ALGEBRAIC AND FORMAL GEOMETRY

0EI5

Contents

1. Introduction

0EI6 This chapter continues the study of formal algebraic geometry and in particular the question of whether a formal object is the completion of an algebraic one. A fundamental reference is [Gro68]. Here is a list of results we have already discussed in the Stacks project:

(1) The theorem on formal functions, see Cohomology of Schemes, Section 20.
(2) Coherent formal modules, see Cohomology of Schemes, Section 23.

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(3) Grothendieck's existence theorem, see Cohomology of Schemes, Sections 24, 25, and 27.
(4) Grothendieck's algebraization theorem, see Cohomology of Schemes, Section 28.
(5) Grothendieck's existence theorem more generally, see More on Flatness, Sections 28 and 29.

Let us give an overview of the contents of this chapter.

Let X be a scheme and let $\mathcal{I} \subset \mathcal{O}_X$ be a finite type quasi-coherent sheaf of ideals. Many questions in this chapter have to do with inverse systems $(\mathcal{F}_n)$ of quasi-coherent $\mathcal{O}_X$-modules such that $\mathcal{F}_n = \mathcal{F}_{n+1}/\mathcal{I}^n\mathcal{F}_{n+1}$. An important special case is where X is a scheme over a Noetherian ring A and $\mathcal{I} = I\mathcal{O}_X$ for some ideal $I \subset A$. In Section 2 we prove some elementary results on such systems of coherent modules. In Section 3 we discuss additional results when $I = (f)$ is a principal. In Section 4 we work in the slightly more general setting where $\text{cd}(A, I) = 1$. One of the themes of this chapter will be to show that results proven in the case $I = (f)$ also hold true when we only assume $\text{cd}(A, I) = 1$.

In Section 6 we discuss derived completion of modules on a ringed site $(\mathcal{C}, \mathcal{O})$ with respect to a finite type sheaf of ideals $\mathcal{I}$. This section is the natural continuation of the theory of derived completion in commutative algebra as described in More on Algebra, Section 82. The first main result is that derived completion exists. The second main result is that for a morphism f if ringed sites derived completion commutes with derived pushforward:

$$(Rf_*K)^\wedge = Rf_*(K^\wedge)$$

if the ideal sheaf upstairs is locally generated by sections coming from the ideal downstairs, see Lemma 6.19. We stress that both main results are very elementary in case the ideals in question are globally finitely generated which will be true for all applications of this theory in this chapter. The displayed equality is the "correct" version of the theorem on formal functions, see discussion in Section 7.

Let A be a Noetherian ring and let I, J be two ideals of A. Let M be a finite A-module. The next topic in this chapter is the map

$$R\Gamma_J(M) \longrightarrow R\Gamma_J(M)^\wedge$$

from local cohomology of M into the derived I-adic completion of the same. It turns out that if we impose suitable depth conditions this map becomes an isomorphism on cohomology in a range of degrees. In Section 8 we work essentially in the generality just mentioned. In Section 9 we assume A is a local ring and $J = \mathfrak{m}$ is a maximal ideal. We encourage the reader to read this section before the other two in this part of the chapter. Finally, in Section 10 we bootstrap the local case to obtain stronger results back in the general case.

In the next part of this chapter we use the results on completion of local cohomology to get a nonexhaustive list of results on cohomology of the completion of coherent modules. More precisely, let A be a Noetherian ring, let $I \subset A$ be an ideal, and let $U \subset \text{Spec}(A)$ be an open subscheme. If $\mathcal{F}$ is a coherent $\mathcal{O}_U$-module, then we may consider the maps

$$H^i(U, \mathcal{F}) \longrightarrow \lim H^i(U, \mathcal{F}/I^n\mathcal{F})$$

and ask if we get an isomorphism in a certain range of degrees. In Section 11 we work out some examples where U is the punctured spectrum of a local ring. In Section 12 we discuss the general case. In Section 14 we apply some of the results obtained to questions of connectedness in algebraic geometry.

The remaining sections of this chapter are devoted to a discussion of algebraization of coherent formal modules. In other words, given an inverse system of coherent modules $(\mathcal{F}_n)$ on U as above with $\mathcal{F}_n = \mathcal{F}_{n+1}/I^n\mathcal{F}_{n+1}$ we ask whether there exists a coherent $\mathcal{O}_U$-module $\mathcal{F}$ such that $\mathcal{F}_n = \mathcal{F}/I^n\mathcal{F}$ for all n. We encourage the reader to read Section 16 for a precise statement of the question, a useful general result (Lemma 16.10), and a nontrivial application (Lemma 16.11). To prove a result going essentially beyond this case quite a bit more theory has to be developed. Please see Section 22 for the strongest results of this type obtained in this chapter.

2. Formal sections, I

0EH3 Let A be a ring and $I \subset A$ an ideal. Let X be a scheme over $\text{Spec}(A)$. In this section we prove some general facts on inverse systems of $\mathcal{O}_X$-modules $\{\mathcal{F}_n\}$ such that $\mathcal{F}_n = \mathcal{F}_{n+1}/I^n\mathcal{F}_{n+1}$. In particular, we prove two lemmas on the behaviour of the inverse system $\{H^0(X, \mathcal{F}_n)\}$. These results have generalizations to higher cohomology groups which we will add here if we need them.

0EH4 **Lemma 2.1.** *Let I be an ideal of a ring A. Let X be a scheme over* $\text{Spec}(A)$. *Let*

$$\ldots \to \mathcal{F}_3 \to \mathcal{F}_2 \to \mathcal{F}_1$$

be an inverse system of $\mathcal{O}_X$-modules such that $\mathcal{F}_n = \mathcal{F}_{n+1}/I^n\mathcal{F}_{n+1}$. Assume

$$\bigoplus\nolimits_{n \geq 0} H^1(X, I^n\mathcal{F}_{n+1})$$

satisfies the ascending chain condition as a graded $\bigoplus_{n \geq 0} I^n/I^{n+1}$-module. Then the inverse system $M_n = \Gamma(X, \mathcal{F}_n)$ satisfies the Mittag-Leffler condition.

Proof. Set $H^1_n = H^1(X, I^n\mathcal{F}_{n+1})$ and let $\delta_n : M_n \to H^1_n$ be the boundary map on cohomology. Then $\bigoplus \text{Im}(\delta_n) \subset \bigoplus H^1_n$ is a graded submodule. Namely, if $s \in M_n$ and $f \in I^m$, then we have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & I^n\mathcal{F}_{n+1} \arrow[r] \arrow[d, "f"] & \mathcal{F}_{n+1} \arrow[r] \arrow[d, "f"] & \mathcal{F}_n \arrow[r] \arrow[d, "f"] & 0 \\
0 \arrow[r] & I^{n+m}\mathcal{F}_{n+m+1} \arrow[r] & \mathcal{F}_{n+m+1} \arrow[r] & \mathcal{F}_{n+m} \arrow[r] & 0
\end{tikzcd}$$

The middle vertical map is given by lifting a local section of $\mathcal{F}_{n+1}$ to a section of $\mathcal{F}_{n+m+1}$ and then multiplying by f; similarly for the other vertical arrows. We conclude that $\delta_{n+m}(fs) = f\delta_n(s)$. By assumption we can find $s_j \in M_{n_j}$, $j = 1, \ldots, N$ such that $\delta_{n_j}(s_j)$ generate $\bigoplus \text{Im}(\delta_n)$ as a graded module. Let $n > c = \max(n_j)$. Let $s \in M_n$. Then we can find $f_j \in I^{n-n_j}$ such that $\delta_n(s) = \sum f_j\delta_{n_j}(s_j)$. We conclude that $\delta(s - \sum f_js_j) = 0$, i.e., we can find $s' \in M_{n+1}$ mapping to $s - \sum f_js_j$ in M_n. It follows that

$$\text{Im}(M_{n+1} \to M_{n-c}) = \text{Im}(M_n \to M_{n-c})$$

This proves the lemma. $\square$

0EH5 **Lemma 2.2.** *Let I be an ideal of a ring A. Let X be a scheme over $\mathrm{Spec}(A)$. Let*

$$\ldots \to \mathcal{F}_3 \to \mathcal{F}_2 \to \mathcal{F}_1$$

be an inverse system of $\mathcal{O}_X$-modules such that $\mathcal{F}_n = \mathcal{F}_{n+1}/I^n\mathcal{F}_{n+1}$. Given n define

$$H^1_n = \bigcap\nolimits_{m \geq n} \mathrm{Im}\left(H^1(X, I^n\mathcal{F}_{m+1}) \to H^1(X, I^n\mathcal{F}_{n+1})\right)$$

If $\bigoplus H^1_n$ satisfies the ascending chain condition as a graded $\bigoplus_{n\geq 0} I^n/I^{n+1}$-module, then the inverse system $M_n = \Gamma(X, \mathcal{F}_n)$ satisfies the Mittag-Leffler condition.

Proof. The proof is exactly the same as the proof of Lemma 2.1. In fact, the result will follow from the arguments given there as soon as we show that $\bigoplus H^1_n$ is a graded $\bigoplus_{n\geq 0} I^n/I^{n+1}$-submodule of $\bigoplus H^1(X, I^n\mathcal{F}_{n+1})$ and that the boundary maps δ_n have image contained in H^1_n.

Suppose that $\xi \in H^1_n$ and $f \in I^k$. Choose $m \gg n+k$. Choose $\xi' \in H^1(X, I^n\mathcal{F}_{m+1})$ lifting ξ. We consider the diagram

$$\begin{tikzcd}
0 \arrow[r] & I^n\mathcal{F}_{m+1} \arrow[r] \arrow[d, "f"'] & \mathcal{F}_{m+1} \arrow[r] \arrow[d, "f"'] & \mathcal{F}_n \arrow[r] \arrow[d, "f"'] & 0 \\
0 \arrow[r] & I^{n+k}\mathcal{F}_{m+1} \arrow[r] & \mathcal{F}_{m+1} \arrow[r] & \mathcal{F}_{n+k} \arrow[r] & 0
\end{tikzcd}$$

constructed as in the proof of Lemma 2.1. We get an induced map on cohomology and we see that $f\xi' \in H^1(X, I^{n+k}\mathcal{F}_{m+1})$ maps to $f\xi$. Since this is true for all $m \gg n+k$ we see that $f\xi$ is in H^1_{n+k} as desired.

To see the boundary maps δ_n have image contained in H^1_n we consider the diagrams

$$\begin{tikzcd}
0 \arrow[r] & I^n\mathcal{F}_{m+1} \arrow[r] \arrow[d] & \mathcal{F}_{m+1} \arrow[r] \arrow[d] & \mathcal{F}_n \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & I^n\mathcal{F}_{n+1} \arrow[r] & \mathcal{F}_{n+1} \arrow[r] & \mathcal{F}_n \arrow[r] & 0
\end{tikzcd}$$

for $m \geq n$. Looking at the induced maps on cohomology we conclude. □

0EI7 **Lemma 2.3.** *Let I be a finitely generated ideal of a ring A. Let X be a scheme over $\mathrm{Spec}(A)$. Let*

$$\ldots \to \mathcal{F}_3 \to \mathcal{F}_2 \to \mathcal{F}_1$$

be an inverse system of $\mathcal{O}_X$-modules such that $\mathcal{F}_n = \mathcal{F}_{n+1}/I^n\mathcal{F}_{n+1}$. Assume

$$\bigoplus\nolimits_{n\geq 0} H^0(X, I^n\mathcal{F}_{n+1})$$

satisfies the ascending chain condition as a graded $\bigoplus_{n\geq 0} I^n/I^{n+1}$-module. Then the limit topology on $M = \lim \Gamma(X, \mathcal{F}_n)$ is the I-adic topology.

Proof. Set $F^n = \mathrm{Ker}(M \to H^0(X, \mathcal{F}_n))$ for $n \geq 1$ and $F^0 = M$. Observe that $IF^n \subset F^{n+1}$. In particular $I^nM \subset F^n$ and we are trying to show that given n there exists an $m \geq n$ such that $F^m \subset I^nM$. We have an injective map of graded modules

$$\bigoplus\nolimits_{n\geq 0} F^n/F^{n+1} \longrightarrow \bigoplus\nolimits_{n\geq 0} H^0(X, I^n\mathcal{F}_{n+1})$$

By assumption the left hand side is generated by finitely many homogeneous elements. Hence we can find r and $c_1, \ldots, c_r \geq 0$ and $a_i \in F^{c_i}$ whose images in $\bigoplus F^n/F^{n+1}$ generate. Set $c = \max(c_i)$.

For $n \geq c$ we claim that $IF^n = F^{n+1}$. Namely, suppose $a \in F^{n+1}$. The image of a in F^{n+1}/F^{n+2} is a linear combination of our a_i. Therefore $a - \sum f_i a_i \in F^{n+2}$ for some $f_i \in I^{n+1-c_i}$. Since $I^{n+1-c_i} = I \cdot I^{n-c_i}$ as $n \geq c_i$ we can write $f_i = \sum g_{i,j} h_{i,j}$ with $g_{i,j} \in I$ and $h_{i,j} a_i \in F^n$. Thus we see that $F^{n+1} = F^{n+2} + IF^n$. A simple induction argument gives $F^{n+1} = F^{n+e} + IF^n$ for all $e > 0$. It follows that IF^n is dense in F^{n+1}. Choose generators $k_1, \ldots, k_r$ of I and consider the continuous map

$$u : (F^n)^{\oplus r} \longrightarrow F^{n+1}, \quad (x_1, \ldots, x_r) \mapsto \sum k_i x_i$$

(in the limit topology). By the above the image of $(F^m)^{\oplus r}$ under u is dense in F^{m+1} for all $m \geq n$. By the open mapping lemma (More on Algebra, Lemma 35.5) we find that u is open. Hence u is surjective. Hence $IF^n = F^{n+1}$ for $n \geq c$. This concludes the proof. □

0EI8 **Lemma 2.4.** *Let X be a scheme. Let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. Let*

$$\ldots \to \mathcal{F}_3 \to \mathcal{F}_2 \to \mathcal{F}_1$$

be an inverse system of quasi-coherent $\mathcal{O}_X$-modules such that $\mathcal{F}_n = \mathcal{F}_{n+1}/\mathcal{I}^n\mathcal{F}_{n+1}$. Set $\mathcal{F} = \lim \mathcal{F}_n$. Then

1. *$\mathcal{F} = R\lim \mathcal{F}_n$,*
2. *for any affine open $U \subset X$ we have $H^p(U, \mathcal{F}) = 0$ for $p > 0$, and*
3. *for each p there is a short exact sequence $0 \to R^1 \lim H^{p-1}(X, \mathcal{F}_n) \to H^p(X, \mathcal{F}) \to \lim H^p(X, \mathcal{F}_n) \to 0$.*

If moreover $\mathcal{I}$ is of finite type, then

4. *$\mathcal{F}_n = \mathcal{F}/\mathcal{I}^n\mathcal{F}$, and*
5. *$\mathcal{I}^n\mathcal{F} = \lim_{m \geq n} \mathcal{I}^n\mathcal{F}_m$.*

Proof. Parts (1), (2), and (3) are general facts about inverse systems of quasi-coherent modules with surjective transition maps, see Derived Categories of Schemes, Lemma 3.2 and Cohomology, Lemma 32.1. Next, assume $\mathcal{I}$ is of finite type. Let $U \subset X$ be affine open. Say $U = \mathrm{Spec}(A)$ and $\mathcal{I}|_U$ corresponds to $I \subset A$. Observe that I is a finitely generated ideal. By the equivalence of categories between quasi-coherent $\mathcal{O}_U$-modules and A-modules (Schemes, Lemma 7.5) we find that $M_n = \mathcal{F}_n(U)$ is an inverse system of A-modules with $M_n = M_{n+1}/I^n M_{n+1}$. Thus

$$M = \mathcal{F}(U) = \lim \mathcal{F}_n(U) = \lim M_n$$

is an I-adically complete module with $M/I^nM = M_n$ by Algebra, Lemma 97.1. This proves (4). Part (5) translates into the statement that $\lim_{m \geq n} I^nM/I^mM = I^nM$. Since $I^mM = I^{m-n} \cdot I^nM$ this is just the statement that I^mM is I-adically complete. This follows from Algebra, Lemma 95.3 and the fact that M is complete. □

3. Formal sections, II

0BLA In this section we ask if completion and taking cohomology commute for sheaves of modules on schemes over an affine base A when completion is with respect to a principal ideal in A. Of course, we have already discussed the theorem on formal functions in Cohomology of Schemes, Section 20. Moreover, we will see in Section 7 that derived completion commutes with derived cohomology in great generality. In this section we just collect a few simple special cases of this material that will help us with future developments.

0EH9 **Lemma 3.1.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $f \in \Gamma(X, \mathcal{O}_X)$. Let*

$$\ldots \to \mathcal{F}_3 \to \mathcal{F}_2 \to \mathcal{F}_1$$

be inverse system of $\mathcal{O}_X$-modules. The following are equivalent

(1) *for all $n \geq 1$ the map $f : \mathcal{F}_{n+1} \to \mathcal{F}_{n+1}$ factors through $\mathcal{F}_{n+1} \to \mathcal{F}_n$ to give a short exact sequence $0 \to \mathcal{F}_n \to \mathcal{F}_{n+1} \to \mathcal{F}_1 \to 0$,*
(2) *for all $n \geq 1$ the map $f^n : \mathcal{F}_{n+1} \to \mathcal{F}_{n+1}$ factors through $\mathcal{F}_{n+1} \to \mathcal{F}_1$ to give a short exact sequence $0 \to \mathcal{F}_1 \to \mathcal{F}_{n+1} \to \mathcal{F}_n \to 0$*
(3) *there exists an $\mathcal{O}_X$-module $\mathcal{G}$ which is f-divisible such that $\mathcal{F}_n = \mathcal{G}[f^n]$.*

If X is a scheme and $\mathcal{F}_n$ is quasi-coherent, then these are also equivalent to

(4) *there exists an $\mathcal{O}_X$-module $\mathcal{F}$ which is f-torsion free such that $\mathcal{F}_n = \mathcal{F}/f^n\mathcal{F}$.*

Proof. We omit the proof of the equivalence of (1) and (2). The condition that $\mathcal{G}$ is f-divisible means that $f : \mathcal{G} \to \mathcal{G}$ is surjective. Thus given $\mathcal{F}_n$ as in (1) we set $\mathcal{G} = \operatorname{colim} \mathcal{F}_n$ where the maps $\mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to \ldots$ are as in (1). This produces an f-divisible $\mathcal{O}_X$-module with $\mathcal{F}_n = \mathcal{G}[f^n]$ as can be seen by checking on stalks. The condition that $\mathcal{F}$ is f-torsion free means that $f : \mathcal{F} \to \mathcal{F}$ is injective. If X is a scheme and $\mathcal{F}_n$ is quasi-coherent, then we set $\mathcal{F} = \lim \mathcal{F}_n$. Namely, for an affine open $U \subset X$ the transition maps $\mathcal{F}_{n+1}(U) \to \mathcal{F}_n(U)$ are surjective by vanishing of higher cohomology. This produces an f-torsion free $\mathcal{O}_X$-module with $\mathcal{F}_n = \mathcal{F}/f^n\mathcal{F}$ (Lemma 2.4). $\square$

0EHA **Lemma 3.2.** *Suppose X, f, $(\mathcal{F}_n)$ is as in Lemma 3.1. Then the limit topology on $H^p = \lim H^p(X, \mathcal{F}_n)$ is the f-adic topology.*

Proof. Namely, it is clear that $f^t H^p$ maps to zero in $H^p(X, \mathcal{F}_t)$. On the other hand, let $c \geq 1$. If $\xi = (\xi_n) \in H^p$ is small in the limit topology, then $\xi_c = 0$, and hence ξ_n maps to zero in $H^p(X, \mathcal{F}_c)$ for $n \geq c$. Consider the inverse system of short exact sequences

$$0 \to \mathcal{F}_{n-c} \xrightarrow{f^c} \mathcal{F}_n \to \mathcal{F}_c \to 0$$

and the corresponding inverse system of long exact cohomology sequences

$$H^{p-1}(X, \mathcal{F}_c) \to H^p(X, \mathcal{F}_{n-c}) \to H^p(X, \mathcal{F}_n) \to H^p(X, \mathcal{F}_c)$$

Since the term $H^{p-1}(X, \mathcal{F}_c)$ is independent of n we can choose a compatible sequence of elements $\xi'_n \in H^1(X, \mathcal{F}_{n-c})$ lifting ξ_n. Setting $\xi' = (\xi'_n)$ we see that $\xi = f^{c+1}\xi'$. This even shows that $f^c H^p = \operatorname{Ker}(H^p \to H^p(X, \mathcal{F}_c))$ on the nose. $\square$

0BLB **Lemma 3.3.** *Let A be a Noetherian ring complete with respect to a principal ideal (f). Let X be a scheme over* $\operatorname{Spec}(A)$. *Let*

$$\ldots \to \mathcal{F}_3 \to \mathcal{F}_2 \to \mathcal{F}_1$$

be an inverse system of $\mathcal{O}_X$-modules. Assume

(1) $\Gamma(X, \mathcal{F}_1)$ *is a finite A-module,*
(2) *the equivalent conditions of Lemma 3.1 hold.*

Then

$$M = \lim \Gamma(X, \mathcal{F}_n)$$

is a finite A-module, f is a nonzerodivisor on M, and M/fM is the image of M in $\Gamma(X, \mathcal{F}_1)$.

Proof. By Lemma 3.2 and its proof we have $M/fM \subset H^0(X, \mathcal{F}_1)$. From (1) and the Noetherian property of A we get that M/fM is a finite A-module. Observe that $\bigcap f^n M = 0$ as $f^n M$ maps to zero in $H^0(X, \mathcal{F}_n)$. By Algebra, Lemma 95.12 we conclude that M is finite over A. $\square$

0BLC **Lemma 3.4.** *Let A be a ring. Let $f \in A$. Let X be a scheme over* $\mathrm{Spec}(A)$. *Let*

$$\ldots \to \mathcal{F}_3 \to \mathcal{F}_2 \to \mathcal{F}_1$$

be an inverse system of $\mathcal{O}_X$-modules. Assume

1. *either $H^1(X, \mathcal{F}_1)$ is an A-module of finite length or A is Noetherian and $H^1(X, \mathcal{F}_1)$ is a finite A-module,*
2. *the equivalent conditions of Lemma 3.1 hold.*

Then the inverse system $M_n = \Gamma(X, \mathcal{F}_n)$ satisfies the Mittag-Leffler condition.

Proof. Set $I = (f)$. We will use the criterion of Lemma 2.1. Observe that $f^n : \mathcal{F}_0 \to I^n\mathcal{F}_{n+1}$ is an isomorphism for all $n \geq 0$. Thus it suffices to show that

$$\bigoplus\nolimits_{n \geq 1} H^1(X, \mathcal{F}_1) \cdot f^{n+1}$$

is a graded $S = \bigoplus_{n \geq 0} A/(f) \cdot f^n$-module satisfying the ascending chain condition. If A is not Noetherian, then $H^1(X, \mathcal{F}_1)$ has finite length and the result holds. If A is Noetherian, then S is a Noetherian ring and the result holds as the module is finite over S by the assumed finiteness of $H^1(X, \mathcal{F}_1)$. Some details omitted. $\square$

0DXG **Lemma 3.5.** *Let A be a ring. Let $f \in A$. Let X be a scheme over* $\mathrm{Spec}(A)$. *Let*

$$\ldots \to \mathcal{F}_3 \to \mathcal{F}_2 \to \mathcal{F}_1$$

be an inverse system of $\mathcal{O}_X$-modules. Assume

1. *either there is an $m \geq 1$ such that the image of $H^1(X, \mathcal{F}_m) \to H^1(X, \mathcal{F}_1)$ is an A-module of finite length or A is Noetherian and the intersection of the images of $H^1(X, \mathcal{F}_m) \to H^1(X, \mathcal{F}_1)$ is a finite A-module,*
2. *the equivalent conditions of Lemma 3.1 hold.*

Then the inverse system $M_n = \Gamma(X, \mathcal{F}_n)$ satisfies the Mittag-Leffler condition.

Proof. Set $I = (f)$. We will use the criterion of Lemma 2.2 involving the modules H^1_n. For $m \geq n$ we have $I^n\mathcal{F}_{m+1} = \mathcal{F}_{m+1-n}$. Thus we see that

$$H^1_n = \bigcap\nolimits_{m \geq 1} \mathrm{Im}\left(H^1(X, \mathcal{F}_m) \to H^1(X, \mathcal{F}_1)\right)$$

is independent of n and $\bigoplus H^1_n = \bigoplus H^1_1 \cdot f^{n+1}$. Thus we conclude exactly as in the proof of Lemma 3.4. $\square$

0BLD **Lemma 3.6.** *Let A be a ring and $f \in A$. Let X be a scheme over A. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Assume that $\mathcal{F}[f^n] = \mathrm{Ker}(f^n : \mathcal{F} \to \mathcal{F})$ stabilizes. Then* [BdJ14, Lemma 1.6]

$$R\Gamma(X, \lim \mathcal{F}/f^n\mathcal{F}) = R\Gamma(X, \mathcal{F})^\wedge$$

where the right hand side indicates the derived completion with respect to the ideal $(f) \subset A$. Let H^p be the pth cohomology group of this complex. Then there are short exact sequences

$$0 \to R^1 \lim H^{p-1}(X, \mathcal{F}/f^n\mathcal{F}) \to H^p \to \lim H^p(X, \mathcal{F}/f^n\mathcal{F}) \to 0$$

and

$$0 \to H^0(H^p(X, \mathcal{F})^\wedge) \to H^p \to T_f(H^{p+1}(X, \mathcal{F})) \to 0$$

where $T_f(-)$ denote the f-adic Tate module as in More on Algebra, Example 83.4.

Proof. We start with the canonical identifications

$$\begin{aligned} R\Gamma(X, \mathcal{F})^\wedge &= R\lim R\Gamma(X, \mathcal{F}) \otimes_A^{\mathbf{L}} (A \xrightarrow{f^n} A) \\ &= R\lim R\Gamma(X, \mathcal{F} \xrightarrow{f^n} \mathcal{F}) \\ &= R\Gamma(X, R\lim(\mathcal{F} \xrightarrow{f^n} \mathcal{F})) \end{aligned}$$

The first equality holds by More on Algebra, Lemma 82.17. The second by the projection formula, see Cohomology, Lemma 45.3. The third by Cohomology, Lemma 32.2. Note that by Derived Categories of Schemes, Lemma 3.2 we have $\lim \mathcal{F}/f^n\mathcal{F} = R\lim \mathcal{F}/f^n\mathcal{F}$. Thus to finish the proof of the first statement of the lemma it suffices to show that the pro-objects $(f^n : \mathcal{F} \to \mathcal{F})$ and $(\mathcal{F}/f^n\mathcal{F})$ are isomorphic. There is clearly a map from the first inverse system to the second. Suppose that $\mathcal{F}[f^c] = \mathcal{F}[f^{c+1}] = \mathcal{F}[f^{c+2}] = \ldots$. Then we can define an arrow of inverse systems in $D(\mathcal{O}_X)$ in the other direction by the diagrams

$$\begin{tikzcd} \mathcal{F}/\mathcal{F}[f^c] \arrow[r, "f^{n+c}"'] \arrow[d, "f^c"'] & \mathcal{F} \arrow[d, "1"] \\ \mathcal{F} \arrow[r, "f^n"] & \mathcal{F} \end{tikzcd}$$

Since the top horizontal arrow is injective the complex in the top row is quasi-isomorphic to $\mathcal{F}/f^{n+c}\mathcal{F}$. Some details omitted.

Since $R\Gamma(X, -)$ commutes with derived limits (Injectives, Lemma 13.6) we see that

$$R\Gamma(X, \lim \mathcal{F}/f^n\mathcal{F}) = R\Gamma(X, R\lim \mathcal{F}/f^n\mathcal{F}) = R\lim R\Gamma(X, \mathcal{F}/f^n\mathcal{F})$$

(for first equality see first paragraph of proof). By More on Algebra, Remark 77.9 we obtain exact sequences

$$0 \to R^1\lim H^{p-1}(X, \mathcal{F}/f^n\mathcal{F}) \to H^p(X, \lim \mathcal{F}/I^n\mathcal{F}) \to \lim H^p(X, \mathcal{F}/I^n\mathcal{F}) \to 0$$

of A-modules. The second set of short exact sequences follow immediately from the discussion in More on Algebra, Example 83.4. □

4. Formal sections, III

0EI9 In this section we generalize some of the results of Section 3 to the case of an ideal $I \subset A$ of cohomological dimension 1.

0EIA **Lemma 4.1.** *Let $I = (f_1, \ldots, f_r)$ be an ideal of a Noetherian ring A. If $cd(A, I) = 1$, then there exist $c \geq 1$ and maps $\varphi_j : I^c \to A$ such that $\sum f_j\varphi_j : I^c \to I$ is the inclusion map.*

Proof. Since $\text{cd}(A, I) = 1$ the complement $U = \text{Spec}(A) \setminus V(I)$ is affine (Local Cohomology, Lemma 3.8). Say $U = \text{Spec}(B)$. Then $IB = B$ and we can write $1 = \sum_{j=1,\ldots,r} f_jb_j$ for some $b_j \in B$. By Cohomology of Schemes, Lemma 10.4 we can represent b_j by maps $\varphi_j : I^c \to A$ for some $c \geq 0$. Then $\sum f_j\varphi_j : I^c \to I \subset A$ is the canonical embedding, after possibly replacing c by a larger integer, by the same lemma. □

0EIB **Lemma 4.2.** *Let $I = (f_1, \ldots, f_r)$ be an ideal of a Noetherian ring A with $cd(A, I) = 1$. Let $c \geq 1$ and $\varphi_j : I^c \to A$, $j = 1, \ldots, r$ be as in Lemma 4.1. Then there is a unique graded A-algebra map*

$$\Phi : \bigoplus\nolimits_{n \geq 0} I^{nc} \to A[T_1, \ldots, T_r]$$

with $\Phi(g) = \sum \varphi_j(g)T_j$ for $g \in I^c$. Moreover, the composition of Φ with the map $A[T_1, \ldots, T_r] \to \bigoplus_{n \geq 0} I^n$, $T_j \mapsto f_j$ is the inclusion map $\bigoplus_{n \geq 0} I^{nc} \to \bigoplus_{n \geq 0} I^n$.

Proof. For each j and $m \geq c$ the restriction of φ_j to I^m is a map $\varphi_j : I^m \to I^{m-c}$. Given $j_1, \ldots, j_n \in \{1, \ldots, r\}$ we claim that the composition

$$\varphi_{j_1} \ldots \varphi_{j_n} : I^{nc} \to I^{(n-1)c} \to \ldots \to I^c \to A$$

is independent of the order of the indices $j_1, \ldots, j_n$. Namely, if $g = g_1 \ldots g_n$ with $g_i \in I^c$, then we see that

$$(\varphi_{j_1} \ldots \varphi_{j_n})(g) = \varphi_{j_1}(g_1) \ldots \varphi_{j_n}(g_n)$$

is independent of the ordering as multiplication in A is commutative. Thus we can define Φ by sending $g \in I^{nc}$ to

$$\Phi(g) = \sum\nolimits_{e_1 + \ldots + e_r = n} (\varphi_1^{e_1} \circ \ldots \circ \varphi_r^{e_r})(g) T_1^{e_1} \ldots T_r^{e_r}$$

It is straightforward to prove that this is a graded A-algebra homomorphism with the desired property. Uniqueness is immediate as is the final property. This proves the lemma. □

0EIC **Lemma 4.3.** *Let $I = (f_1, \ldots, f_r)$ be an ideal of a Noetherian ring A with $cd(A, I) = 1$. Let $c \geq 1$ and $\varphi_j : I^c \to A$, $j = 1, \ldots, r$ be as in Lemma 4.1. Let $A \to B$ be a ring map with B Noetherian and let N be a finite B-module. Then, after possibly increasing c and adjusting φ_j accordingly, there is a unique unique graded B-module map*

$$\Phi_N : \bigoplus\nolimits_{n \geq 0} I^{nc} N \to N[T_1, \ldots, T_r]$$

with $\Phi_N(gx) = \Phi(g)x$ for $g \in I^{nc}$ and $x \in N$ where Φ is as in Lemma 4.2. The composition of Φ_N with the map $N[T_1, \ldots, T_r] \to \bigoplus_{n \geq 0} I^n N$, $T_j \mapsto f_j$ is the inclusion map $\bigoplus_{n \geq 0} I^{nc} N \to \bigoplus_{n \geq 0} I^n N$.

Proof. The uniqueness is clear from the formula and the uniqueness of Φ in Lemma 4.2. Consider the Noetherian A-algebra $B' = B \oplus N$ where N is an ideal of square zero. To show the existence of Φ_N it is enough (via Lemma 4.1) to show that φ_j extends to a map $\varphi'_j : I^c B' \to B'$ after possibly increasing c to some c' (and replacing φ_j by the composition of the inclusion $I^{c'} \to I^c$ with φ_j). Recall that φ_j corresponds to a section

$$h_j \in \Gamma(\mathrm{Spec}(A) \setminus V(I), \mathcal{O}_{\mathrm{Spec}(A)})$$

see Cohomology of Schemes, Lemma 10.4. (This is in fact how we chose our φ_j in the proof of Lemma 4.1.) Let us use the same lemma to represent the pullback

$$h'_j \in \Gamma(\mathrm{Spec}(B') \setminus V(IB'), \mathcal{O}_{\mathrm{Spec}(B')})$$

of h_j by a B'-linear map $\varphi'_j : I^{c'} B' \to B'$ for some $c' \geq c$. The agreement with φ_j will hold for c' sufficiently large by a further application of the lemma: namely we can test agreement on a finite list of generators of $I^{c'}$. Small detail omitted. □

0EH6 **Lemma 4.4.** *Let $I = (f_1, \ldots, f_r)$ be an ideal of a Noetherian ring A with $cd(A, I) = 1$. Let $c \geq 1$ and $\varphi_j : I^c \to A$, $j = 1, \ldots, r$ be as in Lemma 4.1. Let X be a Noetherian scheme over* $\mathrm{Spec}(A)$. *Let*

$$\ldots \to \mathcal{F}_3 \to \mathcal{F}_2 \to \mathcal{F}_1$$

be an inverse system of coherent $\mathcal{O}_X$-modules such that $\mathcal{F}_n = \mathcal{F}_{n+1}/I^n\mathcal{F}_{n+1}$. Set $\mathcal{F} = \lim \mathcal{F}_n$. Then, after possibly increasing c and adjusting φ_j accordingly, there exists a unique graded $\mathcal{O}_X$-module map

$$\Phi_{\mathcal{F}} : \bigoplus\nolimits_{n \geq 0} I^{nc}\mathcal{F} \longrightarrow \mathcal{F}[T_1, \ldots, T_r]$$

with $\Phi_{\mathcal{F}}(gs) = \Phi(g)s$ for $g \in I^{nc}$ and s a local section of $\mathcal{F}$ where Φ is as in Lemma 4.2. The composition of $\Phi_{\mathcal{F}}$ with the map $\mathcal{F}[T_1, \ldots, T_r] \to \bigoplus_{n \geq 0} I^n\mathcal{F}$, $T_j \mapsto f_j$ is the canonical inclusion $\bigoplus_{n \geq 0} I^{nc}\mathcal{F} \to \bigoplus_{n \geq 0} I^n\mathcal{F}$.

Proof. The uniqueness is immediate from the $\mathcal{O}_X$-linearity and the requirement that $\Phi_{\mathcal{F}}(gs) = \Phi(g)s$ for $g \in I^{nc}$ and s a local section of $\mathcal{F}$. Thus we may assume $X = \mathrm{Spec}(B)$ is affine. Observe that $(\mathcal{F}_n)$ is an object of the category $Coh(X, I\mathcal{O}_X)$ introduced in Cohomology of Schemes, Section 23. Let $B' = B^\wedge$ be the I-adic completion of B. By Cohomology of Schemes, Lemma 23.1 the object $(\mathcal{F}_n)$ corresponds to a finite B'-module N in the sense that $\mathcal{F}_n$ is the coherent module associated to the finite B-module N/I^nN. Applying Lemma 4.3 to $I \subset A \to B'$ and N we see that, after possibly increasing c and adjusting φ_j accordingly, we get unique maps

$$\Phi_N : \bigoplus\nolimits_{n \geq 0} I^{nc}N \to N[T_1, \ldots, T_r]$$

with the corresponding properties. Note that in degree n we obtain an inverse system of maps $N/I^mN \to \bigoplus_{e_1+\ldots+e_r=n} N/I^{m-nc}N \cdot T_1^{e_1} \ldots T_r^{e_r}$ for $m \geq nc$. Translating back into coherent sheaves we see that Φ_N corresponds to a system of maps

$$\Phi^n_m : I^{nc}\mathcal{F}_m \longrightarrow \bigoplus\nolimits_{e_1+\ldots+e_r=n} \mathcal{F}_{m-nc} \cdot T_1^{e_1} \ldots T_r^{e_r}$$

for varying $m \geq nc$ and $n \geq 1$. Taking the inverse limit of these maps over m we obtain $\Phi_{\mathcal{F}} = \bigoplus_n \lim_m \Phi^n_m$. Note that $\lim_m I^t\mathcal{F}_m = I^t\mathcal{F}$ as can be seen by evaluating on affines for example, but in fact we don't need this because it is clear there is a map $I^t\mathcal{F} \to \lim_m I^t\mathcal{F}_m$. □

0EH7 **Lemma 4.5.** *Let I be an ideal of a Noetherian ring A. Let X be a Noetherian scheme over* $\mathrm{Spec}(A)$. *Let*

$$\ldots \to \mathcal{F}_3 \to \mathcal{F}_2 \to \mathcal{F}_1$$

be an inverse system of coherent $\mathcal{O}_X$-modules such that $\mathcal{F}_n = \mathcal{F}_{n+1}/I^n\mathcal{F}_{n+1}$. If $cd(A, I) = 1$, then for all $p \in \mathbf{Z}$ the limit topology on $\lim H^p(X, \mathcal{F}_n)$ is I-adic.

Proof. First it is clear that $I^t \lim H^p(X, \mathcal{F}_n)$ maps to zero in $H^p(X, \mathcal{F}_t)$. Thus the I-adic topology is finer than the limit topology. For the converse we set $\mathcal{F} = \lim \mathcal{F}_n$, we pick generators $f_1, \ldots, f_r$ of I, we pick $c \geq 1$, and we choose $\Phi_{\mathcal{F}}$ as in Lemma 4.4. We will use the results of Lemma 2.4 without further mention. In particular we have a short exact sequence

$$0 \to R^1 \lim H^{p-1}(X, \mathcal{F}_n) \to H^p(X, \mathcal{F}) \to \lim H^p(X, \mathcal{F}_n) \to 0$$

Thus we can lift any element ξ of $\lim H^p(X, \mathcal{F}_n)$ to an element $\xi' \in H^p(X, \mathcal{F})$. Suppose ξ maps to zero in $H^p(X, \mathcal{F}_{nc})$ for some n, in other words, suppose ξ is "small" in the limit topology. We have a short exact sequence

$$0 \to I^{nc}\mathcal{F} \to \mathcal{F} \to \mathcal{F}_{nc} \to 0$$

and hence the assumption means we can lift ξ' to an element $\xi'' \in H^p(X, I^{nc}\mathcal{F})$. Applying $\Phi_\mathcal{F}$ we get

$$\Phi_\mathcal{F}(\xi'') = \sum\nolimits_{e_1+\ldots+e_r=n} \xi'_{e_1,\ldots,e_r} \cdot T_1^{e_1} \ldots T_r^{e_r}$$

for some $\xi'_{e_1,\ldots,e_r} \in H^p(X, \mathcal{F})$. Letting $\xi_{e_1,\ldots,e_r} \in \lim H^p(X, \mathcal{F}_n)$ be the images and using the final assertion of Lemma 4.4 we conclude that

$$\xi = \sum f_1^{e_1} \ldots f_r^{e_r} \xi_{e_1,\ldots,e_r}$$

is in $I^n \lim H^p(X, \mathcal{F}_n)$ as desired. $\square$

0EH8 **Example 4.6.** Let k be a field. Let $A = k[x, y][[s, t]]/(xs - yt)$. Let $I = (s, t)$ and $\mathfrak{a} = (x, y, s, t)$. Let $X = \mathrm{Spec}(A) - V(\mathfrak{a})$ and $\mathcal{F}_n = \mathcal{O}_X/I^n\mathcal{O}_X$. Observe that the rational function

$$g = \frac{t}{x} = \frac{s}{y}$$

is regular in an open neighbourhood $V \subset X$ of $V(I\mathcal{O}_X)$. Hence every power g^e determines a section $g^e \in M = \lim H^0(X, \mathcal{F}_n)$. Observe that $g^e \to 0$ as $e \to \infty$ in the limit topology on M since g^e maps to zero in $\mathcal{F}_e$. On the other hand, $g^e \notin IM$ for any e as the reader can see by computing $H^0(U, \mathcal{F}_n)$; computation omitted. Observe that $\mathrm{cd}(A, I) = 2$. Thus the result of Lemma 4.5 is sharp.

5. Mittag-Leffler conditions

0EFN When taking local cohomology with respect to the maximal ideal of a local Noetherian ring, we often get the Mittag-Leffler condition for free. This implies the same thing is true for higher cohomology groups of an inverse system of coherent sheaves with surjective transition maps on the puncture spectrum.

0DX0 **Lemma 5.1.** *Let $(A, \mathfrak{m})$ be a Noetherian local ring.*

1. *Let M be a finite A-module. Then the A-module $H^i_\mathfrak{m}(M)$ satisfies the descending chain condition for any i.*
2. *Let $U = \mathrm{Spec}(A) \setminus \{\mathfrak{m}\}$ be the punctured spectrum of A. Let $\mathcal{F}$ be a coherent $\mathcal{O}_U$-module. Then the A-module $H^i(U, \mathcal{F})$ satisfies the descending chain condition for $i > 0$.*

Proof. Proof of (1). Let $A^\wedge$ be the completion of A. Since $H^i_\mathfrak{m}(M)$ is $\mathfrak{m}$-power torsion, we see that $H^i_\mathfrak{m}(M) = H^i_\mathfrak{m}(M) \otimes_A A^\wedge$. Moreover, we have $H^i_\mathfrak{m}(M) \otimes_A A^\wedge = H^i_{\mathfrak{m}A^\wedge}(M \otimes_A A^\wedge)$ by Dualizing Complexes, Lemma 9.3. Thus

$$H^i_\mathfrak{m}(M) = H^i_{\mathfrak{m}A^\wedge}(M \otimes_A A^\wedge)$$

and A-submodules of the left hand side are the same thing as $A^\wedge$-submodules of the right hand side. Thus we reduce to the case discussed in the next paragraph.

Assume A is complete. Then A has a normalized dualizing complex $\omega_A^\bullet$ (Dualizing Complexes, Lemma 22.4). By the local duality theorem (Dualizing Complexes, Lemma 18.4) we find an isomorphism

$$\mathrm{Hom}_A(H^i_\mathfrak{m}(M), E) = \mathrm{Ext}_A^{-i}(M, \omega_A^\bullet)^\wedge$$

where E is an injective hull of the residue field of A. The module $\text{Ext}_A^{-i}(M, \omega_A^\bullet)$ on the right hand side is a finite A-module by Dualizing Complexes, Lemma 15.2. Since A is complete, the completion isn't necessary. Thus $H^i_\mathfrak{m}(M)$ has the descending chain condition by Matlis duality, see Dualizing Complexes, Proposition 7.8 and its addendum Remark 7.9.

Part (2) follows from (1) via Local Cohomology, Lemma 7.2. □

0DX1 **Lemma 5.2.** *Let $(A, \mathfrak{m})$ be a Noetherian local ring.*

1. *Let (M_n) be an inverse system of finite A-modules. Then the inverse system $H^i_\mathfrak{m}(M_n)$ satisfies the Mittag-Leffler condition for any i.*
2. *Let $U = \text{Spec}(A) \setminus \{\mathfrak{m}\}$ be the punctured spectrum of A. Let $\mathcal{F}_n$ be an inverse system of coherent $\mathcal{O}_U$-modules. Then the inverse system $H^i(U, \mathcal{F}_n)$ satisfies the Mittag-Leffler condition for $i > 0$.*

Proof. Follows immediately from Lemma 5.1. □

0EHB **Lemma 5.3.** *Let $(A, \mathfrak{m})$ be a Noetherian local ring. Let (M_n) be an inverse system of finite A-modules. Let $M \to \lim M_n$ be a map where M is a finite A-module such that for some i the map $H^i_\mathfrak{m}(M) \to \lim H^i_\mathfrak{m}(M_n)$ is an isomorphism. Then the inverse system $H^i_\mathfrak{m}(M_n)$ is essentially constant with value $H^i_\mathfrak{m}(M)$.*

Proof. By Lemma 5.2 the inverse system $H^i_\mathfrak{m}(M_n)$ satisfies the Mittag-Leffler condition. Let $E_n \subset H^i_\mathfrak{m}(M_n)$ be the image of $H^i_\mathfrak{m}(M_{n'})$ for $n' \gg n$. Then (E_n) is an inverse system with surjective transition maps and $H^i_\mathfrak{m}(M) = \lim E_n$. Since $H^i_\mathfrak{m}(M)$ has the descending chain condition by Lemma 5.1 we find there can only be a finite number of nontrivial kernels of the surjections $H^i_\mathfrak{m}(M) \to E_n$. Thus $E_n \to E_{n-1}$ is an isomorphism for all $n \gg 0$ as desired. □

0DXJ **Lemma 5.4.** *Let $(A, \mathfrak{m})$ be a Noetherian local ring. Let $I \subset A$ be an ideal. Let M be a finite A-module. Then*

$$H^i(R\Gamma_\mathfrak{m}(M)^\wedge) = \lim H^i_\mathfrak{m}(M/I^nM)$$

for all i where $R\Gamma_\mathfrak{m}(M)^\wedge$ denotes the derived I-adic completion.

Proof. Apply Dualizing Complexes, Lemma 12.4 and Lemma 5.2 to see the vanishing of the $R^1 \lim$ terms. □

6. Derived completion on a ringed site

0995 We urge the reader to skip this section on a first reading.

The algebra version of this material can be found in More on Algebra, Section 82. Let $\mathcal{O}$ be a sheaf of rings on a site $\mathcal{C}$. Let f be a global section of $\mathcal{O}$. We denote $\mathcal{O}_f$ the sheaf associated to the presheaf of localizations $U \mapsto \mathcal{O}(U)_f$.

0996 **Lemma 6.1.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let f be a global section of $\mathcal{O}$.*

1. *For $L, N \in D(\mathcal{O}_f)$ we have $R\mathcal{H}om_\mathcal{O}(L, N) = R\mathcal{H}om_{\mathcal{O}_f}(L, N)$. In particular the two $\mathcal{O}_f$-structures on $R\mathcal{H}om_\mathcal{O}(L, N)$ agree.*
2. *For $K \in D(\mathcal{O})$ and $L \in D(\mathcal{O}_f)$ we have*
$$R\mathcal{H}om_\mathcal{O}(L, K) = R\mathcal{H}om_{\mathcal{O}_f}(L, R\mathcal{H}om_\mathcal{O}(\mathcal{O}_f, K))$$
In particular $R\mathcal{H}om_\mathcal{O}(\mathcal{O}_f, R\mathcal{H}om_\mathcal{O}(\mathcal{O}_f, K)) = R\mathcal{H}om_\mathcal{O}(\mathcal{O}_f, K)$.

