BRAUER GROUPS

073W

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1. Introduction

073X A reference is the lectures by Serre in the Seminaire Cartan, see [Ser55]. Serre in turn refers to [Deu68] and [ANT44]. We changed some of the proofs, in particular we used a fun argument of Rieffel to prove Wedderburn's theorem. Very likely this change is not an improvement and we strongly encourage the reader to read the original exposition by Serre.

2. Noncommutative algebras

073Y Let k be a field. In this chapter an *algebra* A over k is a possibly noncommutative ring A together with a ring map $k \to A$ such that k maps into the center of A and such that 1 maps to an identity element of A. An *A-module* is a right A-module such that the identity of A acts as the identity.

073Z **Definition 2.1.** Let A be a k-algebra. We say A is *finite* if $\dim_k(A) < \infty$. In this case we write $[A : k] = \dim_k(A)$.

0740 **Definition 2.2.** A *skew field* is a possibly noncommutative ring with an identity element 1, with $1 \neq 0$, in which every nonzero element has a multiplicative inverse.

A skew field is a k-algebra for some k (e.g., for the prime field contained in it). We will use below that any module over a skew field is free because a maximal linearly independent set of vectors forms a basis and exists by Zorn's lemma.

0741 **Definition 2.3.** Let A be a k-algebra. We say an A-module M is *simple* if it is nonzero and the only A-submodules are 0 and M. We say A is *simple* if the only two-sided ideals of A are 0 and A.

0742 **Definition 2.4.** A k-algebra A is *central* if the center of A is the image of $k \to A$.

0743 **Definition 2.5.** Given a k-algebra A we denote A^{op} the k-algebra we get by reversing the order of multiplication in A. This is called the *opposite algebra*.

3. Wedderburn's theorem

0744 The following cute argument can be found in a paper of Rieffel, see [Rie65]. The proof could not be simpler (quote from Carl Faith's review).

0745 **Lemma 3.1.** *Let A be a possibly noncommutative ring with 1 which contains no nontrivial two-sided ideal. Let M be a nonzero right ideal in A, and view M as a right A-module. Then A coincides with the bicommutant of M.*

Proof. Let $A' = \mathrm{End}_A(M)$, so M is a left A'-module. Set $A'' = \mathrm{End}_{A'}(M)$ (the bicommutant of M). We view M as a right A''-module[1]. Let $R : A \to A''$ be the natural homomorphism such that $mR(a) = ma$. Then R is injective, since $R(1) = \mathrm{id}_M$ and A contains no nontrivial two-sided ideal. We claim that $R(M)$ is a right ideal in A''. Namely, $R(m)a'' = R(ma'')$ for $a'' \in A''$ and m in M, because *left* multiplication of M by any element n of M represents an element of A', and so $(nm)a'' = n(ma'')$ for all n in M. Finally, the product ideal AM is a two-sided ideal, and so $A = AM$. Thus $R(A) = R(A)R(M)$, so that $R(A)$ is a right ideal in A''. But $R(A)$ contains the identity element of A'', and so $R(A) = A''$. □

0746 **Lemma 3.2.** *Let A be a k-algebra. If A is finite, then*

(1) *A has a simple module,*
(2) *any nonzero module contains a simple submodule,*
(3) *a simple module over A has finite dimension over k, and*
(4) *if M is a simple A-module, then $End_A(M)$ is a skew field.*

Proof. Of course (1) follows from (2) since A is a nonzero A-module. For (2), any submodule of minimal (finite) dimension as a k-vector space will be simple. There exists a finite dimensional one because a cyclic submodule is one. If M is simple, then $mA \subset M$ is a sub-module, hence we see (3). Any nonzero element of $\mathrm{End}_A(M)$ is an isomorphism, hence (4) holds. □

0747 **Theorem 3.3.** *Let A be a simple finite k-algebra. Then A is a matrix algebra over a finite k-algebra K which is a skew field.*

Proof. We may choose a simple submodule $M \subset A$ and then the k-algebra $K = \mathrm{End}_A(M)$ is a skew field, see Lemma 3.2. By Lemma 3.1 we see that $A = \mathrm{End}_K(M)$. Since K is a skew field and M is finitely generated (since $\dim_k(M) < \infty$) we see that M is finite free as a left K-module. It follows immediately that $A \cong \mathrm{Mat}(n \times n, K^{op})$. □

4. Lemmas on algebras

0748 Let A be a k-algebra. Let $B \subset A$ be a subalgebra. The *centralizer of B in A* is the subalgebra

$$C = \{y \in A \mid xy = yx \text{ for all } x \in B\}.$$

It is a k-algebra.

