

ARTIN'S AXIOMS

07SZ

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1. Introduction

07T0 In this chapter we discuss Artin's axioms for the representability of functors by algebraic spaces. As references we suggest the papers [Art69], [Art70], [Art74].

Some of the notation, conventions, and terminology in this chapter is awkward and may seem backwards to the more experienced reader. This is intentional. Please see Quot, Section 2 for an explanation.

2. Conventions

07T1 The conventions we use in this chapter are the same as those in the chapter on algebraic stacks, see Algebraic Stacks, Section 2. In this chapter the base scheme S will often be locally Noetherian (although we will always reiterate this condition when stating results).

3. Predeformation categories

07T2 Let S be a locally Noetherian base scheme. Let

$$p : \mathcal{X} \longrightarrow (Sch/S)_{fppf}$$

be a category fibred in groupoids. Let k be a field and let $\text{Spec}(k) \rightarrow S$ be a morphism of finite type (see Morphisms, Lemma 15.1). We will sometimes simply say that k is a field of finite type over S . Let x_0 be an object of \mathcal{X} lying over $\text{Spec}(k)$. Given S , \mathcal{X} , k , and x_0 we will construct a predeformation category, as defined in Formal Deformation Theory, Definition 6.2. The construction will resemble the construction of Formal Deformation Theory, Remark 6.4.

First, by Morphisms, Lemma 15.1 we may pick an affine open $\text{Spec}(\Lambda) \subset S$ such that $\text{Spec}(k) \rightarrow S$ factors through $\text{Spec}(\Lambda)$ and the associated ring map $\Lambda \rightarrow k$ is finite. This provides us with the category \mathcal{C}_Λ , see Formal Deformation Theory, Definition 3.1. The category \mathcal{C}_Λ , up to canonical equivalence, does not depend on the choice of the affine open $\text{Spec}(\Lambda)$ of S . Namely, \mathcal{C}_Λ is equivalent to the opposite of the category of factorizations

07T3 (3.0.1)
$$\text{Spec}(k) \rightarrow \text{Spec}(A) \rightarrow S$$

of the structure morphism such that A is an Artinian local ring and such that $\text{Spec}(k) \rightarrow \text{Spec}(A)$ corresponds to a ring map $A \rightarrow k$ which identifies k with the residue field of A .

We let $\mathcal{F} = \mathcal{F}_{\mathcal{X},k,x_0}$ be the category whose

- (1) objects are morphisms $x_0 \rightarrow x$ of \mathcal{X} where $p(x) = \text{Spec}(A)$ with A an Artinian local ring and $p(x_0) \rightarrow p(x) \rightarrow S$ a factorization as in (3.0.1), and
- (2) morphisms $(x_0 \rightarrow x) \rightarrow (x_0 \rightarrow x')$ are commutative diagrams

$$\begin{array}{ccc} x & \longleftarrow & x' \\ & \nearrow & \nwarrow \\ & x_0 & \end{array}$$

in \mathcal{X} . (Note the reversal of arrows.)

If $x_0 \rightarrow x$ is an object of \mathcal{F} then writing $p(x) = \text{Spec}(A)$ we obtain an object A of \mathcal{C}_Λ . We often say that $x_0 \rightarrow x$ or x lies over A . A morphism of \mathcal{F} between objects $x_0 \rightarrow x$ lying over A and $x_0 \rightarrow x'$ lying over A' corresponds to a morphism $x' \rightarrow x$ of \mathcal{X} , hence a morphism $p(x' \rightarrow x) : \text{Spec}(A') \rightarrow \text{Spec}(A)$ which in turn corresponds to a ring map $A \rightarrow A'$. As \mathcal{X} is a category over the category of schemes over S we see that $A \rightarrow A'$ is Λ -algebra homomorphism. Thus we obtain a functor

07T4 (3.0.2)
$$p : \mathcal{F} = \mathcal{F}_{\mathcal{X},k,x_0} \longrightarrow \mathcal{C}_\Lambda.$$

We will use the notation $\mathcal{F}(A)$ to denote the fibre category over an object A of \mathcal{C}_Λ . An object of $\mathcal{F}(A)$ is simply a morphism $x_0 \rightarrow x$ of \mathcal{X} such that x lies over $\text{Spec}(A)$ and $x_0 \rightarrow x$ lies over $\text{Spec}(k) \rightarrow \text{Spec}(A)$.

07T5 **Lemma 3.1.** *The functor $p : \mathcal{F} \rightarrow \mathcal{C}_\Lambda$ defined above is a predeformation category.*

Proof. We have to show that \mathcal{F} is (a) cofibred in groupoids over \mathcal{C}_Λ and (b) that $\mathcal{F}(k)$ is a category equivalent to a category with a single object and a single morphism.

Proof of (a). The fibre categories of \mathcal{F} over \mathcal{C}_Λ are groupoids as the fibre categories of \mathcal{X} are groupoids. Let $A \rightarrow A'$ be a morphism of \mathcal{C}_Λ and let $x_0 \rightarrow x$ be an object of $\mathcal{F}(A)$. Because \mathcal{X} is fibred in groupoids, we can find a morphism $x' \rightarrow x$ lying over $\text{Spec}(A') \rightarrow \text{Spec}(A)$. Since the composition $A \rightarrow A' \rightarrow k$ is equal to the given map $A \rightarrow k$ we see (by uniqueness of pullbacks up to isomorphism) that the pullback via $\text{Spec}(k) \rightarrow \text{Spec}(A')$ of x' is x_0 , i.e., that there exists a morphism $x_0 \rightarrow x'$ lying over $\text{Spec}(k) \rightarrow \text{Spec}(A')$ compatible with $x_0 \rightarrow x$ and $x' \rightarrow x$. This proves that \mathcal{F} has pushforwards. We conclude by (the dual of) Categories, Lemma 34.2.

Proof of (b). If $A = k$, then $\text{Spec}(k) = \text{Spec}(A)$ and since \mathcal{X} is fibred in groupoids over $(\text{Sch}/S)_{fppf}$ we see that given any object $x_0 \rightarrow x$ in $\mathcal{F}(k)$ the morphism $x_0 \rightarrow x$ is an isomorphism. Hence every object of $\mathcal{F}(k)$ is isomorphic to $x_0 \rightarrow x_0$. Clearly the only self morphism of $x_0 \rightarrow x_0$ in \mathcal{F} is the identity. \square

Let S be a locally Noetherian base scheme. Let $F : \mathcal{X} \rightarrow \mathcal{Y}$ be a 1-morphism between categories fibred in groupoids over $(\text{Sch}/S)_{fppf}$. Let k be a field of finite type over S . Let x_0 be an object of \mathcal{X} lying over $\text{Spec}(k)$. Set $y_0 = F(x_0)$ which is an object of \mathcal{Y} lying over $\text{Spec}(k)$. Then F induces a functor

$$07WJ \quad (3.1.1) \quad F : \mathcal{F}_{\mathcal{X},k,x_0} \longrightarrow \mathcal{F}_{\mathcal{Y},k,y_0}$$

of categories cofibred over \mathcal{C}_Λ . Namely, to the object $x_0 \rightarrow x$ of $\mathcal{F}_{\mathcal{X},k,x_0}(A)$ we associate the object $F(x_0) \rightarrow F(x)$ of $\mathcal{F}_{\mathcal{Y},k,y_0}(A)$.

07WK **Lemma 3.2.** *Let S be a locally Noetherian scheme. Let $F : \mathcal{X} \rightarrow \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $(\text{Sch}/S)_{fppf}$. Assume either*

- (1) *F is formally smooth on objects (Criteria for Representability, Section 6),*
- (2) *F is representable by algebraic spaces and formally smooth, or*
- (3) *F is representable by algebraic spaces and smooth.*

Then for every finite type field k over S and object x_0 of \mathcal{X} over k the functor (3.1.1) is smooth in the sense of Formal Deformation Theory, Definition 8.1.

Proof. Case (1) is a matter of unwinding the definitions. Assumption (2) implies (1) by Criteria for Representability, Lemma 6.3. Assumption (3) implies (2) by More on Morphisms of Spaces, Lemma 19.6 and the principle of Algebraic Stacks, Lemma 10.9. \square

07WL **Lemma 3.3.** *Let S be a locally Noetherian scheme. Let*

$$\begin{array}{ccc} \mathcal{W} & \longrightarrow & \mathcal{Z} \\ \downarrow & & \downarrow \\ \mathcal{X} & \longrightarrow & \mathcal{Y} \end{array}$$

be a 2-fibre product of categories fibred in groupoids over $(Sch/S)_{fppf}$. Let k be a finite type field over S and w_0 an object of \mathcal{W} over k . Let x_0, z_0, y_0 be the images of w_0 under the morphisms in the diagram. Then

$$\begin{array}{ccc} \mathcal{F}_{\mathcal{W}, k, w_0} & \longrightarrow & \mathcal{F}_{\mathcal{Z}, k, z_0} \\ \downarrow & & \downarrow \\ \mathcal{F}_{\mathcal{X}, k, x_0} & \longrightarrow & \mathcal{F}_{\mathcal{Y}, k, y_0} \end{array}$$

is a fibre product of predeformation categories.

Proof. This is a matter of unwinding the definitions. Details omitted. \square

4. Pushouts and stacks

07WM In this section we show that algebraic stacks behave well with respect to certain pushouts. The results in this section hold over any base scheme.

The following lemma is also correct when Y, X', X, Y' are algebraic spaces, see (insert future reference here).

07WN **Lemma 4.1.** *Let S be a scheme. Let*

$$\begin{array}{ccc} X & \longrightarrow & X' \\ \downarrow & & \downarrow \\ Y & \longrightarrow & Y' \end{array}$$

be a pushout in the category of schemes over S where $X \rightarrow X'$ is a thickening and $X \rightarrow Y$ is affine, see *More on Morphisms*, Lemma 14.3. Let \mathcal{Z} be an algebraic stack over S . Then the functor of fibre categories

$$\mathcal{Z}_{Y'} \longrightarrow \mathcal{Z}_Y \times_{\mathcal{Z}_X} \mathcal{Z}_{X'}$$

is an equivalence of categories.

Proof. Let y' be an object of left hand side. The sheaf $Isom(y', y')$ on the category of schemes over Y' is representable by an algebraic space I over Y' , see *Algebraic Stacks*, Lemma 10.11. We conclude that the functor of the lemma is fully faithful as Y' is the pushout in the category of algebraic spaces as well as the category of schemes, see *Pushouts of Spaces*, Lemma 2.2.

Let (y, x', f) be an object of the right hand side. Here $f : y|_X \rightarrow x'|_X$ is an isomorphism. To finish the proof we have to construct an object y' of $\mathcal{Z}_{Y'}$, whose restrictions to Y and X' agree with y and x' in a manner compatible with φ . In fact, it suffices to construct y' fppf locally on Y' , see *Stacks*, Lemma 4.8. Choose a representable algebraic stack \mathcal{W} and a surjective smooth morphism $\mathcal{W} \rightarrow \mathcal{Z}$. Then

$$(Sch/Y)_{fppf} \times_{y, \mathcal{Z}} \mathcal{W} \quad \text{and} \quad (Sch/X')_{fppf} \times_{x', \mathcal{Z}} \mathcal{W}$$

are algebraic stacks representable by algebraic spaces V and U' smooth over Y and X' . The isomorphism f induces an isomorphism $\varphi : V \times_Y X \rightarrow U' \times_{X'} X$ over X . By *Pushouts of Spaces*, Lemmas 2.4 and 2.9 we see that the pushout $V' = V \amalg_{V \times_Y X} U'$ is an algebraic space smooth over Y' whose base change to Y and X' recovers V and U' in a manner compatible with φ .

Let W be the algebraic space representing \mathcal{W} . The projections $V \rightarrow W$ and $U' \rightarrow W$ agree as morphisms over $V \times_Y X \cong U' \times_{X'} X$ hence the universal property of the pushout determines a morphism of algebraic spaces $V' \rightarrow W$. Choose a scheme Y'_1 and a surjective étale morphism $Y'_1 \rightarrow V'$. Set $Y_1 = Y \times_{Y'} Y'_1$, $X'_1 = X' \times_{Y'} Y'_1$, $X_1 = X \times_{Y'} Y'_1$. The composition

$$(Sch/Y'_1) \rightarrow (Sch/V') \rightarrow (Sch/W) = \mathcal{W} \rightarrow \mathcal{Z}$$

corresponds by the 2-Yoneda lemma to an object y'_1 of \mathcal{Z} over Y'_1 whose restriction to Y_1 and X'_1 agrees with $y|_{Y_1}$ and $x'|_{X'_1}$ in a manner compatible with $f|_{X_1}$. Thus we have constructed our desired object smooth locally over Y' and we win. \square

5. The Rim-Schlessinger condition

06L9 The motivation for the following definition comes from Lemma 4.1 and Formal Deformation Theory, Definition 16.1 and Lemma 16.4.

07WP **Definition 5.1.** Let S be a locally Noetherian scheme. Let \mathcal{Z} be a category fibred in groupoids over $(Sch/S)_{fppf}$. We say \mathcal{Z} satisfies *condition (RS)* if for every pushout

$$\begin{array}{ccc} X & \longrightarrow & X' \\ \downarrow & & \downarrow \\ Y & \longrightarrow & Y' = Y \amalg_X X' \end{array}$$

in the category of schemes over S where

- (1) X, X', Y, Y' are spectra of local Artinian rings,
- (2) X, X', Y, Y' are of finite type over S , and
- (3) $X \rightarrow X'$ (and hence $Y \rightarrow Y'$) is a closed immersion

the functor of fibre categories

$$\mathcal{Z}_{Y'} \longrightarrow \mathcal{Z}_Y \times_{\mathcal{Z}_X} \mathcal{Z}_{X'}$$

is an equivalence of categories.

If A is an Artinian local ring with residue field k , then any morphism $\text{Spec}(A) \rightarrow S$ is affine and of finite type if and only if the induced morphism $\text{Spec}(k) \rightarrow S$ is of finite type, see Morphisms, Lemmas 11.13 and 15.2.

07WQ **Lemma 5.2.** Let \mathcal{X} be an algebraic stack over a locally Noetherian base S . Then \mathcal{X} satisfies (RS).

Proof. Immediate from the definitions and Lemma 4.1. \square

07WR **Lemma 5.3.** Let S be a scheme. Let $p : \mathcal{X} \rightarrow \mathcal{Y}$ and $q : \mathcal{Z} \rightarrow \mathcal{Y}$ be 1-morphisms of categories fibred in groupoids over $(Sch/S)_{fppf}$. If \mathcal{X}, \mathcal{Y} , and \mathcal{Z} satisfy (RS), then so does $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z}$.

Proof. This is formal. Let

$$\begin{array}{ccc} X & \longrightarrow & X' \\ \downarrow & & \downarrow \\ Y & \longrightarrow & Y' = Y \amalg_X X' \end{array}$$

be a diagram as in Definition 5.1. We have to show that

$$(\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z})_{Y'} \longrightarrow (\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z})_Y \times_{(\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z})_X} (\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z})_{X'}$$

is an equivalence. Using the definition of the 2-fibre product this becomes

$$07WS \quad (5.3.1) \quad \mathcal{X}_{Y'} \times_{\mathcal{Y}_{Y'}} \mathcal{Z}_{Y'} \longrightarrow (\mathcal{X}_Y \times_{\mathcal{Y}_Y} \mathcal{Z}_Y) \times_{(\mathcal{X}_X \times_{\mathcal{Y}_X} \mathcal{Z}_X)} (\mathcal{X}_{X'} \times_{\mathcal{Y}_{X'}} \mathcal{Z}_{X'}).$$

We are given that each of the functors

$$\mathcal{X}_{Y'} \rightarrow \mathcal{X}_Y \times_{\mathcal{Y}_Y} \mathcal{Z}_Y, \quad \mathcal{Y}_{Y'} \rightarrow \mathcal{X}_X \times_{\mathcal{Y}_X} \mathcal{Z}_X, \quad \mathcal{Z}_{Y'} \rightarrow \mathcal{X}_{X'} \times_{\mathcal{Y}_{X'}} \mathcal{Z}_{X'}$$

are equivalences. An object of the right hand side of (5.3.1) is a system

$$((x_Y, z_Y, \phi_Y), (x_{X'}, z_{X'}, \phi_{X'}), (\alpha, \beta)).$$

Then $(x_Y, x_{Y'}, \alpha)$ is isomorphic to the image of an object $x_{Y'}$ in $\mathcal{X}_{Y'}$ and $(z_Y, z_{Y'}, \beta)$ is isomorphic to the image of an object $z_{Y'}$ of $\mathcal{Z}_{Y'}$. The pair of morphisms $(\phi_Y, \phi_{X'})$ corresponds to a morphism ψ between the images of $x_{Y'}$ and $z_{Y'}$ in $\mathcal{Y}_{Y'}$. Then $(x_{Y'}, z_{Y'}, \psi)$ is an object of the left hand side of (5.3.1) mapping to the given object of the right hand side. This proves that (5.3.1) is essentially surjective. We omit the proof that it is fully faithful. \square

6. Deformation categories

07WT We match the notation introduced above with the notation from the chapter ‘‘Formal Deformation Theory’’.

07WU **Lemma 6.1.** *Let S be a locally Noetherian scheme. Let \mathcal{X} be a category fibred in groupoids over $(Sch/S)_{fppf}$ satisfying (RS). For any field k of finite type over S and any object x_0 of \mathcal{X} lying over k the predeformation category $p : \mathcal{F}_{\mathcal{X}, k, x_0} \rightarrow \mathcal{C}_\Lambda$ (3.0.2) is a deformation category, see Formal Deformation Theory, Definition 16.8.*

Proof. Set $\mathcal{F} = \mathcal{F}_{\mathcal{X}, k, x_0}$. Let $f_1 : A_1 \rightarrow A$ and $f_2 : A_2 \rightarrow A$ be ring maps in \mathcal{C}_Λ with f_2 surjective. We have to show that the functor

$$\mathcal{F}(A_1 \times_A A_2) \longrightarrow \mathcal{F}(A_1) \times_{\mathcal{F}(A)} \mathcal{F}(A_2)$$

is an equivalence, see Formal Deformation Theory, Lemma 16.4. Set $X = \text{Spec}(A)$, $X' = \text{Spec}(A_2)$, $Y = \text{Spec}(A_1)$ and $Y' = \text{Spec}(A_1 \times_A A_2)$. Note that $Y' = Y \amalg_X X'$ in the category of schemes, see More on Morphisms, Lemma 14.3. We know that in the diagram of functors of fibre categories

$$\begin{array}{ccc} \mathcal{X}_{Y'} & \longrightarrow & \mathcal{X}_Y \times_{\mathcal{X}_X} \mathcal{X}_{X'} \\ \downarrow & & \downarrow \\ \mathcal{X}_{\text{Spec}(k)} & \xlongequal{\quad} & \mathcal{X}_{\text{Spec}(k)} \end{array}$$

the top horizontal arrow is an equivalence by Definition 5.1. Since $\mathcal{F}(B)$ is the category of objects of $\mathcal{X}_{\text{Spec}(B)}$ with an identification with x_0 over k we win. \square

07WV **Remark 6.2.** Let S be a locally Noetherian scheme. Let \mathcal{X} be fibred in groupoids over $(Sch/S)_{fppf}$. Let k be a field of finite type over S and x_0 an object of \mathcal{X} over k . Let $p : \mathcal{F} \rightarrow \mathcal{C}_\Lambda$ be as in (3.0.2). If \mathcal{F} is a deformation category, i.e., if \mathcal{F} satisfies the Rim-Schlessinger condition (RS), then we see that \mathcal{F} satisfies Schlessinger’s conditions (S1) and (S2) by Formal Deformation Theory, Lemma 16.6. Let $\bar{\mathcal{F}}$ be the functor of isomorphism classes, see Formal Deformation Theory, Remarks 5.2

(10). Then $\overline{\mathcal{F}}$ satisfies (S1) and (S2) as well, see Formal Deformation Theory, Lemma 10.5. This holds in particular in the situation of Lemma 6.1.

7. Change of field

07WW This section is the analogue of Formal Deformation Theory, Section 29. As pointed out there, to discuss what happens under change of field we need to write $\mathcal{C}_{\Lambda,k}$ instead of \mathcal{C}_{Λ} . In the following lemma we use the notation $\mathcal{F}_{l/k}$ introduced in Formal Deformation Theory, Situation 29.1.

07WX **Lemma 7.1.** *Let S be a locally Noetherian scheme. Let \mathcal{X} be a category fibred in groupoids over $(Sch/S)_{fppf}$. Let k be a field of finite type over S and let l/k be a finite extension. Let x_0 be an object of \mathcal{F} lying over $\text{Spec}(k)$. Denote $x_{l,0}$ the restriction of x_0 to $\text{Spec}(l)$. Then there is a canonical functor*

$$(\mathcal{F}_{\mathcal{X},k,x_0})_{l/k} \longrightarrow \mathcal{F}_{\mathcal{X},l,x_{l,0}}$$

of categories cofibred in groupoids over $\mathcal{C}_{\Lambda,l}$. If \mathcal{X} satisfies (RS), then this functor is an equivalence.

Proof. Consider a factorization

$$\text{Spec}(l) \rightarrow \text{Spec}(B) \rightarrow S$$

as in (3.0.1). By definition we have

$$(\mathcal{F}_{\mathcal{X},k,x_0})_{l/k}(B) = \mathcal{F}_{\mathcal{X},k,x_0}(B \times_l k)$$

see Formal Deformation Theory, Situation 29.1. Thus an object of this is a morphism $x_0 \rightarrow x$ of \mathcal{X} lying over the morphism $\text{Spec}(k) \rightarrow \text{Spec}(B \times_l k)$. Choosing pullback functor for \mathcal{X} we can associate to $x_0 \rightarrow x$ the morphism $x_{l,0} \rightarrow x_B$ where x_B is the restriction of x to $\text{Spec}(B)$ (via the morphism $\text{Spec}(B) \rightarrow \text{Spec}(B \times_l k)$ coming from $B \times_l k \subset B$). This construction is functorial in B and compatible with morphisms.

