ARTIN'S AXIOMS

07SZ

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1. Introduction

07T0 In this chapter we discuss Artin's axioms for the representability of functors by algebraic spaces. As references we suggest the papers [Art69], [Art70], [Art74].

Some of the notation, conventions, and terminology in this chapter is awkward and may seem backwards to the more experienced reader. This is intentional. Please see Quot, Section 2 for an explanation.

Let S be a locally Noetherian base scheme. Let

$$p : \mathcal{X} \longrightarrow (Sch/S)_{fppf}$$

be a category fibred in groupoids. Let x_0 be an object of $\mathcal{X}$ over a field k of finite type over S. Throughout this chapter an important role is played by the predeformation category (see Formal Deformation Theory, Definition 6.2)

$$\mathcal{F}_{\mathcal{X},k,x_0} \longrightarrow \{\text{Artinian local } S\text{-algebras with residue field } k\}$$

associated to x_0 over k. We introduce the Rim-Schlessinger condition (RS) for $\mathcal{X}$ and show it guarantees that $\mathcal{F}_{\mathcal{X},k,x_0}$ is a deformation category, i.e., $\mathcal{F}_{\mathcal{X},k,x_0}$ satisfies (RS) itself. We discuss how $\mathcal{F}_{\mathcal{X},k,x_0}$ changes if one replaces k by a finite extension and we discuss tangent spaces.

Next, we discuss formal objects $\xi = (\xi_n)$ of $\mathcal{X}$ which are inverse systems of objects lying over the quotients $R/\mathfrak{m}^n$ where R is a Noetherian complete local S-algebra whose residue field is of finite type over S. This is the same thing as having a formal object in $\mathcal{F}_{\mathcal{X},k,x_0}$ for some x_0 and k. A formal object is called effective when there is an object of $\mathcal{X}$ over R which gives rise to the inverse system. A formal object of $\mathcal{X}$ is called versal if it gives rise to a versal formal object of $\mathcal{F}_{\mathcal{X},k,x_0}$. Finally, given a finite type S-scheme U, an object x of $\mathcal{X}$ over U, and a closed point $u_0 \in U$ we say x is versal at u_0 if the induced formal object over the complete local ring $\mathcal{O}^\wedge_{U,u_0}$ is versal.

Having worked through this material we can state Artin's celebrated theorem: our $\mathcal{X}$ is an algebraic stack if the following are true

(1) $\mathcal{O}_{S,s}$ is a G-ring for all $s \in S$[1],
(2) $\Delta : \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is representable by algebraic spaces,
(3) $\mathcal{X}$ is a stack for the étale topology,
(4) $\mathcal{X}$ is limit preserving,
(5) $\mathcal{X}$ satisfies (RS),
(6) tangent spaces and spaces of infinitesimal automorphisms of the deformation categories $\mathcal{F}_{\mathcal{X},k,x_0}$ are finite dimensional,
(7) formal objects are effective,
(8) $\mathcal{X}$ satisfies openness of versality.

This is Lemma 17.1; see also Proposition 17.2 for a slight improvement. There is an analogous proposition characterizing which functors $F : (Sch/S)^{opp}_{fppf} \to Sets$ are algebraic spaces, see Section 16.

Here is a rough outline of the proof of Artin's theorem. First we show that there are plenty of versal formal objects using (RS) and the finite dimensionality of tangent and aut spaces, see for example Formal Deformation Theory, Lemma 27.6. These formal objects are effective by assumption. Effective formal objects can be "approximated" by objects x over finite type S-schemes U, see Lemma 10.1. This approximation uses the local rings of S are G-rings and that $\mathcal{X}$ is limit preserving; it is perhaps the most difficult part of the proof relying as it does on general Néron desingularization to approximate formal solutions of algebraic equations over

[1]This means that S is a G-scheme, see Properties, Section 14.

a Noetherian local G-ring by solutions in the henselization. Next openness of versality implies we may (after shrinking U) assume x is versal at every closed point of U. Having done all of this we show that $U \to \mathcal{X}$ is a smooth morphism. Taking sufficiently many $U \to \mathcal{X}$ we show that we obtain a "smooth atlas" for $\mathcal{X}$ which shows that $\mathcal{X}$ is an algebraic stack.

In checking Artin's axioms for a given category $\mathcal{X}$ fibred in groupoids, the most difficult step is often to verify openness of versality. For the discussion that follows, assume that $\mathcal{X}/S$ already satisfies the other conditions listed above. In this chapter we offer two methods that will allow the reader to prove $\mathcal{X}$ satisfies openness of versality:

(1) The first is to assume a stronger Rim-Schlessinger condition, called (RS*) and to assume a stronger version of formal effectiveness, essentially requiring objects over inverse systems of thickenings to be effective. It turns out that under these assumptions, openness of versality comes for free, see Lemma 20.3. Please observe that here we are using in an essential manner that $\mathcal{X}$ is defined on that category of all schemes over S, not just the category of Noetherian schemes!
(2) The second, following Artin, is to require $\mathcal{X}$ to come equipped with an obstruction theory. If said obstruction theory "commutes with products" in a suitable sense, then $\mathcal{X}$ satisfies openness of versality, see Lemma 22.2.

Obstruction theories can be axiomatized in many different ways and indeed many variants (often adapted to specific moduli stacks) can be found in the literature. We explain a variant using the derived category (which often arises naturally from deformation theory computations done in the literature) in Lemma 24.4.

In Section 26 we discuss what needs to be modified to make things work for functors defined on the category $(\mathit{Noetherian}/S)_{\mathit{étale}}$ of locally Noetherian schemes over S.

In the final section of this chapter as an application of Artin's axioms we prove Artin's theorem on the existence of contractions, see Section 27. The theorem says roughly that given an algebraic space X' separated of finite type over S, a closed subset $T' \subset |X'|$, and a formal modification

$$\mathfrak{f} : X'_{/T'} \longrightarrow \mathfrak{X}$$

where $\mathfrak{X}$ is a Noetherian formal algebraic space over S, there exists a proper morphism $f : X' \to X$ which "realizes the contraction". By this we mean that there exists an identification $\mathfrak{X} = X_{/T}$ such that $\mathfrak{f} = f_{/T'} : X'_{/T'} \to X_{/T}$ where $T = f(T')$ and moreover f is an isomorphism over $X \setminus T$. The proof proceeds by defining a functor F on the category of locally Noetherian schemes over S and proving Artin's axioms for F. Amusingly, in this application of Artin's axioms, openness of versality is not the hardest thing to prove, instead the proof that F is limit preserving requires a lot of work and preliminary results.

2. Conventions

07T1 The conventions we use in this chapter are the same as those in the chapter on algebraic stacks, see Algebraic Stacks, Section 2. In this chapter the base scheme S will often be locally Noetherian (although we will always reiterate this condition when stating results).

3. Predeformation categories

07T2 Let S be a locally Noetherian base scheme. Let

$$p : \mathcal{X} \longrightarrow (Sch/S)_{fppf}$$

be a category fibred in groupoids. Let k be a field and let $\mathrm{Spec}(k) \to S$ be a morphism of finite type (see Morphisms, Lemma 16.1). We will sometimes simply say that k *is a field of finite type over* S. Let x_0 be an object of $\mathcal{X}$ lying over $\mathrm{Spec}(k)$. Given S, $\mathcal{X}$, k, and x_0 we will construct a predeformation category, as defined in Formal Deformation Theory, Definition 6.2. The construction will resemble the construction of Formal Deformation Theory, Remark 6.4.

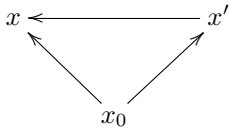
First, by Morphisms, Lemma 16.1 we may pick an affine open $\mathrm{Spec}(\Lambda) \subset S$ such that $\mathrm{Spec}(k) \to S$ factors through $\mathrm{Spec}(\Lambda)$ and the associated ring map $\Lambda \to k$ is finite. This provides us with the category $\mathcal{C}_\Lambda$, see Formal Deformation Theory, Definition 3.1. The category $\mathcal{C}_\Lambda$, up to canonical equivalence, does not depend on the choice of the affine open $\mathrm{Spec}(\Lambda)$ of S. Namely, $\mathcal{C}_\Lambda$ is equivalent to the opposite of the category of factorizations

07T3 (3.0.1) $$\mathrm{Spec}(k) \to \mathrm{Spec}(A) \to S$$

of the structure morphism such that A is an Artinian local ring and such that $\mathrm{Spec}(k) \to \mathrm{Spec}(A)$ corresponds to a ring map $A \to k$ which identifies k with the residue field of A.

We let $\mathcal{F} = \mathcal{F}_{\mathcal{X},k,x_0}$ be the category whose

(1) objects are morphisms $x_0 \to x$ of $\mathcal{X}$ where $p(x) = \mathrm{Spec}(A)$ with A an Artinian local ring and $p(x_0) \to p(x) \to S$ a factorization as in (3.0.1), and
(2) morphisms $(x_0 \to x) \to (x_0 \to x')$ are commutative diagrams



in $\mathcal{X}$. (Note the reversal of arrows.)

If $x_0 \to x$ is an object of $\mathcal{F}$ then writing $p(x) = \mathrm{Spec}(A)$ we obtain an object A of $\mathcal{C}_\Lambda$. We often say that $x_0 \to x$ or x lies over A. A morphism of $\mathcal{F}$ between objects $x_0 \to x$ lying over A and $x_0 \to x'$ lying over A' corresponds to a morphism $x' \to x$ of $\mathcal{X}$, hence a morphism $p(x' \to x) : \mathrm{Spec}(A') \to \mathrm{Spec}(A)$ which in turn corresponds to a ring map $A \to A'$. As $\mathcal{X}$ is a category over the category of schemes over S we see that $A \to A'$ is Λ-algebra homomorphism. Thus we obtain a functor

07T4 (3.0.2) $$p : \mathcal{F} = \mathcal{F}_{\mathcal{X},k,x_0} \longrightarrow \mathcal{C}_\Lambda.$$

We will use the notation $\mathcal{F}(A)$ to denote the fibre category over an object A of $\mathcal{C}_\Lambda$. An object of $\mathcal{F}(A)$ is simply a morphism $x_0 \to x$ of $\mathcal{X}$ such that x lies over $\mathrm{Spec}(A)$ and $x_0 \to x$ lies over $\mathrm{Spec}(k) \to \mathrm{Spec}(A)$.

07T5 **Lemma 3.1.** *The functor $p : \mathcal{F} \to \mathcal{C}_\Lambda$ defined above is a predeformation category.*

Proof. We have to show that $\mathcal{F}$ is (a) cofibred in groupoids over $\mathcal{C}_\Lambda$ and (b) that $\mathcal{F}(k)$ is a category equivalent to a category with a single object and a single morphism.

Proof of (a). The fibre categories of $\mathcal{F}$ over $\mathcal{C}_\Lambda$ are groupoids as the fibre categories of $\mathcal{X}$ are groupoids. Let $A \to A'$ be a morphism of $\mathcal{C}_\Lambda$ and let $x_0 \to x$ be an object of $\mathcal{F}(A)$. Because $\mathcal{X}$ is fibred in groupoids, we can find a morphism $x' \to x$ lying over $\mathrm{Spec}(A') \to \mathrm{Spec}(A)$. Since the composition $A \to A' \to k$ is equal the given map $A \to k$ we see (by uniqueness of pullbacks up to isomorphism) that the pullback via $\mathrm{Spec}(k) \to \mathrm{Spec}(A')$ of x' is x_0, i.e., that there exists a morphism $x_0 \to x'$ lying over $\mathrm{Spec}(k) \to \mathrm{Spec}(A')$ compatible with $x_0 \to x$ and $x' \to x$. This proves that $\mathcal{F}$ has pushforwards. We conclude by (the dual of) Categories, Lemma 35.2.

Proof of (b). If $A = k$, then $\mathrm{Spec}(k) = \mathrm{Spec}(A)$ and since $\mathcal{X}$ is fibred in groupoids over $(Sch/S)_{fppf}$ we see that given any object $x_0 \to x$ in $\mathcal{F}(k)$ the morphism $x_0 \to x$ is an isomorphism. Hence every object of $\mathcal{F}(k)$ is isomorphic to $x_0 \to x_0$. Clearly the only self morphism of $x_0 \to x_0$ in $\mathcal{F}$ is the identity. □

Let S be a locally Noetherian base scheme. Let $F : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism between categories fibred in groupoids over $(Sch/S)_{fppf}$. Let k is a field of finite type over S. Let x_0 be an object of $\mathcal{X}$ lying over $\mathrm{Spec}(k)$. Set $y_0 = F(x_0)$ which is an object of $\mathcal{Y}$ lying over $\mathrm{Spec}(k)$. Then F induces a functor

07WJ (3.1.1) $$F : \mathcal{F}_{\mathcal{X},k,x_0} \longrightarrow \mathcal{F}_{\mathcal{Y},k,y_0}$$

of categories cofibred over $\mathcal{C}_\Lambda$. Namely, to the object $x_0 \to x$ of $\mathcal{F}_{\mathcal{X},k,x_0}(A)$ we associate the object $F(x_0) \to F(x)$ of $\mathcal{F}_{\mathcal{Y},k,y_0}(A)$.

07WK **Lemma 3.2.** *Let S be a locally Noetherian scheme. Let $F : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. Assume either*

1. *F is formally smooth on objects (Criteria for Representability, Section 6),*
2. *F is representable by algebraic spaces and formally smooth, or*
3. *F is representable by algebraic spaces and smooth.*

Then for every finite type field k over S and object x_0 of $\mathcal{X}$ over k the functor (3.1.1) is smooth in the sense of Formal Deformation Theory, Definition 8.1.

Proof. Case (1) is a matter of unwinding the definitions. Assumption (2) implies (1) by Criteria for Representability, Lemma 6.3. Assumption (3) implies (2) by More on Morphisms of Spaces, Lemma 19.6 and the principle of Algebraic Stacks, Lemma 10.9. □

07WL **Lemma 3.3.** *Let S be a locally Noetherian scheme. Let*

$$\begin{array}{ccc} \mathcal{W} & \longrightarrow & \mathcal{Z} \\ \downarrow & & \downarrow \\ \mathcal{X} & \longrightarrow & \mathcal{Y} \end{array}$$

be a 2-fibre product of categories fibred in groupoids over $(Sch/S)_{fppf}$. Let k be a finite type field over S and w_0 an object of $\mathcal{W}$ over k. Let x_0, z_0, y_0 be the images of w_0 under the morphisms in the diagram. Then

$$\begin{array}{ccc} \mathcal{F}_{\mathcal{W},k,w_0} & \longrightarrow & \mathcal{F}_{\mathcal{Z},k,z_0} \\ \downarrow & & \downarrow \\ \mathcal{F}_{\mathcal{X},k,x_0} & \longrightarrow & \mathcal{F}_{\mathcal{Y},k,y_0} \end{array}$$

is a fibre product of predeformation categories.

Proof. This is a matter of unwinding the definitions. Details omitted. □

4. Pushouts and stacks

07WM In this section we show that algebraic stacks behave well with respect to certain pushouts. The results in this section hold over any base scheme.

The following lemma is also correct when Y, X', X, Y' are algebraic spaces, see (insert future reference here).

07WN **Lemma 4.1.** *Let S be a scheme. Let*

$$\begin{array}{ccc} X & \longrightarrow & X' \\ \downarrow & & \downarrow \\ Y & \longrightarrow & Y' \end{array}$$

be a pushout in the category of schemes over S where $X \to X'$ is a thickening and $X \to Y$ is affine, see More on Morphisms, Lemma 14.3. Let $\mathcal{Z}$ be an algebraic stack over S. Then the functor of fibre categories

$$\mathcal{Z}_{Y'} \longrightarrow \mathcal{Z}_Y \times_{\mathcal{Z}_X} \mathcal{Z}_{X'}$$

is an equivalence of categories.

Proof. Let y' be an object of left hand side. The sheaf $Isom(y', y')$ on the category of schemes over Y' is representable by an algebraic space I over Y', see Algebraic Stacks, Lemma 10.11. We conclude that the functor of the lemma is fully faithful as Y' is the pushout in the category of algebraic spaces as well as the category of schemes, see Pushouts of Spaces, Lemma 6.1.

Let (y, x', f) be an object of the right hand side. Here $f : y|_X \to x'|_X$ is an isomorphism. To finish the proof we have to construct an object y' of $\mathcal{Z}_{Y'}$ whose restrictions to Y and X' agree with y and x' in a manner compatible with f. In fact, it suffices to construct y' fppf locally on Y', see Stacks, Lemma 4.8. Choose a representable algebraic stack $\mathcal{W}$ and a surjective smooth morphism $\mathcal{W} \to \mathcal{Z}$. Then

$$(Sch/Y)_{fppf} \times_{y,\mathcal{Z}} \mathcal{W} \quad \text{and} \quad (Sch/X')_{fppf} \times_{x',\mathcal{Z}} \mathcal{W}$$

are algebraic stacks representable by algebraic spaces V and U' smooth over Y and X'. The isomorphism f induces an isomorphism $\varphi : V \times_Y X \to U' \times_{X'} X$ over X. By Pushouts of Spaces, Lemmas 6.2 and 6.7 we see that the pushout $V' = V \amalg_{V \times_Y X} U'$ is an algebraic space smooth over Y' whose base change to Y and X' recovers V and U' in a manner compatible with φ.

Let W be the algebraic space representing $\mathcal{W}$. The projections $V \to W$ and $U' \to W$ agree as morphisms over $V \times_Y X \cong U' \times_{X'} X$ hence the universal property of the pushout determines a morphism of algebraic spaces $V' \to W$. Choose a scheme Y'_1 and a surjective étale morphism $Y'_1 \to V'$. Set $Y_1 = Y \times_{Y'} Y'_1$, $X'_1 = X' \times_{Y'} Y'_1$, $X_1 = X \times_{Y'} Y'_1$. The composition

$$(Sch/Y'_1) \to (Sch/V') \to (Sch/W) = \mathcal{W} \to \mathcal{Z}$$

corresponds by the 2-Yoneda lemma to an object y'_1 of $\mathcal{Z}$ over Y'_1 whose restriction to Y_1 and X'_1 agrees with $y|_{Y_1}$ and $x'|_{X'_1}$ in a manner compatible with $f|_{X_1}$. Thus we have constructed our desired object smooth locally over Y' and we win. □

5. The Rim-Schlessinger condition

06L9 The motivation for the following definition comes from Lemma 4.1 and Formal Deformation Theory, Definition 16.1 and Lemma 16.4.

07WP **Definition 5.1.** Let S be a locally Noetherian scheme. Let $\mathcal{Z}$ be a category fibred in groupoids over $(Sch/S)_{fppf}$. We say $\mathcal{Z}$ satisfies *condition (RS)* if for every pushout

$$\begin{array}{ccc} X & \longrightarrow & X' \\ \downarrow & & \downarrow \\ Y & \longrightarrow & Y' = Y \amalg_X X' \end{array}$$

in the category of schemes over S where

(1) X, X', Y, Y' are spectra of local Artinian rings,
(2) X, X', Y, Y' are of finite type over S, and
(3) $X \to X'$ (and hence $Y \to Y'$) is a closed immersion

the functor of fibre categories

$$\mathcal{Z}_{Y'} \longrightarrow \mathcal{Z}_Y \times_{\mathcal{Z}_X} \mathcal{Z}_{X'}$$

is an equivalence of categories.

If A is an Artinian local ring with residue field k, then any morphism $\mathrm{Spec}(A) \to S$ is affine and of finite type if and only if the induced morphism $\mathrm{Spec}(k) \to S$ is of finite type, see Morphisms, Lemmas 11.14 and 16.2.

07WQ **Lemma 5.2.** *Let $\mathcal{X}$ be an algebraic stack over a locally Noetherian base S. Then $\mathcal{X}$ satisfies (RS).*

Proof. Immediate from the definitions and Lemma 4.1. □

07WR **Lemma 5.3.** *Let S be a scheme. Let $p : \mathcal{X} \to \mathcal{Y}$ and $q : \mathcal{Z} \to \mathcal{Y}$ be 1-morphisms of categories fibred in groupoids over $(Sch/S)_{fppf}$. If $\mathcal{X}$, $\mathcal{Y}$, and $\mathcal{Z}$ satisfy (RS), then so does $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z}$.*

Proof. This is formal. Let

$$\begin{array}{ccc} X & \longrightarrow & X' \\ \downarrow & & \downarrow \\ Y & \longrightarrow & Y' = Y \amalg_X X' \end{array}$$

be a diagram as in Definition 5.1. We have to show that

$$(\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z})_{Y'} \longrightarrow (\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z})_Y \times_{(\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z})_X} (\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z})_{X'}$$

is an equivalence. Using the definition of the 2-fibre product this becomes

07WS (5.3.1) $$\mathcal{X}_{Y'} \times_{\mathcal{Y}_{Y'}} \mathcal{Z}_{Y'} \longrightarrow (\mathcal{X}_Y \times_{\mathcal{Y}_Y} \mathcal{Z}_Y) \times_{(\mathcal{X}_X \times_{\mathcal{Y}_X} \mathcal{Z}_X)} (\mathcal{X}_{X'} \times_{\mathcal{Y}_{X'}} \mathcal{Z}_{X'}).$$

We are given that each of the functors

$$\mathcal{X}_{Y'} \to \mathcal{X}_Y \times_{\mathcal{Y}_Y} \mathcal{Z}_Y, \quad \mathcal{Y}_{Y'} \to \mathcal{X}_X \times_{\mathcal{Y}_X} \mathcal{Z}_X, \quad \mathcal{Z}_{Y'} \to \mathcal{X}_{X'} \times_{\mathcal{Y}_{X'}} \mathcal{Z}_{X'}$$

are equivalences. An object of the right hand side of (5.3.1) is a system

$$((x_Y, z_Y, \phi_Y), (x_{X'}, z_{X'}, \phi_{X'}), (\alpha, \beta)).$$

Then $(x_Y, x_{Y'}, \alpha)$ is isomorphic to the image of an object $x_{Y'}$ in $\mathcal{X}_{Y'}$ and $(z_Y, z_{Y'}, \beta)$ is isomorphic to the image of an object $z_{Y'}$ of $\mathcal{Z}_{Y'}$. The pair of morphisms $(\phi_Y, \phi_{X'})$ corresponds to a morphism ψ between the images of $x_{Y'}$ and $z_{Y'}$ in $\mathcal{Y}_{Y'}$. Then $(x_{Y'}, z_{Y'}, \psi)$ is an object of the left hand side of (5.3.1) mapping to the given object of the right hand side. This proves that (5.3.1) is essentially surjective. We omit the proof that it is fully faithful. □

6. Deformation categories

07WT We match the notation introduced above with the notation from the chapter "Formal Deformation Theory".

07WU **Lemma 6.1.** *Let S be a locally Noetherian scheme. Let $\mathcal{X}$ be a category fibred in groupoids over $(Sch/S)_{fppf}$ satisfying (RS). For any field k of finite type over S and any object x_0 of $\mathcal{X}$ lying over k the predeformation category $p : \mathcal{F}_{\mathcal{X},k,x_0} \to \mathcal{C}_\Lambda$ (3.0.2) is a deformation category, see Formal Deformation Theory, Definition 16.8.*

Proof. Set $\mathcal{F} = \mathcal{F}_{\mathcal{X},k,x_0}$. Let $f_1 : A_1 \to A$ and $f_2 : A_2 \to A$ be ring maps in $\mathcal{C}_\Lambda$ with f_2 surjective. We have to show that the functor

$$\mathcal{F}(A_1 \times_A A_2) \longrightarrow \mathcal{F}(A_1) \times_{\mathcal{F}(A)} \mathcal{F}(A_2)$$

is an equivalence, see Formal Deformation Theory, Lemma 16.4. Set $X = \mathrm{Spec}(A)$, $X' = \mathrm{Spec}(A_2)$, $Y = \mathrm{Spec}(A_1)$ and $Y' = \mathrm{Spec}(A_1 \times_A A_2)$. Note that $Y' = Y \amalg_X X'$ in the category of schemes, see More on Morphisms, Lemma 14.3. We know that in the diagram of functors of fibre categories

$$\begin{array}{ccc} \mathcal{X}_{Y'} & \longrightarrow & \mathcal{X}_Y \times_{\mathcal{X}_X} \mathcal{X}_{X'} \\ \downarrow & & \downarrow \\ \mathcal{X}_{\mathrm{Spec}(k)} & = & \mathcal{X}_{\mathrm{Spec}(k)} \end{array}$$

the top horizontal arrow is an equivalence by Definition 5.1. Since $\mathcal{F}(B)$ is the category of objects of $\mathcal{X}_{\mathrm{Spec}(B)}$ with an identification with x_0 over k we win. □

07WV **Remark 6.2.** Let S be a locally Noetherian scheme. Let $\mathcal{X}$ be fibred in groupoids over $(Sch/S)_{fppf}$. Let k be a field of finite type over S and x_0 an object of $\mathcal{X}$ over k. Let $p : \mathcal{F} \to \mathcal{C}_\Lambda$ be as in (3.0.2). If $\mathcal{F}$ is a deformation category, i.e., if $\mathcal{F}$ satisfies the Rim-Schlessinger condition (RS), then we see that $\mathcal{F}$ satisfies Schlessinger's conditions (S1) and (S2) by Formal Deformation Theory, Lemma 16.6. Let $\overline{\mathcal{F}}$ be the functor of isomorphism classes, see Formal Deformation Theory, Remarks 5.2 (10). Then $\overline{\mathcal{F}}$ satisfies (S1) and (S2) as well, see Formal Deformation Theory, Lemma 10.5. This holds in particular in the situation of Lemma 6.1.

7. Change of field

07WW This section is the analogue of Formal Deformation Theory, Section 29. As pointed out there, to discuss what happens under change of field we need to write $\mathcal{C}_{\Lambda,k}$ instead of $\mathcal{C}_\Lambda$. In the following lemma we use the notation $\mathcal{F}_{l/k}$ introduced in Formal Deformation Theory, Situation 29.1.

07WX **Lemma 7.1.** *Let S be a locally Noetherian scheme. Let $\mathcal{X}$ be a category fibred in groupoids over $(Sch/S)_{fppf}$. Let k be a field of finite type over S and let l/k be*

a finite extension. Let x_0 be an object of $\mathcal{F}$ lying over $\mathrm{Spec}(k)$. Denote $x_{l,0}$ the restriction of x_0 to $\mathrm{Spec}(l)$. Then there is a canonical functor

$$(\mathcal{F}_{\mathcal{X},k,x_0})_{l/k} \longrightarrow \mathcal{F}_{\mathcal{X},l,x_{l,0}}$$

of categories cofibred in groupoids over $\mathcal{C}_{\Lambda,l}$. If $\mathcal{X}$ satisfies (RS), then this functor is an equivalence.

Proof. Consider a factorization

$$\mathrm{Spec}(l) \to \mathrm{Spec}(B) \to S$$

as in (3.0.1). By definition we have

$$(\mathcal{F}_{\mathcal{X},k,x_0})_{l/k}(B) = \mathcal{F}_{\mathcal{X},k,x_0}(B \times_l k)$$

see Formal Deformation Theory, Situation 29.1. Thus an object of this is a morphism $x_0 \to x$ of $\mathcal{X}$ lying over the morphism $\mathrm{Spec}(k) \to \mathrm{Spec}(B \times_l k)$. Choosing pullback functor for $\mathcal{X}$ we can associate to $x_0 \to x$ the morphism $x_{l,0} \to x_B$ where x_B is the restriction of x to $\mathrm{Spec}(B)$ (via the morphism $\mathrm{Spec}(B) \to \mathrm{Spec}(B \times_l k)$ coming from $B \times_l k \subset B$). This construction is functorial in B and compatible with morphisms.

Next, assume $\mathcal{X}$ satisfies (RS). Consider the diagrams

$$\begin{array}{ccc} l & \longleftarrow & B \\ \uparrow & & \uparrow \\ k & \longleftarrow & B \times_l k \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathrm{Spec}(l) & \longrightarrow & \mathrm{Spec}(B) \\ \downarrow & & \downarrow \\ \mathrm{Spec}(k) & \longrightarrow & \mathrm{Spec}(B \times_l k) \end{array}$$

The diagram on the left is a fibre product of rings. The diagram on the right is a pushout in the category of schemes, see More on Morphisms, Lemma 14.3. These schemes are all of finite type over S (see remarks following Definition 5.1). Hence (RS) kicks in to give an equivalence of fibre categories

$$\mathcal{X}_{\mathrm{Spec}(B \times_l k)} \longrightarrow \mathcal{X}_{\mathrm{Spec}(k)} \times_{\mathcal{X}_{\mathrm{Spec}(l)}} \mathcal{X}_{\mathrm{Spec}(B)}$$

This implies that the functor defined above gives an equivalence of fibre categories. Hence the functor is an equivalence on categories cofibred in groupoids by (the dual of) Categories, Lemma 35.9. □

8. Tangent spaces

07WY Let S be a locally Noetherian scheme. Let $\mathcal{X}$ be a category fibred in groupoids over $(Sch/S)_{fppf}$. Let k be a field of finite type over S and let x_0 be an object of $\mathcal{X}$ over k. In Formal Deformation Theory, Section 12 we have defined the *tangent space*

07WZ (8.0.1) $$T\mathcal{F}_{\mathcal{X},k,x_0} = \left\{ \begin{array}{l} \text{isomorphism classes of morphisms} \\ x_0 \to x \text{ over } \mathrm{Spec}(k) \to \mathrm{Spec}(k[\epsilon]) \end{array} \right\}$$

of the predeformation category $\mathcal{F}_{\mathcal{X},k,x_0}$. In Formal Deformation Theory, Section 19 we have defined

07X0 (8.0.2) $$\mathrm{Inf}(\mathcal{F}_{\mathcal{X},k,x_0}) = \mathrm{Ker}\left(\mathrm{Aut}_{\mathrm{Spec}(k[\epsilon])}(x_0') \to \mathrm{Aut}_{\mathrm{Spec}(k)}(x_0)\right)$$

where x_0' is the pullback of x_0 to $\mathrm{Spec}(k[\epsilon])$. If $\mathcal{X}$ satisfies the Rim-Schlessinger condition (RS), then $T\mathcal{F}_{\mathcal{X},k,x_0}$ comes equipped with a natural k-vector space structure by Formal Deformation Theory, Lemma 12.2 (assumptions hold by Lemma 6.1 and Remark 6.2). Moreover, Formal Deformation Theory, Lemma 19.9 shows that

$\mathrm{Inf}(\mathcal{F}_{\mathcal{X},k,x_0})$ has a natural k-vector space structure such that addition agrees with composition of automorphisms. A natural condition is to ask these vector spaces to have finite dimension.

The following lemma tells us this is true if $\mathcal{X}$ is locally of finite type over S (see Morphisms of Stacks, Section 17).

07X1 **Lemma 8.1.** *Let S be a locally Noetherian scheme. Assume*

(1) *$\mathcal{X}$ is an algebraic stack,*
(2) *U is a scheme locally of finite type over S, and*
(3) *$(Sch/U)_{fppf} \to \mathcal{X}$ is a smooth surjective morphism.*

Then, for any $\mathcal{F} = \mathcal{F}_{\mathcal{X},k,x_0}$ as in Section 3 the tangent space $T\mathcal{F}$ and infinitesimal automorphism space $Inf(\mathcal{F})$ have finite dimension over k.

Proof. Let us write $\mathcal{U} = (Sch/U)_{fppf}$. By our definition of algebraic stacks the 1-morphism $\mathcal{U} \to \mathcal{X}$ is representable by algebraic spaces. Hence in particular the 2-fibre product

$$\mathcal{U}_{x_0} = (Sch/\operatorname{Spec}(k))_{fppf} \times_{\mathcal{X}} \mathcal{U}$$

is representable by an algebraic space U_{x_0} over $\operatorname{Spec}(k)$. Then $U_{x_0} \to \operatorname{Spec}(k)$ is smooth and surjective (in particular U_{x_0} is nonempty). By Spaces over Fields, Lemma 16.2 we can find a finite extension l/k and a point $\operatorname{Spec}(l) \to U_{x_0}$ over k. We have

$$(\mathcal{F}_{\mathcal{X},k,x_0})_{l/k} = \mathcal{F}_{\mathcal{X},l,x_{l,0}}$$

by Lemma 7.1 and the fact that $\mathcal{X}$ satisfies (RS). Thus we see that

$$T\mathcal{F} \otimes_k l \cong T\mathcal{F}_{\mathcal{X},l,x_{l,0}} \quad \text{and} \quad \mathrm{Inf}(\mathcal{F}) \otimes_k l \cong \mathrm{Inf}(\mathcal{F}_{\mathcal{X},l,x_{l,0}})$$

by Formal Deformation Theory, Lemmas 29.3 and 29.4 (these are applicable by Lemmas 5.2 and 6.1 and Remark 6.2). Hence it suffices to prove that $T\mathcal{F}_{\mathcal{X},l,x_{l,0}}$ and $\mathrm{Inf}(\mathcal{F}_{\mathcal{X},l,x_{l,0}})$ have finite dimension over l. Note that $x_{l,0}$ comes from a point u_0 of $\mathcal{U}$ over l.

