DE RHAM COHOMOLOGY

0FK4

Contents

1. Introduction

0FK5 In this chapter we start with a discussion of the de Rham complex of a morphism of schemes and we end with a proof that de Rham cohomology defines a Weil cohomology theory when the base field has characteristic zero.

2. The de Rham complex

07HX Let $p : X \to S$ be a morphism of schemes. There is a complex

$$\Omega^\bullet_{X/S} = \mathcal{O}_{X/S} \to \Omega^1_{X/S} \to \Omega^2_{X/S} \to \ldots$$

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of $p^{-1}\mathcal{O}_S$-modules with $\Omega^i_{X/S} = \wedge^i(\Omega_{X/S})$ placed in degree i and differential determined by the rule $\mathrm{d}(g_0\mathrm{d}g_1 \wedge \ldots \wedge \mathrm{d}g_p) = \mathrm{d}g_0 \wedge \mathrm{d}g_1 \wedge \ldots \wedge \mathrm{d}g_p$ on local sections. See Modules, Section 27.

Given a commutative diagram

$$\begin{tikzcd} X' \arrow[r, "f"'] \arrow[d] & X \arrow[d] \\ S' \arrow[r] & S \end{tikzcd}$$

of schemes, there are canonical maps of complexes $f^{-1}\Omega^\bullet_{X/S} \to \Omega^\bullet_{X'/S'}$ and $\Omega^\bullet_{X/S} \to f_*\Omega^\bullet_{X'/S'}$. See Modules, Section 27. Linearizing, for every p we obtain a linear map $f^*\Omega^p_{X/S} \to \Omega^p_{X'/S'}$.

In particular, if $f : Y \to X$ be a morphism of schemes over a base scheme S, then there is a map of complexes

$$\Omega^\bullet_{X/S} \longrightarrow f_*\Omega^\bullet_{Y/S}$$

Linearizing, we see that for every $p \geq 0$ we obtain a canonical map

$$\Omega^p_{X/S} \otimes_{\mathcal{O}_X} f_*\mathcal{O}_Y \longrightarrow f_*\Omega^p_{Y/S}$$

0FL5 **Lemma 2.1.** *Let*

$$\begin{tikzcd} X' \arrow[r, "f"'] \arrow[d] & X \arrow[d] \\ S' \arrow[r] & S \end{tikzcd}$$

be a cartesian diagram of schemes. Then the maps discussed above induce isomorphisms $f^*\Omega^p_{X/S} \to \Omega^p_{X'/S'}$.

Proof. Combine Morphisms, Lemma 31.10 with the fact that formation of exterior power commutes with base change. □

0FLV **Lemma 2.2.** *Consider a commutative diagram of schemss*

$$\begin{tikzcd} X' \arrow[r, "f"'] \arrow[d] & X \arrow[d] \\ S' \arrow[r] & S \end{tikzcd}$$

If $X' \to X$ and $S' \to S$ are étale, then the maps discussed above induce isomorphisms $f^*\Omega^p_{X/S} \to \Omega^p_{X'/S'}$.

Proof. We have $\Omega_{S'/S} = 0$ and $\Omega_{X'/X} = 0$, see for example Morphisms, Lemma 34.15. Then by the short exact sequences of Morphisms, Lemmas 31.9 and 32.16 we see that $\Omega_{X'/S'} = \Omega_{X'/S} = f^*\Omega_{X/S}$. Taking exterior powers we conclude. □

3. de Rham cohomology

0FL6 Let $p : X \to S$ be a morphism of schemes. We define the *de Rham cohomology of X over S* to be the cohomology groups

$$H^i_{dR}(X/S) = H^i(R\Gamma(X, \Omega^\bullet_{X/S}))$$

Since $\Omega^\bullet_{X/S}$ is a complex of $p^{-1}\mathcal{O}_S$-modules, these cohomology groups are naturally modules over $H^0(S, \mathcal{O}_S)$.

Given a commutative diagram

$$\begin{tikzcd} X' \arrow[r, "f"'] \arrow[d] & X \arrow[d] \\ S' \arrow[r] & S \end{tikzcd}$$

of schemes, using the canonical maps of Section 2 we obtain pullback maps

$$f^* : R\Gamma(X, \Omega^\bullet_{X/S}) \longrightarrow R\Gamma(X', \Omega^\bullet_{X'/S'})$$

and

$$f^* : H^i_{dR}(X/S) \longrightarrow H^i_{dR}(X'/S')$$

These pullbacks satisfy an obvious composition law. In particular, if we work over a fixed base scheme S, then de Rham cohomology is a contravariant functor on the category of schemes over S.

0FLW **Lemma 3.1.** *Let $X \to S$ be a morphism of affine schemes given by the ring map $R \to A$. Then $R\Gamma(X, \Omega^\bullet_{X/S}) = \Omega^\bullet_{A/R}$ in $D(R)$ and $H^i_{dR}(X/S) = H^i(\Omega^\bullet_{A/R})$.*

Proof. This follows from Cohomology of Schemes, Lemma 2.2 and Leray's acyclicity lemma (Derived Categories, Lemma 16.7). $\square$

0FLX **Lemma 3.2.** *Let $p : X \to S$ be a morphism of schemes. If p is quasi-compact and quasi-separated, then $Rp_*\Omega^\bullet_{X/S}$ is an object of $D_{QCoh}(\mathcal{O}_S)$.*

Proof. There is a spectral sequence with first page $E_1^{a,b} = R^a p_* \Omega^q_{X/S}$ converging to $Rp_*\Omega^\bullet_{X/S}$ (see Derived Categories, Lemma 21.3). Hence by Homology, Lemma 25.3 it suffices to show that $R^a p_* \Omega^q_{X/S}$ is quasi-coherent. This follows from Cohomology of Schemes, Lemma 4.5. $\square$

0FLY **Lemma 3.3.** *Let $p : X \to S$ be a proper morphism of schemes with S locally Noetherian. Then $Rp_*\Omega^\bullet_{X/S}$ is an object of $D_{Coh}(\mathcal{O}_S)$.*

Proof. In this case by Morphisms, Lemma 31.12 the modules $\Omega^i_{X/S}$ are coherent. Hence we can use exactly the same argument as in the proof of Lemma 3.2 using Cohomology of Schemes, Proposition 19.1. $\square$

0FLZ **Lemma 3.4.** *Let A be a Noetherian ring. Let X be a proper scheme over $S = \mathrm{Spec}(A)$. Then $H^i_{dR}(X/S)$ is a finite A-module for all i.*

Proof. This is a special case of Lemma 3.3. $\square$

0FM0 **Lemma 3.5.** *Let $f : X \to S$ be a proper smooth morphism of schemes. Then $Rf_*\Omega^p_{X/S}$, $p \geq 0$ and $Rf_*\Omega^\bullet_{X/S}$ are perfect objects of $D(\mathcal{O}_S)$ whose formation commutes with arbitrary change of base.*

Proof. Since f is smooth the modules $\Omega^p_{X/S}$ are finite locally free $\mathcal{O}_X$-modules, see Morphisms, Lemma 32.12. Their formation commutes with arbitrary change of base by Lemma 2.1. Hence $Rf_*\Omega^p_{X/S}$ is a perfect object of $D(\mathcal{O}_S)$ whose formation commutes with abitrary base change, see Derived Categories of Schemes, Lemma 27.4. This proves the first assertion of the lemma.

To prove that $Rf_*\Omega^\bullet_{X/S}$ is perfect on S we may work locally on S. Thus we may assume S is quasi-compact. This means we may assume that $\Omega^n_{X/S}$ is zero for n large enough. For every $p \geq 0$ we claim that $Rf_*\sigma_{\geq p}\Omega^\bullet_{X/S}$ is a perfect object of $D(\mathcal{O}_S)$ whose formation commutes with arbitrary change of base. By the above we see that this is true for $p \gg 0$. Suppose the claim holds for p and consider the distinguished triangle

$$\sigma_{\geq p}\Omega^\bullet_{X/S} \to \sigma_{\geq p-1}\Omega^\bullet_{X/S} \to \Omega^{p-1}_{X/S}[-(p-1)] \to (\sigma_{\geq p}\Omega^\bullet_{X/S})[1]$$

in $D(f^{-1}\mathcal{O}_S)$. Applying the exact functor Rf_* we obtain a distinguished triangle in $D(\mathcal{O}_S)$. Since we have the 2-out-of-3 property for being perfect (Cohomology, Lemma 45.7) we conclude $Rf_*\sigma_{\geq p-1}\Omega^\bullet_{X/S}$ is a perfect object of $D(\mathcal{O}_S)$. Similarly for the commutation with arbitrary base change. $\square$

4. Cup product

0FM1 Consider the maps $\Omega^p_{X/S} \times \Omega^q_{X/S} \to \Omega^{p+q}_{X/S}$ given by $(\omega, \eta) \longmapsto \omega \wedge \eta$. Using the formula for d given in Section 2 and the Leibniz rule for $\mathrm{d} : \mathcal{O}_X \to \Omega_{X/S}$ we see that $\mathrm{d}(\omega \wedge \eta) = \mathrm{d}(\omega) \wedge \eta + (-1)^{\deg(\omega)}\omega \wedge \mathrm{d}(\eta)$. This means that $\wedge$ defines a morphism

0FM2 (4.0.1)
$$\wedge : \mathrm{Tot}(\Omega^\bullet_{X/S} \otimes_{p^{-1}\mathcal{O}_S} \Omega^\bullet_{X/S}) \longrightarrow \Omega^\bullet_{X/S}$$

of complexes of $p^{-1}\mathcal{O}_S$-modules.

Combining the cup product of Cohomology, Section 31 with (4.0.1) we find a $H^0(S, \mathcal{O}_S)$-bilinear cup product map

$$\cup : H^i_{dR}(X/S) \times H^j_{dR}(X/S) \longrightarrow H^{i+j}_{dR}(X/S)$$

For example, if $\omega \in \Gamma(X, \Omega^i_{X/S})$ and $\eta \in \Gamma(X, \Omega^j_{X/S})$ are closed, then the cup product of the de Rham cohomology classes of ω and η is the de Rham cohomology class of $\omega \wedge \eta$, see discussion in Cohomology, Section 31.

Given a commutative diagram

$$\begin{tikzcd} X' \arrow[r, "f"] \arrow[d] & X \arrow[d] \\ S' \arrow[r] & S \end{tikzcd}$$

of schemes, the pullback maps $f^* : R\Gamma(X, \Omega^\bullet_{X/S}) \to R\Gamma(X', \Omega^\bullet_{X'/S'})$ and $f^* : H^i_{dR}(X/S) \longrightarrow H^i_{dR}(X'/S')$ are compatible with the cup product defined above.

0FM3 **Lemma 4.1.** *Let $p : X \to S$ be a morphism of schemes. The cup product on $H^*_{dR}(X/S)$ is associative and graded commutative.*

Proof. This follows from Cohomology, Lemmas 31.4 and 31.5 and the fact that $\wedge$ is associative and graded commutative. $\square$

0FU6 **Remark 4.2.** Let $p : X \to S$ be a morphism of schemes. Then we can think of $\Omega^\bullet_{X/S}$ as a sheaf of differential graded $p^{-1}\mathcal{O}_S$-algebras, see Differential Graded Sheaves, Definition 12.1. In particular, the discussion in Differential Graded Sheaves, Section 32 applies. For example, this means that for any commutative diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "p"'] & Y \arrow[d, "q"] \\ S \arrow[r, "h"] & T \end{tikzcd}$$

of schemes there is a canonical relative cup product

$$\mu : Rf_*\Omega^\bullet_{X/S} \otimes^{\mathbf{L}}_{q^{-1}\mathcal{O}_T} Rf_*\Omega^\bullet_{X/S} \longrightarrow Rf_*\Omega^\bullet_{X/S}$$

in $D(Y, q^{-1}\mathcal{O}_T)$ which is associative and which on cohomology reproduces the cup product discussed above.

0FU7 **Remark 4.3.** Let $f : X \to S$ be a morphism of schemes. Let $\xi \in H^n_{dR}(X/S)$. According to the discussion Differential Graded Sheaves, Section 32 there exists a canonical morphism

$$\xi' : \Omega^\bullet_{X/S} \to \Omega^\bullet_{X/S}[n]$$

in $D(f^{-1}\mathcal{O}_S)$ uniquely characterized by (1) and (2) of the following list of properties:

1. ξ' can be lifted to a map in the derived category of right differential graded $\Omega^\bullet_{X/S}$-modules, and
2. $\xi'(1) = \xi$ in $H^0(X, \Omega^\bullet_{X/S}[n]) = H^n_{dR}(X/S)$,
3. the map ξ' sends $\eta \in H^m_{dR}(X/S)$ to $\xi \cup \eta$ in $H^{n+m}_{dR}(X/S)$,
4. the construction of ξ' commutes with restrictions to opens: for $U \subset X$ open the restriction $\xi'|_U$ is the map corresponding to the image $\xi|_U \in H^n_{dR}(U/S)$,
5. for any diagram as in Remark 4.2 we obtain a commutative diagram

$$\begin{tikzcd} Rf_*\Omega^\bullet_{X/S} \otimes^{\mathbf{L}}_{q^{-1}\mathcal{O}_T} Rf_*\Omega^\bullet_{X/S} \arrow[r, "\mu"'] \arrow[d, "\xi'\otimes\mathrm{id}"'] & Rf_*\Omega^\bullet_{X/S} \arrow[d, "\xi'"] \\ Rf_*\Omega^\bullet_{X/S}[n] \otimes^{\mathbf{L}}_{q^{-1}\mathcal{O}_T} Rf_*\Omega^\bullet_{X/S} \arrow[r, "\mu"] & Rf_*\Omega^\bullet_{X/S}[n] \end{tikzcd}$$

in $D(Y, q^{-1}\mathcal{O}_T)$.

