

## CHOW HOMOLOGY AND CHERN CLASSES

02P3

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## 1. Introduction

02P4 In this chapter we discuss Chow homology groups and the construction of chern classes of vector bundles as elements of operational Chow cohomology groups (everything with  $\mathbf{Z}$ -coefficients).

We start this chapter by giving the shortest possible algebraic proof of the Key Lemma 6.3. We first define the Herbrand quotient (Section 2) and we compute it in some cases (Section 3). Next, we prove some simple algebra lemmas on existence of suitable factorizations after modifications (Section 4). Using these we construct/define the tame symbol in Section 5. Only the most basic properties of the tame symbol are needed to prove the Key Lemma, which we do in Section 6.

Next, we introduce the basic setup we work with in the rest of this chapter in Section 7. To make the material a little bit more challenging we decided to treat a somewhat more general case than is usually done. Namely we assume our schemes  $X$  are locally of finite type over a fixed locally Noetherian base scheme which is universally catenary and is endowed with a dimension function. These assumptions suffice to be able to define the Chow homology groups  $A_*(X)$  and the action of capping with chern classes on them. This is an indication that we should be able to define these also for algebraic stacks locally of finite type over such a base.

Next, we follow the first few chapters of [Ful98] in order to define cycles, flat pull-back, proper pushforward, and rational equivalence, except that we have been less precise about the supports of the cycles involved.

We diverge from the presentation given in [Ful98] by using the Key lemma mentioned above to prove a basic commutativity relation in Section 25. Using this we prove that the operation of intersecting with an invertible sheaf passes through rational equivalence and is commutative, see Section 26. One more application of the Key lemma proves that the Gysin map of an effective Cartier divisor passes

through rational equivalence, see Section 28. Having proved this, it is straightforward to define chern classes of vector bundles, prove additivity, prove the splitting principle, introduce chern characters, Todd classes, and state the Grothendieck-Riemann-Roch theorem.

There are two appendices. In Appendix A (Section 43) we discuss an alternative (longer) construction of the tame symbol and corresponding proof of the Key Lemma. Finally, in Appendix B (Section 44) we briefly discuss the relationship with  $K$ -theory of coherent sheaves and we discuss some blowup lemmas. We suggest the reader look at their introductions for more information.

We will return to the Chow groups  $A_*(X)$  for smooth projective varieties over algebraically closed fields in the next chapter. Using a moving lemma as in [Sam56], [Che58a], and [Che58b] and Serre's Tor-formula (see [Ser00] or [Ser65]) we will define a ring structure on  $A_*(X)$ . See Intersection Theory, Section 1 ff.

## 2. Periodic complexes and Herbrand quotients

02PF Of course there is a very general notion of periodic complexes. We can require periodicity of the maps, or periodicity of the objects. We will add these here as needed. For the moment we only need the following cases.

02PG **Definition 2.1.** Let  $R$  be a ring.

- (1) A *2-periodic complex* over  $R$  is given by a quadruple  $(M, N, \varphi, \psi)$  consisting of  $R$ -modules  $M, N$  and  $R$ -module maps  $\varphi : M \rightarrow N, \psi : N \rightarrow M$  such that

$$\dots \longrightarrow M \xrightarrow{\varphi} N \xrightarrow{\psi} M \xrightarrow{\varphi} N \longrightarrow \dots$$

is a complex. In this setting we define the *cohomology modules* of the complex to be the  $R$ -modules

$$H^0(M, N, \varphi, \psi) = \text{Ker}(\varphi)/\text{Im}(\psi) \quad \text{and} \quad H^1(M, N, \varphi, \psi) = \text{Ker}(\psi)/\text{Im}(\varphi).$$

We say the 2-periodic complex is *exact* if the cohomology groups are zero.

- (2) A *(2, 1)-periodic complex* over  $R$  is given by a triple  $(M, \varphi, \psi)$  consisting of an  $R$ -module  $M$  and  $R$ -module maps  $\varphi : M \rightarrow M, \psi : M \rightarrow M$  such that

$$\dots \longrightarrow M \xrightarrow{\varphi} M \xrightarrow{\psi} M \xrightarrow{\varphi} M \longrightarrow \dots$$

is a complex. Since this is a special case of a 2-periodic complex we have its *cohomology modules*  $H^0(M, \varphi, \psi), H^1(M, \varphi, \psi)$  and a notion of exactness.

In the following we will use any result proved for 2-periodic complexes without further mention for (2, 1)-periodic complexes. It is clear that the collection of 2-periodic complexes forms a category with morphisms  $(f, g) : (M, N, \varphi, \psi) \rightarrow (M', N', \varphi', \psi')$  pairs of morphisms  $f : M \rightarrow M'$  and  $g : N \rightarrow N'$  such that  $\varphi' \circ f = f \circ \varphi$  and  $\psi' \circ g = g \circ \psi$ . We obtain an abelian category, with kernels and cokernels as in Homology, Lemma 12.3.

02PH **Definition 2.2.** Let  $(M, N, \varphi, \psi)$  be a 2-periodic complex over a ring  $R$  whose cohomology modules have finite length. In this case we define the *multiplicity* of  $(M, N, \varphi, \psi)$  to be the integer

$$e_R(M, N, \varphi, \psi) = \text{length}_R(H^0(M, N, \varphi, \psi)) - \text{length}_R(H^1(M, N, \varphi, \psi))$$

In the case of a  $(2, 1)$ -periodic complex  $(M, \varphi, \psi)$ , we denote this by  $e_R(M, \varphi, \psi)$  and we will sometimes call this the *(additive) Herbrand quotient*.

If the cohomology groups of  $(M, \varphi, \psi)$  are finite abelian groups, then it is customary to call the *(multiplicative) Herbrand quotient*

$$q(M, \varphi, \psi) = \frac{\#H^0(M, \varphi, \psi)}{\#H^1(M, \varphi, \psi)}$$

In words: the multiplicative Herbrand quotient is the number of elements of  $H^0$  divided by the number of elements of  $H^1$ . If  $R$  is local and if the residue field of  $R$  is finite with  $q$  elements, then we see that

$$q(M, \varphi, \psi) = q^{e_R(M, \varphi, \psi)}$$

An example of a  $(2, 1)$ -periodic complex over a ring  $R$  is any triple of the form  $(M, 0, \psi)$  where  $M$  is an  $R$ -module and  $\psi$  is an  $R$ -linear map. If the kernel and cokernel of  $\psi$  have finite length, then we obtain

$$0\text{EA6} \quad (2.2.1) \quad e_R(M, 0, \psi) = \text{length}_R(\text{Coker}(\psi)) - \text{length}_R(\text{Ker}(\psi))$$

We state and prove the obligatory lemmas on these notations.

0EA7 **Lemma 2.3.** *Let  $R$  be a ring. Suppose that we have a short exact sequence of 2-periodic complexes*

$$0 \rightarrow (M_1, N_1, \varphi_1, \psi_1) \rightarrow (M_2, N_2, \varphi_2, \psi_2) \rightarrow (M_3, N_3, \varphi_3, \psi_3) \rightarrow 0$$

*If two out of three have cohomology modules of finite length so does the third and we have*

$$e_R(M_2, N_2, \varphi_2, \psi_2) = e_R(M_1, N_1, \varphi_1, \psi_1) + e_R(M_3, N_3, \varphi_3, \psi_3).$$

**Proof.** We abbreviate  $A = (M_1, N_1, \varphi_1, \psi_1)$ ,  $B = (M_2, N_2, \varphi_2, \psi_2)$  and  $C = (M_3, N_3, \varphi_3, \psi_3)$ . We have a long exact cohomology sequence

$$\dots \rightarrow H^1(C) \rightarrow H^0(A) \rightarrow H^0(B) \rightarrow H^0(C) \rightarrow H^1(A) \rightarrow H^1(B) \rightarrow H^1(C) \rightarrow \dots$$

This gives a finite exact sequence

$$0 \rightarrow I \rightarrow H^0(A) \rightarrow H^0(B) \rightarrow H^0(C) \rightarrow H^1(A) \rightarrow H^1(B) \rightarrow K \rightarrow 0$$

with  $0 \rightarrow K \rightarrow H^1(C) \rightarrow I \rightarrow 0$  a filtration. By additivity of the length function (Algebra, Lemma 51.3) we see the result.  $\square$

0EA8 **Lemma 2.4.** *Let  $R$  be a ring. If  $(M, N, \varphi, \psi)$  is a 2-periodic complex such that  $M, N$  have finite length, then  $e_R(M, N, \varphi, \psi) = \text{length}_R(M) - \text{length}_R(N)$ . In particular, if  $(M, \varphi, \psi)$  is a  $(2, 1)$ -periodic complex such that  $M$  has finite length, then  $e_R(M, \varphi, \psi) = 0$ .*

**Proof.** This follows from the additivity of Lemma 2.3 and the short exact sequence  $0 \rightarrow (M, 0, 0, 0) \rightarrow (M, N, \varphi, \psi) \rightarrow (0, N, 0, 0) \rightarrow 0$ .  $\square$

0EA9 **Lemma 2.5.** *Let  $R$  be a ring. Let  $f : (M, \varphi, \psi) \rightarrow (M', \varphi', \psi')$  be a map of  $(2, 1)$ -periodic complexes whose cohomology modules have finite length. If  $\text{Ker}(f)$  and  $\text{Coker}(f)$  have finite length, then  $e_R(M, \varphi, \psi) = e_R(M', \varphi', \psi')$ .*

**Proof.** Apply the additivity of Lemma 2.3 and observe that  $(\text{Ker}(f), \varphi, \psi)$  and  $(\text{Coker}(f), \varphi', \psi')$  have vanishing multiplicity by Lemma 2.4.  $\square$

### 3. Calculation of some multiplicities

0EAA To prove equality of certain cycles later on we need to compute some multiplicities. Our main tool, besides the elementary lemmas on multiplicities given in the previous section, will be Algebra, Lemma 120.7.

02QF **Lemma 3.1.** *Let  $R$  be a Noetherian local ring. Let  $M$  be a finite  $R$ -module. Let  $x \in R$ . Assume that*

- (1)  $\dim(\text{Supp}(M)) \leq 1$ , and
- (2)  $\dim(\text{Supp}(M/xM)) \leq 0$ .

Write  $\text{Supp}(M) = \{\mathfrak{m}, \mathfrak{q}_1, \dots, \mathfrak{q}_t\}$ . Then

$$e_R(M, 0, x) = \sum_{i=1, \dots, t} \text{ord}_{R/\mathfrak{q}_i}(x) \text{length}_{R_{\mathfrak{q}_i}}(M_{\mathfrak{q}_i}).$$

**Proof.** We first make some preparatory remarks. The result of the lemma holds if  $M$  has finite length, i.e., if  $t = 0$ , because both the left hand side and the right hand side are zero in this case, see Lemma 2.4. Also, if we have a short exact sequence  $0 \rightarrow M \rightarrow M' \rightarrow M'' \rightarrow 0$  of modules satisfying (1) and (2), then lemma for 2 out of 3 of these implies the lemma for the third by the additivity of length (Algebra, Lemma 51.3) and additivity of multiplicities (Lemma 2.3).

Denote  $M_i$  the image of  $M$  in  $M_{\mathfrak{q}_i}$ , so  $\text{Supp}(M_i) = \{\mathfrak{m}, \mathfrak{q}_i\}$ . The kernel and cokernel of the map  $M \rightarrow \bigoplus M_i$  have support  $\{\mathfrak{m}\}$  and hence have finite length. By our preparatory remarks, it follows that it suffices to prove the lemma for each  $M_i$ . Thus we may assume that  $\text{Supp}(M) = \{\mathfrak{m}, \mathfrak{q}\}$ . In this case we can filter  $M$  by powers of  $\mathfrak{q}$ . Again additivity shows that it suffices to prove the lemma in the case  $M$  is annihilated by  $\mathfrak{q}$ . In this case we can view  $M$  as a  $R/\mathfrak{q}$ -module, i.e., we may assume that  $R$  is a Noetherian local domain of dimension 1 with fraction field  $K$ . Dividing by the torsion submodule, i.e., by the kernel of  $M \rightarrow M \otimes_R K = V$  (the torsion has finite length hence is handled by our preliminary remarks) we may assume that  $M \subset V$  is a lattice (Algebra, Definition 120.3). Then  $x : M \rightarrow M$  is injective and  $\text{length}_R(M/xM) = d(M, xM)$  (Algebra, Definition 120.5). Since  $\text{length}_K(V) = \dim_K(V)$  we see that  $\det(x : V \rightarrow V) = x^{\dim_K(V)}$  and  $\text{ord}_R(\det(x : V \rightarrow V)) = \dim_K(V) \text{ord}_R(x)$ . Thus the desired equality follows from Algebra, Lemma 120.7 in this case.  $\square$

02QG **Lemma 3.2.** *Let  $R$  be a Noetherian local ring. Let  $x \in R$ . If  $M$  is a finite Cohen-Macaulay module over  $R$  with  $\dim(\text{Supp}(M)) = 1$  and  $\dim(\text{Supp}(M/xM)) = 0$ , then*

$$\text{length}_R(M/xM) = \sum_i \text{length}_R(R/(x, \mathfrak{q}_i)) \text{length}_{R_{\mathfrak{q}_i}}(M_{\mathfrak{q}_i}).$$

where  $\mathfrak{q}_1, \dots, \mathfrak{q}_t$  are the minimal primes of the support of  $M$ . If  $I \subset R$  is an ideal such that  $x$  is a nonzerodivisor on  $R/I$  and  $\dim(R/I) = 1$ , then

$$\text{length}_R(R/(x, I)) = \sum_i \text{length}_R(R/(x, \mathfrak{q}_i)) \text{length}_{R_{\mathfrak{q}_i}}((R/I)_{\mathfrak{q}_i})$$

where  $\mathfrak{q}_1, \dots, \mathfrak{q}_n$  are the minimal primes over  $I$ .

**Proof.** These are special cases of Lemma 3.1.  $\square$

Here is another case where we can determine the value of a multiplicity.

0EAB **Lemma 3.3.** *Let  $R$  be a ring. Let  $M$  be an  $R$ -module. Let  $\varphi : M \rightarrow M$  be an endomorphism and  $n > 0$  such that  $\varphi^n = 0$  and such that  $\text{Ker}(\varphi)/\text{Im}(\varphi^{n-1})$  has finite length as an  $R$ -module. Then*

$$e_R(M, \varphi^i, \varphi^{n-i}) = 0$$

for  $i = 0, \dots, n$ .

**Proof.** The cases  $i = 0, n$  are trivial as  $\pi^0 = \text{id}_M$  by convention. Let us think of  $M$  as an  $R[t]$ -module where multiplication by  $t$  is given by  $\varphi$ . Let us write  $K_i = \text{Ker}(t^i : M \rightarrow M)$  and

$$a_i = \text{length}_R(K_i/t^{n-i}M), \quad b_i = \text{length}_R(K_i/tK_{i+1}), \quad c_i = \text{length}_R(K_1/t^iK_{i+1})$$

Boundary values are  $a_0 = a_n = b_0 = c_0 = 0$ . The  $c_i$  are integers for  $i < n$  as  $K_1/t^iK_{i+1}$  is a quotient of  $K_1/t^{n-1}M$  which is assumed to have finite length. We will use frequently that  $K_i \cap t^jM = t^jK_{i+j}$ . For  $0 < i < n-1$  we have an exact sequence

$$0 \rightarrow K_1/t^{n-i-1}K_{n-i} \rightarrow K_{i+1}/t^{n-i-1}M \xrightarrow{t} K_i/t^{n-i}M \rightarrow K_i/tK_{i+1} \rightarrow 0$$

By induction on  $i$  we conclude that  $a_i$  and  $b_i$  are integers for  $i < n$  and that

$$c_{n-i-1} - a_{i+1} + a_i - b_i = 0$$

For  $0 < i < n-1$  there is a short exact sequence

$$0 \rightarrow K_i/tK_{i+1} \rightarrow K_{i+1}/tK_{i+2} \xrightarrow{t^i} K_1/t^{i+1}K_{i+2} \rightarrow K_1/t^iK_{i+1} \rightarrow 0$$

which gives

$$b_i - b_{i+1} + c_{i+1} - c_i = 0$$

Since  $b_0 = c_0$  we conclude that  $b_i = c_i$  for  $i < n$ . Then we see that

$$a_2 = a_1 + b_{n-2} - b_1, \quad a_3 = a_2 + b_{n-3} - b_2, \quad \dots$$

It is straightforward to see that this implies  $a_i = a_{n-i}$  as desired.  $\square$

0EAC **Lemma 3.4.** *Let  $(R, \mathfrak{m})$  be a Noetherian local ring. Let  $(M, \varphi, \psi)$  be a  $(2, 1)$ -periodic complex over  $R$  with  $M$  finite and with cohomology groups of finite length over  $R$ . Let  $x \in R$  be such that  $\dim(\text{Supp}(M/xM)) \leq 0$ . Then*

$$e_R(M, x\varphi, \psi) = e_R(M, \varphi, \psi) - e_R(\text{Im}(\varphi), 0, x)$$

and

$$e_R(M, \varphi, x\psi) = e_R(M, \varphi, \psi) + e_R(\text{Im}(\psi), 0, x)$$

**Proof.** We will only prove the first formula as the second is proved in exactly the same manner. Let  $M' = M[x^\infty]$  be the  $x$ -power torsion submodule of  $M$ . Consider the short exact sequence  $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$ . Then  $M''$  is  $x$ -power torsion free (More on Algebra, Lemma 79.4). Since  $\varphi, \psi$  map  $M'$  into  $M'$  we obtain a short exact sequence

$$0 \rightarrow (M', \varphi', \psi') \rightarrow (M, \varphi, \psi) \rightarrow (M'', \varphi'', \psi'') \rightarrow 0$$

of  $(2, 1)$ -periodic complexes. Also, we get a short exact sequence  $0 \rightarrow M' \cap \text{Im}(\varphi) \rightarrow \text{Im}(\varphi) \rightarrow \text{Im}(\varphi'') \rightarrow 0$ . We have  $e_R(M', \varphi, \psi) = e_R(M', x\varphi, \psi) = e_R(M' \cap \text{Im}(\varphi), 0, x) = 0$  by Lemma 2.5. By additivity (Lemma 2.3) we see that it suffices to prove the lemma for  $(M'', \varphi'', \psi'')$ . This reduces us to the case discussed in the next paragraph.

Assume  $x : M \rightarrow M$  is injective. In this case  $\text{Ker}(x\varphi) = \text{Ker}(\varphi)$ . On the other hand we have a short exact sequence

$$0 \rightarrow \text{Im}(\varphi)/x\text{Im}(\varphi) \rightarrow \text{Ker}(\psi)/\text{Im}(x\varphi) \rightarrow \text{Ker}(\psi)/\text{Im}(\varphi) \rightarrow 0$$

This together with (2.2.1) proves the formula.  $\square$

#### 4. Preparation for tame symbols

0EAD In this section we put some lemma that will help us define the tame symbol in the next section.

0EAE **Lemma 4.1.** *Let  $A$  be a Noetherian ring. Let  $\mathfrak{m}_1, \dots, \mathfrak{m}_r$  be pairwise distinct maximal ideals of  $A$ . For  $i = 1, \dots, r$  let  $\varphi_i : A_{\mathfrak{m}_i} \rightarrow B_i$  be a ring map whose kernel and cokernel are annihilated by a power of  $\mathfrak{m}_i$ . Then there exists a ring map  $\varphi : A \rightarrow B$  such that*

- (1) *the localization of  $\varphi$  at  $\mathfrak{m}_i$  is isomorphic to  $\varphi_i$ , and*
- (2)  *$\text{Ker}(\varphi)$  and  $\text{Coker}(\varphi)$  are annihilated by a power of  $\mathfrak{m}_1 \cap \dots \cap \mathfrak{m}_r$ .*

*Moreover, if each  $\varphi_i$  is finite, injective, or surjective then so is  $\varphi$ .*

**Proof.** Set  $I = \mathfrak{m}_1 \cap \dots \cap \mathfrak{m}_r$ . Set  $A_i = A_{\mathfrak{m}_i}$  and  $A' = \prod A_i$ . Then  $IA' = \prod \mathfrak{m}_i A_i$  and  $A \rightarrow A'$  is a flat ring map such that  $A/I \cong A'/IA'$ . Thus we may use More on Algebra, Lemma 80.16 to see that there exists an  $A$ -module map  $\varphi : A \rightarrow B$  with  $\varphi_i$  isomorphic to the localization of  $\varphi$  at  $\mathfrak{m}_i$ . Then we can use the discussion in More on Algebra, Remark 80.19 to endow  $B$  with an  $A$ -algebra structure matching the given  $A$ -algebra structure on  $B_i$ . The final statement of the lemma follows easily from the fact that  $\text{Ker}(\varphi)_{\mathfrak{m}_i} \cong \text{Ker}(\varphi_i)$  and  $\text{Coker}(\varphi)_{\mathfrak{m}_i} \cong \text{Coker}(\varphi_i)$ .  $\square$

The following lemma is very similar to Algebra, Lemma 118.3.

02Q7 **Lemma 4.2.** *Let  $(R, \mathfrak{m})$  be a Noetherian local ring of dimension 1. Let  $a, b \in R$  be nonzerodivisors. There exists a finite ring extension  $R \subset R'$  with  $R'/R$  annihilated by a power of  $\mathfrak{m}$  and nonzerodivisors  $t, a', b' \in R'$  such that  $a = ta'$  and  $b = tb'$  and  $R' = a'R' + b'R'$ .*

**Proof.** If  $a$  or  $b$  is a unit, then the lemma is true with  $R = R'$ . Thus we may assume  $a, b \in \mathfrak{m}$ . Set  $I = (a, b)$ . The idea is to blow up  $R$  in  $I$ . Instead of doing the algebraic argument we work geometrically. Let  $X = \text{Proj}(\bigoplus_{d \geq 0} I^d)$ . By Divisors, Lemma 32.4 the morphism  $X \rightarrow \text{Spec}(R)$  is an isomorphism over the punctured spectrum  $U = \text{Spec}(R) \setminus \{\mathfrak{m}\}$ . Thus we may and do view  $U$  as an open subscheme of  $X$ . The morphism  $X \rightarrow \text{Spec}(R)$  is projective by Divisors, Lemma 32.13. Also, every generic point of  $X$  lies in  $U$ , for example by Divisors, Lemma 32.10. It follows from Varieties, Lemma 17.2 that  $X \rightarrow \text{Spec}(R)$  is finite. Thus  $X = \text{Spec}(R')$  is affine and  $R \rightarrow R'$  is finite. We have  $R_a \cong R'_a$  as  $U = D(a)$ . Hence a power of  $a$  annihilates the finite  $R$ -module  $R'/R$ . As  $\mathfrak{m} = \sqrt{(a)}$  we see that  $R'/R$  is annihilated by a power of  $\mathfrak{m}$ . By Divisors, Lemma 32.4 we see that  $IR'$  is a locally principal ideal. Since  $R'$  is semi-local we see that  $IR'$  is principal, see Algebra, Lemma 77.6, say  $IR' = (t)$ . Then we have  $a = a't$  and  $b = b't$  and everything is clear.  $\square$

0EAF **Lemma 4.3.** *Let  $(R, \mathfrak{m})$  be a Noetherian local ring of dimension 1. Let  $a, b \in R$  be nonzerodivisors with  $a \in \mathfrak{m}$ . There exists an integer  $n = n(R, a, b)$  such that for a finite ring extension  $R \subset R'$  if  $b = a^m c$  for some  $c \in R'$ , then  $m \leq n$ .*

**Proof.** Choose a minimal prime  $\mathfrak{q} \subset R$ . Observe that  $\dim(R/\mathfrak{q}) = 1$ , in particular  $R/\mathfrak{q}$  is not a field. We can choose a discrete valuation ring  $A$  dominating  $R/\mathfrak{q}$  with the same fraction field, see Algebra, Lemma 118.1. Observe that  $a$  and  $b$  map to nonzero elements of  $A$  as nonzerodivisors in  $R$  are not contained in  $\mathfrak{q}$ . Let  $v$  be the discrete valuation on  $A$ . Then  $v(a) > 0$  as  $a \in \mathfrak{m}$ . We claim  $n = v(b)/v(a)$  works.

Let  $R \subset R'$  be given. Set  $A' = A \otimes_R R'$ . Since  $\text{Spec}(R') \rightarrow \text{Spec}(R)$  is surjective (Algebra, Lemma 35.17) also  $\text{Spec}(A') \rightarrow \text{Spec}(A)$  is surjective (Algebra, Lemma 29.3). Pick a prime  $\mathfrak{q}' \subset A'$  lying over  $(0) \subset A$ . Then  $A \subset A'' = A'/\mathfrak{q}'$  is a finite extension of rings (again inducing a surjection on spectra). Pick a maximal ideal  $\mathfrak{m}'' \subset A''$  lying over the maximal ideal of  $A$  and a discrete valuation ring  $A'''$  dominating  $A''_{\mathfrak{m}''}$  (see lemma cited above). Then  $A \rightarrow A'''$  is an extension of discrete valuation rings and we have  $b = a^m c$  in  $A'''$ . Thus  $v'''(b) \geq m v'''(a)$ . Since  $v''' = ev$  where  $e$  is the ramification index of  $A'''/A$ , we find that  $m \leq n$  as desired.  $\square$

0EAG **Lemma 4.4.** *Let  $(A, \mathfrak{m})$  be a Noetherian local ring of dimension 1. Let  $r \geq 2$  and let  $a_1, \dots, a_r \in A$  be nonzerodivisors not all units. Then there exist*

- (1) *a finite ring extension  $A \subset B$  with  $B/A$  annihilated by a power of  $\mathfrak{m}$ ,*
- (2) *for each of maximal ideal  $\mathfrak{m}_j \subset B$  a nonzerodivisor  $\pi_j \in B_j = B_{\mathfrak{m}_j}$ , and*
- (3) *factorizations  $a_i = u_{i,j} \pi_j^{e_{i,j}}$  in  $B_j$  with  $u_{i,j} \in B_j$  units and  $e_{i,j} \geq 0$ .*

**Proof.** Since at least one  $a_i$  is not a unit and we find that  $\mathfrak{m}$  is not an associated prime of  $A$ . Moreover, for any  $A \subset B$  as in the statement  $\mathfrak{m}$  is not an associated prime of  $B$  and  $\mathfrak{m}_j$  is not an associate prime of  $B_j$ . Keeping this in mind will help check the arguments below.

First, we claim that it suffices to prove the lemma for  $r = 2$ . We will argue this by induction on  $r$ ; we suggest the reader skip the proof. Suppose we are given  $A \subset B$  and  $\pi_j$  in  $B_j = B_{\mathfrak{m}_j}$  and factorizations  $a_i = u_{i,j} \pi_j^{e_{i,j}}$  for  $i = 1, \dots, r-1$  in  $B_j$  with  $u_{i,j} \in B_j$  units and  $e_{i,j} \geq 0$ . Then by the case  $r = 2$  for  $\pi_j$  and  $a_r$  in  $B_j$  we can find extensions  $B_j \subset C_j$  and for every maximal ideal  $\mathfrak{m}_{j,k}$  of  $C_j$  a nonzerodivisor  $\pi_{j,k} \in C_{j,k} = (C_j)_{\mathfrak{m}_{j,k}}$  and factorizations

$$\pi_j = v_{j,k} \pi_{j,k}^{f_{j,k}} \quad \text{and} \quad a_r = w_{j,k} \pi_{j,k}^{g_{j,k}}$$

as in the lemma. There exists a unique finite extension  $B \subset C$  with  $C/B$  annihilated by a power of  $\mathfrak{m}$  such that  $C_j \cong C_{\mathfrak{m}_j}$  for all  $j$ , see Lemma 4.1. The maximal ideals of  $C$  correspond 1-to-1 to the maximal ideals  $\mathfrak{m}_{j,k}$  in the localizations and in these localizations we have

$$a_i = u_{i,j} \pi_j^{e_{i,j}} = u_{i,j} v_{j,k}^{e_{i,j}} \pi_{j,k}^{e_{i,j} f_{j,k}}$$

for  $i \leq r-1$ . Since  $a_r$  factors correctly too the proof of the induction step is complete.

Proof of the case  $r = 2$ . We will use induction on

$$\ell = \min(\text{length}_A(A/a_1A), \text{length}_A(A/a_2A)).$$

If  $\ell = 0$ , then either  $a_1$  or  $a_2$  is a unit and the lemma holds with  $A = B$ . Thus we may and do assume  $\ell > 0$ .

Suppose we have a finite extension of rings  $A \subset A'$  such that  $A'/A$  is annihilated by a power of  $\mathfrak{m}$  and such that  $\mathfrak{m}$  is not an associated prime of  $A'$ . Let  $\mathfrak{m}_1, \dots, \mathfrak{m}_r \subset A'$  be the maximal ideals and set  $A'_i = A'_{\mathfrak{m}_i}$ . If we can solve the problem for  $a_1, a_2$  in each  $A'_i$ , then we can apply Lemma 4.1 to produce a solution for  $a_1, a_2$  in  $A$ .



Choose  $x \in \{a_1, a_2\}$  such that  $\ell = \text{length}_A(A/xA)$ . By Lemma 2.5 and (2.2.1) we have  $\text{length}_A(A/xA) = \text{length}_A(A'/xA')$ . On the other hand, we have

$$\text{length}_A(A'/xA') = \sum [\kappa(\mathfrak{m}_i) : \kappa(\mathfrak{m})] \text{length}_{A'_i}(A'_i/xA'_i)$$

by Algebra, Lemma 51.12. Since  $x \in \mathfrak{m}$  we see that each term on the right hand side is positive. We conclude that the induction hypothesis applies to  $a_1, a_2$  in each  $A'_i$  if  $r > 1$  or if  $r = 1$  and  $[\kappa(\mathfrak{m}_1) : \kappa(\mathfrak{m})] > 1$ . We conclude that we may assume each  $A'$  as above is local with the same residue field as  $A$ .

Applying the discussion of the previous paragraph, we may replace  $A$  by the ring constructed in Lemma 4.2 for  $a_1, a_2 \in A$ . Then since  $A$  is local we find, after possibly switching  $a_1$  and  $a_2$ , that  $a_2 \in (a_1)$ . Write  $a_2 = a_1^m c$  with  $m > 0$  maximal. In fact, by Lemma 4.3 we may assume  $m$  is maximal even after replacing  $A$  by any finite extension  $A \subset A'$  as in the previous paragraph. If  $c$  is a unit, then we are done. If not, then we replace  $A$  by the ring constructed in Lemma 4.2 for  $a_1, c \in A$ . Then either (1)  $c = a_1 c'$  or (2)  $a_1 = c a'_1$ . The first case cannot happen since it would give  $a_2 = a_1^{m+1} c'$  contradicting the maximality of  $m$ . In the second case we get  $a_1 = c a'_1$  and  $a_2 = c^{m+1} (a'_1)^m$ . Then it suffices to prove the lemma for  $A$  and  $c, a'_1$ . If  $a'_1$  is a unit we're done and if not, then  $\text{length}_A(A/cA) < \ell$  because  $cA$  is a strictly bigger ideal than  $a_1 A$ . Thus we win by induction hypothesis.  $\square$

## 5. Tame symbols

0EAH Consider a Noetherian local ring  $(A, \mathfrak{m})$  of dimension 1. We denote  $Q(A)$  the total ring of fractions of  $A$ , see Algebra, Example 9.8. The *tame symbol* will be a map

$$\partial_A(-, -) : Q(A)^* \times Q(A)^* \longrightarrow \kappa(\mathfrak{m})^*$$

satisfying the following properties:

- 0EAI (1)  $\partial_A(f, gh) = \partial_A(f, g)\partial_A(f, h)$  for  $f, g, h \in Q(A)^*$ ,
- 0EAJ (2)  $\partial_A(f, g)\partial_A(g, f) = 1$  for  $f, g \in Q(A)^*$ ,
- 0EAK (3)  $\partial_A(f, 1-f) = 1$  for  $f \in Q(A)^*$  such that  $1-f \in Q(A)^*$ ,
- 0EAL (4)  $\partial_A(aa', b) = \partial_A(a, b)\partial_A(a', b)$  and  $\partial_A(a, bb') = \partial_A(a, b)\partial_A(a, b')$  for  $a, a', b, b' \in A$  nonzerodivisors,
- 0EAM (5)  $\partial_A(b, b) = (-1)^m$  with  $m = \text{length}_A(A/bA)$  for  $b \in A$  a nonzerodivisor,
- 0EAN (6)  $\partial_A(u, b) = u^m \pmod{\mathfrak{m}}$  with  $m = \text{length}_A(A/bA)$  for  $u \in A$  a unit and  $b \in A$  a nonzerodivisor, and
- 0EAP (7)  $\partial_A(a, b-a)\partial_A(b, b) = \partial_A(b, b-a)\partial_A(a, b)$  for  $a, b \in A$  such that  $a, b, b-a$  are nonzerodivisors.

Since it is easier to work with elements of  $A$  we will often think of  $\partial_A$  as a map defined on pairs of nonzerodivisors of  $A$  satisfying (4), (5), (6), (7). It is an exercise to see that setting

$$\partial_A\left(\frac{a}{b}, \frac{c}{d}\right) = \partial_A(a, b)\partial_A(a, d)^{-1}\partial_A(b, c)^{-1}\partial_A(b, d)$$

we get a well defined map  $Q(A)^* \times Q(A)^* \rightarrow \kappa(\mathfrak{m})^*$  satisfying (1), (2), (3) as well as the other properties.

We do not claim there is a unique map with these properties. Instead, we will give a recipe for constructing such a map. Namely, given  $a_1, a_2 \in A$  nonzerodivisors, we

choose a ring extension  $A \subset B$  and local factorizations as in Lemma 4.4. Then we define

$$\text{0EAQ} \quad (5.0.1) \quad \partial_A(a_1, a_2) = \prod_j \text{Norm}_{\kappa(\mathfrak{m}_j)/\kappa(\mathfrak{m})}((-1)^{e_{1,j}e_{2,j}} u_{1,j}^{e_{2,j}} u_{2,j}^{-e_{1,j}} \bmod \mathfrak{m}_j)^{m_j}$$

where  $m_j = \text{length}_{B_j}(B_j/\pi_j B_j)$  and the product is taken over the maximal ideals  $\mathfrak{m}_1, \dots, \mathfrak{m}_r$  of  $B$ .

**0EAR** **Lemma 5.1.** *The formula (5.0.1) determines a well defined element of  $\kappa(\mathfrak{m})^*$ . In other words, the right hand side does not depend on the choice of the local factorizations or the choice of  $B$ .*

**Proof.** Independence of choice of factorizations. Suppose we have a Noetherian 1-dimensional local ring  $B$ , elements  $a_1, a_2 \in B$ , and nonzerodivisors  $\pi, \theta$  such that we can write

$$a_1 = u_1 \pi^{e_1} = v_1 \theta^{f_1}, \quad a_2 = u_2 \pi^{e_2} = v_2 \theta^{f_2}$$

with  $e_i, f_i \geq 0$  integers and  $u_i, v_i$  units in  $B$ . Observe that this implies

$$a_1^{e_2} = u_1^{e_2} u_2^{-e_1} a_2^{e_1}, \quad a_1^{f_2} = v_1^{f_2} v_2^{-f_1} a_2^{f_1}$$

On the other hand, setting  $m = \text{length}_B(B/\pi B)$  and  $k = \text{length}_B(B/\theta B)$  we find  $e_2 m = \text{length}_B(B/a_2 B) = f_2 k$ . Expanding  $a_1^{e_2 m} = a_1^{f_2 k}$  using the above we find

$$(u_1^{e_2} u_2^{-e_1})^m = (v_1^{f_2} v_2^{-f_1})^k$$

This proves the desired equality up to signs. To see the signs work out we have to show  $m e_1 e_2$  is even if and only if  $k f_1 f_2$  is even. This follows as both  $m e_2 = k f_2$  and  $m e_1 = k f_1$  (same argument as above).

Independence of choice of  $B$ . Suppose given two extensions  $A \subset B$  and  $A \subset B'$  as in Lemma 4.4. Then

$$C = (B \otimes_A B') / (\mathfrak{m}\text{-power torsion})$$

will be a third one. Thus we may assume we have  $A \subset B \subset C$  and factorizations over the local rings of  $B$  and we have to show that using the same factorizations over the local rings of  $C$  gives the same element of  $\kappa(\mathfrak{m})$ . By transitivity of norms (Fields, Lemma 20.5) this comes down to the following problem: if  $B$  is a Noetherian local ring of dimension 1 and  $\pi \in B$  is a nonzerodivisor, then

$$\lambda^m = \prod \text{Norm}_{\kappa_k/\kappa}(\lambda)^{m_k}$$

Here we have used the following notation: (1)  $\kappa$  is the residue field of  $B$ , (2)  $\lambda$  is an element of  $\kappa$ , (3)  $\mathfrak{m}_k \subset C$  are the maximal ideals of  $C$ , (4)  $\kappa_k = \kappa(\mathfrak{m}_k)$  is the residue field of  $C_k = C_{\mathfrak{m}_k}$ , (5)  $m = \text{length}_B(B/\pi B)$ , and (6)  $m_k = \text{length}_{C_k}(C_k/\pi C_k)$ . The displayed equality holds because  $\text{Norm}_{\kappa_k/\kappa}(\lambda) = \lambda^{[\kappa_k:\kappa]}$  as  $\lambda \in \kappa$  and because  $m = \sum m_k [\kappa_k : \kappa]$ . First, we have  $m = \text{length}_B(B/xB) = \text{length}_B(C/\pi C)$  by Lemma 2.5 and (2.2.1). Finally, we have  $\text{length}_B(C/\pi C) = \sum m_k [\kappa_k : \kappa]$  by Algebra, Lemma 51.12.  $\square$

**0EAS** **Lemma 5.2.** *The tame symbol (5.0.1) satisfies (4), (5), (6), (7) and hence gives a map  $\partial_A : Q(A)^* \times Q(A)^* \rightarrow \kappa(\mathfrak{m})^*$  satisfying (1), (2), (3).*

**Proof.** Let us prove (4). Let  $a_1, a_2, a_3 \in A$  be nonzerodivisors. Choose  $A \subset B$  as in Lemma 4.4 for  $a_1, a_2, a_3$ . Then the equality

$$\partial_A(a_1 a_2, a_3) = \partial_A(a_1, a_3) \partial_A(a_2, a_3)$$

follows from the equality

$$(-1)^{(e_{1,j}+e_{2,j})e_{3,j}} (u_{1,j} u_{2,j})^{e_{3,j}} u_{3,j}^{-e_{1,j}-e_{2,j}} = (-1)^{e_{1,j}e_{3,j}} u_{1,j}^{e_{3,j}} u_{3,j}^{-e_{1,j}} (-1)^{e_{2,j}e_{3,j}} u_{2,j}^{e_{3,j}} u_{3,j}^{-e_{2,j}}$$

in  $B_j$ . Properties (5) and (6) are equally immediate.

Let us prove (7). Let  $a_1, a_2, a_1 - a_2 \in A$  be nonzerodivisors and set  $a_3 = a_1 - a_2$ . Choose  $A \subset B$  as in Lemma 4.4 for  $a_1, a_2, a_3$ . Then it suffices to show

$$(-1)^{e_{1,j}e_{2,j}+e_{1,j}e_{3,j}+e_{2,j}e_{3,j}+e_{2,j}} u_{1,j}^{e_{2,j}-e_{3,j}} u_{2,j}^{e_{3,j}-e_{1,j}} u_{3,j}^{e_{1,j}-e_{2,j}} \pmod{\mathfrak{m}_j} = 1$$

This is clear if  $e_{1,j} = e_{2,j} = e_{3,j}$ . Say  $e_{1,j} > e_{2,j}$ . Then we see that  $e_{3,j} = e_{2,j}$  because  $a_3 = a_1 - a_2$  and we see that  $u_{3,j}$  has the same residue class as  $-u_{2,j}$ . Hence the formula is true – the signs work out as well and this verification is the reason for the choice of signs in (5.0.1). The other cases are handled in exactly the same manner.  $\square$

OEAT **Lemma 5.3.** *Let  $(A, \mathfrak{m})$  be a Noetherian local ring of dimension 1. Let  $A \subset B$  be a finite ring extension with  $B/A$  annihilated by a power of  $\mathfrak{m}$  and  $\mathfrak{m}$  not an associated prime of  $B$ . For  $a, b \in A$  nonzerodivisors we have*

$$\partial_A(a, b) = \prod \text{Norm}_{\kappa(\mathfrak{m}_j)/\kappa(\mathfrak{m})}(\partial_{B_j}(a, b))$$

where the product is over the maximal ideals  $\mathfrak{m}_j$  of  $B$  and  $B_j = B_{\mathfrak{m}_j}$ .

**Proof.** Choose  $B_j \subset C_j$  as in Lemma 4.4 for  $a, b$ . By Lemma 4.1 we can choose a finite ring extension  $B \subset C$  with  $C_j \cong C_{\mathfrak{m}_j}$  for all  $j$ . Let  $\mathfrak{m}_{j,k} \subset C$  be the maximal ideals of  $C$  lying over  $\mathfrak{m}_j$ . Let

$$a = u_{j,k} \pi_{j,k}^{f_{j,k}}, \quad b = v_{j,k} \pi_{j,k}^{g_{j,k}}$$

be the local factorizations which exist by our choice of  $C_j \cong C_{\mathfrak{m}_j}$ . By definition we have

$$\partial_A(a, b) = \prod_{j,k} \text{Norm}_{\kappa(\mathfrak{m}_{j,k})/\kappa(\mathfrak{m})}((-1)^{f_{j,k}g_{j,k}} u_{j,k}^{g_{j,k}} v_{j,k}^{-f_{j,k}} \pmod{\mathfrak{m}_{j,k}})^{m_{j,k}}$$

and

$$\partial_{B_j}(a, b) = \prod_k \text{Norm}_{\kappa(\mathfrak{m}_{j,k})/\kappa(\mathfrak{m}_j)}((-1)^{f_{j,k}g_{j,k}} u_{j,k}^{g_{j,k}} v_{j,k}^{-f_{j,k}} \pmod{\mathfrak{m}_{j,k}})^{m_{j,k}}$$

The result follows by transitivity of norms for  $\kappa(\mathfrak{m}_{j,k})/\kappa(\mathfrak{m}_j)/\kappa(\mathfrak{m})$ , see Fields, Lemma 20.5.  $\square$

0EPG **Lemma 5.4.** *Let  $(A, \mathfrak{m}, \kappa) \rightarrow (A', \mathfrak{m}', \kappa')$  be a local homomorphism of Noetherian local rings of dimension 1. If  $A \rightarrow A'$  is flat,  $\mathfrak{m}' = \mathfrak{m}A'$ , and  $\kappa'/\kappa$  is separable, then for  $a_1, a_2 \in A$  nonzerodivisors the tame symbol  $\partial_A(a_1, a_2)$  maps to  $\partial_{A'}(a_1, a_2)$ .*

**Proof.** If  $a_1, a_2$  are both units, then  $\partial_A(a_1, a_2) = 1$  and  $\partial_{A'}(a_1, a_2) = 1$  and the result is true. If not, then we can choose a ring extension  $A \subset B$  and local factorizations as in Lemma 4.4. Set  $B' = A' \otimes_A B$ . Since  $A'$  is flat over  $A$  we see that  $A' \subset B'$  is a ring extension with  $B'/A'$  annihilated by a power of  $\mathfrak{m}'$ . Let  $\mathfrak{m}_1, \dots, \mathfrak{m}_m$  be the maximal ideals of  $B$ . For each  $j \in \{1, \dots, m\}$  denote  $\kappa_j = \kappa(\mathfrak{m}_j)$  the residue field. Then

$$\kappa_j \otimes_{\kappa} \kappa' = \prod_{l=1, \dots, n_j} \kappa'_{j,l}$$

is a product of fields each finite over  $\kappa'$  because  $\kappa'/\kappa$  is a separable field extension (Algebra, Lemma 42.6). It follows that  $B'$  has corresponding maximal ideals  $\mathfrak{m}'_{j,l}$  lying over  $\mathfrak{m}_j$ . As factorizations in  $B'_{j,l} = B'_{\mathfrak{m}'_{j,l}}$  we use the image of the factorizations  $a_i = u_{i,j}\pi_j^{e_{i,j}}$  given to us in  $B_j$ . Thus we obtain

$$\partial_A(a_1, a_2) = \prod_j \text{Norm}_{\kappa_j/\kappa}((-1)^{e_{1,j}e_{2,j}} u_{1,j}^{e_{2,j}} u_{2,j}^{-e_{1,j}} \bmod \mathfrak{m}_j)^{m_j}$$

by definition and similarly

$$\partial_{A'}(a_1, a_2) = \prod_{j,l} \text{Norm}_{\kappa'_{j,l}/\kappa'}((-1)^{e_{1,j}e_{2,j}} u_{1,j}^{e_{2,j}} u_{2,j}^{-e_{1,j}} \bmod \mathfrak{m}'_{j,l})^{m_j}$$

To finish the proof we observe that if  $u \in \kappa_j$  has image  $u_l \in \kappa'_{j,l}$ , then  $\text{Norm}_{\kappa_j/\kappa}(u)$  in  $\kappa$  maps to  $\prod_l \text{Norm}_{\kappa'_{j,l}/\kappa'}(u_l)$  in  $\kappa'$ . This follows from the fact that taking determinants of linear maps commutes with ground field extension.  $\square$

### 6. A key lemma

0EAU In this section we apply the results above to prove Lemma 6.3. This lemma is a low degree case of the statement that there is a complex for Milnor K-theory similar to the Gersten-Quillen complex in Quillen's K-theory. See Remark 6.4.

0EAV **Lemma 6.1.** *Let  $(A, \mathfrak{m})$  be a 2-dimensional Noetherian local ring. Let  $t \in \mathfrak{m}$  be a nonzerodivisor. Say  $V(t) = \{\mathfrak{m}, \mathfrak{q}_1, \dots, \mathfrak{q}_r\}$ . Let  $A_{\mathfrak{q}_i} \subset B_i$  be a finite ring extension with  $B_i/A_{\mathfrak{q}_i}$  annihilated by a power of  $t$ . Then there exists a finite extension  $A \subset B$  of local rings identifying residue fields with  $B_i \cong B_{\mathfrak{q}_i}$  and  $B/A$  annihilated by a power of  $t$ .*

**Proof.** Choose  $n > 0$  such that  $B_i \subset t^{-n}A_{\mathfrak{q}_i}$ . Let  $M \subset t^{-n}A$ , resp.  $M' \subset t^{-2n}A$  be the  $A$ -submodule consisting of elements mapping to  $B_i$  in  $t^{-n}A_{\mathfrak{q}_i}$ , resp.  $t^{-2n}A_{\mathfrak{q}_i}$ . Then  $M \subset M'$  are finite  $A$ -modules as  $A$  is Noetherian and  $M_{\mathfrak{q}_i} = M'_{\mathfrak{q}_i} = B_i$  as localization is exact. Thus  $M'/M$  is annihilated by  $\mathfrak{m}^c$  for some  $c > 0$ . Observe that  $M \cdot M \subset M'$  under the multiplication  $t^{-n}A \times t^{-n}A \rightarrow t^{-2n}A$ . Hence  $B = A + \mathfrak{m}^{c+1}M$  is a finite  $A$ -algebra with the correct localizations. We omit the verification that  $B$  is local with maximal ideal  $\mathfrak{m} + \mathfrak{m}^{c+1}M$ .  $\square$

0EAW **Lemma 6.2.** *Let  $(A, \mathfrak{m})$  be a 2-dimensional Noetherian local ring. Let  $a, b \in A$  be nonzerodivisors. Then we have*

$$\sum \text{ord}_{A/\mathfrak{q}}(\partial_{A_{\mathfrak{q}}}(a, b)) = 0$$

where the sum is over the height 1 primes  $\mathfrak{q}$  of  $A$ .

**Proof.** If  $\mathfrak{q}$  is a height 1 prime of  $A$  such that  $a, b$  map to a unit of  $A_{\mathfrak{q}}$ , then  $\partial_{A_{\mathfrak{q}}}(a, b) = 1$ . Thus the sum is finite. In fact, if  $V(ab) = \{\mathfrak{m}, \mathfrak{q}_1, \dots, \mathfrak{q}_r\}$  then the sum is over  $i = 1, \dots, r$ . For each  $i$  we pick an extension  $A_{\mathfrak{q}_i} \subset B_i$  as in Lemma 4.4 for  $a, b$ . By Lemma 6.1 with  $t = ab$  and the given list of primes we may assume we have a finite local extension  $A \subset B$  with  $B/A$  annihilated by a power of  $ab$  and such that for each  $i$  the  $B_{\mathfrak{q}_i} \cong B_i$ . Observe that if  $\mathfrak{q}_{i,j}$  are the primes of  $B$  lying over  $\mathfrak{q}_i$  then we have

$$\text{ord}_{A/\mathfrak{q}_i}(\partial_{A_{\mathfrak{q}_i}}(a, b)) = \sum_j \text{ord}_{B/\mathfrak{q}_{i,j}}(\partial_{B_{\mathfrak{q}_{i,j}}}(a, b))$$

by Lemma 5.3 and Algebra, Lemma 120.8. Thus we may replace  $A$  by  $B$  and reduce to the case discussed in the next paragraph.

Assume for each  $i$  there is a nonzerodivisor  $\pi_i \in A_{\mathfrak{q}_i}$  and units  $u_i, v_i \in A_{\mathfrak{q}_i}$  such that for some integers  $e_i, f_i \geq 0$  we have

$$a = u_i \pi_i^{e_i}, \quad b = v_i \pi_i^{f_i}$$

in  $A_{\mathfrak{q}_i}$ . Setting  $m_i = \text{length}_{A_{\mathfrak{q}_i}}(A_{\mathfrak{q}_i}/\pi_i)$  we have  $\partial_{A_{\mathfrak{q}_i}}(a, b) = ((-1)^{e_i f_i} u_i^{f_i} v_i^{-e_i})^{m_i}$  by definition. Since  $a, b$  are nonzerodivisors the  $(2, 1)$ -periodic complex  $(A/(ab), a, b)$  has vanishing cohomology. Denote  $M_i$  the image of  $A/(ab)$  in  $A_{\mathfrak{q}_i}/(ab)$ . Then we have a map

$$A/(ab) \longrightarrow \bigoplus M_i$$

whose kernel and cokernel are supported in  $\{\mathfrak{m}\}$  and hence have finite length. Thus we see that

$$\sum e_A(M_i, a, b) = 0$$

by Lemma 2.5. Hence it suffices to show  $e_A(M_i, a, b) = -\text{ord}_{A/\mathfrak{q}_i}(\partial_{A_{\mathfrak{q}_i}}(a, b))$ .

Let us prove this first, in case  $\pi_i, u_i, v_i$  are the images of elements  $\pi_i, u_i, v_i \in A$  (using the same symbols should not cause any confusion). In this case we get

$$\begin{aligned} e_A(M_i, a, b) &= e_A(M_i, u_i \pi_i^{e_i}, v_i \pi_i^{f_i}) \\ &= e_A(M_i, \pi_i^{e_i}, \pi_i^{f_i}) - e_A(\pi_i^{e_i} M_i, 0, u_i) + e_A(\pi_i^{f_i} M_i, 0, v_i) \\ &= 0 - f_i m_i \text{ord}_{A/\mathfrak{q}_i}(u_i) + e_i m_i \text{ord}_{A/\mathfrak{q}_i}(v_i) \\ &= -m_i \text{ord}_{A/\mathfrak{q}_i}(u_i^{f_i} v_i^{-e_i}) = -\text{ord}_{A/\mathfrak{q}_i}(\partial_{A_{\mathfrak{q}_i}}(a, b)) \end{aligned}$$

The second equality holds by Lemma 3.4. Observe that  $M_i \subset (M_i)_{\mathfrak{q}_i} = A_{\mathfrak{q}_i}/(\pi_i^{e_i+f_i})$  and  $(\pi_i^{e_i} M_i)_{\mathfrak{q}_i} \cong A_{\mathfrak{q}_i}/\pi_i^{f_i}$  and  $(\pi_i^{f_i} M_i)_{\mathfrak{q}_i} \cong A_{\mathfrak{q}_i}/\pi_i^{e_i}$ . The 0 in the third equality comes from Lemma 3.3 and the other two terms come from Lemma 3.1. The last two equalities follow from multiplicativity of the order function and from the definition of our tame symbol.

In general, we may first choose  $c \in A$ ,  $c \notin \mathfrak{q}_i$  such that  $c\pi_i \in A$ . After replacing  $\pi_i$  by  $c\pi_i$  and  $u_i$  by  $c^{-e_i}u_i$  and  $v_i$  by  $c^{-f_i}v_i$  we may and do assume  $\pi_i$  is in  $A$ . Next, choose an  $c \in A$ ,  $c \notin \mathfrak{q}_i$  with  $cu_i, cv_i \in A$ . Then we observe that

$$e_A(M_i, ca, cb) = e_A(M_i, a, b) - e_A(aM_i, 0, c) + e_A(bM_i, 0, c)$$

by Lemma 3.1. On the other hand, we have

$$\partial_{A_{\mathfrak{q}_i}}(ca, cb) = c^{m_i(f_i - e_i)} \partial_{A_{\mathfrak{q}_i}}(a, b)$$

in  $\kappa(\mathfrak{q}_i)^*$  because  $c$  is a unit in  $A_{\mathfrak{q}_i}$ . The arguments in the previous paragraph show that  $e_A(M_i, ca, cb) = -\text{ord}_{A/\mathfrak{q}_i}(\partial_{A_{\mathfrak{q}_i}}(ca, cb))$ . Thus it suffices to prove

$$e_A(aM_i, 0, c) = \text{ord}_{A/\mathfrak{q}_i}(c^{m_i f_i}) \quad \text{and} \quad e_A(bM_i, 0, c) = \text{ord}_{A/\mathfrak{q}_i}(c^{m_i e_i})$$

and this follows from Lemma 3.1 by the description (see above) of what happens when we localize at  $\mathfrak{q}_i$ .  $\square$

**OEAX Lemma 6.3** (Key Lemma). *Let  $A$  be a 2-dimensional Noetherian local domain with fraction field  $K$ . Let  $f, g \in K^*$ . Let  $\mathfrak{q}_1, \dots, \mathfrak{q}_t$  be the height 1 primes  $\mathfrak{q}$  of  $A$  such that either  $f$  or  $g$  is not an element of  $A_{\mathfrak{q}}^*$ . Then we have*

$$\sum_{i=1, \dots, t} \text{ord}_{A/\mathfrak{q}_i}(\partial_{A_{\mathfrak{q}_i}}(f, g)) = 0$$

When  $A$  is an excellent ring this is [Kat86, Proposition 1].

We can also write this as

$$\sum_{\text{height}(\mathfrak{q})=1} \text{ord}_{A/\mathfrak{q}}(\partial_{A_{\mathfrak{q}}}(f, g)) = 0$$

since at any height 1 prime  $\mathfrak{q}$  of  $A$  where  $f, g \in A_{\mathfrak{q}}^*$  we have  $\partial_{A_{\mathfrak{q}}}(f, g) = 1$ .

**Proof.** Since the tame symbols  $\partial_{A_{\mathfrak{q}}}(f, g)$  are bilinear and the order functions  $\text{ord}_{A/\mathfrak{q}}$  are additive it suffices to prove the formula when  $f$  and  $g$  are elements of  $A$ . This case is proven in Lemma 6.2.  $\square$

0EAY **Remark 6.4** (Milnor K-theory). For a field  $k$  let us denote  $K_*^M(k)$  the quotient of the tensor algebra on  $k^*$  divided by the two-sided ideal generated by the elements  $x \otimes 1 - x$ . Thus  $K_0^M(k) = \mathbf{Z}$ ,  $K_1^M(k) = k^*$ , and

$$K_2^M(k) = k^* \otimes_{\mathbf{Z}} k^* / \langle x \otimes 1 - x \rangle$$

If  $(A, \mathfrak{m})$  is a 1-dimensional Noetherian local domain with fraction field  $Q(A)$  and residue field  $\kappa$  there is a tame symbol

$$\partial_A : K_{i+1}^M(Q(A)) \rightarrow K_i^M(\kappa(\mathfrak{m}))$$

You can use the method of Section 5 to define these maps, provided you extend the norm map to  $K_i^M$  for all  $i$ . Next, let  $X$  be a Noetherian scheme with a dimension function  $\delta$ . Then we can use these tame symbols to get the arrows in the following:

$$\bigoplus_{\delta(x)=j+1} K_{i+1}^M(\kappa(x)) \longrightarrow \bigoplus_{\delta(x)=j} K_i^M(\kappa(x)) \longrightarrow \bigoplus_{\delta(x)=j-1} K_{i-1}^M(\kappa(x))$$

However, it is not clear, if you define the maps as suggested above, that the composition is zero. When  $i = 1$  and  $j$  arbitrary, this follows from Lemma 6.3. For excellent  $X$  this follows from [Kat86] modulo the verification that Kato's maps are the same as ours.

## 7. Setup

02QK We will throughout work over a locally Noetherian universally catenary base  $S$  endowed with a dimension function  $\delta$ . Although it is likely possible to generalize (parts of) the discussion in the chapter, it seems that this is a good first approximation. It is exactly the generality discussed in [Tho90]. We usually do not assume our schemes are separated or quasi-compact. Many interesting algebraic stacks are non-separated and/or non-quasi-compact and this is a good case study to see how to develop a reasonable theory for those as well. In order to reference these hypotheses we give it a number.

02QL **Situation 7.1.** Here  $S$  is a locally Noetherian, and universally catenary scheme. Moreover, we assume  $S$  is endowed with a dimension function  $\delta : S \rightarrow \mathbf{Z}$ .

See Morphisms, Definition 16.1 for the notion of a universally catenary scheme, and see Topology, Definition 20.1 for the notion of a dimension function. Recall that any locally Noetherian catenary scheme locally has a dimension function, see Properties, Lemma 11.3. Moreover, there are lots of schemes which are universally catenary, see Morphisms, Lemma 16.4.

Let  $(S, \delta)$  be as in Situation 7.1. Any scheme  $X$  locally of finite type over  $S$  is locally Noetherian and catenary. In fact,  $X$  has a canonical dimension function

$$\delta = \delta_{X/S} : X \rightarrow \mathbf{Z}$$

associated to  $(f : X \rightarrow S, \delta)$  given by the rule  $\delta_{X/S}(x) = \delta(f(x)) + \text{trdeg}_{\kappa(f(x))} \kappa(x)$ . See Morphisms, Lemma 50.3. Moreover, if  $h : X \rightarrow Y$  is a morphism of schemes locally of finite type over  $S$ , and  $x \in X$ ,  $y = h(x)$ , then obviously  $\delta_{X/S}(x) = \delta_{Y/S}(y) + \text{trdeg}_{\kappa(y)} \kappa(x)$ . We will freely use this function and its properties in the following.

Here are the basic examples of setups as above. In fact, the main interest lies in the case where the base is the spectrum of a field, or the case where the base is the spectrum of a Dedekind ring (e.g.  $\mathbf{Z}$ , or a discrete valuation ring).

02QM **Example 7.2.** Here  $S = \text{Spec}(k)$  and  $k$  is a field. We set  $\delta(pt) = 0$  where  $pt$  indicates the unique point of  $S$ . The pair  $(S, \delta)$  is an example of a situation as in Situation 7.1 by Morphisms, Lemma 16.4.

02QN **Example 7.3.** Here  $S = \text{Spec}(A)$ , where  $A$  is a Noetherian domain of dimension 1. For example we could consider  $A = \mathbf{Z}$ . We set  $\delta(\mathfrak{p}) = 0$  if  $\mathfrak{p}$  is a maximal ideal and  $\delta(\mathfrak{p}) = 1$  if  $\mathfrak{p} = (0)$  corresponds to the generic point. This is an example of Situation 7.1 by Morphisms, Lemma 16.4.