We interrupt the flow of the argument to show that the lemma for infinitesimal automorphisms follows from the lemma for tangent spaces. Namely, let $\mathcal{R} = \mathcal{U} \times_{\mathcal{X}} \mathcal{U}$. Let r_0 be the l-valued point $(u_0, u_0, \mathrm{id}_{x_0})$ of $\mathcal{R}$. Combining Lemma 3.3 and Formal Deformation Theory, Lemma 26.2 we see that

$$\mathrm{Inf}(\mathcal{F}_{\mathcal{X},l,x_{l,0}}) \subset T\mathcal{F}_{\mathcal{R},l,r_0}$$

Note that $\mathcal{R}$ is an algebraic stack, see Algebraic Stacks, Lemma 14.2. Also, $\mathcal{R}$ is representable by an algebraic space R smooth over U (via either projection, see Algebraic Stacks, Lemma 16.2). Hence, choose an scheme U' and a surjective étale morphism $U' \to R$ we see that U' is smooth over U, hence locally of finite type over S. As $(Sch/U')_{fppf} \to \mathcal{R}$ is surjective and smooth, we have reduced the question to the case of tangent spaces.

The functor (3.1.1)

$$\mathcal{F}_{\mathcal{U},l,u_0} \longrightarrow \mathcal{F}_{\mathcal{X},l,x_{l,0}}$$

is smooth by Lemma 3.2. The induced map on tangent spaces

$$T\mathcal{F}_{\mathcal{U},l,u_0} \longrightarrow T\mathcal{F}_{\mathcal{X},l,x_{l,0}}$$

is l-linear (by Formal Deformation Theory, Lemma 12.4) and surjective (as smooth maps of predeformation categories induce surjective maps on tangent spaces by

Formal Deformation Theory, Lemma 8.8). Hence it suffices to prove that the tangent space of the deformation space associated to the representable algebraic stack $\mathcal{U}$ at the point u_0 is finite dimensional. Let $\mathrm{Spec}(R) \subset U$ be an affine open such that $u_0 : \mathrm{Spec}(l) \to U$ factors through $\mathrm{Spec}(R)$ and such that $\mathrm{Spec}(R) \to S$ factors through $\mathrm{Spec}(\Lambda) \subset S$. Let $\mathfrak{m}_R \subset R$ be the kernel of the Λ-algebra map $\varphi_0 : R \to l$ corresponding to u_0. Note that R, being of finite type over the Noetherian ring Λ, is a Noetherian ring. Hence $\mathfrak{m}_R = (f_1, \ldots, f_n)$ is a finitely generated ideal. We have

$$T\mathcal{F}_{\mathcal{U},l,u_0} = \{\varphi : R \to l[\epsilon] \mid \varphi \text{ is a } \Lambda\text{-algebra map and } \varphi \bmod \epsilon = \varphi_0\}$$

An element of the right hand side is determined by its values on $f_1, \ldots, f_n$ hence the dimension is at most n and we win. Some details omitted. □

07X2 **Lemma 8.2.** *Let S be a locally Noetherian scheme. Let $p : \mathcal{X} \to \mathcal{Y}$ and $q : \mathcal{Z} \to \mathcal{Y}$ be 1-morphisms of categories fibred in groupoids over $(Sch/S)_{fppf}$. Assume $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{Z}$ satisfy (RS). Let k be a field of finite type over S and let w_0 be an object of $\mathcal{W} = \mathcal{X} \times_{\mathcal{Y}} \mathcal{Z}$ over k. Denote x_0, y_0, z_0 the objects of $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ you get from w_0. Then there is a 6-term exact sequence*

$$0 \longrightarrow Inf(\mathcal{F}_{\mathcal{W},k,w_0}) \longrightarrow Inf(\mathcal{F}_{\mathcal{X},k,x_0}) \oplus Inf(\mathcal{F}_{\mathcal{Z},k,z_0}) \longrightarrow Inf(\mathcal{F}_{\mathcal{Y},k,y_0})$$
$$T\mathcal{F}_{\mathcal{W},k,w_0} \longrightarrow T\mathcal{F}_{\mathcal{X},k,x_0} \oplus T\mathcal{F}_{\mathcal{Z},k,z_0} \longrightarrow T\mathcal{F}_{\mathcal{Y},k,y_0}$$

of k-vector spaces.

Proof. By Lemma 5.3 we see that $\mathcal{W}$ satisfies (RS) and hence the lemma makes sense. To see the lemma is true, apply Lemmas 3.3 and 6.1 and Formal Deformation Theory, Lemma 20.1. □

9. Formal objects

07X3 In this section we transfer some of the notions already defined in the chapter "Formal Deformation Theory" to the current setting. In the following we will say "R is an S-algebra" to indicate that R is a ring endowed with a morphism of schemes $\mathrm{Spec}(R) \to S$.

07X4 **Definition 9.1.** Let S be a locally Noetherian scheme. Let $p : \mathcal{X} \to (Sch/S)_{fppf}$ be a category fibred in groupoids.

(1) A *formal object* $\xi = (R, \xi_n, f_n)$ of $\mathcal{X}$ consists of a Noetherian complete local S-algebra R, objects ξ_n of $\mathcal{X}$ lying over $\mathrm{Spec}(R/\mathfrak{m}_R^n)$, and morphisms $f_n : \xi_n \to \xi_{n+1}$ of $\mathcal{X}$ lying over $\mathrm{Spec}(R/\mathfrak{m}^n) \to \mathrm{Spec}(R/\mathfrak{m}^{n+1})$ such that $R/\mathfrak{m}$ is a field of finite type over S.

(2) A *morphism of formal objects* $a : \xi = (R, \xi_n, f_n) \to \eta = (T, \eta_n, g_n)$ is given by morphisms $a_n : \xi_n \to \eta_n$ such that for every n the diagram

$$\begin{array}{ccc} \xi_n & \xrightarrow[f_n]{} & \xi_{n+1} \\ {\scriptstyle a_n}\downarrow & & \downarrow{\scriptstyle a_{n+1}} \\ \eta_n & \xrightarrow{g_n} & \eta_{n+1} \end{array}$$

is commutative. Applying the functor p we obtain a compatible collection of morphisms $\mathrm{Spec}(R/\mathfrak{m}_R^n) \to \mathrm{Spec}(T/\mathfrak{m}_T^n)$ and hence a morphism $a_0 : \mathrm{Spec}(R) \to \mathrm{Spec}(T)$ over S. We say that *a lies over a_0*.

Thus we obtain a category of formal objects of $\mathcal{X}$.

0CXH **Remark 9.2.** Let S be a locally Noetherian scheme. Let $p : \mathcal{X} \to (Sch/S)_{fppf}$ be a category fibred in groupoids. Let $\xi = (R, \xi_n, f_n)$ be a formal object. Set $k = R/\mathfrak{m}$ and $x_0 = \xi_1$. The formal object ξ defines a formal object ξ of the predeformation category $\mathcal{F}_{\mathcal{X},k,x_0}$. This follows immediately from Definition 9.1 above, Formal Deformation Theory, Definition 7.1, and our construction of the predeformation category $\mathcal{F}_{\mathcal{X},k,x_0}$ in Section 3.

If $F : \mathcal{X} \to \mathcal{Y}$ is a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$, then F induces a functor between categories of formal objects as well.

07X5 **Lemma 9.3.** *Let S be a locally Noetherian scheme. Let $F : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. Let $\eta = (R, \eta_n, g_n)$ be a formal object of $\mathcal{Y}$ and let ξ_1 be an object of $\mathcal{X}$ with $F(\xi_1) \cong \eta_1$. If F is formally smooth on objects (see Criteria for Representability, Section 6), then there exists a formal object $\xi = (R, \xi_n, f_n)$ of $\mathcal{X}$ such that $F(\xi) \cong \eta$.*

Proof. Note that each of the morphisms $\mathrm{Spec}(R/\mathfrak{m}^n) \to \mathrm{Spec}(R/\mathfrak{m}^{n+1})$ is a first order thickening of affine schemes over S. Hence the assumption on F means that we can successively lift ξ_1 to objects $\xi_2, \xi_3, \ldots$ of $\mathcal{X}$ endowed with compatible isomorphisms $\eta_n|_{\mathrm{Spec}(R/\mathfrak{m}^{n-1})} \cong \eta_{n-1}$ and $F(\eta_n) \cong \xi_n$. □

Let S be a locally Noetherian scheme. Let $p : \mathcal{X} \to (Sch/S)_{fppf}$ be a category fibred in groupoids. Suppose that x is an object of $\mathcal{X}$ over R, where R is a Noetherian complete local S-algebra with residue field of finite type over S. Then we can consider the system of restrictions $\xi_n = x|_{\mathrm{Spec}(R/\mathfrak{m}^n)}$ endowed with the natural morphisms $\xi_1 \to \xi_2 \to \ldots$ coming from transitivity of restriction. Thus $\xi = (R, \xi_n, \xi_n \to \xi_{n+1})$ is a formal object of $\mathcal{X}$. This construction is functorial in the object x. Thus we obtain a functor

07X6 (9.3.1)
$$\left\{\begin{array}{c}\text{objects } x \text{ of } \mathcal{X} \text{ such that } p(x) = \mathrm{Spec}(R) \\ \text{where } R \text{ is Noetherian complete local} \\ \text{with } R/\mathfrak{m} \text{ of finite type over } S\end{array}\right\} \longrightarrow \left\{\text{formal objects of } \mathcal{X}\right\}$$

To be precise the left hand side is the full subcategory of $\mathcal{X}$ consisting of objects as indicated and the right hand side is the category of formal objects of $\mathcal{X}$ as in Definition 9.1.

07X7 **Definition 9.4.** Let S be a locally Noetherian scheme. Let $\mathcal{X}$ be a category fibred in groupoids over $(Sch/S)_{fppf}$. A formal object $\xi = (R, \xi_n, f_n)$ of $\mathcal{X}$ is called *effective* if it is in the essential image of the functor (9.3.1).

If the category fibred in groupoids is an algebraic stack, then every formal object is effective as follows from the next lemma.

07X8 **Lemma 9.5.** *Let S be a locally Noetherian scheme. Let $\mathcal{X}$ be an algebraic stack over S. The functor (9.3.1) is an equivalence.*

Proof. Case I: $\mathcal{X}$ is representable (by a scheme). Say $\mathcal{X} = (Sch/X)_{fppf}$ for some scheme X over S. Unwinding the definitions we have to prove the following: Given a Noetherian complete local S-algebra R with $R/\mathfrak{m}$ of finite type over S we have

$$\mathrm{Mor}_S(\mathrm{Spec}(R), X) \longrightarrow \lim \mathrm{Mor}_S(\mathrm{Spec}(R/\mathfrak{m}^n), X)$$

is bijective. This follows from Formal Spaces, Lemma 33.2.

Case II. $\mathcal{X}$ is representable by an algebraic space. Say $\mathcal{X}$ is representable by X. Again we have to show that

$$\mathrm{Mor}_S(\mathrm{Spec}(R), X) \longrightarrow \lim \mathrm{Mor}_S(\mathrm{Spec}(R/\mathfrak{m}^n), X)$$

is bijective for R as above. This is Formal Spaces, Lemma 33.3.

Case III: General case of an algebraic stack. A general remark is that the left and right hand side of (9.3.1) are categories fibred in groupoids over the category of affine schemes over S which are spectra of Noetherian complete local rings with residue field of finite type over S. We will also see in the proof below that they form stacks for a certain topology on this category.

We first prove fully faithfulness. Let R be a Noetherian complete local S-algebra with $k = R/\mathfrak{m}$ of finite type over S. Let x, x' be objects of $\mathcal{X}$ over R. As $\mathcal{X}$ is an algebraic stack $Isom(x, x')$ is representable by an algebraic space I over $\mathrm{Spec}(R)$, see Algebraic Stacks, Lemma 10.11. Applying Case II to I over $\mathrm{Spec}(R)$ implies immediately that (9.3.1) is fully faithful on fibre categories over $\mathrm{Spec}(R)$. Hence the functor is fully faithful by Categories, Lemma 35.9.

Essential surjectivity. Let $\xi = (R, \xi_n, f_n)$ be a formal object of $\mathcal{X}$. Choose a scheme U over S and a surjective smooth morphism $f : (Sch/U)_{fppf} \to \mathcal{X}$. For every n consider the fibre product

$$(Sch/\mathrm{Spec}(R/\mathfrak{m}^n))_{fppf} \times_{\xi_n, \mathcal{X}, f} (Sch/U)_{fppf}$$

By assumption this is representable by an algebraic space V_n surjective and smooth over $\mathrm{Spec}(R/\mathfrak{m}^n)$. The morphisms $f_n : \xi_n \to \xi_{n+1}$ induce cartesian squares

$$\begin{array}{ccc} V_{n+1} & \longleftarrow & V_n \\ \downarrow & & \downarrow \\ \mathrm{Spec}(R/\mathfrak{m}^{n+1}) & \longleftarrow & \mathrm{Spec}(R/\mathfrak{m}^n) \end{array}$$

of algebraic spaces. By Spaces over Fields, Lemma 16.2 we can find a finite separable extension k'/k and a point $v'_1 : \mathrm{Spec}(k') \to V_1$ over k. Let $R \subset R'$ be the finite étale extension whose residue field extension is k'/k (exists and is unique by Algebra, Lemmas 153.7 and 153.9). By the infinitesimal lifting criterion of smoothness (see More on Morphisms of Spaces, Lemma 19.6) applied to $V_n \to \mathrm{Spec}(R/\mathfrak{m}^n)$ for $n = 2, 3, 4, \ldots$ we can successively find morphisms $v'_n : \mathrm{Spec}(R'/(\mathfrak{m}')^n) \to V_n$ over $\mathrm{Spec}(R/\mathfrak{m}^n)$ fitting into commutative diagrams

$$\begin{array}{ccc} \mathrm{Spec}(R'/(\mathfrak{m}')^{n+1}) & \longleftarrow & \mathrm{Spec}(R'/(\mathfrak{m}')^n) \\ {\scriptstyle v'_{n+1}}\downarrow & & \downarrow{\scriptstyle v'_n} \\ V_{n+1} & \longleftarrow & V_n \end{array}$$

Composing with the projection morphisms $V_n \to U$ we obtain a compatible system of morphisms $u'_n : \mathrm{Spec}(R'/(\mathfrak{m}')^n) \to U$. By Case I the family (u'_n) comes from a unique morphism $u' : \mathrm{Spec}(R') \to U$. Denote x' the object of $\mathcal{X}$ over $\mathrm{Spec}(R')$ we get by applying the 1-morphism f to u'. By construction, there exists a morphism of formal objects

$$(9.3.1)(x') = (R', x'|_{\mathrm{Spec}(R'/(\mathfrak{m}')^n)}, \ldots) \longrightarrow (R, \xi_n, f_n)$$

lying over $\mathrm{Spec}(R') \to \mathrm{Spec}(R)$. Note that $R' \otimes_R R'$ is a finite product of spectra of Noetherian complete local rings to which our current discussion applies. Denote $p_0, p_1 : \mathrm{Spec}(R' \otimes_R R') \to \mathrm{Spec}(R')$ the two projections. By the fully faithfulness shown above there exists a canonical isomorphism $\varphi : p_0^* x' \to p_1^* x'$ because we have such isomorphisms over $\mathrm{Spec}((R' \otimes_R R')/\mathfrak{m}^n(R' \otimes_R R'))$. We omit the proof that the isomorphism φ satisfies the cocycle condition (see Stacks, Definition 3.1). Since $\{\mathrm{Spec}(R') \to \mathrm{Spec}(R)\}$ is an fppf covering we conclude that x' descends to an object x of $\mathcal{X}$ over $\mathrm{Spec}(R)$. We omit the proof that x_n is the restriction of x to $\mathrm{Spec}(R/\mathfrak{m}^n)$. □

07X9 **Lemma 9.6.** *Let S be a scheme. Let $p : \mathcal{X} \to \mathcal{Y}$ and $q : \mathcal{Z} \to \mathcal{Y}$ be 1-morphisms of categories fibred in groupoids over $(Sch/S)_{fppf}$. If the functor (9.3.1) is an equivalence for $\mathcal{X}$, $\mathcal{Y}$, and $\mathcal{Z}$, then it is an equivalence for $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z}$.*

Proof. The left and the right hand side of (9.3.1) for $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z}$ are simply the 2-fibre products of the left and the right hand side of (9.3.1) for $\mathcal{X}$, $\mathcal{Z}$ over $\mathcal{Y}$. Hence the result follows as taking 2-fibre products is compatible with equivalences of categories, see Categories, Lemma 31.7. □

10. Approximation

07XA A fundamental insight of Michael Artin is that you can approximate objects of a limit preserving stack. Namely, given an object x of the stack over a Noetherian complete local ring, you can find an object x_A over an algebraic ring which is "close to" x. Here an algebraic ring means a finite type S-algebra and close means adically close. In this section we present this in a simple, yet general form.

To formulate the result we need to pull together some definitions from different places in the Stacks project. First, in Criteria for Representability, Section 5 we introduced *limit preserving on objects* for 1-morphisms of categories fibred in groupoids over the category of schemes. In More on Algebra, Definition 51.1 we defined the notion of a *G-ring*. Let S be a locally Noetherian scheme. Let A be an S-algebra. We say that A is *of finite type over S* or is a *finite type S-algebra* if $\mathrm{Spec}(A) \to S$ is of finite type. In this case A is a Noetherian ring. Finally, given a ring A and ideal I we denote $\mathrm{Gr}_I(A) = \bigoplus I^n/I^{n+1}$.

07XB **Lemma 10.1.** *Let S be a locally Noetherian scheme. Let $p : \mathcal{X} \to (Sch/S)_{fppf}$ be a category fibred in groupoids. Let x be an object of $\mathcal{X}$ lying over $\mathrm{Spec}(R)$ where R is a Noetherian complete local ring with residue field k of finite type over S. Let $s \in S$ be the image of $\mathrm{Spec}(k) \to S$. Assume that (a) $\mathcal{O}_{S,s}$ is a G-ring and (b) p is limit preserving on objects. Then for every integer $N \geq 1$ there exist*

1. *a finite type S-algebra A,*
2. *a maximal ideal $\mathfrak{m}_A \subset A$,*
3. *an object x_A of $\mathcal{X}$ over $\mathrm{Spec}(A)$,*
4. *an S-isomorphism $R/\mathfrak{m}_R^N \cong A/\mathfrak{m}_A^N$,*

(5) *an isomorphism* $x|_{\mathrm{Spec}(R/\mathfrak{m}_R^N)} \cong x_A|_{\mathrm{Spec}(A/\mathfrak{m}_A^N)}$ *compatible with (4), and*
(6) *an isomorphism* $Gr_{\mathfrak{m}_R}(R) \cong Gr_{\mathfrak{m}_A}(A)$ *of graded* k*-algebras.*

Proof. Choose an affine open $\mathrm{Spec}(\Lambda) \subset S$ such that k is a finite Λ-algebra, see Morphisms, Lemma 16.1. We may and do replace S by $\mathrm{Spec}(\Lambda)$.

We may write R as a directed colimit $R = \mathrm{colim}\, C_j$ where each C_j is a finite type Λ-algebra (see Algebra, Lemma 127.2). By assumption (b) the object x is isomorphic to the restriction of an object over one of the C_j. Hence we may choose a finite type Λ-algebra C, a Λ-algebra map $C \to R$, and an object x_C of $\mathcal{X}$ over $\mathrm{Spec}(C)$ such that $x = x_C|_{\mathrm{Spec}(R)}$. The choice of C is a bookkeeping device and could be avoided. For later use, let us write $C = \Lambda[y_1, \ldots, y_u]/(f_1, \ldots, f_v)$ and we denote $\overline{a}_i \in R$ the image of y_i under the map $C \to R$. Set $\mathfrak{m}_C = C \cap \mathfrak{m}_R$.

Choose a Λ-algebra surjection $\Lambda[x_1, \ldots, x_s] \to k$ and denote $\mathfrak{m}'$ the kernel. By the universal property of polynomial rings we may lift this to a Λ-algebra map $\Lambda[x_1, \ldots, x_s] \to R$. We add some variables (i.e., we increase s a bit) mapping to generators of $\mathfrak{m}_R$. Having done this we see that $\Lambda[x_1, \ldots, x_s] \to R/\mathfrak{m}_R^2$ is surjective. Then we see that

07XC (10.1.1) $$P = \Lambda[x_1, \ldots, x_s]^\wedge_{\mathfrak{m}'} \longrightarrow R$$

is a surjective map of Noetherian complete local rings, see for example Formal Deformation Theory, Lemma 4.2.

Choose lifts $a_i \in P$ of $\overline{a}_i$ we found above. Choose generators $b_1, \ldots, b_r \in P$ for the kernel of (10.1.1). Choose $c_{ji} \in P$ such that

$$f_j(a_1, \ldots, a_u) = \sum c_{ji} b_i$$

in P which is possible by the choices made so far. Choose generators

$$k_1, \ldots, k_t \in \mathrm{Ker}(P^{\oplus r} \xrightarrow{(b_1, \ldots, b_r)} P)$$

and write $k_i = (k_{i1}, \ldots, k_{ir})$ and $K = (k_{ij})$ so that

$$P^{\oplus t} \xrightarrow{K} P^{\oplus r} \xrightarrow{(b_1, \ldots, b_r)} P \to R \to 0$$

is an exact sequence of P-modules. In particular we have $\sum k_{ij} b_j = 0$. After possibly increasing N we may assume $N - 1$ works in the Artin-Rees lemma for the first two maps of this exact sequence (see More on Algebra, Section 4 for terminology).

By assumption $\mathcal{O}_{S,s} = \Lambda_{\Lambda \cap \mathfrak{m}'}$ is a G-ring. Hence by More on Algebra, Proposition 51.10 the ring $\Lambda[x_1, \ldots, x_s]_{\mathfrak{m}'}$ is a G-ring. Hence by Smoothing Ring Maps, Theorem 13.2 there exist an étale ring map

$$\Lambda[x_1, \ldots, x_s]_{\mathfrak{m}'} \to B,$$

a maximal ideal $\mathfrak{m}_B$ of B lying over $\mathfrak{m}'$, and elements $a_i', b_i', c_{ij}', k_{ij}' \in B'$ such that

(1) $\kappa(\mathfrak{m}') = \kappa(\mathfrak{m}_B)$ which implies that $\Lambda[x_1, \ldots, x_s]_{\mathfrak{m}'} \subset B_{\mathfrak{m}_B} \subset P$ and P is identified with the completion of B at $\mathfrak{m}_B$, see remark preceding Smoothing Ring Maps, Theorem 13.2,
(2) $a_i - a_i', b_i - b_i', c_{ij} - c_{ij}', k_{ij} - k_{ij}' \in (\mathfrak{m}')^N P$, and
(3) $f_j(a_1', \ldots, a_u') = \sum c_{ji}' b_i'$ and $\sum k_{ij}' b_j' = 0$.

Set $A = B/(b'_1, \ldots, b'_r)$ and denote $\mathfrak{m}_A$ the image of $\mathfrak{m}_B$ in A. (Note that A is essentially of finite type over Λ; at the end of the proof we will show how to obtain an A which is of finite type over Λ.) There is a ring map $C \to A$ sending $y_i \mapsto a'_i$ because the a'_i satisfy the desired equations modulo $(b'_1, \ldots, b'_r)$. Note that $A/\mathfrak{m}_A^N = R/\mathfrak{m}_R^N$ as quotients of $P = B^\wedge$ by property (2) above. Set $x_A = x_C|_{\mathrm{Spec}(A)}$. Since the maps

$$C \to A \to A/\mathfrak{m}_A^N \cong R/\mathfrak{m}_R^N \quad \text{and} \quad C \to R \to R/\mathfrak{m}_R^N$$

are equal we see that x_A and x agree modulo $\mathfrak{m}_R^N$ via the isomorphism $A/\mathfrak{m}_A^N = R/\mathfrak{m}_R^N$. At this point we have shown properties (1) – (5) of the statement of the lemma. To see (6) note that

$$P^{\oplus t} \xrightarrow{K} P^{\oplus r} \xrightarrow{(b_1, \ldots, b_r)} P \quad \text{and} \quad P^{\oplus t} \xrightarrow{K'} P^{\oplus r} \xrightarrow{(b'_1, \ldots, b'_r)} P$$

are two complexes of P-modules which are congruent modulo $(\mathfrak{m}')^N$ with the first one being exact. By our choice of N above we see from More on Algebra, Lemma 4.2 that $R = P/(b_1, \ldots, b_r)$ and $P/(b'_1, \ldots, b'_r) = B^\wedge/(b'_1, \ldots, b'_r) = A^\wedge$ have isomorphic associated graded algebras, which is what we wanted to show.

This last paragraph of the proof serves to clean up the issue that A is essentially of finite type over S and not yet of finite type. The construction above gives $A = B/(b'_1, \ldots, b'_r)$ and $\mathfrak{m}_A \subset A$ with B étale over $\Lambda[x_1, \ldots, x_s]_{\mathfrak{m}'}$. Hence A is of finite type over the Noetherian ring $\Lambda[x_1, \ldots, x_s]_{\mathfrak{m}'}$. Thus we can write $A = (A_0)_{\mathfrak{m}'}$ for some finite type $\Lambda[x_1, \ldots, x_n]$ algebra A_0. Then $A = \mathrm{colim}(A_0)_f$ where $f \in \Lambda[x_1, \ldots, x_n] \setminus \mathfrak{m}'$, see Algebra, Lemma 9.9. Because $p : \mathcal{X} \to (Sch/S)_{fppf}$ is limit preserving on objects, we see that x_A comes from some object $x_{(A_0)_f}$ over $\mathrm{Spec}((A_0)_f)$ for an f as above. After replacing A by $(A_0)_f$ and x_A by $x_{(A_0)_f}$ and $\mathfrak{m}_A$ by $(A_0)_f \cap \mathfrak{m}_A$ the proof is finished. □

11. Limit preserving

07XK The morphism $p : \mathcal{X} \to (Sch/S)_{fppf}$ is limit preserving on objects, as defined in Criteria for Representability, Section 5, if the functor of the definition below is essentially surjective. However, the example in Examples, Section 54 shows that this isn't equivalent to being limit preserving.

07XL **Definition 11.1.** Let S be a scheme. Let $\mathcal{X}$ be a category fibred in groupoids over $(Sch/S)_{fppf}$. We say $\mathcal{X}$ is *limit preserving* if for every affine scheme T over S which is a limit $T = \lim T_i$ of a directed inverse system of affine schemes T_i over S, we have an equivalence

$$\mathrm{colim}\, \mathcal{X}_{T_i} \longrightarrow \mathcal{X}_T$$

of fibre categories.

We spell out what this means. First, given objects x, y of $\mathcal{X}$ over T_i we should have

$$\mathrm{Mor}_{\mathcal{X}_T}(x|_T, y|_T) = \mathrm{colim}_{i' \geq i} \mathrm{Mor}_{\mathcal{X}_{T_{i'}}}(x|_{T_{i'}}, y|_{T_{i'}})$$

and second every object of $\mathcal{X}_T$ is isomorphic to the restriction of an object over T_i for some i. Note that the first condition means that the presheaves $Isom_{\mathcal{X}}(x, y)$ (see Stacks, Definition 2.2) are limit preserving.

07XM **Lemma 11.2.** *Let S be a scheme. Let $p : \mathcal{X} \to \mathcal{Y}$ and $q : \mathcal{Z} \to \mathcal{Y}$ be 1-morphisms of categories fibred in groupoids over $(Sch/S)_{fppf}$.*

(1) *If $\mathcal{X} \to (Sch/S)_{fppf}$ and $\mathcal{Z} \to (Sch/S)_{fppf}$ are limit preserving on objects and $\mathcal{Y}$ is limit preserving, then $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z} \to (Sch/S)_{fppf}$ is limit preserving on objects.*
(2) *If $\mathcal{X}$, $\mathcal{Y}$, and $\mathcal{Z}$ are limit preserving, then so is $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z}$.*

Proof. This is formal. Proof of (1). Let $T = \lim_{i \in I} T_i$ be the directed limit of affine schemes T_i over S. We will prove that the functor $\operatorname{colim} \mathcal{X}_{T_i} \to \mathcal{X}_T$ is essentially surjective. Recall that an object of the fibre product over T is a quadruple (T, x, z, α) where x is an object of $\mathcal{X}$ lying over T, z is an object of $\mathcal{Z}$ lying over T, and $\alpha : p(x) \to q(z)$ is a morphism in the fibre category of $\mathcal{Y}$ over T. By assumption on $\mathcal{X}$ and $\mathcal{Z}$ we can find an i and objects x_i and z_i over T_i such that $x_i|_T \cong T$ and $z_i|_T \cong z$. Then α corresponds to an isomorphism $p(x_i)|_T \to q(z_i)|_T$ which comes from an isomorphism $\alpha_{i'} : p(x_i)|_{T_{i'}} \to q(z_i)|_{T_{i'}}$ by our assumption on $\mathcal{Y}$. After replacing i by i', x_i by $x_i|_{T_{i'}}$, and z_i by $z_i|_{T_{i'}}$ we see that $(T_i, x_i, z_i, \alpha_i)$ is an object of the fibre product over T_i which restricts to an object isomorphic to (T, x, z, α) over T as desired.

We omit the arguments showing that $\operatorname{colim} \mathcal{X}_{T_i} \to \mathcal{X}_T$ is fully faithful in (2). □

07XN **Lemma 11.3.** *Let S be a scheme. Let $\mathcal{X}$ be an algebraic stack over S. Then the following are equivalent*

(1) *$\mathcal{X}$ is a stack in setoids and $\mathcal{X} \to (Sch/S)_{fppf}$ is limit preserving on objects,*
(2) *$\mathcal{X}$ is a stack in setoids and limit preserving,*
(3) *$\mathcal{X}$ is representable by an algebraic space locally of finite presentation.*

Proof. Under each of the three assumptions $\mathcal{X}$ is representable by an algebraic space X over S, see Algebraic Stacks, Proposition 13.3. It is clear that (1) and (2) are equivalent as a functor between setoids is an equivalence if and only if it is surjective on isomorphism classes. Finally, (1) and (3) are equivalent by Limits of Spaces, Proposition 3.10. □

0CXI **Lemma 11.4.** *Let S be a scheme. Let $\mathcal{X}$ be a category fibred in groupoids over $(Sch/S)_{fppf}$. Assume $\Delta : \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is representable by algebraic spaces and $\mathcal{X}$ is limit preserving. Then Δ is locally of finite type.*

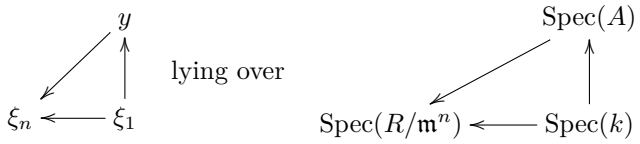
Proof. We apply Criteria for Representability, Lemma 5.6. Let V be an affine scheme V locally of finite presentation over S and let θ be an object of $\mathcal{X} \times \mathcal{X}$ over V. Let F_θ be an algebraic space representing $\mathcal{X} \times_{\Delta, \mathcal{X} \times \mathcal{X}, \theta} (Sch/V)_{fppf}$ and let $f_\theta : F_\theta \to V$ be the canonical morphism (see Algebraic Stacks, Section 9). It suffices to show that $F_\theta \to V$ has the corresponding properties. By Lemmas 11.2 and 11.3 we see that $F_\theta \to S$ is locally of finite presentation. It follows that $F_\theta \to V$ is locally of finite type by Morphisms of Spaces, Lemma 23.6. □

12. Versality

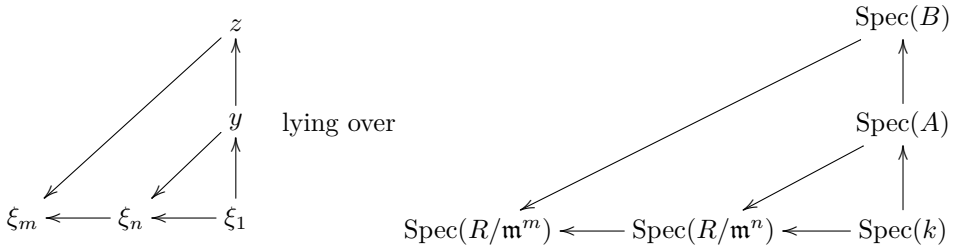
07XD In the previous section we explained how to approximate objects over complete local rings by algebraic objects. But in order to show that a stack $\mathcal{X}$ is an algebraic stack, we need to find smooth 1-morphisms from schemes towards $\mathcal{X}$. Since we are not going to assume a priori that $\mathcal{X}$ has a representable diagonal, we cannot even speak about smooth morphisms towards $\mathcal{X}$. Instead, borrowing terminology from deformation theory, we will introduce versal objects.

0CXJ **Definition 12.1.** Let S be a locally Noetherian scheme. Let $p : \mathcal{X} \to (Sch/S)_{fppf}$ be a category fibred in groupoids. Let $\xi = (R, \xi_n, f_n)$ be a formal object. Set $k = R/\mathfrak{m}$ and $x_0 = \xi_1$. We will say that ξ is *versal* if ξ as a formal object of $\mathcal{F}_{\mathcal{X},k,x_0}$ (Remark 9.2) is versal in the sense of Formal Deformation Theory, Definition 8.9.