Next, assume \mathcal{X} satisfies (RS). Consider the diagrams

$$\begin{array}{ccc} l & \longleftarrow & B \\ \uparrow & & \uparrow \\ k & \longleftarrow & B \times_l k \end{array} \quad \text{and} \quad \begin{array}{ccc} \text{Spec}(l) & \longrightarrow & \text{Spec}(B) \\ \downarrow & & \downarrow \\ \text{Spec}(k) & \longrightarrow & \text{Spec}(B \times_l k) \end{array}$$

The diagram on the left is a fibre product of rings. The diagram on the right is a pushout in the category of schemes, see More on Morphisms, Lemma 14.3. These schemes are all of finite type over S (see remarks following Definition 5.1). Hence (RS) kicks in to give an equivalence of fibre categories

$$\mathcal{X}_{\text{Spec}(B \times_l k)} \longrightarrow \mathcal{X}_{\text{Spec}(k)} \times_{\mathcal{X}_{\text{Spec}(l)}} \mathcal{X}_{\text{Spec}(B)}$$

This implies that the functor defined above gives an equivalence of fibre categories. Hence the functor is an equivalence on categories cofibred in groupoids by (the dual of) Categories, Lemma 34.8. \square

8. Tangent spaces

07WY Let S be a locally Noetherian scheme. Let \mathcal{X} be a category fibred in groupoids over $(Sch/S)_{fppf}$. Let k be a field of finite type over S and let x_0 be an object of \mathcal{X} over k . In Formal Deformation Theory, Section 12 we have defined the *tangent space*

$$07WZ \quad (8.0.1) \quad T\mathcal{F}_{\mathcal{X},k,x_0} = \left\{ \begin{array}{l} \text{isomorphism classes of morphisms} \\ x_0 \rightarrow x \text{ over } \text{Spec}(k) \rightarrow \text{Spec}(k[\epsilon]) \end{array} \right\}$$

of the predeformation category $\mathcal{F}_{\mathcal{X},k,x_0}$. In Formal Deformation Theory, Section 19 we have defined

$$07X0 \quad (8.0.2) \quad \text{Inf}(\mathcal{F}_{\mathcal{X},k,x_0}) = \text{Ker}(\text{Aut}_{\text{Spec}(k[\epsilon])}(x'_0) \rightarrow \text{Aut}_{\text{Spec}(k)}(x_0))$$

where x'_0 is the pullback of x_0 to $\text{Spec}(k[\epsilon])$. If \mathcal{X} satisfies the Rim-Schlessinger condition (RS), then $T\mathcal{F}_{\mathcal{X},k,x_0}$ comes equipped with a natural k -vector space structure by Formal Deformation Theory, Lemma 12.2 (assumptions hold by Lemma 6.1 and Remark 6.2). Moreover, Formal Deformation Theory, Lemma 19.9 shows that $\text{Inf}(\mathcal{F}_{\mathcal{X},k,x_0})$ has a natural k -vector space structure such that addition agrees with composition of automorphisms. A natural condition is to ask these vector spaces to have finite dimension.

The following lemma tells us this is true if \mathcal{X} is locally of finite type over S (see Morphisms of Stacks, Section 17).

07X1 **Lemma 8.1.** *Let S be a locally Noetherian scheme. Assume*

- (1) \mathcal{X} is an algebraic stack,
- (2) U is a scheme locally of finite type over S , and
- (3) $(Sch/U)_{fppf} \rightarrow \mathcal{X}$ is a smooth surjective morphism.

Then, for any $\mathcal{F} = \mathcal{F}_{\mathcal{X},k,x_0}$ as in Section 3 the tangent space $T\mathcal{F}$ and infinitesimal automorphism space $\text{Inf}(\mathcal{F})$ have finite dimension over k

Proof. Let us write $\mathcal{U} = (Sch/U)_{fppf}$. By our definition of algebraic stacks the 1-morphism $\mathcal{U} \rightarrow \mathcal{X}$ is representable by algebraic spaces. Hence in particular the 2-fibre product

$$\mathcal{U}_{x_0} = (Sch/\text{Spec}(k))_{fppf} \times_{\mathcal{X}} \mathcal{U}$$

is representable by an algebraic space U_{x_0} over $\text{Spec}(k)$. Then $U_{x_0} \rightarrow \text{Spec}(k)$ is smooth and surjective (in particular U_{x_0} is nonempty). By Spaces over Fields, Lemma 16.2 we can find a finite extension $l \supset k$ and a point $\text{Spec}(l) \rightarrow U_{x_0}$ over k . We have

$$(\mathcal{F}_{\mathcal{X},k,x_0})_{l/k} = \mathcal{F}_{\mathcal{X},l,x_{l,0}}$$

by Lemma 7.1 and the fact that \mathcal{X} satisfies (RS). Thus we see that

$$T\mathcal{F} \otimes_k l \cong T\mathcal{F}_{\mathcal{X},l,x_{l,0}} \quad \text{and} \quad \text{Inf}(\mathcal{F}) \otimes_k l \cong \text{Inf}(\mathcal{F}_{\mathcal{X},l,x_{l,0}})$$

by Formal Deformation Theory, Lemmas 29.3 and 29.4 (these are applicable by Lemmas 5.2 and 6.1 and Remark 6.2). Hence it suffices to prove that $T\mathcal{F}_{\mathcal{X},l,x_{l,0}}$ and $\text{Inf}(\mathcal{F}_{\mathcal{X},l,x_{l,0}})$ have finite dimension over l . Note that $x_{l,0}$ comes from a point u_0 of \mathcal{U} over l .

We interrupt the flow of the argument to show that the lemma for infinitesimal automorphisms follows from the lemma for tangent spaces. Namely, let $\mathcal{R} = \mathcal{U} \times_{\mathcal{X}} \mathcal{U}$.

Let r_0 be the l -valued point $(u_0, u_0, \text{id}_{x_0})$ of \mathcal{R} . Combining Lemma 3.3 and Formal Deformation Theory, Lemma 26.2 we see that

$$\text{Inf}(\mathcal{F}_{\mathcal{X}, l, x_{l,0}}) \subset T\mathcal{F}_{\mathcal{R}, l, r_0}$$

Note that \mathcal{R} is an algebraic stack, see Algebraic Stacks, Lemma 14.2. Also, \mathcal{R} is representable by an algebraic space R smooth over U (via either projection, see Algebraic Stacks, Lemma 16.2). Hence, choose an scheme U' and a surjective étale morphism $U' \rightarrow R$ we see that U' is smooth over U , hence locally of finite type over S . As $(\text{Sch}/U')_{fppf} \rightarrow \mathcal{R}$ is surjective and smooth, we have reduced the question to the case of tangent spaces.

The functor (3.1.1)

$$\mathcal{F}_{\mathcal{U}, l, u_0} \longrightarrow \mathcal{F}_{\mathcal{X}, l, x_{l,0}}$$

is smooth by Lemma 3.2. The induced map on tangent spaces

$$T\mathcal{F}_{\mathcal{U}, l, u_0} \longrightarrow T\mathcal{F}_{\mathcal{X}, l, x_{l,0}}$$

is l -linear (by Formal Deformation Theory, Lemma 12.4) and surjective (as smooth maps of predeformation categories induce surjective maps on tangent spaces by Formal Deformation Theory, Lemma 8.8). Hence it suffices to prove that the tangent space of the deformation space associated to the representable algebraic stack \mathcal{U} at the point u_0 is finite dimensional. Let $\text{Spec}(R) \subset U$ be an affine open such that $u_0 : \text{Spec}(l) \rightarrow U$ factors through $\text{Spec}(R)$ and such that $\text{Spec}(R) \rightarrow S$ factors through $\text{Spec}(\Lambda) \subset S$. Let $\mathfrak{m}_R \subset R$ be the kernel of the Λ -algebra map $\varphi_0 : R \rightarrow l$ corresponding to u_0 . Note that R , being of finite type over the Noetherian ring Λ , is a Noetherian ring. Hence $\mathfrak{m}_R = (f_1, \dots, f_n)$ is a finitely generated ideal. We have

$$T\mathcal{F}_{\mathcal{U}, l, u_0} = \{\varphi : R \rightarrow l[\epsilon] \mid \varphi \text{ is a } \Lambda\text{-algebra map and } \varphi \bmod \epsilon = \varphi_0\}$$

An element of the right hand side is determined by its values on f_1, \dots, f_n hence the dimension is at most n and we win. \square

07X2 **Lemma 8.2.** *Let S be a locally Noetherian scheme. Let $p : \mathcal{X} \rightarrow \mathcal{Y}$ and $q : \mathcal{Z} \rightarrow \mathcal{Y}$ be 1-morphisms of categories fibred in groupoids over $(\text{Sch}/S)_{fppf}$. Assume $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ satisfy (RS). Let k be a field of finite type over S and let w_0 be an object of $\mathcal{W} = \mathcal{X} \times_{\mathcal{Y}} \mathcal{Z}$ over k . Denote x_0, y_0, z_0 the objects of $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ you get from w_0 . Then there is a 6-term exact sequence*

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{Inf}(\mathcal{F}_{\mathcal{W}, k, w_0}) & \longrightarrow & \text{Inf}(\mathcal{F}_{\mathcal{X}, k, x_0}) \oplus \text{Inf}(\mathcal{F}_{\mathcal{Z}, k, z_0}) & \longrightarrow & \text{Inf}(\mathcal{F}_{\mathcal{Y}, k, y_0}) \\ & & & & \swarrow & & \\ & & T\mathcal{F}_{\mathcal{W}, k, w_0} & \longrightarrow & T\mathcal{F}_{\mathcal{X}, k, x_0} \oplus T\mathcal{F}_{\mathcal{Z}, k, z_0} & \longrightarrow & T\mathcal{F}_{\mathcal{Y}, k, y_0} \end{array}$$

of k -vector spaces.

Proof. By Lemma 5.3 we see that \mathcal{W} satisfies (RS) and hence the lemma makes sense. To see the lemma is true, apply Lemmas 3.3 and 6.1 and Formal Deformation Theory, Lemma 20.1. \square

9. Formal objects

07X3 In this section we transfer some of the notions already defined in the chapter “Formal Deformation Theory” to the current setting. In the following we will say “ R is an S -algebra” to indicate that R is a ring endowed with a morphism of schemes $\text{Spec}(R) \rightarrow S$.

07X4 **Definition 9.1.** Let S be a locally Noetherian scheme. Let $p : \mathcal{X} \rightarrow (\text{Sch}/S)_{fppf}$ be a category fibred in groupoids.

- (1) A *formal object* $\xi = (R, \xi_n, f_n)$ of \mathcal{X} consists of a Noetherian complete local S -algebra R , objects ξ_n of \mathcal{X} lying over $\text{Spec}(R/\mathfrak{m}_R^n)$, and morphisms $f_n : \xi_n \rightarrow \xi_{n+1}$ of \mathcal{X} lying over $\text{Spec}(R/\mathfrak{m}^n) \rightarrow \text{Spec}(R/\mathfrak{m}^{n+1})$ such that R/\mathfrak{m} is a field of finite type over S .
- (2) A *morphism of formal objects* $a : \xi = (R, \xi_n, f_n) \rightarrow \eta = (T, \eta_n, g_n)$ is given by morphisms $a_n : \xi_n \rightarrow \eta_n$ such that for every n the diagram

$$\begin{array}{ccc} \xi_{n+1} & \xrightarrow{f_n} & \xi_n \\ a_{n+1} \downarrow & & \downarrow a_n \\ \eta_{n+1} & \xrightarrow{g_n} & \eta_n \end{array}$$

is commutative. Applying the functor p we obtain a compatible collection of morphisms $\text{Spec}(R/\mathfrak{m}_R^n) \rightarrow \text{Spec}(T/\mathfrak{m}_T^n)$ and hence a morphism $a_0 : \text{Spec}(R) \rightarrow \text{Spec}(T)$ over S . We say that a *lies over* a_0 .

Thus we obtain a category of formal objects of \mathcal{X} .

0CXH **Remark 9.2.** Let S be a locally Noetherian scheme. Let $p : \mathcal{X} \rightarrow (\text{Sch}/S)_{fppf}$ be a category fibred in groupoids. Let $\xi = (R, \xi_n, f_n)$ be a formal object. Set $k = R/\mathfrak{m}$ and $x_0 = \xi_1$. The formal object ξ defines a formal object ξ of the predeformation category $\mathcal{F}_{\mathcal{X}, k, x_0}$. This follows immediately from Definition 9.1 above, Formal Deformation Theory, Definition 7.1, and our construction of the predeformation category $\mathcal{F}_{\mathcal{X}, k, x_0}$ in Section 3.

If $F : \mathcal{X} \rightarrow \mathcal{Y}$ is a 1-morphism of categories fibred in groupoids over $(\text{Sch}/S)_{fppf}$, then F induces a functor between categories of formal objects as well.

07X5 **Lemma 9.3.** *Let S be a locally Noetherian scheme. Let $F : \mathcal{X} \rightarrow \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $(\text{Sch}/S)_{fppf}$. Let $\eta = (R, \eta_n, g_n)$ be a formal object of \mathcal{Y} and let ξ_1 be an object of \mathcal{X} with $F(\xi_1) \cong \eta_1$. If F is formally smooth on objects (see *Criteria for Representability*, Section 6), then there exists a formal object $\xi = (R, \xi_n, f_n)$ of \mathcal{X} such that $F(\xi) \cong \eta$.*

Proof. Note that each of the morphisms $\text{Spec}(R/\mathfrak{m}^n) \rightarrow \text{Spec}(R/\mathfrak{m}^{n+1})$ is a first order thickening of affine schemes over S . Hence the assumption on F means that we can successively lift ξ_1 to objects ξ_2, ξ_3, \dots of \mathcal{X} endowed with compatible isomorphisms $\eta_n|_{\text{Spec}(R/\mathfrak{m}^{n-1})} \cong \eta_{n-1}$ and $F(\eta_n) \cong \xi_n$. \square

Let S be a locally Noetherian scheme. Let $p : \mathcal{X} \rightarrow (\text{Sch}/S)_{fppf}$ be a category fibred in groupoids. Suppose that x is an object of \mathcal{X} over R , where R is a Noetherian complete local S -algebra with residue field of finite type over S . Then we can consider the system of restrictions $\xi_n = x|_{\text{Spec}(R/\mathfrak{m}^n)}$ endowed with the natural morphisms $\xi_1 \rightarrow \xi_2 \rightarrow \dots$ coming from transitivity of restriction. Thus

$\xi = (R, \xi_n, \xi_n \rightarrow \xi_{n+1})$ is a formal object of \mathcal{X} . This construction is functorial in the object x . Thus we obtain a functor

$$07X6 \quad (9.3.1) \quad \left\{ \begin{array}{l} \text{objects } x \text{ of } \mathcal{X} \text{ such that } p(x) = \text{Spec}(R) \\ \text{where } R \text{ is Noetherian complete local} \\ \text{with } R/\mathfrak{m} \text{ of finite type over } S \end{array} \right\} \longrightarrow \{\text{formal objects of } \mathcal{X}\}$$

To be precise the left hand side is the full subcategory of \mathcal{X} consisting of objects as indicated and the right hand side is the category of formal objects of \mathcal{X} as in Definition 9.1.

07X7 **Definition 9.4.** Let S be a locally Noetherian scheme. Let \mathcal{X} be a category fibred in groupoids over $(Sch/S)_{fppf}$. A formal object $\xi = (R, \xi_n, f_n)$ of \mathcal{X} is called *effective* if it is in the essential image of the functor (9.3.1).

If the category fibred in groupoids is an algebraic stack, then every formal object is effective as follows from the next lemma.

07X8 **Lemma 9.5.** *Let S be a locally Noetherian scheme. Let \mathcal{X} be an algebraic stack over S . The functor (9.3.1) is an equivalence.*

Proof. Case I: \mathcal{X} is representable (by a scheme). Say $\mathcal{X} = (Sch/X)_{fppf}$ for some scheme X over S . Unwinding the definitions we have to prove the following: Given a Noetherian complete local S -algebra R with R/\mathfrak{m} of finite type over S we have

$$\text{Mor}_S(\text{Spec}(R), X) \longrightarrow \lim \text{Mor}_S(\text{Spec}(R/\mathfrak{m}^n), X)$$

is bijective. This follows from Formal Spaces, Lemma 26.2.

Case II. \mathcal{X} is representable by an algebraic space. Say \mathcal{X} is representable by X . Again we have to show that

$$\text{Mor}_S(\text{Spec}(R), X) \longrightarrow \lim \text{Mor}_S(\text{Spec}(R/\mathfrak{m}^n), X)$$

is bijective for R as above. This is Formal Spaces, Lemma 26.3.

Case III: General case of an algebraic stack. A general remark is that the left and right hand side of (9.3.1) are categories fibred in groupoids over the category of affine schemes over S which are spectra of Noetherian complete local rings with residue field of finite type over S . We will also see in the proof below that they form stacks for a certain topology on this category.

We first prove fully faithfulness. Let R be a Noetherian complete local S -algebra with $k = R/\mathfrak{m}$ of finite type over S . Let x, x' be objects of \mathcal{X} over R . As \mathcal{X} is an algebraic stack $\text{Isom}(x, x')$ is representable by an algebraic space I over $\text{Spec}(R)$, see Algebraic Stacks, Lemma 10.11. Applying Case II to I over $\text{Spec}(R)$ implies immediately that (9.3.1) is fully faithful on fibre categories over $\text{Spec}(R)$. Hence the functor is fully faithful by Categories, Lemma 34.8.

Essential surjectivity. Let $\xi = (R, \xi_n, f_n)$ be a formal object of \mathcal{X} . Choose a scheme U over S and a surjective smooth morphism $f : (Sch/U)_{fppf} \rightarrow \mathcal{X}$. For every n consider the fibre product

$$(Sch/\text{Spec}(R/\mathfrak{m}^n))_{fppf} \times_{\xi_n, \mathcal{X}, f} (Sch/U)_{fppf}$$

By assumption this is representable by an algebraic space V_n surjective and smooth over $\mathrm{Spec}(R/\mathfrak{m}^n)$. The morphisms $f_n : \xi_n \rightarrow \xi_{n+1}$ induce cartesian squares

$$\begin{array}{ccc} V_{n+1} & \longleftarrow & V_n \\ \downarrow & & \downarrow \\ \mathrm{Spec}(R/\mathfrak{m}^{n+1}) & \longleftarrow & \mathrm{Spec}(R/\mathfrak{m}^n) \end{array}$$

of algebraic spaces. By Spaces over Fields, Lemma 16.2 we can find a finite separable extension $k \subset k'$ and a point $v'_1 : \mathrm{Spec}(k') \rightarrow V_1$ over k . Let $R \subset R'$ be the finite étale extension whose residue field extension is $k \subset k'$ (exists and is unique by Algebra, Lemmas 149.7 and 149.9). By the infinitesimal lifting criterion of smoothness (see More on Morphisms of Spaces, Lemma 19.6) applied to $V_n \rightarrow \mathrm{Spec}(R/\mathfrak{m}^n)$ for $n = 2, 3, 4, \dots$ we can successively find morphisms $v'_n : \mathrm{Spec}(R'/(\mathfrak{m}')^n) \rightarrow V_n$ over $\mathrm{Spec}(R/\mathfrak{m}^n)$ fitting into commutative diagrams

$$\begin{array}{ccc} \mathrm{Spec}(R'/(\mathfrak{m}')^{n+1}) & \longleftarrow & \mathrm{Spec}(R'/(\mathfrak{m}')^n) \\ \downarrow v'_{n+1} & & \downarrow v'_n \\ V_{n+1} & \longleftarrow & V_n \end{array}$$

Composing with the projection morphisms $V_n \rightarrow U$ we obtain a compatible system of morphisms $u'_n : \mathrm{Spec}(R'/(\mathfrak{m}')^n) \rightarrow U$. By Case I the family (u'_n) comes from a unique morphism $u' : \mathrm{Spec}(R') \rightarrow U$. Denote x' the object of \mathcal{X} over $\mathrm{Spec}(R')$ we get by applying the 1-morphism f to u' . By construction, there exists a morphism of formal objects

$$(9.3.1)(x') = (R', x'|_{\mathrm{Spec}(R'/(\mathfrak{m}')^n), \dots}) \longrightarrow (R, \xi_n, f_n)$$

lying over $\mathrm{Spec}(R') \rightarrow \mathrm{Spec}(R)$. Note that $R' \otimes_R R'$ is a finite product of spectra of Noetherian complete local rings to which our current discussion applies. Denote $p_0, p_1 : \mathrm{Spec}(R' \otimes_R R') \rightarrow \mathrm{Spec}(R')$ the two projections. By the fully faithfulness shown above there exists a canonical isomorphism $\varphi : p_0^* x' \rightarrow p_1^* x'$ because we have such isomorphisms over $\mathrm{Spec}((R' \otimes_R R')/\mathfrak{m}^n(R' \otimes_R R'))$. We omit the proof that the isomorphism φ satisfies the cocycle condition (see Stacks, Definition 3.1). Since $\{\mathrm{Spec}(R') \rightarrow \mathrm{Spec}(R)\}$ is an fppf covering we conclude that x' descends to an object x of \mathcal{X} over $\mathrm{Spec}(R)$. We omit the proof that x_n is the restriction of x to $\mathrm{Spec}(R/\mathfrak{m}^n)$. \square

07X9 Lemma 9.6. *Let S be a scheme. Let $p : \mathcal{X} \rightarrow \mathcal{Y}$ and $q : \mathcal{Z} \rightarrow \mathcal{Y}$ be 1-morphisms of categories fibred in groupoids over $(\mathrm{Sch}/S)_{\mathrm{fppf}}$. If the functor (9.3.1) is an equivalence for \mathcal{X} , \mathcal{Y} , and \mathcal{Z} , then it is an equivalence for $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z}$.*

Proof. The left and the right hand side of (9.3.1) for $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z}$ are simply the 2-fibre products of the left and the right hand side of (9.3.1) for \mathcal{X} , \mathcal{Z} over \mathcal{Y} . Hence the result follows as taking 2-fibre products is compatible with equivalences of categories, see Categories, Lemma 30.7. \square

10. Approximation

07XA A fundamental insight of Michael Artin is that you can approximate objects of a limit preserving stack. Namely, given an object x of the stack over a Noetherian complete local ring, you can find an object x_A over an algebraic ring which is “close

to" x . Here an algebraic ring means a finite type S -algebra and close means adically close. In this section we present this in a simple, yet general form.