If  $S$  is Jacobson and  $\delta$  sends closed points to zero, then  $\delta$  is the function sending a point to the dimension of its closure.

02QO **Lemma 7.4.** *Let  $(S, \delta)$  be as in Situation 7.1. Assume in addition  $S$  is a Jacobson scheme, and  $\delta(s) = 0$  for every closed point  $s$  of  $S$ . Let  $X$  be locally of finite type over  $S$ . Let  $Z \subset X$  be an integral closed subscheme and let  $\xi \in Z$  be its generic point. The following integers are the same:*

- (1)  $\delta_{X/S}(\xi)$ ,
- (2)  $\dim(Z)$ , and
- (3)  $\dim(\mathcal{O}_{Z,z})$  where  $z$  is a closed point of  $Z$ .

**Proof.** Let  $X \rightarrow S$ ,  $\xi \in Z \subset X$  be as in the lemma. Since  $X$  is locally of finite type over  $S$  we see that  $X$  is Jacobson, see Morphisms, Lemma 15.9. Hence closed points of  $X$  are dense in every closed subset of  $Z$  and map to closed points of  $S$ . Hence given any chain of irreducible closed subsets of  $Z$  we can end it with a closed point of  $Z$ . It follows that  $\dim(Z) = \sup_z (\dim(\mathcal{O}_{Z,z}))$  (see Properties, Lemma 10.3) where  $z \in Z$  runs over the closed points of  $Z$ . Note that  $\dim(\mathcal{O}_{Z,z}) = \delta(\xi) - \delta(z)$  by the properties of a dimension function. For each closed  $z \in Z$  the field extension  $\kappa(z) \supset \kappa(f(z))$  is finite, see Morphisms, Lemma 15.8. Hence  $\delta_{X/S}(z) = \delta(f(z)) = 0$  for  $z \in Z$  closed. It follows that all three integers are equal.  $\square$

In the situation of the lemma above the value of  $\delta$  at the generic point of a closed irreducible subset is the dimension of the irreducible closed subset. However, in general we cannot expect the equality to hold. For example if  $S = \text{Spec}(\mathbf{C}[[t]])$  and  $X = \text{Spec}(\mathbf{C}((t)))$  then we would get  $\delta(x) = 1$  for the unique point of  $X$ , but  $\dim(X) = 0$ . Still we want to think of  $\delta_{X/S}$  as giving the dimension of the irreducible closed subschemes. Thus we introduce the following terminology.

02QP **Definition 7.5.** Let  $(S, \delta)$  as in Situation 7.1. For any scheme  $X$  locally of finite type over  $S$  and any irreducible closed subset  $Z \subset X$  we define

$$\dim_{\delta}(Z) = \delta(\xi)$$

where  $\xi \in Z$  is the generic point of  $Z$ . We will call this the  $\delta$ -dimension of  $Z$ . If  $Z$  is a closed subscheme of  $X$ , then we define  $\dim_{\delta}(Z)$  as the supremum of the  $\delta$ -dimensions of its irreducible components.

### 8. Cycles

02QQ Since we are not assuming our schemes are quasi-compact we have to be a little careful when defining cycles. We have to allow infinite sums because a rational function may have infinitely many poles for example. In any case, if  $X$  is quasi-compact then a cycle is a finite sum as usual.

02QR **Definition 8.1.** Let  $(S, \delta)$  be as in Situation 7.1. Let  $X$  be locally of finite type over  $S$ . Let  $k \in \mathbf{Z}$ .

(1) A *cycle on  $X$*  is a formal sum

$$\alpha = \sum n_Z [Z]$$

where the sum is over integral closed subschemes  $Z \subset X$ , each  $n_Z \in \mathbf{Z}$ , and the collection  $\{Z; n_Z \neq 0\}$  is locally finite (Topology, Definition 28.4).

(2) A  *$k$ -cycle on  $X$*  is a cycle

$$\alpha = \sum n_Z [Z]$$

where  $n_Z \neq 0 \Rightarrow \dim_\delta(Z) = k$ .

(3) The abelian group of all  $k$ -cycles on  $X$  is denoted  $Z_k(X)$ .

In other words, a  $k$ -cycle on  $X$  is a locally finite formal  $\mathbf{Z}$ -linear combination of integral closed subschemes of  $\delta$ -dimension  $k$ . Addition of  $k$ -cycles  $\alpha = \sum n_Z [Z]$  and  $\beta = \sum m_Z [Z]$  is given by

$$\alpha + \beta = \sum (n_Z + m_Z) [Z],$$

i.e., by adding the coefficients.

### 9. Cycle associated to a closed subscheme

02QS

02QT **Lemma 9.1.** Let  $(S, \delta)$  be as in Situation 7.1. Let  $X$  be locally of finite type over  $S$ . Let  $Z \subset X$  be a closed subscheme.

(1) Let  $Z' \subset Z$  be an irreducible component and let  $\xi \in Z'$  be its generic point. Then

$$\text{length}_{\mathcal{O}_{X, \xi}} \mathcal{O}_{Z, \xi} < \infty$$

(2) If  $\dim_\delta(Z) \leq k$  and  $\xi \in Z$  with  $\delta(\xi) = k$ , then  $\xi$  is a generic point of an irreducible component of  $Z$ .

**Proof.** Let  $Z' \subset Z$ ,  $\xi \in Z'$  be as in (1). Then  $\dim(\mathcal{O}_{Z, \xi}) = 0$  (for example by Properties, Lemma 10.3). Hence  $\mathcal{O}_{Z, \xi}$  is Noetherian local ring of dimension zero, and hence has finite length over itself (see Algebra, Proposition 59.6). Hence, it also has finite length over  $\mathcal{O}_{X, \xi}$ , see Algebra, Lemma 51.12.

Assume  $\xi \in Z$  and  $\delta(\xi) = k$ . Consider the closure  $Z' = \overline{\{\xi\}}$ . It is an irreducible closed subscheme with  $\dim_\delta(Z') = k$  by definition. Since  $\dim_\delta(Z) = k$  it must be an irreducible component of  $Z$ . Hence we see (2) holds.  $\square$

02QU **Definition 9.2.** Let  $(S, \delta)$  be as in Situation 7.1. Let  $X$  be locally of finite type over  $S$ . Let  $Z \subset X$  be a closed subscheme.

(1) For any irreducible component  $Z' \subset Z$  with generic point  $\xi$  the integer  $m_{Z', Z} = \text{length}_{\mathcal{O}_{X, \xi}} \mathcal{O}_{Z, \xi}$  (Lemma 9.1) is called the *multiplicity of  $Z'$  in  $Z$* .



- (2) Assume  $\dim_\delta(Z) \leq k$ . The  $k$ -cycle associated to  $Z$  is

$$[Z]_k = \sum m_{Z',Z} [Z']$$

where the sum is over the irreducible components of  $Z$  of  $\delta$ -dimension  $k$ . (This is a  $k$ -cycle by Divisors, Lemma 26.1.)

It is important to note that we only define  $[Z]_k$  if the  $\delta$ -dimension of  $Z$  does not exceed  $k$ . In other words, by convention, if we write  $[Z]_k$  then this implies that  $\dim_\delta(Z) \leq k$ .

### 10. Cycle associated to a coherent sheaf

02QV

**Lemma 10.1.** *Let  $(S, \delta)$  be as in Situation 7.1. Let  $X$  be locally of finite type over  $S$ . Let  $\mathcal{F}$  be a coherent  $\mathcal{O}_X$ -module.*

- (1) *The collection of irreducible components of the support of  $\mathcal{F}$  is locally finite.*
- (2) *Let  $Z' \subset \text{Supp}(\mathcal{F})$  be an irreducible component and let  $\xi \in Z'$  be its generic point. Then*

$$\text{length}_{\mathcal{O}_{X,\xi}} \mathcal{F}_\xi < \infty$$

- (3) *If  $\dim_\delta(\text{Supp}(\mathcal{F})) \leq k$  and  $\xi \in Z$  with  $\delta(\xi) = k$ , then  $\xi$  is a generic point of an irreducible component of  $\text{Supp}(\mathcal{F})$ .*

**Proof.** By Cohomology of Schemes, Lemma 9.7 the support  $Z$  of  $\mathcal{F}$  is a closed subset of  $X$ . We may think of  $Z$  as a reduced closed subscheme of  $X$  (Schemes, Lemma 12.4). Hence (1) follows from Divisors, Lemma 26.1 applied to  $Z$  and (3) follows from Lemma 9.1 applied to  $Z$ .

Let  $\xi \in Z'$  be as in (2). In this case for any specialization  $\xi' \rightsquigarrow \xi$  in  $X$  we have  $\mathcal{F}_{\xi'} = 0$ . Recall that the non-maximal primes of  $\mathcal{O}_{X,\xi}$  correspond to the points of  $X$  specializing to  $\xi$  (Schemes, Lemma 13.2). Hence  $\mathcal{F}_\xi$  is a finite  $\mathcal{O}_{X,\xi}$ -module whose support is  $\{\mathfrak{m}_\xi\}$ . Hence it has finite length by Algebra, Lemma 61.3.  $\square$

02QX

**Definition 10.2.** *Let  $(S, \delta)$  be as in Situation 7.1. Let  $X$  be locally of finite type over  $S$ . Let  $\mathcal{F}$  be a coherent  $\mathcal{O}_X$ -module.*

- (1) *For any irreducible component  $Z' \subset \text{Supp}(\mathcal{F})$  with generic point  $\xi$  the integer  $m_{Z',\mathcal{F}} = \text{length}_{\mathcal{O}_{X,\xi}} \mathcal{F}_\xi$  (Lemma 10.1) is called the *multiplicity of  $Z'$  in  $\mathcal{F}$* .*
- (2) *Assume  $\dim_\delta(\text{Supp}(\mathcal{F})) \leq k$ . The  $k$ -cycle associated to  $\mathcal{F}$  is*

$$[\mathcal{F}]_k = \sum m_{Z',\mathcal{F}} [Z']$$

where the sum is over the irreducible components of  $\text{Supp}(\mathcal{F})$  of  $\delta$ -dimension  $k$ . (This is a  $k$ -cycle by Lemma 10.1.)

It is important to note that we only define  $[\mathcal{F}]_k$  if  $\mathcal{F}$  is coherent and the  $\delta$ -dimension of  $\text{Supp}(\mathcal{F})$  does not exceed  $k$ . In other words, by convention, if we write  $[\mathcal{F}]_k$  then this implies that  $\mathcal{F}$  is coherent on  $X$  and  $\dim_\delta(\text{Supp}(\mathcal{F})) \leq k$ .

02QY

**Lemma 10.3.** *Let  $(S, \delta)$  be as in Situation 7.1. Let  $X$  be locally of finite type over  $S$ . Let  $Z \subset X$  be a closed subscheme. If  $\dim_\delta(Z) \leq k$ , then  $[Z]_k = [\mathcal{O}_Z]_k$ .*

**Proof.** This is because in this case the multiplicities  $m_{Z',Z}$  and  $m_{Z',\mathcal{O}_Z}$  agree by definition.  $\square$

02QZ **Lemma 10.4.** *Let  $(S, \delta)$  be as in Situation 7.1. Let  $X$  be locally of finite type over  $S$ . Let  $0 \rightarrow \mathcal{F} \rightarrow \mathcal{G} \rightarrow \mathcal{H} \rightarrow 0$  be a short exact sequence of coherent sheaves on  $X$ . Assume that the  $\delta$ -dimension of the supports of  $\mathcal{F}$ ,  $\mathcal{G}$ , and  $\mathcal{H}$  is  $\leq k$ . Then  $[\mathcal{G}]_k = [\mathcal{F}]_k + [\mathcal{H}]_k$ .*

**Proof.** Follows immediately from additivity of lengths, see Algebra, Lemma 51.3.  $\square$

## 11. Preparation for proper pushforward

02R0

02R1 **Lemma 11.1.** *Let  $(S, \delta)$  be as in Situation 7.1. Let  $X, Y$  be locally of finite type over  $S$ . Let  $f : X \rightarrow Y$  be a morphism. Assume  $X, Y$  integral and  $\dim_\delta(X) = \dim_\delta(Y)$ . Then either  $f(X)$  is contained in a proper closed subscheme of  $Y$ , or  $f$  is dominant and the extension of function fields  $R(Y) \subset R(X)$  is finite.*

**Proof.** The closure  $\overline{f(X)} \subset Y$  is irreducible as  $X$  is irreducible (Topology, Lemmas 8.2 and 8.3). If  $\overline{f(X)} \neq Y$ , then we are done. If  $\overline{f(X)} = Y$ , then  $f$  is dominant and by Morphisms, Lemma 8.5 we see that the generic point  $\eta_Y$  of  $Y$  is in the image of  $f$ . Of course this implies that  $f(\eta_X) = \eta_Y$ , where  $\eta_X \in X$  is the generic point of  $X$ . Since  $\delta(\eta_X) = \delta(\eta_Y)$  we see that  $R(Y) = \kappa(\eta_Y) \subset \kappa(\eta_X) = R(X)$  is an extension of transcendence degree 0. Hence  $R(Y) \subset R(X)$  is a finite extension by Morphisms, Lemma 49.7 (which applies by Morphisms, Lemma 14.8).  $\square$

02R2

**Lemma 11.2.** *Let  $(S, \delta)$  be as in Situation 7.1. Let  $X, Y$  be locally of finite type over  $S$ . Let  $f : X \rightarrow Y$  be a morphism. Assume  $f$  is quasi-compact, and  $\{Z_i\}_{i \in I}$  is a locally finite collection of closed subsets of  $X$ . Then  $\{\overline{f(Z_i)}\}_{i \in I}$  is a locally finite collection of closed subsets of  $Y$ .*

**Proof.** Let  $V \subset Y$  be a quasi-compact open subset. Since  $f$  is quasi-compact the open  $f^{-1}(V)$  is quasi-compact. Hence the set  $\{i \in I \mid Z_i \cap f^{-1}(V) \neq \emptyset\}$  is finite by a simple topological argument which we omit. Since this is the same as the set

$$\{i \in I \mid f(Z_i) \cap V \neq \emptyset\} = \{i \in I \mid \overline{f(Z_i)} \cap V \neq \emptyset\}$$

the lemma is proved.  $\square$

## 12. Proper pushforward

02R3

02R4 **Definition 12.1.** Let  $(S, \delta)$  be as in Situation 7.1. Let  $X, Y$  be locally of finite type over  $S$ . Let  $f : X \rightarrow Y$  be a morphism. Assume  $f$  is proper.

(1) Let  $Z \subset X$  be an integral closed subscheme with  $\dim_\delta(Z) = k$ . We define

$$f_*[Z] = \begin{cases} 0 & \text{if } \dim_\delta(f(Z)) < k, \\ \deg(Z/f(Z))[f(Z)] & \text{if } \dim_\delta(f(Z)) = k. \end{cases}$$

Here we think of  $f(Z) \subset Y$  as an integral closed subscheme. The degree of  $Z$  over  $f(Z)$  is finite if  $\dim_\delta(f(Z)) = \dim_\delta(Z)$  by Lemma 11.1.

(2) Let  $\alpha = \sum n_Z[Z]$  be a  $k$ -cycle on  $X$ . The *pushforward* of  $\alpha$  as the sum

$$f_*\alpha = \sum n_Z f_*[Z]$$

where each  $f_*[Z]$  is defined as above. The sum is locally finite by Lemma 11.2 above.

By definition the proper pushforward of cycles

$$f_* : Z_k(X) \longrightarrow Z_k(Y)$$

is a homomorphism of abelian groups. It turns  $X \mapsto Z_k(X)$  into a covariant functor on the category of schemes locally of finite type over  $S$  with morphisms equal to proper morphisms.

02R5 **Lemma 12.2.** *Let  $(S, \delta)$  be as in Situation 7.1. Let  $X, Y$ , and  $Z$  be locally of finite type over  $S$ . Let  $f : X \rightarrow Y$  and  $g : Y \rightarrow Z$  be proper morphisms. Then  $g_* \circ f_* = (g \circ f)_*$  as maps  $Z_k(X) \rightarrow Z_k(Z)$ .*

**Proof.** Let  $W \subset X$  be an integral closed subscheme of dimension  $k$ . Consider  $W' = f(W) \subset Y$  and  $W'' = g(W') \subset Z$ . Since  $f, g$  are proper we see that  $W'$  (resp.  $W''$ ) is an integral closed subscheme of  $Y$  (resp.  $Z$ ). We have to show that  $g_*(f_*[W]) = (f \circ g)_*[W]$ . If  $\dim_\delta(W'') < k$ , then both sides are zero. If  $\dim_\delta(W'') = k$ , then we see the induced morphisms

$$W \longrightarrow W' \longrightarrow W''$$

both satisfy the hypotheses of Lemma 11.1. Hence

$$g_*(f_*[W]) = \deg(W/W') \deg(W'/W'')[W''], \quad (f \circ g)_*[W] = \deg(W/W'')[W''].$$

Then we can apply Morphisms, Lemma 49.9 to conclude.  $\square$

02R6 **Lemma 12.3.** *Let  $(S, \delta)$  be as in Situation 7.1. Let  $f : X \rightarrow Y$  be a proper morphism of schemes which are locally of finite type over  $S$ .*

(1) *Let  $Z \subset X$  be a closed subscheme with  $\dim_\delta(Z) \leq k$ . Then*

$$f_*[Z]_k = [f_*\mathcal{O}_Z]_k.$$

(2) *Let  $\mathcal{F}$  be a coherent sheaf on  $X$  such that  $\dim_\delta(\text{Supp}(\mathcal{F})) \leq k$ . Then*

$$f_*[\mathcal{F}]_k = [f_*\mathcal{F}]_k.$$

*Note that the statement makes sense since  $f_*\mathcal{F}$  and  $f_*\mathcal{O}_Z$  are coherent  $\mathcal{O}_Y$ -modules by Cohomology of Schemes, Proposition 19.1.*

**Proof.** Part (1) follows from (2) and Lemma 10.3. Let  $\mathcal{F}$  be a coherent sheaf on  $X$ . Assume that  $\dim_\delta(\text{Supp}(\mathcal{F})) \leq k$ . By Cohomology of Schemes, Lemma 9.7 there exists a closed subscheme  $i : Z \rightarrow X$  and a coherent  $\mathcal{O}_Z$ -module  $\mathcal{G}$  such that  $i_*\mathcal{G} \cong \mathcal{F}$  and such that the support of  $\mathcal{F}$  is  $Z$ . Let  $Z' \subset Y$  be the scheme theoretic image of  $f|_Z : Z \rightarrow Y$ . Consider the commutative diagram of schemes

$$\begin{array}{ccc} Z & \xrightarrow{i} & X \\ f|_Z \downarrow & & \downarrow f \\ Z' & \xrightarrow{i'} & Y \end{array}$$

We have  $f_*\mathcal{F} = f_*i_*\mathcal{G} = i'_*(f|_Z)_*\mathcal{G}$  by going around the diagram in two ways. Suppose we know the result holds for closed immersions and for  $f|_Z$ . Then we see that

$$f_*[\mathcal{F}]_k = f_*i_*[\mathcal{G}]_k = (i')_*(f|_Z)_*[\mathcal{G}]_k = (i')_*[(f|_Z)_*\mathcal{G}]_k = [(i')_*(f|_Z)_*\mathcal{G}]_k = [f_*\mathcal{F}]_k$$

as desired. The case of a closed immersion is straightforward (omitted). Note that  $f|_Z : Z \rightarrow Z'$  is a dominant morphism (see Morphisms, Lemma 6.3). Thus we have reduced to the case where  $\dim_\delta(X) \leq k$  and  $f : X \rightarrow Y$  is proper and dominant.

Assume  $\dim_\delta(X) \leq k$  and  $f : X \rightarrow Y$  is proper and dominant. Since  $f$  is dominant, for every irreducible component  $Z \subset Y$  with generic point  $\eta$  there exists a point  $\xi \in X$  such that  $f(\xi) = \eta$ . Hence  $\delta(\eta) \leq \delta(\xi) \leq k$ . Thus we see that in the expressions

$$f_*[\mathcal{F}]_k = \sum n_Z[Z], \quad \text{and} \quad [f_*\mathcal{F}]_k = \sum m_Z[Z].$$

whenever  $n_Z \neq 0$ , or  $m_Z \neq 0$  the integral closed subscheme  $Z$  is actually an irreducible component of  $Y$  of  $\delta$ -dimension  $k$ . Pick such an integral closed subscheme  $Z \subset Y$  and denote  $\eta$  its generic point. Note that for any  $\xi \in X$  with  $f(\xi) = \eta$  we have  $\delta(\xi) \geq k$  and hence  $\xi$  is a generic point of an irreducible component of  $X$  of  $\delta$ -dimension  $k$  as well (see Lemma 9.1). Since  $f$  is quasi-compact and  $X$  is locally Noetherian, there can be only finitely many of these and hence  $f^{-1}(\{\eta\})$  is finite. By Morphisms, Lemma 49.1 there exists an open neighbourhood  $\eta \in V \subset Y$  such that  $f^{-1}(V) \rightarrow V$  is finite. Replacing  $Y$  by  $V$  and  $X$  by  $f^{-1}(V)$  we reduce to the case where  $Y$  is affine, and  $f$  is finite.

Write  $Y = \text{Spec}(R)$  and  $X = \text{Spec}(A)$  (possible as a finite morphism is affine). Then  $R$  and  $A$  are Noetherian rings and  $A$  is finite over  $R$ . Moreover  $\mathcal{F} = \widetilde{M}$  for some finite  $A$ -module  $M$ . Note that  $f_*\mathcal{F}$  corresponds to  $M$  viewed as an  $R$ -module. Let  $\mathfrak{p} \subset R$  be the minimal prime corresponding to  $\eta \in Y$ . The coefficient of  $Z$  in  $[f_*\mathcal{F}]_k$  is clearly  $\text{length}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}})$ . Let  $\mathfrak{q}_i, i = 1, \dots, t$  be the primes of  $A$  lying over  $\mathfrak{p}$ . Then  $A_{\mathfrak{p}} = \prod A_{\mathfrak{q}_i}$  since  $A_{\mathfrak{p}}$  is an Artinian ring being finite over the dimension zero local Noetherian ring  $R_{\mathfrak{p}}$ . Clearly the coefficient of  $Z$  in  $f_*[\mathcal{F}]_k$  is

$$\sum_{i=1, \dots, t} [\kappa(\mathfrak{q}_i) : \kappa(\mathfrak{p})] \text{length}_{A_{\mathfrak{q}_i}}(M_{\mathfrak{q}_i})$$

Hence the desired equality follows from Algebra, Lemma 51.12.  $\square$

### 13. Preparation for flat pullback

02R7 Recall that a morphism  $f : X \rightarrow Y$  which is locally of finite type is said to have relative dimension  $r$  if every nonempty fibre is equidimensional of dimension  $r$ . See Morphisms, Definition 28.1.

02R8 **Lemma 13.1.** *Let  $(S, \delta)$  be as in Situation 7.1. Let  $X, Y$  be locally of finite type over  $S$ . Let  $f : X \rightarrow Y$  be a morphism. Assume  $f$  is flat of relative dimension  $r$ . For any closed subset  $Z \subset Y$  we have*

$$\dim_\delta(f^{-1}(Z)) = \dim_\delta(Z) + r.$$

*provided  $f^{-1}(Z)$  is nonempty. If  $Z$  is irreducible and  $Z' \subset f^{-1}(Z)$  is an irreducible component, then  $Z'$  dominates  $Z$  and  $\dim_\delta(Z') = \dim_\delta(Z) + r$ .*

**Proof.** It suffices to prove the final statement. We may replace  $Y$  by the integral closed subscheme  $Z$  and  $X$  by the scheme theoretic inverse image  $f^{-1}(Z) = Z \times_Y X$ . Hence we may assume  $Z = Y$  is integral and  $f$  is a flat morphism of relative dimension  $r$ . Since  $Y$  is locally Noetherian the morphism  $f$  which is locally of finite type, is actually locally of finite presentation. Hence Morphisms, Lemma 24.9 applies and we see that  $f$  is open. Let  $\xi \in X$  be a generic point of an irreducible component of  $X$ . By the openness of  $f$  we see that  $f(\xi)$  is the generic point  $\eta$  of  $Z = Y$ . Note that  $\dim_\xi(X_\eta) = r$  by assumption that  $f$  has relative dimension  $r$ . On the other hand, since  $\xi$  is a generic point of  $X$  we see that  $\mathcal{O}_{X, \xi} = \mathcal{O}_{X_\eta, \xi}$  has only one prime ideal and hence has dimension 0. Thus by Morphisms, Lemma 27.1

we conclude that the transcendence degree of  $\kappa(\xi)$  over  $\kappa(\eta)$  is  $r$ . In other words,  $\delta(\xi) = \delta(\eta) + r$  as desired.  $\square$

Here is the lemma that we will use to prove that the flat pullback of a locally finite collection of closed subschemes is locally finite.

02R9 **Lemma 13.2.** *Let  $(S, \delta)$  be as in Situation 7.1. Let  $X, Y$  be locally of finite type over  $S$ . Let  $f : X \rightarrow Y$  be a morphism. Assume  $\{Z_i\}_{i \in I}$  is a locally finite collection of closed subsets of  $Y$ . Then  $\{f^{-1}(Z_i)\}_{i \in I}$  is a locally finite collection of closed subsets of  $X$ .*

**Proof.** Let  $U \subset X$  be a quasi-compact open subset. Since the image  $f(U) \subset Y$  is a quasi-compact subset there exists a quasi-compact open  $V \subset Y$  such that  $f(U) \subset V$ . Note that

$$\{i \in I \mid f^{-1}(Z_i) \cap U \neq \emptyset\} \subset \{i \in I \mid Z_i \cap V \neq \emptyset\}.$$

Since the right hand side is finite by assumption we win.  $\square$

#### 14. Flat pullback

02RA In the following we use  $f^{-1}(Z)$  to denote the *scheme theoretic inverse image* of a closed subscheme  $Z \subset Y$  for a morphism of schemes  $f : X \rightarrow Y$ . We recall that the scheme theoretic inverse image is the fibre product

$$\begin{array}{ccc} f^{-1}(Z) & \longrightarrow & X \\ \downarrow & & \downarrow \\ Z & \longrightarrow & Y \end{array}$$

and it is also the closed subscheme of  $X$  cut out by the quasi-coherent sheaf of ideals  $f^{-1}(\mathcal{I})\mathcal{O}_X$ , if  $\mathcal{I} \subset \mathcal{O}_Y$  is the quasi-coherent sheaf of ideals corresponding to  $Z$  in  $Y$ . (This is discussed in Schemes, Section 4 and Lemma 17.6 and Definition 17.7.)

02RB **Definition 14.1.** Let  $(S, \delta)$  be as in Situation 7.1. Let  $X, Y$  be locally of finite type over  $S$ . Let  $f : X \rightarrow Y$  be a morphism. Assume  $f$  is flat of relative dimension  $r$ .

- (1) Let  $Z \subset Y$  be an integral closed subscheme of  $\delta$ -dimension  $k$ . We define  $f^*[Z]$  to be the  $(k+r)$ -cycle on  $X$  to the scheme theoretic inverse image

$$f^*[Z] = [f^{-1}(Z)]_{k+r}.$$

This makes sense since  $\dim_\delta(f^{-1}(Z)) = k+r$  by Lemma 13.1.

- (2) Let  $\alpha = \sum n_i [Z_i]$  be a  $k$ -cycle on  $Y$ . The *flat pullback of  $\alpha$  by  $f$*  is the sum

$$f^*\alpha = \sum n_i f^*[Z_i]$$

where each  $f^*[Z_i]$  is defined as above. The sum is locally finite by Lemma 13.2.

- (3) We denote  $f^* : Z_k(Y) \rightarrow Z_{k+r}(X)$  the map of abelian groups so obtained.

An open immersion is flat. This is an important though trivial special case of a flat morphism. If  $U \subset X$  is open then sometimes the pullback by  $j : U \rightarrow X$  of a cycle is called the *restriction* of the cycle to  $U$ . Note that in this case the maps

$$j^* : Z_k(X) \longrightarrow Z_k(U)$$

are all *surjective*. The reason is that given any integral closed subscheme  $Z' \subset U$ , we can take the closure of  $Z'$  in  $X$  and think of it as a reduced closed subscheme of  $X$  (see Schemes, Lemma 12.4). And clearly  $Z \cap U = Z'$ , in other words  $j^*[Z] = [Z']$  whence the surjectivity. In fact a little bit more is true.

02RC **Lemma 14.2.** *Let  $(S, \delta)$  be as in Situation 7.1. Let  $X$  be locally of finite type over  $S$ . Let  $U \subset X$  be an open subscheme, and denote  $i : Y = X \setminus U \rightarrow X$  as a reduced closed subscheme of  $X$ . For every  $k \in \mathbf{Z}$  the sequence*

$$Z_k(Y) \xrightarrow{i_*} Z_k(X) \xrightarrow{j^*} Z_k(U) \longrightarrow 0$$

*is an exact complex of abelian groups.*

**Proof.** First assume  $X$  is quasi-compact. Then  $Z_k(X)$  is a free  $\mathbf{Z}$ -module with basis given by the elements  $[Z]$  where  $Z \subset X$  is integral closed of  $\delta$ -dimension  $k$ . Such a basis element maps either to the basis element  $[Z \cap U]$  or to zero if  $Z \subset Y$ . Hence the lemma is clear in this case. The general case is similar and the proof is omitted.  $\square$

02RD **Lemma 14.3.** *Let  $(S, \delta)$  be as in Situation 7.1. Let  $X, Y, Z$  be locally of finite type over  $S$ . Let  $f : X \rightarrow Y$  and  $g : Y \rightarrow Z$  be flat morphisms of relative dimensions  $r$  and  $s$ . Then  $g \circ f$  is flat of relative dimension  $r + s$  and*

$$f^* \circ g^* = (g \circ f)^*$$

*as maps  $Z_k(Z) \rightarrow Z_{k+r+s}(X)$ .*

**Proof.** The composition is flat of relative dimension  $r + s$  by Morphisms, Lemma 28.3. Suppose that

- (1)  $W \subset Z$  is a closed integral subscheme of  $\delta$ -dimension  $k$ ,
- (2)  $W' \subset Y$  is a closed integral subscheme of  $\delta$ -dimension  $k + s$  with  $W' \subset g^{-1}(W)$ , and
- (3)  $W'' \subset Y$  is a closed integral subscheme of  $\delta$ -dimension  $k + s + r$  with  $W'' \subset f^{-1}(W')$ .

We have to show that the coefficient  $n$  of  $[W'']$  in  $(g \circ f)^*[W]$  agrees with the coefficient  $m$  of  $[W'']$  in  $f^*(g^*[W])$ . That it suffices to check the lemma in these cases follows from Lemma 13.1. Let  $\xi'' \in W''$ ,  $\xi' \in W'$  and  $\xi \in W$  be the generic points. Consider the local rings  $A = \mathcal{O}_{Z, \xi}$ ,  $B = \mathcal{O}_{Y, \xi'}$  and  $C = \mathcal{O}_{X, \xi''}$ . Then we have local flat ring maps  $A \rightarrow B$ ,  $B \rightarrow C$  and moreover

$$n = \text{length}_C(C/\mathfrak{m}_A C), \quad \text{and} \quad m = \text{length}_C(C/\mathfrak{m}_B C) \text{length}_B(B/\mathfrak{m}_A B)$$

Hence the equality follows from Algebra, Lemma 51.14.  $\square$

02RE **Lemma 14.4.** *Let  $(S, \delta)$  be as in Situation 7.1. Let  $X, Y$  be locally of finite type over  $S$ . Let  $f : X \rightarrow Y$  be a flat morphism of relative dimension  $r$ .*

- (1) *Let  $Z \subset Y$  be a closed subscheme with  $\dim_\delta(Z) \leq k$ . Then we have  $\dim_\delta(f^{-1}(Z)) \leq k + r$  and  $[f^{-1}(Z)]_{k+r} = f^*[Z]_k$  in  $Z_{k+r}(X)$ .*
- (2) *Let  $\mathcal{F}$  be a coherent sheaf on  $Y$  with  $\dim_\delta(\text{Supp}(\mathcal{F})) \leq k$ . Then we have  $\dim_\delta(\text{Supp}(f^*\mathcal{F})) \leq k + r$  and*

$$f^*[\mathcal{F}]_k = [f^*\mathcal{F}]_{k+r}$$

*in  $Z_{k+r}(X)$ .*

**Proof.** Part (1) follows from part (2) by Lemma 10.3 and the fact that  $f^*\mathcal{O}_Z = \mathcal{O}_{f^{-1}(Z)}$ .

Proof of (2). As  $X, Y$  are locally Noetherian we may apply Cohomology of Schemes, Lemma 9.1 to see that  $\mathcal{F}$  is of finite type, hence  $f^*\mathcal{F}$  is of finite type (Modules, Lemma 9.2), hence  $f^*\mathcal{F}$  is coherent (Cohomology of Schemes, Lemma 9.1 again). Thus the lemma makes sense. Let  $W \subset Y$  be an integral closed subscheme of  $\delta$ -dimension  $k$ , and let  $W' \subset X$  be an integral closed subscheme of dimension  $k+r$  mapping into  $W$  under  $f$ . We have to show that the coefficient  $n$  of  $[W']$  in  $f^*[\mathcal{F}]_k$  agrees with the coefficient  $m$  of  $[W']$  in  $[f^*\mathcal{F}]_{k+r}$ . Let  $\xi \in W$  and  $\xi' \in W'$  be the generic points. Let  $A = \mathcal{O}_{Y,\xi}$ ,  $B = \mathcal{O}_{X,\xi'}$  and set  $M = \mathcal{F}_\xi$  as an  $A$ -module. (Note that  $M$  has finite length by our dimension assumptions, but we actually do not need to verify this. See Lemma 10.1.) We have  $f^*\mathcal{F}_{\xi'} = B \otimes_A M$ . Thus we see that

$$n = \text{length}_B(B \otimes_A M) \quad \text{and} \quad m = \text{length}_A(M)\text{length}_B(B/\mathfrak{m}_A B)$$

Thus the equality follows from Algebra, Lemma 51.13. □

### 15. Push and pull

02RF In this section we verify that proper pushforward and flat pullback are compatible when this makes sense. By the work we did above this is a consequence of cohomology and base change.

02RG **Lemma 15.1.** *Let  $(S, \delta)$  be as in Situation 7.1. Let*

$$\begin{array}{ccc} X' & \xrightarrow{\quad} & X \\ f' \downarrow & & \downarrow f \\ Y' & \xrightarrow{\quad} & Y \end{array}$$

*be a fibre product diagram of schemes locally of finite type over  $S$ . Assume  $f : X \rightarrow Y$  proper and  $g : Y' \rightarrow Y$  flat of relative dimension  $r$ . Then also  $f'$  is proper and  $g'$  is flat of relative dimension  $r$ . For any  $k$ -cycle  $\alpha$  on  $X$  we have*

$$g^* f_* \alpha = f'_*(g')^* \alpha$$

*in  $Z_{k+r}(Y')$ .*

**Proof.** The assertion that  $f'$  is proper follows from Morphisms, Lemma 39.5. The assertion that  $g'$  is flat of relative dimension  $r$  follows from Morphisms, Lemmas 28.2 and 24.7. It suffices to prove the equality of cycles when  $\alpha = [W]$  for some integral closed subscheme  $W \subset X$  of  $\delta$ -dimension  $k$ . Note that in this case we have  $\alpha = [\mathcal{O}_W]_k$ , see Lemma 10.3. By Lemmas 12.3 and 14.4 it therefore suffices to show that  $f'_*(g')^*\mathcal{O}_W$  is isomorphic to  $g^*f_*\mathcal{O}_W$ . This follows from cohomology and base change, see Cohomology of Schemes, Lemma 5.2. □

02RH **Lemma 15.2.** *Let  $(S, \delta)$  be as in Situation 7.1. Let  $X, Y$  be locally of finite type over  $S$ . Let  $f : X \rightarrow Y$  be a finite locally free morphism of degree  $d$  (see Morphisms, Definition 46.1). Then  $f$  is both proper and flat of relative dimension 0, and*

$$f_* f^* \alpha = d\alpha$$

*for every  $\alpha \in Z_k(Y)$ .*

**Proof.** A finite locally free morphism is flat and finite by Morphisms, Lemma 46.2, and a finite morphism is proper by Morphisms, Lemma 42.11. We omit showing that a finite morphism has relative dimension 0. Thus the formula makes sense. To prove it, let  $Z \subset Y$  be an integral closed subscheme of  $\delta$ -dimension  $k$ . It suffices to prove the formula for  $\alpha = [Z]$ . Since the base change of a finite locally free morphism is finite locally free (Morphisms, Lemma 46.4) we see that  $f_*f^*\mathcal{O}_Z$  is a finite locally free sheaf of rank  $d$  on  $Z$ . Hence

$$f_*f^*[Z] = f_*f^*[\mathcal{O}_Z]_k = [f_*f^*\mathcal{O}_Z]_k = d[Z]$$

where we have used Lemmas 14.4 and 12.3.  $\square$

## 16. Preparation for principal divisors

02RI Some of the material in this section partially overlaps with the discussion in Divisors, Section 26.

02RK **Lemma 16.1.** *Let  $(S, \delta)$  be as in Situation 7.1. Let  $X$  be locally of finite type over  $S$ . Assume  $X$  is integral.*

- (1) *If  $Z \subset X$  is an integral closed subscheme, then the following are equivalent:*
  - (a)  *$Z$  is a prime divisor,*
  - (b)  *$Z$  has codimension 1 in  $X$ , and*
  - (c)  *$\dim_\delta(Z) = \dim_\delta(X) - 1$ .*
- (2) *If  $Z$  is an irreducible component of an effective Cartier divisor on  $X$ , then  $\dim_\delta(Z) = \dim_\delta(X) - 1$ .*

**Proof.** Part (1) follows from the definition of a prime divisor (Divisors, Definition 26.2) and the definition of a dimension function (Topology, Definition 20.1). Let  $\xi \in Z$  be the generic point of an irreducible component  $Z$  of an effective Cartier divisor  $D \subset X$ . Then  $\dim(\mathcal{O}_{D, \xi}) = 0$  and  $\mathcal{O}_{D, \xi} = \mathcal{O}_{X, \xi}/(f)$  for some nonzerodivisor  $f \in \mathcal{O}_{X, \xi}$  (Divisors, Lemma 15.2). Then  $\dim(\mathcal{O}_{X, \xi}) = 1$  by Algebra, Lemma 59.12. Hence  $Z$  is as in (1) by Properties, Lemma 10.3 and the proof is complete.  $\square$

02RM **Lemma 16.2.** *Let  $f : X \rightarrow Y$  be a morphism of schemes. Let  $\xi \in Y$  be a point. Assume that*

- (1)  *$X, Y$  are integral,*
- (2)  *$Y$  is locally Noetherian*
- (3)  *$f$  is proper, dominant and  $R(X) \subset R(Y)$  is finite, and*
- (4)  *$\dim(\mathcal{O}_{Y, \xi}) = 1$ .*

*Then there exists an open neighbourhood  $V \subset Y$  of  $\xi$  such that  $f|_{f^{-1}(V)} : f^{-1}(V) \rightarrow V$  is finite.*

**Proof.** This lemma is a special case of Varieties, Lemma 17.2. Here is a direct argument in this case. By Cohomology of Schemes, Lemma 21.2 it suffices to prove that  $f^{-1}(\{\xi\})$  is finite. We replace  $Y$  by an affine open, say  $Y = \text{Spec}(R)$ . Note that  $R$  is Noetherian, as  $Y$  is assumed locally Noetherian. Since  $f$  is proper it is quasi-compact. Hence we can find a finite affine open covering  $X = U_1 \cup \dots \cup U_n$  with each  $U_i = \text{Spec}(A_i)$ . Note that  $R \rightarrow A_i$  is a finite type injective homomorphism of domains such that the induced extension of fraction fields is finite. Thus the lemma follows from Algebra, Lemma 112.2.  $\square$



### 17. Principal divisors

02RN The following definition is the analogue of Divisors, Definition 26.5 in our current setup.

02RO **Definition 17.1.** Let  $(S, \delta)$  be as in Situation 7.1. Let  $X$  be locally of finite type over  $S$ . Assume  $X$  is integral with  $\dim_\delta(X) = n$ . Let  $f \in R(X)^*$ . The *principal divisor associated to  $f$*  is the  $(n - 1)$ -cycle

$$\operatorname{div}(f) = \operatorname{div}_X(f) = \sum \operatorname{ord}_Z(f)[Z]$$

defined in Divisors, Definition 26.5. This makes sense because prime divisors have  $\delta$ -dimension  $n - 1$  by Lemma 16.1.

In the situation of the definition for  $f, g \in R(X)^*$  we have

$$\operatorname{div}_X(fg) = \operatorname{div}_X(f) + \operatorname{div}_X(g)$$

in  $Z_{n-1}(X)$ . See Divisors, Lemma 26.6. The following lemma will be superseded by the more general Lemma 20.2.

02RR **Lemma 17.2.** *Let  $(S, \delta)$  be as in Situation 7.1. Let  $X, Y$  be locally of finite type over  $S$ . Assume  $X, Y$  are integral and  $n = \dim_\delta(Y)$ . Let  $f : X \rightarrow Y$  be a flat morphism of relative dimension  $r$ . Let  $g \in R(Y)^*$ . Then*

$$f^*(\operatorname{div}_Y(g)) = \operatorname{div}_X(g)$$

in  $Z_{n+r-1}(X)$ .

**Proof.** Note that since  $f$  is flat it is dominant so that  $f$  induces an embedding  $R(Y) \subset R(X)$ , and hence we may think of  $g$  as an element of  $R(X)^*$ . Let  $Z \subset X$  be an integral closed subscheme of  $\delta$ -dimension  $n + r - 1$ . Let  $\xi \in Z$  be its generic point. If  $\dim_\delta(f(Z)) > n - 1$ , then we see that the coefficient of  $[Z]$  in the left and right hand side of the equation is zero. Hence we may assume that  $Z' = \overline{f(Z)}$  is an integral closed subscheme of  $Y$  of  $\delta$ -dimension  $n - 1$ . Let  $\xi' = f(\xi)$ . It is the generic point of  $Z'$ . Set  $A = \mathcal{O}_{Y, \xi'}$ ,  $B = \mathcal{O}_{X, \xi}$ . The ring map  $A \rightarrow B$  is a flat local homomorphism of Noetherian local domains of dimension 1. We have  $g$  in the fraction field of  $A$ . What we have to show is that

$$\operatorname{ord}_A(g) \operatorname{length}_B(B/\mathfrak{m}_A B) = \operatorname{ord}_B(g).$$

This follows from Algebra, Lemma 51.13 (details omitted). □

### 18. Principal divisors and pushforward

02RS The first lemma implies that the pushforward of a principal divisor along a generically finite morphism is a principal divisor.

02RT **Lemma 18.1.** *Let  $(S, \delta)$  be as in Situation 7.1. Let  $X, Y$  be locally of finite type over  $S$ . Assume  $X, Y$  are integral and  $n = \dim_\delta(X) = \dim_\delta(Y)$ . Let  $p : X \rightarrow Y$  be a dominant proper morphism. Let  $f \in R(X)^*$ . Set*

$$g = Nm_{R(X)/R(Y)}(f).$$

Then we have  $p_* \operatorname{div}(f) = \operatorname{div}(g)$ .

**Proof.** Let  $Z \subset Y$  be an integral closed subscheme of  $\delta$ -dimension  $n - 1$ . We want to show that the coefficient of  $[Z]$  in  $p_*\text{div}(f)$  and  $\text{div}(g)$  are equal. We may apply Lemma 16.2 to the morphism  $p : X \rightarrow Y$  and the generic point  $\xi \in Z$ . Hence we may replace  $Y$  by an affine open neighbourhood of  $\xi$  and assume that  $p : X \rightarrow Y$  is finite. Write  $Y = \text{Spec}(R)$  and  $X = \text{Spec}(A)$  with  $p$  induced by a finite homomorphism  $R \rightarrow A$  of Noetherian domains which induces a finite field extension  $L/K$  of fraction fields. Now we have  $f \in L$ ,  $g = \text{Nm}(f) \in K$ , and a prime  $\mathfrak{p} \subset R$  with  $\dim(R_{\mathfrak{p}}) = 1$ . The coefficient of  $[Z]$  in  $\text{div}_Y(g)$  is  $\text{ord}_{R_{\mathfrak{p}}}(g)$ . The coefficient of  $[Z]$  in  $p_*\text{div}_X(f)$  is

$$\sum_{\mathfrak{q} \text{ lying over } \mathfrak{p}} [\kappa(\mathfrak{q}) : \kappa(\mathfrak{p})] \text{ord}_{A_{\mathfrak{q}}}(f)$$

The desired equality therefore follows from Algebra, Lemma 120.8.  $\square$

An important role in the discussion of principal divisors is played by the “universal” principal divisor  $[0] - [\infty]$  on  $\mathbf{P}_S^1$ . To make this more precise, let us denote

$$D_0, D_\infty \subset \mathbf{P}_S^1 = \underline{\text{Proj}}_S(\mathcal{O}_S[T_0, T_1])$$

the closed subscheme cut out by the section  $T_1$ , resp.  $T_0$  of  $\mathcal{O}(1)$ . These are effective Cartier divisors, see Divisors, Definition 13.1 and Lemma 14.10. The following lemma says that loosely speaking we have “ $\text{div}(T_1/T_0) = [D_0] - [D_1]$ ” and that this is the universal principal divisor.

02RQ **Lemma 18.2.** *Let  $(S, \delta)$  be as in Situation 7.1. Let  $X$  be locally of finite type over  $S$ . Assume  $X$  is integral and  $n = \dim_\delta(X)$ . Let  $f \in R(X)^*$ . Let  $U \subset X$  be a nonempty open such that  $f$  corresponds to a section  $f \in \Gamma(U, \mathcal{O}_X^*)$ . Let  $Y \subset X \times_S \mathbf{P}_S^1$  be the closure of the graph of  $f : U \rightarrow \mathbf{P}_S^1$ . Then*

- (1) *the projection morphism  $p : Y \rightarrow X$  is proper,*
- (2)  *$p|_{p^{-1}(U)} : p^{-1}(U) \rightarrow U$  is an isomorphism,*
- (3) *the pullbacks  $Y_0 = q^{-1}D_0$  and  $Y_\infty = q^{-1}D_\infty$  via the morphism  $q : Y \rightarrow \mathbf{P}_S^1$  are defined (Divisors, Definition 13.12),*
- (4) *we have*

$$\text{div}_Y(f) = [Y_0]_{n-1} - [Y_\infty]_{n-1}$$

- (5) *we have*

$$\text{div}_X(f) = p_*\text{div}_Y(f)$$

- (6) *if we view  $Y_0$  and  $Y_\infty$  as closed subschemes of  $X$  via the morphism  $p$  then we have*

$$\text{div}_X(f) = [Y_0]_{n-1} - [Y_\infty]_{n-1}$$

**Proof.** Since  $X$  is integral, we see that  $U$  is integral. Hence  $Y$  is integral, and  $(1, f)(U) \subset Y$  is an open dense subscheme. Also, note that the closed subscheme  $Y \subset X \times_S \mathbf{P}_S^1$  does not depend on the choice of the open  $U$ , since after all it is the closure of the one point set  $\{\eta'\} = \{(1, f)(\eta)\}$  where  $\eta \in X$  is the generic point. Having said this let us prove the assertions of the lemma.

For (1) note that  $p$  is the composition of the closed immersion  $Y \rightarrow X \times_S \mathbf{P}_S^1 = \mathbf{P}_X^1$  with the proper morphism  $\mathbf{P}_X^1 \rightarrow X$ . As a composition of proper morphisms is proper (Morphisms, Lemma 39.4) we conclude.

It is clear that  $Y \cap U \times_S \mathbf{P}_S^1 = (1, f)(U)$ . Thus (2) follows. It also follows that  $\dim_\delta(Y) = n$ .

Note that  $q(\eta') = f(\eta)$  is not contained in  $D_0$  or  $D_\infty$  since  $f \in R(X)^*$ . Hence (3) by Divisors, Lemma 13.13. We obtain  $\dim_\delta(Y_0) = n - 1$  and  $\dim_\delta(Y_\infty) = n - 1$  from Lemma 16.1.

Consider the effective Cartier divisor  $Y_0$ . At every point  $\xi \in Y_0$  we have  $f \in \mathcal{O}_{Y,\xi}$  and the local equation for  $Y_0$  is given by  $f$ . In particular, if  $\delta(\xi) = n - 1$  so  $\xi$  is the generic point of a integral closed subscheme  $Z$  of  $\delta$ -dimension  $n - 1$ , then we see that the coefficient of  $[Z]$  in  $\text{div}_Y(f)$  is

$$\text{ord}_Z(f) = \text{length}_{\mathcal{O}_{Y,\xi}}(\mathcal{O}_{Y,\xi}/f\mathcal{O}_{Y,\xi}) = \text{length}_{\mathcal{O}_{Y,\xi}}(\mathcal{O}_{Y_0,\xi})$$

which is the coefficient of  $[Z]$  in  $[Y_0]_{n-1}$ . A similar argument using the rational function  $1/f$  shows that  $-[Y_\infty]$  agrees with the terms with negative coefficients in the expression for  $\text{div}_Y(f)$ . Hence (4) follows.

Note that  $D_0 \rightarrow S$  is an isomorphism. Hence we see that  $X \times_S D_0 \rightarrow X$  is an isomorphism as well. Clearly we have  $Y_0 = Y \cap X \times_S D_0$  (scheme theoretic intersection) inside  $X \times_S \mathbf{P}_S^1$ . Hence it is really the case that  $Y_0 \rightarrow X$  is a closed immersion. It follows that

$$p_*\mathcal{O}_{Y_0} = \mathcal{O}_{Y'_0}$$

where  $Y'_0 \subset X$  is the image of  $Y_0 \rightarrow X$ . By Lemma 12.3 we have  $p_*[Y_0]_{n-1} = [Y'_0]_{n-1}$ . The same is true for  $D_\infty$  and  $Y_\infty$ . Hence (6) is a consequence of (5). Finally, (5) follows immediately from Lemma 18.1.  $\square$

The following lemma says that the degree of a principal divisor on a proper curve is zero.

02RU **Lemma 18.3.** *Let  $K$  be any field. Let  $X$  be a 1-dimensional integral scheme endowed with a proper morphism  $c : X \rightarrow \text{Spec}(K)$ . Let  $f \in K(X)^*$  be an invertible rational function. Then*

$$\sum_{x \in X \text{ closed}} [\kappa(x) : K] \text{ord}_{\mathcal{O}_{X,x}}(f) = 0$$

where  $\text{ord}$  is as in Algebra, Definition 120.2. In other words,  $c_*\text{div}(f) = 0$ .

**Proof.** Consider the diagram

$$\begin{array}{ccc} Y & \xrightarrow{\quad} & X \\ q \downarrow & & \downarrow c \\ \mathbf{P}_K^1 & \xrightarrow{c'} & \text{Spec}(K) \end{array}$$

that we constructed in Lemma 18.2 starting with  $X$  and the rational function  $f$  over  $S = \text{Spec}(K)$ . We will use all the results of this lemma without further mention. We have to show that  $c_*\text{div}_X(f) = c_*p_*\text{div}_Y(f) = 0$ . This is the same as proving that  $c'_*q_*\text{div}_Y(f) = 0$ . If  $q(Y)$  is a closed point of  $\mathbf{P}_K^1$  then we see that  $\text{div}_X(f) = 0$  and the lemma holds. Thus we may assume that  $q$  is dominant. Suppose we can show that  $q : Y \rightarrow \mathbf{P}_K^1$  is finite locally free of degree  $d$  (see Morphisms, Definition 46.1). Since  $\text{div}_Y(f) = [q^{-1}D_0]_0 - [q^{-1}D_\infty]_0$  we see (by definition of flat pullback) that  $\text{div}_Y(f) = q^*([D_0]_0 - [D_\infty]_0)$ . Then by Lemma 15.2 we get  $q_*\text{div}_Y(f) = d([D_0]_0 - [D_\infty]_0)$ . Since clearly  $c'_*[D_0]_0 = c'_*[D_\infty]_0$  we win.

It remains to show that  $q$  is finite locally free. (It will automatically have some given degree as  $\mathbf{P}_K^1$  is connected.) Since  $\dim(\mathbf{P}_K^1) = 1$  we see that  $q$  is finite for example by Lemma 16.2. All local rings of  $\mathbf{P}_K^1$  at closed points are regular local rings of

dimension 1 (in other words discrete valuation rings), since they are localizations of  $K[T]$  (see Algebra, Lemma 113.1). Hence for  $y \in Y$  closed the local ring  $\mathcal{O}_{Y,y}$  will be flat over  $\mathcal{O}_{\mathbf{P}^1, q(y)}$  as soon as it is torsion free (More on Algebra, Lemma 22.11). This is obviously the case as  $\mathcal{O}_{Y,y}$  is a domain and  $q$  is dominant. Thus  $q$  is flat. Hence  $q$  is finite locally free by Morphisms, Lemma 46.2.  $\square$

### 19. Rational equivalence

02RV In this section we define *rational equivalence* on  $k$ -cycles. We will allow locally finite sums of images of principal divisors (under closed immersions). This leads to some pretty strange phenomena, see Example 19.3. However, if we do not allow these then we do not know how to prove that capping with chern classes of line bundles factors through rational equivalence.

02RW **Definition 19.1.** Let  $(S, \delta)$  be as in Situation 7.1. Let  $X$  be a scheme locally of finite type over  $S$ . Let  $k \in \mathbf{Z}$ .

- (1) Given any locally finite collection  $\{W_j \subset X\}$  of integral closed subschemes with  $\dim_\delta(W_j) = k + 1$ , and any  $f_j \in R(W_j)^*$  we may consider

$$\sum (i_j)_* \operatorname{div}(f_j) \in Z_k(X)$$

where  $i_j : W_j \rightarrow X$  is the inclusion morphism. This makes sense as the morphism  $\coprod i_j : \coprod W_j \rightarrow X$  is proper.

- (2) We say that  $\alpha \in Z_k(X)$  is *rationally equivalent to zero* if  $\alpha$  is a cycle of the form displayed above.  
 (3) We say  $\alpha, \beta \in Z_k(X)$  are *rationally equivalent* and we write  $\alpha \sim_{\text{rat}} \beta$  if  $\alpha - \beta$  is rationally equivalent to zero.  
 (4) We define

$$A_k(X) = Z_k(X) / \sim_{\text{rat}}$$

to be the *Chow group of  $k$ -cycles on  $X$* . This is sometimes called the *Chow group of  $k$ -cycles modulo rational equivalence on  $X$* .

There are many other interesting (adequate) equivalence relations. Rational equivalence is the coarsest one of them all. A very simple but important lemma is the following.

02RX **Lemma 19.2.** Let  $(S, \delta)$  be as in Situation 7.1. Let  $X$  be a scheme locally of finite type over  $S$ . Let  $U \subset X$  be an open subscheme, and denote  $i : Y = X \setminus U \rightarrow X$  as a reduced closed subscheme of  $X$ . Let  $k \in \mathbf{Z}$ . Suppose  $\alpha, \beta \in Z_k(X)$ . If  $\alpha|_U \sim_{\text{rat}} \beta|_U$  then there exist a cycle  $\gamma \in Z_k(Y)$  such that

$$\alpha \sim_{\text{rat}} \beta + i_* \gamma.$$

In other words, the sequence

$$A_k(Y) \xrightarrow{i_*} A_k(X) \xrightarrow{j^*} A_k(U) \longrightarrow 0$$

is an exact complex of abelian groups.

**Proof.** Let  $\{W_j\}_{j \in J}$  be a locally finite collection of integral closed subschemes of  $U$  of  $\delta$ -dimension  $k + 1$ , and let  $f_j \in R(W_j)^*$  be elements such that  $(\alpha - \beta)|_U = \sum (i_j)_* \operatorname{div}(f_j)$  as in the definition. Set  $W'_j \subset X$  equal to the closure of  $W_j$ . Suppose that  $V \subset X$  is a quasi-compact open. Then also  $V \cap U$  is quasi-compact open in  $U$  as  $V$  is Noetherian. Hence the set  $\{j \in J \mid W_j \cap V \neq \emptyset\} = \{j \in J \mid W'_j \cap V \neq \emptyset\}$  is

finite since  $\{W_j\}$  is locally finite. In other words we see that  $\{W'_j\}$  is also locally finite. Since  $R(W_j) = R(W'_j)$  we see that

$$\alpha - \beta - \sum (i'_j)_* \operatorname{div}(f_j)$$

is a cycle supported on  $Y$  and the lemma follows (see Lemma 14.2).  $\square$

02RY **Example 19.3.** Here is a “strange” example. Suppose that  $S$  is the spectrum of a field  $k$  with  $\delta$  as in Example 7.2. Suppose that  $X = C_1 \cup C_2 \cup \dots$  is an infinite union of curves  $C_j \cong \mathbf{P}_k^1$  glued together in the following way: The point  $\infty \in C_j$  is glued transversally to the point  $0 \in C_{j+1}$  for  $j = 1, 2, 3, \dots$ . Take the point  $0 \in C_1$ . This gives a zero cycle  $[0] \in Z_0(X)$ . The “strangeness” in this situation is that actually  $[0] \sim_{\text{rat}} 0!$  Namely we can choose the rational function  $f_j \in R(C_j)$  to be the function which has a simple zero at 0 and a simple pole at  $\infty$  and no other zeros or poles. Then we see that the sum  $\sum (i_j)_* \operatorname{div}(f_j)$  is exactly the 0-cycle  $[0]$ . In fact it turns out that  $A_0(X) = 0$  in this example. If you find this too bizarre, then you can just make sure your spaces are always quasi-compact (so  $X$  does not even exist for you).

02RZ **Remark 19.4.** Let  $(S, \delta)$  be as in Situation 7.1. Let  $X$  be a scheme locally of finite type over  $S$ . Suppose we have infinite collections  $\alpha_i, \beta_i \in Z_k(X)$ ,  $i \in I$  of  $k$ -cycles on  $X$ . Suppose that the supports of  $\alpha_i$  and  $\beta_i$  form locally finite collections of closed subsets of  $X$  so that  $\sum \alpha_i$  and  $\sum \beta_i$  are defined as cycles. Moreover, assume that  $\alpha_i \sim_{\text{rat}} \beta_i$  for each  $i$ . Then it is not clear that  $\sum \alpha_i \sim_{\text{rat}} \sum \beta_i$ . Namely, the problem is that the rational equivalences may be given by locally finite families  $\{W_{i,j}, f_{i,j} \in R(W_{i,j})^*\}_{j \in J_i}$  but the union  $\{W_{i,j}\}_{i \in I, j \in J_i}$  may not be locally finite.

In many cases in practice, one has a locally finite family of closed subsets  $\{T_i\}_{i \in I}$  such that  $\alpha_i, \beta_i$  are supported on  $T_i$  and such that  $\alpha_i = \beta_i$  in  $A_k(T_i)$ , in other words, the families  $\{W_{i,j}, f_{i,j} \in R(W_{i,j})^*\}_{j \in J_i}$  consist of subschemes  $W_{i,j} \subset T_i$ . In this case it is true that  $\sum \alpha_i \sim_{\text{rat}} \sum \beta_i$  on  $X$ , simply because the family  $\{W_{i,j}\}_{i \in I, j \in J_i}$  is automatically locally finite in this case.

## 20. Rational equivalence and push and pull

02S0 In this section we show that flat pullback and proper pushforward commute with rational equivalence.

