral. This finishes the proof. □

**Lemma 22.5.** Let $X$ be a spectral space. Let $E \subset X$ be a constructible subset.

1. If $x \in E$, then $x$ is the specialization of a point of $E$.
2. If $E$ is stable under specialization, then $E$ is closed.
3. If $E$ is stable under generalization, then $E$ is open.

**Proof.** Proof of (1). Let $x \in E$. Let $\{U_i\}$ be the set of quasi-compact open neighbourhoods of $x$. A finite intersection of the $U_i$ is another one. The intersection $U_i \cap E$ is nonempty for all $i$. Since the subsets $U_i \cap E$ are closed in the constructible topology we see that $\bigcap(U_i \cap E)$ is nonempty by Lemma 22.2 and Lemma 11.6. Since $X$ is a sober space and $\{U_i\}$ is a fundamental system of open neighbourhoods of $x$, we see that $\bigcap U_i$ is the set of generalizations of $x$. Thus $x$ is a specialization of a point of $E$.

Part (2) is immediate from (1).

Proof of (3). Assume $E$ is stable under generalization. The complement of $E$ is constructible (Lemma 14.2) and closed under specialization (Lemma 18.2). Hence the complement is closed by (2), i.e., $E$ is open. □

**Lemma 22.6.** Let $X$ be a spectral space. Let $x, y \in X$. Then either there exists a third point specializing to both $x$ and $y$, or there exist disjoint open neighbourhoods containing $x$ and $y$.

**Proof.** Let $\{U_i\}$ be the set of quasi-compact open neighbourhoods of $x$. A finite intersection of the $U_i$ is another one. Let $\{V_j\}$ be the set of quasi-compact open neighbourhoods of $y$. A finite intersection of the $V_j$ is another one. If $U_i \cap V_j$ is empty for some $i,j$ we are done. If not, then the intersection $U_i \cap V_j$ is nonempty for all $i$ and $j$. The sets $U_i \cap V_j$ are closed in the constructible topology on $X$. By Lemma 22.2 we see that $\bigcap(U_i \cap V_j)$ is nonempty (Lemma 11.6). Since $X$ is a sober space and $\{U_i\}$ is a fundamental system of open neighbourhoods of $x$, we see that $\bigcap U_i$ is the set of generalizations of $x$. Similarly, $\bigcap V_j$ is the set of generalizations of $y$. Thus any element of $\bigcap(U_i \cap V_j)$ specializes to both $x$ and $y$. □
Lemma 22.7. Let $X$ be a spectral space. The following are equivalent:

1. $X$ is profinite,
2. $X$ is Hausdorff,
3. $X$ is totally disconnected,
4. every quasi-compact open is closed,
5. there are no nontrivial specializations between points,
6. every point of $X$ is closed,
7. every point of $X$ is the generic point of an irreducible component of $X$,
8. add more here.

Proof. Lemma 21.2 shows the implication (1) $\Rightarrow$ (3). Irreducible components are closed, so if $X$ is totally disconnected, then every point is closed. So (3) implies (6). The equivalence of (6) and (5) is immediate, and (6) $\Leftrightarrow$ (7) holds because $X$ is sober. Assume (5). Then all constructible subsets of $X$ are closed (Lemma 22.5), in particular all quasi-compact opens are closed. So (5) implies (4). Since $X$ is sober, for any two points there is a quasi-compact open containing exactly one of them, hence (4) implies (2). It remains to prove (2) implies (1). Suppose $X$ is Hausdorff. Every quasi-compact open is also closed (Lemma 11.4). This implies $X$ is totally disconnected. Hence it is profinite, by Lemma 21.2. □

Lemma 22.8. If $X$ is a spectral space, then $\pi_0(X)$ is a profinite space.


Lemma 22.9. The product of two spectral spaces is spectral.

Proof. Let $X, Y$ be spectral spaces. Denote $p : X \times Y \to X$ and $q : X \times Y \to Y$ the projections. Let $Z \subset X \times Y$ be a closed irreducible subset. Then $p(Z) \subset X$ is irreducible and $q(Z) \subset Y$ is irreducible. Let $x \in X$ be the generic point of the closure of $p(X)$ and let $y \in Y$ be the generic point of the closure of $q(Y)$. If $(x, y) \not\in Z$, then there exist opens $x \in U \subset X$, $y \in V \subset Y$ such that $Z \cap U \times V = \emptyset$. Hence $Z$ is contained in $(X \setminus U) \times Y \cup X \times (Y \setminus V)$. Since $Z$ is irreducible, we see that either $Z \subset (X \setminus U) \times Y$ or $Z \subset X \times (Y \setminus V)$. In the first case $p(Z) \subset (X \setminus U)$ and in the second case $q(Z) \subset (Y \setminus V)$. Both cases are absurd as $x$ is in the closure of $p(Z)$ and $y$ is in the closure of $q(Z)$. Thus we conclude that $(x, y) \in Z$, which means that $(x, y)$ is the generic point for $Z$.

A basis of the topology of $X \times Y$ are the opens of the form $U \times V$ with $U \subset X$ and $V \subset Y$ quasi-compact open (here we use that $X$ and $Y$ are spectral). Then $U \times V$ is quasi-compact as the product of quasi-compact spaces is quasi-compact. Moreover, any quasi-compact open of $X \times Y$ is a finite union of such quasi-compact rectangles $U \times V$. It follows that the intersection of two such is again quasi-compact (since $X$ and $Y$ are spectral). This concludes the proof. □

Lemma 22.10. Let $f : X \to Y$ be a continuous map of topological spaces. If

1. $X$ and $Y$ are spectral,
2. $f$ is spectral and bijective, and
3. generalizations (resp. specializations) lift along $f$.

Then $f$ is a homeomorphism.

Proof. Since $f$ is spectral it defines a continuous map between $X$ and $Y$ in the constructible topology. By Lemmas 22.2 and 16.8 it follows that $X \to Y$ is a
homeomorphism in the constructible topology. Let \( U \subset X \) be quasi-compact open. Then \( f(U) \) is constructible in \( Y \). Let \( y \in Y \) specialize to a point in \( f(U) \). By the last assumption we see that \( f^{-1}(y) \) specializes to a point of \( U \). Hence \( f^{-1}(y) \in U \). Thus \( y \in f(U) \). It follows that \( f(U) \) is open, see Lemma \([22.5]\). Whence \( f \) is a homeomorphism. To prove the lemma in case specializations lift along \( f \) one shows instead that \( f(Z) \) is closed if \( X \setminus Z \) is a quasi-compact open of \( X \). \( \square \)

**Lemma 22.11.** The inverse limit of a directed inverse system of finite sober topological spaces is a spectral topological space.