To formulate the result we need to pull together some definitions from different places in the Stacks project. First, in Criteria for Representability, Section 5 we introduced *limit preserving on objects* for 1-morphisms of categories fibred in groupoids over the category of schemes. In More on Algebra, Definition 49.1 we defined the notion of a G -ring. Let S be a locally Noetherian scheme. Let A be an S -algebra. We say that A is *of finite type over S* or is a *finite type S -algebra* if $\mathrm{Spec}(A) \rightarrow S$ is of finite type. In this case A is a Noetherian ring. Finally, given a ring A and ideal I we denote $\mathrm{Gr}_I(A) = \bigoplus I^n/I^{n+1}$.

07XB **Lemma 10.1.** *Let S be a locally Noetherian scheme. Let $p : \mathcal{X} \rightarrow (\mathrm{Sch}/S)_{\mathrm{fppf}}$ be a category fibred in groupoids. Let x be an object of \mathcal{X} lying over $\mathrm{Spec}(R)$ where R is a Noetherian complete local ring with residue field k of finite type over S . Let $s \in S$ be the image of $\mathrm{Spec}(k) \rightarrow S$. Assume that (a) $\mathcal{O}_{S,s}$ is a G -ring and (b) p is limit preserving on objects. Then for every integer $N \geq 1$ there exist*

- (1) a finite type S -algebra A ,
- (2) a maximal ideal $\mathfrak{m}_A \subset A$,
- (3) an object x_A of \mathcal{X} over $\mathrm{Spec}(A)$,
- (4) an S -isomorphism $R/\mathfrak{m}_R^N \cong A/\mathfrak{m}_A^N$,
- (5) an isomorphism $x|_{\mathrm{Spec}(R/\mathfrak{m}_R^N)} \cong x_A|_{\mathrm{Spec}(A/\mathfrak{m}_A^N)}$ compatible with (4), and
- (6) an isomorphism $\mathrm{Gr}_{\mathfrak{m}_R}(R) \cong \mathrm{Gr}_{\mathfrak{m}_A}(A)$ of graded k -algebras.

Proof. Choose an affine open $\mathrm{Spec}(\Lambda) \subset S$ such that k is a finite Λ -algebra, see Morphisms, Lemma 15.1. We may and do replace S by $\mathrm{Spec}(\Lambda)$.

We may write R as a directed colimit $R = \mathrm{colim} C_j$ where each C_j is a finite type Λ -algebra (see Algebra, Lemma 126.2). By assumption (b) the object x is isomorphic to the restriction of an object over one of the C_j . Hence we may choose a finite type Λ -algebra C , a Λ -algebra map $C \rightarrow R$, and an object x_C of \mathcal{X} over $\mathrm{Spec}(C)$ such that $x = x_C|_{\mathrm{Spec}(R)}$. The choice of C is a bookkeeping device and could be avoided. For later use, let us write $C = \Lambda[y_1, \dots, y_u]/(f_1, \dots, f_v)$ and we denote $\bar{a}_i \in R$ the image of y_i under the map $C \rightarrow R$. Set $\mathfrak{m}_C = C \cap \mathfrak{m}_R$.

Choose a Λ -algebra surjection $\Lambda[x_1, \dots, x_s] \rightarrow k$ and denote \mathfrak{m}' the kernel. By the universal property of polynomial rings we may lift this to a Λ -algebra map $\Lambda[x_1, \dots, x_s] \rightarrow R$. We add some variables (i.e., we increase s a bit) mapping to generators of \mathfrak{m}_R . Having done this we see that $\Lambda[x_1, \dots, x_s] \rightarrow R/\mathfrak{m}_R^2$ is surjective. Then we see that

07XC (10.1.1)
$$P = \Lambda[x_1, \dots, x_s]_{\mathfrak{m}'}^{\wedge} \longrightarrow R$$

is a surjective map of Noetherian complete local rings, see for example Formal Deformation Theory, Lemma 4.2.

Choose lifts $a_i \in P$ of \bar{a}_i we found above. Choose generators $b_1, \dots, b_r \in P$ for the kernel of (10.1.1). Choose $c_{ji} \in P$ such that

$$f_j(a_1, \dots, a_u) = \sum c_{ji} b_i$$

in P which is possible by the choices made so far. Choose generators

$$k_1, \dots, k_t \in \mathrm{Ker}(P^{\oplus r} \xrightarrow{(b_1, \dots, b_r)} P)$$

and write $k_i = (k_{i1}, \dots, k_{ir})$ and $K = (k_{ij})$ so that

$$P^{\oplus t} \xrightarrow{K} P^{\oplus r} \xrightarrow{(b_1, \dots, b_r)} P \rightarrow R \rightarrow 0$$

is an exact sequence of P -modules. In particular we have $\sum k_{ij}b_j = 0$. After possibly increasing N we may assume $N - 1$ works in the Artin-Rees lemma for the first two maps of this exact sequence (see More on Algebra, Section 4 for terminology).

By assumption $\mathcal{O}_{S,s} = \Lambda_{\Lambda \cap \mathfrak{m}'}$ is a G-ring. Hence by More on Algebra, Proposition 49.10 the ring $\Lambda[x_1, \dots, x_s]_{\mathfrak{m}'}$ is a G-ring. Hence by Smoothing Ring Maps, Theorem 13.2 there exist an étale ring map

$$\Lambda[x_1, \dots, x_s]_{\mathfrak{m}'} \rightarrow B,$$

a maximal ideal \mathfrak{m}_B of B lying over \mathfrak{m}' , and elements $a'_i, b'_i, c'_{ij}, k'_{ij} \in B'$ such that

- (1) $\kappa(\mathfrak{m}') = \kappa(\mathfrak{m}_B)$ which implies that $\Lambda[x_1, \dots, x_s]_{\mathfrak{m}'} \subset B_{\mathfrak{m}_B} \subset P$ and P is identified with the completion of B at \mathfrak{m}_B , see remark preceding Smoothing Ring Maps, Theorem 13.2,
- (2) $a_i - a'_i, b_i - b'_i, c_{ij} - c'_{ij}, k_{ij} - k'_{ij} \in (\mathfrak{m}')^N P$, and
- (3) $f_j(a'_1, \dots, a'_u) = \sum c'_{ji}b'_i$ and $\sum k'_{ij}b'_j = 0$.

Set $A = B/(b'_1, \dots, b'_r)$ and denote \mathfrak{m}_A the image of \mathfrak{m}_B in A . (Note that A is essentially of finite type over Λ ; at the end of the proof we will show how to obtain an A which is of finite type over Λ .) There is a ring map $C \rightarrow A$ sending $y_i \mapsto a'_i$ because the a'_i satisfy the desired equations modulo (b'_1, \dots, b'_r) . Note that $A/\mathfrak{m}_A^N = R/\mathfrak{m}_R^N$ as quotients of $P = B^\wedge$ by property (2) above. Set $x_A = x_C|_{\text{Spec}(A)}$. Since the maps

$$C \rightarrow A \rightarrow A/\mathfrak{m}_A^N \cong R/\mathfrak{m}_R^N \quad \text{and} \quad C \rightarrow R \rightarrow R/\mathfrak{m}_R^N$$

are equal we see that x_A and x agree modulo \mathfrak{m}_R^N via the isomorphism $A/\mathfrak{m}_A^N = R/\mathfrak{m}_R^N$. At this point we have shown properties (1) – (5) of the statement of the lemma. To see (6) note that

$$P^{\oplus t} \xrightarrow{K} P^{\oplus r} \xrightarrow{(b_1, \dots, b_r)} P \quad \text{and} \quad P^{\oplus t} \xrightarrow{K'} P^{\oplus r} \xrightarrow{(b'_1, \dots, b'_r)} P$$

are two complexes of P -modules which are congruent modulo $(\mathfrak{m}')^N$ with the first one being exact. By our choice of N above we see from More on Algebra, Lemma 4.2 that $R = P/(b_1, \dots, b_r)$ and $P/(b'_1, \dots, b'_r) = B^\wedge/(b'_1, \dots, b'_r) = A^\wedge$ have isomorphic associated graded algebras, which is what we wanted to show.

This last paragraph of the proof serves to clean up the issue that A is essentially of finite type over S and not yet of finite type. The construction above gives $A = B/(b'_1, \dots, b'_r)$ and $\mathfrak{m}_A \subset A$ with B étale over $\Lambda[x_1, \dots, x_s]_{\mathfrak{m}'}$. Hence A is of finite type over the Noetherian ring $\Lambda[x_1, \dots, x_s]_{\mathfrak{m}'}$. Thus we can write $A = (A_0)_{\mathfrak{m}'}$ for some finite type $\Lambda[x_1, \dots, x_n]$ algebra A_0 . Then $A = \text{colim}(A_0)_f$ where $f \in \Lambda[x_1, \dots, x_n] \setminus \mathfrak{m}'$, see Algebra, Lemma 9.9. Because $p : \mathcal{X} \rightarrow (\text{Sch}/S)_{fppf}$ is limit preserving on objects, we see that x_A comes from some object $x_{(A_0)_f}$ over $\text{Spec}((A_0)_f)$ for an f as above. After replacing A by $(A_0)_f$ and x_A by $x_{(A_0)_f}$ and \mathfrak{m}_A by $(A_0)_f \cap \mathfrak{m}_A$ the proof is finished. \square

11. Limit preserving

07XK The morphism $p : \mathcal{X} \rightarrow (Sch/S)_{fppf}$ is limit preserving on objects, as defined in Criteria for Representability, Section 5, if the functor of the definition below is essentially surjective. However, the example in Examples, Section 47 shows that this isn't equivalent to being limit preserving.

07XL **Definition 11.1.** Let S be a scheme. Let \mathcal{X} be a category fibred in groupoids over $(Sch/S)_{fppf}$. We say \mathcal{X} is *limit preserving* if for every affine scheme T over S which is a limit $T = \lim T_i$ of a directed inverse system of affine schemes T_i over S , we have an equivalence

$$\operatorname{colim} \mathcal{X}_{T_i} \longrightarrow \mathcal{X}_T$$

of fibre categories.

We spell out what this means. First, given objects x, y of \mathcal{X} over T_i we should have

$$\operatorname{Mor}_{\mathcal{X}_T}(x|_T, y|_T) = \operatorname{colim}_{i' \geq i} \operatorname{Mor}_{\mathcal{X}_{T_{i'}}}(x|_{T_{i'}}, y|_{T_{i'}})$$

and second every object of \mathcal{X}_T is isomorphic to the restriction of an object over T_i for some i . Note that the first condition means that the presheaves $\operatorname{Isom}_{\mathcal{X}}(x, y)$ (see Stacks, Definition 2.2) are limit preserving.

07XM **Lemma 11.2.** *Let S be a scheme. Let $p : \mathcal{X} \rightarrow \mathcal{Y}$ and $q : \mathcal{Z} \rightarrow \mathcal{Y}$ be 1-morphisms of categories fibred in groupoids over $(Sch/S)_{fppf}$.*

- (1) *If $\mathcal{X} \rightarrow (Sch/S)_{fppf}$ and $\mathcal{Z} \rightarrow (Sch/S)_{fppf}$ are limit preserving on objects and \mathcal{Y} is limit preserving, then $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z} \rightarrow (Sch/S)_{fppf}$ is limit preserving on objects.*
- (2) *If \mathcal{X} , \mathcal{Y} , and \mathcal{Z} are limit preserving, then so is $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z}$.*

Proof. This is formal. Proof of (1). Let $T = \lim_{i \in I} T_i$ be the directed limit of affine schemes T_i over S . We will prove that the functor $\operatorname{colim} \mathcal{X}_{T_i} \rightarrow \mathcal{X}_T$ is essentially surjective. Recall that an object of the fibre product over T is a quadruple (T, x, z, α) where x is an object of \mathcal{X} lying over T , z is an object of \mathcal{Z} lying over T , and $\alpha : p(x) \rightarrow q(z)$ is a morphism in the fibre category of \mathcal{Y} over T . By assumption on \mathcal{X} and \mathcal{Z} we can find an i and objects x_i and z_i over T_i such that $x_i|_T \cong x$ and $z_i|_T \cong z$. Then α corresponds to an isomorphism $p(x_i)|_T \rightarrow q(z_i)|_T$ which comes from an isomorphism $\alpha_{i'} : p(x_i)|_{T_{i'}} \rightarrow q(z_i)|_{T_{i'}}$ by our assumption on \mathcal{Y} . After replacing i by i' , x_i by $x_i|_{T_{i'}}$, and z_i by $z_i|_{T_{i'}}$, we see that $(T_{i'}, x_i|_{T_{i'}}, z_i|_{T_{i'}}, \alpha_{i'})$ is an object of the fibre product over $T_{i'}$ which restricts to an object isomorphic to (T, x, z, α) over T as desired.

We omit the arguments showing that $\operatorname{colim} \mathcal{X}_{T_i} \rightarrow \mathcal{X}_T$ is fully faithful in (2). \square

07XN **Lemma 11.3.** *Let S be a scheme. Let \mathcal{X} be an algebraic stack over S . Then the following are equivalent*

- (1) *\mathcal{X} is a stack in setoids and $\mathcal{X} \rightarrow (Sch/S)_{fppf}$ is limit preserving on objects,*
- (2) *\mathcal{X} is a stack in setoids and limit preserving,*
- (3) *\mathcal{X} is representable by an algebraic space locally of finite presentation.*

Proof. Under each of the three assumptions \mathcal{X} is representable by an algebraic space X over S , see Algebraic Stacks, Proposition 13.3. It is clear that (1) and (2) are equivalent as a functor between setoids is an equivalence if and only if it is surjective on isomorphism classes. Finally, (1) and (3) are equivalent by Limits of Spaces, Proposition 3.8. \square

0CXI **Lemma 11.4.** *Let S be a scheme. Let \mathcal{X} be a category fibred in groupoids over $(Sch/S)_{fppf}$. Assume $\Delta : \mathcal{X} \rightarrow \mathcal{X} \times \mathcal{X}$ is representable by algebraic spaces and \mathcal{X} is limit preserving. Then Δ is locally of finite type.*

Proof. We apply Criteria for Representability, Lemma 5.6. Let V be an affine scheme of finite type over S and let θ be an object of $\mathcal{X} \times \mathcal{X}$ over V . Let F_θ be an algebraic space representing $\mathcal{X} \times_{\Delta, \mathcal{X} \times \mathcal{X}, \theta} (Sch/V)_{fppf}$ and let $f_\theta : F_\theta \rightarrow V$ be the canonical morphism (see Algebraic Stacks, Section 9). It suffices to show that $F_\theta \rightarrow V$ has the corresponding properties. By Lemmas 11.2 and 11.3 we see that $F_\theta \rightarrow S$ is locally of finite presentation. It follows that $F_\theta \rightarrow V$ is locally of finite type by Morphisms of Spaces, Lemma 23.6. \square

12. Versality

07XD In the previous section we explained how to approximate objects over complete local rings by algebraic objects. But in order to show that a stack \mathcal{X} is an algebraic stack, we need to find smooth 1-morphisms from schemes towards \mathcal{X} . Since we are not going to assume a priori that \mathcal{X} has a representable diagonal, we cannot even speak about smooth morphisms towards \mathcal{X} . Instead, borrowing terminology from deformation theory, we will introduce versal objects.

0CXJ **Definition 12.1.** Let S be a locally Noetherian scheme. Let $p : \mathcal{X} \rightarrow (Sch/S)_{fppf}$ be a category fibred in groupoids. Let $\xi = (R, \xi_n, f_n)$ be a formal object. Set $k = R/\mathfrak{m}$ and $x_0 = \xi_1$. We will say that ξ is *versal* if ξ as a formal object of $\mathcal{F}_{\mathcal{X}, k, x_0}$ (Remark 9.2) is versal in the sense of Formal Deformation Theory, Definition 8.9.

We briefly spell out what this means. With notation as in the definition, suppose given morphisms $\xi_1 = x_0 \rightarrow y \rightarrow z$ of \mathcal{X} lying over closed immersions $\text{Spec}(k) \rightarrow \text{Spec}(A) \rightarrow \text{Spec}(B)$ where A, B are Artinian local rings with residue field k . Suppose given an $n \geq 1$ and a commutative diagram

$$\begin{array}{ccc}
 & y & \\
 \swarrow & \uparrow & \\
 \xi_n & \longleftarrow \xi_1 &
 \end{array}
 \quad \text{lying over} \quad
 \begin{array}{ccc}
 & \text{Spec}(A) & \\
 \swarrow & \uparrow & \\
 \text{Spec}(R/\mathfrak{m}^n) & \longleftarrow \text{Spec}(k) &
 \end{array}$$

Versality means that for any data as above there exists an $m \geq n$ and a commutative diagram

$$\begin{array}{ccc}
 & z & \\
 \swarrow & \uparrow & \\
 \xi_m & \longleftarrow \xi_n & \longleftarrow \xi_1
 \end{array}
 \quad \text{lying over} \quad
 \begin{array}{ccc}
 & \text{Spec}(B) & \\
 \swarrow & \uparrow & \\
 \text{Spec}(R/\mathfrak{m}^m) & \longleftarrow \text{Spec}(R/\mathfrak{m}^n) & \longleftarrow \text{Spec}(k)
 \end{array}$$

Please compare with Formal Deformation Theory, Remark 8.10.

Let S be a locally Noetherian scheme. Let U be a scheme over S with structure morphism $U \rightarrow S$ locally of finite type. Let $u_0 \in U$ be a finite type point of U , see Morphisms, Definition 15.3. Set $k = \kappa(u_0)$. Note that the composition $\text{Spec}(k) \rightarrow S$ is also of finite type, see Morphisms, Lemma 14.3. Let $p : \mathcal{X} \rightarrow (Sch/S)_{fppf}$ be a

category fibred in groupoids. Let x be an object of \mathcal{X} which lies over U . Denote x_0 the pullback of x by u_0 . By the 2-Yoneda lemma x corresponds to a 1-morphism

$$x : (Sch/U)_{fppf} \longrightarrow \mathcal{X},$$

see Algebraic Stacks, Section 5. We obtain a morphism of predeformation categories

$$07XE \quad (12.1.1) \quad \hat{x} : \mathcal{F}_{(Sch/U)_{fppf}, k, u_0} \longrightarrow \mathcal{F}_{\mathcal{X}, k, x_0},$$

over \mathcal{C}_Λ see (3.1.1).

07XF **Definition 12.2.** Let S be a locally Noetherian scheme. Let \mathcal{X} be fibred in groupoids over $(Sch/S)_{fppf}$. Let U be a scheme locally of finite type over S . Let x be an object of \mathcal{X} lying over U . Let u_0 be finite type point of U . We say x is *versal* at u_0 if the morphism \hat{x} (12.1.1) is smooth, see Formal Deformation Theory, Definition 8.1.

This definition matches our notion of versality for formal objects of \mathcal{X} .

0CXK **Lemma 12.3.** *With notation as in Definition 12.2. Let $R = \mathcal{O}_{U, u_0}^\wedge$. Let ξ be the formal object of \mathcal{X} over R associated to $x|_{\text{Spec}(R)}$, see (9.3.1). Then*

$$x \text{ is versal at } u_0 \Leftrightarrow \xi \text{ is versal}$$

Proof. Observe that \mathcal{O}_{U, u_0} is a Noetherian local S -algebra with residue field k . Hence $R = \mathcal{O}_{U, u_0}^\wedge$ is an object of $\mathcal{C}_\Lambda^\wedge$, see Formal Deformation Theory, Definition 4.1. Recall that ξ is versal if $\underline{\xi} : \underline{R}|_{\mathcal{C}_\Lambda} \rightarrow \mathcal{F}_{\mathcal{X}, k, x_0}$ is smooth and x is versal at u_0 if $\hat{x} : \mathcal{F}_{(Sch/U)_{fppf}, k, u_0} \rightarrow \mathcal{F}_{\mathcal{X}, k, x_0}$ is smooth. There is an identification of predeformation categories

$$\underline{R}|_{\mathcal{C}_\Lambda} = \mathcal{F}_{(Sch/U)_{fppf}, k, u_0},$$

see Formal Deformation Theory, Remark 7.12 for notation. Namely, given an Artinian local S -algebra A with residue field identified with k we have

$$\text{Mor}_{\mathcal{C}_\Lambda^\wedge}(R, A) = \{\varphi \in \text{Mor}_S(\text{Spec}(A), U) \mid \varphi|_{\text{Spec}(k)} = u_0\}$$

Unwinding the definitions the reader verifies that the resulting map

$$\underline{R}|_{\mathcal{C}_\Lambda} = \mathcal{F}_{(Sch/U)_{fppf}, k, u_0} \xrightarrow{\hat{x}} \mathcal{F}_{\mathcal{X}, k, x_0},$$

is equal to $\underline{\xi}$ and we see that the lemma is true. \square

Here is a sanity check.

0CXL **Lemma 12.4.** *Let S be a locally Noetherian scheme. Let $f : U \rightarrow V$ be a morphism of schemes locally of finite type over S . Let $u_0 \in U$ be a finite type point. The following are equivalent*

- (1) f is smooth at u_0 ,
- (2) f viewed as an object of $(Sch/V)_{fppf}$ over U is versal at u_0 .

Proof. This is a restatement of More on Morphisms, Lemma 12.1. \square

It turns out that this notion is well behaved with respect to field extensions.

07XG **Lemma 12.5.** *Let $S, \mathcal{X}, U, x, u_0$ be as in Definition 12.2. Let l be a field and let $u_{l,0} : \text{Spec}(l) \rightarrow U$ be a morphism with image u_0 such that $l/k = \kappa(u_0)$ is finite. Set $x_{l,0} = x_0|_{\text{Spec}(l)}$. If \mathcal{X} satisfies (RS) and x is versal at u_0 , then*

$$\mathcal{F}_{(Sch/U)_{fppf}, l, u_{l,0}} \longrightarrow \mathcal{F}_{\mathcal{X}, l, x_{l,0}}$$

is smooth.