(3) *If g is a second global section of $\mathcal{O}$, then*

$$R\mathcal{H}om_{\mathcal{O}}(\mathcal{O}_f, R\mathcal{H}om_{\mathcal{O}}(\mathcal{O}_g, K)) = R\mathcal{H}om_{\mathcal{O}}(\mathcal{O}_{gf}, K).$$

Proof. Proof of (1). Let $\mathcal{J}^\bullet$ be a K-injective complex of $\mathcal{O}_f$-modules representing N. By Cohomology on Sites, Lemma 21.9 it follows that $\mathcal{J}^\bullet$ is a K-injective complex of $\mathcal{O}$-modules as well. Let $\mathcal{F}^\bullet$ be a complex of $\mathcal{O}_f$-modules representing L. Then

$$R\mathcal{H}om_{\mathcal{O}}(L, N) = R\mathcal{H}om_{\mathcal{O}}(\mathcal{F}^\bullet, \mathcal{J}^\bullet) = R\mathcal{H}om_{\mathcal{O}_f}(\mathcal{F}^\bullet, \mathcal{J}^\bullet)$$

by Modules on Sites, Lemma 11.4 because $\mathcal{J}^\bullet$ is a K-injective complex of $\mathcal{O}$ and of $\mathcal{O}_f$-modules.

Proof of (2). Let $\mathcal{I}^\bullet$ be a K-injective complex of $\mathcal{O}$-modules representing K. Then $R\mathcal{H}om_{\mathcal{O}}(\mathcal{O}_f, K)$ is represented by $\mathcal{H}om_{\mathcal{O}}(\mathcal{O}_f, \mathcal{I}^\bullet)$ which is a K-injective complex of $\mathcal{O}_f$-modules and of $\mathcal{O}$-modules by Cohomology on Sites, Lemmas 21.10 and 21.9. Let $\mathcal{F}^\bullet$ be a complex of $\mathcal{O}_f$-modules representing L. Then

$$R\mathcal{H}om_{\mathcal{O}}(L, K) = R\mathcal{H}om_{\mathcal{O}}(\mathcal{F}^\bullet, \mathcal{I}^\bullet) = R\mathcal{H}om_{\mathcal{O}_f}(\mathcal{F}^\bullet, \mathcal{H}om_{\mathcal{O}}(\mathcal{O}_f, \mathcal{I}^\bullet))$$

by Modules on Sites, Lemma 27.6 and because $\mathcal{H}om_{\mathcal{O}}(\mathcal{O}_f, \mathcal{I}^\bullet)$ is a K-injective complex of $\mathcal{O}_f$-modules.

Proof of (3). This follows from the fact that $R\mathcal{H}om_{\mathcal{O}}(\mathcal{O}_g, \mathcal{I}^\bullet)$ is K-injective as a complex of $\mathcal{O}$-modules and the fact that $\mathcal{H}om_{\mathcal{O}}(\mathcal{O}_f, \mathcal{H}om_{\mathcal{O}}(\mathcal{O}_g, \mathcal{H})) = \mathcal{H}om_{\mathcal{O}}(\mathcal{O}_{gf}, \mathcal{H})$ for all sheaves of $\mathcal{O}$-modules $\mathcal{H}$. $\square$

Let $K \in D(\mathcal{O})$. We denote $T(K, f)$ a derived limit (Derived Categories, Definition 32.1) of the inverse system

$$\ldots \to K \xrightarrow{f} K \xrightarrow{f} K$$

in $D(\mathcal{O})$.

0997 **Lemma 6.2.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let f be a global section of $\mathcal{O}$. Let $K \in D(\mathcal{O})$. The following are equivalent*

(1) $R\mathcal{H}om_{\mathcal{O}}(\mathcal{O}_f, K) = 0$,
(2) $R\mathcal{H}om_{\mathcal{O}}(L, K) = 0$ *for all L in $D(\mathcal{O}_f)$*,
(3) $T(K, f) = 0$.

Proof. It is clear that (2) implies (1). The implication (1) $\Rightarrow$ (2) follows from Lemma 6.1. A free resolution of the $\mathcal{O}$-module $\mathcal{O}_f$ is given by

$$0 \to \bigoplus\nolimits_{n \in \mathbf{N}} \mathcal{O} \to \bigoplus\nolimits_{n \in \mathbf{N}} \mathcal{O} \to \mathcal{O}_f \to 0$$

where the first map sends a local section $(x_0, x_1, \ldots)$ to $(fx_0 - x_1, fx_1 - x_2, \ldots)$ and the second map sends $(x_0, x_1, \ldots)$ to $x_0 + x_1/f + x_2/f^2 + \ldots$. Applying $\mathcal{H}om_{\mathcal{O}}(-, \mathcal{I}^\bullet)$ where $\mathcal{I}^\bullet$ is a K-injective complex of $\mathcal{O}$-modules representing K we get a short exact sequence of complexes

$$0 \to \mathcal{H}om_{\mathcal{O}}(\mathcal{O}_f, \mathcal{I}^\bullet) \to \prod \mathcal{I}^\bullet \to \prod \mathcal{I}^\bullet \to 0$$

because $\mathcal{I}^n$ is an injective $\mathcal{O}$-module. The products are products in $D(\mathcal{O})$, see Injectives, Lemma 13.4. This means that the object $T(K, f)$ is a representative of $R\mathcal{H}om_{\mathcal{O}}(\mathcal{O}_f, K)$ in $D(\mathcal{O})$. Thus the equivalence of (1) and (3). $\square$

0998 **Lemma 6.3.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $K \in D(\mathcal{O})$. The rule which associates to U the set $\mathcal{I}(U)$ of sections $f \in \mathcal{O}(U)$ such that $T(K|_U, f) = 0$ is a sheaf of ideals in $\mathcal{O}$.*

Proof. We will use the results of Lemma 6.2 without further mention. If $f \in \mathcal{I}(U)$, and $g \in \mathcal{O}(U)$, then $\mathcal{O}_{U,gf}$ is an $\mathcal{O}_{U,f}$-module hence $R\mathcal{H}om_{\mathcal{O}}(\mathcal{O}_{U,gf}, K|_U) = 0$, hence $gf \in \mathcal{I}(U)$. Suppose $f, g \in \mathcal{O}(U)$. Then there is a short exact sequence

$$0 \to \mathcal{O}_{U,f+g} \to \mathcal{O}_{U,f(f+g)} \oplus \mathcal{O}_{U,g(f+g)} \to \mathcal{O}_{U,gf(f+g)} \to 0$$

because f, g generate the unit ideal in $\mathcal{O}(U)_{f+g}$. This follows from Algebra, Lemma 23.2 and the easy fact that the last arrow is surjective. Because $R\mathcal{H}om_{\mathcal{O}}(-, K|_U)$ is an exact functor of triangulated categories the vanishing of $R\mathcal{H}om_{\mathcal{O}_U}(\mathcal{O}_{U,f(f+g)}, K|_U)$, $R\mathcal{H}om_{\mathcal{O}_U}(\mathcal{O}_{U,g(f+g)}, K|_U)$, and $R\mathcal{H}om_{\mathcal{O}_U}(\mathcal{O}_{U,gf(f+g)}, K|_U)$, implies the vanishing of $R\mathcal{H}om_{\mathcal{O}_U}(\mathcal{O}_{U,f+g}, K|_U)$. We omit the verification of the sheaf condition. □

We can make the following definition for any ringed site.

0999 **Definition 6.4.** Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{I} \subset \mathcal{O}$ be a sheaf of ideals. Let $K \in D(\mathcal{O})$. We say that K is *derived complete with respect to $\mathcal{I}$* if for every object U of $\mathcal{C}$ and $f \in \mathcal{I}(U)$ the object $T(K|_U, f)$ of $D(\mathcal{O}_U)$ is zero.

It is clear that the full subcategory $D_{comp}(\mathcal{O}) = D_{comp}(\mathcal{O}, \mathcal{I}) \subset D(\mathcal{O})$ consisting of derived complete objects is a saturated triangulated subcategory, see Derived Categories, Definitions 3.4 and 6.1. This subcategory is preserved under products and homotopy limits in $D(\mathcal{O})$. But it is not preserved under countable direct sums in general.

099A **Lemma 6.5.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{I} \subset \mathcal{O}$ be a sheaf of ideals. If $K \in D(\mathcal{O})$ and $L \in D_{comp}(\mathcal{O})$, then $R\mathcal{H}om_{\mathcal{O}}(K, L) \in D_{comp}(\mathcal{O})$.*

Proof. Let U be an object of $\mathcal{C}$ and let $f \in \mathcal{I}(U)$. Recall that

$$\mathrm{Hom}_{D(\mathcal{O}_U)}(\mathcal{O}_{U,f}, R\mathcal{H}om_{\mathcal{O}}(K, L)|_U) = \mathrm{Hom}_{D(\mathcal{O}_U)}(K|_U \otimes^{\mathbf{L}}_{\mathcal{O}_U} \mathcal{O}_{U,f}, L|_U)$$

by Cohomology on Sites, Lemma 34.2. The right hand side is zero by Lemma 6.2 and the relationship between internal hom and actual hom, see Cohomology on Sites, Lemma 34.1. The same vanishing holds for all U'/U. Thus the object $R\mathcal{H}om_{\mathcal{O}_U}(\mathcal{O}_{U,f}, R\mathcal{H}om_{\mathcal{O}}(K, L)|_U)$ of $D(\mathcal{O}_U)$ has vanishing 0th cohomology sheaf (by locus citatus). Similarly for the other cohomology sheaves, i.e., $R\mathcal{H}om_{\mathcal{O}_U}(\mathcal{O}_{U,f}, R\mathcal{H}om_{\mathcal{O}}(K, L)|_U)$ is zero in $D(\mathcal{O}_U)$. By Lemma 6.2 we conclude. □

099C **Lemma 6.6.** *Let $\mathcal{C}$ be a site. Let $\mathcal{O} \to \mathcal{O}'$ be a homomorphism of sheaves of rings. Let $\mathcal{I} \subset \mathcal{O}$ be a sheaf of ideals. The inverse image of $D_{comp}(\mathcal{O}, \mathcal{I})$ under the restriction functor $D(\mathcal{O}') \to D(\mathcal{O})$ is $D_{comp}(\mathcal{O}', \mathcal{I}\mathcal{O}')$.*

Proof. Using Lemma 6.3 we see that $K' \in D(\mathcal{O}')$ is in $D_{comp}(\mathcal{O}', \mathcal{I}\mathcal{O}')$ if and only if $T(K'|_U, f)$ is zero for every local section $f \in \mathcal{I}(U)$. Observe that the cohomology sheaves of $T(K'|_U, f)$ are computed in the category of abelian sheaves, so it doesn't matter whether we think of f as a section of $\mathcal{O}$ or take the image of f as a section of $\mathcal{O}'$. The lemma follows immediately from this and the definition of derived complete objects. □

099J **Lemma 6.7.** *Let $f : (Sh(\mathcal{D}), \mathcal{O}') \to (Sh(\mathcal{C}), \mathcal{O})$ be a morphism of ringed topoi. Let $\mathcal{I} \subset \mathcal{O}$ and $\mathcal{I}' \subset \mathcal{O}'$ be sheaves of ideals such that $f^\sharp$ sends $f^{-1}\mathcal{I}$ into $\mathcal{I}'$. Then Rf_* sends $D_{comp}(\mathcal{O}', \mathcal{I}')$ into $D_{comp}(\mathcal{O}, \mathcal{I})$.*

Proof. We may assume f is given by a morphism of ringed sites corresponding to a continuous functor $\mathcal{C} \to \mathcal{D}$ (Modules on Sites, Lemma 7.2). Let U be an object of $\mathcal{C}$ and let g be a section of $\mathcal{I}$ over U. We have to show that $\mathrm{Hom}_{D(\mathcal{O}_U)}(\mathcal{O}_{U,g}, Rf_*K|_U) = 0$ whenever K is derived complete with respect to $\mathcal{I}'$. Namely, by Cohomology on Sites, Lemma 34.1 this, applied to all objects over U and all shifts of K, will imply that $R\mathcal{H}om_{\mathcal{O}_U}(\mathcal{O}_{U,g}, Rf_*K|_U)$ is zero, which implies that $T(Rf_*K|_U, g)$ is zero (Lemma 6.2) which is what we have to show (Definition 6.4). Let V in $\mathcal{D}$ be the image of U. Then

$$\mathrm{Hom}_{D(\mathcal{O}_U)}(\mathcal{O}_{U,g}, Rf_*K|_U) = \mathrm{Hom}_{D(\mathcal{O}'_V)}(\mathcal{O}'_{V,g'}, K|_V) = 0$$

where $g' = f^\sharp(g) \in \mathcal{I}'(V)$. The second equality because K is derived complete and the first equality because the derived pullback of $\mathcal{O}_{U,g}$ is $\mathcal{O}'_{V,g'}$ and Cohomology on Sites, Lemma 20.1. □

The following lemma is the simplest case where one has derived completion.

099B **Lemma 6.8.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed on a site. Let $f_1, \ldots, f_r$ be global sections of $\mathcal{O}$. Let $\mathcal{I} \subset \mathcal{O}$ be the ideal sheaf generated by $f_1, \ldots, f_r$. Then the inclusion functor $D_{comp}(\mathcal{O}) \to D(\mathcal{O})$ has a left adjoint, i.e., given any object K of $D(\mathcal{O})$ there exists a map $K \to K^\wedge$ with $K^\wedge$ in $D_{comp}(\mathcal{O})$ such that the map*

$$\mathrm{Hom}_{D(\mathcal{O})}(K^\wedge, E) \longrightarrow \mathrm{Hom}_{D(\mathcal{O})}(K, E)$$

is bijective whenever E is in $D_{comp}(\mathcal{O})$. In fact we have

$$K^\wedge = R\mathcal{H}om_{\mathcal{O}}(\mathcal{O} \to \prod\nolimits_{i_0} \mathcal{O}_{f_{i_0}} \to \prod\nolimits_{i_0 < i_1} \mathcal{O}_{f_{i_0} f_{i_1}} \to \ldots \to \mathcal{O}_{f_1 \ldots f_r}, K)$$

functorially in K.

Proof. Define $K^\wedge$ by the last displayed formula of the lemma. There is a map of complexes

$$(\mathcal{O} \to \prod\nolimits_{i_0} \mathcal{O}_{f_{i_0}} \to \prod\nolimits_{i_0 < i_1} \mathcal{O}_{f_{i_0} f_{i_1}} \to \ldots \to \mathcal{O}_{f_1 \ldots f_r}) \longrightarrow \mathcal{O}$$

which induces a map $K \to K^\wedge$. It suffices to prove that $K^\wedge$ is derived complete and that $K \to K^\wedge$ is an isomorphism if K is derived complete.

Let f be a global section of $\mathcal{O}$. By Lemma 6.1 the object $R\mathcal{H}om_{\mathcal{O}}(\mathcal{O}_f, K^\wedge)$ is equal to

$$R\mathcal{H}om_{\mathcal{O}}((\mathcal{O}_f \to \prod\nolimits_{i_0} \mathcal{O}_{f f_{i_0}} \to \prod\nolimits_{i_0 < i_1} \mathcal{O}_{f f_{i_0} f_{i_1}} \to \ldots \to \mathcal{O}_{f f_1 \ldots f_r}), K)$$

If $f = f_i$ for some i, then $f_1, \ldots, f_r$ generate the unit ideal in $\mathcal{O}_f$, hence the extended alternating Čech complex

$$\mathcal{O}_f \to \prod\nolimits_{i_0} \mathcal{O}_{f f_{i_0}} \to \prod\nolimits_{i_0 < i_1} \mathcal{O}_{f f_{i_0} f_{i_1}} \to \ldots \to \mathcal{O}_{f f_1 \ldots f_r}$$

is zero (even homotopic to zero). In this way we see that $K^\wedge$ is derived complete.

If K is derived complete, then $R\mathcal{H}om_{\mathcal{O}}(\mathcal{O}_f, K)$ is zero for all $f = f_{i_0} \ldots f_{i_p}$, $p \geq 0$. Thus $K \to K^\wedge$ is an isomorphism in $D(\mathcal{O})$. □

Next we explain why derived completion is a completion.

0A0E **Lemma 6.9.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed on a site. Let $f_1, \ldots, f_r$ be global sections of $\mathcal{O}$. Let $\mathcal{I} \subset \mathcal{O}$ be the ideal sheaf generated by $f_1, \ldots, f_r$. Let $K \in D(\mathcal{O})$. The derived completion $K^\wedge$ of Lemma 6.8 is given by the formula*

$$K^\wedge = R\lim K \otimes_{\mathcal{O}}^{\mathbf{L}} K_n$$

where $K_n = K(\mathcal{O}, f_1^n, \ldots, f_r^n)$ is the Koszul complex on $f_1^n, \ldots, f_r^n$ over $\mathcal{O}$.

Proof. In More on Algebra, Lemma 28.13 we have seen that the extended alternating Čech complex

$$\mathcal{O} \to \prod\nolimits_{i_0} \mathcal{O}_{f_{i_0}} \to \prod\nolimits_{i_0 < i_1} \mathcal{O}_{f_{i_0} f_{i_1}} \to \ldots \to \mathcal{O}_{f_1 \ldots f_r}$$

is a colimit of the Koszul complexes $K^n = K(\mathcal{O}, f_1^n, \ldots, f_r^n)$ sitting in degrees $0, \ldots, r$. Note that K^n is a finite chain complex of finite free $\mathcal{O}$-modules with dual $\mathcal{H}om_{\mathcal{O}}(K^n, \mathcal{O}) = K_n$ where K_n is the Koszul cochain complex sitting in degrees $-r, \ldots, 0$ (as usual). By Lemma 6.8 the functor $E \mapsto E^\wedge$ is gotten by taking $R\mathcal{H}om$ from the extended alternating Čech complex into E:

$$E^\wedge = R\mathcal{H}om(\operatorname{colim} K^n, E)$$

This is equal to $R\lim(E \otimes_{\mathcal{O}}^{\mathbf{L}} K_n)$ by Cohomology on Sites, Lemma 45.10. $\square$

099D **Lemma 6.10.** *There exist a way to construct*

1. *for every pair (A, I) consisting of a ring A and a finitely generated ideal $I \subset A$ a complex $K(A, I)$ of A-modules,*
2. *a map $K(A, I) \to A$ of complexes of A-modules,*
3. *for every ring map $A \to B$ and finitely generated ideal $I \subset A$ a map of complexes $K(A, I) \to K(B, IB)$,*

such that

(a) *for $A \to B$ and $I \subset A$ finitely generated the diagram*

$$\begin{tikzcd} K(A,I) \arrow[r] \arrow[d] & A \arrow[d] \\ K(B,IB) \arrow[r] & B \end{tikzcd}$$

commutes,

(b) *for $A \to B \to C$ and $I \subset A$ finitely generated the composition of the maps $K(A, I) \to K(B, IB) \to K(C, IC)$ is the map $K(A, I) \to K(C, IC)$.*

(c) *for $A \to B$ and a finitely generated ideal $I \subset A$ the induced map $K(A, I) \otimes_A^{\mathbf{L}} B \to K(B, IB)$ is an isomorphism in $D(B)$, and*

(d) *if $I = (f_1, \ldots, f_r) \subset A$ then there is a commutative diagram*

$$\begin{tikzcd} (A \to \prod_{i_0} A_{f_{i_0}} \to \prod_{i_0 < i_1} A_{f_{i_0} f_{i_1}} \to \ldots \to A_{f_1 \ldots f_r}) \arrow[r] \arrow[d] & K(A,I) \arrow[d] \\ A \arrow[r, "1"] & A \end{tikzcd}$$

in $D(A)$ whose horizontal arrows are isomorphisms.

Proof. Let S be the set of rings A_0 of the form $A_0 = \mathbf{Z}[x_1, \ldots, x_n]/J$. Every finite type $\mathbf{Z}$-algebra is isomorphic to an element of S. Let $\mathcal{A}_0$ be the category whose objects are pairs (A_0, I_0) where $A_0 \in S$ and $I_0 \subset A_0$ is an ideal and whose morphisms $(A_0, I_0) \to (B_0, J_0)$ are ring maps $\varphi : A_0 \to B_0$ such that $J_0 = \varphi(I_0)B_0$.

Suppose we can construct $K(A_0, I_0) \to A_0$ functorially for objects of $\mathcal{A}_0$ having properties (a), (b), (c), and (d). Then we take

$$K(A, I) = \operatorname{colim}_{\varphi:(A_0,I_0)\to(A,I)} K(A_0, I_0)$$

where the colimit is over ring maps $\varphi : A_0 \to A$ such that $\varphi(I_0)A = I$ with (A_0, I_0) in $\mathcal{A}_0$. A morphism between $(A_0, I_0) \to (A, I)$ and $(A'_0, I'_0) \to (A, I)$ are given by maps $(A_0, I_0) \to (A'_0, I'_0)$ in $\mathcal{A}_0$ commuting with maps to A. The category of these $(A_0, I_0) \to (A, I)$ is filtered (details omitted). Moreover, $\operatorname{colim}_{\varphi:(A_0,I_0)\to(A,I)} A_0 = A$ so that $K(A, I)$ is a complex of A-modules. Finally, given $\varphi : A \to B$ and $I \subset A$ for every $(A_0, I_0) \to (A, I)$ in the colimit, the composition $(A_0, I_0) \to (B, IB)$ lives in the colimit for (B, IB). In this way we get a map on colimits. Properties (a), (b), (c), and (d) follow readily from this and the corresponding properties of the complexes $K(A_0, I_0)$.

Endow $\mathcal{C}_0 = \mathcal{A}_0^{opp}$ with the chaotic topology. We equip $\mathcal{C}_0$ with the sheaf of rings $\mathcal{O} : (A, I) \mapsto A$. The ideals I fit together to give a sheaf of ideals $\mathcal{I} \subset \mathcal{O}$. Choose an injective resolution $\mathcal{O} \to \mathcal{J}^\bullet$. Consider the object

$$\mathcal{F}^\bullet = \bigcup\nolimits_n \mathcal{J}^\bullet[\mathcal{I}^n]$$

Let $U = (A, I) \in \operatorname{Ob}(\mathcal{C}_0)$. Since the topology in $\mathcal{C}_0$ is chaotic, the value $\mathcal{J}^\bullet(U)$ is a resolution of A by injective A-modules. Hence the value $\mathcal{F}^\bullet(U)$ is an object of $D(A)$ representing the image of $R\Gamma_I(A)$ in $D(A)$, see Dualizing Complexes, Section 9. Choose a complex of $\mathcal{O}$-modules $\mathcal{K}^\bullet$ and a commutative diagram

$$\begin{tikzcd} \mathcal{O} \arrow[r] & \mathcal{J}^\bullet \\ \mathcal{K}^\bullet \arrow[u] \arrow[r] & \mathcal{F}^\bullet \arrow[u] \end{tikzcd}$$

where the horizontal arrows are quasi-isomorphisms. This is possible by the construction of the derived category $D(\mathcal{O})$. Set $K(A, I) = \mathcal{K}^\bullet(U)$ where $U = (A, I)$. Properties (a) and (b) are clear and properties (c) and (d) follow from Dualizing Complexes, Lemmas 10.2 and 10.3. $\square$

099E **Lemma 6.11.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{I} \subset \mathcal{O}$ be a finite type sheaf of ideals. There exists a map $K \to \mathcal{O}$ in $D(\mathcal{O})$ such that for every $U \in \operatorname{Ob}(\mathcal{C})$ such that $\mathcal{I}|_U$ is generated by $f_1, \ldots, f_r \in \mathcal{I}(U)$ there is an isomorphism*

$$(\mathcal{O}_U \to \prod\nolimits_{i_0} \mathcal{O}_{U,f_{i_0}} \to \prod\nolimits_{i_0<i_1} \mathcal{O}_{U,f_{i_0}f_{i_1}} \to \ldots \to \mathcal{O}_{U,f_1\ldots f_r}) \longrightarrow K|_U$$

compatible with maps to $\mathcal{O}_U$.

Proof. Let $\mathcal{C}' \subset \mathcal{C}$ be the full subcategory of objects U such that $\mathcal{I}|_U$ is generated by finitely many sections. Then $\mathcal{C}' \to \mathcal{C}$ is a special cocontinuous functor (Sites, Definition 29.2). Hence it suffices to work with $\mathcal{C}'$, see Sites, Lemma 29.1. In other words we may assume that for every object U of $\mathcal{C}$ there exists a finitely generated ideal $I \subset \mathcal{I}(U)$ such that $\mathcal{I}|_U = \operatorname{Im}(I \otimes \mathcal{O}_U \to \mathcal{O}_U)$. We will say that I generates $\mathcal{I}|_U$. Warning: We do not know that $\mathcal{I}(U)$ is a finitely generated ideal in $\mathcal{O}(U)$.

Let U be an object and $I \subset \mathcal{O}(U)$ a finitely generated ideal which generates $\mathcal{I}|_U$. On the category $\mathcal{C}/U$ consider the complex of presheaves

$$K_{U,I}^\bullet : U'/U \longmapsto K(\mathcal{O}(U'), I\mathcal{O}(U'))$$

with $K(-,-)$ as in Lemma 6.10. We claim that the sheafification of this is independent of the choice of I. Indeed, if $I' \subset \mathcal{O}(U)$ is a finitely generated ideal which also generates $\mathcal{I}|_U$, then there exists a covering $\{U_j \to U\}$ such that $I\mathcal{O}(U_j) = I'\mathcal{O}(U_j)$. (Hint: this works because both I and I' are finitely generated and generate $\mathcal{I}|_U$.) Hence $K_{U,I}^\bullet$ and $K_{U,I'}^\bullet$ are the *same* for any object lying over one of the U_j. The statement on sheafifications follows. Denote $K_U^\bullet$ the common value.

The independence of choice of I also shows that $K_U^\bullet|_{\mathcal{C}/U'} = K_{U'}^\bullet$ whenever we are given a morphism $U' \to U$ and hence a localization morphism $\mathcal{C}/U' \to \mathcal{C}/U$. Thus the complexes $K_U^\bullet$ glue to give a single well defined complex $K^\bullet$ of $\mathcal{O}$-modules. The existence of the map $K^\bullet \to \mathcal{O}$ and the quasi-isomorphism of the lemma follow immediately from the corresponding properties of the complexes $K(-,-)$ in Lemma 6.10. □

099F **Proposition 6.12.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{I} \subset \mathcal{O}$ be a finite type sheaf of ideals. There exists a left adjoint to the inclusion functor $D_{comp}(\mathcal{O}) \to D(\mathcal{O})$.*

Proof. Let $K \to \mathcal{O}$ in $D(\mathcal{O})$ be as constructed in Lemma 6.11. Let $E \in D(\mathcal{O})$. Then $E^\wedge = R\mathcal{H}om(K, E)$ together with the map $E \to E^\wedge$ will do the job. Namely, locally on the site $\mathcal{C}$ we recover the adjoint of Lemma 6.8. This shows that $E^\wedge$ is always derived complete and that $E \to E^\wedge$ is an isomorphism if E is derived complete. □

0CQH **Remark 6.13** (Comparison with completion)**.** Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{I} \subset \mathcal{O}$ be a finite type sheaf of ideals. Let $K \mapsto K^\wedge$ be the derived completion functor of Proposition 6.12. For any $n \geq 1$ the object $K \otimes_\mathcal{O}^\mathbf{L} \mathcal{O}/\mathcal{I}^n$ is derived complete as it is annihilated by powers of local sections of $\mathcal{I}$. Hence there is a canonical factorization

$$K \to K^\wedge \to K \otimes_\mathcal{O}^\mathbf{L} \mathcal{O}/\mathcal{I}^n$$

of the canonical map $K \to K \otimes_\mathcal{O}^\mathbf{L} \mathcal{O}/\mathcal{I}^n$. These maps are compatible for varying n and we obtain a comparison map

$$K^\wedge \longrightarrow R\lim \left(K \otimes_\mathcal{O}^\mathbf{L} \mathcal{O}/\mathcal{I}^n\right)$$

The right hand side is more recognizable as a kind of completion. In general this comparison map is not an isomorphism.

0A0F **Remark 6.14** (Localization and derived completion)**.** Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{I} \subset \mathcal{O}$ be a finite type sheaf of ideals. Let $K \mapsto K^\wedge$ be the derived completion functor of Proposition 6.12. It follows from the construction in the proof of the proposition that $K^\wedge|_U$ is the derived completion of $K|_U$ for any $U \in \mathrm{Ob}(\mathcal{C})$. But we can also prove this as follows. From the definition of derived complete objects it follows that $K^\wedge|_U$ is derived complete. Thus we obtain a canonical map $a : (K|_U)^\wedge \to K^\wedge|_U$. On the other hand, if E is a derived complete object of $D(\mathcal{O}_U)$, then Rj_*E is a derived complete object of $D(\mathcal{O})$ by Lemma 6.7. Here j is the localization morphism (Modules on Sites, Section 19). Hence we also obtain a canonical map $b : K^\wedge \to Rj_*((K|_U)^\wedge)$. We omit the (formal) verification that the adjoint of b is the inverse of a.

099G **Remark 6.15** (Completed tensor product)**.** Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{I} \subset \mathcal{O}$ be a finite type sheaf of ideals. Denote $K \mapsto K^\wedge$ the adjoint of Proposition 6.12. Then we set

$$K \otimes_\mathcal{O}^\wedge L = (K \otimes_\mathcal{O}^\mathbf{L} L)^\wedge$$

This *completed tensor product* defines a functor $D_{comp}(\mathcal{O}) \times D_{comp}(\mathcal{O}) \to D_{comp}(\mathcal{O})$ such that we have

$$\mathrm{Hom}_{D_{comp}(\mathcal{O})}(K, R\mathcal{H}om_\mathcal{O}(L, M)) = \mathrm{Hom}_{D_{comp}(\mathcal{O})}(K \otimes^\wedge_\mathcal{O} L, M)$$

for $K, L, M \in D_{comp}(\mathcal{O})$. Note that $R\mathcal{H}om_\mathcal{O}(L, M) \in D_{comp}(\mathcal{O})$ by Lemma 6.5.

099H **Lemma 6.16.** *Let $\mathcal{C}$ be a site. Assume $\varphi : \mathcal{O} \to \mathcal{O}'$ is a flat homomorphism of sheaves of rings. Let $f_1, \ldots, f_r$ be global sections of $\mathcal{O}$ such that $\mathcal{O}/(f_1, \ldots, f_r) \cong \mathcal{O}'/(f_1, \ldots, f_r)$. Then the map of extended alternating Čech complexes*

$$\begin{array}{c} \mathcal{O} \to \prod_{i_0} \mathcal{O}_{f_{i_0}} \to \prod_{i_0 < i_1} \mathcal{O}_{f_{i_0} f_{i_1}} \to \ldots \to \mathcal{O}_{f_1 \ldots f_r} \\ \downarrow \\ \mathcal{O}' \to \prod_{i_0} \mathcal{O}'_{f_{i_0}} \to \prod_{i_0 < i_1} \mathcal{O}'_{f_{i_0} f_{i_1}} \to \ldots \to \mathcal{O}'_{f_1 \ldots f_r} \end{array}$$

is a quasi-isomorphism.

Proof. Observe that the second complex is the tensor product of the first complex with $\mathcal{O}'$. We can write the first extended alternating Čech complex as a colimit of the Koszul complexes $K_n = K(\mathcal{O}, f_1^n, \ldots, f_r^n)$, see More on Algebra, Lemma 28.13. Hence it suffices to prove $K_n \to K_n \otimes_\mathcal{O} \mathcal{O}'$ is a quasi-isomorphism. Since $\mathcal{O} \to \mathcal{O}'$ is flat it suffices to show that $H^i \to H^i \otimes_\mathcal{O} \mathcal{O}'$ is an isomorphism where H^i is the ith cohomology sheaf $H^i = H^i(K_n)$. These sheaves are annihilated by $f_1^n, \ldots, f_r^n$, see More on Algebra, Lemma 28.6. Thus it suffices to show that $\mathcal{O}/(f_1^n, \ldots, f_r^n) \to \mathcal{O}'/(f_1^n, \ldots, f_r^n)$ is an isomorphism. Equivalently, we will show that $\mathcal{O}/(f_1, \ldots, f_r)^n \to \mathcal{O}'/(f_1, \ldots, f_r)^n$ is an isomorphism for all n. This holds for $n = 1$ by assumption. It follows for all n by induction using Modules on Sites, Lemma 28.14 applied to the ring map $\mathcal{O}/(f_1, \ldots, f_r)^{n+1} \to \mathcal{O}/(f_1, \ldots, f_r)^n$ and the module $\mathcal{O}'/(f_1, \ldots, f_r)^{n+1}$. $\square$

099I **Lemma 6.17.** *Let $\mathcal{C}$ be a site. Let $\mathcal{O} \to \mathcal{O}'$ be a homomorphism of sheaves of rings. Let $\mathcal{I} \subset \mathcal{O}$ be a finite type sheaf of ideals. If $\mathcal{O} \to \mathcal{O}'$ is flat and $\mathcal{O}/\mathcal{I} \cong \mathcal{O}'/\mathcal{I}\mathcal{O}'$, then the restriction functor $D(\mathcal{O}') \to D(\mathcal{O})$ induces an equivalence $D_{comp}(\mathcal{O}', \mathcal{I}\mathcal{O}') \to D_{comp}(\mathcal{O}, \mathcal{I})$.*

Proof. Lemma 6.7 implies restriction $r : D(\mathcal{O}') \to D(\mathcal{O})$ sends $D_{comp}(\mathcal{O}', \mathcal{I}\mathcal{O}')$ into $D_{comp}(\mathcal{O}, \mathcal{I})$. We will construct a quasi-inverse $E \mapsto E'$.

Let $K \to \mathcal{O}$ be the morphism of $D(\mathcal{O})$ constructed in Lemma 6.11. Set $K' = K \otimes_\mathcal{O}^\mathbf{L} \mathcal{O}'$ in $D(\mathcal{O}')$. Then $K' \to \mathcal{O}'$ is a map in $D(\mathcal{O}')$ which satisfies the conclusions of Lemma 6.11 with respect to $\mathcal{I}' = \mathcal{I}\mathcal{O}'$. The map $K \to r(K')$ is a quasi-isomorphism by Lemma 6.16. Now, for $E \in D_{comp}(\mathcal{O}, \mathcal{I})$ we set

$$E' = R\mathcal{H}om_\mathcal{O}(r(K'), E)$$

viewed as an object in $D(\mathcal{O}')$ using the $\mathcal{O}'$-module structure on K'. Since E is derived complete we have $E = R\mathcal{H}om_\mathcal{O}(K, E)$, see proof of Proposition 6.12. On the other hand, since $K \to r(K')$ is an isomorphism in we see that there is an isomorphism $E \to r(E')$ in $D(\mathcal{O})$. To finish the proof we have to show that, if $E = r(M')$ for an object M' of $D_{comp}(\mathcal{O}', \mathcal{I}')$, then $E' \cong M'$. To get a map we use

$$M' = R\mathcal{H}om_{\mathcal{O}'}(\mathcal{O}', M') \to R\mathcal{H}om_\mathcal{O}(r(\mathcal{O}'), r(M')) \to R\mathcal{H}om_\mathcal{O}(r(K'), r(M')) = E'$$

where the second arrow uses the map $K' \to \mathcal{O}'$. To see that this is an isomorphism, one shows that r applied to this arrow is the same as the isomorphism $E \to r(E')$ above. Details omitted. $\square$

099K **Lemma 6.18.** *Let $f : (Sh(\mathcal{D}), \mathcal{O}') \to (Sh(\mathcal{C}), \mathcal{O})$ be a morphism of ringed topoi. Let $\mathcal{I} \subset \mathcal{O}$ and $\mathcal{I}' \subset \mathcal{O}'$ be finite type sheaves of ideals such that $f^\sharp$ sends $f^{-1}\mathcal{I}$ into $\mathcal{I}'$. Then Rf_* sends $D_{comp}(\mathcal{O}', \mathcal{I}')$ into $D_{comp}(\mathcal{O}, \mathcal{I})$ and has a left adjoint Lf^*_{comp} which is Lf^* followed by derived completion.*

Proof. The first statement we have seen in Lemma 6.7. Note that the second statement makes sense as we have a derived completion functor $D(\mathcal{O}') \to D_{comp}(\mathcal{O}', \mathcal{I}')$ by Proposition 6.12. OK, so now let $K \in D_{comp}(\mathcal{O}, \mathcal{I})$ and $M \in D_{comp}(\mathcal{O}', \mathcal{I}')$. Then we have

$$\mathrm{Hom}(K, Rf_*M) = \mathrm{Hom}(Lf^*K, M) = \mathrm{Hom}(Lf^*_{comp}K, M)$$

by the universal property of derived completion. $\square$

0A0G **Lemma 6.19.** *Let $f : (Sh(\mathcal{D}), \mathcal{O}') \to (Sh(\mathcal{C}), \mathcal{O})$ be a morphism of ringed topoi. Let $\mathcal{I} \subset \mathcal{O}$ be a finite type sheaf of ideals. Let $\mathcal{I}' \subset \mathcal{O}'$ be the ideal generated by $f^\sharp(f^{-1}\mathcal{I})$. Then Rf_* commutes with derived completion, i.e., $Rf_*(K^\wedge) = (Rf_*K)^\wedge$.*

Generalization of [BS13, Lemma 6.5.9 (2)]. Compare with [HLP14, Theorem 6.5] in the setting of quasi-coherent modules and morphisms of (derived) algebraic stacks.

Proof. By Proposition 6.12 the derived completion functors exist. By Lemma 6.7 the object $Rf_*(K^\wedge)$ is derived complete, and hence we obtain a canonical map $(Rf_*K)^\wedge \to Rf_*(K^\wedge)$ by the universal property of derived completion. We may check this map is an isomorphism locally on $\mathcal{C}$. Thus, since derived completion commutes with localization (Remark 6.14) we may assume that $\mathcal{I}$ is generated by global sections $f_1, \ldots, f_r$. Then $\mathcal{I}'$ is generated by $g_i = f^\sharp(f_i)$. By Lemma 6.9 we have to prove that

$$R\lim \left(Rf_*K \otimes^{\mathbf{L}}_{\mathcal{O}} K(\mathcal{O}, f_1^n, \ldots, f_r^n)\right) = Rf_* \left(R\lim K \otimes^{\mathbf{L}}_{\mathcal{O}'} K(\mathcal{O}', g_1^n, \ldots, g_r^n)\right)$$

Because Rf_* commutes with $R\lim$ (Cohomology on Sites, Lemma 23.3) it suffices to prove that

$$Rf_*K \otimes^{\mathbf{L}}_{\mathcal{O}} K(\mathcal{O}, f_1^n, \ldots, f_r^n) = Rf_* \left(K \otimes^{\mathbf{L}}_{\mathcal{O}'} K(\mathcal{O}', g_1^n, \ldots, g_r^n)\right)$$

This follows from the projection formula (Cohomology on Sites, Lemma 46.1) and the fact that $Lf^*K(\mathcal{O}, f_1^n, \ldots, f_r^n) = K(\mathcal{O}', g_1^n, \ldots, g_r^n)$. $\square$

0BLX **Lemma 6.20.** *Let A be a ring and let $I \subset A$ be a finitely generated ideal. Let $\mathcal{C}$ be a site and let $\mathcal{O}$ be a sheaf of A-algebras. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}$-modules. Then we have*

$$R\Gamma(\mathcal{C}, \mathcal{F})^\wedge = R\Gamma(\mathcal{C}, \mathcal{F}^\wedge)$$

in $D(A)$ where $\mathcal{F}^\wedge$ is the derived completion of $\mathcal{F}$ with respect to $I\mathcal{O}$ and on the left hand wide we have the derived completion with respect to I. This produces two spectral sequences

$$E_2^{i,j} = H^i(H^j(\mathcal{C}, \mathcal{F})^\wedge) \quad \text{and} \quad E_2^{p,q} = H^p(\mathcal{C}, H^q(\mathcal{F}^\wedge))$$

both converging to $H^*(R\Gamma(\mathcal{C}, \mathcal{F})^\wedge) = H^*(\mathcal{C}, \mathcal{F}^\wedge)$

Proof. Apply Lemma 6.19 to the morphism of ringed topoi $(\mathcal{C}, \mathcal{O}) \to (pt, A)$ and take cohomology to get the first statement. The second spectral sequence is just the Leray spectral sequence for this morphism, see Cohomology on Sites, Lemma 15.5. The first spectral sequence is the spectral sequence of More on Algebra, Example 82.20 applied to $R\Gamma(\mathcal{C}, \mathcal{F})^\wedge$. $\square$

0CQI **Remark 6.21.** Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{I} \subset \mathcal{O}$ be a finite type sheaf of ideals. Let $K \mapsto K^\wedge$ be the derived completion of Proposition 6.12. Let $U \in \text{Ob}(\mathcal{C})$ be an object such that $\mathcal{I}$ is generated as an ideal sheaf by $f_1, \ldots, f_r \in \mathcal{I}(U)$. Set $A = \mathcal{O}(U)$ and $I = (f_1, \ldots, f_r) \subset A$. Warning: it may not be the case that $I = \mathcal{I}(U)$. Then we have

$$R\Gamma(U, K^\wedge) = R\Gamma(U, K)^\wedge$$

where the right hand side is the derived completion of the object $R\Gamma(U, K)$ of $D(A)$ with respect to I. This is true because derived completion commutes with localization (Remark 6.14) and Lemma 6.20.

7. The theorem on formal functions

0A0H We interrupt the flow of the exposition to talk a little bit about derived completion in the setting of quasi-coherent modules on schemes and to use this to give a somewhat different proof of the theorem on formal functions. We give some pointers to the literature in Remark 7.4.

Lemma 6.19 is a (very formal) derived version of the theorem on formal functions (Cohomology of Schemes, Theorem 20.5). To make this more explicit, suppose $f : X \to S$ is a morphism of schemes, $\mathcal{I} \subset \mathcal{O}_S$ is a quasi-coherent sheaf of ideals of finite type, and $\mathcal{F}$ is a quasi-coherent sheaf on X. Then the lemma says that

0A0I (7.0.1) $$Rf_*(\mathcal{F}^\wedge) = (Rf_*\mathcal{F})^\wedge$$

where $\mathcal{F}^\wedge$ is the derived completion of $\mathcal{F}$ with respect to $f^{-1}\mathcal{I} \cdot \mathcal{O}_X$ and the right hand side is the derived completion of $Rf_*\mathcal{F}$ with respect to $\mathcal{I}$. To see that this gives back the theorem on formal functions we have to do a bit of work.

0A0L **Lemma 7.1.** *Let X be a locally Noetherian scheme. Let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. Let K be a pseudo-coherent object of $D(\mathcal{O}_X)$ with derived completion $K^\wedge$. Then*

$$H^p(U, K^\wedge) = \lim H^p(U, K)/I^n H^p(U, K) = H^p(U, K)^\wedge$$

for any affine open $U \subset X$ where $I = \mathcal{I}(U)$ and where on the right we have the derived completion with respect to I.

Proof. Write $U = \text{Spec}(A)$. The ring A is Noetherian and hence $I \subset A$ is finitely generated. Then we have

$$R\Gamma(U, K^\wedge) = R\Gamma(U, K)^\wedge$$

by Remark 6.21. Now $R\Gamma(U, K)$ is a pseudo-coherent complex of A-modules (Derived Categories of Schemes, Lemma 9.2). By More on Algebra, Lemma 84.4 we conclude that the pth cohomology module of $R\Gamma(U, K^\wedge)$ is equal to the I-adic completion of $H^p(U, K)$. This proves the first equality. The second (less important) equality follows immediately from a second application of the lemma just used. $\square$

0A0K **Lemma 7.2.** *Let X be a locally Noetherian scheme. Let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. Let K be an object of $D(\mathcal{O}_X)$. Then*

1. *the derived completion $K^\wedge$ is equal to $R\lim(K \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{O}_X/\mathcal{I}^n)$.*

Let K is a pseudo-coherent object of $D(\mathcal{O}_X)$. Then

2. *the cohomology sheaf $H^q(K^\wedge)$ is equal to $\lim H^q(K)/\mathcal{I}^n H^q(K)$.*

Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module[1]*. Then*

3. *the derived completion $\mathcal{F}^\wedge$ is equal to $\lim \mathcal{F}/\mathcal{I}^n\mathcal{F}$,*
4. $\lim \mathcal{F}/I^n\mathcal{F} = R\lim \mathcal{F}/I^n\mathcal{F}$,
5. $H^p(U, \mathcal{F}^\wedge) = 0$ *for $p \neq 0$ for all affine opens $U \subset X$.*

Proof. Proof of (1). There is a canonical map

$$K \longrightarrow R\lim(K \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{O}_X/\mathcal{I}^n),$$

see Remark 6.13. Derived completion commutes with passing to open subschemes (Remark 6.14). Formation of $R\lim$ commutes with passsing to open subschemes. It follows that to check our map is an isomorphism, we may work locally. Thus we may assume $X = U = \mathrm{Spec}(A)$. Say $I = (f_1, \ldots, f_r)$. Let $K_n = K(A, f_1^n, \ldots, f_r^n)$ be the Koszul complex. By More on Algebra, Lemma 84.1 we have seen that the pro-systems $\{K_n\}$ and $\{A/I^n\}$ of $D(A)$ are isomorphic. Using the equivalence $D(A) = D_{QCoh}(\mathcal{O}_X)$ of Derived Categories of Schemes, Lemma 3.5 we see that the pro-systems $\{K(\mathcal{O}_X, f_1^n, \ldots, f_r^n)\}$ and $\{\mathcal{O}_X/\mathcal{I}^n\}$ are isomorphic in $D(\mathcal{O}_X)$. This proves the second equality in

$$K^\wedge = R\lim\left(K \otimes^{\mathbf{L}}_{\mathcal{O}_X} K(\mathcal{O}_X, f_1^n, \ldots, f_r^n)\right) = R\lim(K \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{O}_X/\mathcal{I}^n)$$

The first equality is Lemma 6.9.

Assume K is pseudo-coherent. For $U \subset X$ affine open we have $H^q(U, K^\wedge) = \lim H^q(U, K)/\mathcal{I}^n(U)H^q(U, K)$ by Lemma 7.1. As this is true for every U we see that $H^q(K^\wedge) = \lim H^q(K)/\mathcal{I}^n H^q(K)$ as sheaves. This proves (2).

Part (3) is a special case of (2). Parts (4) and (5) follow from Derived Categories of Schemes, Lemma 3.2. $\square$

0A0M **Lemma 7.3.** *Let A be a Noetherian ring and let $I \subset A$ be an ideal. Let X be a Noetherian scheme over A. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Assume that $H^p(X, \mathcal{F})$ is a finite A-module for all p. Then there are short exact sequences*

$$0 \to R^1\lim H^{p-1}(X, \mathcal{F}/I^n\mathcal{F}) \to H^p(X, \mathcal{F})^\wedge \to \lim H^p(X, \mathcal{F}/I^n\mathcal{F}) \to 0$$

of A-modules where $H^p(X, \mathcal{F})^\wedge$ is the usual I-adic completion. If f is proper, then the $R^1\lim$ term is zero.