**Proof.** Let \( I \) be a directed partially ordered set. Let \( X_i \) be an inverse system of finite sober spaces over \( I \). Let \( X = \lim X_i \) which exists by Lemma \([13.1]\). As a set \( X = \lim X_i \). Denote \( p_i : X \to X_i \) the projection. Because \( I \) is directed we may apply Lemma \([13.2]\). A basis for the topology is given by the opens \( p_i^{-1}(U_i) \) for \( U_i \subset X_i \) open. Since an open covering of \( p_i^{-1}(U_i) \) is in particular an open covering in the profinite topology, we conclude that \( p_i^{-1}(U_i) \) is quasi-compact. Given \( U_i \subset X_i \) and \( U_j \subset X_j \), then \( p_i^{-1}(U_i) \cap p_j^{-1}(U_j) = p_k^{-1}(U_k) \) for some \( k \geq i, j \) and open \( U_k \subset X_k \). Finally, if \( Z \subset X \) is irreducible and closed, then \( p_i(Z) \subset X_i \) is irreducible and therefore has a unique generic point \( \xi_i \) (because \( X_i \) is a finite sober topological space). Then \( \xi = \lim \xi_i \) is a generic point of \( Z \) (it is a point of \( Z \) as \( Z \) is closed). This finishes the proof. \( \square \)

**Lemma 22.12.** Let \( W \) be the topological space with two points, one closed, the other not. A topological space is spectral if and only if it is homeomorphic to a subspace of a product of copies of \( W \) which is closed in the constructible topology.

**Proof.** Write \( W = \{0, 1\} \) where 0 is a specialization of 1 but not vice versa. Let \( I \) be a set. The space \( \prod_{i \in I} W \) is spectral by Lemma \([22.11]\). Thus we see that a subspace of \( \prod_{i \in I} W \) closed in the constructible topology is a spectral space by Lemma \([22.4]\).

For the converse, let \( X \) be a spectral space. Let \( U \subset X \) be a quasi-compact open. Consider the continuous map

\[
f_U : X \to W
\]

which maps every point in \( U \) to 1 and every point in \( X \setminus U \) to 0. Taking the product of these maps we obtain a continuous map

\[
f = \prod f_U : X \to \prod_{U} W
\]

By construction the map \( f : X \to Y \) is spectral. By Lemma \([22.3]\) the image of \( f \) is closed in the constructible topology. If \( x', x \in X \) are distinct, then since \( X \) is sober either \( x' \) is not a specialization of \( x \) or conversely. In either case (as the quasi-compact opens form a basis for the topology of \( X \)) there exists a quasi-compact open \( U \subset X \) such that \( f_U(x') \neq f_U(x) \). Thus \( f \) is injective. Let \( Y = f(X) \) endowed with the induced topology. Let \( y' \to y \) be a specialization in \( Y \) and say \( f(x') = y' \) and \( f(x) = y \). Arguing as above we see that \( x' \to x \), since otherwise there is a \( U \) such that \( x \in U \) and \( x' \notin U \), which would imply \( f_U(x') \not\to f_U(x) \). We conclude that \( f : X \to Y \) is a homeomorphism by Lemma \([22.10]\). \( \square \)

**Lemma 22.13.** A topological space is spectral if and only if it is a directed inverse limit of finite sober topological spaces.
Proof. One direction is given by Lemma 22.11. For the converse, assume $X$ is spectral. Then we may assume $X \subset \prod_{i \in I} W$ is a subset closed in the constructible topology where $W = \{0, 1\}$ as in Lemma 22.12. We can write

$$\prod_{i \in I} W = \lim_{J \subset I \text{ finite}} \prod_{j \in J} W$$

as a cofiltered limit. For each $J$, let $X_J \subset \prod_{j \in J} W$ be the image of $X$. Then we see that $X = \lim X_J$ as sets because $X$ is closed in the product with the constructible topology (detail omitted). A formal argument (omitted) on limits shows that $X = \lim X_J$ as topological spaces. □

Lemma 22.14. Let $X$ be a topological space and let $c : X \to X'$ be the universal map from $X$ to a sober topological space, see Lemma 7.14.

1. If $X$ is quasi-compact, so is $X'$.
2. If $X$ is quasi-compact, has a basis of quasi-compact opens, and the intersection of two quasi-compact opens is quasi-compact, then $X'$ is spectral.
3. If $X$ is Noetherian, then $X'$ is a Noetherian spectral space.

Proof. Let $U \subset X$ be open and let $U' \subset X'$ be the corresponding open, i.e., the open such that $c^{-1}(U') = U$. Then $U$ is quasi-compact if and only if $U'$ is quasi-compact, as pulling back by $c$ is a bijection between the opens of $X$ and $X'$ which commutes with unions. This in particular proves (1).

Proof of (2). It follows from the above that $X'$ has a basis of quasi-compact opens. Since $c^{-1}$ also commutes with intersections of pairs of opens, we see that the intersection of two quasi-compact opens $X'$ is quasi-compact. Finally, $X'$ is quasi-compact by (1) and sober by construction. Hence $X'$ is spectral.

Proof of (3). It is immediate that $X'$ is Noetherian as this is defined in terms of the acc for open subsets which holds for $X$. We have already seen in (2) that $X'$ is spectral. □

23. Limits of spectral spaces

Lemma 22.13 tells us that every spectral space is a cofiltered limit of finite sober spaces. Every finite sober space is a spectral space and every continuous map of finite sober spaces is a spectral map of spectral spaces. In this section we prove some lemmas concerning limits of systems of spectral topological spaces along spectral maps.

Lemma 23.1. Let $\mathcal{I}$ be a category. Let $i \mapsto X_i$ be a diagram of spectral spaces such that for $a : j \to i$ in $\mathcal{I}$ the corresponding map $f_a : X_j \to X_i$ is spectral.

1. Given subsets $Z_i \subset X_i$ closed in the constructible topology with $f_a(Z_j) \subset Z_i$ for all $a : j \to i$ in $\mathcal{I}$, then $\lim Z_i$ is quasi-compact.
2. The space $X = \lim X_i$ is quasi-compact.

Proof. The limit $Z = \lim Z_i$ exists by Lemma 13.1. Denote $X'_i$ the space $X_i$ endowed with the constructible topology and $Z'_i$ the corresponding subspace of $X'_i$. Let $a : j \to i$ in $\mathcal{I}$ be a morphism. As $f_a$ is spectral it defines a continuous map $f_a : X'_j \to X'_i$. Thus $f_a|_{Z_j} : Z'_j \to Z'_i$ is a continuous map of quasi-compact Hausdorff spaces (by Lemmas 22.2 and 11.3). Thus $Z' = \lim Z_i$ is quasi-compact by Lemma 13.5. The maps $Z'_i \to Z_i$ are continuous, hence $Z' \to Z$ is continuous.
and a bijection on underlying sets. Hence $\mathcal{Z}$ is quasi-compact as the image of the surjective continuous map $Z' \to Z$ (Lemma \[11.7\]). □

**Lemma 23.2.** Let $\mathcal{I}$ be a cofiltered category. Let $i \mapsto X_i$ be a diagram of spectral spaces such that for $a : j \to i$ in $\mathcal{I}$ the corresponding map $f_a : X_j \to X_i$ is spectral.