5. Hodge cohomology

0FM4 Let $p : X \to S$ be a morphism of schemes. We define the *de Hodge cohomology of X over S* to be the cohomology groups

$$H^n_{Hodge}(X/S) = \bigoplus\nolimits_{n=p+q} H^q(X, \Omega^p_{X/S})$$

viewed as a graded $H^0(X, \mathcal{O}_X)$-module. The wedge product of forms combined with the cup product of Cohomology, Section 31 defines a $H^0(X, \mathcal{O}_X)$-bilinear cup product

$$\cup : H^i_{Hodge}(X/S) \times H^j_{Hodge}(X/S) \longrightarrow H^{i+j}_{Hodge}(X/S)$$

Of course if $\xi \in H^q(X, \Omega^p_{X/S})$ and $\xi' \in H^{q'}(X, \Omega^{p'}_{X/S})$ then $\xi \cup \xi' \in H^{q+q'}(X, \Omega^{p+p'}_{X/S})$.

0FM5 **Lemma 5.1.** *Let $p : X \to S$ be a morphism of schemes. The cup product on $H^*_{Hodge}(X/S)$ is associative and graded commutative.*

Proof. The proof is identical to the proof of Lemma 4.1. □

Given a commutative diagram

$$\begin{tikzcd} X' \arrow[r, "f"'] \arrow[d] & X \arrow[d] \\ S' \arrow[r] & S \end{tikzcd}$$

of schemes, there are pullback maps $f^* : H^i_{Hodge}(X/S) \longrightarrow H^i_{Hodge}(X'/S')$ compatible with gradings and with the cup product defined above.

6. Two spectral sequences

0FM6 Let $p : X \to S$ be a morphism of schemes. Since the category of $p^{-1}\mathcal{O}_S$-modules on X has enough injectives there exist a Cartan-Eilenberg resolution for $\Omega^\bullet_{X/S}$. See Derived Categories, Lemma 21.2. Hence we can apply Derived Categories, Lemma 21.3 to get two spectral sequences both converging to the de Rham cohomology of X over S.

The first is customarily called *the Hodge-to-de Rham spectral sequence*. The first page of this spectral sequence has

$$E_1^{p,q} = H^q(X, \Omega^p_{X/S})$$

which are the Hodge cohomology groups of X/S (whence the name). The differential d_1 on this page is given by the maps $d_1^{p,q} : H^q(X, \Omega^p_{X/S}) \to H^q(X.\Omega^{p+1}_{X/S})$ induced by the differential $\mathrm{d} : \Omega^p_{X/S} \to \Omega^{p+1}_{X/S}$. Here is a picture

$$\begin{tikzcd} H^2(X, \mathcal{O}_X) \arrow[r] \arrow[drr, dashed] \arrow[ddrrr, dotted] & H^2(X, \Omega^1_{X/S}) \arrow[r] \arrow[drr, dashed] & H^2(X, \Omega^2_{X/S}) \arrow[r] & H^2(X, \Omega^3_{X/S}) \\ H^1(X, \mathcal{O}_X) \arrow[r] \arrow[drr, dashed] & H^1(X, \Omega^1_{X/S}) \arrow[r] \arrow[drr, dashed] & H^1(X, \Omega^2_{X/S}) \arrow[r] & H^1(X, \Omega^3_{X/S}) \\ H^0(X, \mathcal{O}_X) \arrow[r] & H^0(X, \Omega^1_{X/S}) \arrow[r] & H^0(X, \Omega^2_{X/S}) \arrow[r] & H^0(X, \Omega^3_{X/S}) \end{tikzcd}$$

where we have drawn striped arrows to indicate the source and target of the differentials on the E_2 page and a dotted arrow for a differential on the E_3 page. Looking in degree 0 we conclude that

$$H^0_{dR}(X/S) = \mathrm{Ker}(\mathrm{d} : H^0(X, \mathcal{O}_X) \to H^0(X, \Omega^1_{X/S}))$$

Of course, this is also immediately clear from the fact that the de Rham complex starts in degree 0 with $\mathcal{O}_X \to \Omega^1_{X/S}$.

The second spectral sequence is usually called *the conjugate spectral sequence*. The second page of this spectral sequence has

$$E_2^{p,q} = H^p(X, H^q(\Omega^\bullet_{X/S})) = H^p(X, \mathcal{H}^q)$$

where $\mathcal{H}^q = H^q(\Omega^\bullet_{X/S})$ is the qth cohomology sheaf of the de Rham complex of X/S. The differentials on this page are given by $E_2^{p,q} \to E_2^{p+2,q-1}$. Here is a picture

$$
\begin{tikzcd}
H^0(X,\mathcal{H}^2) \arrow[ddrrr, dotted] \arrow[drr] & H^1(X,\mathcal{H}^2) \arrow[drr] & H^2(X,\mathcal{H}^2) & H^3(X,\mathcal{H}^2) \\
H^0(X,\mathcal{H}^1) \arrow[drr] & H^1(X,\mathcal{H}^1) \arrow[drr] & H^2(X,\mathcal{H}^1) & H^3(X,\mathcal{H}^1) \\
H^0(X,\mathcal{H}^0) & H^1(X,\mathcal{H}^0) & H^2(X,\mathcal{H}^0) & H^3(X,\mathcal{H}^0)
\end{tikzcd}
$$

Looking in degree 0 we conclude that

$$H^0_{dR}(X/S) = H^0(X, \mathcal{H}^0)$$

which is obvious if you think about it. In degree 1 we get an exact sequence

$$0 \to H^1(X, \mathcal{H}^0) \to H^1_{dR}(X/S) \to H^0(X, \mathcal{H}^1) \to H^2(X, \mathcal{H}^0) \to H^2_{dR}(X/S)$$

It turns out that if $X \to S$ is smooth and S lives in characteristic p, then the sheaves $\mathcal{H}^q$ are computable (in terms of a certain sheaves of differentials) and the conjugate spectral sequence is a valuable tool (insert future reference here).

7. The Hodge filtration

0FM7 Let $X \to S$ be a morphism of schemes. The Hodge filtration on $H^n_{dR}(X/S)$ is the filtration induced by the Hodge-to-de Rham spectral sequence (Homology, Definition 24.5). To avoid misunderstanding, we explicitly define it as follows.

0FM8 **Definition 7.1.** Let $X \to S$ be a morphism of schemes. The *Hodge filtration* on $H^n_{dR}(X/S)$ is the filtration with terms

$$F^p H^n_{dR}(X/S) = \mathrm{Im}\left(H^n(X, \sigma_{\geq p}\Omega^\bullet_{X/S}) \longrightarrow H^n_{dR}(X/S)\right)$$

where $\sigma_{\geq p}\Omega^\bullet_{X/S}$ is as in Homology, Section 15.

Of course $\sigma_{\geq p}\Omega^\bullet_{X/S}$ is a subcomplex of the relative de Rham complex and we obtain a filtration

$$\Omega^\bullet_{X/S} = \sigma_{\geq 0}\Omega^\bullet_{X/S} \supset \sigma_{\geq 1}\Omega^\bullet_{X/S} \supset \sigma_{\geq 2}\Omega^\bullet_{X/S} \supset \sigma_{\geq 3}\Omega^\bullet_{X/S} \supset \ldots$$

of the relative de Rham complex with $\mathrm{gr}^p(\Omega^\bullet_{X/S}) = \Omega^p_{X/S}[-p]$. The spectral sequence constructed in Cohomology, Lemma 29.1 for $\Omega^\bullet_{X/S}$ viewed as a filtered complex of sheaves is the same as the Hodge-to-de Rham spectral sequence constructed in Section 6 by Cohomology, Example 29.4. Further the wedge product (4.0.1) sends $\mathrm{Tot}(\sigma_{\geq i}\Omega^\bullet_{X/S} \otimes_{p^{-1}\mathcal{O}_S} \sigma_{\geq j}\Omega^\bullet_{X/S})$ into $\sigma_{\geq i+j}\Omega^\bullet_{X/S}$. Hence we get commutative diagrams

$$
\begin{tikzcd}
H^n(X, \sigma_{\geq j}\Omega^\bullet_{X/S})) \times H^m(X, \sigma_{\geq j}\Omega^\bullet_{X/S})) \arrow[r] \arrow[d] & H^{n+m}(X, \sigma_{\geq i+j}\Omega^\bullet_{X/S})) \arrow[d] \\
H^n_{dR}(X/S) \times H^m_{dR}(X/S) \arrow[r, "\cup"] & H^{n+m}_{dR}(X/S)
\end{tikzcd}
$$

In particular we find that

$$F^i H^n_{dR}(X/S) \cup F^j H^m_{dR}(X/S) \subset F^{i+j} H^{n+m}_{dR}(X/S)$$

8. Künneth formula

0FM9 An important feature of de Rham cohomology is that there is a Künneth formula.

Let $a : X \to S$ and $b : Y \to S$ be morphisms of schemes with the same target. Let $p : X \times_S Y \to X$ and $q : X \times_S Y \to Y$ be the projection morphisms and $f = a \circ p = b \circ q$. Here is a picture

$$\begin{tikzcd} & X \times_S Y \arrow[ld, "p"'] \arrow[rd, "q"] \arrow[dd, "f"] & \\ X \arrow[rd, "a"'] & & Y \arrow[ld, "b"] \\ & S & \end{tikzcd}$$

0FMA **Lemma 8.1.** *In the situation above there is a canonical isomorphism*

$$Tot(p^{-1}\Omega^\bullet_{X/S} \otimes_{f^{-1}\mathcal{O}_S} q^{-1}\Omega^\bullet_{Y/S}) \longrightarrow \Omega^\bullet_{X \times_S Y/S}$$

of complexes of $f^{-1}\mathcal{O}_S$-modules.

Proof. By Derived Categories of Schemes, Remark 22.2 we have

$$p^{-1}\Omega^i_{X/S} \otimes_{f^{-1}\mathcal{O}_S} q^{-1}\Omega^j_{Y/S} = p^*\Omega^i_{X/S} \otimes_{\mathcal{O}_{X \times_S Y}} q^*\Omega^j_{Y/S}$$

for all i, j. On the other hand, we know that $\Omega_{X \times_S Y/S} = p^*\Omega_{X/S} \oplus q^*\Omega_{Y/S}$ by Morphisms, Lemma 31.11. Taking exterior powers we obtain

$$\Omega^n_{X \times_S Y/S} = \bigoplus\nolimits_{i+j=n} p^*\Omega^i_{X/S} \otimes_{\mathcal{O}_{X \times_S Y}} q^*\Omega^j_{Y/S} = \bigoplus\nolimits_{i+j=n} p^{-1}\Omega^i_{X/S} \otimes_{f^{-1}\mathcal{O}_S} q^{-1}\Omega^j_{Y/S}$$

by elementary properties of exterior powers. This finishes the proof. □

If $S = \mathrm{Spec}(A)$ is affine, then combining the result of Lemma 8.1 with the cup product map of Derived Categories of Schemes, Equation (22.4.1) we obtain a cup product

$$R\Gamma(X, \Omega^\bullet_{X/S}) \otimes_A^{\mathbf{L}} R\Gamma(Y, \Omega^\bullet_{Y/S}) \longrightarrow R\Gamma(X \times_S Y, \Omega^\bullet_{X \times_S Y/S})$$

On the level of cohomology, using the discussion in More on Algebra, Section 61, we obtain a canonical map

$$H^i_{dR}(X/S) \otimes_A H^j_{dR}(Y/S) \longrightarrow H^{i+j}_{dR}(X \times_S Y/S), \quad (\xi, \zeta) \longmapsto p^*\xi \cup q^*\zeta$$

We note that the construction above indeed proceeds by first pulling back and then taking the cup product.

0FMB **Lemma 8.2.** *Assume X and Y are smooth, quasi-compact, and quasi-separated over $S = \mathrm{Spec}(A)$. Then the map*

$$R\Gamma(X, \Omega^\bullet_{X/S}) \otimes_A^{\mathbf{L}} R\Gamma(Y, \Omega^\bullet_{Y/S}) \longrightarrow R\Gamma(X \times_S Y, \Omega^\bullet_{X \times_S Y/S})$$

is an isomorphism in $D(A)$.