[1] This means that given $a'' \in A''$ and $m \in M$ we have a product $ma'' \in M$. In particular, the multiplication in A'' is the opposite of what you'd get if you wrote elements of A'' as endomorphisms acting on the left.

0749 **Lemma 4.1.** *Let A, A' be k-algebras. Let $B \subset A$, $B' \subset A'$ be subalgebras with centralizers C, C'. Then the centralizer of $B \otimes_k B'$ in $A \otimes_k A'$ is $C \otimes_k C'$.*

Proof. Denote $C'' \subset A \otimes_k A'$ the centralizer of $B \otimes_k B'$. It is clear that $C \otimes_k C' \subset C''$. Conversely, every element of C'' commutes with $B \otimes 1$ hence is contained in $C \otimes_k A'$. Similarly $C'' \subset A \otimes_k C'$. Thus $C'' \subset C \otimes_k A' \cap A \otimes_k C' = C \otimes_k C'$. □

074A **Lemma 4.2.** *Let A be a finite simple k-algebra. Then the center k' of A is a finite field extension of k.*

Proof. Write $A = \mathrm{Mat}(n \times n, K)$ for some skew field K finite over k, see Theorem 3.3. By Lemma 4.1 the center of A is $k \otimes_k k'$ where $k' \subset K$ is the center of K. Since the center of a skew field is a field, we win. □

074B **Lemma 4.3.** *Let V be a k vector space. Let K be a central k-algebra which is a skew field. Let $W \subset V \otimes_k K$ be a two-sided K-sub vector space. Then W is generated as a left K-vector space by $W \cap (V \otimes 1)$.*

Proof. Let $V' \subset V$ be the k-sub vector space generated by $v \in V$ such that $v \otimes 1 \in W$. Then $V' \otimes_k K \subset W$ and we have

$$W/(V' \otimes_k K) \subset (V/V') \otimes_k K.$$

If $\overline{v} \in V/V'$ is a nonzero vector such that $\overline{v} \otimes 1$ is contained in $W/(V' \otimes_k K)$, then we see that $v \otimes 1 \in W$ where $v \in V$ lifts $\overline{v}$. This contradicts our construction of V'. Hence we may replace V by V/V' and W by $W/(V' \otimes_k K)$ and it suffices to prove that $W \cap (V \otimes 1)$ is nonzero if W is nonzero.

To see this let $w \in W$ be a nonzero element which can be written as $w = \sum_{i=1,\ldots,n} v_i \otimes k_i$ with n minimal. We may right multiply with k_1^{-1} and assume that $k_1 = 1$. If $n = 1$, then we win because $v_1 \otimes 1 \in W$. If $n > 1$, then we see that for any $c \in K$

$$cw - wc = \sum\nolimits_{i=2,\ldots,n} v_i \otimes (ck_i - k_i c) \in W$$

and hence $ck_i - k_i c = 0$ by minimality of n. This implies that k_i is in the center of K which is k by assumption. Hence $w = (v_1 + \sum k_i v_i) \otimes 1$ contradicting the minimality of n. □

074C **Lemma 4.4.** *Let A be a k-algebra. Let K be a central k-algebra which is a skew field. Then any two-sided ideal $I \subset A \otimes_k K$ is of the form $J \otimes_k K$ for some two-sided ideal $J \subset A$. In particular, if A is simple, then so is $A \otimes_k K$.*

Proof. Set $J = \{a \in A \mid a \otimes 1 \in I\}$. This is a two-sided ideal of A. And $I = J \otimes_k K$ by Lemma 4.3. □

074D **Lemma 4.5.** *Let R be a possibly noncommutative ring. Let $n \geq 1$ be an integer. Let $R_n = Mat(n \times n, R)$.*

1. *The functors $M \mapsto M^{\oplus n}$ and $N \mapsto Ne_{11}$ define quasi-inverse equivalences of categories $Mod_R \leftrightarrow Mod_{R_n}$.*
2. *A two-sided ideal of R_n is of the form IR_n for some two-sided ideal I of R.*
3. *The center of R_n is equal to the center of R.*