Proof. Consider the two spectral sequences of Lemma 6.20. The first degenerates by More on Algebra, Lemma 84.4. We obtain $H^p(X, \mathcal{F})^\wedge$ in degree p. This is where we use the assumption that $H^p(X, \mathcal{F})$ is a finite A-module. The second degenerates because

$$\mathcal{F}^\wedge = \lim \mathcal{F}/I^n\mathcal{F} = R\lim \mathcal{F}/I^n\mathcal{F}$$

is a sheaf by Lemma 7.2. We obtain $H^p(X, \lim \mathcal{F}/I^n\mathcal{F})$ in degree p. Since $R\Gamma(X, -)$ commutes with derived limits (Injectives, Lemma 13.6) we also get

$$R\Gamma(X, \lim \mathcal{F}/I^n\mathcal{F}) = R\Gamma(X, R\lim \mathcal{F}/I^n\mathcal{F}) = R\lim R\Gamma(X, \mathcal{F}/I^n\mathcal{F})$$

[1] For example $H^q(K)$ for K pseudo-coherent on our locally Noetherian X.

By More on Algebra, Remark 78.6 we obtain exact sequences

$$0 \to R^1 \lim H^{p-1}(X, \mathcal{F}/I^n\mathcal{F}) \to H^p(X, \lim \mathcal{F}/I^n\mathcal{F}) \to \lim H^p(X, \mathcal{F}/I^n\mathcal{F}) \to 0$$

of A-modules. Combining the above we get the first statement of the lemma. The vanishing of the $R^1 \lim$ term follows from Cohomology of Schemes, Lemma 20.4. $\square$

0AKL **Remark 7.4.** Here are some references to discussions of related material the literature. It seems that a "derived formal functions theorem" for proper maps goes back to [Lur04, Theorem 6.3.1]. There is the discussion in [Lur11], especially Chapter 4 which discusses the affine story, see More on Algebra, Section 82. In [GR13, Section 2.9] one finds a discussion of proper base change and derived completion using (ind) coherent modules. An analogue of (7.0.1) for complexes of quasi-coherent modules can be found as [HLP14, Theorem 6.5]

8. Algebraization of local cohomology, I

0EFF Let A be a Noetherian ring and let I and J be two ideals of A. Let M be a finite A-module. In this section we study the cohomology groups of the object

$$R\Gamma_J(M)^\wedge \quad \text{of} \quad D(A)$$

where $^\wedge$ denotes derived I-adic completion. Observe that in Dualizing Complexes, Lemma 12.5 we have shown, if A is complete with respect to I, that there is an isomorphism

$$\operatorname{colim} H^0_Z(M) \longrightarrow H^0(R\Gamma_J(M)^\wedge)$$

where the (directed) colimit is over the closed subsets $Z = V(J')$ with $J' \subset J$ and $V(J') \cap V(I) = V(J) \cap V(I)$. The union of these closed subsets is

0EFG (8.0.1) $$T = \{\mathfrak{p} \in \operatorname{Spec}(A) : V(\mathfrak{p}) \cap V(I) \subset V(J) \cap V(I)\}$$

This is a subset of $\operatorname{Spec}(A)$ stable under specialization. The result above becomes the statement that

$$H^0_T(M) \longrightarrow H^0(R\Gamma_J(M)^\wedge)$$

is an isomorphism provided A is complete with respect to I, see Local Cohomology, Lemma 4.3 and Remark 4.6. Our method to extend this isomorphism to higher cohomology groups rests on the following lemma.

0EFH **Lemma 8.1.** *Let I, J be ideals of a Noetherian ring A. Let M be a finite A-module. Let $\mathfrak{p} \subset A$ be a prime. Let s and d be integers. Assume*

1. *A has a dualizing complex,*
2. *$\mathfrak{p} \notin V(J) \cap V(I)$,*
3. *$cd(A, I) \leq d$, and*
4. *for all primes $\mathfrak{p}' \subset \mathfrak{p}$ we have $depth_{A_{\mathfrak{p}'}}(M_{\mathfrak{p}'}) + \dim((A/\mathfrak{p}')_{\mathfrak{q}}) > d + s$ for all $\mathfrak{q} \in V(\mathfrak{p}') \cap V(J) \cap V(I)$.*

Then there exists an $f \in A$, $f \notin \mathfrak{p}$ which annihilates $H^i(R\Gamma_J(M)^\wedge)$ for $i \leq s$ where $^\wedge$ indicates I-adic completion.

Proof. We will use that $R\Gamma_J = R\Gamma_{V(J)}$ and similarly for $I + J$, see Dualizing Complexes, Lemma 10.1. Observe that $R\Gamma_J(M)^\wedge = R\Gamma_I(R\Gamma_J(M))^\wedge = R\Gamma_{I+J}(M)^\wedge$, see Dualizing Complexes, Lemmas 12.1 and 9.6. Thus we may replace J by $I + J$ and assume $I \subset J$ and $\mathfrak{p} \notin V(J)$. Recall that

$$R\Gamma_J(M)^\wedge = R\operatorname{Hom}_A(R\Gamma_I(A), R\Gamma_J(M))$$

by the description of derived completion in More on Algebra, Lemma 82.10 combined with the description of local cohomology in Dualizing Complexes, Lemma 10.2. Assumption (3) means that $R\Gamma_I(A)$ has nonzero cohomology only in degrees $\leq d$. Using the canonical truncations of $R\Gamma_I(A)$ we find it suffices to show that

$$\mathrm{Ext}^i(N, R\Gamma_J(M))$$

is annihilated by an $f \in A$, $f \notin \mathfrak p$ for $i \leq s + d$ and any A-module N. In turn using the canonical truncations for $R\Gamma_J(M)$ we see that it suffices to show $H^i_J(M)$ is annihilated by an $f \in A$, $f \notin \mathfrak p$ for $i \leq s + d$. This follows from Local Cohomology, Lemma 9.2. □

0EFI **Lemma 8.2.** *Let I, J be ideals of a Noetherian ring. Let M be a finite A-module. Let s and d be integers. With T as in (8.0.1) assume*

1. *A has a dualizing complex,*
2. *if $\mathfrak p \in V(I)$, then no condition,*
3. *if $\mathfrak p \notin V(I)$, $\mathfrak p \in T$, then $\dim((A/\mathfrak p)_\mathfrak q) \leq d$ for some $\mathfrak q \in V(\mathfrak p) \cap V(J) \cap V(I)$,*
4. *if $\mathfrak p \notin V(I)$, $\mathfrak p \notin T$, then*

$$\mathit{depth}_{A_\mathfrak p}(M_\mathfrak p) \geq s \quad \mathit{or} \quad \mathit{depth}_{A_\mathfrak p}(M_\mathfrak p) + \dim((A/\mathfrak p)_\mathfrak q) > d + s$$

for all $\mathfrak q \in V(\mathfrak p) \cap V(J) \cap V(I)$.

Then there exists an ideal $J_0 \subset J$ with $V(J_0) \cap V(I) = V(J) \cap V(I)$ such that for any $J' \subset J_0$ with $V(J') \cap V(I) = V(J) \cap V(I)$ the map

$$R\Gamma_{J'}(M) \longrightarrow R\Gamma_{J_0}(M)$$

induces an isomorphism in cohomology in degrees $\leq s$ and moreover these modules are annihilated by a power of $J_0 I$.

Proof. Let us consider the set

$$B = \{\mathfrak p \notin V(I),\ \mathfrak p \in T,\ \text{and } \mathrm{depth}(M_\mathfrak p) \leq s\}$$

Choose $J_0 \subset J$ such that $V(J_0)$ is the closure of $B \cup V(J)$.

Claim I: $V(J_0) \cap V(I) = V(J) \cap V(I)$.

Proof of Claim I. The inclusion $\supset$ holds by construction. Let $\mathfrak p$ be a minimal prime of $V(J_0)$. If $\mathfrak p \in B \cup V(J)$, then either $\mathfrak p \in T$ or $\mathfrak p \in V(J)$ and in both cases $V(\mathfrak p) \cap V(I) \subset V(J) \cap V(I)$ as desired. If $\mathfrak p \notin B \cup V(J)$, then $V(\mathfrak p) \cap B$ is dense, hence infinite, and we conclude that $\mathrm{depth}(M_\mathfrak p) < s$ by Local Cohomology, Lemma 8.2. In fact, let $V(\mathfrak p) \cap B = \{\mathfrak p_\lambda\}_{\lambda \in \Lambda}$. Pick $\mathfrak q_\lambda \in V(\mathfrak p_\lambda) \cap V(J) \cap V(I)$ as in (3). Let $\delta : \mathrm{Spec}(A) \to \mathbf{Z}$ be the dimension function associated to a dualizing complex $\omega_A^\bullet$ for A. Since Λ is infinite and δ is bounded, there exists an infinite subset $\Lambda' \subset \Lambda$ on which $\delta(\mathfrak q_\lambda)$ is constant. For $\lambda \in \Lambda'$ we have

$$\mathrm{depth}(M_{\mathfrak p_\lambda}) + \delta(\mathfrak p_\lambda) - \delta(\mathfrak q_\lambda) = \mathrm{depth}(M_{\mathfrak p_\lambda}) + \dim((A/\mathfrak p_\lambda)_{\mathfrak q_\lambda}) \leq d + s$$

by (3) and the definition of B. By the semi-continuity of the function depth $+\ \delta$ proved in Duality for Schemes, Lemma 2.7 we conclude that

$$\mathrm{depth}(M_\mathfrak p) + \dim((A/\mathfrak p)_{\mathfrak q_\lambda}) = \mathrm{depth}(M_\mathfrak p) + \delta(\mathfrak p) - \delta(\mathfrak q_\lambda) \leq d + s$$

Since also $\mathfrak p \notin V(I)$ we read off from (4) that $\mathfrak p \in T$, i.e., $V(\mathfrak p) \cap V(I) \subset V(J) \cap V(I)$. This finishes the proof of Claim I.

Claim II: $H^i_{J_0}(M) \to H^i_{J'}(M)$ is an isomorphism for $i \leq s$ and $J' \subset J_0$ with $V(J') \cap V(I) = V(J) \cap V(I)$.

Proof of claim II. Choose $\mathfrak p \in V(J')$ not in $V(J_0)$. It suffices to show that $H^i_{\mathfrak p A_\mathfrak p}(M_\mathfrak p) = 0$ for $i \leq s$, see Local Cohomology, Lemma 2.6. Observe that $\mathfrak p \in T$. Hence since $\mathfrak p$ is not in B we see that $\text{depth}(M_\mathfrak p) > s$ and the groups vanish by Dualizing Complexes, Lemma 11.1.

Claim III. The final statement of the lemma is true.

By Claim II for $i \leq s$ we have

$$H^i_T(M) = H^i_{J_0}(M) = H^i_{J'}(M)$$

for all ideals $J' \subset J_0$ with $V(J') \cap V(I) = V(J) \cap V(I)$. See Local Cohomology, Lemma 4.3. Let us check the hypotheses of Local Cohomology, Proposition 9.1 for the subsets $T \subset T \cup V(I)$, the module M, and the integer s. We have to show that given $\mathfrak p \subset \mathfrak q$ with $\mathfrak p \not\in T \cup V(I)$ and $\mathfrak q \in T$ we have

$$\text{depth}_{A_\mathfrak p}(M_\mathfrak p) + \dim((A/\mathfrak p)_\mathfrak q) > s$$

If $\text{depth}(M_\mathfrak p) \geq s$, then this is true because the dimension of $(A/\mathfrak p)_\mathfrak q$ is at least 1. Thus we may assume $\text{depth}(M_\mathfrak p) < s$. If $\mathfrak q \in V(I)$, then $\mathfrak q \in V(J) \cap V(I)$ and the inequality holds by (4). If $\mathfrak q \not\in V(I)$, then we can use (3) to pick $\mathfrak q' \in V(\mathfrak q) \cap V(J) \cap V(I)$ with $\dim((A/\mathfrak q)_{\mathfrak q'}) \leq d$. Then assumption (4) gives

$$\text{depth}_{A_\mathfrak p}(M_\mathfrak p) + \dim((A/\mathfrak p)_{\mathfrak q'}) > s + d$$

Since A is catenary this implies the inequality we want. Applying Local Cohomology, Proposition 9.1 we find $J'' \subset A$ with $V(J'') \subset T \cup V(I)$ such that J'' annihilates $H^i_T(M)$ for $i \leq s$. Then we can write $V(J'') \cup V(J_0) \cup V(I) = V(J'I)$ for some $J' \subset J_0$ with $V(J') \cap V(I) = V(J) \cap V(I)$. Replacing J_0 by J' the proof is complete. □

0EFJ **Lemma 8.3.** *In Lemma 8.2 if instead of the empty condition (2) we assume*

(2') *if* $\mathfrak p \in V(I)$, $\mathfrak p \not\in V(J) \cap V(I)$, *then* $\text{depth}_{A_\mathfrak p}(M_\mathfrak p) + \dim((A/\mathfrak p)_\mathfrak q) > s$ *for all* $\mathfrak q \in V(\mathfrak p) \cap V(J) \cap V(I)$,

then the conditions also imply that $H^i_{J_0}(M)$ *is a finite* A*-module for* $i \leq s$.

Proof. Recall that $H^i_{J_0}(M) = H^i_T(M)$, see proof of Lemma 8.2. Thus it suffices to check that for $\mathfrak p \not\in T$ and $\mathfrak q \in T$ with $\mathfrak p \subset \mathfrak q$ we have $\text{depth}_{A_\mathfrak p}(M_\mathfrak p) + \dim((A/\mathfrak p)_\mathfrak q) > s$, see Local Cohomology, Proposition 10.1. Condition (2') tells us this is true for $\mathfrak p \in V(I)$. Since we know $H^i_T(M)$ is annihilated by a power of IJ_0 we know the condition holds if $\mathfrak p \not\in V(IJ_0)$ by Local Cohomology, Proposition 9.1. This covers all cases and the proof is complete. □

0EFK **Lemma 8.4.** *If in Lemma 8.2 we additionally assume*

(6) *if* $\mathfrak p \not\in V(I)$, $\mathfrak p \in T$, *then* $\text{depth}_{A_\mathfrak p}(M_\mathfrak p) > s$,

then $H^i_{J_0}(M) = H^i_J(M) = H^i_{J+I}(M)$ *for* $i \leq s$ *and these modules are annihilated by a power of* I.

Proof. Choose $\mathfrak p \in V(J)$ or $\mathfrak p \in V(J_0)$ but $\mathfrak p \not\in V(J + I) = V(J_0 + I)$. It suffices to show that $H^i_{\mathfrak p A_\mathfrak p}(M_\mathfrak p) = 0$ for $i \leq s$, see Local Cohomology, Lemma 2.6. These groups vanish by condition (6) and Dualizing Complexes, Lemma 11.1. The final statement follows from Local Cohomology, Proposition 9.1. □

0EFL **Lemma 8.5.** *Let* I, J *be ideals of a Noetherian ring* A. *Let* M *be a finite* A*-module. Let* s *and* d *be integers. With* T *as in (8.0.1) assume*

(1) *A is I-adically complete and has a dualizing complex,*
(2) *if $\mathfrak p \in V(I)$ no condition,*
(3) *$cd(A, I) \leq d$,*
(4) *if $\mathfrak p \not\in V(I)$, $\mathfrak p \not\in T$ then*

$$depth_{A_\mathfrak p}(M_\mathfrak p) \geq s \quad or \quad depth_{A_\mathfrak p}(M_\mathfrak p) + \dim((A/\mathfrak p)_\mathfrak q) > d + s$$

for all $\mathfrak q \in V(\mathfrak p) \cap V(J) \cap V(I)$,
(5) *if $\mathfrak p \not\in V(I)$, $\mathfrak p \not\in T$, $V(\mathfrak p) \cap V(J) \cap V(I) \neq \emptyset$, and $depth(M_\mathfrak p) < s$, then one of the following holds[2]:*
   (a) *$\dim(Supp(M_\mathfrak p)) < s + 2^3$, or*
   (b) *$\delta(\mathfrak p) > d + \delta_{max} - 1$ where δ is a dimension function and δ_{max} is the maximum of δ on $V(J) \cap V(I)$, or*
   (c) *$depth_{A_\mathfrak p}(M_\mathfrak p) + \dim((A/\mathfrak p)_\mathfrak q) > d + s + \delta_{max} - \delta_{min} - 2$ for all $\mathfrak q \in V(\mathfrak p) \cap V(J) \cap V(I)$.*

Then there exists an ideal $J_0 \subset J$ with $V(J_0) \cap V(I) = V(J) \cap V(I)$ such that for any $J' \subset J_0$ with $V(J') \cap V(I) = V(J) \cap V(I)$ the map

$$R\Gamma_{J'}(M) \longrightarrow R\Gamma_J(M)^\wedge$$

induces an isomorphism on cohomology in degrees $\leq s$. Here $^\wedge$ denotes derived I-adic completion.

We encourage the reader to read the proof in the local case first (Lemma 9.5) as it explains the structure of the proof without having to deal with all the inequalities.

Proof. For an ideal $\mathfrak a \subset A$ we have $R\Gamma_\mathfrak a = R\Gamma_{V(\mathfrak a)}$, see Dualizing Complexes, Lemma 10.1. Next, we observe that

$$R\Gamma_J(M)^\wedge = R\Gamma_I(R\Gamma_J(M))^\wedge = R\Gamma_{I+J}(M)^\wedge = R\Gamma_{I+J'}(M)^\wedge = R\Gamma_I(R\Gamma_{J'}(M))^\wedge = R\Gamma_{J'}(M)^\wedge$$

by Dualizing Complexes, Lemmas 9.6 and 12.1. This explains how we define the arrow in the statement of the lemma.

We claim that the hypotheses of Lemma 8.2 are implied by our current hypotheses on M. The only thing to verify is hypothesis (3). Thus let $\mathfrak p \not\in V(I)$, $\mathfrak p \in T$. Then $V(\mathfrak p) \cap V(I)$ is nonempty as I is contained in the Jacobson radical of A (Algebra, Lemma 95.6). Since $\mathfrak p \in T$ we have $V(\mathfrak p) \cap V(I) = V(\mathfrak p) \cap V(J) \cap V(I)$. Let $\mathfrak q \in V(\mathfrak p) \cap V(I)$ be the generic point of an irreducible component. We have $\text{cd}(A_\mathfrak q, I_\mathfrak q) \leq d$ by Local Cohomology, Lemma 3.6. We have $V(\mathfrak p A_\mathfrak q) \cap V(I_\mathfrak q) = \{\mathfrak q A_\mathfrak q\}$ by our choice of $\mathfrak q$ and we conclude $\dim((A/\mathfrak p)_\mathfrak q) \leq d$ by Local Cohomology, Lemma 3.10.

Observe that the lemma holds for $s < 0$. This is not a trivial case because it is not a priori clear that $H^i(R\Gamma_J(M)^\wedge)$ is zero for $i < 0$. However, this vanishing was esthablished in Dualizing Complexes, Lemma 12.4. We will prove the lemma by induction for $s \geq 0$.

The lemma for $s = 0$ follows immediately from the conclusion of Lemma 8.2 and Dualizing Complexes, Lemma 12.5.

Assume $s > 0$ and the lemma has been shown for smaller values of s. Let $M' \subset M$ be the maximal submodule whose support is contained in $V(I) \cup T$. Then M' is a

[2]Our method forces this additonal condition. We will return to this (insert future reference).
[3]For example if M satisfies Serre's condition (S_s) on the complement of $V(I) \cup T$.

finite A-module whose support is contained in $V(J') \cup V(I)$ for some ideal $J' \subset J$ with $V(J') \cap V(I) = V(J) \cap V(I)$. We claim that

$$R\Gamma_{J'}(M') \to R\Gamma_J(M')^\wedge$$

is an isomorphism for any choice of J'. Namely, we can choose a short exact sequence $0 \to M_1 \oplus M_2 \to M' \to N \to 0$ with M_1 annihilated by a power of J', with M_2 annihilated by a power of I, and with N annihilated by a power of $I + J'$. Thus it suffices to show that the claim holds for M_1, M_2, and N. In the case of M_1 we see that $R\Gamma_{J'}(M_1) = M_1$ and since M_1 is a finite A-module and I-adically complete we have $M_1^\wedge = M_1$. This proves the claim for M_1 by the initial remarks of the proof. In the case of M_2 we see that $H^i_J(M_2) = H^i_{I+J}(M) = H^i_{I+J'}(M) = H^i_{J'}(M_2)$ are annihilated by a power of I and hence derived complete. Thus the claim in this case also. For N we can use either of the arguments just given. Considering the short exact sequence $0 \to M' \to M \to M/M' \to 0$ we see that it suffices to prove the lemma for M/M'. Thus we may assume $\text{Ass}(M) \cap (V(I) \cup T) = \emptyset$.

Let $\mathfrak p \in \text{Ass}(M)$ be such that $V(\mathfrak p) \cap V(J) \cap V(I) = \emptyset$. Since I is contained in the Jacobson radical of A this implies that $V(\mathfrak p) \cap V(J') = \emptyset$ for any $J' \subset J$ with $V(J') \cap V(I) = V(J) \cap V(I)$. Thus setting $N = H^0_{\mathfrak p}(M)$ we see that $R\Gamma_J(N) = R\Gamma_{J'}(N) = 0$ for all $J' \subset J$ with $V(J') \cap V(I) = V(J) \cap V(I)$. In particular $R\Gamma_J(N)^\wedge = 0$. Thus we may replace M by M/N as this changes the structure of M only in primes which do not play a role in conditions (4) or (5). Repeating we may assume that $V(\mathfrak p) \cap V(J) \cap V(I) \neq \emptyset$ for all $\mathfrak p \in \text{Ass}(M)$.

Assume $\text{Ass}(M) \cap (V(I) \cup T) = \emptyset$ and that $V(\mathfrak p) \cap V(J) \cap V(I) \neq \emptyset$ for all $\mathfrak p \in \text{Ass}(M)$. Let $\mathfrak p \in \text{Ass}(M)$. We want to show that we may apply Lemma 8.1. It is in the verification of this that we will use the supplemental condition (5). Choose $\mathfrak p' \subset \mathfrak p$ and $\mathfrak q' \subset V(\mathfrak p) \cap V(J) \cap V(I)$.

1. If $M_{\mathfrak p'} = 0$, then $\text{depth}(M_{\mathfrak p'}) = \infty$ and $\text{depth}(M_{\mathfrak p'}) + \dim((A/\mathfrak p')_{\mathfrak q'}) > d + s$.
2. If $\text{depth}(M_{\mathfrak p'}) < s$, then $\text{depth}(M_{\mathfrak p'}) + \dim((A/\mathfrak p')_{\mathfrak q'}) > d + s$ by (4).

In the remaining cases we have $M_{\mathfrak p'} \neq 0$ and $\text{depth}(M_{\mathfrak p'}) \geq s$. In particular, we see that $\mathfrak p'$ is in the support of M and we can choose $\mathfrak p'' \subset \mathfrak p'$ with $\mathfrak p'' \in \text{Ass}(M)$.

(a) Observe that $\dim((A/\mathfrak p'')_{\mathfrak p'}) \geq \text{depth}(M_{\mathfrak p'})$ by Algebra, Lemma 71.9. If equality holds, then we have

$$\text{depth}(M_{\mathfrak p'}) + \dim((A/\mathfrak p')_{\mathfrak q'}) = \text{depth}(M_{\mathfrak p''}) + \dim((A/\mathfrak p'')_{\mathfrak q'}) > s + d$$

by (4) applied to $\mathfrak p''$ and we are done. This means we are only in trouble if $\dim((A/\mathfrak p'')_{\mathfrak p'}) > \text{depth}(M_{\mathfrak p'})$. This implies that $\dim(M_{\mathfrak p}) \geq s + 2$. Thus if (5)(a) holds, then this does not occur.

(b) If (5)(b) holds, then we get

$$\text{depth}(M_{\mathfrak p'}) + \dim((A/\mathfrak p')_{\mathfrak q'}) \geq s + \delta(\mathfrak p') - \delta(\mathfrak q') \geq s + 1 + \delta(\mathfrak p) - \delta_{max} > s + d$$

as desired.

(c) If (5)(c) holds, then we get

$$\begin{aligned}
\text{depth}(M_{\mathfrak p'}) + \dim((A/\mathfrak p')_{\mathfrak q'}) &\geq s + \delta(\mathfrak p') - \delta(\mathfrak q') \\
&\geq s + 1 + \delta(\mathfrak p) - \delta(\mathfrak q') \\
&= s + 1 + \delta(\mathfrak p) - \delta(\mathfrak q) + \delta(\mathfrak q) - \delta(\mathfrak q') \\
&> s + 1 + (s + d + \delta_{max} - \delta_{min} - 2) + \delta(\mathfrak q) - \delta(\mathfrak q') \\
&\geq 2s + d - 1 \geq s + d
\end{aligned}$$

as desired. Observe that this argument works because we know that a prime $\mathfrak q \in V(\mathfrak p) \cap V(J) \cap V(I)$ exists.

Now we are ready to do the induction step.

Choose an ideal J_0 as in Lemma 8.2 and an integer $t > 0$ such that $(J_0 I)^t$ annihilates $H^s_J(M)$. The assumptions of Lemma 8.1 are satisfied for every $\mathfrak p \in \text{Ass}(M)$ (see previous paragraph). Thus the annihilator $\mathfrak a \subset A$ of $H^s(R\Gamma_J(M)^\wedge)$ is not contained in $\mathfrak p$ for $\mathfrak p \in \text{Ass}(M)$. Thus we can find an $f \in \mathfrak a (J_0 I)^t$ not in any associated prime of M which is an annihilator of both $H^s(R\Gamma_J(M)^\wedge)$ and $H^s_J(M)$. Then f is a nonzerodivisor on M and we can consider the short exact sequence

$$0 \to M \xrightarrow{f} M \to M/fM \to 0$$

Our choice of f shows that we obtain

$$\begin{tikzcd}
H^{s-1}_{J'}(M) \arrow[r] \arrow[d] & H^{s-1}_{J'}(M/fM) \arrow[r] \arrow[d] & H^s_{J'}(M) \arrow[r] \arrow[d] & 0 \\
H^{s-1}(R\Gamma_J(M)^\wedge) \arrow[r] & H^{s-1}(R\Gamma_J(M/fM)^\wedge) \arrow[r] & H^s(R\Gamma_J(M)^\wedge) \arrow[r] & 0
\end{tikzcd}$$

for any $J' \subset J_0$ with $V(J') \cap V(I) = V(J) \cap V(I)$. Thus if we choose J' such that it works for M and M/fM and $s-1$ (possible by induction hypothesis – see next paragraph), then we conclude that the lemma is true.

To finish the proof we have to show that the module M/fM satisfies the hypotheses (4) and (5) for $s-1$. Thus we let $\mathfrak p$ be a prime in the support of M/fM with $\text{depth}((M/fM)_\mathfrak p) < s-1$ and with $V(\mathfrak p) \cap V(J) \cap V(I)$ nonempty. Then $\dim(M_\mathfrak p) = \dim((M/fM)_\mathfrak p) + 1$ and $\text{depth}(M_\mathfrak p) = \text{depth}((M/fM)_\mathfrak p) + 1$. In particular, we know (4) and (5) hold for $\mathfrak p$ and M with the original value s. The desired inequalities then follow by inspection. $\square$

0EFM **Example 8.6.** In Lemma 8.5 we do not know that the inverse systems $H^i_J(M/I^nM)$ satisfy the Mittag-Leffler condition. For example, suppose that $A = \mathbf{Z}_p[[x,y]]$, $I = (p)$, $J = (p, x)$, and $M = A/(xy - p)$. Then the image of $H^0_J(M/p^nM) \to H^0_J(M/pM)$ is the ideal generated by y^n in $M/pM = A/(p, xy)$.

9. Algebraization of local cohomology, II

0EFP In this section we redo the arguments of Section 8 when $(A, \mathfrak m)$ is a local ring and we take local cohomology $R\Gamma_\mathfrak m$ with respect to $\mathfrak m$. As before our main tool is the following lemma.

0DXK **Lemma 9.1.** *Let $(A, \mathfrak m)$ be a Noetherian local ring. Let $I \subset A$ be an ideal. Let M be a finite A-module and let $\mathfrak p \subset A$ be a prime. Let s and d be integers. Assume*

(1) *A has a dualizing complex,*

(2) *$cd(A, I) \leq d$, and*
(3) *$depth_{A_\mathfrak{p}}(M_\mathfrak{p}) + \dim(A/\mathfrak{p}) > d + s$.*

Then there exists an $f \in A \setminus \mathfrak{p}$ which annihilates $H^i(R\Gamma_\mathfrak{m}(M)^\wedge)$ for $i \leq s$ where $^\wedge$ indicates I-adic completion.

Proof. According to Local Cohomology, Lemma 8.4 the function

$$\mathfrak{p}' \longmapsto \text{depth}_{A_{\mathfrak{p}'}}(M_{\mathfrak{p}'}) + \dim(A/\mathfrak{p}')$$

is lower semi-continuous on $\text{Spec}(A)$. Thus the value of this function on $\mathfrak{p}' \subset \mathfrak{p}$ is $> s + d$. Thus our lemma is a special case of Lemma 8.1 provided that $\mathfrak{p} \neq \mathfrak{m}$. If $\mathfrak{p} = \mathfrak{m}$, then we have $H^i_\mathfrak{m}(M) = 0$ for $i \leq s + d$ by the relationship between depth and local cohomology (Dualizing Complexes, Lemma 11.1). Thus the argument given in the proof of Lemma 8.1 shows that $H^i(R\Gamma_\mathfrak{m}(M)^\wedge) = 0$ for $i \leq s$ in this (degenerate) case. ☐

0DXM **Lemma 9.2.** *Let $(A, \mathfrak{m})$ be a Noetherian local ring. Let $I \subset A$ be an ideal. Let M be a finite A-module. Let s and d be integers. Assume*

(1) *A has a dualizing complex,*
(2) *if $\mathfrak{p} \in V(I)$, then no condition,*
(3) *if $\mathfrak{p} \notin V(I)$ and $V(\mathfrak{p}) \cap V(I) = \{\mathfrak{m}\}$, then $\dim(A/\mathfrak{p}) \leq d$,*
(4) *if $\mathfrak{p} \notin V(I)$ and $V(\mathfrak{p}) \cap V(I) \neq \{\mathfrak{m}\}$, then*

$$depth_{A_\mathfrak{p}}(M_\mathfrak{p}) \geq s \quad \text{or} \quad depth_{A_\mathfrak{p}}(M_\mathfrak{p}) + \dim(A/\mathfrak{p}) > d + s$$

Then there exists an ideal $J_0 \subset A$ with $V(J_0) \cap V(I) = \{\mathfrak{m}\}$ such that for any $J \subset J_0$ with $V(J) \cap V(I) = \{\mathfrak{m}\}$ the map

$$R\Gamma_J(M) \longrightarrow R\Gamma_{J_0}(M)$$

induces an isomorphism in cohomology in degrees $\leq s$ and moreover these modules are annihilated by a power of $J_0 I$.

Proof. This is a special case of Lemma 8.2. ☐

0DXN **Lemma 9.3.** *In Lemma 9.2 if instead of the empty condition (2) we assume*

(2') *if $\mathfrak{p} \in V(I)$ and $\mathfrak{p} \neq \mathfrak{m}$, then $depth_{A_\mathfrak{p}}(M_\mathfrak{p}) + \dim(A/\mathfrak{p}) > s$,*

then the conditions also imply that $H^i_{J_0}(M)$ is a finite A-module for $i \leq s$.

Proof. This is a special case of Lemma 8.3. ☐

0EFQ **Lemma 9.4.** *If in Lemma 9.2 we additionally assume*

(6) *if $\mathfrak{p} \notin V(I)$ and $V(\mathfrak{p}) \cap V(I) = \{\mathfrak{m}\}$, then $depth_{A_\mathfrak{p}}(M_\mathfrak{p}) > s$,*

then $H^i_{J_0}(M) = H^i_J(M) = H^i_\mathfrak{m}(M)$ for $i \leq s$ and these modules are annihilated by a power of I.

Proof. This is a special case of Lemma 8.4. ☐

0DXP **Lemma 9.5.** *Let $(A, \mathfrak{m})$ be a Noetherian local ring. Let $I \subset A$ be an ideal. Let M be a finite A-module. Let s and d be integers. Assume*

(1) *A is I-adically complete and has a dualizing complex,*
(2) *if $\mathfrak{p} \in V(I)$, no condition,*
(3) *$cd(A, I) \leq d$,*

(4) *if* $\mathfrak{p} \not\in V(I)$ *and* $V(\mathfrak{p}) \cap V(I) \neq \{\mathfrak{m}\}$ *then*

$$depth_{A_\mathfrak{p}}(M_\mathfrak{p}) \geq s \quad or \quad depth_{A_\mathfrak{p}}(M_\mathfrak{p}) + \dim(A/\mathfrak{p}) > d + s$$

Then there exists an ideal $J_0 \subset A$ *with* $V(J_0) \cap V(I) = \{\mathfrak{m}\}$ *such that for any* $J \subset J_0$ *with* $V(J) \cap V(I) = \{\mathfrak{m}\}$ *the map*

$$R\Gamma_J(M) \longrightarrow R\Gamma_J(M)^\wedge = R\Gamma_\mathfrak{m}(M)^\wedge$$

induces an isomorphism in cohomology in degrees $\leq s$. *Here* $^\wedge$ *denotes derived* I*-adic completion.*

Proof. This lemma is a special case of Lemma 8.5 since condition (5)(c) is implied by condition (4) as $\delta_{max} = \delta_{min} = \delta(\mathfrak{m})$. We will give the proof of this important special case as it is somewhat easier (fewer things to check).

There is no difference between $R\Gamma_\mathfrak{a}$ and $R\Gamma_{V(\mathfrak{a})}$ in our current situation, see Dualizing Complexes, Lemma 10.1. Next, we observe that

$$R\Gamma_\mathfrak{m}(M)^\wedge = R\Gamma_I(R\Gamma_J(M))^\wedge = R\Gamma_J(M)^\wedge$$

by Dualizing Complexes, Lemmas 9.6 and 12.1 which explains the equality sign in the statement of the lemma.

Observe that the lemma holds for $s < 0$. This is not a trivial case because it is not a priori clear that $H^s(R\Gamma_\mathfrak{m}(M)^\wedge)$ is zero for negative s. However, this vanishing was esthablished in Lemma 5.4. We will prove the lemma by induction for $s \geq 0$.

The assumptions of Lemma 9.2 are satisfied by Local Cohomology, Lemma 3.10. The lemma for $s = 0$ follows from Lemma 9.2 and Dualizing Complexes, Lemma 12.5.

Assume $s > 0$ and the lemma holds for smaller values of s. Let $M' \subset M$ be the submodule of elements whose support is condained in $V(I) \cup V(J)$ for some ideal J with $V(J) \cap V(I) = \{\mathfrak{m}\}$. Then M' is a finite A-module. We claim that

$$R\Gamma_J(M') \to R\Gamma_\mathfrak{m}(M')^\wedge$$

is an isomorphism for any choice of J. Namely, for any such module there is a short exact sequence $0 \to M_1 \oplus M_2 \to M' \to N \to 0$ with M_1 annihilated by a power of J, with M_2 annihilated by a power of I and with N annihilated by a power of $\mathfrak{m}$. In the case of M_1 we see that $R\Gamma_J(M_1) = M_1$ and since M_1 is a finite A-module and I-adically complete we have $M_1^\wedge = M_1$. Thus the claim holds for M_1. In the case of M_2 we see that $H^i_J(M_2)$ is annihilated by a power of I and hence derived complete. Thus the claim for M_2. By the same arguments the claim holds for N and we conclude that the claim holds. Considering the short exact sequence $0 \to M' \to M \to M/M' \to 0$ we see that it suffices to prove the lemma for M/M'. This we may assume $\mathfrak{p} \in \text{Ass}(M)$ implies $V(\mathfrak{p}) \cap V(I) \neq \{\mathfrak{m}\}$, i.e., $\mathfrak{p}$ is a prime as in (4).

Choose an ideal J_0 as in Lemma 9.2 and an integer $t > 0$ such that $(J_0I)^t$ annihilates $H^s_J(M)$. Here J denotes an arbitrary ideal $J \subset J_0$ with $V(J) \cap V(I) = \{\mathfrak{m}\}$. The assumptions of Lemma 9.1 are satisfied for every $\mathfrak{p} \in \text{Ass}(M)$ (see previous paragraph). Thus the annihilator $\mathfrak{a} \subset A$ of $H^s(R\Gamma_\mathfrak{m}(M)^\wedge)$ is not contained in $\mathfrak{p}$ for $\mathfrak{p} \in \text{Ass}(M)$. Thus we can find an $f \in \mathfrak{a}(J_0I)^t$ not in any associated prime of M which is an annihilator of both $H^s(R\Gamma_\mathfrak{m}(M)^\wedge)$ and $H^s_J(M)$. Then f is a nonzerodivisor on M and we can consider the short exact sequence

$$0 \to M \xrightarrow{f} M \to M/fM \to 0$$

Our choice of f shows that we obtain

$$\begin{tikzcd} H^{s-1}_J(M) \arrow[r] \arrow[d] & H^{s-1}_J(M/fM) \arrow[r] \arrow[d] & H^s_J(M) \arrow[r] \arrow[d] & 0 \\ H^{s-1}(R\Gamma_\mathfrak{m}(M)^\wedge) \arrow[r] & H^{s-1}(R\Gamma_\mathfrak{m}(M/fM)^\wedge) \arrow[r] & H^s(R\Gamma_\mathfrak{m}(M)^\wedge) \arrow[r] & 0 \end{tikzcd}$$

for any $J \subset J_0$ with $V(J) \cap V(I) = \{\mathfrak{m}\}$. Thus if we choose J such that it works for M and M/fM and $s-1$ (possible by induction hypothesis), then we conclude that the lemma is true. $\square$

10. Algebraization of local cohomology, III

0EFT In this section we bootstrap the material in Sections 8 and 11 to give a stronger result the following situation.

0EFU **Situation 10.1.** Here A is a Noetherian ring. We have an ideal $I \subset A$, a finite A-module M, and a subset $T \subset V(I)$ stable under specialization. We have integers s and d. We assume

(1) A has a dualizing complex,
(3) $\text{cd}(A, I) \leq d$,
(4) given primes $\mathfrak{p} \subset \mathfrak{r} \subset \mathfrak{q}$ with $\mathfrak{p} \notin V(I)$, $\mathfrak{r} \in V(I) \setminus T$, $\mathfrak{q} \in T$ we have
$$\text{depth}_{A_\mathfrak{p}}(M_\mathfrak{p}) \geq s \quad \text{or} \quad \text{depth}_{A_\mathfrak{p}}(M_\mathfrak{p}) + \dim((A/\mathfrak{p})_\mathfrak{q}) > d + s$$
(6) given $\mathfrak{q} \in T$ denoting $A', \mathfrak{m}', I', M'$ are the usual I-adic completions of $A_\mathfrak{q}, \mathfrak{q}A_\mathfrak{q}, I_\mathfrak{q}, M_\mathfrak{q}$ we have
$$\text{depth}(M'_{\mathfrak{p}'}) > s$$
for all $\mathfrak{p}' \in \text{Spec}(A') \setminus V(I')$ with $V(\mathfrak{p}') \cap V(I') = \{\mathfrak{m}'\}$.

The following lemma explains why in Situation 10.1 it suffices to look at triples $\mathfrak{p} \subset \mathfrak{r} \subset \mathfrak{q}$ of primes in (4) even though the actual assumption only involves $\mathfrak{p}$ and $\mathfrak{q}$.

0EID **Lemma 10.2.** *In Situation 10.1 let $\mathfrak{p} \subset \mathfrak{q}$ be primes of A with $\mathfrak{p} \notin V(I)$ and $\mathfrak{q} \in T$. If there does not exist an $\mathfrak{r} \in V(I) \setminus T$ with $\mathfrak{p} \subset \mathfrak{r} \subset \mathfrak{q}$ then depth$(M_\mathfrak{p}) > s$.*

Proof. Choose $\mathfrak{q}' \in T$ with $\mathfrak{p} \subset \mathfrak{q}' \subset \mathfrak{q}$ such that there is no prime in T strictly in between $\mathfrak{p}$ and $\mathfrak{q}'$. To prove the lemma we may and do replace $\mathfrak{q}$ by $\mathfrak{q}'$. Next, let $\mathfrak{p}' \subset A_\mathfrak{q}$ be the prime corresponding to $\mathfrak{p}$. After doing this we obtain that $V(\mathfrak{p}') \cap V(IA_\mathfrak{q}) = \{\mathfrak{q}A_\mathfrak{q}\}$ because of the nonexistence of a prime $\mathfrak{r}$ as in the lemma. Let $A', I', \mathfrak{m}', M'$ be the I-adic completions of $A_\mathfrak{q}, I_\mathfrak{q}, \mathfrak{q}A_\mathfrak{q}, M_\mathfrak{q}$. Since $A_\mathfrak{q} \to A'$ is faithfully flat (Algebra, Lemma 96.3) we can choose $\mathfrak{p}'' \subset A'$ lying over $\mathfrak{p}'$ with $\dim(A'_{\mathfrak{p}''}/\mathfrak{p}'A'_{\mathfrak{p}''}) = 0$. Then we see that

$$\text{depth}(M'_{\mathfrak{p}''}) = \text{depth}((M_\mathfrak{q} \otimes_{A_\mathfrak{q}} A')_{\mathfrak{p}''}) = \text{depth}(M_\mathfrak{p} \otimes_{A_\mathfrak{p}} A'_{\mathfrak{p}''}) = \text{depth}(M_\mathfrak{p})$$

by flatness of $A \to A'$ and our choice of $\mathfrak{p}''$, see Algebra, Lemma 157.1. Since $\mathfrak{p}''$ lies over $\mathfrak{p}'$ we have $V(\mathfrak{p}'') \cap V(I') = \{\mathfrak{m}'\}$. Thus condition (6) in Situation 10.1 implies depth$(M'_{\mathfrak{p}''}) > s$ which finishes the proof. $\square$

The following tedious lemma explains the relationships between various collections of conditions one might impose.

0EFV **Lemma 10.3.** *In Situation 10.1 we have*

(E) *if $T' \subset T$ is a smaller specialization stable subset, then A, I, T', M satisfies the assumptions of Situation 10.1,*
(F) *if $S \subset A$ is a multiplicative subset, then $S^{-1}A, S^{-1}I, T', S^{-1}M$ satisfies the assumptions of Situation 10.1 where $T' \subset V(S^{-1}I)$ is the inverse image of T,*
(G) *the quadruple A', I', T', M' satisfies the assumptions of Situation 10.1 where A', I', M' are the usual I-adic completions of A, I, M and $T' \subset V(I')$ is the inverse image of T.*

Let $I \subset \mathfrak{a} \subset A$ be an ideal such that $V(\mathfrak{a}) \subset T$. Then

(A) *if I is contained in the Jacobson radical of A, then all hypotheses of Lemmas 8.2 and 8.4 are satisfied for $A, I, \mathfrak{a}, M$,*
(B) *if A is complete with respect to I, then all hypotheses except for possibly (5) of Lemma 8.5 are satisfied for $A, I, \mathfrak{a}, M$,*
(C) *if A is local with maximal ideal $\mathfrak{m} = \mathfrak{a}$, then all hypotheses of Lemmas 9.2 and 9.4 hold for $A, \mathfrak{m}, I, M$,*
(D) *if A is local with maximal ideal $\mathfrak{m} = \mathfrak{a}$ and I-adically complete, then all hypotheses of Lemma 9.5 hold for $A, \mathfrak{m}, I, M$,*

Proof. Proof of (E). We have to prove assumptions (1), (3), (4), (6) of Situation 10.1 hold for A, I, T, M. Shrinking T to T' weakens assumption (6) and strengthens assumption (4). However, if we have $\mathfrak{p} \subset \mathfrak{r} \subset \mathfrak{q}$ with $\mathfrak{p} \not\in V(I)$, $\mathfrak{r} \in V(I) \setminus T'$, $\mathfrak{q} \in T'$ as in assumption (4) for A, I, T', M, then either we can pick $\mathfrak{r} \in V(I) \setminus T$ and condition (4) for A, I, T, M kicks in or we cannot find such an $\mathfrak{r}$ in which case we get $\text{depth}(M_\mathfrak{p}) > s$ by Lemma 10.2. This proves (4) holds for A, I, T', M as desired.

Proof of (F). This is straightforward and we omit the details.

Proof of (G). We have to prove assumptions (1), (3), (4), (6) of Situation 10.1 hold for the I-adic completions A', I', T', M'. Please keep in mind that $\text{Spec}(A') \to \text{Spec}(A)$ induces an isomorphism $V(I') \to V(I)$.

Assumption (1): The ring A' has a dualizing complex, see Dualizing Complexes, Lemma 22.4.

Assumption (3): Since $I' = IA'$ this follows from Local Cohomology, Lemma 3.5.

Assumption (4): If we have primes $\mathfrak{p}' \subset \mathfrak{r}' \subset \mathfrak{q}'$ in A' with $\mathfrak{p}' \not\in V(I')$, $\mathfrak{r}' \in V(I') \setminus T'$, $\mathfrak{q}' \in T'$ then their images $\mathfrak{p} \subset \mathfrak{r} \subset \mathfrak{q}$ in the spectrum of A satisfy $\mathfrak{p} \not\in V(I)$, $\mathfrak{r} \in V(I) \setminus T$, $\mathfrak{q} \in T$. Then we have

$$\text{depth}_{A_\mathfrak{p}}(M_\mathfrak{p}) \geq s \quad \text{or} \quad \text{depth}_{A_\mathfrak{p}}(M_\mathfrak{p}) + \dim((A/\mathfrak{p})_\mathfrak{q}) > d + s$$

by assumption (4) for A, I, T, M. We have $\text{depth}(M'_{\mathfrak{p}'}) \geq \text{depth}(M_\mathfrak{p})$ and $\text{depth}(M'_{\mathfrak{p}'}) + \dim((A'/\mathfrak{p}')_{\mathfrak{q}'}) = \text{depth}(M_\mathfrak{p}) + \dim((A/\mathfrak{p})_\mathfrak{q})$ by Local Cohomology, Lemma 10.3. Thus assumption (4) holds for A', I', T', M'.

Assumption (6): Let $\mathfrak{q}' \in T'$ lying over the prime $\mathfrak{q} \in T$. Then $A'_{\mathfrak{q}'}$ and $A_\mathfrak{q}$ have isomorphic I-adic completions and similarly for $M_\mathfrak{q}$ and $M'_{\mathfrak{q}'}$. Thus assumption (6) for A', I', T', M' is equivalent to assumption (6) for A, I, T, M.

Proof of (A). We have to check conditions (1), (2), (3), (4), and (6) of Lemmas 8.2 and 8.4 for $(A, I, \mathfrak{a}, M)$. Warning: the set T in the statement of these lemmas is not the same as the set T above.

Condition (1): This holds because we have assumed A has a dualizing complex in Situation 10.1.

Condition (2): This is empty.

Condition (3): Let $\mathfrak{p} \subset A$ with $V(\mathfrak{p}) \cap V(I) \subset V(\mathfrak{a})$. Since I is contained in the Jacobson radical of A we see that $V(\mathfrak{p}) \cap V(I) \neq \emptyset$. Let $\mathfrak{q} \in V(\mathfrak{p}) \cap V(I)$ be a generic point. Since $\text{cd}(A_\mathfrak{q}, I_\mathfrak{q}) \leq d$ (Local Cohomology, Lemma 3.6) and since $V(\mathfrak{p}A_\mathfrak{q}) \cap V(I_\mathfrak{q}) = \{\mathfrak{q}A_\mathfrak{q}\}$ we get $\dim((A/\mathfrak{p})_\mathfrak{q}) \leq d$ by Local Cohomology, Lemma 3.10 which proves (3).