Proof. By Morphisms, Lemma 32.12 the sheaves $\Omega^n_{X/S}$ and $\Omega^m_{Y/S}$ are finite locally free $\mathcal{O}_X$ and $\mathcal{O}_Y$-modules. On the other hand, X and Y are flat over S (Morphisms, Lemma 32.9) and hence we find that $\Omega^n_{X/S}$ and $\Omega^m_{Y/S}$ are flat over S. Also, observe that $\Omega^\bullet_{X/S}$ is a locally bounded. Thus the result by Lemma 8.1 and Derived Categories of Schemes, Lemma 22.6. □

Given a possibly non-affine base scheme S we can do this construction over all affine opens and upon sheafification we obtain a relative cup product

$$Ra_*\Omega^\bullet_{X/S} \otimes^{\mathbf{L}}_{\mathcal{O}_S} Rb_*\Omega^\bullet_{Y/S} \longrightarrow Rf_*\Omega^\bullet_{X \times_S Y/S}$$

in $D(\mathcal{O}_S)$. We can also define this as the composition of the maps

$$\begin{tikzcd}
Ra_*\Omega^\bullet_{X/S} \otimes^{\mathbf{L}}_{\mathcal{O}_S} Rb_*\Omega^\bullet_{Y/S} \arrow[d, "\text{units of adjunction}"] & \\
Rf_*(p^{-1}\Omega^\bullet_{X/S}) \otimes^{\mathbf{L}}_{\mathcal{O}_S} Rf_*(q^{-1}\Omega^\bullet_{Y/S}) \arrow[r] \arrow[d, "\text{relative cup product}"] & Rf_*(\Omega^\bullet_{X \times_S Y/S}) \otimes^{\mathbf{L}}_{\mathcal{O}_S} Rf_*(\Omega^\bullet_{X \times_S Y/S}) \arrow[d, "\text{relative cup product}"] \\
Rf_*(p^{-1}\Omega^\bullet_{X/S} \otimes^{\mathbf{L}}_{f^{-1}\mathcal{O}_S} q^{-1}\Omega^\bullet_{Y/S}) \arrow[r] \arrow[d, "\text{from derived to usual}"] & Rf_*(\Omega^\bullet_{X \times_S Y/S} \otimes^{\mathbf{L}}_{f^{-1}\mathcal{O}_S} \Omega^\bullet_{X \times_S Y/S}) \arrow[d, "\text{from derived to usual}"] \\
Rf_*\mathrm{Tot}(p^{-1}\Omega^\bullet_{X/S} \otimes_{f^{-1}\mathcal{O}_S} q^{-1}\Omega^\bullet_{Y/S}) \arrow[r] \arrow[dr, equal] & Rf_*\mathrm{Tot}(\Omega^\bullet_{X \times_S Y/S} \otimes_{f^{-1}\mathcal{O}_S} \Omega^\bullet_{X \times_S Y/S}) \arrow[d] \\
& Rf_*\Omega^\bullet_{X \times_S Y/S}
\end{tikzcd}$$

Here the first arrow uses the units $\mathrm{id} \to Rp_*p^{-1}$ and $\mathrm{id} \to Rq_*q^{-1}$ of adjunction as well as the identifications $Rf_*p^{-1} = Ra_*Rp_*p^{-1}$ and $Rf_*q^{-1} = Rb_*Rq_*q^{-1}$. The second arrow is the relative cup product of Cohomology, Remark 28.7. The third arrow is the map sending a derived tensor product of complexes to the totalization of the tensor product of complexes. The final equality is Lemma 8.1. Using the identifications $R\Gamma(X, \Omega^\bullet_{X/S}) = R\Gamma(S, Ra_*\Omega^\bullet_{X/S})$ and $R\Gamma(Y, \Omega^\bullet_{Y/S}) = R\Gamma(S, Rb_*\Omega^\bullet_{Y/S})$ we obtain a map

$$R\Gamma(X, \Omega^\bullet_{X/S}) \otimes^{\mathbf{L}}_{H^0(S,\mathcal{O}_S)} R\Gamma(Y, \Omega^\bullet_{Y/S}) \to R\Gamma(S, Ra_*\Omega^\bullet_{X/S} \otimes^{\mathbf{L}}_{\mathcal{O}_S} Rb_*\Omega^\bullet_{Y/S})$$

by using the cup product of Cohomology, Section 31 on S. Using the relative cup product for de Rham cohomology constructed by the large diagram above and taking $R\Gamma(S, -)$ this produces a cup product

$$H^i_{dR}(X/S) \otimes_{H^0(S,\mathcal{O}_S)} H^j_{dR}(Y/S) \longrightarrow H^{i+j}_{dR}(X \times_S Y/S), \quad (\xi, \zeta) \longmapsto p^*\xi \cup q^*\zeta$$

which as indicated is given by pulling back and then cupping. The reader can deduce this from the commutativity of the diagram and the compatibility of relative cup product with composition of morphisms given in Cohomology, Lemma 31.6 (take the second morphism equal to the morphism to a point).

0FMC **Lemma 8.3.** *Assume X and Y are smooth, quasi-compact, and quasi-separated over S. Then the relative cup product*

$$Ra_*\Omega^\bullet_{X/S} \otimes^{\mathbf{L}}_{\mathcal{O}_S} Rb_*\Omega^\bullet_{Y/S} \longrightarrow Rf_*\Omega^\bullet_{X \times_S Y/S}$$

is an isomorphism in $D(\mathcal{O}_S)$.

Proof. Immediate consequence of Lemma 8.2. $\square$

9. First chern class in de Rham cohomology

0FLE Let $X \to S$ be a morphism of schemes. There is a map of complexes

$$\mathrm{d}\log : \mathcal{O}_X^*[-1] \longrightarrow \Omega^\bullet_{X/S}$$

which sends the section $g \in \mathcal{O}_X^*(U)$ to the section $\mathrm{d}\log(g) = g^{-1}\mathrm{d}g$ of $\Omega^1_{X/S}(U)$. Thus we can consider the map

$$\mathrm{Pic}(X) = H^1(X, \mathcal{O}_X^*) = H^2(X, \mathcal{O}_X^*[-1]) \longrightarrow H^2_{dR}(X/S)$$

where the first equality is Cohomology, Lemma 6.1. The image of the isomorphism class of the invertible module $\mathcal{L}$ is denoted $c_1^{dR}(\mathcal{L}) \in H^2_{dR}(X/S)$.

We can also use the map $\mathrm{d}\log : \mathcal{O}_X^* \to \Omega^1_{X/S}$ to define a chern class in Hodge cohomology

$$c_1^{Hodge} : \mathrm{Pic}(X) \longrightarrow H^1(X, \Omega^1_{X/S}) \subset H^2_{Hodge}(X/S)$$

These constructions are compatible with pullbacks.

0FMD **Lemma 9.1.** *Given a commutative diagram*

$$\begin{tikzcd} X' \arrow[r, "f"] \arrow[d] & X \arrow[d] \\ S' \arrow[r] & S \end{tikzcd}$$

of schemes the diagrams

$$\begin{tikzcd} \mathrm{Pic}(X') \arrow[d, "c_1^{dR}"'] & \mathrm{Pic}(X) \arrow[l, "f^*"'] \arrow[d, "c_1^{dR}"] \\ H^2_{dR}(X'/S') & H^2_{dR}(X/S) \arrow[l, "f^*"] \end{tikzcd} \qquad \begin{tikzcd} \mathrm{Pic}(X') \arrow[d, "c_1^{Hodge}"'] & \mathrm{Pic}(X) \arrow[l, "f^*"'] \arrow[d, "c_1^{Hodge}"] \\ H^1(X', \Omega^1_{X'/S'}) & H^1(X, \Omega^1_{X/S}) \arrow[l, "f^*"] \end{tikzcd}$$

commute.

Proof. Omitted. ☐

Let us "compute" the element $c_1^{dR}(\mathcal{L})$ in Čech cohomology (with sign rules for Čech differentials as in Cohomology, Section 25). Namely, choose an open covering $\mathcal{U} : X = \bigcup_{i \in I} U_i$ such that we have a trivializing section s_i of $\mathcal{L}|_{U_i}$ for all i. On the overlaps $U_{i_0i_1} = U_{i_0} \cap U_{i_1}$ we have an invertible function $f_{i_0i_1}$ such that $f_{i_0i_1} = s_{i_1}|_{U_{i_0i_1}} s_{i_0}|_{U_{i_0i_1}}^{-1}$ [1]. Of course we have

$$f_{i_1i_2}|_{U_{i_0i_1i_2}} f_{i_0i_2}^{-1}|_{U_{i_0i_1i_2}} f_{i_0i_1}|_{U_{i_0i_1i_2}} = 1$$

The cohomology class of $\mathcal{L}$ in $H^1(X, \mathcal{O}_X^*)$ is the image of the Čech cohomology class of the cocycle $\{f_{i_0i_1}\}$ in $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{O}_X^*)$. Therefore we see that $c_1^{dR}(\mathcal{L})$ is the image of the cohomology class associated to the Čech cocycle $\{\alpha_{i_0\ldots i_p}\}$ in $\mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \Omega^\bullet_{X/S}))$ of degree 2 given by

1. $\alpha_{i_0} = 0$ in $\Omega^2_{X/S}(U_{i_0})$,
2. $\alpha_{i_0i_1} = f_{i_0i_1}^{-1}\mathrm{d}f_{i_0i_1}$ in $\Omega^1_{X/S}(U_{i_0i_1})$, and
3. $\alpha_{i_0i_1i_2} = 0$ in $\mathcal{O}_{X/S}(U_{i_0i_1i_2})$.

[1] The Čech differential of a 0-cycle $\{a_{i_0}\}$ has $a_{i_1} - a_{i_0}$ over $U_{i_0i_1}$.

Suppose we have invertible modules $\mathcal{L}_k$, $k = 1, \ldots, a$ each trivialized over U_i for all $i \in I$ giving rise to cocycles f_{k,i_0i_1} and $\alpha_k = \{\alpha_{k,i_0\ldots i_p}\}$ as above. Using the rule in Cohomology, Section 25 we can compute

$$\beta = \alpha_1 \cup \alpha_2 \cup \ldots \cup \alpha_a$$

to be given by the cocycle $\beta = \{\beta_{i_0\ldots i_p}\}$ described as follows

(1) $\beta_{i_0\ldots i_p} = 0$ in $\Omega^{2a-p}_{X/S}(U_{i_0\ldots i_p})$ unless $p = a$, and
(2) $\beta_{i_0\ldots i_a} = (-1)^{a(a-1)/2}\alpha_{1,i_0i_1} \wedge \alpha_{2,i_1i_2} \wedge \ldots \wedge \alpha_{a,i_{a-1}i_a}$ in $\Omega^a_{X/S}(U_{i_0\ldots i_a})$.

Thus this is a cocycle representing $c_1^{dR}(\mathcal{L}_1) \cup \ldots \cup c_1^{dR}(\mathcal{L}_a)$ Of course, the same computation shows that the cocycle $\{\beta_{i_0\ldots i_a}\}$ in $\check{\mathcal{C}}^a(\mathcal{U}, \Omega^a_{X/S}))$ represents the cohomology class $c_1^{Hodge}(\mathcal{L}_1) \cup \ldots \cup c_1^{Hodge}(\mathcal{L}_a)$

0FME **Remark 9.2.** Here is a reformulation of the calculations above in more abstract terms. Let $p : X \to S$ be a morphism of schemes. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. If we view $\mathrm{d}\log$ as a map

$$\mathcal{O}_X^*[-1] \to \sigma_{\geq 1}\Omega^\bullet_{X/S}$$

then using $\mathrm{Pic}(X) = H^1(X, \mathcal{O}_X^*)$ as above we find a cohomology class

$$\gamma_1(\mathcal{L}) \in H^2(X, \sigma_{\geq 1}\Omega^\bullet_{X/S})$$

The image of $\gamma_1(\mathcal{L})$ under the map $\sigma_{\geq 1}\Omega^\bullet_{X/S} \to \Omega^\bullet_{X/S}$ recovers $c_1^{dR}(\mathcal{L})$. In particular we see that $c_1^{dR}(\mathcal{L}) \in F^1H^2_{dR}(X/S)$, see Section 7. The image of $\gamma_1(\mathcal{L})$ under the map $\sigma_{\geq 1}\Omega^\bullet_{X/S} \to \Omega^1_{X/S}[-1]$ recovers $c_1^{Hodge}(\mathcal{L})$. Taking the cup product (see Section 7) we obtain

$$\xi = \gamma_1(\mathcal{L}_1) \cup \ldots \cup \gamma_1(\mathcal{L}_a) \in H^{2a}(X, \sigma_{\geq a}\Omega^\bullet_{X/S})$$

The commutative diagrams in Section 7 show that ξ is mapped to $c_1^{dR}(\mathcal{L}_1) \cup \ldots \cup c_1^{dR}(\mathcal{L}_a)$ in $H^{2a}_{dR}(X/S)$ by the map $\sigma_{\geq a}\Omega^\bullet_{X/S} \to \Omega^\bullet_{X/S}$. Also, it follows $c_1^{dR}(\mathcal{L}_1) \cup \ldots \cup c_1^{dR}(\mathcal{L}_a)$ is contained in $F^aH^{2a}_{dR}(X/S)$. Similarly, the map $\sigma_{\geq a}\Omega^\bullet_{X/S} \to \Omega^a_{X/S}[-a]$ sends ξ to $c_1^{Hodge}(\mathcal{L}_1) \cup \ldots \cup c_1^{Hodge}(\mathcal{L}_a)$ in $H^a(X, \Omega^a_{X/S})$.

0FMF **Remark 9.3.** Let $p : X \to S$ be a morphism of schemes. For $i > 0$ denote $\Omega^i_{X/S,log} \subset \Omega^i_{X/S}$ the abelian subsheaf generated by local sections of the form

$$\mathrm{d}\log(u_1) \wedge \ldots \wedge \mathrm{d}\log(u_i)$$

where $u_1, \ldots, u_n$ are invertible local sections of $\mathcal{O}_X$. For $i = 0$ the subsheaf $\Omega^0_{X/S,log} \subset \mathcal{O}_X$ is the image of $\mathbf{Z} \to \mathcal{O}_X$. For every $i \geq 0$ we have a map of complexes

$$\Omega^i_{X/S,log}[-i] \longrightarrow \Omega^\bullet_{X/S}$$

because the derivative of a logarithmic form is zero. Moreover, wedging logarithmic forms gives another, hence we find bilinear maps

$$\wedge : \Omega^i_{X/S,log} \times \Omega^j_{X/S,log} \longrightarrow \Omega^{i+j}_{X/S,log}$$

compatible with (4.0.1) and the maps above. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Using the map of abelian sheaves $\mathrm{d}\log : \mathcal{O}_X^* \to \Omega^1_{X/S,log}$ and the identification $\mathrm{Pic}(X) = H^1(X, \mathcal{O}_X^*)$ we find a canonical cohomology class

$$\tilde{\gamma}_1(\mathcal{L}) \in H^1(X, \Omega^1_{X/S,log})$$

These classes have the following properties

(1) the image of $\tilde{\gamma}_1(\mathcal{L})$ under the canonical map $\Omega^1_{X/S,log}[-1] \to \sigma_{\geq 1}\Omega^\bullet_{X/S}$ sends $\tilde{\gamma}_1(\mathcal{L})$ to the class $\gamma_1(\mathcal{L}) \in H^2(X, \sigma_{\geq 1}\Omega^\bullet_{X/S})$ of Remark 9.2,
(2) the image of $\tilde{\gamma}_1(\mathcal{L})$ under the canonical map $\Omega^1_{X/S,log}[-1] \to \Omega^\bullet_{X/S}$ sends $\tilde{\gamma}_1(\mathcal{L})$ to $c_1^{dR}(\mathcal{L})$ in $H^2_{dR}(X/S)$,
(3) the image of $\tilde{\gamma}_1(\mathcal{L})$ under the canonical map $\Omega^1_{X/S,log} \to \Omega^1_{X/S}$ sends $\tilde{\gamma}_1(\mathcal{L})$ to $c_1^{Hodge}(\mathcal{L})$ in $H^1(X, \Omega^1_{X/S})$,
(4) the construction of these classes is compatible with pullbacks,
(5) add more here.

