1. Introduction

A reference is the lectures by Serre in the Seminaire Cartan, see [Ser55]. Serre in turn refers to [Deu68] and [ANT44]. We changed some of the proofs, in particular we used a fun argument of Rieffel to prove Wedderburn’s theorem. Very likely this change is not an improvement and we strongly encourage the reader to read the original exposition by Serre.

2. Noncommutative algebras

Let $k$ be a field. In this chapter an algebra $A$ over $k$ is a possibly noncommutative ring $A$ together with a ring map $k \to A$ such that $k$ maps into the center of $A$ and such that 1 maps to an identity element of $A$. An $A$-module is a right $A$-module such that the identity of $A$ acts as the identity.

- **Definition 2.1.** Let $A$ be a $k$-algebra. We say $A$ is *finite* if $\dim_k(A) < \infty$. In this case we write $[A : k] = \dim_k(A)$.

- **Definition 2.2.** A *skew field* is a possibly noncommutative ring with an identity element 1, with $1 \neq 0$, in which every nonzero element has a multiplicative inverse.

  A skew field is a $k$-algebra for some $k$ (e.g., for the prime field contained in it). We will use below that any module over a skew field is free because a maximal linearly independent set of vectors forms a basis and exists by Zorn’s lemma.

- **Definition 2.3.** Let $A$ be a $k$-algebra. We say an $A$-module $M$ is *simple* if it is nonzero and the only $A$-submodules are 0 and $M$. We say $A$ is *simple* if the only two-sided ideals of $A$ are 0 and $A$. 

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Definition 2.4. A $k$-algebra $A$ is central if the center of $A$ is the image of $k \to A$.

Definition 2.5. Given a $k$-algebra $A$ we denote $A^{op}$ the $k$-algebra we get by reversing the order of multiplication in $A$. This is called the opposite algebra.

3. Wedderburn’s theorem

The following cute argument can be found in a paper of Rieffel, see [Rie65]. The proof could not be simpler (quote from Carl Faith’s review).

Lemma 3.1. Let $A$ be a possibly noncommutative ring with 1 which contains no nontrivial two-sided ideal. Let $M$ be a nonzero right ideal in $A$, and view $M$ as a right $A$-module. Then $A$ coincides with the bicommutant of $M$.

Proof. Let $A' = \text{End}_A(M)$, so $M$ is a left $A'$-module. Set $A'' = \text{End}_{A'}(M)$ (the bicommutant of $M$). We view $A''$ as an algebra so that $M$ is a right $A''$-module. Let $R : A \to A''$ be the natural homomorphism such that $mR(a) = ma$. Then $R$ is injective, since $R(1) = \text{id}_M$ and $A$ contains no nontrivial two-sided ideal. We claim that $R(M)$ is a right ideal in $A''$. Namely, $R(m) a'' = R(ma'')$ for $a'' \in A''$ and $m$ in $M$, because left multiplication of $M$ by any element $n$ of $M$ represents an element of $A'$, and so $(nm)a'' = n(ma'')$ for all $n$ in $M$. Finally, the product ideal $AM$ is a two-sided ideal, and so $A = AM$. Thus $R(A) = R(A)R(M)$, so that $R(A)$ is a right ideal in $A''$. But $R(A)$ contains the identity element of $A''$, and so $R(A) = A''$. □

Lemma 3.2. Let $A$ be a $k$-algebra. If $A$ is finite, then

1. $A$ has a simple module,
2. any nonzero module contains a simple submodule,
3. a simple module over $A$ has finite dimension over $k$, and
4. if $M$ is a simple $A$-module, then $\text{End}_A(M)$ is a skew field.

Proof. Of course (1) follows from (2) since $A$ is a nonzero $A$-module. For (2), any submodule of minimal (finite) dimension as a $k$-vector space will be simple. There exists a finite dimensional one because a cyclic submodule is one. If $M$ is simple, then $mA \subseteq M$ is a sub-module, hence we see (3). Any nonzero element of $\text{End}_A(M)$ is an isomorphism, hence (4) holds. □

Theorem 3.3. Let $A$ be a simple finite $k$-algebra. Then $A$ is a matrix algebra over a finite $k$-algebra $K$ which is a skew field.

