1. Introduction

In this chapter we study “local” properties of general algebraic spaces, i.e., those algebraic spaces which aren’t quasi-separated. Quasi-separated algebraic spaces are studied in [Knu71]. It turns out that essentially new phenomena happen, especially regarding points and specializations of points, on more general algebraic spaces. On the other hand, for most basic results on algebraic spaces, one needn’t worry about these phenomena, which is why we have decided to have this material in a separate chapter following the standard development of the theory.
2. Conventions

06NM The standing assumption is that all schemes are contained in a big fppf site $\text{Sch}_{\text{fppf}}$. And all rings $A$ considered have the property that $\text{Spec}(A)$ is (isomorphic) to an object of this big site.

Let $S$ be a scheme and let $X$ be an algebraic space over $S$. In this chapter and the following we will write $X \times_S X$ for the product of $X$ with itself (in the category of algebraic spaces over $S$), instead of $X \times X$.

3. Universally bounded fibres

03JK We briefly discuss what it means for a morphism from a scheme to an algebraic space to have universally bounded fibres. Please refer to Morphisms, Section 56 for similar definitions and results on morphisms of schemes.

Definition 3.1. Let $S$ be a scheme. Let $X$ be an algebraic space over $S$, and let $U$ be a scheme over $S$. Let $f : U \to X$ be a morphism over $S$. We say the fibres of $f$ are universally bounded\(^1\) if there exists an integer $n$ such that for all fields $k$ and all morphisms $\text{Spec}(k) \to X$ the fibre product $\text{Spec}(k) \times_X U$ is a finite scheme over $k$ whose degree over $k$ is $\leq n$.

This definition makes sense because the fibre product $\text{Spec}(k) \times_X U$ is a scheme. Moreover, if $Y$ is a scheme we recover the notion of Morphisms, Definition 56.1 by virtue of Morphisms, Lemma 56.2.

Lemma 3.2. Let $S$ be a scheme. Let $X$ be an algebraic space over $S$. Let $V \to U$ be a morphism of schemes over $S$, and let $U \to X$ be a morphism from $U$ to $X$. If the fibres of $V \to U$ and $U \to X$ are universally bounded, then so are the fibres of $V \to X$.

Proof. Let $n$ be an integer which works for $V \to U$, and let $m$ be an integer which works for $U \to X$ in Definition 3.1. Let $\text{Spec}(k) \to X$ be a morphism, where $k$ is a field. Consider the morphisms

$$\text{Spec}(k) \times_X V \longrightarrow \text{Spec}(k) \times_X U \longrightarrow \text{Spec}(k).$$

By assumption the scheme $\text{Spec}(k) \times_X U$ is finite of degree at most $m$ over $k$, and $n$ is an integer which bounds the degree of the fibres of the first morphism. Hence by Morphisms, Lemma 56.4 we conclude that $\text{Spec}(k) \times_X V$ is finite over $k$ of degree at most $nm$. \qed

Lemma 3.3. Let $S$ be a scheme. Let $Y \to X$ be a representable morphism of algebraic spaces over $S$. Let $U \to X$ be a morphism from a scheme to $X$. If the fibres of $U \to X$ are universally bounded, then the fibres of $U \times_X Y \to Y$ are universally bounded.

Proof. This is clear from the definition, and properties of fibre products. (Note that $U \times_X Y$ is a scheme as we assumed $Y \to X$ representable, so the definition applies.) \qed

\(^1\)This is probably nonstandard notation.
Lemma 3.4. Let $S$ be a scheme. Let $g : Y \to X$ be a representable morphism of algebraic spaces over $S$. Let $f : U \to X$ be a morphism from a scheme towards $X$. Let $f' : U \times_X Y \to Y$ be the base change of $f$. If

$$\text{Im}(|f| : |U| \to |X|) \subseteq \text{Im}(|g| : |Y| \to |X|)$$

and $f'$ has universally bounded fibres, then $f$ has universally bounded fibres.

Proof. Let $n \geq 0$ be an integer bounding the degrees of the fibre products $\text{Spec}(k) \times_Y (U \times_X Y)$ as in Definition 3.1 for the morphism $f'$. We claim that $n$ works for $f$ also. Namely, suppose that $x : \text{Spec}(k) \to X$ is a morphism from the spectrum of a field. Then either $\text{Spec}(k) \times_X U$ is empty (and there is nothing to prove), or $x$ is in the image of $|f|$. By Properties of Spaces, Lemma 4.3 and the assumption of the lemma we see that this means there exists a field extension $k \subset k'$ and a commutative diagram

$$\begin{array}{ccc}
\text{Spec}(k') & \longrightarrow & Y \\
\downarrow & & \downarrow \\
\text{Spec}(k) & \longrightarrow & X
\end{array}$$

Hence we see that

$$\text{Spec}(k') \times_Y (U \times_X Y) = \text{Spec}(k') \times_{\text{Spec}(k)} (\text{Spec}(k) \times_X U)$$

Since the scheme $\text{Spec}(k') \times_Y (U \times_X Y)$ is assumed finite of degree $\leq n$ over $k'$ it follows that also $\text{Spec}(k) \times_X U$ is finite of degree $\leq n$ over $k$ as desired. (Some details omitted.)

Lemma 3.5. Let $S$ be a scheme. Let $X$ be an algebraic space over $S$. Consider a commutative diagram

$$\begin{array}{ccc}
U & \xrightarrow{f} & V \\
\downarrow{g} & & \downarrow{h} \\
X & \xrightarrow{} &
\end{array}$$

where $U$ and $V$ are schemes. If $g$ has universally bounded fibres, and $f$ is surjective and flat, then also $h$ has universally bounded fibres.

Proof. Assume $g$ has universally bounded fibres, and $f$ is surjective and flat. Say $n \geq 0$ is an integer which bounds the degrees of the schemes $\text{Spec}(k) \times_X U$ as in Definition 3.1. We claim $n$ also works for $h$. Let $\text{Spec}(k) \to X$ be a morphism from the spectrum of a field to $X$. Consider the morphism of schemes

$$\text{Spec}(k) \times_X V \to \text{Spec}(k) \times_X U$$

It is flat and surjective. By assumption the scheme on the left is finite of degree $\leq n$ over $\text{Spec}(k)$. It follows from Morphisms, Lemma 56.10 that the degree of the scheme on the right is also bounded by $n$ as desired.

Lemma 3.6. Let $S$ be a scheme. Let $X$ be a scheme over $S$, and let $U$ be a scheme over $S$. Let $\varphi : U \to X$ be a morphism. If the fibres of $\varphi$ are universally bounded, then there exists an integer $n$ such that each fibre of $|U| \to |X|$ has at most $n$ elements.
Proof. The integer $n$ of Definition 3.1 works. Namely, pick $x \in |X|$. Represent $x$ by a morphism $x : \text{Spec}(k) \to X$. Then we get a commutative diagram

\[
\begin{array}{ccc}
\text{Spec}(k) \times_X U & \longrightarrow & U \\
\downarrow & & \downarrow \\
\text{Spec}(k) & \xrightarrow{x} & X
\end{array}
\]

which shows (via Properties of Spaces, Lemma 4.3) that the inverse image of $x$ in $|U|$ is the image of the top horizontal arrow. Since $\text{Spec}(k) \times_X U$ is finite of degree $\leq n$ over $k$ it has at most $n$ points. \qed

4. Finiteness conditions and points

In this section we elaborate on the question of when points can be represented by monomorphisms from spectra of fields into the space.

Before we give the proof of the next lemma let us recall some facts about étale morphisms of schemes:

(1) An étale morphism is flat and hence generalizations lift along an étale morphism (Morphisms, Lemmas 36.12 and 25.9).

(2) An étale morphism is unramified, an unramified morphism is locally quasi-finite, hence fibres are discrete (Morphisms, Lemmas 36.16, 35.10, and 20.6).

(3) A quasi-compact étale morphism is quasi-finite and in particular has finite fibres (Morphisms, Lemmas 20.9 and 20.10).

(4) An étale scheme over a field $k$ is a disjoint union of spectra of finite separable field extension of $k$ (Morphisms, Lemma 36.7).

For a general discussion of étale morphisms, please see Étale Morphisms, Section 11.

Lemma 4.2. Let $S$ be a scheme. Let $X$ be an algebraic space over $S$. Let $x \in |X|$. The following are equivalent:

(1) there exists a family of schemes $U_i$ and étale morphisms $\varphi_i : U_i \to X$ such that $\coprod \varphi_i : \coprod U_i \to X$ is surjective, and such that for each $i$ the fibre of $|U_i| \to |X|$ over $x$ is finite, and

(2) for every affine scheme $U$ and étale morphism $\varphi : U \to X$ the fibre of $|U| \to |X|$ over $x$ is finite.

Proof. The implication (2) $\Rightarrow$ (1) is trivial. Let $\varphi_i : U_i \to X$ be a family of étale morphisms as in (1). Let $\varphi : U \to X$ be an étale morphism from an affine scheme towards $X$. Consider the fibre product diagrams

\[
\begin{array}{ccc}
U \times_X U_i & \overset{p_i}{\longrightarrow} & U_i \\
\downarrow \varphi & & \downarrow \varphi_i \\
U & \xrightarrow{\varphi} & X
\end{array}
\quad \quad \quad
\begin{array}{ccc}
\coprod U \times_X U_i & \overset{\coprod p_i}{\longrightarrow} & \coprod U_i \\
\downarrow \coprod \varphi_i & & \downarrow \coprod \varphi \\
\coprod U & \xrightarrow{\coprod \varphi_i} & \coprod X
\end{array}
\]

Since $q_i$ is étale it is open (see Remark 11). Moreover, the morphism $\coprod q_i$ is surjective. Hence there exist finitely many indices $i_1, \ldots, i_n$ and a quasi-compact opens $W_{i_j} \subseteq U \times_X U_{i_j}$ which surject onto $U$. The morphism $p_{i_j}$ is étale, hence locally quasi-finite (see remark on étale morphisms above). Thus we may apply
Morphisms, Lemma \[56.9\] to see the fibres of $p_i|_{W_{ij}} : W_{ij} \to U_i$ are finite. Hence by Properties of Spaces, Lemma \[4.3\], and the assumption on $\varphi_i$ we conclude that the fibre of $\varphi$ over $x$ is finite. In other words (2) holds. □

03JU \[Lemma 4.3.\] Let $S$ be a scheme. Let $X$ be an algebraic space over $S$. Let $x \in |X|$. The following are equivalent:

(1) there exists a scheme $U$, an étale morphism $\varphi : U \to X$, and points $u, u' \in U$ mapping to $x$ such that setting $R = U \times_X U$ the fibre of

$$|R| \to |U| \times_{|X|} |U|$$

over $(u, u')$ is finite,

(2) for every scheme $U$, étale morphism $\varphi : U \to X$ and any points $u, u' \in U$ mapping to $x$ setting $R = U \times_X U$ the fibre of

$$|R| \to |U| \times_{|X|} |U|$$

over $(u, u')$ is finite,

(3) there exists a morphism $\text{Spec}(k) \to X$ with $k$ a field in the equivalence class of $x$ such that the projections $\text{Spec}(k) \times_X \text{Spec}(k) \to \text{Spec}(k)$ are étale and quasi-compact, and

(4) there exists a monomorphism $\text{Spec}(k) \to X$ with $k$ a field in the equivalence class of $x$.

Proof. Assume (1), i.e., let $\varphi : U \to X$ be an étale morphism from a scheme towards $X$, and let $u, u'$ be points of $U$ lying over $x$ such that the fibre of $|R| \to |U| \times_{|X|} |U|$ over $(u, u')$ is a finite set. In this proof we think of a point $u = \text{Spec}(\kappa(u))$ as a scheme. Note that $u \to U$, $u' \to U$ are monomorphisms (see Schemes, Lemma \[23.7\]), hence $u \times_X u' \to R = U \times_X U$ is a monomorphism. In this language the assumption really means that $u \times_X u'$ is a scheme whose underlying topological space has finitely many points. Let $\psi : W \to X$ be an étale morphism from a scheme towards $X$. Let $w, w' \in W$ be points of $W$ mapping to $x$. We have to show that $w \times_X w'$ is a scheme whose underlying topological space has finitely many points. Consider the fibre product diagram

$$
\begin{array}{ccc}
W \times_X U & \longrightarrow & U \\
\downarrow & & \downarrow \varphi \\
W & \longrightarrow & X \\
\end{array}
$$

As $x$ is the image of $u$ and $u'$ we may pick points $\tilde{w}, \tilde{w}'$ in $W \times_X U$ with $q(\tilde{w}) = w$, $q(\tilde{w}') = w'$, $u = p(\tilde{w})$ and $u' = p(\tilde{w}')$, see Properties of Spaces, Lemma \[4.3\]. As $p$, $q$ are étale the field extensions $\kappa(w) \subset \kappa(\tilde{w}) \supset \kappa(u)$ and $\kappa(w') \subset \kappa(\tilde{w}') \supset \kappa(u')$ are finite separable, see Remark \[4.1\]. Then we get a commutative diagram

$$
\begin{array}{ccc}
w \times_X w' & \hookrightarrow & \tilde{w} \times_X \tilde{w}' \\
\downarrow & & \downarrow \\
w \times_X w' & \hookrightarrow & \tilde{w} \times_S \tilde{w}' \\
\end{array}
$$

where the squares are fibre product squares. The lower horizontal morphisms are étale and quasi-compact, as any scheme of the form $\text{Spec}(k) \times_S \text{Spec}(k')$ is affine, and by our observations about the field extensions above. Thus we see that the top horizontal arrows are étale and quasi-compact and hence have finite fibres. We
have seen above that $|u \times_X u'|$ is finite, so we conclude that $|w \times_X u'|$ is finite. In other words, (2) holds.

Assume (2). Let $U \to X$ be an étale morphism from a scheme $U$ such that $x$ is in the image of $|U| \to |X|$. Let $u \in U$ be a point mapping to $x$. Then we have seen in the previous paragraph that $u = \text{Spec}(\kappa(u)) \to X$ has the property that $u \times_X u$ has a finite underlying topological space. On the other hand, the projection maps $u \times_X u \to u$ are the composition

$$u \times_X u \to u \times_X U \to u \times_X X = u,$$

i.e., the composition of a monomorphism (the base change of the monomorphism $u \to U$) by an étale morphism (the base change of the étale morphism $U \to X$). Hence $u \times_X U$ is a disjoint union of spectra of fields finite separable over $\kappa(u)$ (see Remark 4.1). Since $u \times_X u$ is finite the image of it in $u \times_X U$ is a finite disjoint union of spectra of fields finite separable over $\kappa(u)$. By Schemes, Lemma 23.11 we conclude that $u \times_X u$ is a finite disjoint union of spectra of fields finite separable over $\kappa(u)$. In other words, we see that $u \times_X u \to u$ is quasi-compact and étale. This means that (3) holds.

Let us prove that (3) implies (4). Let $\text{Spec}(k) \to X$ be a morphism from the spectrum of a field into $X$, in the equivalence class of $x$ such that the two projections $t, s : R = \text{Spec}(k) \times_X \text{Spec}(k) \to \text{Spec}(k)$ are quasi-compact and étale. This means in particular that $R$ is an étale equivalence relation on $\text{Spec}(k)$. By Spaces, Theorem 10.5 we know that the quotient sheaf $X' = \text{Spec}(k)/R$ is an algebraic space. By Groupoids, Lemma 20.6 the map $X' \to X$ is a monomorphism. Since $s, t$ are quasi-compact, we see that $R$ is quasi-compact and hence Properties of Spaces, Lemma 15.3 applies to $X'$, and we see that $X' = \text{Spec}(k')$ for some field $k'$. Hence we get a factorization

$$\text{Spec}(k) \to \text{Spec}(k') \to X$$

which shows that $\text{Spec}(k') \to X$ is a monomorphism mapping to $x \in |X|$. In other words (4) holds.

Finally, we prove that (4) implies (1). Let $\text{Spec}(k) \to X$ be a monomorphism with $k$ a field in the equivalence class of $x$. Let $U \to X$ be a surjective étale morphism from a scheme $U$ to $X$. Let $u \in U$ be a point over $x$. Since $\text{Spec}(k) \times_X u$ is nonempty, and since $\text{Spec}(k) \times_X u \to u$ is a monomorphism we conclude that $\text{Spec}(k) \times_X u = u$ (see Schemes, Lemma 23.11). Hence $u \to U \to X$ factors through $\text{Spec}(k) \to X$, here is a picture

\[
\begin{array}{ccc}
\text{Spec}(k) & \to & \text{Spec}(k') \\
\downarrow & & \downarrow \\
\text{Spec}(k) & \to & X \\
\end{array}
\]

Since the right vertical arrow is étale this implies that $k \subset \kappa(u)$ is a finite separable extension. Hence we conclude that

$$u \times_X u = u \times_{\text{Spec}(k)} u$$

is a finite scheme, and we win by the discussion of the meaning of property (1) in the first paragraph of this proof. \qed
Lemma 4.4. Let $S$ be a scheme. Let $X$ be an algebraic space over $S$. Let $x \in |X|$. Let $U$ be a scheme and let $\varphi : U \to X$ be an étale morphism. The following are equivalent:

1. $x$ is in the image of $|U| \to |X|$, and setting $R = U \times_X U$ the fibres of both $|U| \to |X|$ and $|R| \to |X|$ over $x$ are finite,

2. there exists a monomorphism $\text{Spec}(k) \to X$ with $k$ a field in the equivalence class of $x$, and the fibre product $\text{Spec}(k) \times_X U$ is a finite nonempty scheme over $k$.

Proof. Assume (1). This clearly implies the first condition of Lemma 4.3 and hence we obtain a monomorphism $\text{Spec}(k) \to X$ in the class of $x$. Taking the fibre product we see that $\text{Spec}(k) \times_X U \to \text{Spec}(k)$ is a scheme étale over $\text{Spec}(k)$ with finitely many points, hence a finite nonempty scheme over $k$, i.e., (2) holds.

Assume (2). By assumption $x$ is in the image of $|U| \to |X|$. The finiteness of the fibre of $|U| \to |X|$ over $x$ is clear since this fibre is equal to $|\text{Spec}(k) \times_X U|$ by Properties of Spaces, Lemma 4.3. The finiteness of the fibre of $|R| \to |X|$ above $x$ is also clear since it is equal to the set underlying the scheme

$$(\text{Spec}(k) \times_X U) \times \text{Spec}(k) \text{ (Spec}(k) \times_X U)$$

which is finite over $k$. Thus (1) holds.

Lemma 4.5. Let $S$ be a scheme. Let $X$ be an algebraic space over $S$. Let $x \in |X|$. The following are equivalent:

1. for every affine scheme $U$, any étale morphism $\varphi : U \to X$ setting $R = U \times_X U$ the fibres of both $|U| \to |X|$ and $|R| \to |X|$ over $x$ are finite,

2. there exist schemes $U_i$ and étale morphisms $U_i \to X$ such that $\coprod U_i \to X$ is surjective and for each $i$, setting $R_i = U_i \times_X U_i$ the fibres of both $|U_i| \to |X|$ and $|R_i| \to |X|$ over $x$ are finite,

3. there exists a monomorphism $\text{Spec}(k) \to X$ with $k$ a field in the equivalence class of $x$, and for any affine scheme $U$ and étale morphism $U \to X$ the fibre product $\text{Spec}(k) \times_X U$ is a finite scheme over $k$,

4. there exists a quasi-compact monomorphism $\text{Spec}(k) \to X$ with $k$ a field in the equivalence class of $x$, and

5. there exists a quasi-compact morphism $\text{Spec}(k) \to X$ with $k$ a field in the equivalence class of $x$.

Proof. The equivalence of (1) and (3) follows on applying Lemma 4.4 to every étale morphism $U \to X$ with $U$ affine. It is clear that (3) implies (2). Assume $U_i \to X$ and $R_i$ are as in (2). We conclude from Lemma 4.2 that for any affine scheme $U$ and étale morphism $U \to X$ the fibre of $|U| \to |X|$ over $x$ is finite. Say this fibre is $\{u_1, \ldots, u_n\}$. Then, as Lemma 4.3 (1) applies to $U_i \to X$ for some $i$ such that $x$ is in the image of $|U_i| \to |X|$, we see that the fibre of $|R = U \times_X U| \to |U| \times |X|$ $|U|$ is finite over $(u_a, u_b)$, $a, b \in \{1, \ldots, n\}$. Hence the fibre of $|R| \to |X|$ over $x$ is finite.
In this way we see that (1) holds. At this point we know that (1), (2), and (3) are equivalent.

If (4) holds, then for any affine scheme $U$ and étale morphism $U \to X$ the scheme $\Spec(k) \times_X U$ is on the one hand étale over $k$ (hence a disjoint union of spectra of finite separable extensions of $k$ by Remark 4.1) and on the other hand quasi-compact over $U$ (hence quasi-compact). Thus we see that (3) holds. Conversely, if $U_i \to X$ is as in (2) and $\Spec(k) \to X$ is a monomorphism as in (3), then

$$\coprod \Spec(k) \times_X U_i \to \coprod U_i$$

is quasi-compact (because over each $U_i$ we see that $\Spec(k) \times_X U_i$ is a finite disjoint union spectra of fields). Thus $\Spec(k) \to X$ is quasi-compact by Morphisms of Spaces, Lemma 8.8.

It is immediate that (4) implies (5). Conversely, let $\Spec(k) \to X$ be a quasi-compact morphism in the equivalence class of $x$. Let $U \to X$ be an étale morphism with $U$ affine. Consider the fibre product

$$
\begin{array}{ccc}
F & \longrightarrow & U \\
\downarrow & & \downarrow \\
\Spec(k) & \longrightarrow & X
\end{array}
$$

Then $F \to U$ is quasi-compact, hence $F$ is quasi-compact. On the other hand, $F \to \Spec(k)$ is étale, hence $F$ is a finite disjoint union of spectra of finite separable extensions of $k$ (Remark 4.1). Since the image of $|F| \to |U|$ is the fibre of $|U| \to |X|$ over $x$ (Properties of Spaces, Lemma 4.3), we conclude that the fibre of $|U| \to |X|$ over $x$ is finite. The scheme $F \times_{\Spec(k)} \Spec(k)$ is also a finite union of spectra of fields because it is also quasi-compact and étale over $\Spec(k)$. There is a monomorphism $F \times_X F \to F \times_{\Spec(k)} \Spec(k)$, hence $F \times_X F$ is a finite disjoint union of spectra of fields (Schemes, Lemma 23.11). Thus the image of $F \times_X F \to U \times_X U = R$ is finite. Since this image is the fibre of $|R| \to |X|$ over $x$ by Properties of Spaces, Lemma 4.3 we conclude that (1) holds.

\[\square\]

**Lemma 4.6.** Let $S$ be a scheme. Let $X$ be an algebraic space over $S$. The following are equivalent:

1. there exist schemes $U_i$ and étale morphisms $U_i \to X$ such that $\coprod U_i \to X$ is surjective and each $U_i \to X$ has universally bounded fibres, and
2. for every affine scheme $U$ and étale morphism $\varphi : U \to X$ the fibres of $U \to X$ are universally bounded.

**Proof.** The implication (2) $\Rightarrow$ (1) is trivial. Assume (1). Let $(\varphi_i : U_i \to X)_{i \in I}$ be a collection of étale morphisms from schemes towards $X$, covering $X$, such that each $\varphi_i$ has universally bounded fibres. Let $\psi : U \to X$ be an étale morphism from an affine scheme towards $X$. For each $i$ consider the fibre product diagram

$$
\begin{array}{ccc}
U \times_X U_i & \longrightarrow & U_i \\
\downarrow \varphi_i & & \downarrow \\
\psi & \longrightarrow & X
\end{array}
$$

Since $\varphi_i$ is étale it is open (see Remark 4.1). Moreover, we have $U = \bigcup \text{Im}(q_i)$, since the family $(\varphi_i)_{i \in I}$ is surjective. Since $U$ is affine, hence quasi-compact we
can finite finitely many $i_1,\ldots,i_n \in I$ and quasi-compact opens $W_j \subset U \times_X U_i$, such that $U = \bigcup p_{i_j}(W_j)$. The morphism $p_{i_j}$ is étale, hence locally quasi-finite (see remark on étale morphisms above). Thus we may apply Morphisms, Lemma 56.9 to see the fibres of $p_{i_j}|_{W_j} : W_j \to U_{i_j}$ are universally bounded. Hence by Lemma 3.2 we see that the fibres of $W_j \to X$ are universally bounded. Thus also $\coprod_{j=1,\ldots,n} W_j \to X$ has universally bounded fibres. Since $\coprod_{j=1,\ldots,n} W_j \to X$ factors through the surjective étale map $\coprod q_{i_j}|_{W_j} : \coprod_{j=1,\ldots,n} W_j \to U$ we see that the fibres of $U \to X$ are universally bounded by Lemma 3.5. In other words (2) holds.