1. Given nonempty subsets $\mathcal{Z}_i \subset X_i$ closed in the constructible topology with $f_a(\mathcal{Z}_j) \subset \mathcal{Z}_i$ for all $a : j \to i$ in $\mathcal{I}$, then $\lim_{\mathcal{I}} \mathcal{Z}_i$ is nonempty.
2. If each $X_i$ is nonempty, then $X = \lim_{\mathcal{I}} X_i$ is nonempty.

**Proof.** Denote $X'_i$ the space $X_i$ endowed with the constructible topology and $Z'_i$ the corresponding subspace of $X'_i$. Let $a : j \to i$ in $\mathcal{I}$ be a morphism. As $f_a$ is spectral it defines a continuous map $f_a : X'_j \to X'_i$. Thus $f_a|_{Z'_j} : Z'_j \to Z'_i$ is a continuous map of quasi-compact Hausdorff spaces (by Lemmas \[22.2\] and \[11.3\]). By Lemma \[13.6\] the space $Z'_i$ is nonempty. Since $\lim_{\mathcal{I}} Z'_i = \lim_{\mathcal{I}} Z_i$ as sets we conclude. □

**Lemma 23.3.** Let $\mathcal{I}$ be a cofiltered category. Let $i \mapsto X_i$ be a diagram of spectral spaces such that for $a : j \to i$ in $\mathcal{I}$ the corresponding map $f_a : X_j \to X_i$ is spectral. Let $X = \lim_{\mathcal{I}} X_i$ with projections $p_i : X \to X_i$. Let $i \in \text{Ob}(\mathcal{I})$ and let $E, F \subset X_i$ be subsets with $E$ closed in the constructible topology and $F$ open in the constructible topology. Then $p_i^{-1}(E) \subset p_i^{-1}(F)$ if and only if there is a morphism $a : j \to i$ in $\mathcal{I}$ such that $f_a^{-1}(E) \subset f_a^{-1}(F)$.

**Proof.** Observe that $$p_i^{-1}(E) \setminus p_i^{-1}(F) = \lim_{a : j \to i} f_a^{-1}(E) \setminus f_a^{-1}(F).$$ Since $f_a$ is a spectral map, it is continuous in the constructible topology hence the set $f_a^{-1}(E) \setminus f_a^{-1}(F)$ is closed in the constructible topology. Hence Lemma \[23.2\] applies to show that the LHS is nonempty if and only if each of the spaces of the RHS is nonempty. □

**Lemma 23.4.** Let $\mathcal{I}$ be a cofiltered category. Let $i \mapsto X_i$ be a diagram of spectral spaces such that for $a : j \to i$ in $\mathcal{I}$ the corresponding map $f_a : X_j \to X_i$ is spectral. Let $X = \lim_{\mathcal{I}} X_i$ with projections $p_i : X \to X_i$. Let $E \subset X$ be a constructible subset. Then there exists an $i \in \text{Ob}(\mathcal{I})$ and a constructible subset $E_i \subset X_i$ such that $p_i^{-1}(E_i) = E$. If $E$ is open, resp. closed, we may choose $E_i$ open, resp. closed.

**Proof.** Assume $E$ is a quasi-compact open of $X$. By Lemma \[13.2\] we can write $E = p_i^{-1}(U_i)$ for some $i$ and some open $U_i \subset X_i$. Write $U_i = \bigcup U_{i,\alpha}$ as a union of quasi-compact opens. As $E$ is quasi-compact we can find $\alpha_1, \ldots, \alpha_n$ such that $E = p_i^{-1}(U_{i,\alpha_1} \cup \ldots \cup U_{i,\alpha_n})$. Hence $E_i = U_{i,\alpha_1} \cup \ldots \cup U_{i,\alpha_n}$ works.

Assume $E$ is a constructible closed subset. Then $E^c$ is quasi-compact open. So $E^c = p_i^{-1}(F_i)$ for some $i$ and quasi-compact open $F_i \subset X_i$ by the result of the previous paragraph. Then $E = p_i^{-1}(F_i^c)$ as desired.

If $E$ is general we can write $E = \bigcup_{l=1}^n U_l \cap Z_l$ with $U_l$ constructible open and $Z_l$ constructible closed. By the result of the previous paragraphs we may write $U_l = p_i^{-1}(U_{i,l})$ and $Z_l = p_i^{-1}(Z_{l,i})$ with $U_{i,l} \subset X_i$ constructible open and $Z_{l,i} \subset X_i$ constructible closed. As $\mathcal{I}$ is cofiltered we may choose an object $k$ of $\mathcal{I}$ and morphism $a_l : k \to i_l$ and $b_l : k \to j_l$. Then taking $E_k = \bigcup_{l=1}^n f_{a_l}^{-1}(U_{i,l}) \cap f_{b_l}^{-1}(Z_{j,l})$ we obtain a constructible subset of $X_k$ whose inverse image in $X$ is $E$. □
Lemma 23.5. Let $\mathcal{I}$ be a cofiltered index category. Let $i \mapsto X_i$ be a diagram of spectral spaces such that for $a : j \to i$ in $\mathcal{I}$ the corresponding map $f_a : X_j \to X_i$ is spectral. Then the inverse limit $X = \lim X_i$ is a spectral topological space and the projection maps $p_i : X \to X_i$ are spectral.

Proof. The limit $X = \lim X_i$ exists (Lemma 13.1) and is quasi-compact by Lemma 23.1.

Denote $p_i : X \to X_i$ the projection. Because $\mathcal{I}$ is cofiltered we can apply Lemma 13.2. Hence a basis for the topology on $X$ is given by the opens $p_i^{-1}(U_i)$ for $U_i \subset X_i$ open. Since a basis for the topology of $X_i$ is given by the quasi-compact open, we conclude that a basis for the topology on $X$ is given by $p_i^{-1}(U_i)$ with $U_i \subset X_i$ quasi-compact open. A formal argument shows that

$$p_i^{-1}(U_i) = \lim_{\alpha : j \to i} f_{a}^{-1}(U_i)$$

as topological spaces. Since each $f_a$ is spectral the sets $f_a^{-1}(U_i)$ are closed in the constructible topology of $X_j$ and hence $p_i^{-1}(U_i)$ is quasi-compact by Lemma 23.1.

Thus $X$ has a basis for the topology consisting of quasi-compact opens.

Any quasi-compact open $U$ of $X$ is of the form $U = p_i^{-1}(U_i)$ for some $i$ and some quasi-compact open $U_i \subset X_i$ (see Lemma 23.4). Given $U_i \subset X_i$ and $U_j \subset X_j$ quasi-compact open, then $p_i^{-1}(U_i) \cap p_j^{-1}(U_j) = p_k^{-1}(U_k)$ for some $k$ and quasi-compact open $U_k \subset X_k$. Namely, choose $k$ and morphisms $k \to i$ and $k \to j$ and let $U_k$ be the intersection of the pullbacks of $U_i$ and $U_j$ to $X_k$. Thus we see that the intersection of two quasi-compact opens of $X$ is quasi-compact open.