We briefly spell out what this means. With notation as in the definition, suppose given morphisms $\xi_1 = x_0 \to y \to z$ of $\mathcal{X}$ lying over closed immersions $\mathrm{Spec}(k) \to \mathrm{Spec}(A) \to \mathrm{Spec}(B)$ where A, B are Artinian local rings with residue field k. Suppose given an $n \geq 1$ and a commutative diagram



Versality means that for any data as above there exists an $m \geq n$ and a commutative diagram



Please compare with Formal Deformation Theory, Remark 8.10.

Let S be a locally Noetherian scheme. Let U be a scheme over S with structure morphism $U \to S$ locally of finite type. Let $u_0 \in U$ be a finite type point of U, see Morphisms, Definition 16.3. Set $k = \kappa(u_0)$. Note that the composition $\mathrm{Spec}(k) \to S$ is also of finite type, see Morphisms, Lemma 15.3. Let $p : \mathcal{X} \to (Sch/S)_{fppf}$ be a category fibred in groupoids. Let x be an object of $\mathcal{X}$ which lies over U. Denote x_0 the pullback of x by u_0. By the 2-Yoneda lemma x corresponds to a 1-morphism

$$x : (Sch/U)_{fppf} \longrightarrow \mathcal{X},$$

see Algebraic Stacks, Section 5. We obtain a morphism of predeformation categories

07XE (12.1.1) $$\hat{x} : \mathcal{F}_{(Sch/U)_{fppf},k,u_0} \longrightarrow \mathcal{F}_{\mathcal{X},k,x_0},$$

over $\mathcal{C}_\Lambda$ see (3.1.1).

07XF **Definition 12.2.** Let S be a locally Noetherian scheme. Let $\mathcal{X}$ be fibred in groupoids over $(Sch/S)_{fppf}$. Let U be a scheme locally of finite type over S. Let x be an object of $\mathcal{X}$ lying over U. Let u_0 be a finite type point of U. We say x is *versal* at u_0 if the morphism $\hat{x}$ (12.1.1) is smooth, see Formal Deformation Theory, Definition 8.1.

This definition matches our notion of versality for formal objects of $\mathcal{X}$.

0CXK **Lemma 12.3.** *With notation as in Definition 12.2. Let $R = \mathcal{O}^\wedge_{U,u_0}$. Let ξ be the formal object of $\mathcal{X}$ over R associated to $x|_{\mathrm{Spec}(R)}$, see (9.3.1). Then*

$$x \text{ is versal at } u_0 \Leftrightarrow \xi \text{ is versal}$$

Proof. Observe that $\mathcal{O}_{U,u_0}$ is a Noetherian local S-algebra with residue field k. Hence $R = \mathcal{O}^\wedge_{U,u_0}$ is an object of $\mathcal{C}^\wedge_\Lambda$, see Formal Deformation Theory, Definition 4.1. Recall that ξ is versal if $\underline{\xi} : \underline{R}|_{\mathcal{C}_\Lambda} \to \mathcal{F}_{\mathcal{X},k,x_0}$ is smooth and x is versal at u_0 if $\hat{x} : \mathcal{F}_{(Sch/U)_{fppf},k,u_0} \to \mathcal{F}_{\mathcal{X},k,x_0}$ is smooth. There is an identification of predeformation categories

$$\underline{R}|_{\mathcal{C}_\Lambda} = \mathcal{F}_{(Sch/U)_{fppf},k,u_0},$$

see Formal Deformation Theory, Remark 7.12 for notation. Namely, given an Artinian local S-algebra A with residue field identified with k we have

$$\mathrm{Mor}_{\mathcal{C}^\wedge_\Lambda}(R, A) = \{\varphi \in \mathrm{Mor}_S(\mathrm{Spec}(A), U) \mid \varphi|_{\mathrm{Spec}(k)} = u_0\}$$

Unwinding the definitions the reader verifies that the resulting map

$$\underline{R}|_{\mathcal{C}_\Lambda} = \mathcal{F}_{(Sch/U)_{fppf},k,u_0} \xrightarrow{\hat{x}} \mathcal{F}_{\mathcal{X},k,x_0},$$

is equal to $\underline{\xi}$ and we see that the lemma is true. □

Here is a sanity check.

0CXL **Lemma 12.4.** *Let S be a locally Noetherian scheme. Let $f : U \to V$ be a morphism of schemes locally of finite type over S. Let $u_0 \in U$ be a finite type point. The following are equivalent*

1. *f is smooth at u_0,*
2. *f viewed as an object of $(Sch/V)_{fppf}$ over U is versal at u_0.*

Proof. This is a restatement of More on Morphisms, Lemma 12.1. □

It turns out that this notion is well behaved with respect to field extensions.

07XG **Lemma 12.5.** *Let S, $\mathcal{X}$, U, x, u_0 be as in Definition 12.2. Let l be a field and let $u_{l,0} : \mathrm{Spec}(l) \to U$ be a morphism with image u_0 such that $l/k = \kappa(u_0)$ is finite. Set $x_{l,0} = x_0|_{\mathrm{Spec}(l)}$. If $\mathcal{X}$ satisfies (RS) and x is versal at u_0, then*

$$\mathcal{F}_{(Sch/U)_{fppf},l,u_{l,0}} \longrightarrow \mathcal{F}_{\mathcal{X},l,x_{l,0}}$$

is smooth.

Proof. Note that $(Sch/U)_{fppf}$ satisfies (RS) by Lemma 5.2. Hence the functor of the lemma is the functor

$$(\mathcal{F}_{(Sch/U)_{fppf},k,u_0})_{l/k} \longrightarrow (\mathcal{F}_{\mathcal{X},k,x_0})_{l/k}$$

associated to $\hat{x}$, see Lemma 7.1. Hence the lemma follows from Formal Deformation Theory, Lemma 29.5. □

The following lemma is another sanity check. It more or less signifies that if x is versal at u_0 as in Definition 12.2, then x viewed as a morphism from U to $\mathcal{X}$ is smooth whenever we make a base change by a scheme.

0CXM **Lemma 12.6.** *Let S, $\mathcal{X}$, U, x, u_0 be as in Definition 12.2. Assume*

1. *$\Delta : \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is representable by algebraic spaces,*
2. *Δ is locally of finite type (for example if $\mathcal{X}$ is limit preserving), and*
3. *$\mathcal{X}$ has (RS).*

Let V be a scheme locally of finite type over S and let y be an object of $\mathcal{X}$ over V. Form the 2-fibre product

$$\begin{array}{ccc} \mathcal{Z} & \longrightarrow & (Sch/U)_{fppf} \\ \downarrow & & \downarrow{\scriptstyle x} \\ (Sch/V)_{fppf} & \xrightarrow{y} & \mathcal{X} \end{array}$$

Let Z be the algebraic space representing $\mathcal{Z}$ and let $z_0 \in |Z|$ be a finite type point lying over u_0. If x is versal at u_0, then the morphism $Z \to V$ is smooth at z_0.

Proof. (The parenthetical remark in the statement holds by Lemma 11.4.) Observe that Z exists by assumption (1) and Algebraic Stacks, Lemma 10.11. By assumption (2) we see that $Z \to V \times_S U$ is locally of finite type. Choose a scheme W, a closed point $w_0 \in W$, and an étale morphism $W \to Z$ mapping w_0 to z_0, see Morphisms of Spaces, Definition 25.2. Then W is locally of finite type over S and w_0 is a finite type point of W. Let $l = \kappa(z_0)$. Denote $z_{l,0}$, $v_{l,0}$, $u_{l,0}$, and $x_{l,0}$ the objects of $\mathcal{Z}$, $(Sch/V)_{fppf}$, $(Sch/U)_{fppf}$, and $\mathcal{X}$ over $\mathrm{Spec}(l)$ obtained by pullback to $\mathrm{Spec}(l) = w_0$. Consider

$$\begin{array}{ccccc} \mathcal{F}_{(Sch/W)_{fppf},l,w_0} & \longrightarrow & \mathcal{F}_{\mathcal{Z},l,z_{l,0}} & \longrightarrow & \mathcal{F}_{(Sch/U)_{fppf},l,u_{l,0}} \\ & & \downarrow & & \downarrow \\ & & \mathcal{F}_{(Sch/V)_{fppf},l,v_{l,0}} & \longrightarrow & \mathcal{F}_{\mathcal{X},l,x_{l,0}} \end{array}$$

By Lemma 3.3 the square is a fibre product of predeformation categories. By Lemma 12.5 we see that the right vertical arrow is smooth. By Formal Deformation Theory, Lemma 8.7 the left vertical arrow is smooth. By Lemma 3.2 we see that the left horizontal arrow is smooth. We conclude that the map

$$\mathcal{F}_{(Sch/W)_{fppf},l,w_0} \to \mathcal{F}_{(Sch/V)_{fppf},l,v_{l,0}}$$

is smooth by Formal Deformation Theory, Lemma 8.7. Thus we conclude that $W \to V$ is smooth at w_0 by More on Morphisms, Lemma 12.1. This exactly means that $Z \to V$ is smooth at z_0 and the proof is complete. □

We restate the approximation result in terms of versal objects.

07XH **Lemma 12.7.** *Let S be a locally Noetherian scheme. Let $p : \mathcal{X} \to (Sch/S)_{fppf}$ be a category fibred in groupoids. Let $\xi = (R, \xi_n, f_n)$ be a formal object of $\mathcal{X}$ with ξ_1 lying over $\mathrm{Spec}(k) \to S$ with image $s \in S$. Assume*

(1) *ξ is versal,*
(2) *ξ is effective,*
(3) *$\mathcal{O}_{S,s}$ is a G-ring, and*
(4) *$p : \mathcal{X} \to (Sch/S)_{fppf}$ is limit preserving on objects.*

Then there exist a morphism of finite type $U \to S$, a finite type point $u_0 \in U$ with residue field k, and an object x of $\mathcal{X}$ over U such that x is versal at u_0 and such that $x|_{\mathrm{Spec}(\mathcal{O}_{U,u_0}/\mathfrak{m}_{u_0}^n)} \cong \xi_n$.

Proof. Choose an object x_R of $\mathcal{X}$ lying over $\mathrm{Spec}(R)$ whose associated formal object is ξ. Let $N = 2$ and apply Lemma 10.1. We obtain $A, \mathfrak{m}_A, x_A, \ldots$. Let

$\eta = (A^\wedge, \eta_n, g_n)$ be the formal object associated to $x_A|_{\mathrm{Spec}(A^\wedge)}$. We have a diagram

$$\begin{array}{ccc} & & \eta \\ & \nearrow & \downarrow \\ \xi & \longrightarrow & \xi_2 = \eta_2 \end{array} \quad \text{lying over} \quad \begin{array}{ccc} & & A^\wedge \\ & \nearrow & \downarrow \\ R & \longrightarrow & R/\mathfrak{m}_R^2 = A/\mathfrak{m}_A^2 \end{array}$$

The versality of ξ means exactly that we can find the dotted arrows in the diagrams, because we can successively find morphisms $\xi \to \eta_3$, $\xi \to \eta_4$, and so on by Formal Deformation Theory, Remark 8.10. The corresponding ring map $R \to A^\wedge$ is surjective by Formal Deformation Theory, Lemma 4.2. On the other hand, we have $\dim_k \mathfrak{m}_R^n/\mathfrak{m}_R^{n+1} = \dim_k \mathfrak{m}_A^n/\mathfrak{m}_A^{n+1}$ for all n by construction. Hence $R/\mathfrak{m}_R^n$ and $A/\mathfrak{m}_A^n$ have the same (finite) length as Λ-modules by additivity of length and Formal Deformation Theory, Lemma 3.4. It follows that $R/\mathfrak{m}_R^n \to A/\mathfrak{m}_A^n$ is an isomorphism for all n, hence $R \to A^\wedge$ is an isomorphism. Thus η is isomorphic to a versal object, hence versal itself. By Lemma 12.3 we conclude that x_A is versal at the point u_0 of $U = \mathrm{Spec}(A)$ corresponding to $\mathfrak{m}_A$. □

07XI **Example 12.8.** In this example we show that the local ring $\mathcal{O}_{S,s}$ has to be a G-ring in order for the result of Lemma 12.7 to be true. Namely, let Λ be a Noetherian ring and let $\mathfrak{m}$ be a maximal ideal of Λ. Set $R = \Lambda_\mathfrak{m}^\wedge$. Let $\Lambda \to C \to R$ be a factorization with C of finite type over Λ. Set $S = \mathrm{Spec}(\Lambda)$, $U = S \setminus \{\mathfrak{m}\}$, and $S' = U \amalg \mathrm{Spec}(C)$. Consider the functor $F : (Sch/S)_{fppf}^{opp} \to Sets$ defined by the rule

$$F(T) = \begin{cases} * & \text{if } T \to S \text{ factors through } S' \\ \emptyset & \text{else} \end{cases}$$

Let $\mathcal{X} = \mathcal{S}_F$ is the category fibred in sets associated to F, see Algebraic Stacks, Section 7. Then $\mathcal{X} \to (Sch/S)_{fppf}$ is limit preserving on objects and there exists an effective, versal formal object ξ over R. Hence if the conclusion of Lemma 12.7 holds for $\mathcal{X}$, then there exists a finite type ring map $\Lambda \to A$ and a maximal ideal $\mathfrak{m}_A$ lying over $\mathfrak{m}$ such that

(1) $\kappa(\mathfrak{m}) = \kappa(\mathfrak{m}_A)$,
(2) $\Lambda \to A$ and $\mathfrak{m}_A$ satisfy condition (4) of Algebra, Lemma 141.2, and
(3) there exists a Λ-algebra map $C \to A$.

Thus $\Lambda \to A$ is smooth at $\mathfrak{m}_A$ by the lemma cited. Slicing A we may assume that $\Lambda \to A$ is étale at $\mathfrak{m}_A$, see for example More on Morphisms, Lemma 38.5 or argue directly. Write $C = \Lambda[y_1, \ldots, y_n]/(f_1, \ldots, f_m)$. Then $C \to R$ corresponds to a solution in R of the system of equations $f_1 = \ldots = f_m = 0$, see Smoothing Ring Maps, Section 13. Thus if the conclusion of Lemma 12.7 holds for every $\mathcal{X}$ as above, then a system of equations which has a solution in R has a solution in the henselization of $\Lambda_\mathfrak{m}$. In other words, the approximation property holds for $\Lambda_\mathfrak{m}^h$. This implies that $\Lambda_\mathfrak{m}^h$ is a G-ring (insert future reference here; see also discussion in Smoothing Ring Maps, Section 1) which in turn implies that $\Lambda_\mathfrak{m}$ is a G-ring.

13. Openness of versality

07XP Next, we come to openness of versality.

07XQ **Definition 13.1.** Let S be a locally Noetherian scheme.

(1) Let $\mathcal{X}$ be a category fibred in groupoids over $(Sch/S)_{fppf}$. We say $\mathcal{X}$ satisfies *openness of versality* if given a scheme U locally of finite type over S, an object x of $\mathcal{X}$ over U, and a finite type point $u_0 \in U$ such that x is versal at u_0, then there exists an open neighbourhood $u_0 \in U' \subset U$ such that x is versal at every finite type point of U'.
(2) Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. We say f satisfies *openness of versality* if given a scheme U locally of finite type over S, an object y of $\mathcal{Y}$ over U, openness of versality holds for $(Sch/U)_{fppf} \times_{\mathcal{Y}} \mathcal{X}$.

Openness of versality is often the hardest to check. The following example shows that requiring this is necessary however.

07XR **Example 13.2.** Let k be a field and set $\Lambda = k[s,t]$. Consider the functor $F : \Lambda\text{-algebras} \longrightarrow Sets$ defined by the rule

$$F(A) = \begin{cases} * & \text{if there exist } f_1, \ldots, f_n \in A \text{ such that} \\ & A = (s, t, f_1, \ldots, f_n) \text{ and } f_i s = 0 \ \forall i \\ \emptyset & \text{else} \end{cases}$$

Geometrically $F(A) = *$ means there exists a quasi-compact open neighbourhood W of $V(s,t) \subset \mathrm{Spec}(A)$ such that $s|_W = 0$. Let $\mathcal{X} \subset (Sch/\mathrm{Spec}(\Lambda))_{fppf}$ be the full subcategory consisting of schemes T which have an affine open covering $T = \bigcup \mathrm{Spec}(A_j)$ with $F(A_j) = *$ for all j. Then $\mathcal{X}$ satisfies [0], [1], [2], [3], and [4] but not [5]. Namely, over $U = \mathrm{Spec}(k[s,t]/(s))$ there exists an object x which is versal at $u_0 = (s,t)$ but not at any other point. Details omitted.

Let S be a locally Noetherian scheme. Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. Consider the following property

07XS (13.2.1) for all fields k of finite type over S and all $x_0 \in \mathrm{Ob}(\mathcal{X}_{\mathrm{Spec}(k)})$ the map $\mathcal{F}_{\mathcal{X},k,x_0} \to \mathcal{F}_{\mathcal{Y},k,f(x_0)}$ of predeformation categories is smooth

We formulate some lemmas around this concept. First we link it with (openness of) versality.

07XT **Lemma 13.3.** *Let S be a locally Noetherian scheme. Let $\mathcal{X}$ be a category fibred in groupoids over $(Sch/S)_{fppf}$. Let U be a scheme locally of finite type over S. Let x be an object of $\mathcal{X}$ over U. Assume that x is versal at every finite type point of U and that $\mathcal{X}$ satisfies (RS). Then $x : (Sch/U)_{fppf} \to \mathcal{X}$ satisfies (13.2.1).*

Proof. Let $\mathrm{Spec}(l) \to U$ be a morphism with l of finite type over S. Then the image $u_0 \in U$ is a finite type point of U and $l/\kappa(u_0)$ is a finite extension, see discussion in Morphisms, Section 16. Hence we see that $\mathcal{F}_{(Sch/U)_{fppf},l,u_{l,0}} \to \mathcal{F}_{\mathcal{X},l,x_{l,0}}$ is smooth by Lemma 12.5. □

07XU **Lemma 13.4.** *Let S be a locally Noetherian scheme. Let $f : \mathcal{X} \to \mathcal{Y}$ and $g : \mathcal{Y} \to \mathcal{Z}$ be composable 1-morphisms of categories fibred in groupoids over $(Sch/S)_{fppf}$. If f and g satisfy (13.2.1) so does $g \circ f$.*

Proof. This follows formally from Formal Deformation Theory, Lemma 8.7. □

07XV **Lemma 13.5.** *Let S be a locally Noetherian scheme. Let $f : \mathcal{X} \to \mathcal{Y}$ and $\mathcal{Z} \to \mathcal{Y}$ be 1-morphisms of categories fibred in groupoids over $(Sch/S)_{fppf}$. If f satisfies (13.2.1) so does the projection $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z} \to \mathcal{Z}$.*

Proof. Follows immediately from Lemma 3.3 and Formal Deformation Theory, Lemma 8.7. □

07XW **Lemma 13.6.** *Let S be a locally Noetherian scheme. Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphisms of categories fibred in groupoids over $(Sch/S)_{fppf}$. If f is formally smooth on objects, then f satisfies (13.2.1). If f is representable by algebraic spaces and smooth, then f satisfies (13.2.1).*

Proof. A reformulation of Lemma 3.2. □

07XX **Lemma 13.7.** *Let S be a locally Noetherian scheme. Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. Assume*

1. *f is representable by algebraic spaces,*
2. *f satisfies (13.2.1),*
3. *$\mathcal{X} \to (Sch/S)_{fppf}$ is limit preserving on objects, and*
4. *$\mathcal{Y}$ is limit preserving.*

Then f is smooth.

Proof. The key ingredient of the proof is More on Morphisms, Lemma 12.1 which (almost) says that a morphism of schemes of finite type over S satisfying (13.2.1) is a smooth morphism. The other arguments of the proof are essentially bookkeeping.

Let V be a scheme over S and let y be an object of $\mathcal{Y}$ over V. Let Z be an algebraic space representing the 2-fibre product $\mathcal{Z} = \mathcal{X} \times_{f,\mathcal{X},y} (Sch/V)_{fppf}$. We have to show that the projection morphism $Z \to V$ is smooth, see Algebraic Stacks, Definition 10.1. In fact, it suffices to do this when V is an affine scheme locally of finite presentation over S, see Criteria for Representability, Lemma 5.6. Then $(Sch/V)_{fppf}$ is limit preserving by Lemma 11.3. Hence $Z \to S$ is locally of finite presentation by Lemmas 11.2 and 11.3. Choose a scheme W and a surjective étale morphism $W \to Z$. Then W is locally of finite presentation over S.

Since f satisfies (13.2.1) we see that so does $\mathcal{Z} \to (Sch/V)_{fppf}$, see Lemma 13.5. Next, we see that $(Sch/W)_{fppf} \to \mathcal{Z}$ satisfies (13.2.1) by Lemma 13.6. Thus the composition

$$(Sch/W)_{fppf} \to \mathcal{Z} \to (Sch/V)_{fppf}$$

satisfies (13.2.1) by Lemma 13.4. More on Morphisms, Lemma 12.1 shows that the composition $W \to Z \to V$ is smooth at every finite type point w_0 of W. Since the smooth locus is open we conclude that $W \to V$ is a smooth morphism of schemes by Morphisms, Lemma 16.7. Thus we conclude that $Z \to V$ is a smooth morphism of algebraic spaces by definition. □

The lemma below is how we will use openness of versality.

07XY **Lemma 13.8.** *Let S be a locally Noetherian scheme. Let $p : \mathcal{X} \to (Sch/S)_{fppf}$ be a category fibred in groupoids. Let k be a finite type field over S and let x_0 be an object of $\mathcal{X}$ over* $\mathrm{Spec}(k)$ *with image $s \in S$. Assume*

1. *$\Delta : \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is representable by algebraic spaces,*
2. *$\mathcal{X}$ satisfies axioms [1], [2], [3] (see Section 14),*
3. *every formal object of $\mathcal{X}$ is effective,*
4. *openness of versality holds for $\mathcal{X}$, and*
5. *$\mathcal{O}_{S,s}$ is a G-ring.*

Then there exist a morphism of finite type $U \to S$ and an object x of $\mathcal{X}$ over U such that

$$x : (Sch/U)_{fppf} \longrightarrow \mathcal{X}$$

is smooth and such that there exists a finite type point $u_0 \in U$ whose residue field is k and such that $x|_{u_0} \cong x_0$.

Proof. By axiom [2], Lemma 6.1, and Remark 6.2 we see that $\mathcal{F}_{\mathcal{X},k,x_0}$ satisfies (S1) and (S2). Since also the tangent space has finite dimension by axiom [3] we deduce from Formal Deformation Theory, Lemma 13.4 that $\mathcal{F}_{\mathcal{X},k,x_0}$ has a versal formal object ξ. Assumption (3) says ξ is effective. By axiom [1] and Lemma 12.7 there exists a morphism of finite type $U \to S$, an object x of $\mathcal{X}$ over U, and a finite type point u_0 of U with residue field k such that x is versal at u_0 and such that $x|_{\mathrm{Spec}(k)} \cong x_0$. By openness of versality we may shrink U and assume that x is versal at every finite type point of U. We claim that

$$x : (Sch/U)_{fppf} \longrightarrow \mathcal{X}$$

is smooth which proves the lemma. Namely, by Lemma 13.3 x satisfies (13.2.1) whereupon Lemma 13.7 finishes the proof. □

14. Axioms

07XJ Let S be a locally Noetherian scheme. Let $p : \mathcal{X} \to (Sch/S)_{fppf}$ be a category fibred in groupoids. Here are the axioms we will consider on $\mathcal{X}$.

- [-1] a set theoretic condition[2] to be ignored by readers who are not interested in set theoretical issues,
- [0] $\mathcal{X}$ is a stack in groupoids for the étale topology,
- [1] $\mathcal{X}$ is limit preserving,
- [2] $\mathcal{X}$ satisfies the Rim-Schlessinger condition (RS),
- [3] the spaces $T\mathcal{F}_{\mathcal{X},k,x_0}$ and $\mathrm{Inf}(\mathcal{F}_{\mathcal{X},k,x_0})$ are finite dimensional for every k and x_0, see (8.0.1) and (8.0.2),
- [4] the functor (9.3.1) is an equivalence,
- [5] $\mathcal{X}$ and $\Delta : \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ satisfy openness of versality.

15. Axioms for functors

07XZ Let S be a scheme. Let $F : (Sch/S)^{opp}_{fppf} \to Sets$ be a functor. Denote $\mathcal{X} = \mathcal{S}_F$ the category fibred in sets associated to F, see Algebraic Stacks, Section 7. In this section we provide a translation between the material above as it applies to $\mathcal{X}$, to statements about F.

Let S be a locally Noetherian scheme. Let $F : (Sch/S)^{opp}_{fppf} \to Sets$ be a functor. Let k be a field of finite type over S. Let $x_0 \in F(\mathrm{Spec}(k))$. The associated predeformation category (3.0.2) corresponds to the functor

$$F_{k,x_0} : \mathcal{C}_\Lambda \longrightarrow Sets, \quad A \longmapsto \{x \in F(\mathrm{Spec}(A)) \mid x|_{\mathrm{Spec}(k)} = x_0\}.$$

[2]The condition is the following: the supremum of all the cardinalities $|\mathrm{Ob}(\mathcal{X}_{\mathrm{Spec}(k)})/\cong|$ and $|\mathrm{Arrows}(\mathcal{X}_{\mathrm{Spec}(k)})|$ where k runs over the finite type fields over S is $\leq$ than the size of some object of $(Sch/S)_{fppf}$.

Recall that we do not distinguish between categories cofibred in sets over $\mathcal{C}_\Lambda$ and functor $\mathcal{C}_\Lambda \to Sets$, see Formal Deformation Theory, Remarks 5.2 (11). Given a transformation of functors $a : F \to G$, setting $y_0 = a(x_0)$ we obtain a morphism

$$F_{k,x_0} \longrightarrow G_{k,y_0}$$

see (3.1.1). Lemma 3.2 tells us that if $a : F \to G$ is formally smooth (in the sense of More on Morphisms of Spaces, Definition 13.1), then $F_{k,x_0} \longrightarrow G_{k,y_0}$ is smooth as in Formal Deformation Theory, Remark 8.4.

Lemma 4.1 says that if $Y' = Y \amalg_X X'$ in the category of schemes over S where $X \to X'$ is a thickening and $X \to Y$ is affine, then the map

$$F(Y \amalg_X X') \to F(Y) \times_{F(X)} F(X')$$

is a bijection, provided that F is an algebraic space. We say a general functor F satisfies the *Rim-Schlessinger condition* or we say F *satisfies (RS)* if given any pushout $Y' = Y \amalg_X X'$ where Y, X, X' are spectra of Artinian local rings of finite type over S, then

$$F(Y \amalg_X X') \to F(Y) \times_{F(X)} F(X')$$

is a bijection. Thus every algebraic space satisfies (RS).

Lemma 6.1 says that given a functor F which satisfies (RS), then all F_{k,x_0} are deformation functors as in Formal Deformation Theory, Definition 16.8, i.e., they satisfy (RS) as in Formal Deformation Theory, Remark 16.5. In particular the tangent space

$$TF_{k,x_0} = \{x \in F(\mathrm{Spec}(k[\epsilon])) \mid x|_{\mathrm{Spec}(k)} = x_0\}$$

has the structure of a k-vector space by Formal Deformation Theory, Lemma 12.2.

Lemma 8.1 says that an algebraic space F locally of finite type over S gives rise to deformation functors F_{k,x_0} with finite dimensional tangent spaces TF_{k,x_0}.

A *formal object*[3] $\xi = (R, \xi_n)$ of F consists of a Noetherian complete local S-algebra R whose residue field is of finite type over S, together with elements $\xi_n \in F(\mathrm{Spec}(R/\mathfrak{m}^n))$ such that $\xi_{n+1}|_{\mathrm{Spec}(R/\mathfrak{m}^n)} = \xi_n$. A formal object ξ defines a formal object ξ of $F_{R/\mathfrak{m},\xi_1}$. We say ξ is *versal* if and only if it is versal in the sense of Formal Deformation Theory, Definition 8.9. A formal object $\xi = (R, \xi_n)$ is called *effective* if there exists an $x \in F(\mathrm{Spec}(R))$ such that $\xi_n = x|_{\mathrm{Spec}(R/\mathfrak{m}^n)}$ for all $n \geq 1$. Lemma 9.5 says that if F is an algebraic space, then every formal object is effective.

Let U be a scheme locally of finite type over S and let $x \in F(U)$. Let $u_0 \in U$ be a finite type point. We say that x is versal at u_0 if and only if $\xi = (\mathcal{O}^\wedge_{U,u_0}, x|_{\mathrm{Spec}(\mathcal{O}_{U,u_0}/\mathfrak{m}^n_{u_0})})$ is a versal formal object in the sense described above.

Let S be a locally Noetherian scheme. Let $F : (Sch/S)^{opp}_{fppf} \to Sets$ be a functor. Here are the axioms we will consider on F.

- [-1] a set theoretic condition[4] to be ignored by readers who are not interested in set theoretical issues,
- [0] F is a sheaf for the étale topology,
- [1] F is limit preserving,

[3] This is what Artin calls a formal deformation.

[4] The condition is the following: the supremum of all the cardinalities $|F(\mathrm{Spec}(k))|$ where k runs over the finite type fields over S is $\leq$ than the size of some object of $(Sch/S)_{fppf}$.

[2] F satisfies the Rim-Schlessinger condition (RS),
[3] every tangent space TF_{k,x_0} is finite dimensional,
[4] every formal object is effective,
[5] F satisfies openness of versality.

Here *limit preserving* is the notion defined in Limits of Spaces, Definition 3.1 and *openness of versality* means the following: Given a scheme U locally of finite type over S, given $x \in F(U)$, and given a finite type point $u_0 \in U$ such that x is versal at u_0, then there exists an open neighbourhood $u_0 \in U' \subset U$ such that x is versal at every finite type point of U'.

16. Algebraic spaces

07Y0 The following is our first main result on algebraic spaces.

07Y1 **Proposition 16.1.** *Let S be a locally Noetherian scheme. Let $F : (Sch/S)^{opp}_{fppf} \to Sets$ be a functor. Assume that*

(1) *$\Delta : F \to F \times F$ is representable by algebraic spaces,*
(2) *F satisfies axioms [-1], [0], [1], [2], [3], [4], [5] (see Section 15), and*
(3) *$\mathcal{O}_{S,s}$ is a G-ring for all finite type points s of S.*

Then F is an algebraic space.

Proof. Lemma 13.8 applies to F. Using this we choose, for every finite type field k over S and $x_0 \in F(\mathrm{Spec}(k))$, an affine scheme U_{k,x_0} of finite type over S and a smooth morphism $U_{k,x_0} \to F$ such that there exists a finite type point $u_{k,x_0} \in U_{k,x_0}$ with residue field k such that x_0 is the image of u_{k,x_0}. Then

$$U = \coprod\nolimits_{k,x_0} U_{k,x_0} \longrightarrow F$$

is smooth[5]. To finish the proof it suffices to show this map is surjective, see Bootstrap, Lemma 12.3 (this is where we use axiom [0]). By Criteria for Representability, Lemma 5.6 it suffices to show that $U \times_F V \to V$ is surjective for those $V \to F$ where V is an affine scheme locally of finite presentation over S. Since $U \times_F V \to V$ is smooth the image is open. Hence it suffices to show that the image of $U \times_F V \to V$ contains all finite type points of V, see Morphisms, Lemma 16.7. Let $v_0 \in V$ be a finite type point. Then $k = \kappa(v_0)$ is a finite type field over S. Denote x_0 the composition $\mathrm{Spec}(k) \xrightarrow{v_0} V \to F$. Then $(u_{k,x_0}, v_0) : \mathrm{Spec}(k) \to U \times_F V$ is a point mapping to v_0 and we win. □

07Y2 **Lemma 16.2.** *Let S be a locally Noetherian scheme. Let $a : F \to G$ be a transformation of functors $(Sch/S)^{opp}_{fppf} \to Sets$. Assume that*

(1) *a is injective,*
(2) *F satisfies axioms [0], [1], [2], [4], and [5],*
(3) *$\mathcal{O}_{S,s}$ is a G-ring for all finite type points s of S,*
(4) *G is an algebraic space locally of finite type over S,*

Then F is an algebraic space.

[5]Set theoretical remark: This coproduct is (isomorphic) to an object of $(Sch/S)_{fppf}$ as we have a bound on the index set by axiom [-1], see Sets, Lemma 9.9.

Proof. By Lemma 8.1 the functor G satisfies [3]. As $F \to G$ is injective, we conclude that F also satisfies [3]. Moreover, as $F \to G$ is injective, we see that given schemes U, V and morphisms $U \to F$ and $V \to F$, then $U \times_F V = U \times_G V$. Hence $\Delta : F \to F \times F$ is representable (by schemes) as this holds for G by assumption. Thus Proposition 16.1 applies[6]. □

17. Algebraic stacks

07Y3 Proposition 17.2 is our first main result on algebraic stacks.