0EPH **Lemma 20.1.** *Let  $(S, \delta)$  be as in Situation 7.1. Let  $X, Y$  be schemes locally of finite type over  $S$ . Assume  $Y$  integral with  $\dim_\delta(Y) = k$ . Let  $f : X \rightarrow Y$  be a flat morphism of relative dimension  $r$ . Then for  $g \in R(Y)^*$  we have*

$$f^* \operatorname{div}_Y(g) = \sum n_j i_{j,*} \operatorname{div}_{X_j}(g \circ f|_{X_j})$$

as  $(k+r-1)$ -cycles on  $X$  where the sum is over the irreducible components  $X_j$  of  $X$  and  $n_j$  is the multiplicity of  $X_j$  in  $X$ .

**Proof.** Let  $Z \subset X$  be an integral closed subscheme of  $\delta$ -dimension  $k+r-1$ . We have to show that the coefficient  $n$  of  $[Z]$  in  $f^* \operatorname{div}(g)$  is equal to the coefficient  $m$  of  $[Z]$  in  $\sum i_{j,*} \operatorname{div}(g \circ f|_{X_j})$ . Let  $Z'$  be the closure of  $f(Z)$  which is an integral closed subscheme of  $Y$ . By Lemma 13.1 we have  $\dim_\delta(Z') \geq k-1$ . Thus either  $Z' = Y$  or  $Z'$  is a prime divisor on  $Y$ . If  $Z' = Y$ , then the coefficients  $n$  and  $m$  are both zero: this is clear for  $n$  by definition of  $f^*$  and follows for  $m$  because  $g \circ f|_{X_j}$  is a

unit in any point of  $X_j$  mapping to the generic point of  $Y$ . Hence we may assume that  $Z' \subset Y$  is a prime divisor.

We are going to translate the equality of  $n$  and  $m$  into algebra. Namely, let  $\xi' \in Z'$  and  $\xi \in Z$  be the generic points. Set  $A = \mathcal{O}_{Y, \xi'}$  and  $B = \mathcal{O}_{X, \xi}$ . Note that  $A, B$  are Noetherian,  $A \rightarrow B$  is flat, local,  $A$  is a domain, and  $\mathfrak{m}_A B$  is an ideal of definition of the local ring  $B$ . The rational function  $g$  is an element of the fraction field  $Q(A)$  of  $A$ . By construction, the closed subschemes  $X_j$  which meet  $\xi$  correspond 1-to-1 with minimal primes

$$\mathfrak{q}_1, \dots, \mathfrak{q}_s \subset B$$

The integers  $n_j$  are the corresponding lengths

$$n_i = \text{length}_{B_{\mathfrak{q}_i}}(B_{\mathfrak{q}_i})$$

The rational functions  $g \circ f|_{X_j}$  correspond to the image  $g_i \in \kappa(\mathfrak{q}_i)^*$  of  $g \in Q(A)$ . Putting everything together we see that

$$n = \text{ord}_A(g) \text{length}_B(B/\mathfrak{m}_A B)$$

and that

$$m = \sum \text{ord}_{B/\mathfrak{q}_i}(g_i) \text{length}_{B_{\mathfrak{q}_i}}(B_{\mathfrak{q}_i})$$

Writing  $g = x/y$  for some nonzero  $x, y \in A$  we see that it suffices to prove

$$\text{length}_A(A/(x)) \text{length}_B(B/\mathfrak{m}_A B) = \text{length}_B(B/xB)$$

(equality uses Algebra, Lemma 51.13) equals

$$\sum_{i=1, \dots, s} \text{length}_{B/\mathfrak{q}_i}(B/(x, \mathfrak{q}_i)) \text{length}_{B_{\mathfrak{q}_i}}(B_{\mathfrak{q}_i})$$

and similarly for  $y$ . As  $A \rightarrow B$  is flat it follows that  $x$  is a nonzerodivisor in  $B$ . Hence the desired equality follows from Lemma 3.2.  $\square$

02S1 **Lemma 20.2.** *Let  $(S, \delta)$  be as in Situation 7.1. Let  $X, Y$  be schemes locally of finite type over  $S$ . Let  $f : X \rightarrow Y$  be a flat morphism of relative dimension  $r$ . Let  $\alpha \sim_{\text{rat}} \beta$  be rationally equivalent  $k$ -cycles on  $Y$ . Then  $f^* \alpha \sim_{\text{rat}} f^* \beta$  as  $(k+r)$ -cycles on  $X$ .*

**Proof.** What do we have to show? Well, suppose we are given a collection

$$i_j : W_j \longrightarrow Y$$

of closed immersions, with each  $W_j$  integral of  $\delta$ -dimension  $k+1$  and rational functions  $g_j \in R(W_j)^*$ . Moreover, assume that the collection  $\{i_j(W_j)\}_{j \in J}$  is locally finite on  $Y$ . Then we have to show that

$$f^*(\sum i_{j,*} \text{div}(g_j)) = \sum f^* i_{j,*} \text{div}(g_j)$$

is rationally equivalent to zero on  $X$ . The sum on the right makes sense as  $\{W_j\}$  is locally finite in  $X$  by Lemma 13.2.

Consider the fibre products

$$i'_j : W'_j = W_j \times_Y X \longrightarrow X.$$

and denote  $f_j : W'_j \rightarrow W_j$  the first projection. By Lemma 15.1 we can write the sum above as

$$\sum i'_{j,*} (f_j^* \text{div}(g_j))$$

By Lemma 20.1 we see that each  $f_j^* \operatorname{div}(g_j)$  is rationally equivalent to zero on  $W'_j$ . Hence each  $i'_{j,*}(f_j^* \operatorname{div}(g_j))$  is rationally equivalent to zero. Then the same is true for the displayed sum by the discussion in Remark 19.4.  $\square$

02S2 **Lemma 20.3.** *Let  $(S, \delta)$  be as in Situation 7.1. Let  $X, Y$  be schemes locally of finite type over  $S$ . Let  $p : X \rightarrow Y$  be a proper morphism. Suppose  $\alpha, \beta \in Z_k(X)$  are rationally equivalent. Then  $p_*\alpha$  is rationally equivalent to  $p_*\beta$ .*

**Proof.** What do we have to show? Well, suppose we are given a collection

$$i_j : W_j \longrightarrow X$$

of closed immersions, with each  $W_j$  integral of  $\delta$ -dimension  $k + 1$  and rational functions  $f_j \in R(W_j)^*$ . Moreover, assume that the collection  $\{i_j(W_j)\}_{j \in J}$  is locally finite on  $X$ . Then we have to show that

$$p_* \left( \sum i_{j,*} \operatorname{div}(f_j) \right)$$

is rationally equivalent to zero on  $X$ .

Note that the sum is equal to

$$\sum p_* i_{j,*} \operatorname{div}(f_j).$$

Let  $W'_j \subset Y$  be the integral closed subscheme which is the image of  $p \circ i_j$ . The collection  $\{W'_j\}$  is locally finite in  $Y$  by Lemma 11.2. Hence it suffices to show, for a given  $j$ , that either  $p_* i_{j,*} \operatorname{div}(f_j) = 0$  or that it is equal to  $i'_{j,*} \operatorname{div}(g_j)$  for some  $g_j \in R(W'_j)^*$ .

The arguments above therefore reduce us to the case of a single integral closed subscheme  $W \subset X$  of  $\delta$ -dimension  $k + 1$ . Let  $f \in R(W)^*$ . Let  $W' = p(W)$  as above. We get a commutative diagram of morphisms

$$\begin{array}{ccc} W & \xrightarrow{i} & X \\ p' \downarrow & & \downarrow p \\ W' & \xrightarrow{i'} & Y \end{array}$$

Note that  $p_* i_* \operatorname{div}(f) = i'_*(p')_* \operatorname{div}(f)$  by Lemma 12.2. As explained above we have to show that  $(p')_* \operatorname{div}(f)$  is the divisor of a rational function on  $W'$  or zero. There are three cases to distinguish.

The case  $\dim_\delta(W') < k$ . In this case automatically  $(p')_* \operatorname{div}(f) = 0$  and there is nothing to prove.

The case  $\dim_\delta(W') = k$ . Let us show that  $(p')_* \operatorname{div}(f) = 0$  in this case. Let  $\eta \in W'$  be the generic point. Note that  $c : W_\eta \rightarrow \operatorname{Spec}(K)$  is a proper integral curve over  $K = \kappa(\eta)$  whose function field  $K(W_\eta)$  is identified with  $R(W)$ . Here is a diagram

$$\begin{array}{ccc} W_\eta & \longrightarrow & W \\ c \downarrow & & \downarrow p' \\ \operatorname{Spec}(K) & \longrightarrow & W' \end{array}$$

Let us denote  $f_\eta \in K(W_\eta)^*$  the rational function corresponding to  $f \in R(W)^*$ . Moreover, the closed points  $\xi$  of  $W_\eta$  correspond 1 – 1 to the closed integral subschemes  $Z = Z_\xi \subset W$  of  $\delta$ -dimension  $k$  with  $p'(Z) = W'$ . Note that the multiplicity

of  $Z_\xi$  in  $\text{div}(f)$  is equal to  $\text{ord}_{\mathcal{O}_{W_\eta, \xi}}(f_\eta)$  simply because the local rings  $\mathcal{O}_{W_\eta, \xi}$  and  $\mathcal{O}_{W, \xi}$  are identified (as subrings of their fraction fields). Hence we see that the multiplicity of  $[W']$  in  $(p')_*\text{div}(f)$  is equal to the multiplicity of  $[\text{Spec}(K)]$  in  $c_*\text{div}(f_\eta)$ . By Lemma 18.3 this is zero.

The case  $\dim_\delta(W') = k + 1$ . In this case Lemma 18.1 applies, and we see that indeed  $p'_*\text{div}(f) = \text{div}(g)$  for some  $g \in R(W')^*$  as desired.  $\square$

## 21. Rational equivalence and the projective line

02S3 Let  $(S, \delta)$  be as in Situation 7.1. Let  $X$  be a scheme locally of finite type over  $S$ . Given any closed subscheme  $Z \subset X \times_S \mathbf{P}_S^1 = X \times \mathbf{P}^1$  we let  $Z_0$ , resp.  $Z_\infty$  be the scheme theoretic closed subscheme  $Z_0 = \text{pr}_2^{-1}(D_0)$ , resp.  $Z_\infty = \text{pr}_2^{-1}(D_\infty)$ . Here  $D_0, D_\infty$  are as defined just above Lemma 18.2.

02S4 **Lemma 21.1.** *Let  $(S, \delta)$  be as in Situation 7.1. Let  $X$  be a scheme locally of finite type over  $S$ . Let  $W \subset X \times_S \mathbf{P}_S^1$  be an integral closed subscheme of  $\delta$ -dimension  $k + 1$ . Assume  $W \neq W_0$ , and  $W \neq W_\infty$ . Then*

- (1)  $W_0, W_\infty$  are effective Cartier divisors of  $W$ ,
- (2)  $W_0, W_\infty$  can be viewed as closed subschemes of  $X$  and

$$[W_0]_k \sim_{\text{rat}} [W_\infty]_k,$$

- (3) for any locally finite family of integral closed subschemes  $W_i \subset X \times_S \mathbf{P}_S^1$  of  $\delta$ -dimension  $k + 1$  with  $W_i \neq (W_i)_0$  and  $W_i \neq (W_i)_\infty$  we have  $\sum([ (W_i)_0 ]_k - [ (W_i)_\infty ]_k) \sim_{\text{rat}} 0$  on  $X$ , and
- (4) for any  $\alpha \in Z_k(X)$  with  $\alpha \sim_{\text{rat}} 0$  there exists a locally finite family of integral closed subschemes  $W_i \subset X \times_S \mathbf{P}_S^1$  as above such that  $\alpha = \sum([ (W_i)_0 ]_k - [ (W_i)_\infty ]_k)$ .

**Proof.** Part (1) follows from Divisors, Lemma 13.13 since the generic point of  $W$  is not mapped into  $D_0$  or  $D_\infty$  under the projection  $X \times_S \mathbf{P}_S^1 \rightarrow \mathbf{P}_S^1$  by assumption.

Since  $X \times_S D_0 \rightarrow X$  is an isomorphism we see that  $W_0$  is isomorphic to a closed subscheme of  $X$ . Similarly for  $W_\infty$ . Consider the morphism  $p : W \rightarrow X$ . It is proper and on  $W$  we have  $[W_0]_k \sim_{\text{rat}} [W_\infty]_k$ . Hence part (2) follows from Lemma 20.3 as clearly  $p_*[W_0]_k = [W_0]_k$  and similarly for  $W_\infty$ .

The only content of statement (3) is, given parts (1) and (2), that the collection  $\{(W_i)_0, (W_i)_\infty\}$  is a locally finite collection of closed subschemes of  $X$ . This is clear.

Suppose that  $\alpha \sim_{\text{rat}} 0$ . By definition this means there exist integral closed subschemes  $V_i \subset X$  of  $\delta$ -dimension  $k + 1$  and rational functions  $f_i \in R(V_i)^*$  such that the family  $\{V_i\}_{i \in I}$  is locally finite in  $X$  and such that  $\alpha = \sum(V_i \rightarrow X)_*\text{div}(f_i)$ . Let

$$W_i \subset V_i \times_S \mathbf{P}_S^1 \subset X \times_S \mathbf{P}_S^1$$

be the closure of the graph of the rational map  $f_i$  as in Lemma 18.2. Then we have that  $(V_i \rightarrow X)_*\text{div}(f_i)$  is equal to  $[ (W_i)_0 ]_k - [ (W_i)_\infty ]_k$  by that same lemma. Hence the result is clear.  $\square$

02S5 **Lemma 21.2.** *Let  $(S, \delta)$  be as in Situation 7.1. Let  $X$  be a scheme locally of finite type over  $S$ . Let  $Z$  be a closed subscheme of  $X \times \mathbf{P}^1$ . Assume  $\dim_\delta(Z) \leq$*



$k + 1$ ,  $\dim_\delta(Z_0) \leq k$ ,  $\dim_\delta(Z_\infty) \leq k$  and assume any embedded point  $\xi$  (Divisors, Definition 4.1) of  $Z$  has  $\delta(\xi) < k$ . Then

$$[Z_0]_k \sim_{\text{rat}} [Z_\infty]_k$$

as  $k$ -cycles on  $X$ .

**Proof.** Let  $\{W_i\}_{i \in I}$  be the collection of irreducible components of  $Z$  which have  $\delta$ -dimension  $k + 1$ . Write

$$[Z]_{k+1} = \sum n_i [W_i]$$

with  $n_i > 0$  as per definition. Note that  $\{W_i\}$  is a locally finite collection of closed subsets of  $X \times_S \mathbf{P}_S^1$  by Divisors, Lemma 26.1. We claim that

$$[Z_0]_k = \sum n_i [(W_i)_0]_k$$

and similarly for  $[Z_\infty]_k$ . If we prove this then the lemma follows from Lemma 21.1.

Let  $Z' \subset X$  be an integral closed subscheme of  $\delta$ -dimension  $k$ . To prove the equality above it suffices to show that the coefficient  $n$  of  $[Z']$  in  $[Z_0]_k$  is the same as the coefficient  $m$  of  $[Z']$  in  $\sum n_i [(W_i)_0]_k$ . Let  $\xi' \in Z'$  be the generic point. Set  $\xi = (\xi', 0) \in X \times_S \mathbf{P}_S^1$ . Consider the local ring  $A = \mathcal{O}_{X \times_S \mathbf{P}_S^1, \xi}$ . Let  $I \subset A$  be the ideal cutting out  $Z$ , in other words so that  $A/I = \mathcal{O}_{Z, \xi}$ . Let  $t \in A$  be the element cutting out  $X \times_S D_0$  (i.e., the coordinate of  $\mathbf{P}^1$  at zero pulled back). By our choice of  $\xi' \in Z'$  we have  $\delta(\xi) = k$  and hence  $\dim(A/I) = 1$ . Since  $\xi$  is not an embedded point by definition we see that  $A/I$  is Cohen-Macaulay. Since  $\dim_\delta(Z_0) = k$  we see that  $\dim(A/(t, I)) = 0$  which implies that  $t$  is a nonzerodivisor on  $A/I$ . Finally, the irreducible closed subschemes  $W_i$  passing through  $\xi$  correspond to the minimal primes  $I \subset \mathfrak{q}_i$  over  $I$ . The multiplicities  $n_i$  correspond to the lengths  $\text{length}_{A_{\mathfrak{q}_i}}(A/I)_{\mathfrak{q}_i}$ . Hence we see that

$$n = \text{length}_A(A/(t, I))$$

and

$$m = \sum \text{length}_A(A/(t, \mathfrak{q}_i)) \text{length}_{A_{\mathfrak{q}_i}}(A/I)_{\mathfrak{q}_i}$$

Thus the result follows from Lemma 3.2.  $\square$

02S6 **Lemma 21.3.** *Let  $(S, \delta)$  be as in Situation 7.1. Let  $X$  be a scheme locally of finite type over  $S$ . Let  $\mathcal{F}$  be a coherent sheaf on  $X \times \mathbf{P}^1$ . Let  $i_0, i_\infty : X \rightarrow X \times \mathbf{P}^1$  be the closed immersion such that  $i_t(x) = (x, t)$ . Denote  $\mathcal{F}_0 = i_0^* \mathcal{F}$  and  $\mathcal{F}_\infty = i_\infty^* \mathcal{F}$ . Assume*

- (1)  $\dim_\delta(\text{Supp}(\mathcal{F})) \leq k + 1$ ,
- (2)  $\dim_\delta(\text{Supp}(\mathcal{F}_0)) \leq k$ ,  $\dim_\delta(\text{Supp}(\mathcal{F}_\infty)) \leq k$ , and
- (3) any embedded associated point  $\xi$  of  $\mathcal{F}$  has  $\delta(\xi) < k$ .

Then

$$[\mathcal{F}_0]_k \sim_{\text{rat}} [\mathcal{F}_\infty]_k$$

as  $k$ -cycles on  $X$ .

**Proof.** Let  $\{W_i\}_{i \in I}$  be the collection of irreducible components of  $\text{Supp}(\mathcal{F})$  which have  $\delta$ -dimension  $k + 1$ . Write

$$[\mathcal{F}]_{k+1} = \sum n_i [W_i]$$

with  $n_i > 0$  as per definition. Note that  $\{W_i\}$  is a locally finite collection of closed subsets of  $X \times_S \mathbf{P}_S^1$  by Lemma 10.1. We claim that

$$[\mathcal{F}_0]_k = \sum n_i [(W_i)_0]_k$$

and similarly for  $[\mathcal{F}_\infty]_k$ . If we prove this then the lemma follows from Lemma 21.1.

Let  $Z' \subset X$  be an integral closed subscheme of  $\delta$ -dimension  $k$ . To prove the equality above it suffices to show that the coefficient  $n$  of  $[Z']$  in  $[\mathcal{F}_0]_k$  is the same as the coefficient  $m$  of  $[Z']$  in  $\sum n_i [(W_i)_0]_k$ . Let  $\xi' \in Z'$  be the generic point. Set  $\xi = (\xi', 0) \in X \times_S \mathbf{P}_S^1$ . Consider the local ring  $A = \mathcal{O}_{X \times_S \mathbf{P}_S^1, \xi}$ . Let  $M = \mathcal{F}_\xi$  as an  $A$ -module. Let  $t \in A$  be the element cutting out  $X \times_S D_0$  (i.e., the coordinate of  $\mathbf{P}^1$  at zero pulled back). By our choice of  $\xi' \in Z'$  we have  $\delta(\xi) = k$  and hence  $\dim(\text{Supp}(M)) = 1$ . Since  $\xi$  is not an associated point of  $\mathcal{F}$  by definition we see that  $M$  is Cohen-Macaulay module. Since  $\dim_\delta(\text{Supp}(\mathcal{F}_0)) = k$  we see that  $\dim(\text{Supp}(M/tM)) = 0$  which implies that  $t$  is a nonzerodivisor on  $M$ . Finally, the irreducible closed subschemes  $W_i$  passing through  $\xi$  correspond to the minimal primes  $\mathfrak{q}_i$  of  $\text{Ass}(M)$ . The multiplicities  $n_i$  correspond to the lengths  $\text{length}_{A_{\mathfrak{q}_i}} M_{\mathfrak{q}_i}$ . Hence we see that

$$n = \text{length}_A(M/tM)$$

and

$$m = \sum \text{length}_A(A/(t, \mathfrak{q}_i)A) \text{length}_{A_{\mathfrak{q}_i}} M_{\mathfrak{q}_i}$$

Thus the result follows from Lemma 3.2.  $\square$

## 22. The divisor associated to an invertible sheaf

02SI The following definition is the analogue of Divisors, Definition 27.4 in our current setup.

02SJ **Definition 22.1.** Let  $(S, \delta)$  be as in Situation 7.1. Let  $X$  be locally of finite type over  $S$ . Assume  $X$  is integral and  $n = \dim_\delta(X)$ . Let  $\mathcal{L}$  be an invertible  $\mathcal{O}_X$ -module.

- (1) For any nonzero meromorphic section  $s$  of  $\mathcal{L}$  we define the *Weil divisor associated to  $s$*  is the  $(n - 1)$ -cycle

$$\text{div}_\mathcal{L}(s) = \sum \text{ord}_{Z, \mathcal{L}}(s) [Z]$$

defined in Divisors, Definition 27.4. This makes sense because Weil divisors have  $\delta$ -dimension  $n - 1$  by Lemma 16.1.

- (2) We define *Weil divisor associated to  $\mathcal{L}$*  as

$$c_1(\mathcal{L}) \cap [X] = \text{class of } \text{div}_\mathcal{L}(s) \in A_{n-1}(X)$$

where  $s$  is any nonzero meromorphic section of  $\mathcal{L}$  over  $X$ . This is well defined by Divisors, Lemma 27.3.

There are some cases where it is easy to compute the Weil divisor associated to an invertible sheaf.

02SK **Lemma 22.2.** Let  $(S, \delta)$  be as in Situation 7.1. Let  $X$  be locally of finite type over  $S$ . Assume  $X$  is integral and  $n = \dim_\delta(X)$ . Let  $\mathcal{L}$  be an invertible  $\mathcal{O}_X$ -module. Let  $s \in \Gamma(X, \mathcal{L})$  be a nonzero global section. Then

$$\text{div}_\mathcal{L}(s) = [Z(s)]_{n-1}$$

in  $Z_{n-1}(X)$  and

$$c_1(\mathcal{L}) \cap [X] = [Z(s)]_{n-1}$$

in  $A_{n-1}(X)$ .

**Proof.** Let  $Z \subset X$  be an integral closed subscheme of  $\delta$ -dimension  $n - 1$ . Let  $\xi \in Z$  be its generic point. Choose a generator  $s_\xi \in \mathcal{L}_\xi$ . Write  $s = f s_\xi$  for some  $f \in \mathcal{O}_{X,\xi}$ . By definition of  $Z(s)$ , see Divisors, Definition 14.8 we see that  $Z(s)$  is cut out by a quasi-coherent sheaf of ideals  $\mathcal{I} \subset \mathcal{O}_X$  such that  $\mathcal{I}_\xi = (f)$ . Hence  $\text{length}_{\mathcal{O}_{X,x}}(\mathcal{O}_{Z(s),\xi}) = \text{length}_{\mathcal{O}_{X,x}}(\mathcal{O}_{X,\xi}/(f)) = \text{ord}_{\mathcal{O}_{X,x}}(f)$  as desired.  $\square$

The following lemma will be superseded by the more general Lemma 24.2.

02SM **Lemma 22.3.** *Let  $(S, \delta)$  be as in Situation 7.1. Let  $X, Y$  be locally of finite type over  $S$ . Assume  $X, Y$  are integral and  $n = \dim_\delta(Y)$ . Let  $\mathcal{L}$  be an invertible  $\mathcal{O}_Y$ -module. Let  $f : X \rightarrow Y$  be a flat morphism of relative dimension  $r$ . Let  $\mathcal{L}$  be an invertible sheaf on  $Y$ . Then*

$$f^*(c_1(\mathcal{L}) \cap [Y]) = c_1(f^*\mathcal{L}) \cap [X]$$

in  $A_{n+r-1}(X)$ .

**Proof.** Let  $s$  be a nonzero meromorphic section of  $\mathcal{L}$ . We will show that actually  $f^*\text{div}_{\mathcal{L}}(s) = \text{div}_{f^*\mathcal{L}}(f^*s)$  and hence the lemma holds. To see this let  $\xi \in Y$  be a point and let  $s_\xi \in \mathcal{L}_\xi$  be a generator. Write  $s = g s_\xi$  with  $g \in R(X)^*$ . Then there is an open neighbourhood  $V \subset Y$  of  $\xi$  such that  $s_\xi \in \mathcal{L}(V)$  and such that  $s_\xi$  generates  $\mathcal{L}|_V$ . Hence we see that

$$\text{div}_{\mathcal{L}}(s)|_V = \text{div}(g)|_V.$$

In exactly the same way, since  $f^*s_\xi$  generates  $\mathcal{L}$  over  $f^{-1}(V)$  and since  $f^*s = g f^*s_\xi$  we also have

$$\text{div}_{\mathcal{L}}(f^*s)|_{f^{-1}(V)} = \text{div}(g)|_{f^{-1}(V)}.$$

Thus the desired equality of cycles over  $f^{-1}(V)$  follows from the corresponding result for pullbacks of principal divisors, see Lemma 17.2.  $\square$

### 23. Intersecting with an invertible sheaf

02SN In this section we study the following construction.

02SO **Definition 23.1.** Let  $(S, \delta)$  be as in Situation 7.1. Let  $X$  be locally of finite type over  $S$ . Let  $\mathcal{L}$  be an invertible  $\mathcal{O}_X$ -module. We define, for every integer  $k$ , an operation

$$c_1(\mathcal{L}) \cap - : Z_{k+1}(X) \rightarrow A_k(X)$$

called *intersection with the first chern class of  $\mathcal{L}$* .

- (1) Given an integral closed subscheme  $i : W \rightarrow X$  with  $\dim_\delta(W) = k + 1$  we define

$$c_1(\mathcal{L}) \cap [W] = i_*(c_1(i^*\mathcal{L}) \cap [W])$$

where the right hand side is defined in Definition 22.1.

- (2) For a general  $(k + 1)$ -cycle  $\alpha = \sum n_i [W_i]$  we set

$$c_1(\mathcal{L}) \cap \alpha = \sum n_i c_1(\mathcal{L}) \cap [W_i]$$

Write each  $c_1(\mathcal{L}) \cap W_i = \sum_j n_{i,j} [Z_{i,j}]$  with  $\{Z_{i,j}\}_j$  a locally finite sum of integral closed subschemes of  $W_i$ . Since  $\{W_i\}$  is a locally finite collection of integral closed subschemes on  $X$ , it follows easily that  $\{Z_{i,j}\}_{i,j}$  is a locally finite collection of closed subschemes of  $X$ . Hence  $c_1(\mathcal{L}) \cap \alpha = \sum n_i n_{i,j} [Z_{i,j}]$  is a cycle. Another, more convenient, way to think about this is to observe that the morphism  $\coprod W_i \rightarrow X$  is proper. Hence  $c_1(\mathcal{L}) \cap \alpha$  can be viewed as the pushforward of a class in  $A_k(\coprod W_i) = \coprod A_k(W_i)$ . This also explains why the result is well defined up to rational equivalence on  $X$ .

The main goal for the next few sections is to show that intersecting with  $c_1(\mathcal{L})$  factors through rational equivalence. This is not a triviality.

02SP **Lemma 23.2.** *Let  $(S, \delta)$  be as in Situation 7.1. Let  $X$  be locally of finite type over  $S$ . Let  $\mathcal{L}, \mathcal{N}$  be an invertible sheaves on  $X$ . Then*

$$c_1(\mathcal{L}) \cap \alpha + c_1(\mathcal{N}) \cap \alpha = c_1(\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{N}) \cap \alpha$$

in  $A_k(X)$  for every  $\alpha \in Z_{k-1}(X)$ . Moreover,  $c_1(\mathcal{O}_X) \cap \alpha = 0$  for all  $\alpha$ .

**Proof.** The additivity follows directly from Divisors, Lemma 27.5 and the definitions. To see that  $c_1(\mathcal{O}_X) \cap \alpha = 0$  consider the section  $1 \in \Gamma(X, \mathcal{O}_X)$ . This restricts to an everywhere nonzero section on any integral closed subscheme  $W \subset X$ . Hence  $c_1(\mathcal{O}_X) \cap [W] = 0$  as desired.  $\square$

Recall that  $Z(s) \subset X$  denotes the zero scheme of a global section  $s$  of an invertible sheaf on a scheme  $X$ , see Divisors, Definition 14.8.

0EPI **Lemma 23.3.** *Let  $(S, \delta)$  be as in Situation 7.1. Let  $Y$  be locally of finite type over  $S$ . Let  $\mathcal{L}$  be an invertible  $\mathcal{O}_Y$ -module. Let  $s \in \Gamma(Y, \mathcal{L})$ . Assume*

- (1)  $\dim_\delta(Y) \leq k + 1$ ,
- (2)  $\dim_\delta(Z(s)) \leq k$ , and
- (3) for every generic point  $\xi$  of an irreducible component of  $Z(s)$  of  $\delta$ -dimension  $k$  the multiplication by  $s$  induces an injection  $\mathcal{O}_{Y,\xi} \rightarrow \mathcal{L}_\xi$ .

Write  $[Y]_{k+1} = \sum n_i [Y_i]$  where  $Y_i \subset Y$  are the irreducible components of  $Y$  of  $\delta$ -dimension  $k + 1$ . Set  $s_i = s|_{Y_i} \in \Gamma(Y_i, \mathcal{L}|_{Y_i})$ . Then

02SR (23.3.1) 
$$[Z(s)]_k = \sum n_i [Z(s_i)]_k$$

as  $k$ -cycles on  $Y$ .

**Proof.** Let  $Z \subset Y$  be an integral closed subscheme of  $\delta$ -dimension  $k$ . Let  $\xi \in Z$  be its generic point. We want to compare the coefficient  $n$  of  $[Z]$  in the expression  $\sum n_i [Z(s_i)]_k$  with the coefficient  $m$  of  $[Z]$  in the expression  $[Z(s)]_k$ . Choose a generator  $s_\xi \in \mathcal{L}_\xi$ . Write  $A = \mathcal{O}_{Y,\xi}$ ,  $L = \mathcal{L}_\xi$ . Then  $L = As_\xi$ . Write  $s = fs_\xi$  for some (unique)  $f \in A$ . Hypothesis (3) means that  $f : A \rightarrow A$  is injective. Since  $\dim_\delta(Y) \leq k + 1$  and  $\dim_\delta(Z) = k$  we have  $\dim(A) = 0$  or  $1$ . We have

$$m = \text{length}_A(A/(f))$$

which is finite in either case.

If  $\dim(A) = 0$ , then  $f : A \rightarrow A$  being injective implies that  $f \in A^*$ . Hence in this case  $m$  is zero. Moreover, the condition  $\dim(A) = 0$  means that  $\xi$  does not lie on any irreducible component of  $\delta$ -dimension  $k + 1$ , i.e.,  $n = 0$  as well.

Now, let  $\dim(A) = 1$ . Since  $A$  is a Noetherian local ring it has finitely many minimal primes  $\mathfrak{q}_1, \dots, \mathfrak{q}_t$ . These correspond 1-1 with the  $Y_i$  passing through  $\xi'$ . Moreover  $n_i = \text{length}_{A_{\mathfrak{q}_i}}(A_{\mathfrak{q}_i})$ . Also, the multiplicity of  $[Z]$  in  $[Z(s_i)]_k$  is  $\text{length}_A(A/(f, \mathfrak{q}_i))$ . Hence the equation to prove in this case is

$$\text{length}_A(A/(f)) = \sum \text{length}_{A_{\mathfrak{q}_i}}(A_{\mathfrak{q}_i}) \text{length}_A(A/(f, \mathfrak{q}_i))$$

which follows from Lemma 3.2.  $\square$

The following lemma is a useful result in order to compute the intersection product of the  $c_1$  of an invertible sheaf and the cycle associated to a closed subscheme. Recall that  $Z(s) \subset X$  denotes the zero scheme of a global section  $s$  of an invertible sheaf on a scheme  $X$ , see Divisors, Definition 14.8.

02SQ **Lemma 23.4.** *Let  $(S, \delta)$  be as in Situation 7.1. Let  $X$  be locally of finite type over  $S$ . Let  $\mathcal{L}$  be an invertible  $\mathcal{O}_X$ -module. Let  $Y \subset X$  be a closed subscheme. Let  $s \in \Gamma(Y, \mathcal{L}|_Y)$ . Assume*

- (1)  $\dim_\delta(Y) \leq k + 1$ ,
- (2)  $\dim_\delta(Z(s)) \leq k$ , and
- (3) *for every generic point  $\xi$  of an irreducible component of  $Z(s)$  of  $\delta$ -dimension  $k$  the multiplication by  $s$  induces an injection  $\mathcal{O}_{Y, \xi} \rightarrow (\mathcal{L}|_Y)_\xi^1$ .*

Then

$$c_1(\mathcal{L}) \cap [Y]_{k+1} = [Z(s)]_k$$

in  $A_k(X)$ .

**Proof.** Write

$$[Y]_{k+1} = \sum n_i [Y_i]$$

where  $Y_i \subset Y$  are the irreducible components of  $Y$  of  $\delta$ -dimension  $k + 1$  and  $n_i > 0$ . By assumption the restriction  $s_i = s|_{Y_i} \in \Gamma(Y_i, \mathcal{L}|_{Y_i})$  is not zero, and hence is a regular section. By Lemma 22.2 we see that  $[Z(s_i)]_k$  represents  $c_1(\mathcal{L}|_{Y_i})$ . Hence by definition

$$c_1(\mathcal{L}) \cap [Y]_{k+1} = \sum n_i [Z(s_i)]_k$$

Thus the result follows from Lemma 23.3.  $\square$

## 24. Intersecting with an invertible sheaf and push and pull

0AYA In this section we prove that the operation  $c_1(\mathcal{L}) \cap -$  commutes with flat pullback and proper pushforward.

0EPJ **Lemma 24.1.** *Let  $(S, \delta)$  be as in Situation 7.1. Let  $X, Y$  be locally of finite type over  $S$ . Let  $f : X \rightarrow Y$  be a flat morphism of relative dimension  $r$ . Let  $\mathcal{L}$  be an invertible sheaf on  $Y$ . Assume  $Y$  is integral and  $n = \dim_\delta(Y)$ . Let  $s$  be a nonzero meromorphic section of  $\mathcal{L}$ . Then we have*

$$f^* \text{div}_{\mathcal{L}}(s) = \sum n_i \text{div}_{f^* \mathcal{L}|_{X_i}}(s_i)$$

in  $Z_{n+r-1}(X)$ . Here the sum is over the irreducible components  $X_i \subset X$  of  $\delta$ -dimension  $n + r$ , the section  $s_i = f|_{X_i}^*(s)$  is the pullback of  $s$ , and  $n_i = m_{X_i, X}$  is the multiplicity of  $X_i$  in  $X$ .

<sup>1</sup>For example, this holds if  $s$  is a regular section of  $\mathcal{L}|_Y$ .

**Proof.** To prove this equality of cycles, we may work locally on  $Y$ . Hence we may assume  $Y$  is affine and  $s = p/q$  for some nonzero sections  $p \in \Gamma(Y, \mathcal{L})$  and  $q \in \Gamma(Y, \mathcal{O})$ . If we can show both

$$f^* \operatorname{div}_{\mathcal{L}}(p) = \sum n_i \operatorname{div}_{f^* \mathcal{L}|_{X_i}}(p_i) \quad \text{and} \quad f^* \operatorname{div}_{\mathcal{O}}(q) = \sum n_i \operatorname{div}_{\mathcal{O}_{X_i}}(q_i)$$

(with obvious notations) then we win by the additivity, see Divisors, Lemma 27.5. Thus we may assume that  $s \in \Gamma(Y, \mathcal{L})$ . In this case we may apply the equality (23.3.1) to see that

$$[Z(f^*(s))]_{k+r-1} = \sum n_i \operatorname{div}_{f^* \mathcal{L}|_{X_i}}(s_i)$$

where  $f^*(s) \in f^* \mathcal{L}$  denotes the pullback of  $s$  to  $X$ . On the other hand we have

$$f^* \operatorname{div}_{\mathcal{L}}(s) = f^*[Z(s)]_{k-1} = [f^{-1}(Z(s))]_{k+r-1},$$

by Lemmas 22.2 and 14.4. Since  $Z(f^*(s)) = f^{-1}(Z(s))$  we win.  $\square$

02SS **Lemma 24.2.** *Let  $(S, \delta)$  be as in Situation 7.1. Let  $X, Y$  be locally of finite type over  $S$ . Let  $f : X \rightarrow Y$  be a flat morphism of relative dimension  $r$ . Let  $\mathcal{L}$  be an invertible sheaf on  $Y$ . Let  $\alpha$  be a  $k$ -cycle on  $Y$ . Then*

$$f^*(c_1(\mathcal{L}) \cap \alpha) = c_1(f^* \mathcal{L}) \cap f^* \alpha$$

in  $A_{k+r-1}(X)$ .

**Proof.** Write  $\alpha = \sum n_i [W_i]$ . We will show that

$$f^*(c_1(\mathcal{L}) \cap [W_i]) = c_1(f^* \mathcal{L}) \cap f^*[W_i]$$

in  $A_{k+r-1}(X)$  by producing a rational equivalence on the closed subscheme  $f^{-1}(W_i)$  of  $X$ . By the discussion in Remark 19.4 this will prove the equality of the lemma is true.

Let  $W \subset Y$  be an integral closed subscheme of  $\delta$ -dimension  $k$ . Consider the closed subscheme  $W' = f^{-1}(W) = W \times_Y X$  so that we have the fibre product diagram

$$\begin{array}{ccc} W' & \longrightarrow & X \\ h \downarrow & & \downarrow f \\ W & \longrightarrow & Y \end{array}$$

We have to show that  $f^*(c_1(\mathcal{L}) \cap [W]) = c_1(f^* \mathcal{L}) \cap f^*[W]$ . Choose a nonzero meromorphic section  $s$  of  $\mathcal{L}|_W$ . Let  $W'_i \subset W'$  be the irreducible components of  $\delta$ -dimension  $k+r$ . Write  $[W']_{k+r} = \sum n_i [W'_i]$  with  $n_i$  the multiplicity of  $W'_i$  in  $W'$  as per definition. So  $f^*[W] = \sum n_i [W'_i]$  in  $Z_{k+r}(X)$ . Since each  $W'_i \rightarrow W$  is dominant we see that  $s_i = s|_{W'_i}$  is a nonzero meromorphic section for each  $i$ . By Lemma 24.1 we have the following equality of cycles

$$h^* \operatorname{div}_{\mathcal{L}|_W}(s) = \sum n_i \operatorname{div}_{f^* \mathcal{L}|_{W'_i}}(s_i)$$

in  $Z_{k+r-1}(W')$ . This finishes the proof since the left hand side is a cycle on  $W'$  which pushes to  $f^*(c_1(\mathcal{L}) \cap [W])$  in  $A_{k+r-1}(X)$  and the right hand side is a cycle on  $W'$  which pushes to  $c_1(f^* \mathcal{L}) \cap f^*[W]$  in  $A_{k+r-1}(X)$ .  $\square$

02ST **Lemma 24.3.** *Let  $(S, \delta)$  be as in Situation 7.1. Let  $X, Y$  be locally of finite type over  $S$ . Let  $f : X \rightarrow Y$  be a proper morphism. Let  $\mathcal{L}$  be an invertible sheaf on  $Y$ . Let  $s$  be a nonzero meromorphic section  $s$  of  $\mathcal{L}$  on  $Y$ . Assume  $X, Y$  integral,  $f$  dominant, and  $\dim_\delta(X) = \dim_\delta(Y)$ . Then*

$$f_*(\operatorname{div}_{f^*\mathcal{L}}(f^*s)) = [R(X) : R(Y)]\operatorname{div}_{\mathcal{L}}(s).$$

as cycles on  $Y$ . In particular

$$f_*(c_1(f^*\mathcal{L}) \cap [X]) = c_1(\mathcal{L}) \cap f_*[Y].$$

**Proof.** The last equation follows from the first since  $f_*[X] = [R(X) : R(Y)][Y]$  by definition. It turns out that we can re-use Lemma 18.1 to prove this. Namely, since we are trying to prove an equality of cycles, we may work locally on  $Y$ . Hence we may assume that  $\mathcal{L} = \mathcal{O}_Y$ . In this case  $s$  corresponds to a rational function  $g \in R(Y)$ , and we are simply trying to prove

$$f_*(\operatorname{div}_X(g)) = [R(X) : R(Y)]\operatorname{div}_Y(g).$$

Comparing with the result of the aforementioned Lemma 18.1 we see this true since  $\operatorname{Nm}_{R(X)/R(Y)}(g) = g^{[R(X):R(Y)]}$  as  $g \in R(Y)^*$ .  $\square$

02SU **Lemma 24.4.** *Let  $(S, \delta)$  be as in Situation 7.1. Let  $X, Y$  be locally of finite type over  $S$ . Let  $p : X \rightarrow Y$  be a proper morphism. Let  $\alpha \in Z_{k+1}(X)$ . Let  $\mathcal{L}$  be an invertible sheaf on  $Y$ . Then*

$$p_*(c_1(p^*\mathcal{L}) \cap \alpha) = c_1(\mathcal{L}) \cap p_*\alpha$$

in  $A_k(Y)$ .

**Proof.** Suppose that  $p$  has the property that for every integral closed subscheme  $W \subset X$  the map  $p|_W : W \rightarrow Y$  is a closed immersion. Then, by definition of capping with  $c_1(\mathcal{L})$  the lemma holds.

We will use this remark to reduce to a special case. Namely, write  $\alpha = \sum n_i[W_i]$  with  $n_i \neq 0$  and  $W_i$  pairwise distinct. Let  $W'_i \subset Y$  be the image of  $W_i$  (as an integral closed subscheme). Consider the diagram

$$\begin{array}{ccc} X' = \coprod W_i & \xrightarrow{q} & X \\ p' \downarrow & & \downarrow p \\ Y' = \coprod W'_i & \xrightarrow{q'} & Y. \end{array}$$

Since  $\{W_i\}$  is locally finite on  $X$ , and  $p$  is proper we see that  $\{W'_i\}$  is locally finite on  $Y$  and that  $q, q', p'$  are also proper morphisms. We may think of  $\sum n_i[W_i]$  also as a  $k$ -cycle  $\alpha' \in Z_k(X')$ . Clearly  $q_*\alpha' = \alpha$ . We have  $q_*(c_1(q^*p^*\mathcal{L}) \cap \alpha') = c_1(p^*\mathcal{L}) \cap q_*\alpha'$  and  $(q')_*(c_1((q')^*\mathcal{L}) \cap p'_*\alpha') = c_1(\mathcal{L}) \cap q'_*p'_*\alpha'$  by the initial remark of the proof. Hence it suffices to prove the lemma for the morphism  $p'$  and the cycle  $\sum n_i[W_i]$ . Clearly, this means we may assume  $X, Y$  integral,  $f : X \rightarrow Y$  dominant and  $\alpha = [X]$ . In this case the result follows from Lemma 24.3.  $\square$

## 25. The key formula

0AYB Let  $(S, \delta)$  be as in Situation 7.1. Let  $X$  be locally of finite type over  $S$ . Assume  $X$  is integral and  $\dim_\delta(X) = n$ . Let  $\mathcal{L}$  and  $\mathcal{N}$  be invertible sheaves on  $X$ . Let  $s$  be a nonzero meromorphic section of  $\mathcal{L}$  and let  $t$  be a nonzero meromorphic section of  $\mathcal{N}$ . Let  $Z_i \subset X$ ,  $i \in I$  be a locally finite set of irreducible closed subsets of codimension 1 with the following property: If  $Z \notin \{Z_i\}$  with generic point  $\xi$ , then  $s$  is a generator for  $\mathcal{L}_\xi$  and  $t$  is a generator for  $\mathcal{N}_\xi$ . Such a set exists by Divisors, Lemma 27.2. Then

$$\operatorname{div}_{\mathcal{L}}(s) = \sum \operatorname{ord}_{Z_i, \mathcal{L}}(s)[Z_i]$$

and similarly

$$\operatorname{div}_{\mathcal{N}}(t) = \sum \operatorname{ord}_{Z_i, \mathcal{N}}(t)[Z_i]$$

Unwinding the definitions more, we pick for each  $i$  generators  $s_i \in \mathcal{L}_{\xi_i}$  and  $t_i \in \mathcal{N}_{\xi_i}$  where  $\xi_i$  is the generic point of  $Z_i$ . Then we can write

$$s = f_i s_i \quad \text{and} \quad t = g_i t_i$$

Set  $B_i = \mathcal{O}_{X, \xi_i}$ . Then by definition

$$\operatorname{ord}_{Z_i, \mathcal{L}}(s) = \operatorname{ord}_{B_i}(f_i) \quad \text{and} \quad \operatorname{ord}_{Z_i, \mathcal{N}}(t) = \operatorname{ord}_{B_i}(g_i)$$

Since  $t_i$  is a generator of  $\mathcal{N}_{\xi_i}$  we see that its image in the fibre  $\mathcal{N}_{\xi_i} \otimes \kappa(\xi_i)$  is a nonzero meromorphic section of  $\mathcal{N}|_{Z_i}$ . We will denote this image  $t_i|_{Z_i}$ . From our definitions it follows that

$$c_1(\mathcal{N}) \cap \operatorname{div}_{\mathcal{L}}(s) = \sum \operatorname{ord}_{B_i}(f_i)(Z_i \rightarrow X)_* \operatorname{div}_{\mathcal{N}|_{Z_i}}(t_i|_{Z_i})$$

and similarly

$$c_1(\mathcal{L}) \cap \operatorname{div}_{\mathcal{N}}(t) = \sum \operatorname{ord}_{B_i}(g_i)(Z_i \rightarrow X)_* \operatorname{div}_{\mathcal{L}|_{Z_i}}(s_i|_{Z_i})$$

in  $A_{n-2}(X)$ . We are going to find a rational equivalence between these two cycles. To do this we consider the tame symbol

$$\partial_{B_i}(f_i, g_i) \in \kappa(\xi_i)^*$$

see Section 5.

0AYC **Lemma 25.1** (Key formula). *In the situation above the cycle*

$$\sum (Z_i \rightarrow X)_* \left( \operatorname{ord}_{B_i}(f_i) \operatorname{div}_{\mathcal{N}|_{Z_i}}(t_i|_{Z_i}) - \operatorname{ord}_{B_i}(g_i) \operatorname{div}_{\mathcal{L}|_{Z_i}}(s_i|_{Z_i}) \right)$$

*is equal to the cycle*

$$\sum (Z_i \rightarrow X)_* \operatorname{div}(\partial_{B_i}(f_i, g_i))$$

**Proof.** First, let us examine what happens if we replace  $s_i$  by  $us_i$  for some unit  $u$  in  $B_i$ . Then  $f_i$  gets replaced by  $u^{-1}f_i$ . Thus the first part of the first expression of the lemma is unchanged and in the second part we add

$$-\operatorname{ord}_{B_i}(g_i) \operatorname{div}(u|_{Z_i})$$

(where  $u|_{Z_i}$  is the image of  $u$  in the residue field) by Divisors, Lemma 27.3 and in the second expression we add

$$\operatorname{div}(\partial_{B_i}(u^{-1}, g_i))$$

by bi-linearity of the tame symbol. These terms agree by property (6) of the tame symbol.



Let  $Z \subset X$  be an irreducible closed with  $\dim_\delta(Z) = n - 2$ . To show that the coefficients of  $Z$  of the two cycles of the lemma is the same, we may do a replacement  $s_i \mapsto us_i$  as in the previous paragraph. In exactly the same way one shows that we may do a replacement  $t_i \mapsto vt_i$  for some unit  $v$  of  $B_i$ .

Since we are proving the equality of cycles we may argue one coefficient at a time. Thus we choose an irreducible closed  $Z \subset X$  with  $\dim_\delta(Z) = n - 2$  and compare coefficients. Let  $\xi \in Z$  be the generic point and set  $A = \mathcal{O}_{X,\xi}$ . This is a Noetherian local domain of dimension 2. Choose generators  $\sigma$  and  $\tau$  for  $\mathcal{L}_\xi$  and  $\mathcal{N}_\xi$ . After shrinking  $X$ , we may and do assume  $\sigma$  and  $\tau$  define trivializations of the invertible sheaves  $\mathcal{L}$  and  $\mathcal{N}$  over all of  $X$ . Because  $Z_i$  is locally finite after shrinking  $X$  we may assume  $Z \subset Z_i$  for all  $i \in I$  and that  $I$  is finite. Then  $\xi_i$  corresponds to a prime  $\mathfrak{q}_i \subset A$  of height 1. We may write  $s_i = a_i\sigma$  and  $t_i = b_i\tau$  for some  $a_i$  and  $b_i$  units in  $A_{\mathfrak{q}_i}$ . By the remarks above, it suffices to prove the lemma when  $a_i = b_i = 1$  for all  $i$ .

Assume  $a_i = b_i = 1$  for all  $i$ . Then the first expression of the lemma is zero, because we choose  $\sigma$  and  $\tau$  to be trivializing sections. Write  $s = f\sigma$  and  $t = g\tau$  with  $f$  and  $g$  in the fraction field of  $A$ . By the previous paragraph we have reduced to the case  $f_i = f$  and  $g_i = g$  for all  $i$ . Moreover, for a height 1 prime  $\mathfrak{q}$  of  $A$  which is not in  $\{\mathfrak{q}_i\}$  we have that both  $f$  and  $g$  are units in  $A_{\mathfrak{q}}$  (by our choice of the family  $\{Z_i\}$  in the discussion preceding the lemma). Thus the coefficient of  $Z$  in the second expression of the lemma is

$$\sum_i \text{ord}_{A/\mathfrak{q}_i}(\partial_{B_i}(f, g))$$

which is zero by the key Lemma 6.3. □

## 26. Intersecting with an invertible sheaf and rational equivalence

02TG Applying the key lemma we obtain the fundamental properties of intersecting with invertible sheaves. In particular, we will see that  $c_1(\mathcal{L}) \cap -$  factors through rational equivalence and that these operations for different invertible sheaves commute.

02TH **Lemma 26.1.** *Let  $(S, \delta)$  be as in Situation 7.1. Let  $X$  be locally of finite type over  $S$ . Assume  $X$  integral and  $\dim_\delta(X) = n$ . Let  $\mathcal{L}, \mathcal{N}$  be invertible on  $X$ . Choose a nonzero meromorphic section  $s$  of  $\mathcal{L}$  and a nonzero meromorphic section  $t$  of  $\mathcal{N}$ . Set  $\alpha = \text{div}_{\mathcal{L}}(s)$  and  $\beta = \text{div}_{\mathcal{N}}(t)$ . Then*

$$c_1(\mathcal{N}) \cap \alpha = c_1(\mathcal{L}) \cap \beta$$

in  $A_{n-2}(X)$ .

**Proof.** Immediate from the key Lemma 25.1 and the discussion preceding it. □

02TI **Lemma 26.2.** *Let  $(S, \delta)$  be as in Situation 7.1. Let  $X$  be locally of finite type over  $S$ . Let  $\mathcal{L}$  be invertible on  $X$ . The operation  $\alpha \mapsto c_1(\mathcal{L}) \cap \alpha$  factors through rational equivalence to give an operation*

$$c_1(\mathcal{L}) \cap - : A_{k+1}(X) \rightarrow A_k(X)$$

**Proof.** Let  $\alpha \in Z_{k+1}(X)$ , and  $\alpha \sim_{rat} 0$ . We have to show that  $c_1(\mathcal{L}) \cap \alpha$  as defined in Definition 23.1 is zero. By Definition 19.1 there exists a locally finite family

$\{W_j\}$  of integral closed subschemes with  $\dim_\delta(W_j) = k + 2$  and rational functions  $f_j \in R(W_j)^*$  such that

$$\alpha = \sum (i_j)_* \operatorname{div}_{W_j}(f_j)$$

Note that  $p : \coprod W_j \rightarrow X$  is a proper morphism, and hence  $\alpha = p_* \alpha'$  where  $\alpha' \in Z_{k+1}(\coprod W_j)$  is the sum of the principal divisors  $\operatorname{div}_{W_j}(f_j)$ . By Lemma 24.4 we have  $c_1(\mathcal{L}) \cap \alpha = p_*(c_1(p^* \mathcal{L}) \cap \alpha')$ . Hence it suffices to show that each  $c_1(\mathcal{L}|_{W_j}) \cap \operatorname{div}_{W_j}(f_j)$  is zero. In other words we may assume that  $X$  is integral and  $\alpha = \operatorname{div}_X(f)$  for some  $f \in R(X)^*$ .

Assume  $X$  is integral and  $\alpha = \operatorname{div}_X(f)$  for some  $f \in R(X)^*$ . We can think of  $f$  as a regular meromorphic section of the invertible sheaf  $\mathcal{N} = \mathcal{O}_X$ . Choose a meromorphic section  $s$  of  $\mathcal{L}$  and denote  $\beta = \operatorname{div}_{\mathcal{L}}(s)$ . By Lemma 26.1 we conclude that

$$c_1(\mathcal{L}) \cap \alpha = c_1(\mathcal{O}_X) \cap \beta.$$

However, by Lemma 23.2 we see that the right hand side is zero in  $A_k(X)$  as desired.  $\square$

Let  $(S, \delta)$  be as in Situation 7.1. Let  $X$  be locally of finite type over  $S$ . Let  $\mathcal{L}$  be invertible on  $X$ . We will denote

$$c_1(\mathcal{L})^s \cap - : A_{k+s}(X) \rightarrow A_k(X)$$

the operation  $c_1(\mathcal{L}) \cap -$ . This makes sense by Lemma 26.2. We will denote  $c_1(\mathcal{L}^s \cap -$  the  $s$ -fold iterate of this operation for all  $s \geq 0$ .

02TJ **Lemma 26.3.** *Let  $(S, \delta)$  be as in Situation 7.1. Let  $X$  be locally of finite type over  $S$ . Let  $\mathcal{L}, \mathcal{N}$  be invertible on  $X$ . For any  $\alpha \in A_{k+2}(X)$  we have*

$$c_1(\mathcal{L}) \cap c_1(\mathcal{N}) \cap \alpha = c_1(\mathcal{N}) \cap c_1(\mathcal{L}) \cap \alpha$$

as elements of  $A_k(X)$ .

**Proof.** Write  $\alpha = \sum m_j [Z_j]$  for some locally finite collection of integral closed subschemes  $Z_j \subset X$  with  $\dim_\delta(Z_j) = k + 2$ . Consider the proper morphism  $p : \coprod Z_j \rightarrow X$ . Set  $\alpha' = \sum m_j [Z_j]$  as a  $(k + 2)$ -cycle on  $\coprod Z_j$ . By several applications of Lemma 24.4 we see that  $c_1(\mathcal{L}) \cap c_1(\mathcal{N}) \cap \alpha = p_*(c_1(p^* \mathcal{L}) \cap c_1(p^* \mathcal{N}) \cap \alpha')$  and  $c_1(\mathcal{N}) \cap c_1(\mathcal{L}) \cap \alpha = p_*(c_1(p^* \mathcal{N}) \cap c_1(p^* \mathcal{L}) \cap \alpha')$ . Hence it suffices to prove the formula in case  $X$  is integral and  $\alpha = [X]$ . In this case the result follows from Lemma 26.1 and the definitions.  $\square$

## 27. Intersecting with effective Cartier divisors

02T7 In this section we define the gysin map for the zero locus of a section of an invertible sheaf. The most interesting case is that of an effective Cartier divisor; the reason for the generalization is to be able to formulate various compatibilities, see Remark 27.2 and Lemmas 27.7, 27.8, and 28.4. These results can be generalized to deal with locally principal closed subschemes with a virtual normal bundle (Remark 27.4). A generalization in a different direction comes from looking at pseudo-divisors (Remark 27.5).

Recall that effective Cartier divisors correspond 1-to-1 to isomorphism classes of pairs  $(\mathcal{L}, s)$  where  $\mathcal{L}$  is an invertible sheaf and  $s$  is a global section, see Divisors, Lemma 14.10. If  $D$  corresponds to  $(\mathcal{L}, s)$ , then  $\mathcal{L} = \mathcal{O}_X(D)$ . Please keep this in mind while reading this section.

02T8 **Definition 27.1.** Let  $(S, \delta)$  be as in Situation 7.1. Let  $X$  be locally of finite type over  $S$ . Let  $(\mathcal{L}, s)$  be a pair consisting of an invertible sheaf and a global section  $s \in \Gamma(X, \mathcal{L})$ . Let  $D = Z(s)$  be the zero scheme of  $s$ , and denote  $i : D \rightarrow X$  the closed immersion. We define, for every integer  $k$ , a (refined) *Gysin homomorphism*

$$i^* : Z_{k+1}(X) \rightarrow A_k(D).$$

by the following rules:

- (1) Given an integral closed subscheme  $W \subset X$  with  $\dim_\delta(W) = k + 1$  we define
  - (a) if  $W \not\subset D$ , then  $i^*[W] = [D \cap W]_k$  as a  $k$ -cycle on  $D$ , and
  - (b) if  $W \subset D$ , then  $i^*[W] = i'_*(c_1(\mathcal{L}|_W) \cap [W])$ , where  $i' : W \rightarrow D$  is the induced closed immersion.
- (2) For a general  $(k + 1)$ -cycle  $\alpha = \sum n_j [W_j]$  we set

$$i^*\alpha = \sum n_j i^*[W_j]$$

- (3) If  $D$  is an effective Cartier divisor, then we denote  $D \cdot \alpha = i_* i^* \alpha$  the pushforward of the class to a class on  $X$ .

In fact, as we will see later, this Gysin homomorphism  $i^*$  can be viewed as an example of a non-flat pullback. Thus we will sometimes informally call the class  $i^*\alpha$  the *pullback* of the class  $\alpha$ .

0B6Y **Remark 27.2.** Let  $f : X' \rightarrow X$  be a morphism of schemes locally of finite type over  $S$  as in Situation 7.1. Let  $(\mathcal{L}, s, i : D \rightarrow X)$  be a triple as in Definition 27.1. Then we can set  $\mathcal{L}' = f^*\mathcal{L}$ ,  $s' = f^*s$ , and  $D' = X' \times_X D = Z(s')$ . This gives a commutative diagram

$$\begin{array}{ccc} D' & \longrightarrow & X' \\ g \downarrow & & \downarrow f \\ D & \xrightarrow{i} & X \end{array}$$

and we can ask for various compatibilities between  $i^*$  and  $(i')^*$ .

0B6Z **Remark 27.3.** Let  $X \rightarrow S$ ,  $\mathcal{L}$ ,  $s$ ,  $i : D \rightarrow X$  be as in Definition 27.1 and assume that  $\mathcal{L}|_D \cong \mathcal{O}_D$ . In this case we can define a canonical map  $i^* : Z_{k+1}(X) \rightarrow Z_k(D)$  on cycles, by requiring that  $i^*[W] = 0$  whenever  $W \subset D$ . The possibility to do this will be useful later on.

0B70 **Remark 27.4.** Let  $X$  be a scheme locally of finite type over  $S$  as in Situation 7.1. Let  $(D, \mathcal{N}, \sigma)$  be a triple consisting of a locally principal (Divisors, Definition 13.1) closed subscheme  $i : D \rightarrow X$ , an invertible  $\mathcal{O}_D$ -module  $\mathcal{N}$ , and a surjection  $\sigma : \mathcal{N}^{\otimes -1} \rightarrow i^*\mathcal{I}_D$  of  $\mathcal{O}_D$ -modules. Here  $\mathcal{N}$  should be thought of as a *virtual normal bundle of  $D$  in  $X$* . The construction of  $i^* : Z_{k+1}(X) \rightarrow A_k(D)$  in Definition 27.1 generalizes to such triples and it is perhaps the correct generality for the definition.