Proof. Part (1) proves itself. If $J \subset R_n$ is a two-sided ideal, then $J = \bigoplus e_{ii} J e_{jj}$ and all of the summands $e_{ii} J e_{jj}$ are equal to each other and are a two-sided ideal I of R. This proves (2). Part (3) is clear. □

074E **Lemma 4.6.** *Let A be a finite simple k-algebra.*

(1) *There exists exactly one simple A-module M up to isomorphism.*
(2) *Any finite A-module is a direct sum of copies of a simple module.*
(3) *Two finite A-modules are isomorphic if and only if they have the same dimension over k.*
(4) *If $A = Mat(n \times n, K)$ with K a finite skew field extension of k, then $M = K^{\oplus n}$ is a simple A-module and $End_A(M) = K^{op}$.*
(5) *If M is a simple A-module, then $L = End_A(M)$ is a skew field finite over k acting on the left on M, we have $A = End_L(M)$, and the centers of A and L agree. Also $[A : k][L : k] = \dim_k(M)^2$.*
(6) *For a finite A-module N the algebra $B = End_A(N)$ is a matrix algebra over the skew field L of (5). Moreover $End_B(N) = A$.*

Proof. By Theorem 3.3 we can write $A = \mathrm{Mat}(n \times n, K)$ for some finite skew field extension K of k. By Lemma 4.5 the category of modules over A is equivalent to the category of modules over K. Thus (1), (2), and (3) hold because every module over K is free. Part (4) holds because the equivalence transforms the K-module K to $M = K^{\oplus n}$. Using $M = K^{\oplus n}$ in (5) we see that $L = K^{op}$. The statement about the center of $L = K^{op}$ follows from Lemma 4.5. The statement about $\mathrm{End}_L(M)$ follows from the explicit form of M. The formula of dimensions is clear. Part (6) follows as N is isomorphic to a direct sum of copies of a simple module. □

074F **Lemma 4.7.** *Let A, A' be two simple k-algebras one of which is finite and central over k. Then $A \otimes_k A'$ is simple.*

Proof. Suppose that A' is finite and central over k. Write $A' = \mathrm{Mat}(n \times n, K')$, see Theorem 3.3. Then the center of K' is k and we conclude that $A \otimes_k K'$ is simple by Lemma 4.4. Hence $A \otimes_k A' = \mathrm{Mat}(n \times n, A \otimes_k K')$ is simple by Lemma 4.5. □

074G **Lemma 4.8.** *The tensor product of finite central simple algebras over k is finite, central, and simple.*

Proof. Combine Lemmas 4.1 and 4.7. □

074H **Lemma 4.9.** *Let A be a finite central simple algebra over k. Let k'/k be a field extension. Then $A' = A \otimes_k k'$ is a finite central simple algebra over k'.*

Proof. Combine Lemmas 4.1 and 4.7. □

074I **Lemma 4.10.** *Let A be a finite central simple algebra over k. Then $A \otimes_k A^{op} \cong Mat(n \times n, k)$ where $n = [A : k]$.*

Proof. By Lemma 4.8 the algebra $A \otimes_k A^{op}$ is simple. Hence the map

$$A \otimes_k A^{op} \longrightarrow \mathrm{End}_k(A), \quad a \otimes a' \longmapsto (x \mapsto axa')$$

is injective. Since both sides of the arrow have the same dimension we win. □

5. The Brauer group of a field

074J Let k be a field. Consider two finite central simple algebras A and B over k. We say A and B are *similar* if there exist $n, m > 0$ such that $\mathrm{Mat}(n \times n, A) \cong \mathrm{Mat}(m \times m, B)$ as k-algebras.

074K **Lemma 5.1.** *Similarity.*

(1) *Similarity defines an equivalence relation on the set of isomorphism classes of finite central simple algebras over k.*
(2) *Every similarity class contains a unique (up to isomorphism) finite central skew field extension of k.*
(3) *If $A = Mat(n \times n, K)$ and $B = Mat(m \times m, K')$ for some finite central skew fields K, K' over k then A and B are similar if and only if $K \cong K'$ as k-algebras.*

Proof. Note that by Wedderburn's theorem (Theorem 3.3) we can always write a finite central simple algebra as a matrix algebra over a finite central skew field. Hence it suffices to prove the third assertion. To see this it suffices to show that if $A = \mathrm{Mat}(n \times n, K) \cong \mathrm{Mat}(m \times m, K') = B$ then $K \cong K'$. To see this note that for a simple module M of A we have $\mathrm{End}_A(M) = K^{op}$, see Lemma 4.6. Hence $A \cong B$ implies $K^{op} \cong (K')^{op}$ and we win. □