Proof. Note that $(Sch/U)_{fppf}$ satisfies (RS) by Lemma 5.2. Hence the functor of the lemma is the functor

$$(\mathcal{F}_{(Sch/U)_{fppf}, k, u_0})_{l/k} \longrightarrow (\mathcal{F}_{\mathcal{X}, k, x_0})_{l/k}$$

associated to \hat{x} , see Lemma 7.1. Hence the lemma follows from Formal Deformation Theory, Lemma 29.5. \square

The following lemma is another sanity check. It more or less signifies that if x is versal at u_0 as in Definition 12.2, then x viewed as a morphism from U to \mathcal{X} is smooth whenever we make a base change by a scheme.

0CXM **Lemma 12.6.** *Let $S, \mathcal{X}, U, x, u_0$ be as in Definition 12.2. Assume*

- (1) $\Delta : \mathcal{X} \rightarrow \mathcal{X} \times \mathcal{X}$ is representable by algebraic spaces,
- (2) \mathcal{X} is limit preserving, and
- (3) \mathcal{X} has (RS).

Let V be a scheme locally of finite type over S and let y be an object of \mathcal{X} over V . Form the 2-fibre product

$$\begin{array}{ccc} \mathcal{Z} & \longrightarrow & (Sch/U)_{fppf} \\ \downarrow & & \downarrow x \\ (Sch/V)_{fppf} & \xrightarrow{y} & \mathcal{X} \end{array}$$

Let Z be the algebraic space representing \mathcal{Z} and let $z_0 \in |Z|$ be a finite type point lying over u_0 . If x is versal at u_0 , then the morphism $Z \rightarrow V$ is smooth at z_0 .

Proof. Observe that Z exists by Algebraic Stacks, Lemma 10.11. By Lemma 11.4 we see that $Z \rightarrow V \times_S U$ is locally of finite type. Choose a scheme W , a closed point $w_0 \in W$, and an étale morphism $W \rightarrow Z$ mapping w_0 to z_0 , see Morphisms of Spaces, Definition 25.2. Then W is locally of finite type over S and w_0 is a finite type point of W . Let $l = \kappa(z_0)$. Denote $z_{l,0}, v_{l,0}, u_{l,0}$, and $x_{l,0}$ the objects of $\mathcal{Z}, (Sch/V)_{fppf}, (Sch/U)_{fppf}$, and \mathcal{X} over $\text{Spec}(l)$ obtained by pullback to $\text{Spec}(l) = w_0$. Consider

$$\begin{array}{ccccc} \mathcal{F}_{(Sch/W)_{fppf}, l, w_0} & \longrightarrow & \mathcal{F}_{\mathcal{Z}, l, z_{l,0}} & \longrightarrow & \mathcal{F}_{(Sch/U)_{fppf}, l, u_{l,0}} \\ & & \downarrow & & \downarrow \\ & & \mathcal{F}_{(Sch/V)_{fppf}, l, v_{l,0}} & \longrightarrow & \mathcal{F}_{\mathcal{X}, l, x_{l,0}} \end{array}$$

By Lemma 3.3 the square is a fibre product of predeformation categories. By Lemma 12.5 we see that the right vertical arrow is smooth. By Formal Deformation Theory, Lemma 8.7 the left vertical arrow is smooth. By Lemma 3.2 we see that the left horizontal arrow is smooth. We conclude that the map

$$\mathcal{F}_{(Sch/W)_{fppf}, l, w_0} \rightarrow \mathcal{F}_{(Sch/V)_{fppf}, l, v_{l,0}}$$

is smooth by Formal Deformation Theory, Lemma 8.7. Thus we conclude that $W \rightarrow V$ is smooth at w_0 by More on Morphisms, Lemma 12.1. This exactly means that $Z \rightarrow V$ is smooth at z_0 and the proof is complete. \square

We restate the approximation result in terms of versal objects.

07XH **Lemma 12.7.** *Let S be a locally Noetherian scheme. Let $p : \mathcal{X} \rightarrow (Sch/S)_{fppf}$ be a category fibred in groupoids. Let $\xi = (R, \xi_n, f_n)$ be a formal object of \mathcal{X} with ξ_1 lying over $\text{Spec}(k) \rightarrow S$ with image $s \in S$. Assume*

- (1) ξ is versal,
- (2) ξ is effective,
- (3) $\mathcal{O}_{S,s}$ is a G-ring, and
- (4) $p : \mathcal{X} \rightarrow (Sch/S)_{fppf}$ is limit preserving on objects.

Then there exist a morphism of finite type $U \rightarrow S$, a finite type point $u_0 \in U$ with residue field k , and an object x of \mathcal{X} over U such that x is versal at u_0 and such that $x|_{\text{Spec}(\mathcal{O}_{U,u_0}/\mathfrak{m}_{u_0}^n)} \cong \xi_n$.

Proof. Choose an object x_R of \mathcal{X} lying over $\text{Spec}(R)$ whose associated formal object is ξ . Let $N = 2$ and apply Lemma 10.1. We obtain $A, \mathfrak{m}_A, x_A, \dots$. Let $\eta = (A^\wedge, \eta_n, g_n)$ be the formal object associated to $x_A|_{\text{Spec}(A^\wedge)}$. We have a diagram

$$\begin{array}{ccc}
 & \nearrow \eta & \\
 \xi & \longrightarrow & \xi_2 = \eta_2 \\
 & \searrow & \\
 & & \downarrow \\
 & & \eta
 \end{array}
 \quad \text{lying over} \quad
 \begin{array}{ccc}
 & \nearrow A^\wedge & \\
 R & \longrightarrow & R/\mathfrak{m}_R^2 = A/\mathfrak{m}_A^2 \\
 & \searrow & \\
 & & \downarrow \\
 & & A^\wedge
 \end{array}$$

The versality of ξ means exactly that we can find the dotted arrows in the diagrams, because we can successively find morphisms $\xi \rightarrow \eta_3$, $\xi \rightarrow \eta_4$, and so on by Formal Deformation Theory, Remark 8.10. The corresponding ring map $R \rightarrow A^\wedge$ is surjective by Formal Deformation Theory, Lemma 4.2. On the other hand, we have $\dim_k \mathfrak{m}_R^n/\mathfrak{m}_R^{n+1} = \dim_k \mathfrak{m}_A^n/\mathfrak{m}_A^{n+1}$ for all n by construction. Hence R/\mathfrak{m}_R^n and A/\mathfrak{m}_A^n have the same (finite) length as Λ -modules by additivity of length and Formal Deformation Theory, Lemma 3.4. It follows that $R/\mathfrak{m}_R^n \rightarrow A/\mathfrak{m}_A^n$ is an isomorphism for all n , hence $R \rightarrow A^\wedge$ is an isomorphism. Thus η is isomorphic to a versal object, hence versal itself. By Lemma 12.3 we conclude that x_A is versal at the point u_0 of $U = \text{Spec}(A)$ corresponding to \mathfrak{m}_A . \square

07XI **Example 12.8.** In this example we show that the local ring $\mathcal{O}_{S,s}$ has to be a G-ring in order for the result of Lemma 12.7 to be true. Namely, let Λ be a Noetherian ring and let \mathfrak{m} be a maximal ideal of Λ . Set $R = \Lambda_\mathfrak{m}^\wedge$. Let $\Lambda \rightarrow C \rightarrow R$ be a factorization with C of finite type over Λ . Set $S = \text{Spec}(\Lambda)$, $U = S \setminus \{\mathfrak{m}\}$, and $S' = U \amalg \text{Spec}(C)$. Consider the functor $F : (Sch/S)_{fppf}^{opp} \rightarrow Sets$ defined by the rule

$$F(T) = \begin{cases} * & \text{if } T \rightarrow S \text{ factors through } S' \\ \emptyset & \text{else} \end{cases}$$

Let $\mathcal{X} = \mathcal{S}_F$ is the category fibred in sets associated to F , see Algebraic Stacks, Section 7. Then $\mathcal{X} \rightarrow (Sch/S)_{fppf}$ is limit preserving on objects and there exists an effective, versal formal object ξ over R . Hence if the conclusion of Lemma 12.7 holds for \mathcal{X} , then there exists a finite type ring map $\Lambda \rightarrow A$ and a maximal ideal \mathfrak{m}_A lying over \mathfrak{m} such that

- (1) $\kappa(\mathfrak{m}) = \kappa(\mathfrak{m}_A)$,
- (2) $\Lambda \rightarrow A$ and \mathfrak{m}_A satisfy condition (4) of Algebra, Lemma 140.2, and
- (3) there exists a Λ -algebra map $C \rightarrow A$.

Thus $\Lambda \rightarrow A$ is smooth at \mathfrak{m}_A by the lemma cited. Slicing A we may assume that $\Lambda \rightarrow A$ is étale at \mathfrak{m}_A , see for example More on Morphisms, Lemma 34.5 or argue directly. Write $C = \Lambda[y_1, \dots, y_n]/(f_1, \dots, f_m)$. Then $C \rightarrow R$ corresponds to a solution in R of the system of equations $f_1 = \dots = f_m = 0$, see Smoothing Ring Maps, Section 13. Thus if the conclusion of Lemma 12.7 holds for every \mathcal{X} as above, then a system of equations which has a solution in R has a solution in the henselization of $\Lambda_{\mathfrak{m}}$. In other words, the approximation property holds for $\Lambda_{\mathfrak{m}}^h$. This implies that $\Lambda_{\mathfrak{m}}^h$ is a G-ring (insert future reference here; see also discussion in Smoothing Ring Maps, Section 1) which in turn implies that $\Lambda_{\mathfrak{m}}$ is a G-ring.

13. Openness of versality

07XP Next, we come to openness of versality.

07XQ **Definition 13.1.** Let S be a locally Noetherian scheme.

- (1) Let \mathcal{X} be a category fibred in groupoids over $(Sch/S)_{fppf}$. We say \mathcal{X} satisfies *openness of versality* if given a scheme U locally of finite type over S , an object x of \mathcal{X} over U , and a finite type point $u_0 \in U$ such that x is versal at u_0 , then there exists an open neighbourhood $u_0 \in U' \subset U$ such that x is versal at every finite type point of U' .
- (2) Let $f : \mathcal{X} \rightarrow \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. We say f satisfies *openness of versality* if given a scheme U locally of finite type over S , an object y of \mathcal{Y} over U , openness of versality holds for $(Sch/U)_{fppf} \times_{\mathcal{Y}} \mathcal{X}$.

Openness of versality is often the hardest to check. The following example shows that requiring this is necessary however.

07XR **Example 13.2.** Let k be a field and set $\Lambda = k[s, t]$. Consider the functor $F : \Lambda\text{-algebras} \rightarrow Sets$ defined by the rule

$$F(A) = \begin{cases} * & \text{if there exist } f_1, \dots, f_n \in A \text{ such that} \\ & A = (s, t, f_1, \dots, f_n) \text{ and } f_i s = 0 \forall i \\ \emptyset & \text{else} \end{cases}$$

Geometrically $F(A) = *$ means there exists a quasi-compact open neighbourhood W of $V(s, t) \subset \text{Spec}(A)$ such that $s|_W = 0$. Let $\mathcal{X} \subset (Sch/\text{Spec}(\Lambda))_{fppf}$ be the full subcategory consisting of schemes T which have an affine open covering $T = \bigcup \text{Spec}(A_j)$ with $F(A_j) = *$ for all j . Then \mathcal{X} satisfies [0], [1], [2], [3], and [4] but not [5]. Namely, over $U = \text{Spec}(k[s, t]/(s))$ there exists an object x which is versal at $u_0 = (s, t)$ but not at any other point. Details omitted.

Let S be a locally Noetherian scheme. Let $f : \mathcal{X} \rightarrow \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. Consider the following property

07XS (13.2.1) for all fields k of finite type over S and all $x_0 \in \text{Ob}(\mathcal{X}_{\text{Spec}(k)})$ the map $\mathcal{F}_{\mathcal{X}, k, x_0} \rightarrow \mathcal{F}_{\mathcal{Y}, k, f(x_0)}$ of predeformation categories is smooth

We formulate some lemmas around this concept. First we link it with (openness of) versality.

07XT **Lemma 13.3.** *Let S be a locally Noetherian scheme. Let \mathcal{X} be a category fibred in groupoids over $(Sch/S)_{fppf}$. Let U be a scheme locally of finite type over S . Let x be an object of \mathcal{X} over U . Assume that x is versal at every finite type point of U and that \mathcal{X} satisfies (RS). Then $x : (Sch/U)_{fppf} \rightarrow \mathcal{X}$ satisfies (13.2.1).*

Proof. Let $\text{Spec}(l) \rightarrow U$ be a morphism with l of finite type over S . Then the image $u_0 \in U$ is a finite type point of U and $\kappa(u_0) \subset l$ is a finite extension, see discussion in Morphisms, Section 15. Hence we see that $\mathcal{F}_{(Sch/U)_{fppf}, l, u_{1,0}} \rightarrow \mathcal{F}_{\mathcal{X}, l, x_{1,0}}$ is smooth by Lemma 12.5. \square

07XU **Lemma 13.4.** *Let S be a locally Noetherian scheme. Let $f : \mathcal{X} \rightarrow \mathcal{Y}$ and $g : \mathcal{Y} \rightarrow \mathcal{Z}$ be composable 1-morphisms of categories fibred in groupoids over $(Sch/S)_{fppf}$. If f and g satisfy (13.2.1) so does $g \circ f$.*

Proof. This follows formally from Formal Deformation Theory, Lemma 8.7. \square

07XV **Lemma 13.5.** *Let S be a locally Noetherian scheme. Let $f : \mathcal{X} \rightarrow \mathcal{Y}$ and $\mathcal{Z} \rightarrow \mathcal{Y}$ be 1-morphisms of categories fibred in groupoids over $(Sch/S)_{fppf}$. If f satisfies (13.2.1) so does the projection $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z} \rightarrow \mathcal{Z}$.*

Proof. Follows immediately from Lemma 3.3 and Formal Deformation Theory, Lemma 8.7. \square

07XW **Lemma 13.6.** *Let S be a locally Noetherian scheme. Let $f : \mathcal{X} \rightarrow \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. If f is formally smooth on objects, then f satisfies (13.2.1). If f is representable by algebraic spaces and smooth, then f satisfies (13.2.1).*

Proof. A reformulation of Lemma 3.2. \square

07XX **Lemma 13.7.** *Let S be a locally Noetherian scheme. Let $f : \mathcal{X} \rightarrow \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. Assume*

- (1) f is representable by algebraic spaces,
- (2) f satisfies (13.2.1),
- (3) $\mathcal{X} \rightarrow (Sch/S)_{fppf}$ is limit preserving on objects, and
- (4) \mathcal{Y} is limit preserving.

Then f is smooth.

Proof. The key ingredient of the proof is More on Morphisms, Lemma 12.1 which (almost) says that a morphism of schemes of finite type over S satisfying (13.2.1) is a smooth morphism. The other arguments of the proof are essentially bookkeeping.

Let V be a scheme over S and let y be an object of \mathcal{Y} over V . Let Z be an algebraic space representing the 2-fibre product $Z = \mathcal{X} \times_{f, \mathcal{X}, y} (Sch/V)_{fppf}$. We have to show that the projection morphism $Z \rightarrow V$ is smooth, see Algebraic Stacks, Definition 10.1. In fact, it suffices to do this when V is an affine scheme locally of finite presentation over S , see Criteria for Representability, Lemma 5.6. Then $(Sch/V)_{fppf}$ is limit preserving by Lemma 11.3. Hence $Z \rightarrow S$ is locally of finite presentation by Lemmas 11.2 and 11.3. Choose a scheme W and a surjective étale morphism $W \rightarrow Z$. Then W is locally of finite presentation over S .

Since f satisfies (13.2.1) we see that so does $Z \rightarrow (Sch/V)_{fppf}$, see Lemma 13.5. Next, we see that $(Sch/W)_{fppf} \rightarrow Z$ satisfies (13.2.1) by Lemma 13.6. Thus the composition

$$(Sch/W)_{fppf} \rightarrow Z \rightarrow (Sch/V)_{fppf}$$

satisfies (13.2.1) by Lemma 13.4. More on Morphisms, Lemma 12.1 shows that the composition $W \rightarrow Z \rightarrow V$ is smooth at every finite type point w_0 of W . Since the smooth locus is open we conclude that $W \rightarrow V$ is a smooth morphism of schemes by Morphisms, Lemma 15.7. Thus we conclude that $Z \rightarrow V$ is a smooth morphism of algebraic spaces by definition. \square

The lemma below is how we will use openness of versality.

07XY **Lemma 13.8.** *Let S be a locally Noetherian scheme. Let $p : \mathcal{X} \rightarrow (Sch/S)_{fppf}$ be a category fibred in groupoids. Let k be a finite type field over S and let x_0 be an object of \mathcal{X} over $\text{Spec}(k)$ with image $s \in S$. Assume*

- (1) $\Delta : \mathcal{X} \rightarrow \mathcal{X} \times \mathcal{X}$ is representable by algebraic spaces,
- (2) \mathcal{X} satisfies axioms [1], [2], [3] (see Section 14),
- (3) every formal object of \mathcal{X} is effective,
- (4) openness of versality holds for \mathcal{X} , and
- (5) $\mathcal{O}_{S,s}$ is a G -ring.

Then there exist a morphism of finite type $U \rightarrow S$ and an object x of \mathcal{X} over U such that

$$x : (Sch/U)_{fppf} \longrightarrow \mathcal{X}$$

is smooth and such that there exists a finite type point $u_0 \in U$ whose residue field is k and such that $x|_{u_0} \cong x_0$.

Proof. By axiom [2], Lemma 6.1, and Remark 6.2 we see that $\mathcal{F}_{\mathcal{X},k,x_0}$ satisfies (S1) and (S2). Since also the tangent space has finite dimension by axiom [3] we deduce from Formal Deformation Theory, Lemma 13.4 that $\mathcal{F}_{\mathcal{X},k,x_0}$ has a versal formal object ξ . Assumption (3) says ξ is effective. By axiom [1] and Lemma 12.7 there exists a morphism of finite type $U \rightarrow S$, an object x of \mathcal{X} over U , and a finite type point u_0 of U with residue field k such that x is versal at u_0 and such that $x|_{\text{Spec}(k)} \cong x_0$. By openness of versality we may shrink U and assume that x is versal at every finite type point of U . We claim that

$$x : (Sch/U)_{fppf} \longrightarrow \mathcal{X}$$

is smooth which proves the lemma. Namely, by Lemma 13.3 x satisfies (13.2.1) whereupon Lemma 13.7 finishes the proof. \square

14. Axioms

07XJ Let S be a locally Noetherian scheme. Let $p : \mathcal{X} \rightarrow (Sch/S)_{fppf}$ be a category fibred in groupoids. Here are the axioms we will consider on \mathcal{X} .

- [-1] a set theoretic condition¹ to be ignored by readers who are not interested in set theoretical issues,
- [0] \mathcal{X} is a stack in groupoids for the étale topology,
- [1] \mathcal{X} is limit preserving,
- [2] \mathcal{X} satisfies the Rim-Schlessinger condition (RS),
- [3] the spaces $T\mathcal{F}_{\mathcal{X},k,x_0}$ and $\text{Inf}(\mathcal{F}_{\mathcal{X},k,x_0})$ are finite dimensional for every k and x_0 , see (8.0.1) and (8.0.2),
- [4] the functor (9.3.1) is an equivalence,

¹The condition is the following: the supremum of all the cardinalities $|\text{Ob}(\mathcal{X}_{\text{Spec}(k)})| \cong |\text{Arrows}(\mathcal{X}_{\text{Spec}(k)})|$ where k runs over the finite type fields over S is \leq than the size of some object of $(Sch/S)_{fppf}$.

[5] \mathcal{X} and $\Delta : \mathcal{X} \rightarrow \mathcal{X} \times \mathcal{X}$ satisfy openness of versality.

15. Axioms for functors

07XZ Let S be a scheme. Let $F : (Sch/S)_{fppf}^{opp} \rightarrow Sets$ be a functor. Denote $\mathcal{X} = \mathcal{S}_F$ the category fibred in sets associated to F , see Algebraic Stacks, Section 7. In this section we provide a translation between the material above as it applies to \mathcal{X} , to statements about F .

Let S be a locally Noetherian scheme. Let $F : (Sch/S)_{fppf}^{opp} \rightarrow Sets$ be a functor. Let k be a field of finite type over S . Let $x_0 \in F(\text{Spec}(k))$. The associated predeformation category (3.0.2) corresponds to the functor

$$F_{k,x_0} : \mathcal{C}_\Lambda \longrightarrow Sets, \quad A \longmapsto \{x \in F(\text{Spec}(A)) \mid x|_{\text{Spec}(k)} = x_0\}.$$

Recall that we do not distinguish between categories cofibred in sets over \mathcal{C}_Λ and functor $\mathcal{C}_\Lambda \rightarrow Sets$, see Formal Deformation Theory, Remarks 5.2 (11). Given a transformation of functors $a : F \rightarrow G$, setting $y_0 = a(x_0)$ we obtain a morphism

$$F_{k,x_0} \longrightarrow G_{k,y_0}$$

see (3.1.1). Lemma 3.2 tells us that if $a : F \rightarrow G$ is formally smooth (in the sense of More on Morphisms of Spaces, Definition 13.1), then $F_{k,x_0} \rightarrow G_{k,y_0}$ is smooth as in Formal Deformation Theory, Remark 8.4.

Lemma 4.1 says that if $Y' = Y \amalg_X X'$ in the category of schemes over S where $X \rightarrow X'$ is a thickening and $X \rightarrow Y$ is affine, then the map

$$F(Y \amalg_X X') \rightarrow F(Y) \times_{F(X)} F(X')$$

is a bijection, provided that F is an algebraic space. We say a general functor F satisfies the *Rim-Schlessinger condition* or we say F *satisfies (RS)* if given any pushout $Y' = Y \amalg_X X'$ where Y, X, X' are spectra of Artinian local rings of finite type over S , then

$$F(Y \amalg_X X') \rightarrow F(Y) \times_{F(X)} F(X')$$

is a bijection. Thus every algebraic space satisfies (RS).