0B7D **Remark 27.5.** Let  $X$  be a scheme locally of finite type over  $S$  as in Situation 7.1. In [Ful98] a *pseudo-divisor* on  $X$  is defined as a triple  $D = (\mathcal{L}, Z, s)$  where  $\mathcal{L}$  is an invertible  $\mathcal{O}_X$ -module,  $Z \subset X$  is a closed subset, and  $s \in \Gamma(X \setminus Z, \mathcal{L})$  is a nowhere vanishing section. Similarly to the above, one can define for every  $\alpha$  in  $A_{k+1}(X)$  a product  $D \cdot \alpha$  in  $A_k(Z \cap |\alpha|)$  where  $|\alpha|$  is the support of  $\alpha$ .

02T9 **Lemma 27.6.** *Let  $(S, \delta)$  be as in Situation 7.1. Let  $X$  be locally of finite type over  $S$ . Let  $(\mathcal{L}, s, i : D \rightarrow X)$  be as in Definition 27.1. Let  $\alpha$  be a  $(k + 1)$ -cycle on*

$X$ . Then  $i_*i^*\alpha = c_1(\mathcal{L}) \cap \alpha$  in  $A_k(X)$ . In particular, if  $D$  is an effective Cartier divisor, then  $D \cdot \alpha = c_1(\mathcal{O}_X(D)) \cap \alpha$ .

**Proof.** Write  $\alpha = \sum n_j[W_j]$  where  $i_j : W_j \rightarrow X$  are integral closed subschemes with  $\dim_\delta(W_j) = k$ . Since  $D$  is the zero scheme of  $s$  we see that  $D \cap W_j$  is the zero scheme of the restriction  $s|_{W_j}$ . Hence for each  $j$  such that  $W_j \not\subset D$  we have  $c_1(\mathcal{L}) \cap [W_j] = [D \cap W_j]_k$  by Lemma 23.4. So we have

$$c_1(\mathcal{L}) \cap \alpha = \sum_{W_j \not\subset D} n_j [D \cap W_j]_k + \sum_{W_j \subset D} n_j i_{j,*}(c_1(\mathcal{L})|_{W_j}) \cap [W_j]$$

in  $A_k(X)$  by Definition 23.1. The right hand side matches (termwise) the pushforward of the class  $i^*\alpha$  on  $D$  from Definition 27.1. Hence we win.  $\square$

02TA **Lemma 27.7.** *Let  $(S, \delta)$  be as in Situation 7.1. Let  $f : X' \rightarrow X$  be a proper morphism of schemes locally of finite type over  $S$ . Let  $(\mathcal{L}, s, i : D \rightarrow X)$  be as in Definition 27.1. Form the diagram*

$$\begin{array}{ccc} D' & \longrightarrow & X' \\ g \downarrow & & \downarrow f \\ D & \xrightarrow{i} & X \end{array}$$

as in Remark 27.2. For any  $(k+1)$ -cycle  $\alpha'$  on  $X'$  we have  $i^*f_*\alpha' = g_*(i')^*\alpha'$  in  $A_k(D)$  (this makes sense as  $f_*$  is defined on the level of cycles).

**Proof.** Suppose  $\alpha = [W']$  for some integral closed subscheme  $W' \subset X'$ . Let  $W = f(W') \subset X$ . In case  $W' \not\subset D'$ , then  $W \not\subset D$  and we see that

$$[W' \cap D']_k = \operatorname{div}_{\mathcal{L}'|_{W'}}(s'|_{W'}) \quad \text{and} \quad [W \cap D]_k = \operatorname{div}_{\mathcal{L}|_W}(s|_W)$$

and hence  $f_*$  of the first cycle equals the second cycle by Lemma 24.3. Hence the equality holds as cycles. In case  $W' \subset D'$ , then  $W \subset D$  and  $f_*(c_1(\mathcal{L}|_{W'}) \cap [W'])$  is equal to  $c_1(\mathcal{L}|_W) \cap [W]$  in  $A_k(W)$  by the second assertion of Lemma 24.3. By Remark 19.4 the result follows for general  $\alpha'$ .  $\square$

0B71 **Lemma 27.8.** *Let  $(S, \delta)$  be as in Situation 7.1. Let  $f : X' \rightarrow X$  be a flat morphism of relative dimension  $r$  of schemes locally of finite type over  $S$ . Let  $(\mathcal{L}, s, i : D \rightarrow X)$  be as in Definition 27.1. Form the diagram*

$$\begin{array}{ccc} D' & \longrightarrow & X' \\ g \downarrow & & \downarrow f \\ D & \xrightarrow{i} & X \end{array}$$

as in Remark 27.2. For any  $(k+1)$ -cycle  $\alpha$  on  $X$  we have  $(i')^*f^*\alpha = g^*i^*\alpha$  in  $A_{k+r}(D)$  (this makes sense as  $f^*$  is defined on the level of cycles).

**Proof.** Suppose  $\alpha = [W]$  for some integral closed subscheme  $W \subset X$ . Let  $W' = f^{-1}(W) \subset X'$ . In case  $W \not\subset D$ , then  $W' \not\subset D'$  and we see that

$$W' \cap D' = g^{-1}(W \cap D)$$

as closed subschemes of  $D'$ . Hence the equality holds as cycles, see Lemma 14.4. In case  $W \subset D$ , then  $W' \subset D'$  and  $W' = g^{-1}(W)$  with  $[W']_{k+1+r} = g^*[W]$  and equality holds in  $A_{k+r}(D')$  by Lemma 24.2. By Remark 19.4 the result follows for general  $\alpha$ .  $\square$

02TB **Lemma 27.9.** *Let  $(S, \delta)$  be as in Situation 7.1. Let  $X$  be locally of finite type over  $S$ . Let  $(\mathcal{L}, s, i : D \rightarrow X)$  be as in Definition 27.1.*

- (1) *Let  $Z \subset X$  be a closed subscheme such that  $\dim_\delta(Z) \leq k + 1$  and such that  $D \cap Z$  is an effective Cartier divisor on  $Z$ . Then  $i^*[Z]_{k+1} = [D \cap Z]_k$ .*
- (2) *Let  $\mathcal{F}$  be a coherent sheaf on  $X$  such that  $\dim_\delta(\text{Supp}(\mathcal{F})) \leq k + 1$  and  $s : \mathcal{F} \rightarrow \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}$  is injective. Then*

$$i^*[\mathcal{F}]_{k+1} = [i^*\mathcal{F}]_k$$

*in  $A_k(D)$ .*

**Proof.** Assume  $Z \subset X$  as in (1). Then set  $\mathcal{F} = \mathcal{O}_Z$ . The assumption that  $D \cap Z$  is an effective Cartier divisor is equivalent to the assumption that  $s : \mathcal{F} \rightarrow \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}$  is injective. Moreover  $[Z]_{k+1} = [\mathcal{F}]_{k+1}$  and  $[D \cap Z]_k = [\mathcal{O}_{D \cap Z}]_k = [i^*\mathcal{F}]_k$ . See Lemma 10.3. Hence part (1) follows from part (2).

Write  $[\mathcal{F}]_{k+1} = \sum m_j[W_j]$  with  $m_j > 0$  and pairwise distinct integral closed subschemes  $W_j \subset X$  of  $\delta$ -dimension  $k + 1$ . The assumption that  $s : \mathcal{F} \rightarrow \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}$  is injective implies that  $W_j \not\subset D$  for all  $j$ . By definition we see that

$$i^*[\mathcal{F}]_{k+1} = \sum [D \cap W_j]_k.$$

We claim that

$$\sum [D \cap W_j]_k = [i^*\mathcal{F}]_k$$

as cycles. Let  $Z \subset D$  be an integral closed subscheme of  $\delta$ -dimension  $k$ . Let  $\xi \in Z$  be its generic point. Let  $A = \mathcal{O}_{X, \xi}$ . Let  $M = \mathcal{F}_\xi$ . Let  $f \in A$  be an element generating the ideal of  $D$ , i.e., such that  $\mathcal{O}_{D, \xi} = A/fA$ . By assumption  $\dim(\text{Supp}(M)) = 1$ , the map  $f : M \rightarrow M$  is injective, and  $\text{length}_A(M/fM) < \infty$ . Moreover,  $\text{length}_A(M/fM)$  is the coefficient of  $[Z]$  in  $[i^*\mathcal{F}]_k$ . On the other hand, let  $\mathfrak{q}_1, \dots, \mathfrak{q}_t$  be the minimal primes in the support of  $M$ . Then

$$\sum \text{length}_{A_{\mathfrak{q}_i}}(M_{\mathfrak{q}_i}) \text{ord}_{A/\mathfrak{q}_i}(f)$$

is the coefficient of  $[Z]$  in  $\sum [D \cap W_j]_k$ . Hence we see the equality by Lemma 3.2.  $\square$

## 28. Gysin homomorphisms

02TK In this section we use the key formula to show the Gysin homomorphism factor through rational equivalence.

02TM **Lemma 28.1.** *Let  $(S, \delta)$  be as in Situation 7.1. Let  $X$  be locally of finite type over  $S$ . Let  $X$  be integral and  $n = \dim_\delta(X)$ . Let  $i : D \rightarrow X$  be an effective Cartier divisor. Let  $\mathcal{N}$  be an invertible  $\mathcal{O}_X$ -module and let  $t$  be a nonzero meromorphic section of  $\mathcal{N}$ . Then  $i^* \text{div}_{\mathcal{N}}(t) = c_1(\mathcal{N}) \cap [D]_{n-1}$  in  $A_{n-2}(D)$ .*

**Proof.** Write  $\text{div}_{\mathcal{N}}(t) = \sum \text{ord}_{Z_i, \mathcal{N}}(t)[Z_i]$  for some integral closed subschemes  $Z_i \subset X$  of  $\delta$ -dimension  $n - 1$ . We may assume that the family  $\{Z_i\}$  is locally finite, that  $t \in \Gamma(U, \mathcal{N}|_U)$  is a generator where  $U = X \setminus \bigcup Z_i$ , and that every irreducible component of  $D$  is one of the  $Z_i$ , see Divisors, Lemmas 26.1, 26.4, and 27.2.

Set  $\mathcal{L} = \mathcal{O}_X(D)$ . Denote  $s \in \Gamma(X, \mathcal{O}_X(D)) = \Gamma(X, \mathcal{L})$  the canonical section. We will apply the discussion of Section 25 to our current situation. For each  $i$  let  $\xi_i \in Z_i$  be its generic point. Let  $B_i = \mathcal{O}_{X, \xi_i}$ . For each  $i$  we pick generators  $s_i \in \mathcal{L}_{\xi_i}$  and  $t_i \in \mathcal{N}_{\xi_i}$  over  $B_i$  but we insist that we pick  $s_i = s$  if  $Z_i \not\subset D$ . Write  $s = f_i s_i$  and

$t = g_i t_i$  with  $f_i, g_i \in B_i$ . Then  $\text{ord}_{Z_i, \mathcal{N}}(t) = \text{ord}_{B_i}(g_i)$ . On the other hand, we have  $f_i \in B_i$  and

$$[D]_{n-1} = \sum \text{ord}_{B_i}(f_i)[Z_i]$$

because of our choices of  $s_i$ . We claim that

$$i^* \text{div}_{\mathcal{N}}(t) = \sum \text{ord}_{B_i}(g_i) \text{div}_{\mathcal{L}|_{Z_i}}(s_i|_{Z_i})$$

as cycles. More precisely, the right hand side is a cycle representing the left hand side. Namely, this is clear by our formula for  $\text{div}_{\mathcal{N}}(t)$  and the fact that  $\text{div}_{\mathcal{L}|_{Z_i}}(s_i|_{Z_i}) = [Z(s_i|_{Z_i})]_{n-2} = [Z_i \cap D]_{n-2}$  when  $Z_i \not\subset D$  because in that case  $s_i|_{Z_i} = s|_{Z_i}$  is a regular section, see Lemma 22.2. Similarly,

$$c_1(\mathcal{N}) \cap [D]_{n-1} = \sum \text{ord}_{B_i}(f_i) \text{div}_{\mathcal{N}|_{Z_i}}(t_i|_{Z_i})$$

The key formula (Lemma 25.1) gives the equality

$$\sum \left( \text{ord}_{B_i}(f_i) \text{div}_{\mathcal{N}|_{Z_i}}(t_i|_{Z_i}) - \text{ord}_{B_i}(g_i) \text{div}_{\mathcal{L}|_{Z_i}}(s_i|_{Z_i}) \right) = \sum \text{div}_{Z_i}(\partial_{B_i}(f_i, g_i))$$

of cycles. If  $Z_i \not\subset D$ , then  $f_i = 1$  and hence  $\text{div}_{Z_i}(\partial_{B_i}(f_i, g_i)) = 0$ . Thus we get a rational equivalence between our specific cycles representing  $i^* \text{div}_{\mathcal{N}}(t)$  and  $c_1(\mathcal{N}) \cap [D]_{n-1}$  on  $D$ . This finishes the proof.  $\square$

02T0 **Lemma 28.2.** *Let  $(S, \delta)$  be as in Situation 7.1. Let  $X$  be locally of finite type over  $S$ . Let  $(\mathcal{L}, s, i : D \rightarrow X)$  be as in Definition 27.1. The Gysin homomorphism factors through rational equivalence to give a map  $i^* : A_{k+1}(X) \rightarrow A_k(D)$ .*

**Proof.** Let  $\alpha \in Z_{k+1}(X)$  and assume that  $\alpha \sim_{\text{rat}} 0$ . This means there exists a locally finite collection of integral closed subschemes  $W_j \subset X$  of  $\delta$ -dimension  $k+2$  and  $f_j \in R(W_j)^*$  such that  $\alpha = \sum i_{j,*} \text{div}_{W_j}(f_j)$ . Set  $X' = \coprod W_i$  and consider the diagram

$$\begin{array}{ccc} D' & \xrightarrow{\quad} & X' \\ q \downarrow & i' & \downarrow p \\ D & \xrightarrow{\quad i \quad} & X \end{array}$$

of Remark 27.2. Since  $X' \rightarrow X$  is proper we see that  $i^* p_* = q_*(i')^*$  by Lemma 27.7. As we know that  $q_*$  factors through rational equivalence (Lemma 20.3), it suffices to prove the result for  $\alpha' = \sum \text{div}_{W_j}(f_j)$  on  $X'$ . Clearly this reduces us to the case where  $X$  is integral and  $\alpha = \text{div}(f)$  for some  $f \in R(X)^*$ .

Assume  $X$  is integral and  $\alpha = \text{div}(f)$  for some  $f \in R(X)^*$ . If  $X = D$ , then we see that  $i^* \alpha$  is equal to  $c_1(\mathcal{L}) \cap \alpha$ . This is rationally equivalent to zero by Lemma 26.2. If  $D \neq X$ , then we see that  $i^* \text{div}_X(f)$  is equal to  $c_1(\mathcal{O}_D) \cap [D]_{n-1}$  in  $A_k(D)$  by Lemma 28.1. Of course capping with  $c_1(\mathcal{O}_D)$  is the zero map.  $\square$

0B72 **Lemma 28.3.** *Let  $(S, \delta)$  be as in Situation 7.1. Let  $X$  be locally of finite type over  $S$ . Let  $(\mathcal{L}, s, i : D \rightarrow X)$  be a triple as in Definition 27.1. Let  $\mathcal{N}$  be an invertible  $\mathcal{O}_X$ -module. Then  $i^*(c_1(\mathcal{N}) \cap \alpha) = c_1(i^* \mathcal{N}) \cap i^* \alpha$  in  $A_{k-2}(D)$  for all  $\alpha \in A_k(Z)$ .*

**Proof.** With exactly the same proof as in Lemma 28.2 this follows from Lemmas 24.4, 26.3, and 28.1.  $\square$

0B73 **Lemma 28.4.** *Let  $(S, \delta)$  be as in Situation 7.1. Let  $X$  be locally of finite type over  $S$ . Let  $(\mathcal{L}, s, i : D \rightarrow X)$  and  $(\mathcal{L}', s', i' : D' \rightarrow X)$  be two triples as in Definition 27.1. Then the diagram*

$$\begin{array}{ccc} A_k(X) & \xrightarrow{\quad i^* \quad} & A_{k-1}(D) \\ (i')^* \downarrow & & \downarrow \\ A_{k-1}(D') & \xrightarrow{\quad} & A_{k-2}(D \cap D') \end{array}$$

*commutes where each of the maps is a gysin map.*

**Proof.** Denote  $j : D \cap D' \rightarrow D$  and  $j' : D \cap D' \rightarrow D'$  the closed immersions corresponding to  $(\mathcal{L}|_{D'}, s|_{D'})$  and  $(\mathcal{L}'_D, s|_D)$ . We have to show that  $(j')^*i^*\alpha = j^*(i')^*\alpha$  for all  $\alpha \in A_k(X)$ . Let  $W \subset X$  be an integral closed subscheme of dimension  $k$ . Let us prove the equality in case  $\alpha = [W]$ . We will deduce it from the key formula.

We let  $\sigma$  be a nonzero meromorphic section of  $\mathcal{L}|_W$  which we require to be equal to  $s|_W$  if  $W \not\subset D$ . We let  $\sigma'$  be a nonzero meromorphic section of  $\mathcal{L}'|_W$  which we require to be equal to  $s'|_W$  if  $W \not\subset D'$ . Write

$$\operatorname{div}_{\mathcal{L}|_W}(\sigma) = \sum \operatorname{ord}_{Z_i, \mathcal{L}|_W}(\sigma)[Z_i] = \sum n_i[Z_i]$$

and similarly

$$\operatorname{div}_{\mathcal{L}'|_W}(\sigma') = \sum \operatorname{ord}_{Z_i, \mathcal{L}'|_W}(\sigma')[Z_i] = \sum n'_i[Z_i]$$

as in the discussion in Section 25. Then we see that  $Z_i \subset D$  if  $n_i \neq 0$  and  $Z'_i \subset D'$  if  $n'_i \neq 0$ . For each  $i$ , let  $\xi_i \in Z_i$  be the generic point. As in Section 25 we choose for each  $i$  an element  $\sigma_i \in \mathcal{L}_{\xi_i}$ , resp.  $\sigma'_i \in \mathcal{L}'_{\xi_i}$  which generates over  $B_i = \mathcal{O}_{W, \xi_i}$  and which is equal to the image of  $s$ , resp.  $s'$  if  $Z_i \not\subset D$ , resp.  $Z_i \not\subset D'$ . Write  $\sigma = f_i \sigma_i$  and  $\sigma' = f'_i \sigma'_i$  so that  $n_i = \operatorname{ord}_{B_i}(f_i)$  and  $n'_i = \operatorname{ord}_{B_i}(f'_i)$ . From our definitions it follows that

$$(j')^*i^*[W] = \sum \operatorname{ord}_{B_i}(f_i) \operatorname{div}_{\mathcal{L}'|_{Z_i}}(\sigma'_i|_{Z_i})$$

as cycles and

$$j^*(i')^*[W] = \sum \operatorname{ord}_{B_i}(f'_i) \operatorname{div}_{\mathcal{L}|_{Z_i}}(\sigma_i|_{Z_i})$$

The key formula (Lemma 25.1) now gives the equality

$$\sum \left( \operatorname{ord}_{B_i}(f_i) \operatorname{div}_{\mathcal{L}'|_{Z_i}}(\sigma'_i|_{Z_i}) - \operatorname{ord}_{B_i}(f'_i) \operatorname{div}_{\mathcal{L}|_{Z_i}}(\sigma_i|_{Z_i}) \right) = \sum \operatorname{div}_{Z_i}(\partial_{B_i}(f_i, f'_i))$$

of cycles. Note that  $\operatorname{div}_{Z_i}(\partial_{B_i}(f_i, f'_i)) = 0$  if  $Z_i \not\subset D \cap D'$  because in this case either  $f_i = 1$  or  $f'_i = 1$ . Thus we get a rational equivalence between our specific cycles representing  $(j')^*i^*[W]$  and  $j^*(i')^*[W]$  on  $D \cap D' \cap W$ . By Remark 19.4 the result follows for general  $\alpha$ .  $\square$

### 29. Relative effective Cartier divisors

02TP Relative effective Cartier divisors are defined and studied in Divisors, Section 18. To develop the basic results on chern classes of vector bundles we only need the case where both the ambient scheme and the effective Cartier divisor are flat over the base.

02TR **Lemma 29.1.** *Let  $(S, \delta)$  be as in Situation 7.1. Let  $X, Y$  be locally of finite type over  $S$ . Let  $p : X \rightarrow Y$  be a flat morphism of relative dimension  $r$ . Let  $i : D \rightarrow X$  be a relative effective Cartier divisor (Divisors, Definition 18.2). Let  $\mathcal{L} = \mathcal{O}_X(D)$ . For any  $\alpha \in A_{k+1}(Y)$  we have*

$$i^* p^* \alpha = (p|_D)^* \alpha$$

in  $A_{k+r}(D)$  and

$$c_1(\mathcal{L}) \cap p^* \alpha = i_*((p|_D)^* \alpha)$$

in  $A_{k+r}(X)$ .

**Proof.** Let  $W \subset Y$  be an integral closed subscheme of  $\delta$ -dimension  $k + 1$ . By Divisors, Lemma 18.1 we see that  $D \cap p^{-1}W$  is an effective Cartier divisor on  $p^{-1}W$ . By Lemma 27.9 we get the first equality in

$$i^*[p^{-1}W]_{k+r+1} = [D \cap p^{-1}W]_{k+r} = [(p|_D)^{-1}(W)]_{k+r}.$$

and the second because  $D \cap p^{-1}(W) = (p|_D)^{-1}(W)$  as schemes. Since by definition  $p^*[W] = [p^{-1}W]_{k+r+1}$  we see that  $i^* p^*[W] = (p|_D)^*[W]$  as cycles. If  $\alpha = \sum m_j [W_j]$  is a general  $k + 1$  cycle, then we get  $i^* \alpha = \sum m_j i^* p^*[W_j] = \sum m_j (p|_D)^*[W_j]$  as cycles. This proves then first equality. To deduce the second from the first apply Lemma 27.6.  $\square$

### 30. Affine bundles

02TS For an affine bundle the pullback map is surjective on Chow groups.

02TT **Lemma 30.1.** *Let  $(S, \delta)$  be as in Situation 7.1. Let  $X, Y$  be locally of finite type over  $S$ . Let  $f : X \rightarrow Y$  be a flat morphism of relative dimension  $r$ . Assume that for every  $y \in Y$ , there exists an open neighbourhood  $U \subset Y$  such that  $f|_{f^{-1}(U)} : f^{-1}(U) \rightarrow U$  is identified with the morphism  $U \times \mathbf{A}^r \rightarrow U$ . Then  $f^* : A_k(Y) \rightarrow A_{k+r}(X)$  is surjective for all  $k \in \mathbf{Z}$ .*

**Proof.** Let  $\alpha \in A_{k+r}(X)$ . Write  $\alpha = \sum m_j [W_j]$  with  $m_j \neq 0$  and  $W_j$  pairwise distinct integral closed subschemes of  $\delta$ -dimension  $k + r$ . Then the family  $\{W_j\}$  is locally finite in  $X$ . For any quasi-compact open  $V \subset Y$  we see that  $f^{-1}(V) \cap W_j$  is nonempty only for finitely many  $j$ . Hence the collection  $Z_j = \overline{f(W_j)}$  of closures of images is a locally finite collection of integral closed subschemes of  $Y$ .

Consider the fibre product diagrams

$$\begin{array}{ccc} f^{-1}(Z_j) & \longrightarrow & X \\ f_j \downarrow & & \downarrow f \\ Z_j & \longrightarrow & Y \end{array}$$

Suppose that  $[W_j] \in Z_{k+r}(f^{-1}(Z_j))$  is rationally equivalent to  $f_j^* \beta_j$  for some  $k$ -cycle  $\beta_j \in A_k(Z_j)$ . Then  $\beta = \sum m_j \beta_j$  will be a  $k$ -cycle on  $Y$  and  $f^* \beta = \sum m_j f_j^* \beta_j$  will be rationally equivalent to  $\alpha$  (see Remark 19.4). This reduces us to the case  $Y$  integral, and  $\alpha = [W]$  for some integral closed subscheme of  $X$  dominating  $Y$ . In particular we may assume that  $d = \dim_\delta(Y) < \infty$ .

Hence we can use induction on  $d = \dim_\delta(Y)$ . If  $d < k$ , then  $A_{k+r}(X) = 0$  and the lemma holds. By assumption there exists a dense open  $V \subset Y$  such that  $f^{-1}(V) \cong V \times \mathbf{A}^r$  as schemes over  $V$ . Suppose that we can show that  $\alpha|_{f^{-1}(V)} = f^* \beta$  for



some  $\beta \in Z_k(V)$ . By Lemma 14.2 we see that  $\beta = \beta'|_V$  for some  $\beta' \in Z_k(Y)$ . By the exact sequence  $A_k(f^{-1}(Y \setminus V)) \rightarrow A_k(X) \rightarrow A_k(f^{-1}(V))$  of Lemma 19.2 we see that  $\alpha - f^*\beta'$  comes from a cycle  $\alpha' \in A_{k+r}(f^{-1}(Y \setminus V))$ . Since  $\dim_\delta(Y \setminus V) < d$  we win by induction on  $d$ .

Thus we may assume that  $X = Y \times \mathbf{A}^r$ . In this case we can factor  $f$  as

$$X = Y \times \mathbf{A}^r \rightarrow Y \times \mathbf{A}^{r-1} \rightarrow \dots \rightarrow Y \times \mathbf{A}^1 \rightarrow Y.$$

Hence it suffices to do the case  $r = 1$ . By the argument in the second paragraph of the proof we are reduced to the case  $\alpha = [W]$ ,  $Y$  integral, and  $W \rightarrow Y$  dominant. Again we can do induction on  $d = \dim_\delta(Y)$ . If  $W = Y \times \mathbf{A}^1$ , then  $[W] = f^*[Y]$ . Lastly,  $W \subset Y \times \mathbf{A}^1$  is a proper inclusion, then  $W \rightarrow Y$  induces a finite field extension  $R(Y) \subset R(W)$ . Let  $P(T) \in R(Y)[T]$  be the monic irreducible polynomial such that the generic fibre of  $W \rightarrow Y$  is cut out by  $P$  in  $\mathbf{A}_{R(Y)}^1$ . Let  $V \subset Y$  be a nonempty open such that  $P \in \Gamma(V, \mathcal{O}_Y)[T]$ , and such that  $W \cap f^{-1}(V)$  is still cut out by  $P$ . Then we see that  $\alpha|_{f^{-1}(V)} \sim_{rat} 0$  and hence  $\alpha \sim_{rat} \alpha'$  for some cycle  $\alpha'$  on  $(Y \setminus V) \times \mathbf{A}^1$ . By induction on the dimension we win.  $\square$

0B74 **Lemma 30.2.** *Let  $(S, \delta)$  be as in Situation 7.1. Let  $X$  be locally of finite type over  $S$ . Let  $\mathcal{L}$  be an invertible  $\mathcal{O}_X$ -module. Let*

$$p : L = \underline{\text{Spec}}(\text{Sym}^*(\mathcal{L})) \longrightarrow X$$

*be the associated vector bundle over  $X$ . Then  $p^* : A_k(X) \rightarrow A_{k+1}(L)$  is an isomorphism for all  $k$ .*

**Proof.** For surjectivity see Lemma 30.1. Let  $o : X \rightarrow L$  be the zero section of  $L \rightarrow X$ , i.e., the morphism corresponding to the surjection  $\text{Sym}^*(\mathcal{L}) \rightarrow \mathcal{O}_X$  which maps  $\mathcal{L}^{\otimes n}$  to zero for all  $n > 0$ . Then  $p \circ o = \text{id}_X$  and  $o(X)$  is an effective Cartier divisor on  $L$ . Hence by Lemma 29.1 we see that  $o^* \circ p^* = \text{id}$  and we conclude that  $p^*$  is injective too.  $\square$

02TU **Remark 30.3.** We will see later (Lemma 32.3) that if  $X$  is a vector bundle of rank  $r$  over  $Y$  then the pullback map  $A_k(Y) \rightarrow A_{k+r}(X)$  is an isomorphism. This is true whenever  $X \rightarrow Y$  satisfies the assumptions of Lemma 30.1, see [Tot14, Lemma 2.2].

### 31. Bivariant intersection theory

0B75 In order to intelligently talk about higher chern classes of vector bundles we introduce the following notion, following [FM81]. It follows from [Ful98, Theorem 17.1] that our definition agrees with that of [Ful98] modulo the caveat that we are working in different settings.

0B76 **Definition 31.1.** Let  $(S, \delta)$  be as in Situation 7.1. Let  $f : X \rightarrow Y$  be a morphism of schemes locally of finite type over  $S$ . Let  $p \in \mathbf{Z}$ . A *bivariant class  $c$  of degree  $p$  for  $f$*  is given by a rule which assigns to every locally of finite type morphism  $Y' \rightarrow Y$  and every  $k$  a map

Similar to [Ful98, Definition 17.1]

$$c \cap - : A_k(Y') \longrightarrow A_{k-p}(X')$$

where  $X' = Y' \times_Y X$ , satisfying the following conditions

- (1) if  $Y'' \rightarrow Y'$  is a proper, then  $c \cap (Y'' \rightarrow Y')_* \alpha'' = (X'' \rightarrow X')_*(c \cap \alpha'')$  for all  $\alpha''$  on  $Y''$  where  $X'' = Y'' \times_Y X$ ,

- (2) if  $Y'' \rightarrow Y'$  is flat locally of finite type of fixed relative dimension, then  $c \cap (Y'' \rightarrow Y')^* \alpha' = (X'' \rightarrow X')^*(c \cap \alpha')$  for all  $\alpha'$  on  $Y'$ , and
- (3) if  $(\mathcal{L}', s', i' : D' \rightarrow Y')$  is as in Definition 27.1 with pullback  $(\mathcal{N}', t', j' : E' \rightarrow X')$  to  $X'$ , then we have  $c \cap (i')^* \alpha' = (j')^*(c \cap \alpha')$  for all  $\alpha'$  on  $Y'$ .

The collection of all bivariant classes of degree  $p$  for  $f$  is denoted  $A^p(X \rightarrow Y)$ .

Let  $(S, \delta)$  be as in Situation 7.1. Let  $X \rightarrow Y$  and  $Y \rightarrow Z$  be morphisms of schemes locally of finite type over  $S$ . Let  $p \in \mathbf{Z}$ . It is clear that  $A^p(X \rightarrow Y)$  is an abelian group. Moreover, it is clear that we have a bilinear composition

$$A^p(X \rightarrow Y) \times A^q(Y \rightarrow Z) \rightarrow A^{p+q}(X \rightarrow Z)$$

which is associative. We will be most interested in  $A^p(X) = A^p(X \rightarrow X)$ , which will always mean the bivariant cohomology classes for  $\text{id}_X$ . Namely, that is where chern classes will live.

0B7E **Definition 31.2.** Let  $(S, \delta)$  be as in Situation 7.1. Let  $X$  be locally of finite type over  $S$ . The *Chow cohomology* of  $X$  is the graded  $\mathbf{Z}$ -algebra  $A^*(X)$  whose degree  $p$  component is  $A^p(X \rightarrow X)$ .

Warning: It is not clear that the  $\mathbf{Z}$ -algebra structure on  $A^*(X)$  is commutative, but we will see that chern classes live in its center.

0B7F **Remark 31.3.** Let  $(S, \delta)$  be as in Situation 7.1. Let  $f : X \rightarrow Y$  be a morphism of schemes locally of finite type over  $S$ . Then there is a canonical  $\mathbf{Z}$ -algebra map  $A^*(Y) \rightarrow A^*(X)$ . Namely, given  $c \in A^p(Y)$  and  $X' \rightarrow X$ , then we can let  $f^*c$  be defined by the map  $c \cap - : A_k(X') \rightarrow A_{k-p}(X')$  which is given by thinking of  $X'$  as a scheme over  $Y$ .

0B77 **Lemma 31.4.** Let  $(S, \delta)$  be as in Situation 7.1. Let  $X$  be locally of finite type over  $S$ . Let  $\mathcal{L}$  be an invertible  $\mathcal{O}_X$ -module. Then the rule that to  $f : X' \rightarrow X$  assigns  $c_1(f^*\mathcal{L}) \cap - : A_k(X') \rightarrow A_{k-1}(X')$  is a bivariant class of degree 1.

**Proof.** This follows from Lemmas 26.2, 24.4, 24.2, and 28.3.  $\square$

Having said this we see that we can define  $c_1(\mathcal{L})$  as the element of  $A^1(X)$  constructed in Lemma 31.4. We will return to this in Section 35.

0B78 **Lemma 31.5.** Let  $(S, \delta)$  be as in Situation 7.1. Let  $f : X \rightarrow Y$  be a flat morphism of relative dimension  $r$  between schemes locally of finite type over  $S$ . Then the rule that to  $Y' \rightarrow Y$  assigns  $(f')^* : A_k(Y') \rightarrow A_{k+r}(X')$  where  $X' = X \times_Y Y'$  is a bivariant class of degree  $-r$ .

**Proof.** This follows from Lemmas 20.2, 14.3, 15.1, and 27.8.  $\square$

0B79 **Lemma 31.6.** Let  $(S, \delta)$  be as in Situation 7.1. Let  $X$  be locally of finite type over  $S$ . Let  $(\mathcal{L}, s, i : D \rightarrow X)$  be a triple as in Definition 27.1. Then the rule that to  $f : X' \rightarrow X$  assigns  $(i')^* : A_k(X') \rightarrow A_{k-1}(D')$  where  $D' = D \times_X X'$  is a bivariant class of degree 1.

**Proof.** This follows from Lemmas 28.2, 27.7, 27.8, and 28.4.  $\square$

0EPK **Lemma 31.7.** Let  $(S, \delta)$  be as in Situation 7.1. Let  $f : X \rightarrow Y$  and  $g : Y \rightarrow Z$  be morphisms of schemes locally of finite type over  $S$ . Let  $c \in A^p(X \rightarrow Z)$  and assume  $f$  is proper. Then the rule that to  $Z' \rightarrow Z$  assigns  $\alpha \mapsto f'_*(c \cap \alpha)$  is a bivariant class of degree  $p$ .

**Proof.** This follows from Lemmas 12.2, 15.1, and 27.7.  $\square$

Here is a criterion to see that an operation passes through rational equivalence.

0B7A **Lemma 31.8.** *Let  $(S, \delta)$  be as in Situation 7.1. Let  $f : X \rightarrow Y$  be a morphism of schemes locally of finite type over  $S$ . Let  $p \in \mathbf{Z}$ . Suppose given a rule which assigns to every locally of finite type morphism  $Y' \rightarrow Y$  and every  $k$  a map* Very weak form of [Ful98, Theorem 17.1]

$$c \cap - : Z_k(Y') \longrightarrow A_{k-p}(X')$$

where  $Y' = X' \times_X Y$ , satisfying condition (3) of Definition 31.1 whenever  $\mathcal{L}'|_{D'} \cong \mathcal{O}_{D'}$ . Then  $c \cap -$  factors through rational equivalence.

**Proof.** The statement makes sense because given a triple  $(\mathcal{L}, s, i : D \rightarrow X)$  as in Definition 27.1 such that  $\mathcal{L}|_D \cong \mathcal{O}_D$ , then the operation  $i^*$  is defined on the level of cycles, see Remark 27.3. Let  $\alpha \in Z_k(X')$  be a cycle which is rationally equivalent to zero. We have to show that  $c \cap \alpha = 0$ . By Lemma 21.1 there exists a cycle  $\beta \in Z_{k+1}(X' \times \mathbf{P}^1)$  such that  $\alpha = i_0^* \beta - i_\infty^* \beta$  where  $i_0, i_\infty : X' \rightarrow X' \times \mathbf{P}^1$  are the closed immersions of  $X'$  over  $0, \infty$ . Since these are examples of effective Cartier divisors with trivial normal bundles, we see that  $c \cap i_0^* \beta = j_0^*(c \cap \beta)$  and  $c \cap i_\infty^* \beta = j_\infty^*(c \cap \beta)$  where  $j_0, j_\infty : Y' \rightarrow Y' \times \mathbf{P}^1$  are closed immersions as before. Since  $j_0^*(c \cap \beta) \sim_{\text{rat}} j_\infty^*(c \cap \beta)$  (follows from Lemma 21.1) we conclude.  $\square$

Here we see that  $c_1(\mathcal{L})$  is in the center of  $A^*(X)$ .

0B7B **Lemma 31.9.** *Let  $(S, \delta)$  be as in Situation 7.1. Let  $X$  be locally of finite type over  $S$ . Let  $\mathcal{L}$  be an invertible  $\mathcal{O}_X$ -module. Then  $c_1(\mathcal{L}) \in A^1(X)$  commutes with every element  $c \in A^p(X)$ .*

**Proof.** Let  $p : L \rightarrow X$  be as in Lemma 30.2 and let  $o : X \rightarrow L$  be the zero section. Observe that  $p^* \mathcal{L}^{\otimes -1}$  has a canonical section whose zero scheme is exactly the effective Cartier divisor  $o(X)$ . Let  $\alpha \in A_k(X)$ . Then we see that

$$p^*(c_1(\mathcal{L}^{\otimes -1}) \cap \alpha) = c_1(p^* \mathcal{L}^{\otimes -1}) \cap p^* \alpha = o_* o^* p^* \alpha$$

by Lemmas 24.2 and 29.1. Since  $c$  is a bivariant class we have

$$\begin{aligned} p^*(c \cap c_1(\mathcal{L}^{\otimes -1}) \cap \alpha) &= c \cap p^*(c_1(\mathcal{L}^{\otimes -1}) \cap \alpha) \\ &= c \cap o_* o^* p^* \alpha \\ &= o_* o^* p^*(c \cap \alpha) \\ &= p^*(c_1(\mathcal{L}^{\otimes -1}) \cap c \cap \alpha) \end{aligned}$$

(last equality by the above applied to  $c \cap \alpha$ ). Since  $p^*$  is injective by a lemma cited above we get that  $c_1(\mathcal{L}^{\otimes -1})$  is in the center of  $A^*(X)$ . This proves the lemma.  $\square$

Here is a criterion for when a bivariant class is zero.

02UC **Lemma 31.10.** *Let  $(S, \delta)$  be as in Situation 7.1. Let  $f : X \rightarrow Y$  be a morphism of schemes locally of finite type over  $S$ . Let  $c \in A^p(X \rightarrow Y)$ . For  $Y'' \rightarrow Y' \rightarrow Y$  set  $X'' = Y'' \times_Y X$  and  $X' = Y' \times_Y X$ . The following are equivalent*

- (1)  $c$  is zero,
- (2)  $c \cap [Y'] = 0$  in  $A_*(X')$  for every integral scheme  $Y'$  locally of finite type over  $Y$ , and
- (3) for every integral scheme  $Y'$  locally of finite type over  $Y$ , there exists a proper birational morphism  $Y'' \rightarrow Y'$  such that  $c \cap [Y''] = 0$  in  $A_*(X'')$ .

**Proof.** The implications (1)  $\Rightarrow$  (2)  $\Rightarrow$  (3) are clear. Assumption (3) implies (2) because  $(Y'' \rightarrow Y')_*[Y''] = [Y']$  and hence  $c \cap [Y'] = (X'' \rightarrow X')_*(c \cap [Y''])$  as  $c$  is a bivariant class. Assume (2). Let  $Y' \rightarrow Y$  be locally of finite type. Let  $\alpha \in A_k(Y')$ . Write  $\alpha = \sum n_i[Y'_i]$  with  $Y'_i \subset Y'$  a locally finite collection of integral closed subschemes of  $\delta$ -dimension  $k$ . Then we see that  $\alpha$  is pushforward of the cycle  $\alpha' = \sum n_i[Y'_i]$  on  $Y'' = \coprod Y'_i$  under the proper morphism  $Y'' \rightarrow Y'$ . By the properties of bivariant classes it suffices to prove that  $c \cap \alpha' = 0$  in  $A_{k-p}(X'')$ . We have  $A_{k-p}(X'') = \prod A_{k-p}(X'_i)$  where  $X'_i = Y'_i \times_Y X$ . This follows immediately from the definitions. The projection maps  $A_{k-p}(X'') \rightarrow A_{k-p}(X'_i)$  are given by flat pullback. Since capping with  $c$  commutes with flat pullback, we see that it suffices to show that  $c \cap [Y'_i]$  is zero in  $A_{k-p}(X'_i)$  which is true by assumption.  $\square$

### 32. Projective space bundle formula

02TV Let  $(S, \delta)$  be as in Situation 7.1. Let  $X$  be locally of finite type over  $S$ . Consider a finite locally free  $\mathcal{O}_X$ -module  $\mathcal{E}$  of rank  $r$ . Our convention is that the *projective bundle associated to  $\mathcal{E}$*  is the morphism

$$\mathbf{P}(\mathcal{E}) = \underline{\text{Proj}}_X(\text{Sym}^*(\mathcal{E})) \xrightarrow{\pi} X$$

over  $X$  with  $\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$  normalized so that  $\pi_*(\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)) = \mathcal{E}$ . In particular there is a surjection  $\pi^*\mathcal{E} \rightarrow \mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$ . We will say informally “let  $(\pi : P \rightarrow X, \mathcal{O}_P(1))$  be the projective bundle associated to  $\mathcal{E}$ ” to denote the situation where  $P = \mathbf{P}(\mathcal{E})$  and  $\mathcal{O}_P(1) = \mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$ .

02TW **Lemma 32.1.** *Let  $(S, \delta)$  be as in Situation 7.1. Let  $X$  be locally of finite type over  $S$ . Let  $\mathcal{E}$  be a finite locally free  $\mathcal{O}_X$ -module  $\mathcal{E}$  of rank  $r$ . Let  $(\pi : P \rightarrow X, \mathcal{O}_P(1))$  be the projective bundle associated to  $\mathcal{E}$ . For any  $\alpha \in A_k(X)$  the element*

$$\pi_*(c_1(\mathcal{O}_P(1))^s \cap \pi^*\alpha) \in A_{k+r-1-s}(X)$$

is 0 if  $s < r - 1$  and is equal to  $\alpha$  when  $s = r - 1$ .

**Proof.** Let  $Z \subset X$  be an integral closed subscheme of  $\delta$ -dimension  $k$ . Note that  $\pi^*[Z] = [\pi^{-1}(Z)]$  as  $\pi^{-1}(Z)$  is integral of  $\delta$ -dimension  $r - 1$ . If  $s < r - 1$ , then by construction  $c_1(\mathcal{O}_P(1))^s \cap \pi^*[Z]$  is represented by a  $(k + r - 1 - s)$ -cycle supported on  $\pi^{-1}(Z)$ . Hence the pushforward of this cycle is zero for dimension reasons.

Let  $s = r - 1$ . By the argument given above we see that  $\pi_*(c_1(\mathcal{O}_P(1))^s \cap \pi^*\alpha) = n[Z]$  for some  $n \in \mathbf{Z}$ . We want to show that  $n = 1$ . For the same dimension reasons as above it suffices to prove this result after replacing  $X$  by  $X \setminus T$  where  $T \subset Z$  is a proper closed subset. Let  $\xi$  be the generic point of  $Z$ . We can choose elements  $e_1, \dots, e_{r-1} \in \mathcal{E}_\xi$  which form part of a basis of  $\mathcal{E}_\xi$ . These give rational sections  $s_1, \dots, s_{r-1}$  of  $\mathcal{O}_P(1)|_{\pi^{-1}(Z)}$  whose common zero set is the closure of the image a rational section of  $\mathbf{P}(\mathcal{E}|_Z) \rightarrow Z$  union a closed subset whose support maps to a proper closed subset  $T$  of  $Z$ . After removing  $T$  from  $X$  (and correspondingly  $\pi^{-1}(T)$  from  $P$ ), we see that  $s_1, \dots, s_n$  form a sequence of global sections  $s_i \in \Gamma(\pi^{-1}(Z), \mathcal{O}_{\pi^{-1}(Z)}(1))$  whose common zero set is the image of a section

$Z \rightarrow \pi^{-1}(Z)$ . Hence we see successively that

$$\begin{aligned} \pi^*[Z] &= [\pi^{-1}(Z)] \\ c_1(\mathcal{O}_P(1)) \cap \pi^*[Z] &= [Z(s_1)] \\ c_1(\mathcal{O}_P(1))^2 \cap \pi^*[Z] &= [Z(s_1) \cap Z(s_2)] \\ &\dots = \dots \\ c_1(\mathcal{O}_P(1))^{r-1} \cap \pi^*[Z] &= [Z(s_1) \cap \dots \cap Z(s_{r-1})] \end{aligned}$$

by repeated applications of Lemma 23.4. Since the pushforward by  $\pi$  of the image of a section of  $\pi$  over  $Z$  is clearly  $[Z]$  we see the result when  $\alpha = [Z]$ . We omit the verification that these arguments imply the result for a general cycle  $\alpha = \sum n_j [Z_j]$ .  $\square$

02TX **Lemma 32.2** (Projective space bundle formula). *Let  $(S, \delta)$  be as in Situation 7.1. Let  $X$  be locally of finite type over  $S$ . Let  $\mathcal{E}$  be a finite locally free  $\mathcal{O}_X$ -module  $\mathcal{E}$  of rank  $r$ . Let  $(\pi : P \rightarrow X, \mathcal{O}_P(1))$  be the projective bundle associated to  $\mathcal{E}$ . The map*

$$\bigoplus_{i=0}^{r-1} A_{k+i}(X) \longrightarrow A_{k+r-1}(P),$$

$$(\alpha_0, \dots, \alpha_{r-1}) \longmapsto \pi^* \alpha_0 + c_1(\mathcal{O}_P(1)) \cap \pi^* \alpha_1 + \dots + c_1(\mathcal{O}_P(1))^{r-1} \cap \pi^* \alpha_{r-1}$$

is an isomorphism.

**Proof.** Fix  $k \in \mathbf{Z}$ . We first show the map is injective. Suppose that  $(\alpha_0, \dots, \alpha_{r-1})$  is an element of the left hand side that maps to zero. By Lemma 32.1 we see that

$$0 = \pi_*(\pi^* \alpha_0 + c_1(\mathcal{O}_P(1)) \cap \pi^* \alpha_1 + \dots + c_1(\mathcal{O}_P(1))^{r-1} \cap \pi^* \alpha_{r-1}) = \alpha_{r-1}$$

Next, we see that

$$0 = \pi_*(c_1(\mathcal{O}_P(1)) \cap (\pi^* \alpha_0 + c_1(\mathcal{O}_P(1)) \cap \pi^* \alpha_1 + \dots + c_1(\mathcal{O}_P(1))^{r-2} \cap \pi^* \alpha_{r-2})) = \alpha_{r-2}$$

and so on. Hence the map is injective.

It remains to show the map is surjective. Let  $X_i$ ,  $i \in I$  be the irreducible components of  $X$ . Then  $P_i = \mathbf{P}(\mathcal{E}|_{X_i})$ ,  $i \in I$  are the irreducible components of  $P$ . Consider the commutative diagram

$$\begin{array}{ccc} \coprod P_i & \xrightarrow{p} & P \\ \prod \pi_i \downarrow & & \downarrow \pi \\ \coprod X_i & \xrightarrow{q} & X \end{array}$$

Observe that  $p_*$  is surjective. If  $\beta \in A_k(\coprod X_i)$  then  $\pi^* q_* \beta = p_*(\prod \pi_i)^* \beta$ , see Lemma 15.1. Similarly for capping with  $c_1(\mathcal{O}(1))$  by Lemma 24.4. Hence, if the map of the lemma is surjective for each of the morphisms  $\pi_i : P_i \rightarrow X_i$ , then the map is surjective for  $\pi : P \rightarrow X$ . Hence we may assume  $X$  is irreducible. Thus  $\dim_\delta(X) < \infty$  and in particular we may use induction on  $\dim_\delta(X)$ .

The result is clear if  $\dim_\delta(X) < k$ . Let  $\alpha \in A_{k+r-1}(P)$ . For any locally closed subscheme  $T \subset X$  denote  $\gamma_T : \bigoplus A_{k+i}(T) \rightarrow A_{k+r-1}(\pi^{-1}(T))$  the map

$$\gamma_T(\alpha_0, \dots, \alpha_{r-1}) = \pi^* \alpha_0 + \dots + c_1(\mathcal{O}_{\pi^{-1}(T)}(1))^{r-1} \cap \pi^* \alpha_{r-1}.$$

Suppose for some nonempty open  $U \subset X$  we have  $\alpha|_{\pi^{-1}(U)} = \gamma_U(\alpha_0, \dots, \alpha_{r-1})$ . Then we may choose lifts  $\alpha'_i \in A_{k+i}(X)$  and we see that  $\alpha - \gamma_X(\alpha'_0, \dots, \alpha'_{r-1})$  is by Lemma 19.2 rationally equivalent to a  $k$ -cycle on  $P_Y = \mathbf{P}(\mathcal{E}|_Y)$  where  $Y = X \setminus U$

as a reduced closed subscheme. Note that  $\dim_\delta(Y) < \dim_\delta(X)$ . By induction the result holds for  $P_Y \rightarrow Y$  and hence the result holds for  $\alpha$ . Hence we may replace  $X$  by any nonempty open of  $X$ .

In particular we may assume that  $\mathcal{E} \cong \mathcal{O}_X^{\oplus r}$ . In this case  $\mathbf{P}(\mathcal{E}) = X \times \mathbf{P}^{r-1}$ . Let us use the stratification

$$\mathbf{P}^{r-1} = \mathbf{A}^{r-1} \amalg \mathbf{A}^{r-2} \amalg \dots \amalg \mathbf{A}^0$$

The closure of each stratum is a  $\mathbf{P}^{r-1-i}$  which is a representative of  $c_1(\mathcal{O}(1))^i \cap [\mathbf{P}^{r-1}]$ . Hence  $P$  has a similar stratification

$$P = U^{r-1} \amalg U^{r-2} \amalg \dots \amalg U^0$$

Let  $P^i$  be the closure of  $U^i$ . Let  $\pi^i : P^i \rightarrow X$  be the restriction of  $\pi$  to  $P^i$ . Let  $\alpha \in A_{k+r-1}(P)$ . By Lemma 30.1 we can write  $\alpha|_{U^{r-1}} = \pi^* \alpha_0|_{U^{r-1}}$  for some  $\alpha_0 \in A_k(X)$ . Hence the difference  $\alpha - \pi^* \alpha_0$  is the image of some  $\alpha' \in A_{k+r-1}(P^{r-2})$ . By Lemma 30.1 again we can write  $\alpha'|_{U^{r-2}} = (\pi^{r-2})^* \alpha_1|_{U^{r-2}}$  for some  $\alpha_1 \in A_{k+1}(X)$ . By Lemma 29.1 we see that the image of  $(\pi^{r-2})^* \alpha_1$  represents  $c_1(\mathcal{O}_P(1)) \cap \pi^* \alpha_1$ . We also see that  $\alpha - \pi^* \alpha_0 - c_1(\mathcal{O}_P(1)) \cap \pi^* \alpha_1$  is the image of some  $\alpha'' \in A_{k+r-1}(P^{r-3})$ . And so on.  $\square$

02TY **Lemma 32.3.** *Let  $(S, \delta)$  be as in Situation 7.1. Let  $X$  be locally of finite type over  $S$ . Let  $\mathcal{E}$  be a finite locally free sheaf of rank  $r$  on  $X$ . Let*

$$p : E = \underline{\text{Spec}}(\text{Sym}^*(\mathcal{E})) \longrightarrow X$$

*be the associated vector bundle over  $X$ . Then  $p^* : A_k(X) \rightarrow A_{k+r}(E)$  is an isomorphism for all  $k$ .*

**Proof.** (For the case of linebundles, see Lemma 30.2.) For surjectivity see Lemma 30.1. Let  $(\pi : P \rightarrow X, \mathcal{O}_P(1))$  be the projective space bundle associated to the finite locally free sheaf  $\mathcal{E} \oplus \mathcal{O}_X$ . Let  $s \in \Gamma(P, \mathcal{O}_P(1))$  correspond to the global section  $(0, 1) \in \Gamma(X, \mathcal{E} \oplus \mathcal{O}_X)$ . Let  $D = Z(s) \subset P$ . Note that  $(\pi|_D : D \rightarrow X, \mathcal{O}_P(1)|_D)$  is the projective space bundle associated to  $\mathcal{E}$ . We denote  $\pi_D = \pi|_D$  and  $\mathcal{O}_D(1) = \mathcal{O}_P(1)|_D$ . Moreover,  $D$  is an effective Cartier divisor on  $P$ . Hence  $\mathcal{O}_P(D) = \mathcal{O}_P(1)$  (see Divisors, Lemma 14.10). Also there is an isomorphism  $E \cong P \setminus D$ . Denote  $j : E \rightarrow P$  the corresponding open immersion. For injectivity we use that the kernel of

$$j^* : A_{k+r}(P) \longrightarrow A_{k+r}(E)$$

are the cycles supported in the effective Cartier divisor  $D$ , see Lemma 19.2. So if  $p^* \alpha = 0$ , then  $\pi^* \alpha = i_* \beta$  for some  $\beta \in A_{k+r}(D)$ . By Lemma 32.2 we may write

$$\beta = \pi_D^* \beta_0 + \dots + c_1(\mathcal{O}_D(1))^{r-1} \cap \pi_D^* \beta_{r-1}.$$

for some  $\beta_i \in A_{k+i}(X)$ . By Lemmas 29.1 and 24.4 this implies

$$\pi^* \alpha = i_* \beta = c_1(\mathcal{O}_P(1)) \cap \pi^* \beta_0 + \dots + c_1(\mathcal{O}_D(1))^r \cap \pi^* \beta_{r-1}.$$

Since the rank of  $\mathcal{E} \oplus \mathcal{O}_X$  is  $r + 1$  this contradicts Lemma 24.4 unless all  $\alpha$  and all  $\beta_i$  are zero.  $\square$

### 33. The Chern classes of a vector bundle

02TZ We can use the projective space bundle formula to define the chern classes of a rank  $r$  vector bundle in terms of the expansion of  $c_1(\mathcal{O}(1))^r$  in terms of the lower powers, see formula (33.1.1). The reason for the signs will be explained later.

02U0 **Definition 33.1.** Let  $(S, \delta)$  be as in Situation 7.1. Let  $X$  be locally of finite type over  $S$ . Assume  $X$  is integral and  $n = \dim_\delta(X)$ . Let  $\mathcal{E}$  be a finite locally free sheaf of rank  $r$  on  $X$ . Let  $(\pi : P \rightarrow X, \mathcal{O}_P(1))$  be the projective space bundle associated to  $\mathcal{E}$ .

(1) By Lemma 32.2 there are elements  $c_i \in A_{n-i}(X)$ ,  $i = 0, \dots, r$  such that  $c_0 = [X]$ , and

02U1 (33.1.1) 
$$\sum_{i=0}^r (-1)^i c_1(\mathcal{O}_P(1))^i \cap \pi^* c_{r-i} = 0.$$

(2) With notation as above we set  $c_i(\mathcal{E}) \cap [X] = c_i$  as an element of  $A_{n-i}(X)$ . We call these the *chern classes of  $\mathcal{E}$  on  $X$* .

(3) The *total chern class of  $\mathcal{E}$  on  $X$*  is the combination

$$c(\mathcal{E}) \cap [X] = c_0(\mathcal{E}) \cap [X] + c_1(\mathcal{E}) \cap [X] + \dots + c_r(\mathcal{E}) \cap [X]$$

which is an element of  $A_*(X) = \bigoplus_{k \in \mathbf{Z}} A_k(X)$ .

Let us check that this does not give a new notion in case the vector bundle has rank 1.

02U2 **Lemma 33.2.** *Let  $(S, \delta)$  be as in Situation 7.1. Let  $X$  be locally of finite type over  $S$ . Assume  $X$  is integral and  $n = \dim_\delta(X)$ . Let  $\mathcal{L}$  be an invertible  $\mathcal{O}_X$ -module. The first chern class of  $\mathcal{L}$  on  $X$  of Definition 33.1 is equal to the Weil divisor associated to  $\mathcal{L}$  by Definition 22.1.*

**Proof.** In this proof we use  $c_1(\mathcal{L}) \cap [X]$  to denote the construction of Definition 22.1. Since  $\mathcal{L}$  has rank 1 we have  $\mathbf{P}(\mathcal{L}) = X$  and  $\mathcal{O}_{\mathbf{P}(\mathcal{L})}(1) = \mathcal{L}$  by our normalizations. Hence (33.1.1) reads

$$(-1)^1 c_1(\mathcal{L}) \cap c_0 + (-1)^0 c_1 = 0$$

Since  $c_0 = [X]$ , we conclude  $c_1 = c_1(\mathcal{L}) \cap [X]$  as desired.  $\square$

02U3 **Remark 33.3.** We could also rewrite equation 33.1.1 as

05M8 (33.3.1) 
$$\sum_{i=0}^r c_1(\mathcal{O}_P(-1))^i \cap \pi^* c_{r-i} = 0.$$

but we find it easier to work with the tautological quotient sheaf  $\mathcal{O}_P(1)$  instead of its dual.

### 34. Intersecting with chern classes

02U4 In this section we define chern classes of vector bundles on  $X$  as bivariant classes on  $X$ , see Lemma 34.7 and the discussion following this lemma. Our construction follows the familiar pattern of first defining the operation on prime cycles and then summing. In Lemma 34.2 we show that the result is determined by the usual formula on the associated projective bundle. Next, we show that capping with chern classes passes through rational equivalence, commutes with proper pushforward, commutes with flat pullback, and commutes with the gysin maps for inclusions of effective Cartier divisors. These lemmas could have been avoided by directly using

the characterization in Lemma 34.2 and using Lemma 31.7; the reader who wishes to see this worked out should consult Chow Groups of Spaces, Lemma 28.1.

02U5 **Definition 34.1.** Let  $(S, \delta)$  be as in Situation 7.1. Let  $X$  be locally of finite type over  $S$ . Let  $\mathcal{E}$  be a finite locally free sheaf of rank  $r$  on  $X$ . We define, for every integer  $k$  and any  $0 \leq j \leq r$ , an operation

$$c_j(\mathcal{E}) \cap - : Z_k(X) \rightarrow A_{k-j}(X)$$

called *intersection with the  $j$ th chern class of  $\mathcal{E}$* .

(1) Given an integral closed subscheme  $i : W \rightarrow X$  of  $\delta$ -dimension  $k$  we define

$$c_j(\mathcal{E}) \cap [W] = i_*(c_j(i^*\mathcal{E}) \cap [W]) \in A_{k-j}(X)$$

where  $c_j(i^*\mathcal{E}) \cap [W]$  is as defined in Definition 33.1.

(2) For a general  $k$ -cycle  $\alpha = \sum n_i [W_i]$  we set

$$c_j(\mathcal{E}) \cap \alpha = \sum n_i c_j(\mathcal{E}) \cap [W_i]$$

Again, if  $\mathcal{E}$  has rank 1 then this agrees with our previous definition.

02U6 **Lemma 34.2.** Let  $(S, \delta)$  be as in Situation 7.1. Let  $X$  be locally of finite type over  $S$ . Let  $\mathcal{E}$  be a finite locally free sheaf of rank  $r$  on  $X$ . Let  $(\pi : P \rightarrow X, \mathcal{O}_P(1))$  be the projective bundle associated to  $\mathcal{E}$ . For  $\alpha \in Z_k(X)$  the elements  $c_j(\mathcal{E}) \cap \alpha$  are the unique elements  $\alpha_j$  of  $A_{k-j}(X)$  such that  $\alpha_0 = \alpha$  and

$$\sum_{i=0}^r (-1)^i c_1(\mathcal{O}_P(1))^i \cap \pi^*(\alpha_{r-i}) = 0$$

holds in the Chow group of  $P$ .