10. de Rham cohomology of a line bundle

0FU8 A line bundle is a special case of a vector bundle, which in turn is a cone endowed with some extra structure. To intelligently talk about the de Rham complex of these, it makes sense to discuss the de Rham complex of a graded ring.

0FU9 **Remark 10.1** (de Rham complex of a graded ring)**.** Let G be an abelian monoid written additively with neutral element 0. Let $R \to A$ be a ring map and assume A comes with a grading $A = \bigoplus_{g \in G} A_g$ by R-modules such that R maps into A_0 and $A_g \cdot A_{g'} \subset A_{g+g'}$. Then the module of differentials comes with a grading

$$\Omega_{A/R} = \bigoplus\nolimits_{g \in G} \Omega_{A/R,g}$$

where $\Omega_{A/R,g}$ is the R-submodule of $\Omega_{A/R}$ generated by $a_0 \mathrm{d}a_1$ with $a_i \in A_{g_i}$ such that $g = g_0 + g_1$. Similarly, we obtain

$$\Omega^p_{A/R} = \bigoplus\nolimits_{g \in G} \Omega^p_{A/R,g}$$

where $\Omega^p_{A/R,g}$ is the R-submodule of $\Omega^p_{A/R}$ generated by $a_0 \mathrm{d}a_1 \wedge \ldots \wedge \mathrm{d}a_p$ with $a_i \in A_{g_i}$ such that $g = g_0 + g_1 + \ldots + g_p$. Of course the differentials preserve the grading and the wedge product is compatible with the gradings in the obvious manner.

Let $f : X \to S$ be a morphism of schemes. Let $\pi : C \to X$ be a cone, see Constructions, Definition 7.2. Recall that this means π is affine and we have a grading $\pi_*\mathcal{O}_C = \bigoplus_{n \geq 0} \mathcal{A}_n$ with $\mathcal{A}_0 = \mathcal{O}_X$. Using the discussion in Remark 10.1 over affine opens we find that[2]

$$\pi_*(\Omega^\bullet_{C/S}) = \bigoplus\nolimits_{n \geq 0} \Omega^\bullet_{C/S,n}$$

is canonically a direct sum of subcomplexes. Moreover, we have a factorization

$$\Omega^\bullet_{X/S} \to \Omega^\bullet_{C/S,0} \to \pi_*(\Omega^\bullet_{C/S})$$

and we know that $\omega \wedge \eta \in \Omega^{p+q}_{C/S,n+m}$ if $\omega \in \Omega^p_{C/S,n}$ and $\eta \in \Omega^q_{C/S,m}$.

Let $f : X \to S$ be a morphism of schemes. Let $\pi : L \to X$ be the line bundle associated to the invertible $\mathcal{O}_X$-module $\mathcal{L}$. This means that π is the unique affine morphism such that

$$\pi_*\mathcal{O}_L = \bigoplus\nolimits_{n \geq 0} \mathcal{L}^{\otimes n}$$

[2] With excuses for the notation!

as $\mathcal{O}_X$-algebras. Thus L is a cone over X. By the discussion above we find a canonical direct sum decomposition

$$\pi_*(\Omega^\bullet_{L/S}) = \bigoplus\nolimits_{n \geq 0} \Omega^\bullet_{L/S,n}$$

compatible with wedge product, compatible with the decomposition of $\pi_*\mathcal{O}_L$ above, and such that $\Omega_{X/S}$ maps into the part $\Omega_{L/S,0}$ of degree 0.

There is another case which will be useful to us. Namely, consider the complement[3] $L^\star \subset L$ of the zero section $o : X \to L$ in our line bundle L. A local computation shows we have a canonical isomorphism

$$(L^\star \to X)_*\mathcal{O}_{L^\star} = \bigoplus\nolimits_{n \in \mathbf{Z}} \mathcal{L}^{\otimes n}$$

of $\mathcal{O}_X$-algebras. The right hand side is a $\mathbf{Z}$-graded quasi-coherent $\mathcal{O}_X$-algebra. Using the discussion in Remark 10.1 over affine opens we find that

$$(L^\star \to X)_*(\Omega^\bullet_{L^\star/S}) = \bigoplus\nolimits_{n \in \mathbf{Z}} \Omega^\bullet_{L^\star/S,n}$$

compatible with wedge product, compatible with the decomposition of $(L^\star \to X)_*\mathcal{O}_{L^\star}$ above, and such that $\Omega_{X/S}$ maps into the part $\Omega_{L^\star/S,0}$ of degree 0. The complex $\Omega^\bullet_{L^\star/S,0}$ will be of particular interest to us.

0FUF **Lemma 10.2.** *With notation as above, there is a short exact sequence of complexes*

$$0 \to \Omega^\bullet_{X/S} \to \Omega^\bullet_{L^\star/S,0} \to \Omega^\bullet_{X/S}[-1] \to 0$$

Proof. We have constructed the map $\Omega^\bullet_{X/S} \to \Omega^\bullet_{L^\star/S,0}$ above.

Construction of $\text{Res} : \Omega^\bullet_{L^\star/S,0} \to \Omega^\bullet_{X/S}[-1]$. Let $U \subset X$ be an open and let $s \in \mathcal{L}(U)$ and $s' \in \mathcal{L}^{\otimes -1}(U)$ be sections such that $s's = 1$. Then s gives an invertible section of the sheaf of algebras $(L^\star \to X)_*\mathcal{O}_{L^\star}$ over U with inverse $s' = s^{-1}$. Then we can consider the 1-form $\text{d}\log(s) = s'\text{d}(s)$ which is an element of $\Omega^1_{L^\star/S,0}(U)$ by our construction of the grading on $\Omega^1_{L^\star/S}$. Our computations on affines given below will show that 1 and $\text{d}\log(s)$ freely generate $\Omega^\bullet_{L^\star/S,0}|_U$ as a right module over $\Omega^\bullet_{X/S}|_U$. Thus we can define Res over U by the rule

$$\text{Res}(\omega' + \text{d}\log(s) \wedge \omega) = \omega$$

for all $\omega', \omega \in \Omega^\bullet_{X/S}(U)$. This map is independent of the choice of local generator s and hence glues to give a global map. Namely, another choice of s would be of the form gs for some invertible $g \in \mathcal{O}_X(U)$ and we would get $\text{d}\log(gs) = g^{-1}\text{d}(g) + \text{d}\log(s)$ from which the independence easily follows. Finally, observe that our rule for Res is compatible with differentials as $\text{d}(\omega' + \text{d}\log(s) \wedge \omega) = \text{d}(\omega') - \text{d}\log(s) \wedge \text{d}(\omega)$ and because the differential on $\Omega^\bullet_{X/S}[-1]$ sends ω' to $-\text{d}(\omega')$ by our sign convention in Homology, Definition 14.7.

Local computation. We can cover X by affine opens $U \subset X$ such that $\mathcal{L}|_U \cong \mathcal{O}_U$ which moreover map into an affine open $V \subset S$. Write $U = \text{Spec}(A)$, $V = \text{Spec}(R)$ and choose a generator s of $\mathcal{L}$. We find that we have

$$L^\star \times_X U = \text{Spec}(A[s, s^{-1}])$$

[3]The scheme $L^\star$ is the $\mathbf{G}_m$-torsor over X associated to L. This is why the grading we get below is a $\mathbf{Z}$-grading, compare with Groupoids, Example 12.3 and Lemmas 12.4 and 12.5.

Computing differentials we see that

$$\Omega^1_{A[s,s^{-1}]/R} = A[s,s^{-1}] \otimes_A \Omega^1_{A/R} \oplus A[s,s^{-1}]\mathrm{d}\log(s)$$

and therefore taking exterior powers we obtain

$$\Omega^p_{A[s,s^{-1}]/R} = A[s,s^{-1}] \otimes_A \Omega^p_{A/R} \oplus A[s,s^{-1}]\mathrm{d}\log(s) \otimes_A \Omega^{p-1}_{A/R}$$

Taking degree 0 parts we find

$$\Omega^p_{A[s,s^{-1}]/R,0} = \Omega^p_{A/R} \oplus \mathrm{d}\log(s) \otimes_A \Omega^{p-1}_{A/R}$$

and the proof of the lemma is complete. □

0FUG **Lemma 10.3.** *The "boundary" map $\delta : \Omega^\bullet_{X/S} \to \Omega^\bullet_{X/S}[2]$ in $D(X, f^{-1}\mathcal{O}_S)$ coming from the short exact sequence in Lemma 10.2 is the map of Remark 4.3 for $\xi = c_1^{dR}(\mathcal{L})$.*

Proof. To be precise we consider the shift

$$0 \to \Omega^\bullet_{X/S}[1] \to \Omega^\bullet_{L^\star/S,0}[1] \to \Omega^\bullet_{X/S} \to 0$$

of the short exact sequence of Lemma 10.2. As the degree zero part of a grading on $(L^\star \to X)_*\Omega^\bullet_{L^\star/S}$ we see that $\Omega^\bullet_{L^\star/S,0}$ is a differential graded $\mathcal{O}_X$-algebra and that the map $\Omega^\bullet_{X/S} \to \Omega^\bullet_{L^\star/S,0}$ is a homomorphism of differential graded $\mathcal{O}_X$-algebras. Hence we may view $\Omega^\bullet_{X/S}[1] \to \Omega^\bullet_{L^\star/S,0}[1]$ as a map of right differential graded $\Omega^\bullet_{X/S}$-modules on X. The map $\mathrm{Res} : \Omega^\bullet_{L^\star/S,0}[1] \to \Omega^\bullet_{X/S}$ is a map of right differential graded $\Omega^\bullet_{X/S}$-modules since it is locally defined by the rule $\mathrm{Res}(\omega' + \mathrm{d}\log(s) \wedge \omega) = \omega$, see proof of Lemma 10.2. Thus by the discussion in Differential Graded Sheaves, Section 32 we see that δ comes from a map $\delta' : \Omega^\bullet_{X/S} \to \Omega^\bullet_{X/S}[2]$ in the derived category $D(\Omega^\bullet_{X/S}, \mathrm{d})$ of right differential graded modules over the de Rham complex. The uniqueness averted in Remark 4.3 shows it suffices to prove that $\delta(1) = c_1^{dR}(\mathcal{L})$.

We claim that there is a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}_X^* \arrow[r] \arrow[d, "\mathrm{d}\log"'] & E \arrow[r] \arrow[d] & \underline{\mathbf{Z}} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Omega^\bullet_{X/S}[1] \arrow[r] & \Omega^\bullet_{L^\star/S,0}[1] \arrow[r] & \Omega^\bullet_{X/S} \arrow[r] & 0
\end{tikzcd}$$

where the top row is a short exact sequence of abelian sheaves whose boundary map sends 1 to the class of $\mathcal{L}$ in $H^1(X, \mathcal{O}_X^*)$. It suffices to prove the claim by the compatibility of boundary maps with maps between short exact sequences. We define E as the sheafification of the rule

$$U \longmapsto \{(s,n) \mid n \in \mathbf{Z},\ s \in \mathcal{L}^{\otimes n}(U) \text{ generator}\}$$

with group structure given by $(s,n) \cdot (t,m) = (s \otimes t, n+m)$. The middle vertical map sends (s,n) to $\mathrm{d}\log(s)$. This produces a map of short exact sequences because the map $Res : \Omega^1_{L^\star/S,0} \to \mathcal{O}_X$ constructed in the proof of Lemma 10.2 sends $\mathrm{d}\log(s)$ to 1 if s is a local generator of $\mathcal{L}$. To calculate the boundary of 1 in the top row, choose local trivializations s_i of $\mathcal{L}$ over opens U_i as in Section 9. On the overlaps $U_{i_0i_1} = U_{i_0} \cap U_{i_1}$ we have an invertible function $f_{i_0i_1}$ such that

$f_{i_0i_1} = s_{i_1}|_{U_{i_0i_1}} s_{i_0}|_{U_{i_0i_1}}^{-1}$ and the cohomology class of $\mathcal{L}$ is given by the Čech cocycle $\{f_{i_0i_1}\}$. Then of course we have

$$(f_{i_0i_1}, 0) = (s_{i_1}, 1)|_{U_{i_0i_1}} \cdot (s_{i_0}, 1)|_{U_{i_0i_1}}^{-1}$$

as sections of E which finishes the proof. □

0FUH **Lemma 10.4.** *With notation as above we have*

(1) $\Omega^p_{L^\star/S,n} = \Omega^p_{L^\star/S,0} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}$ *for all* $n \in \mathbf{Z}$ *as quasi-coherent* $\mathcal{O}_X$*-modules,*
(2) $\Omega^\bullet_{X/S} = \Omega^\bullet_{L/X,0}$ *as complexes, and*
(3) *for* $n > 0$ *and* $p \geq 0$ *we have* $\Omega^p_{L/X,n} = \Omega^p_{L^\star/S,n}$.