Lemma 6.1 says that given a functor F which satisfies (RS), then all F_{k,x_0} are deformation functors as in Formal Deformation Theory, Definition 16.8, i.e., they satisfy (RS) as in Formal Deformation Theory, Remark 16.5. In particular the tangent space

$$TF_{k,x_0} = \{x \in F(\text{Spec}(k[\epsilon])) \mid x|_{\text{Spec}(k)} = x_0\}$$

has the structure of a k -vector space by Formal Deformation Theory, Lemma 12.2.

Lemma 8.1 says that an algebraic space F locally of finite type over S gives rise to deformation functors F_{k,x_0} with finite dimensional tangent spaces TF_{k,x_0} .

A *formal object*² $\xi = (R, \xi_n)$ of F consists of a Noetherian complete local S -algebra R whose residue field is of finite type over S , together with elements $\xi_n \in F(\text{Spec}(R/\mathfrak{m}^n))$ such that $\xi_{n+1}|_{\text{Spec}(R/\mathfrak{m}^n)} = \xi_n$. A formal object ξ defines a formal object ξ of $F_{R/\mathfrak{m}, \xi_1}$. We say ξ is *versal* if and only if it is versal in the sense of Formal Deformation Theory, Definition 8.9. A formal object $\xi = (R, \xi_n)$ is called *effective* if there exists an $x \in F(\text{Spec}(R))$ such that $\xi_n = x|_{\text{Spec}(R/\mathfrak{m}^n)}$ for

²This is what Artin calls a formal deformation.

all $n \geq 1$. Lemma 9.5 says that if F is an algebraic space, then every formal object is effective.

Let U be a scheme locally of finite type over S and let $x \in F(U)$. Let $u_0 \in U$ be a finite type point. We say that x is versal at u_0 if and only if $\xi = (\mathcal{O}_{U, u_0}^\wedge, x|_{\text{Spec}(\mathcal{O}_{U, u_0}/\mathfrak{m}_{u_0}^n)})$ is a versal formal object in the sense described above.

Let S be a locally Noetherian scheme. Let $F : (Sch/S)_{fppf}^{opp} \rightarrow Sch$ be a functor. Here are the axioms we will consider on F .

- [-1] a set theoretic condition³ to be ignored by readers who are not interested in set theoretical issues,
- [0] F is a sheaf for the étale topology,
- [1] F is limit preserving,
- [2] F satisfies the Rim-Schlessinger condition (RS),
- [3] every tangent space TF_{k, x_0} is finite dimensional,
- [4] every formal object is effective,
- [5] F satisfies openness of versality.

Here *limit preserving* is the notion defined in Limits of Spaces, Definition 3.1 and *openness of versality* means the following: Given a scheme U locally of finite type over S , given $x \in F(U)$, and given a finite type point $u_0 \in U$ such that x is versal at u_0 , then there exists an open neighbourhood $u_0 \in U' \subset U$ such that x is versal at every finite type point of U' .

16. Algebraic spaces

07Y0 The following is our first main result on algebraic spaces.

07Y1 **Proposition 16.1.** *Let S be a locally Noetherian scheme. Let $F : (Sch/S)_{fppf}^{opp} \rightarrow Sets$ be a functor. Assume that*

- (1) $\Delta : F \rightarrow F \times F$ is representable by algebraic spaces,
- (2) F satisfies axioms [-1], [0], [1], [2], [3], [4], [5] (see Section 15), and
- (3) $\mathcal{O}_{S, s}$ is a G -ring for all finite type points s of S .

Then F is an algebraic space.

Proof. Lemma 13.8 applies to F . Using this we choose, for every finite type field k over S and $x_0 \in F(\text{Spec}(k))$, an affine scheme U_{k, x_0} of finite type over S and a smooth morphism $U_{k, x_0} \rightarrow F$ such that there exists a finite type point $u_{k, x_0} \in U_{k, x_0}$ with residue field k such that x_0 is the image of u_{k, x_0} . Then

$$U = \coprod_{k, x_0} U_{k, x_0} \longrightarrow F$$

is smooth⁴. To finish the proof it suffices to show this map is surjective, see Bootstrap, Lemma 12.3 (this is where we use axiom [0]). By Criteria for Representability, Lemma 5.6 it suffices to show that $U \times_F V \rightarrow V$ is surjective for those $V \rightarrow F$ where V is an affine scheme locally of finite presentation over S . Since $U \times_F V \rightarrow V$ is smooth the image is open. Hence it suffices to show that the image of $U \times_F V \rightarrow V$ contains all finite type points of V , see Morphisms, Lemma 15.7. Let $v_0 \in V$ be

³The condition is the following: the supremum of all the cardinalities $|F(\text{Spec}(k))|$ where k runs over the finite type fields over S is \leq than the size of some object of $(Sch/S)_{fppf}$.

⁴Set theoretical remark: This coproduct is (isomorphic) to an object of $(Sch/S)_{fppf}$ as we have a bound on the index set by axiom [-1], see Sets, Lemma 9.9.

a finite type point. Then $k = \kappa(v_0)$ is a finite type field over S . Denote x_0 the composition $\text{Spec}(k) \xrightarrow{v_0} V \rightarrow F$. Then $(u_{k,x_0}, v_0) : \text{Spec}(k) \rightarrow U \times_F V$ is a point mapping to v_0 and we win. \square

07Y2 **Lemma 16.2.** *Let S be a locally Noetherian scheme. Let $a : F \rightarrow G$ be a transformation of functors $(\text{Sch}/S)_{fppf}^{opp} \rightarrow \text{Sets}$. Assume that*

- (1) a is injective,
- (2) F satisfies axioms [0], [1], [2], [4], and [5],
- (3) $\mathcal{O}_{S,s}$ is a G -ring for all finite type points s of S ,
- (4) G is an algebraic space locally of finite type over S ,

Then F is an algebraic space.

Proof. By Lemma 8.1 the functor G satisfies [3]. As $F \rightarrow G$ is injective, we conclude that F also satisfies [3]. Moreover, as $F \rightarrow G$ is injective, we see that given schemes U, V and morphisms $U \rightarrow F$ and $V \rightarrow F$, then $U \times_F V = U \times_G V$. Hence $\Delta : F \rightarrow F \times F$ is representable (by schemes) as this holds for G by assumption. Thus Proposition 16.1 applies⁵. \square

17. Algebraic stacks

07Y3 Proposition 17.2 is our first main result on algebraic stacks.

07Y4 **Lemma 17.1.** *Let S be a locally Noetherian scheme. Let $p : \mathcal{X} \rightarrow (\text{Sch}/S)_{fppf}$ be a category fibred in groupoids. Assume that*

- (1) $\Delta : \mathcal{X} \rightarrow \mathcal{X} \times \mathcal{X}$ is representable by algebraic spaces,
- (2) \mathcal{X} satisfies axioms [-1], [0], [1], [2], [3] (see Section 14),
- (3) every formal object of \mathcal{X} is effective,
- (4) \mathcal{X} satisfies openness of versality, and
- (5) $\mathcal{O}_{S,s}$ is a G -ring for all finite type points s of S .

Then \mathcal{X} is an algebraic stack.

Proof. Lemma 13.8 applies to \mathcal{X} . Using this we choose, for every finite type field k over S and every isomorphism class of object $x_0 \in \text{Ob}(\mathcal{X}_{\text{Spec}(k)})$, an affine scheme U_{k,x_0} of finite type over S and a smooth morphism $(\text{Sch}/U_{k,x_0})_{fppf} \rightarrow \mathcal{X}$ such that there exists a finite type point $u_{k,x_0} \in U_{k,x_0}$ with residue field k such that x_0 is the image of u_{k,x_0} . Then

$$(\text{Sch}/U)_{fppf} \rightarrow \mathcal{X}, \quad \text{with } U = \coprod_{k,x_0} U_{k,x_0}$$

is smooth⁶. To finish the proof it suffices to show this map is surjective, see Criteria for Representability, Lemma 19.1 (this is where we use axiom [0]). By Criteria for Representability, Lemma 5.6 it suffices to show that $(\text{Sch}/U)_{fppf} \times_{\mathcal{X}} (\text{Sch}/V)_{fppf} \rightarrow (\text{Sch}/V)_{fppf}$ is surjective for those $y : (\text{Sch}/V)_{fppf} \rightarrow \mathcal{X}$ where V is an affine scheme locally of finite presentation over S . By assumption (1) the fibre product $(\text{Sch}/U)_{fppf} \times_{\mathcal{X}} (\text{Sch}/V)_{fppf}$ is representable by an algebraic space W . Then $W \rightarrow V$ is smooth, hence the image is open. Hence it suffices to show that the image of $W \rightarrow V$ contains all finite type points of V , see Morphisms, Lemma 15.7. Let $v_0 \in V$ be a finite type point. Then $k = \kappa(v_0)$ is a finite type field over S . Denote

⁵The set theoretic condition [-1] holds for F as it holds for G . Details omitted.

⁶Set theoretical remark: This coproduct is (isomorphic to) an object of $(\text{Sch}/S)_{fppf}$ as we have a bound on the index set by axiom [-1], see Sets, Lemma 9.9.

$x_0 = y|_{\mathrm{Spec}(k)}$ the pullback of y by v_0 . Then (u_{k,x_0}, v_0) will give a morphism $\mathrm{Spec}(k) \rightarrow W$ whose composition with $W \rightarrow V$ is v_0 and we win. \square

07Y5 **Proposition 17.2.** *Let S be a locally Noetherian scheme. Let $p : \mathcal{X} \rightarrow (\mathrm{Sch}/S)_{\mathrm{fppf}}$ be a category fibred in groupoids. Assume that*

- (1) $\Delta_{\Delta} : \mathcal{X} \rightarrow \mathcal{X} \times_{\mathcal{X} \times \mathcal{X}} \mathcal{X}$ is representable by algebraic spaces,
- (2) \mathcal{X} satisfies axioms [-1], [0], [1], [2], [3], [4], and [5] (see Section 14),
- (3) $\mathcal{O}_{S,s}$ is a G -ring for all finite type points s of S .

Then \mathcal{X} is an algebraic stack.

Proof. We first prove that $\Delta : \mathcal{X} \rightarrow \mathcal{X} \times \mathcal{X}$ is representable by algebraic spaces. To do this it suffices to show that

$$\mathcal{Y} = \mathcal{X} \times_{\Delta, \mathcal{X} \times \mathcal{X}, y} (\mathrm{Sch}/V)_{\mathrm{fppf}}$$

is representable by an algebraic space for any affine scheme V locally of finite presentation over S and object y of $\mathcal{X} \times \mathcal{X}$ over V , see Criteria for Representability, Lemma 5.5⁷. Observe that \mathcal{Y} is fibred in setoids (Stacks, Lemma 2.5) and let $Y : (\mathrm{Sch}/S)_{\mathrm{fppf}}^{\mathrm{opp}} \rightarrow \mathrm{Sets}$, $T \mapsto \mathrm{Ob}(\mathcal{Y}_T)/\cong$ be the functor of isomorphism classes. We will apply Proposition 16.1 to see that Y is an algebraic space.

Note that $\Delta_{\mathcal{Y}} : \mathcal{Y} \rightarrow \mathcal{Y} \times \mathcal{Y}$ (and hence also $Y \rightarrow Y \times Y$) is representable by algebraic spaces by condition (1) and Criteria for Representability, Lemma 4.4. Observe that Y is a sheaf for the étale topology by Stacks, Lemmas 6.3 and 6.7, i.e., axiom [0] holds. Also Y is limit preserving by Lemma 11.2, i.e., we have [1]. Note that Y has (RS), i.e., axiom [2] holds, by Lemmas 5.2 and 5.3. Axiom [3] for Y follows from Lemmas 8.1 and 8.2. Axiom [4] follows from Lemmas 9.5 and 9.6. Axiom [5] for Y follows directly from openness of versality for $\Delta_{\mathcal{X}}$ which is part of axiom [5] for \mathcal{X} . Thus all the assumptions of Proposition 16.1 are satisfied and Y is an algebraic space.

At this point it follows from Lemma 17.1 that \mathcal{X} is an algebraic stack. \square

18. Strong Rim-Schlessinger

0CXN In the rest of this chapter the following strictly stronger version of the Rim-Schlessinger conditions will play an important role.

07Y8 **Definition 18.1.** Let S be a scheme. Let \mathcal{X} be a category fibred in groupoids over $(\mathrm{Sch}/S)_{\mathrm{fppf}}$. We say \mathcal{X} satisfies *condition (RS*)* if given a fibre product diagram

$$\begin{array}{ccc} B' & \longrightarrow & B \\ \uparrow & & \uparrow \\ A' = A \times_B B' & \longrightarrow & A \end{array}$$

of S -algebras, with $B' \rightarrow B$ surjective with square zero kernel, the functor of fibre categories

$$\mathcal{X}_{\mathrm{Spec}(A')} \longrightarrow \mathcal{X}_{\mathrm{Spec}(A)} \times_{\mathcal{X}_{\mathrm{Spec}(B)}} \mathcal{X}_{\mathrm{Spec}(B')}$$

is an equivalence of categories.

⁷The set theoretic condition in Criteria for Representability, Lemma 5.5 will hold: the size of the algebraic space Y representing \mathcal{Y} is suitably bounded. Namely, $Y \rightarrow S$ will be locally of finite type and Y will satisfy axiom [-1]. Details omitted.

We make some observations: with $A \rightarrow B \leftarrow B'$ as in Definition 18.1

- (1) we have $\mathrm{Spec}(A') = \mathrm{Spec}(A) \amalg_{\mathrm{Spec}(B)} \mathrm{Spec}(B')$ in the category of schemes, see More on Morphisms, Lemma 14.3, and
- (2) if \mathcal{X} is an algebraic stack, then \mathcal{X} satisfies (RS*) by Lemma 4.1.

If S is locally Noetherian, then

- (3) if A, B, B' are of finite type over S and B is finite over A , then A' is of finite type over S ⁸, and
- (4) if \mathcal{X} satisfies (RS*), then \mathcal{X} satisfies (RS) because (RS) covers exactly those cases of (RS*) where A, B, B' are Artinian local.

0CXP **Lemma 18.2.** *Let \mathcal{X} be an algebraic stack over a base S . Then \mathcal{X} satisfies (RS*).*

Proof. This is implied by Lemma 4.1, see remarks following Definition 18.1. \square

0CXQ **Lemma 18.3.** *Let S be a scheme. Let $p : \mathcal{X} \rightarrow \mathcal{Y}$ and $q : \mathcal{Z} \rightarrow \mathcal{Y}$ be 1-morphisms of categories fibred in groupoids over $(\mathrm{Sch}/S)_{\mathrm{fppf}}$. If \mathcal{X}, \mathcal{Y} , and \mathcal{Z} satisfy (RS*), then so does $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z}$.*

Proof. The proof is exactly the same as the proof of Lemma 5.3. \square

19. Strong formal effectiveness

0CXR In this section we demonstrate how a strong version of effectiveness of formal objects implies openness of versality.

0CXS **Lemma 19.1.** *Let S be a locally Noetherian scheme. Let \mathcal{X} be a category fibred in groupoids over $(\mathrm{Sch}/S)_{\mathrm{fppf}}$ having (RS*). Let x be an object of \mathcal{X} over an affine scheme U of finite type over S . Let $u_n \in U$, $n \geq 1$ be pairwise distinct finite type points such that x is not versal at u_n for all n . After replacing u_n by a subsequence, there exist morphisms*

$$x \rightarrow x_1 \rightarrow x_2 \rightarrow \dots \quad \text{in } \mathcal{X} \text{ lying over } U \rightarrow U_1 \rightarrow U_2 \rightarrow \dots$$

over S such that

- (1) for each n the morphism $U \rightarrow U_n$ is a first order thickening,
- (2) for each n we have a short exact sequence

$$0 \rightarrow \kappa(u_n) \rightarrow \mathcal{O}_{U_n} \rightarrow \mathcal{O}_{U_{n-1}} \rightarrow 0$$

with $U_0 = U$ for $n = 1$,

- (3) for each n there does **not** exist a pair (W, α) consisting of an open neighbourhood $W \subset U_n$ of u_n and a morphism $\alpha : x_n|_W \rightarrow x$ such that the composition

$$x|_{U \cap W} \xrightarrow{\text{restriction of } x \rightarrow x_n} x_n|_W \xrightarrow{\alpha} x$$

is the canonical morphism $x|_{U \cap W} \rightarrow x$.

Proof. After replacing u_n , $n \geq 1$ by a subsequence we may and do assume that there are no specializations among these points, see Properties, Lemma 5.11. In particular, for every n we can find an open $U' \subset U$ such that $u_n \in U'$ and $u_i \notin U'$ for $i = 1, \dots, n-1$. This means that the problem of constructing our system

⁸If $\mathrm{Spec}(A)$ maps into an affine open of S this follows from More on Algebra, Lemma 5.1. The general case follows using More on Algebra, Lemma 5.3.

decomposes into a separate problem for each n . More precisely, suppose that for each $n \geq 1$ we can find

$$x \rightarrow y_n \quad \text{in } \mathcal{X} \text{ lying over } U \rightarrow T_n$$

such that

- (1) the morphism $U \rightarrow T_n$ is a first order thickening,
- (2) we have a short exact sequence

$$0 \rightarrow \kappa(u_n) \rightarrow \mathcal{O}_{T_n} \rightarrow \mathcal{O}_U \rightarrow 0$$

- (3) for each n there does **not** exist a pair (W, α) consisting of an open neighbourhood $W \subset T_n$ of u_n and a morphism $\beta : y_n|_W \rightarrow x$ such that the composition

$$x|_{U \cap W} \xrightarrow{\text{restriction of } x \rightarrow y_n} y_n|_W \xrightarrow{\beta} x$$

is the canonical morphism $x|_{U \cap W} \rightarrow x$.

Then we can define inductively

$$U_1 = T_1, \quad U_{n+1} = U_n \amalg_U T_{n+1}$$

Setting $x_1 = y_1$ and using (RS*) we find inductively x_{n+1} over U_{n+1} restricting to x_n over U_n and y_{n+1} over T_{n+1} . Property (1) for $U \rightarrow U_n$ follows from the construction of the pushout in More on Morphisms, Lemma 14.3. Property (2) for U_n similarly follows from property (2) for T_n by the construction of the pushout. After shrinking to an open neighbourhood U' of u_n as discussed above, property (3) for (U_n, x_n) follows from property (3) for (T_n, y_n) simply because the corresponding open subschemes of T_n and U_n are isomorphic. Details omitted.

This reduces us to the following: suppose given a single finite type point $u \in U$ such that x is not versal at u , we need to construct a morphism $x \rightarrow y$ of \mathcal{X} lying over $U \rightarrow T$ satisfying properties (1), (2), and (3) formulated above. Let $R = \mathcal{O}_{U,u}^\wedge$. Let $k = \kappa(u)$ be the residue field of R . Let ξ be the formal object of \mathcal{X} over R associated to x . Since x is not versal at u , we see that ξ is not versal, see Lemma 12.3. By the discussion following Definition 12.1 this means we can find morphisms $\xi_1 \rightarrow x_A \rightarrow x_B$ of \mathcal{X} lying over closed immersions $\text{Spec}(k) \rightarrow \text{Spec}(A) \rightarrow \text{Spec}(B)$ where A, B are Artinian local rings with residue field k , an $n \geq 1$ and a commutative diagram

$$\begin{array}{ccc}
 & x_A & \\
 \swarrow & \uparrow & \\
 \xi_n & \longleftarrow \xi_1 &
 \end{array}
 \quad \text{lying over} \quad
 \begin{array}{ccc}
 & \text{Spec}(A) & \\
 \swarrow & \uparrow & \\
 \text{Spec}(R/\mathfrak{m}^n) & \longleftarrow \text{Spec}(k) &
 \end{array}$$

such that there does **not** exist an $m \geq n$ and a commutative diagram

$$\begin{array}{ccc}
 & x_B & \\
 \swarrow & \uparrow & \\
 \xi_m & \longleftarrow \xi_n & \longleftarrow \xi_1
 \end{array}
 \quad \text{lying over} \quad
 \begin{array}{ccc}
 & \text{Spec}(B) & \\
 \swarrow & \uparrow & \\
 \text{Spec}(R/\mathfrak{m}^m) & \longleftarrow \text{Spec}(R/\mathfrak{m}^n) & \longleftarrow \text{Spec}(k)
 \end{array}$$

We may moreover assume that $B \rightarrow A$ is a small extension, i.e., that the kernel I of the surjection $B \rightarrow A$ is isomorphic to k as an A -module. This follows from Formal Deformation Theory, Remark 8.10. Then we simply define

$$T = U \amalg_{\text{Spec}(A)} \text{Spec}(B)$$

By property (RS*) we find y over T whose restriction to $\text{Spec}(B)$ is x_B and whose restriction to U is x (this gives the arrow $x \rightarrow y$ lying over $U \rightarrow T$). To finish the proof we verify conditions (1), (2), and (3).