07Y4 **Lemma 17.1.** *Let S be a locally Noetherian scheme. Let $p : \mathcal{X} \to (Sch/S)_{fppf}$ be a category fibred in groupoids. Assume that*

(1) *$\Delta : \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is representable by algebraic spaces,*
(2) *$\mathcal{X}$ satisfies axioms [-1], [0], [1], [2], [3] (see Section 14),*
(3) *every formal object of $\mathcal{X}$ is effective,*
(4) *$\mathcal{X}$ satisfies openness of versality, and*
(5) *$\mathcal{O}_{S,s}$ is a G-ring for all finite type points s of S.*

Then $\mathcal{X}$ is an algebraic stack.

Proof. Lemma 13.8 applies to $\mathcal{X}$. Using this we choose, for every finite type field k over S and every isomorphism class of object $x_0 \in \mathrm{Ob}(\mathcal{X}_{\mathrm{Spec}(k)})$, an affine scheme U_{k,x_0} of finite type over S and a smooth morphism $(Sch/U_{k,x_0})_{fppf} \to \mathcal{X}$ such that there exists a finite type point $u_{k,x_0} \in U_{k,x_0}$ with residue field k such that x_0 is the image of u_{k,x_0}. Then

$$(Sch/U)_{fppf} \to \mathcal{X}, \quad \text{with} \quad U = \coprod\nolimits_{k,x_0} U_{k,x_0}$$

is smooth[7]. To finish the proof it suffices to show this map is surjective, see Criteria for Representability, Lemma 19.1 (this is where we use axiom [0]). By Criteria for Representability, Lemma 5.6 it suffices to show that $(Sch/U)_{fppf} \times_{\mathcal{X}} (Sch/V)_{fppf} \to (Sch/V)_{fppf}$ is surjective for those $y : (Sch/V)_{fppf} \to \mathcal{X}$ where V is an affine scheme locally of finite presentation over S. By assumption (1) the fibre product $(Sch/U)_{fppf} \times_{\mathcal{X}} (Sch/V)_{fppf}$ is representable by an algebraic space W. Then $W \to V$ is smooth, hence the image is open. Hence it suffices to show that the image of $W \to V$ contains all finite type points of V, see Morphisms, Lemma 16.7. Let $v_0 \in V$ be a finite type point. Then $k = \kappa(v_0)$ is a finite type field over S. Denote $x_0 = y|_{\mathrm{Spec}(k)}$ the pullback of y by v_0. Then (u_{k,x_0}, v_0) will give a morphism $\mathrm{Spec}(k) \to W$ whose composition with $W \to V$ is v_0 and we win. □

07Y5 **Proposition 17.2.** *Let S be a locally Noetherian scheme. Let $p : \mathcal{X} \to (Sch/S)_{fppf}$ be a category fibred in groupoids. Assume that*

(1) *$\Delta_\Delta : \mathcal{X} \to \mathcal{X} \times_{\mathcal{X} \times \mathcal{X}} \mathcal{X}$ is representable by algebraic spaces,*
(2) *$\mathcal{X}$ satisfies axioms [-1], [0], [1], [2], [3], [4], and [5] (see Section 14),*
(3) *$\mathcal{O}_{S,s}$ is a G-ring for all finite type points s of S.*

Then $\mathcal{X}$ is an algebraic stack.

---

[6] The set theoretic condition [-1] holds for F as it holds for G. Details omitted.

[7] Set theoretical remark: This coproduct is (isomorphic to) an object of $(Sch/S)_{fppf}$ as we have a bound on the index set by axiom [-1], see Sets, Lemma 9.9.

Proof. We first prove that $\Delta : \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is representable by algebraic spaces. To do this it suffices to show that

$$\mathcal{Y} = \mathcal{X} \times_{\Delta, \mathcal{X} \times \mathcal{X}, y} (Sch/V)_{fppf}$$

is representable by an algebraic space for any affine scheme V locally of finite presentation over S and object y of $\mathcal{X} \times \mathcal{X}$ over V, see Criteria for Representability, Lemma 5.5[8]. Observe that $\mathcal{Y}$ is fibred in setoids (Stacks, Lemma 2.5) and let $Y : (Sch/S)^{opp}_{fppf} \to Sets$, $T \mapsto \mathrm{Ob}(\mathcal{Y}_T)/\cong$ be the functor of isomorphism classes. We will apply Proposition 16.1 to see that Y is an algebraic space.

Note that $\Delta_{\mathcal{Y}} : \mathcal{Y} \to \mathcal{Y} \times \mathcal{Y}$ (and hence also $Y \to Y \times Y$) is representable by algebraic spaces by condition (1) and Criteria for Representability, Lemma 4.4. Observe that Y is a sheaf for the étale topology by Stacks, Lemmas 6.3 and 6.7, i.e., axiom [0] holds. Also Y is limit preserving by Lemma 11.2, i.e., we have [1]. Note that Y has (RS), i.e., axiom [2] holds, by Lemmas 5.2 and 5.3. Axiom [3] for Y follows from Lemmas 8.1 and 8.2. Axiom [4] follows from Lemmas 9.5 and 9.6. Axiom [5] for Y follows directly from openness of versality for $\Delta_{\mathcal{X}}$ which is part of axiom [5] for $\mathcal{X}$. Thus all the assumptions of Proposition 16.1 are satisfied and Y is an algebraic space.

At this point it follows from Lemma 17.1 that $\mathcal{X}$ is an algebraic stack. □

18. Strong Rim-Schlessinger

0CXN In the rest of this chapter the following strictly stronger version of the Rim-Schlessinger conditions will play an important role.

07Y8 **Definition 18.1.** Let S be a scheme. Let $\mathcal{X}$ be a category fibred in groupoids over $(Sch/S)_{fppf}$. We say $\mathcal{X}$ satisfies *condition (RS*)* if given a fibre product diagram

$$\begin{array}{ccc} B' & \longrightarrow & B \\ \uparrow & & \uparrow \\ A' = A \times_B B' & \longrightarrow & A \end{array}$$

of S-algebras, with $B' \to B$ surjective with square zero kernel, the functor of fibre categories

$$\mathcal{X}_{\mathrm{Spec}(A')} \longrightarrow \mathcal{X}_{\mathrm{Spec}(A)} \times_{\mathcal{X}_{\mathrm{Spec}(B)}} \mathcal{X}_{\mathrm{Spec}(B')}$$

is an equivalence of categories.

We make some observations: with $A \to B \leftarrow B'$ as in Definition 18.1

(1) we have $\mathrm{Spec}(A') = \mathrm{Spec}(A) \amalg_{\mathrm{Spec}(B)} \mathrm{Spec}(B')$ in the category of schemes, see More on Morphisms, Lemma 14.3, and
(2) if $\mathcal{X}$ is an algebraic stack, then $\mathcal{X}$ satisfies (RS*) by Lemma 18.2.

If S is locally Noetherian, then

(3) if A, B, B' are of finite type over S and B is finite over A, then A' is of finite type over S[9], and

[8] The set theoretic condition in Criteria for Representability, Lemma 5.5 will hold: the size of the algebraic space Y representing $\mathcal{Y}$ is suitably bounded. Namely, $Y \to S$ will be locally of finite type and Y will satisfy axiom [-1]. Details omitted.

[9] If $\mathrm{Spec}(A)$ maps into an affine open of S this follows from More on Algebra, Lemma 5.1. The general case follows using More on Algebra, Lemma 5.3.

(4) if $\mathcal{X}$ satisfies (RS*), then $\mathcal{X}$ satisfies (RS) because (RS) covers exactly those cases of (RS*) where A, B, B' are Artinian local.

0CXP **Lemma 18.2.** *Let $\mathcal{X}$ be an algebraic stack over a base S. Then $\mathcal{X}$ satisfies (RS*).*

Proof. This is implied by Lemma 4.1, see remarks following Definition 18.1. □

0CXQ **Lemma 18.3.** *Let S be a scheme. Let $p : \mathcal{X} \to \mathcal{Y}$ and $q : \mathcal{Z} \to \mathcal{Y}$ be 1-morphisms of categories fibred in groupoids over $(Sch/S)_{fppf}$. If $\mathcal{X}$, $\mathcal{Y}$, and $\mathcal{Z}$ satisfy (RS*), then so does $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z}$.*

Proof. The proof is exactly the same as the proof of Lemma 5.3. □

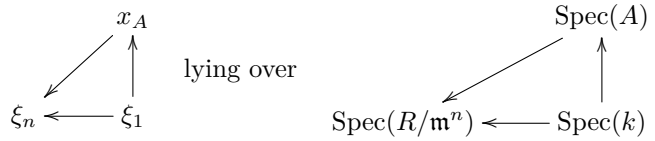
19. Versality and generalizations

0G2I We prove that versality is preserved under generalizations for stacks which have (RS*) and are limit preserving. We suggest skipping this section on a first reading.

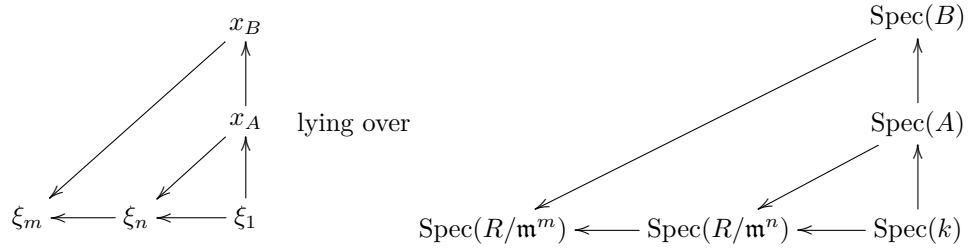
0G2J **Lemma 19.1.** *Let S be a locally Noetherian scheme. Let $\mathcal{X}$ be a category fibred in groupoids over $(Sch/S)_{fppf}$ having (RS*). Let x be an object of $\mathcal{X}$ over an affine scheme U of finite type over S. Let $u \in U$ be a finite type point such that x is not versal at u. Then there exists a morphism $x \to y$ of $\mathcal{X}$ lying over $U \to T$ satisfying*

(1) *the morphism $U \to T$ is a first order thickening,*
(2) *we have a short exact sequence*
$$0 \to \kappa(u) \to \mathcal{O}_T \to \mathcal{O}_U \to 0$$
(3) *there does* **not** *exist a pair (W, α) consisting of an open neighbourhood $W \subset T$ of u and a morphism $\beta : y|_W \to x$ such that the composition*
$$x|_{U \cap W} \xrightarrow{\text{restriction of } x \to y} y|_W \xrightarrow{\beta} x$$
is the canonical morphism $x|_{U\cap W} \to x$.

Proof. Let $R = \mathcal{O}^\wedge_{U,u}$. Let $k = \kappa(u)$ be the residue field of R. Let ξ be the formal object of $\mathcal{X}$ over R associated to x. Since x is not versal at u, we see that ξ is not versal, see Lemma 12.3. By the discussion following Definition 12.1 this means we can find morphisms $\xi_1 \to x_A \to x_B$ of $\mathcal{X}$ lying over closed immersions $\mathrm{Spec}(k) \to \mathrm{Spec}(A) \to \mathrm{Spec}(B)$ where A, B are Artinian local rings with residue field k, an $n \geq 1$ and a commutative diagram



such that there does **not** exist an $m \geq n$ and a commutative diagram



We may moreover assume that $B \to A$ is a small extension, i.e., that the kernel I of the surjection $B \to A$ is isomorphic to k as an A-module. This follows from Formal Deformation Theory, Remark 8.10. Then we simply define

$$T = U \amalg_{\mathrm{Spec}(A)} \mathrm{Spec}(B)$$

By property (RS*) we find y over T whose restriction to $\mathrm{Spec}(B)$ is x_B and whose restriction to U is x (this gives the arrow $x \to y$ lying over $U \to T$). To finish the proof we verify conditions (1), (2), and (3).

By the construction of the pushout we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I & \longrightarrow & B & \longrightarrow & A & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & I & \longrightarrow & \Gamma(T, \mathcal{O}_T) & \longrightarrow & \Gamma(U, \mathcal{O}_U) & \longrightarrow & 0 \end{array}$$

with exact rows. This immediately proves (1) and (2). To finish the proof we will argue by contradiction. Assume we have a pair (W, β) as in (3). Since $\mathrm{Spec}(B) \to T$ factors through W we get the morphism

$$x_B \to y|_W \xrightarrow{\beta} x$$

Since B is Artinian local with residue field $k = \kappa(u)$ we see that $x_B \to x$ lies over a morphism $\mathrm{Spec}(B) \to U$ which factors through $\mathrm{Spec}(\mathcal{O}_{U,u}/\mathfrak{m}_u^m)$ for some $m \geq n$. In other words, $x_B \to x$ factors through ξ_m giving a map $x_B \to \xi_m$. The compatibility condition on the morphism α in condition (3) translates into the condition that

$$\begin{array}{ccc} x_B & \longleftarrow & x_A \\ \downarrow & & \downarrow \\ \xi_m & \longleftarrow & \xi_n \end{array}$$

is commutative. This gives the contradiction we were looking for. □

0G2K **Lemma 19.2.** *Let S be a locally Noetherian scheme. Let $\mathcal{X}$ be a category fibred in groupoids over $(Sch/S)_{fppf}$. Assume*

(1) *$\Delta : \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is representable by algebraic spaces,*
(2) *$\mathcal{X}$ has (RS*),*
(3) *$\mathcal{X}$ is limit preserving.*

Let x be an object of $\mathcal{X}$ over a scheme U of finite type over S. Let $u \rightsquigarrow u_0$ be a specialization of finite type points of U such that x is versal at u_0. Then x is versal at u.

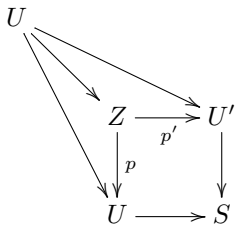
Proof. After shrinking U we may assume U is affine and U maps into an affine open $\mathrm{Spec}(\Lambda)$ of S. If x is not versal at u then we may pick $x \to y$ lying over $U \to T$ as in Lemma 19.1. Write $U = \mathrm{Spec}(R_0)$ and $T = \mathrm{Spec}(R)$. The morphism $U \to T$ corresponds to a surjective ring map $R \to R_0$ whose kernel is an ideal of square zero. By assumption (3) we get that y comes from an object x' over $U' = \mathrm{Spec}(R')$ for some finite type Λ-subalgebra $R' \subset R$. After increasing R' we may and do assume that $R' \to R_0$ is surjective, so that $U \subset U'$ is a first order thickening. Thus we now have

$$x \to y \to x' \text{ lying over } U \to T \to U'$$

By assumption (1) there is an algebraic space Z over S representing

$$(Sch/U)_{fppf} \times_{x,\mathcal{X},x'} (Sch/U')_{fppf}$$

see Algebraic Stacks, Lemma 10.11. By construction of 2-fibre products, a V-valued point of Z corresponds to a triple (a, a', α) consisting of morphisms $a : V \to U$, $a' : V \to U'$ and a morphism $\alpha : a^*x \to (a')^*x'$. We obtain a commutative diagram



The morphism $i : U \to Z$ comes the isomorphism $x \to x'|_U$. Let $z_0 = i(u_0) \in Z$. By Lemma 12.6 we see that $Z \to U'$ is smooth at z_0. After replacing U by an affine open neighbourhood of u_0, replacing U' by the corresponding open, and replacing Z by the intersection of the inverse images of these opens by p and p', we reach the situation where $Z \to U'$ is smooth along $i(U)$. Since $u \rightsquigarrow u_0$ the point u is in this open. Condition (3) of Lemma 19.1 is clearly preserved by shrinking U (all of the schemes U, T, U' have the same underlying topological space). Since $U \to U'$ is a first order thickening of affine schemes, we can choose a morphism $i' : U' \to Z$ such that $p' \circ i' = \mathrm{id}_{U'}$ and whose restriction to U is i (More on Morphisms of Spaces, Lemma 19.6). Pulling back the universal morphism $p^*x \to (p')^*x'$ by i' we obtain a morphism

$$x' \to x$$

lying over $p \circ i' : U' \to U$ such that the composition

$$x \to x' \to x$$

is the identity. Recall that we have $y \to x'$ lying over the morphism $T \to U'$. Composing we get a morphism $y \to x$ whose existence contradicts condition (3) of Lemma 19.1. This contradiction finishes the proof. □

20. Strong formal effectiveness

0CXR In this section we demonstrate how a strong version of effectiveness of formal objects implies openness of versality. The proof of [Bha16, Theorem 1.1] shows that quasi-compact and quasi-separated algebraic spaces satisfy the strong formal effectiveness discussed in Remark 20.2. In addition, the theory we develop is nonempty: we use it later to show openness of versality for the stack of coherent sheaves and for moduli of complexes, see Quot, Theorems 6.1 and 16.12.

0G2S **Lemma 20.1.** *Let S be a locally Noetherian scheme. Let $\mathcal{X}$ be a category fibred in groupoids over $(Sch/S)_{fppf}$ having (RS*). Let x be an object of $\mathcal{X}$ over an affine scheme U of finite type over S. Let $u_n \in U$, $n \geq 1$ be a finite type points such that (a) there are no specializations $u_n \rightsquigarrow u_m$ for $n \neq m$, and (b) x is not versal at u_n for all n. Then there exist morphisms*

$$x \to x_1 \to x_2 \to \ldots \quad \text{in } \mathcal{X} \text{ lying over} \quad U \to U_1 \to U_2 \to \ldots$$

over S such that

(1) *for each n the morphism $U \to U_n$ is a first order thickening,*
(2) *for each n we have a short exact sequence*
$$0 \to \kappa(u_n) \to \mathcal{O}_{U_n} \to \mathcal{O}_{U_{n-1}} \to 0$$
with $U_0 = U$ for $n = 1$,
(3) *for each n there does* **not** *exist a pair (W, α) consisting of an open neighbourhood $W \subset U_n$ of u_n and a morphism $\alpha : x_n|_W \to x$ such that the composition*
$$x|_{U \cap W} \xrightarrow{\text{restriction of } x \to x_n} x_n|_W \xrightarrow{\alpha} x$$
is the canonical morphism $x|_{U\cap W} \to x$.

Proof. Since there are no specializations among the points u_n (and in particular the u_n are pairwise distinct), for every n we can find an open $U' \subset U$ such that $u_n \in U'$ and $u_i \notin U'$ for $i = 1, \ldots, n-1$. By Lemma 19.1 for each $n \geq 1$ we can find
$$x \to y_n \quad \text{in } \mathcal{X} \text{ lying over } \quad U \to T_n$$
such that

(1) the morphism $U \to T_n$ is a first order thickening,
(2) we have a short exact sequence
$$0 \to \kappa(u_n) \to \mathcal{O}_{T_n} \to \mathcal{O}_U \to 0$$
(3) there does **not** exist a pair (W, α) consisting of an open neighbourhood $W \subset T_n$ of u_n and a morphism $\beta : y_n|_W \to x$ such that the composition
$$x|_{U \cap W} \xrightarrow{\text{restriction of } x \to y_n} y_n|_W \xrightarrow{\beta} x$$
is the canonical morphism $x|_{U\cap W} \to x$.

Thus we can define inductively
$$U_1 = T_1, \quad U_{n+1} = U_n \amalg_U T_{n+1}$$
Setting $x_1 = y_1$ and using (RS*) we find inductively x_{n+1} over U_{n+1} restricting to x_n over U_n and y_{n+1} over T_{n+1}. Property (1) for $U \to U_n$ follows from the construction of the pushout in More on Morphisms, Lemma 14.3. Property (2) for U_n similarly follows from property (2) for T_n by the construction of the pushout. After shrinking to an open neighbourhood U' of u_n as discussed above, property (3) for (U_n, x_n) follows from property (3) for (T_n, y_n) simply because the corresponding open subschemes of T_n and U_n are isomorphic. Some details omitted. □

0CXT **Remark 20.2** (Strong effectiveness)**.** Let S be a locally Noetherian scheme. Let $\mathcal{X}$ be a category fibred in groupoids over $(Sch/S)_{fppf}$. Assume we have

(1) an affine open $\mathrm{Spec}(\Lambda) \subset S$,
(2) an inverse system (R_n) of Λ-algebras with surjective transition maps whose kernels are locally nilpotent,
(3) a system (ξ_n) of objects of $\mathcal{X}$ lying over the system $(\mathrm{Spec}(R_n))$.

In this situation, set $R = \lim R_n$. We say that (ξ_n) is *effective* if there exists an object ξ of $\mathcal{X}$ over $\mathrm{Spec}(R)$ whose restriction to $\mathrm{Spec}(R_n)$ gives the system (ξ_n).

It is not the case that every algebraic stack $\mathcal{X}$ over S satisfies a strong effectiveness axiom of the form: every system (ξ_n) as in Remark 20.2 is effective. An example is given in Examples, Section 75.

0CXU **Lemma 20.3.** *Let S be a locally Noetherian scheme. Let $\mathcal{X}$ be a category fibred in groupoids over $(Sch/S)_{fppf}$. Assume*

(1) *$\Delta : \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is representable by algebraic spaces,*
(2) *$\mathcal{X}$ has (RS*),*
(3) *$\mathcal{X}$ is limit preserving,*
(4) *systems (ξ_n) as in Remark 20.2 where $\mathrm{Ker}(R_m \to R_n)$ is an ideal of square zero for all $m \geq n$ are effective.*

Then $\mathcal{X}$ satisfies openness of versality.

Proof. Choose a scheme U locally of finite type over S, a finite type point u_0 of U, and an object x of $\mathcal{X}$ over U such that x is versal at u_0. After shrinking U we may assume U is affine and U maps into an affine open $\mathrm{Spec}(\Lambda)$ of S. Let $E \subset U$ be the set of finite type points u such that x is not versal at u. By Lemma 19.2 if $u \in E$ then u_0 is not a specialization of u. If openness of versality does not hold, then u_0 is in the closure $\overline{E}$ of E. By Properties, Lemma 5.13 we may choose a countable subset $E' \subset E$ with the same closure as E. By Properties, Lemma 5.12 we may assume there are no specializations among the points of E'. Observe that E' has to be (countably) infinite as u_0 isn't the specialization of any point of E' as pointed out above. Thus we can write $E' = \{u_1, u_2, u_3, \ldots\}$, there are no specializations among the u_i, and u_0 is in the closure of E'.

Choose $x \to x_1 \to x_2 \to \ldots$ lying over $U \to U_1 \to U_2 \to \ldots$ as in Lemma 20.1. Write $U_n = \mathrm{Spec}(R_n)$ and $U = \mathrm{Spec}(R_0)$. Set $R = \lim R_n$. Observe that $R \to R_0$ is surjective with kernel an ideal of square zero. By assumption (4) we get ξ over $\mathrm{Spec}(R)$ whose base change to R_n is x_n. By assumption (3) we get that ξ comes from an object ξ' over $U' = \mathrm{Spec}(R')$ for some finite type Λ-subalgebra $R' \subset R$. After increasing R' we may and do assume that $R' \to R_0$ is surjective, so that $U \subset U'$ is a first order thickening. Thus we now have

$$x \to x_1 \to x_2 \to \ldots \to \xi' \text{ lying over } U \to U_1 \to U_2 \to \ldots \to U'$$

By assumption (1) there is an algebraic space Z over S representing

$$(Sch/U)_{fppf} \times_{x,\mathcal{X},\xi'} (Sch/U')_{fppf}$$

see Algebraic Stacks, Lemma 10.11. By construction of 2-fibre products, a T-valued point of Z corresponds to a triple (a, a', α) consisting of morphisms $a : T \to U$, $a' : T \to U'$ and a morphism $\alpha : a^*x \to (a')^*\xi'$. We obtain a commutative diagram

$$\begin{array}{ccc} U & & \\ & \searrow & \\ & Z \xrightarrow[p']{} U' & \\ & \downarrow p \quad \downarrow & \\ & U \longrightarrow S & \end{array}$$

The morphism $i : U \to Z$ comes the isomorphism $x \to \xi'|_U$. Let $z_0 = i(u_0) \in Z$. By Lemma 12.6 we see that $Z \to U'$ is smooth at z_0. After replacing U by an affine open neighbourhood of u_0, replacing U' by the corresponding open, and replacing Z by the intersection of the inverse images of these opens by p and p', we reach the situation where $Z \to U'$ is smooth along $i(U)$. Note that this also involves replacing u_n by a subsequence, namely by those indices such that u_n is in the open.

Moreover, condition (3) of Lemma 20.1 is clearly preserved by shrinking U (all of the schemes U, U_n, U' have the same underlying topological space). Since $U \to U'$ is a first order thickening of affine schemes, we can choose a morphism $i' : U' \to Z$ such that $p' \circ i' = \mathrm{id}_{U'}$ and whose restriction to U is i (More on Morphisms of Spaces, Lemma 19.6). Pulling back the universal morphism $p^*x \to (p')^*\xi'$ by i' we obtain a morphism

$$\xi' \to x$$

lying over $p \circ i' : U' \to U$ such that the composition

$$x \to \xi' \to x$$

is the identity. Recall that we have $x_1 \to \xi'$ lying over the morphism $U_1 \to U'$. Composing we get a morphism $x_1 \to x$ whose existence contradicts condition (3) of Lemma 20.1. This contradiction finishes the proof. □

0CXV **Remark 20.4.** There is a way to deduce openness of versality of the diagonal of an category fibred in groupoids from a strong formal effectiveness axiom. Let S be a locally Noetherian scheme. Let $\mathcal{X}$ be a category fibred in groupoids over $(Sch/S)_{fppf}$. Assume

(1) $\Delta_\Delta : \mathcal{X} \to \mathcal{X} \times_{\mathcal{X}\times\mathcal{X}} \mathcal{X}$ is representable by algebraic spaces,
(2) $\mathcal{X}$ has (RS*),
(3) $\mathcal{X}$ is limit preserving,
(4) given an inverse system (R_n) of S-algebras as in Remark 20.2 where $\mathrm{Ker}(R_m \to R_n)$ is an ideal of square zero for all $m \geq n$ the functor

$$\mathcal{X}_{\mathrm{Spec}(\lim R_n)} \longrightarrow \lim_n \mathcal{X}_{\mathrm{Spec}(R_n)}$$

is fully faithful.

Then $\Delta : \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ satisfies openness of versality. This follows by applying Lemma 20.3 to fibre products of the form $\mathcal{X} \times_{\Delta,\mathcal{X}\times\mathcal{X},y} (Sch/V)_{fppf}$ for any affine scheme V locally of finite presentation over S and object y of $\mathcal{X} \times \mathcal{X}$ over V. If we ever need this, we will change this remark into a lemma and provide a detailed proof.

21. Infinitesimal deformations

07Y6 In this section we discuss a generalization of the notion of the tangent space introduced in Section 8. To do this intelligently, we borrow some notation from Formal Deformation Theory, Sections 11, 17, and 19.

Let S be a scheme. Let $\mathcal{X}$ be a category fibred in groupoids over $(Sch/S)_{fppf}$. Given a homomorphism $A' \to A$ of S-algebras and an object x of $\mathcal{X}$ over $\mathrm{Spec}(A)$ we write $\mathit{Lift}(x, A')$ for the category of lifts of x to $\mathrm{Spec}(A')$. An object of $\mathit{Lift}(x, A')$ is a morphism $x \to x'$ of $\mathcal{X}$ lying over $\mathrm{Spec}(A) \to \mathrm{Spec}(A')$ and morphisms of $\mathit{Lift}(x, A')$ are defined as commutative diagrams. The set of isomorphism classes of $\mathit{Lift}(x, A')$ is denoted $\mathrm{Lift}(x, A')$. See Formal Deformation Theory, Definition 17.1 and Remark 17.2. If $A' \to A$ is surjective with locally nilpotent kernel we call an element x' of $\mathrm{Lift}(x, A')$ a *(infinitesimal) deformation* of x. In this case the *group of infinitesimal automorphisms of x' over x* is the kernel

$$\mathrm{Inf}(x'/x) = \mathrm{Ker}\left(\mathrm{Aut}_{\mathcal{X}_{\mathrm{Spec}(A')}}(x') \to \mathrm{Aut}_{\mathcal{X}_{\mathrm{Spec}(A)}}(x)\right)$$

Note that an element of $\mathrm{Inf}(x'/x)$ is the same thing as a lift of id_x over $\mathrm{Spec}(A')$ for (the category fibred in sets associated to) $Aut_{\mathcal{X}}(x')$. Compare with Formal Deformation Theory, Definition 19.1 and Formal Deformation Theory, Remark 19.8.

If M is an A-module we denote $A[M]$ the A-algebra whose underlying A-module is $A \oplus M$ and whose multiplication is given by $(a, m) \cdot (a', m') = (aa', am' + a'm)$. When $M = A$ this is the ring of dual numbers over A, which we denote $A[\epsilon]$ as is customary. There is an A-algebra map $A[M] \to A$. The pullback of x to $\mathrm{Spec}(A[M])$ is called the *trivial deformation* of x to $\mathrm{Spec}(A[M])$.

07Y7 **Lemma 21.1.** *Let S be a scheme. Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. Let*

$$\begin{array}{ccc} B' & \longrightarrow & B \\ \uparrow & & \uparrow \\ A' & \longrightarrow & A \end{array}$$

be a commutative diagram of S-algebras. Let x be an object of $\mathcal{X}$ over $\mathrm{Spec}(A)$, let y be an object of $\mathcal{Y}$ over $\mathrm{Spec}(B)$, and let $\phi : f(x)|_{\mathrm{Spec}(B)} \to y$ be a morphism of $\mathcal{Y}$ over $\mathrm{Spec}(B)$. Then there is a canonical functor

$$Lift(x, A') \longrightarrow Lift(y, B')$$

of categories of lifts induced by f and ϕ. The construction is compatible with compositions of 1-morphisms of categories fibred in groupoids in an obvious manner.

Proof. This lemma proves itself. □

Let S be a base scheme. Let $\mathcal{X}$ be a category fibred in groupoids over $(Sch/S)_{fppf}$. We define a category whose objects are pairs $(x, A' \to A)$ where

(1) $A' \to A$ is a surjection of S-algebras whose kernel is an ideal of square zero,
(2) x is an object of $\mathcal{X}$ lying over $\mathrm{Spec}(A)$.

A morphism $(y, B' \to B) \to (x, A' \to A)$ is given by a commutative diagram

$$\begin{array}{ccc} B' & \longrightarrow & B \\ \uparrow & & \uparrow \\ A' & \longrightarrow & A \end{array}$$

of S-algebras together with a morphism $x|_{\mathrm{Spec}(B)} \to y$ over $\mathrm{Spec}(B)$. Let us call this the category of *deformation situations*.

07Y9 **Lemma 21.2.** *Let S be a scheme. Let $\mathcal{X}$ be a category fibred in groupoids over $(Sch/S)_{fppf}$. Assume $\mathcal{X}$ satisfies condition (RS*). Let A be an S-algebra and let x be an object of $\mathcal{X}$ over $\mathrm{Spec}(A)$.*

(1) *There exists an A-linear functor $Inf_x : Mod_A \to Mod_A$ such that given a deformation situation $(x, A' \to A)$ and a lift x' there is an isomorphism $Inf_x(I) \to Inf(x'/x)$ where $I = \mathrm{Ker}(A' \to A)$.*
(2) *There exists an A-linear functor $T_x : Mod_A \to Mod_A$ such that*
   (a) *given M in Mod_A there is a bijection $T_x(M) \to Lift(x, A[M])$,*
   (b) *given a deformation situation $(x, A' \to A)$ there is an action*

$$T_x(I) \times Lift(x, A') \to Lift(x, A')$$

where $I = \mathrm{Ker}(A' \to A)$. It is simply transitive if $Lift(x, A') \neq \emptyset$.

Proof. We define Inf_x as the functor

$$\mathrm{Mod}_A \longrightarrow \mathit{Sets}, \quad M \longrightarrow \mathrm{Inf}(x'_M/x) = \mathrm{Lift}(\mathrm{id}_x, A[M])$$

mapping M to the group of infinitesimal automorphisms of the trivial deformation x'_M of x to $\mathrm{Spec}(A[M])$ or equivalently the group of lifts of id_x in $Aut_{\mathcal{X}}(x'_M)$. We define T_x as the functor

$$\mathrm{Mod}_A \longrightarrow \mathit{Sets}, \quad M \longrightarrow \mathrm{Lift}(x, A[M])$$

of isomorphism classes of infinitesimal deformations of x to $\mathrm{Spec}(A[M])$. We apply Formal Deformation Theory, Lemma 11.4 to Inf_x and T_x. This lemma is applicable, since (RS*) tells us that

$$\mathit{Lift}(x, A[M \times N]) = \mathit{Lift}(x, A[M]) \times \mathit{Lift}(x, A[N])$$

as categories (and trivial deformations match up too).