Finally, let $Z \subset X$ be irreducible and closed. Then $p_i(Z) \subset X_i$ is irreducible and therefore $Z_i = p_i(Z)$ has a unique generic point $\xi_i$ (because $X_i$ is a spectral space). Then $f_a(\xi_j) = \xi_i$ for $a : j \to i$ in $\mathcal{I}$ because $f_a(Z_j) = Z_i$. Hence $\xi = \lim \xi_i$ is a point of $X$. Claim: $\xi \in Z$. Namely, if not we can find a quasi-compact open containing $\xi$ disjoint from $Z$. This would be of the form $p_i^{-1}(U_i)$ for some $i$ and quasi-compact open $U_i \subset X_i$. Then $\xi_i \in U_i$ but $p_i(Z) \cap U_i = \emptyset$ which contradicts $\xi_i \in p_i(Z)$. So $\xi \in Z$ and hence $\{\xi\} \subset Z$. Conversely, every $z \in Z$ is in the closure of $\xi$. Namely, given a quasi-compact open neighbourhood $U$ of $z$ we write $U = p_i^{-1}(U_i)$ for some $i$ and quasi-compact open $U_i \subset X_i$. We see that $p_i(z) \in U_i$ hence $\xi_i \in U_i$, hence $\xi \in U$. Thus $\xi$ is the generic point of $Z$. This finishes the proof.

Lemma 23.6. Let $\mathcal{I}$ be a cofiltered index category. Let $i \mapsto X_i$ be a diagram of spectral spaces such that for $a : j \to i$ in $\mathcal{I}$ the corresponding map $f_a : X_j \to X_i$ is spectral. Set $X = \lim X_i$ and denote $p_i : X \to X_i$ the projection.

1. Given any quasi-compact open $U \subset X$ there exists an $i \in \text{Ob}(\mathcal{I})$ and a quasi-compact open $U_i \subset X_i$ such that $p_i^{-1}(U_i) = U$.

2. Given $U_i \subset X_i$ and $U_j \subset X_j$ quasi-compact opens such that $p_i^{-1}(U_i) \subset p_j^{-1}(U_j)$ there exist $k \in \text{Ob}(\mathcal{I})$ and morphisms $a : k \to i$ and $b : k \to j$ such that $f_a^{-1}(U_i) \subset f_b^{-1}(U_j)$.

3. If $U_{i_1}, \ldots, U_{i_n} \subset X_i$ are quasi-compact opens and $p_i^{-1}(U_i) = p_{i_1}^{-1}(U_{i_1}) \cup \ldots \cup p_{i_n}^{-1}(U_{i_n})$ then $f_a^{-1}(U_i) = f_a^{-1}(U_{i_1}) \cup \ldots \cup f_a^{-1}(U_{i_n})$ for some morphism $a : j \to i$ in $\mathcal{I}$.

4. Same statement as in (3) but for intersections.
Lemma 23.7. Let $W$ be a subset of a spectral space $X$. The following are equivalent:

1. $W$ is an intersection of constructible sets and closed under generalizations,
2. $W$ is quasi-compact and closed under generalizations,
3. there exists a quasi-compact subset $E \subset X$ such that $W$ is the set of points specializing to $E$,
4. $W$ is an intersection of quasi-compact open subsets,
5. there exists a nonempty set $I$ and quasi-compact opens $U_i \subset X$, $i \in I$ such that $W = \bigcap U_i$ and for all $i, j \in I$ there exists a $k \in I$ with $U_k \subset U_i \cap U_j$.

In this case we have (a) $W$ is a spectral space, (b) $W = \lim U_i$ as topological spaces, and (c) for any open $U$ containing $W$ there exists an $i$ with $U_i \subset U$.

Proof. Let $E \subset X$ satisfy (1). Then $E$ is closed in the constructible topology, hence quasi-compact in the constructible topology (by Lemmas 22.2 and 11.3), hence quasi-compact in the topology of $X$ (because opens in $X$ are open in the constructible topology). Thus (2) holds.

It is clear that (2) implies (3) by taking $E = W$.

Let $X$ be a spectral space and let $E \subset W$ be as in (3). Since every point of $W$ specializes to a point of $E$ we see that an open of $W$ which contains $E$ is equal to $W$. Hence since $E$ is quasi-compact, so is $W$. If $x \in X$, $x \notin W$, then $Z = \{x\}$ is disjoint from $W$. Since $W$ is quasi-compact we can find a quasi-compact open $U$ with $W \subset U$ and $U \cap Z = \emptyset$. We conclude that (4) holds.

If $W = \bigcap_{j \in J} U_j$ then setting $I$ equal to the set of finite subsets of $J$ and $U_i = U_{j_1} \cap \ldots \cap U_{j_r}$ for $i = \{j_1, \ldots, j_r\}$ shows that (4) implies (5). It is immediate that (5) implies (1).

Let $I$ and $U_i$ be as in (5). Since $W = \bigcap U_i$ we have $W = \lim U_i$ by the universal property of limits. Then $W$ is a spectral space by Lemma 23.5. Let $U \subset X$ be an open neighbourhood of $W$. Then $E_i = U_i \cap (X \setminus U)$ is a family of constructible subsets of the spectral space $Z = X \setminus U$ with empty intersection. Using that the spectral topology on $Z$ is quasi-compact (Lemma 22.2) we conclude from Lemma 11.6 that $E_i = \emptyset$ for some $i$.

Lemma 23.8. Let $X$ be a spectral space. Let $E \subset X$ be a constructible subset. Let $W \subset X$ be the set of points of $X$ which specialize to a point of $E$. Then $W \setminus E$ is a spectral space. If $W = \lim U_i$ with $U_i$ as in Lemma 23.7 then $W \setminus E = \lim (U_i \setminus E)$.

Proof. Since $E$ is constructible, it is quasi-compact and hence Lemma 23.7 applies to $W$. If $E$ is constructible, then $W$ is constructible in $U_i$ for all $i \in I$. Hence $U_i \setminus E$ is spectral by Lemma 22.4. Since $W \setminus E = \bigcap (U_i \setminus E)$ we have $W \setminus E = \lim (U_i \setminus E)$ by the universal property of limits. Then $W \setminus E$ is a spectral space by Lemma 23.5.
24. Stone-$reve{C}$ech compactification

The Stone-$reve{C}$ech compactification of a topological space $X$ is a map $X \to \beta(X)$ from $X$ to a Hausdorff quasi-compact space $\beta(X)$ which is universal for such maps. We prove this exists by a standard argument using the following simple lemma.

**Lemma 24.1.** Let $f : X \to Y$ be a continuous map of topological spaces. Assume that $f(X)$ is dense in $Y$ and that $Y$ is Hausdorff. Then the cardinality of $Y$ is at most the cardinality of $P(P(X))$ where $P$ is the power set operation.