By the construction of the pushout we have a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & I & \longrightarrow & B & \longrightarrow & A \longrightarrow 0 \\ & & \uparrow & & \uparrow & & \uparrow \\ 0 & \longrightarrow & I & \longrightarrow & \Gamma(T, \mathcal{O}_T) & \longrightarrow & \Gamma(U, \mathcal{O}_U) \longrightarrow 0 \end{array}$$

with exact rows. This immediately proves (1) and (2). To finish the proof we will argue by contradiction. Assume we have a pair (W, β) as in (3). Since $\text{Spec}(B) \rightarrow T$ factors through W we get the morphism

$$x_B \rightarrow y|_W \xrightarrow{\beta} x$$

Since B is Artinian local with residue field $k = \kappa(u)$ we see that $x_B \rightarrow x$ lies over a morphism $\text{Spec}(B) \rightarrow U$ which factors through $\text{Spec}(\mathcal{O}_{U,u}/\mathfrak{m}_u^m)$ for some $m \geq n$. In other words, $x_B \rightarrow x$ factors through ξ_m giving a map $x_B \rightarrow \xi_m$. The compatibility condition on the morphism α in condition (3) translates into the condition that

$$\begin{array}{ccc} x_B & \longleftarrow & x_A \\ \downarrow & & \downarrow \\ \xi_m & \longleftarrow & \xi_n \end{array}$$

is commutative. This gives the contradiction we were looking for. \square

0CXT **Remark 19.2** (Strong effectiveness). Let S be a locally Noetherian scheme. Let \mathcal{X} be a category fibred in groupoids over $(\text{Sch}/S)_{fppf}$. Assume we have

- (1) an affine open $\text{Spec}(\Lambda) \subset S$,
- (2) an inverse system (R_n) of Λ -algebras with surjective transition maps whose kernels are locally nilpotent,
- (3) a system (ξ_n) of objects of \mathcal{X} lying over the system $(\text{Spec}(R_n))$.

In this situation, set $R = \lim R_n$. We say that (ξ_n) is *effective* if there exists an object ξ of \mathcal{X} over $\text{Spec}(R)$ whose restriction to $\text{Spec}(R_n)$ gives the system (ξ_n) .

It is not the case that every algebraic stack \mathcal{X} over S satisfies a strong effectiveness axiom of the form: every system (ξ_n) as in Remark 19.2 is effective. An example is given in Examples, Section 65.

0CXU **Lemma 19.3.** Let S be a locally Noetherian scheme. Let \mathcal{X} be a category fibred in groupoids over $(\text{Sch}/S)_{fppf}$. Assume

- (1) $\Delta : \mathcal{X} \rightarrow \mathcal{X} \times \mathcal{X}$ is representable by algebraic spaces,
- (2) \mathcal{X} has (RS*),
- (3) \mathcal{X} is limit preserving,

- (4) systems (ξ_n) as in Remark 19.2 where $\text{Ker}(R_m \rightarrow R_n)$ is an ideal of square zero for all $m \geq n$ are effective.

Then \mathcal{X} satisfies openness of versality.

Proof. Choose a scheme U locally of finite type over S , a finite type point u_0 of U , and an object x of \mathcal{X} over U such that x is versal at u_0 . After shrinking U we may assume U is affine and U maps into an affine open $\text{Spec}(\Lambda)$ of S . If openness of versality does not hold, then we get an infinite sequence of finite type points u_n such that u_0 is a limit point of the sequence and such that x is not versal at u_n for $n \geq 1$. After passing to a subsequence we get $x \rightarrow x_1 \rightarrow x_2 \rightarrow \dots$ lying over $U \rightarrow U_1 \rightarrow U_2 \rightarrow \dots$ as in Lemma 19.1. Write $U_n = \text{Spec}(R_n)$ and $U = \text{Spec}(R_0)$. Set $R = \varinjlim R_n$. Observe that $R \rightarrow R_0$ is surjective with kernel an ideal of square zero. By assumption (4) we get ξ over $\text{Spec}(R)$ whose base change to R_n is x_n . By assumption (3) we get that ξ comes from an object ξ' over $U' = \text{Spec}(R')$ for some finite type Λ -subalgebra $R' \subset R$. After increasing R' we may and do assume that $R' \rightarrow R_0$ is surjective, so that $U \subset U'$ is a first order thickening. Thus we now have

$$x \rightarrow x_1 \rightarrow x_2 \rightarrow \dots \rightarrow \xi' \text{ lying over } U \rightarrow U_1 \rightarrow U_2 \rightarrow \dots \rightarrow U'$$

By assumption (1) there is an algebraic space Z over S representing

$$(Sch/U)_{fppf} \times_{x, \mathcal{X}, \xi'} (Sch/U')_{fppf}$$

see Algebraic Stacks, Lemma 10.11. By construction of 2-fibre products, a T -valued point of Z corresponds to a triple (a, a', α) consisting of morphisms $a : T \rightarrow U$, $a' : T \rightarrow U'$ and a morphism $\alpha : a^*x \rightarrow (a')^*\xi'$. We obtain a commutative diagram

$$\begin{array}{ccc} U & & \\ \searrow & & \searrow \\ & Z & \xrightarrow{p'} U' \\ \searrow & \downarrow p & \downarrow \\ & U & \longrightarrow S \end{array}$$

The morphism $i : U \rightarrow Z$ comes from the isomorphism $x \rightarrow \xi'|_U$. Let $z_0 = i(u_0) \in Z$. By Lemma 12.6 we see that $Z \rightarrow U'$ is smooth at z_0 . After replacing U by an affine open neighbourhood of u_0 , replacing U' by the corresponding open, and replacing W by the intersection of the inverse images of these opens by p and p' , we reach the situation where $Z \rightarrow U'$ is smooth along $i(U)$. Note that this also involves replacing u_n by a subsequence, namely by those indices such that u_n is in the open. Moreover, condition (3) of Lemma 19.1 is clearly preserved by shrinking U (all of the schemes U, U_n, U' have the same underlying topological space). Since $U \rightarrow U'$ is a first order thickening of affine schemes, we can choose a morphism $i' : U' \rightarrow Z$ such that $p' \circ i' = \text{id}_{U'}$ and whose restriction to U is i (More on Morphisms of Spaces, Lemma 19.6). Pulling back the universal morphism $p^*x \rightarrow (p')^*\xi'$ by i' we obtain a morphism

$$\xi' \rightarrow x$$

lying over $p \circ i' : U' \rightarrow U$ such that the composition

$$x \rightarrow \xi' \rightarrow x$$

is the identity. Recall that we have $x_1 \rightarrow \xi'$ lying over the morphism $U_1 \rightarrow U'$. Composing we get a morphism $x_1 \rightarrow x$ whose existence contradicts condition (3) of Lemma 19.1. This contradiction finishes the proof. \square

0CXV **Remark 19.4.** There is a way to deduce openness of versality of the diagonal of an category fibred in groupoids from a strong formal effectiveness axiom. Let S be a locally Noetherian scheme. Let \mathcal{X} be a category fibred in groupoids over $(Sch/S)_{fppf}$. Assume

- (1) $\Delta_{\Delta} : \mathcal{X} \rightarrow \mathcal{X} \times_{\mathcal{X} \times \mathcal{X}} \mathcal{X}$ is representable by algebraic spaces,
- (2) \mathcal{X} has (RS*),
- (3) \mathcal{X} is limit preserving,
- (4) given an inverse system (R_n) of S -algebras as in Remark 19.2 where $\text{Ker}(R_m \rightarrow R_n)$ is an ideal of square zero for all $m \geq n$ the functor

$$\mathcal{X}_{\text{Spec}(\lim R_n)} \longrightarrow \lim_n \mathcal{X}_{\text{Spec}(R_n)}$$

is fully faithful.

Then $\Delta : \mathcal{X} \rightarrow \mathcal{X} \times \mathcal{X}$ satisfies openness of versality. This follows by applying Lemma 19.3 to fibre products of the form $\mathcal{X} \times_{\Delta, \mathcal{X} \times \mathcal{X}, y} (Sch/V)_{fppf}$ for any affine scheme V locally of finite presentation over S and object y of $\mathcal{X} \times \mathcal{X}$ over V . If we ever need this, we will change this remark into a lemma and provide a detailed proof.

20. Infinitesimal deformations

07Y6 In this section we discuss a generalization of the notion of the tangent space introduced in Section 8. To do this intelligently, we borrow some notation from Formal Deformation Theory, Sections 11, 17, and 19.

Let S be a scheme. Let \mathcal{X} be a category fibred in groupoids over $(Sch/S)_{fppf}$. Given a homomorphism $A' \rightarrow A$ of S -algebras and an object x of \mathcal{X} over $\text{Spec}(A)$ we write $Lift(x, A')$ for the category of lifts of x to $\text{Spec}(A')$. An object of $Lift(x, A')$ is a morphism $x \rightarrow x'$ of \mathcal{X} lying over $\text{Spec}(A) \rightarrow \text{Spec}(A')$ and morphisms of $Lift(x, A')$ are defined as commutative diagrams. The set of isomorphism classes of $Lift(x, A')$ is denoted $\text{Lift}(x, A')$. See Formal Deformation Theory, Definition 17.1 and Remark 17.2. If $A' \rightarrow A$ is surjective with locally nilpotent kernel we call an element x' of $\text{Lift}(x, A')$ a (*infinitesimal*) *deformation* of x . In this case the *group of infinitesimal automorphisms of x' over x* is the kernel

$$\text{Inf}(x'/x) = \text{Ker} \left(\text{Aut}_{\mathcal{X}_{\text{Spec}(A')}}(x') \rightarrow \text{Aut}_{\mathcal{X}_{\text{Spec}(A)}}(x) \right)$$

Note that an element of $\text{Inf}(x'/x)$ is the same thing as a lift of id_x over $\text{Spec}(A')$ for (the category fibred in sets associated to) $\text{Aut}_{\mathcal{X}}(x')$. Compare with Formal Deformation Theory, Definition 19.1 and Formal Deformation Theory, Remark 19.8.

If M is an A -module we denote $A[M]$ the A -algebra whose underlying A -module is $A \oplus M$ and whose multiplication is given by $(a, m) \cdot (a', m') = (aa', am' + a'm)$. When $M = A$ this is the ring of dual numbers over A , which we denote $A[\epsilon]$ as is customary. There is an A -algebra map $A[M] \rightarrow A$. The pullback of x to $\text{Spec}(A[M])$ is called the *trivial deformation* of x to $\text{Spec}(A[M])$.

07Y7 **Lemma 20.1.** *Let S be a scheme. Let $f : \mathcal{X} \rightarrow \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. Let*

$$\begin{array}{ccc} B' & \longrightarrow & B \\ \uparrow & & \uparrow \\ A' & \longrightarrow & A \end{array}$$

be a commutative diagram of S -algebras. Let x be an object of \mathcal{X} over $\text{Spec}(A)$, let y be an object of \mathcal{Y} over $\text{Spec}(B)$, and let $\phi : f(x)|_{\text{Spec}(B)} \rightarrow y$ be a morphism of \mathcal{Y} over $\text{Spec}(B)$. Then there is a canonical functor

$$\text{Lift}(x, A') \longrightarrow \text{Lift}(y, B')$$

of categories of lifts induced by f and ϕ . The construction is compatible with compositions of 1-morphisms of categories fibred in groupoids in an obvious manner.

Proof. This lemma proves itself. \square

Let S be a base scheme. Let \mathcal{X} be a category fibred in groupoids over $(Sch/S)_{fppf}$. We define a category whose objects are pairs $(x, A' \rightarrow A)$ where

- (1) $A' \rightarrow A$ is a surjection of S -algebras whose kernel is an ideal of square zero,
- (2) x is an object of \mathcal{X} lying over $\text{Spec}(A)$.

A morphism $(y, B' \rightarrow B) \rightarrow (x, A' \rightarrow A)$ is given by a commutative diagram

$$\begin{array}{ccc} B' & \longrightarrow & B \\ \uparrow & & \uparrow \\ A' & \longrightarrow & A \end{array}$$

of S -algebras together with a morphism $x|_{\text{Spec}(B)} \rightarrow y$ over $\text{Spec}(B)$. Let us call this the category of *deformation situations*.

07Y9 **Lemma 20.2.** *Let S be a scheme. Let \mathcal{X} be a category fibred in groupoids over $(Sch/S)_{fppf}$. Assume \mathcal{X} satisfies condition (RS^*) . Let A be an S -algebra and let x be an object of \mathcal{X} over $\text{Spec}(A)$.*

- (1) *There exists an A -linear functor $\text{Inf}_x : \text{Mod}_A \rightarrow \text{Mod}_A$ such that given a deformation situation $(x, A' \rightarrow A)$ and a lift x' there is an isomorphism $\text{Inf}_x(I) \rightarrow \text{Inf}(x'/x)$ where $I = \text{Ker}(A' \rightarrow A)$.*
- (2) *There exists an A -linear functor $T_x : \text{Mod}_A \rightarrow \text{Mod}_A$ such that*
 - (a) *given M in Mod_A there is a bijection $T_x(M) \rightarrow \text{Lift}(x, A[M])$,*
 - (b) *given a deformation situation $(x, A' \rightarrow A)$ there is an action*

$$T_x(I) \times \text{Lift}(x, A') \rightarrow \text{Lift}(x, A')$$

where $I = \text{Ker}(A' \rightarrow A)$. It is simply transitive if $\text{Lift}(x, A') \neq \emptyset$.

Proof. We define Inf_x as the functor

$$\text{Mod}_A \longrightarrow \text{Sets}, \quad M \longrightarrow \text{Inf}(x'_M/x) = \text{Lift}(\text{id}_x, A[M])$$

mapping M to the group of infinitesimal automorphisms of the trivial deformation x'_M of x to $\text{Spec}(A[M])$ or equivalently the group of lifts of id_x in $\text{Aut}_{\mathcal{X}}(x'_M)$. We define T_x as the functor

$$\text{Mod}_A \longrightarrow \text{Sets}, \quad M \longrightarrow \text{Lift}(x, A[M])$$

of isomorphism classes of infinitesimal deformations of x to $\text{Spec}(A[M])$. We apply Formal Deformation Theory, Lemma 11.4 to Inf_x and T_x . This lemma is applicable, since (RS*) tells us that

$$\text{Lift}(x, A[M \times N]) = \text{Lift}(x, A[M]) \times \text{Lift}(x, A[N])$$

as categories (and trivial deformations match up too).

Let $(x, A' \rightarrow A)$ be a deformation situation. Consider the ring map $g : A' \times_A A' \rightarrow A[I]$ defined by the rule $g(a_1, a_2) = \bar{a}_1 \oplus a_2 - a_1$. There is an isomorphism

$$A' \times_A A' \longrightarrow A' \times_A A[I]$$

given by $(a_1, a_2) \mapsto (a_1, g(a_1, a_2))$. This isomorphism commutes with the projections to A' on the first factor, and hence with the projections to A . Thus applying (RS*) twice we find equivalences of categories

$$\begin{aligned} \text{Lift}(x, A') \times \text{Lift}(x, A') &= \text{Lift}(x, A' \times_A A') \\ &= \text{Lift}(x, A' \times_A A[I]) \\ &= \text{Lift}(x, A') \times \text{Lift}(x, A[I]) \end{aligned}$$

Using these maps and projection onto the last factor of the last product we see that we obtain “difference maps”

$$\text{Inf}(x'/x) \times \text{Inf}(x'/x) \longrightarrow \text{Inf}_x(I) \quad \text{and} \quad \text{Lift}(x, A') \times \text{Lift}(x, A') \longrightarrow T_x(I)$$

These difference maps satisfy the transitivity rule “ $(x'_1 - x'_2) + (x'_2 - x'_3) = x'_1 - x'_3$ ” because

$$\begin{array}{ccc} A' \times_A A' \times_A A' & \xrightarrow{(a_1, a_2, a_3) \mapsto (g(a_1, a_2), g(a_2, a_3))} & A[I] \times_A A[I] = A[I \times I] \\ & \searrow_{(a_1, a_2, a_3) \mapsto g(a_1, a_3)} & \downarrow + \\ & & A[I] \end{array}$$

is commutative. Inverting the string of equivalences above we obtain an action which is free and transitive provided $\text{Inf}(x'/x)$, resp. $\text{Lift}(x, A')$ is nonempty. Note that $\text{Inf}(x'/x)$ is always nonempty as it is a group. \square

07YA **Remark 20.3** (Functoriality). Assumptions and notation as in Lemma 20.2. Suppose $A \rightarrow B$ is a ring map and $y = x|_{\text{Spec}(B)}$. Let $M \in \text{Mod}_A$, $N \in \text{Mod}_B$ and let $M \rightarrow N$ an A -linear map. Then there are canonical maps $\text{Inf}_x(M) \rightarrow \text{Inf}_y(N)$ and $T_x(M) \rightarrow T_y(N)$ simply because there is a pullback functor

$$\text{Lift}(x, A[M]) \rightarrow \text{Lift}(y, B[N])$$

coming from the ring map $A[M] \rightarrow B[N]$. Similarly, given a morphism of deformation situations $(y, B' \rightarrow B) \rightarrow (x, A' \rightarrow A)$ we obtain a pullback functor $\text{Lift}(x, A') \rightarrow \text{Lift}(y, B')$. Since the construction of the action, the addition, and the scalar multiplication on Inf_x and T_x use only morphisms in the categories of lifts (see proof of Formal Deformation Theory, Lemma 11.4) we see that the constructions above are functorial. In other words we obtain A -linear maps

$$\text{Inf}_x(M) \rightarrow \text{Inf}_y(N) \quad \text{and} \quad T_x(M) \rightarrow T_y(N)$$

such that the diagrams

$$\begin{array}{ccc} \text{Inf}_y(J) & \longrightarrow & \text{Inf}(y'/y) & & T_y(J) \times \text{Lift}(y, B') & \longrightarrow & \text{Lift}(y, B') \\ \uparrow & & \uparrow & \text{and} & \uparrow & & \uparrow \\ \text{Inf}_x(I) & \longrightarrow & \text{Inf}(x'/x) & & T_x(I) \times \text{Lift}(x, A') & \longrightarrow & \text{Lift}(x, A') \end{array}$$

commute. Here $I = \text{Ker}(A' \rightarrow A)$, $J = \text{Ker}(B' \rightarrow B)$, x' is a lift of x to A' (which may not always exist) and $y' = x'|_{\text{Spec}(B')}$.

07YB **Remark 20.4** (Automorphisms). Assumptions and notation as in Lemma 20.2. Let x', x'' be lifts of x to A' . Then we have a composition map

$$\text{Inf}(x'/x) \times \text{Mor}_{\text{Lift}(x, A')}(x', x'') \times \text{Inf}(x''/x) \longrightarrow \text{Mor}_{\text{Lift}(x, A')}(x', x'').$$

Since $\text{Lift}(x, A')$ is a groupoid, if $\text{Mor}_{\text{Lift}(x, A')}(x', x'')$ is nonempty, then this defines a simply transitive left action of $\text{Inf}(x'/x)$ on $\text{Mor}_{\text{Lift}(x, A')}(x', x'')$ and a simply transitive right action by $\text{Inf}(x''/x)$. Now the lemma says that $\text{Inf}(x'/x) = \text{Inf}_x(I) = \text{Inf}(x''/x)$. We claim that the two actions described above agree via these identifications. Namely, either $x' \not\cong x''$ in which the claim is clear, or $x' \cong x''$ and in that case we may assume that $x'' = x'$ in which case the result follows from the fact that $\text{Inf}(x'/x)$ is commutative. In particular, we obtain a well defined action

$$\text{Inf}_x(I) \times \text{Mor}_{\text{Lift}(x, A')}(x', x'') \longrightarrow \text{Mor}_{\text{Lift}(x, A')}(x', x'')$$

which is simply transitive as soon as $\text{Mor}_{\text{Lift}(x, A')}(x', x'')$ is nonempty.

07YE **Remark 20.5.** Let S be a scheme. Let \mathcal{X} be a category fibred in groupoids over $(\text{Sch}/S)_{\text{fppf}}$. Let A be an S -algebra. There is a notion of a *short exact sequence*

$$(x, A'_1 \rightarrow A) \rightarrow (x, A'_2 \rightarrow A) \rightarrow (x, A'_3 \rightarrow A)$$

of deformation situations: we ask the corresponding maps between the kernels $I_i = \text{Ker}(A'_i \rightarrow A)$ give a short exact sequence

$$0 \rightarrow I_3 \rightarrow I_2 \rightarrow I_1 \rightarrow 0$$

of A -modules. Note that in this case the map $A'_3 \rightarrow A'_1$ factors through A , hence there is a canonical isomorphism $A'_1 = A[I_1]$.

0DNN **Lemma 20.6.** *Let S be a scheme. Let $p : \mathcal{X} \rightarrow \mathcal{Y}$ and $q : \mathcal{Z} \rightarrow \mathcal{Y}$ be 1-morphisms of categories fibred in groupoids over $(\text{Sch}/S)_{\text{fppf}}$. Assume $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ satisfy (RS^*) . Let A be an S -algebra and let w be an object of $\mathcal{W} = \mathcal{X} \times_{\mathcal{Y}} \mathcal{Z}$ over A . Denote x, y, z the objects of $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ you get from w . For any A -module M there is a 6-term exact sequence*

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{Inf}_w(M) & \longrightarrow & \text{Inf}_x(M) \oplus \text{Inf}_z(M) & \longrightarrow & \text{Inf}_y(M) \\ & & & & & \searrow & \\ & & & & & & T_w(M) & \longleftarrow & T_x(M) \oplus T_z(M) & \longrightarrow & T_y(M) \end{array}$$

of A -modules.

Proof. By Lemma 18.3 we see that \mathcal{W} satisfies (RS^*) and hence $T_w(M)$ and $\text{Inf}_w(M)$ are defined. The horizontal arrows are defined using the functoriality of Lemma 20.1.

Definition of the “boundary” map $\delta : \text{Inf}_y(M) \rightarrow T_w(M)$. Choose isomorphisms $p(x) \rightarrow y$ and $y \rightarrow q(z)$ such that $w = (x, z, p(x) \rightarrow y \rightarrow q(z))$ in the description of the 2-fibre product of Categories, Lemma 34.7 and more precisely Categories, Lemma 31.3. Let x', y', z', w' denote the trivial deformation of x, y, z, w over $A[M]$. By pullback we get isomorphisms $y' \rightarrow p(x')$ and $q(z') \rightarrow y'$. An element $\alpha \in \text{Inf}_y(M)$ is the same thing as an automorphism $\alpha : y' \rightarrow y'$ over $A[M]$ which restricts to the identity on y over A . Thus setting

$$\delta(\alpha) = (x', z', p(x') \rightarrow y' \xrightarrow{\alpha} y' \rightarrow q(z'))$$

we obtain an object of $T_w(M)$. This is a map of A -modules by Formal Deformation Theory, Lemma 11.5.