Condition (4): Suppose $\mathfrak{p} \not\in V(I)$ and $\mathfrak{q} \in V(\mathfrak{p}) \cap V(\mathfrak{a})$. It suffices to show

$$\text{depth}_{A_\mathfrak{p}}(M_\mathfrak{p}) \geq s \quad \text{or} \quad \text{depth}_{A_\mathfrak{p}}(M_\mathfrak{p}) + \dim((A/\mathfrak{p})_\mathfrak{q}) > d + s$$

If there exists a prime $\mathfrak{p} \subset \mathfrak{r} \subset \mathfrak{q}$ with $\mathfrak{r} \in V(I) \setminus T$, then this follows immediately from assumption (4) in Situation 10.1. If not, then $\text{depth}(M_\mathfrak{p}) > s$ by Lemma 10.2.

Condition (6): Let $\mathfrak{p} \not\in V(I)$ with $V(\mathfrak{p}) \cap V(I) \subset V(\mathfrak{a})$. Since I is contained in the Jacobson radical of A we see that $V(\mathfrak{p}) \cap V(I) \neq \emptyset$. Choose $\mathfrak{q} \in V(\mathfrak{p}) \cap V(I) \subset V(\mathfrak{a})$. It is clear there does not exist a prime $\mathfrak{p} \subset \mathfrak{r} \subset \mathfrak{q}$ with $\mathfrak{r} \in V(I) \setminus T$. By Lemma 10.2 we have $\text{depth}(M_\mathfrak{p}) > s$ which proves (6).

Proof of (B). We have to check conditions (1), (2), (3), (4) of Lemma 8.5. Warning: the set T in the statement of this lemma is not the same as the set T above.

Condition (1): This holds because A is complete and has a dualizing complex.

Condition (2): This is empty.

Condition (3): This is the same as assumption (3) in Situation 10.1.

Condition (4): This is the same as assumption (4) in Lemma 8.2 which we proved in (A).

Proof of (C). This is true because the assumptions in Lemmas 9.2 and 9.4 are the same as the assumptions in Lemmas 8.2 and 8.4 in the local case and we proved these hold in (A).

Proof of (D). This is true because the assumptions in Lemma 9.5 are the same as the assumptions (1), (2), (3), (4) in Lemma 8.5 and we proved these hold in (B). $\square$

0EFR **Lemma 10.4.** *In Situation 10.1 assume A is local with maximal ideal $\mathfrak{m}$ and $T = \{\mathfrak{m}\}$. Then $H^i_\mathfrak{m}(M) \to \lim H^i_\mathfrak{m}(M/I^nM)$ is an isomorphism for $i \leq s$ and these modules are annihilated by a power of I.*

Proof. Let $A', I', \mathfrak{m}', M'$ be the usual I-adic completions of $A, I, \mathfrak{m}, M$. Recall that we have $H^i_\mathfrak{m}(M) \otimes_A A' = H^i_{\mathfrak{m}'}(M')$ by flatness of $A \to A'$ and Dualizing Complexes, Lemma 9.3. Since $H^i_\mathfrak{m}(M)$ is $\mathfrak{m}$-power torsion we have $H^i_\mathfrak{m}(M) = H^i_\mathfrak{m}(M) \otimes_A A'$, see More on Algebra, Lemma 80.3. We conclude that $H^i_\mathfrak{m}(M) = H^i_{\mathfrak{m}'}(M')$. The exact same arguments will show that $H^i_\mathfrak{m}(M/I^nM) = H^i_{\mathfrak{m}'}(M'/(I')^nM')$ for all n and i.

Lemmas 9.5, 9.2, and 9.4 apply to $A', \mathfrak{m}', I', M'$ by Lemma 10.3 parts (C) and (D). Thus we get an isomorphism

$$H^i_{\mathfrak{m}'}(M') \longrightarrow H^i(R\Gamma_{\mathfrak{m}'}(M')^\wedge)$$

for $i \leq s$ where $^\wedge$ is derived I'-adic completion and these modules are annihilated by a power of I'. By Lemma 5.4 we obtain isomorphisms

$$H^i_{\mathfrak{m}'}(M') \longrightarrow \lim H^i_{\mathfrak{m}'}(M'/(I')^n M'))$$

for $i \leq s$. Combined with the already esthablished comparison with local cohomology over A we conclude the lemma is true. □

0EFW **Lemma 10.5.** *Let $I \subset \mathfrak{a}$ be ideals of a Noetherian ring A. Let M be a finite A-module. Let s and d be integers. If we assume*

(a) *A has a dualizing complex,*
(b) *$cd(A, I) \leq d$,*
(c) *if $\mathfrak{p} \notin V(I)$ and $\mathfrak{q} \in V(\mathfrak{p}) \cap V(\mathfrak{a})$ then $depth_{A_\mathfrak{p}}(M_\mathfrak{p}) > s$ or $depth_{A_\mathfrak{p}}(M_\mathfrak{p}) + \dim((A/\mathfrak{p})_\mathfrak{q}) > d + s$.*

Then $A, I, V(\mathfrak{a}), M, s, d$ are as in Situation 10.1.

Proof. We have to show that assumptions (1), (3), (4), and (6) of Situation 10.1 hold. It is clear that (a) ⇒ (1), (b) ⇒ (3), and (c) ⇒ (4). To finish the proof in the next paragraph we show (6) holds.

Let $\mathfrak{q} \in V(\mathfrak{a})$. Denote $A', I', \mathfrak{m}', M'$ the I-adic completions of $A_\mathfrak{q}, I_\mathfrak{q}, \mathfrak{q}A_\mathfrak{q}, M_\mathfrak{q}$. Let $\mathfrak{p}' \subset A'$ be a nonmaximal prime with $V(\mathfrak{p}') \cap V(I') = \{\mathfrak{m}'\}$. Observe that this implies $\dim(A'/\mathfrak{p}') \leq d$ by Local Cohomology, Lemma 3.10. Denote $\mathfrak{p} \subset A$ the image of $\mathfrak{p}'$. We have $\text{depth}(M'_{\mathfrak{p}'}) \geq \text{depth}(M_\mathfrak{p})$ and $\text{depth}(M'_{\mathfrak{p}'}) + \dim(A'/\mathfrak{p}') = \text{depth}(M_\mathfrak{p}) + \dim((A/\mathfrak{p})_\mathfrak{q})$ by Local Cohomology, Lemma 10.3. By assumption (c) either we have $\text{depth}(M'_{\mathfrak{p}'}) \geq \text{depth}(M_\mathfrak{p}) > s$ and we're done or we have $\text{depth}(M'_{\mathfrak{p}'}) + \dim(A'/\mathfrak{p}') > s + d$ which implies $\text{depth}(M'_{\mathfrak{p}'}) > s$ because of the already shown inequality $\dim(A'/\mathfrak{p}') \leq d$. In both cases we obtain what we want. □

0EFX **Lemma 10.6.** *In Situation 10.1 the inverse systems $\{H^i_T(I^n M)\}_{n \geq 0}$ are pro-zero for $i \leq s$. Moreover, there exists an integer m_0 such that for all $m \geq m_0$ there exists an integer $m'(m) \geq m$ such that for $k \geq m'(m)$ the image of $H^{s+1}_T(I^k M) \to H^{s+1}_T(I^m M)$ maps injectively to $H^{s+1}_T(I^{m_0} M)$.*

Proof. Fix m. Let $\mathfrak{q} \in T$. By Lemmas 10.3 and 10.4 we see that

$$H^i_\mathfrak{q}(M_\mathfrak{q}) \longrightarrow \lim H^i_\mathfrak{q}(M_\mathfrak{q}/I^n M_\mathfrak{q})$$

is an isomorphism for $i \leq s$. The inverse systems $\{H^i_\mathfrak{q}(I^n M_\mathfrak{q})\}_{n \geq 0}$ and $\{H^i_\mathfrak{q}(M/I^n M)\}_{n \geq 0}$ satisfy the Mittag-Leffler condition for all i, see Lemma 5.2. Thus looking at the inverse system of long exact sequences

$$0 \to H^0_\mathfrak{q}(I^n M_\mathfrak{q}) \to H^0_\mathfrak{q}(M_\mathfrak{q}) \to H^0_\mathfrak{q}(M_\mathfrak{q}/I^n M_\mathfrak{q}) \to H^1_\mathfrak{q}(I^n M_\mathfrak{q}) \to H^1_\mathfrak{q}(M_\mathfrak{q}) \to \ldots$$

we conclude (some details omitted) that there exists an integer $m'(m, \mathfrak{q}) \geq m$ such that for all $k \geq m'(m, \mathfrak{q})$ the map $H^i_\mathfrak{q}(I^k M_\mathfrak{q}) \to H^i_\mathfrak{q}(I^m M_\mathfrak{q})$ is zero for $i \leq s$ and the image of $H^{s+1}_\mathfrak{q}(I^k M_\mathfrak{q}) \to H^{s+1}_\mathfrak{q}(I^m M_\mathfrak{q})$ is independent of $k \geq m'(m, \mathfrak{q})$ and maps injectively into $H^{s+1}_\mathfrak{q}(M_\mathfrak{q})$.

Suppose we can show that $m'(m, \mathfrak{q})$ can be chosen independently of $\mathfrak{q} \in T$. Then the lemma follows immediately from Local Cohomology, Lemmas 5.2 and 5.3.

Let $\omega_A^\bullet$ be a dualizing complex. Let $\delta : \mathrm{Spec}(A) \to \mathbf{Z}$ be the corresponding dimension function. Recall that δ attains only a finite number of values, see Dualizing Complexes, Lemma 17.4. Claim: for each $d \in \mathbf{Z}$ the integer $m'(m, \mathfrak q)$ can be chosen independently of $\mathfrak q \in T$ with $\delta(\mathfrak q) = d$. Clearly the claim implies the lemma by what we said above.

Pick $\mathfrak q \in T$ with $\delta(\mathfrak q) = d$. Consider the ext modules

$$E(n, j) = \mathrm{Ext}^j_A(I^nM, \omega_A^\bullet)$$

A key feature we will use is that these are finite A-modules. Recall that $(\omega_A^\bullet)_{\mathfrak q}[-d]$ is a normalized dualizing complex for $A_{\mathfrak q}$ by definition of the dimension function associated to a dualizing complex, see Dualizing Complexes, Section 17. The local duality theorem (Dualizing Complexes, Lemma 18.4) tells us that the $\mathfrak q A_{\mathfrak q}$-adic completion of $E(n, -d-i)_{\mathfrak q}$ is Matlis dual to $H^i_{\mathfrak q}(I^nM_{\mathfrak q})$. Thus the choice of $m'(m, \mathfrak q)$ for $i \leq s$ in the first paragraph tells us that for $k \geq m'(m, \mathfrak q)$ and $j \geq -d - s$ the map

$$E(m, j)_{\mathfrak q} \to E(k, j)_{\mathfrak q}$$

is zero. Since these modules are finite and nonzero only for a finite number of possible j (small detail omitted), we can find an open neighbourhood $W \subset \mathrm{Spec}(A)$ of $\mathfrak q$ such that

$$E(m, j)_{\mathfrak q'} \to E(m'(m, \mathfrak q), j)_{\mathfrak q'}$$

is zero for $j \geq -d-s$ for all $\mathfrak q' \in W$. Then of course the maps $E(m, j)_{\mathfrak q'} \to E(k, j)_{\mathfrak q'}$ for $k \geq m'(m, \mathfrak q)$ are zero as well.

For $i = s + 1$ corresponding to $j = -d - s - 1$ we obtain from local duality and the results of the first paragraph that

$$K_{k,\mathfrak q} = \mathrm{Ker}(E(m, -d - s - 1)_{\mathfrak q} \to E(k, -d - s - 1)_{\mathfrak q})$$

is independent of $k \geq m'(m, \mathfrak q)$ and that

$$E(0, -d - s - 1)_{\mathfrak q} \to E(m, -d - s - 1)_{\mathfrak q}/K_{m'(m,\mathfrak q),\mathfrak q}$$

is surjective. For $k \geq m'(m, \mathfrak q)$ set

$$K_k = \mathrm{Ker}(E(m, -d - s - 1) \to E(k, -d - s - 1))$$

Since K_k is an increasing sequence of submodules of the finite module $E(m, -d - s - 1)$ we see that, at the cost of increasing $m'(m, \mathfrak q)$ a little bit, we may assume $K_{m'(m,\mathfrak q)} = K_k$ for $k \geq m'(m, \mathfrak q)$. After shrinking W further if necessary, we may also assume that

$$E(0, -d - s - 1)_{\mathfrak q'} \to E(m, -d - s - 1)_{\mathfrak q'}/K_{m'(m,\mathfrak q),\mathfrak q'}$$

is surjective for all $\mathfrak q' \in W$ (as before use that these modules are finite and that the map is surjective after localization at $\mathfrak q$).

Any subset, in particular $T_d = \{\mathfrak q \in T \text{ with } \delta(\mathfrak q) = d\}$, of the Noetherian topological space $\mathrm{Spec}(A)$ with the endowed topology is Noetherian and hence quasi-compact. Above we have seen that for every $\mathfrak q \in T_d$ there is an open neighbourhood W where $m'(m, \mathfrak q)$ works for all $\mathfrak q' \in T_d \cap W$. We conclude that we can find an integer $m'(m, d)$ such that for all $\mathfrak q \in T_d$ we have

$$E(m, j)_{\mathfrak q} \to E(m'(m, d), j)_{\mathfrak q}$$

is zero for $j \geq -d-s$ and with $K_{m'(m,d)} = \text{Ker}(E(m, -d-s-1) \to E(m'(m,d), -d-s-1))$ we have

$$K_{m'(m,d),\mathfrak q} = \text{Ker}(E(m, -d-s-1)_\mathfrak q \to E(k, -d-s-1)_\mathfrak q)$$

for all $k \geq m'(m,d)$ and the map

$$E(0, -d-s-1)_\mathfrak q \to E(m, -d-s-1)_\mathfrak q / K_{m'(m,d),\mathfrak q}$$

is surjective. Using the local duality theorem again (in the opposite direction) we conclude that the claim is correct. This finishes the proof. □

0EFY **Lemma 10.7.** *In Situation 10.1 there exists an integer $m_0 \geq 0$ such that*

1. $\{H^i_T(M/I^nM)\}_{n\geq 0}$ *satisfies the Mittag-Leffler condition for $i < s$.*
2. $\{H^i_T(I^{m_0}M/I^nM)\}_{n\geq m_0}$ *satisfies the Mittag-Leffler condition for $i \leq s$,*
3. $H^i_T(M) \to \lim H^i_T(M/I^nM)$ *is an isomorphism for $i < s$,*
4. $H^s_T(I^{m_0}M) \to \lim H^s_T(I^{m_0}M/I^nM)$ *is an isomorphism for $i \leq s$,*
5. $H^s_T(M) \to \lim H^s_T(M/I^nM)$ *is injective with cokernel killed by I^{m_0}, and*
6. $R^1 \lim H^s_T(M/I^nM)$ *is killed by I^{m_0}.*

Proof. Consider the long exact sequences

$$0 \to H^0_T(I^nM) \to H^0_T(M) \to H^0_T(M/I^nM) \to H^1_T(I^nM) \to H^1_T(M) \to \ldots$$

Parts (1) and (3) follows easily from this and Lemma 10.6.

Let m_0 and $m'(-)$ be as in Lemma 10.6. For $m \geq m_0$ consider the long exact sequence

$$H^s_T(I^mM) \to H^s_T(I^{m_0}M) \to H^s_T(I^{m_0}M/I^mM) \to H^{s+1}_T(I^mM) \to H^1_T(I^{m_0}M)$$

Then for $k \geq m'(m)$ the image of $H^{s+1}_T(I^kM) \to H^{s+1}_T(I^mM)$ maps injectively to $H^{s+1}_T(I^{m_0}M)$. Hence the image of $H^s_T(I^{m_0}M/I^kM) \to H^s_T(I^{m_0}M/I^mM)$ maps to zero in $H^{s+1}_T(I^mM)$ for all $k \geq m'(m)$. We conclude that (2) and (4) hold.

Consider the short exact sequences $0 \to I^{m_0}M \to M \to M/I^{m_0}M \to 0$ and $0 \to I^{m_0}M/I^nM \to M/I^nM \to M/I^{m_0}M \to 0$. We obtain a diagram

$$\begin{tikzcd}
H^{s-1}_T(M/I^{m_0}M) \arrow[r] & \lim H^s_T(I^{m_0}M/I^nM) \arrow[r] & \lim H^s_T(M/I^nM) \arrow[r] & H^s_T(M/I^{m_0}M) \\
H^{s-1}_T(M/I^{m_0}M) \arrow[r] \arrow[u, equal] & H^s_T(I^{m_0}M) \arrow[r] \arrow[u, "\cong"'] & H^s_T(M) \arrow[r] \arrow[u] & H^s_T(M/I^{m_0}M) \arrow[u, equal]
\end{tikzcd}$$

whose lower row is exact. The top row is also exact (at the middle two spots) by Homology, Lemma 28.4. Part (5) follows.

Write $B_n = H^s_T(M/I^nM)$. Let $A_n \subset B_n$ be the image of $H^s_T(I^{m_0}M/I^nM) \to H^s_T(M/I^nM)$. Then (A_n) satisfies the Mittag-Leffler condition by (2) and Homology, Lemma 28.3. Also $C_n = B_n/A_n$ is killed by I^{m_0}. Thus $R^1 \lim B_n \cong R^1 \lim C_n$ is killed by I^{m_0} and we get (6). □

0EIE **Theorem 10.8.** *In Situation 10.1 the inverse system $\{H^i_T(M/I^nM)\}_{n\geq 0}$ satisfies the Mittag-Leffler condition for $i \leq s$, the map*

$$H^i_T(M) \longrightarrow \lim H^i_T(M/I^nM)$$

is an isomorphism for $i \leq s$, and $H^i_T(M)$ is annihilated by a power of I for $i \leq s$.

Proof. To prove the final assertion of the theorem we apply Local Cohomology, Proposition 9.1 with $T \subset V(I) \subset \text{Spec}(A)$. Namely, suppose that $\mathfrak{p} \not\in V(I)$, $\mathfrak{q} \in T$ with $\mathfrak{p} \subset \mathfrak{q}$. Then either there exists a prime $\mathfrak{p} \subset \mathfrak{r} \subset \mathfrak{q}$ with $\mathfrak{r} \in V(I) \setminus T$ and we get

$$\text{depth}_{A_\mathfrak{p}}(M_\mathfrak{p}) \geq s \quad \text{or} \quad \text{depth}_{A_\mathfrak{p}}(M_\mathfrak{p}) + \dim((A/\mathfrak{p})_\mathfrak{q}) > d + s$$

by (4) in Situation 10.1 or there does not exist an $\mathfrak{r}$ and we get $\text{depth}_{A_\mathfrak{p}}(M_\mathfrak{p}) > s$ by Lemma 10.2. In all three cases we see that $\text{depth}_{A_\mathfrak{p}}(M_\mathfrak{p}) + \dim((A/\mathfrak{p})_\mathfrak{q}) > s$. Thus Local Cohomology, Proposition 9.1 (2) holds and we find that a power of I annihilates $H^i_T(M)$ for $i \leq s$.

We already know the other two assertions of the theorem hold for $i < s$ by Lemma 10.7 and for the module $I^{m_0}M$ for $i = s$ and m_0 large enough. To finish of the proof we will show that in fact these assertions for $i = s$ holds for M.

Let $M' = H^0_I(M)$ and $M'' = M/M'$ so that we have a short exact sequence

$$0 \to M' \to M \to M'' \to 0$$

and M'' has $H^0_I(M') = 0$ by Dualizing Complexes, Lemma 11.6. By Artin-Rees (Algebra, Lemma 50.2) we get short exact sequences

$$0 \to M' \to M/I^nM \to M''/I^nM'' \to 0$$

for n large enough. Consider the long exact sequences

$$H^s_T(M') \to H^s_T(M/I^nM) \to H^s_T(M''/I^nM'') \to H^{s+1}_T(M')$$

Now it is a simple matter to see that if we have Mittag-Leffler for the inverse system $\{H^s_T(M''/I^nM'')\}_{n\geq 0}$ then we have Mittag-Leffler for the inverse system $\{H^s_T(M/I^nM)\}_{n\geq 0}$. (Note that the ML condition for an inverse system of groups G_n only depends on the values of the inverse system for sufficiently large n.) Moreover the sequence

$$H^s_T(M') \to \lim H^s_T(M/I^nM) \to \lim H^s_T(M''/I^nM'') \to H^{s+1}_T(M')$$

is exact because we have ML in the required spots, see Homology, Lemma 28.4. Hence, if $H^s_T(M'') \to \lim H^s_T(M''/I^nM'')$ is an isomorphism, then $H^s_T(M) \to \lim H^s_T(M/I^nM)$ is an isomorphism too by the five lemma (Homology, Lemma 5.20). This reduces us to the case discussed in the next paragraph.

Assume that $H^0_I(M) = 0$. Choose generators $f_1, \ldots, f_r$ of I^{m_0} where m_0 is the integer found for M in Lemma 10.7. Then we consider the exact sequence

$$0 \to M \xrightarrow{f_1,\ldots,f_r} (I^{m_0}M)^{\oplus r} \to Q \to 0$$

defining Q. Some observations: the first map is injective exactly because $H^0_I(M) = 0$. The cokernel Q of this injection is a finite A-module such that for every $1 \leq j \leq r$ we have $Q_{f_j} \cong (M_{f_j})^{\oplus r-1}$. In particular, for a prime $\mathfrak{p} \subset A$ with $\mathfrak{p} \not\in V(I)$ we have $Q_\mathfrak{p} \cong (M_\mathfrak{p})^{\oplus r-1}$. Similarly, given $\mathfrak{q} \in T$ and $\mathfrak{p}' \subset A' = (A_\mathfrak{q})^\wedge$ not contained in $V(IA')$, we have $Q'_{\mathfrak{p}'} \cong (M'_{\mathfrak{p}'})^{\oplus r-1}$ where $Q' = (Q_\mathfrak{q})^\wedge$ and $M' = (M_\mathfrak{q})^\wedge$. Thus the conditions in Situation 10.1 hold for A, I, T, Q. (Observe that Q may have nonvanishing $H^0_I(Q)$ but this won't matter.)

For any $n \geq 0$ we set $F^nM = M \cap I^n(I^{m_0}M)^{\oplus r}$ so that we get short exact sequences

$$0 \to F^nM \to I^n(I^{m_0}M)^{\oplus r} \to I^nQ \to 0$$

By Artin-Rees (Algebra, Lemma 50.2) there exists a $c \geq 0$ such that $I^nM \subset F^nM \subset I^{n-c}M$ for all $n \geq c$. Let m_0 be the integer and let $m'(m)$ be the function defined for $m \geq m_0$ found in Lemma 10.6 applied to M. Note that the integer m_0 is the same as our integer m_0 chosen above (you don't need to check this: you can just take the maximum of the two integers if you like). Finally, by Lemma 10.6 applied to Q for every integer m there exists an integer $m''(m) \geq m$ such that $H^s_T(I^kQ) \to H^s_T(I^mQ)$ is zero for all $k \geq m''(m)$.

Fix $m \geq m_0$. Choose $k \geq m'(m''(m+c))$. Choose $\xi \in H^{s+1}_T(I^kM)$ which maps to zero in $H^{s+1}_T(M)$. We want to show that ξ maps to zero in $H^{s+1}_T(I^mM)$. Namely, this will show that $\{H^s_T(M/I^nM)\}_{n\geq 0}$ is Mittag-Leffler exactly as in the proof of Lemma 10.7. Picture to help vizualize the argument:

$$\begin{tikzcd}
 & H^{s+1}_T(I^kM) \arrow[r] \arrow[d] & H^{s+1}_T(I^k(I^{m_0}M)^{\oplus r}) \arrow[d] \\
H^s_T(I^{m''(m+c)}Q) \arrow[r, "\delta"'] \arrow[d] & H^{s+1}_T(F^{m''(m+c)}M) \arrow[r] \arrow[d] & H^{s+1}_T(I^{m''(m+c)}(I^{m_0}M)^{\oplus r}) \\
H^s_T(I^{m+c}Q) \arrow[r] & H^{s+1}_T(F^{m+c}M) \arrow[d] & \\
 & H^{s+1}_T(I^mM) &
\end{tikzcd}$$

The image of ξ in $H^{s+1}_T(I^k(I^{m_0}M)^{\oplus r})$ maps to zero in $H^{s+1}_T((I^{m_0}M)^{\oplus r})$ and hence maps to zero in $H^{s+1}_T(I^{m''(m+c)}(I^{m_0}M)^{\oplus r})$ by choice of $m'(-)$. Thus the image $\xi' \in H^{s+1}_T(F^{m''(m+c)}M)$ maps to zero in $H^{s+1}_T(I^{m''(m+c)}(I^{m_0}M)^{\oplus r})$ and hence $\xi' = \delta(\eta)$ for some $\eta \in H^s_T(I^{m''(m+c)}Q)$. By our choice of $m''(-)$ we find that η maps to zero in $H^s_T(I^{m+c}Q)$. This in turn means that ξ' maps to zero in $H^{s+1}_T(F^{m+c}M)$. Since $F^{m+c}M \subset I^mM$ we conclude.

Finally, we prove the statement on limits. Consider the short exact sequences

$$0 \to M/F^nM \to (I^{m_0}M)^{\oplus r}/I^n(I^{m_0}M)^{\oplus r} \to Q/I^nQ \to 0$$

We have $\lim H^s_T(M/I^nM) = \lim H^s_T(M/F^nM)$ as these inverse systems are pro-isomorphic. We obtain a commutative diagram

$$\begin{tikzcd}
H^{s-1}_T(Q) \arrow[r] \arrow[d] & \lim H^{s-1}_T(Q/I^nQ) \arrow[d] \\
H^s_T(M) \arrow[r] \arrow[d] & \lim H^s_T(M/I^nM) \arrow[d] \\
H^s_T((I^{m_0}M)^{\oplus r}) \arrow[r] \arrow[d] & \lim H^s_T((I^{m_0}M)^{\oplus r}/I^n(I^{m_0}M)^{\oplus r}) \arrow[d] \\
H^s_T(Q) \arrow[r] & \lim H^s_T(Q/I^nQ)
\end{tikzcd}$$

The right column is exact because we have ML in the required spots, see Homology, Lemma 28.4. The lowest horizontal arrow is injective (!) by part (5) of Lemma 10.7. The horizontal arrow above it is bijective by part (4) of Lemma 10.7. The arrows in cohomological degrees $\leq s-1$ are isomorphisms. Thus we conclude $H^s_T(M) \to \lim H^s_T(M/I^nM)$ is an isomorphism by the five lemma (Homology, Lemma 5.20). This finishes the proof of the theorem. $\square$

0EG0 **Lemma 10.9.** *Let $I \subset \mathfrak{a} \subset A$ be ideals of a Noetherian ring A and let M be a finite A-module. Let s and d be integers. Suppose that*

(1) *$A, I, V(\mathfrak{a}), M$ satisfy the assumptions of Situation 10.1 for s and d, and*
(2) *$A, I, \mathfrak{a}, M$ satisfy the conditions of Lemma 8.5 for $s+1$ and d with $J = \mathfrak{a}$.*

Then there exists an ideal $J_0 \subset \mathfrak{a}$ with $V(J_0) \cap V(I) = V(\mathfrak{a})$ such that for any $J \subset J_0$ with $V(J) \cap V(I) = V(\mathfrak{a})$ the map

$$H^{s+1}_J(M) \longrightarrow \lim H^{s+1}_{\mathfrak{a}}(M/I^nM)$$

is an isomorphism.

Proof. Namely, we have the existence of J_0 and the isomorphism $H^{s+1}_J(M) = H^{s+1}(R\Gamma_{\mathfrak{a}}(M)^\wedge)$ by Lemma 8.5, we have a short exact sequence

$$0 \to R^1 \lim H^s_{\mathfrak{a}}(M/I^nM) \to H^{s+1}(R\Gamma_{\mathfrak{a}}(M)^\wedge) \to \lim H^{s+1}_{\mathfrak{a}}(M/I^nM) \to 0$$

by Dualizing Complexes, Lemma 12.4, and the module $R^1 \lim H^s_{\mathfrak{a}}(M/I^nM)$ is zero because $\{H^s_{\mathfrak{a}}(M/I^nM)\}_{n \geq 0}$ has Mittag-Leffler by Theorem 10.8. $\square$

11. Algebraization of formal sections, I

0DXH In this section we study the problem of algebraization of formal sections in the local case. Let $(A, \mathfrak{m})$ be a Noetherian local ring. Let $I \subset A$ be an ideal. Let

$$X = \mathrm{Spec}(A) \supset U = \mathrm{Spec}(A) \setminus \{\mathfrak{m}\}$$

and denote $Y = V(I)$ the closed subscheme corresponding to I. Let $\mathcal{F}$ be a coherent $\mathcal{O}_U$-module. In this section we consider the limits

$$\lim_n H^i(U, \mathcal{F}/I^n\mathcal{F})$$

This is closely related to the cohomology of the pullback of $\mathcal{F}$ to the formal completion of U along Y; however, since we have not yet introduced formal schemes, we cannot use this terminology here.

0DXI **Lemma 11.1.** *Let U be the punctured spectrum of a Noetherian local ring A. Let $\mathcal{F}$ be a coherent $\mathcal{O}_U$-module. Let $I \subset A$ be an ideal. Then*

$$H^i(R\Gamma(U, \mathcal{F})^\wedge) = \lim H^i(U, \mathcal{F}/I^n\mathcal{F})$$

for all i where $R\Gamma(U, \mathcal{F})^\wedge$ denotes the derived I-adic completion.

Proof. By Lemmas 6.20 and 7.2 we have

$$R\Gamma(U, \mathcal{F})^\wedge = R\Gamma(U, \mathcal{F}^\wedge) = R\Gamma(U, R\lim \mathcal{F}/I^n\mathcal{F})$$

Thus we obtain short exact sequences

$$0 \to R^1 \lim H^{i-1}(U, \mathcal{F}/I^n\mathcal{F}) \to H^i(R\Gamma(U, \mathcal{F})^\wedge) \to \lim H^i(U, \mathcal{F}/I^n\mathcal{F}) \to 0$$

by Cohomology, Lemma 32.1. The $R^1 \lim$ terms vanish because the inverse systems of groups $H^i(U, \mathcal{F}/I^n\mathcal{F})$ satisfy the Mittag-Leffler condition by Lemma 5.2. $\square$

0DXQ **Theorem 11.2.** *Let $(A, \mathfrak{m})$ be a Noetherian local ring which has a dualizing complex and is complete with respect to an ideal I. Set $X = \mathrm{Spec}(A)$, $Y = V(I)$, and $U = X \setminus \{\mathfrak{m}\}$. Let $\mathcal{F}$ be a coherent sheaf on U. Assume*

(1) *$cd(A, I) \leq d$, i.e., $H^i(X \setminus Y, \mathcal{G}) = 0$ for $i \geq d$ and quasi-coherent $\mathcal{G}$ on X,*
(2) *for any $x \in X \setminus Y$ whose closure $\overline{\{x\}}$ in X meets $U \cap Y$ we have*

$$depth_{\mathcal{O}_{X,x}}(\mathcal{F}_x) \geq s \quad or \quad depth_{\mathcal{O}_{X,x}}(\mathcal{F}_x) + \dim(\overline{\{x\}}) > d + s$$

Then there exists an open $V_0 \subset U$ containing $U \cap Y$ such that for any open $V \subset V_0$ containing $U \cap Y$ the map

$$H^i(V, \mathcal{F}) \to \lim H^i(U, \mathcal{F}/I^n\mathcal{F})$$

is an isomorphism for $i < s$. If in addition $depth_{\mathcal{O}_{X,x}}(\mathcal{F}_x) + \dim(\overline{\{x\}}) > s$ for all $x \in U \cap Y$, then these cohomology groups are finite A-modules.

The method of proof follows roughly the method of proof of [Fal79, Theorem 1] and [Fal80b, Satz 2]. The result is almost the same as [Ray74, Theorem 1.1] (affine complement case) and [Ray75, Theorem 3.9] (complement is union of few affines).

Proof. Choose a finite A-module M such that $\mathcal{F}$ is the restriction to U of the coherent $\mathcal{O}_X$-module associated to M, see Local Cohomology, Lemma 7.2. Then the assumptions of Lemma 9.5 are satisfied. Pick J_0 as in that lemma and set $V_0 = X \setminus V(J_0)$. Then opens $V \subset V_0$ containing $U \cap Y$ correspond 1-to-1 with ideals $J \subset J_0$ with $V(J) \cap V(I) = \{\mathfrak{m}\}$. Moreover, for such a choice we have a distinguished triangle

$$R\Gamma_J(M) \to M \to R\Gamma(V, \mathcal{F}) \to R\Gamma_J(M)[1]$$

We similarly have a distinguished triangle

$$R\Gamma_{\mathfrak{m}}(M)^\wedge \to M \to R\Gamma(U, \mathcal{F})^\wedge \to R\Gamma_{\mathfrak{m}}(M)^\wedge[1]$$

involving derived I-adic completions. The cohomology groups of $R\Gamma(U, \mathcal{F})^\wedge$ are equal to the limits in the statement of the theorem by Lemma 11.1. The canonical map between these triangles and some easy arguments show that our theorem follows from the main Lemma 9.5 (note that we have $i < s$ here whereas we have $i \leq s$ in the lemma; this is because of the shift). The finiteness of the cohomology groups (under the additional assumption) follows from Lemma 9.3. ☐

0DXR **Lemma 11.3.** *Let $(A, \mathfrak{m})$ be a Noetherian local ring which has a dualizing complex and is complete with respect to an ideal I. Set $X = \mathrm{Spec}(A)$, $Y = V(I)$, and $U = X \setminus \{\mathfrak{m}\}$. Let $\mathcal{F}$ be a coherent sheaf on U. Assume for any associated point $x \in U$ of $\mathcal{F}$ we have $\dim(\overline{\{x\}}) > cd(A, I) + 1$ where $\overline{\{x\}}$ is the closure in X. Then the map*

$$\mathrm{colim}\, H^0(V, \mathcal{F}) \longrightarrow \lim H^0(U, \mathcal{F}/I^n\mathcal{F})$$

is an isomorphism of finite A-modules where the colimit is over opens $V \subset U$ containing $U \cap Y$.

Proof. Apply Theorem 11.2 with $s = 1$ (we get finiteness too). ☐

12. Algebraization of formal sections, II

0EG1 It is a bit difficult to succintly state all possible consequences of the results in Sections 8 and 10 for cohomology of coherent sheaves on quasi-affine schemes and their completion with respect to an ideal. This section gives a nonexhaustive list of applications to H^0. The next section contains applications to higher cohomology.

The following lemma will be superceded by Proposition 12.2.

0EIF **Lemma 12.1.** *Let $I \subset \mathfrak{a}$ be ideals of a Noetherian ring A. Let $\mathcal{F}$ be a coherent module on $U = \mathrm{Spec}(A) \setminus V(\mathfrak{a})$. Assume*

1. *A is I-adically complete and has a dualizing complex,*
2. *if $x \in Ass(\mathcal{F})$, $x \notin V(I)$, $\overline{\{x\}} \cap V(I) \not\subset V(\mathfrak{a})$ and $z \in \overline{\{x\}} \cap V(\mathfrak{a})$, then $\dim(\mathcal{O}_{\overline{\{x\}},z}) > cd(A, I) + 1$,*
3. *one of the following holds:*
   (a) *the restriction of $\mathcal{F}$ to $U \setminus V(I)$ is (S_1)*
   (b) *the dimension of $V(\mathfrak{a})$ is at most 2*[4]*.*

Then we obtain an isomorphism

$$\mathrm{colim}\, H^0(V, \mathcal{F}) \longrightarrow \lim H^0(U, \mathcal{F}/I^n\mathcal{F})$$

where the colimit is over opens $V \subset U$ containing $U \cap V(I)$.

Proof. Choose a finite A-module M such that $\mathcal{F}$ is the restriction to U of the coherent module associated to M, see Local Cohomology, Lemma 7.2. Set $d = \mathrm{cd}(A, I)$. Let $\mathfrak{p}$ be a prime of A not contained in $V(I)$ and let $\mathfrak{q} \in V(\mathfrak{p}) \cap V(\mathfrak{a})$. Then either $\mathfrak{p}$ is not an associated prime of M and hence $\mathrm{depth}(M_\mathfrak{p}) \geq 1$ or we have $\dim((A/\mathfrak{p})_\mathfrak{q}) > d + 1$ by (2). Thus the hypotheses of Lemma 8.5 are satisfied for $s = 1$ and d; here we use condition (3). Thus we find there exists an ideal $J_0 \subset \mathfrak{a}$ with $V(J_0) \cap V(I) = V(\mathfrak{a})$ such that for any $J \subset J_0$ with $V(J) \cap V(I) = V(\mathfrak{a})$ the maps

$$H^i_J(M) \longrightarrow H^i(R\Gamma_\mathfrak{a}(M)^\wedge)$$

are isomorphisms for $i = 0, 1$. Consider the morphisms of exact triangles

$$\begin{tikzcd}
R\Gamma_J(M) \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & R\Gamma(V, \mathcal{F}) \arrow[r] \arrow[d] & R\Gamma_J(M)[1] \arrow[d] \\
R\Gamma_J(M)^\wedge \arrow[r] & M \arrow[r] & R\Gamma(V, \mathcal{F})^\wedge \arrow[r] & R\Gamma_J(M)^\wedge[1] \\
R\Gamma_\mathfrak{a}(M)^\wedge \arrow[r] \arrow[u] & M \arrow[r] \arrow[u] & R\Gamma(U, \mathcal{F})^\wedge \arrow[r] \arrow[u] & R\Gamma_\mathfrak{a}(M)^\wedge[1] \arrow[u]
\end{tikzcd}$$

where $V = \mathrm{Spec}(A) \setminus V(J)$. Recall that $R\Gamma_\mathfrak{a}(M)^\wedge \to R\Gamma_J(M)^\wedge$ is an isomorphism (because $\mathfrak{a}$, $\mathfrak{a}+I$, and $J+I$ cut out the same closed subscheme, for example see proof of Lemma 8.5). Hence $R\Gamma(U, \mathcal{F})^\wedge = R\Gamma(V, \mathcal{F})^\wedge$. This produces a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & H^0_J(M) \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & \Gamma(V, \mathcal{F}) \arrow[r] \arrow[d] & H^1_J(M) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & H^0(R\Gamma_J(M)^\wedge) \arrow[r] & M \arrow[r] & H^0(R\Gamma(V, \mathcal{F})^\wedge) \arrow[r] & H^1(R\Gamma_J(M)^\wedge) \arrow[r] & 0 \\
0 \arrow[r] & H^0(R\Gamma_\mathfrak{a}(M)^\wedge) \arrow[r] \arrow[u] & M \arrow[r] \arrow[u] & H^0(R\Gamma(U, \mathcal{F})^\wedge) \arrow[r] \arrow[u] & H^1(R\Gamma_\mathfrak{a}(M)^\wedge) \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

[4]In the sense that the difference of the maximal and minimal values on $V(\mathfrak{a})$ of a dimension function on $\mathrm{Spec}(A)$ is at most 2.

with exact rows and isomorphisms for the lower vertical arrows. Hence we obtain an isomorphism $\Gamma(V, \mathcal{F}) \to H^0(R\Gamma(U, \mathcal{F})^\wedge)$. By Lemmas 6.20 and 7.2 we have

$$R\Gamma(U, \mathcal{F})^\wedge = R\Gamma(U, \mathcal{F}^\wedge) = R\Gamma(U, R\lim \mathcal{F}/I^n\mathcal{F})$$

and we find $H^0(R\Gamma(U, \mathcal{F})^\wedge) = \lim H^0(U, \mathcal{F}/I^n\mathcal{F})$ by Cohomology, Lemma 32.1. □

Now we bootstrap the preceding lemma to get rid of condition (3).

0EG2 **Proposition 12.2.** *Let $I \subset \mathfrak{a}$ be ideals of a Noetherian ring A. Let $\mathcal{F}$ be a coherent module on $U = \mathrm{Spec}(A) \setminus V(\mathfrak{a})$. Assume*

1. *A is I-adically complete and has a dualizing complex,*
2. *if $x \in Ass(\mathcal{F})$, $x \notin V(I)$, $\overline{\{x\}} \cap V(I) \not\subset V(\mathfrak{a})$ and $z \in \overline{\{x\}} \cap V(\mathfrak{a})$, then $\dim(\mathcal{O}_{\overline{\{x\}},z}) > cd(A, I) + 1$.*

Then we obtain an isomorphism

$$\mathrm{colim}\, H^0(V, \mathcal{F}) \longrightarrow \lim H^0(U, \mathcal{F}/I^n\mathcal{F})$$

where the colimit is over opens $V \subset U$ containing $U \cap V(I)$.

Proof. Let $T \subset U$ be the set of points x with $\overline{\{x\}} \cap V(I) \subset V(\mathfrak{a})$. Let $\mathcal{F} \to \mathcal{F}'$ be the surjection of coherent modules on U constructed in Local Cohomology, Lemma 14.1. Since $\mathcal{F} \to \mathcal{F}'$ is an isomorphism over an open $V \subset U$ containing $U \cap V(I)$ it suffices to prove the lemma with $\mathcal{F}$ replaced by $\mathcal{F}'$. Hence we may and do assume for $x \in U$ with $\overline{\{x\}} \cap V(I) \subset V(\mathfrak{a})$ we have $\mathrm{depth}(\mathcal{F}_x) \geq 1$.

Let $\mathcal{V}$ be the set of open subschemes $V \subset U$ containing $U \cap V(I)$ ordered by reverse inclusion. This is a directed set. We first claim that

$$\mathcal{F}(V) \longrightarrow \lim H^0(U, \mathcal{F}/I^n\mathcal{F})$$

is injective for any $V \in \mathcal{F}$ (and in particular the map of the lemma is injective). Namely, an associated point x of $\mathcal{F}$ must have $\overline{\{x\}} \cap U \cap Y \neq \emptyset$ by the previous paragraph. If $y \in \overline{\{x\}} \cap U \cap Y$ then $\mathcal{F}_x$ is a localization of $\mathcal{F}_y$ and $\mathcal{F}_y \subset \lim \mathcal{F}_y/I^n\mathcal{F}_y$ by Krull's intersection theorem (Algebra, Lemma 50.4). This proves the claim as a section $s \in \mathcal{F}(V)$ in the kernel would have to have empty support, hence would have to be zero.

Choose a finite A-module M such that $\mathcal{F}$ is the restriction of $\widetilde{M}$ to U, see Local Cohomology, Lemma 7.2. We may and do assume that $H^0_\mathfrak{a}(M) = 0$. Let $\mathrm{Ass}(M) \setminus V(I) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$. We will prove the lemma by induction on n. After reordering we may assume that $\mathfrak{p}_n$ is a minimal element of the set $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$ with respect to inclusion, i.e, $\mathfrak{p}_n$ is a generic point of the support of M. Set

$$M' = H^0_{\mathfrak{p}_1 \ldots \mathfrak{p}_{n-1} I}(M)$$

and $M'' = M/M'$. Let $\mathcal{F}'$ and $\mathcal{F}''$ be the coherent $\mathcal{O}_U$-modules corresponding to M' and M''. Dualizing Complexes, Lemma 11.6 implies that M'' has only one associated prime, namely $\mathfrak{p}_n$. On the other hand, since $\mathfrak{p}_n \notin V(\mathfrak{p}_1 \ldots \mathfrak{p}_{n-1} I)$ we see that $\mathfrak{p}_n$ is not an associated prime of M'. Hence the induction hypothesis applies to M'; note that since $\mathcal{F}' \subset \mathcal{F}$ the condition $\mathrm{depth}(\mathcal{F}'_x) \geq 1$ at points x with $\overline{\{x\}} \cap V(I) \subset V(\mathfrak{a})$ holds, see Algebra, Lemma 71.6.

Let $\hat{s}$ be an element of $\lim H^0(U, \mathcal{F}/I^n\mathcal{F})$. Let $\hat{s}''$ be the image in $\lim H^0(U, \mathcal{F}''/I^n\mathcal{F}'')$. Since $\mathcal{F}''$ has only one associated point, namely the point corresponding to $\mathfrak{p}_n$, we see that Lemma 12.1 applies and we find an open $U \cap V(I) \subset V \subset U$ and a section

$s'' \in \mathcal{F}''(V)$ mapping to $\hat{s}''$. Let $J \subset A$ be an ideal such that $V(J) = \text{Spec}(A) \setminus V$. By Cohomology of Schemes, Lemma 10.4 after replacing J by a power, we may assume there is an A-linear map $\varphi : J \to M''$ corresponding to s''. Since $M \to M''$ is surjective, for each $g \in J$ we can choose $m_g \in M$ mapping to $\varphi(g) \in M''$. Then $\hat{s}'_g = g\hat{s} - m_g$ is in $\lim H^0(U, \mathcal{F}'/I^n\mathcal{F}')$. By induction hypothesis there is a $V' \geq V$ section $s'_g \in \mathcal{F}'(V')$ mapping to $\hat{s}'_g$. All in all we conclude that $g\hat{s}$ is in the image of $\mathcal{F}(V') \to \lim H^0(U, \mathcal{F}/I^n\mathcal{F})$ for some $V' \subset V$ possibly depending on g. However, since J is finitely generated we can find a single $V' \in \mathcal{V}$ which works for each of the generators and it follows that V' works for all g.

Combining the previous paragraph with the injectivity shown in the second paragraph we find there exists a $V' \geq V$ and an A-module map $\psi : J \to \mathcal{F}(V')$ such that $\psi(g)$ maps to $g\hat{s}$. This determines a map $\widetilde{J} \to (V' \to \text{Spec}(A))_*\mathcal{F}|_{V'}$ whose restriction to V' provides an element $s \in \mathcal{F}(V')$ mapping to $\hat{s}$. This finishes the proof. □

0EIG **Lemma 12.3.** *Let $I \subset \mathfrak{a}$ be ideals of a Noetherian ring A. Let $\mathcal{F}$ be a coherent module on $U = \text{Spec}(A) \setminus V(\mathfrak{a})$. Assume*

1. *A is I-adically complete and has a dualizing complex,*
2. *if $x \in Ass(\mathcal{F})$, $x \notin V(I)$, $z \in V(\mathfrak{a}) \cap \overline{\{x\}}$, then $\dim(\mathcal{O}_{\overline{\{x\}},z}) > cd(A, I) + 1$,*
3. *for $x \in U$ with $\overline{\{x\}} \cap V(I) \subset V(\mathfrak{a})$ we have $depth(\mathcal{F}_x) \geq 2$,*

Then we obtain an isomorphism

$$H^0(U, \mathcal{F}) \longrightarrow \lim H^0(U, \mathcal{F}/I^n\mathcal{F})$$

Proof. Let $\hat{s} \in \lim H^0(U, \mathcal{F}/I^n\mathcal{F})$. By Proposition 12.2 we find that $\hat{s}$ is the image of an element $s \in \mathcal{F}(V)$ for some $V \subset U$ open containing $U \cap V(I)$. However, condition (3) shows that $\text{depth}(\mathcal{F}_x) \geq 2$ for all $x \in U \setminus V$ and hence we find that $\mathcal{F}(V) = \mathcal{F}(U)$ by Divisors, Lemma 5.11 and the proof is complete. □

0EG3 **Example 12.4.** Let A be a Noetherian domain which has a dualizing complex and which is complete with respect to a nonzero $f \in A$. Let $f \in \mathfrak{a} \subset A$ be an ideal. Assume every irreducible component of $Z = V(\mathfrak{a})$ has codimension > 2 in $X = \text{Spec}(A)$. Equivalently, assume every irreducible component of Z has codimension > 1 in $Y = V(f)$. Then with $U = X \setminus Z$ every element of

$$\lim_n \Gamma(U, \mathcal{O}_U/f^n\mathcal{O}_U)$$

is the restriction of a section of $\mathcal{O}_U$ defined on an open neighbourhood of

$$V(f) \setminus Z = V(f) \cap U = Y \setminus Z = U \cap Y$$

In particular we see that $Y \setminus Z$ is connected. See Lemma 14.2 below.