**Proof.** Let $S = f(X) \subset Y$. Let $D$ be the set of all closed domains of $Y$, i.e., subsets $D \subset Y$ which equal the closure of its interior. Note that the closure of an open subset of $Y$ is a closed domain. For $y \in Y$ consider the set

$$I_y = \{ T \subset S \mid \text{there exists } D \in D \text{ with } T = S \cap D \text{ and } y \in D \}.$$ 

Since $S$ is dense in $Y$ for every closed domain $D$ we see that $S \cap D$ is dense in $D$. Hence, if $D \cap S = D' \cap S$ for $D, D' \in D$, then $D = D'$. Thus $I_y = I_y'$ implies that $y = y'$ because the Hausdorff condition assures us that we can find a closed domain containing $y$ but not $y'$. The result follows. \[\square\]

Let $X$ be a topological space. By Lemma 24.1 there is a set $I$ of isomorphism classes of continuous maps $f : X \to Y$ which have dense image and where $Y$ is Hausdorff and quasi-compact. For $i \in I$ choose a representative $f_i : X \to Y_i$. Consider the map

$$\prod f_i : X \to \prod_{i \in I} Y_i$$

and denote $\beta(X)$ the closure of the image. Since each $Y_i$ is Hausdorff, so is $\beta(X)$. Since each $Y_i$ is quasi-compact, so is $\beta(X)$ (use Theorem 13.4 and Lemma 11.3).

Let us show the canonical map $X \to \beta(X)$ satisfies the universal property with respect to maps to Hausdorff, quasi-compact spaces. Namely, let $f : X \to Y$ be such a morphism. Let $Z \subset Y$ be the closure of $f(X)$. Then $X \to Z$ is isomorphic to one of the maps $f_i : X \to Y_i$, say $f_{i_0} : X \to Y_{i_0}$. Thus $f$ factors as $X \to \beta(X) \to \prod Y_i \to Y_{i_0} \cong Z \to Y$ as desired.

**Lemma 24.2.** Let $X$ be a Hausdorff, locally quasi-compact space. There exists a map $X \to X^*$ which identifies $X$ as an open subspace of a quasi-compact Hausdorff space $X^*$ such that $X^* \setminus X$ is a singleton (one point compactification). In particular, the map $X \to \beta(X)$ identifies $X$ with an open subspace of $\beta(X)$.

**Proof.** Set $X^* = X \amalg \{ \infty \}$. We declare a subset $V$ of $X^*$ to be open if either $V \subset X$ is open in $X$, or $\infty \in V$ and $U = V \cap X$ is an open of $X$ such that $X \setminus U$ is quasi-compact. We omit the verification that this defines a topology. It is clear that $X \to X^*$ identifies $X$ with an open subspace of $X$.

Since $X$ is locally quasi-compact, every point $x \in X$ has a quasi-compact neighbourhood $x \in E \subset X$. Then $E$ is closed (Lemma 11.3) and $V = (X \setminus E) \amalg \{ \infty \}$ is an open neighbourhood of $\infty$ disjoint from the interior of $E$. Thus $X^*$ is Hausdorff.

Let $X^* = \bigcup V_i$ be an open covering. Then for some $i$, say $i_0$, we have $\infty \in V_{i_0}$. By construction $Z = X^* \setminus V_{i_0}$ is quasi-compact. Hence the covering $Z \subset \bigcup_{i \neq i_0} Z \cap V_i$ has a finite refinement which implies that the given covering of $X^*$ has a finite refinement. Thus $X^*$ is quasi-compact.
The map $X \to X^*$ factors as $X \to \beta(X) \to X^*$ by the universal property of the Stone-Čech compactification. Let $\varphi : \beta(X) \to X^*$ be this factorization. Then $X \to \varphi^{-1}(X)$ is a section to $\varphi^{-1}(X) \to X$ hence has closed image (Lemma 3.3). Since the image of $X \to \beta(X)$ is dense we conclude that $X = \varphi^{-1}(X)$. □

25. Extremally disconnected spaces

The material in this section is taken from [Gle58] (with a slight modification as in [Rai59]). In Gleason’s paper it is shown that in the category of quasi-compact Hausdorff spaces, the “projective objects” are exactly the extremally disconnected spaces.

**Definition 25.1.** A topological space $X$ is called *extremally disconnected* if the closure of every open subset of $X$ is open.

If $X$ is Hausdorff and extremally disconnected, then $X$ is totally disconnected (this isn’t true in general). If $X$ is quasi-compact, Hausdorff, and extremally disconnected, then $X$ is profinite by Lemma 21.2 but the converse does not hold in general. Namely, Gleason shows that in an extremally disconnected Hausdorff space $X$ a convergent sequence $x_1,x_2,x_3,\ldots$ is eventually constant. Hence for example the $p$-adic integers $\mathbb{Z}_p = \varprojlim \mathbb{Z}/p^n\mathbb{Z}$ is a profinite space which is not extremally disconnected.

**Lemma 25.2.** Let $f : X \to Y$ be a continuous map of topological spaces. Assume $f$ is surjective and $f(E) \neq Y$ for all proper closed subsets $E \subset X$. Then for $U \subset X$ open the subset $f(U)$ is contained in the closure of $Y \setminus f(X \setminus U)$.

**Proof.** Pick $y \in f(U)$ and let $V \subset Y$ be any open neighbourhood of $y$. We will show that $V$ intersects $Y \setminus f(X \setminus U)$. Note that $W = U \cap f^{-1}(V)$ is a nonempty open subset of $X$, hence $f(X \setminus W) \neq Y$. Take $y' \in Y$, $y' \notin f(X \setminus W)$. It is elementary to show that $y' \in V$ and $y' \in Y \setminus f(X \setminus U)$. □

**Lemma 25.3.** Let $X$ be an extremally disconnected space. If $U,V \subset X$ are disjoint open subsets, then $\overline{U}$ and $\overline{V}$ are disjoint too.

**Proof.** By assumption $\overline{U}$ is open, hence $V \setminus \overline{U}$ is open and disjoint from $U$, hence empty because $\overline{U}$ is the intersection of all the closed subsets of $X$ containing $U$. This means the open $\overline{V} \cap \overline{U}$ avoids $V$ hence is empty by the same argument. □

**Lemma 25.4.** Let $f : X \to Y$ be a continuous map of Hausdorff quasi-compact topological spaces. If $Y$ is extremally disconnected, $f$ is surjective, and $f(Z) \neq Y$ for every proper closed subset $Z$ of $X$, then $f$ is a homeomorphism.

**Proof.** By Lemma 16.8 it suffices to show that $f$ is injective. Suppose that $x,x' \in X$ are distinct points with $y = f(x) = f(x')$. Choose disjoint open neighbourhoods $U,U' \subset X$ of $x,x'$. Observe that $f$ is closed (Lemma 16.7) hence $T = f(X \setminus U)$ and $T' = f(X \setminus U')$ are closed in $Y$. Since $X$ is the union of $X \setminus U$ and $X \setminus U'$ we see that $Y = T \cup T'$. By Lemma 25.2 we see that $y$ is contained in the closure of $Y \setminus T$ and the closure of $Y \setminus T'$. On the other hand, by Lemma 25.3 this intersection is empty. In this way we obtain the desired contradiction. □

**Lemma 25.5.** Let $f : X \to Y$ be a continuous surjective map of Hausdorff quasi-compact topological spaces. There exists a quasi-compact subset $E \subset X$ such that $f(E) = Y$ but $f(E') \neq Y$ for all proper closed subsets $E' \subset E$. 