**Proof.** The uniqueness of  $\alpha_0, \dots, \alpha_r$  such that  $\alpha_0 = \alpha$  and such that the displayed equation holds follows from the projective space bundle formula Lemma 32.2. The identity holds by definition for  $\alpha = [W]$  where  $W$  is an integral closed subscheme of  $X$ . For a general  $k$ -cycle  $\alpha$  on  $X$  write  $\alpha = \sum n_a [W_a]$  with  $n_a \neq 0$ , and  $i_a : W_a \rightarrow X$  pairwise distinct integral closed subschemes. Then the family  $\{W_a\}$  is locally finite on  $X$ . Set  $P_a = \pi^{-1}(W_a) = \mathbf{P}(\mathcal{E}|_{W_a})$ . Denote  $i'_a : P_a \rightarrow P$  the corresponding closed immersions. Consider the fibre product diagram

$$\begin{array}{ccc} P' & \xlongequal{\quad} & \coprod P_a \xrightarrow{\quad} P \\ \pi' \downarrow & & \coprod i'_a \downarrow \\ X' & \xlongequal{\quad} & \coprod W_a \xrightarrow{\quad} X \end{array}$$

The morphism  $p : X' \rightarrow X$  is proper. Moreover  $\pi' : P' \rightarrow X'$  together with the invertible sheaf  $\mathcal{O}_{P'}(1) = \coprod \mathcal{O}_{P_a}(1)$  which is also the pullback of  $\mathcal{O}_P(1)$  is the projective bundle associated to  $\mathcal{E}' = p^*\mathcal{E}$ . By definition

$$c_j(\mathcal{E}) \cap [\alpha] = \sum i_{a,*}(c_j(\mathcal{E}|_{W_a}) \cap [W_a]).$$

Write  $\beta_{a,j} = c_j(\mathcal{E}|_{W_a}) \cap [W_a]$  which is an element of  $A_{k-j}(W_a)$ . We have

$$\sum_{i=0}^r (-1)^i c_1(\mathcal{O}_{P_a}(1))^i \cap \pi_a^*(\beta_{a,r-i}) = 0$$

for each  $a$  by definition. Thus clearly we have

$$\sum_{i=0}^r (-1)^i c_1(\mathcal{O}_{P'}(1))^i \cap (\pi')^*(\beta_{r-i}) = 0$$



with  $\beta_j = \sum n_a \beta_{a,j} \in A_{k-j}(X')$ . Denote  $p' : P' \rightarrow P$  the morphism  $\coprod i'_a$ . We have  $\pi^* p_* \beta_j = p'_*(\pi')^* \beta_j$  by Lemma 15.1. By the projection formula of Lemma 24.4 we conclude that

$$\sum_{i=0}^r (-1)^i c_1(\mathcal{O}_P(1))^i \cap \pi^*(p_* \beta_j) = 0$$

Since  $p_* \beta_j$  is a representative of  $c_j(\mathcal{E}) \cap \alpha$  we win.  $\square$

We will consistently use this characterization of chern classes to prove many more properties.

02U7 **Lemma 34.3.** *Let  $(S, \delta)$  be as in Situation 7.1. Let  $X$  be locally of finite type over  $S$ . Let  $\mathcal{E}$  be a finite locally free sheaf of rank  $r$  on  $X$ . If  $\alpha \sim_{\text{rat}} \beta$  are rationally equivalent  $k$ -cycles on  $X$  then  $c_j(\mathcal{E}) \cap \alpha = c_j(\mathcal{E}) \cap \beta$  in  $A_{k-j}(X)$ .*

**Proof.** By Lemma 34.2 the elements  $\alpha_j = c_j(\mathcal{E}) \cap \alpha$ ,  $j \geq 1$  and  $\beta_j = c_j(\mathcal{E}) \cap \beta$ ,  $j \geq 1$  are uniquely determined by the *same* equation in the chow group of the projective bundle associated to  $\mathcal{E}$ . (This of course relies on the fact that flat pullback is compatible with rational equivalence, see Lemma 20.2.) Hence they are equal.  $\square$

In other words capping with chern classes of finite locally free sheaves factors through rational equivalence to give maps

$$c_j(\mathcal{E}) \cap - : A_k(X) \rightarrow A_{k-j}(X).$$

Our next task is to show that chern classes are bivariant classes, see Definition 31.1.

02U9 **Lemma 34.4.** *Let  $(S, \delta)$  be as in Situation 7.1. Let  $X, Y$  be locally of finite type over  $S$ . Let  $\mathcal{E}$  be a finite locally free sheaf of rank  $r$  on  $X$ . Let  $p : X \rightarrow Y$  be a proper morphism. Let  $\alpha$  be a  $k$ -cycle on  $X$ . Let  $\mathcal{E}$  be a finite locally free sheaf on  $Y$ . Then*

$$p_*(c_j(p^* \mathcal{E}) \cap \alpha) = c_j(\mathcal{E}) \cap p_* \alpha$$

**Proof.** Let  $(\pi : P \rightarrow Y, \mathcal{O}_P(1))$  be the projective bundle associated to  $\mathcal{E}$ . Then  $P_X = X \times_Y P$  is the projective bundle associated to  $p^* \mathcal{E}$  and  $\mathcal{O}_{P_X}(1)$  is the pullback of  $\mathcal{O}_P(1)$ . Write  $\alpha_j = c_j(p^* \mathcal{E}) \cap \alpha$ , so  $\alpha_0 = \alpha$ . By Lemma 34.2 we have

$$\sum_{i=0}^r (-1)^i c_1(\mathcal{O}_P(1))^i \cap \pi_X^*(\alpha_{r-i}) = 0$$

in the chow group of  $P_X$ . Consider the fibre product diagram

$$\begin{array}{ccc} P_X & \xrightarrow{p'} & P \\ \pi_X \downarrow & & \downarrow \pi \\ X & \xrightarrow{p} & Y \end{array}$$

Apply proper pushforward  $p'_*$  (Lemma 20.3) to the displayed equality above. Using Lemmas 24.4 and 15.1 we obtain

$$\sum_{i=0}^r (-1)^i c_1(\mathcal{O}_P(1))^i \cap \pi^*(p_* \alpha_{r-i}) = 0$$

in the chow group of  $P$ . By the characterization of Lemma 34.2 we conclude.  $\square$

02U8 **Lemma 34.5.** *Let  $(S, \delta)$  be as in Situation 7.1. Let  $X, Y$  be locally of finite type over  $S$ . Let  $\mathcal{E}$  be a finite locally free sheaf of rank  $r$  on  $Y$ . Let  $f : X \rightarrow Y$  be a flat morphism of relative dimension  $r$ . Let  $\alpha$  be a  $k$ -cycle on  $Y$ . Then*

$$f^*(c_j(\mathcal{E}) \cap \alpha) = c_j(f^* \mathcal{E}) \cap f^* \alpha$$

**Proof.** Write  $\alpha_j = c_j(\mathcal{E}) \cap \alpha$ , so  $\alpha_0 = \alpha$ . By Lemma 34.2 we have

$$\sum_{i=0}^r (-1)^i c_1(\mathcal{O}_P(1))^i \cap \pi^*(\alpha_{r-i}) = 0$$

in the chow group of the projective bundle  $(\pi : P \rightarrow Y, \mathcal{O}_P(1))$  associated to  $\mathcal{E}$ . Consider the fibre product diagram

$$\begin{array}{ccc} P_X = \mathbf{P}(f^*\mathcal{E}) & \longrightarrow & P \\ \pi_X \downarrow & & \downarrow \pi \\ X & \xrightarrow{f} & Y \end{array}$$

Note that  $\mathcal{O}_{P_X}(1)$  is the pullback of  $\mathcal{O}_P(1)$ . Apply flat pullback  $(f')^*$  (Lemma 20.2) to the displayed equation above. By Lemmas 24.2 and 14.3 we see that

$$\sum_{i=0}^r (-1)^i c_1(\mathcal{O}_{P_X}(1))^i \cap \pi_X^*(f^*\alpha_{r-i}) = 0$$

holds in the chow group of  $P_X$ . By the characterization of Lemma 34.2 we conclude.  $\square$

0B7G **Lemma 34.6.** *Let  $(S, \delta)$  be as in Situation 7.1. Let  $X$  be locally of finite type over  $S$ . Let  $\mathcal{E}$  be a finite locally free sheaf of rank  $r$  on  $X$ . Let  $(\mathcal{L}, s, i : D \rightarrow X)$  be as in Definition 27.1. Then  $c_j(\mathcal{E}|_D) \cap i^*\alpha = i^*(c_j(\mathcal{E}) \cap \alpha)$  for all  $\alpha \in A_k(X)$ .*

**Proof.** Write  $\alpha_j = c_j(\mathcal{E}) \cap \alpha$ , so  $\alpha_0 = \alpha$ . By Lemma 34.2 we have

$$\sum_{i=0}^r (-1)^i c_1(\mathcal{O}_P(1))^i \cap \pi^*(\alpha_{r-i}) = 0$$

in the chow group of the projective bundle  $(\pi : P \rightarrow X, \mathcal{O}_P(1))$  associated to  $\mathcal{E}$ . Consider the fibre product diagram

$$\begin{array}{ccc} P_D = \mathbf{P}(\mathcal{E}|_D) & \longrightarrow & P \\ \pi_D \downarrow & & \downarrow \pi \\ D & \xrightarrow{i} & X \end{array}$$

Note that  $\mathcal{O}_{P_D}(1)$  is the pullback of  $\mathcal{O}_P(1)$ . Apply the gysin map  $(i')^*$  (Lemma 28.2) to the displayed equation above. Applying Lemmas 28.3 and 27.8 we obtain

$$\sum_{i=0}^r (-1)^i c_1(\mathcal{O}_{P_D}(1))^i \cap \pi_D^*(i^*\alpha_{r-i}) = 0$$

in the chow group of  $P_D$ . By the characterization of Lemma 34.2 we conclude.  $\square$

At this point we have enough material to be able to prove that capping with chern classes defines a bivariant class.

0B7H **Lemma 34.7.** *Let  $(S, \delta)$  be as in Situation 7.1. Let  $X$  be locally of finite type over  $S$ . Let  $\mathcal{E}$  be a locally free  $\mathcal{O}_X$ -module of rank  $r$ . Let  $0 \leq p \leq r$ . Then the rule that to  $f : X' \rightarrow X$  assigns  $c_p(f^*\mathcal{E}) \cap - : A_k(X') \rightarrow A_{k-1}(X')$  is a bivariant class of degree  $p$ .*

**Proof.** Immediate from Lemmas 34.3, 34.4, 34.5, and 34.6 and Definition 31.1.  $\square$

Let  $(S, \delta)$  be as in Situation 7.1. Let  $X$  be locally of finite type over  $S$ . Let  $\mathcal{E}$  be a locally free  $\mathcal{O}_X$ -module of rank  $r$ . At this point we define the *chern classes* of  $\mathcal{E}$  to be the elements

$$c_j(\mathcal{E}) \in A^j(X)$$

constructed in Lemma 34.7. The *total chern class* of  $\mathcal{E}$  is the element

$$c(\mathcal{E}) = c_0(\mathcal{E}) + c_1(\mathcal{E}) + \dots + c_r(\mathcal{E}) \in A^*(X)$$

Next we see that chern classes are in the center of the bivariant Chow cohomology ring  $A^*(X)$ .

02UA **Lemma 34.8.** *Let  $(S, \delta)$  be as in Situation 7.1. Let  $X$  be locally of finite type over  $S$ . Let  $\mathcal{E}$  be a locally free  $\mathcal{O}_X$ -module of rank  $r$ . Then  $c_j(\mathcal{L}) \in A^j(X)$  commutes with every element  $c \in A^p(X)$ . In particular, if  $\mathcal{F}$  is a second locally free  $\mathcal{O}_X$ -module on  $X$  of rank  $s$ , then*

$$c_i(\mathcal{E}) \cap c_j(\mathcal{F}) \cap \alpha = c_j(\mathcal{F}) \cap c_i(\mathcal{E}) \cap \alpha$$

as elements of  $A_{k-i-j}(X)$  for all  $\alpha \in A_k(X)$ .

**Proof.** Let  $\alpha \in A_k(X)$ . Write  $\alpha_j = c_j(\mathcal{E}) \cap \alpha$ , so  $\alpha_0 = \alpha$ . By Lemma 34.2 we have

$$\sum_{i=0}^r (-1)^i c_1(\mathcal{O}_P(1))^i \cap \pi^*(\alpha_{r-i}) = 0$$

in the chow group of the projective bundle  $(\pi : P \rightarrow Y, \mathcal{O}_P(1))$  associated to  $\mathcal{E}$ . Applying  $c \cap -$  and using Lemma 31.9 and the properties of bivariant classes we obtain

$$\sum_{i=0}^r (-1)^i c_1(\mathcal{O}_P(1))^i \cap \pi^*(c \cap \alpha_{r-i}) = 0$$

in the Chow group of  $P$ . Hence we see that  $c \cap \alpha_j$  is equal to  $c_j(\mathcal{E}) \cap (c \cap \alpha)$  by the characterization of Lemma 34.2. This proves the lemma.  $\square$

0ESW **Remark 34.9.** Let  $(S, \delta)$  be as in Situation 7.1. Let  $X$  be locally of finite type over  $S$ . Let  $\mathcal{E}$  be a finite locally free  $\mathcal{O}_X$ -module. If the rank of  $\mathcal{E}$  is not constant then we can still define the chern classes of  $\mathcal{E}$ . Namely, in this case we can write

$$X = X_0 \amalg X_1 \amalg X_2 \amalg \dots$$

where  $X_r \subset X$  is the open and closed subspace where the rank of  $\mathcal{E}$  is  $r$ . If  $X' \rightarrow X$  is a morphism which is locally of finite type, then we obtain by pullback a corresponding decomposition of  $X'$  and we find that

$$A_*(X') = \prod_{r \geq 0} A_*(X'_r)$$

by our definitions. Then we simply define  $c_i(\mathcal{E})$  to be the bivariant class which preserves these direct product decompositions and acts by the already defined operations  $c_i(\mathcal{E}|_{X_r}) \cap -$  on the factors. Observe that in this setting it may happen that  $c_i(\mathcal{E})$  is nonzero for infinitely many  $i$ .

### 35. Polynomial relations among chern classes

02UB Let  $(S, \delta)$  be as in Situation 7.1. Let  $X$  be locally of finite type over  $S$ . Let  $\mathcal{E}_i$  be a finite collection of finite locally free sheaves on  $X$ . By Lemma 34.8 we see that the chern classes

$$c_j(\mathcal{E}_i) \in A^*(X)$$

generate a commutative (and even central)  $\mathbf{Z}$ -subalgebra of the Chow cohomology algebra  $A^*(X)$ . Thus we can say what it means for a polynomial in these chern classes to be zero, or for two polynomials to be the same. As an example, saying that  $c_1(\mathcal{E}_1)^5 + c_2(\mathcal{E}_2)c_3(\mathcal{E}_3) = 0$  means that the operations

$$A_k(Y) \longrightarrow A_{k-5}(Y), \quad \alpha \longmapsto c_1(\mathcal{E}_1)^5 \cap \alpha + c_2(\mathcal{E}_2) \cap c_3(\mathcal{E}_3) \cap \alpha$$

are zero for all morphisms  $f : Y \rightarrow X$  which are locally of finite type. By Lemma 31.10 this is equivalent to the requirement that given any morphism  $f : Y \rightarrow X$  where  $Y$  is an integral scheme locally of finite type over  $S$  the cycle

$$c_1(\mathcal{E}_1)^5 \cap [Y] + c_2(\mathcal{E}_2) \cap c_3(\mathcal{E}_3) \cap [Y]$$

is zero in  $A_{\dim(Y)-5}(Y)$ .

A specific example is the relation

$$c_1(\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{N}) = c_1(\mathcal{L}) + c_1(\mathcal{N})$$

proved in Lemma 23.2. More generally, here is what happens when we tensor an arbitrary locally free sheaf by an invertible sheaf.

02UD **Lemma 35.1.** *Let  $(S, \delta)$  be as in Situation 7.1. Let  $X$  be locally of finite type over  $S$ . Let  $\mathcal{E}$  be a finite locally free sheaf of rank  $r$  on  $X$ . Let  $\mathcal{L}$  be an invertible sheaf on  $X$ . Then we have*

02UE (35.1.1) 
$$c_i(\mathcal{E} \otimes \mathcal{L}) = \sum_{j=0}^i \binom{r-i+j}{j} c_{i-j}(\mathcal{E}) c_1(\mathcal{L})^j$$

in  $A^*(X)$ .

**Proof.** This should hold for any triple  $(X, \mathcal{E}, \mathcal{L})$ . In particular it should hold when  $X$  is integral and by Lemma 31.10 it is enough to prove it holds when capping with  $[X]$  for such  $X$ . Thus assume that  $X$  is integral. Let  $(\pi : P \rightarrow X, \mathcal{O}_P(1))$ , resp.  $(\pi' : P' \rightarrow X, \mathcal{O}_{P'}(1))$  be the projective space bundle associated to  $\mathcal{E}$ , resp.  $\mathcal{E} \otimes \mathcal{L}$ . Consider the canonical morphism

$$\begin{array}{ccc} P & \xrightarrow{g} & P' \\ & \searrow \pi & \swarrow \pi' \\ & X & \end{array}$$

see Constructions, Lemma 20.1. It has the property that  $g^* \mathcal{O}_{P'}(1) = \mathcal{O}_P(1) \otimes \pi^* \mathcal{L}$ . This means that we have

$$\sum_{i=0}^r (-1)^i (\xi + x)^i \cap \pi^* (c_{r-i}(\mathcal{E} \otimes \mathcal{L}) \cap [X]) = 0$$

in  $A_*(P)$ , where  $\xi$  represents  $c_1(\mathcal{O}_P(1))$  and  $x$  represents  $c_1(\pi^* \mathcal{L})$ . By simple algebra this is equivalent to

$$\sum_{i=0}^r (-1)^i \xi^i \left( \sum_{j=i}^r (-1)^{j-i} \binom{j}{i} x^{j-i} \cap \pi^* (c_{r-j}(\mathcal{E} \otimes \mathcal{L}) \cap [X]) \right) = 0$$

Comparing with Equation (33.1.1) it follows from this that

$$c_{r-i}(\mathcal{E}) \cap [X] = \sum_{j=i}^r \binom{j}{i} (-c_1(\mathcal{L}))^{j-i} \cap c_{r-j}(\mathcal{E} \otimes \mathcal{L}) \cap [X]$$

Reworking this (getting rid of minus signs, and renumbering) we get the desired relation.  $\square$

Some example cases of (35.1.1) are

$$c_1(\mathcal{E} \otimes \mathcal{L}) = c_1(\mathcal{E}) + rc_1(\mathcal{L})$$

$$c_2(\mathcal{E} \otimes \mathcal{L}) = c_2(\mathcal{E}) + (r-1)c_1(\mathcal{E})c_1(\mathcal{L}) + \binom{r}{2}c_1(\mathcal{L})^2$$

$$c_3(\mathcal{E} \otimes \mathcal{L}) = c_3(\mathcal{E}) + (r-2)c_2(\mathcal{E})c_1(\mathcal{L}) + \binom{r-1}{2}c_1(\mathcal{E})c_1(\mathcal{L})^2 + \binom{r}{3}c_1(\mathcal{L})^3$$

### 36. Additivity of chern classes

02UF All of the preliminary lemmas follow trivially from the final result.

02UG **Lemma 36.1.** *Let  $(S, \delta)$  be as in Situation 7.1. Let  $X$  be locally of finite type over  $S$ . Let  $\mathcal{E}, \mathcal{F}$  be finite locally free sheaves on  $X$  of ranks  $r, r-1$  which fit into a short exact sequence*

$$0 \rightarrow \mathcal{O}_X \rightarrow \mathcal{E} \rightarrow \mathcal{F} \rightarrow 0$$

Then we have

$$c_r(\mathcal{E}) = 0, \quad c_j(\mathcal{E}) = c_j(\mathcal{F}), \quad j = 0, \dots, r-1$$

in  $A^*(X)$ .

**Proof.** By Lemma 31.10 it suffices to show that if  $X$  is integral then  $c_j(\mathcal{E}) \cap [X] = c_j(\mathcal{F}) \cap [X]$ . Let  $(\pi : P \rightarrow X, \mathcal{O}_P(1))$ , resp.  $(\pi' : P' \rightarrow X, \mathcal{O}_{P'}(1))$  denote the projective space bundle associated to  $\mathcal{E}$ , resp.  $\mathcal{F}$ . The surjection  $\mathcal{E} \rightarrow \mathcal{F}$  gives rise to a closed immersion

$$i : P' \longrightarrow P$$

over  $X$ . Moreover, the element  $1 \in \Gamma(X, \mathcal{O}_X) \subset \Gamma(X, \mathcal{E})$  gives rise to a global section  $s \in \Gamma(P, \mathcal{O}_P(1))$  whose zero set is exactly  $P'$ . Hence  $P'$  is an effective Cartier divisor on  $P$  such that  $\mathcal{O}_P(P') \cong \mathcal{O}_P(1)$ . Hence we see that

$$c_1(\mathcal{O}_P(1)) \cap \pi^* \alpha = i_*((\pi')^* \alpha)$$

for any cycle class  $\alpha$  on  $X$  by Lemma 29.1. By Lemma 34.2 we see that  $\alpha_j = c_j(\mathcal{F}) \cap [X]$ ,  $j = 0, \dots, r-1$  satisfy

$$\sum_{j=0}^{r-1} (-1)^j c_1(\mathcal{O}_{P'}(1))^j \cap (\pi')^* \alpha_j = 0$$

Pushing this to  $P$  and using the remark above as well as Lemma 24.4 we get

$$\sum_{j=0}^{r-1} (-1)^j c_1(\mathcal{O}_P(1))^{j+1} \cap \pi^* \alpha_j = 0$$

By the uniqueness of Lemma 34.2 we conclude that  $c_r(\mathcal{E}) \cap [X] = 0$  and  $c_j(\mathcal{E}) \cap [X] = \alpha_j = c_j(\mathcal{F}) \cap [X]$  for  $j = 0, \dots, r-1$ . Hence the lemma holds.  $\square$

02UH **Lemma 36.2.** *Let  $(S, \delta)$  be as in Situation 7.1. Let  $X$  be locally of finite type over  $S$ . Let  $\mathcal{E}, \mathcal{F}$  be finite locally free sheaves on  $X$  of ranks  $r, r-1$  which fit into a short exact sequence*

$$0 \rightarrow \mathcal{L} \rightarrow \mathcal{E} \rightarrow \mathcal{F} \rightarrow 0$$

where  $\mathcal{L}$  is an invertible sheaf. Then

$$c(\mathcal{E}) = c(\mathcal{L})c(\mathcal{F})$$

in  $A^*(X)$ .

**Proof.** This relation really just says that  $c_i(\mathcal{E}) = c_i(\mathcal{F}) + c_1(\mathcal{L})c_{i-1}(\mathcal{F})$ . By Lemma 36.1 we have  $c_j(\mathcal{E} \otimes \mathcal{L}^{\otimes -1}) = c_j(\mathcal{E} \otimes \mathcal{L}^{\otimes -1})$  for  $j = 0, \dots, r$  (where we set  $c_r(\mathcal{F}) = 0$  by convention). Applying Lemma 35.1 we deduce

$$\sum_{j=0}^i \binom{r-i+j}{j} (-1)^j c_{i-j}(\mathcal{E}) c_1(\mathcal{L})^j = \sum_{j=0}^i \binom{r-1-i+j}{j} (-1)^j c_{i-j}(\mathcal{F}) c_1(\mathcal{L})^j$$

Setting  $c_i(\mathcal{E}) = c_i(\mathcal{F}) + c_1(\mathcal{L})c_{i-1}(\mathcal{F})$  gives a “solution” of this equation. The lemma follows if we show that this is the only possible solution. We omit the verification.  $\square$

02UI **Lemma 36.3.** *Let  $(S, \delta)$  be as in Situation 7.1. Let  $X$  be a scheme locally of finite type over  $S$ . Suppose that  $\mathcal{E}$  sits in an exact sequence*

$$0 \rightarrow \mathcal{E}_1 \rightarrow \mathcal{E} \rightarrow \mathcal{E}_2 \rightarrow 0$$

*of finite locally free sheaves  $\mathcal{E}_i$  of rank  $r_i$ . The total chern classes satisfy*

$$c(\mathcal{E}) = c(\mathcal{E}_1)c(\mathcal{E}_2)$$

*in  $A^*(X)$ .*

**Proof.** By Lemma 31.10 we may assume that  $X$  is integral and we have to show the identity when capping against  $[X]$ . By induction on  $r_1$ . The case  $r_1 = 1$  is Lemma 36.2. Assume  $r_1 > 1$ . Let  $(\pi : P \rightarrow X, \mathcal{O}_P(1))$  denote the projective space bundle associated to  $\mathcal{E}_1$ . Note that

- (1)  $\pi^* : A_*(X) \rightarrow A_*(P)$  is injective, and
- (2)  $\pi^*\mathcal{E}_1$  sits in a short exact sequence  $0 \rightarrow \mathcal{F} \rightarrow \pi^*\mathcal{E}_1 \rightarrow \mathcal{L} \rightarrow 0$  where  $\mathcal{L}$  is invertible.

The first assertion follows from the projective space bundle formula and the second follows from the definition of a projective space bundle. (In fact  $\mathcal{L} = \mathcal{O}_P(1)$ .) Let  $Q = \pi^*\mathcal{E}/\mathcal{F}$ , which sits in an exact sequence  $0 \rightarrow \mathcal{L} \rightarrow Q \rightarrow \pi^*\mathcal{E}_2 \rightarrow 0$ . By induction we have

$$\begin{aligned} c(\pi^*\mathcal{E}) \cap [P] &= c(\mathcal{F}) \cap c(\pi^*\mathcal{E}/\mathcal{F}) \cap [P] \\ &= c(\mathcal{F}) \cap c(\mathcal{L}) \cap c(\pi^*\mathcal{E}_2) \cap [P] \\ &= c(\pi^*\mathcal{E}_1) \cap c(\pi^*\mathcal{E}_2) \cap [P] \end{aligned}$$

Since  $[P] = \pi^*[X]$  we win by Lemma 34.5.  $\square$

02UJ **Lemma 36.4.** *Let  $(S, \delta)$  be as in Situation 7.1. Let  $X$  be locally of finite type over  $S$ . Let  $\mathcal{L}_i$ ,  $i = 1, \dots, r$  be invertible  $\mathcal{O}_X$ -modules on  $X$ . Let  $\mathcal{E}$  be a locally free rank  $\mathcal{O}_X$ -module endowed with a filtration*

$$0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \mathcal{E}_2 \subset \dots \subset \mathcal{E}_r = \mathcal{E}$$

*such that  $\mathcal{E}_i/\mathcal{E}_{i-1} \cong \mathcal{L}_i$ . Set  $c_1(\mathcal{L}_i) = x_i$ . Then*

$$c(\mathcal{E}) = \prod_{i=1}^r (1 + x_i)$$

*in  $A^*(X)$ .*

**Proof.** Apply Lemma 36.2 and induction.  $\square$

### 37. The splitting principle

02UK In our setting it is not so easy to say what the splitting principle exactly says/is. Here is a possible formulation.

02UL **Lemma 37.1.** *Let  $(S, \delta)$  be as in Situation 7.1. Let  $X$  be locally of finite type over  $S$ . Let  $\mathcal{E}_i$  be a finite collection of locally free  $\mathcal{O}_X$ -modules of rank  $r_i$ . There exists a projective flat morphism  $\pi : P \rightarrow X$  of relative dimension  $d$  such that*

- (1) *for any morphism  $f : Y \rightarrow X$  the map  $\pi_Y^* : A_*(Y) \rightarrow A_{*+d}(Y \times_X P)$  is injective, and*
- (2) *each  $\pi^*\mathcal{E}_i$  has a filtration whose successive quotients  $\mathcal{L}_{i,1}, \dots, \mathcal{L}_{i,r_i}$  are invertible  $\mathcal{O}_P$ -modules.*

**Proof.** Omitted. Hint: Use a composition of projective space bundles. □

Let  $(S, \delta)$ ,  $X$ , and  $\mathcal{E}_i$  be as in Lemma 37.1. The *splitting principle* refers to the practice of symbolically writing

$$c(\mathcal{E}_i) = \prod (1 + x_{i,j})$$

The symbols  $x_{i,1}, \dots, x_{i,r_i}$  are called the *Chern roots* of  $\mathcal{E}_i$ . We think of  $x_{i,j}$  as the first chern classes of some (unknown) invertible sheaves whose direct sum equals  $\mathcal{E}_i$ . The usefulness of the splitting principle comes from the assertion that in order to prove a polynomial relation among chern classes of the  $\mathcal{E}_i$  it is enough to prove the corresponding relation among the chern roots.

Namely, let  $\pi : P \rightarrow X$  be as in Lemma 37.1. Recall that there is a canonical  $\mathbf{Z}$ -algebra map  $\pi^* : A^*(X) \rightarrow A^*(P)$ , see Remark 31.3. The injectivity of  $\pi_Y^*$  on Chow groups for every  $Y$  over  $X$ , implies that the map  $\pi^* : A^*(X) \rightarrow A^*(P)$  is injective (details omitted). We have

$$\pi^*c(\mathcal{E}_i) = \prod (1 + c_1(\mathcal{L}_{i,j}))$$

by Lemma 36.4. Thus we may identify the chern roots  $x_{i,j}$  with  $c_1(\mathcal{L}_{i,j})$  at least after applying the injective map  $\pi^* : A^*(X) \rightarrow A^*(P)$ .

To see how this works, it is best to give an example. Let us calculate the chern classes of the dual  $\mathcal{E}^\vee$  of a locally free  $\mathcal{O}_X$ -module  $\mathcal{E}$  of rank  $r$ . Note that if  $\pi^*\mathcal{E}$  has a filtration with subquotients the invertible modules  $\mathcal{L}_1, \dots, \mathcal{L}_r$ , then  $\pi^*\mathcal{E}^\vee$  has a filtration with subquotients invertible sheaves  $\mathcal{L}_r^{-1}, \dots, \mathcal{L}_1^{\otimes -1}$ . Hence if  $x_i$  are the chern roots of  $\mathcal{E}$ , in other words, if  $x_i = c_1(\mathcal{L}_i)$ , then the  $-x_i$  are the chern roots of  $\mathcal{E}^\vee$  by Lemma 23.2. It follows that

$$\pi^*c(\mathcal{E}^\vee) = \prod (1 - x_i)$$

in  $A^*(P)$  and hence by elementary algebra that

$$c_j(\mathcal{E}^\vee) = (-1)^j c_j(\mathcal{E})$$

in  $A^*(X)$  by the injectivity above.

It should be said here that in any application of the splitting principle it is no longer necessary to choose an actual  $\pi : P \rightarrow X$  and to use the pullback map; it suffices to know that one exists. In a way this is an abuse of language, more than anything else. In the following paragraph we give an example.

Let us compute the chern classes of a tensor product of vector bundles. Namely, suppose that  $\mathcal{E}, \mathcal{F}$  are finite locally free of ranks  $r, s$ . Write

$$c(\mathcal{E}) = \prod_{i=1}^r (1 + x_i), \quad c(\mathcal{F}) = \prod_{j=1}^s (1 + y_j)$$

where  $x_i, y_j$  are the chern roots of  $\mathcal{E}, \mathcal{F}$ . Then we see that

$$c(\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F}) = \prod_{i,j} (1 + x_i + y_j)$$

because if  $\mathcal{E}$  is the direct sum of invertible sheaves  $\mathcal{L}_i$  and  $\mathcal{F}$  is the direct sum of invertible sheaves  $\mathcal{N}_j$ , then  $\mathcal{E} \otimes \mathcal{F}$  is the direct sum of the invertible sheaves  $\mathcal{L}_i \otimes \mathcal{N}_j$ . Here are some examples of what this means in terms of chern classes

$$c_1(\mathcal{E} \otimes \mathcal{F}) = rc_1(\mathcal{F}) + sc_1(\mathcal{E})$$

$$c_2(\mathcal{E} \otimes \mathcal{F}) = r^2c_2(\mathcal{F}) + rsc_1(\mathcal{F})c_1(\mathcal{E}) + s^2c_2(\mathcal{E})$$

### 38. The Chern character and tensor products

02UM We define the *Chern character* of a finite locally free sheaf of rank  $r$  to be the formal expression

$$ch(\mathcal{E}) = \sum_{i=1}^r e^{x_i}$$

if the  $x_i$  are the chern roots of  $\mathcal{E}$ . Writing this in terms of chern classes  $c_i = c_i(\mathcal{E})$  we see that

$$ch(\mathcal{E}) = r + c_1 + \frac{1}{2}(c_1^2 - 2c_2) + \frac{1}{6}(c_1^3 - 3c_1c_2 + 3c_3) + \frac{1}{24}(c_1^4 - 4c_1^2c_2 + 4c_1c_3 + 2c_2^2 - 4c_4) + \dots$$

What does it mean that the coefficients are rational numbers? Well this simply means that we think of  $ch_j(\mathcal{E})$  as an element of  $A^j(X) \otimes \mathbf{Q}$ . By the above we have in case of an exact sequence

$$0 \rightarrow \mathcal{E}_1 \rightarrow \mathcal{E} \rightarrow \mathcal{E}_2 \rightarrow 0$$

that

$$ch(\mathcal{E}) = ch(\mathcal{E}_1) + ch(\mathcal{E}_2)$$

in  $A^*(X) \otimes \mathbf{Q}$ . Using the Chern character we can express the compatibility of the chern classes and tensor product as follows:

$$ch(\mathcal{E}_1 \otimes_{\mathcal{O}_X} \mathcal{E}_2) = ch(\mathcal{E}_1)ch(\mathcal{E}_2)$$

in  $A^*(X) \otimes \mathbf{Q}$ . This follows directly from the discussion of the chern roots of the tensor product in the previous section.

0ESX **Remark 38.1.** Let  $(S, \delta)$  be as in Situation 7.1. Let  $X$  be locally of finite type over  $S$ . Let  $\mathcal{E}$  be a finite locally free  $\mathcal{O}_X$ -module. If the rank of  $\mathcal{E}$  is not constant then we can still define the Chern character  $ch(\mathcal{E})$  of  $\mathcal{E}$ , exactly as in Remark 34.9. It is still the case that  $ch_i(\mathcal{E})$  is in  $A^i(X) \otimes \mathbf{Q}$  with denominator at worst  $i!$ .



### 39. Chern classes and the derived category

0ESY In this section we define the total chern class of an object of the derived category which may be represented globally by a finite complex of finite locally free modules.

Let  $(S, \delta)$  be as in Situation 7.1. Let  $X$  be locally of finite type over  $S$ . Let

$$\mathcal{E}^a \rightarrow \mathcal{E}^{a+1} \rightarrow \dots \rightarrow \mathcal{E}^b$$

be a finite complex of finite locally free  $\mathcal{O}_X$ -modules. Then we define the *total chern class of the complex* by the formula

$$c(\mathcal{E}^\bullet) = \prod_{p=a, \dots, b} c(\mathcal{E}^p)^{(-1)^p}$$

in  $A^*(X)$ . Here the inverse is the formal inverse, so

$$(1 + c_1 + c_2 + c_3 + \dots)^{-1} = 1 - c_1 + c_1^2 - c_2 - c_1^3 + 2c_1c_2 - c_3 + \dots$$

We similarly define the *Chern character of the complex* by the formula

$$ch(\mathcal{E}^\bullet) = \sum_{p=a, \dots, b} (-1)^p ch(\mathcal{E}^p)$$

in  $A^*(X) \otimes \mathbf{Q}$ . Let us prove that  $c(\mathcal{E}^\bullet)$  only depends on the image of the complex in the derived category.

0ESZ **Lemma 39.1.** *Let  $(S, \delta)$  be as in Situation 7.1. Let  $X$  be locally of finite type over  $S$ . Let  $E \in D(\mathcal{O}_X)$  be an object such that there exists a finite complex  $\mathcal{E}^\bullet$  of finite locally free  $\mathcal{O}_X$ -modules representing  $E$ . Then  $c(\mathcal{E}^\bullet) \in A^*(X)$  is independent of the choice of the complex. Similarly for  $ch(\mathcal{E}^\bullet)$ .*

**Proof.** Suppose we have a second finite complex  $\mathcal{F}^\bullet$  of finite locally free  $\mathcal{O}_X$ -modules representing  $E$ . Choose  $a \leq b$  such that  $\mathcal{F}^p$  and  $\mathcal{E}^p$  are zero for  $p \notin [a, b]$ . We will prove the lemma by induction on  $b - a$ . If  $b - a = 0$ , then we have  $\mathcal{F}^a \cong \mathcal{E}^a \cong E$  and the result is clear.

Induction step. Assume  $b > a$ . Let  $g : Y \rightarrow X$  be a morphism locally of finite type with  $Y$  integral. By Lemma 31.10 it suffices to show that with  $c(g^*\mathcal{E}^\bullet) \cap [Y]$  is the same as  $c(g^*\mathcal{F}^\bullet) \cap [Y]$  and it even suffices to prove this after replacing  $Y$  by an integral scheme proper and birational over  $Y$ . By More on Flatness, Lemma 38.4 we may assume that  $H^b(Lg^*E)$  is perfect of tor dimension  $\leq 1$ . This reduces us to the case discussed in the next paragraph.

Assume  $X$  is integral and  $H^b(E)$  is a perfect  $\mathcal{O}_X$ -module of tor dimension  $\leq 1$ . Let

$$\mathcal{G} = \text{Ker}(\mathcal{E}^b \oplus \mathcal{F}^b \rightarrow H^b(E))$$

Since  $H^b(E)$  has tor dimension  $\leq 1$  we see that  $\mathcal{G}$  is finite locally free. Then there is a commutative diagram

$$\begin{array}{ccc} \mathcal{G}[-b] & \xrightarrow{\alpha} & \mathcal{E}^\bullet \\ \downarrow \beta & & \downarrow \\ \mathcal{F}^\bullet & \longrightarrow & E \end{array}$$

in  $D(\mathcal{O}_X)$ . (Warning: you have to choose the negative of the canonical map for one of the arrows to make this diagram commute.) Choose a distinguished triangle

$$\mathcal{G}[-b] \rightarrow E \rightarrow E' \rightarrow \mathcal{G}[-b+1]$$

in  $D(\mathcal{O}_X)$ . On the other hand, the cone on  $\alpha : \mathcal{G}[-b] \rightarrow \mathcal{E}^\bullet$  gives a distinguished triangle

$$\mathcal{G}[-b] \rightarrow \mathcal{E}^\bullet \rightarrow C(\alpha) \rightarrow \mathcal{G}[-b+1]$$

and similarly for  $\mathcal{F}^\bullet$  and  $\beta$ . Since  $\mathcal{G} \rightarrow \mathcal{E}^b$  is surjective, it follows that  $C(\alpha)$  has vanishing cohomology in degree  $b$  and hence

$$\tau_{\leq b-1}C(\alpha) \longrightarrow C(\alpha)$$

is an isomorphism in  $D(\mathcal{O}_X)$ . On the other hand, the displayed arrow determines an isomorphism of complexes except in degrees  $b-1$  and  $b$  where we have

$$(\tau_{\leq b-1}C(\alpha))^{b-1} = \text{Ker}(C(\alpha)^{b-1} \rightarrow C(\alpha)^b) \quad \text{and} \quad (\tau_{\leq b-1}C(\alpha))^b = 0$$

Since  $C(\alpha)^{b-1} \rightarrow C(\alpha)^b$  is a surjection of finite locally free  $\mathcal{O}_X$ -modules, we conclude from multiplicativity of total chern classes (Lemma 36.3)

$$c(\tau_{\leq b-1}C(\alpha)) = c(C(\alpha))$$

and similarly for  $C(\beta)$ . By the axioms of a triangulated category we obtain an isomorphism  $C(\alpha) \rightarrow E'$  in  $D(\mathcal{O}_X)$  and similarly of  $\mathcal{F}^\bullet$ . By induction hypothesis we obtain

$$c(\tau_{\leq b-1}C(\alpha)) \cap [X] = c(\tau_{\leq b-1}C(\beta)) \cap [X]$$

We conclude that

$$c(\tau_{\leq b-1}C(\alpha)) \cap [X] = c(C(\alpha)) \cap [X] = c(\mathcal{E}^\bullet)c(\mathcal{G})^{(-1)^{b-1}} \cap [X]$$

and similarly for  $\mathcal{F}^\bullet$ . The second equality follows because the terms of  $C(\alpha)$  are identical to the terms of the complex  $\mathcal{E}^\bullet$ , except in degree  $b-1$  we've added  $\mathcal{G}$  (plus we use Lemma 36.3 again). We conclude that

$$c(\mathcal{E}^\bullet)c(\mathcal{G})^{(-1)^{b-1}} \cap [X] = c(\mathcal{F}^\bullet)c(\mathcal{G})^{(-1)^{b-1}} \cap [X]$$

and we win since multiplying by a total chern class is an invertible operation.  $\square$

#### 40. Todd classes

02UN A final class associated to a vector bundle  $\mathcal{E}$  of rank  $r$  is its *Todd class*  $Todd(\mathcal{E})$ . In terms of the chern roots  $x_1, \dots, x_r$  it is defined as

$$Todd(\mathcal{E}) = \prod_{i=1}^r \frac{x_i}{1 - e^{-x_i}}$$

In terms of the chern classes  $c_i = c_i(\mathcal{E})$  we have

$$Todd(\mathcal{E}) = 1 + \frac{1}{2}c_1 + \frac{1}{12}(c_1^2 + c_2) + \frac{1}{24}c_1c_2 + \frac{1}{720}(-c_1^4 + 4c_1^2c_2 + 3c_2^2 + c_1c_3 - c_4) + \dots$$

We have made the appropriate remarks about denominators in the previous section. It is the case that given an exact sequence

$$0 \rightarrow \mathcal{E}_1 \rightarrow \mathcal{E} \rightarrow \mathcal{E}_2 \rightarrow 0$$

we have

$$Todd(\mathcal{E}) = Todd(\mathcal{E}_1)Todd(\mathcal{E}_2).$$

### 41. Degrees of zero cycles

0AZ0 We start defining the degree of a zero cycle on a proper scheme over a field. One approach is to define it directly as in Lemma 41.2 and then show it is well defined by Lemma 18.3. Instead we define it as follows.

0AZ1 **Definition 41.1.** Let  $k$  be a field (Example 7.2). Let  $p : X \rightarrow \text{Spec}(k)$  be proper. The *degree of a zero cycle* on  $X$  is given by proper pushforward

$$p_* : A_0(X) \rightarrow A_0(\text{Spec}(k))$$

(Lemma 20.3) combined with the natural isomorphism  $A_0(\text{Spec}(k)) = \mathbf{Z}$  which maps  $[\text{Spec}(k)]$  to 1. Notation:  $\deg(\alpha)$ .

Let us spell this out further.

0AZ2 **Lemma 41.2.** Let  $k$  be a field. Let  $X$  be proper over  $k$ . Let  $\alpha = \sum n_i [Z_i]$  be in  $Z_0(X)$ . Then

$$\deg(\alpha) = \sum n_i \deg(Z_i)$$

where  $\deg(Z_i)$  is the degree of  $Z_i \rightarrow \text{Spec}(k)$ , i.e.,  $\deg(Z_i) = \dim_k \Gamma(Z_i, \mathcal{O}_{Z_i})$ .

**Proof.** This is the definition of proper pushforward (Definition 12.1). □

Next, we make the connection with degrees of vector bundles over 1-dimensional proper schemes over fields as defined in Varieties, Section 43.

0AZ3 **Lemma 41.3.** Let  $k$  be a field. Let  $X$  be a proper scheme over  $k$  of dimension  $\leq 1$ . Let  $\mathcal{E}$  be a finite locally free  $\mathcal{O}_X$ -module of constant rank. Then

$$\deg(\mathcal{E}) = \deg(c_1(\mathcal{E}) \cap [X]_1)$$

where the left hand side is defined in Varieties, Definition 43.1.

**Proof.** Let  $C_i \subset X$ ,  $i = 1, \dots, t$  be the irreducible components of dimension 1 with reduced induced scheme structure and let  $m_i$  be the multiplicity of  $C_i$  in  $X$ . Then  $[X]_1 = \sum m_i [C_i]$  and  $c_1(\mathcal{E}) \cap [X]_1$  is the sum of the pushforwards of the cycles  $m_i c_1(\mathcal{E}|_{C_i}) \cap [C_i]$ . Since we have a similar decomposition of the degree of  $\mathcal{E}$  by Varieties, Lemma 43.6 it suffices to prove the lemma in case  $X$  is a proper curve over  $k$ .

Assume  $X$  is a proper curve over  $k$ . By Divisors, Lemma 35.1 there exists a modification  $f : X' \rightarrow X$  such that  $f^* \mathcal{E}$  has a filtration whose successive quotients are invertible  $\mathcal{O}_{X'}$ -modules. Since  $f_* [X']_1 = [X]_1$  we conclude from Lemma 34.4 that

$$\deg(c_1(\mathcal{E}) \cap [X]_1) = \deg(c_1(f^* \mathcal{E}) \cap [X']_1)$$

Since we have a similar relationship for the degree by Varieties, Lemma 43.4 we reduce to the case where  $\mathcal{E}$  has a filtration whose successive quotients are invertible  $\mathcal{O}_X$ -modules. In this case, we may use additivity of the degree (Varieties, Lemma 43.3) and of first chern classes (Lemma 36.3) to reduce to the case discussed in the next paragraph.

Assume  $X$  is a proper curve over  $k$  and  $\mathcal{E}$  is an invertible  $\mathcal{O}_X$ -module. By Divisors, Lemma 15.12 we see that  $\mathcal{E}$  is isomorphic to  $\mathcal{O}_X(D) \otimes \mathcal{O}_X(D')^{\otimes -1}$  for some effective Cartier divisors  $D, D'$  on  $X$  (this also uses that  $X$  is projective, see Varieties, Lemma 42.4 for example). By additivity of degree under tensor product of invertible sheaves (Varieties, Lemma 43.7) and additivity of  $c_1$  under tensor product of

invertible sheaves (Lemma 23.2 or 35.1) we reduce to the case  $\mathcal{E} = \mathcal{O}_X(D)$ . In this case the left hand side gives  $\deg(D)$  (Varieties, Lemma 43.9) and the right hand side gives  $\deg([D]_0)$  by Lemma 23.4. Since

$$[D]_0 = \sum_{x \in D} \text{length}_{\mathcal{O}_{X,x}}(\mathcal{O}_{D,x})[x] = \sum_{x \in D} \text{length}_{\mathcal{O}_{D,x}}(\mathcal{O}_{D,x})[x]$$

by definition, we see

$$\deg([D]_0) = \sum_{x \in D} \text{length}_{\mathcal{O}_{D,x}}(\mathcal{O}_{D,x}[\kappa(x) : k]) = \dim_k \Gamma(D, \mathcal{O}_D) = \deg(D)$$

The penultimate equality by Algebra, Lemma 51.12 using that  $D$  is affine.  $\square$

Finally, we can tie everything up with the numerical intersections defined in Varieties, Section 44.

**0BFI Lemma 41.4.** *Let  $k$  be a field. Let  $X$  be a proper scheme over  $k$ . Let  $Z \subset X$  be a closed subscheme of dimension  $d$ . Let  $\mathcal{L}_1, \dots, \mathcal{L}_d$  be invertible  $\mathcal{O}_X$ -modules. Then*

$$(\mathcal{L}_1 \cdots \mathcal{L}_d \cdot Z) = \deg(c_1(\mathcal{L}_1) \cap \dots \cap c_1(\mathcal{L}_d) \cap [Z]_d)$$

where the left hand side is defined in Varieties, Definition 44.3. In particular,

$$\deg_{\mathcal{L}}(Z) = \deg(c_1(\mathcal{L})^d \cap [Z]_d)$$

if  $\mathcal{L}$  is an ample invertible  $\mathcal{O}_X$ -module.

**Proof.** We will prove this by induction on  $d$ . If  $d = 0$ , then the result is true by Varieties, Lemma 32.3. Assume  $d > 0$ .

Let  $Z_i \subset Z$ ,  $i = 1, \dots, t$  be the irreducible components of dimension  $d$  with reduced induced scheme structure and let  $m_i$  be the multiplicity of  $Z_i$  in  $Z$ . Then  $[Z]_d = \sum m_i [Z_i]$  and  $c_1(\mathcal{L}_1) \cap \dots \cap c_1(\mathcal{L}_d) \cap [Z]_d$  is the sum of the cycles  $m_i c_1(\mathcal{L}_1) \cap \dots \cap c_1(\mathcal{L}_d) \cap [Z_i]$ . Since we have a similar decomposition for  $(\mathcal{L}_1 \cdots \mathcal{L}_d \cdot Z)$  by Varieties, Lemma 44.2 it suffices to prove the lemma in case  $Z = X$  is a proper variety of dimension  $d$  over  $k$ .

By Chow's lemma there exists a birational proper morphism  $f : Y \rightarrow X$  with  $Y$  H-projective over  $k$ . See Cohomology of Schemes, Lemma 18.1 and Remark 18.2. Then

$$(f^* \mathcal{L}_1 \cdots f^* \mathcal{L}_d \cdot Y) = (\mathcal{L}_1 \cdots \mathcal{L}_d \cdot X)$$

by Varieties, Lemma 44.7 and we have

$$f_*(c_1(f^* \mathcal{L}_1) \cap \dots \cap c_1(f^* \mathcal{L}_d) \cap [Y]) = c_1(\mathcal{L}_1) \cap \dots \cap c_1(\mathcal{L}_d) \cap [X]$$

by Lemma 24.4. Thus we may replace  $X$  by  $Y$  and assume that  $X$  is projective over  $k$ .

If  $X$  is a proper  $d$ -dimensional projective variety, then we can write  $\mathcal{L}_1 = \mathcal{O}_X(D) \otimes \mathcal{O}_X(D')^{\otimes -1}$  for some effective Cartier divisors  $D, D' \subset X$  by Divisors, Lemma 15.12. By additivity for both sides of the equation (Varieties, Lemma 44.5 and Lemma 23.2) we reduce to the case  $\mathcal{L}_1 = \mathcal{O}_X(D)$  for some effective Cartier divisor  $D$ . By Varieties, Lemma 44.8 we have

$$(\mathcal{L}_1 \cdots \mathcal{L}_d \cdot X) = (\mathcal{L}_2 \cdots \mathcal{L}_d \cdot D)$$

and by Lemma 23.4 we have

$$c_1(\mathcal{L}_1) \cap \dots \cap c_1(\mathcal{L}_d) \cap [X] = c_1(\mathcal{L}_2) \cap \dots \cap c_1(\mathcal{L}_d) \cap [D]_{d-1}$$

Thus we obtain the result from our induction hypothesis.  $\square$

**42. Grothendieck-Riemann-Roch**

02UO Let  $(S, \delta)$  be as in Situation 7.1. Let  $X, Y$  be locally of finite type over  $S$ . Let  $\mathcal{E}$  be a finite locally free sheaf  $\mathcal{E}$  on  $X$  of rank  $r$ . Let  $f : X \rightarrow Y$  be a proper smooth morphism. Assume that  $R^i f_* \mathcal{E}$  are locally free sheaves on  $Y$  of finite rank. The Grothendieck-Riemann-Roch theorem say in this case that

$$f_*(\text{Todd}(T_{X/Y})\text{ch}(\mathcal{E})) = \sum (-1)^i \text{ch}(R^i f_* \mathcal{E})$$

Here

$$T_{X/Y} = \mathcal{H}om_{\mathcal{O}_X}(\Omega_{X/Y}, \mathcal{O}_X)$$

is the relative tangent bundle of  $X$  over  $Y$ . If  $Y = \text{Spec}(k)$  where  $k$  is a field, then we can restate this as

$$\chi(X, \mathcal{E}) = \text{deg}(\text{Todd}(T_{X/k})\text{ch}(\mathcal{E}))$$

The theorem is more general and becomes easier to prove when formulated in correct generality. We will return to this elsewhere (insert future reference here).

**43. Appendix A: Alternative approach to key lemma**

0EAZ In this appendix we first define determinants  $\det_\kappa(M)$  of finite length modules  $M$  over local rings  $(R, \mathfrak{m}, \kappa)$ , see Subsection 43.1. The determinant  $\det_\kappa(M)$  is a 1-dimensional  $\kappa$ -vector space. We use this in Subsection 43.12 to define the determinant  $\det_\kappa(M, \varphi, \psi) \in \kappa^*$  of an exact  $(2, 1)$ -periodic complex  $(M, \varphi, \psi)$  with  $M$  of finite length. In Subsection 43.26 we use these determinants to construct a tame symbol  $d_R(a, b) = \det_\kappa(R/ab, a, b)$  for a pair of nonzerodivisors  $a, b \in R$  when  $R$  is Noetherian of dimension 1. Although there is no doubt that

$$d_R(a, b) = \partial_R(a, b)$$

where  $\partial_R$  is as in Section 5, we have not (yet) added the verification. The advantage of the tame symbol as constructed in this appendix is that it extends (for example) to pairs of injective endomorphisms  $\varphi, \psi$  of a finite  $R$ -module  $M$  of dimension 1 such that  $\varphi(\psi(M)) = \psi(\varphi(M))$ . In Subsection 43.40 we relate Herbrand quotients and determinants. An easy to state version of main the main result (Proposition 43.43) is the formula

$$-e_R(M, \varphi, \psi) = \text{ord}_R(\det_K(M_K, \varphi, \psi))$$

when  $(M, \varphi, \psi)$  is a  $(2, 1)$ -periodic complex whose Herbrand quotient  $e_R$  (Definition 2.2) is defined over a 1-dimensional Noetherian local domain  $R$  with fraction field  $K$ . We use this proposition to give an alternative proof of the key lemma (Lemma 6.3) for the tame symbol constructed in this appendix, see Lemma 43.46.

02P5 **43.1. Determinants of finite length modules.** The material in this section is related to the material in the paper [KM76] and to the material in the thesis [Ros09].

Given any field  $\kappa$  and any finite dimensional  $\kappa$ -vector space  $V$  we set  $\det_\kappa(V) = \wedge^n(V)$  where  $n = \dim_\kappa(V)$ . We will generalize this to finite length modules over local rings. If the local ring contains a field, then the determinant constructed below is a “usual” determinant, see Remark 43.9.

02P6 **Definition 43.2.** Let  $R$  be a local ring with maximal ideal  $\mathfrak{m}$  and residue field  $\kappa$ . Let  $M$  be a finite length  $R$ -module. Say  $l = \text{length}_R(M)$ .

- (1) Given elements  $x_1, \dots, x_r \in M$  we denote  $\langle x_1, \dots, x_r \rangle = Rx_1 + \dots + Rx_r$  the  $R$ -submodule of  $M$  generated by  $x_1, \dots, x_r$ .
- (2) We will say an  $l$ -tuple of elements  $(e_1, \dots, e_l)$  of  $M$  is *admissible* if  $\mathfrak{m}e_i \subset \langle e_1, \dots, e_{i-1} \rangle$  for  $i = 1, \dots, l$ .
- (3) A *symbol*  $[e_1, \dots, e_l]$  will mean  $(e_1, \dots, e_l)$  is an admissible  $l$ -tuple.
- (4) An *admissible relation* between symbols is one of the following:
  - (a) if  $(e_1, \dots, e_l)$  is an admissible sequence and for some  $1 \leq a \leq l$  we have  $e_a \in \langle e_1, \dots, e_{a-1} \rangle$ , then  $[e_1, \dots, e_l] = 0$ ,
  - (b) if  $(e_1, \dots, e_l)$  is an admissible sequence and for some  $1 \leq a \leq l$  we have  $e_a = \lambda e'_a + x$  with  $\lambda \in R^*$ , and  $x \in \langle e_1, \dots, e_{a-1} \rangle$ , then

$$[e_1, \dots, e_l] = \bar{\lambda}[e_1, \dots, e_{a-1}, e'_a, e_{a+1}, \dots, e_l]$$

where  $\bar{\lambda} \in \kappa^*$  is the image of  $\lambda$  in the residue field, and

- (c) if  $(e_1, \dots, e_l)$  is an admissible sequence and  $\mathfrak{m}e_a \subset \langle e_1, \dots, e_{a-2} \rangle$  then

$$[e_1, \dots, e_l] = -[e_1, \dots, e_{a-2}, e_a, e_{a-1}, e_{a+1}, \dots, e_l].$$

- (5) We define the *determinant of the finite length  $R$ -module  $M$*  to be

$$\det_{\kappa}(M) = \left\{ \frac{\kappa\text{-vector space generated by symbols}}{\kappa\text{-linear combinations of admissible relations}} \right\}$$

We stress that always  $l = \text{length}_R(M)$ . We also stress that it does not follow that the symbol  $[e_1, \dots, e_l]$  is additive in the entries (this will typically not be the case). Before we can show that the determinant  $\det_{\kappa}(M)$  actually has dimension 1 we have to show that it has dimension at most 1.

02P7 **Lemma 43.3.** *With notations as above we have  $\dim_{\kappa}(\det_{\kappa}(M)) \leq 1$ .*

**Proof.** Fix an admissible sequence  $(f_1, \dots, f_l)$  of  $M$  such that

$$\text{length}_R(\langle f_1, \dots, f_i \rangle) = i$$

for  $i = 1, \dots, l$ . Such an admissible sequence exists exactly because  $M$  has length  $l$ . We will show that any element of  $\det_{\kappa}(M)$  is a  $\kappa$ -multiple of the symbol  $[f_1, \dots, f_l]$ . This will prove the lemma.

Let  $(e_1, \dots, e_l)$  be an admissible sequence of  $M$ . It suffices to show that  $[e_1, \dots, e_l]$  is a multiple of  $[f_1, \dots, f_l]$ . First assume that  $\langle e_1, \dots, e_l \rangle \neq M$ . Then there exists an  $i \in [1, \dots, l]$  such that  $e_i \in \langle e_1, \dots, e_{i-1} \rangle$ . It immediately follows from the first admissible relation that  $[e_1, \dots, e_n] = 0$  in  $\det_{\kappa}(M)$ . Hence we may assume that  $\langle e_1, \dots, e_l \rangle = M$ . In particular there exists a smallest index  $i \in \{1, \dots, l\}$  such that  $f_1 \in \langle e_1, \dots, e_i \rangle$ . This means that  $e_i = \lambda f_1 + x$  with  $x \in \langle e_1, \dots, e_{i-1} \rangle$  and  $\lambda \in R^*$ . By the second admissible relation this means that  $[e_1, \dots, e_l] = \bar{\lambda}[e_1, \dots, e_{i-1}, f_1, e_{i+1}, \dots, e_l]$ . Note that  $\mathfrak{m}f_1 = 0$ . Hence by applying the third admissible relation  $i - 1$  times we see that

$$[e_1, \dots, e_l] = (-1)^{i-1} \bar{\lambda} [f_1, e_1, \dots, e_{i-1}, e_{i+1}, \dots, e_l].$$

Note that it is also the case that  $\langle f_1, e_1, \dots, e_{i-1}, e_{i+1}, \dots, e_l \rangle = M$ . By induction suppose we have proven that our original symbol is equal to a scalar times

$$[f_1, \dots, f_j, e_{j+1}, \dots, e_l]$$

for some admissible sequence  $(f_1, \dots, f_j, e_{j+1}, \dots, e_l)$  whose elements generate  $M$ , i.e., with  $\langle f_1, \dots, f_j, e_{j+1}, \dots, e_l \rangle = M$ . Then we find the smallest  $i$  such that

$f_{j+1} \in \langle f_1, \dots, f_j, e_{j+1}, \dots, e_i \rangle$  and we go through the same process as above to see that

$$[f_1, \dots, f_j, e_{j+1}, \dots, e_i] = (\text{scalar})[f_1, \dots, f_j, f_{j+1}, e_{j+1}, \dots, \hat{e}_i, \dots, e_i]$$

Continuing in this vein we obtain the desired result.  $\square$

Before we show that  $\det_\kappa(M)$  always has dimension 1, let us show that it agrees with the usual top exterior power in the case the module is a vector space over  $\kappa$ .