□

Lemma 4.7. Let $S$ be a scheme. Let $X$ be an algebraic space over $S$. The following are equivalent:

(1) there exists a Zariski covering $X = \bigcup X_i$ and for each $i$ a scheme $U_i$ and a quasi-compact surjective étale morphism $U_i \to X_i$, and

(2) there exist schemes $U_i$ and étale morphisms $U_i \to X$ such that the projections $U_i \times_X U_i \to U_i$ are quasi-compact and $\coprod U_i \to X$ is surjective.

Proof. If (1) holds then the morphisms $U_i \to X_i \to X$ are étale (combine Morphisms, Lemma 36.3 and Spaces, Lemmas 5.4 and 5.3). Moreover, as $U_i \times_X U_i = U_i \times_X U_i$, both projections $U_i \times_X U_i \to U_i$ are quasi-compact.

If (2) holds then let $X_i \subset X$ be the open subspace corresponding to the image of the open map $|U_i| \to |X|$, see Properties of Spaces, Lemma 4.10. The morphisms $U_i \to X_i$ are surjective. Hence $U_i \to X_i$ is surjective étale, and the projections $U_i \times_X U_i \to U_i$ are quasi-compact, because $U_i \times_X U_i = U_i \times_X U_i$. Thus by Spaces, Lemma 11.3 the morphisms $U_i \to X_i$ are quasi-compact. □

5. Conditions on algebraic spaces

In this section we discuss the relationship between various natural conditions on algebraic spaces we have seen above. Please read Section 6 to get a feeling for the meaning of these conditions.

Lemma 5.1. Let $S$ be a scheme. Let $X$ be an algebraic space over $S$. Consider the following conditions on $X$:

(a) For every $x \in |X|$, the equivalent conditions of Lemma 4.3 hold.
(b) For every $x \in |X|$, the equivalent conditions of Lemma 4.3 hold.
(c) The space $X$ is Zariski locally quasi-separated.
(d) The space $X$ is quasi-separated
(e) The space $X$ is representable, i.e., $X$ is a scheme.
(f) The space $X$ is a quasi-separated scheme.
We have

\[
(\theta) \downarrow \downarrow (\iota) \uparrow \uparrow (\zeta) \rightarrow (\epsilon) \rightarrow (\delta) \rightarrow (\gamma) \leftrightarrow (\alpha) + (\beta)
\]

**Proof.** The implication \((\gamma) \Leftrightarrow (\alpha) + (\beta)\) is immediate. The implications in the diamond on the left are clear from the definitions.

Assume \((\zeta)\), i.e., that \(X\) is Zariski locally quasi-separated. Then \((\epsilon)\) holds by Properties of Spaces, Lemma 6.6.

Assume \((\epsilon)\). By Lemma 4.7 there exists a Zariski open covering \(X = \bigcup X_i\) such that for each \(i\) there exists a scheme \(U_i\) and a quasi-compact surjective étale morphism \(U_i \to X_i\). Choose an \(i\) and an affine open subscheme \(W \subset U_i\). It suffices to show that \(W \to X\) has universally bounded fibres, since then the family of all these morphisms \(W \to X\) covers \(X\). To do this we consider the diagram

\[
W \times X U_i \xrightarrow{p} U_i \\
q \downarrow \downarrow \downarrow \downarrow \\
W \longrightarrow X
\]

Since \(W \to X\) factors through \(X_i\) we see that \(W \times X U_i = W \times X_i U_i\), and hence \(q\) is quasi-compact. Since \(W\) is affine this implies that the scheme \(W \times X U_i\) is quasi-compact. Thus we may apply Morphisms, Lemma 56.9 and we conclude that \(p\) has universally bounded fibres. From Lemma 3.4 we conclude that \(W \to X\) has universally bounded fibres as well.

Assume \((\delta)\). Let \(U\) be an affine scheme, and let \(U \to X\) be an étale morphism. By assumption the fibres of the morphism \(U \to X\) are universally bounded. Thus also the fibres of both projections \(R = U \times X U \to U\) are universally bounded, see Lemma 3.3. And by Lemma 3.2 also the fibres of \(R \to X\) are universally bounded. Hence for any \(x \in X\) the fibres of \(|U| \to |X|\) and \(|R| \to |X|\) over \(x\) are finite, see Lemma 3.6. In other words, the equivalent conditions of Lemma 4.5 hold. This proves that \((\delta) \Rightarrow (\gamma)\).

\[\square\]

**Lemma 5.2.** Let \(S\) be a scheme. Let \(\mathcal{P}\) be one of the properties \((\alpha), (\beta), (\gamma), (\delta), (\epsilon), (\zeta),\) or \((\theta)\) of algebraic spaces listed in Lemma 5.1. Then if \(X\) is an algebraic space over \(S\), and \(X = \bigcup X_i\) is a Zariski open covering such that each \(X_i\) has \(\mathcal{P}\), then \(X\) has \(\mathcal{P}\).

**Proof.** Let \(X\) be an algebraic space over \(S\), and let \(X = \bigcup X_i\) is a Zariski open covering such that each \(X_i\) has \(\mathcal{P}\).

The case \(\mathcal{P} = (\alpha)\). The condition \((\alpha)\) for \(X_i\) means that for every \(x \in |X_i|\) and every affine scheme \(U\), and étale morphism \(\varphi : U \to X_i\) the fibre of \(\varphi : |U| \to |X_i|\) over \(x\) is finite. Consider \(x \in X\), an affine scheme \(U\) and an étale morphism \(U \to X\). Since \(X = \bigcup X_i\) is a Zariski open covering there exists a finite affine open covering \(U = U_1 \cup \ldots \cup U_n\) such that each \(U_j \to X\) factors through some \(X_i\). By assumption
the fibres of \(|U_j| \to |X_i|\) over \(x\) are finite for \(j = 1, \ldots, n\). Clearly this means that the fibre of \(|U| \to |X|\) over \(x\) is finite. This proves the result for (α).

The case \(\mathcal{P} = (\beta)\). The condition (β) for \(X_i\) means that every \(x \in |X_i|\) is represented by a monomorphism from the spectrum of a field towards \(X_i\). Hence the same follows for \(X\) as \(X_i \to X\) is a monomorphism and \(X = \bigcup X_i\).

The case \(\mathcal{P} = (\gamma)\). Note that \((\gamma) = (\alpha) + (\beta)\) by Lemma 5.1 hence the lemma for (γ) follows from the cases treated above.

The case \(\mathcal{P} = (\delta)\). The condition (δ) for \(X_i\) means there exist schemes \(U_{ij}\) and étale morphisms \(U_{ij} \to X_i\) with universally bounded fibres which cover \(X_i\). These schemes also give an étale surjective morphism \(\coprod U_{ij} \to X\) and \(U_{ij} \to X\) still has universally bounded fibres.

The case \(\mathcal{P} = (\epsilon)\). The condition (ε) for \(X_i\) means we can find a set \(J_i\) and morphisms \(\varphi_{ij} : U_{ij} \to X_i\) such that each \(\varphi_{ij}\) is étale, both projections \(U_{ij} \times X_i\), \(U_{ij} \to U_{ij}\) are quasi-compact, and \(\coprod_{j \in J_i} U_{ij} \to X_i\) is surjective. In this case the compositions \(U_{ij} \to X_i \to X\) are étale (combine Morphisms, Lemmas 36.3 and 36.9 and Spaces, Lemmas 5.4 and 5.3). Since \(X_i \subset X\) is a subspace we see that \(U_{ij} \times X, U_{ij} = U_{ij} \times X U_{ij}\), and hence the condition on fibre products is preserved. And clearly \(\coprod_{i,j} U_{ij} \to X\) is surjective. Hence \(X\) satisfies (ε).

The case \(\mathcal{P} = (\zeta)\). The condition (ζ) for \(X_i\) means that \(X_i\) is Zariski locally quasi-separated. It is immediately clear that this means \(X\) is Zariski locally quasi-separated.

For (θ), see Properties of Spaces, Lemma 13.1 \(\square\)

**Lemma 5.3.** Let \(S\) be a scheme. Let \(\mathcal{P}\) be one of the properties (β), (γ), (δ), (ε), or (θ) of algebraic spaces listed in Lemma 5.1. Let \(X, Y\) be algebraic spaces over \(S\). Let \(X \to Y\) be a representable morphism. If \(Y\) has property \(\mathcal{P}\), so does \(X\).

**Proof.** Assume \(f : X \to Y\) is a representable morphism of algebraic spaces, and assume that \(Y\) has \(\mathcal{P}\). Let \(x \in |X|\), and set \(y = f(x) \in |Y|\).

The case \(\mathcal{P} = (\beta)\). Condition (β) for \(Y\) means there exists a monomorphism \(\text{Spec}(k) \to Y\) representing \(y\). The fibre product \(X_y = \text{Spec}(k) \times_Y X\) is a scheme, and \(x\) corresponds to a point of \(X_y\), i.e., to a monomorphism \(\text{Spec}(k') \to X_y\). As \(X_y \to X\) is a monomorphism also we see that \(x\) is represented by the monomorphism \(\text{Spec}(k') \to X_y \to X\). In other words (β) holds for \(X\).

The case \(\mathcal{P} = (\gamma)\). Since (γ) ⇒ (β) we have seen in the preceding paragraph that \(y\) and \(x\) can be represented by monomorphisms as in the following diagram

\[
\begin{array}{ccc}
\text{Spec}(k') & \xrightarrow{x} & X \\
\downarrow & & \downarrow \\
\text{Spec}(k) & \xrightarrow{y} & Y
\end{array}
\]

Also, by definition of property (γ) via Lemma 4.5 (2) there exist schemes \(V_i\) and étale morphisms \(V_i \to Y\) such that \(\coprod V_i \to Y\) is surjective and for each \(i\), setting \(R_i = V_i \times_Y V_i\) the fibres of both

\(|V_i| \to |Y|\) and \(|R_i| \to |Y|\)
over $y$ are finite. This means that the schemes $(V_i)_y$ and $(R_i)_y$ are finite schemes over $y = \text{Spec}(k)$. As $X \to Y$ is representable, the fibre products $U_i = V_i \times_Y X$ are schemes. The morphisms $U_i \to X$ are étale, and $\coprod U_i \to X$ is surjective. Finally, for each $i$ we have

$$(U_i)_x = (V_i \times_Y X)_x = (V_i)_y \times_{\text{Spec}(k)} \text{Spec}(k')$$

and

$$(U_i \times_X U_i)_x = (((V_i \times_Y X) \times_X (V_i \times_Y X))_x = (R_i)_y \times_{\text{Spec}(k)} \text{Spec}(k')$$

hence these are finite over $k'$ as base changes of the finite schemes $(V_i)_y$ and $(R_i)_y$. This implies that $(\gamma)$ holds for $X$, again via the second condition of Lemma 4.5.

The case $\mathcal{P} = (\delta)$. Let $V \to Y$ be an étale morphism with $V$ an affine scheme. Since $Y$ has property $(\delta)$ this morphism has universally bounded fibres. By Lemma 3.3 the base change $V \times_Y X \to X$ also has universally bounded fibres. Hence the first part of Lemma 4.6 applies and we see that $Y$ also has property $(\delta)$.

The case $\mathcal{P} = (\epsilon)$. We will repeatedly use Spaces, Lemma 5.5. Let $V_i \to Y$ be as in Lemma 4.7 (2). Set $U_i = X \times_Y V_i$. The morphisms $U_i \to X$ are étale, and $\coprod U_i \to X$ is surjective. Because $U_i \times_X U_i = X \times_Y (V_i \times_Y V_i)$ we see that the projections $U_i \times_Y U_i \to U_i$ are base changes of the projections $V_i \times_Y V_i \to V_i$, and so quasi-compact as well. Hence $X$ satisfies Lemma 4.7 (2).

The case $\mathcal{P} = (\theta)$. In this case the result is Categories, Lemma 8.3.

6. Reasonable and decent algebraic spaces

In Lemma 5.1 we have seen a number of conditions on algebraic spaces related to the behaviour of étale morphisms from affine schemes into $X$ and related to the existence of special étale coverings of $X$ by schemes. We tabulate the different types of conditions here:

<table>
<thead>
<tr>
<th>Condition</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(\alpha)$</td>
<td>fibres of étale morphisms from affines are finite</td>
</tr>
<tr>
<td>$(\beta)$</td>
<td>points come from monomorphisms of spectra of fields</td>
</tr>
<tr>
<td>$(\gamma)$</td>
<td>points come from quasi-compact monomorphisms of spectra of fields</td>
</tr>
<tr>
<td>$(\delta)$</td>
<td>fibres of étale morphisms from affines are universally bounded</td>
</tr>
<tr>
<td>$(\epsilon)$</td>
<td>cover by étale morphisms from schemes quasi-compact onto their image</td>
</tr>
</tbody>
</table>

The conditions in the following definition are not exactly conditions on the diagonal of $X$, but they are in some sense separation conditions on $X$.

**Definition 6.1.** Let $S$ be a scheme. Let $X$ be an algebraic space over $S$.

1. We say $X$ is **decent** if for every point $x \in X$ the equivalent conditions of Lemma 4.5 hold, in other words property $(\gamma)$ of Lemma 5.1 holds.
2. We say $X$ is **reasonable** if the equivalent conditions of Lemma 4.6 hold, in other words property $(\delta)$ of Lemma 5.1 holds.
3. We say $X$ is **very reasonable** if the equivalent conditions of Lemma 4.7 hold, i.e., property $(\epsilon)$ of Lemma 5.1 holds.
We have the following implications among these conditions on algebraic spaces:

\[
\text{representable} \quad \Downarrow \quad \text{very reasonable} \quad \Rightarrow \quad \text{reasonable} \quad \Rightarrow \quad \text{decent}
\]

The notion of a very reasonable algebraic space is obsolete. It was introduced because the assumption was needed to prove some results which are now proven for the class of decent spaces. The class of decent spaces is the largest class of spaces \( X \) where one has a good relationship between the topology of \(|X|\) and properties of \( X \) itself.

**Example 6.2.** The algebraic space \( \mathbb{A}^1_\mathbb{Q}/\mathbb{Z} \) constructed in Spaces, Example 14.8 is not decent as its “generic point” cannot be represented by a monomorphism from the spectrum of a field.

**Remark 6.3.** Reasonable algebraic spaces are technically easier to work with than very reasonable algebraic spaces. For example, if \( X \to Y \) is a quasi-compact étale surjective morphism of algebraic spaces and \( X \) is reasonable, then so is \( Y \), see Lemma 17.8 but we don’t know if this is true for the property “very reasonable.” Below we give another technical property enjoyed by reasonable algebraic spaces.

**Lemma 6.4.** Let \( S \) be a scheme. Let \( X \) be a quasi-compact reasonable algebraic space. Then there exists a directed system of quasi-compact and quasi-separated algebraic spaces \( X_i \) such that \( X = \text{colim}_i X_i \) (colimit in the category of sheaves). Moreover we can arrange it such that

1. for every quasi-compact scheme \( T \) over \( S \) we have \( \text{colim}_i X_i(T) = X(T) \),
2. the transition morphisms \( X_i \to X_i' \) of the system and the coprojections \( X_i \to X \) are surjective and étale, and
3. if \( X \) is a scheme, then the algebraic spaces \( X_i \) are schemes and the transition morphisms \( X_i \to X_i' \) and the coprojections \( X_i \to X \) are local isomorphisms.

**Proof.** We sketch the proof. By Properties of Spaces, Lemma 6.3 we have \( X = U/R \) with \( U \) affine. In this case, reasonable means \( U \to X \) is universally bounded. Hence there exists an integer \( N \) such that the “fibres” of \( U \to X \) have degree at most \( N \), see Definition 3.1. Denote \( s,t : R \to U \) and \( c : R \times_{s,U,t} R \to R \) the groupoid structural maps.

Claim: for every quasi-compact open \( A \subset R \) there exists an open \( R' \subset R \) such that

1. \( A \subset R' \),
2. \( R' \) is quasi-compact, and
3. \( (U,R',s|_{R'},t|_{R'},c|_{R' \times_{s,U,t} R'}) \) is a groupoid scheme.

Note that \( e : U \to R \) is open as it is a section of the étale morphism \( s : R \to U \), see Étale Morphisms, Proposition 6.1 Moreover \( U \) is affine hence quasi-compact. Hence we may replace \( A \) by \( A \cup e(U) \subset R \), and assume that \( A \) contains \( e(U) \). Next, we define inductively \( A_1 = A \), and

\[
A^n = c(A^{n-1} \times_{s,U,t} A) \subset R
\]
for \( n \geq 2 \). Arguing inductively, we see that \( A^n \) is quasi-compact for all \( n \geq 2 \), as the image of the quasi-compact fibre product \( A^{n-1} \times_{s,U,t} A \). If \( k \) is an algebraically closed field over \( S \), and we consider \( k \)-points then

\[
A^n(k) = \left\{ (u, u') \in U(k) : \begin{array}{l}
\text{there exist } u = u_1, u_2, \ldots, u_n \in U(k) \\
\text{with } (u_i, u_{i+1}) \in A \text{ for all } i = 1, \ldots, n - 1.
\end{array} \right\}
\]

But as the fibres of \( U(k) \to X(k) \) have size at most \( N \) we see that if \( n > N \) then we get a repeat in the sequence above, and we can shorten it proving \( A^N = A^n \) for all \( n \geq N \). This implies that \( R' = A^N \) gives a groupoid scheme \((U, R', s|_{R'}, t|_{R'}, c|_{R'} \times_{s, U, t} R')\), proving the claim above.

Consider the map of sheaves on \((\text{Sch}/S)_{fppf}\)

\[
\operatorname{colim}_{R' \subset R} U/R' \to U/R
\]

where \( R' \subset R \) runs over the quasi-compact open subschemes of \( R \) which give étale equivalence relations as above. Each of the quotients \( U/R' \) is an algebraic space (see Spaces, Theorem \[10.5\]). Since \( R' \) is quasi-compact, and \( U \) affine the morphism \( R' \to U \times_{\text{Spec}(\mathcal{O})} U \) is quasi-compact, and hence \( U/R' \) is quasi-separated. Finally, if \( T \) is a quasi-compact scheme, then

\[
\operatorname{colim}_{R' \subset R} U(T)/R'(T) \to U(T)/R(T)
\]

is a bijection, since every morphism from \( T \) into \( R \) ends up in one of the open subrelations \( R' \) by the claim above. This clearly implies that the colimit of the sheaves \( U/R' \) is \( U/R \). In other words the algebraic space \( X = U/R \) is the colimit of the quasi-separated algebraic spaces \( U/R' \).

Properties (1) and (2) follow from the discussion above. If \( X \) is a scheme, then if we choose \( U \) to be a finite disjoint union of affine opens of \( X \) we will obtain (3).

Details omitted. \( \square \)

\textbf{Lemma 6.5.} \textit{Let } \( S \) \textit{be a scheme. Let } \( X, Y \) \textit{be algebraic spaces over } \( S \). \textit{Let } \( X \to Y \) \textit{be a representable morphism. If } \( Y \) \textit{is decent (resp. reasonable), then so is } \( X \).

\textbf{Proof.} Translation of Lemma \[5.3\]. \( \square \)

\textbf{Lemma 6.6.} \textit{Let } \( S \) \textit{be a scheme. Let } \( X \to Y \) \textit{be an étale morphism of algebraic spaces over } \( S \). \textit{If } \( Y \) \textit{is decent, resp. reasonable, then so is } \( X \).

\textbf{Proof.} Let \( U \) be an affine scheme and \( U \to X \) an étale morphism. Set \( R = U \times_X U \) and \( R' = U \times_Y U \). Note that \( R \to R' \) is a monomorphism.

Let \( x \in |X| \). To show that \( X \) is decent, we have to show that the fibres of \( |U| \to |X| \) and \( |R| \to |X| \) over \( x \) are finite. But if \( Y \) is decent, then the fibres of \( |U| \to |Y| \) and \( |R'| \to |Y| \) are finite. Hence the result for “decent”.

To show that \( X \) is reasonable, we have to show that the fibres of \( U \to X \) are universally bounded. However, if \( Y \) is reasonable, then the fibres of \( U \to Y \) are universally bounded, which immediately implies the same thing for the fibres of \( U \to X \). Hence the result for “reasonable”. \( \square \)
7. Points and specializations

There exists an étale morphism of algebraic spaces $f : X \to Y$ and a nontrivial specialization between points in a fibre of $|f| : |X| \to |Y|$, see Examples, Lemma 49.1. If the source of the morphism is a scheme we can avoid this by imposing condition (α) on $Y$.

**Lemma 7.1.** Let $S$ be a scheme. Let $X$ be an algebraic space over $S$. Let $U \to X$ be an étale morphism from a scheme to $X$. Assume $u, u' \in |U|$ map to the same point $x$ of $|X|$, and $u' \rightsquigarrow u$. If the pair $(X, x)$ satisfies the equivalent conditions of Lemma 4.2 then $u = u'$.

**Proof.** Assume the pair $(X, x)$ satisfies the equivalent conditions for Lemma 4.2. Let $U$ be a scheme, $U \to X$ étale, and let $u, u' \in |U|$ map to $x$ of $|X|$, and $u' \rightsquigarrow u$. We may and do replace $U$ by an affine neighbourhood of $u$. Let $t, s : R = U \times_X U \to U$ be the étale projection maps.

Pick a point $r \in R$ with $t(r) = u$ and $s(r) = u'$. This is possible by Properties of Spaces, Lemma 4.15. Because generalizations lift along the étale morphism $t$ (Remark 4.11) we can find a specialization $r' \rightsquigarrow r$ with $t(r') = u'$. Set $u'' = s(r')$. Then $u'' \rightsquigarrow u'$. Thus we may repeat and find $r'' \rightsquigarrow r'$ with $t(r'') = u''$. Set $u''' = s(r'')$, and so on. Here is a picture:

![Diagram](image)

In Remark 4.11 we have seen that there are no specializations among points in the fibres of the étale morphism $s$. Hence if $u^{(n+1)} = u^{(n)}$ for some $n$, then also $r^{(n)} = r^{(n-1)}$ and hence also (by taking $t$) $u^{(n)} = u^{(n-1)}$. This then forces the whole tower to collapse, in particular $u = u'$. Thus we see that if $u \neq u'$, then all the specializations are strict and $\{u, u', u'', \ldots\}$ is an infinite set of points in $U$ which map to the point $x$ in $|X|$. As we chose $U$ affine this contradicts the second part of Lemma 4.2 as desired.

**Lemma 7.2.** Let $S$ be a scheme. Let $X$ be an algebraic space over $S$. Let $x, x' \in |X|$ and assume $x' \rightsquigarrow x$, i.e., $x$ is a specialization of $x'$. Assume the pair $(X, x')$ satisfies the equivalent conditions of Lemma 4.5. Then for every étale morphism $\varphi : U \to X$ from a scheme $U$ and any $u \in U$ with $\varphi(u) = x$, exists a point $u' \in U$, $u' \rightsquigarrow u$ with $\varphi(u') = x'$.

**Proof.** We may replace $U$ by an affine open neighbourhood of $u$. Hence we may assume that $U$ is affine. As $x$ is in the image of the open map $|U| \to |X|$, so is $x'$. Thus we may replace $X$ by the Zariski open subspace corresponding to the image of $|U| \to |X|$, see Properties of Spaces, Lemma 4.10. In other words we may assume that $U \to X$ is surjective and étale. Let $s, t : R = U \times_X U \to U$ be the
In this section we prove that a quasi-compact and quasi-separated algebraic space and \( \phi \) can find a specialization \( x \in U \to W \). As generalizations lift along maps to the given specialization \( x \). By our assumption that \((X, x')\) satisfies the equivalent conditions of Lemma 4.5 we can find \( u \in U \) and \( \{r'_1, \ldots, r'_m\} \subset R \) form the complete inverse image of \( \{x'\} \). Consider the closed sets

\[
T = \{u'_1\} \cup \ldots \cup \{u'_n\} \subset |U|, \quad T' = \{r'_1\} \cup \ldots \cup \{r'_m\} \subset |R|.
\]

Trivially we have \( s(T') \subset T \). Because \( R \) is an equivalence relation we also have \( t(T') = s(T') \) as the set \( \{r'_i\} \) is invariant under the inverse of \( R \) by construction. Let \( w \in T \) be any point. Then \( u'_i \sim w \) for some \( i \). Choose \( r \in R \) with \( s(r) = w \).