Given two finite central simple k-algebras A, B the tensor product $A \otimes_k B$ is another, see Lemma 4.8. Moreover if A is similar to A', then $A \otimes_k B$ is similar to $A' \otimes_k B$ because tensor products and taking matrix algebras commute. Hence tensor product defines an operation on equivalence classes of finite central simple algebras which is clearly associative and commutative. Finally, Lemma 4.10 shows that $A \otimes_k A^{op}$ is isomorphic to a matrix algebra, i.e., that $A \otimes_k A^{op}$ is in the similarity class of k. Thus we obtain an abelian group.

074L **Definition 5.2.** Let k be a field. The *Brauer group* of k is the abelian group of similarity classes of finite central simple k-algebras defined above. Notation $\mathrm{Br}(k)$.

For any map of fields $k \to k'$ we obtain a group homomorphism

$$\mathrm{Br}(k) \longrightarrow \mathrm{Br}(k'), \quad A \longmapsto A \otimes_k k'$$

see Lemma 4.9. In other words, $\mathrm{Br}(-)$ is a functor from the category of fields to the category of abelian groups. Observe that the Brauer group of a field is zero if and only if every finite central skew field extension $k \subset K$ is trivial.

074M **Lemma 5.3.** *The Brauer group of an algebraically closed field is zero.*

Proof. Let $k \subset K$ be a finite central skew field extension. For any element $x \in K$ the subring $k[x] \subset K$ is a commutative finite integral k-sub algebra, hence a field, see Algebra, Lemma 36.19. Since k is algebraically closed we conclude that $k[x] = k$. Since x was arbitrary we conclude $k = K$. □

074N **Lemma 5.4.** *Let A be a finite central simple algebra over a field k. Then $[A : k]$ is a square.*

Proof. This is true because $A \otimes_k \overline{k}$ is a matrix algebra over $\overline{k}$ by Lemma 5.3. □

6. Skolem-Noether

074P

074Q **Theorem 6.1.** *Let A be a finite central simple k-algebra. Let B be a simple k-algebra. Let $f, g : B \to A$ be two k-algebra homomorphisms. Then there exists an invertible element $x \in A$ such that $f(b) = xg(b)x^{-1}$ for all $b \in B$.*

Proof. Choose a simple A-module M. Set $L = \mathrm{End}_A(M)$. Then L is a skew field with center k which acts on the left on M, see Lemmas 3.2 and 4.6. Then M has two $B \otimes_k L^{op}$-module structures defined by $m \cdot_1 (b \otimes l) = lmf(b)$ and $m \cdot_2 (b \otimes l) = lmg(b)$. The k-algebra $B \otimes_k L^{op}$ is simple by Lemma 4.7. Since B is simple, the existence of a k-algebra homomorphism $B \to A$ implies that B is finite. Thus $B \otimes_k L^{op}$ is finite simple and we conclude the two $B \otimes_k L^{op}$-module structures on M are isomorphic by Lemma 4.6. Hence we find $\varphi : M \to M$ intertwining these operations. In particular φ is in the commutant of L which implies that φ is multiplication by some $x \in A$, see Lemma 4.6. Working out the definitions we see that x is a solution to our problem. □

074R **Lemma 6.2.** *Let A be a finite central simple k-algebra. Any automorphism of A is inner. In particular, any automorphism of $Mat(n \times n, k)$ is inner.*

Proof. The Skolem-Noether theorem (Theorem 6.1) applies. □

7. The centralizer theorem

074S

074T **Theorem 7.1.** *Let A be a finite central simple algebra over k, and let B be a simple subalgebra of A. Then*

1. *the centralizer C of B in A is simple,*
2. *$[A : k] = [B : k][C : k]$, and*
3. *the centralizer of C in A is B.*

Proof. Throughout this proof we use the results of Lemma 4.6 freely. Choose a simple A-module M. Set $L = \mathrm{End}_A(M)$. Then L is a skew field with center k which acts on the left on M and $A = \mathrm{End}_L(M)$. Then M is a right $B \otimes_k L^{op}$-module and $C = \mathrm{End}_{B \otimes_k L^{op}}(M)$. Since the algebra $B \otimes_k L^{op}$ is simple by Lemma 4.7 we see that C is simple (by Lemma 4.6 again).