02P8 **Lemma 43.4.** *Let  $R$  be a local ring with maximal ideal  $\mathfrak{m}$  and residue field  $\kappa$ . Let  $M$  be a finite length  $R$ -module which is annihilated by  $\mathfrak{m}$ . Let  $l = \dim_\kappa(M)$ . Then the map*

$$\det_\kappa(M) \longrightarrow \wedge_\kappa^l(M), \quad [e_1, \dots, e_l] \longmapsto e_1 \wedge \dots \wedge e_l$$

*is an isomorphism.*

**Proof.** It is clear that the rule described in the lemma gives a  $\kappa$ -linear map since all of the admissible relations are satisfied by the usual symbols  $e_1 \wedge \dots \wedge e_l$ . It is also clearly a surjective map. Since by Lemma 43.3 the left hand side has dimension at most one we see that the map is an isomorphism.  $\square$

02P9 **Lemma 43.5.** *Let  $R$  be a local ring with maximal ideal  $\mathfrak{m}$  and residue field  $\kappa$ . Let  $M$  be a finite length  $R$ -module. The determinant  $\det_\kappa(M)$  defined above is a  $\kappa$ -vector space of dimension 1. It is generated by the symbol  $[f_1, \dots, f_l]$  for any admissible sequence such that  $\langle f_1, \dots, f_l \rangle = M$ .*

**Proof.** We know  $\det_\kappa(M)$  has dimension at most 1, and in fact that it is generated by  $[f_1, \dots, f_l]$ , by Lemma 43.3 and its proof. We will show by induction on  $l = \text{length}(M)$  that it is nonzero. For  $l = 1$  it follows from Lemma 43.4. Choose a nonzero element  $f \in M$  with  $\mathfrak{m}f = 0$ . Set  $\overline{M} = M/\langle f \rangle$ , and denote the quotient map  $x \mapsto \bar{x}$ . We will define a surjective map

$$\psi : \det_\kappa(M) \rightarrow \det_\kappa(\overline{M})$$

which will prove the lemma since by induction the determinant of  $\overline{M}$  is nonzero.

We define  $\psi$  on symbols as follows. Let  $(e_1, \dots, e_l)$  be an admissible sequence. If  $f \notin \langle e_1, \dots, e_l \rangle$  then we simply set  $\psi([e_1, \dots, e_l]) = 0$ . If  $f \in \langle e_1, \dots, e_l \rangle$  then we choose an  $i$  minimal such that  $f \in \langle e_1, \dots, e_i \rangle$ . We may write  $e_i = \lambda f + x$  for some unit  $\lambda \in R$  and  $x \in \langle e_1, \dots, e_{i-1} \rangle$ . In this case we set

$$\psi([e_1, \dots, e_l]) = (-1)^i \bar{\lambda} [\bar{e}_1, \dots, \bar{e}_{i-1}, \bar{e}_{i+1}, \dots, \bar{e}_l].$$

Note that it is indeed the case that  $(\bar{e}_1, \dots, \bar{e}_{i-1}, \bar{e}_{i+1}, \dots, \bar{e}_l)$  is an admissible sequence in  $\overline{M}$ , so this makes sense. Let us show that extending this rule  $\kappa$ -linearly to linear combinations of symbols does indeed lead to a map on determinants. To do this we have to show that the admissible relations are mapped to zero.

Type (a) relations. Suppose we have  $(e_1, \dots, e_l)$  an admissible sequence and for some  $1 \leq a \leq l$  we have  $e_a \in \langle e_1, \dots, e_{a-1} \rangle$ . Suppose that  $f \in \langle e_1, \dots, e_i \rangle$  with  $i$  minimal. Then  $i \neq a$  and  $\bar{e}_a \in \langle \bar{e}_1, \dots, \bar{e}_i, \dots, \bar{e}_{a-1} \rangle$  if  $i < a$  or  $\bar{e}_a \in \langle \bar{e}_1, \dots, \bar{e}_{a-1} \rangle$  if  $i > a$ . Thus the same admissible relation for  $\det_\kappa(\overline{M})$  forces the symbol  $[\bar{e}_1, \dots, \bar{e}_{i-1}, \bar{e}_{i+1}, \dots, \bar{e}_l]$  to be zero as desired.

Type (b) relations. Suppose we have  $(e_1, \dots, e_l)$  an admissible sequence and for some  $1 \leq a \leq l$  we have  $e_a = \lambda e'_a + x$  with  $\lambda \in R^*$ , and  $x \in \langle e_1, \dots, e_{a-1} \rangle$ . Suppose

that  $f \in \langle e_1, \dots, e_i \rangle$  with  $i$  minimal. Say  $e_i = \mu f + y$  with  $y \in \langle e_1, \dots, e_{i-1} \rangle$ . If  $i < a$  then the desired equality is

$$(-1)^i \bar{\lambda}[\bar{e}_1, \dots, \bar{e}_{i-1}, \bar{e}_{i+1}, \dots, \bar{e}_l] = (-1)^i \bar{\lambda}[\bar{e}_1, \dots, \bar{e}_{i-1}, \bar{e}_{i+1}, \dots, \bar{e}_{a-1}, \bar{e}'_a, \bar{e}_{a+1}, \dots, \bar{e}_l]$$

which follows from  $\bar{e}_a = \lambda \bar{e}'_a + \bar{x}$  and the corresponding admissible relation for  $\det_\kappa(\bar{M})$ . If  $i > a$  then the desired equality is

$$(-1)^i \bar{\lambda}[\bar{e}_1, \dots, \bar{e}_{i-1}, \bar{e}_{i+1}, \dots, \bar{e}_l] = (-1)^i \bar{\lambda}[\bar{e}_1, \dots, \bar{e}_{a-1}, \bar{e}'_a, \bar{e}_{a+1}, \dots, \bar{e}_{i-1}, \bar{e}_{i+1}, \dots, \bar{e}_l]$$

which follows from  $\bar{e}_a = \lambda \bar{e}'_a + \bar{x}$  and the corresponding admissible relation for  $\det_\kappa(\bar{M})$ . The interesting case is when  $i = a$ . In this case we have  $e_a = \lambda e'_a + x = \mu f + y$ . Hence also  $e'_a = \lambda^{-1}(\mu f + y - x)$ . Thus we see that

$$\psi([e_1, \dots, e_l]) = (-1)^i \bar{\mu}[\bar{e}_1, \dots, \bar{e}_{i-1}, \bar{e}_{i+1}, \dots, \bar{e}_l] = \psi(\bar{\lambda}[e_1, \dots, e_{a-1}, e'_a, e_{a+1}, \dots, e_l])$$

as desired.

Type (c) relations. Suppose that  $(e_1, \dots, e_l)$  is an admissible sequence and  $\mathfrak{m}e_a \subset \langle e_1, \dots, e_{a-2} \rangle$ . Suppose that  $f \in \langle e_1, \dots, e_i \rangle$  with  $i$  minimal. Say  $e_i = \lambda f + x$  with  $x \in \langle e_1, \dots, e_{i-1} \rangle$ . We distinguish 4 cases:

Case 1:  $i < a - 1$ . The desired equality is

$$\begin{aligned} & (-1)^i \bar{\lambda}[\bar{e}_1, \dots, \bar{e}_{i-1}, \bar{e}_{i+1}, \dots, \bar{e}_l] \\ &= (-1)^{i+1} \bar{\lambda}[\bar{e}_1, \dots, \bar{e}_{i-1}, \bar{e}_{i+1}, \dots, \bar{e}_{a-2}, \bar{e}_a, \bar{e}_{a-1}, \bar{e}_{a+1}, \dots, \bar{e}_l] \end{aligned}$$

which follows from the type (c) admissible relation for  $\det_\kappa(\bar{M})$ .

Case 2:  $i > a$ . The desired equality is

$$\begin{aligned} & (-1)^i \bar{\lambda}[\bar{e}_1, \dots, \bar{e}_{i-1}, \bar{e}_{i+1}, \dots, \bar{e}_l] \\ &= (-1)^{i+1} \bar{\lambda}[\bar{e}_1, \dots, \bar{e}_{a-2}, \bar{e}_a, \bar{e}_{a-1}, \bar{e}_{a+1}, \dots, \bar{e}_{i-1}, \bar{e}_{i+1}, \dots, \bar{e}_l] \end{aligned}$$

which follows from the type (c) admissible relation for  $\det_\kappa(\bar{M})$ .

Case 3:  $i = a$ . We write  $e_a = \lambda f + \mu e_{a-1} + y$  with  $y \in \langle e_1, \dots, e_{a-2} \rangle$ . Then

$$\psi([e_1, \dots, e_l]) = (-1)^a \bar{\lambda}[\bar{e}_1, \dots, \bar{e}_{a-1}, \bar{e}_{a+1}, \dots, \bar{e}_l]$$

by definition. If  $\bar{\mu}$  is nonzero, then we have  $e_{a-1} = -\mu^{-1} \lambda f + \mu^{-1} e_a - \mu^{-1} y$  and we obtain

$$\psi(-[e_1, \dots, e_{a-2}, e_a, e_{a-1}, e_{a+1}, \dots, e_l]) = (-1)^a \overline{\mu^{-1} \lambda}[\bar{e}_1, \dots, \bar{e}_{a-2}, \bar{e}_a, \bar{e}_{a+1}, \dots, \bar{e}_l]$$

by definition. Since in  $\bar{M}$  we have  $\bar{e}_a = \mu \bar{e}_{a-1} + \bar{y}$  we see the two outcomes are equal by relation (a) for  $\det_\kappa(\bar{M})$ . If on the other hand  $\bar{\mu}$  is zero, then we can write  $e_a = \lambda f + y$  with  $y \in \langle e_1, \dots, e_{a-2} \rangle$  and we have

$$\psi(-[e_1, \dots, e_{a-2}, e_a, e_{a-1}, e_{a+1}, \dots, e_l]) = (-1)^a \bar{\lambda}[\bar{e}_1, \dots, \bar{e}_{a-1}, \bar{e}_{a+1}, \dots, \bar{e}_l]$$

which is equal to  $\psi([e_1, \dots, e_l])$ .

Case 4:  $i = a - 1$ . Here we have

$$\psi([e_1, \dots, e_l]) = (-1)^{a-1} \bar{\lambda}[\bar{e}_1, \dots, \bar{e}_{a-2}, \bar{e}_a, \dots, \bar{e}_l]$$

by definition. If  $f \notin \langle e_1, \dots, e_{a-2}, e_a \rangle$  then

$$\psi(-[e_1, \dots, e_{a-2}, e_a, e_{a-1}, e_{a+1}, \dots, e_l]) = (-1)^{a+1} \bar{\lambda}[\bar{e}_1, \dots, \bar{e}_{a-2}, \bar{e}_a, \dots, \bar{e}_l]$$

Since  $(-1)^{a-1} = (-1)^{a+1}$  the two expressions are the same. Finally, assume  $f \in \langle e_1, \dots, e_{a-2}, e_a \rangle$ . In this case we see that  $e_{a-1} = \lambda f + x$  with  $x \in \langle e_1, \dots, e_{a-2} \rangle$



and  $e_a = \mu f + y$  with  $y \in \langle e_1, \dots, e_{a-2} \rangle$  for units  $\lambda, \mu \in R$ . We conclude that both  $e_a \in \langle e_1, \dots, e_{a-1} \rangle$  and  $e_{a-1} \in \langle e_1, \dots, e_{a-2}, e_a \rangle$ . In this case a relation of type (a) applies to both  $[e_1, \dots, e_l]$  and  $[e_1, \dots, e_{a-2}, e_a, e_{a-1}, e_{a+1}, \dots, e_l]$  and the compatibility of  $\psi$  with these shown above to see that both

$$\psi([e_1, \dots, e_l]) \quad \text{and} \quad \psi([e_1, \dots, e_{a-2}, e_a, e_{a-1}, e_{a+1}, \dots, e_l])$$

are zero, as desired.

At this point we have shown that  $\psi$  is well defined, and all that remains is to show that it is surjective. To see this let  $(\bar{f}_2, \dots, \bar{f}_l)$  be an admissible sequence in  $\bar{M}$ . We can choose lifts  $f_2, \dots, f_l \in M$ , and then  $(f, f_2, \dots, f_l)$  is an admissible sequence in  $M$ . Since  $\psi([f, f_2, \dots, f_l]) = [f_2, \dots, f_l]$  we win.  $\square$

Let  $R$  be a local ring with maximal ideal  $\mathfrak{m}$  and residue field  $\kappa$ . Note that if  $\varphi : M \rightarrow N$  is an isomorphism of finite length  $R$ -modules, then we get an isomorphism

$$\det_\kappa(\varphi) : \det_\kappa(M) \rightarrow \det_\kappa(N)$$

simply by the rule

$$\det_\kappa(\varphi)([e_1, \dots, e_l]) = [\varphi(e_1), \dots, \varphi(e_l)]$$

for any symbol  $[e_1, \dots, e_l]$  for  $M$ . Hence we see that  $\det_\kappa$  is a functor

05M7 (43.5.1) 
$$\left\{ \begin{array}{l} \text{finite length } R\text{-modules} \\ \text{with isomorphisms} \end{array} \right\} \longrightarrow \left\{ \begin{array}{l} \text{1-dimensional } \kappa\text{-vector spaces} \\ \text{with isomorphisms} \end{array} \right\}$$

This is typical for a “determinant functor” (see [Knu02]), as is the following additivity property.

02PA **Lemma 43.6.** *Let  $(R, \mathfrak{m}, \kappa)$  be a local ring. For every short exact sequence*

$$0 \rightarrow K \rightarrow L \rightarrow M \rightarrow 0$$

*of finite length  $R$ -modules there exists a canonical isomorphism*

$$\gamma_{K \rightarrow L \rightarrow M} : \det_\kappa(K) \otimes_\kappa \det_\kappa(M) \longrightarrow \det_\kappa(L)$$

*defined by the rule on nonzero symbols*

$$[e_1, \dots, e_k] \otimes [\bar{f}_1, \dots, \bar{f}_m] \longrightarrow [e_1, \dots, e_k, f_1, \dots, f_m]$$

*with the following properties:*

- (1) *For every isomorphism of short exact sequences, i.e., for every commutative diagram*

$$\begin{array}{ccccccc} 0 & \longrightarrow & K & \longrightarrow & L & \longrightarrow & M \longrightarrow 0 \\ & & \downarrow u & & \downarrow v & & \downarrow w \\ 0 & \longrightarrow & K' & \longrightarrow & L' & \longrightarrow & M' \longrightarrow 0 \end{array}$$

*with short exact rows and isomorphisms  $u, v, w$  we have*

$$\gamma_{K' \rightarrow L' \rightarrow M'} \circ (\det_\kappa(u) \otimes \det_\kappa(w)) = \det_\kappa(v) \circ \gamma_{K \rightarrow L \rightarrow M},$$

- (2) for every commutative square of finite length  $R$ -modules with exact rows and columns

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & D & \longrightarrow & E & \longrightarrow & F \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & G & \longrightarrow & H & \longrightarrow & I \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0
 \end{array}$$

the following diagram is commutative

$$\begin{array}{ccc}
 \det_{\kappa}(A) \otimes \det_{\kappa}(C) \otimes \det_{\kappa}(G) \otimes \det_{\kappa}(I) & \xrightarrow{\gamma_{A \rightarrow B \rightarrow C} \otimes \gamma_{G \rightarrow H \rightarrow I}} & \det_{\kappa}(B) \otimes \det_{\kappa}(H) \\
 \downarrow \epsilon & & \downarrow \gamma_{B \rightarrow E \rightarrow H} \\
 & & \det_{\kappa}(E) \\
 & & \uparrow \gamma_{D \rightarrow E \rightarrow F} \\
 \det_{\kappa}(A) \otimes \det_{\kappa}(G) \otimes \det_{\kappa}(C) \otimes \det_{\kappa}(I) & \xrightarrow{\gamma_{A \rightarrow D \rightarrow G} \otimes \gamma_{C \rightarrow F \rightarrow I}} & \det_{\kappa}(D) \otimes \det_{\kappa}(F)
 \end{array}$$

where  $\epsilon$  is the switch of the factors in the tensor product times  $(-1)^{cg}$  with  $c = \text{length}_R(C)$  and  $g = \text{length}_R(G)$ , and

- (3) the map  $\gamma_{K \rightarrow L \rightarrow M}$  agrees with the usual isomorphism if  $0 \rightarrow K \rightarrow L \rightarrow M \rightarrow 0$  is actually a short exact sequence of  $\kappa$ -vector spaces.

**Proof.** The significance of taking nonzero symbols in the explicit description of the map  $\gamma_{K \rightarrow L \rightarrow M}$  is simply that if  $(e_1, \dots, e_l)$  is an admissible sequence in  $K$ , and  $(\bar{f}_1, \dots, \bar{f}_m)$  is an admissible sequence in  $M$ , then it is not guaranteed that  $(e_1, \dots, e_l, f_1, \dots, f_m)$  is an admissible sequence in  $L$  (where of course  $f_i \in L$  signifies a lift of  $\bar{f}_i$ ). However, if the symbol  $[e_1, \dots, e_l]$  is nonzero in  $\det_{\kappa}(K)$ , then necessarily  $K = \langle e_1, \dots, e_k \rangle$  (see proof of Lemma 43.3), and in this case it is true that  $(e_1, \dots, e_k, f_1, \dots, f_m)$  is an admissible sequence. Moreover, by the admissible relations of type (b) for  $\det_{\kappa}(L)$  we see that the value of  $[e_1, \dots, e_k, f_1, \dots, f_m]$  in  $\det_{\kappa}(L)$  is independent of the choice of the lifts  $f_i$  in this case also. Given this remark, it is clear that an admissible relation for  $e_1, \dots, e_k$  in  $K$  translates into an admissible relation among  $e_1, \dots, e_k, f_1, \dots, f_m$  in  $L$ , and similarly for an admissible relation among the  $\bar{f}_1, \dots, \bar{f}_m$ . Thus  $\gamma$  defines a linear map of vector spaces as claimed in the lemma.

By Lemma 43.5 we know  $\det_{\kappa}(L)$  is generated by any single symbol  $[x_1, \dots, x_{k+m}]$  such that  $(x_1, \dots, x_{k+m})$  is an admissible sequence with  $L = \langle x_1, \dots, x_{k+m} \rangle$ . Hence it is clear that the map  $\gamma_{K \rightarrow L \rightarrow M}$  is surjective and hence an isomorphism.

Property (1) holds because

$$\begin{aligned} & \det_{\kappa}(v)([e_1, \dots, e_k, f_1, \dots, f_m]) \\ &= [v(e_1), \dots, v(e_k), v(f_1), \dots, v(f_m)] \\ &= \gamma_{K' \rightarrow L' \rightarrow M'}([u(e_1), \dots, u(e_k)] \otimes [w(f_1), \dots, w(f_m)]). \end{aligned}$$

Property (2) means that given a symbol  $[\alpha_1, \dots, \alpha_a]$  generating  $\det_{\kappa}(A)$ , a symbol  $[\gamma_1, \dots, \gamma_c]$  generating  $\det_{\kappa}(C)$ , a symbol  $[\zeta_1, \dots, \zeta_g]$  generating  $\det_{\kappa}(G)$ , and a symbol  $[\tilde{l}_1, \dots, \tilde{l}_i]$  generating  $\det_{\kappa}(I)$  we have

$$\begin{aligned} & [\alpha_1, \dots, \alpha_a, \tilde{\gamma}_1, \dots, \tilde{\gamma}_c, \tilde{\zeta}_1, \dots, \tilde{\zeta}_g, \tilde{l}_1, \dots, \tilde{l}_i] \\ &= (-1)^{cg} [\alpha_1, \dots, \alpha_a, \tilde{\zeta}_1, \dots, \tilde{\zeta}_g, \tilde{\gamma}_1, \dots, \tilde{\gamma}_c, \tilde{l}_1, \dots, \tilde{l}_i] \end{aligned}$$

(for suitable lifts  $\tilde{x}$  in  $E$ ) in  $\det_{\kappa}(E)$ . This holds because we may use the admissible relations of type (c)  $cg$  times in the following order: move the  $\tilde{\zeta}_1$  past the elements  $\tilde{\gamma}_c, \dots, \tilde{\gamma}_1$  (allowed since  $\mathfrak{m}\tilde{\zeta}_1 \subset A$ ), then move  $\tilde{\zeta}_2$  past the elements  $\tilde{\gamma}_c, \dots, \tilde{\gamma}_1$  (allowed since  $\mathfrak{m}\tilde{\zeta}_2 \subset A + R\tilde{\zeta}_1$ ), and so on.

Part (3) of the lemma is obvious. This finishes the proof.  $\square$

We can use the maps  $\gamma$  of the lemma to define more general maps  $\gamma$  as follows. Suppose that  $(R, \mathfrak{m}, \kappa)$  is a local ring. Let  $M$  be a finite length  $R$ -module and suppose we are given a finite filtration (see Homology, Definition 16.1)

$$M = F^n \supset F^{n+1} \supset \dots \supset F^{m-1} \supset F^m = 0.$$

Then there is a canonical isomorphism

$$\gamma_{(M,F)} : \bigotimes_i \det_{\kappa}(F^i/F^{i+1}) \longrightarrow \det_{\kappa}(M)$$

well defined up to sign(!). One can make the sign explicit either by giving a well defined order of the terms in the tensor product (starting with higher indices unfortunately), and by thinking of the target category for the functor  $\det_{\kappa}$  as the category of 1-dimensional super vector spaces. See [KM76, Section 1].

Here is another typical result for determinant functors. It is not hard to show. The tricky part is usually to show the existence of a determinant functor.

02PB **Lemma 43.7.** *Let  $(R, \mathfrak{m}, \kappa)$  be any local ring. The functor*

$$\det_{\kappa} : \left\{ \begin{array}{l} \text{finite length } R\text{-modules} \\ \text{with isomorphisms} \end{array} \right\} \longrightarrow \left\{ \begin{array}{l} \text{1-dimensional } \kappa\text{-vector spaces} \\ \text{with isomorphisms} \end{array} \right\}$$

*endowed with the maps  $\gamma_{K \rightarrow L \rightarrow M}$  is characterized by the following properties*

- (1) *its restriction to the subcategory of modules annihilated by  $\mathfrak{m}$  is isomorphic to the usual determinant functor (see Lemma 43.4), and*
- (2) *(1), (2) and (3) of Lemma 43.6 hold.*

**Proof.** Omitted.  $\square$

02PC **Lemma 43.8.** *Let  $(R', \mathfrak{m}') \rightarrow (R, \mathfrak{m})$  be a local ring homomorphism which induces an isomorphism on residue fields  $\kappa$ . Then for every finite length  $R$ -module the restriction  $M_{R'}$  is a finite length  $R'$ -module and there is a canonical isomorphism*

$$\det_{R,\kappa}(M) \longrightarrow \det_{R',\kappa}(M_{R'})$$

*This isomorphism is functorial in  $M$  and compatible with the isomorphisms  $\gamma_{K \rightarrow L \rightarrow M}$  of Lemma 43.6 defined for  $\det_{R,\kappa}$  and  $\det_{R',\kappa}$ .*

**Proof.** If the length of  $M$  as an  $R$ -module is  $l$ , then the length of  $M$  as an  $R'$ -module (i.e.,  $M_{R'}$ ) is  $l$  as well, see Algebra, Lemma 51.12. Note that an admissible sequence  $x_1, \dots, x_l$  of  $M$  over  $R$  is an admissible sequence of  $M$  over  $R'$  as  $\mathfrak{m}'$  maps into  $\mathfrak{m}$ . The isomorphism is obtained by mapping the symbol  $[x_1, \dots, x_l] \in \det_{R, \kappa}(M)$  to the corresponding symbol  $[x_1, \dots, x_l] \in \det_{R', \kappa}(M)$ . It is immediate to verify that this is functorial for isomorphisms and compatible with the isomorphisms  $\gamma$  of Lemma 43.6.  $\square$

0BDQ **Remark 43.9.** Let  $(R, \mathfrak{m}, \kappa)$  be a local ring and assume either the characteristic of  $\kappa$  is zero or it is  $p$  and  $pR = 0$ . Let  $M_1, \dots, M_n$  be finite length  $R$ -modules. We will show below that there exists an ideal  $I \subset \mathfrak{m}$  annihilating  $M_i$  for  $i = 1, \dots, n$  and a section  $\sigma : \kappa \rightarrow R/I$  of the canonical surjection  $R/I \rightarrow \kappa$ . The restriction  $M_{i, \kappa}$  of  $M_i$  via  $\sigma$  is a  $\kappa$ -vector space of dimension  $l_i = \text{length}_R(M_i)$  and using Lemma 43.8 we see that

$$\det_{\kappa}(M_i) = \wedge_{\kappa}^{l_i}(M_{i, \kappa})$$

These isomorphisms are compatible with the isomorphisms  $\gamma_{K \rightarrow M \rightarrow L}$  of Lemma 43.6 for short exact sequences of finite length  $R$ -modules annihilated by  $I$ . The conclusion is that verifying a property of  $\det_{\kappa}$  often reduces to verifying corresponding properties of the usual determinant on the category finite dimensional vector spaces.

For  $I$  we can take the annihilator (Algebra, Definition 39.3) of the module  $M = \bigoplus M_i$ . In this case we see that  $R/I \subset \text{End}_R(M)$  hence has finite length. Thus  $R/I$  is an Artinian local ring with residue field  $\kappa$ . Since an Artinian local ring is complete we see that  $R/I$  has a coefficient ring by the Cohen structure theorem (Algebra, Theorem 154.8) which is a field by our assumption on  $R$ .

Here is a case where we can compute the determinant of a linear map. In fact there is nothing mysterious about this in any case, see Example 43.11 for a random example.

02PD **Lemma 43.10.** *Let  $R$  be a local ring with residue field  $\kappa$ . Let  $u \in R^*$  be a unit. Let  $M$  be a module of finite length over  $R$ . Denote  $u_M : M \rightarrow M$  the map multiplication by  $u$ . Then*

$$\det_{\kappa}(u_M) : \det_{\kappa}(M) \longrightarrow \det_{\kappa}(M)$$

*is multiplication by  $\bar{u}^l$  where  $l = \text{length}_R(M)$  and  $\bar{u} \in \kappa^*$  is the image of  $u$ .*

**Proof.** Denote  $f_M \in \kappa^*$  the element such that  $\det_{\kappa}(u_M) = f_M \text{id}_{\det_{\kappa}(M)}$ . Suppose that  $0 \rightarrow K \rightarrow L \rightarrow M \rightarrow 0$  is a short exact sequence of finite  $R$ -modules. Then we see that  $u_K, u_L, u_M$  give an isomorphism of short exact sequences. Hence by Lemma 43.6 (1) we conclude that  $f_K f_M = f_L$ . This means that by induction on length it suffices to prove the lemma in the case of length 1 where it is trivial.  $\square$

02PE **Example 43.11.** Consider the local ring  $R = \mathbf{Z}_p$ . Set  $M = \mathbf{Z}_p/(p^2) \oplus \mathbf{Z}_p/(p^3)$ . Let  $u : M \rightarrow M$  be the map given by the matrix

$$u = \begin{pmatrix} a & b \\ pc & d \end{pmatrix}$$

where  $a, b, c, d \in \mathbf{Z}_p$ , and  $a, d \in \mathbf{Z}_p^*$ . In this case  $\det_{\kappa}(u)$  equals multiplication by  $a^2 d^3 \text{ mod } p \in \mathbf{F}_p^*$ . This can easily be seen by consider the effect of  $u$  on the symbol  $[p^2 e, pe, pf, e, f]$  where  $e = (0, 1) \in M$  and  $f = (1, 0) \in M$ .

0BDR **43.12. Periodic complexes and determinants.** Let  $R$  be a local ring with residue field  $\kappa$ . Let  $(M, \varphi, \psi)$  be a  $(2, 1)$ -periodic complex over  $R$ . Assume that  $M$  has finite length and that  $(M, \varphi, \psi)$  is exact. We are going to use the determinant construction to define an invariant of this situation. See Subsection 43.1. Let us abbreviate  $K_\varphi = \text{Ker}(\varphi)$ ,  $I_\varphi = \text{Im}(\varphi)$ ,  $K_\psi = \text{Ker}(\psi)$ , and  $I_\psi = \text{Im}(\psi)$ . The short exact sequences

$$0 \rightarrow K_\varphi \rightarrow M \rightarrow I_\varphi \rightarrow 0, \quad 0 \rightarrow K_\psi \rightarrow M \rightarrow I_\psi \rightarrow 0$$

give isomorphisms

$$\gamma_\varphi : \det_\kappa(K_\varphi) \otimes \det_\kappa(I_\varphi) \longrightarrow \det_\kappa(M), \quad \gamma_\psi : \det_\kappa(K_\psi) \otimes \det_\kappa(I_\psi) \longrightarrow \det_\kappa(M),$$

see Lemma 43.6. On the other hand the exactness of the complex gives equalities  $K_\varphi = I_\psi$ , and  $K_\psi = I_\varphi$  and hence an isomorphism

$$\sigma : \det_\kappa(K_\varphi) \otimes \det_\kappa(I_\varphi) \longrightarrow \det_\kappa(K_\psi) \otimes \det_\kappa(I_\psi)$$

by switching the factors. Using this notation we can define our invariant.

02PJ **Definition 43.13.** Let  $R$  be a local ring with residue field  $\kappa$ . Let  $(M, \varphi, \psi)$  be a  $(2, 1)$ -periodic complex over  $R$ . Assume that  $M$  has finite length and that  $(M, \varphi, \psi)$  is exact. The *determinant* of  $(M, \varphi, \psi)$  is the element

$$\det_\kappa(M, \varphi, \psi) \in \kappa^*$$

such that the composition

$$\det_\kappa(M) \xrightarrow{\gamma_\psi \circ \sigma \circ \gamma_\varphi^{-1}} \det_\kappa(M)$$

is multiplication by  $(-1)^{\text{length}_R(I_\varphi)\text{length}_R(I_\psi)} \det_\kappa(M, \varphi, \psi)$ .

02PK **Remark 43.14.** Here is a more down to earth description of the determinant introduced above. Let  $R$  be a local ring with residue field  $\kappa$ . Let  $(M, \varphi, \psi)$  be a  $(2, 1)$ -periodic complex over  $R$ . Assume that  $M$  has finite length and that  $(M, \varphi, \psi)$  is exact. Let us abbreviate  $I_\varphi = \text{Im}(\varphi)$ ,  $I_\psi = \text{Im}(\psi)$  as above. Assume that  $\text{length}_R(I_\varphi) = a$  and  $\text{length}_R(I_\psi) = b$ , so that  $a + b = \text{length}_R(M)$  by exactness. Choose admissible sequences  $x_1, \dots, x_a \in I_\varphi$  and  $y_1, \dots, y_b \in I_\psi$  such that the symbol  $[x_1, \dots, x_a]$  generates  $\det_\kappa(I_\varphi)$  and the symbol  $[y_1, \dots, y_b]$  generates  $\det_\kappa(I_\psi)$ . Choose  $\tilde{x}_i \in M$  such that  $\varphi(\tilde{x}_i) = x_i$ . Choose  $\tilde{y}_j \in M$  such that  $\psi(\tilde{y}_j) = y_j$ . Then  $\det_\kappa(M, \varphi, \psi)$  is characterized by the equality

$$[x_1, \dots, x_a, \tilde{y}_1, \dots, \tilde{y}_b] = (-1)^{ab} \det_\kappa(M, \varphi, \psi) [y_1, \dots, y_b, \tilde{x}_1, \dots, \tilde{x}_a]$$

in  $\det_\kappa(M)$ . This also explains the sign.

02PL **Lemma 43.15.** *Let  $R$  be a local ring with residue field  $\kappa$ . Let  $(M, \varphi, \psi)$  be a  $(2, 1)$ -periodic complex over  $R$ . Assume that  $M$  has finite length and that  $(M, \varphi, \psi)$  is exact. Then*

$$\det_\kappa(M, \varphi, \psi) \det_\kappa(M, \psi, \varphi) = 1.$$

**Proof.** Omitted. □

02PM **Lemma 43.16.** *Let  $R$  be a local ring with residue field  $\kappa$ . Let  $(M, \varphi, \varphi)$  be a  $(2, 1)$ -periodic complex over  $R$ . Assume that  $M$  has finite length and that  $(M, \varphi, \varphi)$  is exact. Then  $\text{length}_R(M) = 2\text{length}_R(\text{Im}(\varphi))$  and*

$$\det_\kappa(M, \varphi, \varphi) = (-1)^{\text{length}_R(\text{Im}(\varphi))} = (-1)^{\frac{1}{2}\text{length}_R(M)}$$

**Proof.** Follows directly from the sign rule in the definitions.  $\square$

02PN **Lemma 43.17.** *Let  $R$  be a local ring with residue field  $\kappa$ . Let  $M$  be a finite length  $R$ -module.*

- (1) *if  $\varphi : M \rightarrow M$  is an isomorphism then  $\det_{\kappa}(M, \varphi, 0) = \det_{\kappa}(\varphi)$ .*
- (2) *if  $\psi : M \rightarrow M$  is an isomorphism then  $\det_{\kappa}(M, 0, \psi) = \det_{\kappa}(\psi)^{-1}$ .*

**Proof.** Let us prove (1). Set  $\psi = 0$ . Then we may, with notation as above Definition 43.13, identify  $K_{\varphi} = I_{\psi} = 0$ ,  $I_{\varphi} = K_{\psi} = M$ . With these identifications, the map

$$\gamma_{\varphi} : \kappa \otimes \det_{\kappa}(M) = \det_{\kappa}(K_{\varphi}) \otimes \det_{\kappa}(I_{\varphi}) \longrightarrow \det_{\kappa}(M)$$

is identified with  $\det_{\kappa}(\varphi^{-1})$ . On the other hand the map  $\gamma_{\psi}$  is identified with the identity map. Hence  $\gamma_{\psi} \circ \sigma \circ \gamma_{\varphi}^{-1}$  is equal to  $\det_{\kappa}(\varphi)$  in this case. Whence the result. We omit the proof of (2).  $\square$

02PO **Lemma 43.18.** *Let  $R$  be a local ring with residue field  $\kappa$ . Suppose that we have a short exact sequence of  $(2, 1)$ -periodic complexes*

$$0 \rightarrow (M_1, \varphi_1, \psi_1) \rightarrow (M_2, \varphi_2, \psi_2) \rightarrow (M_3, \varphi_3, \psi_3) \rightarrow 0$$

*with all  $M_i$  of finite length, and each  $(M_1, \varphi_1, \psi_1)$  exact. Then*

$$\det_{\kappa}(M_2, \varphi_2, \psi_2) = \det_{\kappa}(M_1, \varphi_1, \psi_1) \det_{\kappa}(M_3, \varphi_3, \psi_3).$$

*in  $\kappa^*$ .*

**Proof.** Let us abbreviate  $I_{\varphi,i} = \text{Im}(\varphi_i)$ ,  $K_{\varphi,i} = \text{Ker}(\varphi_i)$ ,  $I_{\psi,i} = \text{Im}(\psi_i)$ , and  $K_{\psi,i} = \text{Ker}(\psi_i)$ . Observe that we have a commutative square

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & K_{\varphi,1} & \longrightarrow & K_{\varphi,2} & \longrightarrow & K_{\varphi,3} \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & M_1 & \longrightarrow & M_2 & \longrightarrow & M_3 \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & I_{\varphi,1} & \longrightarrow & I_{\varphi,2} & \longrightarrow & I_{\varphi,3} \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0
 \end{array}$$

of finite length  $R$ -modules with exact rows and columns. The top row is exact since it can be identified with the sequence  $I_{\psi,1} \rightarrow I_{\psi,2} \rightarrow I_{\psi,3} \rightarrow 0$  of images, and similarly for the bottom row. There is a similar diagram involving the modules  $I_{\psi,i}$  and  $K_{\psi,i}$ . By definition  $\det_{\kappa}(M_2, \varphi_2, \psi_2)$  corresponds, up to a sign, to the

composition of the left vertical maps in the following diagram

$$\begin{array}{ccc}
 \det_{\kappa}(M_1) \otimes \det_{\kappa}(M_3) & \xrightarrow{\gamma} & \det_{\kappa}(M_2) \\
 \downarrow \gamma^{-1} \otimes \gamma^{-1} & & \downarrow \gamma^{-1} \\
 \det_{\kappa}(K_{\varphi,1}) \otimes \det_{\kappa}(I_{\varphi,1}) \otimes \det_{\kappa}(K_{\varphi,3}) \otimes \det_{\kappa}(I_{\varphi,3}) & \xrightarrow{\gamma \otimes \gamma} & \det_{\kappa}(K_{\varphi,2}) \otimes \det_{\kappa}(I_{\varphi,2}) \\
 \downarrow \sigma \otimes \sigma & & \downarrow \sigma \\
 \det_{\kappa}(K_{\psi,1}) \otimes \det_{\kappa}(I_{\psi,1}) \otimes \det_{\kappa}(K_{\psi,3}) \otimes \det_{\kappa}(I_{\psi,3}) & \xrightarrow{\gamma \otimes \gamma} & \det_{\kappa}(K_{\psi,2}) \otimes \det_{\kappa}(I_{\psi,2}) \\
 \downarrow \gamma \otimes \gamma & & \downarrow \gamma \\
 \det_{\kappa}(M_1) \otimes \det_{\kappa}(M_3) & \xrightarrow{\gamma} & \det_{\kappa}(M_2)
 \end{array}$$

The top and bottom squares are commutative up to sign by applying Lemma 43.6 (2). The middle square is trivially commutative (we are just switching factors). Hence we see that  $\det_{\kappa}(M_2, \varphi_2, \psi_2) = \epsilon \det_{\kappa}(M_1, \varphi_1, \psi_1) \det_{\kappa}(M_3, \varphi_3, \psi_3)$  for some sign  $\epsilon$ . And the sign can be worked out, namely the outer rectangle in the diagram above commutes up to

$$\begin{aligned}
 \epsilon &= (-1)^{\text{length}(I_{\varphi,1})\text{length}(K_{\varphi,3}) + \text{length}(I_{\psi,1})\text{length}(K_{\psi,3})} \\
 &= (-1)^{\text{length}(I_{\varphi,1})\text{length}(I_{\psi,3}) + \text{length}(I_{\psi,1})\text{length}(I_{\varphi,3})}
 \end{aligned}$$

(proof omitted). It follows easily from this that the signs work out as well.  $\square$

02PP **Example 43.19.** Let  $k$  be a field. Consider the ring  $R = k[T]/(T^2)$  of dual numbers over  $k$ . Denote  $t$  the class of  $T$  in  $R$ . Let  $M = R$  and  $\varphi = ut$ ,  $\psi = vt$  with  $u, v \in k^*$ . In this case  $\det_k(M)$  has generator  $e = [t, 1]$ . We identify  $I_{\varphi} = K_{\varphi} = I_{\psi} = K_{\psi} = (t)$ . Then  $\gamma_{\varphi}(t \otimes t) = u^{-1}[t, 1]$  (since  $u^{-1} \in M$  is a lift of  $t \in I_{\varphi}$ ) and  $\gamma_{\psi}(t \otimes t) = v^{-1}[t, 1]$  (same reason). Hence we see that  $\det_k(M, \varphi, \psi) = -u/v \in k^*$ .

02PQ **Example 43.20.** Let  $R = \mathbf{Z}_p$  and let  $M = \mathbf{Z}_p/(p^l)$ . Let  $\varphi = p^b u$  and  $\psi = p^a v$  with  $a, b \geq 0$ ,  $a + b = l$  and  $u, v \in \mathbf{Z}_p^*$ . Then a computation as in Example 43.19 shows that

$$\begin{aligned}
 \det_{\mathbf{F}_p}(\mathbf{Z}_p/(p^l), p^b u, p^a v) &= (-1)^{ab} u^a / v^b \pmod{p} \\
 &= (-1)^{\text{ord}_p(\alpha)\text{ord}_p(\beta)} \frac{\alpha^{\text{ord}_p(\beta)}}{\beta^{\text{ord}_p(\alpha)}} \pmod{p}
 \end{aligned}$$

with  $\alpha = p^b u, \beta = p^a v \in \mathbf{Z}_p$ . See Lemma 43.37 for a more general case (and a proof).

02PR **Example 43.21.** Let  $R = k$  be a field. Let  $M = k^{\oplus a} \oplus k^{\oplus b}$  be  $l = a + b$  dimensional. Let  $\varphi$  and  $\psi$  be the following diagonal matrices

$$\varphi = \text{diag}(u_1, \dots, u_a, 0, \dots, 0), \quad \psi = \text{diag}(0, \dots, 0, v_1, \dots, v_b)$$

with  $u_i, v_j \in k^*$ . In this case we have

$$\det_k(M, \varphi, \psi) = \frac{u_1 \dots u_a}{v_1 \dots v_b}.$$

This can be seen by a direct computation or by computing in case  $l = 1$  and using the additivity of Lemma 43.18.

02PS **Example 43.22.** Let  $R = k$  be a field. Let  $M = k^{\oplus a} \oplus k^{\oplus a}$  be  $l = 2a$  dimensional. Let  $\varphi$  and  $\psi$  be the following block matrices

$$\varphi = \begin{pmatrix} 0 & U \\ 0 & 0 \end{pmatrix}, \quad \psi = \begin{pmatrix} 0 & V \\ 0 & 0 \end{pmatrix},$$

with  $U, V \in \text{Mat}(a \times a, k)$  invertible. In this case we have

$$\det_k(M, \varphi, \psi) = (-1)^a \frac{\det(U)}{\det(V)}.$$

This can be seen by a direct computation. The case  $a = 1$  is similar to the computation in Example 43.19.

02PT **Example 43.23.** Let  $R = k$  be a field. Let  $M = k^{\oplus 4}$ . Let

$$\varphi = \begin{pmatrix} 0 & 0 & 0 & 0 \\ u_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & u_2 & 0 \end{pmatrix} \quad \psi = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & v_2 & 0 \\ 0 & 0 & 0 & 0 \\ v_1 & 0 & 0 & 0 \end{pmatrix}$$

with  $u_1, u_2, v_1, v_2 \in k^*$ . Then we have

$$\det_k(M, \varphi, \psi) = -\frac{u_1 u_2}{v_1 v_2}.$$

Next we come to the analogue of the fact that the determinant of a composition of linear endomorphisms is the product of the determinants. To avoid very long formulae we write  $I_\varphi = \text{Im}(\varphi)$ , and  $K_\varphi = \text{Ker}(\varphi)$  for any  $R$ -module map  $\varphi : M \rightarrow M$ . We also denote  $\varphi\psi = \varphi \circ \psi$  for a pair of morphisms  $\varphi, \psi : M \rightarrow M$ .

02PU **Lemma 43.24.** *Let  $R$  be a local ring with residue field  $\kappa$ . Let  $M$  be a finite length  $R$ -module. Let  $\alpha, \beta, \gamma$  be endomorphisms of  $M$ . Assume that*

- (1)  $I_\alpha = K_{\beta\gamma}$ , and similarly for any permutation of  $\alpha, \beta, \gamma$ ,
- (2)  $K_\alpha = I_{\beta\gamma}$ , and similarly for any permutation of  $\alpha, \beta, \gamma$ .

Then

- (1) The triple  $(M, \alpha, \beta\gamma)$  is an exact  $(2, 1)$ -periodic complex.
- (2) The triple  $(I_\gamma, \alpha, \beta)$  is an exact  $(2, 1)$ -periodic complex.
- (3) The triple  $(M/K_\beta, \alpha, \gamma)$  is an exact  $(2, 1)$ -periodic complex.
- (4) We have

$$\det_\kappa(M, \alpha, \beta\gamma) = \det_\kappa(I_\gamma, \alpha, \beta) \det_\kappa(M/K_\beta, \alpha, \gamma).$$

**Proof.** It is clear that the assumptions imply part (1) of the lemma.

To see part (1) note that the assumptions imply that  $I_{\gamma\alpha} = I_{\alpha\gamma}$ , and similarly for kernels and any other pair of morphisms. Moreover, we see that  $I_{\gamma\beta} = I_{\beta\gamma} = K_\alpha \subset I_\gamma$  and similarly for any other pair. In particular we get a short exact sequence

$$0 \rightarrow I_{\beta\gamma} \rightarrow I_\gamma \xrightarrow{\alpha} I_{\alpha\gamma} \rightarrow 0$$

and similarly we get a short exact sequence

$$0 \rightarrow I_{\alpha\gamma} \rightarrow I_\gamma \xrightarrow{\beta} I_{\beta\gamma} \rightarrow 0.$$

This proves  $(I_\gamma, \alpha, \beta)$  is an exact  $(2, 1)$ -periodic complex. Hence part (2) of the lemma holds.



To see that  $\alpha, \gamma$  give well defined endomorphisms of  $M/K_\beta$  we have to check that  $\alpha(K_\beta) \subset K_\beta$  and  $\gamma(K_\beta) \subset K_\beta$ . This is true because  $\alpha(K_\beta) = \alpha(I_{\gamma\alpha}) = I_{\alpha\gamma\alpha} \subset I_{\alpha\gamma} = K_\beta$ , and similarly in the other case. The kernel of the map  $\alpha : M/K_\beta \rightarrow M/K_\beta$  is  $K_{\beta\alpha}/K_\beta = I_\gamma/K_\beta$ . Similarly, the kernel of  $\gamma : M/K_\beta \rightarrow M/K_\beta$  is equal to  $I_\alpha/K_\beta$ . Hence we conclude that (3) holds.

We introduce  $r = \text{length}_R(K_\alpha)$ ,  $s = \text{length}_R(K_\beta)$  and  $t = \text{length}_R(K_\gamma)$ . By the exact sequences above and our hypotheses we have  $\text{length}_R(I_\alpha) = s + t$ ,  $\text{length}_R(I_\beta) = r + t$ ,  $\text{length}_R(I_\gamma) = r + s$ , and  $\text{length}(M) = r + s + t$ . Choose

- (1) an admissible sequence  $x_1, \dots, x_r \in K_\alpha$  generating  $K_\alpha$
- (2) an admissible sequence  $y_1, \dots, y_s \in K_\beta$  generating  $K_\beta$ ,
- (3) an admissible sequence  $z_1, \dots, z_t \in K_\gamma$  generating  $K_\gamma$ ,
- (4) elements  $\tilde{x}_i \in M$  such that  $\beta\gamma\tilde{x}_i = x_i$ ,
- (5) elements  $\tilde{y}_i \in M$  such that  $\alpha\gamma\tilde{y}_i = y_i$ ,
- (6) elements  $\tilde{z}_i \in M$  such that  $\beta\alpha\tilde{z}_i = z_i$ .

With these choices the sequence  $y_1, \dots, y_s, \alpha\tilde{z}_1, \dots, \alpha\tilde{z}_t$  is an admissible sequence in  $I_\alpha$  generating it. Hence, by Remark 43.14 the determinant  $D = \det_\kappa(M, \alpha, \beta\gamma)$  is the unique element of  $\kappa^*$  such that

$$\begin{aligned} & [y_1, \dots, y_s, \alpha\tilde{z}_1, \dots, \alpha\tilde{z}_t, \tilde{x}_1, \dots, \tilde{x}_r] \\ &= (-1)^{r(s+t)} D [x_1, \dots, x_r, \gamma\tilde{y}_1, \dots, \gamma\tilde{y}_s, \tilde{z}_1, \dots, \tilde{z}_t] \end{aligned}$$

By the same remark, we see that  $D_1 = \det_\kappa(M/K_\beta, \alpha, \gamma)$  is characterized by

$$[y_1, \dots, y_s, \alpha\tilde{z}_1, \dots, \alpha\tilde{z}_t, \tilde{x}_1, \dots, \tilde{x}_r] = (-1)^{rt} D_1 [y_1, \dots, y_s, \gamma\tilde{x}_1, \dots, \gamma\tilde{x}_r, \tilde{z}_1, \dots, \tilde{z}_t]$$

By the same remark, we see that  $D_2 = \det_\kappa(I_\gamma, \alpha, \beta)$  is characterized by

$$[y_1, \dots, y_s, \gamma\tilde{x}_1, \dots, \gamma\tilde{x}_r, \tilde{z}_1, \dots, \tilde{z}_t] = (-1)^{rs} D_2 [x_1, \dots, x_r, \gamma\tilde{y}_1, \dots, \gamma\tilde{y}_s, \tilde{z}_1, \dots, \tilde{z}_t]$$

Combining the formulas above we see that  $D = D_1 D_2$  as desired.  $\square$

02PV **Lemma 43.25.** *Let  $R$  be a local ring with residue field  $\kappa$ . Let  $\alpha : (M, \varphi, \psi) \rightarrow (M', \varphi', \psi')$  be a morphism of  $(2, 1)$ -periodic complexes over  $R$ . Assume*

- (1)  $M, M'$  have finite length,
- (2)  $(M, \varphi, \psi), (M', \varphi', \psi')$  are exact,
- (3) the maps  $\varphi, \psi$  induce the zero map on  $K = \text{Ker}(\alpha)$ , and
- (4) the maps  $\varphi, \psi$  induce the zero map on  $Q = \text{Coker}(\alpha)$ .

Denote  $N = \alpha(M) \subset M'$ . We obtain two short exact sequences of  $(2, 1)$ -periodic complexes

$$\begin{aligned} 0 &\rightarrow (N, \varphi', \psi') \rightarrow (M', \varphi', \psi') \rightarrow (Q, 0, 0) \rightarrow 0 \\ 0 &\rightarrow (K, 0, 0) \rightarrow (M, \varphi, \psi) \rightarrow (N, \varphi', \psi') \rightarrow 0 \end{aligned}$$

which induce two isomorphisms  $\alpha_i : Q \rightarrow K$ ,  $i = 0, 1$ . Then

$$\det_\kappa(M, \varphi, \psi) = \det_\kappa(\alpha_0^{-1} \circ \alpha_1) \det_\kappa(M', \varphi', \psi')$$

In particular, if  $\alpha_0 = \alpha_1$ , then  $\det_\kappa(M, \varphi, \psi) = \det_\kappa(M', \varphi', \psi')$ .

**Proof.** There are (at least) two ways to prove this lemma. One is to produce an enormous commutative diagram using the properties of the determinants. The other is to use the characterization of the determinants in terms of admissible sequences of elements. It is the second approach that we will use.

First let us explain precisely what the maps  $\alpha_i$  are. Namely,  $\alpha_0$  is the composition

$$\alpha_0 : Q = H^0(Q, 0, 0) \rightarrow H^1(N, \varphi', \psi') \rightarrow H^2(K, 0, 0) = K$$

and  $\alpha_1$  is the composition

$$\alpha_1 : Q = H^1(Q, 0, 0) \rightarrow H^2(N, \varphi', \psi') \rightarrow H^3(K, 0, 0) = K$$

coming from the boundary maps of the short exact sequences of complexes displayed in the lemma. The fact that the complexes  $(M, \varphi, \psi)$ ,  $(M', \varphi', \psi')$  are exact implies these maps are isomorphisms.

We will use the notation  $I_\varphi = \text{Im}(\varphi)$ ,  $K_\varphi = \text{Ker}(\varphi)$  and similarly for the other maps. Exactness for  $M$  and  $M'$  means that  $K_\varphi = I_\psi$  and three similar equalities. We introduce  $k = \text{length}_R(K)$ ,  $a = \text{length}_R(I_\varphi)$ ,  $b = \text{length}_R(I_\psi)$ . Then we see that  $\text{length}_R(M) = a + b$ , and  $\text{length}_R(N) = a + b - k$ ,  $\text{length}_R(Q) = k$  and  $\text{length}_R(M') = a + b$ . The exact sequences below will show that also  $\text{length}_R(I_{\varphi'}) = a$  and  $\text{length}_R(I_{\psi'}) = b$ .

The assumption that  $K \subset K_\varphi = I_\psi$  means that  $\varphi$  factors through  $N$  to give an exact sequence

$$0 \rightarrow \alpha(I_\psi) \rightarrow N \xrightarrow{\varphi\alpha^{-1}} I_\psi \rightarrow 0.$$

Here  $\varphi\alpha^{-1}(x') = y$  means  $x' = \alpha(x)$  and  $y = \varphi(x)$ . Similarly, we have

$$0 \rightarrow \alpha(I_\varphi) \rightarrow N \xrightarrow{\psi\alpha^{-1}} I_\varphi \rightarrow 0.$$

The assumption that  $\psi'$  induces the zero map on  $Q$  means that  $I_{\psi'} = K_{\varphi'} \subset N$ . This means the quotient  $\varphi'(N) \subset I_{\varphi'}$  is identified with  $Q$ . Note that  $\varphi'(N) = \alpha(I_\varphi)$ . Hence we conclude there is an isomorphism

$$\varphi' : Q \rightarrow I_{\varphi'}/\alpha(I_\varphi)$$

simply described by  $\varphi'(x' \bmod N) = \varphi'(x') \bmod \alpha(I_\varphi)$ . In exactly the same way we get

$$\psi' : Q \rightarrow I_{\psi'}/\alpha(I_\psi)$$

Finally, note that  $\alpha_0$  is the composition

$$Q \xrightarrow{\varphi'} I_{\varphi'}/\alpha(I_\varphi) \xrightarrow{\psi\alpha^{-1}|_{I_{\varphi'}/\alpha(I_\varphi)}} K$$

and similarly  $\alpha_1 = \varphi\alpha^{-1}|_{I_{\psi'}/\alpha(I_\psi)} \circ \psi'$ .

To shorten the formulas below we are going to write  $\alpha x$  instead of  $\alpha(x)$  in the following. No confusion should result since all maps are indicated by Greek letters and elements by Roman letters. We are going to choose

- (1) an admissible sequence  $z_1, \dots, z_k \in K$  generating  $K$ ,
- (2) elements  $z'_i \in M$  such that  $\varphi z'_i = z_i$ ,
- (3) elements  $z''_i \in M$  such that  $\psi z''_i = z_i$ ,
- (4) elements  $x_{k+1}, \dots, x_a \in I_\varphi$  such that  $z_1, \dots, z_k, x_{k+1}, \dots, x_a$  is an admissible sequence generating  $I_\varphi$ ,
- (5) elements  $\tilde{x}_i \in M$  such that  $\varphi \tilde{x}_i = x_i$ ,
- (6) elements  $y_{k+1}, \dots, y_b \in I_\psi$  such that  $z_1, \dots, z_k, y_{k+1}, \dots, y_b$  is an admissible sequence generating  $I_\psi$ ,
- (7) elements  $\tilde{y}_i \in M$  such that  $\psi \tilde{y}_i = y_i$ , and

(8) elements  $w_1, \dots, w_k \in M'$  such that  $w_1 \bmod N, \dots, w_k \bmod N$  are an admissible sequence in  $Q$  generating  $Q$ .

By Remark 43.14 the element  $D = \det_{\kappa}(M, \varphi, \psi) \in \kappa^*$  is characterized by

$$\begin{aligned} & [z_1, \dots, z_k, x_{k+1}, \dots, x_a, z'_1, \dots, z'_k, \tilde{y}_{k+1}, \dots, \tilde{y}_b] \\ = & (-1)^{ab} D [z_1, \dots, z_k, y_{k+1}, \dots, y_b, z'_1, \dots, z'_k, \tilde{x}_{k+1}, \dots, \tilde{x}_a] \end{aligned}$$

Note that by the discussion above  $\alpha x_{k+1}, \dots, \alpha x_a, \varphi w_1, \dots, \varphi w_k$  is an admissible sequence generating  $I_{\varphi'}$  and  $\alpha y_{k+1}, \dots, \alpha y_b, \psi w_1, \dots, \psi w_k$  is an admissible sequence generating  $I_{\psi'}$ . Hence by Remark 43.14 the element  $D' = \det_{\kappa}(M', \varphi', \psi') \in \kappa^*$  is characterized by

$$\begin{aligned} & [\alpha x_{k+1}, \dots, \alpha x_a, \varphi' w_1, \dots, \varphi' w_k, \alpha \tilde{y}_{k+1}, \dots, \alpha \tilde{y}_b, w_1, \dots, w_k] \\ = & (-1)^{ab} D' [\alpha y_{k+1}, \dots, \alpha y_b, \psi' w_1, \dots, \psi' w_k, \alpha \tilde{x}_{k+1}, \dots, \alpha \tilde{x}_a, w_1, \dots, w_k] \end{aligned}$$

Note how in the first, resp. second displayed formula the first, resp. last  $k$  entries of the symbols on both sides are the same. Hence these formulas are really equivalent to the equalities

$$\begin{aligned} & [\alpha x_{k+1}, \dots, \alpha x_a, \alpha z'_1, \dots, \alpha z'_k, \alpha \tilde{y}_{k+1}, \dots, \alpha \tilde{y}_b] \\ = & (-1)^{ab} D [\alpha y_{k+1}, \dots, \alpha y_b, \alpha z'_1, \dots, \alpha z'_k, \alpha \tilde{x}_{k+1}, \dots, \alpha \tilde{x}_a] \end{aligned}$$

and

$$\begin{aligned} & [\alpha x_{k+1}, \dots, \alpha x_a, \varphi' w_1, \dots, \varphi' w_k, \alpha \tilde{y}_{k+1}, \dots, \alpha \tilde{y}_b] \\ = & (-1)^{ab} D' [\alpha y_{k+1}, \dots, \alpha y_b, \psi' w_1, \dots, \psi' w_k, \alpha \tilde{x}_{k+1}, \dots, \alpha \tilde{x}_a] \end{aligned}$$

in  $\det_{\kappa}(N)$ . Note that  $\varphi' w_1, \dots, \varphi' w_k$  and  $\alpha z'_1, \dots, \alpha z'_k$  are admissible sequences generating the module  $I_{\varphi'}/\alpha(I_{\varphi})$ . Write

$$[\varphi' w_1, \dots, \varphi' w_k] = \lambda_0 [\alpha z'_1, \dots, \alpha z'_k]$$

in  $\det_{\kappa}(I_{\varphi'}/\alpha(I_{\varphi}))$  for some  $\lambda_0 \in \kappa^*$ . Similarly, write

$$[\psi' w_1, \dots, \psi' w_k] = \lambda_1 [\alpha z'_1, \dots, \alpha z'_k]$$

in  $\det_{\kappa}(I_{\psi'}/\alpha(I_{\psi}))$  for some  $\lambda_1 \in \kappa^*$ . On the one hand it is clear that

$$\alpha_i([w_1, \dots, w_k]) = \lambda_i [z_1, \dots, z_k]$$

for  $i = 0, 1$  by our description of  $\alpha_i$  above, which means that

$$\det_{\kappa}(\alpha_0^{-1} \circ \alpha_1) = \lambda_1 / \lambda_0$$

and on the other hand it is clear that

$$\begin{aligned} & \lambda_0 [\alpha x_{k+1}, \dots, \alpha x_a, \alpha z'_1, \dots, \alpha z'_k, \alpha \tilde{y}_{k+1}, \dots, \alpha \tilde{y}_b] \\ = & [\alpha x_{k+1}, \dots, \alpha x_a, \varphi' w_1, \dots, \varphi' w_k, \alpha \tilde{y}_{k+1}, \dots, \alpha \tilde{y}_b] \end{aligned}$$

and

$$\begin{aligned} & \lambda_1 [\alpha y_{k+1}, \dots, \alpha y_b, \alpha z'_1, \dots, \alpha z'_k, \alpha \tilde{x}_{k+1}, \dots, \alpha \tilde{x}_a] \\ = & [\alpha y_{k+1}, \dots, \alpha y_b, \psi' w_1, \dots, \psi' w_k, \alpha \tilde{x}_{k+1}, \dots, \alpha \tilde{x}_a] \end{aligned}$$

which imply  $\lambda_0 D = \lambda_1 D'$ . The lemma follows.  $\square$

02PW **43.26. Symbols.** The correct generality for this construction is perhaps the situation of the following lemma.