Let $(x, A' \to A)$ be a deformation situation. Consider the ring map $g : A' \times_A A' \to A[I]$ defined by the rule $g(a_1, a_2) = \overline{a_1} \oplus a_2 - a_1$. There is an isomorphism

$$A' \times_A A' \longrightarrow A' \times_A A[I]$$

given by $(a_1, a_2) \mapsto (a_1, g(a_1, a_2))$. This isomorphism commutes with the projections to A' on the first factor, and hence with the projections to A. Thus applying (RS*) twice we find equivalences of categories

$$\begin{aligned} \mathit{Lift}(x, A') \times \mathit{Lift}(x, A') &= \mathit{Lift}(x, A' \times_A A') \\ &= \mathit{Lift}(x, A' \times_A A[I]) \\ &= \mathit{Lift}(x, A') \times \mathit{Lift}(x, A[I]) \end{aligned}$$

Using these maps and projection onto the last factor of the last product we see that we obtain "difference maps"

$$\mathrm{Inf}(x'/x) \times \mathrm{Inf}(x'/x) \longrightarrow \mathrm{Inf}_x(I) \quad \text{and} \quad \mathrm{Lift}(x, A') \times \mathrm{Lift}(x, A') \longrightarrow T_x(I)$$

These difference maps satisfy the transitivity rule "$(x'_1 - x'_2) + (x'_2 - x'_3) = x'_1 - x'_3$" because

$$\begin{array}{ccc} A' \times_A A' \times_A A' & \xrightarrow[(a_1,a_2,a_3) \mapsto (g(a_1,a_2), g(a_2,a_3))]{} & A[I] \times_A A[I] = A[I \times I] \\ & \searrow^{(a_1,a_2,a_3) \mapsto g(a_1,a_3)} & \downarrow + \\ & & A[I] \end{array}$$

is commutative. Inverting the string of equivalences above we obtain an action which is free and transitive provided $\mathrm{Inf}(x'/x)$, resp. $\mathrm{Lift}(x, A')$ is nonempty. Note that $\mathrm{Inf}(x'/x)$ is always nonempty as it is a group. □

07YA **Remark 21.3** (Functoriality)**.** Assumptions and notation as in Lemma 21.2. Suppose $A \to B$ is a ring map and $y = x|_{\mathrm{Spec}(B)}$. Let $M \in \mathrm{Mod}_A$, $N \in \mathrm{Mod}_B$ and let $M \to N$ an A-linear map. Then there are canonical maps $\mathrm{Inf}_x(M) \to \mathrm{Inf}_y(N)$ and $T_x(M) \to T_y(N)$ simply because there is a pullback functor

$$\mathit{Lift}(x, A[M]) \to \mathit{Lift}(y, B[N])$$

coming from the ring map $A[M] \to B[N]$. Similarly, given a morphism of deformation situations $(y, B' \to B) \to (x, A' \to A)$ we obtain a pullback functor $\mathit{Lift}(x, A') \to \mathit{Lift}(y, B')$. Since the construction of the action, the addition, and the scalar multiplication on Inf_x and T_x use only morphisms in the categories of

lifts (see proof of Formal Deformation Theory, Lemma 11.4) we see that the constructions above are functorial. In other words we obtain A-linear maps

$$\mathrm{Inf}_x(M) \to \mathrm{Inf}_y(N) \quad \text{and} \quad T_x(M) \to T_y(N)$$

such that the diagrams

$$\begin{array}{ccc} \mathrm{Inf}_y(J) & \longrightarrow & \mathrm{Inf}(y'/y) \\ \uparrow & & \uparrow \\ \mathrm{Inf}_x(I) & \longrightarrow & \mathrm{Inf}(x'/x) \end{array} \quad \text{and} \quad \begin{array}{ccc} T_y(J) \times \mathrm{Lift}(y, B') & \longrightarrow & \mathrm{Lift}(y, B') \\ \uparrow & & \uparrow \\ T_x(I) \times \mathrm{Lift}(x, A') & \longrightarrow & \mathrm{Lift}(x, A') \end{array}$$

commute. Here $I = \mathrm{Ker}(A' \to A)$, $J = \mathrm{Ker}(B' \to B)$, x' is a lift of x to A' (which may not always exist) and $y' = x'|_{\mathrm{Spec}(B')}$.

07YB **Remark 21.4** (Automorphisms)**.** Assumptions and notation as in Lemma 21.2. Let x', x'' be lifts of x to A'. Then we have a composition map

$$\mathrm{Inf}(x'/x) \times \mathrm{Mor}_{\mathit{Lift}(x,A')}(x', x'') \times \mathrm{Inf}(x''/x) \longrightarrow \mathrm{Mor}_{\mathit{Lift}(x,A')}(x', x'').$$

Since $\mathit{Lift}(x, A')$ is a groupoid, if $\mathrm{Mor}_{\mathit{Lift}(x,A')}(x', x'')$ is nonempty, then this defines a simply transitive left action of $\mathrm{Inf}(x'/x)$ on $\mathrm{Mor}_{\mathit{Lift}(x,A')}(x', x'')$ and a simply transitive right action by $\mathrm{Inf}(x''/x)$. Now the lemma says that $\mathrm{Inf}(x'/x) = \mathrm{Inf}_x(I) = \mathrm{Inf}(x''/x)$. We claim that the two actions described above agree via these identifications. Namely, either $x' \not\cong x''$ in which the claim is clear, or $x' \cong x''$ and in that case we may assume that $x'' = x'$ in which case the result follows from the fact that $\mathrm{Inf}(x'/x)$ is commutative. In particular, we obtain a well defined action

$$\mathrm{Inf}_x(I) \times \mathrm{Mor}_{\mathit{Lift}(x,A')}(x', x'') \longrightarrow \mathrm{Mor}_{\mathit{Lift}(x,A')}(x', x'')$$

which is simply transitive as soon as $\mathrm{Mor}_{\mathit{Lift}(x,A')}(x', x'')$ is nonempty.

07YE **Remark 21.5.** Let S be a scheme. Let $\mathcal{X}$ be a category fibred in groupoids over $(\mathit{Sch}/S)_{\mathit{fppf}}$. Let A be an S-algebra. There is a notion of a *short exact sequence*

$$(x, A'_1 \to A) \to (x, A'_2 \to A) \to (x, A'_3 \to A)$$

of deformation situations: we ask the corresponding maps between the kernels $I_i = \mathrm{Ker}(A'_i \to A)$ give a short exact sequence

$$0 \to I_3 \to I_2 \to I_1 \to 0$$

of A-modules. Note that in this case the map $A'_3 \to A'_1$ factors through A, hence there is a canonical isomorphism $A'_1 = A[I_1]$.

0DNN **Lemma 21.6.** *Let S be a scheme. Let $p : \mathcal{X} \to \mathcal{Y}$ and $q : \mathcal{Z} \to \mathcal{Y}$ be 1-morphisms of categories fibred in groupoids over $(\mathit{Sch}/S)_{\mathit{fppf}}$. Assume $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{Z}$ satisfy (RS*). Let A be an S-algebra and let w be an object of $\mathcal{W} = \mathcal{X} \times_{\mathcal{Y}} \mathcal{Z}$ over A. Denote x, y, z the objects of $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ you get from w. For any A-module M there is a 6-term exact sequence*

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathit{Inf}_w(M) & \longrightarrow & \mathit{Inf}_x(M) \oplus \mathit{Inf}_z(M) & \longrightarrow & \mathit{Inf}_y(M) \\ & & & & & \searrow & \\ & & T_w(M) & \longrightarrow & T_x(M) \oplus T_z(M) & \longrightarrow & T_y(M) \end{array}$$

of A-modules.

Proof. By Lemma 18.3 we see that $\mathcal{W}$ satisfies (RS*) and hence $T_w(M)$ and $\mathrm{Inf}_w(M)$ are defined. The horizontal arrows are defined using the functoriality of Lemma 21.1.

Definition of the "boundary" map $\delta : \mathrm{Inf}_y(M) \to T_w(M)$. Choose isomorphisms $p(x) \to y$ and $y \to q(z)$ such that $w = (x, z, p(x) \to y \to q(z))$ in the description of the 2-fibre product of Categories, Lemma 35.7 and more precisely Categories, Lemma 32.3. Let x', y', z', w' denote the trivial deformation of x, y, z, w over $A[M]$. By pullback we get isomorphisms $y' \to p(x')$ and $q(z') \to y'$. An element $\alpha \in \mathrm{Inf}_y(M)$ is the same thing as an automorphism $\alpha : y' \to y'$ over $A[M]$ which restricts to the identity on y over A. Thus setting

$$\delta(\alpha) = (x', z', p(x') \to y' \xrightarrow{\alpha} y' \to q(z'))$$

we obtain an object of $T_w(M)$. This is a map of A-modules by Formal Deformation Theory, Lemma 11.5.

The rest of the proof is exactly the same as the proof of Formal Deformation Theory, Lemma 20.1. □

0D18 **Remark 21.7** (Compatibility with previous tangent spaces)**.** Let S be a locally Noetherian scheme. Let $\mathcal{X}$ be a category fibred in groupoids over $(Sch/S)_{fppf}$. Assume $\mathcal{X}$ has (RS*). Let k be a field of finite type over S and let x_0 be an object of $\mathcal{X}$ over $\mathrm{Spec}(k)$. Then we have equalities of k-vector spaces

$$T\mathcal{F}_{\mathcal{X},k,x_0} = T_{x_0}(k) \quad \text{and} \quad \mathrm{Inf}(\mathcal{F}_{\mathcal{X},k,x_0}) = \mathrm{Inf}_{x_0}(k)$$

where the spaces on the left hand side of the equality signs are given in (8.0.1) and (8.0.2) and the spaces on the right hand side are given by Lemma 21.2.

07YC **Remark 21.8** (Canonical element)**.** Assumptions and notation as in Lemma 21.2. Choose an affine open $\mathrm{Spec}(\Lambda) \subset S$ such that $\mathrm{Spec}(A) \to S$ corresponds to a ring map $\Lambda \to A$. Consider the ring map

$$A \longrightarrow A[\Omega_{A/\Lambda}], \quad a \longmapsto (a, \mathrm{d}_{A/\Lambda}(a))$$

Pulling back x along the corresponding morphism $\mathrm{Spec}(A[\Omega_{A/\Lambda}]) \to \mathrm{Spec}(A)$ we obtain a deformation x_{can} of x over $A[\Omega_{A/\Lambda}]$. We call this the *canonical element*

$$x_{can} \in T_x(\Omega_{A/\Lambda}) = \mathrm{Lift}(x, A[\Omega_{A/\Lambda}]).$$

Next, assume that Λ is Noetherian and $\Lambda \to A$ is of finite type. Let $k = \kappa(\mathfrak{p})$ be a residue field at a finite type point u_0 of $U = \mathrm{Spec}(A)$. Let $x_0 = x|_{u_0}$. By (RS*) and the fact that $A[k] = A \times_k k[k]$ the space $T_x(k)$ is the tangent space to the deformation functor $\mathcal{F}_{\mathcal{X},k,x_0}$. Via

$$T\mathcal{F}_{U,k,u_0} = \mathrm{Der}_\Lambda(A, k) = \mathrm{Hom}_A(\Omega_{A/\Lambda}, k)$$

(see Formal Deformation Theory, Example 11.11) and functoriality of T_x the canonical element produces the map on tangent spaces induced by the object x over U. Namely, $\theta \in T\mathcal{F}_{U,k,u_0}$ maps to $T_x(\theta)(x_{can})$ in $T_x(k) = T\mathcal{F}_{\mathcal{X},k,x_0}$.

07YD **Remark 21.9** (Canonical automorphism)**.** Let S be a locally Noetherian scheme. Let $\mathcal{X}$ be a category fibred in groupoids over $(Sch/S)_{fppf}$. Assume $\mathcal{X}$ satisfies condition (RS*). Let A be an S-algebra such that $\mathrm{Spec}(A) \to S$ maps into an affine open and let x, y be objects of $\mathcal{X}$ over $\mathrm{Spec}(A)$. Further, let $A \to B$ be a ring

map and let $\alpha : x|_{\mathrm{Spec}(B)} \to y|_{\mathrm{Spec}(B)}$ be a morphism of $\mathcal{X}$ over $\mathrm{Spec}(B)$. Consider the ring map

$$B \longrightarrow B[\Omega_{B/A}], \quad b \longmapsto (b, \mathrm{d}_{B/A}(b))$$

Pulling back α along the corresponding morphism $\mathrm{Spec}(B[\Omega_{B/A}]) \to \mathrm{Spec}(B)$ we obtain a morphism α_{can} between the pullbacks of x and y over $B[\Omega_{B/A}]$. On the other hand, we can pullback α by the morphism $\mathrm{Spec}(B[\Omega_{B/A}]) \to \mathrm{Spec}(B)$ corresponding to the injection of B into the first summand of $B[\Omega_{B/A}]$. By the discussion of Remark 21.4 we can take the difference

$$\varphi(x, y, \alpha) = \alpha_{can} - \alpha|_{\mathrm{Spec}(B[\Omega_{B/A}])} \in \mathrm{Inf}_{x|_{\mathrm{Spec}(B)}}(\Omega_{B/A}).$$

We will call this the *canonical automorphism*. It depends on all the ingredients A, x, y, $A \to B$ and α.

22. Obstruction theories

07YF In this section we describe what an obstruction theory is. Contrary to the spaces of infinitesimal deformations and infinitesimal automorphisms, an obstruction theory is an additional piece of data. The formulation is motivated by the results of Lemma 21.2 and Remark 21.3.

07YG **Definition 22.1.** Let S be a locally Noetherian base. Let $\mathcal{X}$ be a category fibred in groupoids over $(Sch/S)_{fppf}$. An *obstruction theory* is given by the following data

(1) for every S-algebra A such that $\mathrm{Spec}(A) \to S$ maps into an affine open and every object x of $\mathcal{X}$ over $\mathrm{Spec}(A)$ an A-linear functor
$$\mathcal{O}_x : \mathrm{Mod}_A \to \mathrm{Mod}_A$$
of *obstruction modules*,
(2) for (x, A) as in (1), a ring map $A \to B$, $M \in \mathrm{Mod}_A$, $N \in \mathrm{Mod}_B$, and an A-linear map $M \to N$ an induced A-linear map $\mathcal{O}_x(M) \to \mathcal{O}_y(N)$ where $y = x|_{\mathrm{Spec}(B)}$, and
(3) for every deformation situation $(x, A' \to A)$ an *obstruction* element $o_x(A') \in \mathcal{O}_x(I)$ where $I = \mathrm{Ker}(A' \to A)$.

These data are subject to the following conditions

(i) the functoriality maps turn the obstruction modules into a functor from the category of triples (x, A, M) to sets,
(ii) for every morphism of deformation situations $(y, B' \to B) \to (x, A' \to A)$ the element $o_x(A')$ maps to $o_y(B')$, and
(iii) we have
$$\mathrm{Lift}(x, A') \neq \emptyset \Leftrightarrow o_x(A') = 0$$
for every deformation situation $(x, A' \to A)$.

This last condition explains the terminology. The module $\mathcal{O}_x(A')$ is called the *obstruction module*. The element $o_x(A')$ is the *obstruction*. Most obstruction theories have additional properties, and in order to make them useful additional conditions are needed. Moreover, this is just a sample definition, for example in the definition we could consider only deformation situations of finite type over S.

One of the main reasons for introducing obstruction theories is to check openness of versality. An example of this type of result is Lemma 22.2 below. The initial idea to do this is due to Artin, see the papers of Artin mentioned in the introduction. It has been taken up for example in the work by Flenner [Fle81], Hall [Hal17], Hall and

Rydh [HR12], Olsson [Ols06], Olsson and Starr [OS03], and Lieblich [Lie06] (random order of references). Moreover, for particular categories fibred in groupoids, often authors develop a little bit of theory adapted to the problem at hand. We will develop this theory later (insert future reference here).

0CYF **Lemma 22.2.** *Let S be a locally Noetherian scheme. Let $\mathcal{X}$ be a category fibred in groupoids over $(Sch/S)_{fppf}$. Assume* This is [Hal17, Theorem 4.4]

(1) *$\Delta : \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is representable by algebraic spaces,*
(2) *$\mathcal{X}$ has (RS*),*
(3) *$\mathcal{X}$ is limit preserving,*
(4) *there exists an obstruction theory*[10]*,*
(5) *for an object x of $\mathcal{X}$ over $\mathrm{Spec}(A)$ and A-modules M_n, $n \geq 1$ we have*
   (a) *$T_x(\prod M_n) = \prod T_x(M_n)$,*
   (b) *$\mathcal{O}_x(\prod M_n) \to \prod \mathcal{O}_x(M_n)$ is injective.*

Then $\mathcal{X}$ satisfies openness of versality.

Proof. We prove this by verifying condition (4) of Lemma 20.3. Let (ξ_n) and (R_n) be as in Remark 20.2 such that $\mathrm{Ker}(R_m \to R_n)$ is an ideal of square zero for all $m \geq n$. Set $A = R_1$ and $x = \xi_1$. Denote $M_n = \mathrm{Ker}(R_n \to R_1)$. Then M_n is an A-module. Set $R = \lim R_n$. Let

$$\tilde{R} = \{(r_1, r_2, r_3 \ldots) \in \prod R_n \text{ such that all have the same image in } A\}$$

Then $\tilde{R} \to A$ is surjective with kernel $M = \prod M_n$. There is a map $R \to \tilde{R}$ and a map $\tilde{R} \to A[M]$, $(r_1, r_2, r_3, \ldots) \mapsto (r_1, r_2 - r_1, r_3 - r_2, \ldots)$. Together these give a short exact sequence

$$(x, R \to A) \to (x, \tilde{R} \to A) \to (x, A[M])$$

of deformation situations, see Remark 21.5. The associated sequence of kernels $0 \to \lim M_n \to M \to M \to 0$ is the canonical sequence computing the limit of the system of modules (M_n).

Let $o_x(\tilde{R}) \in \mathcal{O}_x(M)$ be the obstruction element. Since we have the lifts ξ_n we see that $o_x(\tilde{R})$ maps to zero in $\mathcal{O}_x(M_n)$. By assumption (5)(b) we see that $o_x(\tilde{R}) = 0$. Choose a lift $\tilde{\xi}$ of x to $\mathrm{Spec}(\tilde{R})$. Let $\tilde{\xi}_n$ be the restriction of $\tilde{\xi}$ to $\mathrm{Spec}(R_n)$. There exists elements $t_n \in T_x(M_n)$ such that $t_n \cdot \tilde{\xi}_n = \xi_n$ by Lemma 21.2 part (2)(b). By assumption (5)(a) we can find $t \in T_x(M)$ mapping to t_n in $T_x(M_n)$. After replacing $\tilde{\xi}$ by $t \cdot \tilde{\xi}$ we find that $\tilde{\xi}$ restricts to ξ_n over $\mathrm{Spec}(R_n)$ for all n. In particular, since ξ_{n+1} restricts to ξ_n over $\mathrm{Spec}(R_n)$, the restriction $\overline{\xi}$ of $\tilde{\xi}$ to $\mathrm{Spec}(A[M])$ has the property that it restricts to the trivial deformation over $\mathrm{Spec}(A[M_n])$ for all n. Hence by assumption (5)(a) we find that $\overline{\xi}$ is the trivial deformation of x. By axiom (RS*) applied to $R = \tilde{R} \times_{A[M]} A$ this implies that $\tilde{\xi}$ is the pullback of a deformation ξ of x over R. This finishes the proof. □

07YH **Example 22.3.** Let $S = \mathrm{Spec}(\Lambda)$ for some Noetherian ring Λ. Let $W \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_W$-module flat over S. Consider the functor

$$F : (Sch/S)^{opp}_{fppf} \longrightarrow Sets, \quad T/S \longrightarrow H^0(W_T, \mathcal{F}_T)$$

[10] Analyzing the proof the reader sees that in fact it suffices to check the functoriality (ii) of obstruction classes in Definition 22.1 for maps $(y, B' \to B) \to (x, A' \to A)$ with $B = A$ and $y = x$.

where $W_T = T \times_S W$ is the base change and $\mathcal{F}_T$ is the pullback of $\mathcal{F}$ to W_T. If $T = \mathrm{Spec}(A)$ we will write $W_T = W_A$, etc. Let $\mathcal{X} \to (Sch/S)_{fppf}$ be the category fibred in groupoids associated to F. Then $\mathcal{X}$ has an obstruction theory. Namely,

(1) given A over Λ and $x \in H^0(W_A, \mathcal{F}_A)$ we set $\mathcal{O}_x(M) = H^1(W_A, \mathcal{F}_A \otimes_A M)$,
(2) given a deformation situation $(x, A' \to A)$ we let $o_x(A') \in \mathcal{O}_x(A)$ be the image of x under the boundary map

$$H^0(W_A, \mathcal{F}_A) \longrightarrow H^1(W_A, \mathcal{F}_A \otimes_A I)$$

coming from the short exact sequence of modules

$$0 \to \mathcal{F}_A \otimes_A I \to \mathcal{F}_{A'} \to \mathcal{F}_A \to 0.$$

We have omitted some details, in particular the construction of the short exact sequence above (it uses that W_A and $W_{A'}$ have the same underlying topological space) and the explanation for why flatness of $\mathcal{F}$ over S implies that the sequence above is short exact.

07YI **Example 22.4** (Key example)**.** Let $S = \mathrm{Spec}(\Lambda)$ for some Noetherian ring Λ. Say $\mathcal{X} = (Sch/X)_{fppf}$ with $X = \mathrm{Spec}(R)$ and $R = \Lambda[x_1, \ldots, x_n]/J$. The naive cotangent complex $NL_{R/\Lambda}$ is (canonically) homotopy equivalent to

$$J/J^2 \longrightarrow \bigoplus\nolimits_{i=1,\ldots,n} R\mathrm{d}x_i,$$

see Algebra, Lemma 134.2. Consider a deformation situation $(x, A' \to A)$. Denote I the kernel of $A' \to A$. The object x corresponds to $(a_1, \ldots, a_n)$ with $a_i \in A$ such that $f(a_1, \ldots, a_n) = 0$ in A for all $f \in J$. Set

$$\begin{aligned} \mathcal{O}_x(A') &= \mathrm{Hom}_R(J/J^2, I)/\mathrm{Hom}_R(R^{\oplus n}, I) \\ &= \mathrm{Ext}^1_R(NL_{R/\Lambda}, I) \\ &= \mathrm{Ext}^1_A(NL_{R/\Lambda} \otimes_R A, I). \end{aligned}$$

Choose lifts $a'_i \in A'$ of a_i in A. Then $o_x(A')$ is the class of the map $J/J^2 \to I$ defined by sending $f \in J$ to $f(a'_1, \ldots, a'_n) \in I$. We omit the verification that $o_x(A')$ is independent of choices. It is clear that if $o_x(A') = 0$ then the map lifts. Finally, functoriality is straightforward. Thus we obtain an obstruction theory. We observe that $o_x(A')$ can be described a bit more canonically as the composition

$$NL_{R/\Lambda} \to NL_{A/\Lambda} \to NL_{A/A'} = I[1]$$

in $D(A)$, see Algebra, Lemma 134.6 for the last identification.

23. Naive obstruction theories

07YJ The title of this section refers to the fact that we will use the naive cotangent complex in this section. Let $(x, A' \to A)$ be a deformation situation for a given category fibred in groupoids over a locally Noetherian scheme S. The key Example 22.4 suggests that any obstruction theory should be closely related to maps in $D(A)$ with target the naive cotangent complex of A. Working this out we find a criterion for versality in Lemma 23.3 which leads to a criterion for openness of versality in Lemma 23.4. We introduce a notion of a naive obstruction theory in Definition 23.5 to try to formalize the notion a bit further.

In the following we will use the naive cotangent complex as defined in Algebra, Section 134. In particular, if $A' \to A$ is a surjection of Λ-algebras with square zero kernel I, then there are maps

$$NL_{A'/\Lambda} \to NL_{A/\Lambda} \to NL_{A/A'}$$

whose composition is homotopy equivalent to zero (see Algebra, Remark 134.5). This doesn't form a distinguished triangle in general as we are using the naive cotangent complex and not the full one. There is a homotopy equivalence $NL_{A/A'} \to I[1]$ (the complex consisting of I placed in degree -1, see Algebra, Lemma 134.6). Finally, note that there is a canonical map $NL_{A/\Lambda} \to \Omega_{A/\Lambda}$.

07YK **Lemma 23.1.** *Let $A \to k$ be a ring map with k a field. Let $E \in D^-(A)$. Then $\mathrm{Ext}^i_A(E, k) = \mathrm{Hom}_k(H^{-i}(E \otimes^{\mathbf{L}} k), k)$.*

Proof. Omitted. Hint: Replace E by a bounded above complex of free A-modules and compute both sides. □

07YL **Lemma 23.2.** *Let $\Lambda \to A \to k$ be finite type ring maps of Noetherian rings with $k = \kappa(\mathfrak{p})$ for some prime $\mathfrak{p}$ of A. Let $\xi : E \to NL_{A/\Lambda}$ be morphism of $D^-(A)$ such that $H^{-1}(\xi \otimes^{\mathbf{L}} k)$ is not surjective. Then there exists a surjection $A' \to A$ of Λ-algebras such that*

(a) *$I = \mathrm{Ker}(A' \to A)$ has square zero and is isomorphic to k as an A-module,*
(b) *$\Omega_{A'/\Lambda} \otimes k = \Omega_{A/\Lambda} \otimes k$, and*
(c) *$E \to NL_{A/A'}$ is zero.*

Proof. Let $f \in A$, $f \notin \mathfrak{p}$. Suppose that $A'' \to A_f$ satisfies (a), (b), (c) for the induced map $E \otimes_A A_f \to NL_{A_f/\Lambda}$, see Algebra, Lemma 134.13. Then we can set $A' = A'' \times_{A_f} A$ and get a solution. Namely, it is clear that $A' \to A$ satisfies (a) because $\mathrm{Ker}(A' \to A) = \mathrm{Ker}(A'' \to A) = I$. Pick $f'' \in A''$ lifting f. Then the localization of A' at (f'', f) is isomorphic to A'' (for example by More on Algebra, Lemma 5.3). Thus (b) and (c) are clear for A' too. In this way we see that we may replace A by the localization A_f (finitely many times). In particular (after such a replacement) we may assume that $\mathfrak{p}$ is a maximal ideal of A, see Morphisms, Lemma 16.1.

Choose a presentation $A = \Lambda[x_1, \ldots, x_n]/J$. Then $NL_{A/\Lambda}$ is (canonically) homotopy equivalent to

$$J/J^2 \longrightarrow \bigoplus\nolimits_{i=1,\ldots,n} A\mathrm{d}x_i,$$

see Algebra, Lemma 134.2. After localizing if necessary (using Nakayama's lemma) we can choose generators $f_1, \ldots, f_m$ of J such that $f_j \otimes 1$ form a basis for $J/J^2 \otimes_A k$. Moreover, after renumbering, we can assume that the images of $\mathrm{d}f_1, \ldots, \mathrm{d}f_r$ form a basis for the image of $J/J^2 \otimes k \to \bigoplus k\mathrm{d}x_i$ and that $\mathrm{d}f_{r+1}, \ldots, \mathrm{d}f_m$ map to zero in $\bigoplus k\mathrm{d}x_i$. With these choices the space

$$H^{-1}(NL_{A/\Lambda} \otimes^{\mathbf{L}}_A k) = H^{-1}(NL_{A/\Lambda} \otimes_A k)$$

has basis $f_{r+1} \otimes 1, \ldots, f_m \otimes 1$. Changing basis once again we may assume that the image of $H^{-1}(\xi \otimes^{\mathbf{L}} k)$ is contained in the k-span of $f_{r+1} \otimes 1, \ldots, f_{m-1} \otimes 1$. Set

$$A' = \Lambda[x_1, \ldots, x_n]/(f_1, \ldots, f_{m-1}, \mathfrak{p}f_m)$$

By construction $A' \to A$ satisfies (a). Since $\mathrm{d}f_m$ maps to zero in $\bigoplus k\mathrm{d}x_i$ we see that (b) holds. Finally, by construction the induced map $E \to NL_{A/A'} = I[1]$ induces

the zero map $H^{-1}(E \otimes^{\mathbf{L}}_A k) \to I \otimes_A k$. By Lemma 23.1 we see that the composition is zero. □

The following lemma is our key technical result.

07YM **Lemma 23.3.** *Let S be a locally Noetherian scheme. Let $\mathcal{X}$ be a category fibred in groupoids over $(Sch/S)_{fppf}$ satisfying (RS*). Let $U = \mathrm{Spec}(A)$ be an affine scheme of finite type over S which maps into an affine open $\mathrm{Spec}(\Lambda)$. Let x be an object of $\mathcal{X}$ over U. Let $\xi : E \to NL_{A/\Lambda}$ be a morphism of $D^-(A)$. Assume*

(i) *for every deformation situation $(x, A' \to A)$ we have: x lifts to $\mathrm{Spec}(A')$ if and only if $E \to NL_{A/\Lambda} \to NL_{A/A'}$ is zero, and*
(ii) *there is an isomorphism of functors $T_x(-) \to \mathrm{Ext}^0_A(E, -)$ such that $E \to NL_{A/\Lambda} \to \Omega^1_{A/\Lambda}$ corresponds to the canonical element (see Remark 21.8).*

Let $u_0 \in U$ be a finite type point with residue field $k = \kappa(u_0)$. Consider the following statements

(1) *x is versal at u_0, and*
(2) *$\xi : E \to NL_{A/\Lambda}$ induces a surjection $H^{-1}(E \otimes^{\mathbf{L}}_A k) \to H^{-1}(NL_{A/\Lambda} \otimes^{\mathbf{L}}_A k)$ and an injection $H^0(E \otimes^{\mathbf{L}}_A k) \to H^0(NL_{A/\Lambda} \otimes^{\mathbf{L}}_A k)$.*

Then we always have (2) $\Rightarrow$ (1) and we have (1) $\Rightarrow$ (2) if u_0 is a closed point.

Proof. Let $\mathfrak{p} = \mathrm{Ker}(A \to k)$ be the prime corresponding to u_0.

Assume that x versal at u_0 and that u_0 is a closed point of U. If $H^{-1}(\xi \otimes^{\mathbf{L}}_A k)$ is not surjective, then let $A' \to A$ be an extension with kernel I as in Lemma 23.2. Because u_0 is a closed point, we see that I is a finite A-module, hence that A' is a finite type Λ-algebra (this fails if u_0 is not closed). In particular A' is Noetherian. By property (c) for A' and (i) for ξ we see that x lifts to an object x' over A'. Let $\mathfrak{p}' \subset A'$ be kernel of the surjective map to k. By Artin-Rees (Algebra, Lemma 51.2) there exists an $n > 1$ such that $(\mathfrak{p}')^n \cap I = 0$. Then we see that

$$B' = A'/(\mathfrak{p}')^n \longrightarrow A/\mathfrak{p}^n = B$$

is a small, essential extension of local Artinian rings, see Formal Deformation Theory, Lemma 3.12. On the other hand, as x is versal at u_0 and as $x'|_{\mathrm{Spec}(B')}$ is a lift of $x|_{\mathrm{Spec}(B)}$, there exists an integer $m \geq n$ and a map $q : A/\mathfrak{p}^m \to B'$ such that the composition $A/\mathfrak{p}^m \to B' \to B$ is the quotient map. Since the maximal ideal of B' has nth power equal to zero, this q factors through B which contradicts the fact that $B' \to B$ is an essential surjection. This contradiction shows that $H^{-1}(\xi \otimes^{\mathbf{L}}_A k)$ is surjective.

Assume that x versal at u_0. By Lemma 23.1 the map $H^0(\xi \otimes^{\mathbf{L}}_A k)$ is dual to the map $\mathrm{Ext}^0_A(NL_{A/\Lambda}, k) \to \mathrm{Ext}^0_A(E, k)$. Note that

$$\mathrm{Ext}^0_A(NL_{A/\Lambda}, k) = \mathrm{Der}_\Lambda(A, k) \quad \text{and} \quad T_x(k) = \mathrm{Ext}^0_A(E, k)$$

Condition (ii) assures us the map $\mathrm{Ext}^0_A(NL_{A/\Lambda}, k) \to \mathrm{Ext}^0_A(E, k)$ sends a tangent vector θ to U at u_0 to the corresponding infinitesimal deformation of x_0, see Remark 21.8. Hence if x is versal, then this map is surjective, see Formal Deformation Theory, Lemma 13.2. Hence $H^0(\xi \otimes^{\mathbf{L}}_A k)$ is injective. This finishes the proof of (1) $\Rightarrow$ (2) in case u_0 is a closed point.

For the rest of the proof assume $H^{-1}(E \otimes^{\mathbf{L}}_A k) \to H^{-1}(NL_{A/\Lambda} \otimes^{\mathbf{L}}_A k)$ is surjective and $H^0(E \otimes^{\mathbf{L}}_A k) \to H^0(NL_{A/\Lambda} \otimes^{\mathbf{L}}_A k)$ injective. Set $R = A^\wedge_\mathfrak{p}$ and let η be the

formal object over R associated to $x|_{\mathrm{Spec}(R)}$. The map $d\underline{\eta}$ on tangent spaces is surjective because it is identified with the dual of the injective map $H^0(E \otimes^{\mathbf{L}}_A k) \to H^0(NL_{A/\Lambda} \otimes^{\mathbf{L}}_A k)$ (see previous paragraph). According to Formal Deformation Theory, Lemma 13.2 it suffices to prove the following: Let $C' \to C$ be a small extension of finite type Artinian local Λ-algebras with residue field k. Let $R \to C$ be a Λ-algebra map compatible with identifications of residue fields. Let $y = x|_{\mathrm{Spec}(C)}$ and let y' be a lift of y to C'. To show: we can lift the Λ-algebra map $R \to C$ to $R \to C'$.