Since generalizations lift along \( s : R \to U \), see Remark 4.1, we can find \( r' \sim r \) with \( s(r') = u'_i \). Then \( r' = r'_j \) for some \( j \) and we conclude that \( w \in s(T') \). Hence \( T = s(T') = t(T') \) is an \(|R|\)-invariant closed set in \(|U|\). This means \( T \) is the inverse image of a closed (!) subset \( T'' = \varphi(T) \) of \(|X|\), see Properties of Spaces, Lemmas 4.5 and 4.6. Hence \( T'' = \{x'\} \). Thus \( T \) contains some point \( u_1 \) mapping to \( x \) as \( x \in T'' \). I.e., we see that for some \( i \) there exists a specialization \( u'_i \sim u_1 \) which maps to the given specialization \( x' \sim x \).

To finish the proof, choose a point \( r \in R \) such that \( s(r) = u \) and \( t(r) = u_1 \) (using Properties of Spaces, Lemma 4.3). As generalizations lift along \( t \), and \( u'_i \sim u \) we can find a specialization \( r' \sim r \) such that \( t(r') = u'_i \). Set \( u' = s(r') \). Then \( u' \sim u \) and \( \varphi(u') = x' \) as desired. \( \square \)

**Lemma 7.3.** Let \( S \) be a scheme. Let \( f : Y \to X \) be a flat morphism of algebraic spaces over \( S \). Let \( x, x' \in |X| \) and assume \( x' \sim x \), i.e., \( x \) is a specialization of \( x' \). Assume the pair \((X, x')\) satisfies the equivalent conditions of Lemma 4.7 (for example if \( X \) is decent, \( X \) is quasi-separated, or \( X \) is representable). Then for every \( y \in |Y| \) with \( f(y) = x \), there exists a point \( y' \in |Y| \), \( y' \sim y \) with \( f(y') = x' \).

**Proof.** (The parenthetical statement holds by the definition of decent spaces and the implications between the different separation conditions mentioned in Section 6) Choose a scheme \( V \) and a surjective étale morphism \( V \to Y \). Choose \( v \in V \) mapping to \( y \). Then we see that it suffices to prove the lemma for \( V \to X \). Thus we may assume \( Y \) is a scheme. Choose a scheme \( U \) and a surjective étale morphism \( U \to X \). Choose \( u \in U \) mapping to \( x \). By Lemma 7.2 we may choose \( u' \sim u \) mapping to \( x' \). By Properties of Spaces, Lemma 4.3 we may choose \( z \in U \times_X Y \) mapping to \( y \) and \( u \). Thus we reduce to the case of the flat morphism of schemes \( U \times_X Y \to U \) which is Morphisms, Lemma 25.9. \( \square \)

### 8. Stratifying algebraic spaces by schemes

In this section we prove that a quasi-compact and quasi-separated algebraic space has a finite stratification by locally closed subspaces each of which is a scheme and such that the glueing of the parts is by elementary distinguished squares. We first prove a slightly weaker result for reasonable algebraic spaces.

**Lemma 8.1.** Let \( S \) be a scheme. Let \( W \to X \) be a morphism of a scheme \( W \) to an algebraic space \( X \) which is flat, locally of finite presentation, separated, locally quasi-finite with universally bounded fibres. There exist reduced closed subspaces

\[
\emptyset = Z_{-1} \subset Z_0 \subset Z_1 \subset Z_2 \subset \ldots \subset Z_n = X
\]

0B7W  

0A4I  

07S8
such that with $X_r = Z_r \setminus Z_{r-1}$ the stratification $X = \bigsqcup_{r=0, \ldots, n} X_r$ is characterized by the following universal property: Given $g : T \to X$ the projection $W \times_X T \to T$ is finite locally free of degree $r$ if and only if $g(|T|) \subset |X_r|$.

**Proof.** Let $n$ be an integer bounding the degrees of the fibres of $W \to X$. Choose a scheme $U$ and a surjective étale morphism $U \to X$. Apply More on Morphisms, Lemma 41.3 to $W \times_X U \to U$. We obtain closed subsets

$$\emptyset = Y_{-1} \subset Y_0 \subset Y_1 \subset \ldots \subset Y_n = U$$

characterized by the property stated in the lemma for the morphism $W \times_X U \to U$. Clearly, the formation of these closed subsets commutes with base change. Setting $R = U \times_X U$ with projection maps $s, t : R \to U$ we conclude that

$$s^{-1}(Y_r) = t^{-1}(Y_r)$$

as closed subsets of $R$. In other words the closed subsets $Y_r \subset U$ are $R$-invariant. This means that $|Y_r|$ is the inverse image of a closed subset $Z_r \subset |X|$. Denote $Z_r \subset X$ also the reduced induced algebraic space structure, see Properties of Spaces, Definition 12.5.

Let $g : T \to X$ be a morphism of algebraic spaces. Choose a scheme $V$ and a surjective étale morphism $V \to T$. To prove the final assertion of the lemma it suffices to prove the assertion for the composition $V \to X$ (by our definition of finite locally free morphisms, see Morphisms of Spaces, Section 46). Similarly, the morphism of schemes $W \times_X V \to V$ is finite locally free of degree $r$ if and only if the morphism of schemes

$$W \times_X (U \times_X V) \to U \times_X V$$

is finite locally free of degree $r$ (see Descent, Lemma 20.30). By construction this happens if and only if $|U \times_X V| \to |U|$ maps into $|Y_r|$, which is true if and only if $|V| \to |X|$ maps into $|Z_r|$. □

086T **Lemma 8.2.** Let $S$ be a scheme. Let $W \to X$ be a morphism of a scheme $W$ to an algebraic space $X$ which is flat, locally of finite presentation, separated, and locally quasi-finite. Then there exist open subspaces

$$X = X_0 \supset X_1 \supset X_2 \supset \ldots$$

such that a morphism $\text{Spec}(k) \to X$ factors through $X_d$ if and only if $W \times_X \text{Spec}(k)$ has degree $\geq d$ over $k$.

**Proof.** Choose a scheme $U$ and a surjective étale morphism $U \to X$. Apply More on Morphisms, Lemma 41.3 to $W \times_X U \to U$. We obtain open subschemes

$$U = U_0 \supset U_1 \supset U_2 \supset \ldots$$

characterized by the property stated in the lemma for the morphism $W \times_X U \to U$. Clearly, the formation of these closed subsets commutes with base change. Setting $R = U \times_X U$ with projection maps $s, t : R \to U$ we conclude that

$$s^{-1}(U_d) = t^{-1}(U_d)$$

as open subschemes of $R$. In other words the open subschemes $U_d \subset U$ are $R$-invariant. This means that $U_d$ is the inverse image of an open subspace $X_d \subset X$ (Properties of Spaces, Lemma 12.2). □
Let $S$ be a scheme. Let $X$ be a quasi-compact algebraic space over $S$. There exist open subspaces

\[ \ldots \subset U_4 \subset U_3 \subset U_2 \subset U_1 = X \]

with the following properties:

1. setting $T_p = U_p \setminus U_{p+1}$ (with reduced induced subspace structure) there exists a separated scheme $V_p$ and a surjective étale morphism $f_p : V_p \to U_p$ such that $f_p^{-1}(T_p) \to T_p$ is an isomorphism,

2. if $x \in |X|$ can be represented by a quasi-compact morphism $\text{Spec}(k) \to X$ from a field, then $x \in T_p$ for some $p$.

**Proof.** By Properties of Spaces, Lemma 6.3 we can choose an affine scheme $U$ and a surjective étale morphism $U \to X$. For $p \geq 0$ set

\[ W_p = U \times_X \ldots \times_X U \setminus \text{all diagonals} \]

where the fibre product has $p$ factors. Since $U$ is separated, the morphism $U \to X$ is separated and all fibre products $U \times_X \ldots \times_X U$ are separated schemes. Since $U \to X$ is separated the diagonal $U \to U \times_U U$ is a closed immersion. Since $U \to X$ is étale the diagonal $U \to U \times_U U$ is an open immersion, see Morphisms of Spaces, Lemmas 39.10 and 38.9. Similarly, all the diagonal morphisms are open and closed immersions and $W_p$ is an open and closed subscheme of $U \times_X \ldots \times_X U$. Moreover, the morphism

\[ U \times_X \ldots \times_X U \longrightarrow U \times_{\text{Spec}(k)} \ldots \times_{\text{Spec}(k)} U \]

is locally quasi-finite and separated (Morphisms of Spaces, Lemma 4.5) and its target is an affine scheme. Hence every finite set of points of $U \times_X \ldots \times_X U$ is contained in an affine open, see More on Morphisms, Lemma 41.1. Therefore, the same is true for $W_p$. There is a free action of the symmetric group $S_p$ on $W_p$ over $X$ (because we threw out the fix point locus from $U \times_X \ldots \times_X U$). By the above and Properties of Spaces, Proposition 14.1 the quotient $V_p = W_p / S_p$ is a scheme. Since the action of $S_p$ on $W_p$ was over $X$, there is a morphism $V_p \to X$. Since $W_p \to X$ is étale and since $W_p \to V_p$ is surjective étale, it follows that also $V_p \to X$ is étale, see Properties of Spaces, Lemma 16.3. Observe that $V_p$ is a separated scheme by Properties of Spaces, Lemma 14.3.

We let $U_p \subset X$ be the open subspace which is the image of $V_p \to X$. By construction a morphism $\text{Spec}(k) \to X$ with $k$ algebraically closed, factors through $U_p$ if and only if $U \times_X \text{Spec}(k)$ has $\geq p$ points; as usual observe that $U \times_X \text{Spec}(k)$ is scheme theoretically a disjoint union of (possibly infinitely many) copies of $\text{Spec}(k)$, see Remark 4.1. It follows that the $U_p$ give a filtration of $X$ as stated in the lemma. Moreover, our morphism $\text{Spec}(k) \to X$ factors through $T_p$ if and only if $U \times_X \text{Spec}(k)$ has exactly $p$ points. In this case we see that $V_p \times_X \text{Spec}(k)$ has exactly one point. Set $Z_p = f_p^{-1}(T_p) \subset V_p$. This is a closed subscheme of $V_p$. Then $Z_p \to T_p$ is an étale morphism between algebraic spaces which induces a bijection on $k$-valued points for any algebraically closed field $k$. To be sure this implies that $Z_p \to T_p$ is universally injective, whence an open immersion by Morphisms of Spaces, Lemma 51.2 hence an isomorphism and (1) has been proved.

Let $x : \text{Spec}(k) \to X$ be a quasi-compact morphism where $k$ is a field. Then the composition $\text{Spec}(\overline{k}) \to \text{Spec}(k) \to X$ is quasi-compact as well (Morphisms of Spaces, Lemma 8.5). In this case the scheme $U \times_X \text{Spec}(\overline{k})$ is quasi-compact. In
view of the fact (seen above) that it is a disjoint union of copies of Spec($k$) we find that it has finitely many points. If the number of points is $p$, then we see that indeed $x \in T_p$ and the proof is finished. \hfill \qed

\textbf{Lemma 8.4.} Let $S$ be a scheme. Let $X$ be a quasi-compact, reasonable algebraic space over $S$. There exist an integer $n$ and open subspaces

$$0 = U_{n+1} \subset U_n \subset U_{n-1} \subset \ldots \subset U_1 = X$$

with the following property: setting $T_p = U_p \setminus U_{p+1}$ (with reduced induced subspace structure) there exists a separated scheme $V_p$ and a surjective étale morphism $f_p : V_p \rightarrow U_p$ such that $f_p^{-1}(T_p) \rightarrow T_p$ is an isomorphism.

\textbf{Proof.} The proof of this lemma is identical to the proof of Lemma \textcolor{red}{8.3}. Let $n$ be an integer bounding the degrees of the fibres of $U \rightarrow X$ which exists as $X$ is reasonable, see Definition \textcolor{red}{6.1}. Then we see that $U_{n+1} = \emptyset$ and the proof is complete. \hfill \qed

\textbf{Lemma 8.5.} Let $S$ be a scheme. Let $X$ be a quasi-compact, reasonable algebraic space over $S$. There exist an integer $n$ and open subspaces

$$0 = U_{n+1} \subset U_n \subset U_{n-1} \subset \ldots \subset U_1 = X$$

such that each $T_p = U_p \setminus U_{p+1}$ (with reduced induced subspace structure) is a scheme.

\textbf{Proof.} Immediate consequence of Lemma \textcolor{red}{8.4}. \hfill \qed

The following result is almost identical to [GR71, Proposition 5.7.8].

\textbf{Lemma 8.6.} Let $X$ be a quasi-compact and quasi-separated algebraic space over $\text{Spec}(\mathbb{Z})$. There exist an integer $n$ and open subspaces

$$0 = U_{n+1} \subset U_n \subset U_{n-1} \subset \ldots \subset U_1 = X$$

with the following property: setting $T_p = U_p \setminus U_{p+1}$ (with reduced induced subspace structure) there exists a quasi-compact separated scheme $V_p$ and a surjective étale morphism $f_p : V_p \rightarrow U_p$ such that $f_p^{-1}(T_p) \rightarrow T_p$ is an isomorphism.

\textbf{Proof.} The proof of this lemma is identical to the proof of Lemma \textcolor{red}{8.3}. Observe that a quasi-separated space is reasonable, see Lemma \textcolor{red}{6.1} and Definition \textcolor{red}{6.1}. Hence we find that $U_{n+1} = \emptyset$ as in Lemma \textcolor{red}{8.4}. At the end of the argument we add that since $X$ is quasi-separated the schemes $U \times X \ldots \times X U$ are all quasi-compact. Hence the schemes $W_p$ are quasi-compact. Hence the quotients $V_p = W_p/S_p$ by the symmetric group $S_p$ are quasi-compact schemes. \hfill \qed

The following lemma probably belongs somewhere else.

\textbf{Lemma 8.7.} Let $S$ be a scheme. Let $X$ be a quasi-separated algebraic space over $S$. Let $E \subset |X|$ be a subset. Then $E$ is étale locally constructible (Properties of Spaces, Definition 8.2) if and only if $E$ is a locally constructible subset of the topological space $|X|$ (Topology, Definition 15.1).

\textbf{Proof.} Assume $E \subset |X|$ is a locally constructible subset of the topological space $|X|$. Let $f : U \rightarrow X$ be an étale morphism where $U$ is a scheme. We have to show that $f^{-1}(E)$ is locally constructible in $U$. The question is local on $U$ and $X$, hence we may assume that $X$ is quasi-compact, $E \subset |X|$ is constructible, and $U$ is affine. In this case $U \rightarrow X$ is quasi-compact, hence $f : |U| \rightarrow |X|$ is quasi-compact. Observe that retrocompact opens of $|X|$, resp. $U$ are the same thing as...
quasi-compact opens of $|X|$, resp. $U$, see Topology, Lemma \[27.1\] Thus $f^{-1}(E)$ is constructible by Topology, Lemma \[15.3\]

Conversely, assume $E$ is étale locally constructible. We want to show that $E$ is locally constructible in the topological space $|X|$. The question is local on $X$, hence we may assume that $X$ is quasi-compact as well as quasi-separated. We will show that in this case $E$ is constructible in $|X|$. Choose open subspaces

$$0 = U_{n+1} \subset U_n \subset U_{n-1} \subset \ldots \subset U_1 = X$$

and surjective étale morphisms $f_p : V_p \to U_p$ inducing isomorphisms $f_p^{-1}(T_p) \to T_p = U_p \setminus U_{p+1}$ where $V_p$ is a quasi-compact separated scheme as in Lemma \[8.6\]. By definition the inverse image $E_p \subset V_p$ of $E$ is locally constructible in $V_p$. Then $E_p$ is constructible in $V_p$ by Properties, Lemma \[2.5\]. Thus $E_p \cap |f_p^{-1}(T_p)| = E \cap |T_p|$ is constructible in $|T_p|$ by Topology, Lemma \[15.7\] (observe that $V_p \setminus f_p^{-1}(T_p)$ is quasi-compact as it is the inverse image of the quasi-compact space $U_{p+1}$ by the quasi-compact morphism $f_p$).

Thus

$$E = (|T_n| \cap E) \cup (|T_{n-1}| \cap E) \cup \ldots \cup (|T_1| \cap E)$$

is constructible by Topology, Lemma \[15.14\]. Here we use that $|T_p|$ is constructible in $|X|$ which is clear from what was said above. \[\square\]

9. Integral cover by a scheme

0D2T Here we prove that given any quasi-compact and quasi-separated algebraic space $X$, there is a scheme $Y$ and a surjective, integral morphism $Y \to X$. After we develop some theory about limits of algebraic spaces, we will prove that one can do this with a finite morphism, see Limits of Spaces, Section \[16\]

0G2D Lemma 9.1. Let $S$ be a scheme. Let $j : V \to Y$ be a quasi-compact open immersion of algebraic spaces over $S$. Let $\pi : Z \to V$ be an integral morphism. Then there exists an integral morphism $\nu : Y' \to Y$ such that $Z$ is $V$-isomorphic to the inverse image of $V$ in $Y'$.

Proof. Since both $j$ and $\pi$ are quasi-compact and separated, so is $j \circ \pi$. Let $\nu : Y' \to Y$ be the normalization of $Y$ in $Z$, see Morphisms of Spaces, Section \[48\]. Of course $\nu$ is integral, see Morphisms of Spaces, Lemma \[48.5\]. The final statement follows formally from Morphisms of Spaces, Lemmas \[48.4\] and \[48.10\]. \[\square\]

09YB Lemma 9.2. Let $S$ be a scheme. Let $X$ be a quasi-compact and quasi-separated algebraic space over $S$.

1. There exists a surjective integral morphism $Y \to X$ where $Y$ is a scheme,

2. given a surjective étale morphism $U \to X$ we may choose $Y \to X$ such that for every $y \in Y$ there is an open neighbourhood $V \subset Y$ such that $V \to X$ factors through $U$.

Proof. Part (1) is the special case of part (2) where $U = X$. Choose a surjective étale morphism $U' \to U$ where $U'$ is a scheme. It is clear that we may replace $U$ by $U'$ and hence we may assume $U$ is a scheme. Since $X$ is quasi-compact, there exist finitely many affine opens $U_i \subset U$ such that $U' = \coprod U_i \to X$ is surjective. After replacing $U$ by $U'$ again, we see that we may assume $U$ is affine. Since $X$ is quasi-separated, hence reasonable, there exists an integer $d$ bounding the degree of the geometric fibres of $U \to X$ (see Lemma \[5.1\]). We will prove the lemma by
induction on $d$ for all quasi-compact and separated schemes $U$ mapping surjective and étale onto $X$. If $d = 1$, then $U = X$ and the result holds with $Y = U$. Assume $d > 1$.

We apply Morphisms of Spaces, Lemma 52.2 and we obtain a factorization

$$
\begin{array}{c}
U \\
\downarrow j \\
X
\end{array}
\quad \rightarrow 
\begin{array}{c}
Y \\
\downarrow \pi
\end{array}
$$

with $\pi$ integral and $j$ a quasi-compact open immersion. We may and do assume that $j(U)$ is scheme theoretically dense in $Y$. Note that

$$U \times_X Y = U \amalg W$$

where the first summand is the image of $U \rightarrow U \times_X Y$ (which is closed by Morphisms of Spaces, Lemma 4.6 and open because it is étale as a morphism between algebraic spaces étale over $Y$) and the second summand is the (open and closed) complement. The image $V \subset Y$ of $W$ is an open subspace containing $Y \setminus U$.

The étale morphism $W \rightarrow Y$ has geometric fibres of cardinality $< d$. Namely, this is clear for geometric points of $U \subset Y$ by inspection. Since $|U| \subset |Y|$ is dense, it holds for all geometric points of $Y$ by Lemma 8.1 (the degree of the fibres of a quasi-compact étale morphism does not go up under specialization). Thus we may apply the induction hypothesis to $W \rightarrow V$ and find a surjective integral morphism $Z \rightarrow V$ with $Z$ a scheme, which Zariski locally factors through $W$. Choose a factorization $Z \rightarrow Z' \rightarrow Y$ with $Z' \rightarrow Y$ integral and $Z \rightarrow Z'$ open immersion (Lemma 9.1). After replacing $Z'$ by the scheme theoretic closure of $Z$ in $Z'$ we may assume that $Z$ is scheme theoretically dense in $Z'$.

Finally, let $T \subset Y$ be the induced closed subspace structure on $Y \setminus V$. Consider the morphism

$$Z' \amalg T \rightarrow X$$

This is a surjective integral morphism by construction. Since $T \subset U$ it is clear that the morphism $T \rightarrow X$ factors through $U$. On the other hand, let $z \in Z'$ be a point. If $z \notin Z$, then $z$ maps to a point of $Y \setminus V \subset U$ and we find a neighbourhood of $z$ on which the morphism factors through $U$. If $z \in Z$, then we have an open neighbourhood of $z$ in $Z$ (which is also an open neighbourhood of $z$ in $Z'$) which factors through $W \subset U \times_X Y$ and hence through $U$. □

10. Schematic locus

06NN In this section we prove that a decent algebraic space has a dense open subspace which is a scheme. We first prove this for reasonable algebraic spaces.

03JI **Proposition 10.1.** Let $S$ be a scheme. Let $X$ be an algebraic space over $S$. If $X$ is reasonable, then there exists a dense open subspace of $X$ which is a scheme.

**Proof.** By Properties of Spaces, Lemma 13.1 the question is local on $X$. Hence we may assume there exists an affine scheme $U$ and a surjective étale morphism $U \rightarrow X$ (Properties of Spaces, Lemma 6.1). Let $n$ be an integer bounding the degrees of the fibres of $U \rightarrow X$ which exists as $X$ is reasonable, see Definition 6.1.

We will argue by induction on $n$ that whenever

(1) $U \rightarrow X$ is a surjective étale morphism whose fibres have degree $\leq n$, and
(2) $U$ is isomorphic to a locally closed subscheme of an affine scheme
then the schematic locus is dense in $X$.

Let $X_n \subset X$ be the open subspace which is the complement of the closed subspace $Z_{n-1} \subset X$ constructed in Lemma 8.1 using the morphism $U \to X$. Let $U_n \subset U$ be the inverse image of $X_n$. Then $U_n \to X_n$ is finite locally free of degree $n$. Hence $X_n$ is a scheme by Properties of Spaces, Proposition 14.1 (and the fact that any finite set of points of $U_n$ is contained in an affine open of $U_n$, see Properties, Lemma 29.5).

Let $X' \subset X$ be the open subspace such that $|X'|$ is the interior of $|Z_{n-1}|$ in $|X|$ (see Topology, Definition 21.1). Let $U' \subset U$ be the inverse image. Then $U' \to X'$ is surjective étale and has degrees of fibres bounded by $n-1$. By induction we see that the schematic locus of $X'$ is an open dense $X'' \subset X'$. By elementary topology we see that $X'' \cup X_n \subset X$ is open and dense and we win. □

**Theorem 10.2** (David Rydh). Let $S$ be a scheme. Let $X$ be an algebraic space over $S$. If $X$ is decent, then there exists a dense open subspace of $X$ which is a scheme.

**Proof.** Assume $X$ is a decent algebraic space for which the theorem is false. By Properties of Spaces, Lemma 13.1 there exists a largest open subspace $X' \subset X$ which is a scheme. Since $X'$ is not dense in $X$, there exists an open subspace $X'' \subset X$ such that $|X''| \cap |X'| = \emptyset$. Replacing $X$ by $X''$ we get a nonempty decent algebraic space $X$ which does not contain any open subspace which is a scheme.

Choose a nonempty affine scheme $U$ and an étale morphism $U \to X$. We may and do replace $X$ by the open subscheme corresponding to the image of $|U| \to |X|$. Consider the sequence of open subspaces

$$X = X_0 \supset X_1 \supset X_2 \ldots$$

constructed in Lemma 8.2 for the morphism $U \to X$. Note that $X_0 = X_1$ as $U \to X$ is surjective. Let $U = U_0 = U_1 \supset U_2 \ldots$ be the induced sequence of open subschemes of $U$.

Choose a nonempty open affine $V_1 \subset U_1$ (for example $V_1 = U_1$). By induction we will construct a sequence of nonempty affine opens $V_1 \supset V_2 \supset \ldots$ with $V_n \subset U_n$. Namely, having constructed $V_1, \ldots, V_{n-1}$ we can always choose $V_n$ unless $V_{n-1} \cap U_n = \emptyset$. But if $V_{n-1} \cap U_n = \emptyset$, then the open subspace $X' \subset X$ with $|X'| = \text{Im}(|V_{n-1}| \to |X|)$ is contained in $|X|\setminus|X_n|$. Hence $V_{n-1} \to X'$ is an étale morphism whose fibres have degree bounded by $n-1$. In other words, $X'$ is reasonable (by definition), hence $X'$ contains a nonempty open subscheme by Proposition 10.1. This is a contradiction which shows that we can pick $V_n$.