02PX **Lemma 43.27.** *Let  $A$  be a Noetherian local ring. Let  $M$  be a finite  $A$ -module of dimension 1. Assume  $\varphi, \psi : M \rightarrow M$  are two injective  $A$ -module maps, and assume  $\varphi(\psi(M)) = \psi(\varphi(M))$ , for example if  $\varphi$  and  $\psi$  commute. Then  $\text{length}_R(M/\varphi\psi M) < \infty$  and  $(M/\varphi\psi M, \varphi, \psi)$  is an exact  $(2, 1)$ -periodic complex.*

**Proof.** Let  $\mathfrak{q}$  be a minimal prime of the support of  $M$ . Then  $M_{\mathfrak{q}}$  is a finite length  $A_{\mathfrak{q}}$ -module, see Algebra, Lemma 61.3. Hence both  $\varphi$  and  $\psi$  induce isomorphisms  $M_{\mathfrak{q}} \rightarrow M_{\mathfrak{q}}$ . Thus the support of  $M/\varphi\psi M$  is  $\{\mathfrak{m}_A\}$  and hence it has finite length (see lemma cited above). Finally, the kernel of  $\varphi$  on  $M/\varphi\psi M$  is clearly  $\psi M/\varphi\psi M$ , and hence the kernel of  $\varphi$  is the image of  $\psi$  on  $M/\varphi\psi M$ . Similarly the other way since  $M/\varphi\psi M = M/\psi\varphi M$  by assumption.  $\square$

02PY **Lemma 43.28.** *Let  $A$  be a Noetherian local ring. Let  $a, b \in A$ .*

- (1) *If  $M$  is a finite  $A$ -module of dimension 1 such that  $a, b$  are nonzerodivisors on  $M$ , then  $\text{length}_A(M/abM) < \infty$  and  $(M/abM, a, b)$  is a  $(2, 1)$ -periodic exact complex.*
- (2) *If  $a, b$  are nonzerodivisors and  $\dim(A) = 1$  then  $\text{length}_A(A/(ab)) < \infty$  and  $(A/(ab), a, b)$  is a  $(2, 1)$ -periodic exact complex.*

*In particular, in these cases  $\det_{\kappa}(M/abM, a, b) \in \kappa^*$ , resp.  $\det_{\kappa}(A/(ab), a, b) \in \kappa^*$  are defined.*

**Proof.** Follows from Lemma 43.27.  $\square$

02PZ **Definition 43.29.** Let  $A$  be a Noetherian local ring with residue field  $\kappa$ . Let  $a, b \in A$ . Let  $M$  be a finite  $A$ -module of dimension 1 such that  $a, b$  are nonzerodivisors on  $M$ . We define the *symbol associated to  $M, a, b$*  to be the element

$$d_M(a, b) = \det_{\kappa}(M/abM, a, b) \in \kappa^*$$

02Q0 **Lemma 43.30.** *Let  $A$  be a Noetherian local ring. Let  $a, b, c \in A$ . Let  $M$  be a finite  $A$ -module with  $\dim(\text{Supp}(M)) = 1$ . Assume  $a, b, c$  are nonzerodivisors on  $M$ . Then*

$$d_M(a, bc) = d_M(a, b)d_M(a, c)$$

*and  $d_M(a, b)d_M(b, a) = 1$ .*

**Proof.** The first statement follows from Lemma 43.24 applied to  $M/abcM$  and endomorphisms  $\alpha, \beta, \gamma$  given by multiplication by  $a, b, c$ . The second comes from Lemma 43.15.  $\square$

02Q1 **Definition 43.31.** Let  $A$  be a Noetherian local domain of dimension 1 with residue field  $\kappa$ . Let  $K$  be the fraction field of  $A$ . We define the *tame symbol* of  $A$  to be the map

$$K^* \times K^* \longrightarrow \kappa^*, \quad (x, y) \longmapsto d_A(x, y)$$

where  $d_A(x, y)$  is extended to  $K^* \times K^*$  by the multiplicativity of Lemma 43.30.

It is clear that we may extend more generally  $d_M(-, -)$  to certain rings of fractions of  $A$  (even if  $A$  is not a domain).

0AY9 **Lemma 43.32.** *Let  $A$  be a Noetherian local ring and  $M$  a finite  $A$ -module of dimension 1. Let  $a \in A$  be a nonzerodivisor on  $M$ . Then  $d_M(a, a) = (-1)^{\text{length}_A(M/aM)}$ .*

**Proof.** Immediate from Lemma 43.16. □

02Q2 **Lemma 43.33.** *Let  $A$  be a Noetherian local ring. Let  $M$  be a finite  $A$ -module of dimension 1. Let  $b \in A$  be a nonzerodivisor on  $M$ , and let  $u \in A^*$ . Then*

$$d_M(u, b) = u^{\text{length}_A(M/bM)} \pmod{\mathfrak{m}_A}.$$

*In particular, if  $M = A$ , then  $d_A(u, b) = u^{\text{ord}_A(b)} \pmod{\mathfrak{m}_A}$ .*

**Proof.** Note that in this case  $M/ubM = M/bM$  on which multiplication by  $b$  is zero. Hence  $d_M(u, b) = \det_{\kappa}(u|_{M/bM})$  by Lemma 43.17. The lemma then follows from Lemma 43.10. □

02Q3 **Lemma 43.34.** *Let  $A$  be a Noetherian local ring. Let  $a, b \in A$ . Let*

$$0 \rightarrow M \rightarrow M' \rightarrow M'' \rightarrow 0$$

*be a short exact sequence of  $A$ -modules of dimension 1 such that  $a, b$  are nonzerodivisors on all three  $A$ -modules. Then*

$$d_{M'}(a, b) = d_M(a, b)d_{M''}(a, b)$$

*in  $\kappa^*$ .*

**Proof.** It is easy to see that this leads to a short exact sequence of exact  $(2, 1)$ -periodic complexes

$$0 \rightarrow (M/abM, a, b) \rightarrow (M'/abM', a, b) \rightarrow (M''/abM'', a, b) \rightarrow 0$$

Hence the lemma follows from Lemma 43.18. □

02Q4 **Lemma 43.35.** *Let  $A$  be a Noetherian local ring. Let  $\alpha : M \rightarrow M'$  be a homomorphism of finite  $A$ -modules of dimension 1. Let  $a, b \in A$ . Assume*

- (1)  $a, b$  are nonzerodivisors on both  $M$  and  $M'$ , and
- (2)  $\dim(\text{Ker}(\alpha)), \dim(\text{Coker}(\alpha)) \leq 0$ .

*Then  $d_M(a, b) = d_{M'}(a, b)$ .*

**Proof.** If  $a \in A^*$ , then the equality follows from the equality  $\text{length}(M/bM) = \text{length}(M'/bM')$  and Lemma 43.33. Similarly if  $b$  is a unit the lemma holds as well (by the symmetry of Lemma 43.30). Hence we may assume that  $a, b \in \mathfrak{m}_A$ . This in particular implies that  $\mathfrak{m}$  is not an associated prime of  $M$ , and hence  $\alpha : M \rightarrow M'$  is injective. This permits us to think of  $M$  as a submodule of  $M'$ . By assumption  $M'/M$  is a finite  $A$ -module with support  $\{\mathfrak{m}_A\}$  and hence has finite length. Note that for any third module  $M''$  with  $M \subset M'' \subset M'$  the maps  $M \rightarrow M''$  and  $M'' \rightarrow M'$  satisfy the assumptions of the lemma as well. This reduces us, by induction on the length of  $M'/M$ , to the case where  $\text{length}_A(M'/M) = 1$ . Finally, in this case consider the map

$$\bar{\alpha} : M/abM \rightarrow M'/abM'.$$

By construction the cokernel  $Q$  of  $\bar{\alpha}$  has length 1. Since  $a, b \in \mathfrak{m}_A$ , they act trivially on  $Q$ . It also follows that the kernel  $K$  of  $\bar{\alpha}$  has length 1 and hence also  $a, b$  act trivially on  $K$ . Hence we may apply Lemma 43.25. Thus it suffices to see that the two maps  $\alpha_i : Q \rightarrow K$  are the same. In fact, both maps are equal to the map  $q = x' \pmod{\text{Im}(\bar{\alpha})} \mapsto abx' \in K$ . We omit the verification. □

02Q5 **Lemma 43.36.** *Let  $A$  be a Noetherian local ring. Let  $M$  be a finite  $A$ -module with  $\dim(\text{Supp}(M)) = 1$ . Let  $a, b \in A$  nonzerodivisors on  $M$ . Let  $\mathfrak{q}_1, \dots, \mathfrak{q}_t$  be the minimal primes in the support of  $M$ . Then*

$$d_M(a, b) = \prod_{i=1, \dots, t} d_{A/\mathfrak{q}_i}(a, b)^{\text{length}_{A/\mathfrak{q}_i}(M_{\mathfrak{q}_i})}$$

as elements of  $\kappa^*$ .

**Proof.** Choose a filtration by  $A$ -submodules

$$0 = M_0 \subset M_1 \subset \dots \subset M_n = M$$

such that each quotient  $M_j/M_{j-1}$  is isomorphic to  $A/\mathfrak{p}_j$  for some prime ideal  $\mathfrak{p}_j$  of  $A$ . See Algebra, Lemma 61.1. For each  $j$  we have either  $\mathfrak{p}_j = \mathfrak{q}_i$  for some  $i$ , or  $\mathfrak{p}_j = \mathfrak{m}_A$ . Moreover, for a fixed  $i$ , the number of  $j$  such that  $\mathfrak{p}_j = \mathfrak{q}_i$  is equal to  $\text{length}_{A/\mathfrak{q}_i}(M_{\mathfrak{q}_i})$  by Algebra, Lemma 61.5. Hence  $d_{M_j}(a, b)$  is defined for each  $j$  and

$$d_{M_j}(a, b) = \begin{cases} d_{M_{j-1}}(a, b)d_{A/\mathfrak{q}_i}(a, b) & \text{if } \mathfrak{p}_j = \mathfrak{q}_i \\ d_{M_{j-1}}(a, b) & \text{if } \mathfrak{p}_j = \mathfrak{m}_A \end{cases}$$

by Lemma 43.34 in the first instance and Lemma 43.35 in the second. Hence the lemma.  $\square$

02Q6 **Lemma 43.37.** *Let  $A$  be a discrete valuation ring with fraction field  $K$ . For nonzero  $x, y \in K$  we have*

$$d_A(x, y) = (-1)^{\text{ord}_A(x)\text{ord}_A(y)} \frac{x^{\text{ord}_A(y)}}{y^{\text{ord}_A(x)}} \bmod \mathfrak{m}_A,$$

in other words the symbol is equal to the usual tame symbol.

**Proof.** By multiplicativity it suffices to prove this when  $x, y \in A$ . Let  $t \in A$  be a uniformizer. Write  $x = t^b u$  and  $y = t^a v$  for some  $a, b \geq 0$  and  $u, v \in A^*$ . Set  $l = a + b$ . Then  $t^{l-1}, \dots, t^b$  is an admissible sequence in  $(x)/(xy)$  and  $t^{l-1}, \dots, t^a$  is an admissible sequence in  $(y)/(xy)$ . Hence by Remark 43.14 we see that  $d_A(x, y)$  is characterized by the equation

$$[t^{l-1}, \dots, t^b, v^{-1}t^{b-1}, \dots, v^{-1}] = (-1)^{ab} d_A(x, y) [t^{l-1}, \dots, t^a, u^{-1}t^{a-1}, \dots, u^{-1}].$$

Hence by the admissible relations for the symbols  $[x_1, \dots, x_l]$  we see that

$$d_A(x, y) = (-1)^{ab} u^a/v^b \bmod \mathfrak{m}_A$$

as desired.  $\square$

02Q8 **Lemma 43.38.** *Let  $A$  be a Noetherian local ring. Let  $a, b \in A$ . Let  $M$  be a finite  $A$ -module of dimension 1 on which each of  $a, b, b - a$  are nonzerodivisors. Then*

$$d_M(a, b - a)d_M(b, b) = d_M(b, b - a)d_M(a, b)$$

in  $\kappa^*$ .

**Proof.** By Lemma 43.36 it suffices to show the relation when  $M = A/\mathfrak{q}$  for some prime  $\mathfrak{q} \subset A$  with  $\dim(A/\mathfrak{q}) = 1$ .

In case  $M = A/\mathfrak{q}$  we may replace  $A$  by  $A/\mathfrak{q}$  and  $a, b$  by their images in  $A/\mathfrak{q}$ . Hence we may assume  $A = M$  and  $A$  a local Noetherian domain of dimension 1. The reason is that the residue field  $\kappa$  of  $A$  and  $A/\mathfrak{q}$  are the same and that for any  $A/\mathfrak{q}$ -module  $M$  the determinant taken over  $A$  or over  $A/\mathfrak{q}$  are canonically identified. See Lemma 43.8.

It suffices to show the relation when both  $a, b$  are in the maximal ideal. Namely, the case where one or both are units follows from Lemmas 43.33 and 43.32.

Choose an extension  $A \subset A'$  and factorizations  $a = ta'$ ,  $b = tb'$  as in Lemma 4.2. Note that also  $b - a = t(b' - a')$  and that  $A' = (a', b') = (a', b' - a') = (b' - a', b')$ . Here and in the following we think of  $A'$  as an  $A$ -module and  $a, b, a', b', t$  as  $A$ -module endomorphisms of  $A'$ . We will use the notation  $d_{A'}^A(a', b')$  and so on to indicate

$$d_{A'}^A(a', b') = \det_{\kappa}(A'/a'b'A', a', b')$$

which is defined by Lemma 43.27. The upper index  $A$  is used to distinguish this from the already defined symbol  $d_{A'}(a', b')$  which is different (for example because it has values in the residue field of  $A'$  which may be different from  $\kappa$ ). By Lemma 43.35 we see that  $d_A(a, b) = d_{A'}^A(a, b)$ , and similarly for the other combinations. Using this and multiplicativity we see that it suffices to prove

$$d_{A'}^A(a', b' - a')d_{A'}^A(b', b') = d_{A'}^A(b', b' - a')d_{A'}^A(a', b')$$

Now, since  $(a', b') = A'$  and so on we have

$$\begin{aligned} A'/(a'(b' - a')) &\cong A'/(a') \oplus A'/(b' - a') \\ A'/(b'(b' - a')) &\cong A'/(b') \oplus A'/(b' - a') \\ A'/(a'b') &\cong A'/(a') \oplus A'/(b') \end{aligned}$$

Moreover, note that multiplication by  $b' - a'$  on  $A/(a')$  is equal to multiplication by  $b'$ , and that multiplication by  $b' - a'$  on  $A/(b')$  is equal to multiplication by  $-a'$ . Using Lemmas 43.17 and 43.18 we conclude

$$\begin{aligned} d_{A'}^A(a', b' - a') &= \det_{\kappa}(b'|_{A'/(a')})^{-1} \det_{\kappa}(a'|_{A'/(b' - a')}) \\ d_{A'}^A(b', b' - a') &= \det_{\kappa}(-a'|_{A'/(b')})^{-1} \det_{\kappa}(b'|_{A'/(b' - a')}) \\ d_{A'}^A(a', b') &= \det_{\kappa}(b'|_{A'/(a')})^{-1} \det_{\kappa}(a'|_{A'/(b')}) \end{aligned}$$

Hence we conclude that

$$(-1)^{\text{length}_A(A'/(b'))} d_{A'}^A(a', b' - a') = d_{A'}^A(b', b' - a') d_{A'}^A(a', b')$$

the sign coming from the  $-a'$  in the second equality above. On the other hand, by Lemma 43.16 we have  $d_{A'}^A(b', b') = (-1)^{\text{length}_A(A'/(b'))}$  and the lemma is proved.  $\square$

The tame symbol is a Steinberg symbol.

02Q9 **Lemma 43.39.** *Let  $A$  be a Noetherian local domain of dimension 1 with fraction field  $K$ . For  $x \in K \setminus \{0, 1\}$  we have*

$$d_A(x, 1 - x) = 1$$

**Proof.** Write  $x = a/b$  with  $a, b \in A$ . The hypothesis implies, since  $1 - x = (b - a)/b$ , that also  $b - a \neq 0$ . Hence we compute

$$d_A(x, 1 - x) = d_A(a, b - a) d_A(a, b)^{-1} d_A(b, b - a)^{-1} d_A(b, b)$$

Thus we have to show that  $d_A(a, b - a) d_A(b, b) = d_A(b, b - a) d_A(a, b)$ . This is Lemma 43.38.  $\square$

02QA **43.40. Lengths and determinants.** In this section we use the determinant to compare lattices. The key lemma is the following.

02QB **Lemma 43.41.** *Let  $R$  be a noetherian local ring. Let  $\mathfrak{q} \subset R$  be a prime with  $\dim(R/\mathfrak{q}) = 1$ . Let  $\varphi : M \rightarrow N$  be a homomorphism of finite  $R$ -modules. Assume there exist  $x_1, \dots, x_l \in M$  and  $y_1, \dots, y_l \in N$  with the following properties*

- (1)  $M = \langle x_1, \dots, x_l \rangle$ ,
- (2)  $\langle x_1, \dots, x_i \rangle / \langle x_1, \dots, x_{i-1} \rangle \cong R/\mathfrak{q}$  for  $i = 1, \dots, l$ ,
- (3)  $N = \langle y_1, \dots, y_l \rangle$ , and
- (4)  $\langle y_1, \dots, y_i \rangle / \langle y_1, \dots, y_{i-1} \rangle \cong R/\mathfrak{q}$  for  $i = 1, \dots, l$ .

Then  $\varphi$  is injective if and only if  $\varphi_{\mathfrak{q}}$  is an isomorphism, and in this case we have

$$\text{length}_R(\text{Coker}(\varphi)) = \text{ord}_{R/\mathfrak{q}}(f)$$

where  $f \in \kappa(\mathfrak{q})$  is the element such that

$$[\varphi(x_1), \dots, \varphi(x_l)] = f[y_1, \dots, y_l]$$

in  $\det_{\kappa(\mathfrak{q})}(N_{\mathfrak{q}})$ .

**Proof.** First, note that the lemma holds in case  $l = 1$ . Namely, in this case  $x_1$  is a basis of  $M$  over  $R/\mathfrak{q}$  and  $y_1$  is a basis of  $N$  over  $R/\mathfrak{q}$  and we have  $\varphi(x_1) = fy_1$  for some  $f \in R$ . Thus  $\varphi$  is injective if and only if  $f \notin \mathfrak{q}$ . Moreover,  $\text{Coker}(\varphi) = R/(f, \mathfrak{q})$  and hence the lemma holds by definition of  $\text{ord}_{R/\mathfrak{q}}(f)$  (see Algebra, Definition 120.2).

In fact, suppose more generally that  $\varphi(x_i) = f_i y_i$  for some  $f_i \in R$ ,  $f_i \notin \mathfrak{q}$ . Then the induced maps

$$\langle x_1, \dots, x_i \rangle / \langle x_1, \dots, x_{i-1} \rangle \longrightarrow \langle y_1, \dots, y_i \rangle / \langle y_1, \dots, y_{i-1} \rangle$$

are all injective and have cokernels isomorphic to  $R/(f_i, \mathfrak{q})$ . Hence we see that

$$\text{length}_R(\text{Coker}(\varphi)) = \sum \text{ord}_{R/\mathfrak{q}}(f_i).$$

On the other hand it is clear that

$$[\varphi(x_1), \dots, \varphi(x_l)] = f_1 \dots f_l [y_1, \dots, y_l]$$

in this case from the admissible relation (b) for symbols. Hence we see the result holds in this case also.

We prove the general case by induction on  $l$ . Assume  $l > 1$ . Let  $i \in \{1, \dots, l\}$  be minimal such that  $\varphi(x_1) \in \langle y_1, \dots, y_i \rangle$ . We will argue by induction on  $i$ . If  $i = 1$ , then we get a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \langle x_1 \rangle & \longrightarrow & \langle x_1, \dots, x_l \rangle & \longrightarrow & \langle x_1, \dots, x_l \rangle / \langle x_1 \rangle \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \langle y_1 \rangle & \longrightarrow & \langle y_1, \dots, y_l \rangle & \longrightarrow & \langle y_1, \dots, y_l \rangle / \langle y_1 \rangle \longrightarrow 0 \end{array}$$

and the lemma follows from the snake lemma and induction on  $l$ . Assume now that  $i > 1$ . Write  $\varphi(x_1) = a_1 y_1 + \dots + a_{i-1} y_{i-1} + ay_i$  with  $a_j, a \in R$  and  $a \notin \mathfrak{q}$  (since otherwise  $i$  was not minimal). Set

$$x'_j = \begin{cases} x_j & \text{if } j = 1 \\ ax_j & \text{if } j \geq 2 \end{cases} \quad \text{and} \quad y'_j = \begin{cases} y_j & \text{if } j < i \\ ay_j & \text{if } j \geq i \end{cases}$$



Let  $M' = \langle x'_1, \dots, x'_l \rangle$  and  $N' = \langle y'_1, \dots, y'_l \rangle$ . Since  $\varphi(x'_1) = a_1 y'_1 + \dots + a_{i-1} y'_{i-1} + y'_i$  by construction and since for  $j > 1$  we have  $\varphi(x'_j) = a \varphi(x_j) \in \langle y'_1, \dots, y'_l \rangle$  we get a commutative diagram of  $R$ -modules and maps

$$\begin{array}{ccc} M' & \xrightarrow{\quad} & N' \\ \downarrow & \varphi' & \downarrow \\ M & \xrightarrow{\quad \varphi} & N \end{array}$$

By the result of the second paragraph of the proof we know that  $\text{length}_R(M/M') = (l-1)\text{ord}_{R/\mathfrak{q}}(a)$  and similarly  $\text{length}_R(M/M') = (l-i+1)\text{ord}_{R/\mathfrak{q}}(a)$ . By a diagram chase this implies that

$$\text{length}_R(\text{Coker}(\varphi')) = \text{length}_R(\text{Coker}(\varphi)) + i \text{ord}_{R/\mathfrak{q}}(a).$$

On the other hand, it is clear that writing

$$[\varphi(x_1), \dots, \varphi(x_l)] = f[y_1, \dots, y_l], \quad [\varphi'(x'_1), \dots, \varphi'(x'_l)] = f'[y'_1, \dots, y'_l]$$

we have  $f' = a^i f$ . Hence it suffices to prove the lemma for the case that  $\varphi(x_1) = a_1 y_1 + \dots + a_{i-1} y_{i-1} + y_i$ , i.e., in the case that  $a = 1$ . Next, recall that

$$[y_1, \dots, y_l] = [y_1, \dots, y_{i-1}, a_1 y_1 + \dots + a_{i-1} y_{i-1} + y_i, y_{i+1}, \dots, y_l]$$

by the admissible relations for symbols. The sequence  $y_1, \dots, y_{i-1}, a_1 y_1 + \dots + a_{i-1} y_{i-1} + y_i, y_{i+1}, \dots, y_l$  satisfies the conditions (3), (4) of the lemma also. Hence, we may actually assume that  $\varphi(x_1) = y_i$ . In this case, note that we have  $\mathfrak{q}x_1 = 0$  which implies also  $\mathfrak{q}y_i = 0$ . We have

$$[y_1, \dots, y_l] = -[y_1, \dots, y_{i-2}, y_i, y_{i-1}, y_{i+1}, \dots, y_l]$$

by the third of the admissible relations defining  $\det_{\kappa(\mathfrak{q})}(N_{\mathfrak{q}})$ . Hence we may replace  $y_1, \dots, y_l$  by the sequence  $y'_1, \dots, y'_l = y_1, \dots, y_{i-2}, y_i, y_{i-1}, y_{i+1}, \dots, y_l$  (which also satisfies conditions (3) and (4) of the lemma). Clearly this decreases the invariant  $i$  by 1 and we win by induction on  $i$ .  $\square$

To use the previous lemma we show that often sequences of elements with the required properties exist.

02QC **Lemma 43.42.** *Let  $R$  be a local Noetherian ring. Let  $\mathfrak{q} \subset R$  be a prime ideal. Let  $M$  be a finite  $R$ -module such that  $\mathfrak{q}$  is one of the minimal primes of the support of  $M$ . Then there exist  $x_1, \dots, x_l \in M$  such that*

- (1) *the support of  $M/\langle x_1, \dots, x_l \rangle$  does not contain  $\mathfrak{q}$ , and*
- (2)  *$\langle x_1, \dots, x_i \rangle / \langle x_1, \dots, x_{i-1} \rangle \cong R/\mathfrak{q}$  for  $i = 1, \dots, l$ .*

*Moreover, in this case  $l = \text{length}_{R_{\mathfrak{q}}}(M_{\mathfrak{q}})$ .*

**Proof.** The condition that  $\mathfrak{q}$  is a minimal prime in the support of  $M$  implies that  $l = \text{length}_{R_{\mathfrak{q}}}(M_{\mathfrak{q}})$  is finite (see Algebra, Lemma 61.3). Hence we can find  $y_1, \dots, y_l \in M_{\mathfrak{q}}$  such that  $\langle y_1, \dots, y_i \rangle / \langle y_1, \dots, y_{i-1} \rangle \cong \kappa(\mathfrak{q})$  for  $i = 1, \dots, l$ . We can find  $f_i \in R$ ,  $f_i \notin \mathfrak{q}$  such that  $f_i y_i$  is the image of some element  $z_i \in M$ . Moreover, as  $R$  is Noetherian we can write  $\mathfrak{q} = (g_1, \dots, g_t)$  for some  $g_j \in R$ . By assumption  $g_j y_i \in \langle y_1, \dots, y_{i-1} \rangle$  inside the module  $M_{\mathfrak{q}}$ . By our choice of  $z_i$  we can find some further elements  $f_{ji} \in R$ ,  $f_{ji} \notin \mathfrak{q}$  such that  $f_{ij} g_j z_i \in \langle z_1, \dots, z_{i-1} \rangle$  (equality in the module  $M$ ). The lemma follows by taking

$$x_1 = f_{11} f_{12} \dots f_{1t} z_1, \quad x_2 = f_{11} f_{12} \dots f_{1t} f_{21} f_{22} \dots f_{2t} z_2,$$

and so on. Namely, since all the elements  $f_i, f_{ij}$  are invertible in  $R_{\mathfrak{q}}$  we still have that  $R_{\mathfrak{q}}x_1 + \dots + R_{\mathfrak{q}}x_i / R_{\mathfrak{q}}x_1 + \dots + R_{\mathfrak{q}}x_{i-1} \cong \kappa(\mathfrak{q})$  for  $i = 1, \dots, l$ . By construction,  $\mathfrak{q}x_i \in \langle x_1, \dots, x_{i-1} \rangle$ . Thus  $\langle x_1, \dots, x_i \rangle / \langle x_1, \dots, x_{i-1} \rangle$  is an  $R$ -module generated by one element, annihilated  $\mathfrak{q}$  such that localizing at  $\mathfrak{q}$  gives a  $q$ -dimensional vector space over  $\kappa(\mathfrak{q})$ . Hence it is isomorphic to  $R/\mathfrak{q}$ .  $\square$

Here is the main result of this section. We will see below the various different consequences of this proposition. The reader is encouraged to first prove the easier Lemma 43.44 his/herself.

02QD **Proposition 43.43.** *Let  $R$  be a local Noetherian ring with residue field  $\kappa$ . Suppose that  $(M, \varphi, \psi)$  is a  $(2, 1)$ -periodic complex over  $R$ . Assume*

- (1)  $M$  is a finite  $R$ -module,
- (2) the cohomology modules of  $(M, \varphi, \psi)$  are of finite length, and
- (3)  $\dim(\text{Supp}(M)) = 1$ .

Let  $\mathfrak{q}_i, i = 1, \dots, t$  be the minimal primes of the support of  $M$ . Then we have<sup>2</sup>

$$-e_R(M, \varphi, \psi) = \sum_{i=1, \dots, t} \text{ord}_{R/\mathfrak{q}_i} (\det_{\kappa(\mathfrak{q}_i)}(M_{\mathfrak{q}_i}, \varphi_{\mathfrak{q}_i}, \psi_{\mathfrak{q}_i}))$$

**Proof.** We first reduce to the case  $t = 1$  in the following way. Note that  $\text{Supp}(M) = \{\mathfrak{m}, \mathfrak{q}_1, \dots, \mathfrak{q}_t\}$ , where  $\mathfrak{m} \subset R$  is the maximal ideal. Let  $M_i$  denote the image of  $M \rightarrow M_{\mathfrak{q}_i}$ , so  $\text{Supp}(M_i) = \{\mathfrak{m}, \mathfrak{q}_i\}$ . The map  $\varphi$  (resp.  $\psi$ ) induces an  $R$ -module map  $\varphi_i : M_i \rightarrow M_i$  (resp.  $\psi_i : M_i \rightarrow M_i$ ). Thus we get a morphism of  $(2, 1)$ -periodic complexes

$$(M, \varphi, \psi) \longrightarrow \bigoplus_{i=1, \dots, t} (M_i, \varphi_i, \psi_i).$$

The kernel and cokernel of this map have support contained in  $\{\mathfrak{m}\}$ . Hence by Lemma 2.5 we have

$$e_R(M, \varphi, \psi) = \sum_{i=1, \dots, t} e_R(M_i, \varphi_i, \psi_i)$$

On the other hand we clearly have  $M_{\mathfrak{q}_i} = M_{i, \mathfrak{q}_i}$ , and hence the terms of the right hand side of the formula of the lemma are equal to the expressions

$$\text{ord}_{R/\mathfrak{q}_i} (\det_{\kappa(\mathfrak{q}_i)}(M_{i, \mathfrak{q}_i}, \varphi_{i, \mathfrak{q}_i}, \psi_{i, \mathfrak{q}_i}))$$

In other words, if we can prove the lemma for each of the modules  $M_i$ , then the lemma holds. This reduces us to the case  $t = 1$ .

Assume we have a  $(2, 1)$ -periodic complex  $(M, \varphi, \psi)$  over a Noetherian local ring with  $M$  a finite  $R$ -module,  $\text{Supp}(M) = \{\mathfrak{m}, \mathfrak{q}\}$ , and finite length cohomology modules. The proof in this case follows from Lemma 43.41 and careful bookkeeping. Denote  $K_{\varphi} = \text{Ker}(\varphi)$ ,  $I_{\varphi} = \text{Im}(\varphi)$ ,  $K_{\psi} = \text{Ker}(\psi)$ , and  $I_{\psi} = \text{Im}(\psi)$ . Since  $R$  is Noetherian these are all finite  $R$ -modules. Set

$$a = \text{length}_{R_{\mathfrak{q}}}(I_{\varphi, \mathfrak{q}}) = \text{length}_{R_{\mathfrak{q}}}(K_{\psi, \mathfrak{q}}), \quad b = \text{length}_{R_{\mathfrak{q}}}(I_{\psi, \mathfrak{q}}) = \text{length}_{R_{\mathfrak{q}}}(K_{\varphi, \mathfrak{q}}).$$

Equalities because the complex becomes exact after localizing at  $\mathfrak{q}$ . Note that  $l = \text{length}_{R_{\mathfrak{q}}}(M_{\mathfrak{q}})$  is equal to  $l = a + b$ .

We are going to use Lemma 43.42 to choose sequences of elements in finite  $R$ -modules  $N$  with support contained in  $\{\mathfrak{m}, \mathfrak{q}\}$ . In this case  $N_{\mathfrak{q}}$  has finite length, say  $n \in \mathbf{N}$ . Let us call a sequence  $w_1, \dots, w_n \in N$  with properties (1) and (2)

<sup>2</sup> Obviously we could get rid of the minus sign by redefining  $\det_{\kappa}(M, \varphi, \psi)$  as the inverse of its current value, see Definition 43.13.

of Lemma 43.42 a “good sequence”. Note that the quotient  $N/\langle w_1, \dots, w_n \rangle$  of  $N$  by the submodule generated by a good sequence has support (contained in)  $\{\mathfrak{m}\}$  and hence has finite length (Algebra, Lemma 61.3). Moreover, the symbol  $[w_1, \dots, w_n] \in \det_{\kappa(\mathfrak{q})}(N_{\mathfrak{q}})$  is a generator, see Lemma 43.5.

Having said this we choose good sequences

$$\begin{array}{ll} x_1, \dots, x_b & \text{in } K_{\varphi}, & t_1, \dots, t_a & \text{in } K_{\psi}, \\ y_1, \dots, y_a & \text{in } I_{\varphi} \cap \langle t_1, \dots, t_a \rangle, & s_1, \dots, s_b & \text{in } I_{\psi} \cap \langle x_1, \dots, x_b \rangle. \end{array}$$

We will adjust our choices a little bit as follows. Choose lifts  $\tilde{y}_i \in M$  of  $y_i \in I_{\varphi}$  and  $\tilde{s}_i \in M$  of  $s_i \in I_{\psi}$ . It may not be the case that  $\mathfrak{q}\tilde{y}_1 \subset \langle x_1, \dots, x_b \rangle$  and it may not be the case that  $\mathfrak{q}\tilde{s}_1 \subset \langle t_1, \dots, t_a \rangle$ . However, using that  $\mathfrak{q}$  is finitely generated (as in the proof of Lemma 43.42) we can find a  $d \in R$ ,  $d \notin \mathfrak{q}$  such that  $\mathfrak{q}d\tilde{y}_1 \subset \langle x_1, \dots, x_b \rangle$  and  $\mathfrak{q}d\tilde{s}_1 \subset \langle t_1, \dots, t_a \rangle$ . Thus after replacing  $y_i$  by  $dy_i$ ,  $\tilde{y}_i$  by  $d\tilde{y}_i$ ,  $s_i$  by  $ds_i$  and  $\tilde{s}_i$  by  $d\tilde{s}_i$  we see that we may assume also that  $x_1, \dots, x_b, \tilde{y}_1, \dots, \tilde{y}_a$  and  $t_1, \dots, t_a, \tilde{s}_1, \dots, \tilde{s}_b$  are good sequences in  $M$ .

Finally, we choose a good sequence  $z_1, \dots, z_l$  in the finite  $R$ -module

$$\langle x_1, \dots, x_b, \tilde{y}_1, \dots, \tilde{y}_a \rangle \cap \langle t_1, \dots, t_a, \tilde{s}_1, \dots, \tilde{s}_b \rangle.$$

Note that this is also a good sequence in  $M$ .

Since  $I_{\varphi, \mathfrak{q}} = K_{\psi, \mathfrak{q}}$  there is a unique element  $h \in \kappa(\mathfrak{q})$  such that  $[y_1, \dots, y_a] = h[t_1, \dots, t_a]$  inside  $\det_{\kappa(\mathfrak{q})}(K_{\psi, \mathfrak{q}})$ . Similarly, as  $I_{\psi, \mathfrak{q}} = K_{\varphi, \mathfrak{q}}$  there is a unique element  $h \in \kappa(\mathfrak{q})$  such that  $[s_1, \dots, s_b] = g[x_1, \dots, x_b]$  inside  $\det_{\kappa(\mathfrak{q})}(K_{\varphi, \mathfrak{q}})$ . We can also do this with the three good sequences we have in  $M$ . All in all we get the following identities

$$\begin{aligned} [y_1, \dots, y_a] &= h[t_1, \dots, t_a] \\ [s_1, \dots, s_b] &= g[x_1, \dots, x_b] \\ [z_1, \dots, z_l] &= f_{\varphi}[x_1, \dots, x_b, \tilde{y}_1, \dots, \tilde{y}_a] \\ [z_1, \dots, z_l] &= f_{\psi}[t_1, \dots, t_a, \tilde{s}_1, \dots, \tilde{s}_b] \end{aligned}$$

for some  $g, h, f_{\varphi}, f_{\psi} \in \kappa(\mathfrak{q})$ .

Having set up all this notation let us compute  $\det_{\kappa(\mathfrak{q})}(M, \varphi, \psi)$ . Namely, consider the element  $[z_1, \dots, z_l]$ . Under the map  $\gamma_{\psi} \circ \sigma \circ \gamma_{\varphi}^{-1}$  of Definition 43.13 we have

$$\begin{aligned} [z_1, \dots, z_l] &= f_{\varphi}[x_1, \dots, x_b, \tilde{y}_1, \dots, \tilde{y}_a] \\ &\mapsto f_{\varphi}[x_1, \dots, x_b] \otimes [y_1, \dots, y_a] \\ &\mapsto f_{\varphi}h/g[t_1, \dots, t_a] \otimes [s_1, \dots, s_b] \\ &\mapsto f_{\varphi}h/g[t_1, \dots, t_a, \tilde{s}_1, \dots, \tilde{s}_b] \\ &= f_{\varphi}h/f_{\psi}g[z_1, \dots, z_l] \end{aligned}$$

This means that  $\det_{\kappa(\mathfrak{q})}(M_{\mathfrak{q}}, \varphi_{\mathfrak{q}}, \psi_{\mathfrak{q}})$  is equal to  $f_{\varphi}h/f_{\psi}g$  up to a sign.

We abbreviate the following quantities

$$\begin{aligned}
 k_\varphi &= \text{length}_R(K_\varphi/\langle x_1, \dots, x_b \rangle) \\
 k_\psi &= \text{length}_R(K_\psi/\langle t_1, \dots, t_a \rangle) \\
 i_\varphi &= \text{length}_R(I_\varphi/\langle y_1, \dots, y_a \rangle) \\
 i_\psi &= \text{length}_R(I_\psi/\langle s_1, \dots, s_a \rangle) \\
 m_\varphi &= \text{length}_R(M/\langle x_1, \dots, x_b, \tilde{y}_1, \dots, \tilde{y}_a \rangle) \\
 m_\psi &= \text{length}_R(M/\langle t_1, \dots, t_a, \tilde{s}_1, \dots, \tilde{s}_b \rangle) \\
 \delta_\varphi &= \text{length}_R(\langle x_1, \dots, x_b, \tilde{y}_1, \dots, \tilde{y}_a \rangle/\langle z_1, \dots, z_l \rangle) \\
 \delta_\psi &= \text{length}_R(\langle t_1, \dots, t_a, \tilde{s}_1, \dots, \tilde{s}_b \rangle/\langle z_1, \dots, z_l \rangle)
 \end{aligned}$$

Using the exact sequences  $0 \rightarrow K_\varphi \rightarrow M \rightarrow I_\varphi \rightarrow 0$  we get  $m_\varphi = k_\varphi + i_\varphi$ . Similarly we have  $m_\psi = k_\psi + i_\psi$ . We have  $\delta_\varphi + m_\varphi = \delta_\psi + m_\psi$  since this is equal to the colength of  $\langle z_1, \dots, z_l \rangle$  in  $M$ . Finally, we have

$$\delta_\varphi = \text{ord}_{R/\mathfrak{q}}(f_\varphi), \quad \delta_\psi = \text{ord}_{R/\mathfrak{q}}(f_\psi)$$

by our first application of the key Lemma 43.41.

Next, let us compute the multiplicity of the periodic complex

$$\begin{aligned}
 e_R(M, \varphi, \psi) &= \text{length}_R(K_\varphi/I_\psi) - \text{length}_R(K_\psi/I_\varphi) \\
 &= \text{length}_R(\langle x_1, \dots, x_b \rangle/\langle s_1, \dots, s_b \rangle) + k_\varphi - i_\psi \\
 &\quad - \text{length}_R(\langle t_1, \dots, t_a \rangle/\langle y_1, \dots, y_a \rangle) - k_\psi + i_\varphi \\
 &= \text{ord}_{R/\mathfrak{q}}(g/h) + k_\varphi - i_\psi - k_\psi + i_\varphi \\
 &= \text{ord}_{R/\mathfrak{q}}(g/h) + m_\varphi - m_\psi \\
 &= \text{ord}_{R/\mathfrak{q}}(g/h) + \delta_\psi - \delta_\varphi \\
 &= \text{ord}_{R/\mathfrak{q}}(f_\psi g / f_\varphi h)
 \end{aligned}$$

where we used the key Lemma 43.41 twice in the third equality. By our computation of  $\det_{\kappa(\mathfrak{q})}(M_{\mathfrak{q}}, \varphi_{\mathfrak{q}}, \psi_{\mathfrak{q}})$  this proves the proposition.  $\square$

In most applications the following lemma suffices.

02QE **Lemma 43.44.** *Let  $R$  be a Noetherian local ring with maximal ideal  $\mathfrak{m}$ . Let  $M$  be a finite  $R$ -module, and let  $\psi : M \rightarrow M$  be an  $R$ -module map. Assume that*

- (1)  $\text{Ker}(\psi)$  and  $\text{Coker}(\psi)$  have finite length, and
- (2)  $\dim(\text{Supp}(M)) \leq 1$ .

Write  $\text{Supp}(M) = \{\mathfrak{m}, \mathfrak{q}_1, \dots, \mathfrak{q}_t\}$  and denote  $f_i \in \kappa(\mathfrak{q}_i)^*$  the element such that  $\det_{\kappa(\mathfrak{q}_i)}(\psi_{\mathfrak{q}_i}) : \det_{\kappa(\mathfrak{q}_i)}(M_{\mathfrak{q}_i}) \rightarrow \det_{\kappa(\mathfrak{q}_i)}(M_{\mathfrak{q}_i})$  is multiplication by  $f_i$ . Then we have

$$\text{length}_R(\text{Coker}(\psi)) - \text{length}_R(\text{Ker}(\psi)) = \sum_{i=1, \dots, t} \text{ord}_{R/\mathfrak{q}_i}(f_i).$$

**Proof.** Recall that  $H^0(M, 0, \psi) = \text{Coker}(\psi)$  and  $H^1(M, 0, \psi) = \text{Ker}(\psi)$ , see remarks above Definition 2.2. The lemma follows by combining Proposition 43.43 with Lemma 43.17.

Alternative proof. Reduce to the case  $\text{Supp}(M) = \{\mathfrak{m}, \mathfrak{q}\}$  as in the proof of Proposition 43.43. Then directly combine Lemmas 43.41 and 43.42 to prove this specific case of Proposition 43.43. There is much less bookkeeping in this case, and the reader is encouraged to work this out. Details omitted.  $\square$

02QI **43.45. Application to the key lemma.** In this section we apply the results above to show the analogue of the key lemma (Lemma 6.3) with the tame symbol  $d_A$  constructed above. Please see Remark 6.4 for the relationship with Milnor  $K$ -theory.

02QJ **Lemma 43.46 (Key Lemma).** *Let  $A$  be a 2-dimensional Noetherian local domain with fraction field  $K$ . Let  $f, g \in K^*$ . Let  $\mathfrak{q}_1, \dots, \mathfrak{q}_t$  be the height 1 primes  $\mathfrak{q}$  of  $A$  such that either  $f$  or  $g$  is not an element of  $A_{\mathfrak{q}}^*$ . Then we have*

$$\sum_{i=1, \dots, t} \text{ord}_{A/\mathfrak{q}_i}(d_{A_{\mathfrak{q}_i}}(f, g)) = 0$$

We can also write this as

$$\sum_{\text{height}(\mathfrak{q})=1} \text{ord}_{A/\mathfrak{q}}(d_{A_{\mathfrak{q}}}(f, g)) = 0$$

since at any height one prime  $\mathfrak{q}$  of  $A$  where  $f, g \in A_{\mathfrak{q}}^*$  we have  $d_{A_{\mathfrak{q}}}(f, g) = 1$  by Lemma 43.33.

**Proof.** Since the tame symbols  $d_{A_{\mathfrak{q}}}(f, g)$  are additive (Lemma 43.30) and the order functions  $\text{ord}_{A/\mathfrak{q}}$  are additive (Algebra, Lemma 120.1) it suffices to prove the formula when  $f = a \in A$  and  $g = b \in A$ . In this case we see that we have to show

$$\sum_{\text{height}(\mathfrak{q})=1} \text{ord}_{A/\mathfrak{q}}(\det_{\kappa}(A_{\mathfrak{q}}/(ab), a, b)) = 0$$

By Proposition 43.43 this is equivalent to showing that

$$e_A(A/(ab), a, b) = 0.$$

Since the complex  $A/(ab) \xrightarrow{a} A/(ab) \xrightarrow{b} A/(ab) \xrightarrow{a} A/(ab)$  is exact we win.  $\square$

#### 44. Appendix B: Alternative approaches

0AYD In this appendix we first briefly try to connect the material in the main text with  $K$ -theory of coherent sheaves. In particular we describe how cupping with  $c_1$  of an invertible module is related to tensoring by this invertible module, see Lemma 44.11. This material is obviously very interesting and deserves a much more detailed and expansive exposition. In the final part (Subsections 44.12 and 44.20) we discuss some blowing up lemmas and how this relates to commutativity of intersecting with divisors; we urge the reader to avoid reading these subsections.

02S7 **44.1. Rational equivalence and  $K$ -groups.** In this section we compare the cycle groups  $Z_k(X)$  and the Chow groups  $A_k(X)$  with certain  $K_0$ -groups of abelian categories of coherent sheaves on  $X$ . We avoid having to talk about  $K_1(\mathcal{A})$  for an abelian category  $\mathcal{A}$  by dint of Homology, Lemma 10.3. In particular, the motivation for the precise form of Lemma 44.5 is that lemma.

Let us introduce the following notation. Let  $(S, \delta)$  be as in Situation 7.1. Let  $X$  be a scheme locally of finite type over  $S$ . We denote  $\text{Coh}(X) = \text{Coh}(\mathcal{O}_X)$  the category of coherent sheaves on  $X$ . It is an abelian category, see Cohomology of Schemes, Lemma 9.2. For any  $k \in \mathbf{Z}$  we let  $\text{Coh}_{\leq k}(X)$  be the full subcategory of  $\text{Coh}(X)$  consisting of those coherent sheaves  $\mathcal{F}$  having  $\dim_{\delta}(\text{Supp}(\mathcal{F})) \leq k$ .

02S8 **Lemma 44.2.** *Let  $(S, \delta)$  be as in Situation 7.1. Let  $X$  be a scheme locally of finite type over  $S$ . The categories  $\text{Coh}_{\leq k}(X)$  are Serre subcategories of the abelian category  $\text{Coh}(X)$ .*

When  $A$  is an excellent ring this is [Kat86, Proposition 1].

**Proof.** Omitted. The definition of a Serre subcategory is Homology, Definition 9.1.  $\square$

02S9 **Lemma 44.3.** *Let  $(S, \delta)$  be as in Situation 7.1. Let  $X$  be a scheme locally of finite type over  $S$ . There are maps*

$$Z_k(X) \longrightarrow K_0(\text{Coh}_{\leq k}(X)/\text{Coh}_{\leq k-1}(X)) \longrightarrow Z_k(X)$$

whose composition is the identity. The first is the map

$$\sum n_Z[Z] \mapsto \left[ \bigoplus_{n_Z > 0} \mathcal{O}_Z^{\oplus n_Z} \right] - \left[ \bigoplus_{n_Z < 0} \mathcal{O}_Z^{\oplus -n_Z} \right]$$

and the second comes from the map  $\mathcal{F} \mapsto [\mathcal{F}]_k$ . If  $X$  is quasi-compact, then both maps are isomorphisms.

**Proof.** Note that the direct sum  $\bigoplus_{n_Z > 0} \mathcal{O}_Z^{\oplus n_Z}$  is indeed a coherent sheaf on  $X$  since the family  $\{Z \mid n_Z > 0\}$  is locally finite on  $X$ . The map  $\mathcal{F} \rightarrow [\mathcal{F}]_k$  is additive on  $\text{Coh}_{\leq k}(X)$ , see Lemma 10.4. And  $[\mathcal{F}]_k = 0$  if  $\mathcal{F} \in \text{Coh}_{\leq k-1}(X)$ . This implies we have the left map as shown in the lemma. It is clear that their composition is the identity.

In case  $X$  is quasi-compact we will show that the right arrow is injective. Suppose that  $q \in K_0(\text{Coh}_{\leq k}(X)/\text{Coh}_{\leq k+1}(X))$  maps to zero in  $Z_k(X)$ . By Homology, Lemma 10.3 we can find a  $\tilde{q} \in K_0(\text{Coh}_{\leq k}(X))$  mapping to  $q$ . Write  $\tilde{q} = [\mathcal{F}] - [\mathcal{G}]$  for some  $\mathcal{F}, \mathcal{G} \in K_0(\text{Coh}_{\leq k}(X))$ . Since  $X$  is quasi-compact we may apply Cohomology of Schemes, Lemma 12.3. This shows that there exist integral closed subschemes  $Z_j, T_i \subset X$  and (nonzero) ideal sheaves  $\mathcal{I}_j \subset \mathcal{O}_{Z_j}, \mathcal{I}_i \subset \mathcal{O}_{T_i}$  such that  $\mathcal{F}$ , resp.  $\mathcal{G}$  have filtrations whose successive quotients are the sheaves  $\mathcal{I}_j$ , resp.  $\mathcal{I}_i$ . In particular we see that  $\dim_\delta(Z_j), \dim_\delta(T_i) \leq k$ . In other words we have

$$[\mathcal{F}] = \sum_j [\mathcal{I}_j], \quad [\mathcal{G}] = \sum_i [\mathcal{I}_i],$$

in  $K_0(\text{Coh}_{\leq k}(X))$ . Our assumption is that  $\sum_j [\mathcal{I}_j]_k - \sum_i [\mathcal{I}_i]_k = 0$ . It is clear that we may throw out the indices  $j$ , resp.  $i$  such that  $\dim_\delta(Z_j) < k$ , resp.  $\dim_\delta(T_i) < k$ , since the corresponding sheaves are in  $\text{Coh}_{k-1}(X)$  and also do not contribute to the cycle. Moreover, the exact sequences  $0 \rightarrow \mathcal{I}_j \rightarrow \mathcal{O}_{Z_j} \rightarrow \mathcal{O}_{Z_j}/\mathcal{I}_j \rightarrow 0$  and  $0 \rightarrow \mathcal{I}_i \rightarrow \mathcal{O}_{T_i} \rightarrow \mathcal{O}_{T_i}/\mathcal{I}_i \rightarrow 0$  show similarly that we may replace  $\mathcal{I}_j$ , resp.  $\mathcal{I}_i$  by  $\mathcal{O}_{Z_j}$ , resp.  $\mathcal{O}_{T_i}$ . OK, and finally, at this point it is clear that our assumption

$$\sum_j [\mathcal{O}_{Z_j}]_k - \sum_i [\mathcal{O}_{T_i}]_k = 0$$

implies that in  $K_0(\text{Coh}_k(X))$  we have also  $\sum_j [\mathcal{O}_{Z_j}] - \sum_i [\mathcal{O}_{T_i}] = 0$  as desired.  $\square$

02SA **Remark 44.4.** It seems likely that the arrows of Lemma 44.3 are not isomorphisms if  $X$  is not quasi-compact. For example, suppose  $X$  is an infinite disjoint union  $X = \coprod_{n \in \mathbf{N}} \mathbf{P}_k^1$  over a field  $k$ . Let  $\mathcal{F}$ , resp.  $\mathcal{G}$  be the coherent sheaf on  $X$  whose restriction to the  $n$ th summand is equal to the skyscraper sheaf at 0 associated to  $\mathcal{O}_{\mathbf{P}_k^1, 0}/\mathfrak{m}_0^n$ , resp.  $\kappa(0)^{\oplus n}$ . The cycle associated to  $\mathcal{F}$  is equal to the cycle associated to  $\mathcal{G}$ , namely both are equal to  $\sum n[0_n]$  where  $0_n \in X$  denotes 0 on the  $n$ th component of  $X$ . But there seems to be no way to show that  $[\mathcal{F}] = [\mathcal{G}]$  in  $K_0(\text{Coh}(X))$  since any proof we can envision uses infinitely many relations.

02SB **Lemma 44.5.** *Let  $(S, \delta)$  be as in Situation 7.1. Let  $X$  be a scheme locally of finite type over  $S$ . Let  $\mathcal{F}$  be a coherent sheaf on  $X$ . Let*

$$\dots \longrightarrow \mathcal{F} \xrightarrow{\varphi} \mathcal{F} \xrightarrow{\psi} \mathcal{F} \xrightarrow{\varphi} \mathcal{F} \longrightarrow \dots$$

be a complex as in Homology, Equation (10.2.1). Assume that

- (1)  $\dim_{\delta}(\text{Supp}(\mathcal{F})) \leq k + 1$ .
- (2)  $\dim_{\delta}(\text{Supp}(H^i(\mathcal{F}, \varphi, \psi))) \leq k$  for  $i = 0, 1$ .

Then we have

$$[H^0(\mathcal{F}, \varphi, \psi)]_k \sim_{\text{rat}} [H^1(\mathcal{F}, \varphi, \psi)]_k$$

as  $k$ -cycles on  $X$ .

**Proof.** Let  $\{W_j\}_{j \in J}$  be the collection of irreducible components of  $\text{Supp}(\mathcal{F})$  which have  $\delta$ -dimension  $k+1$ . Note that  $\{W_j\}$  is a locally finite collection of closed subsets of  $X$  by Lemma 10.1. For every  $j$ , let  $\xi_j \in W_j$  be the generic point. Set

$$f_j = \det_{\kappa(\xi_j)}(\mathcal{F}_{\xi_j}, \varphi_{\xi_j}, \psi_{\xi_j}) \in R(W_j)^*.$$

See Definition 43.13 for notation. We claim that

$$-[H^0(\mathcal{F}, \varphi, \psi)]_k + [H^1(\mathcal{F}, \varphi, \psi)]_k = \sum (W_j \rightarrow X)_* \text{div}(f_j)$$

If we prove this then the lemma follows.

Let  $Z \subset X$  be an integral closed subscheme of  $\delta$ -dimension  $k$ . To prove the equality above it suffices to show that the coefficient  $n$  of  $[Z]$  in  $[H^0(\mathcal{F}, \varphi, \psi)]_k - [H^1(\mathcal{F}, \varphi, \psi)]_k$  is the same as the coefficient  $m$  of  $[Z]$  in  $\sum (W_j \rightarrow X)_* \text{div}(f_j)$ . Let  $\xi \in Z$  be the generic point. Consider the local ring  $A = \mathcal{O}_{X, \xi}$ . Let  $M = \mathcal{F}_{\xi}$  as an  $A$ -module. Denote  $\varphi, \psi : M \rightarrow M$  the action of  $\varphi, \psi$  on the stalk. By our choice of  $\xi \in Z$  we have  $\delta(\xi) = k$  and hence  $\dim(\text{Supp}(M)) = 1$ . Finally, the integral closed subschemes  $W_j$  passing through  $\xi$  correspond to the minimal primes  $\mathfrak{q}_i$  of  $\text{Supp}(M)$ . In each case the element  $f_j \in R(W_j)^*$  corresponds to the element  $\det_{\kappa(\mathfrak{q}_i)}(M_{\mathfrak{q}_i}, \varphi, \psi)$  in  $\kappa(\mathfrak{q}_i)^*$ . Hence we see that

$$n = -e_A(M, \varphi, \psi)$$

and

$$m = \sum \text{ord}_{A/\mathfrak{q}_i}(\det_{\kappa(\mathfrak{q}_i)}(M_{\mathfrak{q}_i}, \varphi, \psi))$$

Thus the result follows from Proposition 43.43.  $\square$

02SC **Lemma 44.6.** *Let  $(S, \delta)$  be as in Situation 7.1. Let  $X$  be a scheme locally of finite type over  $S$ . Denote  $B_k(X)$  the image of the map*

$$K_0(\text{Coh}_{\leq k}(X)/\text{Coh}_{\leq k-1}(X)) \longrightarrow K_0(\text{Coh}_{\leq k+1}(X)/\text{Coh}_{\leq k-1}(X)).$$

There is a commutative diagram

$$\begin{array}{ccccc} K_0 \left( \frac{\text{Coh}_{\leq k}(X)}{\text{Coh}_{\leq k-1}(X)} \right) & \longrightarrow & B_k(X) & \hookrightarrow & K_0 \left( \frac{\text{Coh}_{\leq k+1}(X)}{\text{Coh}_{\leq k-1}(X)} \right) \\ \downarrow & & \downarrow & & \\ Z_k(X) & \longrightarrow & A_k(X) & & \end{array}$$

where the left vertical arrow is the one from Lemma 44.3. If  $X$  is quasi-compact then both vertical arrows are isomorphisms.

**Proof.** Suppose we have an element  $[A] - [B]$  of  $K_0(\text{Coh}_{\leq k}(X)/\text{Coh}_{\leq k-1}(X))$  which maps to zero in  $B_k(X)$ , i.e., in  $K_0(\text{Coh}_{\leq k+1}(X)/\text{Coh}_{\leq k-1}(X))$ . Suppose  $[A] = [\mathcal{A}]$  and  $[B] = [\mathcal{B}]$  for some coherent sheaves  $\mathcal{A}, \mathcal{B}$  on  $X$  supported in  $\delta$ -dimension  $\leq k$ . The assumption that  $[A] - [B]$  maps to zero in the group  $K_0(\text{Coh}_{\leq k+1}(X)/\text{Coh}_{\leq k-1}(X))$  means that there exists coherent sheaves  $\mathcal{A}', \mathcal{B}'$  on  $X$  supported in  $\delta$ -dimension  $\leq k-1$  such that  $[\mathcal{A} \oplus \mathcal{A}'] - [\mathcal{B} \oplus \mathcal{B}']$  is zero in  $K_0(\text{Coh}_{\leq k+1}(X))$  (use part (1) of Homology, Lemma 10.3). By part (2) of Homology, Lemma 10.3 this means there exists a  $(2, 1)$ -periodic complex  $(\mathcal{F}, \varphi, \psi)$  in the category  $\text{Coh}_{\leq k+1}(X)$  such that  $\mathcal{A} \oplus \mathcal{A}' = H^0(\mathcal{F}, \varphi, \psi)$  and  $\mathcal{B} \oplus \mathcal{B}' = H^1(\mathcal{F}, \varphi, \psi)$ . By Lemma 44.5 this implies that

$$[\mathcal{A} \oplus \mathcal{A}']_k \sim_{\text{rat}} [\mathcal{B} \oplus \mathcal{B}']_k$$

This proves that  $[A] - [B]$  maps to zero via the composition

$$K_0(\text{Coh}_{\leq k}(X)/\text{Coh}_{\leq k-1}(X)) \longrightarrow Z_k(X) \longrightarrow A_k(X).$$

In other words this proves the commutative diagram exists.