Proof. We may choose a simple submodule $M \subseteq A$ and then the $k$-algebra $K = \text{End}_A(M)$ is a skew field, see Lemma 3.2. By Lemma 3.1 we see that $A = \text{End}_K(M)$. Since $K$ is a skew field and $M$ is finitely generated (since $\dim_k(M) < \infty$) we see that $M$ is finite free as a left $K$-module. It follows immediately that $A \cong \text{Mat}(n \times n, K^{op})$. □

\footnote{This means that given $a'' \in A''$ and $m \in M$ we have a product $ma'' \in M$. In particular, the multiplication in $A''$ is the opposite of what you’d get if you wrote elements of $A''$ as endomorphisms acting on the left.}
4. Lemmas on algebras

Let $A$ be a $k$-algebra. Let $B \subset A$ be a subalgebra. The centralizer of $B$ in $A$ is the subalgebra
\[ C = \{ y \in A \mid xy = yx \text{ for all } x \in B \}. \]

It is a $k$-algebra.

**Lemma 4.1.** Let $A, A'$ be $k$-algebras. Let $B \subset A$, $B' \subset A'$ be subalgebras with centralizers $C, C'$. Then the centralizer of $B \otimes_k B'$ in $A \otimes_k A'$ is $C \otimes_k C'$.

**Proof.** Denote $C'' \subset A \otimes_k A'$ the centralizer of $B \otimes_k B'$. It is clear that $C \otimes_k C'' \subset C''$. Conversely, every element of $C''$ commutes with $B \otimes 1$ hence is contained in $C \otimes_k A'$. Similarly $C'' \subset A \otimes_k C''$. Thus $C'' \subset C \otimes_k A' \cap A \otimes_k C'' = C \otimes_k C''$. □

**Lemma 4.2.** Let $A$ be a finite simple $k$-algebra. Then the center $k'$ of $A$ is a finite field extension of $k$.

**Proof.** Write $A = \text{Mat}(n \times n, K)$ for some skew field $K$ finite over $k$, see Theorem 3.3. By Lemma 4.1 the center of $A$ is $k \otimes_k k'$ where $k' \subset K$ is the center of $K$. Since the center of a skew field is a field, we win. □

**Lemma 4.3.** Let $V$ be a $k$-vector space. Let $K$ be a central $k$-algebra which is a skew field. Let $W \subset V \otimes_k K$ be a two-sided $K$-sub vector space. Then $W$ is generated as a left $K$-vector space by $W \cap (V \otimes 1)$.

**Proof.** Let $V' \subset V$ be the $k$-sub vector space generated by $v \in V$ such that $v \otimes 1 \in W$. Then $V' \otimes_k K \subset W$ and we have
\[ W/(V' \otimes_k K) \subset (V/V') \otimes_k K. \]

If $\overline{v} \in V/V'$ is a nonzero vector such that $\overline{v} \otimes 1$ is contained in $W/(V' \otimes_k K)$, then we see that $v \otimes 1 \in W$ where $v \in V$ lifts $\overline{v}$. This contradicts our construction of $V'$. Hence we may replace $V$ by $V/V'$ and $W$ by $W/(V' \otimes_k K)$ and it suffices to prove that $W \cap (V \otimes 1)$ is nonzero if $W$ is nonzero.

To see this let $w \in W$ be a nonzero element which can be written as $w = \sum_{i=1}^n v_i \otimes k_i$ with $n$ minimal. We may right multiply with $k_1^{-1}$ and assume that $k_1 = 1$. If $n = 1$, then we win because $v_1 \otimes 1 \in W$. If $n > 1$, then we see that for any $c \in K$
\[ cw - wc = \sum_{i=2,\ldots,n} v_i \otimes (ck_i - k_ic) \in W \]

and hence $ck_i - k_ic = 0$ by minimality of $n$. This implies that $k_i$ is in the center of $K$ which is $k$ by assumption. Hence $w = (v_1 + \sum k_i v_i) \otimes 1$ contradicting the minimality of $n$. □

**Lemma 4.4.** Let $A$ be a $k$-algebra. Let $K$ be a central $k$-algebra which is a skew field. Then any two-sided ideal $I \subset A \otimes_k K$ is of the form $J \otimes_k K$ for some two-sided ideal $J \subset A$. In particular, if $A$ is simple, then so is $A \otimes_k K$.