Proof. We will use without further mention that the quasi-compact subsets of $X$ are exactly the closed subsets (Lemma 11.5). Consider the collection $E$ of all quasi-compact subsets $E \subseteq X$ with $f(E) = Y$ ordered by inclusion. We will use Zorn’s lemma to show that $E$ has a minimal element. To do this it suffices to show that given a totally ordered family $E_\lambda$ of elements of $E$ the intersection $\bigcap E_\lambda$ is an element of $E$. It is quasi-compact as it is closed. For every $y \in Y$ the sets $E_\lambda \cap f^{-1}(\{y\})$ are nonempty and closed, hence the intersection $\bigcap E_\lambda \cap f^{-1}(\{y\}) = \bigcap(\bigcap E_\lambda \cap f^{-1}(\{y\}))$ is nonempty by Lemma 11.6. This finishes the proof. □

Proposition 25.6. Let $X$ be a Hausdorff, quasi-compact topological space. The following are equivalent

(1) $X$ is extremally disconnected,
(2) for any surjective continuous map $f : Y \to X$ with $Y$ Hausdorff quasi-compact there exists a continuous section, and
(3) for any solid commutative diagram

\[
\begin{array}{ccc}
Y & \to & Z \\
\downarrow & & \downarrow \\
X & \to & Z
\end{array}
\]

of continuous maps of quasi-compact Hausdorff spaces with $Y \to Z$ surjective, there is a dotted arrow in the category of topological spaces making the diagram commute.

Proof. It is clear that (3) implies (2). On the other hand, if (2) holds and $X \to Z$ and $Y \to Z$ are as in (3), then (2) assures there is a section to the projection $X \times_Z Y \to X$ which implies a suitable dotted arrow exists (details omitted). Thus (3) is equivalent to (2).

Assume $X$ is extremally disconnected and let $f : Y \to X$ be as in (2). By Lemma 25.5 there exists a quasi-compact subset $E \subseteq Y$ such that $f(E) = X$ but $f(E') \neq X$ for all proper closed subsets $E' \subset E$. By Lemma 25.4 we find that $f|_E : E \to X$ is a homeomorphism, the inverse of which gives the desired section.

Assume (2). Let $U \subset X$ be open with complement $Z$. Consider the continuous surjection $f : \overline{U} \amalg Z \to X$. Let $\sigma$ be a section. Then $\overline{U} = \sigma^{-1}(U)$ is open. Thus $X$ is extremally disconnected. □

Lemma 25.7. Let $f : X \to X$ be a continuous selfmap of a Hausdorff topological space. If $f$ is not $\text{id}_X$, then there exists a proper closed subset $E \subset X$ such that $X = E \cup f(E)$.

Proof. Pick $p \in X$ with $f(p) \neq p$. Choose disjoint open neighbourhoods $p \in U$, $f(p) \in V$ and set $E = X \setminus U \cap f^{-1}(V)$. □

Example 25.8. We can use Proposition 25.6 to see that the Stone-Čech compactification $\beta(X)$ of a discrete space $X$ is extremally disconnected. Namely, let $f : Y \to \beta(X)$ be a continuous surjection where $Y$ is quasi-compact and Hausdorff. Then we can lift the map $X \to \beta(X)$ to a continuous (!) map $X \to Y$ as $X$ is discrete. By the universal property of the Stone-Čech compactification we see that we obtain a factorization $X \to \beta(X) \to Y$. Since $\beta(X) \to Y \to \beta(X)$ equals the identity on the dense subset $X$ we conclude that we get a section. In particular,
we conclude that the Stone-Čech compactification of a discrete space is totally disconnected, whence profinite (see discussion following Definition 25.1 and Lemma 21.2).

Using the supply of extremally disconnected spaces given by Example 25.8 we can prove that every quasi-compact Hausdorff space has a “projective cover” in the category of quasi-compact Hausdorff spaces.

**Lemma 25.9.** Let $X$ be a quasi-compact Hausdorff space. There exists a continuous surjection $X' \to X$ with $X'$ quasi-compact, Hausdorff, and extremally disconnected. If we require that every proper closed subset of $X'$ does not map onto $X$, then $X'$ is unique up to isomorphism.

**Proof.** Let $Y = X$ but endowed with the discrete topology. Let $X' = \beta(Y)$. The continuous map $Y \to X$ factors as $Y \to X' \to X$. This proves the first statement of the lemma by Example 25.8.

By Lemma 25.5 we can find a quasi-compact subset $E \subset X'$ such that no proper closed subset of $E$ surjects onto $X$. Because $X'$ is extremally disconnected there exists a continuous map $f : X' \to E$ over $X$ (Proposition 25.6). Composing $f$ with the map $E \to X'$ gives a continuous selfmap $f|_E : E \to E$. This map has to be id$_E$ as otherwise Lemma 25.7 shows that $E$ isn’t minimal. Thus the id$_E$ factors through the extremally disconnected space $X'$. A formal, categorical argument (using the characterization of Proposition 25.6) shows that $E$ is extremally disconnected.

To prove uniqueness, suppose we have a second $X'' \to X$ minimal cover. By the lifting property proven in Proposition 25.6 we can find a continuous map $g : X' \to X''$ over $X$. Observe that $g$ is a closed map (Lemma 16.7). Hence $g(X') \subset X''$ is a closed subset surjecting onto $X$ and we conclude $g(X') = X''$ by minimality of $X''$. On the other hand, if $E \subset X'$ is a proper closed subset, then $g(E) \neq X''$ as $E$ does not map onto $X$ by minimality of $X'$. By Lemma 25.4 we see that $g$ is an isomorphism. □

**Remark 25.10.** Let $X$ be a quasi-compact Hausdorff space. Let $\kappa$ be an infinite cardinal bigger or equal than the cardinality of $X$. Then the cardinality of the minimal quasi-compact, Hausdorff, extremally disconnected cover $X' \to X$ (Lemma 25.9) is at most $2^\kappa$. Namely, choose a subset $S \subset X'$ mapping bijectively to $X$. By minimality of $X'$ the set $S$ is dense in $X'$. Thus $|X'| \leq 2^\kappa$ by Lemma 24.1.

26. Miscellany

The following lemma applies to the underlying topological space associated to a quasi-separated scheme.

**Lemma 26.1.** Let $X$ be a topological space which

1. has a basis of the topology consisting of quasi-compact opens, and
2. has the property that the intersection of any two quasi-compact opens is quasi-compact.

Then

1. $X$ is locally quasi-compact,
2. a quasi-compact open $U \subset X$ is retrocompact,
3. any quasi-compact open $U \subset X$ has a cofinal system of open coverings $U : U = \bigcup_{j \in J} U_j$ with $J$ finite and all $U_j$ and $U_j \cap U_j'$, quasi-compact,
(4) *add more here.*

**Proof.** Omitted. □

**Definition 26.2.** Let $X$ be a topological space. We say $x \in X$ is an *isolated point* of $X$ if $\{x\}$ is open in $X$.

## 27. Partitions and stratifications

Stratifications can be defined in many different ways. We welcome comments on the choice of definitions in this section.