By Limits, Lemma 4.3 the limit $V_\infty = \lim V_n$ is a nonempty scheme. Pick a morphism $\text{Spec}(k) \to V_\infty$. The composition $\text{Spec}(k) \to V_\infty \to U \to X$ has image contained in all $X_n$ by construction. In other words, the fibre $U \times_X \text{Spec}(k)$ has infinite degree which contradicts the definition of a decent space. This contradiction finishes the proof of the theorem. □

**Lemma 10.3.** Let $S$ be a scheme. Let $X \to Y$ be a surjective finite locally free morphism of algebraic spaces over $S$. For $y \in |Y|$ the following are equivalent

(1) $y$ is in the schematic locus of $Y$, and
(2) there exists an affine open \( U \subset X \) containing the preimage of \( y \).

**Proof.** If \( y \in Y \) is in the schematic locus, then it has an affine open neighbourhood \( V \subset Y \) and the inverse image \( U \) of \( V \) in \( X \) is an open finite over \( V \), hence affine. Thus (1) implies (2).

Conversely, assume that \( U \subset X \) as in (2) is given. Set \( R = X \times_Y X \) and denote the projections \( s, t : R \to X \). Consider \( Z = R \setminus s^{-1}(U) \cap t^{-1}(U) \). This is a closed subset of \( R \). The image \( t(Z) \) is a closed subset of \( X \) which can loosely be described as the set of points of \( X \) which are \( R \)-equivalent to a point of \( X \setminus U \). Hence \( U' = X \setminus t(Z) \) is an \( R \)-invariant, open subspace of \( X \) contained in \( U \) which contains the fibre of \( X \to Y \) over \( y \). Since \( X \to Y \) is open (Morphisms of Spaces, Lemma \[30.6\]) the image of \( U' \) is an open subspace \( V' \subset Y \). Since \( U' \) is \( R \)-invariant and \( R = X \times_Y X \), we see that \( U' \) is the inverse image of \( V' \) (use Properties of Spaces, Lemma \[4.3\]). After replacing \( Y \) by \( V' \) and \( X \) by \( U' \) we see that we may assume \( X \) is a scheme isomorphic to an open subscheme of an affine scheme.

Assume \( X \) is a scheme isomorphic to an open subscheme of an affine scheme. In this case the fppf quotient sheaf \( X/R \) is a scheme, see Properties of Spaces, Proposition \[14.1\]. Since \( Y \) is a sheaf in the fppf topology, obtain a canonical map \( X/R \to Y \) factoring \( X \to Y \). Since \( X \to Y \) is surjective finite locally free, it is surjective as a map of sheaves (Spaces, Lemma \[5.9\]). We conclude that \( X/R \to Y \) is surjective as a scheme. On the other hand, since \( R = X \times_Y X \) as sheaves we conclude that \( X/R \to Y \) is injective as a map of sheaves. Hence \( X/R \to Y \) is an isomorphism and we see that \( Y \) is representable. \( \square \)

At this point we have several different ways for proving the following lemma.

**Lemma 10.4.** Let \( S \) be a scheme. Let \( X \) be an algebraic space over \( S \). If there exists a finite, étale, surjective morphism \( U \to X \) where \( U \) is a scheme, then there exists a dense open subspace of \( X \) which is a scheme.

**First proof.** The morphism \( U \to X \) is finite locally free. Hence there is a decomposition of \( U \) into open and closed subspaces \( X_d \subset U \) such that \( U \times_X X_d \to X_d \) is finite locally free of degree \( d \). Thus we may assume \( U \to X \) is finite locally free of degree \( d \). In this case, let \( U_i \subset U, i \in I \) be the set of affine opens. For each \( i \) the morphism \( U_i \to X \) is étale and has universally bounded fibres (namely, bounded by \( d \)). In other words, \( X \) is reasonable and the result follows from Proposition \[10.1\]. \( \square \)

**Second proof.** The question is local on \( X \) (Properties of Spaces, Lemma \[13.1\]), hence may assume \( X \) is quasi-compact. Then \( U \) is quasi-compact. Then there exists a dense open subscheme \( W \subset U \) which is separated (Properties, Lemma \[29.3\]). Set \( Z = U \setminus W \). Let \( R = U \times_X U \) and \( s, t : R \to U \) the projections. Then \( t^{-1}(Z) \) is nowhere dense in \( R \) (Topology, Lemma \[21.6\]) and hence \( \Delta = s(t^{-1}(Z)) \) is an \( R \)-invariant closed nowhere dense subset of \( U \) (Morphisms, Lemma \[48.7\]). Let \( u \in U \setminus \Delta \) be a generic point of an irreducible component. Since these points are dense in \( U \setminus \Delta \) and since \( \Delta \) is nowhere dense, it suffices to show that the image \( x \in X \) of \( u \) is in the schematic locus of \( X \). Observe that \( t(s^{-1}(\{u\})) \subset W \) is a finite set of generic points of irreducible components of \( W \) (compare with Properties of Spaces, Lemma \[11.1\]). By Properties, Lemma \[29.1\] we can find an affine open \( V \subset W \) such that \( t(s^{-1}(\{u\})) \subset V \). Since \( t(s^{-1}(\{u\})) \) is the fibre of \( |U| \to |X| \) over \( x \), we conclude by Lemma \[10.3\]. \( \square \)
Third proof. (This proof is essentially the same as the second proof, but uses fewer references.) Assume $X$ is an algebraic space, $U$ a scheme, and $U \to X$ is a
finite étale surjective morphism. Write $R = U \times_X U$ and denote $s, t : R \to U$
the projections as usual. Note that $s, t$ are surjective, finite and étale. Claim: The
union of the $R$-invariant affine opens of $U$ is topologically dense in $U$.

Proof of the claim. Let $W \subseteq U$ be an affine open. Set $W' = t(s^{-1}(W)) \subseteq U$. Since
$s^{-1}(W)$ is affine (hence quasi-compact) we see that $W' \subseteq U$ is a quasi-compact
open. By Properties, Lemma 29.3 there exists a dense open $W'' \subseteq W'$ which is a
separated scheme. Set $\Delta' = W'' \setminus W''$. This is a nowhere dense closed subset of $W''$. Since $t|_{s^{-1}(W)} : s^{-1}(W) \to W'$ is open (because it is étale) we see that the inverse
image $(t|_{s^{-1}(W)})^{-1}(\Delta') \subseteq s^{-1}(W)$ is a nowhere dense closed subset (see Topology,
Lemma 21.6). Hence, by Morphisms, Lemma 48.7 we see that
$$
\Delta = s \left( (t|_{s^{-1}(W)})^{-1}(\Delta') \right)
$$
is a nowhere dense closed subset of $W$. Pick any point $\eta \in W$, $\eta \not\in \Delta$ which is a
generic point of an irreducible component of $W$ (and hence of $U$). By our choices
above the finite set $t(s^{-1}\{\eta\}) = \{\eta_1, \ldots, \eta_n\}$ is contained in the separated scheme
$W''$. Note that the fibres of $s$ are finite discrete spaces, and that generalizations
lift along the étale morphism $t$, see Morphisms, Lemmas 36.12 and 25.9. In this way we see that each $\eta_i$ is a generic point of an irreducible component of $W''$. Thus, by Properties, Lemma 29.1 we can find an affine open $V \subseteq W''$ such that
$\{\eta_1, \ldots, \eta_n\} \subseteq V$. By Groupoids, Lemma 24.1 this implies that $\eta$ is contained in
an $R$-invariant affine open subscheme of $U$. The claim follows as $W$ was chosen as an
arbitrary affine open of $U$ and because the set of generic points of irreducible
components of $W \setminus \Delta$ is dense in $W$.

Using the claim we can finish the proof. Namely, if $W \subseteq U$ is an $R$-invariant
affine open, then the restriction $R_W$ of $R$ to $W$ equals $R_W = s^{-1}(W) = t^{-1}(W)$
(see Groupoids, Definition 19.1 and discussion following it). In particular the maps
$R_W \to W$ are finite étale also. It follows in particular that $R_W$ is affine. Thus we
see that $W/R_W$ is a scheme, by Groupoids, Proposition 23.9. On the other hand, $W/R_W$ is an open subspace of $X$ by Spaces, Lemma 10.2. Hence having a dense
collection of points contained in $R$-invariant affine open of $U$ certainly implies that
the schematic locus of $X$ (see Properties of Spaces, Lemma 13.1) is open dense in $X$. □

11. Residue fields and henselian local rings

0EMV For a decent algebraic space we can define the residue field and the henselian local
ring at a point. For example, the following lemma tells us the residue field of a
point on a decent space is defined.

03K4 Lemma 11.1. Let $S$ be a scheme. Let $X$ be an algebraic space over $S$. Consider the map

$$\{\text{Spec}(k) \to X \text{ monomorphism where } k \text{ is a field}\} \to |X|$$

This map is always injective. If $X$ is decent then this map is a bijection.

Proof. We have seen in Properties of Spaces, Lemma 4.11 that the map is an
injection in general. By Lemma 5.1 it is surjective when $X$ is decent (actually one
can say this is part of the definition of being decent). □
Let $S$ be a scheme. Let $X$ be an algebraic space over $S$. If a point $x \in |X|$ can be represented by a monomorphism $\text{Spec}(k) \to X$, then the field $k$ is unique up to unique isomorphism. For a decent algebraic space such a monomorphism exists for every point by Lemma 11.1 and hence the following definition makes sense.

**Definition 11.2.** Let $S$ be a scheme. Let $X$ be a decent algebraic space over $S$. Let $x \in |X|$ be a point. The residue field of $X$ at $x$ is the unique field $\kappa(x)$ which comes equipped with a monomorphism $\text{Spec}(\kappa(x)) \to X$ representing $x$.

Let $S$ be a scheme. Let $f : X \to Y$ be a morphism of decent algebraic spaces over $S$. Let $x \in |X|$ be a point. Set $y = f(x) \in |Y|$. Then the composition $\text{Spec}(\kappa(x)) \to Y$ is in the equivalence class defining $y$ and hence factors through $\text{Spec}(\kappa(y)) \to Y$.

![Diagram](image)

The left vertical morphism corresponds to a homomorphism $\kappa(y) \to \kappa(x)$ of fields. We will often simply call this the homomorphism induced by $f$.

**Lemma 11.3.** Let $S$ be a scheme. Let $f : X \to Y$ be a morphism of decent algebraic spaces over $S$. Let $x \in |X|$ be a point with image $y = f(x) \in |Y|$. The following are equivalent

1. $f$ induces an isomorphism $\kappa(y) \to \kappa(x)$,
2. the induced morphism $\text{Spec}(\kappa(x)) \to Y$ is a monomorphism.

**Proof.** Immediate from the discussion above.

The following lemma tells us that the henselian local ring of a point on a decent algebraic space is defined.

**Lemma 11.4.** Let $S$ be a scheme. Let $X$ be a decent algebraic space over $S$. For every point $x \in |X|$ there exists an étale morphism $(U, u) \to (X, x)$ where $U$ is an affine scheme, $u \in U$ is the only point of $U$ lying over $x$, and the induced homomorphism $\kappa(x) \to \kappa(u)$ is an isomorphism.

**Proof.** We may assume that $X$ is quasi-compact by replacing $X$ with a quasi-compact open containing $x$. Recall that $x$ can be represented by a quasi-compact (mono)morphism from the spectrum a field (by definition of decent spaces). Thus the lemma follows from Lemma 8.3.

**Definition 11.5.** Let $S$ be a scheme. Let $X$ be an algebraic space over $S$. Let $x \in X$ be a point. An elementary étale neighbourhood is an étale morphism $(U, u) \to (X, x)$ where $U$ is a scheme, $u \in U$ is a point mapping to $x$, and $\kappa(x) \to \kappa(u)$ is an isomorphism. A morphism of elementary étale neighbourhoods $(U, u) \to (U', u')$ is defined as a morphism $U \to U'$ over $X$ mapping $u$ to $u'$.

If $X$ is not decent then the category of elementary étale neighbourhoods may be empty.
Lemma 11.6. Let $S$ be a scheme. Let $X$ be a decent algebraic space over $S$. Let $x$ be a point of $X$. The category of elementary étale neighborhoods of $(X, x)$ is cofiltered (see Categories, Definition 20.1).

Proof. The category is nonempty by Lemma 11.4. Suppose that we have two elementary étale neighbourhoods $(U_i, u_i) \to (X, x)$. Then consider $U = U_1 \times_X U_2$. Since $\text{Spec}(\kappa(u_i)) \to X$, $i = 1, 2$ are both monomorphisms in the class of $x$ (Lemma 11.3), we see that $u = \text{Spec}(\kappa(u_1)) \times_X \text{Spec}(\kappa(u_2))$ is the spectrum of a field $\kappa(u)$ such that the induced maps $\kappa(u_i) \to \kappa(u)$ are isomorphisms. Then $u \to U$ is a point of $U$ and we see that $(U, u) \to (X, x)$ is an elementary étale neighbourhood dominating $(U_i, u_i)$. If $a, b : (U_1, u_1) \to (U_2, u_2)$ are two morphisms between our elementary étale neighbourhoods, then we consider the scheme $U = U_1 \times (a, b), (U_2 \times X U_2), \Delta U_2$.

Using Properties of Spaces, Lemma 16.6 we see that $U \to X$ is étale. Moreover, in exactly the same manner as before we see that $U$ has a point $u$ such that $(U, u) \to (X, x)$ is an elementary étale neighbourhood. Finally, $U \to U_1$ equalizes $a$ and $b$ and the proof is finished. □

Definition 11.7. Let $S$ be a scheme. Let $X$ be a decent algebraic space over $S$. Let $x \in |X|$. The henselian local ring of $X$ at $x$, is $O_{X,x}^h = \text{colim} \Gamma(U, O_U)$ where the colimit is over the elementary étale neighbourhoods $(U, u) \to (X, x)$.

Here is the analogue of Properties of Spaces, Lemma 22.1.

Lemma 11.8. Let $S$ be a scheme. Let $X$ be a decent algebraic space over $S$. Let $x \in |X|$. Let $(U, u) \to (X, x)$ be an elementary étale neighbourhood. Then $O_{X,x}^h = O_{U,u}^h$.

In words: the henselian local ring of $X$ at $x$ is equal to the henselization $O_{U,u}^h$ of the local ring $O_{U,u}$ of $U$ at $u$.

Proof. Since the category of elementary étale neighbourhood of $(X, x)$ is cofiltered (Lemma 11.6) we see that the category of elementary étale neighbourhoods of $(U, u)$ is initial in the category of elementary étale neighbourhood of $(X, x)$. Then the equality follows from More on Morphisms, Lemma 32.5 and Categories, Lemma 17.2 (initial is turned into cofinal because the colimit defining henselian local rings is over the opposite of the category of elementary étale neighbourhoods). □

Lemma 11.9. Let $S$ be a scheme. Let $X$ be a decent algebraic space over $S$. Let $x$ be a geometric point of $X$ lying over $x \in |X|$. The étale local ring $O_{X,x}$ of $X$ at $x$ (Properties of Spaces, Definition 22.3) is the strict henselization of the henselian local ring $O_{X,x}^h$ of $X$ at $x$.

Proof. Follows from Lemma 11.8, Properties of Spaces, Lemma 22.1 and the fact that $(R^h)^{sh} = R^{sh}$ for a local ring $(R, m, \kappa)$ and a given separable algebraic closure $\kappa^{sep}$ of $\kappa$. This equality follows from Algebra, Lemma 152.7. □
Let $S$ be a scheme. Let $X$ be a decent algebraic space over $S$. Let $x \in |X|$. The residue field of the henselian local ring of $X$ at $x$ (Definition 11.7) is the residue field of $X$ at $x$ (Definition 11.3).

**Proof.** Choose an elementary étale neighbourhood $(U, u) \to (X, x)$. Then $\kappa(u) = \kappa(x)$ and $O^{h}_{X, x} = O^{h}_{U, u}$ (Lemma 11.8). The residue field of $O^{h}_{U, u}$ is $\kappa(u)$ by Algebra, Lemma 155.1 (the output of this lemma is the construction/definition of the henselization of a local ring, see Algebra, Definition 155.3).

**Remark 11.11.** Let $S$ be a scheme. Let $f : X \to Y$ be a morphism of decent algebraic spaces over $S$. Let $x \in |X|$ with image $y \in |Y|$. Choose an elementary étale neighbourhood $(V, v) \to (Y, y)$ (possible by Lemma 11.4). Then $V \times_{Y} X$ is an algebraic space étale over $X$ which has a unique point $x'$ mapping to $x$ in $X$ and to $v$ in $V$. (Details omitted; use that all points can be represented by monomorphisms from spectra of fields.) Choose an elementary étale neighbourhood $(U, u) \to (V \times_{Y} X, x')$. Then we obtain the following commutative diagram

$$
\begin{array}{c}
\text{Spec}(O_{X, x}) \\
\downarrow \\
\text{Spec}(O_{Y, y})
\end{array}
\begin{array}{c}
\longrightarrow \\
\longrightarrow \\
\longrightarrow \\
\longrightarrow
\end{array}
\begin{array}{c}
\text{Spec}(O^{h}_{X, x}) \\
\downarrow \\
\text{Spec}(O^{h}_{Y, y})
\end{array}
\begin{array}{c}
\longrightarrow \\
\longrightarrow \\
\longrightarrow \\
\longrightarrow
\end{array}
\begin{array}{c}
U \\
\downarrow \\
V
\end{array}
\begin{array}{c}
\longrightarrow \\
\longrightarrow \\
\longrightarrow \\
\longrightarrow
\end{array}
\begin{array}{c}
X \\
\downarrow \\
Y
\end{array}
$$

This comes from the identifications $O_{X, x} = O^{h}_{U, u}$, $O^{h}_{X, x} = O^{h}_{U, u}$, $O_{Y, y} = O^{h}_{V, v}$, $O^{h}_{Y, y} = O^{h}_{V, v}$, see in Lemma 11.8 and Properties of Spaces, Lemma 22.1 and the functoriality of the (strict) henselization discussed in Algebra, Sections 154 and 155.

### 12. Points on decent spaces

In this section we prove some properties of points on decent algebraic spaces. The following lemma shows that specialization of points behaves well on decent algebraic spaces. Spaces, Example 14.9 shows that this is not true in general.

**Lemma 12.1.** Let $S$ be a scheme. Let $X$ be a decent algebraic space over $S$. Let $U \to X$ be an étale morphism from a scheme to $X$. If $u, u' \in |U|$ map to the same point of $|X|$, and $u' \rightsquigarrow u$, then $u = u'$.

**Proof.** Combine Lemmas 5.1 and 7.1.

**Lemma 12.2.** Let $S$ be a scheme. Let $X$ be a decent algebraic space over $S$. Let $x, x' \in |X|$ and assume $x' \rightsquigarrow x$, i.e., $x$ is a specialization of $x'$. Then for every étale morphism $\varphi : U \to X$ from a scheme $U$ and any $u \in U$ with $\varphi(u) = x$, exists a point $u' \in U$, $u' \rightsquigarrow u$ with $\varphi(u') = x'$.

**Proof.** Combine Lemmas 5.1 and 7.2.

**Lemma 12.3.** Let $S$ be a scheme. Let $X$ be a decent algebraic space over $S$. Then $|X|$ is Kolmogorov (see Topology, Definition 8.6).

**Proof.** Let $x_{1}, x_{2} \in |X|$ with $x_{1} \rightsquigarrow x_{2}$ and $x_{2} \rightsquigarrow x_{1}$. We have to show that $x_{1} = x_{2}$. Pick a scheme $U$ and an étale morphism $U \to X$ such that $x_{1}, x_{2}$ are both in the image of $|U| \to |X|$. By Lemma 12.2 we can find a specialization $u_{1} \rightsquigarrow u_{2}$ in $U$ mapping to $x_{1} \rightsquigarrow x_{2}$. By Lemma 12.2 we can find $u'_{2} \rightsquigarrow u_{1}$ mapping to $x_{2} \rightsquigarrow x_{1}$.
This means that $u'_2 \leadsto u_2$ is a specialization between points of $U$ mapping to the same point of $X$, namely $x_2$. This is not possible, unless $u'_2 = u_2$, see Lemma 12.1. Hence also $u_3 = u_2$ as desired. □

**Proposition 12.4.** Let $S$ be a scheme. Let $X$ be a decent algebraic space over $S$. Then the topological space $|X|$ is sober (see Topology, Definition 8.6).

**Proof.** We have seen in Lemma 12.3 that $|X|$ is Kolmogorov. Hence it remains to show that every irreducible closed subset $T \subset |X|$ has a generic point. By Properties of Spaces, Lemma 12.3 there exists a closed subspace $Z \subset X$ with $|Z| = |T|$. By definition this means that $Z \to X$ is a representable morphism of algebraic spaces. Hence $Z$ is a decent algebraic space by Lemma 5.3. By Theorem 10.2 we see that there exists an open dense subspace $Z' \subset Z$ which is a scheme. This means that $|Z'| \subset T$ is open dense. Hence the topological space $|Z'|$ is irreducible, which means that $Z'$ is an irreducible scheme. By Schemes, Lemma 11.1, we conclude that $|Z'|$ is the closure of a single point $\eta \in T$ and hence also $T = \{\eta\}$, and we win. □

For decent algebraic spaces dimension works as expected.

**Lemma 12.5.** Let $S$ be a scheme. Dimension as defined in Properties of Spaces, Section 10 behaves well on decent algebraic spaces $X$ over $S$.

(1) If $x \in |X|$, then $\dim_x(|X|) = \dim_x(X)$, and

(2) $\dim(|X|) = \dim(X)$.

**Proof.** Proof of (1). Choose a scheme $U$ with a point $u \in U$ and an étale morphism $h : U \to X$ mapping $u$ to $x$. By definition the dimension of $X$ at $x$ is $\dim_x(|U|)$. Thus we may pick $U$ such that $\dim_x(X) = \dim(|U|)$. Let $d$ be an integer. If $\dim(U) \geq d$, then there exists a sequence of nontrivial specializations $u_d \leadsto \ldots \leadsto u_0$ in $U$. Taking the image we find a corresponding sequence $h(u_d) \leadsto \ldots \leadsto h(u_0)$ each of which is nontrivial by Lemma 12.1. Hence we see that the image of $|U|$ in $|X|$ has dimension at least $d$. Conversely, suppose that $x_d \leadsto \ldots \leadsto x_0$ is a sequence of specializations in $|X|$ with $x_0$ in the image of $|U| \to |X|$. Then we can lift this to a sequence of specializations in $U$ by Lemma 12.2. Part (2) is an immediate consequence of part (1), Topology, Lemma 10.2, and Properties of Spaces, Section 9. □

**Lemma 12.6.** Let $S$ be a scheme. Let $X \to Y$ be a locally quasi-finite morphism of algebraic spaces over $S$. Let $x \in |X|$ with image $y \in |Y|$. Then the dimension of the local ring of $Y$ at $y$ is $\geq$ to the dimension of the local ring of $X$ at $x$.

**Proof.** The definition of the dimension of the local ring of a point on an algebraic space is given in Properties of Spaces, Definition 10.2. Choose an étale morphism $(V,v) \to (Y,y)$ where $V$ is a scheme. Choose an étale morphism $U \to V \times_Y X$ and a point $u \in U$ mapping to $x \in |X|$ and $v \in V$. Then $U \to V$ is locally quasi-finite and we have to prove that

$$\dim(O_{V,v}) \geq \dim(O_{U,u})$$

This is Algebra, Lemma 125.4. □

**Lemma 12.7.** Let $S$ be a scheme. Let $X \to Y$ be a locally quasi-finite morphism of algebraic spaces over $S$. Then $\dim(X) \leq \dim(Y)$.

**Proof.** This follows from Lemma 12.6 and Properties of Spaces, Lemma 10.3. □
The following lemma is a tiny bit stronger than Properties of Spaces, Lemma 15.3. We will improve this lemma in Lemma 14.2.

**Lemma 12.8.** Let $S$ be a scheme. Let $k$ be a field. Let $X$ be an algebraic space over $S$ and assume that there exists a surjective étale morphism $\text{Spec}(k) \to X$. If $X$ is decent, then $X \cong \text{Spec}(k')$ where $k' \subset k$ is a finite separable extension.