**Proof.** Set $J = \{ a \in A \mid a \otimes 1 \in I \}$. This is a two-sided ideal of $A$. And $I = J \otimes_k K$ by Lemma 4.3. □

**Lemma 4.5.** Let $R$ be a possibly noncommutative ring. Let $n \geq 1$ be an integer.

Let $R_n = \text{Mat}(n \times n, R)$.

(1) The functors $M \mapsto M^{\otimes n}$ and $N \mapsto N e_{11}$ define quasi-inverse equivalences of categories $\text{Mod}_R \leftrightarrow \text{Mod}_{R_n}$. 
(2) A two-sided ideal of \( R_n \) is of the form \( tR_n \) for some two-sided ideal \( I \) of \( R \).
(3) The center of \( R_n \) is equal to the center of \( R \).

Proof. Part (1) proves itself. If \( J \subset R_n \) is a two-sided ideal, then \( J = \bigoplus e_{ij}Je_{jj} \) and all of the summands \( e_{ij}Je_{jj} \) are equal to each other and are a two-sided ideal \( I \) of \( R \). This proves (2). Part (3) is clear. \( \square \)

074E Lemma 4.6. Let \( A \) be a finite simple \( k \)-algebra.

(1) There exists exactly one simple \( A \)-module \( M \) up to isomorphism.
(2) Any finite \( A \)-module is a direct sum of copies of a simple module.
(3) Two finite \( A \)-modules are isomorphic if and only if they have the same dimension over \( k \).
(4) If \( A = \text{Mat}(n \times n, K) \) with \( K \) a finite skew field extension of \( k \), then \( M = K^{\oplus n} \) is a simple \( A \)-module and \( \text{End}_A(M) = K^{op} \).
(5) If \( M \) is a simple \( A \)-module, then \( L = \text{End}_A(M) \) is a skew field finite over \( k \) acting on the left on \( M \), we have \( A = \text{End}_L(M) \), and the centers of \( A \) and \( L \) agree. Also \( [A:k] = [L:k] = \dim_k(M)^2 \).
(6) For a finite \( A \)-module \( N \) the algebra \( B = \text{End}_A(N) \) is a matrix algebra over the skew field \( L \) of (5). Moreover \( \text{End}_B(N) = A \).

Proof. By Theorem 3.3 we can write \( A = \text{Mat}(n \times n, K) \) for some finite skew field extension \( K \) of \( k \). By Lemma 4.5 the category of modules over \( A \) is equivalent to the category of modules over \( K \). Thus (1), (2), and (3) hold because every module over \( K \) is free. Part (4) holds because the equivalence transforms the \( K \)-module \( M = K^{\oplus n} \) to \( M = K^{\oplus n} \). Using \( M = K^{\oplus n} \) in (5) we see that \( L = K^{op} \). The statement about the center of \( L = K^{op} \) follows from Lemma 4.5. The statement about \( \text{End}_L(M) \) follows from the explicit form of \( M \). The formula of dimensions is clear. Part (6) follows as \( N \) is isomorphic to a direct sum of copies of a simple module. \( \square \)

074F Lemma 4.7. Let \( A, A' \) be two simple \( k \)-algebras one of which is finite and central over \( k \). Then \( A \otimes_k A' \) is simple.

Proof. Suppose that \( A' \) is finite and central over \( k \). Write \( A' = \text{Mat}(n \times n, K') \), see Theorem 3.3 Then the center of \( K' \) is \( k \) and we conclude that \( A \otimes_k K' \) is simple by Lemma 4.4. Hence \( A \otimes_k A' = \text{Mat}(n \times n, A \otimes_k K') \) is simple by Lemma 4.5. \( \square \)

074G Lemma 4.8. The tensor product of finite central simple algebras over \( k \) is finite, central, and simple.

Proof. Combine Lemmas 4.1 and 4.7. \( \square \)

074H Lemma 4.9. Let \( A \) be a finite central simple algebra over \( k \). Let \( k \subset k' \) be a field extension. Then \( A' = A \otimes_k k' \) is a finite central simple algebra over \( k' \).

Proof. Combine Lemmas 4.1 and 4.7. \( \square \)

074I Lemma 4.10. Let \( A \) be a finite central simple algebra over \( k \). Then \( A \otimes_k A^{op} \cong \text{Mat}(n \times n, k) \) where \( n = [A:k] \).