The rest of the proof is exactly the same as the proof of Formal Deformation Theory, Lemma 20.1. \square

0D18 **Remark 20.7** (Compatibility with previous tangent spaces). Let S be a locally Noetherian scheme. Let \mathcal{X} be a category fibred in groupoids over $(\text{Sch}/S)_{fppf}$. Assume \mathcal{X} has (RS*). Let k be a field of finite type over S and let x_0 be an object of \mathcal{X} over $\text{Spec}(k)$. Then we have equalities of k -vector spaces

$$T\mathcal{F}_{\mathcal{X}, k, x_0} = T_{x_0}(k) \quad \text{and} \quad \text{Inf}(\mathcal{F}_{\mathcal{X}, k, x_0}) = \text{Inf}_{x_0}(k)$$

where the spaces on the left hand side of the equality signs are given in (8.0.1) and (8.0.2) and the spaces on the right hand side are given by Lemma 20.2.

07YC **Remark 20.8** (Canonical element). Assumptions and notation as in Lemma 20.2. Choose an affine open $\text{Spec}(\Lambda) \subset S$ such that $\text{Spec}(A) \rightarrow S$ corresponds to a ring map $\Lambda \rightarrow A$. Consider the ring map

$$A \longrightarrow A[\Omega_{A/\Lambda}], \quad a \longmapsto (a, d_{A/\Lambda}(a))$$

Pulling back x along the corresponding morphism $\text{Spec}(A[\Omega_{A/\Lambda}]) \rightarrow \text{Spec}(A)$ we obtain a deformation x_{can} of x over $A[\Omega_{A/\Lambda}]$. We call this the *canonical element*

$$x_{can} \in T_x(\Omega_{A/\Lambda}) = \text{Lift}(x, A[\Omega_{A/\Lambda}]).$$

Next, assume that Λ is Noetherian and $\Lambda \rightarrow A$ is of finite type. Let $k = \kappa(\mathfrak{p})$ be a residue field at a finite type point u_0 of $U = \text{Spec}(A)$. Let $x_0 = x|_{u_0}$. By (RS*) and the fact that $A[k] = A \times_k k[k]$ the space $T_x(k)$ is the tangent space to the deformation functor $\mathcal{F}_{\mathcal{X}, k, x_0}$. Via

$$T\mathcal{F}_{U, k, u_0} = \text{Der}_\Lambda(A, k) = \text{Hom}_A(\Omega_{A/\Lambda}, k)$$

(see Formal Deformation Theory, Example 11.11) and functoriality of T_x the canonical element produces the map on tangent spaces induced by the object x over U . Namely, $\theta \in T\mathcal{F}_{U, k, u_0}$ maps to $T_x(\theta)(x_{can})$ in $T_x(k) = T\mathcal{F}_{\mathcal{X}, k, x_0}$.

07YD **Remark 20.9** (Canonical automorphism). Let S be a locally Noetherian scheme. Let \mathcal{X} be a category fibred in groupoids over $(\text{Sch}/S)_{fppf}$. Assume \mathcal{X} satisfies condition (RS*). Let A be an S -algebra such that $\text{Spec}(A) \rightarrow S$ maps into an affine open and let x, y be objects of \mathcal{X} over $\text{Spec}(A)$. Further, let $A \rightarrow B$ be a ring map and let $\alpha : x|_{\text{Spec}(B)} \rightarrow y|_{\text{Spec}(B)}$ be a morphism of \mathcal{X} over $\text{Spec}(B)$. Consider the ring map

$$B \longrightarrow B[\Omega_{B/A}], \quad b \longmapsto (b, d_{B/A}(b))$$

Pulling back α along the corresponding morphism $\text{Spec}(B[\Omega_{B/A}]) \rightarrow \text{Spec}(B)$ we obtain a morphism α_{can} between the pullbacks of x and y over $B[\Omega_{B/A}]$. On

the other hand, we can pullback α by the morphism $\mathrm{Spec}(B[\Omega_{B/A}]) \rightarrow \mathrm{Spec}(B)$ corresponding to the injection of B into the first summand of $B[\Omega_{B/A}]$. By the discussion of Remark 20.4 we can take the difference

$$\varphi(x, y, \alpha) = \alpha_{can} - \alpha|_{\mathrm{Spec}(B[\Omega_{B/A}])} \in \mathrm{Inf}_{x|_{\mathrm{Spec}(B)}}(\Omega_{B/A}).$$

We will call this the *canonical automorphism*. It depends on all the ingredients A , x , y , $A \rightarrow B$ and α .

21. Obstruction theories

07YF In this section we describe what an obstruction theory is. Contrary to the spaces of infinitesimal deformations and infinitesimal automorphisms, an obstruction theory is an additional piece of data. The formulation is motivated by the results of Lemma 20.2 and Remark 20.3.

07YG **Definition 21.1.** Let S be a locally Noetherian base. Let \mathcal{X} be a category fibred in groupoids over $(\mathrm{Sch}/S)_{fppf}$. An *obstruction theory* is given by the following data

- (1) for every S -algebra A such that $\mathrm{Spec}(A) \rightarrow S$ maps into an affine open and every object x of \mathcal{X} over $\mathrm{Spec}(A)$ an A -linear functor

$$\mathcal{O}_x : \mathrm{Mod}_A \rightarrow \mathrm{Mod}_A$$

of *obstruction modules*,

- (2) for (x, A) as in (1), a ring map $A \rightarrow B$, $M \in \mathrm{Mod}_A$, $N \in \mathrm{Mod}_B$, and an A -linear map $M \rightarrow N$ an induced A -linear map $\mathcal{O}_x(M) \rightarrow \mathcal{O}_y(N)$ where $y = x|_{\mathrm{Spec}(B)}$, and
- (3) for every deformation situation $(x, A' \rightarrow A)$ an *obstruction element* $o_x(A') \in \mathcal{O}_x(I)$ where $I = \mathrm{Ker}(A' \rightarrow A)$.

These data are subject to the following conditions

- (i) the functoriality maps turn the obstruction modules into a functor from the category of triples (x, A, M) to sets,
- (ii) for every morphism of deformation situations $(y, B' \rightarrow B) \rightarrow (x, A' \rightarrow A)$ the element $o_x(A')$ maps to $o_y(B')$, and
- (iii) we have

$$\mathrm{Lift}(x, A') \neq \emptyset \Leftrightarrow o_x(A') = 0$$

for every deformation situation $(x, A' \rightarrow A)$.

This last condition explains the terminology. The module $\mathcal{O}_x(A')$ is called the *obstruction module*. The element $o_x(A')$ is the *obstruction*. Most obstruction theories have additional properties, and in order to make them useful additional conditions are needed. Moreover, this is just a sample definition, for example in the definition we could consider only deformation situations of finite type over S .

One of the main reasons for introducing obstruction theories is to check openness of versality. An example of this type of result is Lemma 21.2 below. The initial idea to do this is due to Artin, see the papers of Artin mentioned in the introduction. It has been taken up for example in the work by Flenner [Fle81], Hall [Hal17], Hall and Rydh [HR12], Olsson [Ols06], Olsson and Starr [OS03], and Lieblich [Lie06] (random order of references). Moreover, for particular categories fibred in groupoids, often authors develop a little bit of theory adapted to the problem at hand. We will develop this theory later (insert future reference here).

0CYF **Lemma 21.2.** *Let S be a locally Noetherian scheme. Let \mathcal{X} be a category fibred in groupoids over $(Sch/S)_{fppf}$. Assume* This is [Hal17, Theorem 4.4]

- (1) $\Delta : \mathcal{X} \rightarrow \mathcal{X} \times \mathcal{X}$ is representable by algebraic spaces,
- (2) \mathcal{X} has (RS^*) ,
- (3) \mathcal{X} is limit preserving,
- (4) there exists an obstruction theory⁹,
- (5) for an object x of \mathcal{X} over $\text{Spec}(A)$ and A -modules M_n , $n \geq 1$ we have
 - (a) $T_x(\prod M_n) = \prod T_x(M_n)$,
 - (b) $\mathcal{O}_x(\prod M_n) \rightarrow \prod \mathcal{O}_x(M_n)$ is injective.

Then \mathcal{X} satisfies openness of versality.

Proof. We prove this by verifying condition (4) of Lemma 19.3. Let (ξ_n) and (R_n) be as in Remark 19.2 such that $\text{Ker}(R_m \rightarrow R_n)$ is an ideal of square zero for all $m \geq n$. Set $A = R_1$ and $x = \xi_1$. Denote $M_n = \text{Ker}(R_n \rightarrow R_1)$. Then M_n is an A -module. Set $R = \lim R_n$. Let

$$\tilde{R} = \{(r_1, r_2, r_3, \dots) \in \prod R_n \text{ such that all have the same image in } A\}$$

Then $\tilde{R} \rightarrow A$ is surjective with kernel $M = \prod M_n$. There is a map $R \rightarrow \tilde{R}$ and a map $\tilde{R} \rightarrow A[M]$, $(r_1, r_2, r_3, \dots) \mapsto (r_1, r_2 - r_1, r_3 - r_2, \dots)$. Together these give a short exact sequence

$$(x, R \rightarrow A) \rightarrow (x, \tilde{R} \rightarrow A) \rightarrow (x, A[M])$$

of deformation situations, see Remark 20.5. The associated sequence of kernels $0 \rightarrow \lim M_n \rightarrow M \rightarrow M \rightarrow 0$ is the canonical sequence computing the limit of the system of modules (M_n) .

Let $o_x(\tilde{R}) \in \mathcal{O}_x(\tilde{R})$ be the obstruction element. Since we have the lifts ξ_n we see that $o_x(\tilde{R})$ maps to zero in $\mathcal{O}_x(M_n)$. By assumption (5)(b) we see that $o_x(\tilde{R}) = 0$. Choose a lift $\tilde{\xi}$ of x to $\text{Spec}(\tilde{R})$. Let $\tilde{\xi}_n$ be the restriction of $\tilde{\xi}$ to $\text{Spec}(R_n)$. There exists elements $t_n \in T_x(M_n)$ such that $t_n \cdot \tilde{\xi}_n = \xi_n$ by Lemma 20.2 part (2)(b). By assumption (5)(a) we can find $t \in T_x(M)$ mapping to t_n in $T_x(M_n)$. After replacing $\tilde{\xi}$ by $t \cdot \tilde{\xi}$ we find that $\tilde{\xi}$ restricts to ξ_n over $\text{Spec}(R_n)$ for all n . In particular, since ξ_{n+1} restricts to ξ_n over $\text{Spec}(R_n)$, the restriction $\tilde{\xi}$ of $\tilde{\xi}$ to $\text{Spec}(A[M])$ has the property that it restricts to the trivial deformation over $\text{Spec}(A[M_n])$ for all n . Hence by assumption (5)(a) we find that $\tilde{\xi}$ is the trivial deformation of x . By axiom (RS^*) applied to $R = \tilde{R} \times_{A[M]} A$ this implies that $\tilde{\xi}$ is the pullback of a deformation ξ of x over R . This finishes the proof. \square

07YH **Example 21.3.** Let $S = \text{Spec}(\Lambda)$ for some Noetherian ring Λ . Let $W \rightarrow S$ be a morphism of schemes. Let \mathcal{F} be a quasi-coherent \mathcal{O}_W -module flat over S . Consider the functor

$$F : (Sch/S)_{fppf}^{opp} \longrightarrow \text{Sets}, \quad T/S \longrightarrow H^0(W_T, \mathcal{F}_T)$$

where $W_T = T \times_S W$ is the base change and \mathcal{F}_T is the pullback of \mathcal{F} to W_T . If $T = \text{Spec}(A)$ we will write $W_T = W_A$, etc. Let $\mathcal{X} \rightarrow (Sch/S)_{fppf}$ be the category fibred in groupoids associated to F . Then \mathcal{X} has an obstruction theory. Namely,

- (1) given A over Λ and $x \in H^0(W_A, \mathcal{F}_A)$ we set $\mathcal{O}_x(M) = H^1(W_A, \mathcal{F}_A \otimes_A M)$,

⁹Analyzing the proof the reader sees that in fact it suffices to check the functoriality (ii) of obstruction classes in Definition 21.1 for maps $(y, B' \rightarrow B) \rightarrow (x, A' \rightarrow A)$ with $B = A$ and $y = x$.

- (2) given a deformation situation $(x, A' \rightarrow A)$ we let $o_x(A') \in \mathcal{O}_x(A)$ be the image of x under the boundary map

$$H^0(W_A, \mathcal{F}_A) \longrightarrow H^1(W_A, \mathcal{F}_A \otimes_A I)$$

coming from the short exact sequence of modules

$$0 \rightarrow \mathcal{F}_A \otimes_A I \rightarrow \mathcal{F}_{A'} \rightarrow \mathcal{F}_A \rightarrow 0.$$

We have omitted some details, in particular the construction of the short exact sequence above (it uses that W_A and $W_{A'}$ have the same underlying topological space) and the explanation for why flatness of \mathcal{F} over S implies that the sequence above is short exact.

07YI **Example 21.4** (Key example). Let $S = \text{Spec}(\Lambda)$ for some Noetherian ring Λ . Say $\mathcal{X} = (Sch/X)_{fppf}$ with $X = \text{Spec}(R)$ and $R = \Lambda[x_1, \dots, x_n]/J$. The naive cotangent complex $NL_{R/\Lambda}$ is (canonically) homotopy equivalent to

$$J/J^2 \longrightarrow \bigoplus_{i=1, \dots, n} R dx_i,$$

see Algebra, Lemma 133.2. Consider a deformation situation $(x, A' \rightarrow A)$. Denote I the kernel of $A' \rightarrow A$. The object x corresponds to (a_1, \dots, a_n) with $a_i \in A$ such that $f(a_1, \dots, a_n) = 0$ in A for all $f \in J$. Set

$$\begin{aligned} \mathcal{O}_x(A') &= \text{Hom}_R(J/J^2, I) / \text{Hom}_R(R^{\oplus n}, I) \\ &= \text{Ext}_R^1(NL_{R/\Lambda}, I) \\ &= \text{Ext}_A^1(NL_{R/\Lambda} \otimes_R A, I). \end{aligned}$$

Choose lifts $a'_i \in A'$ of a_i in A . Then $o_x(A')$ is the class of the map $J/J^2 \rightarrow I$ defined by sending $f \in J$ to $f(a'_1, \dots, a'_n) \in I$. We omit the verification that $o_x(A')$ is independent of choices. It is clear that if $o_x(A') = 0$ then the map lifts. Finally, functoriality is straightforward. Thus we obtain an obstruction theory. We observe that $o_x(A')$ can be described a bit more canonically as the composition

$$NL_{R/\Lambda} \rightarrow NL_{A/\Lambda} \rightarrow NL_{A/A'} = I[1]$$

in $D(A)$, see Algebra, Lemma 133.6 for the last identification.

22. Naive obstruction theories

07YJ The title of this section refers to the fact that we will use the naive cotangent complex in this section. Let $(x, A' \rightarrow A)$ be a deformation situation for a given category fibred in groupoids over a locally Noetherian scheme S . The key Example 21.4 suggests that any obstruction theory should be closely related to maps in $D(A)$ with target the naive cotangent complex of A . Working this out we find a criterion for versality in Lemma 22.3 which leads to a criterion for openness of versality in Lemma 22.4. We introduce a notion of a naive obstruction theory in Definition 22.5 to try to formalize the notion a bit further.

In the following we will use the naive cotangent complex as defined in Algebra, Section 133. In particular, if $A' \rightarrow A$ is a surjection of Λ -algebras with square zero kernel I , then there are maps

$$NL_{A'/\Lambda} \rightarrow NL_{A/\Lambda} \rightarrow NL_{A/A'}$$

whose composition is homotopy equivalent to zero (see Algebra, Remark 133.5). This doesn't form a distinguished triangle in general as we are using the naive

cotangent complex and not the full one. There is a homotopy equivalence $NL_{A/A'} \rightarrow I[1]$ (the complex consisting of I placed in degree -1 , see Algebra, Lemma 133.6). Finally, note that there is a canonical map $NL_{A/\Lambda} \rightarrow \Omega_{A/\Lambda}$.

07YK **Lemma 22.1.** *Let $A \rightarrow k$ be a ring map with k a field. Let $E \in D^-(A)$. Then $\text{Ext}_A^i(E, k) = \text{Hom}_k(H^{-i}(E \otimes^{\mathbf{L}} k), k)$.*

Proof. Omitted. Hint: Replace E by a bounded above complex of free A -modules and compute both sides. \square

07YL **Lemma 22.2.** *Let $\Lambda \rightarrow A \rightarrow k$ be finite type ring maps of Noetherian rings with $k = \kappa(\mathfrak{p})$ for some prime \mathfrak{p} of A . Let $\xi : E \rightarrow NL_{A/\Lambda}$ be morphism of $D^-(A)$ such that $H^{-1}(\xi \otimes^{\mathbf{L}} k)$ is not surjective. Then there exists a surjection $A' \rightarrow A$ of Λ -algebras such that*

- (a) $I = \text{Ker}(A' \rightarrow A)$ has square zero and is isomorphic to k as an A -module,
- (b) $\Omega_{A'/\Lambda} \otimes k = \Omega_{A/\Lambda} \otimes k$, and
- (c) $E \rightarrow NL_{A'/\Lambda}$ is zero.

Proof. Let $f \in A$, $f \notin \mathfrak{p}$. Suppose that $A'' \rightarrow A_f$ satisfies (a), (b), (c) for the induced map $E \otimes_A A_f \rightarrow NL_{A_f/\Lambda}$, see Algebra, Lemma 133.13. Then we can set $A' = A'' \times_{A_f} A$ and get a solution. Namely, it is clear that $A' \rightarrow A$ satisfies (a) because $\text{Ker}(A' \rightarrow A) = \text{Ker}(A'' \rightarrow A) = I$. Pick $f'' \in A''$ lifting f . Then the localization of A' at (f'', f) is isomorphic to A'' (for example by More on Algebra, Lemma 5.3). Thus (b) and (c) are clear for A' too. In this way we see that we may replace A by the localization A_f (finitely many times). In particular (after such a replacement) we may assume that \mathfrak{p} is a maximal ideal of A , see Morphisms, Lemma 15.1.

Choose a presentation $A = \Lambda[x_1, \dots, x_n]/J$. Then $NL_{A/\Lambda}$ is (canonically) homotopy equivalent to

$$J/J^2 \longrightarrow \bigoplus_{i=1, \dots, n} \text{Ad}x_i,$$

see Algebra, Lemma 133.2. After localizing if necessary (using Nakayama's lemma) we can choose generators f_1, \dots, f_m of J such that $f_j \otimes 1$ form a basis for $J/J^2 \otimes_A k$. Moreover, after renumbering, we can assume that the images of df_1, \dots, df_r form a basis for the image of $J/J^2 \otimes k \rightarrow \bigoplus kdx_i$ and that df_{r+1}, \dots, df_m map to zero in $\bigoplus kdx_i$. With these choices the space

$$H^{-1}(NL_{A/\Lambda} \otimes_A^{\mathbf{L}} k) = H^{-1}(NL_{A/\Lambda} \otimes_A k)$$

has basis $f_{r+1} \otimes 1, \dots, f_m \otimes 1$. Changing basis once again we may assume that the image of $H^{-1}(\xi \otimes^{\mathbf{L}} k)$ is contained in the k -span of $f_{r+1} \otimes 1, \dots, f_{m-1} \otimes 1$. Set

$$A' = \Lambda[x_1, \dots, x_n]/(f_1, \dots, f_{m-1}, \mathfrak{p}f_m)$$

By construction $A' \rightarrow A$ satisfies (a). Since df_m maps to zero in $\bigoplus kdx_i$ we see that (b) holds. Finally, by construction the induced map $E \rightarrow NL_{A'/\Lambda} = I[1]$ induces the zero map $H^{-1}(E \otimes_A^{\mathbf{L}} k) \rightarrow I \otimes_A k$. By Lemma 22.1 we see that the composition is zero. \square

The following lemma is our key technical result.

07YM **Lemma 22.3.** *Let S be a locally Noetherian scheme. Let \mathcal{X} be a category fibred in groupoids over $(Sch/S)_{fppf}$ satisfying (RS^*) . Let $U = \text{Spec}(A)$ be an affine scheme of finite type over S which maps into an affine open $\text{Spec}(\Lambda)$. Let x be an object of \mathcal{X} over U . Let $\xi : E \rightarrow NL_{A/\Lambda}$ be a morphism of $D^-(A)$. Assume*

- (i) *for every deformation situation $(x, A' \rightarrow A)$ we have: x lifts to $\text{Spec}(A')$ if and only if $E \rightarrow NL_{A/\Lambda} \rightarrow NL_{A/A'}$ is zero, and*
- (ii) *there is an isomorphism of functors $T_x(-) \rightarrow \text{Ext}_A^0(E, -)$ such that $E \rightarrow NL_{A/\Lambda} \rightarrow \Omega_{A/\Lambda}^1$ corresponds to the canonical element (see Remark 20.8).*

Let $u_0 \in U$ be a finite type point with residue field $k = \kappa(u_0)$. Consider the following statements

- (1) *x is versal at u_0 , and*
- (2) *$\xi : E \rightarrow NL_{A/\Lambda}$ induces a surjection $H^{-1}(E \otimes_A^{\mathbf{L}} k) \rightarrow H^{-1}(NL_{A/\Lambda} \otimes_A^{\mathbf{L}} k)$ and an injection $H^0(E \otimes_A^{\mathbf{L}} k) \rightarrow H^0(NL_{A/\Lambda} \otimes_A^{\mathbf{L}} k)$.*

Then we always have (2) \Rightarrow (1) and we have (1) \Rightarrow (2) if u_0 is a closed point.

Proof. Let $\mathfrak{p} = \text{Ker}(A \rightarrow k)$ be the prime corresponding to u_0 .