**Proof.** The assumption implies that $|X| = \{x\}$ is a singleton. Since $X$ is decent we can find a quasi-compact monomorphism $\text{Spec}(k') \to X$ whose image is $x$. Then the projection $U = \text{Spec}(k') \times_X \text{Spec}(k) \to \text{Spec}(k)$ is a monomorphism, whence $U = \text{Spec}(k)$, see Schemes, Lemma 23.11. Hence the projection $\text{Spec}(k) = U \to \text{Spec}(k')$ is étale and we win. □

### 13. Reduced singleton spaces

A *singleton space* is an algebraic space $X$ such that $|X|$ is a singleton. It turns out that these can be more interesting than just being the spectrum of a field, see Spaces, Example 14.7. We develop a tiny bit of machinery to be able to talk about these.

**Lemma 13.1.** Let $S$ be a scheme. Let $Z$ be an algebraic space over $S$. Let $k$ be a field and let $\text{Spec}(k) \to Z$ be surjective and flat. Then any morphism $\text{Spec}(k') \to Z$ where $k'$ is a field is surjective and flat.

**Proof.** Consider the fibre square

$$
\begin{array}{ccc}
T & \longrightarrow & \text{Spec}(k) \\
\downarrow & & \downarrow \\
\text{Spec}(k') & \longrightarrow & Z
\end{array}
$$

Note that $T \to \text{Spec}(k')$ is flat and surjective hence $T$ is not empty. On the other hand $T \to \text{Spec}(k)$ is flat as $k$ is a field. Hence $T \to Z$ is flat and surjective. It follows from Morphisms of Spaces, Lemma 31.5 that $\text{Spec}(k') \to Z$ is flat. It is surjective as by assumption $|Z|$ is a singleton. □

**Lemma 13.2.** Let $S$ be a scheme. Let $Z$ be an algebraic space over $S$. The following are equivalent

1. $Z$ is reduced and $|Z|$ is a singleton,
2. there exists a surjective flat morphism $\text{Spec}(k) \to Z$ where $k$ is a field, and
3. there exists a locally of finite type, surjective, flat morphism $\text{Spec}(k) \to Z$ where $k$ is a field.

**Proof.** Assume (1). Let $W$ be a scheme and let $W \to Z$ be a surjective étale morphism. Then $W$ is a reduced scheme. Let $\eta \in W$ be a generic point of an irreducible component of $W$. Since $W$ is reduced we have $\mathcal{O}_{W, \eta} = \kappa(\eta)$. It follows that the canonical morphism $\eta = \text{Spec}(\kappa(\eta)) \to W$ is flat. We see that the composition $\eta \to Z$ is flat (see Morphisms of Spaces, Lemma 30.3). It is also surjective as $|Z|$ is a singleton. In other words (2) holds.

Assume (2). Let $W$ be a scheme and let $W \to Z$ be a surjective étale morphism. Choose a field $k$ and a surjective flat morphism $\text{Spec}(k) \to Z$. Then $W \times_Z \text{Spec}(k)$ is a scheme étale over $k$. Hence $W \times_Z \text{Spec}(k)$ is a disjoint union of spectra of fields (see Remark 4.1), in particular reduced. Since $W \times_Z \text{Spec}(k) \to W$ is surjective
and flat we conclude that $W$ is reduced (Descent, Lemma 16.1). In other words (1) holds.

It is clear that (3) implies (2). Finally, assume (2). Pick a nonempty affine scheme $W$ and an étale morphism $W \to Z$. Pick a closed point $w \in W$ and set $k = \kappa(w)$. The composition

$$\text{Spec}(k) \xrightarrow{w} W \to Z$$

is locally of finite type by Morphisms of Spaces, Lemmas 23.2 and 39.9. It is also flat and surjective by Lemma 13.1. Hence (3) holds.

□

The following lemma singles out a slightly better class of singleton algebraic spaces than the preceding lemma.

**Lemma 13.3.** Let $S$ be a scheme. Let $Z$ be an algebraic space over $S$. The following are equivalent

1. $Z$ is reduced, locally Noetherian, and $|Z|$ is a singleton, and
2. there exists a locally finitely presented, surjective, flat morphism $\text{Spec}(k) \to Z$ where $k$ is a field.

**Proof.** Assume (2) holds. By Lemma 13.2 we see that $Z$ is reduced and $|Z|$ is a singleton. Let $W$ be a scheme and let $W \to Z$ be a surjective étale morphism. Choose a field $k$ and a locally finitely presented, surjective, flat morphism $\text{Spec}(k) \to Z$. Then $W \times_Z \text{Spec}(k)$ is a scheme étale over $k$, hence a disjoint union of spectra of fields (see Remark 4.1), hence locally Noetherian. Since $W \times_Z \text{Spec}(k) \to W$ is flat, surjective, and locally of finite presentation, we see that $\{W \times_Z \text{Spec}(k) \to W\}$ is an fppf covering and we conclude that $W$ is locally Noetherian (Descent, Lemma 13.1). In other words (1) holds.

Assume (1). Pick a nonempty affine scheme $W$ and an étale morphism $W \to Z$. Pick a closed point $w \in W$ and set $k = \kappa(w)$. Because $W$ is locally Noetherian the morphism $w : \text{Spec}(k) \to W$ is of finite presentation, see Morphisms, Lemma 21.7. Hence the composition

$$\text{Spec}(k) \xrightarrow{w} W \to Z$$

is locally of finite presentation by Morphisms of Spaces, Lemmas 28.2 and 39.8. It is also flat and surjective by Lemma 13.1. Hence (2) holds.

□

**Lemma 13.4.** Let $S$ be a scheme. Let $Z' \to Z$ be a monomorphism of algebraic spaces over $S$. Assume there exists a field $k$ and a locally finitely presented, surjective, flat morphism $\text{Spec}(k) \to Z$. Then either $Z'$ is empty or $Z' = Z$.

**Proof.** We may assume that $Z'$ is nonempty. In this case the fibre product $T = Z' \times_Z \text{Spec}(k)$ is nonempty, see Properties of Spaces, Lemma 4.3. Now $T$ is an algebraic space and the projection $T \to \text{Spec}(k)$ is a monomorphism. Hence $T = \text{Spec}(k)$, see Morphisms of Spaces, Lemma 10.8. We conclude that $\text{Spec}(k) \to Z$ factors through $Z'$. But as $\text{Spec}(k) \to Z$ is surjective, flat and locally of finite presentation, we see that $\text{Spec}(k) \to Z$ is surjective as a map of sheaves on $(\text{Sch}/S)_{fppf}$ (see Spaces, Remark 5.2) and we conclude that $Z' = Z$.

□

The following lemma says that to each point of an algebraic space we can associate a canonical reduced, locally Noetherian singleton algebraic space.
Lemma 13.5. Let $S$ be a scheme. Let $X$ be an algebraic space over $S$. Let $x \in |X|$. Then there exists a unique monomorphism $Z \to X$ of algebraic spaces over $S$ such that $Z$ is an algebraic space which satisfies the equivalent conditions of Lemma 13.3 and such that the image of $|Z| \to |X|$ is $\{x\}$.

Proof. Choose a scheme $U$ and a surjective étale morphism $U \to X$. Set $R = U \times_X U$ so that $X = U/R$ is a presentation (see Spaces, Section 9). Set $U' = \coprod_{u \in U \text{ lying over } x} \text{Spec}(k(u))$.

The canonical morphism $U' \to U$ is a monomorphism. Let $R' = U' \times_X U' = R \times_{(U \times_S U)} (U' \times_S U')$. Because $U' \to U$ is a monomorphism we see that the projections $s', t' : R' \to U'$ factor as a monomorphism followed by an étale morphism. Hence, as $U'$ is a disjoint union of spectra of fields, using Remark 4.1 and using Schemes, Lemma 23.11 we conclude that $R'$ is a disjoint union of spectra of fields and that the morphisms $s', t' : R' \to U'$ are étale. Hence $Z = U'/R'$ is an algebraic space by Spaces, Theorem 10.5. As $R'$ is the restriction of $R$ by $U' \to U$ we see $Z \to X$ is a monomorphism by Groupoids, Lemma 20.6. Since $Z \to X$ is a monomorphism we see that $|Z| \to |X|$ is injective, see Morphisms of Spaces, Lemma 10.9. By Properties of Spaces, Lemma 4.3 we see that $|U'| = |Z \times_X U'| \to |Z| \times_{|X|} |U'|$ is surjective which implies (by our choice of $U'$) that $|Z| \to |X|$ has image $\{x\}$. We conclude that $|Z|$ is a singleton. Finally, by construction $U'$ is locally Noetherian and reduced, i.e., we see that $Z$ satisfies the equivalent conditions of Lemma 13.3.

Let us prove uniqueness of $Z \to X$. Suppose that $Z' \to X$ is a second such monomorphism of algebraic spaces. Then the projections $Z' \leftarrow Z' \times_X Z \to Z$ are monomorphisms. The algebraic space in the middle is nonempty by Properties of Spaces, Lemma 4.3. Hence the two projections are isomorphisms by Lemma 13.3 and we win.

We introduce the following terminology which foreshadows the residual gerbes we will introduce later, see Properties of Stacks, Definition 11.8.

Definition 13.6. Let $S$ be a scheme. Let $X$ be an algebraic space over $S$. Let $x \in |X|$. The residual space of $X$ at $x$ is the monomorphism $Z_x \to X$ constructed in Lemma 13.5.

In particular we know that $Z_x$ is a locally Noetherian, reduced, singleton algebraic space and that there exists a field and a surjective, flat, locally finitely presented morphism $\text{Spec}(k) \to Z_x$.

It turns out that $Z_x$ is a regular algebraic space as follows from the following lemma.

Lemma 13.7. A reduced, locally Noetherian singleton algebraic space $Z$ is regular.

\footnote{This is nonstandard notation.}
**Proof.** Let $Z$ be a reduced, locally Noetherian singleton algebraic space over a scheme $S$. Let $W \to Z$ be a surjective étale morphism where $W$ is a scheme. Let $k$ be a field and let $\text{Spec}(k) \to Z$ be surjective, flat, and locally of finite presentation (see Lemma \ref{lem-flat}). The scheme $T = W \times_Z \text{Spec}(k)$ is étale over $k$ in particular regular, see Remark \ref{rem-regular}. Since $T \to W$ is locally of finite presentation, flat, and surjective it follows that $W$ is regular, see Descent, Lemma \ref{lem-descent}. By definition this means that $Z$ is regular. \hfill $\square$

14. Decent spaces

047Y In this section we collect some useful facts on decent spaces.

0BB6 **Lemma 14.1.** Any locally Noetherian decent algebraic space is quasi-separated.

**Proof.** Namely, let $X$ be an algebraic space (over some base scheme, for example over $\mathbb{Z}$) which is decent and locally Noetherian. Let $U \to X$ and $V \to X$ be étale morphisms with $U$ and $V$ affine schemes. We have to show that $W = U \times_X V$ is quasi-compact (Properties of Spaces, Lemma \ref{lem-quasi-compact}). Since $X$ is locally Noetherian, the schemes $U$, $V$ are Noetherian and $W$ is locally Noetherian. Since $X$ is decent, the fibres of the morphism $W \to U$ are finite. Namely, we can represent any $x \in |X|$ by a quasi-compact monomorphism $\text{Spec}(k) \to X$. Then $U_k$ and $V_k$ are finite disjoint unions of spectra of finite separable extensions of $k$ (Remark \ref{rem-finite}) and we see that $W_k = U_k \times_{\text{Spec}(k)} V_k$ is finite. Let $n$ be the maximum degree of a fibre of $W \to U$ at a generic point of an irreducible component of $U$. Consider the stratification

$$U = U_0 \supset U_1 \supset U_2 \supset \ldots$$

associated to $W \to U$ in More on Morphisms, Lemma \ref{lem-stratification}. By our choice of $n$ above we conclude that $U_{n+1}$ is empty. Hence we see that the fibres of $W \to U$ are universally bounded. Then we can apply More on Morphisms, Lemma \ref{lem-stratification} to find a stratification

$$\emptyset = Z_{-1} \subset Z_0 \subset Z_1 \subset Z_2 \subset \ldots \subset Z_n = U$$

by closed subsets such that with $S_r = Z_r \setminus Z_{r-1}$ the morphism $W \times_U S_r \to S_r$ is finite locally free. Since $U$ is Noetherian, the schemes $S_r$ are Noetherian, whence the schemes $W \times_U S_r$ are Noetherian, whence $W = \bigsqcup W \times_U S_r$ is quasi-compact as desired. \hfill $\square$

047Z **Lemma 14.2.** Let $S$ be a scheme. Let $X$ be a decent algebraic space over $S$.

1. If $|X|$ is a singleton then $X$ is a scheme.
2. If $|X|$ is a singleton and $X$ is reduced, then $X \cong \text{Spec}(k)$ for some field $k$.

**Proof.** Assume $|X|$ is a singleton. It follows immediately from Theorem \ref{thm-scheme} that $X$ is a scheme, but we can also argue directly as follows. Choose an affine scheme $U$ and a surjective étale morphism $U \to X$. Set $R = U \times_X U$. Then $U$ and $R$ have finitely many points by Lemma \ref{lem-finitely-many-points} (and the definition of a decent space). All of these points are closed in $U$ and $R$ by Lemma \ref{lem-finitely-many-points}. It follows that $U$ and $R$ are affine schemes. We may shrink $U$ to a singleton space. Then $U$ is the spectrum of a henselian local ring, see Algebra, Lemma \ref{lem-henselian}. The projections $R \to U$ are étale, hence finite étale because $U$ is the spectrum of a $0$-dimensional henselian local ring, see Algebra, Lemma \ref{lem-finite-etale}. It follows that $X$ is a scheme by Groupoids, Proposition \ref{prop-groupoids}.
Part (2) follows from (1) and the fact that a reduced singleton scheme is the spectrum of a field.

\[ \square \]

Remark 14.3. We will see in Limits of Spaces, Lemma \ref{lim15.3} that an algebraic space whose reduction is a scheme is a scheme.

Lemma 14.4. Let \( S \) be a scheme. Let \( X \) be a decent algebraic space over \( S \). Consider a commutative diagram

\[
\begin{array}{ccc}
\text{Spec}(k) & \rightarrow & X \\
\downarrow & & \downarrow \\
S & \rightarrow & X
\end{array}
\]

Assume that the image point \( s \in S \) of \( \text{Spec}(k) \rightarrow S \) is a closed point and that \( \kappa(s) \subset k \) is algebraic. Then the image \( x \) of \( \text{Spec}(k) \rightarrow X \) is a closed point of \( |X| \).

Proof. Suppose that \( x \leadsto x' \) for some \( x' \in |X| \). Choose an étale morphism \( U \rightarrow X \) where \( U \) is a scheme and a point \( u' \in U \) mapping to \( x' \). Choose a specialization \( u \leadsto u' \) in \( U \) with \( u \) mapping to \( x \) in \( X \), see Lemma \ref{12.2}. Then \( u \) is the image of a point \( w \) of the scheme \( W = \text{Spec}(k) \times_X U \). Since the projection \( W \rightarrow \text{Spec}(k) \) is étale we see that \( \kappa(w) \supset k \) is finite. Hence \( \kappa(w) \supset \kappa(s) \) is algebraic. Hence \( \kappa(u) \supset \kappa(s) \) is algebraic. Thus \( u \) is a closed point of \( U \) by Morphisms, Lemma \ref{20.2}. Thus \( u = u' \), whence \( x = x' \).

\[ \square \]

Lemma 14.5. Let \( S \) be a scheme. Let \( X \) be a decent algebraic space over \( S \). Consider a commutative diagram

\[
\begin{array}{ccc}
\text{Spec}(k) & \rightarrow & X \\
\downarrow & & \downarrow \\
S & \rightarrow & X
\end{array}
\]

Assume that the image point \( s \in S \) of \( \text{Spec}(k) \rightarrow S \) is a closed point and that \( \kappa(s) \subset k \) is finite. Then \( \text{Spec}(k) \rightarrow X \) is finite morphism. If \( \kappa(s) = k \) then \( \text{Spec}(k) \rightarrow X \) is closed immersion.

Proof. By Lemma \ref{14.4} the image point \( x \in |X| \) is closed. Let \( Z \subset X \) be the reduced closed subspace with \( |Z| = \{ x \} \) (Properties of Spaces, Lemma \ref{12.3}). Note that \( Z \) is a decent algebraic space by Lemma \ref{6.5}. By Lemma \ref{14.2} we see that \( Z = \text{Spec}(k') \) for some field \( k' \). Of course \( k \supset k' \supset \kappa(s) \). Then \( \text{Spec}(k) \rightarrow Z \) is a finite morphism of schemes and \( Z \rightarrow X \) is a finite morphism as it is a closed immersion. Hence \( \text{Spec}(k) \rightarrow X \) is finite (Morphisms of Spaces, Lemma \ref{45.4}). If \( k = \kappa(s) \), then \( \text{Spec}(k) = Z \) and \( \text{Spec}(k) \rightarrow X \) is a closed immersion.

\[ \square \]

Lemma 14.6. Let \( S \) be a scheme. Suppose \( X \) is a decent algebraic space over \( S \). Let \( x \in |X| \) be a closed point. Then \( x \) can be represented by a closed immersion \( i : \text{Spec}(k) \rightarrow X \) from the spectrum of a field.

Proof. We know that \( x \) can be represented by a quasi-compact monomorphism \( i : \text{Spec}(k) \rightarrow X \) where \( k \) is a field (Definition \ref{6.1}). Let \( U \rightarrow X \) be an étale morphism where \( U \) is an affine scheme. As \( x \) is closed and \( X \) decent, the fibre \( F \) of \( |U| \rightarrow |X| \) over \( x \) consists of closed points (Lemma \ref{12.1}). As \( i \) is a monomorphism, so is \( U_k = U \times_X \text{Spec}(k) \rightarrow U \). In particular, the map \( |U_k| \rightarrow F \) is injective.
Since $U_k$ is quasi-compact and étale over a field, we see that $U_k$ is a finite disjoint union of spectra of fields (Remark 4.1). Say $U_k = \text{Spec}(k_1) \amalg \ldots \amalg \text{Spec}(k_r)$. Since $\text{Spec}(k_i) \to U$ is a monomorphism, we see that its image $u_i$ has residue field $\kappa(u_i) = k_i$. Since $u_i \in F$ is a closed point we conclude the morphism $\text{Spec}(k_i) \to U$ is a closed immersion. As the $u_i$ are pairwise distinct, $U_k \to U$ is a closed immersion. Hence $i$ is a closed immersion (Morphisms of Spaces, Lemma 4.3). This finishes the proof.

15. Locally separated spaces

**Lemma 15.1.** Let $A$ be a ring. Let $k$ be a field. Let $p_n$, $n \geq 1$ be a sequence of pairwise distinct primes of $A$. Moreover, for each $n$ let $k \to \kappa(p_n)$ be an embedding. Then the closure of the image of

\[
\prod_{n \neq m} \text{Spec}(\kappa(p_n) \otimes_k \kappa(p_m)) \to \text{Spec}(A \otimes A)
\]

meets the diagonal.

**Proof.** Set $k_n = \kappa(p_n)$. We may assume that $A = \prod k_n$. Denote $x_n = \text{Spec}(k_n)$ the open and closed point corresponding to $A \to k_n$. Then $\text{Spec}(A) = Z \amalg \{x_n\}$ where $Z$ is a nonempty closed subset. Namely, $Z = V(e_n; n \geq 1)$ where $e_n$ is the idempotent of $A$ corresponding to the factor $k_n$ and $Z$ is nonempty as the ideal generated by the $e_n$ is not equal to $A$. We will show that the closure of the image contains $\Delta(Z)$. The kernel of the map

\[
(\prod k_n) \otimes_k (\prod k_m) \to \prod_{n \neq m} k_n \otimes_k k_m
\]

is the ideal generated by $e_n \otimes e_n$, $n \geq 1$. Hence the closure of the image of the map on spectra is $V(e_n \otimes e_n; n \geq 1)$ whose intersection with $\Delta(\text{Spec}(A))$ is $\Delta(Z)$. Thus it suffices to show that

\[
\prod_{n \neq m} \text{Spec}(k_n \otimes_k k_m) \to \text{Spec}(\prod_{n \neq m} k_n \otimes_k k_m)
\]

has dense image. This follows as the family of ring maps $\prod_{n \neq m} k_n \otimes_k k_m \to k_n \otimes_k k_m$ is jointly injective. \qed

**Lemma 15.2** (David Rydh). A locally separated algebraic space is decent.

**Proof.** Let $S$ be a scheme and let $X$ be a locally separated algebraic space over $S$. We may assume $S = \text{Spec}(\mathbf{Z})$, see Properties of Spaces, Definition 3.1. Unadorned fibre products will be over $\mathbf{Z}$. Let $x \in |X|$. Choose a scheme $U$, an étale morphism $U \to X$, and a point $u \in U$ mapping to $x$ in $|X|$. As usual we identify $u = \text{Spec}(\kappa(u))$. As $X$ is locally separated the morphism

\[
u \times_X u \to u \times u
\]

is an immersion (Morphisms of Spaces, Lemma 4.3). Hence More on Groupoids, Lemma 11.5 tells us that it is a closed immersion (use Schemes, Lemma 10.4). As $\nu \times_X u \to u \times X \to U$ is a monomorphism (base change of $u \to U$) and as $u \times_X U \to u$ is étale we conclude that $\nu \times_X u$ is a disjoint union of spectra of fields (see Remark 4.1 and Schemes, Lemma 23.11). Since it is also closed in the affine scheme $u \times u$ we conclude $\nu \times_X u$ is a finite disjoint union of spectra of fields. Thus $\nu$ can be represented by a monomorphism $\text{Spec}(k) \to X$ where $k$ is a field, see Lemma 4.3.
Next, let $U = \text{Spec}(A)$ be an affine scheme and let $U \to X$ be an étale morphism. To finish the proof it suffices to show that $F = U \times_X \text{Spec}(k)$ is finite. Write $F = \coprod_{i \in I} \text{Spec}(k_i)$ as the disjoint union of finite separable extensions of $k$. We have to show that $I$ is finite. Set $R = U \times_X U$. As $X$ is locally separated, the morphism $j : R \to U \times U$ is an immersion. Let $U' \subset U \times U$ be an open such that $j$ factors through a closed immersion $j' : R \to U'$. Let $e : U \to R$ be the diagonal map. Using that $e$ is a morphism between schemes étale over $U$ such that $\Delta = j \circ e$ is a closed immersion, we conclude that $R = e(U) \amalg W$ for some open and closed subscheme $W \subset R$. Since $j'$ is a closed immersion we conclude that $j'(W) \subset U'$ is closed and disjoint from $j'(e(U))$. Therefore $j(W) \cap \Delta(U) = \emptyset$ in $U \times U$. Note that $W$ contains $\text{Spec}(k_i \otimes_k k_{i'})$ for all $i \neq i'$, $i, i' \in I$. By Lemma 15.1 we conclude that $I$ is finite as desired. 

16. Valuative criterion

06NP For a quasi-compact morphism from a decent space the valuative criterion is necessary in order for the morphism to be universally closed.

03KJ **Proposition 16.1.** Let $S$ be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over $S$. Assume $f$ is quasi-compact, and $X$ is decent. Then $f$ is universally closed if and only if the existence part of the valuative criterion holds.

**Proof.** In Morphisms of Spaces, Lemma 42.1 we have seen one of the implications. To prove the other, assume that $f$ is universally closed. Let

$$
\begin{array}{ccc}
\text{Spec}(K) & \longrightarrow & X \\
\downarrow & & \downarrow \\
\text{Spec}(A) & \longrightarrow & Y
\end{array}
$$

be a diagram as in Morphisms of Spaces, Definition 41.1. Let $X_A = \text{Spec}(A) \times_Y X$, so that we have

$$
\begin{array}{ccc}
\text{Spec}(K) & \longrightarrow & X_A \\
\downarrow & & \downarrow \\
\text{Spec}(A) & & \text{Spec}(A)
\end{array}
$$

By Morphisms of Spaces, Lemma 8.4 we see that $X_A \to \text{Spec}(A)$ is quasi-compact. Since $X_A \to X$ is representable, we see that $X_A$ is decent also, see Lemma 5.3.

Moreover, as $f$ is universally closed, we see that $X_A \to \text{Spec}(A)$ is universally closed. Hence we may and do replace $X$ by $X_A$ and $Y$ by $\text{Spec}(A)$.