Proof. In each case there is a globally defined canonical map which is an isomorphism by local calculations which we omit. □

0FUI **Lemma 10.5.** *In the situation above, assume there is a morphism* $S \to \mathrm{Spec}(\mathbf{Q})$. *Then* $\Omega^\bullet_{X/S} \to \pi_*\Omega^\bullet_{L/S}$ *is a quasi-isomorphism and* $H^*_{dR}(X/S) = H^*_{dR}(L/S)$.

Proof. Let R be a $\mathbf{Q}$-algebra. Let A be an R-algebra. The affine local statement is that the map

$$\Omega^\bullet_{A/R} \longrightarrow \Omega^\bullet_{A[t]/R}$$

is a quasi-isomorphism of complexes of R-modules. In fact it is a homotopy equivalence with homotopy inverse given by the map sending $g\omega + g'\mathrm{d}t \wedge \omega'$ to $g(0)\omega$ for $g, g' \in A[t]$ and $\omega, \omega' \in \Omega^\bullet_{A/R}$. The homotopy sends $g\omega + g'\mathrm{d}t \wedge \omega'$ to $(\int g')\omega'$ were $\int g' \in A[t]$ is the polynomial with vanishing constant term whose derivative with respect to t is g'. Of course, here we use that R contains $\mathbf{Q}$ as $\int t^n = (1/n)t^{n+1}$. □

0FUJ **Example 10.6.** Lemma 10.5 is false in positive characteristic. The de Rham complex of $\mathbf{A}^1_k = \mathrm{Spec}(k[x])$ over a field k looks like a direct sum

$$k \oplus \bigoplus\nolimits_{n \geq 1} (k \cdot t^n \xrightarrow{n} k \cdot t^{n-1}\mathrm{d}t)$$

Hence if the characteristic of k is $p > 0$, then we see that both $H^0_{dR}(\mathbf{A}^1_k/k)$ and $H^1_{dR}(\mathbf{A}^1_k/k)$ are infinite dimensional over k.

11. de Rham cohomology of projective space

0FMG Let A be a ring. Let $n \geq 1$. The structure morphism $\mathbf{P}^n_A \to \mathrm{Spec}(A)$ is a proper smooth of relative dimension n. It is smooth of relative dimension n and of finite type as $\mathbf{P}^n_A$ has a finite affine open covering by schemes each isomorphic to $\mathbf{A}^n_A$, see Constructions, Lemma 13.3. It is proper because it is also separated and universally closed by Constructions, Lemma 13.4. Let us denote $\mathcal{O}$ and $\mathcal{O}(d)$ the structure sheaf $\mathcal{O}_{\mathbf{P}^n_A}$ and the Serre twists $\mathcal{O}_{\mathbf{P}^n_A}(d)$. Let us denote $\Omega = \Omega_{\mathbf{P}^n_A/A}$ the sheaf of relative differentials and Ω^p its exterior powers.

0FMH **Lemma 11.1.** *There exists a short exact sequence*

$$0 \to \Omega \to \mathcal{O}(-1)^{\oplus n+1} \to \mathcal{O} \to 0$$

Proof. To explain this, we recall that $\mathbf{P}^n_A = \mathrm{Proj}(A[T_0, \ldots, T_n])$, and we write symbolically

$$\mathcal{O}(-1)^{\oplus n+1} = \bigoplus\nolimits_{j=0,\ldots,n} \mathcal{O}(-1)\mathrm{d}T_j$$

The first arrow

$$\Omega \to \bigoplus\nolimits_{j=0,\ldots,n} \mathcal{O}(-1)\mathrm{d}T_j$$

in the short exact sequence above is given on each of the standard opens $D_+(T_i) = \mathrm{Spec}(A[T_0/T_i, \ldots, T_n/T_i])$ mentioned above by the rule

$$\sum\nolimits_{j\neq i} g_j \mathrm{d}(T_j/T_i) \longmapsto \sum\nolimits_{j \neq i} g_j/T_i \mathrm{d}T_j - (\sum\nolimits_{j\neq i} g_j T_j/T_i^2)\mathrm{d}T_i$$

This makes sense because $1/T_i$ is a section of $\mathcal{O}(-1)$ over $D_+(T_i)$. The map

$$\bigoplus\nolimits_{j=0,\ldots,n} \mathcal{O}(-1)\mathrm{d}T_j \to \mathcal{O}$$

is given by sending $\mathrm{d}T_j$ to T_j, more precisely, on $D_+(T_i)$ we send the section $\sum g_j \mathrm{d}T_j$ to $\sum T_j g_j$. We omit the verification that this produces a short exact sequence. $\square$

Given an integer $k \in \mathbf{Z}$ and a quasi-coherent $\mathcal{O}_{\mathbf{P}^n_A}$-module $\mathcal{F}$ denote as usual $\mathcal{F}(k)$ the kth Serre twist of $\mathcal{F}$. See Constructions, Definition 10.1.

0FUK **Lemma 11.2.** *In the situation above we have the following cohomology groups*

1. $H^q(\mathbf{P}^n_A, \Omega^p) = 0$ *unless* $0 \leq p = q \leq n$,
2. *for* $0 \leq p \leq n$ *the* A*-module* $H^p(\mathbf{P}^n_A, \Omega^p)$ *free of rank* 1.
3. *for* $q > 0$, $k > 0$, *and* p *arbitrary we have* $H^q(\mathbf{P}^n_A, \Omega^p(k)) = 0$, *and*
4. *add more here.*

Proof. We are going to use the results of Cohomology of Schemes, Lemma 8.1 without further mention. In particular, the statements are true for $H^q(\mathbf{P}^n_A, \mathcal{O}(k))$.

Proof for $p = 1$. Consider the short exact sequence

$$0 \to \Omega \to \mathcal{O}(-1)^{\oplus n+1} \to \mathcal{O} \to 0$$

of Lemma 11.1. Since $\mathcal{O}(-1)$ has vanishing cohomology in all degrees, this gives that $H^q(\mathbf{P}^n_A, \Omega)$ is zero except in degree 1 where it is freely generated by the boundary of 1 in $H^0(\mathbf{P}^n_A, \mathcal{O})$.

Assume $p > 1$. Let us think of the short exact sequence above as defining a 2 step filtration on $\mathcal{O}(-1)^{\oplus n+1}$. The induced filtration on $\wedge^p \mathcal{O}(-1)^{\oplus n+1}$ looks like this

$$0 \to \Omega^p \to \wedge^p \left(\mathcal{O}(-1)^{\oplus n+1}\right) \to \Omega^{p-1} \to 0$$

Observe that $\wedge^p \mathcal{O}(-1)^{\oplus n+1}$ is isomorphic to a direct sum of $n+1$ choose p copies of $\mathcal{O}(-p)$ and hence has vanishing cohomology in all degrees. By induction hypothesis, this shows that $H^q(\mathbf{P}^n_A, \Omega^p)$ is zero unless $q = p$ and $H^p(\mathbf{P}^n_A, \Omega^p)$ is free of rank 1 with generator the boundary of the generator in $H^{p-1}(\mathbf{P}^n_A, \Omega^{p-1})$.

Let $k > 0$. Observe that $\Omega^n = \mathcal{O}(-n-1)$ for example by the short exact sequence above for $p = n+1$. Hence $\Omega^n(k)$ has vanishing cohomology in positive degrees. Using the short exact sequences

$$0 \to \Omega^p(k) \to \wedge^p \left(\mathcal{O}(-1)^{\oplus n+1}\right)(k) \to \Omega^{p-1}(k) \to 0$$

and *descending* induction on p we get the vanishing of cohomology of $\Omega^p(k)$ in positive degrees for all p. $\square$

0FMI **Lemma 11.3.** *We have* $H^q(\mathbf{P}^n_A, \Omega^p) = 0$ *unless* $0 \leq p = q \leq n$. *For* $0 \leq p \leq n$ *the* A*-module* $H^p(\mathbf{P}^n_A, \Omega^p)$ *free of rank* 1 *with basis element* $c_1^{Hodge}(\mathcal{O}(1))^p$.

Proof. We have the vanishing and and freeness by Lemma 11.2. For $p = 0$ it is certainly true that $1 \in H^0(\mathbf{P}^n_A, \mathcal{O})$ is a generator.

Proof for $p = 1$. Consider the short exact sequence

$$0 \to \Omega \to \mathcal{O}(-1)^{\oplus n+1} \to \mathcal{O} \to 0$$

of Lemma 11.1. In the proof of Lemma 11.2 we have seen that the generator of $H^1(\mathbf{P}^n_A, \Omega)$ is the boundary ξ of $1 \in H^0(\mathbf{P}^n_A, \mathcal{O})$. As in the proof of Lemma 11.1 we will identify $\mathcal{O}(-1)^{\oplus n+1}$ with $\bigoplus_{j=0,\ldots,n} \mathcal{O}(-1)\mathrm{d}T_j$. Consider the open covering

$$\mathcal{U} : \mathbf{P}^n_A = \bigcup\nolimits_{i=0,\ldots,n} D_+(T_i)$$

We can lift the restriction of the global section 1 of $\mathcal{O}$ to $U_i = D_+(T_i)$ by the section $T_i^{-1}\mathrm{d}T_i$ of $\bigoplus \mathcal{O}(-1)\mathrm{d}T_j$ over U_i. Thus the cocyle representing ξ is given by

$$T_{i_1}^{-1}\mathrm{d}T_{i_1} - T_{i_0}^{-1}\mathrm{d}T_{i_0} = \mathrm{d}\log(T_{i_1}/T_{i_0}) \in \Omega(U_{i_0i_1})$$

On the other hand, for each i the section T_i is a trivializing section of $\mathcal{O}(1)$ over U_i. Hence we see that $f_{i_0i_1} = T_{i_1}/T_{i_0} \in \mathcal{O}^*(U_{i_0i_1})$ is the cocycle representing $\mathcal{O}(1)$ in $\mathrm{Pic}(\mathbf{P}^n_A)$, see Section 9. Hence $c_1^{Hodge}(\mathcal{O}(1))$ is given by the cocycle $\mathrm{d}\log(T_{i_1}/T_{i_0})$ which agrees with what we got for ξ above.

Proof for general p by induction. The base cases $p = 0, 1$ were handled above. Assume $p > 1$. In the proof of Lemma 11.2 we have seen that the generator of $H^p(\mathbf{P}^n_A, \Omega^p)$ is the boundary of $c_1^{Hodge}(\mathcal{O}(1))^{p-1}$ in the long exact cohomology sequence associated to

$$0 \to \Omega^p \to \wedge^p\left(\mathcal{O}(-1)^{\oplus n+1}\right) \to \Omega^{p-1} \to 0$$

By the calculation in Section 9 the cohomology class $c_1^{Hodge}(\mathcal{O}(1))^{p-1}$ is, up to a sign, represented by the cocycle with terms

$$\beta_{i_0\ldots i_{p-1}} = \mathrm{d}\log(T_{i_1}/T_{i_0}) \wedge \mathrm{d}\log(T_{i_2}/T_{i_1}) \wedge \ldots \wedge \mathrm{d}\log(T_{i_{p-1}}/T_{i_{p-2}})$$

in $\Omega^{p-1}(U_{i_0\ldots i_{p-1}})$. These $\beta_{i_0\ldots i_{p-1}}$ can be lifted to the sections $\tilde\beta_{i_0\ldots i_{p-1}} = T_{i_0}^{-1}\mathrm{d}T_{i_0} \wedge \beta_{i_0\ldots i_{p-1}}$ of $\wedge^p(\bigoplus \mathcal{O}(-1)\mathrm{d}T_j)$ over $U_{i_0\ldots i_{p-1}}$. We conclude that the generator of $H^p(\mathbf{P}^n_A, \Omega^p)$ is given by the cocycle whose components are

$$\begin{aligned}
\sum\nolimits_{a=0}^p (-1)^a \tilde\beta_{i_0\ldots\hat{i_a}\ldots i_p} &= T_{i_1}^{-1}\mathrm{d}T_{i_1} \wedge \beta_{i_1\ldots i_p} + \sum\nolimits_{a=1}^p (-1)^a T_{i_0}^{-1}\mathrm{d}T_{i_0} \wedge \beta_{i_0\ldots\hat{i_a}\ldots i_p} \\
&= (T_{i_1}^{-1}\mathrm{d}T_{i_1} - T_{i_0}^{-1}\mathrm{d}T_{i_0}) \wedge \beta_{i_1\ldots i_p} + T_{i_0}^{-1}\mathrm{d}T_{i_0} \wedge \mathrm{d}(\beta)_{i_0\ldots i_p} \\
&= \mathrm{d}\log(T_{i_1}/T_{i_0}) \wedge \beta_{i_1\ldots i_p}
\end{aligned}$$

viewed as a section of Ω^p over $U_{i_0\ldots i_p}$. This is up to sign the same as the cocycle representing $c_1^{Hodge}(\mathcal{O}(1))^p$ and the proof is complete. □

0FMJ **Lemma 11.4.** *For $0 \leq i \leq n$ the de Rham cohomology $H^{2i}_{dR}(\mathbf{P}^n_A/A)$ is a free A-module of rank 1 with basis element $c_1^{dR}(\mathcal{O}(1))^i$. In all other degrees the de Rham cohomology of $\mathbf{P}^n_A$ over A is zero.*

Proof. Consider the Hodge-to-de Rham spectral sequence of Section 6. By the computation of the Hodge cohomology of $\mathbf{P}^n_A$ over A done in Lemma 11.3 we see that the spectral sequence degenerates on the E_1 page. In this way we see that $H^i_{dR}(\mathbf{P}^n_A/A)$ is a free A-module of rank 1 for $0 \leq i \leq n$ and zero else. Observe that