0EIH **Lemma 12.5.** *Let A be a Noetherian ring. Let $f \in \mathfrak{a} \subset A$ be an element of an ideal of A. Let M be a finite A-module. Assume*

1. *A is f-adically complete,*
2. *f is a nonzerodivisor on M,*
3. *$H^1_{\mathfrak{a}}(M/fM)$ is a finite A-module.*

Then with $U = \text{Spec}(A) \setminus V(\mathfrak{a})$ the map

$$\text{colim}_V \Gamma(V, \widetilde{M}) \longrightarrow \lim \Gamma(U, \widetilde{M/f^nM})$$

is an isomorphism where the colimit is over opens $V \subset U$ containing $U \cap V(f)$.

Proof. Set $\mathcal{F} = \widetilde{M}|_U$. The finiteness of $H^1_\mathfrak{a}(M/fM)$ implies that $H^0(U, \mathcal{F}/f\mathcal{F})$ is finite, see Local Cohomology, Lemma 7.2. By Lemma 3.3 (which applies as f is a nonzerodivisor on $\mathcal{F}$) we see that $N = \lim H^0(U, \mathcal{F}/f^n\mathcal{F})$ is a finite A-module, is f-torsion free, and $N/fN \subset H^0(U, \mathcal{F}/f\mathcal{F})$. On the other hand, we have $M \to N$ and the map

$$M/fM \longrightarrow H^0(U, \mathcal{F}/f\mathcal{F})$$

is an isomorphism upon localization at any prime $\mathfrak{q}$ in $U_0 = V(f) \setminus \{\mathfrak{m}\}$ (details omitted). Thus $M_\mathfrak{q} \to N_\mathfrak{q}$ induces an isomorphism

$$M_\mathfrak{q}/fM_\mathfrak{q} = (M/fM)_\mathfrak{q} \to (N/fN)_\mathfrak{q} = N_\mathfrak{q}/fN_\mathfrak{q}$$

Since f is a nonzerodivisor on both N and M we conclude that $M_\mathfrak{q} \to N_\mathfrak{q}$ is an isomorphism (use Nakayama to see surjectivity). We conclude that M and N determine isomorphic coherent modules over an open V as in the statement of the lemma. This finishes the proof. $\square$

0EII **Lemma 12.6.** *Let A be a Noetherian ring. Let $f \in \mathfrak{a} \subset A$ be an element of an ideal of A. Let M be a finite A-module. Assume*

1. *A is f-adically complete,*
2. *$H^1_\mathfrak{a}(M)$ and $H^2_\mathfrak{a}(M)$ are annihilated by a power of f.*

Then with $U = \text{Spec}(A) \setminus V(\mathfrak{a})$ the map

$$\Gamma(U, \widetilde{M}) \longrightarrow \lim \Gamma(U, \widetilde{M/f^nM})$$

is an isomorphism.

Proof. We may apply Lemma 3.6 to U and $\mathcal{F} = \widetilde{M}|_U$ because $\mathcal{F}$ is a Noetherian object in the category of coherent $\mathcal{O}_U$-modules. Since $H^1(U, \mathcal{F}) = H^2_\mathfrak{a}(M)$ (Local Cohomology, Lemma 7.2) is annihilated by a power of f, we see that its f-adic Tate module is zero. Hence the lemma shows $\lim H^0(U, \mathcal{F}/f^n\mathcal{F})$ is the 0th cohomology group of the derived f-adic completion of $H^0(U, \mathcal{F})$. Consider the exact sequence

$$0 \to H^0_\mathfrak{a}(M) \to M \to \Gamma(U, \mathcal{F}) \to H^1_\mathfrak{a}(M) \to 0$$

of Local Cohomology, Lemma 7.2. Since $H^1_\mathfrak{a}(M)$ is annihilated by a power of f it is derived complete with respect to (f). Since M and $H^0_\mathfrak{a}(M)$ are finite A-modules they are complete (Algebra, Lemma 96.1) hence derived complete (More on Algebra, Proposition 82.5). By More on Algebra, Lemma 82.6 we conclude that $\Gamma(U, \mathcal{F})$ is derived complete as desired. $\square$

13. Algebraization of formal sections, III

0EIJ The next section contains a nonexhaustive list of applications of the material on completion of local cohomology to higher cohomology of coherent modules on quasi-affine schemes and their completion with respect to an ideal.

0EG4 **Proposition 13.1.** *Let $I \subset \mathfrak{a}$ be ideals of a Noetherian ring A. Let $\mathcal{F}$ be a coherent module on $U = \text{Spec}(A) \setminus V(\mathfrak{a})$. Let $s \geq 0$. Assume*

1. *A is I-adically complete and has a dualizing complex,*
2. *if $x \in U \setminus V(I)$ then depth$(\mathcal{F}_x) > s$ or*

   $$\text{depth}(\mathcal{F}_x) + \dim(\mathcal{O}_{\overline{\{x\}}, z}) > \text{cd}(A, I) + s + 1$$

   for all $z \in V(\mathfrak{a}) \cap \overline{\{x\}}$,
3. *one of the following conditions holds:*

(a) *the restriction of $\mathcal{F}$ to $U \setminus V(I)$ is (S_{s+1}), or*
(b) *the dimension of $V(\mathfrak{a})$ is at most 2*[5].

Then the maps

$$H^i(U, \mathcal{F}) \longrightarrow \lim H^i(U, \mathcal{F}/I^n\mathcal{F})$$

are isomorphisms for $i < s$. Moreover we have an isomorphism

$$\operatorname{colim} H^s(V, \mathcal{F}) \longrightarrow \lim H^s(U, \mathcal{F}/I^n\mathcal{F})$$

where the colimit is over opens $V \subset U$ containing $U \cap V(I)$.

Proof. We may assume $s > 0$ as the case $s = 0$ was done in Proposition 12.2.

Choose a finite A-module M such that $\mathcal{F}$ is the restriction to U of the coherent module associated to M, see Local Cohomology, Lemma 7.2. Set $d = \text{cd}(A, I)$. Let $\mathfrak{p}$ be a prime of A not contained in $V(I)$ and let $\mathfrak{q} \in V(\mathfrak{p}) \cap V(\mathfrak{a})$. Then either $\text{depth}(M_\mathfrak{p}) \geq s + 1 > s$ or we have $\dim((A/\mathfrak{p})_\mathfrak{q}) > d + s + 1$ by (2). By Lemma 10.5 we conclude that the assumptions of Situation 10.1 are satisfied for $A, I, V(\mathfrak{a}), M, s, d$. On the other hand, the hypotheses of Lemma 8.5 are satisfied for $s + 1$ and d; this is where condition (3) is used.

Applying Lemma 8.5 we find there exists an ideal $J_0 \subset \mathfrak{a}$ with $V(J_0) \cap V(I) = V(\mathfrak{a})$ such that for any $J \subset J_0$ with $V(J) \cap V(I) = V(\mathfrak{a})$ the maps

$$H^i_J(M) \longrightarrow H^i(R\Gamma_\mathfrak{a}(M)^\wedge)$$

is an isomorphism for $i \leq s + 1$.

For $i \leq s$ the map $H^i_\mathfrak{a}(M) \to H^i_J(M)$ is an isomorphism by Lemmas 10.3 and 8.4. Using the comparison of cohomology and local cohomology (Local Cohomology, Lemma 2.2) we deduce $H^i(U, \mathcal{F}) \to H^i(V, \mathcal{F})$ is an isomorphism for $V = \text{Spec}(A) \setminus V(J)$ and $i < s$.

By Theorem 10.8 we have $H^i_\mathfrak{a}(M) = \lim H^i_\mathfrak{a}(M/I^nM)$ for $i \leq s$. By Lemma 10.9 we have $H^{s+1}_\mathfrak{a}(M) = \lim H^{s+1}_\mathfrak{a}(M/I^nM)$.

The isomorphism $H^0(U, \mathcal{F}) = H^0(V, \mathcal{F}) = \lim H^0(U, \mathcal{F}/I^n\mathcal{F})$ follows from the above and Proposition 12.2. For $0 < i < s$ we get the desired isomorphisms $H^i(U, \mathcal{F}) = H^i(V, \mathcal{F}) = \lim H^i(U, \mathcal{F}/I^n\mathcal{F})$ in the same manner using the relation between local cohomology and cohomology; it is easier than the case $i = 0$ because for $i > 0$ we have

$$H^i(U, \mathcal{F}) = H^{i+1}_\mathfrak{a}(M), \quad H^i(V, \mathcal{F}) = H^{i+1}_J(M), \quad H^i(R\Gamma(U, \mathcal{F})^\wedge) = H^{i+1}(R\Gamma_\mathfrak{a}(M)^\wedge)$$

Similarly for the final statement. □

0EKM **Lemma 13.2.** *Let A be a Noetherian ring. Let $f \in \mathfrak{a} \subset A$ be an element of an ideal of A. Let M be a finite A-module. Let $s \geq 0$. Assume*

(1) *A is f-adically complete,*
(2) *$H^i_\mathfrak{a}(M)$ is annihilated by a power of f for $i \leq s + 1$.*

Then with $U = \text{Spec}(A) \setminus V(\mathfrak{a})$ the map

$$H^i(U, \widetilde{M}) \longrightarrow \lim H^i(U, \widetilde{M/f^nM})$$

is an isomorphism for $i < s$.

[5]In the sense that the difference of the maximal and minimal values on $V(\mathfrak{a})$ of a dimension function on $\text{Spec}(A)$ is at most 2.

Proof. The proof is the same as the proof of Lemma 12.6. We may apply Lemma 3.6 to U and $\mathcal{F} = \widetilde{M}|_U$ because $\mathcal{F}$ is a Noetherian object in the category of coherent $\mathcal{O}_U$-modules. Since $H^i(U, \mathcal{F}) = H^{i+1}_\mathfrak{a}(M)$ (Local Cohomology, Lemma 7.2) is annihilated by a power of f for $i \leq s$, we see that its f-adic Tate module is zero. Hence the lemma shows $\lim H^{i-1}(U, \mathcal{F}/f^n\mathcal{F})$ is the 0th cohomology group of the derived f-adic completion of $H^{i-1}(U, \mathcal{F})$. However, if $s \geq i > 1$, then this equal to the f-power torsion module $H^i_\mathfrak{a}(M)$ and hence equal to its own (derived) completion. For $i = 0$, we refer to Lemma 12.6. □

14. Application to connectedness

0ECQ In this section we discuss Grothendieck's connectedness theorem and variants; the original version can be found as [Gro68, Exposee XIII, Theorem 2.1]. There is a version called Faltings' connectedness theorem in the literature; our guess is that this refers to [Fal80a, Theorem 6]. Let us state and prove the optimal version for complete local rings given in [Var09, Theorem 1.6].

0ECR **Lemma 14.1.** *Let $(A, \mathfrak{m})$ be a Noetherian complete local ring. Let I be a proper ideal of A. Set $X = \text{Spec}(A)$ and $Y = V(I)$. Denote* [Var09, Theorem 1.6]

(1) *d the minimal dimension of an irreducible component of X, and*
(2) *c the minimal dimension of a closed subset $Z \subset X$ such that $X \setminus Z$ is disconnected.*

Then for $Z \subset Y$ closed we have $Y \setminus Z$ is connected if $\dim(Z) < \min(c, d-1) - cd(A, I)$. In particular, the punctured spectrum of A/I is connected if $cd(A, I) < \min(c, d-1)$.

Proof. Let us first prove the final assertion. As a first case, if the punctured spectrum of A/I is empty, then Local Cohomology, Lemma 3.10 shows every irreducible component of X has dimension $\leq \text{cd}(A, I)$ and we get $\min(c, d-1) - \text{cd}(A, I) < 0$ which implies the lemma holds in this case. Thus we may assume $U \cap Y$ is nonempty where $U = X \setminus \{\mathfrak{m}\}$ is the punctured spectrum of A. We may replace A by its reduction. Observe that A has a dualizing complex (Dualizing Complexes, Lemma 22.4) and that A is complete with respect to I (Algebra, Lemma 95.8). If we assume $d - 1 > \text{cd}(A, I)$, then we may apply Lemma 11.3 to see that

$$\text{colim}\, H^0(V, \mathcal{O}_V) \longrightarrow \lim H^0(U, \mathcal{O}_U/I^n\mathcal{O}_U)$$

is an isomorphism where the colimit is over opens $V \subset U$ containing $U \cap Y$. If $U \cap Y$ is disconnected, then its nth infinitesimal neighbourhood in U is disconnected for all n and we find the right hand side has a nontrivial idempotent (here we use that $U \cap Y$ is nonempty). Thus we can find a V which is disconnected. Set $Z = X \setminus V$. By Local Cohomology, Lemma 3.10 we see that every irreducible component of Z has dimension $\leq \text{cd}(A, I)$. Hence $c \leq \text{cd}(A, I)$ and this indeed proves the final statement.

We can deduce the statement of the lemma from what we just proved as follows. Suppose that $Z \subset Y$ closed and $Y \setminus Z$ is disconnected and $\dim(Z) = e$. Recall that a connected space is nonempty by convention. Hence we conclude either (a) $Y = Z$ or (b) $Y \setminus Z = W_1 \amalg W_2$ with W_i nonempty, open, and closed in $Y \setminus Z$. In case (b) we may pick points $w_i \in W_i$ which are closed in U, see Morphisms, Lemma 15.10. Then we can find $f_1, \ldots, f_e \in \mathfrak{m}$ such that $V(f_1, \ldots, f_e) \cap Z = \{\mathfrak{m}\}$ and in case (b) we may assume $w_i \in V(f_1, \ldots, f_e)$. Namely, we can inductively using prime avoidance choose f_i such that $\dim V(f_1, \ldots, f_i) \cap Z = e - i$ and such that in case (b) we have $w_1, w_2 \in V(f_i)$. It follows that the punctured spectrum of $A/I + (f_1, \ldots, f_e)$ is disconnected (small detail omitted). Since $\text{cd}(A, I + (f_1, \ldots, f_e)) \leq \text{cd}(A, I) + e$ by Local Cohomology, Lemmas 3.4 and 3.3 we conclude that

$$\text{cd}(A, I) + e \geq \min(c, d - 1)$$

by the first part of the proof. This implies $e \geq \min(c, d-1) - \text{cd}(A, I)$ which is what we had to show. □

0EG5 **Lemma 14.2.** *Let $I \subset \mathfrak{a}$ be ideals of a Noetherian ring A. Assume*

(1) *A is I-adically complete and has a dualizing complex,*
(2) *if $\mathfrak{p} \subset A$ is a minimal prime not contained in $V(I)$ and $\mathfrak{q} \in V(\mathfrak{p}) \cap V(\mathfrak{a})$, then $\dim((A/\mathfrak{p})_\mathfrak{q}) > cd(A, I) + 1$,*
(3) *any nonempty open $V \subset \text{Spec}(A)$ which contains $V(I) \setminus V(\mathfrak{a})$ is connected*[6].

Then $V(I) \setminus V(\mathfrak{a})$ is either empty or connected.

Proof. We may replace A by its reduction. Then we have the inequality in (2) for all associated primes of A. By Proposition 12.2 we see that

$$\text{colim}\, H^0(V, \mathcal{O}_V) = \lim H^0(T_n, \mathcal{O}_{T_n})$$

where the colimit is over the opens V as in (3) and T_n is the nth infinitesimal neighbourhood of $T = V(I) \setminus V(\mathfrak{a})$ in $U = \text{Spec}(A) \setminus V(\mathfrak{a})$. Thus T is either empty or connected, since if not, then the right hand side would have a nontrivial idempotent and we've assumed the left hand side does not. Some details omitted. □

15. The completion functor

0EKN Let X be a Noetherian scheme. Let $Y \subset X$ be a closed subscheme with quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_X$. In this section we consider inverse systems of coherent $\mathcal{O}_X$-modules $(\mathcal{F}_n)$ with $\mathcal{F}_n$ annihilated by I^n such that the transition maps induce isomorphisms $\mathcal{F}_{n+1}/I^n\mathcal{F}_{n+1} \to \mathcal{F}_n$. The category of these inverse systems was denoted

$$\textit{Coh}(X, \mathcal{I})$$

in Cohomology of Schemes, Section 23. This category is equivalent to the category of coherent modules on the formal completion of X along Y; however, since we have not yet introduced formal schemes or coherent modules on them, we cannot use this terminology here. We are particularly interested in the completion functor

$$\textit{Coh}(\mathcal{O}_X) \longrightarrow \textit{Coh}(X, \mathcal{I}), \quad \mathcal{F} \longmapsto \mathcal{F}^\wedge$$

See Cohomology of Schemes, Equation (23.3.1).

0EKP **Lemma 15.1.** *Let X be a Noetherian scheme and let $Y \subset X$ be a closed subscheme. Let $Y_n \subset X$ be the nth infinitesimal neighbourhood of Y in X. Consider the following conditions*

(1) *X is quasi-affine and $\Gamma(X, \mathcal{O}_X) \to \lim \Gamma(Y_n, \mathcal{O}_{Y_n})$ is an isomorphism,*
(2) *X has an ample invertible module $\mathcal{L}$ and $\Gamma(X, \mathcal{L}^{\otimes m}) \to \lim \Gamma(Y_n, \mathcal{L}^{\otimes m}|_{Y_n})$ is an isomorphism for all $m \gg 0$,*
(3) *for every finite locally free $\mathcal{O}_X$-module $\mathcal{E}$ the map $\Gamma(X, \mathcal{E}) \to \lim \Gamma(Y_n, \mathcal{E}|_{Y_n})$ is an isomorphism, and*

[6] For example if A is a domain.

(4) *the completion functor $Coh(\mathcal{O}_X) \to Coh(X, \mathcal{I})$ is fully faithful on the full subcategory of finite locally free objects.*

Then (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) and (4) $\Rightarrow$ (3).

Proof. Proof of (3) $\Rightarrow$ (4). If $\mathcal{F}$ and $\mathcal{G}$ are finite locally free on X, then considering $\mathcal{H} = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{G}, \mathcal{F})$ and using Cohomology of Schemes, Lemma 23.5 we see that (3) implies (4).

Proof of (2) $\to$ (3). Namely, let $\mathcal{L}$ be ample on X and suppose that $\mathcal{E}$ is a finite locally free $\mathcal{O}_X$-module. We claim we can find a universally exact sequence

$$0 \to \mathcal{E} \to (\mathcal{L}^{\otimes p})^{\oplus r} \to (\mathcal{L}^{\otimes q})^{\oplus s}$$

for some $r, s \geq 0$ and $0 \ll p \ll q$. If this holds, then using the exact sequence

$$0 \to \lim \Gamma(\mathcal{E}|_{Y_n}) \to \lim \Gamma((\mathcal{L}^{\otimes p})^{\oplus r}|_{Y_n}) \to \lim \Gamma((\mathcal{L}^{\otimes q})^{\oplus s}|_{Y_n})$$

and the isomorphisms in (2) we get the isomorphism in (3). To prove the claim, consider the dual locally free module $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{O}_X)$ and apply Properties, Proposition 26.13 to find a surjection

$$(\mathcal{L}^{\otimes -p})^{\oplus r} \longrightarrow \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{O}_X)$$

Taking duals we obtain the first map in the exact sequence (it is universally injective because being a surjection is universal). Repeat with the cokernel to get the second. Some details omitted.

Proof of (1) $\Rightarrow$ (2). This is true because if X is quasi-affine then $\mathcal{O}_X$ is an ample invertible module, see Properties, Lemma 27.1.

We omit the proof of (4) $\Rightarrow$ (3). $\square$

Given a Noetherian scheme and a quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_X$ we will say an object $(\mathcal{F}_n)$ of $Coh(X, \mathcal{I})$ is *finite locally free* if each $\mathcal{F}_n$ is a finite locally free $\mathcal{O}_X/\mathcal{I}^n$-module.

0EK2 **Lemma 15.2.** *Let X be a Noetherian scheme and let $Y \subset X$ be a closed subscheme with ideal sheaf $\mathcal{I} \subset \mathcal{O}_X$. Let $Y_n \subset X$ be the nth infinitesimal neighbourhood of Y in X. Let $\mathcal{V}$ be the set of open subschemes $V \subset X$ containing Y ordered by reverse inclusion.*

(1) *X is quasi-affine and*
$$\text{colim}_{\mathcal{V}} \Gamma(V, \mathcal{O}_V) \longrightarrow \lim \Gamma(Y_n, \mathcal{O}_{Y_n})$$
is an isomorphism,

(2) *X has an ample invertible module $\mathcal{L}$ and*
$$\text{colim}_{\mathcal{V}} \Gamma(V, \mathcal{L}^{\otimes m}) \longrightarrow \lim \Gamma(Y_n, \mathcal{L}^{\otimes m}|_{Y_n})$$
is an isomorphism for all $m \gg 0$,

(3) *for every $V \in \mathcal{V}$ and every finite locally free $\mathcal{O}_V$-module $\mathcal{E}$ the map*
$$\text{colim}_{V' \geq V} \Gamma(V', \mathcal{E}|_{V'}) \longrightarrow \lim \Gamma(Y_n, \mathcal{E}|_{Y_n})$$
is an isomorphism, and

(4) *the completion functor*
$$\text{colim}_{\mathcal{V}} Coh(\mathcal{O}_V) \longrightarrow Coh(X, \mathcal{I}), \quad \mathcal{F} \longmapsto \mathcal{F}^\wedge$$
is fully faithful on the full subcategory of finite locally free objects (see explanation above).

Then (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) and (4) $\Rightarrow$ (3).

Proof. Observe that $\mathcal{V}$ is a directed set, so the colimits are as in Categories, Section 19. The rest of the argument is almost exactly the same as the argument in the proof of Lemma 15.1; we urge the reader to skip it.

Proof of (3) $\Rightarrow$ (4). If $\mathcal{F}$ and $\mathcal{G}$ are finite locally free on $V \in \mathcal{V}$, then considering $\mathcal{H} = \mathcal{H}om_{\mathcal{O}_V}(\mathcal{G}, \mathcal{F})$ and using Cohomology of Schemes, Lemma 23.5 we see that (3) implies (4).

Proof of (2) $\Rightarrow$ (3). Let $\mathcal{L}$ be ample on X and suppose that $\mathcal{E}$ is a finite locally free $\mathcal{O}_V$-module for some $V \in \mathcal{V}$. We claim we can find a universally exact sequence

$$0 \to \mathcal{E} \to (\mathcal{L}^{\otimes p})^{\oplus r}|_V \to (\mathcal{L}^{\otimes q})^{\oplus s}|_V$$

for some $r, s \geq 0$ and $0 \ll p \ll q$. If this is true, then the isomorphism in (2) will imply the isomorphism in (3). To prove the claim, recall that $\mathcal{L}|_V$ is ample, see Properties, Lemma 26.14. Consider the dual locally free module $\mathcal{H}om_{\mathcal{O}_V}(\mathcal{E}, \mathcal{O}_V)$ and apply Properties, Proposition 26.13 to find a surjection

$$(\mathcal{L}^{\otimes -p})^{\oplus r}|_V \longrightarrow \mathcal{H}om_{\mathcal{O}_V}(\mathcal{E}, \mathcal{O}_V)$$

(it is universally injective because being a surjection is universal). Taking duals we obtain the first map in the exact sequence. Repeat with the cokernel to get the second. Some details omitted.

Proof of (1) $\Rightarrow$ (2). This is true because if X is quasi-affine then $\mathcal{O}_X$ is an ample invertible module, see Properties, Lemma 27.1.

We omit the proof of (4) $\Rightarrow$ (3). $\square$

0EIQ **Lemma 15.3.** *Let X be a Noetherian scheme. Let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. The functor*

$$Coh(X, \mathcal{I}) \longrightarrow Pro\text{-}QCoh(\mathcal{O}_X)$$

is fully faithful, see Categories, Remark 22.4.

Proof. Let $(\mathcal{F}_n)$ and $(\mathcal{G}_n)$ be objects of $Coh(X, \mathcal{I})$. A morphism of pro-objects α from $(\mathcal{F}_n)$ to $(\mathcal{G}_n)$ is given by a system of maps $\alpha_n : \mathcal{F}_{n'(n)} \to \mathcal{G}_n$ where $\mathbf{N} \to \mathbf{N}$, $n \mapsto n'(n)$ is an increasing function. Since $\mathcal{F}_n = \mathcal{F}_{n'(n)}/\mathcal{I}^n\mathcal{F}_{n'(n)}$ and since $\mathcal{G}_n$ is annihilated by $\mathcal{I}^n$ we see that α_n induces a map $\mathcal{F}_n \to \mathcal{G}_n$. $\square$

Next we add some examples of the kind of fully faithfulness result we will be able to prove using the work done earlier in this chapter.

0EKQ **Lemma 15.4.** *Let $I \subset \mathfrak{a}$ be ideals of a Noetherian ring A. Let $U = \mathrm{Spec}(A) \setminus V(\mathfrak{a})$. Assume*

1. *A is I-adically complete and has a dualizing complex,*
2. *for any associated prime $\mathfrak{p} \subset A$, $I \not\subset \mathfrak{p}$ and $\mathfrak{q} \in V(\mathfrak{p}) \cap V(\mathfrak{a})$ we have $\dim((A/\mathfrak{p})_\mathfrak{q}) > cd(A, I) + 1$.*
3. *for $\mathfrak{p} \subset A$, $I \not\subset \mathfrak{p}$ with with $V(\mathfrak{p}) \cap V(I) \subset V(\mathfrak{a})$ we have $depth(A_\mathfrak{p}) \geq 2$.*

Then the completion functor

$$Coh(\mathcal{O}_U) \longrightarrow Coh(U, I\mathcal{O}_U), \quad \mathcal{F} \longmapsto \mathcal{F}^\wedge$$

is fully faithful on the full subcategory of finite locally free objects.

Proof. By Lemma 15.1 it suffices to show that

$$\Gamma(U, \mathcal{O}_U) = \lim \Gamma(U, \mathcal{O}_U/I^n\mathcal{O}_U)$$

This follows immediately from Lemma 12.3. $\square$

0EKR **Lemma 15.5.** *Let A be a Noetherian ring. Let $f \in \mathfrak{a}$ be an element of an ideal of A. Let $U = \text{Spec}(A) \setminus V(\mathfrak{a})$. Assume*

(1) *A has a dualizing complex and is complete with respect to f,*
(2) *A_f is (S_2) and for every minimal prime $\mathfrak{p} \subset A$, $f \notin \mathfrak{p}$ and $\mathfrak{q} \in V(\mathfrak{p}) \cap V(\mathfrak{a})$ we have $\dim((A/\mathfrak{p})_\mathfrak{q}) \geq 3$.*

Then the completion functor

$$Coh(\mathcal{O}_U) \longrightarrow Coh(U, I\mathcal{O}_U), \quad \mathcal{F} \longmapsto \mathcal{F}^\wedge$$

is fully faithful on the full subcategory of finite locally free objects.

Proof. We will show that Lemma 15.4 applies. Assumption (1) of Lemma 15.4 holds. Observe that $\text{cd}(A, (f)) \leq 1$, see Local Cohomology, Lemma 3.3. Since A_f is (S_2) we see that every associated prime $\mathfrak{p} \subset A$, $f \notin \mathfrak{p}$ is a minimal prime. Thus we get assumption (2) of Lemma 15.4. If $\mathfrak{p} \subset A$, $f \notin \mathfrak{p}$ satisfies $V(\mathfrak{p}) \cap V(I) \subset V(\mathfrak{a})$ and if $\mathfrak{q} \in V(\mathfrak{p}) \cap V(f)$ is a generic point, then $\dim((A/\mathfrak{p})_\mathfrak{q}) = 1$. Then we obtain $\dim(A_\mathfrak{p}) \geq 2$ by looking at the minimal primes $\mathfrak{p}_0 \subset \mathfrak{p}$ and using that $\dim((A/\mathfrak{p}_0)_\mathfrak{q}) \geq 3$ by assumption. Thus $\text{depth}(A_\mathfrak{p}) \geq 2$ by the (S_2) assumption. This verifies assumption (3) of Lemma 15.4 and the proof is complete. $\square$

0EKS **Lemma 15.6.** *Let A be a Noetherian ring. Let $f \in \mathfrak{a} \subset A$ be an element of an ideal of A. Let $U = \text{Spec}(A) \setminus V(\mathfrak{a})$. Assume*

(1) *A is f-adically complete,*
(2) *$H^1_\mathfrak{a}(A)$ and $H^2_\mathfrak{a}(A)$ are annihilated by a power of f.*

Then the completion functor

$$Coh(\mathcal{O}_U) \longrightarrow Coh(U, I\mathcal{O}_U), \quad \mathcal{F} \longmapsto \mathcal{F}^\wedge$$

is fully faithful on the full subcategory of finite locally free objects.

Proof. By Lemma 15.1 it suffices to show that

$$\Gamma(U, \mathcal{O}_U) = \lim \Gamma(U, \mathcal{O}_U/I^n\mathcal{O}_U)$$

This follows immediately from Lemma 12.6. $\square$

0EKT **Lemma 15.7.** *Let A be a Noetherian ring. Let $f \in \mathfrak{a}$ be an element of an ideal of A. Let $U = \text{Spec}(A) \setminus V(\mathfrak{a})$. Assume*

(1) *A has a dualizing complex and is complete with respect to f,*
(2) *for every prime $\mathfrak{p} \subset A$, $f \notin \mathfrak{p}$ and $\mathfrak{q} \in V(\mathfrak{p}) \cap V(\mathfrak{a})$ we have depth$(A_\mathfrak{p}) + \dim((A/\mathfrak{p})_\mathfrak{q}) > 2$.*

Then the completion functor

$$Coh(\mathcal{O}_U) \longrightarrow Coh(U, I\mathcal{O}_U), \quad \mathcal{F} \longmapsto \mathcal{F}^\wedge$$

is fully faithful on the full subcategory of finite locally free objects.

Proof. This follows from Lemma 15.6 and Local Cohomology, Proposition 9.1. $\square$

0EKU **Lemma 15.8.** *Let $I \subset \mathfrak{a} \subset A$ be ideals of a Noetherian ring A. Let $U = \text{Spec}(A) \setminus V(\mathfrak{a})$. Let $\mathcal{V}$ be the set of open subschemes of U containing $U \cap V(I)$ ordered by reverse inclusion. Assume*

(1) *A is I-adically complete and has a dualizing complex,*
(2) *for any associated prime $\mathfrak{p} \subset A$ with $I \not\subset \mathfrak{p}$ and $V(\mathfrak{p}) \cap V(I) \not\subset V(\mathfrak{a})$ and $\mathfrak{q} \in V(\mathfrak{p}) \cap V(\mathfrak{a})$ we have $\dim((A/\mathfrak{p})_\mathfrak{q}) > cd(A, I) + 1$.*

Then the completion functor

$$\mathrm{colim}_\mathcal{V}\, Coh(\mathcal{O}_V) \longrightarrow Coh(U, I\mathcal{O}_U), \quad \mathcal{F} \longmapsto \mathcal{F}^\wedge$$

is fully faithful on the full subcategory of finite locally free objects.

Proof. By Lemma 15.2 it suffices to show that

$$\mathrm{colim}_\mathcal{V}\, \Gamma(V, \mathcal{O}_V) = \lim \Gamma(U, \mathcal{O}_U/I^n\mathcal{O}_U)$$

This follows immediately from Proposition 12.2. □

0EKV **Lemma 15.9.** *Let A be a Noetherian ring. Let $f \in \mathfrak{a} \subset A$ be an element of an ideal of A. Let $U = \mathrm{Spec}(A) \setminus V(\mathfrak{a})$. Let $\mathcal{V}$ be the set of open subschemes of U containing $U \cap V(f)$ ordered by reverse inclusion. Assume*

(1) *A is f-adically complete,*
(2) *f is a nonzerodivisor,*
(3) *$H^1_\mathfrak{a}(A/fA)$ is a finite A-module.*

Then the completion functor

$$\mathrm{colim}_\mathcal{V}\, Coh(\mathcal{O}_V) \longrightarrow Coh(U, f\mathcal{O}_U), \quad \mathcal{F} \longmapsto \mathcal{F}^\wedge$$

is fully faithful.

Proof. By Lemma 15.2 it suffices to show that

$$\mathrm{colim}_\mathcal{V}\, \Gamma(V, \mathcal{O}_V) = \lim \Gamma(U, \mathcal{O}_U/I^n\mathcal{O}_U)$$

This follows immediately from Lemma 12.5. □

0EIV **Lemma 15.10.** *Let $I \subset \mathfrak{a} \subset A$ be ideals of a Noetherian ring A. Let $U = \mathrm{Spec}(A) \setminus V(\mathfrak{a})$. Let $\mathcal{V}$ be the set of open subschemes of U containing $U \cap V(I)$ ordered by reverse inclusion. Let $\mathcal{F}$ and $\mathcal{G}$ be coherent $\mathcal{O}_V$-modules for some $V \in \mathcal{V}$. The map*

$$\mathrm{colim}_{V' \geq V} \mathrm{Hom}_V(\mathcal{G}|_{V'}, \mathcal{F}|_{V'}) \longrightarrow \mathrm{Hom}_{Coh(U, I\mathcal{O}_U)}(\mathcal{G}^\wedge, \mathcal{F}^\wedge)$$

is bijective if the following assumptions hold:

(1) *A is I-adically complete and has a dualizing complex,*
(2) *if $x \in Ass(\mathcal{F})$, $x \notin V(I)$, $\overline{\{x\}} \cap V(I) \not\subset V(\mathfrak{a})$ and $z \in \overline{\{x\}} \cap V(\mathfrak{a})$, then $\dim(\mathcal{O}_{\overline{\{x\}}, z}) > cd(A, I) + 1$.*

Proof. We may choose coherent $\mathcal{O}_U$-modules $\mathcal{F}'$ and $\mathcal{G}'$ whose restriction to V is $\mathcal{F}$ and $\mathcal{G}$, see Properties, Lemma 22.4. We may modify our choice of $\mathcal{F}'$ to ensure that $\mathrm{Ass}(\mathcal{F}') \subset V$, see for example Local Cohomology, Lemma 14.1. Thus we may and do replace V by U and $\mathcal{F}$ and $\mathcal{G}$ by $\mathcal{F}'$ and $\mathcal{G}'$. Set $\mathcal{H} = \mathcal{H}om_{\mathcal{O}_U}(\mathcal{G}, \mathcal{F})$. This is a coherent $\mathcal{O}_U$-module. We have

$$\mathrm{Hom}_V(\mathcal{G}|_V, \mathcal{F}|_V) = H^0(V, \mathcal{H}) \quad \text{and} \quad \lim H^0(U, \mathcal{H}/\mathcal{I}^n\mathcal{H}) = Mor_{Coh(U, I\mathcal{O}_U)}(\mathcal{G}^\wedge, \mathcal{F}^\wedge)$$

See Cohomology of Schemes, Lemma 23.5. Thus if we can show that the assumptions of Proposition 12.2 hold for $\mathcal{H}$, then the proof is complete. This holds because $\mathrm{Ass}(\mathcal{H}) \subset \mathrm{Ass}(\mathcal{F})$. See Cohomology of Schemes, Lemma 11.2. □

16. Algebraization of coherent formal modules, I

0DXS The essential surjectivity of the completion functor (see below) was studied systematically in [Gro68], [Ray75], and [Ray74]. We work in the following affine situation.

0EHC **Situation 16.1.** Here A is a Noetherian ring and $I \subset \mathfrak{a} \subset A$ are ideals. We set $X = \text{Spec}(A)$, $Y = V(I) = \text{Spec}(A/I)$, and $Z = V(\mathfrak{a}) = \text{Spec}(A/\mathfrak{a})$. Furthermore $U = X \setminus Z$.

In this section we try to find conditions that guarantee an object of $Coh(U, I\mathcal{O}_U)$ is in the image of the completion functor $Coh(\mathcal{O}_U) \to Coh(U, I\mathcal{O}_U)$. See Cohomology of Schemes, Section 23 and Section 15.

0DXT **Lemma 16.2.** *In Situation 16.1. Consider an inverse system (M_n) of A-modules such that*

1. *M_n is a finite A-module,*
2. *M_n is annihilated by I^n,*
3. *the kernel and cokernel of $M_{n+1}/I^n M_{n+1} \to M_n$ are $\mathfrak{a}$-power torsion.*

Then $(\widetilde{M_n}|_U)$ is in $Coh(U, I\mathcal{O}_U)$. Conversely, every object of $Coh(U, I\mathcal{O}_U)$ arises in this manner.

Proof. We omit the verification that $(\widetilde{M_n}|_U)$ is in $Coh(U, I\mathcal{O}_U)$. Let $(\mathcal{F}_n)$ be an object of $Coh(U, I\mathcal{O}_U)$. By Local Cohomology, Lemma 7.2 we see that $\mathcal{F}_n = \widetilde{M_n}$ for some finite A/I^n-module M_n. After dividing M_n by $H^0_{\mathfrak{a}}(M_n)$ we may assume $M_n \subset H^0(U, \mathcal{F}_n)$, see Dualizing Complexes, Lemma 11.6 and the already referenced lemma. After replacing inductively M_{n+1} by the inverse image of M_n under the map $M_{n+1} \to H^0(U, \mathcal{F}_{n+1}) \to H^0(U, \mathcal{F}_n)$, we may assume M_{n+1} maps into M_n. This gives a inverse system (M_n) satisfying (1) and (2) such that $\mathcal{F}_n = \widetilde{M_n}$. To see that (3) holds, use that $M_{n+1}/I^n M_{n+1} \to M_n$ is a map of finite A-modules which induces an isomorphism after applying $\widetilde{\ }$ and restriction to U (here we use the first referenced lemma one more time). ☐

In Situation 16.1 we can study the completion functor Cohomology of Schemes, Equation (23.3.1)

0EIK (16.2.1)
$$Coh(\mathcal{O}_U) \longrightarrow Coh(U, I\mathcal{O}_U), \quad \mathcal{F} \longmapsto \mathcal{F}^\wedge$$

If A is I-adically complete, then this functor is fully faithful on suitable subcategories by our earlier work on algebraization of formal sections, see Section 15 and Lemma 19.6 for some sample results. Next, let $(\mathcal{F}_n)$ be an object of $Coh(U, I\mathcal{O}_U)$. Still assuming A is I-adically complete, we can ask: When is $(\mathcal{F}_n)$ in the essential image of the completion functor displayed above?

0EIL **Lemma 16.3.** *In Situation 16.1 let $(\mathcal{F}_n)$ be an object of $Coh(U, I\mathcal{O}_U)$. Consider the following conditions:*

1. *$(\mathcal{F}_n)$ is in the essential image of the functor (16.2.1),*
2. *$(\mathcal{F}_n)$ is the completion of a coherent $\mathcal{O}_U$-module,*
3. *$(\mathcal{F}_n)$ is the completion of a coherent $\mathcal{O}_V$-module for $U \cap Y \subset V \subset U$ open,*
4. *$(\mathcal{F}_n)$ is the completion of the restriction to U of a coherent $\mathcal{O}_X$-module,*
5. *$(\mathcal{F}_n)$ is the restriction to U of the completion of a coherent $\mathcal{O}_X$-module,*
6. *there exists an object $(\mathcal{G}_n)$ of $Coh(X, I\mathcal{O}_X)$ whose restriction to U is $(\mathcal{F}_n)$.*

Then conditions (1), (2), (3), (4), and (5) are equivalent and imply (6). If A is I-adically complete then condition (6) implies the others.

Proof. Parts (1) and (2) are equivalent, because the completion of a coherent $\mathcal{O}_U$-module $\mathcal{F}$ is by definition the image of $\mathcal{F}$ under the functor (16.2.1). If $V \subset U$ is an open subscheme containing $U \cap Y$, then we have

$$Coh(V, I\mathcal{O}_V) = Coh(U, I\mathcal{O}_U)$$

since the category of coherent $\mathcal{O}_V$-modules supported on $V \cap Y$ is the same as the category of coherent $\mathcal{O}_U$-modules supported on $U \cap Y$. Thus the completion of a coherent $\mathcal{O}_V$-module is an object of $Coh(U, I\mathcal{O}_U)$. Having said this the equivalence of (2), (3), (4), and (5) holds because the functors $Coh(\mathcal{O}_X) \to Coh(\mathcal{O}_U) \to Coh(\mathcal{O}_V)$ are essentially surjective. See Properties, Lemma 22.4.

It is always the case that (5) implies (6). Assume A is I-adically complete. Then any object of $Coh(X, I\mathcal{O}_X)$ corresponds to a finite A-module by Cohomology of Schemes, Lemma 23.1. Thus we see that (6) implies (5) in this case. □

0EHE **Example 16.4.** Let k be a field. Let $A = k[x, y][[t]]$ with $I = (t)$ and $\mathfrak{a} = (x, y, t)$. Let us use notation as in Situation 16.1. Observe that $U \cap Y = (D(x) \cap Y) \cup (D(y) \cap Y)$ is an affine open covering. For $n \geq 1$ consider the invertible module $\mathcal{L}_n$ of $\mathcal{O}_U/t^n\mathcal{O}_U$ given by glueing A_x/t^nA_x and A_y/t^nA_y via the invertible element of A_{xy}/t^nA_{xy} which is the image of any power series of the form

$$u = 1 + \frac{t}{xy} + \sum_{n \geq 2} a_n \frac{t^n}{(xy)^{\varphi(n)}}$$

with $a_n \in k[x, y]$ and $\varphi(n) \in \mathbf{N}$. Then $(\mathcal{L}_n)$ is an invertible object of $Coh(U, I\mathcal{O}_U)$ which is not the completion of a coherent $\mathcal{O}_U$-module $\mathcal{L}$. We only sketch the argument and we omit most of the details. Let $y \in U \cap Y$. Then the completion of the stalk $\mathcal{L}_y$ would be an invertible module hence $\mathcal{L}_y$ is invertible. Thus there would exist an open $V \subset U$ containing $U \cap Y$ such that $\mathcal{L}|_V$ is invertible. By Divisors, Lemma 28.3 we find an invertible A-module M with $\widetilde{M}|_V \cong \mathcal{L}|_V$. However the ring A is a UFD hence we see $M \cong A$ which would imply $\mathcal{L}_n \cong \mathcal{O}_U/I^n\mathcal{O}_U$. Since $\mathcal{L}_2 \not\cong \mathcal{O}_U/I^2\mathcal{O}_U$ by construction we get a contradiction as desired.

Note that if we take $a_n = 0$ for $n \geq 2$, then we see that $\lim H^0(U, \mathcal{L}_n)$ is nonzero: in this case we the function x on $D(x)$ and the function $x + t/y$ on $D(y)$ glue. On the other hand, if we take $a_n = 1$ and $\varphi(n) = 2^n$ or even $\varphi(n) = n^2$ then the reader can show that $\lim H^0(U, \mathcal{L}_n)$ is zero; this gives another proof that $(\mathcal{L}_n)$ is not algebraizable in this case.

If in Situation 16.1 the ring A is not I-adically complete, then Lemma 16.3 suggests the correct thing is to ask whether $(\mathcal{F}_n)$ is in the essential image of the restriction functor

$$Coh(X, I\mathcal{O}_X) \longrightarrow Coh(U, I\mathcal{O}_U)$$

However, we can no longer say that this means $(\mathcal{F}_n)$ is algebraizable. Thus we introduce the following terminology.

0EIM **Definition 16.5.** In Situation 16.1 let $(\mathcal{F}_n)$ be an object of $Coh(U, I\mathcal{O}_U)$. We say $(\mathcal{F}_n)$ *extends to X* if there exists an object $(\mathcal{G}_n)$ of $Coh(X, I\mathcal{O}_X)$ whose restriction to U is isomorphic to $(\mathcal{F}_n)$.

This notion is equivalent to being algebraizable over the completion.

0EIN **Lemma 16.6.** *In Situation 16.1 let $(\mathcal{F}_n)$ be an object of $\textit{Coh}(U, I\mathcal{O}_U)$. Let $A', I', \mathfrak{a}'$ be the I-adic completions of $A, I, \mathfrak{a}$. Set $X' = \mathrm{Spec}(A')$ and $U' = X' \setminus V(\mathfrak{a}')$. The following are equivalent*

1. *$(\mathcal{F}_n)$ extends to X, and*
2. *the pullback of $(\mathcal{F}_n)$ to U' is the completion of a coherent $\mathcal{O}_{U'}$-module.*

Proof. Recall that $A \to A'$ is a flat ring map which induces an isomorphism $A/I \to A'/I'$. See Algebra, Lemmas 96.2 and 96.4. Thus $X' \to X$ is a flat morphism inducing an isomorphism $Y' \to Y$. Thus $U' \to U$ is a flat morphism which induces an isomorphism $U' \cap Y' \to U \cap Y$. This implies that in the commutative diagram

$$\begin{tikzcd} \textit{Coh}(X', I\mathcal{O}_{X'}) \arrow[r] & \textit{Coh}(U', I\mathcal{O}_{U'}) \\ \textit{Coh}(X, I\mathcal{O}_X) \arrow[r] \arrow[u] & \textit{Coh}(U, I\mathcal{O}_U) \arrow[u] \end{tikzcd}$$

the vertical functors are equivalences. See Cohomology of Schemes, Lemma 23.10. The lemma follows formally from this and the results of Lemma 16.3. $\square$

In Situation 16.1 let $(\mathcal{F}_n)$ be an object of $\textit{Coh}(U, I\mathcal{O}_U)$. To figure out if $(\mathcal{F}_n)$ extends to X it makes sense to look at the A-module

0EHD (16.6.1) $$M = \lim H^0(U, \mathcal{F}_n)$$

Observe that M has a limit topology which is (a priori) coarser than the I-adic topology since $M \to H^0(U, \mathcal{F}_n)$ annihilates I^nM. There are canonical maps

$$\widetilde{M}|_U \to \widetilde{M/I^nM}|_U \to \widetilde{H^0(U, \mathcal{F}_n)}|_U \to \mathcal{F}_n$$

One could hope that $\widetilde{M}$ restricts to a coherent module on U and that $(\mathcal{F}_n)$ is the completion of this module. This is naive because this has almost no chance of being true if A is not complete. But even if A is I-adically complete this notion is very difficult to work with. A less naive approach is to consider the following requirement.

0EIP **Definition 16.7.** In Situation 16.1 let $(\mathcal{F}_n)$ be an object of $\textit{Coh}(U, I\mathcal{O}_U)$. We say $(\mathcal{F}_n)$ *canonically extends to* X if the the inverse system

$$\{\widetilde{H^0(U, \mathcal{F}_n)}\}_{n \geq 1}$$

in $\textit{QCoh}(\mathcal{O}_X)$ is pro-isomorphic to an object $(\mathcal{G}_n)$ of $\textit{Coh}(X, I\mathcal{O}_X)$.

We will see in Lemma 16.8 that the condition in Definition 16.7 is stronger than the condition of Definition 16.5.