Observe that it suffices to lift the Λ-algebra map $A \to C$. Let $I = \mathrm{Ker}(C' \to C)$. Note that I is a 1-dimensional k-vector space. The obstruction *ob* to lifting $A \to C$ is an element of $\mathrm{Ext}^1_A(NL_{A/\Lambda}, I)$, see Example 22.4. By Lemma 23.1 and our assumption the map ξ induces an injection

$$\mathrm{Ext}^1_A(NL_{A/\Lambda}, I) \longrightarrow \mathrm{Ext}^1_A(E, I)$$

By the construction of *ob* and (i) the image of *ob* in $\mathrm{Ext}^1_A(E, I)$ is the obstruction to lifting x to $A \times_C C'$. By (RS*) the fact that y/C lifts to y'/C' implies that x lifts to $A \times_C C'$. Hence $ob = 0$ and we are done. □

The key lemma above allows us to conclude that we have openness of versality in some cases.

07YN **Lemma 23.4.** *Let S be a locally Noetherian scheme. Let $\mathcal{X}$ be a category fibred in groupoids over $(Sch/S)_{fppf}$ satisfying (RS*). Let $U = \mathrm{Spec}(A)$ be an affine scheme of finite type over S which maps into an affine open $\mathrm{Spec}(\Lambda)$. Let x be an object of $\mathcal{X}$ over U. Let $\xi : E \to NL_{A/\Lambda}$ be a morphism of $D^-(A)$. Assume*

(i) *for every deformation situation $(x, A' \to A)$ we have: x lifts to $\mathrm{Spec}(A')$ if and only if $E \to NL_{A/\Lambda} \to NL_{A/A'}$ is zero,*
(ii) *there is an isomorphism of functors $T_x(-) \to \mathrm{Ext}^0_A(E, -)$ such that $E \to NL_{A/\Lambda} \to \Omega^1_{A/\Lambda}$ corresponds to the canonical element (see Remark 21.8),*
(iii) *the cohomology groups of E are finite A-modules.*

If x is versal at a closed point $u_0 \in U$, then there exists an open neighbourhood $u_0 \in U' \subset U$ such that x is versal at every finite type point of U'.

Proof. Let C be the cone of ξ so that we have a distinguished triangle

$$E \to NL_{A/\Lambda} \to C \to E[1]$$

in $D^-(A)$. By Lemma 23.3 the assumption that x is versal at u_0 implies that $H^{-1}(C \otimes^{\mathbf{L}} k) = 0$. By More on Algebra, Lemma 78.4 there exists an $f \in A$ not contained in the prime corresponding to u_0 such that $H^{-1}(C \otimes^{\mathbf{L}}_A M) = 0$ for any A_f-module M. Using Lemma 23.3 again we see that we have versality for all finite type points of the open $D(f) \subset U$. □

The technical lemmas above suggest the following definition.

07YP **Definition 23.5.** Let S be a locally Noetherian base. Let $\mathcal{X}$ be a category fibred in groupoids over $(Sch/S)_{fppf}$. Assume that $\mathcal{X}$ satisfies (RS*). A *naive obstruction theory* is given by the following data

07YQ (1) for every S-algebra A such that $\mathrm{Spec}(A) \to S$ maps into an affine open $\mathrm{Spec}(\Lambda) \subset S$ and every object x of $\mathcal{X}$ over $\mathrm{Spec}(A)$ we are given an object $E_x \in D^-(A)$ and a map $\xi_x : E \to NL_{A/\Lambda}$,

07YR (2) given (x, A) as in (1) there are transformations of functors

$$\mathrm{Inf}_x(-) \to \mathrm{Ext}_A^{-1}(E_x, -) \quad \text{and} \quad T_x(-) \to \mathrm{Ext}_A^0(E_x, -)$$

07YS (3) for (x, A) as in (1) and a ring map $A \to B$ setting $y = x|_{\mathrm{Spec}(B)}$ there is a functoriality map $E_x \to E_y$ in $D(A)$.

These data are subject to the following conditions

(i) in the situation of (3) the diagram

$$\begin{array}{ccc} E_y & \xrightarrow[\xi_y]{} & NL_{B/\Lambda} \\ \uparrow & & \uparrow \\ E_x & \xrightarrow{\xi_x} & NL_{A/\Lambda} \end{array}$$

is commutative in $D(A)$,

(ii) given (x, A) as in (1) and $A \to B \to C$ setting $y = x|_{\mathrm{Spec}(B)}$ and $z = x|_{\mathrm{Spec}(C)}$ the composition of the functoriality maps $E_x \to E_y$ and $E_y \to E_z$ is the functoriality map $E_x \to E_z$,

(iii) the maps of (2) are isomorphisms compatible with the functoriality maps and the maps of Remark 21.3,

(iv) the composition $E_x \to NL_{A/\Lambda} \to \Omega_{A/\Lambda}$ corresponds to the canonical element of $T_x(\Omega_{A/\Lambda}) = \mathrm{Ext}^0(E_x, \Omega_{A/\Lambda})$, see Remark 21.8,

(v) given a deformation situation $(x, A' \to A)$ with $I = \mathrm{Ker}(A' \to A)$ the composition $E_x \to NL_{A/\Lambda} \to NL_{A/A'}$ is zero in

$$\mathrm{Hom}_A(E_x, NL_{A/\Lambda}) = \mathrm{Ext}_A^0(E_x, NL_{A/A'}) = \mathrm{Ext}_A^1(E_x, I)$$

if and only if x lifts to A'.

Thus we see in particular that we obtain an obstruction theory as in Section 22 by setting $\mathcal{O}_x(-) = \mathrm{Ext}_A^1(E_x, -)$.

07YT **Lemma 23.6.** *Let S and $\mathcal{X}$ be as in Definition 23.5 and let $\mathcal{X}$ be endowed with a naive obstruction theory. Let $A \to B$ and $y \to x$ be as in (3). Let k be a B-algebra which is a field. Then the functoriality map $E_x \to E_y$ induces bijections*

$$H^i(E_x \otimes_A^{\mathbf{L}} k) \to H^i(E_y \otimes_B^{\mathbf{L}} k)$$

for $i = 0, 1$.

Proof. Let $z = x|_{\mathrm{Spec}(k)}$. Then (RS*) implies that

$$\mathit{Lift}(x, A[k]) = \mathit{Lift}(z, k[k]) \quad \text{and} \quad \mathit{Lift}(y, B[k]) = \mathit{Lift}(z, k[k])$$

because $A[k] = A \times_k k[k]$ and $B[k] = B \times_k k[k]$. Hence the properties of a naive obstruction theory imply that the functoriality map $E_x \to E_y$ induces bijections $\mathrm{Ext}_A^i(E_x, k) \to \mathrm{Ext}_B^i(E_y, k)$ for $i = -1, 0$. By Lemma 23.1 our maps $H^i(E_x \otimes_A^{\mathbf{L}} k) \to H^i(E_y \otimes_B^{\mathbf{L}} k)$, $i = 0, 1$ induce isomorphisms on dual vector spaces hence are isomorphisms. □

07YU **Lemma 23.7.** *Let S be a locally Noetherian scheme. Let $p : \mathcal{X} \to (Sch/S)_{fppf}^{opp}$ be a category fibred in groupoids. Assume that $\mathcal{X}$ satisfies (RS*) and that $\mathcal{X}$ has a naive obstruction theory. Then openness of versality holds for $\mathcal{X}$ provided the complexes E_x of Definition 23.5 have finitely generated cohomology groups for pairs (A, x) where A is of finite type over S.*

Proof. Let U be a scheme locally of finite type over S, let x be an object of $\mathcal{X}$ over U, and let u_0 be a finite type point of U such that x is versal at u_0. We may first shrink U to an affine scheme such that u_0 is a closed point and such that $U \to S$ maps into an affine open $\mathrm{Spec}(\Lambda)$. Say $U = \mathrm{Spec}(A)$. Let $\xi_x : E_x \to NL_{A/\Lambda}$ be the obstruction map. At this point we may apply Lemma 23.4 to conclude. □

24. A dual notion

07YV Let $(x, A' \to A)$ be a deformation situation for a given category $\mathcal{X}$ fibred in groupoids over a locally Noetherian scheme S. Assume $\mathcal{X}$ has an obstruction theory, see Definition 22.1. In practice one often has a complex $K^\bullet$ of A-modules and isomorphisms of functors

$$\mathrm{Inf}_x(-) \to H^0(K^\bullet \otimes^{\mathbf{L}}_A -), \quad T_x(-) \to H^1(K^\bullet \otimes^{\mathbf{L}}_A -), \quad \mathcal{O}_x(-) \to H^2(K^\bullet \otimes^{\mathbf{L}}_A -)$$

In this section we formalize this a little bit and show how this leads to a verification of openness of versality in some cases.

07YW **Example 24.1.** Let $\Lambda, S, W, \mathcal{F}$ be as in Example 22.3. Assume that $W \to S$ is proper and $\mathcal{F}$ coherent. By Cohomology of Schemes, Remark 22.2 there exists a finite complex of finite projective Λ-modules $N^\bullet$ which universally computes the cohomology of $\mathcal{F}$. In particular the obstruction spaces from Example 22.3 are $\mathcal{O}_x(M) = H^1(N^\bullet \otimes_\Lambda M)$. Hence with $K^\bullet = N^\bullet \otimes_\Lambda A[-1]$ we see that $\mathcal{O}_x(M) = H^2(K^\bullet \otimes^{\mathbf{L}}_A M)$.

07YX **Situation 24.2.** Let S be a locally Noetherian scheme. Let $\mathcal{X}$ be a category fibred in groupoids over $(Sch/S)_{fppf}$. Assume that $\mathcal{X}$ has (RS*) so that we can speak of the functor $T_x(-)$, see Lemma 21.2. Let $U = \mathrm{Spec}(A)$ be an affine scheme of finite type over S which maps into an affine open $\mathrm{Spec}(\Lambda)$. Let x be an object of $\mathcal{X}$ over U. Assume we are given

(1) a complex of A-modules $K^\bullet$,
(2) a transformation of functors $T_x(-) \to H^1(K^\bullet \otimes^{\mathbf{L}}_A -)$,
(3) for every deformation situation $(x, A' \to A)$ with kernel $I = \mathrm{Ker}(A' \to A)$ an element $o_x(A') \in H^2(K^\bullet \otimes^{\mathbf{L}}_A I)$

satisfying the following (minimal) conditions

(i) the transformation $T_x(-) \to H^1(K^\bullet \otimes^{\mathbf{L}}_A -)$ is an isomorphism,
(ii) given a morphism $(x, A'' \to A) \to (x, A' \to A)$ of deformation situations the element $o_x(A')$ maps to the element $o_x(A'')$ via the map $H^2(K^\bullet \otimes^{\mathbf{L}}_A I) \to H^2(K^\bullet \otimes^{\mathbf{L}}_A I')$ where $I' = \mathrm{Ker}(A'' \to A)$, and
(iii) x lifts to an object over $\mathrm{Spec}(A')$ if and only if $o_x(A') = 0$.

It is possible to incorporate infinitesimal automorphisms as well, but we refrain from doing so in order to get the sharpest possible result.

In Situation 24.2 an important role will be played by $K^\bullet \otimes^{\mathbf{L}}_A NL_{A/\Lambda}$. Suppose we are given an element $\xi \in H^1(K^\bullet \otimes^{\mathbf{L}}_A NL_{A/\Lambda})$. Then (1) for any surjection $A' \to A$ of Λ-algebras with kernel I of square zero the canonical map $NL_{A/\Lambda} \to NL_{A/A'} = I[1]$ sends ξ to an element $\xi_{A'} \in H^2(K^\bullet \otimes^{\mathbf{L}}_A I)$ and (2) the map $NL_{A/\Lambda} \to \Omega_{A/\Lambda}$ sends ξ to an element ξ_{can} of $H^1(K^\bullet \otimes^{\mathbf{L}}_A \Omega_{A/\Lambda})$.

07YY **Lemma 24.3.** *In Situation 24.2. Assume furthermore that*

(iv) *given a short exact sequence of deformation situations as in Remark 21.5 and a lift* $x'_2 \in Lift(x, A'_2)$ *then* $o_x(A'_3) \in H^2(K^\bullet \otimes^{\mathbf{L}}_A I_3)$ *equals* $\partial\theta$ *where* $\theta \in H^1(K^\bullet \otimes^{\mathbf{L}}_A I_1)$ *is the element corresponding to* $x'_2|_{\mathrm{Spec}(A'_1)}$ *via* $A'_1 = A[I_1]$ *and the given map* $T_x(-) \to H^1(K^\bullet \otimes^{\mathbf{L}}_A -)$.

In this case there exists an element $\xi \in H^1(K^\bullet \otimes^{\mathbf{L}}_A NL_{A/\Lambda})$ *such that*

(1) *for every deformation situation* $(x, A' \to A)$ *we have* $\xi_{A'} = o_x(A')$, *and*
(2) ξ_{can} *matches the canonical element of Remark 21.8 via the given transformation* $T_x(-) \to H^1(K^\bullet \otimes^{\mathbf{L}}_A -)$.

Proof. Choose a $\alpha : \Lambda[x_1, \ldots, x_n] \to A$ with kernel J. Write $P = \Lambda[x_1, \ldots, x_n]$. In the rest of this proof we work with

$$NL(\alpha) = (J/J^2 \longrightarrow \bigoplus A\mathrm{d}x_i)$$

which is permissible by Algebra, Lemma 134.2 and More on Algebra, Lemma 59.2. Consider the element $o_x(P/J^2) \in H^2(K^\bullet \otimes^{\mathbf{L}}_A J/J^2)$ and consider the quotient

$$C = (P/J^2 \times \bigoplus A\mathrm{d}x_i)/(J/J^2)$$

where J/J^2 is embedded diagonally. Note that $C \to A$ is a surjection with kernel $\bigoplus A\mathrm{d}x_i$. Moreover there is a section $A \to C$ to $C \to A$ given by mapping the class of $f \in P$ to the class of $(f, \mathrm{d}f)$ in the pushout. For later use, denote x_C the pullback of x along the corresponding morphism $\mathrm{Spec}(C) \to \mathrm{Spec}(A)$. Thus we see that $o_x(C) = 0$. We conclude that $o_x(P/J^2)$ maps to zero in $H^2(K^\bullet \otimes^{\mathbf{L}}_A \bigoplus A\mathrm{d}x_i)$. It follows that there exists some element $\xi \in H^1(K^\bullet \otimes^{\mathbf{L}}_A NL(\alpha))$ mapping to $o_x(P/J^2)$.

Note that for any deformation situation $(x, A' \to A)$ there exists a Λ-algebra map $P/J^2 \to A'$ compatible with the augmentations to A. Hence the element ξ satisfies the first property of the lemma by construction and property (ii) of Situation 24.2.

Note that our choice of ξ was well defined up to the choice of an element of $H^1(K^\bullet \otimes^{\mathbf{L}}_A \bigoplus A\mathrm{d}x_i)$. We will show that after modifying ξ by an element of the aforementioned group we can arrange it so that the second assertion of the lemma is true. Let $C' \subset C$ be the image of P/J^2 under the Λ-algebra map $P/J^2 \to C$ (inclusion of first factor). Observe that $\mathrm{Ker}(C' \to A) = \mathrm{Im}(J/J^2 \to \bigoplus A\mathrm{d}x_i)$. Set $\overline{C} = A[\Omega_{A/\Lambda}]$. The map $P/J^2 \times \bigoplus A\mathrm{d}x_i \to \overline{C}$, $(f, \sum f_i \mathrm{d}x_i) \mapsto (f \bmod J, \sum f_i \mathrm{d}x_i)$ factors through a surjective map $C \to \overline{C}$. Then

$$(x, \overline{C} \to A) \to (x, C \to A) \to (x, C' \to A)$$

is a short exact sequence of deformation situations. The associated splitting $\overline{C} = A[\Omega_{A/\Lambda}]$ (from Remark 21.5) equals the given splitting above. Moreover, the section $A \to C$ composed with the map $C \to \overline{C}$ is the map $(1, \mathrm{d}) : A \to A[\Omega_{A/\Lambda}]$ of Remark 21.8. Thus x_C restricts to the canonical element x_{can} of $T_x(\Omega_{A/\Lambda}) = \mathrm{Lift}(x, A[\Omega_{A/\Lambda}])$. By condition (iv) we conclude that $o_x(P/J^2)$ maps to ∂x_{can} in

$$H^1(K^\bullet \otimes^{\mathbf{L}}_A \mathrm{Im}(J/J^2 \to \bigoplus A\mathrm{d}x_i))$$

By construction ξ maps to $o_x(P/J^2)$. It follows that x_{can} and ξ_{can} map to the same element in the displayed group which means (by the long exact cohomology sequence) that they differ by an element of $H^1(K^\bullet \otimes^{\mathbf{L}}_A \bigoplus A\mathrm{d}x_i)$ as desired. □

07YZ **Lemma 24.4.** *In Situation 24.2 assume that (iv) of Lemma 24.3 holds and that $K^\bullet$ is a perfect object of $D(A)$. In this case, if x is versal at a closed point $u_0 \in U$ then there exists an open neighbourhood $u_0 \in U' \subset U$ such that x is versal at every finite type point of U'.*

Proof. We may assume that $K^\bullet$ is a finite complex of finite projective A-modules. Thus the derived tensor product with $K^\bullet$ is the same as simply tensoring with $K^\bullet$. Let $E^\bullet$ be the dual perfect complex to $K^\bullet$, see More on Algebra, Lemma 76.15. (So $E^n = \mathrm{Hom}_A(K^{-n}, A)$ with differentials the transpose of the differentials of $K^\bullet$.) Let $E \in D^-(A)$ denote the object represented by the complex $E^\bullet[-1]$. Let $\xi \in H^1(\mathrm{Tot}(K^\bullet \otimes_A NL_{A/\Lambda}))$ be the element constructed in Lemma 24.3 and denote $\xi : E = E^\bullet[-1] \to NL_{A/\Lambda}$ the corresponding map (loc.cit.). We claim that the pair (E, ξ) satisfies all the assumptions of Lemma 23.4 which finishes the proof.

Namely, assumption (i) of Lemma 23.4 follows from conclusion (1) of Lemma 24.3 and the fact that $H^2(K^\bullet \otimes^{\mathbf{L}}_A -) = \mathrm{Ext}^1(E, -)$ by loc.cit. Assumption (ii) of Lemma 23.4 follows from conclusion (2) of Lemma 24.3 and the fact that $H^1(K^\bullet \otimes^{\mathbf{L}}_A -) = \mathrm{Ext}^0(E, -)$ by loc.cit. Assumption (iii) of Lemma 23.4 is clear. □

25. Limit preserving functors on Noetherian schemes

0GE1 It is sometimes convenient to consider functors or stacks defined only on the full subcategory of (locally) Noetherian schemes. In this section we discuss this in the case of algebraic spaces.

Let S be a locally Noetherian scheme. Let us be a bit pedantic in order to line up our categories correctly; people who are ignoring set theoretical issues can just replace the sets of schemes we choose by the collection of all schemes in what follows. As in Topologies, Remark 11.1 we choose a category Sch_α of schemes containing S such that we obtain big sites $(Sch/S)_{Zar}$, $(Sch/S)_{étale}$, $(Sch/S)_{smooth}$, $(Sch/S)_{syntomic}$, and $(Sch/S)_{fppf}$ all with the same underlying category Sch_α/S. Denote

$$Noetherian_\alpha \subset Sch_\alpha$$

the full subcategory consisting of locally Noetherian schemes. This determines a full subcategory

$$Noetherian_\alpha/S \subset Sch_\alpha/S$$

For $\tau \in \{Zariski, étale, smooth, syntomic, fppf\}$ we have

(1) if $f : X \to Y$ is a morphism of Sch_α/S with Y in $Noetherian_\alpha/S$ and f locally of finite type, then X is in $Noetherian_\alpha/S$,
(2) for morphisms $f : X \to Y$ and $g : Z \to Y$ of $Noetherian_\alpha/S$ with f locally of finite type the fibre product $X \times_Y Z$ in $Noetherian_\alpha/S$ exists and agrees with the fibre product in Sch_α/S,
(3) if $\{X_i \to X\}_{i \in I}$ is a covering of $(Sch/S)_\tau$ and X is in $Noetherian_\alpha/S$, then the objects X_i are in $Noetherian_\alpha/S$
(4) the category $Noetherian_\alpha/S$ endowed with the set of coverings of $(Sch/S)_\tau$ whose objects are in $Noetherian_\alpha/S$ is a site we will denote $(Noetherian/S)_\tau$,
(5) the inclusion functor $(Noetherian/S)_\tau \to (Sch/S)_\tau$ is fully faithful, continuous, and cocontinuous.

By Sites, Lemmas 21.1 and 21.5 we obtain a morphism of topoi

$$g_\tau : Sh((Noetherian/S)_\tau) \longrightarrow Sh((Sch/S)_\tau)$$

whose pullback functor is the restriction of sheaves along the inclusion functor $(Noetherian/S)_\tau \to (Sch/S)_\tau$.

0GE2 **Remark 25.1** (Warning)**.** The site $(Noetherian/S)_\tau$ does not have fibre products. Hence we have to be careful in working with sheaves. For example, the continuous inclusion functor $(Noetherian/S)_\tau \to (Sch/S)_\tau$ does not define a morphism of sites. See Examples, Section 61 for an example in case $\tau = fppf$.

Let $F : (Noetherian/S)_\tau^{opp} \to Sets$ be a functor. We say F is *limit preserving* if for any directed limit of affine schemes $X = \lim X_i$ of $(Noetherian/S)_\tau$ we have $F(X) = \operatorname{colim} F(X_i)$.

0GE3 **Lemma 25.2.** *Let $\tau \in \{Zariski, étale, smooth, syntomic, fppf\}$. Restricting along the inclusion functor $(Noetherian/S)_\tau \to (Sch/S)_\tau$ defines an equivalence of categories between*

(1) *the category of limit preserving sheaves on $(Sch/S)_\tau$ and*
(2) *the category of limit preserving sheaves on $(Noetherian/S)_\tau$*

Proof. Let $F : (Noetherian/S)_\tau^{opp} \to Sets$ be a functor which is both limit preserving and a sheaf. By Topologies, Lemmas 13.1 and 13.3 there exists a unique functor $F' : (Sch/S)_\tau^{opp} \to Sets$ which is limit preserving, a sheaf, and restricts to F. In fact, the construction of F' in Topologies, Lemma 13.1 is functorial in F and this construction is a quasi-inverse to restriction. Some details omitted. □

0GE4 **Lemma 25.3.** *Let X be an object of $(Noetherian/S)_\tau$. If the functor of points $h_X : (Noetherian/S)_\tau^{opp} \to Sets$ is limit preserving, then X is locally of finite presentation over S.*

Proof. Let $V \subset X$ be an affine open subscheme which maps into an affine open $U \subset S$. We may write $V = \lim V_i$ as a directed limit of affine schemes V_i of finite presentation over U, see Algebra, Lemma 127.2. By assumption, the arrow $V \to X$ factors as $V \to V_i \to X$ for some i. After increasing i we may assume $V_i \to X$ factors through V as the inverse image of $V \subset X$ in V_i eventually becomes equal to V_i by Limits, Lemma 4.11. Then the identity morphism $V \to V$ factors through V_i for some i in the category of schemes over U. Thus $V \to U$ is of finite presentation; the corresponding algebra fact is that if B is an A-algebra such that $\mathrm{id} : B \to B$ factors through a finitely presented A-algebra, then B is of finite presentation over A (nice exercise). Hence X is locally of finite presentation over S. □

The following lemma has a variant for transformations representable by algebraic spaces.

0GE5 **Lemma 25.4.** *Let $\tau \in \{Zariski, étale, smooth, syntomic, fppf\}$. Let $F', G' : (Sch/S)_\tau^{opp} \to Sets$ be limit preserving and sheaves. Let $a' : F' \to G'$ be a transformation of functors. Denote $a : F \to G$ the restriction of $a' : F' \to G'$ to $(Noetherian/S)_\tau$. The following are equivalent*

(1) *a' is representable (as a transformation of functors, see Categories, Definition 6.4), and*
(2) *for every object V of $(Noetherian/S)_\tau$ and every map $V \to G$ the fibre product $F \times_G V : (Noetherian/S)_\tau^{opp} \to Sets$ is a representable functor, and*
(3) *same as in (2) but only for V affine finite type over S mapping into an affine open of S.*

Proof. Assume (1). By Limits of Spaces, Lemma 3.4 the transformation a' is limit preserving[11]. Take $\xi : V \to G$ as in (2). Denote $V' = V$ but viewed as an object of $(Sch/S)_\tau$. Since G is the restriction of G' to $(Noetherian/S)_\tau$ we see that $\xi \in G(V)$ corresponds to $\xi' \in G'(V')$. By assumption $V' \times_{\xi', G'} F'$ is representable by a scheme U'. The morphism of schemes $U' \to V'$ corresponding to the projection $V' \times_{\xi', G'} F' \to V'$ is locally of finite presentation by Limits of Spaces, Lemma 3.5 and Limits, Proposition 6.1. Hence U' is a locally Noetherian scheme and therefore U' is isomorphic to an object U of $(Noetherian/S)_\tau$. Then U represents $F \times_G V$ as desired.

The implication (2) $\Rightarrow$ (3) is immediate. Assume (3). We will prove (1). Let T be an object of $(Sch/S)_\tau$ and let $T \to G'$ be a morphism. We have to show the functor $F' \times_{G'} T$ is representable by a scheme X over T. Let $\mathcal{B}$ be the set of affine opens of T which map into an affine open of S. This is a basis for the topology of T. Below we will show that for $W \in \mathcal{B}$ the fibre product $F' \times_{G'} W$ is representable by a scheme X_W over W. If $W_1 \subset W_2$ in $\mathcal{B}$, then we obtain an isomorphism $X_{W_1} \to X_{W_2} \times_{W_2} W_1$ because both X_{W_1} and $X_{W_2} \times_{W_2} W_1$ represent the functor $F' \times_{G'} W_1$. These isomorphisms are canonical and satisfy the cocycle condition mentioned in Constructions, Lemma 2.1. Hence we can glue the schemes X_W to a scheme X over T. Compatibility of the glueing maps with the maps $X_W \to F'$ provide us with a map $X \to F'$. The resulting map $X \to F' \times_{G'} T$ is an isomorphism as we may check this locally on T (as source and target of this arrow are sheaves for the Zariski topology).

Let W be an affine scheme which maps into an affine open $U \subset S$. Let $W \to G'$ be a map. Still assuming (3) we have to show that $F' \times_{G'} W$ is representable by a scheme. We may write $W = \lim V_i'$ as a directed limit of affine schemes V_i' of finite presentation over U, see Algebra, Lemma 127.2. Since V_i' is of finite type over an Noetherian scheme, we see that V_i' is a Noetherian scheme. Denote $V_i = V_i'$ but viewed as an object of $(Noetherian/S)_\tau$. As G' is limit preserving can choose an i and a map $V_i' \to G'$ such that $W \to G'$ is the composition $W \to V_i' \to G'$. Since G is the restriction of G' to $(Noetherian/S)_\tau$ the morphism $V_i' \to G'$ is the same thing as a morphism $V_i \to G$ (see above). By assumption (3) the functor $F \times_G V_i$ is representable by an object X_i of $(Noetherian/S)_\tau$. The functor $F \times_G V_i$ is limit preserving as it is the restriction of $F' \times_{G'} V_i'$ and this functor is limit preserving by Limits of Spaces, Lemma 3.6, the assumption that F' and G' are limit preserving, and Limits, Remark 6.2 which tells us that the functor of points of V_i' is limit preserving. By Lemma 25.3 we conclude that X_i is locally of finite presentation over S. Denote $X_i' = X_i$ but viewed as an object of $(Sch/S)_\tau$. Then we see that $F' \times_{G'} V_i'$ and the functors of points $h_{X_i'}$ are both extensions of $h_{X_i} : (Noetherian/S)_\tau^{opp} \to Sets$ to limit preserving sheaves on $(Sch/S)_\tau$. By the equivalence of categories of Lemma 25.2 we deduce that X_i' represents $F' \times_{G'} V_i'$. Then finally

$$F' \times_{G'} W = F' \times_{G'} V_i' \times_{V_i'} W = X_i' \times_{V_i'} W$$

is representable as desired. $\square$

[11]This makes sense even if $\tau \neq fppf$ as the underlying category of $(Sch/S)_\tau$ equals the underlying category of $(Sch/S)_{fppf}$ and the statement doesn't refer to the topology.

26. Algebraic spaces in the Noetherian setting

0GE6 Let S be a locally Noetherian scheme. Let $(\mathit{Noetherian}/S)_{\mathit{étale}} \subset (\mathit{Sch}/S)_{\mathit{étale}}$ denote the site studied in Section 25. Let $F : (\mathit{Noetherian}/S)^{opp}_{\mathit{étale}} \to \mathit{Sets}$ be a functor, i.e., F is a presheaf on $(\mathit{Noetherian}/S)_{\mathit{étale}}$. In this setting all the axioms [-1], [0], [1], [2], [3], [4], [5] of Section 15 make sense. We will review them one by one and make sure the reader knows exactly what we mean.

Axiom [-1]. This is a set theoretic condition to be ignored by readers who are not interested in set theoretic questions. It makes sense for F since it concerns the evaluation of F on spectra of fields of finite type over S which are objects of $(\mathit{Noetherian}/S)_{\mathit{étale}}$.

Axiom [0]. This is the axiom that F is a sheaf on $(\mathit{Noetherian}/S)^{opp}_{\mathit{étale}}$, i.e., satisfies the sheaf condition for étale coverings.

Axiom [1]. This is the axiom that F is limit preserving as defined in Section 25: for any directed limit of affine schemes $X = \lim X_i$ of $(\mathit{Noetherian}/S)_{\mathit{étale}}$ we have $F(X) = \operatorname{colim} F(X_i)$.

Axiom [2]. This is the axiom that F satisfies the Rim-Schlessinger condition (RS). Looking at the definition of condition (RS) in Definition 5.1 and the discussion in Section 15 we see that this means: given any pushout $Y' = Y \amalg_X X'$ of schemes of finite type over S where Y, X, X' are spectra of Artinian local rings, then

$$F(Y \amalg_X X') \to F(Y) \times_{F(X)} F(X')$$

is a bijection. This condition makes sense as the schemes X, X', Y, and Y' are in $(\text{Noetherian}/S)_{\mathit{étale}}$ since they are of finite type over S.

Axiom [3]. This is the axiom that every tangent space TF_{k,x_0} is finite dimensional. This makes sense as the tangent spaces TF_{k,x_0} are constructed from evaluations of F at $\operatorname{Spec}(k)$ and $\operatorname{Spec}(k[\epsilon])$ with k a field of finite type over S and hence are obtained by evaluating at objects of the category $(\mathit{Noetherian}/S)_{\mathit{étale}}$.

Axiom [4]. This is axiom that the every formal object is effective. Looking at the discussion in Sections 9 and 15 we see that this involves evaluating our functor at Noetherian schemes only and hence this condition makes sense for F.

Axiom [5]. This is the axiom stating that F satisfies openness of versality. Recall that this means the following: Given a scheme U locally of finite type over S, given $x \in F(U)$, and given a finite type point $u_0 \in U$ such that x is versal at u_0, then there exists an open neighbourhood $u_0 \in U' \subset U$ such that x is versal at every finite type point of U'. As before, verifying this only involves evaluating our functor at Noetherian schemes.

0GE7 **Proposition 26.1.** *Let S be a locally Noetherian scheme. Let $F : (\mathit{Noetherian}/S)^{opp}_{\mathit{étale}} \to \mathit{Sets}$ be a functor. Assume that*

1. *$\Delta : F \to F \times F$ is representable (as a transformation of functors, see Categories, Definition 6.4),*
2. *F satisfies axioms [-1], [0], [1], [2], [3], [4], [5] (see above), and*
3. *$\mathcal{O}_{S,s}$ is a G-ring for all finite type points s of S.*

Then there exists a unique algebraic space $F' : (\mathit{Sch}/S)^{opp}_{fppf} \to \mathit{Sets}$ whose restriction to $(\mathit{Noetherian}/S)_{\mathit{étale}}$ is F (see proof for elucidation).