Write $B \otimes_k L^{op} = \mathrm{Mat}(m \times m, K)$ for some skew field K finite over k. Then $C = \mathrm{Mat}(n \times n, K^{op})$ if M is isomorphic to a direct sum of n copies of the simple $B \otimes_k L^{op}$-module $K^{\oplus m}$ (the lemma again). Thus we have $\dim_k(M) = nm[K : k]$, $[B : k][L : k] = m^2[K : k]$, $[C : k] = n^2[K : k]$, and $[A : k][L : k] = \dim_k(M)^2$ (by the lemma again). We conclude that (2) holds.

Part (3) follows because of (2) applied to $C \subset A$ shows that $[B : k] = [C' : k]$ where C' is the centralizer of C in A (and the obvious fact that $B \subset C'$). □

074U **Lemma 7.2.** *Let A be a finite central simple algebra over k, and let B be a simple subalgebra of A. If B is a central k-algebra, then $A = B \otimes_k C$ where C is the (central simple) centralizer of B in A.*

Proof. We have $\dim_k(A) = \dim_k(B \otimes_k C)$ by Theorem 7.1. By Lemma 4.7 the tensor product is simple. Hence the natural map $B \otimes_k C \to A$ is injective hence an isomorphism. □

074V **Lemma 7.3.** *Let A be a finite central simple algebra over k. If $K \subset A$ is a subfield, then the following are equivalent*

1. *$[A : k] = [K : k]^2$,*
2. *K is its own centralizer, and*
3. *K is a maximal commutative subring.*

Proof. Theorem 7.1 shows that (1) and (2) are equivalent. It is clear that (3) and (2) are equivalent. □

074W **Lemma 7.4.** *Let A be a finite central skew field over k. Then every maximal subfield $K \subset A$ satisfies $[A : k] = [K : k]^2$.*

Proof. Special case of Lemma 7.3. □

8. Splitting fields

074X

074Y **Definition 8.1.** Let A be a finite central simple k-algebra. We say a field extension k'/k *splits* A, or k' is a *splitting field* for A if $A \otimes_k k'$ is a matrix algebra over k'.

Another way to say this is that the class of A maps to zero under the map $\mathrm{Br}(k) \to \mathrm{Br}(k')$.

074Z **Theorem 8.2.** *Let A be a finite central simple k-algebra. Let k'/k be a finite field extension. The following are equivalent*

1. *k' splits A, and*
2. *there exists a finite central simple algebra B similar to A such that $k' \subset B$ and $[B : k] = [k' : k]^2$.*

Proof. Assume (2). It suffices to show that $B \otimes_k k'$ is a matrix algebra. We know that $B \otimes_k B^{op} \cong \mathrm{End}_k(B)$. Since k' is the centralizer of k' in B^{op} by Lemma 7.3 we see that $B \otimes_k k'$ is the centralizer of $k \otimes k'$ in $B \otimes_k B^{op} = \mathrm{End}_k(B)$. Of course this centralizer is just $\mathrm{End}_{k'}(B)$ where we view B as a k' vector space via the embedding $k' \to B$. Thus the result.

Assume (1). This means that we have an isomorphism $A \otimes_k k' \cong \mathrm{End}_{k'}(V)$ for some k'-vector space V. Let B be the commutant of A in $\mathrm{End}_k(V)$. Note that k' sits in B. By Lemma 7.2 the classes of A and B add up to zero in $\mathrm{Br}(k)$. From the dimension formula in Theorem 7.1 we see that

$$[B : k][A : k] = \dim_k(V)^2 = [k' : k]^2 \dim_{k'}(V)^2 = [k' : k]^2[A : k].$$

Hence $[B : k] = [k' : k]^2$. Thus we have proved the result for the opposite to the Brauer class of A. However, k' splits the Brauer class of A if and only if it splits the Brauer class of the opposite algebra, so we win anyway. □

0750 **Lemma 8.3.** *A maximal subfield of a finite central skew field K over k is a splitting field for K.*

Proof. Combine Lemma 7.4 with Theorem 8.2. □

0751 **Lemma 8.4.** *Consider a finite central skew field K over k. Let $d^2 = [K : k]$. For any finite splitting field k' for K the degree $[k' : k]$ is divisible by d.*

Proof. By Theorem 8.2 there exists a finite central simple algebra B in the Brauer class of K such that $[B : k] = [k' : k]^2$. By Lemma 5.1 we see that $B = \mathrm{Mat}(n \times n, K)$ for some n. Then $[k' : k]^2 = n^2 d^2$ whence the result. □

0752 **Proposition 8.5.** *Consider a finite central skew field K over k. There exists a maximal subfield $k \subset k' \subset K$ which is separable over k. In particular, every Brauer class has a finite separable spitting field.*

Proof. Since every Brauer class is represented by a finite central skew field over k, we see that the second statement follows from the first by Lemma 8.3.