Assume that x versal at u_0 and that u_0 is a closed point of U . If $H^{-1}(\xi \otimes_A^{\mathbf{L}} k)$ is not surjective, then let $A' \rightarrow A$ be an extension with kernel I as in Lemma 22.2. Because u_0 is a closed point, we see that I is a finite A -module, hence that A' is a finite type Λ -algebra (this fails if u_0 is not closed). In particular A' is Noetherian. By property (c) for A' and (i) for ξ we see that x lifts to an object x' over A' . Let $\mathfrak{p}' \subset A'$ be kernel of the surjective map to k . By Artin-Rees (Algebra, Lemma 50.2) there exists an $n > 1$ such that $(\mathfrak{p}')^n \cap I = 0$. Then we see that

$$B' = A'/(\mathfrak{p}')^n \longrightarrow A/\mathfrak{p}^n = B$$

is a small, essential extension of local Artinian rings, see Formal Deformation Theory, Lemma 3.12. On the other hand, as x is versal at u_0 and as $x'|_{\text{Spec}(B')}$ is a lift of $x|_{\text{Spec}(B)}$, there exists an integer $m \geq n$ and a map $q : A/\mathfrak{p}^m \rightarrow B'$ such that the composition $A/\mathfrak{p}^m \rightarrow B' \rightarrow B$ is the quotient map. Since the maximal ideal of B' has n th power equal to zero, this q factors through B which contradicts the fact that $B' \rightarrow B$ is an essential surjection. This contradiction shows that $H^{-1}(\xi \otimes_A^{\mathbf{L}} k)$ is surjective.

Assume that x versal at u_0 . By Lemma 22.1 the map $H^0(\xi \otimes_A^{\mathbf{L}} k)$ is dual to the map $\text{Ext}_A^0(NL_{A/\Lambda}, k) \rightarrow \text{Ext}_A^0(E, k)$. Note that

$$\text{Ext}_A^0(NL_{A/\Lambda}, k) = \text{Der}_\Lambda(A, k) \quad \text{and} \quad T_x(k) = \text{Ext}_A^0(E, k)$$

Condition (ii) assures us the map $\text{Ext}_A^0(NL_{A/\Lambda}, k) \rightarrow \text{Ext}_A^0(E, k)$ sends a tangent vector θ to U at u_0 to the corresponding infinitesimal deformation of x_0 , see Remark 20.8. Hence if x is versal, then this map is surjective, see Formal Deformation Theory, Lemma 13.2. Hence $H^0(\xi \otimes_A^{\mathbf{L}} k)$ is injective. This finishes the proof of (1) \Rightarrow (2) in case u_0 is a closed point.

For the rest of the proof assume $H^{-1}(E \otimes_A^{\mathbf{L}} k) \rightarrow H^{-1}(NL_{A/\Lambda} \otimes_A^{\mathbf{L}} k)$ is surjective and $H^0(E \otimes_A^{\mathbf{L}} k) \rightarrow H^0(NL_{A/\Lambda} \otimes_A^{\mathbf{L}} k)$ injective. Set $R = A_{\mathfrak{p}}^\wedge$ and let η be the formal object over R associated to $x|_{\text{Spec}(R)}$. The map $d\eta$ on tangent spaces is surjective because it is identified with the dual of the injective map $H^0(E \otimes_A^{\mathbf{L}} k) \rightarrow H^0(NL_{A/\Lambda} \otimes_A^{\mathbf{L}} k)$ (see previous paragraph). According to Formal Deformation Theory, Lemma 13.2 it suffices to prove the following: Let $C' \rightarrow C$ be a small

extension of finite type Artinian local Λ -algebras with residue field k . Let $R \rightarrow C$ be a Λ -algebra map compatible with identifications of residue fields. Let $y = x|_{\text{Spec}(C)}$ and let y' be a lift of y to C' . To show: we can lift the Λ -algebra map $R \rightarrow C$ to $R \rightarrow C'$.

Observe that it suffices to lift the Λ -algebra map $A \rightarrow C$. Let $I = \text{Ker}(C' \rightarrow C)$. Note that I is a 1-dimensional k -vector space. The obstruction ob to lifting $A \rightarrow C$ is an element of $\text{Ext}_A^1(NL_{A/\Lambda}, I)$, see Example 21.4. By Lemma 22.1 and our assumption the map ξ induces an injection

$$\text{Ext}_A^1(NL_{A/\Lambda}, I) \longrightarrow \text{Ext}_A^1(E, I)$$

By the construction of ob and (i) the image of ob in $\text{Ext}_A^1(E, I)$ is the obstruction to lifting x to $A \times_C C'$. By (RS*) the fact that y/C lifts to y'/C' implies that x lifts to $A \times_C C'$. Hence $ob = 0$ and we are done. \square

The key lemma above allows us to conclude that we have openness of versality in some cases.

07YN **Lemma 22.4.** *Let S be a locally Noetherian scheme. Let \mathcal{X} be a category fibred in groupoids over $(\text{Sch}/S)_{fppf}$ satisfying (RS*). Let $U = \text{Spec}(A)$ be an affine scheme of finite type over S which maps into an affine open $\text{Spec}(\Lambda)$. Let x be an object of \mathcal{X} over U . Let $\xi : E \rightarrow NL_{A/\Lambda}$ be a morphism of $D^-(A)$. Assume*

- (i) *for every deformation situation $(x, A' \rightarrow A)$ we have: x lifts to $\text{Spec}(A')$ if and only if $E \rightarrow NL_{A/\Lambda} \rightarrow NL_{A/A'}$ is zero,*
- (ii) *there is an isomorphism of functors $T_x(-) \rightarrow \text{Ext}_A^0(E, -)$ such that $E \rightarrow NL_{A/\Lambda} \rightarrow \Omega_{A/\Lambda}^1$ corresponds to the canonical element (see Remark 20.8),*
- (iii) *the cohomology groups of E are finite A -modules.*

If x is versal at a closed point $u_0 \in U$, then there exists an open neighbourhood $u_0 \in U' \subset U$ such that x is versal at every finite type point of U' .

Proof. Let C be the cone of ξ so that we have a distinguished triangle

$$E \rightarrow NL_{A/\Lambda} \rightarrow C \rightarrow E[1]$$

in $D^-(A)$. By Lemma 22.3 the assumption that x is versal at u_0 implies that $H^{-1}(C \otimes^{\mathbf{L}} k) = 0$. By More on Algebra, Lemma 72.4 there exists an $f \in A$ not contained in the prime corresponding to u_0 such that $H^{-1}(C \otimes_A^{\mathbf{L}} M) = 0$ for any A_f -module M . Using Lemma 22.3 again we see that we have versality for all finite type points of the open $D(f) \subset U$. \square

The technical lemmas above suggest the following definition.

07YP **Definition 22.5.** *Let S be a locally Noetherian base. Let \mathcal{X} be a category fibred in groupoids over $(\text{Sch}/S)_{fppf}$. Assume that \mathcal{X} satisfies (RS*). A naive obstruction theory is given by the following data*

- 07YQ (1) *for every S -algebra A such that $\text{Spec}(A) \rightarrow S$ maps into an affine open $\text{Spec}(\Lambda) \subset S$ and every object x of \mathcal{X} over $\text{Spec}(A)$ we are given an object $E_x \in D^-(A)$ and a map $\xi_x : E \rightarrow NL_{A/\Lambda}$,*
- 07YR (2) *given (x, A) as in (1) there are transformations of functors*

$$\text{Inf}_x(-) \rightarrow \text{Ext}_A^{-1}(E_x, -) \quad \text{and} \quad T_x(-) \rightarrow \text{Ext}_A^0(E_x, -)$$

07YS (3) for (x, A) as in (1) and a ring map $A \rightarrow B$ setting $y = x|_{\text{Spec}(B)}$ there is a functoriality map $E_x \rightarrow E_y$ in $D(A)$.

These data are subject to the following conditions

(i) in the situation of (3) the diagram

$$\begin{array}{ccc} E_y & \xrightarrow{\xi_y} & NL_{B/\Lambda} \\ \uparrow & & \uparrow \\ E_x & \xrightarrow{\xi_x} & NL_{A/\Lambda} \end{array}$$

is commutative in $D(A)$,

- (ii) given (x, A) as in (1) and $A \rightarrow B \rightarrow C$ setting $y = x|_{\text{Spec}(B)}$ and $z = x|_{\text{Spec}(C)}$ the composition of the functoriality maps $E_x \rightarrow E_y$ and $E_y \rightarrow E_z$ is the functoriality map $E_x \rightarrow E_z$,
- (iii) the maps of (2) are isomorphisms compatible with the functoriality maps and the maps of Remark 20.3,
- (iv) the composition $E_x \rightarrow NL_{A/\Lambda} \rightarrow \Omega_{A/\Lambda}$ corresponds to the canonical element of $T_x(\Omega_{A/\Lambda}) = \text{Ext}^0(E_x, \Omega_{A/\Lambda})$, see Remark 20.8,
- (v) given a deformation situation $(x, A' \rightarrow A)$ with $I = \text{Ker}(A' \rightarrow A)$ the composition $E_x \rightarrow NL_{A/\Lambda} \rightarrow NL_{A/A'}$ is zero in

$$\text{Hom}_A(E_x, NL_{A/\Lambda}) = \text{Ext}_A^0(E_x, NL_{A/A'}) = \text{Ext}_A^1(E_x, I)$$

if and only if x lifts to A' .

Thus we see in particular that we obtain an obstruction theory as in Section 21 by setting $\mathcal{O}_x(-) = \text{Ext}_A^1(E_x, -)$.

07YT **Lemma 22.6.** *Let S and \mathcal{X} be as in Definition 22.5 and let \mathcal{X} be endowed with a naive obstruction theory. Let $A \rightarrow B$ and $y \rightarrow x$ be as in (3). Let k be a B -algebra which is a field. Then the functoriality map $E_x \rightarrow E_y$ induces bijections*

$$H^i(E_x \otimes_A^{\mathbf{L}} k) \rightarrow H^i(E_y \otimes_B^{\mathbf{L}} k)$$

for $i = 0, 1$.

Proof. Let $z = x|_{\text{Spec}(k)}$. Then (RS*) implies that

$$\text{Lift}(x, A[k]) = \text{Lift}(z, k[k]) \quad \text{and} \quad \text{Lift}(y, B[k]) = \text{Lift}(z, k[k])$$

because $A[k] = A \times_k k[k]$ and $B[k] = B \times_k k[k]$. Hence the properties of a naive obstruction theory imply that the functoriality map $E_x \rightarrow E_y$ induces bijections $\text{Ext}_A^i(E_x, k) \rightarrow \text{Ext}_B^i(E_y, k)$ for $i = -1, 0$. By Lemma 22.1 our maps $H^i(E_x \otimes_A^{\mathbf{L}} k) \rightarrow H^i(E_y \otimes_B^{\mathbf{L}} k)$, $i = 0, 1$ induce isomorphisms on dual vector spaces hence are isomorphisms. \square

07YU **Lemma 22.7.** *Let S be a locally Noetherian scheme. Let $p : \mathcal{X} \rightarrow (\text{Sch}/S)_{\text{fppf}}^{\text{opp}}$ be a category fibred in groupoids. Assume that \mathcal{X} satisfies (RS*) and that \mathcal{X} has a naive obstruction theory. Then openness of versality holds for \mathcal{X} provided the complexes E_x of Definition 22.5 have finitely generated cohomology groups for pairs (A, x) where A is of finite type over S .*

Proof. Let U be a scheme locally of finite type over S , let x be an object of \mathcal{X} over U , and let u_0 be a finite type point of U such that x is versal at u_0 . We may first shrink U to an affine scheme such that u_0 is a closed point and such that $U \rightarrow S$

maps into an affine open $\text{Spec}(\Lambda)$. Say $U = \text{Spec}(A)$. Let $\xi_x : E_x \rightarrow NL_{A/\Lambda}$ be the obstruction map. At this point we may apply Lemma 22.4 to conclude. \square

23. A dual notion

07YV Let $(x, A' \rightarrow A)$ be a deformation situation for a given category \mathcal{X} fibred in groupoids over a locally Noetherian scheme S . Assume \mathcal{X} has an obstruction theory, see Definition 21.1. In practice one often has a complex K^\bullet of A -modules and isomorphisms of functors

$$\text{Inf}_x(-) \rightarrow H^0(K^\bullet \otimes_A^{\mathbf{L}} -), \quad T_x(-) \rightarrow H^1(K^\bullet \otimes_A^{\mathbf{L}} -), \quad \mathcal{O}_x(-) \rightarrow H^2(K^\bullet \otimes_A^{\mathbf{L}} -)$$

In this section we formalize this a little bit and show how this leads to a verification of openness of versality in some cases.

07YW **Example 23.1.** Let $\Lambda, S, W, \mathcal{F}$ be as in Example 21.3. Assume that $W \rightarrow S$ is proper and \mathcal{F} coherent. By Cohomology of Schemes, Remark 22.2 there exists a finite complex of finite projective Λ -modules N^\bullet which universally computes the cohomology of \mathcal{F} . In particular the obstruction spaces from Example 21.3 are $\mathcal{O}_x(M) = H^1(N^\bullet \otimes_\Lambda M)$. Hence with $K^\bullet = N^\bullet \otimes_\Lambda A[-1]$ we see that $\mathcal{O}_x(M) = H^2(K^\bullet \otimes_A^{\mathbf{L}} M)$.

07YX **Situation 23.2.** Let S be a locally Noetherian scheme. Let \mathcal{X} be a category fibred in groupoids over $(\text{Sch}/S)_{fppf}$. Assume that \mathcal{X} has (RS*) so that we can speak of the functor $T_x(-)$, see Lemma 20.2. Let $U = \text{Spec}(A)$ be an affine scheme of finite type over S which maps into an affine open $\text{Spec}(\Lambda)$. Let x be an object of \mathcal{X} over U . Assume we are given

- (1) a complex of A -modules K^\bullet ,
- (2) a transformation of functors $T_x(-) \rightarrow H^1(K^\bullet \otimes_A^{\mathbf{L}} -)$,
- (3) for every deformation situation $(x, A' \rightarrow A)$ with kernel $I = \text{Ker}(A' \rightarrow A)$ an element $o_x(A') \in H^2(K^\bullet \otimes_A^{\mathbf{L}} I)$

satisfying the following (minimal) conditions

- (i) the transformation $T_x(-) \rightarrow H^1(K^\bullet \otimes_A^{\mathbf{L}} -)$ is an isomorphism,
- (ii) given a morphism $(x, A'' \rightarrow A) \rightarrow (x, A' \rightarrow A)$ of deformation situations the element $o_x(A')$ maps to the element $o_x(A'')$ via the map $H^2(K^\bullet \otimes_A^{\mathbf{L}} I) \rightarrow H^2(K^\bullet \otimes_A^{\mathbf{L}} I')$ where $I' = \text{Ker}(A'' \rightarrow A)$, and
- (iii) x lifts to an object over $\text{Spec}(A')$ if and only if $o_x(A') = 0$.

It is possible to incorporate infinitesimal automorphisms as well, but we refrain from doing so in order to get the sharpest possible result.

In Situation 23.2 an important role will be played by $K^\bullet \otimes_A^{\mathbf{L}} NL_{A/\Lambda}$. Suppose we are given an element $\xi \in H^1(K^\bullet \otimes_A^{\mathbf{L}} NL_{A/\Lambda})$. Then (1) for any surjection $A' \rightarrow A$ of Λ -algebras with kernel I of square zero the canonical map $NL_{A/\Lambda} \rightarrow NL_{A/A'} = I[1]$ sends ξ to an element $\xi_{A'} \in H^2(K^\bullet \otimes_A^{\mathbf{L}} I)$ and (2) the map $NL_{A/\Lambda} \rightarrow \Omega_{A/\Lambda}$ sends ξ to an element ξ_{can} of $H^1(K^\bullet \otimes_A^{\mathbf{L}} \Omega_{A/\Lambda})$.

07YY **Lemma 23.3.** *In Situation 23.2. Assume furthermore that*

- (iv) *given a short exact sequence of deformation situations as in Remark 20.5 and a lift $x'_2 \in \text{Lift}(x, A'_2)$ then $o_x(A'_3) \in H^2(K^\bullet \otimes_A^{\mathbf{L}} I_3)$ equals $\partial\theta$ where $\theta \in H^1(K^\bullet \otimes_A^{\mathbf{L}} I_1)$ is the element corresponding to $x'_2|_{\text{Spec}(A'_1)}$ via $A'_1 = A[I_1]$ and the given map $T_x(-) \rightarrow H^1(K^\bullet \otimes_A^{\mathbf{L}} -)$.*

In this case there exists an element $\xi \in H^1(K^\bullet \otimes_A^{\mathbf{L}} NL_{A/\Lambda})$ such that

- (1) for every deformation situation $(x, A' \rightarrow A)$ we have $\xi_{A'} = o_x(A')$, and
- (2) ξ_{can} matches the canonical element of Remark 20.8 via the given transformation $T_x(-) \rightarrow H^1(K^\bullet \otimes_A^{\mathbf{L}} -)$.

Proof. Choose a $\alpha : \Lambda[x_1, \dots, x_n] \rightarrow A$ with kernel J . Write $P = \Lambda[x_1, \dots, x_n]$. In the rest of this proof we work with

$$NL(\alpha) = (J/J^2 \longrightarrow \bigoplus Adx_i)$$

which is permissible by Algebra, Lemma 133.2 and More on Algebra, Lemma 57.1. Consider the element $o_x(P/J^2) \in H^2(K^\bullet \otimes_A^{\mathbf{L}} J/J^2)$ and consider the quotient

$$C = (P/J^2 \times \bigoplus Adx_i)/(J/J^2)$$

where J/J^2 is embedded diagonally. Note that $C \rightarrow A$ is a surjection with kernel $\bigoplus Adx_i$. Moreover there is a section $A \rightarrow C$ to $C \rightarrow A$ given by mapping the class of $f \in P$ to the class of (f, df) in the pushout. For later use, denote x_C the pullback of x along the corresponding morphism $\text{Spec}(C) \rightarrow \text{Spec}(A)$. Thus we see that $o_x(C) = 0$. We conclude that $o_x(P/J^2)$ maps to zero in $H^2(K^\bullet \otimes_A^{\mathbf{L}} \bigoplus Adx_i)$. It follows that there exists some element $\xi \in H^1(K^\bullet \otimes_A^{\mathbf{L}} NL(\alpha))$ mapping to $o_x(P/J^2)$.

Note that for any deformation situation $(x, A' \rightarrow A)$ there exists a Λ -algebra map $P/J^2 \rightarrow A'$ compatible with the augmentations to A . Hence the element ξ satisfies the first property of the lemma by construction and property (ii) of Situation 23.2.

Note that our choice of ξ was well defined up to the choice of an element of $H^1(K^\bullet \otimes_A^{\mathbf{L}} \bigoplus Adx_i)$. We will show that after modifying ξ by an element of the aforementioned group we can arrange it so that the second assertion of the lemma is true. Let $C' \subset C$ be the image of P/J^2 under the Λ -algebra map $P/J^2 \rightarrow C$ (inclusion of first factor). Observe that $\text{Ker}(C' \rightarrow A) = \text{Im}(J/J^2 \rightarrow \bigoplus Adx_i)$. Set $\overline{C} = A[\Omega_{A/\Lambda}]$. The map $P/J^2 \times \bigoplus Adx_i \rightarrow \overline{C}$, $(f, \sum f_i dx_i) \mapsto (f \bmod J, \sum f_i dx_i)$ factors through a surjective map $C \rightarrow \overline{C}$. Then

$$(x, \overline{C} \rightarrow A) \rightarrow (x, C \rightarrow A) \rightarrow (x, C' \rightarrow A)$$

is a short exact sequence of deformation situations. The associated splitting $\overline{C} = A[\Omega_{A/\Lambda}]$ (from Remark 20.5) equals the given splitting above. Moreover, the section $A \rightarrow C$ composed with the map $C \rightarrow \overline{C}$ is the map $(1, d) : A \rightarrow A[\Omega_{A/\Lambda}]$ of Remark 20.8. Thus x_C restricts to the canonical element x_{can} of $T_x(\Omega_{A/\Lambda}) = \text{Lift}(x, A[\Omega_{A/\Lambda}])$. By condition (iv) we conclude that $o_x(P/J^2)$ maps to ∂x_{can} in

$$H^1(K^\bullet \otimes_A^{\mathbf{L}} \text{Im}(J/J^2 \rightarrow \bigoplus Adx_i))$$

By construction ξ maps to $o_x(P/J^2)$. It follows that x_{can} and ξ_{can} map to the same element in the displayed group which means (by the long exact cohomology sequence) that they differ by an element of $H^1(K^\bullet \otimes_A^{\mathbf{L}} \bigoplus Adx_i)$ as desired. \square

07YZ **Lemma 23.4.** *In Situation 23.2 assume that (iv) of Lemma 23.3 holds and that K^\bullet is a perfect object of $D(A)$. In this case, if x is versal at a closed point $u_0 \in U$ then there exists an open neighbourhood $u_0 \in U' \subset U$ such that x is versal at every finite type point of U' .*

Proof. We may assume that K^\bullet is a finite complex of finite projective A -modules. Thus the derived tensor product with K^\bullet is the same as simply tensoring with K^\bullet . Let E^\bullet be the dual perfect complex to K^\bullet , see More on Algebra, Lemma 70.14. (So $E^n = \text{Hom}_A(K^{-n}, A)$ with differentials the transpose of the differentials of K^\bullet .) Let $E \in D^-(A)$ denote the object represented by the complex $E^\bullet[-1]$. Let $\xi \in H^1(\text{Tot}(K^\bullet \otimes_A NL_{A/\Lambda}))$ be the element constructed in Lemma 23.3 and denote $\xi : E = E^\bullet[-1] \rightarrow NL_{A/\Lambda}$ the corresponding map (loc.cit.). We claim that the pair (E, ξ) satisfies all the assumptions of Lemma 22.4 which finishes the proof.

Namely, assumption (i) of Lemma 22.4 follows from conclusion (1) of Lemma 23.3 and the fact that $H^2(K^\bullet \otimes_A^L -) = \text{Ext}^1(E, -)$ by loc.cit. Assumption (ii) of Lemma 22.4 follows from conclusion (2) of Lemma 23.3 and the fact that $H^1(K^\bullet \otimes_A^L -) = \text{Ext}^0(E, -)$ by loc.cit. Assumption (iii) of Lemma 22.4 is clear. \square

24. Other chapters

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