0EIR **Lemma 16.8.** *In Situation 16.1 let $(\mathcal{F}_n)$ be an object of $\textit{Coh}(U, I\mathcal{O}_U)$. If $(\mathcal{F}_n)$ canonically extends to X, then*

1. *$(\widetilde{H^0(U, \mathcal{F}_n)})$ is pro-isomorphic to an object $(\mathcal{G}_n)$ of $\textit{Coh}(X, I\mathcal{O}_X)$ unique up to unique isomorphism,*
2. *the restriction of $(\mathcal{G}_n)$ to U is isomorphic to $(\mathcal{F}_n)$, i.e., $(\mathcal{F}_n)$ extends to X,*
3. *the inverse system $\{H^0(U, \mathcal{F}_n)\}$ satisfies the Mittag-Leffler condition, and*
4. *the module M in (16.6.1) is finite over the I-adic completion of A and the limit topology on M is the I-adic topology.*

Proof. The existence of $(\mathcal{G}_n)$ in (1) follows from Definition 16.7. The uniqueness of $(\mathcal{G}_n)$ in (1) follows from Lemma 15.3. Write $\mathcal{G}_n = \widetilde{M_n}$. Then $\{M_n\}$ is an inverse system of finite A-modules with $M_n = M_{n+1}/I^n M_{n+1}$. By Definition 16.7 the inverse system $\{H^0(U, \mathcal{F}_n)\}$ is pro-isomorphic to $\{M_n\}$. Hence we see that the inverse system $\{H^0(U, \mathcal{F}_n)\}$ satisfies the Mittag-Leffler condition and that $M = \lim M_n$ (as topological modules). Thus the properties of M in (4) follow from Algebra, Lemmas 97.1, 95.12, and 95.3. Since U is quasi-affine the canonical maps

$$\widetilde{H^0(U, \mathcal{F}_n)}|_U \to \mathcal{F}_n$$

are isomorphisms (Properties, Lemma 18.2). We conclude that $(\mathcal{G}_n|_U)$ and $(\mathcal{F}_n)$ are pro-isomorphic and hence isomorphic by Lemma 15.3. □

0EIS **Lemma 16.9.** *In Situation 16.1 let $(\mathcal{F}_n)$ be an object of $Coh(U, I\mathcal{O}_U)$. Let $A \to A'$ be a flat ring map. Set $X' = \mathrm{Spec}(A')$, let $U' \subset X'$ be the inverse image of U, and denote $g : U' \to U$ the induced morphism. Set $(\mathcal{F}'_n) = (g^*\mathcal{F}_n)$, see Cohomology of Schemes, Lemma 23.9. If $(\mathcal{F}_n)$ canonically extends to X, then $(\mathcal{F}'_n)$ canonically extends to X'. Moreover, the extension found in Lemma 16.8 for $(\mathcal{F}_n)$ pulls back to the extension for $(\mathcal{F}'_n)$.*

Proof. Let $f : X' \to X$ be the induced morphism. We have $H^0(U', \mathcal{F}'_n) = H^0(U, \mathcal{F}_n) \otimes_A A'$ by flat base change, see Cohomology of Schemes, Lemma 5.2. Thus if $(\mathcal{G}_n)$ in $Coh(X, I\mathcal{O}_X)$ is pro-isomorphic to $(\widetilde{H^0(U, \mathcal{F}_n)})$, then $(f^*\mathcal{G}_n)$ is pro-isomorphic to

$$(f^*\widetilde{H^0(U, \mathcal{F}_n)}) = (\widetilde{H^0(U, \mathcal{F}_n) \otimes_A A'}) = (\widetilde{H^0(U', \mathcal{F}'_n)})$$

This finishes the proof. □

0EHH **Lemma 16.10.** *In Situation 16.1 let $(\mathcal{F}_n)$ be an object of $Coh(U, I\mathcal{O}_U)$. Let M be as in (16.6.1). Assume*

(a) *the inverse system $H^0(U, \mathcal{F}_n)$ has Mittag-Leffler,*
(b) *the limit topology on M agrees with the I-adic topology, and*
(c) *the image of $M \to H^0(U, \mathcal{F}_n)$ is a finite A-module for all n.*

Then $(\mathcal{F}_n)$ extends canonically to X. In particular, if A is I-adically complete, then $(\mathcal{F}_n)$ is the completion of a coherent $\mathcal{O}_U$-module.

Proof. Since $H^0(U, \mathcal{F}_n)$ has the Mittag-Leffler condition and since the limit topology on M is the I-adic topology we see that $\{M/I^nM\}$ and $\{H^0(U, \mathcal{F}_n)\}$ are pro-isomorphic inverse systems of A-modules. Thus if we set

$$\mathcal{G}_n = \widetilde{M/I^nM}$$

then we see that to verify the condition in Definition 16.7 it suffices to show that M is a finite module over the I-adic completion of A. This follows from the fact that M/I^nM is finite by condition (c) and the above and Algebra, Lemma 95.12. □

The following is in some sense the most straightforward possible application Lemma 16.10 above.

0DXW **Lemma 16.11.** *In Situation 16.1 let $(\mathcal{F}_n)$ be an object of $Coh(U, I\mathcal{O}_U)$. Assume*

(1) *$I = (f)$ is a principal ideal for a nonzerodivisor $f \in \mathfrak{a}$,*
(2) *$\mathcal{F}_n$ is a finite locally free $\mathcal{O}_U/f^n\mathcal{O}_U$-module,*
(3) *$H^1_{\mathfrak{a}}(A/fA)$ and $H^2_{\mathfrak{a}}(A/fA)$ are finite A-modules.*

Then $(\mathcal{F}_n)$ extends canonically to X. In particular, if A is complete, then $(\mathcal{F}_n)$ is the completion of a coherent $\mathcal{O}_U$-module.

Proof. We will prove this by verifying hypotheses (a), (b), and (c) of Lemma 16.10.

Since $\mathcal{F}_n$ is locally free over $\mathcal{O}_U/f^n\mathcal{O}_U$ we see that we have short exact sequences $0 \to \mathcal{F}_n \to \mathcal{F}_{n+1} \to \mathcal{F}_1 \to 0$ for all n. Thus condition (b) holds by Lemma 3.2.

As f is a nonzerodivisor we obtain short exact sequences

$$0 \to A/f^nA \xrightarrow{f} A/f^{n+1}A \to A/fA \to 0$$

and we have corresponding short exact sequences $0 \to \mathcal{F}_n \to \mathcal{F}_{n+1} \to \mathcal{F}_1 \to 0$. We will use Local Cohomology, Lemma 7.2 without further mention. Our assumptions imply that $H^0(U, \mathcal{O}_U/f\mathcal{O}_U)$ and $H^1(U, \mathcal{O}_U/f\mathcal{O}_U)$ are finite A-modules. Hence the same thing is true for $\mathcal{F}_1$, see Local Cohomology, Lemma 11.2. Using induction and the short exact sequences we find that $H^0(U, \mathcal{F}_n)$ are finite A-modules for all n. In this way we see hypothesis (c) is satisfied.

Finally, as $H^1(U, \mathcal{F}_1)$ is a finite A-module we can apply Lemma 3.4 to see hypothesis (a) holds. □

0EHI **Remark 16.12.** In Lemma 16.11 if A is universally catenary with Cohen-Macaulay formal fibres (for example if A has a dualizing complex), then the condition that $H^1_\mathfrak{a}(A/fA)$ and $H^2_\mathfrak{a}(A/fA)$ are finite A-modules, is equivalent with

$$\text{depth}((A/f)_\mathfrak{q}) + \dim((A/\mathfrak{q})_\mathfrak{p}) > 2$$

for all $\mathfrak{q} \in V(f) \setminus V(\mathfrak{a})$ and $\mathfrak{p} \in V(\mathfrak{q}) \cap V(\mathfrak{a})$ by Local Cohomology, Theorem 10.6.

For example, if A/fA is (S_2) and if every irreducible component of $Z = V(\mathfrak{a})$ has codimension ≥ 3 in $Y = \text{Spec}(A/fA)$, then we get the finiteness of $H^1_\mathfrak{a}(A/fA)$ and $H^2_\mathfrak{a}(A/fA)$. This should be contrasted with the slightly weaker conditions found in Lemma 20.1 (see also Remark 20.2).

17. Algebraization of coherent formal modules, II

0EIT We continue the discussion started in Section 16. This section can be skipped on a first reading.

0EIU **Lemma 17.1.** *In Situation 16.1. Let $(\mathcal{F}_n) \to (\mathcal{F}'_n)$ be a morphism of Coh$(U, I\mathcal{O}_U)$ whose kernel and cokernel are annihilated by a power of I. Then*

1. *$(\mathcal{F}_n)$ extends to X if and only if $(\mathcal{F}'_n)$ extends to X, and*
2. *$(\mathcal{F}_n)$ is the completion of a coherent $\mathcal{O}_U$-module if and only if $(\mathcal{F}'_n)$ is.*

Proof. Part (2) follows immediately from Cohomology of Schemes, Lemma 23.6. To see part (1), we first use Lemma 16.6 to reduce to the case where A is I-adically complete. However, in that case (1) reduces to (2) by Lemma 16.3. □

The following two lemmas where originally used in the proof of Lemma 16.10. We keep them here for the reader who is interested to know what intermediate results one can obtain.

0EHF **Lemma 17.2.** *In Situation 16.1 let $(\mathcal{F}_n)$ be an object of Coh$(U, I\mathcal{O}_U)$. If the inverse system $H^0(U, \mathcal{F}_n)$ has Mittag-Leffler, then the canonical maps*

$$\widetilde{M/I^nM}|_U \to \mathcal{F}_n$$

are surjective for all n where M is as in (16.6.1).

Proof. Surjectivity may be checked on the stalk at some point $y \in Y \setminus Z$. If y corresponds to the prime $\mathfrak{q} \subset A$, then we can choose $f \in \mathfrak{a}$, $f \notin \mathfrak{q}$. Then it suffices to show

$$M_f \longrightarrow H^0(U, \mathcal{F}_n)_f = H^0(D(f), \mathcal{F}_n)$$

is surjective as $D(f)$ is affine (equality holds by Properties, Lemma 17.1). Since we have the Mittag-Leffler property, we find that

$$\mathrm{Im}(M \to H^0(U, \mathcal{F}_n)) = \mathrm{Im}(H^0(U, \mathcal{F}_m) \to H^0(U, \mathcal{F}_n))$$

for some $m \geq n$. Using the long exact sequence of cohomology we see that

$$\mathrm{Coker}(H^0(U, \mathcal{F}_m) \to H^0(U, \mathcal{F}_n)) \subset H^1(U, \mathrm{Ker}(\mathcal{F}_m \to \mathcal{F}_n))$$

Since $U = X \setminus V(\mathfrak{a})$ this H^1 is $\mathfrak{a}$-power torsion. Hence after inverting f the cokernel becomes zero. □

0EHG **Lemma 17.3.** *In Situation 16.1 let $(\mathcal{F}_n)$ be an object of $Coh(U, I\mathcal{O}_U)$. Let M be as in (16.6.1). Set*

$$\mathcal{G}_n = \widetilde{M/I^nM}.$$

If the limit topology on M agrees with the I-adic topology, then $\mathcal{G}_n|_U$ is a coherent $\mathcal{O}_U$-module and the map of inverse systems

$$(\mathcal{G}_n|_U) \longrightarrow (\mathcal{F}_n)$$

is injective in the abelian category $Coh(U, I\mathcal{O}_U)$.

Proof. Observe that $\mathcal{G}_n$ is a quasi-coherent $\mathcal{O}_X$-module annihilated by I^n and that $\mathcal{G}_{n+1}/I^n\mathcal{G}_{n+1} = \mathcal{G}_n$. Consider

$$M_n = \mathrm{Im}(M \longrightarrow H^0(U, \mathcal{F}_n))$$

The assumption says that the inverse systems (M_n) and (M/I^nM) are isomorphic as pro-objects of Mod_A. Pick $f \in \mathfrak{a}$ so $D(f) \subset U$ is an affine open. Then we have

$$(M_n)_f \subset H^0(U, \mathcal{F}_n)_f = H^0(D(f), \mathcal{F}_n)$$

Equality holds by Properties, Lemma 17.1. Thus $\widetilde{M_n}|_U \to \mathcal{F}_n$ is injective. It follows that $\widetilde{M_n}|_U$ is a coherent module (Cohomology of Schemes, Lemma 9.3). Since $M \to M/I^nM$ is surjective and factors as $M_{n'} \to M/I^nM$ for some $n' \geq n$ we find that $\mathcal{G}_n|_U$ is coherent as the quotient of a coherent module. Combined with the initical remarks of the proof we conclude that $(\mathcal{G}_n|_U)$ indeed forms an object of $Coh(U, I\mathcal{O}_U)$. Finally, to show the injectivity of the map it suffices to show that

$$\lim (M/I^nM)_f = \lim H^0(D(f), \mathcal{G}_n) \to \lim H^0(D(f), \mathcal{F}_n)$$

is injective, see Cohomology of Schemes, Lemmas 23.2 and 23.1. The injectivity of $\lim (M_n)_f \to \lim H^0(D(f), \mathcal{F}_n)$ is clear (see above) and by our remark on pro-systems we have $\lim (M_n)_f = \lim (M/I^nM)_f$. This finishes the proof. □

18. A distance function

0EIW Let Y be a Noetherian scheme and let $Z \subset Y$ be a closed subset. We define a function

0EIX (18.0.1)
$$\delta_Z^Y = \delta_Z : Y \longrightarrow \mathbf{Z}_{\geq 0} \cup \{\infty\}$$

which measures the "distance" of a point of Y from Z. For an informal discussion, please see Remark 18.3. Let $y \in Y$. We set $\delta_Z(y) = \infty$ if y is contained in a connected component of Y which does not meet Z. If y is contained in a connected component of Y which meets Z, then we can find $k \geq 0$ and a system

$$V_0 \subset W_0 \supset V_1 \subset W_1 \supset \ldots \supset V_k \subset W_k$$

of integral closed subschemes of Y such that $V_0 \subset Z$ and $y \in V_k$ is the generic point. Set $c_i = \text{codim}(V_i, W_i)$ for $i = 0, \ldots, k$ and $b_i = \text{codim}(V_{i+1}, W_i)$ for $i = 0, \ldots, k-1$. For such a system we set

$$\delta(V_0, W_0, V_1, \ldots, W_k) = k + \max_{i=0,1,\ldots,k}(c_i + c_{i+1} + \ldots + c_k - b_i - b_{i+1} - \ldots - b_{k-1})$$

This is $\geq k$ as we can take $i = k$ and we have $c_k \geq 0$. Finally, we set

$$\delta_Z(y) = \min \delta(V_0, W_0, V_1, \ldots, W_k)$$

where the minimum is over all systems of integral closed subschemes of Y as above.

0EIY **Lemma 18.1.** *Let Y be a Noetherian scheme and let $Z \subset Y$ be a closed subset.*

1. *For $y \in Y$ we have $\delta_Z(y) = 0 \Leftrightarrow y \in Z$.*
2. *The subsets $\{y \in Y \mid \delta_Z(y) \leq k\}$ are stable under specialization.*
3. *For $y \in Y$ and $z \in \overline{\{y\}} \cap Z$ we have $\dim(\mathcal{O}_{\overline{\{y\}},z}) \geq \delta_Z(y)$.*
4. *If δ is a dimension function on Y, then $\delta(y) \leq \delta_Z(y) + \delta_{max}$ where δ_{max} is the maximum value of δ on Z.*
5. *If $Y = \text{Spec}(A)$ is the spectrum of a catenary Noetherian local ring with maximal ideal $\mathfrak{m}$ and $Z = \{\mathfrak{m}\}$, then $\delta_Z(y) = \dim(\overline{\{y\}})$.*
6. *Given a pattern of specializations*

$$\begin{tikzcd}
& y'_0 \arrow[dl, rightsquigarrow] \arrow[dr, rightsquigarrow] & & y'_1 \arrow[dl, rightsquigarrow] & \ldots & y'_{k-1} \arrow[dr, rightsquigarrow] & \\
y_0 & & y_1 & & \ldots & & y_k = y
\end{tikzcd}$$

   between points of Y with $y_0 \in Z$ and $y'_i \rightsquigarrow y_i$ an immediate specialization, then $\delta_Z(y_k) \leq k$.
7. *If $Y' \subset Y$ is an open subscheme, then $\delta^{Y'}_{Y' \cap Z}(y') \geq \delta^Y_Z(y')$ for $y' \in Y'$.*

Proof. Part (1) is essentially true by definition. Namely, if $y \in Z$, then we can take $k = 0$ and $V_0 = W_0 = \overline{\{y\}}$.

Proof of (2). Let $y \rightsquigarrow y'$ be a nontrivial specialization and let $V_0 \subset W_0 \supset V_1 \subset W_1 \supset \ldots \subset W_k$ is a system for y. Here there are two cases. Case I: $V_k = W_k$, i.e., $c_k = 0$. In this case we can set $V'_k = W'_k = \overline{\{y'\}}$. An easy computation shows that $\delta(V_0, W_0, \ldots, V'_k, W'_k) \leq \delta(V_0, W_0, \ldots, V_k, W_k)$ because only b_{k-1} is changed into a bigger integer. Case II: $V_k \neq W_k$, i.e., $c_k > 0$. Observe that in this case $\max_{i=0,1,\ldots,k}(c_i + c_{i+1} + \ldots + c_k - b_i - b_{i+1} - \ldots - b_{k-1}) > 0$. Hence if we set

$V'_{k+1} = W_{k+1} = \overline{\{y'\}}$, then although k is replaced by $k+1$, the maximum now looks like

$$\max_{i=0,1,\ldots,k+1}(c_i + c_{i+1} + \ldots + c_k + c_{k+1} - b_i - b_{i+1} - \ldots - b_{k-1} - b_k)$$

with $c_{k+1} = 0$ and $b_k = \text{codim}(V_{k+1}, W_k) > 0$. This is strictly smaller than $\max_{i=0,1,\ldots,k}(c_i+c_{i+1}+\ldots+c_k-b_i-b_{i+1}-\ldots-b_{k-1})$ and hence $\delta(V_0, W_0, \ldots, V'_{k+1}, W'_{k+1}) \leq \delta(V_0, W_0, \ldots, V_k, W_k)$ as desired.

Proof of (3). Given $y \in Y$ and $z \in \overline{\{y\}} \cap Z$ we get the system

$$V_0 = \overline{\{z\}} \subset W_0 = \overline{\{y\}}$$

and $c_0 = \text{codim}(V_0, W_0) = \dim(\mathcal{O}_{\overline{\{y\}},z})$ by Properties, Lemma 10.3. Thus we see that $\delta(V_0, W_0) = 0 + c_0 = c_0$ which proves what we want.

Proof of (4). Let δ be a dimension function on Y. Let $V_0 \subset W_0 \supset V_1 \subset W_1 \supset \ldots \subset W_k$ be a system for y. Let $y'_i \in W_i$ and $y_i \in V_i$ be the generic points, so $y_0 \in Z$ and $y_k = y$. Then we see that

$$\delta(y_i) - \delta(y_{i-1}) = \delta(y'_{i-1}) - \delta(y_{i-1}) - \delta(y'_{i-1}) + \delta(y_i) = c_{i-1} - b_{i-1}$$

Finally, we have $\delta(y'_k) - \delta(y_{k-1}) = c_k$. Thus we see that

$$\delta(y) - \delta(y_0) = c_0 + \ldots + c_k - b_0 - \ldots - b_{k-1}$$

We conclude $\delta(V_0, W_0, \ldots, W_k) \geq k + \delta(y) - \delta(y_0)$ which proves what we want.

Proof of (5). The function $\delta(y) = \dim(\overline{\{y\}})$ is a dimension function. Hence $\delta(y) \leq \delta_Z(y)$ by part (4). By part (3) we have $\delta_Z(y) \leq \delta(y)$ and we are done.

Proof of (6). Given such a sequence of points, we may assume all the specializations $y'_i \rightsquigarrow y_{i+1}$ are nontrivial (otherwise we can shorten the chain of specializations). Then we set $V_i = \overline{\{y_i\}}$ and $W_i = \overline{\{y'_i\}}$ and we compute $\delta(V_0, W_1, V_1, \ldots, W_{k-1}) = k$ because all the codimensions c_i of $V_i \subset W_i$ are 1 and all $b_i > 0$. This implies $\delta_Z(y'_{k-1}) \leq k$ as y'_{k-1} is the generic point of W_k. Then $\delta_Z(y) \leq k$ by part (2) as y is a specialization of y_{k-1}.

Proof of (7). This is clear as their are fewer systems to consider in the computation of $\delta^{Y'}_{Y' \cap Z}$. $\square$

0EIZ **Lemma 18.2.** *Let Y be a universally catenary Noetherian scheme. Let $Z \subset Y$ be a closed subscheme. Let $f : Y' \to Y$ be a finite type morphism all of whose fibres have dimension $\leq e$. Set $Z' = f^{-1}(Z)$. Then*

$$\delta_Z(y) \leq \delta_{Z'}(y') + e - \mathit{trdeg}_{\kappa(y)}(\kappa(y'))$$

for $y' \in Y'$ with image $y \in Y$.

Proof. If $\delta_{Z'}(y') = \infty$, then there is nothing to prove. If $\delta_{Z'}(y') < \infty$, then we choose a system of integral closed subschemes

$$V'_0 \subset W'_0 \supset V'_1 \subset W'_1 \supset \ldots \subset W'_k$$

of Y' with $V'_0 \subset Z'$ and y' the generic point of W'_k such that $\delta_{Z'}(y') = \delta(V'_0, W'_0, \ldots, W'_k)$. Denote

$$V_0 \subset W_0 \supset V_1 \subset W_1 \supset \ldots \subset W_k$$

the scheme theoretic images of the above schemes in Y. Observe that y is the generic point of W_k and that $V_0 \subset Z$. For each i we look at the diagram

$$\begin{tikzcd} V_i' \arrow[r] \arrow[d] & W_i' \arrow[d] & V_{i+1}' \arrow[l] \arrow[d] \\ V_i \arrow[r] & W_i & V_{i+1} \arrow[l] \end{tikzcd}$$

Denote n_i the relative dimension of V_i'/V_i and m_i the relative dimension of W_i'/W_i; more precisely these are the transcendence degrees of the corresponding extensions of the function fields. Set $c_i = \text{codim}(V_i, W_i)$, $c_i' = \text{codim}(V_i', W_i')$, $b_i = \text{codim}(V_{i+1}, W_i)$, and $b_i' = \text{codim}(V_{i+1}', W_i')$. By the dimension formula we have

$$c_i = c_i' + n_i - m_i \quad \text{and} \quad b_i = b_i' + n_{i+1} - m_i$$

See Morphisms, Lemma 50.1. Hence $c_i - b_i = c_i' - b_i' + n_i - n_{i+1}$. Thus we see that

$$\begin{aligned} & c_i + c_{i+1} + \ldots + c_k - b_i - b_{i+1} - \ldots - b_{k-1} \\ & = c_i' + c_{i+1}' + \ldots + c_k' - b_i' - b_{i+1}' - \ldots - b_{k-1}' + n_i - n_k + c_k - c_k' \\ & = c_i' + c_{i+1}' + \ldots + c_k' - b_i' - b_{i+1}' - \ldots - b_{k-1}' + n_i - m_k \end{aligned}$$

Thus we see that

$$\begin{aligned} & \max_{i=0,\ldots,k}(c_i + c_{i+1} + \ldots + c_k - b_i - b_{i+1} - \ldots - b_{k-1}) \\ & \quad = \max_{i=0,\ldots,k}(c_i' + c_{i+1}' + \ldots + c_k' - b_i' - b_{i+1}' - \ldots - b_{k-1}' + n_i - m_k) \\ & \quad = \max_{i=0,\ldots,k}(c_i' + c_{i+1}' + \ldots + c_k' - b_i' - b_{i+1}' - \ldots - b_{k-1}' + n_i) - m_k \\ & \quad \leq \max_{i=0,\ldots,k}(c_i' + c_{i+1}' + \ldots + c_k' - b_i' - b_{i+1}' - \ldots - b_{k-1}') + e - m_k \end{aligned}$$

Since $m_k = \text{trdeg}_{\kappa(y)}(\kappa(y'))$ we conclude that

$$\delta(V_0, W_0, \ldots, W_k) \leq \delta(V_0', W_0', \ldots, W_k') + e - \text{trdeg}_{\kappa(y)}(\kappa(y'))$$

as desired. $\square$

0EJ0 **Remark 18.3.** Let Y be a Noetherian scheme and let $Z \subset Y$ be a closed subset. By Lemma 18.1 we have

$$\delta_Z(y) \leq \min \left\{ k \,\middle|\, \begin{array}{c} \text{there exist specializations in } Y \\ y_0 \leftarrow y_0' \rightarrow y_1 \leftarrow y_1' \rightarrow \ldots \leftarrow y_{k-1}' \rightarrow y_k = y \\ \text{with } y_0 \in Z \text{ and } y_i' \rightsquigarrow y_i \text{ immediate} \end{array} \right\}$$

We claim that if Y is of finite type over a field, then equality holds. If we ever need this result we will formulate a precise result and prove it here. However, in general if we define δ_Z by the right hand side of this inequality, then we don't know if Lemma 18.2 remains true.

0EJ1 **Example 18.4.** Let k be a field and $Y = \mathbf{A}^n_k$. Denote $\delta : Y \to \mathbf{Z}_{\geq 0}$ the usual dimension function.

1. If $Z = \{z\}$ for some closed point z, then
   - (a) $\delta_Z(y) = \delta(y)$ if $y \rightsquigarrow z$ and
   - (b) $\delta_Z(y) = \delta(y) + 1$ if $y \not\rightsquigarrow z$.
2. If Z is a closed subvariety and $W = \overline{\{y\}}$, then
   - (a) $\delta_Z(y) = 0$ if $W \subset Z$,

(b) $\delta_Z(y) = \dim(W) - \dim(Z)$ if Z is contained in W,
(c) $\delta_Z(y) = 1$ if $\dim(W) \leq \dim(Z)$ and $W \not\subset Z$,
(d) $\delta_Z(y) = \dim(W) - \dim(Z) + 1$ if $\dim(W) > \dim(Z)$ and $Z \not\subset W$.

A generalization of case (1) is if Y is of finite type over a field and $Z = \{z\}$ is a closed point. Then $\delta_Z(y) = \delta(y) + t$ where t is the minimum length of a chain of curves connecting z to a closed point of $\overline{\{y\}}$.

19. Algebraization of coherent formal modules, III

0EJ2 We continue the discussion started in Sections 16 and 17. We will use the distance function of Section 18 to formulate a some natural conditions on coherent formal modules in Situation 16.1.

In Situation 16.1 given a point $y \in U \cap Y$ we can consider the I-adic completion

$$\mathcal{O}_{X,y}^\wedge = \lim \mathcal{O}_{X,y}/I^n\mathcal{O}_{X,y}$$

This is a Noetherian local ring complete with respect to $I\mathcal{O}_{X,y}^\wedge$ with maximal ideal $\mathfrak{m}_y^\wedge$, see Algebra, Section 96. Let $(\mathcal{F}_n)$ be an object of $\textit{Coh}(U, I\mathcal{O}_U)$. Let us define the "stalk" of $(\mathcal{F}_n)$ at y by the formula

$$\mathcal{F}_y^\wedge = \lim \mathcal{F}_{n,y}$$

This is a finite module over $\mathcal{O}_{X,y}^\wedge$. See Algebra, Lemmas 97.1 and 95.12.

0EJ3 **Definition 19.1.** In Situation 16.1 let $(\mathcal{F}_n)$ be an object of $\textit{Coh}(U, I\mathcal{O}_U)$. Let a, b be integers. Let δ_Z^Y be as in (18.0.1). We say $(\mathcal{F}_n)$ *satisfies the (a,b)-inequalities* if for $y \in U \cap Y$ and a prime $\mathfrak{p} \subset \mathcal{O}_{X,y}^\wedge$ with $\mathfrak{p} \not\in V(I\mathcal{O}_{X,y}^\wedge)$

(1) if $V(\mathfrak{p}) \cap V(I\mathcal{O}_{X,y}^\wedge) \neq \{\mathfrak{m}_y^\wedge\}$, then

$$\text{depth}((\mathcal{F}_y^\wedge)_\mathfrak{p}) + \delta_Z^Y(y) \geq a \quad \text{or} \quad \text{depth}((\mathcal{F}_y^\wedge)_\mathfrak{p}) + \dim(\mathcal{O}_{X,y}^\wedge/\mathfrak{p}) + \delta_Z^Y(y) > b$$

(2) if $V(\mathfrak{p}) \cap V(I\mathcal{O}_{X,y}^\wedge) = \{\mathfrak{m}_y^\wedge\}$, then

$$\text{depth}((\mathcal{F}_y^\wedge)_\mathfrak{p}) + \delta_Z^Y(y) > a$$

We say $(\mathcal{F}_n)$ *satisfies the strict (a,b)-inequalities* if for $y \in U \cap Y$ and a prime $\mathfrak{p} \subset \mathcal{O}_{X,y}^\wedge$ with $\mathfrak{p} \not\in V(I\mathcal{O}_{X,y}^\wedge)$ we have

$$\text{depth}((\mathcal{F}_y^\wedge)_\mathfrak{p}) + \delta_Z^Y(y) > a \quad \text{or} \quad \text{depth}((\mathcal{F}_y^\wedge)_\mathfrak{p}) + \dim(\mathcal{O}_{X,y}^\wedge/\mathfrak{p}) + \delta_Z^Y(y) > b$$

Here are some elementary observations.

0EJ4 **Lemma 19.2.** *In Situation 16.1 let $(\mathcal{F}_n)$ be an object of $\textit{Coh}(U, I\mathcal{O}_U)$. Let a, b be integers.*

(1) *If $(\mathcal{F}_n)$ is annihilated by a power of I, then $(\mathcal{F}_n)$ satisfies the (a,b)-inequalities for any a, b.*
(2) *If $(\mathcal{F}_n)$ satisfies the $(a+1,b)$-inequalities, then $(\mathcal{F}_n)$ satisfies the strict (a,b)-inequalities.*

If $cd(A, I) \leq d$ and A has a dualizing complex, then

(3) *$(\mathcal{F}_n)$ satisfies the $(s, s+d)$-inequalities if and only if for all $y \in U \cap Y$ the tuple $\mathcal{O}_{X,y}^\wedge, I\mathcal{O}_{X,y}^\wedge, \{\mathfrak{m}_y^\wedge\}, \mathcal{F}_y^\wedge, s - \delta_Z^Y(y), d$ is as in Situation 10.1.*
(4) *If $(\mathcal{F}_n)$ satisfies the strict $(s, s+d)$-inequalities, then $(\mathcal{F}_n)$ satisfies the $(s, s+d)$-inequalities.*

Proof. Immediate except for part (4) which is a consequence of Lemma 10.5 and the translation in (3). $\square$

0EKW **Lemma 19.3.** *In Situation 16.1 let $(\mathcal{F}_n)$ be an object of $Coh(U, I\mathcal{O}_U)$. If $cd(A, I) = 1$, then $\mathcal{F}$ satisfies the $(2,3)$-inequalities if and only if*

$$depth((\mathcal{F}_y^\wedge)_\mathfrak{p}) + \dim(\mathcal{O}_{X,y}^\wedge/\mathfrak{p}) + \delta_Z^Y(y) > 3$$

for all $y \in U \cap Y$ and $\mathfrak{p} \subset \mathcal{O}_{X,y}^\wedge$ with $\mathfrak{p} \not\in V(I\mathcal{O}_{X,y}^\wedge)$.

Proof. Observe that for a prime $\mathfrak{p} \subset \mathcal{O}_{X,y}^\wedge$, $\mathfrak{p} \not\in V(I\mathcal{O}_{X,y}^\wedge)$ we have $V(\mathfrak{p}) \cap V(I\mathcal{O}_{X,y}^\wedge) = \{\mathfrak{m}_y^\wedge\} \Leftrightarrow \dim(\mathcal{O}_{X,y}^\wedge/\mathfrak{p}) = 1$ as $\text{cd}(A, I) = 1$. See Local Cohomology, Lemmas 3.5 and 3.10. OK, consider the three numbers $\alpha = \text{depth}((\mathcal{F}_y^\wedge)_\mathfrak{p}) \geq 0$, $\beta = \dim(\mathcal{O}_{X,y}^\wedge/\mathfrak{p}) \geq 1$, and $\gamma = \delta_Z^Y(y) \geq 1$. Then we see Definition 19.1 requires

(1) if $\beta > 1$, then $\alpha + \gamma \geq 2$ or $\alpha + \beta + \gamma > 3$, and
(2) if $\beta = 1$, then $\alpha + \gamma > 2$.

It is trivial to see that this is equivalent to $\alpha + \beta + \gamma > 3$. $\square$

In the rest of this section, which we suggest the reader skip on a first reading, we will show that, when A is I-adically complete, the category of $(\mathcal{F}_n)$ of $Coh(U, I\mathcal{O}_U)$ which extend to X and satisfy the strict $(1, 1 + \text{cd}(A, I))$-inequalities is equivalent to a full subcategory of the category of coherent $\mathcal{O}_U$-modules.

0EJ5 **Lemma 19.4.** *In Situation 16.1 let $\mathcal{F}$ be a coherent $\mathcal{O}_U$-module and $d \geq 1$. Assume*

(1) *A is I-adically complete, has a dualizing complex, and $cd(A, I) \leq d$,*
(2) *the completion $\mathcal{F}^\wedge$ of $\mathcal{F}$ satisfies the strict $(1, 1 + d)$-inequalities.*

Let $x \in X$ be a point. Let $W = \overline{\{x\}}$. If $W \cap Y$ has an irreducible component contained in Z and one which is not, then $depth(\mathcal{F}_x) \geq 1$.

Proof. Let $W \cap Y = W_1 \cup \ldots \cup W_n$ be the decomposition into irreducible components. By assumption, after renumbering, we can find $0 < m < n$ such that $W_1, \ldots, W_m \subset Z$ and $W_{m+1}, \ldots, W_n \not\subset Z$. We conclude that

$$W \cap Y \setminus ((W_1 \cup \ldots \cup W_m) \cap (W_{m+1} \cup \ldots \cup W_n))$$

is disconnected. By Lemma 14.2 we can find $1 \leq i \leq m < j \leq n$ and $z \in W_i \cap W_j$ such that $\dim(\mathcal{O}_{W,z}) \leq d + 1$. Choose an immediate specialization $y \rightsquigarrow z$ with $y \in W_j$, $y \not\in Z$; existence of y follows from Properties, Lemma 6.4. Observe that $\delta_Z^Y(y) = 1$ and $\dim(\mathcal{O}_{W,y}) \leq d$. Let $\mathfrak{p} \subset \mathcal{O}_{X,y}$ be the prime corresponding to x. Let $\mathfrak{p}' \subset \mathcal{O}_{X,y}^\wedge$ be a minimal prime over $\mathfrak{p}\mathcal{O}_{X,y}^\wedge$. Then we have

$$\text{depth}(\mathcal{F}_x) = \text{depth}((\mathcal{F}_y^\wedge)_{\mathfrak{p}'}) \quad \text{and} \quad \dim(\mathcal{O}_{W,y}) = \dim(\mathcal{O}_{X,y}^\wedge/\mathfrak{p}')$$

See Algebra, Lemma 157.1 and Local Cohomology, Lemma 10.3. Now we read off the conclusion from the inequalities given to us. $\square$

0EJ6 **Lemma 19.5.** *In Situation 16.1 let $\mathcal{F}$ be a coherent $\mathcal{O}_U$-module and $d \geq 1$. Assume*

(1) *A is I-adically complete, has a dualizing complex, and $cd(A, I) \leq d$,*
(2) *the completion $\mathcal{F}^\wedge$ of $\mathcal{F}$ satisfies the strict $(1, 1 + d)$-inequalities, and*
(3) *for $x \in U$ with $\overline{\{x\}} \cap Y \subset Z$ we have $depth(\mathcal{F}_x) \geq 2$.*

Then $H^0(U, \mathcal{F}) \to \lim H^0(U, \mathcal{F}/I^n\mathcal{F})$ is an isomorphism.

Proof. We will prove this by showing that Lemma 12.3 applies. Thus we let $x \in \text{Ass}(\mathcal{F})$ with $x \notin Y$. Set $W = \overline{\{x\}}$. By condition (3) we see that $W \cap Y \not\subset Z$. By Lemma 19.4 we see that no irreducible component of $W \cap Y$ is contained in Z. Thus if $z \in W \cap Z$, then there is an immediate specialization $y \rightsquigarrow z$, $y \in W \cap Y$, $y \notin Z$. For existence of y use Properties, Lemma 6.4. Then $\delta^Y_Z(y) = 1$ and the assumption implies that $\dim(\mathcal{O}_{W,y}) > d$. Hence $\dim(\mathcal{O}_{W,z}) > 1+d$ and we win. □

0EJ7 **Lemma 19.6.** *In Situation 16.1 let $\mathcal{F}$ be a coherent $\mathcal{O}_U$-module and $d \geq 1$. Assume*

(1) *A is I-adically complete, has a dualizing complex, and $cd(A, I) \leq d$,*
(2) *the completion $\mathcal{F}^\wedge$ of $\mathcal{F}$ satisfies the strict $(1, 1 + d)$-inequalities, and*
(3) *for $x \in U$ with $\overline{\{x\}} \cap Y \subset Z$ we have depth$(\mathcal{F}_x) \geq 2$.*

Then the map

$$\text{Hom}_U(\mathcal{G}, \mathcal{F}) \longrightarrow \text{Hom}_{Coh(U,I\mathcal{O}_U)}(\mathcal{G}^\wedge, \mathcal{F}^\wedge)$$

is bijective for every coherent $\mathcal{O}_U$-module $\mathcal{G}$.

Proof. Set $\mathcal{H} = \mathcal{H}om_{\mathcal{O}_U}(\mathcal{G}, \mathcal{F})$. Using Cohomology of Schemes, Lemma 11.2 or More on Algebra, Lemma 23.10 we see that the completion of $\mathcal{H}$ satisfies the strict $(1, 1+d)$-inequalities and that for $x \in U$ with $\overline{\{x\}} \cap Y \subset Z$ we have depth$(\mathcal{H}_x) \geq 2$. Details omitted. Thus by Lemma 19.5 we have

$$\text{Hom}_U(\mathcal{G}, \mathcal{F}) = H^0(U, \mathcal{H}) = \lim H^0(U, \mathcal{H}/\mathcal{I}^n\mathcal{H}) = Mor_{Coh(U,I\mathcal{O}_U)}(\mathcal{G}^\wedge, \mathcal{F}^\wedge)$$

See Cohomology of Schemes, Lemma 23.5 for the final equality. □

0EJ8 **Lemma 19.7.** *In Situation 16.1 let $(\mathcal{F}_n)$ be an object of $Coh(U, I\mathcal{O}_U)$ and $d \geq 1$. Assume*

(1) *A is I-adically complete, has a dualizing complex, and $cd(A, I) \leq d$,*
(2) *$(\mathcal{F}_n)$ is the completion of a coherent $\mathcal{O}_U$-module,*
(3) *$(\mathcal{F}_n)$ satisfies the strict $(1, 1 + d)$-inequalities.*

Then there exists a unique coherent $\mathcal{O}_U$-module $\mathcal{F}$ whose completion is $(\mathcal{F}_n)$ such that for $x \in U$ with $\overline{\{x\}} \cap Y \subset Z$ we have depth$(\mathcal{F}_x) \geq 2$.

Proof. Choose a coherent $\mathcal{O}_U$-module $\mathcal{F}$ whose completion is $(\mathcal{F}_n)$. Let $T = \{x \in U \mid \overline{\{x\}} \cap Y \subset Z\}$. We will construct $\mathcal{F}$ by applying Local Cohomology, Lemma 14.4 with $\mathcal{F}$ and T. Then uniqueness will follow from the mapping property of Lemma 19.6.

Since T is stable under specialization in U the only thing to check is the following. If $x' \rightsquigarrow x$ is an immediate specialization of points of U with $x \in T$ and $x' \notin T$, then depth$(\mathcal{F}_{x'}) \geq 1$. Set $W = \overline{\{x\}}$ and $W' = \overline{\{x'\}}$. Since $x' \notin T$ we see that $W' \cap Y$ is not contained in Z. If $W' \cap Y$ contains an irreducible component contained in Z, then we are done by Lemma 19.4. If not, we choose an irreducible component W_1 of $W \cap Y$ and an irreducible component W'_1 of $W' \cap Y$ with $W_1 \subset W'_1$. Let $z \in W_1$ be the generic point. Let $y \rightsquigarrow z$, $y \in W'_1$ be an immediate specialization with $y \notin Z$; existence of y follows from $W'_1 \not\subset Z$ (see above) and Properties, Lemma 6.4. Then we have the following $z \in Z$, $x \rightsquigarrow z$, $x' \rightsquigarrow y \rightsquigarrow z$, $y \in Y \setminus Z$, and $\delta^Y_Z(y) = 1$. By Local Cohomology, Lemma 3.10 and the fact that z is a generic point of $W \cap Y$ we have $\dim(\mathcal{O}_{W,z}) \leq d$. Since $x' \rightsquigarrow x$ is an immediate specialization we have $\dim(\mathcal{O}_{W',z}) \leq d + 1$. Since $y \neq z$ we conclude $\dim(\mathcal{O}_{W',y}) \leq d$. If depth$(\mathcal{F}_{x'}) = 0$

then we would get a contradiction with assumption (3); details about passage from $\mathcal{O}_{X,y}$ to its completion omitted. This finishes the proof. □

20. Algebraization of coherent formal modules, IV

0EHJ In this section we prove two stronger versions of Lemma 16.11 in the local case, namely, Lemmas 20.1 and 20.4. Although these lemmas will be obsoleted by the more general Proposition 22.2, their proofs are significantly easier.

0DXU **Lemma 20.1.** *In Situation 16.1 let $(\mathcal{F}_n)$ be an object of $Coh(U, I\mathcal{O}_U)$. Assume*

1. *A is local and $\mathfrak{a} = \mathfrak{m}$ is the maximal ideal,*
2. *A has a dualizing complex,*
3. *$I = (f)$ is a principal ideal for a nonzerodivisor $f \in \mathfrak{m}$,*
4. *$\mathcal{F}_n$ is a finite locally free $\mathcal{O}_U/f^n\mathcal{O}_U$-module,*
5. *if $\mathfrak{p} \in V(f) \setminus \{\mathfrak{m}\}$, then $depth((A/f)_\mathfrak{p}) + \dim(A/\mathfrak{p}) > 1$, and*
6. *if $\mathfrak{p} \notin V(f)$ and $V(\mathfrak{p}) \cap V(f) \neq \{\mathfrak{m}\}$, then $depth(A_\mathfrak{p}) + \dim(A/\mathfrak{p}) > 3$.*

Then $(\mathcal{F}_n)$ extends canonically to X. In particular, if A is complete, then $(\mathcal{F}_n)$ is the completion of a coherent $\mathcal{O}_U$-module.

Proof. We will prove this by verifying hypotheses (a), (b), and (c) of Lemma 16.10.

Since $\mathcal{F}_n$ is locally free over $\mathcal{O}_U/f^n\mathcal{O}_U$ we see that we have short exact sequences $0 \to \mathcal{F}_n \to \mathcal{F}_{n+1} \to \mathcal{F}_1 \to 0$ for all n. Thus condition (b) holds by Lemma 3.2.

By induction on n and the short exact sequences $0 \to A/f^n \to A/f^{n+1} \to A/f \to 0$ we see that the associated primes of A/f^nA agree with the associated primes of A/fA. Since the associated points of $\mathcal{F}_n$ correspond to the associated primes of A/f^nA not equal to $\mathfrak{m}$ by assumption (3), we conclude that $M_n = H^0(U, \mathcal{F}_n)$ is a finite A-module by (5) and Local Cohomology, Proposition 7.7. Thus hypothesis (c) holds.

To finish the proof it suffices to show that there exists an $n > 1$ such that the image of

$$H^1(U, \mathcal{F}_n) \longrightarrow H^1(U, \mathcal{F}_1)$$

has finite length as an A-module. Namely, this will imply hypothesis (a) by Lemma 3.5. The image is independent of n for n large enough by Lemma 5.2. Let $\omega_A^\bullet$ be a normalized dualizing complex for A. By the local duality theorem and Matlis duality (Dualizing Complexes, Lemma 18.4 and Proposition 7.8) our claim is equivalent to: the image of

$$\mathrm{Ext}^{-2}_A(M_1, \omega_A^\bullet) \to \mathrm{Ext}^{-2}_A(M_n, \omega_A^\bullet)$$

has finite length for $n \gg 1$. The modules in question are finite A-modules supported at $V(f)$. Thus it suffices to show that this map is zero after localization at a prime $\mathfrak{q}$ containing f and different from $\mathfrak{m}$. Let $\omega_{A_\mathfrak{q}}^\bullet$ be a normalized dualizing complex on $A_\mathfrak{q}$ and recall that $\omega_{A_\mathfrak{q}}^\bullet = (\omega_A^\bullet)_\mathfrak{q}[\dim(A/\mathfrak{q})]$ by Dualizing Complexes, Lemma 17.3. Using the local structure of $\mathcal{F}_n$ given in (4) we find that it suffices to show the vanishing of

$$\mathrm{Ext}^{-2+\dim(A/\mathfrak{q})}_{A_\mathfrak{q}}(A_\mathfrak{q}/f, \omega_{A_\mathfrak{q}}^\bullet) \to \mathrm{Ext}^{-2+\dim(A/\mathfrak{q})}_{A_\mathfrak{q}}(A_\mathfrak{q}/f^n, \omega_{A_\mathfrak{q}}^\bullet)$$

for n large enough. If $\dim(A/\mathfrak{q}) > 3$, then this is immediate from Local Cohomology, Lemma 8.4. For the other cases we will use the long exact sequence

$$\ldots \xrightarrow{f^n} H^{-1}(\omega_{A_\mathfrak{q}}^\bullet) \to \mathrm{Ext}^{-1}_{A_\mathfrak{q}}(A_\mathfrak{q}/f^n, \omega_{A_\mathfrak{q}}^\bullet) \to H^0(\omega_{A_\mathfrak{q}}^\bullet) \xrightarrow{f^n} H^0(\omega_{A_\mathfrak{q}}^\bullet) \to \mathrm{Ext}^0_{A_\mathfrak{q}}(A_\mathfrak{q}/f^n, \omega_{A_\mathfrak{q}}^\bullet) \to 0$$

If $\dim(A/\mathfrak{q}) = 2$, then $H^0(\omega_{A_\mathfrak{q}}^\bullet) = 0$ because $\text{depth}(A_\mathfrak{q}) \geq 1$ as f is a nonzerodivisor. Thus the long exact sequence shows the condition is that

$$f^{n-1} : H^{-1}(\omega_{A_\mathfrak{q}}^\bullet)/f \to H^{-1}(\omega_{A_\mathfrak{q}}^\bullet)/f^n$$

is zero. Now $H^{-1}(\omega_\mathfrak{q}^\bullet)$ is a finite module supported in the primes $\mathfrak{p} \subset A_\mathfrak{q}$ such that $\text{depth}(A_\mathfrak{p}) + \dim((A/\mathfrak{p})_\mathfrak{q}) \leq 1$. Since $\dim((A/\mathfrak{p})_\mathfrak{q}) = \dim(A/\mathfrak{p}) - 2$ condition (6) tells us these primes are contained in $V(f)$. Thus the desired vanishing for n large enough. Finally, if $\dim(A/\mathfrak{q}) = 1$, then condition (5) combined with the fact that f is a nonzerodivisor insures that $A_\mathfrak{q}$ has depth at least 2. Hence $H^0(\omega_{A_\mathfrak{q}}^\bullet) = H^{-1}(\omega_{A_\mathfrak{q}}^\bullet) = 0$ and the long exact sequence shows the claim is equivalent to the vanishing of

$$f^{n-1} : H^{-2}(\omega_{A_\mathfrak{q}}^\bullet)/f \to H^{-2}(\omega_{A_\mathfrak{q}}^\bullet)/f^n$$

Now $H^{-2}(\omega_\mathfrak{q}^\bullet)$ is a finite module supported in the primes $\mathfrak{p} \subset A_\mathfrak{q}$ such that $\text{depth}(A_\mathfrak{p}) + \dim((A/\mathfrak{p})_\mathfrak{q}) \leq 2$. By condition (6) all of these primes are contained in $V(f)$. Thus the desired vanishing for n large enough. □

0DXV **Remark 20.2.** Let $(A, \mathfrak{m})$ be a complete Noetherian normal local domain of dimension ≥ 4 and let $f \in \mathfrak{m}$ be nonzero. Then assumptions (1), (2), (3), (5), and (6) of Lemma 20.1 are satisfied. Thus vectorbundles on the formal completion of U along $U \cap V(f)$ can be algebraized. In Lemma 20.4 we will generalize this to more general coherent formal modules; please also compare with Remark 20.7.