Proof. By Lemma 4.8 the algebra \( A \otimes_k A^{op} \) is simple. Hence the map \( A \otimes_k A^{op} \to \text{End}_k(A), \ a \otimes a' \mapsto (x \mapsto axa') \) is injective. Since both sides of the arrow have the same dimension we win. \( \square \)
5. The Brauer group of a field

Let \( k \) be a field. Consider two finite central simple algebras \( A \) and \( B \) over \( k \). We say \( A \) and \( B \) are similar if there exist \( n, m > 0 \) such that \( \text{Mat}(n \times n, A) \cong \text{Mat}(m \times m, B) \) as \( k \)-algebras.

**Lemma 5.1.** Similarity.

1. Similarity defines an equivalence relation on the set of isomorphism classes of finite central simple algebras over \( k \).
2. Every similarity class contains a unique (up to isomorphism) finite central skew field extension of \( k \).
3. If \( A = \text{Mat}(n \times n, K) \) and \( B = \text{Mat}(m \times m, K') \) for some finite central skew fields \( K, K' \) over \( k \) then \( A \) and \( B \) are similar if and only if \( K \cong K' \) as \( k \)-algebras.

**Proof.** Note that by Wedderburn’s theorem (Theorem 3.3) we can always write a finite central simple algebra as a matrix algebra over a finite central skew field. Hence it suffices to prove the third assertion. To see this it suffices to show that if \( A = \text{Mat}(n \times n, K) \cong \text{Mat}(m \times m, K') = B \) then \( K \cong K' \). To see this note that for a simple module \( M \) of \( A \) we have \( \text{End}_A(M) = K^{\text{op}} \), see Lemma 4.6. Hence \( A \cong B \) implies \( K^{\text{op}} \cong (K')^{\text{op}} \) and we win.

Given two finite central simple \( k \)-algebras \( A, B \) the tensor product \( A \otimes_k B \) is another, see Lemma 4.8. Moreover if \( A \) is similar to \( A' \), then \( A \otimes_k B \) is similar to \( A' \otimes_k B \) because tensor products and taking matrix algebras commute. Hence tensor product defines an operation on equivalence classes of finite central simple algebras which is clearly associative and commutative. Finally, Lemma 4.10 shows that \( A \otimes_k A'^{\text{op}} \) is isomorphic to a matrix algebra, i.e., that \( A \otimes_k A'^{\text{op}} \) is in the similarity class of \( k \). Thus we obtain an abelian group.

**Definition 5.2.** Let \( k \) be a field. The Brauer group of \( k \) is the abelian group of similarity classes of finite central simple \( k \)-algebras defined above. Notation \( \text{Br}(k) \).

For any map of fields \( k \to k' \) we obtain a group homomorphism

\[
\text{Br}(k) \to \text{Br}(k'), \quad A \mapsto A \otimes_k k'
\]

see Lemma 4.9. In other words, \( \text{Br}(-) \) is a functor from the category of fields to the category of abelian groups. Observe that the Brauer group of a field is zero if and only if every finite central skew field extension \( k \subset K \) is trivial.

**Lemma 5.3.** The Brauer group of an algebraically closed field is zero.

**Proof.** Let \( k \subset K \) be a finite central skew field extension. For any element \( x \in K \) the subring \( k[x] \subset K \) is a commutative finite integral \( k \)-sub algebra, hence a field, see Algebra, Lemma 35.19. Since \( k \) is algebraically closed we conclude that \( k[x] = k \).

Since \( x \) was arbitrary we conclude \( k = K \).

**Lemma 5.4.** Let \( A \) be a finite central simple algebra over a field \( k \). Then \( [A : k] \) is a square.

**Proof.** This is true because \( A \otimes_k \overline{k} \) is a matrix algebra over \( \overline{k} \) by Lemma 5.3.
6. Skolem-Noether

**Theorem 6.1.** Let \( A \) be a finite central simple \( k \)-algebra. Let \( B \) be a simple \( k \)-algebra. Let \( f, g : B \to A \) be two \( k \)-algebra homomorphisms. Then there exists an invertible element \( x \in A \) such that \( f(b) = xg(b)x^{-1} \) for all \( b \in B \).