$c_1^{dR}(\mathcal{O}(1))^i \in H^{2i}_{dR}(\mathbf{P}^n_A/A)$ for $i = 0, \ldots, n$ and that for $i = n$ this element is the image of $c_1^{Hodge}(\mathcal{L})^n$ by the map of complexes

$$\Omega^n_{\mathbf{P}^n_A/A}[-n] \longrightarrow \Omega^\bullet_{\mathbf{P}^n_A/A}$$

This follows for example from the discussion in Remark 9.2 or from the explicit description of cocycles representing these classes in Section 9. The spectral sequence shows that the induced map

$$H^n(\mathbf{P}^n_A, \Omega^n_{\mathbf{P}^n_A/A}) \longrightarrow H^{2n}_{dR}(\mathbf{P}^n_A/A)$$

is an isomorphism and since $c_1^{Hodge}(\mathcal{L})^n$ is a generator of of the source (Lemma 11.3), we conclude that $c_1^{dR}(\mathcal{L})^n$ is a generator of the target. By the A-bilinearity of the cup products, it follows that also $c_1^{dR}(\mathcal{L})^i$ is a generator of $H^{2i}_{dR}(\mathbf{P}^n_A/A)$ for $0 \leq i \leq n$. $\square$

12. The spectral sequence for a smooth morphism

0FMK Consider a commutative diagram of schemes

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "p"'] & & Y \arrow[dl, "q"] \\ & S & \end{tikzcd}$$

where f is a smooth morphism. Then we obtain a locally split short exact sequence

$$0 \to f^*\Omega_{Y/S} \to \Omega_{X/S} \to \Omega_{X/Y} \to 0$$

by Morphisms, Lemma 32.16. Let us think of this as a descending filtration F on $\Omega_{X/S}$ with $F^0\Omega_{X/S} = \Omega_{X/S}$, $F^1\Omega_{X/S} = f^*\Omega_{Y/S}$, and $F^2\Omega_{X/S} = 0$. Applying the functor $\wedge^p$ we obtain for every p an induced filtration

$$\Omega^p_{X/S} = F^0\Omega^p_{X/S} \supset F^1\Omega^p_{X/S} \supset F^2\Omega^p_{X/S} \supset \ldots \supset F^{p+1}\Omega^p_{X/S} = 0$$

whose succesive quotients are

$$\mathrm{gr}^k\Omega^p_{X/S} = F^k\Omega^p_{X/S}/F^{k+1}\Omega^p_{X/S} = f^*\Omega^k_{Y/S} \otimes_{\mathcal{O}_X} \Omega^{p-k}_{X/Y} = f^{-1}\Omega^k_{Y/S} \otimes_{f^{-1}\mathcal{O}_Y} \Omega^{p-k}_{X/Y}$$

for $k = 0, \ldots, p$. In fact, the reader can check using the Leibniz rule that $F^k\Omega^\bullet_{X/S}$ is a subcomplex of $\Omega^\bullet_{X/S}$. In this way $\Omega^\bullet_{X/S}$ has the structure of a filtered complex. We can also see this by observing that

$$F^k\Omega^\bullet_{X/S} = \mathrm{Im}\left(\wedge : \mathrm{Tot}(f^{-1}\sigma_{\geq k}\Omega^\bullet_{Y/S} \otimes_{p^{-1}\mathcal{O}_S} \Omega^\bullet_{X/S}) \longrightarrow \Omega^\bullet_{X/S}\right)$$

is the image of a map of complexes on X. The filtered complex

$$\Omega^\bullet_{X/S} = F^0\Omega^\bullet_{X/S} \supset F^1\Omega^\bullet_{X/S} \supset F^2\Omega^\bullet_{X/S} \supset \ldots$$

has the following associated graded parts

$$\mathrm{gr}^k\Omega^\bullet_{X/S} = f^{-1}\Omega^k_{Y/S}[-k] \otimes_{f^{-1}\mathcal{O}_Y} \Omega^\bullet_{X/Y}$$

by what was said above.

0FMM **Lemma 12.1.** *Let $f : X \to Y$ be a quasi-compact, quasi-separated, and smooth morphism of schemes over a base scheme S. There is a bounded spectral sequence with first page*

$$E_1^{p,q} = H^q(\Omega^p_{Y/S} \otimes^{\mathbf{L}}_{\mathcal{O}_Y} Rf_*\Omega^\bullet_{X/Y})$$

converging to $R^{p+q}f_*\Omega^\bullet_{X/S}$.

Proof. Consider $\Omega^\bullet_{X/S}$ as a filtered complex with the filtration introduced above. The spectral sequence is the spectral sequence of Cohomology, Lemma 29.5. By Derived Categories of Schemes, Lemma 22.3 we have

$$Rf_*\mathrm{gr}^k\Omega^\bullet_{X/S} = \Omega^k_{Y/S}[-k] \otimes^{\mathbf{L}}_{\mathcal{O}_Y} Rf_*\Omega^\bullet_{X/Y}$$

and thus we conclude. □

0FMN **Remark 12.2.** In Lemma 12.1 consider the cohomology sheaves

$$\mathcal{H}^q_{dR}(X/Y) = H^q(Rf_*\Omega^\bullet_{X/Y}))$$

If f is proper in addition to being smooth and S is a scheme over $\mathbf{Q}$ then $\mathcal{H}^q_{dR}(X/Y)$ is finite locally free (insert future reference here). If we only assume $\mathcal{H}^q_{dR}(X/Y)$ are flat $\mathcal{O}_Y$-modules, then we obtain (tiny argument omitted)

$$E_1^{p,q} = \Omega^p_{Y/S} \otimes_{\mathcal{O}_Y} \mathcal{H}^q_{dR}(X/Y)$$

and the differentials in the spectral sequence are maps

$$d_1^{p,q} : \Omega^p_{Y/S} \otimes_{\mathcal{O}_Y} \mathcal{H}^q_{dR}(X/Y) \longrightarrow \Omega^{p+1}_{Y/S} \otimes_{\mathcal{O}_Y} \mathcal{H}^q_{dR}(X/Y)$$

In particular, for $p = 0$ we obtain a map $d_1^{0,q} : \mathcal{H}^q_{dR}(X/Y) \to \Omega^1_{Y/S} \otimes_{\mathcal{O}_Y} \mathcal{H}^q_{dR}(X/Y)$ which turns out to be an integrable connection ∇ (insert future reference here) and the complex

$$\mathcal{H}^q_{dR}(X/Y) \to \Omega^1_{Y/S} \otimes_{\mathcal{O}_Y} \mathcal{H}^q_{dR}(X/Y) \to \Omega^2_{Y/S} \otimes_{\mathcal{O}_Y} \mathcal{H}^q_{dR}(X/Y) \to \ldots$$

with differentials given by $d_1^{\bullet,q}$ is the de Rham complex of ∇. The connection ∇ is known as the *Gauss-Manin connection*.

13. Leray-Hirsch type theorems

0FUL In this section we prove that for a smooth proper morphism one can sometimes express the de Rham cohomology upstairs in terms of the de Rham cohomology downstairs.

0FMP **Lemma 13.1.** *Let $f : X \to Y$ be a smooth proper morphism of schemes. Let N and $n_1, \ldots, n_N \geq 0$ be integers and let $\xi_i \in H^{n_i}_{dR}(X/Y)$, $1 \leq i \leq N$. Assume for all points $y \in Y$ the images of $\xi_1, \ldots, \xi_N$ in $H^*_{dR}(X_y/y)$ form a basis over $\kappa(y)$. Then the map*

$$\bigoplus\nolimits_{i=1}^N \mathcal{O}_Y[-n_i] \longrightarrow Rf_*\Omega^\bullet_{X/Y}$$

associated to $\xi_1, \ldots, \xi_N$ is an isomorphism.

Proof. By Lemma 3.5 $Rf_*\Omega^\bullet_{X/Y}$ is a perfect object of $D(\mathcal{O}_Y)$ whose formation commutes with arbitrary base change. Thus the map of the lemma is a map $a : K \to L$ between perfect objects of $D(\mathcal{O}_Y)$ whose derived restriction to any point is an isomorphism by our assumption on fibres. Then the cone C on a is a perfect object of $D(\mathcal{O}_Y)$ (Cohomology, Lemma 45.7) whose derived restriction to any point is zero. It follows that C is zero by More on Algebra, Lemma 71.6 and a is an isomorphism. (This also uses Derived Categories of Schemes, Lemmas 3.5 and 9.7 to translate into algebra.) □

We first prove the main result of this section in the following special case.

0FUM **Lemma 13.2.** *Let $f : X \to Y$ be a smooth proper morphism of schemes over a base S. Assume*

1. *Y and S are affine, and*
2. *there exist integers N and $n_1, \ldots, n_N \geq 0$ and $\xi_i \in H^{n_i}_{dR}(X/S)$, $1 \leq i \leq N$ such that for all points $y \in Y$ the images of $\xi_1, \ldots, \xi_N$ in $H^*_{dR}(X_y/y)$ form a basis over $\kappa(y)$.*

Then the map

$$\bigoplus\nolimits_{i=1}^{N} H^*_{dR}(Y/S) \longrightarrow H^*_{dR}(X/S), \quad (a_1, \ldots, a_N) \longmapsto \sum \xi_i \cup f^*a_i$$

is an isomorphism.

Proof. Say $Y = \mathrm{Spec}(A)$ and $S = \mathrm{Spec}(R)$. In this case $\Omega^\bullet_{A/R}$ computes $R\Gamma(Y, \Omega^\bullet_{Y/S})$ by Lemma 3.1. Choose a finite affine open covering $\mathcal{U} : X = \bigcup_{i \in I} U_i$. Consider the complex

$$K^\bullet = \mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \Omega^\bullet_{X/S}))$$

as in Cohomology, Section 25. Let us collect some facts about this complex most of which can be found in the reference just given:

1. $K^\bullet$ is a complex of R-modules whose terms are A-modules,
2. $K^\bullet$ represents $R\Gamma(X, \Omega^\bullet_{X/S})$ in $D(R)$ (Cohomology of Schemes, Lemma 2.2 and Cohomology, Lemma 25.2),
3. there is a natural map $\Omega^\bullet_{A/R} \to K^\bullet$ of complexes of R-modules which is A-linear on terms and induces the pullback map $H^*_{dR}(Y/S) \to H^*_{dR}(X/S)$ on cohomology,
4. $K^\bullet$ has a multiplication denoted $\wedge$ which turns it into a differential graded R-algebra,
5. the multiplication on $K^\bullet$ induces the cup product on $H^*_{dR}(X/S)$ (Cohomology, Section 31),
6. the filtration F on $\Omega^*_{X/S}$ induces a filtration
$$K^\bullet = F^0K^\bullet \supset F^1K^\bullet \supset F^2K^\bullet \supset \ldots$$
by subcomplexes on $K^\bullet$ such that
   (a) $F^kK^n \subset K^n$ is an A-submmodule,
   (b) $F^kK^\bullet \wedge F^lK^\bullet \subset F^{k+l}K^\bullet$,
   (c) $\mathrm{gr}^kK^\bullet$ is a complex of A-modules,
   (d) $\mathrm{gr}^0K^\bullet = \mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \Omega^\bullet_{X/Y}))$ and represents $R\Gamma(X, \Omega^\bullet_{X/Y})$ in $D(A)$,
   (e) multiplication induces an isomorphism $\Omega^k_{A/R}[-k] \otimes_A \mathrm{gr}^0K^\bullet \to \mathrm{gr}^kK^\bullet$

We omit the detailed proofs of these statements; please see discussion leading up to the construction of the spectral sequence in Lemma 12.1.

For every $i = 1, \ldots, N$ we choose a cocycle $x_i \in K^{n_i}$ representing ξ_i. Next, we look at the map of complexes

$$\tilde{x} : M^\bullet = \bigoplus\nolimits_{i=1,\ldots,N} \Omega^\bullet_{A/R}[-n_i] \longrightarrow K^\bullet$$

which sends ω in the ith summand to $x_i \wedge \omega$. All that remains is to show that this map is a quasi-isomorphism. We endow $M^\bullet$ with the structure of a filtered complex by the rule

$$F^k M^\bullet = \bigoplus\nolimits_{i=1,\ldots,N} (\sigma_{\geq k}\Omega^\bullet_{A/R})[-n_i]$$

With this choice the map $\tilde{x}$ is a morphism of filtered complexes. Observe that $\text{gr}^0 M^\bullet = \bigoplus A[-n_i]$ and multiplication induces an isomorphism $\Omega^k_{A/R}[-k] \otimes_A \text{gr}^0 M^\bullet \to \text{gr}^k M^\bullet$. By construction and Lemma 13.1 we see that

$$\text{gr}^0 \tilde{x} : \text{gr}^0 M^\bullet \longrightarrow \text{gr}^0 K^\bullet$$

is an isomorphism in $D(A)$. It follows that for all $k \geq 0$ we obtain isomorphisms

$$\text{gr}^k \tilde{x} : \text{gr}^k M^\bullet = \Omega^k_{A/R}[-k] \otimes_A \text{gr}^0 M^\bullet \longrightarrow \Omega^k_{A/R}[-k] \otimes_A \text{gr}^0 K^\bullet = \text{gr}^k K^\bullet$$

in $D(A)$. Namely, the complex $\text{gr}^0 K^\bullet = \text{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \Omega^\bullet_{X/Y}))$ is K-flat as a complex of A-modules by Derived Categories of Schemes, Lemma 22.4. Hence the tensor product on the right hand side is the derived tensor product as is true by inspection on the left hand side. Finally, taking the derived tensor product $\Omega^k_{A/R}[-k] \otimes^{\mathbf{L}}_A -$ is a functor on $D(A)$ and therefore sends isomorphisms to isomorphisms. Arguing by induction on k we deduce that

$$\tilde{x} : M^\bullet / F^k M^\bullet \to K^\bullet / F^k K^\bullet$$

is an isomorphism in $D(R)$ since we have the short exact sequences

$$0 \to F^k M^\bullet / F^{k+1} M^\bullet \to M^\bullet / F^{k+1} M^\bullet \to \text{gr}^k M^\bullet \to 0$$

and similarly for $K^\bullet$. This proves that $\tilde{x}$ is a quasi-isomorphism as the filtrations are finite in any given degree. □

0FMR **Proposition 13.3.** *Let $f : X \to Y$ be a smooth proper morphism of schemes over a base S. Let N and $n_1, \ldots, n_N \geq 0$ be integers and let $\xi_i \in H^{n_i}_{dR}(X/S)$, $1 \leq i \leq N$. Assume for all points $y \in Y$ the images of $\xi_1, \ldots, \xi_N$ in $H^*_{dR}(X_y/y)$ form a basis over $\kappa(y)$. The map*

$$\tilde{\xi} = \bigoplus \tilde{\xi}_i[-n_i] : \bigoplus \Omega^\bullet_{Y/S}[-n_i] \longrightarrow Rf_*\Omega^\bullet_{X/S}$$

(see proof) is an isomorphism in $D(Y, (Y \to S)^{-1}\mathcal{O}_S)$ and correspondingly the map

$$\bigoplus\nolimits_{i=1}^N H^*_{dR}(Y/S) \longrightarrow H^*_{dR}(X/S), \quad (a_1, \ldots, a_N) \longmapsto \sum \xi_i \cup f^* a_i$$

is an isomorphism.