0EHK **Lemma 20.3.** *In Situation 16.1 let (M_n) be an inverse system of A-modules as in Lemma 16.2 and let $(\mathcal{F}_n)$ be the corresponding object of $Coh(U, I\mathcal{O}_U)$. Let $d \geq cd(A, I)$ and $s \geq 0$ be integers. With notation as above assume*

1. *A is local with maximal ideal $\mathfrak{m} = \mathfrak{a}$,*
2. *A has a dualizing complex, and*
3. *$(\mathcal{F}_n)$ satisfies the $(s, s+d)$-inequalities (Definition 19.1).*

Let E be an injective hull of the residue field of A. Then for $i \leq s$ there exists a finite A-module N annihilated by a power of I and for $n \gg 0$ compatible maps

$$H^i_\mathfrak{m}(M_n) \to \text{Hom}_A(N, E)$$

whose cokernels are finite length A-modules and whose kernels K_n form an inverse system such that $\text{Im}(K_{n''} \to K_{n'})$ has finite length for $n'' \gg n' \gg 0$.

Proof. Let $\omega_A^\bullet$ be a normalized dualizing complex. Then $\delta_Z^Y = \delta$ is the dimension function associated with this dualizing complex. Observe that $\text{Ext}_A^{-i}(M_n, \omega_A^\bullet)$ is a finite A-module annihilated by I^n. Fix $0 \leq i \leq s$. Below we will find $n_1 > n_0 > 0$ such that if we set

$$N = \text{Im}(\text{Ext}_A^{-i}(M_{n_0}, \omega_A^\bullet) \to \text{Ext}_A^{-i}(M_{n_1}, \omega_A^\bullet))$$

then the kernels of the maps

$$N \to \text{Ext}_A^{-i}(M_n, \omega_A^\bullet), \quad n \geq n_1$$

are finite length A-modules and the cokernels Q_n form a system such that $\text{Im}(Q_{n'} \to Q_{n''})$ has finite length for $n'' \gg n' \gg n_1$. This is equivalent to the statement that the system $\{\text{Ext}_A^{-i}(M_n, \omega_A^\bullet)\}_{n \geq 1}$ is essentially constant in the quotient of the category of finite A-modules modulo the Serre subcategory of finite length A-modules.

By the local duality theorem (Dualizing Complexes, Lemma 18.4) and Matlis duality (Dualizing Complexes, Proposition 7.8) we conclude that there are maps

$$H^i_{\mathfrak m}(M_n) \to \mathrm{Hom}_A(N, E), \quad n \geq n_1$$

as in the statement of the lemma.

Pick $f \in \mathfrak m$. Let $B = A_f^\wedge$ be the I-adic completion of the localization A_f. Recall that $\omega_{A_f}^\bullet = \omega_A^\bullet \otimes_A A_f$ and $\omega_B^\bullet = \omega_A^\bullet \otimes_A B$ are dualizing complexes (Dualizing Complexes, Lemma 15.6 and 22.3). Let M be the finite B-module $\lim M_{n,f}$ (compare with discussion in Cohomology of Schemes, Lemma 23.1). Then

$$\mathrm{Ext}^{-i}_A(M_n, \omega_A^\bullet)_f = \mathrm{Ext}^{-i}_{A_f}(M_{n,f}, \omega_{A_f}^\bullet) = \mathrm{Ext}^{-i}_B(M/I^nM, \omega_B^\bullet)$$

Since $\mathfrak m$ can be generated by finitely many $f \in \mathfrak m$ it suffices to show that for each f the system

$$\{\mathrm{Ext}^{-i}_B(M/I^nM, \omega_B^\bullet)\}_{n \geq 1}$$

is essentially constant. Some details omitted.

Let $\mathfrak q \subset IB$ be a prime ideal. Then $\mathfrak q$ corresponds to a point $y \in U \cap Y$. Observe that $\delta(\mathfrak q) = \dim(\overline{\{y\}})$ is also the value of the dimension function associated to $\omega_B^\bullet$ (we omit the details; use that $\omega_B^\bullet$ is gotten from $\omega_A^\bullet$ by tensoring up with B). Assumption (3) guarantees via Lemma 19.2 that Lemma 10.4 applies to $B_{\mathfrak q}, IB_{\mathfrak q}, \mathfrak q B_{\mathfrak q}, M_{\mathfrak q}$ with s replaced by $s - \delta(y)$. We obtain that

$$H^{i-\delta(\mathfrak q)}_{\mathfrak q B_{\mathfrak q}}(M_{\mathfrak q}) = \lim H^{i-\delta(\mathfrak q)}_{\mathfrak q B_{\mathfrak q}}((M/I^nM)_{\mathfrak q})$$

and this module is annihilated by a power of I. By Lemma 5.3 we find that the inverse systems $H^{i-\delta(\mathfrak q)}_{\mathfrak q B_{\mathfrak q}}((M/I^nM)_{\mathfrak q})$ are essentially constant with value $H^{i-\delta(\mathfrak q)}_{\mathfrak q B_{\mathfrak q}}(M_{\mathfrak q})$. Since $(\omega_B^\bullet)_{\mathfrak q}[-\delta(\mathfrak q)]$ is a normalized dualizing complex on $B_{\mathfrak q}$ the local duality theorem shows that the system

$$\mathrm{Ext}^{-i}_B(M/I^nM, \omega_B^\bullet)_{\mathfrak q}$$

is essentially constant with value $\mathrm{Ext}^{-i}_B(M, \omega_B^\bullet)_{\mathfrak q}$.

To finish the proof we globalize as in the proof of Lemma 10.6; the argument here is easier because we know the value of our system already. Namely, consider the maps

$$\alpha_n : \mathrm{Ext}^{-i}_B(M/I^nM, \omega_B^\bullet) \longrightarrow \mathrm{Ext}^{-i}_B(M, \omega_B^\bullet)$$

for varying n. By the above, for every $\mathfrak q$ we can find an n such that α_n is surjective after localization at $\mathfrak q$. Since B is Noetherian and $\mathrm{Ext}^{-i}_B(M, \omega_B^\bullet)$ a finite module, we can find an n such that α_n is surjective. For any n such that α_n is surjective, given a prime $\mathfrak q \in V(IB)$ we can find an $n' > n$ such that $\mathrm{Ker}(\alpha_n)$ maps to zero in $\mathrm{Ext}^{-i}(M/I^{n'}M, \omega_B^\bullet)$ at least after localizing at $\mathfrak q$. Since $\mathrm{Ker}(\alpha_n)$ is a finite A-module and since supports of sections are quasi-compact, we can find an n' such that $\mathrm{Ker}(\alpha_n)$ maps to zero in $\mathrm{Ext}^{-i}(M/I^{n'}M, \omega_B^\bullet)$. In this way we see that $\mathrm{Ext}^{-i}(M/I^nM, \omega_B^\bullet)$ is essentially constant with value $\mathrm{Ext}^{-i}(M, \omega_B^\bullet)$. This finishes the proof. □

Here is a more general version of Lemma 20.1.

0EJ9 **Lemma 20.4.** *In Situation 16.1 let $(\mathcal F_n)$ be an object of Coh$(U, I\mathcal O_U)$. Assume*

(1) *A is local and $\mathfrak a = \mathfrak m$ is the maximal ideal,*
(2) *A has a dualizing complex,*

(3) *$I = (f)$ is a principal ideal,*
(4) *$(\mathcal{F}_n)$ satisfies the $(2,3)$-inequalities.*

Then $(\mathcal{F}_n)$ extends to X. In particular, if A is I-adically complete, then $(\mathcal{F}_n)$ is the completion of a coherent $\mathcal{O}_U$-module.

Proof. Recall that $Coh(U, I\mathcal{O}_U)$ is an abelian category, see Cohomology of Schemes, Lemma 23.2. Over affine opens of U the object $(\mathcal{F}_n)$ corresponds to a finite module over a Noetherian ring (Cohomology of Schemes, Lemma 23.1). Thus the kernels of the maps $f^N : (\mathcal{F}_n) \to (\mathcal{F}_n)$ stabilize for N large enough. By Lemmas 17.1 and 16.3 in order to prove the lemma we may replace $(\mathcal{F}_n)$ by the image of such a map. Thus we may assume f is injective on $(\mathcal{F}_n)$. After this replacement the equivalent conditions of Lemma 3.1 hold for the inverse system $(\mathcal{F}_n)$ on U. We will use this without further mention in the rest of the proof.

We will check hypotheses (a), (b), and (c) of Lemma 16.10. Hypothesis (b) holds by Lemma 3.2.

Pick a inverse system of modules $\{M_n\}$ as in Lemma 16.2. We may assume $H^0_\mathfrak{m}(M_n) = 0$ by replacing M_n by $M_n/H^0_\mathfrak{m}(M_n)$ if necessary. Then we obtain short exact sequences

$$0 \to M_n \to H^0(U, \mathcal{F}_n) \to H^1_\mathfrak{m}(M_n) \to 0$$

for all n. Let E be an injective hull of the residue field of A. By Lemma 20.3 and our current assumption (4) we can choose, an integer $m \geq 0$, finite A-modules N_1 and N_2 annihilated by f^c for some $c \geq 0$ and compatible systems of maps

$$H^i_\mathfrak{m}(M_n) \to \mathrm{Hom}_A(N_i, E), \quad i = 1, 2$$

for $n \geq m$ with the properties stated in the lemma.

We know that $M = \lim H^0(U, \mathcal{F}_n)$ is an A-module whose limit topology is the f-adic topology. Thus, given n, the module M/f^nM is a subquotient of $H^0(U, \mathcal{F}_N)$ for some $N \gg n$. Looking at the information obtained above we see that f^cM/f^nM is a finite A-module. Since f is a nonzerodivisor on M we conclude that $M/f^{n-c}M$ is a finite A-module. In this way we see that hypothesis (c) of Lemma 16.10 holds.

Next, we study the module

$$Ob = \lim H^1(U, \mathcal{F}_n) = \lim H^2_\mathfrak{m}(M_n)$$

For $n \geq m$ let K_n be the kernel of the map $H^2_\mathfrak{m}(M_n) \to \mathrm{Hom}_A(N_2, E)$. Set $K = \lim K_n$. We obtain an exact sequence

$$0 \to K \to Ob \to \mathrm{Hom}_A(N_2, E)$$

By the above the limit topology on $Ob = \lim H^2_\mathfrak{m}(M_n)$ is the f-adic topology. Since N_2 is annihilated by f^c we conclude the same is true for the limit topology on $K = \lim K_n$. Thus K/fK is a subquotient of K_n for $n \gg 1$. However, since $\{K_n\}$ is pro-isomorphic to a inverse system of finite length A-modules (by the conclusion of Lemma 20.3) we conclude that K/fK is a subquotient of a finite length A-module. It follows that K is a finite A-module, see Algebra, Lemma 95.12. (In fact, we even see that $\dim(\mathrm{Supp}(K)) = 1$ but we will not need this.)

Given $n \geq 1$ consider the boundary map

$$\delta_n : H^0(U, \mathcal{F}_n) \longrightarrow \lim\nolimits_N H^1(U, f^n\mathcal{F}_N) \xrightarrow{f^{-n}} Ob$$

(the second map is an isomorphism) coming from the short exact sequences

$$0 \to f^n \mathcal{F}_N \to \mathcal{F}_N \to \mathcal{F}_n \to 0$$

For each n set

$$P_n = \mathrm{Im}(H^0(U, \mathcal{F}_{n+m}) \to H^0(U, \mathcal{F}_n))$$

where m is as above. Observe that $\{P_n\}$ is an inverse system and that the map $f : \mathcal{F}_n \to \mathcal{F}_{n+1}$ on global sections maps P_n into P_{n+1}. If $p \in P_n$, then $\delta_n(p) \in K \subset Ob$ because $\delta_n(p)$ maps to zero in $H^1(U, f^n\mathcal{F}_{n+m}) = H^2_{\mathfrak m}(M_m)$ and the composition of δ_n and $Ob \to \mathrm{Hom}_A(N_2, E)$ factors through $H^2_{\mathfrak m}(M_m)$ by our choice of m. Hence

$$\bigoplus\nolimits_{n \geq 0} \mathrm{Im}(P_n \to Ob)$$

is a finite graded $A[T]$-module where T acts via multiplication by f. Namely, it is a graded submodule of $K[T]$ and K is finite over A. Arguing as in the proof of Lemma 2.1[7] we find that the inverse system $\{P_n\}$ satisfies ML. Since $\{P_n\}$ is pro-isomorphic to $\{H^0(U, \mathcal{F}_n)\}$ we conclude that $\{H^0(U, \mathcal{F}_n)\}$ has ML. Thus hypothesis (a) of Lemma 16.10 holds and the proof is complete. $\square$

We can unwind condition of Lemma 20.4 as follows.

0EJA **Lemma 20.5.** *In Situation 16.1 let $(\mathcal{F}_n)$ be an object of $Coh(U, I\mathcal{O}_U)$. Assume*

1. *A is local with maximal ideal $\mathfrak a = \mathfrak m$,*
2. *$cd(A, I) = 1$.*

Then $(\mathcal{F}_n)$ satisfies the $(2,3)$-inequalities if and only if for all $y \in U \cap Y$ with $\dim(\{y\}) = 1$ and every prime $\mathfrak p \subset \mathcal{O}^\wedge_{X,y}$, $\mathfrak p \notin V(I\mathcal{O}^\wedge_{X,y})$ we have

$$\mathit{depth}((\mathcal{F}^\wedge_y)_{\mathfrak p}) + \dim(\mathcal{O}^\wedge_{X,y}/\mathfrak p) > 2$$

Proof. We will use Lemma 19.3 without further mention. In particular, we see the condition is necessary. Conversely, suppose the condition is true. Note that $\delta^Y_Z(y) = \dim(\overline{\{y\}})$ by Lemma 18.1. Let us write δ for this function. Let $y \in U \cap Y$. If $\delta(y) > 2$, then the inequality of Lemma 19.3 holds. Finally, suppose $\delta(y) = 2$. We have to show that

$$\mathrm{depth}((\mathcal{F}^\wedge_y)_{\mathfrak p}) + \dim(\mathcal{O}^\wedge_{X,y}/\mathfrak p) > 1$$

Choose a specialization $y \leadsto y'$ with $\delta(y') = 1$. Then there is a ring map $\mathcal{O}^\wedge_{X,y'} \to \mathcal{O}^\wedge_{X,y}$ which identifies the target with the completion of the localization of $\mathcal{O}^\wedge_{X,y'}$ at a prime $\mathfrak q$ with $\dim(\mathcal{O}^\wedge_{X,y'}/\mathfrak q) = 1$. Moreover, we then obtain

$$\mathcal{F}^\wedge_y = \mathcal{F}^\wedge_{y'} \otimes_{\mathcal{O}^\wedge_{X,y'}} \mathcal{O}^\wedge_{X,y}$$

Let $\mathfrak p' \subset \mathcal{O}^\wedge_{X,y'}$ be the image of $\mathfrak p$. By Local Cohomology, Lemma 10.3 we have

$$\begin{aligned} \mathrm{depth}((\mathcal{F}^\wedge_y)_{\mathfrak p}) + \dim(\mathcal{O}^\wedge_{X,y}/\mathfrak p) &= \mathrm{depth}((\mathcal{F}^\wedge_{y'})_{\mathfrak p'}) + \dim((\mathcal{O}^\wedge_{X,y}/\mathfrak p)_{\mathfrak p'}) \\ &= \mathrm{depth}((\mathcal{F}^\wedge_{y'})_{\mathfrak p'}) + \dim(\mathcal{O}^\wedge_{X,y}/\mathfrak p') - 1 \end{aligned}$$

the last equality because the specialization is immediate. Thus the lemma is prove by the assumed inequality for $y', \mathfrak p'$. $\square$

[7] Choose homogeneous generators of the form $\delta_{n_j}(p_j)$ for the displayed module. Then if $k = \max(n_j)$ we find that for $n \geq k$ and any $p \in P_n$ we can find $a_j \in A$ such that $p - \sum a_j f^{n-n_j} p_j$ is in the kernel of δ_n and hence in the image of $P_{n'}$ for all $n' \geq n$. Thus $\mathrm{Im}(P_n \to P_{n-k}) = \mathrm{Im}(P_{n'} \to P_{n-k})$ for all $n' \geq n$.

0EJB **Lemma 20.6.** *In Situation 16.1 let $(\mathcal{F}_n)$ be an object of $Coh(U, I\mathcal{O}_U)$. Assume*

1. *A is local with maximal ideal $\mathfrak{a} = \mathfrak{m}$,*
2. *A has a dualizing complex,*
3. *$cd(A, I) = 1$,*
4. *for $y \in U \cap Y$ the module $\mathcal{F}_y^\wedge$ is finite locally free outside $V(I\mathcal{O}_{X,y}^\wedge)$, for example if $\mathcal{F}_n$ is a finite locally free $\mathcal{O}_U/I^n\mathcal{O}_U$-module, and*
5. *one of the following is true*
   (a) *A_f is (S_2) and every irreducible component of X not contained in Y has dimension ≥ 4, or*
   (b) *if $\mathfrak{p} \notin V(f)$ and $V(\mathfrak{p}) \cap V(f) \neq \{\mathfrak{m}\}$, then $depth(A_\mathfrak{p}) + \dim(A/\mathfrak{p}) > 3$.*

Then $(\mathcal{F}_n)$ satisfies the $(2,3)$-inequalities.

Proof. We will use the criterion of Lemma 20.5. Let $y \in U \cap Y$ with $\dim(\overline{\{y\}} = 1$ and let $\mathfrak{p}$ be a prime $\mathfrak{p} \subset \mathcal{O}_{X,y}^\wedge$ with $\mathfrak{p} \notin V(I\mathcal{O}_{X,y}^\wedge)$. Condition (4) shows that $\text{depth}((\mathcal{F}_y^\wedge)_\mathfrak{p}) = \text{depth}((\mathcal{O}_{X,y}^\wedge)_\mathfrak{p})$. Thus we have to prove

$$\text{depth}((\mathcal{O}_{X,y}^\wedge)_\mathfrak{p}) + \dim(\mathcal{O}_{X,y}^\wedge/\mathfrak{p}) > 2$$

Let $\mathfrak{p}_0 \subset A$ be the image of $\mathfrak{p}$. Let $\mathfrak{q} \subset A$ be the prime corresponding to y. By Local Cohomology, Lemma 10.3 we have

$$\begin{aligned}\text{depth}((\mathcal{O}_{X,y}^\wedge)_\mathfrak{p}) + \dim(\mathcal{O}_{X,y}^\wedge/\mathfrak{p}) &= \text{depth}(A_{\mathfrak{p}_0}) + \dim((A/\mathfrak{p}_0)_\mathfrak{q}) \\ &= \text{depth}(A_{\mathfrak{p}_0}) + \dim(A/\mathfrak{p}_0) - 1\end{aligned}$$

If (5)(a) holds, then we get that this is

$$\geq \min(2, \dim(A_{\mathfrak{p}_0})) + \dim(A/\mathfrak{p}_0) - 1$$

Note that in any case $\dim(A/\mathfrak{p}_0) \geq 2$. Hence if we get 2 for the minimum, then we are done. If not we get

$$\dim(A_{\mathfrak{p}_0}) + \dim(A/\mathfrak{p}_0) - 1 \geq 4 - 1$$

because every component of $\text{Spec}(A)$ passing through $\mathfrak{p}_0$ has dimension ≥ 4. If (5)(b) holds, then we win immediately. $\square$

0EJC **Remark 20.7.** Let $(A, \mathfrak{m})$ be a Noetherian local ring which has a dualizing complex and is complete with respect to $f \in \mathfrak{m}$. Let $(\mathcal{F}_n)$ be an object of $Coh(U, f\mathcal{O}_U)$ where U is the punctured spectrum of A. Set $Y = V(f) \subset X = \text{Spec}(A)$. If for $y \in U \cap V(f)$ closed in U, i.e., with $\dim(\overline{\{y\}}) = 1$, we assume the $\mathcal{O}_{X,y}^\wedge$-module $\mathcal{F}_y^\wedge$ satisfies the following two conditions

1. $\mathcal{F}_y^\wedge[1/f]$ is (S_2) as a $\mathcal{O}_{X,y}^\wedge[1/f]$-module, and
2. for $\mathfrak{p} \in \text{Ass}(\mathcal{F}_y^\wedge[1/f])$ we have $\dim(\mathcal{O}_{X,y}^\wedge/\mathfrak{p}) \geq 3$.

Then $(\mathcal{F}_n)$ is the completion of a coherent module on U. This follows from Lemmas 20.4 and 20.5.

21. Improving coherent formal modules

0EJD Let X be a Noetherian scheme. Let $Y \subset X$ be a closed subscheme with quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_X$. Let $(\mathcal{F}_n)$ be an object of $Coh(X, \mathcal{I})$. In this section we construct maps $(\mathcal{F}_n) \to (\mathcal{F}'_n)$ similar to the maps constructed in Local Cohomology, Section 14 for coherent modules. For a point $y \in Y$ we set

$$\mathcal{O}_{X,y}^\wedge = \lim \mathcal{O}_{X,y}/\mathcal{I}_y^n, \quad \mathcal{I}_y^\wedge = \lim \mathcal{I}_y/\mathcal{I}_y^n \quad \text{and} \quad \mathfrak{m}_y^\wedge = \lim \mathfrak{m}_y/\mathcal{I}_y^n$$

Then $\mathcal{O}^\wedge_{X,y}$ is a Noetherian local ring with maximal ideal $\mathfrak{m}^\wedge_y$ complete with respect to $\mathcal{I}^\wedge_y = \mathcal{I}_y\mathcal{O}^\wedge_{X,y}$. We also set

$$\mathcal{F}^\wedge_y = \lim \mathcal{F}_{n,y}$$

Then $\mathcal{F}^\wedge_y$ is a finite module over $\mathcal{O}^\wedge_{X,y}$ with $\mathcal{F}^\wedge_y/(\mathcal{I}^\wedge_y)^n\mathcal{F}^\wedge_y = \mathcal{F}_{n,y}$ for all n, see Algebra, Lemmas 97.1 and 95.12.

0EJE **Lemma 21.1.** *In the situation above assume X locally has a dualizing complex. Let $T \subset Y$ be a subset stable under specialization. Assume for $y \in T$ and for a nonmaximal prime $\mathfrak{p} \subset \mathcal{O}^\wedge_{X,y}$ with $V(\mathfrak{p}) \cap V(\mathcal{I}^\wedge_y) = \{\mathfrak{m}^\wedge_y\}$ we have*

$$depth_{(\mathcal{O}_{X,y})_\mathfrak{p}}((\mathcal{F}^\wedge_y)_\mathfrak{p}) > 0$$

Then there exists a canonical map $(\mathcal{F}_n) \to (\mathcal{F}'_n)$ of inverse systems of coherent $\mathcal{O}_X$-modules with the following properties

1. *for $y \in T$ we have $depth(\mathcal{F}'_{n,y}) \geq 1$,*
2. *$(\mathcal{F}'_n)$ is isomorphic as a pro-system to an object $(\mathcal{G}_n)$ of $Coh(X, \mathcal{I})$,*
3. *the induced morphism $(\mathcal{F}_n) \to (\mathcal{G}_n)$ of $Coh(X, \mathcal{I})$ is surjective with kernel annihilated by a power of $\mathcal{I}$.*

Proof. For every n we let $\mathcal{F}_n \to \mathcal{F}'_n$ be the surjection constructed in Local Cohomology, Lemma 14.1. Since this is the quotient of $\mathcal{F}_n$ by the subsheaf of sections supported on T we see that we get canonical maps $\mathcal{F}'_{n+1} \to \mathcal{F}'_n$ such that we obtain a map $(\mathcal{F}_n) \to (\mathcal{F}'_n)$ of inverse systems of coherent $\mathcal{O}_X$-modules. Property (1) holds by construction.

To prove properties (2) and (3) we may assume that $X = \mathrm{Spec}(A_0)$ is affine and A_0 has a dualizing complex. Let $I_0 \subset A_0$ be the ideal corresponding to Y. Let A, I be the I-adic completions of A_0, I_0. For later use we observe that A has a dualizing complex (Dualizing Complexes, Lemma 22.4). Let M be the finite A-module corresponding to $(\mathcal{F}_n)$, see Cohomology of Schemes, Lemma 23.1. Then $\mathcal{F}_n$ corresponds to $M_n = M/I^nM$. Recall that $\mathcal{F}'_n$ corresponds to the quotient $M'_n = M_n/H^0_T(M_n)$, see Local Cohomology, Lemma 14.1 and its proof.

Set $s = 0$ and $d = \mathrm{cd}(A, I)$. We claim that A, I, T, M, s, d satisfy assumptions (1), (3), (4), (6) of Situation 10.1. Namely, (1) and (3) are immediate from the above, (4) is the empty condition as $s = 0$, and (6) is the assumption we made in the statement of the lemma.

By Theorem 10.8 we see that $\{H^0_T(M_n)\}$ is Mittag-Leffler, that $\lim H^0_T(M_n) = H^0_T(M)$, and that $H^0_T(M)$ is killed by a power of I. Thus the limit of the short exact sequences $0 \to H^0_T(M_n) \to M_n \to M'_n \to 0$ is the short exact sequence

$$0 \to H^0_T(M) \to M \to \lim M'_n \to 0$$

Setting $M' = \lim M'_n$ we find that $\mathcal{G}_n$ corresponds to the finite A_0-module M'/I^nM'. To finish the prove we have to show that the canonical map $\{M'/I^nM'\} \to \{M'_n\}$ is a pro-isomorphism. This is equivalent to saying that $\{H^0_T(M)+I^nM\} \to \{\ker(M \to M'_n)\}$ is a pro-isomorphism. Which in turn says that $\{H^0_T(M)/H^0_T(M) \cap I^nM\} \to \{H^0_T(M_n)\}$ is a pro-isomorphism. This is true because $\{H^0_T(M_n)\}$ is Mittag-Leffler, $\lim H^0_T(M_n) = H^0_T(M)$, and $H^0_T(M)$ is killed by a power of I (so that Artin-Rees tells us that $H^0_T(M) \cap I^nM = 0$ for n large enough). □

0EJF **Lemma 21.2.** *In the situation above assume X locally has a dualizing complex. Let $T' \subset T \subset Y$ be subsets stable under specialization. Let $d \geq 0$ be an integer. Assume*

(a) *affine locally we have $X = \text{Spec}(A_0)$ and $Y = V(I_0)$ and $cd(A_0, I_0) \leq d$,*
(b) *for $y \in T$ and a nonmaximal prime $\mathfrak{p} \subset \mathcal{O}_{X,y}^\wedge$ with $V(\mathfrak{p}) \cap V(\mathcal{I}_y^\wedge) = \{\mathfrak{m}_y^\wedge\}$ we have*
$$depth_{(\mathcal{O}_{X,y})_\mathfrak{p}}((\mathcal{F}_y^\wedge)_\mathfrak{p}) > 0$$
(c) *for $y \in T'$ and for a prime $\mathfrak{p} \subset \mathcal{O}_{X,y}^\wedge$ with $\mathfrak{p} \notin V(\mathcal{I}_y^\wedge)$ and $V(\mathfrak{p}) \cap V(\mathcal{I}_y^\wedge) \neq \{\mathfrak{m}_y^\wedge\}$ we have*
$$depth_{(\mathcal{O}_{X,y})_\mathfrak{p}}((\mathcal{F}_y^\wedge)_\mathfrak{p}) \geq 1 \quad or \quad depth_{(\mathcal{O}_{X,y})_\mathfrak{p}}((\mathcal{F}_y^\wedge)_\mathfrak{p}) + \dim(\mathcal{O}_{X,y}^\wedge/\mathfrak{p}) > 1 + d$$
(d) *for $y \in T'$ and a nonmaximal prime $\mathfrak{p} \subset \mathcal{O}_{X,y}^\wedge$ with $V(\mathfrak{p}) \cap V(\mathcal{I}_y^\wedge) = \{\mathfrak{m}_y^\wedge\}$ we have*
$$depth_{(\mathcal{O}_{X,y})_\mathfrak{p}}((\mathcal{F}_y^\wedge)_\mathfrak{p}) > 1$$
(e) *if $y \rightsquigarrow y'$ is an immediate specialization and $y' \in T'$, then $y \in T$.*

Then there exists a canonical map $(\mathcal{F}_n) \to (\mathcal{F}_n'')$ of inverse systems of coherent $\mathcal{O}_X$-modules with the following properties

(1) *for $y \in T$ we have $depth(\mathcal{F}_{n,y}'') \geq 1$,*
(2) *for $y' \in T'$ we have $depth(\mathcal{F}_{n,y'}'') \geq 2$,*
(3) *$(\mathcal{F}_n'')$ is isomorphic as a pro-system to an object $(\mathcal{H}_n)$ of $Coh(X, \mathcal{I})$,*
(4) *the induced morphism $(\mathcal{F}_n) \to (\mathcal{H}_n)$ of $Coh(X, \mathcal{I})$ has kernel and cokernel annihilated by a power of $\mathcal{I}$.*

Proof. As in Lemma 21.1 and its proof for every n we let $\mathcal{F}_n \to \mathcal{F}_n'$ be the surjection constructed in Local Cohomology, Lemma 14.1. Next, we let $\mathcal{F}_n' \to \mathcal{F}_n''$ be the injection constructed in Local Cohomology, Lemma 14.5 and its proof. The constructions show that we get canonical maps $\mathcal{F}_{n+1}'' \to \mathcal{F}_n''$ such that we obtain maps
$$(\mathcal{F}_n) \longrightarrow (\mathcal{F}_n') \longrightarrow (\mathcal{F}_n'')$$
of inverse systems of coherent $\mathcal{O}_X$-modules. Properties (1) and (2) hold by construction.

To prove properties (3) and (4) we may assume that $X = \text{Spec}(A_0)$ is affine and A_0 has a dualizing complex. Let $I_0 \subset A_0$ be the ideal corresponding to Y. Let A, I be the I-adic completions of A_0, I_0. For later use we observe that A has a dualizing complex (Dualizing Complexes, Lemma 22.4). Let M be the finite A-module corresponding to $(\mathcal{F}_n)$, see Cohomology of Schemes, Lemma 23.1. Then $\mathcal{F}_n$ corresponds to $M_n = M/I^nM$. Recall that $\mathcal{F}_n'$ corresponds to the quotient $M_n' = M_n/H_T^0(M_n)$. Also, recall that $M' = \lim M_n'$ is the quotient of M by $H_T^0(M)$ and that $\{M_n'\}$ and $\{M'/I^nM'\}$ are isomorphic as pro-systems. Finally, we see that $\mathcal{F}_n''$ corresponds to an extension
$$0 \to M_n' \to M_n'' \to H_{T'}^1(M_n') \to 0$$
see proof of Local Cohomology, Lemma 14.5.

Set $s = 1$. We claim that A, I, T', M', s, d satisfy assumptions (1), (3), (4), (6) of Situation 10.1. Namely, (1) and (3) are immediate, (4) is implied by (c), and (6) follows from (d). We omit the details of the verification (c) $\Rightarrow$ (4).

By Theorem 10.8 we see that $\{H^1_{T'}(M'/I^nM')\}$ is Mittag-Leffler, that $H^1_{T'}(M') = \lim H^1_{T'}(M'/I^nM')$, and that $H^1_{T'}(M')$ is killed by a power of I. We deduce $\{H^1_{T'}(M'_n)\}$ is Mittag-Leffler and $H^1_{T'}(M') = \lim H^1_{T'}(M'_n)$. Thus the limit of the short exact sequences displayed above is the short exact sequence

$$0 \to M' \to \lim M''_n \to H^1_{T'}(M') \to 0$$

Set $M'' = \lim M''_n$. It follows from Local Cohomology, Proposition 10.1 that $H^1_{T'}(M')$ and hence M'' are finite A-modules. Thus we find that $\mathcal{H}_n$ corresponds to the finite A_0-module M''/I^nM''. To finish the prove we have to show that the canonical map $\{M''/I^nM''\} \to \{M''_n\}$ is a pro-isomorphism. Since we already know that $\{M'/I^nM'\}$ is pro-isomorphic to $\{M'_n\}$ the reader verifies (omitted) this is equivalent to asking $\{H^1_{T'}(M')/I^nH^1_{T'}(M')\} \to \{H^1_{T'}(M'_n)\}$ to be a pro-isomorphism. This is true because $\{H^1_{T'}(M'_n)\}$ is Mittag-Leffler, $H^1_{T'}(M') = \lim H^1_{T'}(M'_n)$, and $H^1_{T'}(M')$ is killed by a power of I. □

0EJG **Lemma 21.3.** *In Situation 16.1 assume that A has a dualizing complex. Let $d \geq cd(A, I)$. Let $(\mathcal{F}_n)$ be an object of $Coh(U, I\mathcal{O}_U)$. Assume $(\mathcal{F}_n)$ satisfies the $(2, 2+d)$-inequalities, see Definition 19.1. Then there exists a canonical map $(\mathcal{F}_n) \to (\mathcal{F}''_n)$ of inverse systems of coherent $\mathcal{O}_U$-modules with the following properties*

1. *if $depth(\mathcal{F}''_{n,y}) + \delta^Y_Z(y) \geq 3$ for all $y \in U \cap Y$,*
2. *$(\mathcal{F}''_n)$ is isomorphic as a pro-system to an object $(\mathcal{H}_n)$ of $Coh(U, I\mathcal{O}_U)$,*
3. *the induced morphism $(\mathcal{F}_n) \to (\mathcal{H}_n)$ of $Coh(U, I\mathcal{O}_U)$ has kernel and cokernel annihilated by a power of I,*
4. *the modules $H^0(U, \mathcal{F}''_n)$ and $H^1(U, \mathcal{F}''_n)$ are finite A-modules for all n.*

Proof. The existence and properties (2), (3), (4) follow immediately from Lemma 21.2 applied to U, $U \cap Y$, $T = \{y \in U \cap Y : \delta^Y_Z(y) \leq 2\}$, $T' = \{y \in U \cap Y : \delta^Y_Z(y) \leq 1\}$, and $(\mathcal{F}_n)$. The finiteness of the modules $H^0(U, \mathcal{F}''_n)$ and $H^1(U, \mathcal{F}''_n)$ follows from Local Cohomology, Lemma 11.1 and the elementary properties of the function $\delta^Y_Z(-)$ proved in Lemma 18.1. □

22. Algebraization of coherent formal modules, V

0EJH In this section we prove our most general results on algebraization of coherent formal modules. We first prove it in case the ideal has cohomological dimension 1. Then we apply this to a blowup to prove a more general result.

0EJI **Lemma 22.1.** *In Situation 16.1 let $(\mathcal{F}_n)$ be an object of $Coh(U, I\mathcal{O}_U)$. Assume*

1. *A has a dualizing complex and $cd(A, I) = 1$,*
2. *$(\mathcal{F}_n)$ is pro-isomorphic to an inverse system $(\mathcal{F}''_n)$ of coherent $\mathcal{O}_U$-modules such that $depth(\mathcal{F}''_{n,y}) + \delta^Y_Z(y) \geq 3$ for all $y \in U \cap Y$.*

Then $(\mathcal{F}_n)$ extends canonically to X, see Definition 16.7.

Proof. We will check hypotheses (a), (b), and (c) of Lemma 16.10. Before we start, let us point out that the modules $H^0(U, \mathcal{F}''_n)$ and $H^1(U, \mathcal{F}''_n)$ are finite A-modules for all n by Local Cohomology, Lemma 11.1.

Observe that for each $p \geq 0$ the limit topology on $\lim H^p(U, \mathcal{F}_n)$ is the I-adic topology by Lemma 4.5. In particular, hypothesis (b) holds.

We know that $M = \lim H^0(U, \mathcal{F}_n)$ is an A-module whose limit topology is the I-adic topology. Thus, given n, the module M/I^nM is a subquotient of $H^0(U, \mathcal{F}_N)$

for some $N \gg n$. Since the inverse system $\{H^0(U, \mathcal{F}_N)\}$ is pro-isomorphic to an inverse system of finite A-modules, namely $\{H^0(U, \mathcal{F}''_N)\}$, we conclude that M/I^nM is finite. It follows that M is finite, see Algebra, Lemma 95.12. In particular hypothesis (c) holds.

For each $n \geq 0$ let us write $Ob_n = \lim_N H^1(U, I^n\mathcal{F}_N)$. A special case is $Ob = Ob_0 = \lim_N H^1(U, \mathcal{F}_N)$. Arguing exactly as in the previous paragraph we find that Ob is a finite A-module. (In fact, we also know that Ob/IOb is annihilated by a power of $\mathfrak{a}$, but it seems somewhat difficult to use this.)

We set $\mathcal{F} = \lim \mathcal{F}_n$, we pick generators $f_1, \ldots, f_r$ of I, we pick $c \geq 1$, and we choose $\Phi_\mathcal{F}$ as in Lemma 4.4. We will use the results of Lemma 2.4 without further mention. In particular, for each $n \geq 1$ there are maps

$$\delta_n : H^0(U, \mathcal{F}_n) \longrightarrow H^1(U, I^n\mathcal{F}) \longrightarrow Ob_n$$

The first comes from the short exact sequence $0 \to I^n\mathcal{F} \to \mathcal{F} \to \mathcal{F}_n \to 0$ and the second from $I^n\mathcal{F} = \lim I^n\mathcal{F}_N$. We will later use that if $\delta_n(s) = 0$ for $s \in H^0(U, \mathcal{F}_n)$ then we can for each $n' \geq n$ find $s' \in H^0(U, \mathcal{F}_{n'})$ mapping to s. Observe that there are commutative diagrams

$$\begin{tikzcd}
H^0(U, \mathcal{F}_{nc}) \arrow[r] \arrow[dd] & H^1(U, I^{nc}\mathcal{F}) \arrow[dd] \arrow[dr, "\Phi_\mathcal{F}"] & \\
& & \bigoplus_{e_1+\ldots+e_r=n} H^1(U, \mathcal{F}) \cdot T_1^{e_1} \ldots T_r^{e_r} \arrow[dl] \\
H^0(U, \mathcal{F}_n) \arrow[r] & H^1(U, I^n\mathcal{F}) &
\end{tikzcd}$$

We conclude that the obstruction map $H^0(U, \mathcal{F}_n) \to Ob_n$ sends the image of $H^0(U, \mathcal{F}_{nc}) \to H^0(U, \mathcal{F}_n)$ into the submodule

$$Ob'_n = \operatorname{Im}\left(\bigoplus\nolimits_{e_1+\ldots+e_r=n} Ob \cdot T_1^{e_1} \ldots T_r^{e_r} \to Ob_n\right)$$

where on the summand $Ob \cdot T_1^{e_1} \ldots T_r^{e_r}$ we use the map on cohomology coming from the reductions modulo powers of I of the multiplication map $f_1^{e_1} \ldots f_r^{e_r} : \mathcal{F} \to I^n\mathcal{F}$. By construction

$$\bigoplus\nolimits_{n \geq 0} Ob'_n$$

is a finite graded module over the Rees algebra $\bigoplus_{n\geq 0} I^n$. For each n we set

$$M_n = \{s \in H^0(U, \mathcal{F}_n) \mid \delta_n(s) \in Ob'_n\}$$

Observe that $\{M_n\}$ is an inverse system and that $f_j : \mathcal{F}_n \to \mathcal{F}_{n+1}$ on global sections maps M_n into M_{n+1}. By exactly the same argument as in the proof of Lemma 2.1 we find that $\{M_n\}$ is ML. Namely, because the Rees algebra is Noetherian we can choose a finite number of homogeneous generators of the form $\delta_{n_j}(z_j)$ with $z_j \in M_{n_j}$ for the graded submodule $\bigoplus_{n\geq 0} \operatorname{Im}(M_n \to Ob'_n)$. Then if $k = \max(n_j)$ we find that for $n \geq k$ and any $z \in M_n$ we can find $a_j \in I^{n-n_j}$ such that $z - \sum a_j z_j$ is in the kernel of δ_n and hence in the image of $M_{n'}$ for all $n' \geq n$ (because the vanishing of δ_n means that we can lift $z - \sum a_j z_j$ to an element $z' \in H^0(U, \mathcal{F}_{n'c})$ for all $n' \geq n$ and then the image of z' in $H^0(U, \mathcal{F}_{n'})$ is in $M_{n'}$ by what we proved above). Thus $\operatorname{Im}(M_n \to M_{n-k}) = \operatorname{Im}(M_{n'} \to M_{n-k})$ for all $n' \geq n$.

Choose n. By the Mittag-Leffler property of $\{M_n\}$ we just established we can find an $n' \geq n$ such that the image of $M_{n'} \to M_n$ is the same as the image of $M' \to M_n$. By the above we see that the image of $M' \to M_n$ contains the image of $H^0(U, \mathcal{F}_{n'c}) \to H^0(U, \mathcal{F}_n)$. Thus we see that $\{M_n\}$ and $\{H^0(U, \mathcal{F}_n)\}$ are pro-isomorphic. Therefore $\{H^0(U, \mathcal{F}_n)\}$ has ML and we finally conclude that hypothesis (a) holds. This concludes the proof. □

0EJJ **Proposition 22.2** (Algebraization in cohomological dimension 1)**.** *In Situation 16.1 let $(\mathcal{F}_n)$ be an object of $Coh(U, I\mathcal{O}_U)$. Assume*

(1) *A has a dualizing complex and $cd(A, I) = 1$,*
(2) *$(\mathcal{F}_n)$ satisfies the $(2, 3)$-inequalities, see Definition 19.1.*

Then $(\mathcal{F}_n)$ extends to X. In particular, if A is I-adically complete, then $(\mathcal{F}_n)$ is the completion of a coherent $\mathcal{O}_U$-module.

The local case of this result is [Ray75, IV Corollaire 2.9].

Proof. By Lemma 17.1 we may replace $(\mathcal{F}_n)$ by the object $(\mathcal{H}_n)$ of $Coh(U, I\mathcal{O}_U)$ found in Lemma 21.3. Thus we may assume that $(\mathcal{F}_n)$ is pro-isomorphic to a inverse system $(\mathcal{F}''_n)$ with the properties mentioned in Lemma 21.3. In Lemma 22.1 we proved that $(\mathcal{F}_n)$ canonically extends to X. The final statement follows from Lemma 16.8. □

0EJK **Lemma 22.3.** *In Situation 16.1 let $(\mathcal{F}_n)$ be an object of $Coh(U, I\mathcal{O}_U)$. Assume*

(1) *A has a dualizing complex,*
(2) *all fibres of the blowing up $b : X' \to X$ of I have dimension $\leq d - 1$,*
(3) *one of the following is true*
   (a) *$(\mathcal{F}_n)$ satisfies the $(d + 1, d + 2)$-inequalities (Definition 19.1), or*
   (b) *for $y \in U \cap Y$ and a prime $\mathfrak{p} \subset \mathcal{O}^\wedge_{X,y}$ with $\mathfrak{p} \not\in V(I\mathcal{O}^\wedge_{X,y})$ we have*

$$depth((\mathcal{F}^\wedge_y)_\mathfrak{p}) + \dim(\mathcal{O}^\wedge_{X,y}/\mathfrak{p}) + \delta^Y_Z(y) > d + 2$$

Then $(\mathcal{F}_n)$ extends to X.

Proof. Let $Y' \subset X'$ be the exceptional divisor. Let $Z' \subset Y'$ be the inverse image of $Z \subset Y$. Then $U' = X' \setminus Z'$ is the inverse image of U. With $\delta^{Y'}_{Z'}$ as in (18.0.1) we set

$$T' = \{y' \in Y' \mid \delta^{Y'}_{Z'}(y') = 1 \text{ or } 2\} \subset T = \{y' \in Y' \mid \delta^{Y'}_{Z'}(y') = 1\}$$

These are specialization stable subsets of $U' \cap Y' = Y' \setminus Z'$. Consider the object $(b|^*_{U'}\mathcal{F}_n)$ of $Coh(U', I\mathcal{O}_{U'})$, see Cohomology of Schemes, Lemma 23.9. For $y' \in U' \cap Y'$ let us denote

$$\mathcal{F}^\wedge_{y'} = \lim (b|^*_{U'}\mathcal{F}_n)_{y'}$$

the "stalk" of this pullback at y'. We claim that conditions (a), (b), (c), (d), and (e) of Lemma 21.2 hold for the object $(b|^*_{U'}\mathcal{F}_n)$ on U' with d replaced by 1 and the subsets $T' \subset T \subset U' \cap Y'$. Condition (a) holds because Y' is an effective Cartier divisor and hence locally cut out by 1 equation. Condition (e) holds by Lemma 18.1 parts (1) and (2). To prove (b), (c), and (d) we need some preparation.

Let $y' \in U' \cap Y'$ and let $\mathfrak{p}' \subset \mathcal{O}^\wedge_{X',y'}$ be a prime ideal not contained in $V(I\mathcal{O}^\wedge_{X',y'})$. Denote $y = b(y') \in U \cap Y$. Choose $f \in I$ such that y' is contained in the spectrum of the affine blowup algebra $A[\frac{I}{f}]$, see Divisors, Lemma 32.2. For any A-algebra B denote $B' = B[\frac{IB}{f}]$ the corresponding affine blowup algebra. Denote I-adic completion by $^\wedge$. By our choice of f we get a ring map $(\mathcal{O}^\wedge_{X,y})' \to \mathcal{O}^\wedge_{X',y'}$. If we let $\mathfrak{q}' \subset (\mathcal{O}^\wedge_{X,y})'$ be the inverse image of $\mathfrak{m}^\wedge_{y'}$, then we see that $((\mathcal{O}^\wedge_{X,y})'_{\mathfrak{q}'})^\wedge = \mathcal{O}^\wedge_{X',y'}$.