**Proof.** Choose a simple \( A \)-module \( M \). Set \( L = \text{End}_A(M) \). Then \( L \) is a skew field with center \( k \) which acts on the left on \( M \), see Lemmas 3.2 and 4.6. Then \( M \) has two \( B \otimes_k L^{\text{op}} \)-module structures defined by \( m_1(b \otimes l) = lmf(b) \) and \( m_2(b \otimes l) = lmg(b) \). The \( k \)-algebra \( B \otimes_k L^{\text{op}} \) is simple by Lemma 4.7. Since \( B \) is simple, the existence of a \( k \)-algebra homomorphism \( B \to A \) implies that \( B \) is finite. Thus \( B \otimes_k L^{\text{op}} \) is finite simple and we conclude the two \( B \otimes_k L^{\text{op}} \)-module structures on \( M \) are isomorphic by Lemma 4.6. Hence we find \( \varphi : M \to M \) intertwining these operations. In particular \( \varphi \) is in the commutant of \( L \) which implies that \( \varphi \) is multiplication by some \( x \in A \), see Lemma 4.6. Working out the definitions we see that \( x \) is a solution to our problem. \( \square \)

**Lemma 6.2.** Let \( A \) be a finite central simple \( k \)-algebra. Any automorphism of \( A \) is inner. In particular, any automorphism of \( \text{Mat}(n \times n, k) \) is inner.

**Proof.** Note that \( A \) is a finite central simple algebra over the center of \( A \) which is a finite field extension of \( k \), see Lemma 4.2. Hence the Skolem-Noether theorem (Theorem 6.1) applies. \( \square \)

7. The centralizer theorem

**Theorem 7.1.** Let \( A \) be a finite central simple algebra over \( k \), and let \( B \) be a simple subalgebra of \( A \). Then

1. the centralizer \( C \) of \( B \) in \( A \) is simple,
2. \( [A : k] = [B : k][C : k] \), and
3. the centralizer of \( C \) in \( A \) is \( B \).

**Proof.** Throughout this proof we use the results of Lemma 4.6 freely. Choose a simple \( A \)-module \( M \). Set \( L = \text{End}_A(M) \). Then \( L \) is a skew field with center \( k \) which acts on the left on \( M \) and \( A = \text{End}_L(M) \). Then \( M \) is a right \( B \otimes_k L^{\text{op}} \)-module and \( C = \text{End}_{B \otimes_k L^{\text{op}}}(M) \). Since the algebra \( B \otimes_k L^{\text{op}} \) is simple by Lemma 4.7, we see that \( C \) is simple (by Lemma 4.6 again).

Write \( B \otimes_k L^{\text{op}} = \text{Mat}(m \times m, K) \) for some skew field \( K \) finite over \( k \). Then \( C = \text{Mat}(n \times n, K^{\text{op}}) \) if \( M \) is isomorphic to a direct sum of \( n \) copies of the simple \( B \otimes_k L^{\text{op}} \)-module \( K^{\otimes m} \) (the lemma again). Thus we have \( \dim_k(M) = nm[K : k] \), \( [B : k][L : k] = m^2[K : k] \), \( [C : k] = n^2[K : k] \), and \( [A : k][L : k] = \dim_k(M)^2 \) (by the lemma again). We conclude that (2) holds.

Part (3) follows because of (2) applied to \( C \subset A \) shows that \( [B : k] = [C' : k] \) where \( C' \) is the centralizer of \( C \) in \( A \) (and the obvious fact that \( B \subset C' \)). \( \square \)

**Lemma 7.2.** Let \( A \) be a finite central simple algebra over \( k \), and let \( B \) be a simple subalgebra of \( A \). If \( B \) is a central \( k \)-algebra, then \( A = B \otimes_k C \) where \( C \) is the (central simple) centralizer of \( B \) in \( A \).
Proof. We have \( \dim_k(A) = \dim_k(B \otimes_k C) \) by Theorem 7.1. By Lemma 4.7, the tensor product is simple. Hence the natural map \( B \otimes_k C \to A \) is injective hence an isomorphism. □

Lemma 7.3. Let \( A \) be a finite central simple algebra over \( k \). If \( K \subset A \) is a subfield, then the following are equivalent

1. \( [A : k] = [K : k]^2 \),
2. \( K \) is its own centralizer, and
3. \( K \) is a maximal commutative subring.