Proof. Denote $p : X \to S$ and $q : Y \to S$ be the structure morphisms. Let $\xi'_i : \Omega^\bullet_{X/S} \to \Omega^\bullet_{X/S}[n_i]$ be the map of Remark 4.3 corresponding to ξ_i. Denote

$$\tilde{\xi}_i : \Omega^\bullet_{Y/S} \to Rf_*\Omega^\bullet_{X/S}[n_i]$$

the composition of ξ'_i with the canonical map $\Omega^\bullet_{Y/S} \to Rf_*\Omega^\bullet_{X/S}$. Using

$$R\Gamma(Y, Rf_*\Omega^\bullet_{X/S}) = R\Gamma(X, \Omega^\bullet_{X/S})$$

on cohomology $\tilde{\xi}_i$ is the map $\eta \mapsto \xi_i \cup f^*\eta$ from $H^m_{dR}(Y/S)$ to $H^{m+n}_{dR}(X/S)$. Further, since the formation of ξ'_i commutes with restrictions to opens, so does the formation of $\tilde{\xi}_i$ commute with restriction to opens.

Thus we can consider the map

$$\tilde{\xi} = \bigoplus \tilde{\xi}_i[-n_i] : \bigoplus \Omega^\bullet_{Y/S}[-n_i] \longrightarrow Rf_*\Omega^\bullet_{X/S}$$

To prove the lemma it suffices to show that this is an isomorphism in $D(Y, q^{-1}\mathcal{O}_S)$. If we could show $\tilde{\xi}$ comes from a map of filtered complexes (with suitable filtrations), then we could appeal to the spectral sequence of Lemma 12.1 to finish the proof. This takes more work than is necessary and instead our approach will be to reduce to the affine case (whose proof does in some sense use the spectral sequence).

Indeed, if $Y' \subset Y$ is is any open with inverse image $X' \subset X$, then $\tilde{\xi}|_{X'}$ induces the map

$$\bigoplus\nolimits_{i=1}^N H^*_{dR}(Y'/S) \longrightarrow H^*_{dR}(X'/S), \quad (a_1, \ldots, a_N) \longmapsto \sum \xi_i|_{X'} \cup f^*a_i$$

on cohomology over Y', see discussion above. Thus it suffices to find a basis for the topology on Y such that the proposition holds for the members of the basis (in particular we can forget about the map $\tilde{\xi}$ when we do this). This reduces us to the case where Y and S are affine which is handled by Lemma 13.2 and the proof is complete. □

14. Projective space bundle formula

0FMS The title says it all.

0FMT **Proposition 14.1.** *Let $X \to S$ be a morphism of schemes. Let $\mathcal{E}$ be a locally free $\mathcal{O}_X$-module of constant rank r. Consider the morphism $p : P = \mathbf{P}(\mathcal{E}) \to X$. Then the map*

$$\bigoplus\nolimits_{i=0,\ldots,r-1} H^*_{dR}(X/S) \longrightarrow H^*_{dR}(P/S)$$

given by the rule

$$(a_0, \ldots, a_{r-1}) \longmapsto \sum\nolimits_{i=0,\ldots,r-1} c_1^{dR}(\mathcal{O}_P(1))^i \cup p^*(a_i)$$

is an isomorphism.

Proof. Choose an affine open $\mathrm{Spec}(A) \subset X$ such that $\mathcal{E}$ restricts to the trivial locally free module $\mathcal{O}^{\oplus r}_{\mathrm{Spec}(A)}$. Then $P \times_X \mathrm{Spec}(A) = \mathbf{P}^{r-1}_A$. Thus we see that p is proper and smooth, see Section 11. Moreover, the classes $c_1^{dR}(\mathcal{O}_P(1))^i$, $i = 0, 1, \ldots, r-1$ restricted to a fibre $X_y = \mathbf{P}^{r-1}_y$ freely generate the de Rham cohomology $H^*_{dR}(X_y/y)$ over $\kappa(y)$, see Lemma 11.4. Thus we've verified the conditions of Proposition 13.3 and we win. □

0FUN **Remark 14.2.** In the situation of Proposition 14.1 we get moreover that the map

$$\tilde{\xi} : \bigoplus\nolimits_{t=0,\ldots,r-1} \Omega^\bullet_{X/S}[-2t] \longrightarrow Rp_*\Omega^\bullet_{P/S}$$

is an isomorphism in $D(X, (X \to S)^{-1}\mathcal{O}_X)$ as follows immediately from the application of Proposition 13.3. Note that the arrow for $t = 0$ is simply the canonical map $c_{P/X} : \Omega^\bullet_{X/S} \to Rp_*\Omega^\bullet_{P/S}$ of Section 2. In fact, we can pin down this map further in this particular case. Namely, consider the canonical map

$$\xi' : \Omega^\bullet_{P/S} \to \Omega^\bullet_{P/S}[2]$$

of Remark 4.3 corresponding to $c_1^{dR}(\mathcal{O}_P(1))$. Then

$$\xi'[2(t-1)] \circ \ldots \circ \xi'[2] \circ \xi' : \Omega^\bullet_{P/S} \to \Omega^\bullet_{P/S}[2t]$$

is the map of Remark 4.3 corresponding to $c_1^{dR}(\mathcal{O}_P(1))^t$. Tracing through the choices made in the proof of Proposition 13.3 we find the value

$$\tilde{\xi}|_{\Omega^\bullet_{X/S}[-2t]} = Rp_*\xi'[-2] \circ \ldots \circ Rp_*\xi'[-2(t-1)] \circ Rp_*\xi'[-2t] \circ c_{P/X}[-2t]$$

for the restriction of our isomorphism to the summand $\Omega^\bullet_{X/S}[-2t]$. This has the following simple consequence we will use below: let

$$M = \bigoplus\nolimits_{t=1,\ldots,r-1} \Omega^\bullet_{X/S}[-2t] \quad \text{and} \quad K = \bigoplus\nolimits_{t=0,\ldots,r-2} \Omega^\bullet_{X/S}[-2t]$$

viewed as subcomplexes of the source of the arrow $\tilde{\xi}$. It follows formally from the discussion above that

$$c_{P/X} \oplus \tilde{\xi}|_M : \Omega^\bullet_{X/S} \oplus M \longrightarrow Rp_*\Omega^\bullet_{P/S}$$

is an isomorphism and that the diagram

$$\begin{tikzcd} K \arrow[r, "\text{id}"'] \arrow[d, "\tilde{\xi}|_K"'] & M[2] \arrow[d, "(\tilde{\xi}|_M)[2]"] \\ Rp_*\Omega^\bullet_{P/S} \arrow[r, "Rp_*\xi'"] & Rp_*\Omega^\bullet_{P/S}[2] \end{tikzcd}$$

commutes where $\text{id} : K \to M[2]$ identifies the summand corresponding to t in the deomposition of K to the summand corresponding to $t+1$ in the decomposition of M.

15. Log poles along a divisor

0FMU Let $X \to S$ be a morphism of schemes. Let $Y \subset X$ be an effective Cartier divisor. If X étale locally along Y looks like $Y \times \mathbf{A}^1$, then there is a canonical short exact sequence of complexes

$$0 \to \Omega^\bullet_{X/S} \to \Omega^\bullet_{X/S}(\log Y) \to \Omega^\bullet_{Y/S}[-1] \to 0$$

having many good properties we will discuss in this section. There is a variant of this construction where one starts with a normal crossings divisor (Étale Morphisms, Definition 21.1) which we will discuss elsewhere (insert future reference here).

0FMV **Definition 15.1.** Let $X \to S$ be a morphism of schemes. Let $Y \subset X$ be an effective Cartier divisor. We say the *de Rham complex of log poles is defined for* $Y \subset X$ *over* S if for all $y \in Y$ and local equation $f \in \mathcal{O}_{X,y}$ of Y we have

(1) $\mathcal{O}_{X,y} \to \Omega_{X/S,y}$, $g \mapsto g\text{d}f$ is a split injection, and
(2) $\Omega^p_{X/S,y}$ is f-torsion free for all p.

An easy local calculation shows that it suffices for every $y \in Y$ to find one local equation f for which conditions (1) and (2) hold.

0FMW **Lemma 15.2.** *Let $X \to S$ be a morphism of schemes. Let $Y \subset X$ be an effective Cartier divisor. Assume the de Rham complex of log poles is defined for $Y \subset X$ over S. There is a canonical short exact sequence of complexes*

$$0 \to \Omega^\bullet_{X/S} \to \Omega^\bullet_{X/S}(\log Y) \to \Omega^\bullet_{Y/S}[-1] \to 0$$

Proof. Our assumption is that for every $y \in Y$ and local equation $f \in \mathcal{O}_{X,y}$ of Y we have

$$\Omega_{X/S,y} = \mathcal{O}_{X,y}\mathrm{d}f \oplus M \quad \text{and} \quad \Omega^p_{X/S,y} = \wedge^{p-1}(M)\mathrm{d}f \oplus \wedge^p(M)$$

for some module M with f-torsion free exterior powers $\wedge^p(M)$. It follows that

$$\Omega^p_{Y/S,y} = \wedge^p(M/fM) = \wedge^p(M)/f\wedge^p(M)$$

Below we will tacitly use these facts. In particular the sheaves $\Omega^p_{X/S}$ have no nonzero local sections supported on Y and we have a canonical inclusion

$$\Omega^p_{X/S} \subset \Omega^p_{X/S}(Y)$$

see More on Flatness, Section 42. Let $U = \mathrm{Spec}(A)$ be an affine open subscheme such that $Y \cap U = V(f)$ for some nonzerodivisor $f \in A$. Let us consider the $\mathcal{O}_U$-submodule of $\Omega^p_{X/S}(Y)|_U$ generated by $\Omega^p_{X/S}|_U$ and $\mathrm{d}\log(f) \wedge \Omega^{p-1}_{X/S}$ where $\mathrm{d}\log(f) = f^{-1}\mathrm{d}(f)$. This is independent of the choice of f as another generator of the ideal of Y on U is equal to uf for a unit $u \in A$ and we get

$$\mathrm{d}\log(uf) - \mathrm{d}\log(f) = \mathrm{d}\log(u) = u^{-1}\mathrm{d}u$$

which is a section of $\Omega_{X/S}$ over U. These local sheaves glue to give a quasi-coherent submodule

$$\Omega^p_{X/S} \subset \Omega^p_{X/S}(\log Y) \subset \Omega^p_{X/S}(Y)$$

Let us agree to think of $\Omega^p_{Y/S}$ as a quasi-coherent $\mathcal{O}_X$-module. There is a unique surjective $\mathcal{O}_X$-linear map

$$\mathrm{Res} : \Omega^p_{X/S}(\log Y) \to \Omega^{p-1}_{Y}$$

defined by the rule

$$\mathrm{Res}(\eta' + \mathrm{d}\log(f) \wedge \eta) = \eta|_{Y\cap U}$$

for all opens U as above and all $\eta' \in \Omega^p_{X/S}(U)$ and $\eta \in \Omega^{p-1}_{X/S}(U)$. If a form η over U restricts to zero on $Y \cap U$, then $\eta = \mathrm{d}f \wedge \eta' + f\eta''$ for some forms η' and η'' over U. We conclude that we have a short exact sequence

$$0 \to \Omega^p_{X/S} \to \Omega^p_{X/S}(\log Y) \to \Omega^{p-1}_{Y/S} \to 0$$

for all p. We still have to define the differentials $\Omega^p_{X/S}(\log Y) \to \Omega^{p+1}_{X/S}(\log Y)$. On the subsheaf $\Omega^p_{X/S}$ we use the differential of the de Rham complex of X over S. Finally, we define $\mathrm{d}(\mathrm{d}\log(f) \wedge \eta) = -\mathrm{d}\log(f) \wedge \mathrm{d}\eta$. The sign is forced on us by the Leibniz rule (on $\Omega^\bullet_{X/S}$) and it is compatible with the differential on $\Omega^\bullet_{Y/S}[-1]$ which is after all $-\mathrm{d}_{Y/S}$ by our sign convention in Homology, Definition 14.7. In this way we obtain a short exact sequence of complexes as stated in the lemma. $\square$

0FUA **Definition 15.3.** Let $X \to S$ be a morphism of schemes. Let $Y \subset X$ be an effective Cartier divisor. Assume the de Rham complex of log poles is defined for $Y \subset X$ over S. Then the complex

$$\Omega^\bullet_{X/S}(\log Y)$$

constructed in Lemma 15.2 is the *de Rham complex of log poles for $Y \subset X$ over S*.