Let $\mathfrak p \subset \mathcal{O}^\wedge_{X,y}$ be the corresponding prime. At this point we have a commutative diagram

$$\begin{tikzcd}
\mathcal{O}^\wedge_{X,y} \arrow[r] \arrow[d] & (\mathcal{O}^\wedge_{X,y})' \arrow[r] \arrow[d, "\alpha"'] & (\mathcal{O}^\wedge_{X,y})'_{\mathfrak q'} \arrow[r, "\beta"'] \arrow[d] & \mathcal{O}^\wedge_{X',y'} \arrow[d] \\
\mathcal{O}^\wedge_{X,y}/\mathfrak p \arrow[r] & (\mathcal{O}^\wedge_{X,y}/\mathfrak p)' \arrow[r] & (\mathcal{O}^\wedge_{X,y}/\mathfrak p)'_{\mathfrak q'} \arrow[r, "\gamma"] & ((\mathcal{O}^\wedge_{X,y}/\mathfrak p)'_{\mathfrak q'})^\wedge \arrow[d] \\
 & & & \mathcal{O}^\wedge_{X',y'}/\mathfrak p'
\end{tikzcd}$$

whose vertical arrows are surjective. By More on Algebra, Lemma 42.1 and the dimension formula (Algebra, Lemma 112.1) we have

$$\dim(((\mathcal{O}^\wedge_{X,y}/\mathfrak p)'_{\mathfrak q'})^\wedge) = \dim((\mathcal{O}^\wedge_{X,y}/\mathfrak p)'_{\mathfrak q'}) = \dim(\mathcal{O}^\wedge_{X,y}/\mathfrak p) - \text{trdeg}(\kappa(y')/\kappa(y))$$

Tracing through the definitions of pullbacks, stalks, localizations, and completions we find

$$(\mathcal{F}^\wedge_y)_{\mathfrak p} \otimes_{(\mathcal{O}^\wedge_{X,y})_{\mathfrak p}} (\mathcal{O}^\wedge_{X',y'})_{\mathfrak p'} = (\mathcal{F}^\wedge_{y'})_{\mathfrak p'}$$

Details omitted. The ring maps β and γ in the diagram are flat with Gorenstein (hence Cohen-Macaulay) fibres, as these are completions of rings having a dualizing complex. See Dualizing Complexes, Lemmas 23.1 and 23.2 and the discussion in More on Algebra, Section 50. Observe that $(\mathcal{O}^\wedge_{X,y})_{\mathfrak p} = (\mathcal{O}^\wedge_{X,y})'_{\tilde{\mathfrak p}}$ where $\tilde{\mathfrak p}$ is the kernel of α in the diagram. On the other hand, $(\mathcal{O}^\wedge_{X,y})'_{\tilde{\mathfrak p}} \to (\mathcal{O}^\wedge_{X',y'})_{\mathfrak p'}$ is flat with CM fibres by the above. Whence $(\mathcal{O}^\wedge_{X,y})_{\mathfrak p} \to (\mathcal{O}^\wedge_{X',y'})_{\mathfrak p'}$ is flat with CM fibres. Using Algebra, Lemma 157.1 we see that

$$\text{depth}((\mathcal{F}^\wedge_{y'})_{\mathfrak p'}) = \text{depth}((\mathcal{F}^\wedge_y)_{\mathfrak p}) + \dim(F_{\mathfrak r})$$

where F is the generic formal fibre of $(\mathcal{O}^\wedge_{X,y}/\mathfrak p)'_{\mathfrak q'}$ and $\mathfrak r$ is the prime corresponding to $\mathfrak p'$. Since $(\mathcal{O}^\wedge_{X,y}/\mathfrak p)'_{\mathfrak q'}$ is a universally catenary local domain, its I-adic completion is equidimensional and (universally) catenary by Ratliff's theorem (More on Algebra, Proposition 95.5). It then follows that

$$\dim(((\mathcal{O}^\wedge_{X,y}/\mathfrak p)'_{\mathfrak q'})^\wedge) = \dim(F_{\mathfrak r}) + \dim(\mathcal{O}^\wedge_{X',y'}/\mathfrak p')$$

Combined with Lemma 18.2 we get

0EJL (22.3.1)
$$\begin{aligned}
&\text{depth}((\mathcal{F}^\wedge_{y'})_{\mathfrak p'}) + \delta^{Y'}_{Z'}(y') \\
&= \text{depth}((\mathcal{F}^\wedge_y)_{\mathfrak p}) + \dim(F_{\mathfrak r}) + \delta^{Y'}_{Z'}(y') \\
&\geq \text{depth}((\mathcal{F}^\wedge_y)_{\mathfrak p}) + \delta^Y_Z(y) + \dim(F_{\mathfrak r}) + \text{trdeg}(\kappa(y')/\kappa(y)) - (d-1) \\
&= \text{depth}((\mathcal{F}^\wedge_y)_{\mathfrak p}) + \delta^Y_Z(y) - (d-1) + \dim(\mathcal{O}^\wedge_{X,y}/\mathfrak p) - \dim(\mathcal{O}^\wedge_{X',y'}/\mathfrak p')
\end{aligned} \tag{22.3.1}$$

Please keep in mind that $\dim(\mathcal{O}^\wedge_{X,y}/\mathfrak p) \geq \dim(\mathcal{O}^\wedge_{X',y'}/\mathfrak p')$. Rewriting this we get

0EJM (22.3.2)
$$\begin{aligned}
&\text{depth}((\mathcal{F}^\wedge_{y'})_{\mathfrak p'}) + \dim(\mathcal{O}^\wedge_{X',y'}/\mathfrak p') + \delta^{Y'}_{Z'}(y') \\
&\geq \text{depth}((\mathcal{F}^\wedge_y)_{\mathfrak p}) + \dim(\mathcal{O}^\wedge_{X,y}/\mathfrak p) + \delta^Y_Z(y) - (d-1)
\end{aligned} \tag{22.3.2}$$

This inequality will allow us to check the remaning conditions.

Conditions (b) and (d) of Lemma 21.2. Assume $V(\mathfrak{p}') \cap V(I\mathcal{O}^\wedge_{X',y'}) = \{\mathfrak{m}^\wedge_{y'}\}$. This implies that $\dim(\mathcal{O}^\wedge_{X',y'}/\mathfrak{p}') = 1$ because Z' is an effective Cartier divisor. The combination of (b) and (d) is equivalent with

$$\text{depth}((\mathcal{F}^\wedge_{y'})_{\mathfrak{p}'}) + \delta^{Y'}_{Z'}(y') > 2$$

If $(\mathcal{F}_n)$ satisfies the inequalities in (3)(b) then we immediately conclude this is true by applying (22.3.2). If $(\mathcal{F}_n)$ satisfies (3)(a), i.e., the $(d+1, d+2)$-inequalities, then we see that in any case

$$\text{depth}((\mathcal{F}^\wedge_y)_{\mathfrak{p}}) + \delta^Y_Z(y) \geq d+1 \quad \text{or} \quad \text{depth}((\mathcal{F}^\wedge_y)_{\mathfrak{p}}) + \dim(\mathcal{O}^\wedge_{X,y}/\mathfrak{p}) + \delta^Y_Z(y) > d+2$$

Looking at (22.3.1) and (22.3.2) above this gives what we want except possibly if $\dim(\mathcal{O}^\wedge_{X,y}/\mathfrak{p}) = 1$. However, if $\dim(\mathcal{O}^\wedge_{X,y}/\mathfrak{p}) = 1$, then we have $V(\mathfrak{p}) \cap V(I\mathcal{O}^\wedge_{X,y}) = \{\mathfrak{m}^\wedge_y\}$ and we see that actually

$$\text{depth}((\mathcal{F}^\wedge_y)_{\mathfrak{p}}) + \delta^Y_Z(y) > d+1$$

as $(\mathcal{F}_n)$ satisfies the $(d+1, d+2)$-inequalities and we conclude again.

Condition (c) of Lemma 21.2. Assume $V(\mathfrak{p}') \cap V(I\mathcal{O}^\wedge_{X',y'}) \neq \{\mathfrak{m}^\wedge_{y'}\}$. Then condition (c) is equivalent to

$$\text{depth}((\mathcal{F}^\wedge_{y'})_{\mathfrak{p}'}) + \delta^{Y'}_{Z'}(y') \geq 2 \quad \text{or} \quad \text{depth}((\mathcal{F}^\wedge_{y'})_{\mathfrak{p}'}) + \dim(\mathcal{O}^\wedge_{X',y'}/\mathfrak{p}') + \delta^{Y'}_{Z'}(y') > 3$$

If $(\mathcal{F}_n)$ satisfies the inequalities in (3)(b) then we see the second of the two displayed inequalities holds true by applying (22.3.2). If $(\mathcal{F}_n)$ satisfies (3)(a), i.e., the $(d+1, d+2)$-inequalities, then this follows immediately from (22.3.1) and (22.3.2). This finishes the proof of our claim.

Choose $(b|^*_{U'}\mathcal{F}_n) \to (\mathcal{F}''_n)$ and $(\mathcal{H}_n)$ in $\textit{Coh}(U', I\mathcal{O}_{U'})$ as in Lemma 21.2. For any affine open $W \subset X'$ observe that $\delta^{W\cap Y'}_{W\cap Z'}(y') \geq \delta^{Y'}_{Z'}(y')$ by Lemma 18.1 part (7). Hence we see that $(\mathcal{H}_n|_W)$ satisfies the assumptions of Lemma 22.1. Thus $(\mathcal{H}_n|_W)$ extends canonically to W. Let $(\mathcal{G}_{W,n})$ in $\textit{Coh}(W, I\mathcal{O}_W)$ be the canonical extension as in Lemma 16.8. By Lemma 16.9 we see that for $W' \subset W$ there is a unique isomorphism

$$(\mathcal{G}_{W,n}|_{W'}) \longrightarrow (\mathcal{G}_{W',n})$$

compatible with the given isomorphisms $(\mathcal{G}_{W,n}|_{W\cap U}) \cong (\mathcal{H}_n|_{W\cap U})$. We conclude that there exists an object $(\mathcal{G}_n)$ of $\textit{Coh}(X', I\mathcal{O}_{X'})$ whose restriction to U is isomorphic to $(\mathcal{H}_n)$.

If A is I-adically complete we can finish the proof as follows. By Grothedieck's existence theorem (Cohomology of Schemes, Lemma 24.3) we see that $(\mathcal{G}_n)$ is the completion of a coherent $\mathcal{O}_{X'}$-module. Then by Cohomology of Schemes, Lemma 23.6 we see that $(b|^*_{U'}\mathcal{F}_n)$ is the completion of a coherent $\mathcal{O}_{U'}$-module $\mathcal{F}'$. By Cohomology of Schemes, Lemma 25.3 we see that there is a map

$$(\mathcal{F}_n) \longrightarrow ((b|_{U'})_*\mathcal{F}')^\wedge$$

whose kernel and cokernel is annihilated by a power of I. Then finally, we win by applying Lemma 17.1.

If A is not complete, then, before starting the proof, we may replace A by its completion, see Lemma 16.6. After completion the assumptions still hold: this is immediate for condition (3), follows from Dualizing Complexes, Lemma 22.4 for condition (1), and from Divisors, Lemma 32.3 for condition (2). Thus the complete case implies the general case. $\square$

0EJN **Proposition 22.4** (Algebraization for ideals with few generators)**.** *In Situation 16.1 let $(\mathcal{F}_n)$ be an object of $Coh(U, I\mathcal{O}_U)$. Assume*

1. *A has a dualizing complex,*
2. *$V(I) = V(f_1, \ldots, f_d)$ for some $d \geq 1$ and $f_1, \ldots, f_d \in A$,*
3. *one of the following is true*
   (a) *$(\mathcal{F}_n)$ satisfies the $(d+1, d+2)$-inequalities (Definition 19.1), or*
   (b) *for $y \in U \cap Y$ and a prime $\mathfrak{p} \subset \mathcal{O}_{X,y}^\wedge$ with $\mathfrak{p} \notin V(I\mathcal{O}_{X,y}^\wedge)$ we have*

$$depth((\mathcal{F}_y^\wedge)_\mathfrak{p}) + \dim(\mathcal{O}_{X,y}^\wedge/\mathfrak{p}) + \delta_Z^Y(y) > d + 2$$

Then $(\mathcal{F}_n)$ extends to X. In particular, if A is I-adically complete, then $(\mathcal{F}_n)$ is the completion of a coherent $\mathcal{O}_U$-module.

Proof. We may assume $I = (f_1, \ldots, f_d)$, see Cohomology of Schemes, Lemma 23.11. Then we see that all fibres of the blowup of X in I have dimension at most $d - 1$. Thus we get the extension from Lemma 22.3. The final statement follows from Lemma 16.3. $\square$

Please compare the next lemma with Remarks 16.12, 20.2, 20.7, and 23.2.

0EJP **Lemma 22.5.** *In Situation 16.1 let $(\mathcal{F}_n)$ be an object of $Coh(U, I\mathcal{O}_U)$. Assume*

1. *A is a local ring which has a dualizing complex,*
2. *all irreducible components of X have the same dimension,*
3. *the scheme $X \setminus Y$ is Cohen-Macaulay,*
4. *I is generated by d elements,*
5. *$\dim(X) - \dim(Z) > d + 2$, and*
6. *for $y \in U \cap Y$ the module $\mathcal{F}_y^\wedge$ is finite locally free outside $V(I\mathcal{O}_{X,y}^\wedge)$, for example if $\mathcal{F}_n$ is a finite locally free $\mathcal{O}_U/I^n\mathcal{O}_U$-module.*

Then $(\mathcal{F}_n)$ extends to X. In particular if A is I-adically complete, then $(\mathcal{F}_n)$ is the completion of a coherent $\mathcal{O}_U$-module.

Proof. We will show that the hypotheses (1), (2), (3)(b) of Proposition 22.4 are satisfied. This is clear for (1) and (2).

Let $y \in U \cap Y$ and let $\mathfrak{p}$ be a prime $\mathfrak{p} \subset \mathcal{O}_{X,y}^\wedge$ with $\mathfrak{p} \notin V(I\mathcal{O}_{X,y}^\wedge)$. The last condition shows that $\text{depth}((\mathcal{F}_y^\wedge)_\mathfrak{p}) = \text{depth}((\mathcal{O}_{X,y}^\wedge)_\mathfrak{p})$. Since $X \setminus Y$ is Cohen-Macaulay we see that $(\mathcal{O}_{X,y}^\wedge)_\mathfrak{p}$ is Cohen-Macaulay. Thus we see that

$$\begin{aligned} &\text{depth}((\mathcal{F}_y^\wedge)_\mathfrak{p}) + \dim(\mathcal{O}_{X,y}^\wedge/\mathfrak{p}) + \delta_Z^Y(y) \\ &= \dim((\mathcal{O}_{X,y}^\wedge)_\mathfrak{p}) + \dim(\mathcal{O}_{X,y}^\wedge/\mathfrak{p}) + \delta_Z^Y(y) \\ &= \dim(\mathcal{O}_{X,y}^\wedge) + \delta_Z^Y(y) \end{aligned}$$

The final equality because $\mathcal{O}_{X,y}$ is equidimensional by the second condition. Let $\delta(y) = \dim(\overline{\{y\}})$. This is a dimension function as A is a catenary local ring. By Lemma 18.1 we have $\delta_Z^Y(y) \geq \delta(y) - \dim(Z)$. Since X is equidimensional we get

$$\dim(\mathcal{O}_{X,y}^\wedge) + \delta_Z^Y(y) \geq \dim(\mathcal{O}_{X,y}^\wedge) + \delta(y) - \dim(Z) = \dim(X) - \dim(Z)$$

Thus we get the desired inequality and we win. $\square$

0EJQ **Remark 22.6.** We are unable to prove or disprove the analogue of Proposition 22.4 where the assumption that I has d generators is replaced with the assumption $\text{cd}(A, I) \leq d$. If you know a proof or have a counter example, please email stacks.project@gmail.com. Another obvious question is to what extend the conditions in Proposition 22.4 are necessary.

23. Algebraization of coherent formal modules, VI

0EJR In this section we add a few more easier to prove cases.

0EJS **Proposition 23.1.** *In Situation 16.1 let $(\mathcal{F}_n)$ be an object of $Coh(U, I\mathcal{O}_U)$. Assume*

1. *there exist $f_1, \ldots, f_d \in I$ such that for $y \in U \cap Y$ the ideal $I\mathcal{O}_{X,y}$ is generated by $f_1, \ldots, f_d$ and $f_1, \ldots, f_d$ form a $\mathcal{F}_y^\wedge$-regular sequence,*
2. *$H^0(U, \mathcal{F}_1)$ and $H^1(U, \mathcal{F}_1)$ are finite A-modules.*

Then $(\mathcal{F}_n)$ extends canonically to X. In particular, if A is complete, then $(\mathcal{F}_n)$ is the completion of a coherent $\mathcal{O}_U$-module.

Proof. We will prove this by verifying hypotheses (a), (b), and (c) of Lemma 16.10. For every n we have a short exact sequence

$$0 \to I^n\mathcal{F}_{n+1} \to \mathcal{F}_{n+1} \to \mathcal{F}_n \to 0$$

Since $f_1, \ldots, f_d$ forms a regular sequence (and hence quasi-regular, see Algebra, Lemma 68.2) on each of the "stalks" $\mathcal{F}_y^\wedge$ and since we have $I\mathcal{F}_n = (f_1, \ldots, f_d)\mathcal{F}_n$ for all n, we find that

$$I^n\mathcal{F}_{n+1} = \bigoplus\nolimits_{e_1+\ldots+e_d=n} \mathcal{F}_1 \cdot f_1^{e_1} \ldots f_d^{e_d}$$

by checking on stalks. Using the assumption of finiteness of $H^0(U, \mathcal{F}_1)$ and induction, we first conclude that $M_n = H^0(U, \mathcal{F}_n)$ is a finite A-module for all n. In this way we see that condition (c) of Lemma 16.10 holds. We also see that

$$\bigoplus\nolimits_{n\geq 0} H^1(U, I^n\mathcal{F}_{n+1})$$

is a finite graded $R = \bigoplus I^n/I^{n+1}$-module. By Lemma 2.1 we conclude that condition (a) of Lemma 16.10 is satisfied. Finally, condition (b) of Lemma 16.10 is satisfied because $\bigoplus H^0(U, I^n\mathcal{F}_{n+1})$ is a finite graded R-module and we can apply Lemma 2.3. $\square$

0EJT **Remark 23.2.** In the situation of Proposition 23.1 if we assume A has a dualizing complex, then the condition that $H^0(U, \mathcal{F}_1)$ and $H^1(U, \mathcal{F}_1)$ are finite is equivalent to

$$\text{depth}(\mathcal{F}_{1,y}) + \dim(\mathcal{O}_{\overline{\{y\}},z}) > 2$$

for all $y \in U \cap Y$ and $z \in Z \cap \overline{\{y\}}$. See Local Cohomology, Lemma 11.1. This holds for example if $\mathcal{F}_1$ is a finite locally free $\mathcal{O}_{U\cap Y}$-module, Y is (S_2), and $\text{codim}(Z', Y') \geq 3$ for every pair of irreducible components Y' of Y, Z' of Z with $Z' \subset Y'$.

0EJU **Proposition 23.3.** *In Situation 16.1 let $(\mathcal{F}_n)$ be an object of $Coh(U, I\mathcal{O}_U)$. Assume there is Noetherian local ring $(R, \mathfrak{m})$ and a ring map $R \to A$ such that*

1. *$I = \mathfrak{m}A$,*
2. *for $y \in U \cap Y$ the stalk $\mathcal{F}_y^\wedge$ is R-flat,*
3. *$H^0(U, \mathcal{F}_1)$ and $H^1(U, \mathcal{F}_1)$ are finite A-modules.*

Then $(\mathcal{F}_n)$ extends canonically to X. In particular, if A is complete, then $(\mathcal{F}_n)$ is the completion of a coherent $\mathcal{O}_U$-module.

Proof. The proof is exactly the same as the proof of Proposition 23.1. Namely, if $\kappa = R/\mathfrak{m}$ then for $n \geq 0$ there is an isomorphism

$$I^n\mathcal{F}_{n+1} \cong \mathcal{F}_1 \otimes_\kappa \mathfrak{m}^n/\mathfrak{m}^{n+1}$$

and the right hand side is a finite direct sum of copies of $\mathcal{F}_1$. This can be checked by looking at stalks. Everything else is exactly the same. $\square$

0EJV **Remark 23.4.** Proposition 23.3 is a local version of [Bar10, Theorem 2.10 (i)]. It is straightforward to deduce the global results from the local one; we will sketch the argument. Namely, suppose $(R, \mathfrak{m})$ is a complete Noetherian local ring and $X \to \mathrm{Spec}(R)$ is a proper morphism. For $n \geq 1$ set $X_n = X \times_{\mathrm{Spec}(R)} \mathrm{Spec}(R/\mathfrak{m}^n)$. Let $Z \subset X_1$ be a closed subset of the special fibre. Set $U = X \setminus Z$ and denote $j : U \to X$ the inclusion morphism. Suppose given an object

$$(\mathcal{F}_n) \text{ of } \mathit{Coh}(U, \mathfrak{m}\mathcal{O}_U)$$

which is flat over R in the sense that $\mathcal{F}_n$ is flat over $R/\mathfrak{m}^n$ for all n. Assume that $j_*\mathcal{F}_1$ and $R^1j_*\mathcal{F}_1$ are coherent modules. Then affine locally on X we get a canonical extension of $(\mathcal{F}_n)$ by Proposition 23.3 and formation of this extension commutes with localization (by Lemma 16.11). Thus we get a canonical global object $(\mathcal{G}_n)$ of $\mathit{Coh}(X, \mathfrak{m}\mathcal{O}_X)$ whose restriction of U is $(\mathcal{F}_n)$. By Grothendieck's existence theorem (Cohomology of Schemes, Proposition 25.4) we see there exists a coherent $\mathcal{O}_X$-module $\mathcal{G}$ whose completion is $(\mathcal{G}_n)$. In this way we see that $(\mathcal{F}_n)$ is algebraizable, i.e., it is the completion of a coherent $\mathcal{O}_U$-module.

We add that the coherence of $j_*\mathcal{F}_1$ and $R^1j_*\mathcal{F}_1$ is a condition on the special fibre. Namely, if we denote $j_1 : U_1 \to X_1$ the special fibre of $j : U \to X$, then we can think of $\mathcal{F}_1$ as a coherent sheaf on U_1 and we have $j_*\mathcal{F}_1 = j_{1,*}\mathcal{F}_1$ and $R^1j_*\mathcal{F}_1 = R^1j_{1,*}\mathcal{F}_1$. Hence for example if X_1 is (S_2) and irreducible, we have $\dim(X_1) - \dim(Z) \geq 3$, and $\mathcal{F}_1$ is a locally free $\mathcal{O}_{U_1}$-module, then $j_{1,*}\mathcal{F}_1$ and $R^1j_{1,*}\mathcal{F}_1$ are coherent modules.

24. Application to the completion functor

0EKX In this section we just combine some already obtained results in order to conveniently reference them. There are many (stronger) results we could state here.

0EKY **Lemma 24.1.** *In Situation 16.1 assume*

1. *A has a dualizing complex and is I-adically complete,*
2. *$I = (f)$ generated by a single element,*
3. *A is local with maximal ideal $\mathfrak{a} = \mathfrak{m}$,*
4. *one of the following is true*
   (a) *A_f is (S_2) and for $\mathfrak{p} \subset A$, $f \notin \mathfrak{p}$ minimal we have $\dim(A/\mathfrak{p}) \geq 4$, or*
   (b) *if $\mathfrak{p} \notin V(f)$ and $V(\mathfrak{p}) \cap V(f) \neq \{\mathfrak{m}\}$, then $\mathit{depth}(A_\mathfrak{p}) + \dim(A/\mathfrak{p}) > 3$.*

Then the completion functor

$$\mathrm{colim}_{U_0 \subset U' \subset U \text{ open}} \mathit{Coh}(\mathcal{O}_{U'}) \longrightarrow \mathit{Coh}(U, f\mathcal{O}_U)$$

is an equivalence on the full subcategories of finite locally free objects.

Proof. It follows from Lemma 15.8 that the functor is fully faithful (details omitted). Let us prove essential surjectivity. Let $(\mathcal{F}_n)$ be a finite locally free object of $Coh(U, f\mathcal{O}_U)$. By either Lemma 20.4 or Proposition 22.2 there exists a coherent $\mathcal{O}_U$-module $\mathcal{F}$ such that $(\mathcal{F}_n)$ is the completion of $\mathcal{F}$. Namely, for the application of either result the only thing to check is that $(\mathcal{F}_n)$ satisfies the $(2,3)$-inequalities. This is done in Lemma 20.6. If $y \in U_0$, then the f-adic completion of the stalk $\mathcal{F}_y$ is isomorphic to a finite free module over the f-adic completion of $\mathcal{O}_{U,y}$. Hence $\mathcal{F}$ is finite locally free in an open neighbourhood U' of U_0. This finishes the proof. $\square$

0EKZ **Lemma 24.2.** *In Situation 16.1 assume*

(1) *A is f-adically complete,*
(2) *f is a nonzerodivisor,*
(3) *$H^1_\mathfrak{m}(A/fA)$ and $H^2_\mathfrak{m}(A/fA)$ are finite A-modules.*

Then the completion functor

$$\operatorname{colim}_{U_0 \subset U' \subset U \text{ open}} Coh(\mathcal{O}_{U'}) \longrightarrow Coh(U, f\mathcal{O}_U)$$

is an equivalence on the full subcategories of finite locally free objects.

Proof. The functor is fully faithful by Lemma 15.9. Essential surjectivity follows from Lemma 16.11. $\square$

25. Application to Lefschetz theorems

0EL0 In this section we discuss the relation between coherent sheaves on a projective scheme P and coherent modules on formal completion along an ample divisor Q.

Let k be a field. Let P be a proper scheme over k. Let $\mathcal{L}$ be an ample invertible $\mathcal{O}_P$-module. Let $s \in \Gamma(P, \mathcal{L})$ be a section[8] and let $Q = Z(s)$ be the zero scheme, see Divisors, Definition 14.8. For all $n \geq 1$ we denote $Q_n = Z(s^n)$ the nth infinitesimal neighbourhood of Q. If $\mathcal{F}$ is a coherent $\mathcal{O}_P$-module, then we denote $\mathcal{F}_n = \mathcal{F}|_{Q_n}$ the restriction, i.e., the pullback of $\mathcal{F}$ by the closed immersion $Q_n \to P$.

0EL1 **Proposition 25.1.** *In the situation above assume for all points $p \in P \setminus Q$ we have*

$$depth(\mathcal{F}_p) + \dim(\overline{\{p\}}) > s$$

Then the map

$$H^i(P, \mathcal{F}) \longrightarrow \lim H^i(Q_n, \mathcal{F}_n)$$

is an isomorphism for $0 \leq i < s$.

Proof. We will use More on Morphisms, Lemma 44.1 and we will use the notation used and results found More on Morphisms, Section 44 without further mention; this proof will not make sense without at least understanding the statement of the lemma. Observe that in our case $A = \bigoplus_{m \geq 0} \Gamma(P, \mathcal{L}^{\otimes m})$ is a finite type k-algebra all of whose graded parts are finite dimensional k-vector spaces, see Cohomology of Schemes, Lemma 16.1.

We may and do think of s as an element $f \in A_1 \subset A$, i.e., a homogeneous element of degree 1 of A. Denote $Y = V(f) \subset X$ the closed subscheme defined by f. Then $U \cap Y = (\pi|_U)^{-1}(Q)$ scheme theoretically. Recall the notation $\mathcal{F}_U = \pi^*\mathcal{F}|_U = (\pi|_U)^*\mathcal{F}$. This is a coherent $\mathcal{O}_U$-module. Choose a finite A-module M such that

[8] We do not require s to be a regular section. Correspondingly, Q is only a locally principal closed subscheme of P and not necessarily an effective Cartier divisor.

$\mathcal{F}_U = \widetilde{M}|_U$ (for existence see Local Cohomology, Lemma 7.2). We claim that $H^i_Z(M)$ is annihilated by a power of f for $i \leq s + 1$.

To prove the claim we will apply Local Cohomology, Proposition 9.1. Translating into geometry we see that it suffices to prove for $u \in U$, $u \notin Y$ and $z \in \overline{\{u\}} \cap Z$ that

$$\text{depth}(\mathcal{F}_{U,u}) + \dim(\mathcal{O}_{\overline{\{u\}},z}) > s + 1$$

This requires only a small amount of thought.

Observe that $Z = \text{Spec}(A_0)$ is a finite set of closed points of X because A_0 is a finite dimensional k-algebra. (The reader who would like Z to be a singleton can replace the finite k-algebra A_0 by k; it won't affect anything else in the proof.)

The morphism $\pi : L \to P$ and its restriction $\pi|_U : U \to P$ are smooth of relative dimension 1. Let $u \in U$, $u \notin Y$ and $z \in \overline{\{u\}} \cap Z$. Let $p = \pi(u) \in P \setminus Q$ be its image. Then either u is a generic point of the fibre of π over p or a closed point of the fibre. If u is a generic point of the fibre, then $\text{depth}(\mathcal{F}_{U,u}) = \text{depth}(\mathcal{F}_p)$ and $\dim(\overline{\{u\}}) = \dim(\overline{\{p\}}) + 1$. If u is a closed point of the fibre, then $\text{depth}(\mathcal{F}_{U,u}) = \text{depth}(\mathcal{F}_p) + 1$ and $\dim(\overline{\{u\}}) = \dim(\overline{\{p\}})$. In both cases we have $\dim(\overline{\{u\}}) = \dim(\mathcal{O}_{\overline{\{u\}},z})$ because every point of Z is closed. Thus the desired inequality follows from the assumption in the statement of the lemma.

Let A' be the f-adic completion of A. So $A \to A'$ is flat by Algebra, Lemma 96.2. Denote $U' \subset X' = \text{Spec}(A')$ the inverse image of U and similarly for Y' and Z'. Let $\mathcal{F}'$ on U' be the pullback of $\mathcal{F}_U$ and let $M' = M \otimes_A A'$. By flat base change for local cohomology (Local Cohomology, Lemma 4.7) we have

$$H^i_{Z'}(M') = H^i_Z(M) \otimes_A A'$$

and we find that for $i \leq s + 1$ these are annihilated by a power of f. Consider the diagram

$$\begin{array}{ccccc} & & H^i(U, \mathcal{F}_U) & \longrightarrow & \lim H^i(U, \mathcal{F}_U/f^n\mathcal{F}_U) \\ & \swarrow & \downarrow & & \| \\ H^i(U, \mathcal{F}_U) \otimes_A A' & = & H^i(U', \mathcal{F}') & \longrightarrow & \lim H^i(U', \mathcal{F}'/f^n\mathcal{F}') \end{array}$$

The lower horizontal arrow is an isomorphism for $i < s$ by Lemma 13.2 and the torsion property we just proved. The horizontal equal sign is flat base change (Cohomology of Schemes, Lemma 5.2) and the vertical equal sign is because $U \cap Y$ and $U' \cap Y'$ as well as their nth infinitesimal neighbourhoods are mapped isomorphically onto each other (as we are completing with respect to f).

Applying More on Morphisms, Equation (44.0.2) we have compatible direct sum decompositions

$$\lim H^i(U, \mathcal{F}_U/f^n\mathcal{F}_U) = \lim \left(\bigoplus\nolimits_{m \in \mathbf{Z}} H^i(Q_n, \mathcal{F}_n \otimes \mathcal{L}^{\otimes m})\right)$$

and

$$H^i(U, \mathcal{F}_U) = \bigoplus\nolimits_{m \in \mathbf{Z}} H^i(P, \mathcal{F} \otimes \mathcal{L}^{\otimes m})$$

Thus we conclude by Algebra, Lemma 97.3. $\square$

0EL2 **Lemma 25.2.** *Let k be a field. Let X be a proper scheme over k. Let $\mathcal{L}$ be an ample invertible $\mathcal{O}_X$-module. Let $s \in \Gamma(X, \mathcal{L})$. Let $Y = Z(s)$ be the zero scheme of s with nth infinitesimal neighbourhood $Y_n = Z(s^n)$. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Assume that for all $x \in X \setminus Y$ we have*

$$\text{depth}(\mathcal{F}_x) + \dim(\overline{\{x\}}) > 1$$

Then $\Gamma(V, \mathcal{F}) \to \lim \Gamma(Y_n, \mathcal{F}|_{Y_n})$ is an isomorphism for any open subscheme $V \subset X$ containing Y.

Proof. By Proposition 25.1 this is true for $V = X$. Thus it suffices to show that the map $\Gamma(V, \mathcal{F}) \to \lim \Gamma(Y_n, \mathcal{F}|_{Y_n})$ is injective. If $\sigma \in \Gamma(V, \mathcal{F})$ maps to zero, then its support is disjoint from Y (details omitted; hint: use Krull's intersection theorem). Then the closure $T \subset X$ of $\text{Supp}(\sigma)$ is disjoint from Y. Whence T is proper over k (being closed in X) and affine (being closed in the affine scheme $X \setminus Y$, see Morphisms, Lemma 41.18) and hence finite over k (Morphisms, Lemma 42.11). Thus T is a finite set of closed points of X. Thus $\text{depth}(\mathcal{F}_x) \geq 2$ is at least 1 for $x \in T$ by our assumption. We conclude that $\Gamma(V, \mathcal{F}) \to \Gamma(V \setminus T, \mathcal{F})$ is injective and $\sigma = 0$ as desired. $\square$

0EL3 **Example 25.3.** Let k be a field and let X be a proper variety over k. Let $Y \subset X$ be an effective Cartier divisor such that $\mathcal{O}_X(Y)$ is ample and denote Y_n its nth infinitesimal neighbourhood. Let $\mathcal{E}$ be a finite locally free $\mathcal{O}_X$-module. Here are some special cases of Proposition 25.1.

(1) If X is a curve, we don't learn anything.
(2) If X is a Cohen-Macaulay (for example normal) surface, then
$$H^0(X, \mathcal{E}) \to \lim H^0(Y_n, \mathcal{E}|_{Y_n})$$
is an isomorphism.
(3) If X is a Cohen-Macaulay threefold, then
$$H^0(X, \mathcal{E}) \to \lim H^0(Y_n, \mathcal{E}|_{Y_n}) \quad \text{and} \quad H^1(X, \mathcal{E}) \to \lim H^1(Y_n, \mathcal{E}|_{Y_n})$$
are isomorphisms.

Presumably the pattern is clear. If X is a normal threefold, then we can conclude the result for H^0 but not for H^1.

Before we prove the next main result, we need a lemma.

0EL4 **Lemma 25.4.** *In Situation 16.1 let $(\mathcal{F}_n)$ be an object of $Coh(U, I\mathcal{O}_U)$. Assume*

(1) *A is a graded ring, $\mathfrak{a} = A_+$, and I is a homogeneous ideal,*
(2) *$(\mathcal{F}_n) = (\widetilde{M_n}|_U)$ where (M_n) is an inverse system of graded A-modules, and*
(3) *$(\mathcal{F}_n)$ extends canonically to X.*

Then there is a finite graded A-module N such that

(a) *the inverse systems (N/I^nN) and (M_n) are pro-isomorphic in the category of graded A-modules modulo A_+-power torsion modules, and*
(b) *$(\mathcal{F}_n)$ is the completion of of the coherent module associated to N.*

Proof. Let $(\mathcal{G}_n)$ be the canonical extension as in Lemma 16.8. The grading on A and M_n determines an action

$$a : \mathbf{G}_m \times X \longrightarrow X$$

of the group scheme $\mathbf{G}_m$ on X such that $(\widetilde{M_n})$ becomes an inverse system of $\mathbf{G}_m$-equivariant quasi-coherent $\mathcal{O}_X$-modules, see Groupoids, Example 12.3. Since $\mathfrak{a}$ and I are homogeneous ideals the closed subschemes Z, Y and the open subscheme U are $\mathbf{G}_m$-invariant closed and open subschemes. The restriction $(\mathcal{F}_n)$ of $(\widetilde{M_n})$ is an inverse system of $\mathbf{G}_m$-equivariant coherent $\mathcal{O}_U$-modules. In other words, $(\mathcal{F}_n)$ is a $\mathbf{G}_m$-equivariant coherent formal module, in the sense that there is an isomorphism

$$\alpha : (a^*\mathcal{F}_n) \longrightarrow (p^*\mathcal{F}_n)$$

over $\mathbf{G}_m \times U$ satisfying a suitable cocycle condition. Since a and p are flat morphisms of affine schemes, by Lemma 16.9 we conclude that there exists a unique isomorphism

$$\beta : (a^*\mathcal{G}_n) \longrightarrow (p^*\mathcal{G}_n)$$

over $\mathbf{G}_m \times X$ restricting to α on $\mathbf{G}_m \times U$. The uniqueness guarantees that β satisfies the corresponding cocycle condition. In this way each $\mathcal{G}_n$ becomes a $\mathbf{G}_m$-equivariant coherent $\mathcal{O}_X$-module in a manner compatible with transition maps.

By Groupoids, Lemma 12.5 we see that $\mathcal{G}_n$ with its $\mathbf{G}_m$-equivariant structure corresponds to a graded A-module N_n. The transition maps $N_{n+1} \to N_n$ are graded module maps. Note that N_n is a finite A-module and $N_n = N_{n+1}/I^n N_{n+1}$ because $(\mathcal{G}_n)$ is an object of $\textit{Coh}(X, I\mathcal{O}_X)$. Let N be the finite graded A-module foud in Algebra, Lemma 97.2. Then $N_n = N/I^nN$, whence $(\mathcal{G}_n)$ is the completion of the coherent module associated to N, and a fortiori we see that (b) is true.

To see (a) we have to unwind the situation described above a bit more. First, observe that the kernel and cokernel of $M_n \to H^0(U, \mathcal{F}_n)$ is A_+-power torsion (Local Cohomology, Lemma 7.2). Observe that $H^0(U, \mathcal{F}_n)$ comes with a natural grading such that these maps and the transition maps of the system are graded A-module map; for example we can use that $(U \to X)_*\mathcal{F}_n$ is a $\mathbf{G}_m$-equivariant module on X and use Groupoids, Lemma 12.5. Next, recall that (N_n) and $(H^0(U, \mathcal{F}_n))$ are pro-isomorphic by Definition 16.7 and Lemma 16.8. We omit the verification that the maps defining this pro-isomorphism are graded module maps. Thus (N_n) and (M_n) are pro-isomorphic in the category of graded A-modules modulo A_+-power torsion modules. □

Let k be a field. Let P be a proper scheme over k. Let $\mathcal{L}$ be an ample invertible $\mathcal{O}_P$-module. Let $s \in \Gamma(P, \mathcal{L})$ be a section and let $Q = Z(s)$ be the zero scheme, see Divisors, Definition 14.8. Let $\mathcal{I} \subset \mathcal{O}_P$ be the ideal sheaf of Q. We will use $\textit{Coh}(P, \mathcal{I})$ to denote the category of coherent formal modules introduced in Cohomology of Schemes, Section 23.

0EL5 **Proposition 25.5.** *In the situation above let $(\mathcal{F}_n)$ be an object of $\textit{Coh}(P, \mathcal{I})$. Assume for all $q \in Q$ and for all primes $\mathfrak{p} \in \mathcal{O}^\wedge_{P,q}$, $\mathfrak{p} \not\in V(\mathcal{I}^\wedge_q)$ we have*

$$\textit{depth}((\mathcal{F}^\wedge_q)_\mathfrak{p}) + \dim(\mathcal{O}^\wedge_{P,q}/\mathfrak{p}) + \dim(\overline{\{q\}}) > 2$$

Then $(\mathcal{F}_n)$ is the completion of a coherent $\mathcal{O}_P$-module.

Proof. By Cohomology of Schemes, Lemma 23.6 to prove the lemma, we may replace $(\mathcal{F}_n)$ by an object differing from it by $\mathcal{I}$-torsion (see below for more precision). Let $T' = \{q \in Q \mid \dim(\overline{\{q\}}) = 0\}$ and $T = \{q \in Q \mid \dim(\overline{\{q\}}) \leq 1\}$. The assumption in the proposition is exactly that $Q \subset P$, $(\mathcal{F}_n)$, and $T' \subset T \subset Q$ satisfy the conditions of Lemma 21.2 with $d = 1$; besides trivial manipulations of inequalities, use that $V(\mathfrak{p}) \cap V(\mathcal{I}_y^\wedge) = \{\mathfrak{m}_y^\wedge\} \Leftrightarrow \dim(\mathcal{O}_{P,q}^\wedge/\mathfrak{p}) = 1$ as $\mathcal{I}_y^\wedge$ is generated by 1 element. Combining these two remarks, we may replace $(\mathcal{F}_n)$ by the object $(\mathcal{H}_n)$ of $Coh(P, \mathcal{I})$ found in Lemma 21.2. Thus we may and do assume $(\mathcal{F}_n)$ is pro-isomorphic to an inverse system $(\mathcal{F}_n'')$ of coherent $\mathcal{O}_P$-modules such that $\text{depth}(\mathcal{F}''_{n,q}) + \dim(\overline{\{q\}}) \geq 2$ for all $q \in Q$.

We will use More on Morphisms, Lemma 44.1 and we will use the notation used and results found More on Morphisms, Section 44 without further mention; this proof will not make sense without at least understanding the statement of the lemma. Observe that in our case $A = \bigoplus_{m \geq 0} \Gamma(P, \mathcal{L}^{\otimes m})$ is a finite type k-algebra all of whose graded parts are finite dimensional k-vector spaces, see Cohomology of Schemes, Lemma 16.1.

By Cohomology of Schemes, Lemma 23.9 the pull back by $\pi|_U : U \to P$ is an object $(\pi|_U^*\mathcal{F}_n)$ of $Coh(U, f\mathcal{O}_U)$ which is pro-isomorphic to the inverse system $(\pi|_U^*\mathcal{F}_n'')$ of coherent $\mathcal{O}_U$-modules. We claim

$$\text{depth}(\pi|_U^*\mathcal{F}''_{n,y}) + \delta^Y_Z(y) \geq 3$$

for all $y \in U \cap Y$. Since all the points of Z are closed, we see that $\delta^Y_Z(y) \geq \dim(\overline{\{y\}})$ for all $y \in U \cap Y$, see Lemma 18.1. Let $q \in Q$ be the image of y. Since the morphism $\pi : U \to P$ is smooth of relative dimension 1 we see that either y is a closed point of a fibre of π or a generic point. Thus we see that

$$\text{depth}(\pi^*\mathcal{F}''_{n,y}) + \delta^Y_Z(y) \geq \text{depth}(\pi^*\mathcal{F}''_{n,y}) + \dim(\overline{\{y\}}) = \text{depth}(\mathcal{F}''_{n,q}) + \dim(\overline{\{q\}}) + 1$$

because either the depth goes up by 1 or the dimension. This proves the claim.

By Lemma 22.1 we conclude that $(\pi|_U^*\mathcal{F}_n)$ canonically extends to X. Observe that

$$M_n = \Gamma(U, \pi|_U^*\mathcal{F}_n) = \bigoplus_{m \in \mathbf{Z}} \Gamma(P, \mathcal{F}_n \otimes_{\mathcal{O}_P} \mathcal{L}^{\otimes m})$$

is canonically a graded A-module, see More on Morphisms, Equation (44.0.2). By Properties, Lemma 18.2 we have $\pi|_U^*\mathcal{F}_n = \widetilde{M_n}|_U$. Thus we may apply Lemma 25.4 to find a finite graded A-module N such that (M_n) and (N/I^nN) are pro-isomorphic in the category of graded A-modules modulo A_+-torsion modules. Let $\mathcal{F}$ be the coherent $\mathcal{O}_P$-module associated to N, see Cohomology of Schemes, Proposition 15.3. The same proposition tells us that $(\mathcal{F}/\mathcal{I}^n\mathcal{F})$ is pro-isomorphic to $(\mathcal{F}_n)$. Since both are objects of $Coh(P, \mathcal{I})$ we win by Lemma 15.3. □

0EL6 **Example 25.6.** Let k be a field and let X be a proper variety over k. Let $Y \subset X$ be an effective Cartier divisor such that $\mathcal{O}_X(Y)$ is ample and denote $\mathcal{I} \subset \mathcal{O}_X$ the corresponding sheaf of ideals. Let $(\mathcal{E}_n)$ an object of $Coh(X, \mathcal{I})$ with $\mathcal{E}_n$ finite locally free. Here are some special cases of Proposition 25.5.

(1) If X is a curve or a surface, we don't learn anything.
(2) If X is a Cohen-Macaulay threefold, then $(\mathcal{E}_n)$ is the completion of a coherent $\mathcal{O}_X$-module $\mathcal{E}$.
(3) More generally, if $\dim(X) \geq 3$ and X is (S_3), then $(\mathcal{E}_n)$ is the completion of a coherent $\mathcal{O}_X$-module $\mathcal{E}$.

Of course, if $\mathcal{E}$ exists, then $\mathcal{E}$ is finite locally free in an open neighbourhood of Y.

0EL7 **Proposition 25.7.** *Let k be a field. Let X be a proper scheme over k. Let $\mathcal{L}$ be an ample invertible $\mathcal{O}_X$-module and let $s \in \Gamma(X, \mathcal{L})$. Let $Y = Z(s)$ be the zero scheme of s and denote $\mathcal{I} \subset \mathcal{O}_X$ the corresponding sheaf of ideals. Let $\mathcal{V}$ be the set* *of open subschemes of X containing Y ordered by reverse inclusion. Assume that for all $x \in X \setminus Y$ we have*

$$\text{depth}(\mathcal{O}_{X,x}) + \dim(\overline{\{x\}}) > 2$$

Then the completion functor

$$\text{colim}_{\mathcal{V}} \, Coh(\mathcal{O}_V) \longrightarrow Coh(X, \mathcal{I})$$

is an equivalence on the full subcategories of finite locally free objects.

Proof. To prove fully faithfulness it suffices to prove that

$$\text{colim}_{\mathcal{V}} \Gamma(V, \mathcal{L}^{\otimes m}) \longrightarrow \lim \Gamma(Y_n, \mathcal{L}^{\otimes m}|_{Y_n})$$

is an isomorphism for all m, see Lemma 15.2. This follows from Lemma 25.2.

Essential surjectivity. Let $(\mathcal{F}_n)$ be a finite locally free object of $Coh(X, \mathcal{I})$. Then for $y \in Y$ we have $\mathcal{F}_y^\wedge = \lim \mathcal{F}_{n,y}$ is is a finite free $\mathcal{O}_{X,y}^\wedge$-module. Let $\mathfrak{p} \subset \mathcal{O}_{X,y}^\wedge$ be a prime with $\mathfrak{p} \not\in V(\mathcal{I}_y^\wedge)$. Then $\mathfrak{p}$ lies over a prime $\mathfrak{p}_0 \subset \mathcal{O}_{X,y}$ which corresponds to a specialization $x \rightsquigarrow y$ with $x \not\in Y$. By Local Cohomology, Lemma 10.3 and some dimension theory (see Varieties, Section 20) we have

$$\text{depth}((\mathcal{O}_{X,y}^\wedge)_\mathfrak{p}) + \dim(\mathcal{O}_{X,y}^\wedge/\mathfrak{p}) = \text{depth}(\mathcal{O}_{X,x}) + \dim(\overline{\{x\}}) - \dim(\overline{\{y\}})$$

Thus our assumptions imply the assumptions of Proposition 25.5 are satisfied and we find that $(\mathcal{F}_n)$ is the completion of a coherent $\mathcal{O}_X$-module $\mathcal{F}$. It then follows that $\mathcal{F}_y$ is finite free for all $y \in Y$ and hence $\mathcal{F}$ is finite locally free in an open neighbourhood V of Y. This finishes the proof. $\square$

26. Other chapters

References

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