Proof. Theorem 7.1 shows that (1) and (2) are equivalent. It is clear that (3) and (2) are equivalent. □

Lemma 7.4. Let \( A \) be a finite central skew field over \( k \). Then every maximal subfield \( K \subset A \) satisfies \( [A : k] = [K : k]^2 \).

Proof. Special case of Lemma 7.3. □

8. Splitting fields

Definition 8.1. Let \( A \) be a finite central simple \( k \)-algebra. We say a field extension \( k \subset k' \) splits \( A \), or \( k' \) is a splitting field for \( A \), if \( A \otimes_k k' \) is a matrix algebra over \( k' \). Another way to say this is that the class of \( A \) maps to zero under the map \( Br(k) \to Br(k') \).

Theorem 8.2. Let \( A \) be a finite central simple \( k \)-algebra. Let \( k \subset k' \) be a finite field extension. The following are equivalent

1. \( k' \) splits \( A \), and
2. there exists a finite central simple algebra \( B \) similar to \( A \) such that \( k' \subset B \) and \( [B : k] = [k' : k]^2 \).

Proof. Assume (2). It suffices to show that \( B \otimes_k k' \) is a matrix algebra. We know that \( B \otimes_k B^{op} \cong \text{End}_k(B) \). Since \( k' \) is the centralizer of \( k' \) in \( B^{op} \) by Lemma 7.3, we see that \( B \otimes_k k' \) is the centralizer of \( k \otimes k' \) in \( B \otimes_k B^{op} = \text{End}_k(B) \). Of course this centralizer is just \( \text{End}_k(B) \) where we view \( B \) as a \( k' \)-vector space via the embedding \( k' \to B \). Thus the result.

Assume (1). This means that we have an isomorphism \( A \otimes_k k' \cong \text{End}_{k'}(V) \) for some \( k' \)-vector space \( V \). Let \( B \) be the commutant of \( A \) in \( \text{End}_k(V) \). Note that \( k' \) sits in \( B \). By Lemma 7.2, the classes of \( A \) and \( B \) add up to zero in \( Br(k) \). From the dimension formula in Theorem 7.1 we see that

\[
[B : k][A : k] = \dim_k(V)^2 = [k' : k]^2 \dim_k(V)^2 = [k' : k]^2[A : k].
\]

Hence \( [B : k] = [k' : k]^2 \). Thus we have proved the result for the opposite to the Brauer class of \( A \). However, \( k' \) splits the Brauer class of \( A \) if and only if it splits the Brauer class of the opposite algebra, so we win anyway. □

Lemma 8.3. A maximal subfield of a finite central skew field \( K \) over \( k \) is a splitting field for \( K \).

Proof. Combine Lemma 7.4 with Theorem 8.2 □
Let $k$ be a field. For a $k$-algebra $A$ the following are equivalent

1. $A$ is finite central simple $k$-algebra,
2. $A$ is a finite dimensional $k$-vector space, $k$ is the center of $A$, and $A$ has no nontrivial two-sided ideal,
3. there exists $d \geq 1$ such that $A \otimes_k \bar{k} \cong \text{Mat}(d \times d, \bar{k})$,
4. there exists $d \geq 1$ such that $A \otimes_k k^{\text{sep}} \cong \text{Mat}(d \times d, k^{\text{sep}})$,
5. there exist $d \geq 1$ and a finite Galois extension $k \subset k'$ such that $A \otimes_k k' \cong \text{Mat}(d \times d, k')$,
6. there exist $n \geq 1$ and a finite skew field $K$ over $k$ such that $A \cong \text{Mat}(n \times n, K)$.
The integer \( d \) is called the degree of \( A \).