Proof. Recall that the sites $(Sch/S)_{fppf}$ and $(Sch/S)_{étale}$ have the same underlying category, see discussion in Section 25. Similarly the sites $(Noetherian/S)_{étale}$ and $(Noetherian/S)_{fppf}$ have the same underlying categories. By axioms [0] and [1] the functor F is a sheaf and limit preserving. Let $F' : (Sch/S)^{opp}_{étale} \to Sets$ be the unique extension of F which is a sheaf (for the étale topology) and which is limit preserving, see Lemma 25.2. Then F' satisfies axioms [0] and [1] as given in Section 15. By Lemma 25.4 we see that $\Delta' : F' \to F' \times F'$ is representable (by schemes). On the other hand, it is immediately clear that F' satisfies axioms [-1], [2], [3], [4], [5] of Section 15 as each of these involves only evaluating F' at objects of $(Noetherian/S)_{étale}$ and we've assumed the corresponding conditions for F. Whence F' is an algebraic space by Proposition 16.1. □

27. Artin's theorem on contractions

0GH7 In this section we will freely use the language of formal algebraic spaces, see Formal Spaces, Section 1. Artin's theorem on contractions is one of the two main theorems of Artin's paper [Art70]; the first one is his theorem on dilatations which we stated and proved in Algebraization of Formal Spaces, Section 29.

0GH8 **Situation 27.1.** Let S be a locally Noetherian scheme. Let X' be an algebraic space locally of finite type over S. Let $T' \subset |X'|$ be a closed subset. Let $U' \subset X'$ be the open subspace with $|U'| = |X'| \setminus T'$. Let W be a locally Noetherian formal algebraic space over S with W_{red} locally of finite type over S. Finally, we let

$$g : X'_{/T'} \longrightarrow W$$

be a formal modification, see Algebraization of Formal Spaces, Definition 24.1. Recall that $X'_{/T'}$ denotes the formal completion of X' along T', see Formal Spaces, Section 14.

In the situation above our goal is to prove that there exists a proper morphism $f : X' \to X$ of algebraic spaces over S, a closed subset $T \subset |X|$, and an isomorphism $a : X_{/T} \to W$ of formal algebraic spaces such that

(1) T' is the inverse image of T by $|f| : |X'| \to |X|$,
(2) $f : X' \to X$ maps U' isomorphically to an open subspace U of X, and
(3) $g = a \circ f_{/T}$ where $f_{/T} : X'_{/T'} \to X_{/T}$ is the induced morphism.

Let us say that $(f : X' \to X, T, a)$ is a *solution*.

We will follow Artin's strategy by constructing a functor F on the category of locally Noetherian schemes over S, showing that F is an algebraic space using Proposition 26.1, and proving that setting $X = F$ works.

0GH9 **Remark 27.2.** In particular, we cannot prove that the desired result is true for every Situation 27.1 because we will need to assume the local rings of S are G-rings. If you can prove the result in general or if you have a counter example, please let us know at stacks.project@gmail.com.

In Situation 27.1 let V be a locally Noetherian scheme over S. The value of our functor F on V will be all triples

$$(Z, u' : V \setminus Z \to U', \hat{x} : V_{/Z} \to W)$$

satisfying the following conditions

(1) $Z \subset V$ is a closed subset,

(2) $u' : V \setminus Z \to U'$ is a morphism over S,
(3) $\hat{x} : V_{/Z} \to W$ is an adic morphism of formal algebraic spaces over S,
(4) u' and $\hat{x}$ are compatible (see below).

The compatibility condition is the following: pulling back the formal modification g we obtain a formal modification

$$X'_{/T'} \times_{g,W,\hat{x}} V_{/Z} \longrightarrow V_{/Z}$$

See Algebraization of Formal Spaces, Lemma 24.4. By the main theorem on dilatations (Algebraization of Formal Spaces, Theorem 29.1), there is a unique proper morphism $V' \to V$ of algebraic spaces which is an isomorphism over $V \setminus Z$ such that $V'_{/Z} \to V_{/Z}$ is isomorphic to the displayed arrow. In other words, for some morphism $\hat{x}' : V'_{/Z} \to X'_{/T'}$ we have a cartesian diagram

$$\begin{array}{ccc} V'_{/Z} & \longrightarrow & V_{/Z} \\ {\scriptstyle \hat{x}'}\downarrow & & \downarrow{\scriptstyle \hat{x}} \\ X'_{/T'} & \xrightarrow{g} & W \end{array}$$

of formal algebraic spaces. We will think of $V \setminus Z$ as an open subspace of V' without further mention. The compatibility condition is that there should be a morphism $x' : V' \to X'$ restricting to u' and $\hat{x}$ over $V \setminus Z \subset V'$ and $V'_{/Z}$. In other words, such that the diagram

$$\begin{array}{ccccccc} V \setminus Z & \longrightarrow & V' & \longleftarrow & V'_{/Z} & \longrightarrow & V_{/Z} \\ {\scriptstyle u'}\downarrow & & \downarrow{\scriptstyle x'} & & \downarrow{\scriptstyle \hat{x}'} & & \downarrow{\scriptstyle \hat{x}} \\ U' & \longrightarrow & X' & \longleftarrow & X'_{/T'} & \xrightarrow{g} & W \end{array}$$

is commutative. Observe that by Algebraization of Formal Spaces, Lemma 25.5 the morphism x' is unique if it exists. We will indicate this situation by saying "*$V' \to V$, $\hat{x}'$, and x' witness the compatibility between u' and $\hat{x}$*".

0GID **Remark 27.3.** In Situation 27.1 let V be a locally Noetherian scheme over S. Let $(Z, u', \hat{x})$ be a triple satisfying (1), (2), and (3) above. We want to explain a way to think about the compatibility condition (4). It will not be mathematically precise as we are going use a fictitious category An_S of analytic spaces over S and a fictitious analytification functor

$$\left\{ \begin{array}{c} \text{locally Noetherian formal} \\ \text{algebraic spaces over } S \end{array} \right\} \longrightarrow An_S, \qquad Y \longmapsto Y^{an}$$

For example if $Y = \mathrm{Spf}(k[[t]])$ over $S = \mathrm{Spec}(k)$, then Y^{an} should be thought of as an open unit disc. If $Y = \mathrm{Spec}(k)$, then Y^{an} is a single point. The category An_S should have open and closed immersions and we should be able to take the open complement of a closed. Given Y the morphism $Y_{red} \to Y$ should induces a closed immersion $Y^{an}_{red} \to Y^{an}$. We set $Y^{rig} = Y^{an} \setminus Y^{an}_{red}$ equal to its open complement. If Y is an algebraic space and if $Z \subset Y$ is closed, then the morphism $Y_{/Z} \to Y$ should induce an open immersion $Y^{an}_{/Z} \to Y^{an}$ which in turn should induce an open immersion

$$can : (Y_{/Z})^{rig} \longrightarrow (Y \setminus Z)^{an}$$

Also, given a formal modification $g : Y' \to Y$ of locally Noetherian formal algebraic spaces, the induced morphism $g^{rig} : (Y')^{rig} \to Y^{rig}$ should be an isomorphism. Given An_S and the analytification functor, we can consider the requirement that

$$\begin{array}{ccc} (V_{/Z})^{rig} & \xrightarrow[can]{} & (V \setminus Z)^{an} \\ {\scriptstyle (g^{rig})^{-1} \circ \hat{x}^{an}} \downarrow & & \downarrow {\scriptstyle (u')^{an}} \\ (X'_{/T'})^{rig} & \xrightarrow{can} & (X' \setminus T')^{an} \end{array}$$

commutes. This makes sense as $g^{rig} : (X'_{T'})^{rig} \to W^{rig}$ is an isomorphism and $U' = X' \setminus T'$. Finally, under some assumptions of faithfulness of the analytification functor, this requirement will be equivalent to the compatibility condition formulated above. We hope this will motivate the reader to think of the compatibility of u' and $\hat{x}$ as the requirement that some maps be equal, rather than asking for the existence of a certain commutative diagram.

0GHA **Lemma 27.4.** *In Situation 27.1 the rule F that sends a locally Noetherian scheme V over S to the set of triples $(Z, u', \hat{x})$ satisfying the compatibility condition and which sends a a morphism $\varphi : V_2 \to V_1$ of locally Noetherian schemes over S to the map*

$$F(\varphi) : F(V_1) \longrightarrow F(V_2)$$

sending an element $(Z_1, u'_1, \hat{x}_1)$ of $F(V_1)$ to $(Z_2, u'_2, \hat{x}_2)$ in $F(V_2)$ given by

1. *$Z_2 \subset V_2$ is the inverse image of Z_1 by φ,*
2. *u'_2 is the composition of u'_1 and $\varphi|_{V_2 \setminus Z_2} : V_2 \setminus Z_2 \to V_1 \setminus Z_1$,*
3. *$\hat{x}_2$ is the composition of $\hat{x}_1$ and $\varphi_{/Z_2} : V_{2,/Z_2} \to V_{1,/Z_1}$*

is a contravariant functor.

Proof. To see the compatibility condition between u'_2 and $\hat{x}_2$, let $V'_1 \to V_1$, $\hat{x}'_1$, and x'_1 witness the compatibility between u'_1 and $\hat{x}_1$. Set $V'_2 = V_2 \times_{V_1} V'_1$, set $\hat{x}'_2$ equal to the composition of $\hat{x}'_1$ and $V'_{2,/Z_2} \to V'_{1,/Z_1}$, and set x'_2 equal to the composition of x'_1 and $V'_2 \to V'_1$. Then $V'_2 \to V_2$, $\hat{x}'_2$, and x'_2 witness the compatibility between u'_2 and $\hat{x}_2$. We omit the detailed verification. □

0GHB **Lemma 27.5.** *In Situation 27.1 if there exists a solution $(f : X' \to X, T, a)$ then there is a functorial bijection $F(V) = \mathrm{Mor}_S(V, X)$ on the category of locally Noetherian schemes V over S.*

Proof. Let V be a locally Noetherian scheme over S. Let $x : V \to X$ be a morphism over S. Then we get an element $(Z, u', \hat{x})$ in $F(V)$ as follows

1. $Z \subset V$ is the inverse image of T by x,
2. $u' : V \setminus Z \to U' = U$ is the restriction of x to $V \setminus Z$,
3. $\hat{x} : V_{/Z} \to W$ is the composition of $x_{/Z} : V_{/Z} \to X_{/T}$ with the isomorphism $a : X_{/T} \to W$.

This triple satisfies the compatibility condition because we can take $V' = V \times_{x,X} X'$, we can take $\hat{x}'$ the completion of the projection $x' : V' \to X'$.

Conversely, suppose given an element $(Z, u', \hat{x})$ of $F(V)$. We claim there is a unique morphism $x : V \to X$ compatible with u' and $\hat{x}$. Namely, let $V' \to V$, $\hat{x}'$, and x' witness the compatibility between u' and $\hat{x}$. Then Algebraization of Formal Spaces, Proposition 26.1 is exactly the result we need to find a unique morphism $x : V \to X$

agreeing with $\hat{x}$ over $V_{/Z}$ and with x' over V' (and a fortiori agreeing with u' over $V \setminus Z$).

We omit the verification that the two constructions above define inverse bijections between their respective domains. □

0GHC **Lemma 27.6.** *In Situation 27.1 if there exists an algebraic space X locally of finite type over S and a functorial bijection $F(V) = \mathrm{Mor}_S(V, X)$ on the category of locally Noetherian schemes V over S, then X is a solution.*

Proof. We have to construct a proper morphism $f : X' \to X$, a closed subset $T \subset |X|$, and an isomorphism $a : X_{/T} \to W$ with properties (1), (2), (3) listed just below Situation 27.1.

The discussion in this proof is a bit pedantic because we want to carefully match the underlying categories. In this paragraph we explain how the adventurous reader can proceed less timidly. Namely, the reader may extend our definition of the functor F to all locally Noetherian algebraic spaces over S. Doing so the reader may then conclude that F and X agree as functors on the category of these algebraic spaces, i.e., X represents F. Then one considers the universal object $(T, u', \hat{x})$ in $F(X)$. Then the reader will find that for the triple $X'' \to X$, $\hat{x}'$, x' witnessing the compatibility between u' and $\hat{x}$ the morphism $x' : X'' \to X'$ is an isomorphism and this will produce $f : X' \to X$ by inverting x'. Finally, we already have $T \subset |X|$ and the reader may show that $\hat{x}$ is an isomorphism which can served as the last ingredient namely a.

Denote $h_X(-) = \mathrm{Mor}_S(-, X)$ the functor of points of X restricted to the category $(\mathit{Noetherian}/S)_{\mathit{étale}}$ of Section 25. By Limits of Spaces, Remark 3.11 the algebraic spaces X and X' are limit preserving. Hence so are the restrictions h_X and $h_{X'}$. To construct f it therefore suffices to construct a transformation $h_{X'} \to h_X = F$, see Lemma 25.2. Thus let $V \to S$ be an object of $(\mathit{Noetherian}/S)_{\mathit{étale}}$ and let $\tilde{x} : V \to X'$ be in $h_{X'}(V)$. Then we get an element $(Z, u', \hat{x})$ in $F(V)$ as follows

(1) $Z \subset V$ is the inverse image of T' by $\tilde{x}$,
(2) $u' : V \setminus Z \to U'$ is the restriction of $\tilde{x}$ to $V \setminus Z$,
(3) $\hat{x} : V_{/Z} \to W$ is the composition of $x_{/Z} : V_{/Z} \to X'_{/T'}$ with $g : X'_{/T'} \to W$.

This triple satisfies the compatibility condition: first we always obtain $V' \to V$ and $\hat{x}' : V'_{/Z'} \to X'_{/T'}$ for free (see discussion preceding Lemma 27.4). Then we just define $x' : V' \to X'$ to be the composition of $V' \to V$ and the morphism $\tilde{x} : V \to X'$. We omit the verification that this works.

If $\xi : V \to X$ is an étale morphism where V is a scheme, then we obtain $\xi = (Z, u', \hat{x}) \in F(V) = h_X(V) = X(V)$. Of course, if $\varphi : V' \to V$ is a further étale morphism of schemes, then $(Z, u', \hat{x})$ pulled back to $F(V')$ corresponds to $\xi \circ \varphi$. The closed subset $T \subset |X|$ is just defined as the closed subset such that $\xi : V \to X$ for $\xi = (Z, u', \hat{x})$ pulls T back to Z

Consider Noetherian schemes V over S and a morphism $\xi : V \to X$ corresponding to $(Z, u', \hat{x})$ as above. Then we see that $\xi(V)$ is set theoretically contained in T if and only if $V = Z$ (as topological spaces). Hence we see that $X_{/T}$ agrees with W as a functor. This produces the isomorphism $a : X_{/T} \to W$. (We've omitted a small detail here which is that for the locally Noetherian formal algebraic spaces

$X_{/T}$ and W it suffices to check one gets an isomorphism after evaluating on locally Noetherian schemes over S.)

We omit the proof of conditions (1), (2), and (3). □

0GHD **Remark 27.7.** In Situation 27.1. Let V be a locally Noetherian scheme over S. Let $(Z_i, u'_i, \hat{x}_i) \in F(V)$ for $i = 1, 2$. Let $V'_i \to V$, $\hat{x}'_i$ and x'_i witness the compatibility between u'_i and $\hat{x}_i$ for $i = 1, 2$.

Set $V' = V'_1 \times_V V'_2$. Let $E' \to V'$ denote the equalizer of the morphisms

$$V' \to V'_1 \xrightarrow{x'_1} X' \quad \text{and} \quad V' \to V'_2 \xrightarrow{x'_2} X'$$

Set $Z = Z_1 \cap Z_2$. Let $E_W \to V_{/Z}$ be the equalizer of the morphisms

$$V_{/Z} \to V_{/Z_1} \xrightarrow{\hat{x}_1} W \quad \text{and} \quad V_{/Z} \to V_{/Z_2} \xrightarrow{\hat{x}_2} W$$

Observe that $E' \to V$ is separated and locally of finite type and that E_W is a locally Noetherian formal algebraic space separated over V. The compatibilities between the various morphisms involved show that

(1) $\mathrm{Im}(E' \to V) \cap (Z_1 \cup Z_2)$ is contained in $Z = Z_1 \cap Z_2$,
(2) the morphism $E' \times_V (V \setminus Z) \to V \setminus Z$ is a monomorphism and is equal to the equalizer of the restrictions of u'_1 and u'_2 to $V \setminus (Z_1 \cup Z_2)$,
(3) the morphism $E'_{/Z} \to V_{/Z}$ factors through E_W and the diagram

$$\begin{array}{ccc} E'_{/Z} & \longrightarrow & X'_{/T'} \\ \downarrow & & \downarrow g \\ E_W & \longrightarrow & W \end{array}$$

is cartesian. In particular, the morphism $E'_{/Z} \to E_W$ is a formal modification as the base change of g,
(4) E', $(E' \to V)^{-1}Z$, and $E'_{/Z} \to E_W$ is a triple as in Situation 27.1 with base scheme the locally Noetherian scheme V,
(5) given a morphism $\varphi : A \to V$ of locally Noetherian schemes, the following are equivalent
   (a) $(Z_1, u'_1, \hat{x}_1)$ and $(Z_2, u'_2, \hat{x}_2)$ restrict to the same element of $F(A)$,
   (b) $A \setminus \varphi^{-1}(Z) \to V \setminus Z$ factors through $E' \times_V (V \setminus Z)$ and $A_{/\varphi^{-1}(Z)} \to V_{/Z}$ factors through E_W.

We conclude, using Lemmas 27.5 and 27.6, that if there is a solution $E \to V$ for the triple in (4), then E represents $F \times_{\Delta, F \times F} V$ on the category of locally Noetherian schemes over V.

0GHE **Lemma 27.8.** *In Situation 27.1 assume given a closed subset $Z \subset S$ such that*

(1) *the inverse image of Z in X' is T',*
(2) *$U' \to S \setminus Z$ is a closed immersion,*
(3) *$W \to S_{/Z}$ is a closed immersion.*

Then there exists a solution $(f : X' \to X, T, a)$ and moreover $X \to S$ is a closed immersion.

Proof. Suppose we have a closed subscheme $X \subset S$ such that $X \cap (S \setminus Z) = U'$ and $X_{/Z} = W$. Then X represents the functor F (some details omitted) and hence

is a solution. To find X is clearly a local question on S. In this way we reduce to the case discussed in the next paragraph.

Assume $S = \mathrm{Spec}(A)$ is affine. Let $I \subset A$ be the radical ideal cutting out Z. Write $I = (f_1, \ldots, f_r)$. By assumption we are given

(1) the closed immersion $U' \to S \setminus Z$ determines ideals $J_i \subset A[1/f_i]$ such that J_i and J_j generate the same ideal in $A[1/f_if_j]$,
(2) the closed immersion $W \to S_{/Z}$ is the map $\mathrm{Spf}(A^\wedge/J') \to \mathrm{Spf}(A^\wedge)$ for some ideal $J' \subset A^\wedge$ in the I-adic completion $A^\wedge$ of A.

To finish the proof we need to find an ideal $J \subset A$ such that $J_i = J[1/f_i]$ and $J' = JA^\wedge$. By More on Algebra, Proposition 91.16 it suffices to show that J_i and J' generate the same ideal in $A^\wedge[1/f_i]$ for all i.

Recall that $A' = H^0(X', \mathcal{O})$ is a finite A-algebra whose formation commutes with flat base change (Cohomology of Spaces, Lemmas 20.3 and 11.2). Denote $J'' = \mathrm{Ker}(A \to A')$[12]. We have $J_i = J''A[1/f_i]$ as follows from base change to the spectrum of $A[1/f_i]$. Observe that we have a commutative diagram

$$\begin{array}{ccccc} X' & \longleftarrow & X'_{/T'} \times_{S_{/Z}} \mathrm{Spf}(A^\wedge) & = & X'_{/T'} \times_W \mathrm{Spf}(A^\wedge/J') \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Spec}(A) & \longleftarrow & \mathrm{Spf}(A^\wedge) & \longleftarrow & \mathrm{Spf}(A^\wedge/J') \end{array}$$

The middle vertical arrow is the completion of the left vertical arrow along the obvious closed subsets. By the theorem on formal functions we have

$$(A')^\wedge = \Gamma(X' \times_S \mathrm{Spec}(A^\wedge), \mathcal{O}) = \lim H^0(X' \times_S \mathrm{Spec}(A/I^n), \mathcal{O})$$

See Cohomology of Spaces, Theorem 22.5. From the diagram we conclude that J' maps to zero in $(A')^\wedge$. Hence $J' \subset J''A^\wedge$. Consider the arrows

$$X'_{/T'} \to \mathrm{Spf}(A^\wedge/J''A^\wedge) \to \mathrm{Spf}(A^\wedge/J') = W$$

We know the composition g is a formal modification (in particular rig-étale and rig-surjective) and the second arrow is a closed immersion (in particular an adic monomorphism). Hence $X'_{/T'} \to \mathrm{Spf}(A^\wedge/J''A^\wedge)$ is rig-surjective and rig-étale, see Algebraization of Formal Spaces, Lemmas 21.5 and 20.8. Applying Algebraization of Formal Spaces, Lemmas 21.14 and 21.6 we conclude that $\mathrm{Spf}(A^\wedge/J''A^\wedge) \to W$ is rig-étale and rig-surjective. By Algebraization of Formal Spaces, Lemma 21.13 we conclude that $I^nJ''A^\wedge \subset J'$ for some $n > 0$. It follows that $J''A^\wedge[1/f_i] = J'A^\wedge[1/f_i]$ and we deduce $J_iA^\wedge[1/f_i] = J'A^\wedge[1/f_i]$ for all i as desired. □

0GHF **Lemma 27.9.** *In Situation 27.1 assume $X' \to S$ and $W \to S$ are separated. Then the diagonal $\Delta : F \to F \times F$ is representable by closed immersions.*

Proof. Combine Lemma 27.8 with the discussion in Remark 27.7. □

0GHG **Lemma 27.10.** *In Situation 27.1 the functor F satisfies the sheaf property for all étale coverings of locally Noetherian schemes over S.*

Proof. Omitted. Hint: morphisms may be defined étale locally. □

[12]Contrary to what the reader may expect, the ideals J and J'' won't agree in general.

0GI5 **Lemma 27.11.** *In Situation 27.1 the functor F is limit preserving: for any directed limit $V = \lim V_\lambda$ of Noetherian affine schemes over S we have $F(V) = \operatorname{colim} F(V_\lambda)$.*

Proof. This is an absurdly long proof. Much of it consists of standard arguments on limits and étale localization. We urge the reader to skip ahead to the last part of the proof where something interesting happens.

Let $V = \lim_{\lambda\in\Lambda} V_i$ be a directed limit of schemes over S with V and V_λ Noetherian and with affine transition morphisms. See Limits, Section 2 for material on limits of schemes. We will prove that $\operatorname{colim} F(V_\lambda) \to F(V)$ is bijective.

Proof of injectivity: notation. Let $\lambda \in \Lambda$ and $\xi_{\lambda,1}, \xi_{\lambda,2} \in F(V_\lambda)$ be elements which restrict to the same element of $F(V)$. Write $\xi_{\lambda,1} = (Z_{\lambda,1}, u'_{\lambda,1}, \hat{x}_{\lambda,1})$ and $\xi_{\lambda,2} = (Z_{\lambda,2}, u'_{\lambda,2}, \hat{x}_{\lambda,2})$.

Proof of injectivity: agreement of $Z_{\lambda,i}$. Since $Z_{\lambda,1}$ and $Z_{\lambda,2}$ restrict to the same closed subset of V, we may after increasing i assume $Z_{\lambda,1} = Z_{\lambda,2}$, see Limits, Lemma 4.2 and Topology, Lemma 14.2. Let us denote the common value $Z_\lambda \subset V_\lambda$, for $\mu \geq \lambda$ denote $Z_\mu \subset V_\mu$ the inverse image in V_μ and and denote Z the inverse image in V. We will use below that $Z = \lim_{\mu\geq\lambda} Z_\mu$ as schemes if we view Z and Z_μ as reduced closed subschemes.

Proof of injectivity: agreement of $u'_{\lambda,i}$. Since U' is locally of finite type over S and since the restrictions of $u'_{\lambda,1}$ and $u'_{\lambda,2}$ to $V \setminus Z$ are the same, we may after increasing λ assume $u'_{\lambda,1} = u'_{\lambda,2}$, see Limits, Proposition 6.1. Let us denote the common value u'_λ and denote u' the restriction to $V \setminus Z$.

Proof of injectivity: restatement. At this point we have $\xi_{\lambda,1} = (Z_\lambda, u'_\lambda, \hat{x}_{\lambda,1})$ and $\xi_{\lambda,2} = (Z_\lambda, u'_\lambda, \hat{x}_{\lambda,2})$. The main problem we face in this part of the proof is to show that the morphisms $\hat{x}_{\lambda,1}$ and $\hat{x}_{\lambda,2}$ become the same after increasing λ.

Proof of injectivity: agreement of $\hat{x}_{\lambda,i}|_{Z_\lambda}$. Consider the morpisms $\hat{x}_{\lambda,1}|_{Z_\lambda}, \hat{x}_{\lambda,2}|_{Z_\lambda} : Z_\lambda \to W_{red}$. These morphisms restrict to the same morphism $Z \to W_{red}$. Since W_{red} is a scheme locally of finite type over S we see using Limits, Proposition 6.1 that after replacing λ by a bigger index we may assume $\hat{x}_{\lambda,1}|_{Z_\lambda} = \hat{x}_{\lambda,2}|_{Z_\lambda} : Z_\lambda \to W_{red}$.

Proof of injectivity: end. Next, we are going to apply the discussion in Remark 27.7 to V_λ and the two elements $\xi_{\lambda,1}, \xi_{\lambda,2} \in F(V_\lambda)$. This gives us

(1) $e_\lambda : E'_\lambda \to V_\lambda$ separated and locally of finite type,
(2) $e_\lambda^{-1}(V_\lambda \setminus Z_\lambda) \to V_\lambda \setminus Z_\lambda$ is an isomorphism,
(3) a monomorphism $E_{W,\lambda} \to V_{\lambda,/Z_\lambda}$ which is the equalizer of $\hat{x}_{\lambda,1}$ and $\hat{x}_{\lambda,2}$,
(4) a formal modification $E'_{\lambda,/Z_\lambda} \to E_{W,\lambda}$

Assertion (2) holds by assertion (2) in Remark 27.7 and the preparatory work we did above getting $u'_{\lambda,1} = u'_{\lambda,2} = u'_\lambda$. Since $Z_\lambda = (V_{\lambda,/Z_\lambda})_{red}$ factors through $E_{W,\lambda}$ because $\hat{x}_{\lambda,1}|_{Z_\lambda} = \hat{x}_{\lambda,2}|_{Z_\lambda}$ we see from Formal Spaces, Lemma 27.7 that $E_{W,\lambda} \to V_{\lambda,/Z_\lambda}$ is a closed immersion. Then we see from assertion (4) in Remark 27.7 and Lemma 27.8 applied to the triple E'_λ, $e_\lambda^{-1}(Z_\lambda)$, $E'_{\lambda,/Z_\lambda} \to E_{W,\lambda}$ over V_λ that there exists a closed immersion $E_\lambda \to V_\lambda$ which is a solution for this triple. Next we use assertion (5) in Remark 27.7 which combined with Lemma 27.5 says that E_λ is the "equalizer" of $\xi_{\lambda,1}$ and $\xi_{\lambda,2}$. In particular, we see that $V \to V_\lambda$

factors through E_λ. Then using Limits, Proposition 6.1 once more we find $\mu \geq \lambda$ such that $V_\mu \to V_\lambda$ factors through E_λ and hence the pullbacks of $\xi_{\lambda,1}$ and $\xi_{\lambda,2}$ to V_μ are the same as desired.

Proof of surjectivity: statement. Let $\xi = (Z, u', \hat{x})$ be an element of $F(V)$. We have to find a $\lambda \in \Lambda$ and an element $\xi_\lambda \in F(V_\lambda)$ restricting to ξ.

Proof of surjectivity: the question is étale local. By the unicity proved in the previous part of the proof and by the sheaf property of F in Lemma 27.10, the problem is local on V in the étale topology. More precisely, let $v \in V$. We claim it suffices to find an étale morphism $(\tilde{V}, \tilde{v}) \to (V, v)$ and some λ, some an étale morphism $\tilde{V}_\lambda \to V_\lambda$, and some element $\tilde{\xi}_\lambda \in F(\tilde{V}_\lambda)$ such that $\tilde{V} = \tilde{V}_\lambda \times_{V_\lambda} V$ and $\xi|_U = \tilde{\xi}_\lambda|_U$. We omit a detailed proof of this claim[13].

Proof of surjectivity: rephrasing the problem. Recall that any étale morphism $(\tilde{V}, \tilde{v}) \to (V, v)$ with $\tilde{V}$ affine is the base change of an étale morphism $\tilde{V}_\lambda \to V_\lambda$ with $\tilde{V}_\lambda$ affine for some λ, see for example Topologies, Lemma 13.2. Given $\tilde{V}_\lambda$ we have $\tilde{V} = \lim_{\mu \geq \lambda} \tilde{V}_\lambda \times_{V_\lambda} V_\mu$. Hence given $(\tilde{V}, \tilde{v}) \to (V, v)$ étale with $\tilde{V}$ affine, we may replace (V, v) by $(\tilde{V}, \tilde{v})$ and ξ by the restriction of ξ to $\tilde{V}$.

Proof of surjectivity: reduce to base being affine. In particular, suppose $\tilde{S} \subset S$ is an affine open subscheme such that $v \in V$ maps to a point of $\tilde{S}$. Then we may according to the previous paragraph, replace V by $\tilde{V} = \tilde{S} \times_S V$. Of course, if we do this, it suffices to solve the problem for the functor F restricted to the category of locally Noetherian schemes over $\tilde{S}$. This functor is of course the functor associated to the whole situation base changed to $\tilde{S}$. Thus we may and do assume $S = \mathrm{Spec}(R)$ is a Noetherian affine scheme for the rest of the proof.

Proof of surjectivity: easy case. If $v \in V \setminus Z$, then we can take $\tilde{V} = V \setminus Z$. This descends to an open subscheme $\tilde{V}_\lambda \subset V_\lambda$ for some λ by Limits, Lemma 4.11. Next, after increasing λ we may assume there is a morphism $u'_\lambda : \tilde{V}_\lambda \to U'$ restricting to u'. Taking $\tilde{\xi}_\lambda = (\emptyset, u'_\lambda, \emptyset)$ gives the desired element of $F(\tilde{V}_\lambda)$.

Proof of surjectivity: hard case and reduction to affine W. The most difficult case comes from considering $v \in Z \subset V$. We claim that we can reduce this to the case where W is an affine formal scheme; we urge the reader to skip this argument[14]. Namely, we can choose an étale morphism $\tilde{W} \to W$ where $\tilde{W}$ is an affine formal algebraic space such that the image of v by $\hat{x} : V_{/Z} \to W$ is in the image of $\tilde{W} \to W$ (on reductions). Then the morphisms

$$p : \tilde{W} \times_{W,g} X'_{/T'} \longrightarrow X'_{/T'}$$

and

$$q : \tilde{W} \times_{W,\hat{x}} V_{/Z} \to V_{/Z}$$

are étale morphisms of locally Noetherian formal algebraic spaces. By (an easy case of) Algebraization of Formal Spaces, Theorem 27.4 there exists a morphism

[13] To prove this one assembles a collection of the morphisms $\tilde{V} \to V$ into a finite étale covering and shows that the corresponding morphisms $\tilde{V}_\lambda \to V_\lambda$ form an étale covering as well (after increasing λ). Next one uses the injectivity to see that the elements $\tilde{\xi}_\lambda$ glue (after increasing λ) and one uses the sheaf property for F to descend these elements to an element of $F(V_\lambda)$.

[14] Artin's approach to the proof of this lemma is to work around this and consequently he can avoid proving the injectivity first. Namely, Artin consistently works with a finite affine étale coverings of all spaces in sight keeping track of the maps between them during the proof. In hindsight that might be preferable to what we do here.

$\tilde{X}' \to X'$ of algebraic spaces which is locally of finite type, is an isomorphism over U', and such that $\tilde{X}'_{/T'} \to X'_{/T'}$ is isomorphic to p. By Algebraization of Formal Spaces, Lemma 28.5 the morphism $\tilde{X}' \to X'$ is étale. Denote $\tilde{T}' \subset |\tilde{X}'|$ the inverse image of T'. Denote $\tilde{U}' \subset \tilde{X}'$ the complementary open subspace. Denote $\tilde{g}' : \tilde{X}'_{/\tilde{T}'} \to \tilde{W}$ the formal modification which is the base change of g by $\tilde{W} \to W$. Then we see that

$$\tilde{X}',\ \tilde{T}',\ \tilde{U}',\ \tilde{W},\ \tilde{g} : \tilde{X}'_{/\tilde{T}'} \to \tilde{W}$$

is another example of Situation 27.1. Denote $\tilde{F}$ the functor constructed from this triple. There is a transformation of functors

$$\tilde{F} \longrightarrow F$$

constructed using the morphisms $\tilde{X}' \to X'$ and $\tilde{W} \to W$ in the obvious manner; details omitted.