**Definition 27.1.** Let $X$ be a topological space. A *partition* of $X$ is a decomposition $X = \bigsqcup X_i$ into locally closed subsets $X_i$. The $X_i$ are called the *parts* of the partition. Given two partitions of $X$ we say one *refines* the other if the parts of one are unions of parts of the other.

Any topological space $X$ has a partition into connected components. If $X$ has finitely many irreducible components $Z_1, \ldots, Z_r$, then there is a partition with parts $X_i = \bigcap_{i \in I} Z_i \setminus \bigcup_{i \not\in I} Z_i$ whose indices are subsets $I \subset \{1, \ldots, r\}$ which refines the partition into connected components.

**Definition 27.2.** Let $X$ be a topological space. A *good stratification* of $X$ is a partition $X = \bigsqcup X_i$ such that for all $i, j \in I$ we have $X_i \cap X_j \neq \emptyset \Rightarrow X_i \subset X_j$.

Given a good stratification $X = \bigsqcup_{i \in I} X_i$ we obtain a partial ordering on $I$ by setting $i \leq j$ if and only if $X_i \subset X_j$. Then we see that $X_j = \bigcup_{i \leq j} X_i$.

However, what often happens in algebraic geometry is that one just has that the left hand side is a subset of the right hand side in the last displayed formula. This leads to the following definition.

**Definition 27.3.** Let $X$ be a topological space. A *stratification* of $X$ is given by a partition $X = \bigsqcup_{i \in I} X_i$ and a partial ordering on $I$ such that for each $j \in I$ we have $X_j \subset \bigcup_{i \leq j} X_i$.

The parts $X_i$ are called the *strata* of the stratification.

We often impose additional conditions on the stratification. For example, we say a stratification is *locally finite* if every point has a neighbourhood which meets only finitely many strata.

**Remark 27.4.** Given a locally finite stratification $X = \bigsqcup X_i$ of a topological space $X$, we obtain a family of closed subsets $Z_i = \bigcup_{j \leq i} X_j$ of $X$ indexed by $I$ such that $Z_i \cap Z_j = \bigcup_{k \leq i, j} Z_k$.

Conversely, given closed subsets $Z_i \subset X$ indexed by a partially ordered set $I$ such that $X = \bigcup Z_i$, such that every point has a neighbourhood meeting only finitely many $Z_i$, and such that the displayed formula holds, then we obtain a locally finite stratification of $X$ by setting $X_i = Z_i \setminus \bigcup_{j < i} Z_j$. 
Lemma 27.5. Let $X$ be a topological space. Let $X = \coprod X_i$ be a finite partition of $X$. Then there exists a finite stratification of $X$ refining it.

Proof. Let $T_i = \overline{X_i}$ and $\Delta_i = T_i \setminus X_i$. Let $S$ be the set of all intersections of $T_i$ and $\Delta_i$. (For example $T_1 \cap T_2 \cap \Delta_4$ is an element of $S$.) Then $S = \{Z_s\}$ is a finite collection of closed subsets of $X$ such that $Z_s \cap Z_{s'} \in S$ for all $s, s' \in S$. Define a partial ordering on $S$ by inclusion. Then set $Y_s = Z_s \setminus \bigcup_{s' < s} Z_{s'}$ to get the desired stratification.

Lemma 27.6. Let $X$ be a topological space. Suppose $X = T_1 \cup \ldots \cup T_n$ is written as a union of constructible subsets. There exists a finite stratification $X = \coprod X_i$ with each $X_i$ constructible such that each $T_i$ is a union of strata.

Proof. By definition of constructible subsets, we can write each $T_i$ as a finite union of $U \cap V^c$ with $U, V \subset X$ retrocompact open. Hence we may assume that $T_i = U_i \cap V_i^c$ with $U_i, V_i \subset X$ retrocompact open. Let $S$ be the finite set of closed subsets of $X$ consisting of $\emptyset, X, U_i, V_i^c$ and finite intersections of these. Write $S = \{Z_s\}$. If $s \in S$, then $Z_s$ is constructible (Lemma 14.2). Moreover, $Z_s \cap Z_{s'} \in S$ for all $s, s' \in S$. Define a partial ordering on $S$ by inclusion. Then set $Y_s = Z_s \setminus \bigcup_{s' < s} Z_{s'}$ to get the desired stratification.

Lemma 27.7. Let $X$ be a Noetherian topological space. Any finite partition of $X$ can be refined by a finite good stratification.

Proof. Let $X = \coprod X_i$ be a finite partition of $X$. Let $Z$ be an irreducible component of $X$. Since $X = \bigcup X_i$ with finite index set, there is an $i$ such that $Z \subset X_i$. Since $X_i$ is locally closed this implies that $Z \cap X_i$ contains an open of $Z$. Thus $Z \cap X_i$ contains an open $U$ of $X$ (Lemma 8.2). Write $X_i = U \amalg X_i^1 \amalg X_i^2$ with $X_i^1 = (X_i \setminus U) \cap \overline{U}$ and $X_i^2 = (X_i \setminus U) \cap U^c$. For $i' \neq i$ we set $X_i^{1'} = X_{i'} \cap U$ and $X_i^{2'} = X_{i'} \cap U^c$. Then

$$X \setminus U = \coprod X_i^{k_i}$$

is a partition such that $\overline{U} \setminus U = \bigcup X_i^1$. Note that $X \setminus U$ is closed and strictly smaller than $X$. By Noetherian induction we can refine this partition by a finite good stratification $X \setminus U = \coprod_{\alpha \in A} T_{\alpha}$. Then $X = U \amalg \coprod_{\alpha \in A} T_{\alpha}$ is a finite good stratification of $X$ refining the partition we started with.

28. Colimits of spaces

The category of topological spaces has coproducts. Namely, if $I$ is a set and for $i \in I$ we are given a topological space $X_i$ then we endow the set $\coprod_{i \in I} X_i$ with the coproduct topology. As a basis for this topology we use sets of the form $U_i$ where $U_i \subset X_i$ is open.

The category of topological spaces has coequalizers. Namely, if $a, b : X \rightarrow Y$ are morphisms of topological spaces, then the equalizer of $a$ and $b$ is the coequalizer $Y/ \sim$ in the category of sets endowed with the quotient topology (Section 5).

Lemma 28.1. The category of topological spaces has colimits and the forgetful functor to sets commutes with them.

Proof. This follows from the discussion above and Categories, Lemma 14.11. Another proof of existence of colimits is sketched in Categories, Remark 25.2. It follows from the above that the forgetful functor commutes with colimits. Another way to
see this is to use Categories, Lemma 24.4 and use that the forgetful functor has a
right adjoint, namely the functor which assigns to a set the corresponding chaotic
(or indiscrete) topological space.

29. Topological groups, rings, modules

This is just a short section with definitions and elementary properties.

**Definition 29.1.** A **topological group** is a group \( G \) endowed with a topology such
that multiplication \( G \times G \to G, (x,y) \mapsto xy \) and inverse \( G \to G, x \mapsto x^{-1} \) are
continuous. A **homomorphism of topological groups** is a homomorphism of groups
which is continuous.