Let $x' \in |X|$ be the equivalence class of $\text{Spec}(K) \to X$. Let $y \in |Y| = |\text{Spec}(A)|$ be the closed point. Set $y' = f(x')$; it is the generic point of $\text{Spec}(A)$. Since $f$ is universally closed we see that $f(\{x'\})$ contains $\{y'\}$, and hence contains $y$. Let $x \in \{x'\}$ be a point such that $f(x) = y$. Let $U$ be a scheme, and $\varphi : U \to X$ an étale morphism such that there exists a $u \in U$ with $\varphi(u) = x$. By Lemma 7.2 and our assumption that $X$ is decent there exists a specialization $u' \leadsto u$ on $U$ with $\varphi(u') = x'$. This means that there exists a common field extension $K \subset K' \supset \kappa(u')$.
such that

\[
\begin{array}{ccc}
\text{Spec}(K') & \rightarrow & U \\
\downarrow & & \downarrow \\
\text{Spec}(K) & \rightarrow & X \\
\downarrow & & \downarrow \\
& \rightarrow & \text{Spec}(A)
\end{array}
\]

is commutative. This gives the following commutative diagram of rings

\[
\begin{array}{ccc}
K' & \leftarrow & \mathcal{O}_{U,u} \\
\uparrow & & \uparrow \\
K & \leftarrow & A
\end{array}
\]

By Algebra, Lemma 50.2 we can find a valuation ring \( A' \subset K' \) dominating the image of \( \mathcal{O}_{U,u} \) in \( K' \). Since by construction \( \mathcal{O}_{U,u} \) dominates \( A \) we see that \( A' \) dominates \( A \) also. Hence we obtain a diagram resembling the second diagram of Morphisms of Spaces, Definition 41.1 and the proposition is proved. \( \square \)

17. Relative conditions

This is a (yet another) technical section dealing with conditions on algebraic spaces having to do with points. It is probably a good idea to skip this section.

Definition 17.1. Let \( S \) be a scheme. We say an algebraic space \( X \) over \( S \) has property \((\beta)\) if \( X \) has the corresponding property of Lemma 5.1. Let \( f : X \rightarrow Y \) be a morphism of algebraic spaces over \( S \).

1. We say \( f \) has property \((\beta)\) if for any scheme \( T \) and morphism \( T \rightarrow Y \) the fibre product \( T \times_Y X \) has property \((\beta)\).
2. We say \( f \) is decent if for any scheme \( T \) and morphism \( T \rightarrow Y \) the fibre product \( T \times_Y X \) is a decent algebraic space.
3. We say \( f \) is reasonable if for any scheme \( T \) and morphism \( T \rightarrow Y \) the fibre product \( T \times_Y X \) is a reasonable algebraic space.
4. We say \( f \) is very reasonable if for any scheme \( T \) and morphism \( T \rightarrow Y \) the fibre product \( T \times_Y X \) is a very reasonable algebraic space.

We refer to Remark 17.10 for an informal discussion. It will turn out that the class of very reasonable morphisms is not so useful, but that the classes of decent and reasonable morphisms are useful.
Lemma 17.2. Let $S$ be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over $S$. We have the following implications among the conditions on $f$:

\[
\text{representable} \Rightarrow \text{very reasonable} \Rightarrow \text{reasonable} \Rightarrow \text{decent} \Rightarrow (\beta) \Rightarrow \text{quasi-separated}
\]

Proof. This is clear from the definitions, Lemma 5.1 and Morphisms of Spaces, Lemma 4.12. \hfill \Box

Here is another sanity check.

Lemma 17.3. Let $S$ be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over $S$. If $X$ is decent (resp. is reasonable, resp. has property $(\beta)$ of Lemma 5.1), then $f$ is decent (resp. reasonable, resp. has property $(\beta)$).

Proof. Let $T$ be a scheme and let $T \to Y$ be a morphism. Then $T \to Y$ is representable, hence the base change $T \times_Y X \to X$ is representable. Hence if $X$ is decent (or reasonable), then so is $T \times_Y X$, see Lemma 6.5. Similarly, for property $(\beta)$, see Lemma 5.3. \hfill \Box

Lemma 17.4. Having property $(\beta)$, being decent, or being reasonable is preserved under arbitrary base change.

Proof. This is immediate from the definition. \hfill \Box

Lemma 17.5. Let $S$ be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over $S$. Let $\omega \in \{\beta, \text{decent}, \text{reasonable}\}$. Suppose that $Y$ has property $(\omega)$ and $f : X \to Y$ has $(\omega)$. Then $X$ has $(\omega)$.

Proof. Let us prove the lemma in case $\omega = \beta$. In this case we have to show that any $x \in |X|$ is represented by a monomorphism from the spectrum of a field into $X$. Let $y = f(x) \in |Y|$. By assumption there exists a field $k$ and a monomorphism $\text{Spec}(k) \to Y$ representing $y$. Then $x$ corresponds to a point $x'$ of $\text{Spec}(k) \times_Y X$. By assumption $x'$ is represented by a monomorphism $\text{Spec}(k') \to \text{Spec}(k) \times_Y X$. Clearly the composition $\text{Spec}(k') \to X$ is a monomorphism representing $x$.

Let us prove the lemma in case $\omega = \text{decent}$. Let $x \in |X|$ and $y = f(x) \in |Y|$. By the result of the preceding paragraph we can choose a diagram

\[
\begin{array}{ccc}
\text{Spec}(k') & \xrightarrow{x} & X \\
\downarrow & & \downarrow f \\
\text{Spec}(k) & \xrightarrow{y} & Y
\end{array}
\]

whose horizontal arrows monomorphisms. As $Y$ is decent the morphism $y$ is quasi-compact. As $f$ is decent the algebraic space $\text{Spec}(k) \times_Y X$ is decent. Hence the
monomorphism $\text{Spec}(k') \to \text{Spec}(k) \times_Y X$ is quasi-compact. Then the monomorphism $x : \text{Spec}(k') \to X$ is quasi-compact as a composition of quasi-compact morphisms (use Morphisms of Spaces, Lemmas 8.4 and 8.5). As the point $x$ was arbitrary this implies $X$ is decent.

Let us prove the lemma in case $\omega = \text{reasonable}$. Choose $V \to Y$ étale with $V$ an affine scheme. Choose $U \to V \times_Y X$ étale with $U$ an affine scheme. By assumption $V \to Y$ has universally bounded fibres. By Lemma 5.3 the morphism $V \times_Y X \to X$ has universally bounded fibres. By assumption on $f$ we see that $U \to V \times_Y X$ has universally bounded fibres. By Lemma 3.2 the composition $U \to X$ has universally bounded fibres. Hence there exists sufficiently many étale morphisms $U \to X$ from schemes with universally bounded fibres, and we conclude that $X$ is reasonable. □

03L1 Lemma 17.6. Having property $(\beta)$, being decent, or being reasonable is preserved under compositions.

Proof. Let $\omega \in \{\beta, \text{decent}, \text{reasonable}\}$. Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of algebraic spaces over the scheme $S$. Assume $f$ and $g$ both have property $((\omega))$. Then we have to show that for any scheme $T$ and morphism $T \to Z$ the space $T \times_Z X$ has $((\omega))$. By Lemma 17.4 this reduces us to the following claim: Suppose that $Y$ is an algebraic space having property $((\omega))$, and that $f : X \to Y$ is a morphism with $((\omega))$. Then $X$ has $((\omega))$. This is the content of Lemma 17.5. □

0ABZ Lemma 17.7. Let $S$ be a scheme. Let $f : X \to Y$, $g : Z \to Y$ be morphisms of algebraic spaces over $S$. If $X$ and $Z$ are decent (resp. reasonable, resp. have property $(\beta)$ of Lemma 5.7), then so does $X \times_Y Z$.

Proof. Namely, by Lemma 17.3 the morphism $X \to Y$ has the property. Then the base change $X \times_Y Z \to Z$ has the property by Lemma 17.4. And finally this implies $X \times_Y Z$ has the property by Lemma 17.5. □

03L2 Lemma 17.8. Let $S$ be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over $S$. Let $\mathcal{P} \in \{((\beta)), \text{decent}, \text{reasonable}\}$. Assume

1. $f$ is quasi-compact,
2. $f$ is étale,
3. $|f| : |X| \to |Y|$ is surjective, and
4. the algebraic space $X$ has property $\mathcal{P}$.

Then $Y$ has property $\mathcal{P}$.

Proof. Let us prove this in case $\mathcal{P} = (\beta)$. Let $y \in |Y|$ be a point. We have to show that $y$ can be represented by a monomorphism from a field. Choose a point $x \in |X|$ with $f(x) = y$. By assumption we may represent $x$ by a monomorphism $\text{Spec}(k) \to X$, with $k$ a field. By Lemma 1.3 it suffices to show that the projections $\text{Spec}(k) \times_Y \text{Spec}(k) \to \text{Spec}(k)$ are étale and quasi-compact. We can factor the first projection as

$$
\text{Spec}(k) \times_Y \text{Spec}(k) \to \text{Spec}(k) \times_Y X \to \text{Spec}(k)
$$

The first morphism is a monomorphism, and the second is étale and quasi-compact. By Properties of Spaces, Lemma 16.8 we see that $\text{Spec}(k) \times_Y X$ is a scheme. Hence it is a finite disjoint union of spectra of finite separable field extensions of $k$. By Schemes, Lemma 23.11 we see that the first arrow identifies $\text{Spec}(k) \times_Y \text{Spec}(k)$
Let us prove this in case \( \mathcal{P} = \text{decent} \). We have already seen in the first paragraph of the proof that this implies that every \( y \in |Y| \) can be represented by a monomorphism \( y : \text{Spec}(k) \to Y \). Pick such a \( y \). Pick an affine scheme \( U \) and an étale morphism \( U \to X \) such that the image of \( |U| \to |Y| \) contains \( y \). By Lemma 4.5 it suffices to show that \( U_y \) is a finite scheme over \( k \). The fibre product \( X_y = \text{Spec}(k) \times_Y X \) is a quasi-compact étale algebraic space over \( k \). Hence by Properties of Spaces, Lemma 16.8 it is a scheme. So it is a finite disjoint union of spectra of finite separable extensions of \( k \). Say \( X_y = \{ x_1, \ldots, x_n \} \) so \( x_i \) is given by \( x_i : \text{Spec}(k_i) \to X \) with \( [k_i : k] < \infty \). By assumption \( X \) is decent, so the schemes \( U_{x_i} = \text{Spec}(k_i) \times_X U \) are finite over \( k_i \). Finally, we note that \( U_y = \coprod U_{x_i} \) as a scheme and we conclude that \( U_y \) is finite over \( k \) as desired.

Let us prove this in case \( \mathcal{P} = \text{reasonable} \). Pick an affine scheme \( V \) and an étale morphism \( V \to Y \). We have the show the fibres of \( V \to Y \) are universally bounded. The algebraic space \( V \times_Y X \) is quasi-compact. Thus we can find an affine scheme \( W \) and a surjective étale morphism \( W \to V \times_Y X \), see Properties of Spaces, Lemma 6.3. Here is a picture (solid diagram)

\[
\begin{array}{ccc}
W & \longrightarrow & V \times_Y X \\
\downarrow & & \downarrow \\
V & \overset{f}{\longrightarrow} & X \overset{x}{\longrightarrow} \text{Spec}(k) \\
\downarrow & & \downarrow \quad y \\
\quad & Y & \,
\end{array}
\]

The morphism \( W \to X \) is universally bounded by our assumption that the space \( X \) is reasonable. Let \( n \) be an integer bounding the degrees of the fibres of \( W \to X \). We claim that the same integer works for bounding the fibres of \( V \to Y \). Namely, suppose \( y \in |Y| \) is a point. Then there exists a \( x \in |X| \) with \( f(x) = y \) (see above). This means we can find a field \( k \) and morphisms \( x, y \) given as dotted arrows in the diagram above. In particular we get a surjective étale morphism

\[
\text{Spec}(k) \times_{x,Y} W \to \text{Spec}(k) \times_{x,X} (V \times_Y X) = \text{Spec}(k) \times_{y,Y} V
\]

which shows that the degree of \( \text{Spec}(k) \times_{x,Y} W \) over \( k \) is less than or equal to the degree of \( \text{Spec}(k) \times_{x,X} W \) over \( k \), i.e., \( \leq n \), and we win. (This last part of the argument is the same as the argument in the proof of Lemma 3.3. Unfortunately that lemma is not general enough because it only applies to representable morphisms.)

03L3 **Lemma 17.9.** Let \( S \) be a scheme. Let \( f : X \to Y \) be a morphism of algebraic spaces over \( S \). Let \( \mathcal{P} \in \{ (\beta), \text{decent}, \text{reasonable}, \text{very reasonable} \} \). The following are equivalent

1. \( f \) is \( \mathcal{P} \),
2. for every affine scheme \( Z \) and every morphism \( Z \to Y \) the base change \( Z \times_Y X \to Z \) of \( f \) is \( \mathcal{P} \),
3. for every affine scheme \( Z \) and every morphism \( Z \to Y \) the algebraic space \( Z \times_Y X \) is \( \mathcal{P} \), and
4. there exists a Zariski covering \( Y = \bigcup Y_i \) such that each morphism \( f^{-1}(Y_i) \to Y_i \) has \( \mathcal{P} \).

If \( \mathcal{P} \in \{ (\beta), \text{decent}, \text{reasonable} \} \), then this is also equivalent to
(5) there exists a scheme $V$ and a surjective étale morphism $V \to Y$ such that the base change $V \times_Y X \to V$ has $\mathcal{P}$.

**Proof.** The implications $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4)$ are trivial. The implication $(3) \Rightarrow (1)$ can be seen as follows. Let $Z \to Y$ be a morphism whose source is a scheme over $S$. Consider the algebraic space $Z \times_Y X$. If we assume $(3)$, then for any affine open $W \subset Z$, the open subspace $W \times_Y X$ of $Z \times_Y X$ has property $\mathcal{P}$. Hence by Lemma 5.2 the space $Z \times_Y X$ has property $\mathcal{P}$, i.e., $(1)$ holds. A similar argument (omitted) shows that $(4)$ implies $(1)$.

The implication $(1) \Rightarrow (5)$ is trivial. Let $V \to Y$ be an étale morphism from a scheme as in $(5)$. Let $Z$ be an affine scheme, and let $Z \to Y$ be a morphism. Consider the diagram

$$
\begin{array}{ccc}
Z \times_Y V & \xrightarrow{q} & V \\
p \downarrow & & \downarrow \\
Z & \xrightarrow{p} & Y
\end{array}
$$

Since $p$ is étale, and hence open, we can choose finitely many affine open subschemes $W_i \subset Z \times_Y V$ such that $Z = \bigcup p(W_i)$. Consider the commutative diagram

$$
\begin{array}{ccc}
V \times_Y X & \xleftarrow{(\coprod W_i) \times_Y X} & Z \times_Y X \\
\downarrow & & \downarrow \\
V & \xleftarrow{\coprod W_i} & Z
\end{array}
$$

We know $V \times_Y X$ has property $\mathcal{P}$. By Lemma 5.3 we see that $(\coprod W_i) \times_Y X$ has property $\mathcal{P}$. Note that the morphism $(\coprod W_i) \times_Y X \to Z \times_Y X$ is étale and quasi-compact as the base change of $\coprod W_i \to Z$. Hence by Lemma 17.8 we conclude that $Z \times_Y X$ has property $\mathcal{P}$. □

**Remark 17.10.** An informal description of the properties $(\beta)$, decent, reasonable, very reasonable was given in Section 6. A morphism has one of these properties if (very) loosely speaking the fibres of the morphism have the corresponding properties. Being decent is useful to prove things about specializations of points on $|X|$. Being reasonable is a bit stronger and technically quite easy to work with.

Here is a lemma we promised earlier which uses decent morphisms.

**Lemma 17.11.** Let $S$ be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over $S$. Assume $f$ is quasi-compact and decent. (For example if $f$ is representable, or quasi-separated, see Lemma 17.3.) Then $f$ is universally closed if and only if the existence part of the valuative criterion holds.

**Proof.** In Morphisms of Spaces, Lemma 42.1 we proved that any quasi-compact morphism which satisfies the existence part of the valuative criterion is universally closed. To prove the other, assume that $f$ is universally closed. In the proof of Proposition 16.1 we have seen that it suffices to show, for any valuation ring $A$, and any morphism $\text{Spec}(A) \to Y$, that the base change $f_A : X_A \to \text{Spec}(A)$ satisfies the existence part of the valuative criterion. By definition the algebraic space $X_A$ has property $(\gamma)$ and hence Proposition 16.1 applies to the morphism $f_A$ and we win. □
18. Points of fibres

Let $S$ be a scheme. Consider a cartesian diagram

$$
\begin{array}{ccc}
W & \longrightarrow & Z \\
p & \downarrow & g \\
X & \longrightarrow & Y
\end{array}
$$

of algebraic spaces over $S$. Let $x \in |X|$ and $z \in |Z|$ be points mapping to the same point $y \in |Y|$. We may ask: When is the set

$$
F_{x,z} = \{ w \in |W| \text{ such that } p(w) = x \text{ and } q(w) = z \}
$$

finite?

Example 18.1. If $X, Y, Z$ are schemes, then the set $F_{x,z}$ is equal to the spectrum of $\kappa(x) \otimes_{\kappa(y)} \kappa(z)$ (Schemes, Lemma 17.5). Thus we obtain a finite set if either $\kappa(y) \subset \kappa(x)$ is finite or if $\kappa(y) \subset \kappa(z)$ is finite. In particular, this is always the case if $g$ is quasi-finite at $z$ (Morphisms, Lemma 20.5).

Example 18.2. Let $K$ be a characteristic 0 field endowed with an automorphism $\sigma$ of infinite order. Set $Y = \text{Spec}(K)/\mathbf{Z}$ and $X = \mathbb{A}^1_K/\mathbf{Z}$ where $\mathbf{Z}$ acts on $K$ via $\sigma$ and on $\mathbb{A}^1_K = \text{Spec}(K[t])$ via $t \mapsto t + 1$. Let $Z = \text{Spec}(K)$. Then $W = \mathbb{A}^1_K$. Picture

$$
\begin{array}{ccc}
\mathbb{A}^1_K & \longrightarrow & \text{Spec}(K) \\
p & \downarrow & g \\
\mathbb{A}^1_K/\mathbf{Z} & \longrightarrow & \text{Spec}(K)/\mathbf{Z}
\end{array}
$$

Take $x$ corresponding to $t = 0$ and $z$ the unique point of $\text{Spec}(K)$. Then we see that $F_{x,z} = \mathbf{Z}$ as a set.

Lemma 18.3. In the situation of (18.0.1), if $Z' \to Z$ is a morphism and $z' \in |Z'|$ maps to $z$, then the induced map $F_{x,z'} \to F_{x,z}$ is surjective.

Proof. Set $W' = X \times_Y Z' = W \times_Z Z'$. Then $|W'| \to |W| \times_Z |Z'|$ is surjective by Properties of Spaces, Lemma 4.3. Hence the surjectivity of $F_{x,z'} \to F_{x,z}$. □

Lemma 18.4. In diagram (18.0.1) the set (18.0.2) is finite if $f$ is of finite type and $f$ is quasi-finite at $x$.

Proof. The morphism $q$ is quasi-finite at every $w \in F_{x,z}$, see Morphisms of Spaces, Lemma 27.2. Hence the lemma follows from Morphisms of Spaces, Lemma 27.9. □

Lemma 18.5. In diagram (18.0.1) the set (18.0.2) is finite if $y$ can be represented by a monomorphism $\text{Spec}(k) \to Y$ where $k$ is a field and $g$ is quasi-finite at $z$. (Special case: $Y$ is decent and $g$ is étale.)

Proof. By Lemma 18.3 applied twice we may replace $Z$ by $Z_k = \text{Spec}(k) \times_Y Z$ and $X$ by $X_k = \text{Spec}(k) \times_Y X$. We may and do replace $Y$ by $\text{Spec}(k)$ as well. Note that $Z_k \to \text{Spec}(k)$ is quasi-finite at $z$ by Morphisms of Spaces, Lemma 27.2. Choose a scheme $V$, a point $v \in V$, and an étale morphism $V \to Z_k$ mapping $v$ to
z. Choose a scheme $U$, a point $u \in U$, and an étale morphism $U \to X_k$ mapping $u$ to $x$. Again by Lemma 18.3 it suffices to show $F_{u,v}$ is finite for the diagram

$$
\begin{array}{ccc}
U \times_{\text{Spec}(k)} V & \longrightarrow & V \\
\downarrow & & \downarrow \\
U & \longrightarrow & \text{Spec}(k)
\end{array}
$$

The morphism $V \to \text{Spec}(k)$ is quasi-finite at $v$ (follows from the general discussion in Morphisms of Spaces, Section 22 and the definition of being quasi-finite at a point). At this point the finiteness follows from Example 18.1. The parenthetical remark of the statement of the lemma follows from the fact that on decent spaces points are represented by monomorphisms from fields and from the fact that an étale morphism of algebraic spaces is locally quasi-finite. □

Lemma 18.6. Let $S$ be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over $S$. Let $y \in |Y|$ and assume that $y$ is represented by a quasi-compact monomorphism $\text{Spec}(k) \to Y$. Then $|X_k| \to |X|$ is a homeomorphism onto $f^{-1}(\{y\}) \subset |X|$ with induced topology.

Proof. We will use Properties of Spaces, Lemma 16.7 and Morphisms of Spaces, Lemma 10.9 without further mention. Let $V \to Y$ be an étale morphism with $V$ affine such that there exists a $v \in V$ mapping to $y$. Since $\text{Spec}(k) \to Y$ is quasi-compact there are a finite number of points of $V$ mapping to $y$ (Lemma 4.5). After shrinking $V$ we may assume $v$ is the only one. Choose a scheme $U$ and a surjective étale morphism $U \to X$. Consider the commutative diagram

$$
\begin{array}{ccc}
U & \longleftarrow & U_v \\
\downarrow & & \downarrow \\
X & \longleftarrow & X_v \\
\downarrow & & \downarrow \\
Y & \longleftarrow & V \\
\downarrow & & \downarrow \\
v
\end{array}
$$

Since $U_v \to U_V$ identifies $U_v$ with a subset of $U_V$ with the induced topology (Schemes, Lemma 18.5), and since $|U_V| \to |X_V|$ and $|U_v| \to |X_v|$ are surjective and open, we see that $|X_v| \to |X_V|$ is a homeomorphism onto its image (with induced topology). On the other hand, the inverse image of $f^{-1}(\{y\})$ under the open map $|X_V| \to |X|$ is equal to $|X_v|$. We conclude that $|X_v| \to f^{-1}(\{y\})$ is open. The morphism $X_v \to X$ factors through $X_k$ and $|X_k| \to |X|$ is injective with image $f^{-1}(\{y\})$ by Properties of Spaces, Lemma 4.3. Using $|X_v| \to |X_k| \to f^{-1}(\{y\})$ the lemma follows because $X_v \to X_k$ is surjective. □

Lemma 18.7. Let $X$ be an algebraic space locally of finite type over a field $k$. Let $x \in |X|$. Consider the conditions

1. $\dim_x(|X|) = 0$,
2. $x$ is closed in $|X|$ and if $x' \leadsto x$ in $|X|$ then $x' = x$,
3. $x$ is an isolated point of $|X|$,
4. $\dim_x(X) = 0$,
5. $X \to \text{Spec}(k)$ is quasi-finite at $x$. 

\[0AC8\]
Then (2), (3), (4), and (5) are equivalent. If $X$ is decent, then (1) is equivalent to the others.

**Proof.** Parts (4) and (5) are equivalent for example by Morphisms of Spaces, Lemmas 34.7 and 34.8.

Let $U \to X$ be an étale morphism where $U$ is an affine scheme and let $u \in U$ be a point mapping to $x$. Moreover, if $x$ is a closed point, e.g., in case (2) or (3), then we may and do assume that $u$ is a closed point. Observe that $\dim_u(U) = \dim_x(X)$ by definition and that this is equal to $\dim(O_{U,u})$ if $u$ is a closed point, see Algebra, Lemma 114.6.

If $\dim_x(X) > 0$ and $u$ is closed, by the arguments above we can choose a nontrivial specialization $u' \rightsquigarrow u$ in $U$. Then the transcendence degree of $\kappa(u')$ over $k$ exceeds the transcendence degree of $\kappa(u)$ over $k$. It follows that the images $x$ and $x'$ in $X$ are distinct, because the transcendence degree of $x/k$ and $x'/k$ are well defined, see Morphisms of Spaces, Definition 33.1. This applies in particular in cases (2) and (3) and we conclude that (2) and (3) imply (4).

Conversely, if $X \to \text{Spec}(k)$ is locally quasi-finite at $x$, then $U \to \text{Spec}(k)$ is locally quasi-finite at $u$, hence $u$ is an isolated point of $U$ (Morphisms, Lemma 20.6). It follows that (5) implies (2) and (3) as $|U| \to |X|$ is continuous and open.

Assume $X$ is decent and (1) holds. Then $\dim_x(X) = \dim_x(|X|)$ by Lemma 12.5 and the proof is complete.