To prove the first statement, suppose that we are given a separable subfield $k' \subset K$. Then the centralizer K' of k' in K has center k', and the problem reduces to finding a maximal subfield of K' separable over k'. Thus it suffices to prove, if $k \neq K$, that we can find an element $x \in K$, $x \notin k$ which is separable over k. This statement is clear in characteristic zero. Hence we may assume that k has characteristic $p > 0$. If the ground field k is finite then, the result is clear as well (because extensions of finite fields are always separable). Thus we may assume that k is an infinite field of positive characteristic.

To get a contradiction assume no element of K is separable over k. By the discussion in Fields, Section 28 this means the minimal polynomial of any $x \in K$ is of the form $T^q - a$ where q is a power of p and $a \in k$. Since it is clear that every element of K has a minimal polynomial of degree $\leq \dim_k(K)$ we conclude that there exists a fixed p-power q such that $x^q \in k$ for all $x \in K$.

Consider the map

$$(-)^q : K \longrightarrow K$$

and write it out in terms of a k-basis $\{a_1, \ldots, a_n\}$ of K with $a_1 = 1$. So

$$(\sum x_i a_i)^q = \sum f_i(x_1, \ldots, x_n) a_i.$$

Since multiplication on K is k-bilinear we see that each f_i is a polynomial in $x_1, \ldots, x_n$ (details omitted). The choice of q above and the fact that k is infinite shows that f_i is identically zero for $i \geq 2$. Hence we see that it remains zero on extending k to its algebraic closure $\overline{k}$. But the algebra $K \otimes_k \overline{k}$ is a matrix algebra (for example by Lemmas 4.9 and 5.3), which implies there are some elements whose qth power is not central (e.g., e_{11}). This is the desired contradiction. □

The results above allow us to characterize finite central simple algebras as follows.

0753 **Lemma 8.6.** *Let k be a field. For a k-algebra A the following are equivalent*

1. *A is finite central simple k-algebra,*
2. *A is a finite dimensional k-vector space, k is the center of A, and A has no nontrivial two-sided ideal,*
3. *there exists $d \geq 1$ such that $A \otimes_k \bar{k} \cong Mat(d \times d, \bar{k})$,*
4. *there exists $d \geq 1$ such that $A \otimes_k k^{sep} \cong Mat(d \times d, k^{sep})$,*
5. *there exist $d \geq 1$ and a finite Galois extension k'/k such that $A \otimes_k k' \cong Mat(d \times d, k')$,*
6. *there exist $n \geq 1$ and a finite central skew field K over k such that $A \cong Mat(n \times n, K)$.*

The integer d is called the degree of A.

Proof. The equivalence of (1) and (2) is a consequence of the definitions, see Section 2. Assume (1). By Proposition 8.5 there exists a separable splitting field $k \subset k'$ for A. Of course, then a Galois closure of k'/k is a splitting field also. Thus we see that (1) implies (5). It is clear that (5) $\Rightarrow$ (4) $\Rightarrow$ (3). Assume (3). Then $A \otimes_k \overline{k}$ is a finite central simple $\overline{k}$-algebra for example by Lemma 4.5. This trivially implies that A is a finite central simple k-algebra. Finally, the equivalence of (1) and (6) is Wedderburn's theorem, see Theorem 3.3. □

9. Other chapters

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References

[ANT44] Emil Artin, Cecil James Nesbitt, and Robert McDowell Thrall, *Rings with Minimum Condition*, University of Michigan Publications in Mathematics, no. 1, University of Michigan Press, 1944.

[Deu68] Max Deuring, *Algebren*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Springer-Verlag, Berlin, 1968.

[Rie65] Marc Aristide Rieffel, *A general Wedderburn theorem*, Proc. Nat. Acad. Sci. U.S.A. **54** (1965), 1513.

[Ser55] Jean-Pierre Serre, *Applications algébriques de la cohomologie des groupes. II: Théorie des algèbres simples*, Secrétariat mathématique, 11 rue Pierre Curie, Paris, 1955.