If \( G \) is a topological group and \( H \subset G \) is a subgroup, then \( H \) with the induced
topology is a topological group. If \( G \) is a topological group and \( G \to H \) is a
surjection of groups, then \( H \) endowed with the quotient topology is a topological
group.

**Lemma 29.2.** The category of topological groups has limits and limits commute
with the forgetful functors to (a) the category of topological spaces and (b) the cat-
egory of groups.

**Proof.** It is enough to prove the existence and commutation for products and
equalizers, see Categories, Lemma 14.10. Let \( G_i, i \in I \) be a collection of topological
groups. Take the usual product \( G = \prod G_i \) with the product topology. Since
\( G \times G = \prod (G_i \times G_i) \) as a topological space (because products commutes with
products in any category), we see that multiplication on \( G \) is continuous. Similarly
for the inverse map. Let \( a, b : G \to H \) be two homomorphisms of topological
groups. Then as the equalizer we can simply take the equalizer of \( a \) and \( b \) as maps
topological spaces, which is the same thing as the equalizer as maps of groups
endowed with the induced topology.

If \( G_1 \to G_2 \to G_3 \to \ldots \) is a system of topological groups then the colimi \( G = \)\( \mathrm{colim} G_n \) as a topological group (Lemma 29.3) is in general different from the colimit
as a topological space (Lemma 28.1) even though these have the same underlying
set. See Examples, Section 65.

**Lemma 29.3.** The category of topological groups has colimits and colimits commute
with the forgetful functor to the category of groups.

**Proof.** We will use the argument of Categories, Remark 25.2 to prove existence
of colimits. Namely, suppose that \( I \to \mathbf{Top}, i \mapsto G_i \) is a functor into the category
\textit{TopGroup} of topological groups. Then we can consider

\[ F : \text{TopGroup} \to \text{Sets}, \quad H \mapsto \lim_{i} \text{Mor}_{\text{TopGroup}}(G_i, H) \]

This functor commutes with limits. Moreover, given any topological group \( H \) and
an element \( (\phi_i : G_i \to H) \) of \( F(H) \), there is a subgroup \( H' \subset H \) of cardinality
at most \(|\coprod G_i|\) (coproduct in the category of groups, i.e., the free product on the
\( G_i \)) such that the morphisms \( \phi_i \) map into \( H' \). Namely, we can take the induced
topology on the subgroup generated by the images of the \( \phi_i \). Thus it is clear that
the hypotheses of Categories, Lemma 25.1 are satisfied and we find a topological
group \( G \) representing the functor \( F \), which precisely means that \( G \) is the colimit of
the diagram \( i \mapsto G_i \).
To see the statement on commutation with the forgetful functor to groups we will use Categories, Lemma 24.4. Indeed, the forgetful functor has a right adjoint, namely the functor which assigns to a group the corresponding chaotic (or indiscrete) topological group. □

**Definition 29.4.** A topological ring is a ring $R$ endowed with a topology such that addition $R \times R \to R$, $(x, y) \mapsto x + y$ and multiplication $R \times R \to R$, $(x, y) \mapsto xy$ are continuous. A homomorphism of topological rings is a homomorphism of rings which is continuous.

In the Stacks project rings are commutative with 1. If $R$ is a topological ring, then $(R, +)$ is a topological group since $x \mapsto -x$ is continuous. If $R$ is a topological ring and $R' \subset R$ is a subring, then $R'$ with the induced topology is a topological ring. If $R$ is a topological ring and $R \to R'$ is a surjection of rings, then $R'$ endowed with the quotient topology is a topological ring.

**Lemma 29.5.** The category of topological rings has limits and limits commute with the forgetful functors to (a) the category of topological spaces and (b) the category of rings.

**Proof.** It is enough to prove the existence and commutation for products and equalizers, see Categories, Lemma 14.10. Let $R_i$, $i \in I$ be a collection of topological rings. Take the usual product $R = \prod R_i$ with the product topology. Since $R \times R = \prod (R_i \times R_i)$ as a topological space (because products commutes with products in any category), we see that addition and multiplication on $R$ are continuous. Let $a, b : R \to R'$ be two homomorphisms of topological rings. Then as the equalizer we can simply take the equalizer of $a$ and $b$ as maps of topological spaces, which is the same thing as the equalizer as maps of rings endowed with the induced topology. □

**Lemma 29.6.** The category of topological rings has colimits and colimits commute with the forgetful functor to the category of rings.

**Proof.** The exact same argument as used in the proof of Lemma 29.3 shows existence of colimits. To see the statement on commutation with the forgetful functor to rings we will use Categories, Lemma 24.4. Indeed, the forgetful functor has a right adjoint, namely the functor which assigns to a ring the corresponding chaotic (or indiscrete) topological ring. □

**Definition 29.7.** Let $R$ be a topological ring. A topological module is an $R$-module $M$ endowed with a topology such that addition $M \times M \to M$ and scalar multiplication $R \times M \to M$ are continuous. A homomorphism of topological modules is a homomorphism of modules which is continuous.

If $R$ is a topological ring and $M$ is a topological module, then $(M, +)$ is a topological group since $x \mapsto -x$ is continuous. If $R$ is a topological ring, $M$ is a topological module and $M' \subset M$ is a submodule, then $M'$ with the induced topology is a topological module. If $R$ is a topological ring, $M$ is a topological module, and $M \to M'$ is a surjection of modules, then $M'$ endowed with the quotient topology is a topological module.

**Lemma 29.8.** Let $R$ be a topological ring. The category of topological modules over $R$ has limits and limits commute with the forgetful functors to (a) the category of topological spaces and (b) the category of $R$-modules.
Proof. It is enough to prove the existence and commutation for products and equalizers, see Categories, Lemma 14.10. Let \( M_i, \ i \in I \) be a collection of topological modules over \( R \). Take the usual product \( M = \prod M_i \) with the product topology. Since \( M \times M = \prod (M_i \times M_j) \) as a topological space (because products commutes with products in any category), we see that addition on \( M \) is continuous. Similarly for multiplication \( R \times M \to M \). Let \( a, b : M \to M' \) be two homomorphisms of topological modules over \( R \). Then as the equalizer we can simply take the equalizer of \( a \) and \( b \) as maps of topological spaces, which is the same thing as the equalizer as maps of modules endowed with the induced topology.

Lemma 29.9. Let \( R \) be a topological ring. The category of topological modules over \( R \) has colimits and colimits commute with the forgetful functor to the category of modules over \( R \).

Proof. The exact same argument as used in the proof of Lemma 29.3 shows existence of colimits. To see the statement on commutation with the forgetful functor to \( R \)-modules we will use Categories, Lemma 24.4. Indeed, the forgetful functor has a right adjoint, namely the functor which assigns to a module the corresponding chaotic (or indiscrete) topological module.

30. Other chapters
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- (64) Quotients of Groupoids
- (65) Simplicial Spaces
- (66) Formal Algebraic Spaces
- (67) Restricted Power Series
- (68) Resolution of Surfaces

### Deformation Theory

- (69) Formal Deformation Theory
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