**Lemma 18.8.** Let $X$ be an algebraic space locally of finite type over a field $k$. Consider the conditions

(1) $|X|$ is a finite set,
(2) $|X|$ is a discrete space,
(3) $\dim(|X|) = 0$,
(4) $\dim(X) = 0$,
(5) $X \to \text{Spec}(k)$ is locally quasi-finite.

Then (2), (3), (4), and (5) are equivalent. If $X$ is decent, then (1) implies the others.

**Proof.** Parts (4) and (5) are equivalent for example by Morphisms of Spaces, Lemma 34.7.

Let $U \to X$ be a surjective étale morphism where $U$ is a scheme.

If $\dim(U) > 0$, then choose a nontrivial specialization $u \rightsquigarrow u'$ in $U$ and the transcendence degree of $\kappa(u)$ over $k$ exceeds the transcendence degree of $\kappa(u')$ over $k$. It follows that the images $x$ and $x'$ in $X$ are distinct, because the transcendence degree of $x/k$ and $x'/k$ is well defined, see Morphisms of Spaces, Definition 33.1. We conclude that (2) and (3) imply (4).

Conversely, if $X \to \text{Spec}(k)$ is locally quasi-finite, then $U$ is locally Noetherian (Morphisms, Lemma 15.6) of dimension 0 (Morphisms, Lemma 29.5) and hence is a disjoint union of spectra of Artinian local rings (Properties, Lemma 10.5). Hence $U$ is a discrete topological space, and since $|U| \to |X|$ is continuous and open, the same is true for $|X|$. In other words, (4) implies (2) and (3).

Assume $X$ is decent and (1) holds. Then we may choose $U$ above to be affine. The fibres of $|U| \to |X|$ are finite (this is a part of the defining property of decent
spaces). Hence $U$ is a finite type scheme over $k$ with finitely many points. Hence $U$ is quasi-finite over $k$ (Morphisms, Lemma 20.7) which by definition means that $X \to \text{Spec}(k)$ is locally quasi-finite. □

**Lemma 18.9.** Let $S$ be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over $S$ which is locally of finite type. Let $x \in |X|$ with image $y \in |Y|$. Let $F = f^{-1}(|y|)$ with induced topology from $|X|$. Let $k$ be a field and let $\text{Spec}(k) \to Y$ be in the equivalence class defining $y$. Set $X_k = \text{Spec}(k) \times_Y X$. Let $\tilde{x} \in |X_k|$ map to $x \in |X|$. Consider the following conditions

0ACB (1) $\dim_x(F) = 0$,
0ACC (2) $x$ is isolated in $F$,
0ACD (3) $x$ is closed in $F$ and if $x' \leadsto x$ in $F$, then $x = x'$,
0ACF (4) $\dim_x(|X_k|) = 0$,
0ACG (5) $\tilde{x}$ is isolated in $|X_k|$,
0ACH (6) $\tilde{x}$ is closed in $|X_k|$ and if $\tilde{x}' \leadsto \tilde{x}$ in $|X_k|$, then $\tilde{x} = \tilde{x}'$,
0ACI (7) $\dim_{\tilde{x}}(X_k) = 0$,
0ACJ (8) $f$ is quasi-finite at $x$.

Then we have

$$\begin{align*}
(1) & \xrightarrow{\text{descent}} (5) \xrightarrow{\text{descent}} (6) \xrightarrow{\text{descent}} (7) \xrightarrow{\text{descent}} (8)
\end{align*}$$

If $Y$ is decent, then conditions (2) and (3) are equivalent to each other and to conditions (4), (6), (7), and (8). If $Y$ and $X$ are decent, then all conditions are equivalent.

**Proof.** By Lemma 18.7 conditions (5), (6), and (7) are equivalent to each other and to the condition that $X_k \to \text{Spec}(k)$ is quasi-finite at $\tilde{x}$. Thus by Morphisms of Spaces, Lemma 27.2 they are also equivalent to (8). If $f$ is decent, then $X_k$ is a decent algebraic space and Lemma 18.7 shows that (4) implies (5).

If $Y$ is decent, then we can pick a quasi-compact monomorphism $\text{Spec}(k') \to Y$ in the equivalence class of $y$. In this case Lemma 18.6 tells us that $|X_k| \to F$ is a homeomorphism. Combined with the arguments given above this implies the remaining statements of the lemma; details omitted. □

**Lemma 18.10.** Let $S$ be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over $S$ which is locally of finite type. Let $y \in |Y|$. Let $k$ be a field and let $\text{Spec}(k) \to Y$ be in the equivalence class defining $y$. Set $X_k = \text{Spec}(k) \times_Y X$ and let $F = f^{-1}(|y|)$ with the induced topology from $|X|$. Consider the following conditions

0ACL (1) $F$ is finite,
0ACM (2) $F$ is a discrete topological space,
0ACN (3) $\dim(F) = 0$,
0ACP (4) $|X_k|$ is a finite set,
0ACQ (5) $|X_k|$ is a discrete space,
0ACR (6) $\dim(|X_k|) = 0$,
0ACS (7) $\dim(X_k) = 0$,
0ACT (8) $f$ is quasi-finite at all points of $|X|$ lying over $y$. 
Then we have

\[ (1) \iff (4) \iff \text{decent} \iff (5) \iff (6) \iff (7) \iff (8) \]

If \( Y \) is decent, then conditions (2) and (3) are equivalent to each other and to conditions (5), (6), (7), and (8). If \( Y \) and \( X \) are decent, then (1) implies all the other conditions.

**Proof.** By Lemma 18.8 conditions (5), (6), and (7) are equivalent to each other and to the condition that \( X_k \to \text{Spec}(k) \) is locally quasi-finite. Thus by Morphisms of Spaces, Lemma 27.2 they are also equivalent to (8). If \( f \) is decent, then \( X_k \) is a decent algebraic space and Lemma 18.8 shows that (4) implies (5).

The map \( |X_k| \to F \) is surjective by Properties of Spaces, Lemma 4.3 and we see \( (4) \Rightarrow (1) \).

If \( Y \) is decent, then we can pick a quasi-compact monomorphism \( \text{Spec}(k') \to Y \) in the equivalence class of \( y \). In this case Lemma 18.6 tells us that \( |X_k'| \to F \) is a homeomorphism. Combined with the arguments given above this implies the remaining statements of the lemma; details omitted. \( \square \)

### 19. Monomorphisms

06RY Here is another case where monomorphisms are representable. Please see More on Morphisms of Spaces, Section 04 for more information.

06RZ **Lemma 19.1.** Let \( S \) be a scheme. Let \( Y \) be a disjoint union of spectra of zero dimensional local rings over \( S \). Let \( f : X \to Y \) be a monomorphism of algebraic spaces over \( S \). Then \( f \) is representable, i.e., \( X \) is a scheme.

**Proof.** This immediately reduces to the case \( Y = \text{Spec}(A) \) where \( A \) is a zero dimensional local ring, i.e., \( \text{Spec}(A) = \{ \mathfrak{m}_A \} \) is a singleton. If \( X = \emptyset \), then there is nothing to prove. If not, choose a nonempty affine scheme \( U = \text{Spec}(B) \) and an étale morphism \( U \to X \). As \( |X| \) is a singleton (as a subset of \( |Y| \), see Morphisms of Spaces, Lemma 10.9) we see that \( U \to X \) is surjective. Note that \( U \times_X U = \text{Spec}(B \otimes_A B) \). Thus we see that the ring maps \( B \to B \otimes_A B \) are étale. Since

\[
(B \otimes_A B)/\mathfrak{m}_A(B \otimes_A B) = (B/\mathfrak{m}_A B) \otimes_{A/\mathfrak{m}_A} (B/\mathfrak{m}_A B)
\]

we see that \( B/\mathfrak{m}_A B \to (B \otimes_A B)/\mathfrak{m}_A(B \otimes_A B) \) is flat and in fact free of rank equal to the dimension of \( B/\mathfrak{m}_A B \) as a \( A/\mathfrak{m}_A \)-vector space. Since \( B \to B \otimes_A B \) is étale, this can only happen if this dimension is finite (see for example Morphisms, Lemmas 56.8 and 56.9). Every prime of \( B \) lies over \( \mathfrak{m}_A \) (the unique prime of \( A \)). Hence \( \text{Spec}(B) = \text{Spec}(B/\mathfrak{m}_A) \) as a topological space, and this space is a finite discrete set as \( B/\mathfrak{m}_A B \) is an Artinian ring, see Algebra, Lemmas 53.2 and 53.6. Hence all prime ideals of \( B \) are maximal and \( B = B_1 \times \ldots \times B_n \) is a product of finitely many local rings of dimension zero, see Algebra, Lemma 53.5. Thus \( B \to B \otimes_A B \) is finite étale as all the local rings \( B_i \) are henselian by Algebra, Lemma 153.10. Thus \( X \) is an affine scheme by Groupoids, Proposition 23.9. \( \square \)
20. Generic points

Lemma 20.1. Let $S$ be a scheme. Let $X$ be a decent algebraic space over $S$. Let $x \in |X|$. The following are equivalent

1. $x$ is a generic point of an irreducible component of $|X|$,
2. for any étale morphism $(Y, y) \to (X, x)$ of pointed algebraic spaces, $y$ is a generic point of an irreducible component of $|Y|$,
3. for some étale morphism $(Y, y) \to (X, x)$ of pointed algebraic spaces, $y$ is a generic point of an irreducible component of $|Y|$,
4. the dimension of the local ring of $X$ at $x$ is zero, and
5. $x$ is a point of codimension 0 on $X$.

Proof. Conditions (4) and (5) are equivalent for any algebraic space by definition, see Properties of Spaces, Definition 10.2. Observe that any Conditions (4) and (5) are equivalent for any algebraic space by definition, 12.4. Then codim$_{(X, x)}(U, u, y) = 0$ and we see that (4) holds.

Assume (4). Let $y' \sim y$ be a specialization in $U$. Then $f(y') = f(y) = x$. By Lemma 12.6 we see that $y' = y$. Hence $u$ is a generic point of an irreducible component of $U$. Thus $\dim(\mathcal{O}_{U, u}) = 0$ and we see that (4) holds.

Assume (4). The point $x$ is contained in an irreducible component $T \subset |X|$. Since $|X|$ is sober (Proposition 12.4) we $T$ has a generic point $x'$. Of course $x' \sim x$. Then we can lift this specialization to $u' \sim u$ in $U$ (Lemma 12.2). This contradicts the assumption that $\dim(\mathcal{O}_{U, u}) = 0$ unless $u' = u$, i.e., $x' = x$. □

Lemma 20.2. Let $S$ be a scheme. Let $X$ be a decent algebraic space over $S$. Let $T \subset |X|$ be an irreducible closed subset. Let $\xi \in T$ be the generic point (Proposition 12.4). Then $\text{codim}(T, |X|)$ (Topology, Definition 11.1) is the dimension of the local ring of $X$ at $\xi$ (Properties of Spaces, Definition 10.2).

Proof. Choose a scheme $U$, a point $u \in U$, and an étale morphism $U \to X$ sending $u$ to $\xi$. Then any sequence of nontrivial specializations $\xi_e \sim \cdots \sim \xi_0 = \xi$ can be lifted to a sequence $u_e \sim \cdots \sim u_0 = u$ in $U$ by Lemma 12.2. Conversely, any sequence of nontrivial specializations $u_e \sim \cdots \sim u_0 = u$ in $U$ maps to a sequence of nontrivial specializations $\xi_e \sim \cdots \sim \xi_0 = \xi$ by Lemma 12.1. Because $|X|$ and $U$ are sober topological spaces we conclude that the codimension of $T$ in $|X|$ and of $\{u\}$ in $U$ are the same. In this way the lemma reduces to the schemes case which is Properties, Lemma 10.3. □

Lemma 20.3. Let $S$ be a scheme. Let $X$ be an algebraic space over $S$. Assume

1. every quasi-compact scheme étale over $X$ has finitely many irreducible components, and
2. every $x \in |X|$ of codimension 0 on $X$ can be represented by a monomorphism $\text{Spec}(k) \to X$.

Then $X$ is a reasonable algebraic space.

Proof. Let $U$ be an affine scheme and let $a : U \to X$ be an étale morphism. We have to show that the fibres of $a$ are universally bounded. By assumption (1) the
scheme $U$ has finitely many irreducible components. Let $u_1, \ldots, u_n \in U$ be the
generic points of these irreducible components. Let $\{x_1, \ldots, x_m\} \subset |X|$ be the
image of $\{u_1, \ldots, u_n\}$. Each $x_j$ is a point of codimension 0. By assumption (2)
we may choose a monomorphism $\text{Spec}(k_j) \to X$ representing $x_j$. By Properties of
Spaces, Lemma [11.1] we have
$$U \times_X \text{Spec}(k_j) = \coprod_{a(u_i) = x_j} \text{Spec}(\kappa(u_i))$$
This is a scheme finite over $\text{Spec}(k_j)$ of degree $d_j = \sum_{a(u_i) = x_j} [\kappa(u_i) : k_j]$. Set
$n = \max d_j$.

Observe that $a$ is separated (Properties of Spaces, Lemma [6.4]). Consider the stratification
$$X = X_0 \supset X_1 \supset X_2 \supset \ldots$$
associated to $U \to X$ in Lemma [8.2]. By our choice of $n$ above we conclude that
$X_{n+1}$ is empty. Namely, if not, then $a^{-1}(X_{n+1})$ is a nonempty open of $U$ and hence
would contain one of the $x_i$. This would mean that $X_{n+1}$ contains $x_j = a(u_i)$ which
is impossible. Hence we see that the fibres of $U \to X$ are universally bounded (in fact
by the integer $n$).

0BB9 Lemma 20.4. Let $S$ be a scheme. Let $X$ be an algebraic space over $S$. The
following are equivalent

1. $X$ is decent and $|X|$ has finitely many irreducible components,
2. every quasi-compact scheme étale over $X$ has finitely many irreducible
   components, there are finitely many $x \in |X|$ of codimension 0 on $X$, and each
   of these can be represented by a monomorphism $\text{Spec}(k) \to X$,
3. there exists a dense open $X' \subset X$ which is a scheme, $X'$ has finitely many
   irreducible components with generic points $\{x_1', \ldots, x_m'\}$, and the morphism
   $x_j' \to X$ is quasi-compact for $j = 1, \ldots, m$.

Moreover, if these conditions hold, then $X$ is reasonable and the points $x_j' \in |X|
are the generic points of the irreducible components of $|X|$.

Proof. In the proof we use Properties of Spaces, Lemma [11.1] without further
mention. Assume (1). Then $X$ has a dense open subscheme $X'$ by Theorem [10.2].
Since the closure of an irreducible component of $|X'|$ is an irreducible component
of $|X|$, we see that $|X'|$ has finitely many irreducible components. Thus (3) holds.

Assume $X' \subset X$ is as in (3). Let $\{x_1', \ldots, x_m'\}$ be the generic points of the irreducible
components of $X'$. Let $a : U \to X$ be an étale morphism with $U$ a
quasi-compact scheme. To prove (2) it suffices to show that $U$ has finitely many irreducible
components whose generic points lie over $\{x_1', \ldots, x_m'\}$. It suffices to prove
this for the members of a finite affine open cover of $U$, hence we may and do
assume $U$ is affine. Note that $U' = a^{-1}(X') \subset U$ is a dense open. Since $U' \to X'$ is
an étale morphism of schemes, we see the generic points of irreducible components
of $U'$ are the points lying over $\{x_1', \ldots, x_m'\}$. Since $x_j' \to X$ is quasi-compact there
are finitely many points of $U$ lying over $x_j'$ (Lemma [4.5]). Hence $U'$ has finitely
many irreducible components, which implies that the closures of these irreducible
components are the irreducible components of $U$. Thus (2) holds.

Assume (2). This implies (1) and the final statement by Lemma [20.3]. We also
use that a reasonable algebraic space is decent, see discussion following Definition
[6.1].

□
21. Generically finite morphisms

0BBA This section discusses for morphisms of algebraic spaces the material discussed in Morphisms, Section 51 and Varieties, Section 17 for morphisms of schemes.

0ACZ **Lemma 21.1.** Let $S$ be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over $S$. Assume that $f$ is quasi-separated of finite type. Let $y \in |Y|$ be a point of codimension $0$ on $Y$. The following are equivalent:

1. the space $|X_k|$ is finite where $\text{Spec}(k) \to Y$ represents $y$,
2. $X \to Y$ is quasi-finite at all points of $|X|$ over $y$,
3. there exists an open subspace $Y' \subset Y$ with $y \in |Y'|$ such that $Y' \times_Y X \to Y'$ is finite.

If $Y$ is decent these are also equivalent to

4. the set $f^{-1}(\{y\})$ is finite.

**Proof.** The equivalence of (1) and (2) follows from Lemma 18.10 (and the fact that a quasi-separated morphism is decent by Lemma 17.2).

Assume the equivalent conditions of (1) and (2). Choose an affine scheme $V$ and an étale morphism $V \to Y$ mapping a point $v \in V$ to $y$. Then $v$ is a generic point of an irreducible component of $V$ by Properties of Spaces, Lemma 11.1. Choose an affine scheme $U$ and a surjective étale morphism $U \to V \times_Y X$. Then $U \to V$ is of finite type. The morphism $U \to V$ is quasi-finite at every point lying over $v$ by (2).

It follows that the fibre of $U \to V$ over $v$ is finite (Morphisms, Lemma 20.14). By Morphisms, Lemma 51.1 after shrinking $V$ we may assume that $U \to V$ is finite as well. This of course implies that the two projections $R \to V$ are finite étale. It follows that $V/R = V \times_X Y$ is an affine scheme, see Groupoids, Proposition 23.9. By Morphisms, Lemma 41.9 we conclude that $V \times_Y X \to V$ is proper and by Morphisms, Lemma 44.11 we conclude that $V \times_Y X \to V$ is finite. Finally, we let $Y' \subset Y$ be the open subspace of $Y$ corresponding to the image of $|V| \to |Y|$. By Morphisms of Spaces, Lemma 45.3 we conclude that $Y' \times_Y X \to Y'$ is finite as the base change to $V$ is finite and as $V \to Y'$ is a surjective étale morphism.

If $Y$ is decent and $f$ is quasi-separated, then we see that $X$ is decent too; use Lemmas 17.2 and 17.5. Hence Lemma 18.10 applies to show that (4) implies (1) and (2). On the other hand, we see that (2) implies (4) by Morphisms of Spaces, Lemma 27.9.

0AD0 **Lemma 21.2.** Let $S$ be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over $S$. Assume that $f$ is quasi-separated and locally of finite type and $Y$ quasi-separated. Let $y \in |Y|$ be a point of codimension $0$ on $Y$. The following are equivalent:

1. the set $f^{-1}(\{y\})$ is finite,
2. the space $|X_k|$ is finite where $\text{Spec}(k) \to Y$ represents $y$, 


(3) there exist open subspaces $X' \subset X$ and $Y' \subset Y$ with $f(X') \subset Y'$, $y \in |Y'|$, and $f^{-1}(\{y\}) \subset |X'|$ such that $f|_{X'}: X' \to Y'$ is finite.

**Proof.** Since quasi-separated algebraic spaces are decent, the equivalence of (1) and (2) follows from Lemma 18.10. To prove that (1) and (2) imply (3) we may and do replace $Y$ by a quasi-compact open containing $y$. Since $f^{-1}(\{y\})$ is finite, we can find a quasi-compact open subspace of $X' \subset X$ containing the fibre. The restriction $f|_{X'}: X' \to Y$ is quasi-compact and quasi-separated by Morphisms of Spaces, Lemma 8.10 (this is where we use that $Y$ is quasi-separated). Applying Lemma 21.1 to $f|_{X'}: X' \to Y$ we see that (3) holds. We omit the proof that (3) implies (2). \(\Box\)

**Lemma 21.3.** Let $S$ be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over $S$. Assume $f$ is locally of finite type. Let $X^0 \subset |X|$, resp. $Y^0 \subset |Y|$ denote the set of codimension 0 points of $X$, resp. $Y$. Let $y \in Y^0$. The following are equivalent

1. $f^{-1}(\{y\}) \subset X^0$,
2. $f$ is quasi-finite at all points lying over $y$,
3. $f$ is quasi-finite at all $x \in X^0$ lying over $y$.

**Proof.** Let $V$ be a scheme and let $V \to Y$ be a surjective étale morphism. Let $U$ be a scheme and let $U \to V \times_Y X$ be a surjective étale morphism. Then $f$ is quasi-finite at the image $x$ of a point $u \in U$ if and only if $U \to V$ is quasi-finite at $u$. Moreover, $x \in X^0$ if and only if $u$ is the generic point of an irreducible component of $U$ (Properties of Spaces, Lemma 11.1). Thus the lemma reduces to the case of the morphism $U \to V$, i.e., to Morphisms, Lemma 51.4. \(\Box\)

**Lemma 21.4.** Let $S$ be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over $S$. Assume $f$ is locally of finite type. Let $X^0 \subset |X|$, resp. $Y^0 \subset |Y|$ denote the set of codimension 0 points of $X$, resp. $Y$. Assume

1. $Y$ is decent,
2. $X^0$ and $Y^0$ are finite and $f^{-1}(Y^0) = X^0$,
3. either $f$ is quasi-compact or $f$ is separated.

Then there exists a dense open $V \subset Y$ such that $f^{-1}(V) \to V$ is finite.

**Proof.** By Lemmas 20.4 and 20.1, we may assume $Y$ is a scheme with finitely many irreducible components. Shrinking further we may assume $Y$ is an irreducible affine scheme with generic point $y$. Then the fibre of $f$ over $y$ is finite.

Assume $f$ is quasi-compact and $Y$ affine irreducible. Then $X$ is quasi-compact and we may choose an affine scheme $U$ and a surjective étale morphism $U \to X$. Then $U \to Y$ is of finite type and the fibre of $U \to Y$ over $y$ is the set $U^0$ of generic points of irreducible components of $U$ (Properties of Spaces, Lemma 11.1). Hence $U^0$ is finite (Morphisms, Lemma 20.14) and after shrinking $Y$ we may assume that $U \to Y$ is finite (Morphisms, Lemma 51.1). Next, consider $R = U \times_X U$. Since the projection $s : R \to U$ is étale we see that $R^0 = s^{-1}(U^0)$ lies over $y$. Since $R \to U \times_Y U$ is a monomorphism, we conclude that $R^0$ is finite as $U \times_Y U \to Y$ is finite. And $R$ is separated (Properties of Spaces, Lemma 6.4). Thus we may shrink $Y$ once more to reach the situation where $R$ is finite over $Y$ (Morphisms, Lemma 51.5). In this case it follows that $X = U/R$ is finite over $Y$ by exactly the
same arguments as given in the proof of Lemma 21.1 (or we can simply apply that lemma because it follows immediately that $X$ is quasi-separated as well).

Assume $f$ is separated and $Y$ affine irreducible. Choose $V \subset Y$ and $U \subset X$ as in Lemma 21.2. Since $f|_U : U \to V$ is finite, we see that $U \subset f^{-1}(V)$ is closed as well as open (Morphisms of Spaces, Lemmas 40.6 and 45.9). Thus $f^{-1}(V) = U \amalg W$ for some open subspace $W$ of $X$. However, since $U$ contains all the codimension 0 points of $X$ we conclude that $W = \emptyset$ (Properties of Spaces, Lemma 11.2) as desired.

**22. Birational morphisms**

The following definition of a birational morphism of algebraic spaces seems to be the closest to our definition (Morphisms, Definition 50.1) of a birational morphism of schemes.

**Definition 22.1.** Let $S$ be a scheme. Let $X$ and $Y$ algebraic spaces over $S$. Assume $X$ and $Y$ are decent and that $|X|$ and $|Y|$ have finitely many irreducible components. We say a morphism $f : X \to Y$ is birational if

1. $|f|$ induces a bijection between the set of generic points of irreducible components of $|X|$ and the set of generic points of the irreducible components of $|Y|$, and
2. for every generic point $x \in |X|$ of an irreducible component the local ring map $O_{Y,f(x)} \to O_{X,x}$ is an isomorphism (see clarification below).

Clarification: Since $X$ and $Y$ are decent the topological spaces $|X|$ and $|Y|$ are sober (Proposition 12.4). Hence condition (1) makes sense. Moreover, because we have assumed that $|X|$ and $|Y|$ have finitely many irreducible components, we see that the generic points $x_1, \ldots, x_n \in |X|$, resp. $y_1, \ldots, y_n \in |Y|$ are contained in any dense open of $|X|$, resp. $|Y|$. In particular, they are contained in the schematic locus of $X$, resp. $Y$ by Theorem 10.2. Thus we can define $O_{X,x_i}$, resp. $O_{Y,y_i}$ to be the local ring of this scheme at $x_i$, resp. $y_i$.