Proof of surjectivity: hard case and reduction to affine W, part 2. By the same theorem as used above, there exists a morphism $\tilde{V} \to V$ of algebraic spaces which is locally of finite type, is an isomorphism over $V \setminus Z$ and such that $\tilde{V}_{/Z} \to V_{/Z}$ is isomorphic to q. Denote $\tilde{Z} \subset \tilde{V}$ the inverse image of Z. By Algebraization of Formal Spaces, Lemmas 28.5 and 28.3 the morphism $\tilde{V} \to V$ is étale and separated. In particular $\tilde{V}$ is a (locally Noetherian) scheme, see for example Morphisms of Spaces, Proposition 50.2. We have the morphism u' which we may view as a morphism

$$\tilde{u}' : \tilde{V} \setminus \tilde{Z} \longrightarrow \tilde{U}'$$

where $\tilde{U}' \subset \tilde{X}'$ is the open mapping isomorphically to U'. We have a morphism

$$\tilde{\hat{x}} : \tilde{V}_{/\tilde{Z}} = \tilde{W} \times_{W,\hat{x}} V_{/Z} \longrightarrow \tilde{W}$$

Namely, here we just use the projection. Thus we have the triple

$$\tilde{\xi} = (\tilde{Z}, \tilde{u}', \tilde{\hat{x}}) \in \tilde{F}(\tilde{V})$$

We omit proving the compatibility condition; hints: if $V' \to V$, $\hat{x}'$, and x' witness the compatibility between u' and $\hat{x}$, then one sets $\tilde{V}' = V' \times_V \tilde{V}$ which comes with morphsms $\tilde{\hat{x}}'$ and $\tilde{x}'$ and show this works. The image of $\tilde{\xi}$ under the transformation $\tilde{F} \to F$ is the restriction of ξ to $\tilde{V}$.

Proof of surjectivity: hard case and reduction to affine W, part 3. By our choice of $\tilde{W} \to W$, there is an affine open $\tilde{V}_{open} \subset \tilde{V}$ (we're running out of notation) whose image in V contains our chosen point $v \in V$. Now by the case studied in the next paragraph and the remarks made earlier, we can descend $\tilde{\xi}|_{\tilde{V}_{open}}$ to some element $\tilde{\xi}_\lambda$ of $\tilde{F}$ over $\tilde{V}_{\lambda,open}$ for some étale morphism $\tilde{V}_{\lambda,open} \to V_\lambda$ whose base change to V is $\tilde{V}_{open}$. Applying the transformation of functors $\tilde{F} \to F$ we obtain the element of $F(\tilde{V}_{\lambda,open})$ we were looking for. This reduces us to the case discussed in the next paragraph.

Proof of surjectivity: the case of an affine W. We have $v \in Z \subset V$ and W is an affine formal algebraic space. Recall that

$$\xi = (Z, u', \hat{x}) \in F(V)$$

We may still replace V by an étale neighbourhood of v. In particular we may and do assume V and V_λ are affine.

Proof of surjectivity: descending Z. We can find a λ and a closed subscheme $Z_\lambda \subset V_\lambda$ such that Z is the base change of Z_λ to V. See Limits, Lemma 10.1. Warning: we don't know (and in general it won't be true) that Z_λ is a reduced closed subscheme of V_λ. For $\mu \geq \lambda$ denote $Z_\mu \subset V_\mu$ the scheme theoretic inverse image in V_μ. We will use below that $Z = \lim_{\mu \geq \lambda} Z_\mu$ as schemes.

Proof of surjectivity: descending u'. Since U' is locally of finite type over S we may assume after increasing λ that there exists a morphism $u'_\lambda : V_\lambda \setminus Z_\lambda \to U'$ whose restriction to $V \setminus Z$ is u'. See Limits, Proposition 6.1. For $\mu \geq \lambda$ we will denote u'_μ the restriction of u'_λ to $V_\mu \setminus Z_\mu$.

Proof of surjectivity: descending a witness. Let $V' \to V$, $\hat{x}'$, and x' witness the compatibility between u' and $\hat{x}$. Using the same references as above we may assume (after increasing λ) that there exists a morphism $V'_\lambda \to V_\lambda$ of finite type whose base change to V is $V' \to V$. After increasing λ we may assume $V'_\lambda \to V_\lambda$ is proper (Limits, Lemma 13.1). Next, we may assume $V'_\lambda \to V_\lambda$ is an isomorphism over $V_\lambda \setminus Z_\lambda$ (Limits, Lemma 8.11). Next, we may assume there is a morphism $x'_\lambda : V'_\lambda \to X'$ whose restriction to V' is x'. Increasing λ again we may assume x'_λ agrees with u'_λ over $V_\lambda \setminus Z_\lambda$. For $\mu \geq \lambda$ we denote V'_μ and x'_μ the base change of V'_λ and the restriction of x'_λ.

Proof of surjectivity: algebra. Write $W = \mathrm{Spf}(B)$, $V = \mathrm{Spec}(A)$, and for $\mu \geq \lambda$ write $V_\mu = \mathrm{Spec}(A_\mu)$. Denote $I_\mu \subset A_\mu$ and $I \subset A$ the ideals cutting out Z_μ and Z. Then $I_\lambda A_\mu = I_\mu$ and $I_\lambda A = I$. The morphism $\hat{x}$ determines and is determined by a continuous ring map

$$(\hat{x})^\sharp : B \longrightarrow A^\wedge$$

where $A^\wedge$ is the I-adic completion of A. To finish the proof we have to show that this map descends to a map into $A^\wedge_\mu$ for some sufficiently large μ where $A^\wedge_\mu$ is the I_μ-adic completion of A_μ. This is a nontrivial fact; Artin writes in his paper [Art70]: "Since the data (3.5) involve I-adic completions, which do not commute with direct limits, the verification is somewhat delicate. It is an algebraic analogue of a convergence proof in analysis."

Proof of surjectivity: algebra, more rings. Let us denote

$$C_\mu = \Gamma(V'_\mu, \mathcal{O}) \quad \text{and} \quad C = \Gamma(V', \mathcal{O})$$

Observe that $A \to C$ and $A_\mu \to C_\mu$ are finite ring maps as $V' \to V$ and $V'_\mu \to V_\mu$ are proper morphisms, see Cohomology of Spaces, Lemma 20.3. Since $V = \lim V_\mu$ and $V' = \lim V'_\mu$ we have

$$A = \mathrm{colim}\, A_\mu \quad \text{and} \quad C = \mathrm{colim}\, C_\mu$$

by Limits, Lemma 4.7[15]. For an element $a \in I$, resp. $a \in I_\mu$ the maps $A_a \to C_a$, resp. $(A_\mu)_a \to (C_\mu)_a$ are isomorphisms by flat base change (Cohomology of Spaces, Lemma 11.2). Hence the kernel and cokernel of $A \to C$ is supported on $V(I)$ and similarly for $A_\mu \to C_\mu$. We conclude the kernel and cokernel of $A \to C$ are annihilated by a power of I and the kernel and cokernel of $A_\mu \to C_\mu$ are annihilated by a power of I_μ, see Algebra, Lemma 62.4.

Proof of surjectivity: algebra, more ring maps. Denote $Z_n \subset V$ the nth infinitesimal neighbourhood of Z and denote $Z_{\mu,n} \subset V_\mu$ the nth infinitesimal neighbourhood of

[15]We don't know that $C_\mu = C_\lambda \otimes_{A_\lambda} A_\mu$ as the various morphisms aren't flat.

Z_μ. By the theorem on formal functions (Cohomology of Spaces, Theorem 22.5) we have

$$C^\wedge = \lim_n H^0(V' \times_V Z_n, \mathcal{O}) \quad \text{and} \quad C^\wedge_\mu = \lim_n H^0(V'_\mu \times_{V_\mu} Z_{\mu,n}, \mathcal{O})$$

where $C^\wedge$ and $C^\wedge_\mu$ are the completion with respect to I and I_μ. Combining the completion of the morphism $x'_\mu : V'_\mu \to X'$ with the morphism $g : X'_{/T'} \to W$ we obtain

$$g \circ x'_{\mu,/Z_\mu} : V'_{\mu,/Z_\mu} = \operatorname{colim} V'_\mu \times_{V_\mu} Z_{\mu,n} \longrightarrow W$$

and hence by the description of the completion $C^\wedge_\mu$ above we obtain a continuous ring homomorphism

$$(g \circ x'_{\mu,/Z_\mu})^\sharp : B \longrightarrow C^\wedge_\mu$$

The fact that $V' \to V$, $\hat{x}'$, x' witnesses the compatibility between u' and $\hat{x}$ implies the commutativity of the following diagram

$$\begin{array}{ccc} C^\wedge_\mu & \longrightarrow & C^\wedge \\ {\scriptstyle (g\circ x'_{\mu,/Z_\mu})^\sharp}\uparrow & & \uparrow \\ B & \xrightarrow{(\hat{x})^\sharp} & A^\wedge \end{array}$$

Proof of surjectivity: more algebra arguments. Recall that the finite A-modules $\operatorname{Ker}(A \to C)$ and $\operatorname{Coker}(A \to C)$ are annihilated by a power of I and similarly the finite A_μ-modules $\operatorname{Ker}(A_\mu \to C_\mu)$ and $\operatorname{Coker}(A_\mu \to C_\mu)$ are annihilated by a power of I_μ. This implies that these modules are equal to their completions. Since I-adic completion on the category of finite A-modules is exact (see Algebra, Section 97) it follows that we have

$$\operatorname{Coker}(A^\wedge \to C^\wedge) = \operatorname{Coker}(A \to C)$$

and similarly for kernels and for the maps $A_\mu \to C_\mu$. Of course we also have

$$\operatorname{Ker}(A \to C) = \operatorname{colim} \operatorname{Ker}(A_\mu \to C_\mu) \quad \text{and} \quad \operatorname{Coker}(A \to C) = \operatorname{colim} \operatorname{Coker}(A_\mu \to C_\mu)$$

Recall that $S = \operatorname{Spec}(R)$ is affine. All of the ring maps above are R-algebra homomorphisms as all of the morphisms are morphisms over S. By Algebraization of Formal Spaces, Lemma 12.5 we see that B is topologically of finite type over R. Say B is topologically generated by $b_1, \ldots, b_n$. Pick some μ (for example λ) and consider the elements

$$\text{images of } (g \circ x'_{\mu,/Z_\mu})^\sharp(b_1), \ldots, (g \circ x'_{\mu,/Z_\mu})^\sharp(b_n) \text{ in } \operatorname{Coker}(A_\mu \to C_\mu)$$

The image of these elements in $\operatorname{Coker}(\alpha)$ are zero by the commutativity of the square above. Since $\operatorname{Coker}(A \to C) = \operatorname{colim} \operatorname{Coker}(A_\mu \to C_\mu)$ and these cokernels are equal to their completions we see that after increasing μ we may assume these images are all zero. This means that the continuous homomorphism $(g \circ x'_{\mu,/Z_\mu})^\sharp$ has image contained in $\operatorname{Im}(A_\mu \to C_\mu)$. Choose elements $a_{\mu,j} \in (A_\mu)^\wedge$ mapping to $(g \circ x'_{\mu,/Z_\mu})^\sharp(b_1)$ in $(C_\mu)^\wedge$. Then $a_{\mu,j} \in A^\wedge_\mu$ and $(\hat{x})^\sharp(b_j) \in A^\wedge$ map to the same element of $C^\wedge$ by the commutativity of the square above. Since $\operatorname{Ker}(A \to C) = \operatorname{colim} \operatorname{Ker}(A_\mu \to C_\mu)$ and these kernels are equal to their completions, we may after increasing μ adjust our choices of $a_{\mu,j}$ such that the image of $a_{\mu,j}$ in $A^\wedge$ is equal to $(\hat{x})^\sharp(b_j)$.

Proof of surjectivity: final algebra arguments. Let $\mathfrak{b} \subset B$ be the ideal of topologically nilpotent elements. Let $J \subset R[x_1, \ldots, x_n]$ be the ideal consisting of those

$h(x_1, \ldots, x_n)$ such that $h(b_1, \ldots, b_n) \in \mathfrak{b}$. Then we get a continuous surjection of topological R-algebras

$$\Phi : R[x_1, \ldots, x_n]^\wedge \longrightarrow B, \quad x_j \longmapsto b_j$$

where the completion on the left hand side is with respect to J. Since $R[x_1, \ldots, x_n]$ is Noetherian we can choose generators $h_1, \ldots, h_m$ for J. By the commutativity of the square above we see that $h_j(a_{\mu,1}, \ldots, a_{\mu,n})$ is an element of $A_\mu^\wedge$ whose image in $A^\wedge$ is contained in $IA^\wedge$. Namely, the ring map $(\hat{x})^\sharp$ is continuous and $IA^\wedge$ is the ideal of topological nilpotent elements of $A^\wedge$ because $A^\wedge/IA^\wedge = A/I$ is reduced. (See Algebra, Section 97 for results on completion in Noetherian rings.) Since $A/I = \operatorname{colim} A_\mu/I_\mu$ we conclude that after increasing μ we may assume $h_j(a_{\mu,1}, \ldots, a_{\mu,n})$ is in $I_\mu A_\mu^\wedge$. In particular the elements $h_j(a_{\mu,1}, \ldots, a_{\mu,n})$ of $A_\mu^\wedge$ are topologically nilpotent in $A_\mu^\wedge$. Thus we obtain a continuous R-algebra homomorphism

$$\Psi : R[x_1, \ldots, x_n]^\wedge \longrightarrow A_\mu^\wedge, \quad x_j \longmapsto a_{\mu,j}$$

In order to conclude what we want, we need to see if $\operatorname{Ker}(\Phi)$ is annihilated by Ψ. This may not be true, but we can achieve this after increasing μ. Indeed, since $R[x_1, \ldots, x_n]^\wedge$ is Noetherian, we can choose generators $g_1, \ldots, g_l$ of the ideal $\operatorname{Ker}(\Phi)$. Then we see that

$$\Psi(g_1), \ldots, \Psi(g_l) \in \operatorname{Ker}(A_\mu^\wedge \to C_\mu^\wedge) = \operatorname{Ker}(A_\mu \to C_\mu)$$

map to zero in $\operatorname{Ker}(A \to C) = \operatorname{colim} \operatorname{Ker}(A_\mu \to C_\mu)$. Hence increasing μ as before we get the desired result.

Proof of surjectivity: mopping up. The continuous ring homomorphism $B \to (A_\mu)^\wedge$ constructed above determines a morphism $\hat{x}_\mu : V_{\mu,/Z_\mu} \to W$. The compatibility of $\hat{x}_\mu$ and u'_μ follows from the fact that the ring map $B \to (A_\mu)^\wedge$ is by construction compatible with the ring map $A_\mu \to C_\mu$. In fact, the compatibility will be witnessed by the proper morphism $V'_\mu \to V_\mu$ and the morphisms x'_μ and $\hat{x}'_\mu = x'_{\mu,/Z_\mu}$ we used in the construction. This finishes the proof. □

0GI6 **Lemma 27.12.** *In Situation 27.1 the functor F satisfies the Rim-Schlessinger condition (RS).*

Proof. Recall that the condition only involves the evaluation $F(V)$ of the functor F on schemes V over S which are spectra of Artinian local rings and the restriction maps $F(V_2) \to F(V_1)$ for morphisms $V_1 \to V_2$ of schemes over S which are spectra of Artinian local rings. Thus let V/S be the spetruim of an Artinian local ring. If $\xi = (Z, u', \hat{x}) \in F(V)$ then either $Z = \emptyset$ or $Z = V$ (set theoretically). In the first case we see that $\hat{x}$ is a morphism from the empty formal algebraic space into W. In the second case we see that u' is a morphism from the empty scheme into X' and we see that $\hat{x} : V \to W$ is a morphism into W. We conclude that

$$F(V) = U'(V) \amalg W(V)$$

and moreover for $V_1 \to V_2$ as above the induced map $F(V_2) \to F(V_1)$ is compatible with this decomposition. Hence it suffices to prove that both U' and W satisfy the Rim-Schlessinger condition. For U' this follows from Lemma 5.2. To see that it is true for W, we write $W = \operatorname{colim} W_n$ as in Formal Spaces, Lemma 20.11. Say $V = \operatorname{Spec}(A)$ with $(A, \mathfrak{m})$ an Artinian local ring. Pick $n \geq 1$ such that $\mathfrak{m}^n = 0$. Then we have $W(V) = W_n(V)$. Hence we see that the Rim-Schlessinger condition

for W follows from the Rim-Schlessinger condition for W_n for all n (which in turn follows from Lemma 5.2). □

0GI7 **Lemma 27.13.** *In Situation 27.1 the tangent spaces of the functor F are finite dimensional.*

Proof. In the proof of Lemma 27.12 we have seen that $F(V) = U'(V) \amalg W(V)$ if V is the spectrum of an Artinian local ring. The tangent spaces are computed entirely from evaluations of F on such schemes over S. Hence it suffices to prove that the tangent spaces of the functors U' and W are finite dimensional. For U' this follows from Lemma 8.1. Write $W = \operatorname{colim} W_n$ as in the proof of Lemma 27.12. Then we see that the tangent spaces of W are equal to the tangent spaces of W_2, as to get at the tangent space we only need to evaluate W on spectra of Artinian local rings $(A, \mathfrak{m})$ with $\mathfrak{m}^2 = 0$. Then again we see that the tangent spaces of W_2 have finite dimension by Lemma 8.1. □

0GI8 **Lemma 27.14.** *In Situation 27.1 assume $X' \to S$ is separated. Then every formal object for F is effective.*

Proof. A formal object $\xi = (R, \xi_n)$ of F consists of a Noetherian complete local S-algebra R whose residue field is of finite type over S, together with elements $\xi_n \in F(\operatorname{Spec}(R/\mathfrak{m}^n))$ for all n such that $\xi_{n+1}|_{\operatorname{Spec}(R/\mathfrak{m}^n)} = \xi_n$. By the discussion in the proof of Lemma 27.12 we see that either ξ is a formal object of U' or a formal object of W. In the first case we see that ξ is effective by Lemma 9.5. The second case is the interesting case. Set $V = \operatorname{Spec}(R)$. We will construct an element $(Z, u', \hat{x}) \in F(V)$ whose image in $F(\operatorname{Spec}(R/\mathfrak{m}^n))$ is ξ_n for all $n \geq 1$.

We may view the collection of elements ξ_n as a morphism

$$\xi : \operatorname{Spf}(R) \longrightarrow W$$

of locally Noetherian formal algebraic spaces over S. Observe that ξ is *not* an adic morphism in general. To fix this, let $I \subset R$ be the ideal corresponding to the formal closed subspace

$$\operatorname{Spf}(R) \times_{\xi, W} W_{red} \subset \operatorname{Spf}(R)$$

Note that $I \subset \mathfrak{m}_R$. Set $Z = V(I) \subset V = \operatorname{Spec}(R)$. Since R is $\mathfrak{m}_R$-adically complete it is a fortiori I-adically complete (Algebra, Lemma 96.8). Moreover, we claim that for each $n \geq 1$ the morphism

$$\xi|_{\operatorname{Spf}(R/I^n)} : \operatorname{Spf}(R/I^n) \longrightarrow W$$

actually comes from a morphism

$$\xi'_n : \operatorname{Spec}(R/I^n) \longrightarrow W$$

Namely, this follows from writing $W = \operatorname{colim} W_n$ as in the proof of Lemma 27.12, noticing that $\xi|_{\operatorname{Spf}(R/I^n)}$ maps into W_n, and applying Formal Spaces, Lemma 33.3 to algebraize this to a morphism $\operatorname{Spec}(R/I^n) \to W_n$ as desired. Let us denote $\operatorname{Spf}'(R) = V_{/Z}$ the formal spectrum of R endowed with the I-adic topology – equivalently the formal completion of V along Z. Using the morphisms ξ'_n we obtain an adic morphism

$$\hat{x} = (\xi'_n) : \operatorname{Spf}'(R) \longrightarrow W$$

of locally Noetherian formal algebraic spaces over S. Consider the base change

$$\operatorname{Spf}'(R) \times_{\hat{x}, W, g} X'_{/T'} \longrightarrow \operatorname{Spf}'(R)$$

This is a formal modification by Algebraization of Formal Spaces, Lemma 24.4. Hence by the main theorem on dilatations (Algebraization of Formal Spaces, Theorem 29.1) we obtain a proper morphism

$$V' \longrightarrow V = \mathrm{Spec}(R)$$

which is an isomorphism over $\mathrm{Spec}(R) \setminus V(I)$ and whose completion recovers the formal modification above, in other words

$$V' \times_{\mathrm{Spec}(R)} \mathrm{Spec}(R/I^n) = \mathrm{Spec}(R/I^n) \times_{\xi'_n, W, g} X'_{/T'}$$

This in particular tells us we have a compatible system of morphisms

$$V' \times_{\mathrm{Spec}(R)} \mathrm{Spec}(R/I^n) \longrightarrow X' \times_S \mathrm{Spec}(R/I^n)$$

Hence by Grothendieck's algebraization theorem (in the form of More on Morphisms of Spaces, Lemma 43.3) we obtain a morphism

$$x' : V' \to X'$$

over S recovering the morphisms displayed above. Then finally setting $u' : V \setminus Z \to X'$ the restriction of x' to $V \setminus Z \subset V'$ gives the third component of our desired element $(Z, u', \hat{x}) \in F(V)$. □

0GI9 **Lemma 27.15.** *Let S be a locally Noetherian scheme. Let V be a scheme locally of finite type over S. Let $Z \subset V$ be closed. Let W be a locally Noetherian formal algebraic space over S such that W_{red} is locally of finite type over S. Let $g : V_{/Z} \to W$ be an adic morphism of formal algebraic spaces over S. Let $v \in V$ be a closed point such that g is versal at v (as in Section 15). Then after replacing V by an open neighbourhood of v the morphism g is smooth (see proof).*

Proof. Since g is adic it is representable by algebraic spaces (Formal Spaces, Section 23). Thus by saying g is smooth we mean that g should be smooth in the sense of Bootstrap, Definition 4.1.

Write $W = \mathrm{colim}\, W_n$ as in Formal Spaces, Lemma 20.11. Set $V_n = V_{/Z} \times_{\hat{x}, W} W_n$. Then V_n is a closed subscheme with underlying set Z. Smoothness of $V \to W$ is equivalent to the smoothness of all the morphisms $V_n \to W_n$ (this holds because any morphism $T \to W$ with T a quasi-compact scheme factors through W_n for some n). We know that the morphism $V_n \to W_n$ is smooth at v by Lemma 12.6[16]. Of course this means that given any n we can shrink V such that $V_n \to W_n$ is smooth. The problem is to find an open which works for all n at the same time.

The question is local on V, hence we may assume $S = \mathrm{Spec}(R)$ and $V = \mathrm{Spec}(A)$ are affine.

In this paragraph we reduce to the case where W is an affine formal algebraic space. Choose an affine formal scheme W' and an étale morphism $W' \to W$ such that the image of v in W_{red} is in the image of $W'_{red} \to W_{red}$. Then $V_{/Z} \times_{g,W} W' \to V_{/Z}$ is an adic étale morphism of formal algebraic spaces over S and $V_{/Z} \times_{g,W} W'$ is an affine formal algebraic space. By Algebraization of Formal Spaces, Lemma 25.1 there exists an étale morphism $\varphi : V' \to V$ of affine schemes such that the completion of V' along $Z' = \varphi^{-1}(Z)$ is isomorphic to $V_{/Z} \times_{g,W} W'$ over $V_{/Z}$. Observe that v

[16]The lemma applies since the diagonal of W is representable by algebraic spaces and locally of finite type, see Formal Spaces, Lemma 15.5 and we have seen that W has (RS) in the proof of Lemma 27.12.

is the image of some $v' \in V'$. Since smoothness is preserved under base change we see that $V'_n \to W'_n$ is smooth for all n. In the next paragraph we show that after replacing V' by an open neighbourhood of v' the morphisms $V'_n \to W'_n$ are smooth for all n. Then, after we replace V by the open image of $V' \to V$, we obtain that $V_n \to W_n$ is smooth by étale descent of smoothness. Some details omitted.

Assume $S = \mathrm{Spec}(R)$, $V = \mathrm{Spec}(A)$, $Z = V(I)$, and $W = \mathrm{Spf}(B)$. Let v correspond to the maximal ideal $I \subset \mathfrak{m} \subset A$. We are given an adic continuous R-algebra homomorphism

$$B \longrightarrow A^\wedge$$

Let $\mathfrak{b} \subset B$ be the ideal of topologically nilpotent elements (this is the maximal ideal of definition of the Noetherian adic topological ring B). Observe that $\mathfrak{b}A^\wedge$ and $IA^\wedge$ are both ideals of definition of the Noetherian adic ring $A^\wedge$. Also, $\mathfrak{m}A^\wedge$ is a maximal ideal of $A^\wedge$ containing both $\mathfrak{b}A^\wedge$ and $IA^\wedge$. We are given that

$$B_n = B/\mathfrak{b}^n \to A^\wedge/\mathfrak{b}^n A^\wedge = A_n$$

is smooth at $\mathfrak{m}$ for all n. By the discussion above we may and do assume that $B_1 \to A_1$ is a smooth ring map. Denote $\mathfrak{m}_1 \subset A_1$ the maximal ideal corresponding to $\mathfrak{m}$. Since smoothness implies flatness, we see that: for all $n \geq 1$ the map

$$\mathfrak{b}^n/\mathfrak{b}^{n+1} \otimes_{B_1} (A_1)_{\mathfrak{m}_1} \longrightarrow \left(\mathfrak{b}^n A^\wedge/\mathfrak{b}^{n+1}A^\wedge\right)_{\mathfrak{m}_1}$$

is an isomorphism (see Algebra, Lemma 99.9). Consider the Rees algebra

$$B' = \bigoplus\nolimits_{n\geq 0} \mathfrak{b}^n/\mathfrak{b}^{n+1}$$

which is a finite type graded algebra over the Noetherian ring B_1 and the Rees algebra

$$A' = \bigoplus\nolimits_{n\geq 0} \mathfrak{b}^n A^\wedge/\mathfrak{b}^{n+1}A^\wedge$$

which is a a finite type graded algebra over the Noetherian ring A_1. Consider the homomorphism of graded A_1-algebras

$$\Psi : B' \otimes_{B_1} A_1 \longrightarrow A'$$

By the above this map is an isomorphism after localizing at the maximal ideal $\mathfrak{m}_1$ of A_1. Hence $\mathrm{Ker}(\Psi)$, resp. $\mathrm{Coker}(\Psi)$ is a finite module over $B' \otimes_{B_1} A_1$, resp. A' whose localization at $\mathfrak{m}_1$ is zero. It follows that after replacing A_1 (and correspondingly A) by a principal localization we may assume Ψ is an isomorphism. (This is the key step of the proof.) Then working backwards we see that $B_n \to A_n$ is flat, see Algebra, Lemma 99.9. Hence $A_n \to B_n$ is smooth (as a flat ring map with smooth fibres, see Algebra, Lemma 137.16) and the proof is complete. □

0GIA **Lemma 27.16.** *In Situation 27.1 the functor F satisfies openness of versality.*

Proof. We have to show the following. Given a scheme V locally of finite type over S, given $\xi \in F(V)$, and given a finite type point $v_0 \in V$ such that ξ is versal at v_0, after replacing V by an open neighbourhood of v_0 we have that ξ is versal at every finite type point of V. Write $\xi = (Z, u', \hat{x})$.

First case: $v_0 \notin Z$. Then we can first replace V by $V \setminus Z$. Hence we see that $\xi = (\emptyset, u', \emptyset)$ and the morphism $u' : V \to X'$ is versal at v_0. By More on Morphisms of Spaces, Lemma 20.1 this means that $u' : V \to X'$ is smooth at v_0. Since the set of a points where a morphism is smooth is open, we can after shrinking V assume u' is smooth. Then the same lemma tells us that ξ is versal at every point as desired.

Second case: $v_0 \in Z$. Write $W = \operatorname{colim} W_n$ as in Formal Spaces, Lemma 20.11. By Lemma 27.15 we may assume $\hat{x} : V_{/Z} \to W$ is a smooth morphism of formal algebraic spaces. It follows immediately that $\xi = (Z, u', \hat{x})$ is versal at all finite type points of Z. Let $V' \to V$, $\hat{x}'$, and x' witness the compatibility between u' and $\hat{x}$. We see that $\hat{x}' : V'_{/Z} \to X'_{/T'}$ is smooth as a base change of $\hat{x}$. Since $\hat{x}'$ is the completion of $x' : V' \to X'$ this implies that $x' : V' \to X'$ is smooth at all points of $(V' \to V)^{-1}(Z) = |x'|^{-1}(T') \subset |V'|$ by the already used More on Morphisms of Spaces, Lemma 20.1. Since the set of smooth points of a morphism is open, we see that the closed set of points $B \subset |V'|$ where x' is not smooth does not meet $(V' \to V)^{-1}(Z)$. Since $V' \to V$ is proper and hence closed, we see that $(V' \to V)(B) \subset V$ is a closed subset not meeting Z. Hence after shrinking V we may assume $B = \emptyset$, i.e., x' is smooth. By the discussion in the previous paragraph this exactly means that ξ is versal at all finite type points of V not contained in Z and the proof is complete. □

Here is the final result.

0GIB **Theorem 27.17.** *Let S be a locally Noetherian scheme such that $\mathcal{O}_{S,s}$ is a G-ring for all finite type points $s \in S$. Let X' be an algebraic space locally of finite type over S. Let $T' \subset |X'|$ be a closed subset. Let W be a locally Noetherian formal algebraic space over S with W_{red} locally of finite type over S. Finally, we let* [Art70, Theorem 3.1]

$$g : X'_{/T'} \longrightarrow W$$

be a formal modification, see Algebraization of Formal Spaces, Definition 24.1. If X' and W are separated[17] *over S, then there exists a proper morphism $f : X' \to X$ of algebraic spaces over S, a closed subset $T \subset |X|$, and an isomorphism $a : X_{/T} \to W$ of formal algebraic spaces such that*

1. *T' is the inverse image of T by $|f| : |X'| \to |X|$,*
2. *$f : X' \to X$ maps $X' \setminus T'$ isomorphically to $X \setminus T$, and*
3. *$g = a \circ f_{/T}$ where $f_{/T} : X'_{/T'} \to X_{/T}$ is the induced morphism.*

In other words, $(f : X' \to X, T, a)$ is a solution as defined earlier in this section.

Proof. Let F be the functor constructed using X', T', W, g in this section. By Lemma 27.6 it suffices to show that F corresponds to an algebraic space X locally of finite type over S. In order to do this, we will apply Proposition 26.1. Namely, by Lemma 27.9 the diagonal of F is representable by closed immersions and by Lemmas 27.10, 27.11, 27.12, 27.13, 27.14, and 27.16 we have axioms [0], [1], [2], [3], [4], and [5]. □

0GIC **Remark 27.18.** The proof of Theorem 27.17 uses that X' and W are separated over S in two places. First, the proof uses this in showing $\Delta : F \to F \times F$ is representable by algebraic spaces. This use of the assumption can be entirely avoided by proving that Δ is representable by applying the theorem in the separated case to the triples E', $(E' \to V)^{-1}Z$, and $E'_{/Z} \to E_W$ found in Remark 27.7 (this is the usual bootstrap procedure for the diagonal). Thus the proof of Lemma 27.14 is the only place in our proof of Theorem 27.17 where we really need to use that $X' \to S$ is separated. The reader checks that we use the assumption only to obtain the morphism $x' : V' \to X'$. The existence of x' can be shown, using results in

[17] See Remark 27.18.

the literature, if $X' \to S$ is quasi-separated, see More on Morphisms of Spaces, Remark 43.4. We conclude the theorem holds as stated with "separated" replaced by "quasi-separated". If we ever need this we will precisely state and carefully prove this here.

28. Other chapters

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References

[Art69] Michael Artin, *Algebraization of formal moduli: I*, Global Analysis (Papers in Honor of K. Kodaira), Univ. Tokyo Press, Tokyo, 1969, pp. 21–71.

[Art70] ———, *Algebraization of formal moduli: II – existence of modifications*, Annals of Mathematics **91** (1970), 88–135.

[Art74] ———, *Versal deformations and algebraic stacks*, Inventiones Mathematics **27** (1974), 165–189.

[Bha16] Bhargav Bhatt, *Algebraization and Tannaka duality*, Camb. J. Math. **4** (2016), no. 4, 403–461.

[Fle81] Hubert Flenner, *Ein Kriterium für die Offenheit der Versalität*, Math. Z. **178** (1981), no. 4, 449–473.

[Hal17] Jack Hall, *Openness of versality via coherent functors*, J. Reine Angew. Math. **722** (2017), 137–182.

[HR12] Jack Hall and David Rydh, *Artin's criteria for algebraicity revisited*.

[Lie06] Max Lieblich, *Moduli of complexes on a proper morphism*, J. Algebraic Geom. **15** (2006), no. 1, 175–206.

[Ols06] Martin Christian Olsson, *Deformation theory of representable morphisms of algebraic stacks*, Math. Z. **253** (2006), no. 1, 25–62.

[OS03] Martin Christian Olsson and Jason Starr, *Quot functors for Deligne-Mumford stacks*, Comm. Algebra **31** (2003), no. 8, 4069–4096, Special issue in honor of Steven L. Kleiman.