Next, assume that  $X$  is quasi-compact. By Lemma 44.3 the left vertical arrow is bijective. Hence it suffices to show any  $\alpha \in Z_k(X)$  which is rationally equivalent to zero maps to zero in  $B_k(X)$  via the inverse of the left vertical arrow composed with the horizontal arrow. By Lemma 21.1 we see that  $\alpha = \sum_i ([W_i]_k - [W_i]_\infty)_k$  for some closed integral subschemes  $W_i \subset X \times_S \mathbf{P}^1_S$  of  $\delta$ -dimension  $k+1$ . Moreover the family  $\{W_i\}$  is finite because  $X$  is quasi-compact. Note that the ideal sheaves  $\mathcal{I}_i, \mathcal{J}_i \subset \mathcal{O}_{W_i}$  of the effective Cartier divisors  $(W_i)_0, (W_i)_\infty$  are isomorphic (as  $\mathcal{O}_{W_i}$ -modules). This is true because the ideal sheaves of  $D_0$  and  $D_\infty$  on  $\mathbf{P}^1$  are isomorphic and  $\mathcal{I}_i, \mathcal{J}_i$  are the pullbacks of these. (Some details omitted.) Hence we have short exact sequences

$$0 \rightarrow \mathcal{I}_i \rightarrow \mathcal{O}_{W_i} \rightarrow \mathcal{O}_{(W_i)_0} \rightarrow 0, \quad 0 \rightarrow \mathcal{J}_i \rightarrow \mathcal{O}_{W_i} \rightarrow \mathcal{O}_{(W_i)_\infty} \rightarrow 0$$

of coherent  $\mathcal{O}_{W_i}$ -modules. Also, since  $[W_i]_k = [p_* \mathcal{O}_{(W_i)_0}]_k$  in  $Z_k(X)$  we see that the inverse of the left vertical arrow maps  $[W_i]_k$  to the element  $[p_* \mathcal{O}_{(W_i)_0}]$  in  $K_0(\text{Coh}_{\leq k}(X)/\text{Coh}_{\leq k-1}(X))$ . Thus we have

$$\begin{aligned} \alpha &= \sum_i ([W_i]_k - [W_i]_\infty)_k \\ &\mapsto \sum_i ([p_* \mathcal{O}_{(W_i)_0}] - [p_* \mathcal{O}_{(W_i)_\infty}]) \\ &= \sum_i ([p_* \mathcal{O}_{W_i}] - [p_* \mathcal{I}_i] - [p_* \mathcal{O}_{W_i}] + [p_* \mathcal{J}_i]) \end{aligned}$$

in  $K_0(\text{Coh}_{\leq k+1}(X)/\text{Coh}_{\leq k-1}(X))$ . By what was said above this is zero, and we win.  $\square$

02SD **Remark 44.7.** Let  $(S, \delta)$  be as in Situation 7.1. Let  $X$  be a scheme locally of finite type over  $S$ . Assume  $X$  is quasi-compact. The result of Lemma 44.6 in particular gives a map

$$A_k(X) \longrightarrow K_0(\text{Coh}(X)/\text{Coh}_{\leq k-1}(X)).$$

We have not been able to find a statement or conjecture in the literature as to whether this map is should be injective or not. If  $X$  is connected nonsingular, then, using the isomorphism  $K_0(X) = K^0(X)$  (see insert future reference here) and chern classes (see below), one can show that the map is an isomorphism up to  $(p-1)!$ -torsion where  $p = \dim_\delta(X) - k$ .



02SV **44.8. Cartier divisors and K-groups.** In this section we describe how the intersection with the first chern class of an invertible sheaf  $\mathcal{L}$  corresponds to tensoring with  $\mathcal{L} - \mathcal{O}$  in  $K$ -groups.

02QH **Lemma 44.9.** *Let  $A$  be a Noetherian local ring. Let  $M$  be a finite  $A$ -module. Let  $a, b \in A$ . Assume*

- (1)  $\dim(A) = 1$ ,
- (2) both  $a$  and  $b$  are nonzerodivisors in  $A$ ,
- (3)  $A$  has no embedded primes,
- (4)  $M$  has no embedded associated primes,
- (5)  $\text{Supp}(M) = \text{Spec}(A)$ .

Let  $I = \{x \in A \mid x(a/b) \in A\}$ . Let  $\mathfrak{q}_1, \dots, \mathfrak{q}_t$  be the minimal primes of  $A$ . Then  $(a/b)IM \subset M$  and

$$\text{length}_A(M/(a/b)IM) - \text{length}_A(M/IM) = \sum_i \text{length}_{A_{\mathfrak{q}_i}}(M_{\mathfrak{q}_i}) \text{ord}_{A/\mathfrak{q}_i}(a/b)$$

**Proof.** Since  $M$  has no embedded associated primes, and since the support of  $M$  is  $\text{Spec}(A)$  we see that  $\text{Ass}(M) = \{\mathfrak{q}_1, \dots, \mathfrak{q}_t\}$ . Hence  $a, b$  are nonzerodivisors on  $M$ . Note that

$$\begin{aligned} & \text{length}_A(M/(a/b)IM) \\ &= \text{length}_A(bM/aIM) \\ &= \text{length}_A(M/aIM) - \text{length}_A(M/bM) \\ &= \text{length}_A(M/aM) + \text{length}_A(aM/aIM) - \text{length}_A(M/bM) \\ &= \text{length}_A(M/aM) + \text{length}_A(M/IM) - \text{length}_A(M/bM) \end{aligned}$$

as the injective map  $b : M \rightarrow bM$  maps  $(a/b)IM$  to  $aIM$  and the injective map  $a : M \rightarrow aM$  maps  $IM$  to  $aIM$ . Hence the left hand side of the equation of the lemma is equal to

$$\text{length}_A(M/aM) - \text{length}_A(M/bM).$$

Applying the second formula of Lemma 3.2 with  $x = a, b$  respectively and using Algebra, Definition 120.2 of the ord-functions we get the result.  $\square$

02SW **Lemma 44.10.** *Let  $(S, \delta)$  be as in Situation 7.1. Let  $X$  be locally of finite type over  $S$ . Let  $\mathcal{L}$  be an invertible  $\mathcal{O}_X$ -module. Let  $\mathcal{F}$  be a coherent  $\mathcal{O}_X$ -module. Let  $s \in \Gamma(X, \mathcal{K}_X(\mathcal{L}))$  be a meromorphic section of  $\mathcal{L}$ . Assume*

- (1)  $\dim_\delta(X) \leq k + 1$ ,
- (2)  $X$  has no embedded points,
- (3)  $\mathcal{F}$  has no embedded associated points,
- (4) the support of  $\mathcal{F}$  is  $X$ , and
- (5) the section  $s$  is regular meromorphic.

In this situation let  $\mathcal{I} \subset \mathcal{O}_X$  be the ideal of denominators of  $s$ , see Divisors, Definition 23.10. Then we have the following:

- (1) there are short exact sequences

$$\begin{array}{ccccccc} 0 & \rightarrow & \mathcal{I}\mathcal{F} & \xrightarrow{1} & \mathcal{F} & \rightarrow & \mathcal{Q}_1 \rightarrow 0 \\ 0 & \rightarrow & \mathcal{I}\mathcal{F} & \xrightarrow{s} & \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L} & \rightarrow & \mathcal{Q}_2 \rightarrow 0 \end{array}$$

- (2) the coherent sheaves  $\mathcal{Q}_1, \mathcal{Q}_2$  are supported in  $\delta$ -dimension  $\leq k$ ,

- (3) the section  $s$  restricts to a regular meromorphic section  $s_i$  on every irreducible component  $X_i$  of  $X$  of  $\delta$ -dimension  $k + 1$ , and  
(4) writing  $[\mathcal{F}]_{k+1} = \sum m_i[X_i]$  we have

$$[\mathcal{Q}_2]_k - [\mathcal{Q}_1]_k = \sum m_i(X_i \rightarrow X)_* \operatorname{div}_{\mathcal{L}|_{X_i}}(s_i)$$

in  $Z_k(X)$ , in particular

$$[\mathcal{Q}_2]_k - [\mathcal{Q}_1]_k = c_1(\mathcal{L}) \cap [\mathcal{F}]_{k+1}$$

in  $A_k(X)$ .

**Proof.** Recall from Divisors, Lemma 24.5 the existence of injective maps  $1 : \mathcal{I}\mathcal{F} \rightarrow \mathcal{F}$  and  $s : \mathcal{I}\mathcal{F} \rightarrow \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}$  whose cokernels are supported on a closed nowhere dense subsets  $T$ . Denote  $\mathcal{Q}_i$  there cokernels as in the lemma. We conclude that  $\dim_{\delta}(\operatorname{Supp}(\mathcal{Q}_i)) \leq k$ . By Divisors, Lemmas 23.5 and 23.8 the pullbacks  $s_i$  are defined and are regular meromorphic sections for  $\mathcal{L}|_{X_i}$ . The equality of cycles in (4) implies the equality of cycle classes in (4). Hence the only remaining thing to show is that

$$[\mathcal{Q}_2]_k - [\mathcal{Q}_1]_k = \sum m_i(X_i \rightarrow X)_* \operatorname{div}_{\mathcal{L}|_{X_i}}(s_i)$$

holds in  $Z_k(X)$ . To see this, let  $Z \subset X$  be an integral closed subscheme of  $\delta$ -dimension  $k$ . Let  $\xi \in Z$  be the generic point. Let  $A = \mathcal{O}_{X,\xi}$  and  $M = \mathcal{F}_{\xi}$ . Moreover, choose a generator  $s_{\xi} \in \mathcal{L}_{\xi}$ . Then we can write  $s = (a/b)s_{\xi}$  where  $a, b \in A$  are nonzerodivisors. In this case  $I = \mathcal{I}_{\xi} = \{x \in A \mid x(a/b) \in A\}$ . In this case the coefficient of  $[Z]$  in the left hand side is

$$\operatorname{length}_A(M/(a/b)IM) - \operatorname{length}_A(M/IM)$$

and the coefficient of  $[Z]$  in the right hand side is

$$\sum \operatorname{length}_{A_{\mathfrak{q}_i}}(M_{\mathfrak{q}_i}) \operatorname{ord}_{A/\mathfrak{q}_i}(a/b)$$

where  $\mathfrak{q}_1, \dots, \mathfrak{q}_t$  are the minimal primes of the 1-dimensional local ring  $A$ . Hence the result follows from Lemma 44.9.  $\square$

02SX **Lemma 44.11.** *Let  $(S, \delta)$  be as in Situation 7.1. Let  $X$  be locally of finite type over  $S$ . Let  $\mathcal{L}$  be an invertible  $\mathcal{O}_X$ -module. Let  $\mathcal{F}$  be a coherent  $\mathcal{O}_X$ -module. Assume  $\dim_{\delta}(\operatorname{Supp}(\mathcal{F})) \leq k + 1$ . Then the element*

$$[\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}] - [\mathcal{F}] \in K_0(\operatorname{Coh}_{\leq k+1}(X)/\operatorname{Coh}_{\leq k-1}(X))$$

*lies in the subgroup  $B_k(X)$  of Lemma 44.6 and maps to the element  $c_1(\mathcal{L}) \cap [\mathcal{F}]_{k+1}$  via the map  $B_k(X) \rightarrow A_k(X)$ .*

**Proof.** Let

$$0 \rightarrow \mathcal{K} \rightarrow \mathcal{F} \rightarrow \mathcal{F}' \rightarrow 0$$

be the short exact sequence constructed in Divisors, Lemma 4.6. This in particular means that  $\mathcal{F}'$  has no embedded associated points. Since the support of  $\mathcal{K}$  is nowhere dense in the support of  $\mathcal{F}$  we see that  $\dim_{\delta}(\operatorname{Supp}(\mathcal{K})) \leq k$ . We may re-apply Divisors, Lemma 4.6 starting with  $\mathcal{K}$  to get a short exact sequence

$$0 \rightarrow \mathcal{K}'' \rightarrow \mathcal{K} \rightarrow \mathcal{K}' \rightarrow 0$$

where now  $\dim_\delta(\text{Supp}(\mathcal{K}'')) < k$  and  $\mathcal{K}'$  has no embedded associated points. Suppose we can prove the lemma for the coherent sheaves  $\mathcal{F}'$  and  $\mathcal{K}'$ . Then we see from the equations

$$[\mathcal{F}]_{k+1} = [\mathcal{F}']_{k+1} + [\mathcal{K}']_{k+1} + [\mathcal{K}''']_{k+1}$$

(use Lemma 10.4),

$$[\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}] - [\mathcal{F}] = [\mathcal{F}' \otimes_{\mathcal{O}_X} \mathcal{L}] - [\mathcal{F}'] + [\mathcal{K}' \otimes_{\mathcal{O}_X} \mathcal{L}] - [\mathcal{K}'] + [\mathcal{K}'' \otimes_{\mathcal{O}_X} \mathcal{L}] - [\mathcal{K}''']$$

(use the  $\otimes \mathcal{L}$  is exact) and the trivial vanishing of  $[\mathcal{K}''']_{k+1}$  and  $[\mathcal{K}'' \otimes_{\mathcal{O}_X} \mathcal{L}] - [\mathcal{K}'']$  in  $K_0(\text{Coh}_{\leq k+1}(X)/\text{Coh}_{\leq k-1}(X))$  that the result holds for  $\mathcal{F}$ . What this means is that we may assume that the sheaf  $\mathcal{F}$  has no embedded associated points.

Assume  $X, \mathcal{F}$  as in the lemma, and assume in addition that  $\mathcal{F}$  has no embedded associated points. Consider the sheaf of ideals  $\mathcal{I} \subset \mathcal{O}_X$ , the corresponding closed subscheme  $i : Z \rightarrow X$  and the coherent  $\mathcal{O}_Z$ -module  $\mathcal{G}$  constructed in Divisors, Lemma 4.7. Recall that  $Z$  is a locally Noetherian scheme without embedded points,  $\mathcal{G}$  is a coherent sheaf without embedded associated points, with  $\text{Supp}(\mathcal{G}) = Z$  and such that  $i_*\mathcal{G} = \mathcal{F}$ . Moreover, set  $\mathcal{N} = \mathcal{L}|_Z$ .

By Divisors, Lemma 25.4 the invertible sheaf  $\mathcal{N}$  has a regular meromorphic section  $s$  over  $Z$ . Let us denote  $\mathcal{J} \subset \mathcal{O}_Z$  the sheaf of denominators of  $s$ . By Lemma 44.10 there exist short exact sequences

$$\begin{array}{ccccccc} 0 & \rightarrow & \mathcal{J}\mathcal{G} & \xrightarrow{1} & \mathcal{G} & \rightarrow & \mathcal{Q}_1 \rightarrow 0 \\ 0 & \rightarrow & \mathcal{J}\mathcal{G} & \xrightarrow{s} & \mathcal{G} \otimes_{\mathcal{O}_Z} \mathcal{N} & \rightarrow & \mathcal{Q}_2 \rightarrow 0 \end{array}$$

such that  $\dim_\delta(\text{Supp}(\mathcal{Q}_i)) \leq k$  and such that the cycle  $[\mathcal{Q}_2]_k - [\mathcal{Q}_1]_k$  is a representative of  $c_1(\mathcal{N}) \cap [\mathcal{G}]_{k+1}$ . We see (using the fact that  $i_*(\mathcal{G} \otimes \mathcal{N}) = \mathcal{F} \otimes \mathcal{L}$  by the projection formula, see Cohomology, Lemma 45.2) that

$$[\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}] - [\mathcal{F}] = [i_*\mathcal{Q}_2] - [i_*\mathcal{Q}_1]$$

in  $K_0(\text{Coh}_{\leq k+1}(X)/\text{Coh}_{\leq k-1}(X))$ . This already shows that  $[\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}] - [\mathcal{F}]$  is an element of  $B_k(X)$ . Moreover we have

$$\begin{aligned} [i_*\mathcal{Q}_2]_k - [i_*\mathcal{Q}_1]_k &= i_*([\mathcal{Q}_2]_k - [\mathcal{Q}_1]_k) \\ &= i_*(c_1(\mathcal{N}) \cap [\mathcal{G}]_{k+1}) \\ &= c_1(\mathcal{L}) \cap i_*[\mathcal{G}]_{k+1} \\ &= c_1(\mathcal{L}) \cap [\mathcal{F}]_{k+1} \end{aligned}$$

by the above and Lemmas 24.4 and 12.3. And this agree with the image of the element under  $B_k(X) \rightarrow A_k(X)$  by definition. Hence the lemma is proved.  $\square$

**02SY 44.12. Blowing up lemmas.** In this section we prove some lemmas on representing Cartier divisors by suitable effective Cartier divisors on blowups. These lemmas can be found in [Ful98, Section 2.4]. We have adapted the formulation so they also work in the non-finite type setting. It may happen that the morphism  $b$  of Lemma 44.19 is a composition of infinitely many blowups, but over any given quasi-compact open  $W \subset X$  one needs only finitely many blowups (and this is the result of loc. cit.).

02SZ **Lemma 44.13.** *Let  $(S, \delta)$  be as in Situation 7.1. Let  $X, Y$  be locally of finite type over  $S$ . Let  $f : X \rightarrow Y$  be a proper morphism. Let  $D \subset Y$  be an effective Cartier divisor. Assume  $X, Y$  integral,  $n = \dim_\delta(X) = \dim_\delta(Y)$  and  $f$  dominant. Then*

$$f_*[f^{-1}(D)]_{n-1} = [R(X) : R(Y)][D]_{n-1}.$$

*In particular if  $f$  is birational then  $f_*[f^{-1}(D)]_{n-1} = [D]_{n-1}$ .*

**Proof.** Immediate from Lemma 24.3 and the fact that  $D$  is the zero scheme of the canonical section  $1_D$  of  $\mathcal{O}_X(D)$ .  $\square$

02T0 **Lemma 44.14.** *Let  $(S, \delta)$  be as in Situation 7.1. Let  $X$  be locally of finite type over  $S$ . Assume  $X$  integral with  $\dim_\delta(X) = n$ . Let  $\mathcal{L}$  be an invertible  $\mathcal{O}_X$ -module. Let  $s$  be a nonzero meromorphic section of  $\mathcal{L}$ . Let  $U \subset X$  be the maximal open subscheme such that  $s$  corresponds to a section of  $\mathcal{L}$  over  $U$ . There exists a projective morphism*

$$\pi : X' \longrightarrow X$$

*such that*

- (1)  $X'$  is integral,
- (2)  $\pi|_{\pi^{-1}(U)} : \pi^{-1}(U) \rightarrow U$  is an isomorphism,
- (3) there exist effective Cartier divisors  $D, E \subset X'$  such that

$$\pi^*\mathcal{L} = \mathcal{O}_{X'}(D - E),$$

- (4) the meromorphic section  $s$  corresponds, via the isomorphism above, to the meromorphic section  $1_D \otimes (1_E)^{-1}$  (see Divisors, Definition 14.1),
- (5) we have

$$\pi_*([D]_{n-1} - [E]_{n-1}) = \text{div}_{\mathcal{L}}(s)$$

*in  $Z_{n-1}(X)$ .*

**Proof.** Let  $\mathcal{I} \subset \mathcal{O}_X$  be the quasi-coherent ideal sheaf of denominators of  $s$ , see Divisors, Definition 23.10. By Divisors, Lemma 34.6 we get (2), (3), and (4). By Divisors, Lemma 32.9 we get (1). By Divisors, Lemma 32.13 the morphism  $\pi$  is projective. We still have to prove (5). By Lemma 24.3 we have

$$\pi_*(\text{div}_{\mathcal{L}'}(s')) = \text{div}_{\mathcal{L}}(s).$$

Hence it suffices to show that  $\text{div}_{\mathcal{L}'}(s') = [D]_{n-1} - [E]_{n-1}$ . This follows from the equality  $s' = 1_D \otimes 1_E^{-1}$  and additivity, see Divisors, Lemma 27.5.  $\square$

02T1 **Definition 44.15.** *Let  $(S, \delta)$  be as in Situation 7.1. Let  $X$  be locally of finite type over  $S$ . Assume  $X$  integral and  $\dim_\delta(X) = n$ . Let  $D_1, D_2$  be two effective Cartier divisors in  $X$ . Let  $Z \subset X$  be an integral closed subscheme with  $\dim_\delta(Z) = n - 1$ . The  $\epsilon$ -invariant of this situation is*

$$\epsilon_Z(D_1, D_2) = n_Z \cdot m_Z$$

where  $n_Z$ , resp.  $m_Z$  is the coefficient of  $Z$  in the  $(n-1)$ -cycle  $[D_1]_{n-1}$ , resp.  $[D_2]_{n-1}$ .

02T2 **Lemma 44.16.** *Let  $(S, \delta)$  be as in Situation 7.1. Let  $X$  be locally of finite type over  $S$ . Assume  $X$  integral and  $\dim_\delta(X) = n$ . Let  $D_1, D_2$  be two effective Cartier divisors in  $X$ . Let  $Z$  be an open and closed subscheme of the scheme  $D_1 \cap D_2$ . Assume  $\dim_\delta(D_1 \cap D_2 \setminus Z) \leq n - 2$ . Then there exists a morphism  $b : X' \rightarrow X$ , and Cartier divisors  $D'_1, D'_2, E$  on  $X'$  with the following properties*

- (1)  $X'$  is integral,
- (2)  $b$  is projective,

- (3)  $b$  is the blowup of  $X$  in the closed subscheme  $Z$ ,
- (4)  $E = b^{-1}(Z)$ ,
- (5)  $b^{-1}(D_1) = D'_1 + E$ , and  $b^{-1}D_2 = D'_2 + E$ ,
- (6)  $\dim_\delta(D'_1 \cap D'_2) \leq n - 2$ , and if  $Z = D_1 \cap D_2$  then  $D'_1 \cap D'_2 = \emptyset$ ,
- (7) for every integral closed subscheme  $W'$  with  $\dim_\delta(W') = n - 1$  we have
  - (a) if  $\epsilon_{W'}(D'_1, E) > 0$ , then setting  $W = b(W')$  we have  $\dim_\delta(W) = n - 1$  and
 
$$\epsilon_{W'}(D'_1, E) < \epsilon_W(D_1, D_2),$$
  - (b) if  $\epsilon_{W'}(D'_2, E) > 0$ , then setting  $W = b(W')$  we have  $\dim_\delta(W) = n - 1$  and
 
$$\epsilon_{W'}(D'_2, E) < \epsilon_W(D_1, D_2),$$

**Proof.** Note that the quasi-coherent ideal sheaf  $\mathcal{I} = \mathcal{I}_{D_1} + \mathcal{I}_{D_2}$  defines the scheme theoretic intersection  $D_1 \cap D_2 \subset X$ . Since  $Z$  is a union of connected components of  $D_1 \cap D_2$  we see that for every  $z \in Z$  the kernel of  $\mathcal{O}_{X,z} \rightarrow \mathcal{O}_{Z,z}$  is equal to  $\mathcal{I}_z$ . Let  $b : X' \rightarrow X$  be the blowup of  $X$  in  $Z$ . (So Zariski locally around  $Z$  it is the blowup of  $X$  in  $\mathcal{I}$ .) Denote  $E = b^{-1}(Z)$  the corresponding effective Cartier divisor, see Divisors, Lemma 32.4. Since  $Z \subset D_1$  we have  $E \subset f^{-1}(D_1)$  and hence  $D_1 = D'_1 + E$  for some effective Cartier divisor  $D'_1 \subset X'$ , see Divisors, Lemma 13.8. Similarly  $D_2 = D'_2 + E$ . This takes care of assertions (1) – (5).

Note that if  $W'$  is as in (7) (a) or (7) (b), then the image  $W$  of  $W'$  is contained in  $D_1 \cap D_2$ . If  $W$  is not contained in  $Z$ , then  $b$  is an isomorphism at the generic point of  $W$  and we see that  $\dim_\delta(W) = \dim_\delta(W') = n - 1$  which contradicts the assumption that  $\dim_\delta(D_1 \cap D_2 \setminus Z) \leq n - 2$ . Hence  $W \subset Z$ . This means that to prove (6) and (7) we may work locally around  $Z$  on  $X$ .

Thus we may assume that  $X = \text{Spec}(A)$  with  $A$  a Noetherian domain, and  $D_1 = \text{Spec}(A/a)$ ,  $D_2 = \text{Spec}(A/b)$  and  $Z = D_1 \cap D_2$ . Set  $I = (a, b)$ . Since  $A$  is a domain and  $a, b \neq 0$  we can cover the blowup by two patches, namely  $U = \text{Spec}(A[s]/(as - b))$  and  $V = \text{Spec}(A[t]/(bt - a))$ . These patches are glued using the isomorphism  $A[s, s^{-1}]/(as - b) \cong A[t, t^{-1}]/(bt - a)$  which maps  $s$  to  $t^{-1}$ . The effective Cartier divisor  $E$  is described by  $\text{Spec}(A[s]/(as - b, a)) \subset U$  and  $\text{Spec}(A[t]/(bt - a, b)) \subset V$ . The closed subscheme  $D'_1$  corresponds to  $\text{Spec}(A[t]/(bt - a, t)) \subset U$ . The closed subscheme  $D'_2$  corresponds to  $\text{Spec}(A[s]/(as - b, s)) \subset V$ . Since “ $ts = 1$ ” we see that  $D'_1 \cap D'_2 = \emptyset$ .

Suppose we have a prime  $\mathfrak{q} \subset A[s]/(as - b)$  of height one with  $s, a \in \mathfrak{q}$ . Let  $\mathfrak{p} \subset A$  be the corresponding prime of  $A$ . Observe that  $a, b \in \mathfrak{p}$ . By the dimension formula we see that  $\dim(A_{\mathfrak{p}}) = 1$  as well. The final assertion to be shown is that

$$\text{ord}_{A_{\mathfrak{p}}}(a)\text{ord}_{A_{\mathfrak{p}}}(b) > \text{ord}_{B_{\mathfrak{q}}}(a)\text{ord}_{B_{\mathfrak{q}}}(s)$$

where  $B = A[s]/(as - b)$ . By Algebra, Lemma 123.1 we have  $\text{ord}_{A_{\mathfrak{p}}}(x) \geq \text{ord}_{B_{\mathfrak{q}}}(x)$  for  $x = a, b$ . Since  $\text{ord}_{B_{\mathfrak{q}}}(s) > 0$  we win by additivity of the ord function and the fact that  $as = b$ .  $\square$

02T3 **Definition 44.17.** Let  $X$  be a scheme. Let  $\{D_i\}_{i \in I}$  be a locally finite collection of effective Cartier divisors on  $X$ . Suppose given a function  $I \rightarrow \mathbf{Z}_{\geq 0}$ ,  $i \mapsto n_i$ . The sum of the effective Cartier divisors  $D = \sum n_i D_i$ , is the unique effective Cartier divisor  $D \subset X$  such that on any quasi-compact open  $U \subset X$  we have  $D|_U = \sum_{D_i \cap U \neq \emptyset} n_i D_i|_U$  is the sum as in Divisors, Definition 13.6.

02T4 **Lemma 44.18.** *Let  $(S, \delta)$  be as in Situation 7.1. Let  $X$  be locally of finite type over  $S$ . Assume  $X$  integral and  $\dim_\delta(X) = n$ . Let  $\{D_i\}_{i \in I}$  be a locally finite collection of effective Cartier divisors on  $X$ . Suppose given  $n_i \geq 0$  for  $i \in I$ . Then*

$$[D]_{n-1} = \sum_i n_i [D_i]_{n-1}$$

in  $Z_{n-1}(X)$ .

**Proof.** Since we are proving an equality of cycles we may work locally on  $X$ . Hence this reduces to a finite sum, and by induction to a sum of two effective Cartier divisors  $D = D_1 + D_2$ . By Lemma 22.2 we see that  $D_1 = \operatorname{div}_{\mathcal{O}_X(D_1)}(1_{D_1})$  where  $1_{D_1}$  denotes the canonical section of  $\mathcal{O}_X(D_1)$ . Of course we have the same statement for  $D_2$  and  $D$ . Since  $1_D = 1_{D_1} \otimes 1_{D_2}$  via the identification  $\mathcal{O}_X(D) = \mathcal{O}_X(D_1) \otimes \mathcal{O}_X(D_2)$  we win by Divisors, Lemma 27.5.  $\square$

02T5 **Lemma 44.19.** *Let  $(S, \delta)$  be as in Situation 7.1. Let  $X$  be locally of finite type over  $S$ . Assume  $X$  integral and  $\dim_\delta(X) = d$ . Let  $\{D_i\}_{i \in I}$  be a locally finite collection of effective Cartier divisors on  $X$ . Assume that for all  $\{i, j, k\} \subset I$ ,  $\#\{i, j, k\} = 3$  we have  $D_i \cap D_j \cap D_k = \emptyset$ . Then there exist*

- (1) *an open subscheme  $U \subset X$  with  $\dim_\delta(X \setminus U) \leq d - 3$ ,*
- (2) *a morphism  $b : U' \rightarrow U$ , and*
- (3) *effective Cartier divisors  $\{D'_j\}_{j \in J}$  on  $U'$*

with the following properties:

- (1)  *$b$  is proper morphism  $b : U' \rightarrow U$ ,*
- (2)  *$U'$  is integral,*
- (3)  *$b$  is an isomorphism over the complement of the union of the pairwise intersections of the  $D_i|_U$ ,*
- (4)  *$\{D'_j\}_{j \in J}$  is a locally finite collection of effective Cartier divisors on  $U'$ ,*
- (5)  *$\dim_\delta(D'_j \cap D'_{j'}) \leq d - 2$  if  $j \neq j'$ , and*
- (6)  *$b^{-1}(D_i|_U) = \sum n_{ij} D'_j$  for certain  $n_{ij} \geq 0$ .*

Moreover, if  $X$  is quasi-compact, then we may assume  $U = X$  in the above.

**Proof.** Let us first prove this in the quasi-compact case, since it is perhaps the most interesting case. In this case we produce inductively a sequence of blowups

$$X = X_0 \xleftarrow{b_0} X_1 \xleftarrow{b_1} X_2 \leftarrow \dots$$

and finite sets of effective Cartier divisors  $\{D_{n,i}\}_{i \in I_n}$ . At each stage these will have the property that any triple intersection  $D_{n,i} \cap D_{n,j} \cap D_{n,k}$  is empty. Moreover, for each  $n \geq 0$  we will have  $I_{n+1} = I_n \amalg P(I_n)$  where  $P(I_n)$  denotes the set of pairs of elements of  $I_n$ . Finally, we will have

$$b_n^{-1}(D_{n,i}) = D_{n+1,i} + \sum_{i' \in I_n, i' \neq i} D_{n+1,\{i,i'\}}$$

We conclude that for each  $n \geq 0$  we have  $(b_0 \circ \dots \circ b_n)^{-1}(D_i)$  is a nonnegative integer combination of the divisors  $D_{n+1,j}$ ,  $j \in I_{n+1}$ .

To start the induction we set  $X_0 = X$  and  $I_0 = I$  and  $D_{0,i} = D_i$ .

Given  $(X_n, \{D_{n,i}\}_{i \in I_n})$  let  $X_{n+1}$  be the blowup of  $X_n$  in the closed subscheme  $Z_n = \bigcup_{\{i,i'\} \in P(I_n)} D_{n,i} \cap D_{n,i'}$ . Note that the closed subschemes  $D_{n,i} \cap D_{n,i'}$  are pairwise disjoint by our assumption on triple intersections. In other words we may write  $Z_n = \coprod_{\{i,i'\} \in P(I_n)} D_{n,i} \cap D_{n,i'}$ . Moreover, in a Zariski neighbourhood of  $D_{n,i} \cap D_{n,i'}$

the morphism  $b_n$  is equal to the blowup of the scheme  $X_n$  in the closed subscheme  $D_{n,i} \cap D_{n,i'}$ , and the results of Lemma 44.16 apply. Hence setting  $D_{n+1,\{i,i'\}} = b_n^{-1}(D_i \cap D_{i'})$  we get an effective Cartier divisor. The Cartier divisors  $D_{n+1,\{i,i'\}}$  are pairwise disjoint. Clearly we have  $b_n^{-1}(D_{n,i}) \supset D_{n+1,\{i,i'\}}$  for every  $i' \in I_n$ ,  $i' \neq i$ . Hence, applying Divisors, Lemma 13.8 we see that indeed  $b^{-1}(D_{n,i}) = D_{n+1,i} + \sum_{i' \in I_n, i' \neq i} D_{n+1,\{i,i'\}}$  for some effective Cartier divisor  $D_{n+1,i}$  on  $X_{n+1}$ . In a neighbourhood of  $D_{n+1,\{i,i'\}}$  these divisors  $D_{n+1,i}$  play the role of the primed divisors of Lemma 44.16. In particular we conclude that  $D_{n+1,i} \cap D_{n+1,i'} = \emptyset$  if  $i \neq i'$ ,  $i, i' \in I_n$  by part (6) of Lemma 44.16. This already implies that triple intersections of the divisors  $D_{n+1,i}$  are zero.

OK, and at this point we can use the quasi-compactness of  $X$  to conclude that the invariant

(44.19.1)  
 02T6  $\epsilon(X, \{D_i\}_{i \in I}) = \max\{\epsilon_Z(D_i, D_{i'}) \mid Z \subset X, \dim_\delta(Z) = d - 1, \{i, i'\} \in P(I)\}$

is finite, since after all each  $D_i$  has at most finitely many irreducible components. We claim that for some  $n$  the invariant  $\epsilon(X_n, \{D_{n,i}\}_{i \in I_n})$  is zero. Namely, if not then by Lemma 44.16 we have a strictly decreasing sequence

$$\epsilon(X, \{D_i\}_{i \in I}) = \epsilon(X_0, \{D_{0,i}\}_{i \in I_0}) > \epsilon(X_1, \{D_{1,i}\}_{i \in I_1}) > \dots$$

of positive integers which is a contradiction. Take  $n$  with invariant  $\epsilon(X_n, \{D_{n,i}\}_{i \in I_n})$  equal to zero. This means that there is no integral closed subscheme  $Z \subset X_n$  and no pair of indices  $i, i' \in I_n$  such that  $\epsilon_Z(D_{n,i}, D_{n,i'}) > 0$ . In other words,  $\dim_\delta(D_{n,i}, D_{n,i'}) \leq d - 2$  for all pairs  $\{i, i'\} \in P(I_n)$  as desired.

Next, we come to the general case where we no longer assume that the scheme  $X$  is quasi-compact. The problem with the idea from the first part of the proof is that we may get an infinite sequence of blowups with centers dominating a fixed point of  $X$ . In order to avoid this we cut out suitable closed subsets of codimension  $\geq 3$  at each stage. Namely, we will construct by induction a sequence of morphisms having the following shape

$$\begin{array}{c} X = X_0 \\ \uparrow j_0 \\ U_0 \xleftarrow{b_0} X_1 \\ \uparrow j_1 \\ U_1 \xleftarrow{b_1} X_2 \\ \uparrow j_2 \\ U_2 \xleftarrow{b_2} X_3 \end{array}$$

Each of the morphisms  $j_n : U_n \rightarrow X_n$  will be an open immersion. Each of the morphisms  $b_n : X_{n+1} \rightarrow U_n$  will be a proper birational morphism of integral schemes. As in the quasi-compact case we will have effective Cartier divisors  $\{D_{n,i}\}_{i \in I_n}$  on  $X_n$ . At each stage these will have the property that any triple intersection  $D_{n,i} \cap D_{n,j} \cap D_{n,k}$  is empty. Moreover, for each  $n \geq 0$  we will have  $I_{n+1} = I_n \amalg P(I_n)$  where  $P(I_n)$  denotes the set of pairs of elements of  $I_n$ . Finally, we will arrange it

so that

$$b_n^{-1}(D_{n,i}|_{U_n}) = D_{n+1,i} + \sum_{i' \in I_n, i' \neq i} D_{n+1,\{i,i'\}}$$

We start the induction by setting  $X_0 = X$ ,  $I_0 = I$  and  $D_{0,i} = D_i$ .

Given  $(X_n, \{D_{n,i}\})$  we construct the open subscheme  $U_n$  as follows. For each pair  $\{i, i'\} \in P(I_n)$  consider the closed subscheme  $D_{n,i} \cap D_{n,i'}$ . This has “good” irreducible components which have  $\delta$ -dimension  $d-2$  and “bad” irreducible components which have  $\delta$ -dimension  $d-1$ . Let us set

$$\text{Bad}(i, i') = \bigcup_{W \subset D_{n,i} \cap D_{n,i'} \text{ irred. comp. with } \dim_\delta(W)=d-1} W$$

and similarly

$$\text{Good}(i, i') = \bigcup_{W \subset D_{n,i} \cap D_{n,i'} \text{ irred. comp. with } \dim_\delta(W)=d-2} W.$$

Then  $D_{n,i} \cap D_{n,i'} = \text{Bad}(i, i') \cup \text{Good}(i, i')$  and moreover we have  $\dim_\delta(\text{Bad}(i, i') \cap \text{Good}(i, i')) \leq d-3$ . Here is our choice of  $U_n$ :

$$U_n = X_n \setminus \bigcup_{\{i,i'\} \in P(I_n)} \text{Bad}(i, i') \cap \text{Good}(i, i').$$

By our condition on triple intersections of the divisors  $D_{n,i}$  we see that the union is actually a disjoint union. Moreover, we see that (as a scheme)

$$D_{n,i}|_{U_n} \cap D_{n,i'}|_{U_n} = Z_{n,i,i'} \amalg G_{n,i,i'}$$

where  $Z_{n,i,i'}$  is  $\delta$ -equidimensional of dimension  $d-1$  and  $G_{n,i,i'}$  is  $\delta$ -equidimensional of dimension  $d-2$ . (So topologically  $Z_{n,i,i'}$  is the union of the bad components but throw out intersections with good components.) Finally we set

$$Z_n = \bigcup_{\{i,i'\} \in P(I_n)} Z_{n,i,i'} = \coprod_{\{i,i'\} \in P(I_n)} Z_{n,i,i'},$$

and we let  $b_n : X_{n+1} \rightarrow X_n$  be the blowup in  $Z_n$ . Note that Lemma 44.16 applies to the morphism  $b_n : X_{n+1} \rightarrow X_n$  locally around each of the loci  $D_{n,i}|_{U_n} \cap D_{n,i'}|_{U_n}$ . Hence, exactly as in the first part of the proof we obtain effective Cartier divisors  $D_{n+1,\{i,i'\}}$  for  $\{i, i'\} \in P(I_n)$  and effective Cartier divisors  $D_{n+1,i}$  for  $i \in I_n$  such that  $b_n^{-1}(D_{n,i}|_{U_n}) = D_{n+1,i} + \sum_{i' \in I_n, i' \neq i} D_{n+1,\{i,i'\}}$ . For each  $n$  denote  $\pi_n : X_n \rightarrow X$  the morphism obtained as the composition  $j_0 \circ \dots \circ j_{n-1} \circ b_{n-1}$ .

**Claim:** given any quasi-compact open  $V \subset X$  for all sufficiently large  $n$  the maps

$$\pi_n^{-1}(V) \leftarrow \pi_{n+1}^{-1}(V) \leftarrow \dots$$

are all isomorphisms. Namely, if the map  $\pi_n^{-1}(V) \leftarrow \pi_{n+1}^{-1}(V)$  is not an isomorphism, then  $Z_{n,i,i'} \cap \pi_n^{-1}(V) \neq \emptyset$  for some  $\{i, i'\} \in P(I_n)$ . Hence there exists an irreducible component  $W \subset D_{n,i} \cap D_{n,i'}$  with  $\dim_\delta(W) = d-1$ . In particular we see that  $\epsilon_W(D_{n,i}, D_{n,i'}) > 0$ . Applying Lemma 44.16 repeatedly we see that

$$\epsilon_W(D_{n,i}, D_{n,i'}) < \epsilon(V, \{D_i|_V\}) - n$$

with  $\epsilon(V, \{D_i|_V\})$  as in (44.19.1). Since  $V$  is quasi-compact, we have  $\epsilon(V, \{D_i|_V\}) < \infty$  and taking  $n > \epsilon(V, \{D_i|_V\})$  we see the result.

Note that by construction the difference  $X_n \setminus U_n$  has  $\dim_\delta(X_n \setminus U_n) \leq d-3$ . Let  $T_n = \pi_n(X_n \setminus U_n)$  be its image in  $X$ . Traversing in the diagram of maps above using each  $b_n$  is closed it follows that  $T_0 \cup \dots \cup T_n$  is a closed subset of  $X$  for each  $n$ . Any  $t \in T_n$  satisfies  $\delta(t) \leq d-3$  by construction. Hence  $\overline{T_n} \subset X$  is a closed subset



with  $\dim_\delta(T_n) \leq d-3$ . By the claim above we see that for any quasi-compact open  $V \subset X$  we have  $T_n \cap V \neq \emptyset$  for at most finitely many  $n$ . Hence  $\{\overline{T_n}\}_{n \geq 0}$  is a locally finite collection of closed subsets, and we may set  $U = X \setminus \bigcup \overline{T_n}$ . This will be  $U$  as in the lemma.

Note that  $U_n \cap \pi_n^{-1}(U) = \pi_n^{-1}(U)$  by construction of  $U$ . Hence all the morphisms

$$b_n : \pi_{n+1}^{-1}(U) \longrightarrow \pi_n^{-1}(U)$$

are proper. Moreover, by the claim they eventually become isomorphisms over each quasi-compact open of  $X$ . Hence we can define

$$U' = \lim_n \pi_n^{-1}(U).$$

The induced morphism  $b : U' \rightarrow U$  is proper since this is local on  $U$ , and over each compact open the limit stabilizes. Similarly we set  $J = \bigcup_{n \geq 0} I_n$  using the inclusions  $I_n \rightarrow I_{n+1}$  from the construction. For  $j \in J$  choose an  $n_0$  such that  $j$  corresponds to  $i \in I_{n_0}$  and define  $D'_j = \lim_{n \geq n_0} D_{n,i}$ . Again this makes sense as locally over  $X$  the morphisms stabilize. The other claims of the lemma are verified as in the case of a quasi-compact  $X$ .  $\square$

0AYE **44.20. Commutativity.** The results of this subsection can be used to provide an alternative proof of the lemmas of Section 26 as was done in an earlier version of this chapter. See also the discussion preceding Lemma 44.24.

02TC **Lemma 44.21.** *Let  $(S, \delta)$  be as in Situation 7.1. Let  $X$  be locally of finite type over  $S$ . Let  $\{i_j : D_j \rightarrow X\}_{j \in J}$  be a locally finite collection of effective Cartier divisors on  $X$ . Let  $n_j > 0, j \in J$ . Set  $D = \sum_{j \in J} n_j D_j$ , and denote  $i : D \rightarrow X$  the inclusion morphism. Let  $\alpha \in Z_{k+1}(X)$ . Then*

$$p : \prod_{j \in J} D_j \longrightarrow D$$

is proper and

$$i^* \alpha = p_* \left( \sum n_j i_j^* \alpha \right)$$

in  $A_k(D)$ .

**Proof.** The proof of this lemma is made a bit longer than expected by a subtlety concerning infinite sums of rational equivalences. In the quasi-compact case the family  $D_j$  is finite and the result is altogether easy and a straightforward consequence of Lemmas 22.2 and Divisors, 27.5 and the definitions.

The morphism  $p$  is proper since the family  $\{D_j\}_{j \in J}$  is locally finite. Write  $\alpha = \sum_{a \in A} m_a [W_a]$  with  $W_a \subset X$  an integral closed subscheme of  $\delta$ -dimension  $k+1$ . Denote  $i_a : W_a \rightarrow X$  the closed immersion. We assume that  $m_a \neq 0$  for all  $a \in A$  such that  $\{W_a\}_{a \in A}$  is locally finite on  $X$ .

Observe that by Definition 27.1 the class  $i^* \alpha$  is the class of a cycle  $\sum m_a \beta_a$  for certain  $\beta_a \in Z_k(W_a \cap D)$ . Namely, if  $W_a \not\subset D$  then  $\beta_a = [D \cap W_a]_k$  and if  $W_a \subset D$ , then  $\beta_a$  is a cycle representing  $c_1(\mathcal{O}_X(D)) \cap [W_a]$ .

For each  $a \in A$  write  $J = J_{a,1} \amalg J_{a,2} \amalg J_{a,3}$  where

- (1)  $j \in J_{a,1}$  if and only if  $W_a \cap D_j = \emptyset$ ,
- (2)  $j \in J_{a,2}$  if and only if  $W_a \not\subset W_a \cap D_1 \neq \emptyset$ , and
- (3)  $j \in J_{a,3}$  if and only if  $W_a \subset D_j$ .

Since the family  $\{D_j\}$  is locally finite we see that  $J_{a,3}$  is a finite set. For every  $a \in A$  and  $j \in J$  we choose a cycle  $\beta_{a,j} \in Z_k(W_a \cap D_j)$  as follows

- (1) if  $j \in J_{a,1}$  we set  $\beta_{a,j} = 0$ ,
- (2) if  $j \in J_{a,2}$  we set  $\beta_{a,j} = [D_j \cap W_a]_k$ , and
- (3) if  $j \in J_{a,3}$  we choose  $\beta_{a,j} \in Z_k(W_a)$  representing  $c_1(i_a^* \mathcal{O}_X(D_j)) \cap [W_j]$ .

We claim that

$$\beta_a \sim_{\text{rat}} \sum_{j \in J} n_j \beta_{a,j}$$

in  $A_k(W_a \cap D)$ .

Case I:  $W_a \not\subset D$ . In this case  $J_{a,3} = \emptyset$ . Thus it suffices to show that  $[D \cap W_a]_k = \sum n_j [D_j \cap W_a]_k$  as cycles. This is Lemma 44.18.

Case II:  $W_a \subset D$ . In this case  $\beta_a$  is a cycle representing  $c_1(i_a^* \mathcal{O}_X(D)) \cap [W_a]$ . Write  $D = D_{a,1} + D_{a,2} + D_{a,3}$  with  $D_{a,s} = \sum_{j \in J_{a,s}} n_j D_j$ . By Divisors, Lemma 27.5 we have

$$\begin{aligned} c_1(i_a^* \mathcal{O}_X(D)) \cap [W_a] &= c_1(i_a^* \mathcal{O}_X(D_{a,1})) \cap [W_a] + c_1(i_a^* \mathcal{O}_X(D_{a,2})) \cap [W_a] \\ &\quad + c_1(i_a^* \mathcal{O}_X(D_{a,3})) \cap [W_a]. \end{aligned}$$

It is clear that the first term of the sum is zero. Since  $J_{a,3}$  is finite we see that the last term agrees with  $\sum_{j \in J_{a,3}} n_j c_1(i_a^* \mathcal{L}_j) \cap [W_a]$ , see Divisors, Lemma 27.5. This is represented by  $\sum_{j \in J_{a,3}} n_j \beta_{a,j}$ . Finally, by Case I we see that the middle term is represented by the cycle  $\sum_{j \in J_{a,2}} n_j [D_j \cap W_a]_k = \sum_{j \in J_{a,2}} n_j \beta_{a,j}$ . Whence the claim in this case.

At this point we are ready to finish the proof of the lemma. Namely, we have  $i^* D \sim_{\text{rat}} \sum m_a \beta_a$  by our choice of  $\beta_a$ . For each  $a$  we have  $\beta_a \sim_{\text{rat}} \sum_j \beta_{a,j}$  with the rational equivalence taking place on  $D \cap W_a$ . Since the collection of closed subschemes  $D \cap W_a$  is locally finite on  $D$ , we see that also  $\sum m_a \beta_a \sim_{\text{rat}} \sum_{a,j} m_a \beta_{a,j}$  on  $D$ ! (See Remark 19.4.) Ok, and now it is clear that  $\sum_a m_a \beta_{a,j}$  (viewed as a cycle on  $D_j$ ) represents  $i_j^* \alpha$  and hence  $\sum_{a,j} m_a \beta_{a,j}$  represents  $p_* \sum_j i_j^* \alpha$  and we win.  $\square$

02TD **Lemma 44.22.** *Let  $(S, \delta)$  be as in Situation 7.1. Let  $X$  be locally of finite type over  $S$ . Assume  $X$  integral and  $\dim_\delta(X) = n$ . Let  $D, D'$  be effective Cartier divisors on  $X$ . Assume  $\dim_\delta(D \cap D') = n - 2$ . Let  $i : D \rightarrow X$ , resp.  $i' : D' \rightarrow X$  be the corresponding closed immersions. Then*

- (1) *there exists a cycle  $\alpha \in Z_{n-2}(D \cap D')$  whose pushforward to  $D$  represents  $i^* [D']_{n-1} \in A_{n-2}(D)$  and whose pushforward to  $D'$  represents  $(i')^* [D]_{n-1} \in A_{n-2}(D')$ , and*
- (2) *we have*

$$D \cdot [D']_{n-1} = D' \cdot [D]_{n-1}$$

*in  $A_{n-2}(X)$ .*

**Proof.** Part (2) is a trivial consequence of part (1). Let us write  $[D]_{n-1} = \sum n_a [Z_a]$  and  $[D']_{n-1} = \sum m_b [Z_b]$  with  $Z_a$  the irreducible components of  $D$  and  $[Z_b]$  the irreducible components of  $D'$ . According to Definition 27.1, we have  $i^* D' = \sum m_b i^* [Z_b]$  and  $(i')^* D = \sum n_a (i')^* [Z_a]$ . By assumption, none of the irreducible components

$Z_b$  is contained in  $D$ , and hence  $i^*[Z_b] = [Z_b \cap D]_{n-2}$  by definition. Similarly  $(i')^*[Z_a] = [Z_a \cap D']_{n-2}$ . Hence we are trying to prove the equality of cycles

$$\sum n_a [Z_a \cap D']_{n-2} = \sum m_b [Z_b \cap D]_{n-2}$$

which are indeed supported on  $D \cap D'$ . Let  $W \subset X$  be an integral closed subscheme with  $\dim_\delta(W) = n - 2$ . Let  $\xi \in W$  be its generic point. Set  $R = \mathcal{O}_{X,\xi}$ . It is a Noetherian local domain. Note that  $\dim(R) = 2$ . Let  $f \in R$ , resp.  $f' \in R$  be an element defining the ideal of  $D$ , resp.  $D'$ . By assumption  $\dim(R/(f, f')) = 0$ . Let  $\mathfrak{q}'_1, \dots, \mathfrak{q}'_t \subset R$  be the minimal primes over  $(f')$ , let  $\mathfrak{q}_1, \dots, \mathfrak{q}_s \subset R$  be the minimal primes over  $(f)$ . The equality above comes down to the equality

$$\sum_{i=1, \dots, s} \text{length}_{R_{\mathfrak{q}_i}}(R_{\mathfrak{q}_i}/(f)) \text{ord}_{R/\mathfrak{q}_i}(f') = \sum_{j=1, \dots, t} \text{length}_{R_{\mathfrak{q}'_j}}(R_{\mathfrak{q}'_j}/(f')) \text{ord}_{R/\mathfrak{q}'_j}(f).$$

By Lemma 3.1 applied with  $M = R/(f)$  the left hand side of this equation is equal to

$$\text{length}_R(R/(f, f')) - \text{length}_R(\text{Ker}(f' : R/(f) \rightarrow R/(f)))$$

OK, and now we note that  $\text{Ker}(f' : R/(f) \rightarrow R/(f))$  is canonically isomorphic to  $((f) \cap (f'))/(ff')$  via the map  $x \text{ mod } (f) \mapsto f'x \text{ mod } (ff')$ . Hence the left hand side is

$$\text{length}_R(R/(f, f')) - \text{length}_R((f) \cap (f'))/(ff')$$

Since this is symmetric in  $f$  and  $f'$  we win.  $\square$

02TE **Lemma 44.23.** *Let  $(S, \delta)$  be as in Situation 7.1. Let  $X$  be locally of finite type over  $S$ . Assume  $X$  integral and  $\dim_\delta(X) = n$ . Let  $\{D_j\}_{j \in J}$  be a locally finite collection of effective Cartier divisors on  $X$ . Let  $n_j, m_j \geq 0$  be collections of nonnegative integers. Set  $D = \sum n_j D_j$  and  $D' = \sum m_j D_j$ . Assume that  $\dim_\delta(D_j \cap D_{j'}) = n - 2$  for every  $j \neq j'$ . Then  $D \cdot [D']_{n-1} = D' \cdot [D]_{n-1}$  in  $A_{n-2}(X)$ .*

**Proof.** This lemma is a trivial consequence of Lemmas 44.18 and 44.22 in case the sums are finite, e.g., if  $X$  is quasi-compact. Hence we suggest the reader skip the proof.

Here is the proof in the general case. Let  $i_j : D_j \rightarrow X$  be the closed immersions. Let  $p : \coprod D_j \rightarrow X$  denote coproduct of the morphisms  $i_j$ . Let  $\{Z_a\}_{a \in A}$  be the collection of irreducible components of  $\bigcup D_j$ . For each  $j$  we write

$$[D_j]_{n-1} = \sum d_{j,a} [Z_a].$$

By Lemma 44.18 we have

$$[D]_{n-1} = \sum n_j d_{j,a} [Z_a], \quad [D']_{n-1} = \sum m_j d_{j,a} [Z_a].$$

By Lemma 44.21 we have

$$D \cdot [D']_{n-1} = p_* \left( \sum n_j i_j^* [D']_{n-1} \right), \quad D' \cdot [D]_{n-1} = p_* \left( \sum m_j i_j^* [D]_{n-1} \right).$$

As in the definition of the Gysin homomorphisms (see Definition 27.1) we choose cycles  $\beta_{a,j}$  on  $D_j \cap Z_a$  representing  $i_j^*[Z_a]$ . (Note that in fact  $\beta_{a,j} = [D_j \cap Z_a]_{n-2}$  if  $Z_a$  is not contained in  $D_j$ , i.e., there is no choice in that case.) Now since  $p$  is a closed immersion when restricted to each of the  $D_j$  we can (and we will) view  $\beta_{a,j}$

as a cycle on  $X$ . Plugging in the formulas for  $[D]_{n-1}$  and  $[D']_{n-1}$  obtained above we see that

$$D \cdot [D']_{n-1} = \sum_{j,j',a} n_j m_{j'} d_{j',a} \beta_{a,j}, \quad D' \cdot [D]_{n-1} = \sum_{j,j',a} m_{j'} n_j d_{j,a} \beta_{a,j'}.$$

Moreover, with the same conventions we also have

$$D_j \cdot [D_{j'}]_{n-1} = \sum d_{j',a} \beta_{a,j}.$$

In these terms Lemma 44.22 (see also its proof) says that for  $j \neq j'$  the cycles  $\sum d_{j',a} \beta_{a,j}$  and  $\sum d_{j,a} \beta_{a,j'}$  are equal as cycles! Hence we see that

$$\begin{aligned} D \cdot [D']_{n-1} &= \sum_{j,j',a} n_j m_{j'} d_{j',a} \beta_{a,j} \\ &= \sum_{j \neq j'} n_j m_{j'} \left( \sum_a d_{j',a} \beta_{a,j} \right) + \sum_{j,a} n_j m_j d_{j,a} \beta_{a,j} \\ &= \sum_{j \neq j'} n_j m_{j'} \left( \sum_a d_{j,a} \beta_{a,j'} \right) + \sum_{j,a} n_j m_j d_{j,a} \beta_{a,j} \\ &= \sum_{j,j',a} m_{j'} n_j d_{j,a} \beta_{a,j'} \\ &= D' \cdot [D]_{n-1} \end{aligned}$$

and we win. □

Let  $(S, \delta)$  be as in Situation 7.1. Let  $X$  be locally of finite type over  $S$ . Assume  $X$  integral and  $\dim_\delta(X) = n$ . Let  $D, D'$  be effective Cartier divisors on  $X$ . A stronger (and more useful) version of the following lemma asserts that

$$D \cdot [D']_{n-1} = D' \cdot [D]_{n-1} \quad \text{in } A_{n-2}(D \cap D')$$

for suitable representatives of the dot products involved. The first proof of the lemma together with Lemmas 44.21, 44.22, and 44.23 can be modified to show this (see [Ful98]). It is not so clear how to modify the second proof to prove the refined version. An application of the refined version is a proof that the Gysin homomorphism factors through rational equivalence which we proved by a different method in Lemma 28.2.

02TF **Lemma 44.24.** *Let  $(S, \delta)$  be as in Situation 7.1. Let  $X$  be locally of finite type over  $S$ . Assume  $X$  integral and  $\dim_\delta(X) = n$ . Let  $D, D'$  be effective Cartier divisors on  $X$ . Then*

$$D \cdot [D']_{n-1} = D' \cdot [D]_{n-1}$$

*in  $A_{n-2}(X)$ .*

**First proof of Lemma 44.24.** First, let us prove this in case  $X$  is quasi-compact. In this case, apply Lemma 44.19 to  $X$  and the two element set  $\{D, D'\}$  of effective Cartier divisors. Thus we get a proper morphism  $b : X' \rightarrow X$ , a finite collection of effective Cartier divisors  $D'_j \subset X'$  intersecting pairwise in codimension  $\geq 2$ , with  $b^{-1}(D) = \sum n_j D'_j$ , and  $b^{-1}(D') = \sum m_j D'_j$ . Note that  $b_*[b^{-1}(D)]_{n-1} = [D]_{n-1}$  in  $Z_{n-1}(X)$  and similarly for  $D'$ , see Lemma 44.13. Hence, by Lemma 24.4 we have

$$D \cdot [D']_{n-1} = b_* \left( b^{-1}(D) \cdot [b^{-1}(D')]_{n-1} \right)$$

in  $A_{n-2}(X)$  and similarly for the other term. Hence the lemma follows from the equality  $b^{-1}(D) \cdot [b^{-1}(D')]_{n-1} = b^{-1}(D') \cdot [b^{-1}(D)]_{n-1}$  in  $A_{n-2}(X')$  of Lemma 44.23.

Note that in the proof above, each referenced lemma works also in the general case (when  $X$  is not assumed quasi-compact). The only minor change in the general

case is that the morphism  $b : U' \rightarrow U$  we get from applying Lemma 44.19 has as its target an open  $U \subset X$  whose complement has codimension  $\geq 3$ . Hence by Lemma 19.2 we see that  $A_{n-2}(U) = A_{n-2}(X)$  and after replacing  $X$  by  $U$  the rest of the proof goes through unchanged.  $\square$

**Second proof of Lemma 44.24.** Let  $\mathcal{I} = \mathcal{O}_X(-D)$  and  $\mathcal{I}' = \mathcal{O}_X(-D')$  be the invertible ideal sheaves of  $D$  and  $D'$ . We denote  $\mathcal{I}_{D'} = \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{O}_{D'}$  and  $\mathcal{I}'_D = \mathcal{I}' \otimes_{\mathcal{O}_X} \mathcal{O}_D$ . We can restrict the inclusion map  $\mathcal{I} \rightarrow \mathcal{O}_X$  to  $D'$  to get a map

$$\varphi : \mathcal{I}_{D'} \longrightarrow \mathcal{O}_{D'}$$

and similarly

$$\psi : \mathcal{I}'_D \longrightarrow \mathcal{O}_D$$

It is clear that

$$\text{Coker}(\varphi) \cong \mathcal{O}_{D \cap D'} \cong \text{Coker}(\psi)$$

and

$$\text{Ker}(\varphi) \cong \frac{\mathcal{I} \cap \mathcal{I}'}{\mathcal{I}\mathcal{I}'} \cong \text{Ker}(\psi).$$

Hence we see that

$$\gamma = [\mathcal{I}_{D'}] - [\mathcal{O}_{D'}] = [\mathcal{I}'_D] - [\mathcal{O}_D]$$

in  $K_0(\text{Coh}_{\leq n-1}(X))$ . On the other hand it is clear that

$$[\mathcal{I}'_D]_{n-1} = [D]_{n-1}, \quad [\mathcal{I}_{D'}]_{n-1} = [D']_{n-1}.$$

and that

$$\mathcal{O}_X(D') \otimes \mathcal{I}'_D = \mathcal{O}_D, \quad \mathcal{O}_X(D) \otimes \mathcal{I}_{D'} = \mathcal{O}_{D'}.$$

By Lemma 44.11 (applied two times) this means that the element  $\gamma$  is an element of  $B_{n-2}(X)$ , and maps to both  $c_1(\mathcal{O}_X(D')) \cap [D]_{n-1}$  and to  $c_1(\mathcal{O}_X(D)) \cap [D']_{n-1}$  and we win (since the map  $B_{n-2}(X) \rightarrow A_{n-2}(X)$  is well defined – which is the key to this proof).  $\square$

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