**Proof.** The equivalence of (1) and (2) is a consequence of the definitions, see Section 2. Assume (1). By Proposition 8.5 there exists a separable splitting field \( k \subset k' \) for \( A \). Of course, then a Galois closure of \( k'/k \) is a splitting field also. Thus we see that (1) implies (5). It is clear that (5) \( \Rightarrow \) (4) \( \Rightarrow \) (3). Assume (3). Then \( A \otimes_k k' \) is a finite central simple \( k' \)-algebra for example by Lemma 4.5. This trivially implies that \( A \) is a finite central simple \( k \)-algebra. Finally, the equivalence of (1) and (6) is Wedderburn’s theorem, see Theorem 3.3. \( \square \)

9. Other chapters

Preliminaries

(1) Introduction
(2) Conventions
(3) Set Theory
(4) Categories
(5) Topology
(6) Sheaves on Spaces
(7) Sites and Sheaves
(8) Stacks
(9) Fields
(10) Commutative Algebra
(11) Brauer Groups
(12) Homological Algebra
(13) Derived Categories
(14) Simplicial Methods
(15) More on Algebra
(16) Smoothing Ring Maps
(17) Sheaves of Modules
(18) Modules on Sites
(19) Injectives
(20) Cohomology of Sheaves
(21) Cohomology on Sites
(22) Differential Graded Algebra
(23) Divided Power Algebra
(24) Differential Graded Sheaves
(25) Hypercoverings

Schemes

(25) Schemes
(26) Constructions of Schemes
(27) Properties of Schemes
(28) Morphisms of Schemes
(29) Cohomology of Schemes
(30) Divisors
(31) Limits of Schemes
(32) Varieties
(33) Topologies on Schemes
(34) Descent

(35) Derived Categories of Schemes
(36) More on Morphisms
(37) More on Flatness
(38) Groupoid Schemes
(39) More on Groupoid Schemes
(40) Étale Morphisms of Schemes

Topics in Scheme Theory

(41) Chow Homology
(42) Intersection Theory
(43) Picard Schemes of Curves
(44) Well Cohomology Theories
(45) Adequate Modules
(46) Dualizing Complexes
(47) Duality for Schemes
(48) Discriminants and Differents
(49) de Rham Cohomology
(50) Local Cohomology
(51) Algebraic and Formal Geometry
(52) Algebraic Curves
(53) Resolution of Surfaces
(54) Semistable Reduction
(55) Fundamental Groups of Schemes
(56) Étale Cohomology
(57) Crystalline Cohomology
(58) Pro-étale Cohomology
(59) More Étale Cohomology
(60) The Trace Formula

Algebraic Spaces

(61) Algebraic Spaces
(62) Properties of Algebraic Spaces
(63) Morphisms of Algebraic Spaces
(64) Decent Algebraic Spaces
(65) Cohomology of Algebraic Spaces
(66) Limits of Algebraic Spaces
(67) Divisors on Algebraic Spaces
(68) Algebraic Spaces over Fields
| (69) Topologies on Algebraic Spaces | (91) Examples of Stacks |
| (70) Descent and Algebraic Spaces  | (92) Sheaves on Algebraic Stacks |
| (71) Derived Categories of Spaces  | (93) Criteria for Representability |
| (72) More on Morphisms of Spaces   | (94) Artin’s Axioms |
| (73) Flatness on Algebraic Spaces  | (95) Quot and Hilbert Spaces |
| (74) Groupoids in Algebraic Spaces | (96) Properties of Algebraic Stacks |
| (75) More on Groupoids in Spaces  | (97) Morphisms of Algebraic Stacks |
| (76) Bootstrap                     | (98) Limits of Algebraic Stacks |
| (77) Pushouts of Algebraic Spaces  | (99) Cohomology of Algebraic Stacks |

**Topics in Geometry**

| (78) Chow Groups of Spaces         | (100) Derived Categories of Stacks |
| (79) Quotients of Groupoids        | (101) Introducing Algebraic Stacks |
| (80) More on Cohomology of Spaces  | (102) More on Morphisms of Stacks |
| (81) Simplicial Spaces             | (103) The Geometry of Stacks |
| (82) Duality for Spaces            |                                 |
| (83) Formal Algebraic Spaces       |                                 |
| (84) Restricted Power Series       |                                 |
| (85) Resolution of Surfaces Revisited|                                 |

**Deformation Theory**

| (86) Formal Deformation Theory     | (106) Examples               |
| (87) Deformation Theory            | (107) Exercises              |
| (88) The Cotangent Complex         | (108) Guide to Literature    |
| (89) Deformation Problems          | (109) Desirables             |
|                                   | (110) Coding Style           |
|                                   | (111) Obsolete               |
|                                   | (112) GNU Free Documentation License |

**Algebraic Stacks**

| (90) Algebraic Stacks             | (113) Auto Generated Index |

**References**