We conclude that if the morphism $f : X \to Y$ is birational, then there exist dense open subspaces $X' \subset X$ and $Y' \subset Y$ such that

1. $f(X') \subset Y'$,
2. $X'$ and $Y'$ are representable, and
3. $f|_{X'} : X' \to Y'$ is birational in the sense of Morphisms, Definition 50.1

However, we do insist that $X$ and $Y$ are decent with finitely many irreducible components. Other ways to characterize decent algebraic spaces with finitely many irreducible components are given in Lemma 20.4. In most cases birational morphisms are isomorphisms over dense opens.

**Lemma 22.2.** Let $S$ be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over $S$ which are decent and have finitely many irreducible components. If $f$ is birational then $f$ is dominant.

**Proof.** Follows immediately from the definitions. See Morphisms of Spaces, Definition 18.1. □
Lemma 22.3. Let $S$ be a scheme. Let $f : X \to Y$ be a birational morphism of algebraic spaces over $S$ which are decent and have finitely many irreducible components. If $y \in Y$ is the generic point of an irreducible component, then the base change $X \times_Y \text{Spec}(O_{Y,y}) \to \text{Spec}(O_{Y,y})$ is an isomorphism.

**Proof.** Let $X' \subset X$ and $Y' \subset Y$ be the maximal open subspaces which are representable, see Lemma 20.4. By Lemma 21.3 the fibre of $f$ over $y$ consists of points of codimension $0$ of $X$ and is therefore contained in $X'$. Hence $X \times_Y \text{Spec}(O_{Y,y}) = X' \times_{Y'} \text{Spec}(O_{Y',y})$ and the result follows from Morphisms, Lemma 50.3. □

Lemma 22.4. Let $S$ be a scheme. Let $f : X \to Y$ be a birational morphism of algebraic spaces over $S$ which are decent and have finitely many irreducible components. Assume one of the following conditions is satisfied

1. $f$ is locally of finite type and $Y$ reduced (i.e., integral),
2. $f$ is locally of finite presentation.

Then there exist dense opens $U \subset X$ and $V \subset Y$ such that $f(U) \subset V$ and $f|_U : U \to V$ is an isomorphism.

**Proof.** By Lemma 20.4 we may assume that $X$ and $Y$ are schemes. In this case the result is Morphisms, Lemma 50.5. □

Lemma 22.5. Let $S$ be a scheme. Let $f : X \to Y$ be a birational morphism of algebraic spaces over $S$ which are decent and have finitely many irreducible components. Assume

1. either $f$ is quasi-compact or $f$ is separated, and
2. either $f$ is locally of finite type and $Y$ is reduced or $f$ is locally of finite presentation.

Then there exists a dense open $V \subset Y$ such that $f^{-1}(V) \to V$ is an isomorphism.

**Proof.** By Lemma 20.4 we may assume $Y$ is a scheme. By Lemma 21.4 we may assume that $f$ is finite. Then $X$ is a scheme too and the result follows from Morphisms, Lemma 51.6. □

Lemma 22.6. Let $S$ be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over $S$ which are decent and have finitely many irreducible components. If $f$ is birational and $V \to Y$ is an étale morphism with $V$ affine, then $X \times_Y V$ is decent with finitely many irreducible components and $X \times_Y V \to V$ is birational.

**Proof.** The algebraic space $U = X \times_Y V$ is decent (Lemma 6.6). The generic points of $V$ and $U$ are the elements of $|V|$ and $|U|$ which lie over generic points of $|Y|$ and $|X|$ (Lemma 20.1). Since $Y$ is decent we conclude there are finitely many generic points on $V$. Let $\xi \in |X|$ be a generic point of an irreducible component. By the discussion following Definition 22.1 we have a cartesian square

$$
\begin{array}{ccc}
\text{Spec}(O_{X,\xi}) & \longrightarrow & X \\
\downarrow & & \downarrow \\
\text{Spec}(O_{Y,f(\xi)}) & \longrightarrow & Y
\end{array}
$$
whose horizontal morphisms are monomorphisms identifying local rings and where the left vertical arrow is an isomorphism. It follows that in the diagram

\[
\begin{array}{ccc}
\text{Spec}(O_{X,\xi}) \times_X U & \longrightarrow & U \\
\downarrow & & \downarrow \\
\text{Spec}(O_{Y,f(\xi)}) \times_Y V & \longrightarrow & V
\end{array}
\]

the vertical arrow on the left is an isomorphism. The horizontal arrows have image contained in the schematic locus of $U$ and $V$ and identify local rings (some details omitted). Since the image of the horizontal arrows are the points of $|U|$, resp. $|V|$ lying over $\xi$, resp. $f(\xi)$ we conclude.\hfill\Box

**Lemma 22.7.** Let $S$ be a scheme. Let $f : X \to Y$ be a birational morphism between algebraic spaces over $S$ which are decent and have finitely many irreducible components. Then the normalizations $X' \to X$ and $Y' \to Y$ exist and there is a commutative diagram

\[
\begin{array}{ccc}
X' & \longrightarrow & Y' \\
\downarrow & & \downarrow \\
X & \longrightarrow & Y
\end{array}
\]

of algebraic spaces over $S$. The morphism $X' \to Y'$ is birational.

**Proof.** By Lemma [20.4](#) we see that $X$ and $Y$ satisfy the equivalent conditions of Morphisms of Spaces, Lemma [49.1](#) and the normalizations are defined. By Morphisms of Spaces, Lemma [49.8](#) the algebraic space $X'$ is normal and maps codimension 0 points to codimension 0 points. Since $f$ maps codimension 0 points to codimension 0 points (this is the same as generic points on decent spaces by Lemma [20.1](#)) we obtain from Morphisms of Spaces, Lemma [49.8](#) a factorization of the composition $X' \to X \to Y$ through $Y'$.

Observe that $X'$ and $Y'$ are decent for example by Lemma [6.5](#). Moreover the maps $X' \to X$ and $Y' \to Y$ induce bijections on irreducible components (see references above) hence $X'$ and $Y'$ both have a finite number of irreducible components and the map $X' \to Y'$ induces a bijection between their generic points. To prove that $X' \to Y'$ is birational, it therefore suffices to show it induces an isomorphism on local rings at these points. To do this we may replace $X$ and $Y$ by open neighbourhoods of their generic points, hence we may assume $X$ and $Y$ are affine irreducible schemes with generic points $x$ and $y$. Since $f$ is birational the map $O_{X,x} \to O_{Y,y}$ is an isomorphism. Let $x' \in X'$ and $y' \in Y'$ be the points lying over $x$ and $y$. By construction of the normalization we see that $O_{X',x'} = O_{X,x}/m_x$ and similarly on $Y$. Thus the map $O_{X',x'} \to O_{Y',y'}$ is an isomorphism as well.\hfill\Box

**Lemma 22.8.** Let $S$ be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over $S$. Assume

1. $X$ and $Y$ are decent and have finitely many irreducible components,
2. $f$ is integral and birational,
3. $Y$ is normal, and
4. $X$ is reduced.

Then $f$ is an isomorphism.
Proof. Let $V \to Y$ be an étale morphism with $V$ affine. It suffices to show that $U = X \times_Y V \to V$ is an isomorphism. By Lemma 22.6 and its proof we see that $U$ and $V$ are decent and have finitely many irreducible components and that $U \to V$ is birational. By Properties, Lemma 7.5 $V$ is a finite disjoint union of integral schemes. Thus we may assume $V$ is integral. As $f$ is birational, we see that $U$ is irreducible and reduced, i.e., integral (note that $U$ is a scheme as $f$ is integral, hence representable). Thus we may assume that $X$ and $Y$ are integral schemes and the result follows from the case of schemes, see Morphisms, Lemma 54.8. □

Lemma 22.9. Let $S$ be a scheme. Let $f : X \to Y$ be an integral birational morphism of decent algebraic spaces over $S$ which have finitely many irreducible components. Then there exists a factorization $Y' \to X \to Y$ and $Y' \to X$ is the normalization of $X$.

Proof. Consider the map $X' \to Y'$ of Lemma 22.7. This map is integral by Morphisms of Spaces, Lemma 45.12. Hence it is an isomorphism by Lemma 22.8. □

23. Jacobson spaces

We have defined the Jacobson property for algebraic spaces in Properties of Spaces, Remark 7.3. For representable algebraic spaces it agrees with the property discussed in Properties, Section 6. The relationship between the Jacobson property and the behaviour of the topological space $|X|$ is not evident for general algebraic spaces $|X|$. However, a decent (for example quasi-separated or locally separated) algebraic space $X$ is Jacobson if and only if $|X|$ is Jacobson (see Lemma 23.4).

Lemma 23.1. Let $S$ be a scheme. Let $X$ be a Jacobson algebraic space over $S$. Any algebraic space locally of finite type over $X$ is Jacobson.

Proof. Let $U \to X$ be a surjective étale morphism where $U$ is a scheme. Then $U$ is Jacobson (by definition) and for a morphism of schemes $V \to U$ which is locally of finite type we see that $V$ is Jacobson by the corresponding result for schemes (Morphisms, Lemma 16.9). Thus if $Y \to X$ is a morphism of algebraic spaces which is locally of finite type, then setting $V = U \times_X Y$ we see that $Y$ is Jacobson by definition. □

Lemma 23.2. Let $S$ be a scheme. Let $X$ be a Jacobson algebraic space over $S$. For $x \in X_{\text{ft-pts}}$ and $g : W \to X$ locally of finite type with $W$ a scheme, if $x \in \text{Im}(|g|)$, then there exists a closed point of $W$ mapping to $x$.

Proof. Let $U \to X$ be an étale morphism with $U$ a scheme and with $u \in U$ closed mapping to $x$, see Morphisms of Spaces, Lemma 25.3. Observe that $W, W \times_X U$, and $U$ are Jacobson schemes by Lemma 23.1. Hence finite type points on these schemes are the same thing as closed points by Morphisms, Lemma 16.8. The inverse image $T \subset W \times_X U$ of $u$ is a nonempty (as $x$ in the image of $W \to X$) closed subset. By Morphisms, Lemma 16.7 there is a closed point $t$ of $W \times_X U$ which maps to $u$. As $W \times_X U \to W$ is locally of finite type the image of $t$ in $W$ is closed by Morphisms, Lemma 16.8. □

Lemma 23.3. Let $S$ be a scheme. Let $X$ be a decent Jacobson algebraic space over $S$. Then $X_{\text{ft-pts}} \subset |X|$ is the set of closed points.
Proof. If \( x \in |X| \) is closed, then we can represent \( x \) by a closed immersion \( \text{Spec}(k) \to X \), see Lemma \[14.6\]. Hence \( x \) is certainly a finite type point.

Conversely, let \( x \in |X| \) be a finite type point. We know that \( x \) can be represented by a quasi-compact monomorphism \( \text{Spec}(k) \to X \) where \( k \) is a field (Definition \[6.1\]). On the other hand, by definition, there exists a morphism \( \text{Spec}(k') \to X \) which is locally of finite type and represents \( x \) (Morphisms, Definition \[16.3\]). We obtain a factorization \( \text{Spec}(k') \to \text{Spec}(k) \to X \). Let \( U \to X \) be any étale morphism with \( U \) affine and consider the morphisms

\[
\text{Spec}(k') \times_X U \to \text{Spec}(k) \times_X U \to U
\]

The quasi-compact scheme \( \text{Spec}(k) \times_X U \) is étale over \( \text{Spec}(k) \) hence is a finite disjoint union of spectra of fields (Remark \[4.1\]). Moreover, the first morphism is surjective and locally of finite type (Morphisms, Lemma \[15.8\]) hence surjective on finite type points (Morphisms, Lemma \[16.6\]) and the composition (which is locally of finite type) sends finite type points to closed points as \( U \) is Jacobson (Morphisms, Lemma \[16.8\]). Thus the image of \( \text{Spec}(k) \times_X U \to U \) is a finite set of closed points hence closed. Since this is true for every affine \( U \) and étale morphism \( U \to X \), we conclude that \( x \in |X| \) is closed.

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Lemma 23.4. Let \( S \) be a scheme. Let \( X \) be a decent algebraic space over \( S \). Then \( X \) is Jacobson if and only if \( |X| \) is Jacobson.

Proof. Assume \( X \) is Jacobson and that \( T \subset |X| \) is a closed subset. By Morphisms of Spaces, Lemma \[25.6\] we see that \( T \cap X_{\text{ft-pts}} \) is dense in \( T \). By Lemma \[23.3\] we see that \( X_{\text{ft-pts}} \) is the closed points of \( |X| \). Thus \( |X| \) is indeed Jacobson.

Assume \( |X| \) is Jacobson. Let \( f : U \to X \) be an étale morphism with \( U \) an affine scheme. We have to show that \( U \) is Jacobson. If \( x \in |X| \) is closed, then the fibre \( F = f^{-1}(\{x\}) \) is a finite (by definition of decent) closed (by construction of the topology on \( |X| \)) subset of \( U \). Since there are no specializations between points of \( F \) (Lemma \[2.1\]) we conclude that every point of \( F \) is closed in \( U \). If \( U \) is not Jacobson, then there exists a non-closed point \( u \in U \) such that \( \{u\} \) is locally closed (Topology, Lemma \[18.3\]). We will show that \( f(u) \in |X| \) is closed; by the above \( u \) is closed in \( U \) which is a contradiction and finishes the proof. To prove this we may replace \( U \) by an affine open neighbourhood of \( u \). Thus we may assume that \( \{u\} \) is closed in \( U \). Let \( R = U \times_X U \) with projections \( s, t : R \to U \). Then \( s^{-1}(\{u\}) = \{r_1, \ldots, r_m\} \) is finite (by definition of decent spaces). After replacing \( U \) by a smaller affine open neighbourhood of \( u \) we may assume that \( t(r_j) = u \) for \( j = 1, \ldots, m \). It follows that \( \{u\} \) is an \( R \)-invariant closed subset of \( U \). Hence \( \{f(u)\} \) is a locally closed subset of \( X \) as it is closed in the open \( \{f(|U|)\} \) of \( |X| \). Since \( |X| \) is Jacobson we conclude that \( f(u) \) is closed in \( |X| \) as desired.

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Lemma 23.5. Let \( S \) be a scheme. Let \( X \) be a decent locally Noetherian algebraic space over \( S \). Let \( x \in |X| \). Then

\[
W = \{x' \in |X| : x' \sim x, \ x' \neq x\}
\]

is a Noetherian, spectral, sober, Jacobson topological space.

Proof. We may replace by any open subspace containing \( x \). Thus we may assume that \( X \) is quasi-compact. Then \( |X| \) is a Noetherian topological space (Properties
Thus $W$ is a Noetherian topological space (Topology, Lemma 9.2).

Combining Lemma 14.1 with Properties of Spaces, Lemma 15.2 we see that $|X|$ is a spectral topological space. By Topology, Lemma 24.7 we see that $W \cup \{x\}$ is a spectral topological space. Now $W$ is a quasi-compact open of $W \cup \{x\}$ and hence $W$ is spectral by Topology, Lemma 23.5.

Let $E \subset W$ be an irreducible closed subset. Then if $Z \subset |X|$ is the closure of $E$ we see that $x \in Z$. There is a unique generic point $\eta \in Z$ by Proposition 12.4. Of course $\eta \in W$ and hence $\eta \in E$. We conclude that $E$ has a unique generic point, i.e., $W$ is sober.

Let $x' \in W$ be a point such that $\{x'\}$ is locally closed in $W$. To finish the proof we have to show that $x'$ is a closed point of $W$. If not, then there exists a nontrivial specialization $x' \rightsquigarrow x_1$ in $W$. Let $U$ be an affine scheme, $u \in U$ a point, and let $U \to X$ be an étale morphism mapping $u$ to $x$. By Lemma 12.2 we can choose specializations $u' \rightsquigarrow u_1' \rightsquigarrow u$ mapping to $x' \rightsquigarrow x_1' \rightsquigarrow x$. Let $p' \subset O_{U,u}$ be the prime ideal corresponding to $u'$. The existence of the specializations implies that $\dim(O_{U,u}/p') \geq 2$. Hence every nonempty open of $\Spec(O_{U,u}/p')$ is infinite by Algebra, Lemma 61.1. By Lemma 12.1 we obtain a continuous map

$$\Spec(O_{U,u}/p') \setminus \{m_u/p'\} \to W$$

Since the generic point of the LHS maps to $x'$ the image is contained in $\overline{\{x'\}}$. We conclude the inverse image of $\overline{\{x'\}}$ under the displayed arrow is nonempty open hence infinite. However, the fibres of $U \to X$ are finite as $X$ is decent and we conclude that $\{x'\}$ is infinite. This contradiction finishes the proof. \qed

### 24. Local irreducibility

We have already defined the geometric number of branches of an algebraic space at a point in Properties of Spaces, Section 23. The number of branches of an algebraic space at a point can only be defined for decent algebraic spaces.

**Lemma 24.1.** Let $S$ be a scheme. Let $X$ be a decent algebraic space over $S$. Let $x \in |X|$ be a point. The following are equivalent

1. for any elementary étale neighbourhood $(U,u) \to (X,x)$ the local ring $O_{U,u}$ has a unique minimal prime,
2. for any elementary étale neighbourhood $(U,u) \to (X,x)$ there is a unique irreducible component of $U$ through $u$,
3. for any elementary étale neighbourhood $(U,u) \to (X,x)$ the local ring $O_{U,u}$ is unibranch,
4. the henselian local ring $O_{X,x}$ has a unique minimal prime.

**Proof.** The equivalence of (1) and (2) follows from the fact that irreducible components of $U$ passing through $u$ are in 1-1 correspondence with minimal primes of the local ring of $U$ at $u$. The ring $O_{X,x}$ is the henselization of $O_{U,u}$, see discussion following Definition 11.7. In particular (3) and (4) are equivalent by More on Algebra, Lemma 105.3. The equivalence of (2) and (3) follows from More on Morphisms, Lemma 33.2. \qed
Definition 24.2. Let \( S \) be a scheme. Let \( X \) be a decent algebraic space over \( S \). Let \( x \in |X| \). We say that \( X \) is unibranch at \( x \) if the equivalent conditions of Lemma 24.1 hold. We say that \( X \) is unibranch if \( X \) is unibranch at every \( x \in |X| \).

This is consistent with the definition for schemes (Properties, Definition 15.1).

Lemma 24.3. Let \( S \) be a scheme. Let \( X \) be a decent algebraic space over \( S \). Let \( x \in |X| \) be a point. Let \( n \in \{1, 2, \ldots \} \) be an integer. The following are equivalent

1. for any elementary étale neighbourhood \((U,u) \to (X,x)\) the number of minimal primes of the local ring \(\mathcal{O}_{U,u}\) is \(\leq n\) and for at least one choice of \((U,u)\) it is \(n\),
2. for any elementary étale neighbourhood \((U,u) \to (X,x)\) the number irreducible components of \(U\) passing through \(u\) is \(\leq n\) and for at least one choice of \((U,u)\) it is \(n\),
3. for any elementary étale neighbourhood \((U,u) \to (X,x)\) the number of branches of \(U\) at \(u\) is \(\leq n\) and for at least one choice of \((U,u)\) it is \(n\),
4. the number of minimal prime ideals of \(\mathcal{O}_{X,x}^h\) is \(n\).

Proof. The equivalence of (1) and (2) follows from the fact that irreducible components of \(U\) passing through \(u\) are in 1-1 correspondence with minimal primes of the local ring of \(U\) at \(u\). The ring \(\mathcal{O}_{X,x}^h\) is the henselization of \(\mathcal{O}_{U,u}\), see discussion following Definition 11.7. In particular (3) and (4) are equivalent by More on Algebra, Lemma 105.3.

The equivalence of (2) and (3) follows from More on Morphisms, Lemma 33.2. \(\square\)

Definition 24.4. Let \( S \) be a scheme. Let \( X \) be a decent algebraic space over \( S \). Let \( x \in |X| \). The number of branches of \(X\) at \(x\) is either \(n \in \mathbb{N}\) if the equivalent conditions of Lemma 24.3 hold, or else \(\infty\).

25. Catenary algebraic spaces

This section extends the material in Properties, Section 11 and Morphisms, Section 17 to algebraic spaces.

Definition 25.1. Let \( S \) be a scheme. Let \( X \) be a decent algebraic space over \( S \). We say \( X \) is catenary if \(|X|\) is catenary (Topology, Definition 11.4).

If \(X\) is representable, then this is equivalent to the corresponding notion for the scheme representing \(X\).

Lemma 25.2. Let \( S \) be a locally Noetherian and universally catenary scheme. Let \( \delta : S \to \mathbb{Z} \) be a dimension function. Let \( X \) be a decent algebraic space over \( S \) such that the structure morphism \( X \to S \) is locally of finite type. Let \( \delta_X : |X| \to \mathbb{Z} \) be the map sending \( x \) to \(\delta(f(x)) + \text{the transcendence degree of } x/f(x) \). Then \( \delta_X \) is a dimension function on \(|X|\).

Proof. Let \( \varphi : U \to X \) be a surjective étale morphism where \( U \) is a scheme. Then the similarly defined function \( \delta_U \) is a dimension function on \( U \) by Morphisms, Lemma 32.3. On the other hand, by the definition of relative transcendence degree in (Morphisms of Spaces, Definition 33.1) we see that \( \delta_U(u) = \delta_X(\varphi(u)) \).

Let \( x \leadsto x' \) be a specialization of points in \(|X|\). by Lemma 12.2 we can find a specialization \( u \leadsto u' \) of points of \( U \) with \( \varphi(u) = x \) and \( \varphi(u') = x' \). Moreover, we see that \( x = x' \) if and only if \( u = u' \), see Lemma 12.1. Thus the fact that...
δ_U is a dimension function implies that δ_X is a dimension function, see Topology, Definition 20.1.

Lemma 25.3. Let S be a locally Noetherian and universally catenary scheme. Let X be an algebraic space over S such that X is decent and such that the structure morphism X → S is locally of finite type. Then X is catenary.

Proof. The question is local on S (use Topology, Lemma 11.5). Thus we may assume that S has a dimension function, see Topology, Lemma 20.4. Then we conclude that |X| has a dimension function by Lemma 25.2. Since |X| is sober (Proposition 12.4) we conclude that |X| is catenary by Topology, Lemma 20.2.

By Lemma 25.3, the following definition is compatible with the already existing notion for representable algebraic spaces.

Definition 25.4. Let S be a scheme. Let X be a decent and locally Noetherian algebraic space over S. We say X is universally catenary if for every morphism Y → X of algebraic spaces which is locally of finite type and with Y decent, the algebraic space Y is catenary.

If X is an algebraic space, then the condition “X is decent and locally Noetherian” is equivalent to “X is quasi-separated and locally Noetherian”. This is Lemma 14.1. Thus another way to understand the definition above is that X is universally catenary if and only if Y is catenary for all morphisms Y → X which are quasi-separated and locally of finite type.

Lemma 25.5. Let S be a scheme. Let X be a decent, locally Noetherian, and universally catenary algebraic space over S. Then any decent algebraic space locally of finite type over X is universally catenary.

Proof. This is formal from the definitions and the fact that compositions of morphisms locally of finite type are locally of finite type (Morphisms of Spaces, Lemma 23.2).

Lemma 25.6. Let S be a scheme. Let f : Y → X be a surjective finite morphism of decent and locally Noetherian algebraic spaces. Let δ : |X| → Z be a function. If δ ◦ |f| is a dimension function, then δ is a dimension function.

Proof. Let x ↦ x′, x ≠ x′ be a specialization in |X|. Choose y ∈ |Y| with |f|(y) = x. Since |f| is closed (Morphisms of Spaces, Lemma 45.9) we find a specialization y ↔ y′ with |f|(y′) = x′. Thus we conclude that δ(x) = δ(|f|(y)) > δ(|f|(y′)) = δ(x′) (see Topology, Definition 20.1). If x ↼ x′ is an immediate specialization, then y ↔ y′ is an immediate specialization too: namely if y ↔ y'' ↔ y′, then |f|(y'') must be either x or x′ and there are no nontrivial specializations between points of fibres of |f| by Lemma 18.10.

The discussion will be continued in More on Morphisms of Spaces, Section 32.

26. Other chapters

Preliminaries

(1) Introduction

(2) Conventions

(3) Set Theory

(4) Categories

(5) Topology

(6) Sheaves on Spaces
Decent Algebraic Spaces

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Quot and Hilbert Spaces
Properties of Algebraic Stacks
Morphisms of Algebraic Stacks
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