This complex has many good properties.

0FUP **Lemma 15.4.** *Let $p : X \to S$ be a morphism of schemes. Let $Y \subset X$ be an effective Cartier divisor. Assume the de Rham complex of log poles is defined for $Y \subset X$ over S.*

(1) *The maps $\wedge : \Omega^p_{X/S} \times \Omega^q_{X/S} \to \Omega^{p+q}_{X/S}$ extend uniquely to $\mathcal{O}_X$-bilinear maps*
$$\wedge : \Omega^p_{X/S}(\log Y) \times \Omega^q_{X/S}(\log Y) \to \Omega^{p+q}_{X/S}(\log Y)$$
satisfying the Leibniz rule $d(\omega \wedge \eta) = d(\omega) \wedge \eta + (-1)^{\deg(\omega)}\omega \wedge d(\eta)$,
(2) *with multiplication as in (1) the map $\Omega^\bullet_{X/S} \to \Omega^\bullet_{X/S}(\log(Y)$ is a homomorphism of differential graded $\mathcal{O}_S$-algebras,*
(3) *via the maps in (1) we have $\Omega^p_{X/S}(\log Y) = \wedge^p(\Omega^1_{X/S}(\log Y))$, and*
(4) *the map $Res : \Omega^\bullet_{X/S}(\log Y) \to \Omega^\bullet_{Y/S}[-1]$ satisfies*
$$Res(\omega \wedge \eta) = Res(\omega) \wedge \eta|_Y + (-1)^{\deg(\omega)}\omega|_Y \wedge Res(\eta)$$

Proof. This follows by direct calcuation from the local construction of the complex in the proof of Lemma 15.2. Details omitted. □

Consider a commutative diagram
$$\begin{tikzcd} X' \arrow[r, "f"] \arrow[d] & X \arrow[d] \\ S' \arrow[r] & S \end{tikzcd}$$
of schemes. Let $Y \subset X$ be an effective Cartier divisor whose pullback $Y' = f^*Y$ is defined (Divisors, Definition 13.12). Assume the de Rham complex of log poles is defined for $Y \subset X$ over S and the de Rham complex of log poles is defined for $Y' \subset X'$ over S'. In this case we obtain a map of short exact sequences of complexes
$$\begin{tikzcd} 0 \arrow[r] & f^{-1}\Omega^\bullet_{X/S} \arrow[r] \arrow[d] & f^{-1}\Omega^\bullet_{X/S}(\log Y) \arrow[r] \arrow[d] & f^{-1}\Omega^\bullet_{Y/S}[-1] \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \Omega^\bullet_{X'/S'} \arrow[r] & \Omega^\bullet_{X'/S'}(\log Y') \arrow[r] & \Omega^\bullet_{Y'/S'}[-1] \arrow[r] & 0 \end{tikzcd}$$
Linearizing, for every p we obtain a linear map $f^*\Omega^p_{X/S}(\log Y) \to \Omega^p_{X'/S'}(\log Y')$.

0FUQ **Lemma 15.5.** *Let $f : X \to S$ be a morphism of schemes. Let $Y \subset X$ be an effective Cartier divisor. Assume the de Rham complex of log poles is defined for $Y \subset X$ over S. Denote*
$$\delta : \Omega^\bullet_{Y/S} \to \Omega^\bullet_{X/S}[2]$$
in $D(X, f^{-1}\mathcal{O}_S)$ the "boundary" map coming from the short exact sequence in Lemma 15.2. Denote
$$\xi' : \Omega^\bullet_{X/S} \to \Omega^\bullet_{X/S}[2]$$
in $D(X, f^{-1}\mathcal{O}_S)$ the map of Remark 4.3 corresponding to $\xi = c_1^{dR}(\mathcal{O}_X(-Y))$. Denote
$$\zeta' : \Omega^\bullet_{Y/S} \to \Omega^\bullet_{Y/S}[2]$$

in $D(Y, f|_Y^{-1}\mathcal{O}_S)$ the map of Remark 4.3 corresponding to $\zeta = c_1^{dR}(\mathcal{O}_X(-Y)|_Y)$. Then the diagram

$$\begin{tikzcd}
\Omega^\bullet_{X/S} \arrow[r] \arrow[d, "\xi'"'] & \Omega^\bullet_{Y/S} \arrow[d, "\zeta'"] \arrow[dl, "\delta"'] \\
\Omega^\bullet_{X/S}[2] \arrow[r] & \Omega^\bullet_{Y/S}[2]
\end{tikzcd}$$

is commutative in $D(X, f^{-1}\mathcal{O}_S)$.

Proof. More precisely, we define δ as the boundary map corresponding to the shifted short exact sequence

$$0 \to \Omega^\bullet_{X/S}[1] \to \Omega^\bullet_{X/S}(\log Y)[1] \to \Omega^\bullet_{Y/S} \to 0$$

It suffices to prove each triangle commutes. Set $\mathcal{L} = \mathcal{O}_X(-Y)$. Denote $\pi : L \to X$ the line bundle with $\pi_*\mathcal{O}_L = \bigoplus_{n\geq 0} \mathcal{L}^{\otimes n}$.

Commutativity of the upper left triangle. By Lemma 10.3 the map ξ' is the boundary map of the triangle given in Lemma 10.2. By functoriality it suffices to prove there exists a morphism of short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \Omega^\bullet_{X/S}[1] \arrow[r] \arrow[d] & \Omega^\bullet_{L^\star/S,0}[1] \arrow[r] \arrow[d] & \Omega^\bullet_{X/S} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Omega^\bullet_{X/S}[1] \arrow[r] & \Omega^\bullet_{X/S}(\log Y)[1] \arrow[r] & \Omega^\bullet_{Y/S} \arrow[r] & 0
\end{tikzcd}$$

where the left and right vertical arrows are the obvious ones. We can define the middle vertical arrow by the rule

$$\omega' + \mathrm{d}\log(s) \wedge \omega \longmapsto \omega' + \mathrm{d}\log(f) \wedge \omega$$

where ω', ω are local sections of $\Omega^\bullet_{X/S}$ and where s is a local generator of $\mathcal{L}$ and $f \in \mathcal{O}_X(-Y)$ is the corresponding section of the ideal sheaf of Y in X. Since the constructions of the maps in Lemmas 10.2 and 15.2 match exactly, this works.

Commutativity of the lower right triangle. Denote $\overline{L}$ the restriction of L to Y. By Lemma 10.3 the map ζ' is the boundary map of the triangle given in Lemma 10.2 using the line bundle $\overline{L}$ on Y. By functoriality it suffices to prove there exists a morphism of short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \Omega^\bullet_{X/S}[1] \arrow[r] \arrow[d] & \Omega^\bullet_{X/S}(\log Y)[1] \arrow[r] \arrow[d] & \Omega^\bullet_{Y/S} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Omega^\bullet_{Y/S}[1] \arrow[r] & \Omega^\bullet_{\overline{L}^\star/S,0}[1] \arrow[r] & \Omega^\bullet_{Y/S} \arrow[r] & 0
\end{tikzcd}$$

where the left and right vertical arrows are the obvious ones. We can define the middle vertical arrow by the rule

$$\omega' + \mathrm{d}\log(f) \wedge \omega \longmapsto \omega'|_Y + \mathrm{d}\log(\overline{s}) \wedge \omega|_Y$$

where ω', ω are local sections of $\Omega^\bullet_{X/S}$ and where f is a local generator of $\mathcal{O}_X(-Y)$ viewed as a function on X and where $\overline{s}$ is $f|_Y$ viewed as a section of $\mathcal{L}|_Y = \mathcal{O}_X(-Y)|_Y$. Since the constructions of the maps in Lemmas 10.2 and 15.2 match exactly, this works. $\square$

0FMX **Lemma 15.6.** *Let $X \to S$ be a morphism of schemes. Let $Y \subset X$ be an effective Cartier divisor. Assume the de Rham complex of log poles is defined for $Y \subset X$ over S. Let $b \in H^m_{dR}(X/S)$ be a de Rham cohomology class whose restriction to Y is zero. Then $c_1^{dR}(\mathcal{O}_X(Y)) \cup b = 0$ in $H^{m+2}_{dR}(X/S)$.*

Proof. This follows immediately from Lemma 15.5. Namely, we have

$$c_1^{dR}(\mathcal{O}_X(Y)) \cup b = -c_1^{dR}(\mathcal{O}_X(-Y)) \cup b = -\xi'(b) = -\delta(b|_Y) = 0$$

as desired. For the second equality, see Remark 4.3. $\square$

0FMY **Lemma 15.7.** *Let $X \to T \to S$ be morphisms of schemes. Let $Y \subset X$ be an effective Cartier divisor. If both $X \to T$ and $Y \to T$ are smooth, then the de Rham complex of log poles is defined for $Y \subset X$ over S.*

Proof. Let $y \in Y$ be a point. By More on Morphisms, Lemma 34.9 there exists an integer $0 \geq m$ and a commutative diagram

$$\begin{tikzcd} Y \arrow[d] & V \arrow[l] \arrow[r] \arrow[d] & \mathbf{A}^m_T \arrow[d, "{(a_1,\ldots,a_m)\mapsto(a_1,\ldots,a_m,0)}"] \\ X & U \arrow[l] \arrow[r, "\pi"] & \mathbf{A}^{m+1}_T \end{tikzcd}$$

where $U \subset X$ is open, $V = Y \cap U$, π is étale, $V = \pi^{-1}(\mathbf{A}^m_T)$, and $y \in V$. Denote $z \in \mathbf{A}^m_T$ the image of y. Then we have

$$\Omega^p_{X/S,y} = \Omega^p_{\mathbf{A}^{m+1}_T/S,z} \otimes_{\mathcal{O}_{\mathbf{A}^{m+1}_T,z}} \mathcal{O}_{X,x}$$

by Lemma 2.2. Denote $x_1, \ldots, x_{m+1}$ the coordinate functions on $\mathbf{A}^{m+1}_T$. Since the conditions (1) and (2) in Definition 15.1 do not depend on the choice of the local coordinate, it suffices to check the conditions (1) and (2) when f is the image of x_{m+1} by the flat local ring homomorphism $\mathcal{O}_{\mathbf{A}^{m+1}_T,z} \to \mathcal{O}_{X,x}$. In this way we see that it suffices to check conditions (1) and (2) for $\mathbf{A}^m_T \subset \mathbf{A}^{m+1}_T$ and the point z. To prove this case we may assume $S = \mathrm{Spec}(A)$ and $T = \mathrm{Spec}(B)$ are affine. Let $A \to B$ be the ring map corresponding to the morphism $T \to S$ and set $P = B[x_1, \ldots, x_{m+1}]$ so that $\mathbf{A}^{m+1}_T = \mathrm{Spec}(B)$. We have

$$\Omega_{P/A} = \Omega_{B/A} \otimes_B P \oplus \bigoplus\nolimits_{j=1,\ldots,m} P\mathrm{d}x_j \oplus P\mathrm{d}x_{m+1}$$

Hence the map $P \to \Omega_{P/A}$, $g \mapsto g\mathrm{d}x_{m+1}$ is a split injection and x_{m+1} is a nonzerodivisor on $\Omega^p_{P/A}$ for all $p \geq 0$. Localizing at the prime ideal corresponding to z finishes the proof. $\square$

0FMZ **Remark 15.8.** Let S be a locally Noetherian scheme. Let X be locally of finite type over S. Let $Y \subset X$ be an effective Cartier divisor. If the map

$$\mathcal{O}^\wedge_{X,y} \longrightarrow \mathcal{O}^\wedge_{Y,y}$$

has a section for all $y \in Y$, then the de Rham complex of log poles is defined for $Y \subset X$ over S. If we ever need this result we will formulate a precise statement and add a proof here.

0FN0 **Remark 15.9.** Let S be a locally Noetherian scheme. Let X be locally of finite type over S. Let $Y \subset X$ be an effective Cartier divisor. If for every $y \in Y$ we can find a diagram of schemes over S

$$X \xleftarrow{\varphi} U \xrightarrow{\psi} V$$

with φ étale and $\psi|_{\varphi^{-1}(Y)} : \varphi^{-1}(Y) \to V$ étale, then the de Rham complex of log poles is defined for $Y \subset X$ over S. A special case is when the pair (X, Y) étale locally looks like $(V \times \mathbf{A}^1, V \times \{0\})$. If we ever need this result we will formulate a precise statement and add a proof here.

16. Calculations

0FUB In this section we calculate some Hodge and de Rham cohomology groups for a standard blowing up.

We fix a base ring R. In this section all schemes are schemes over $\text{Spec}(R)$ and all products of schemes are products over $\text{Spec}(R)$. For $s \geq 0$ we denote $\mathbf{A}^s$ and $\mathbf{P}^s$ the affine and projective space over $\text{Spec}(R)$.

Fix integers $0 \leq m$ and $1 \leq n$. Consider the closed immersion $\mathbf{A}^m \subset \mathbf{A}^{m+n}$, $(a_1, \ldots, a_m) \mapsto (a_1, \ldots, a_m, 0, \ldots 0)$. We are going to consider the blowing up L of $\mathbf{A}^{m+n}$ along the closed subscheme $\mathbf{A}^m$. Write

$$\mathbf{A}^{m+n} = \text{Spec}(R[x_1, \ldots, x_m, y_1, \ldots, y_n])$$

We will consider $A = R[x_1, \ldots, x_m, y_1, \ldots, y_n]$ as a graded R-algebra by setting $\deg(x_i) = 0$ and $\deg(y_j) = 1$. With this grading we have

$$P = \text{Proj}(A) = \mathbf{A}^m \times \mathbf{P}^{n-1}$$

Observe that the ideal cutting out $\mathbf{A}^m$ in $\mathbf{A}^{m+n} = \text{Spec}(A)$ is the ideal A_+. Hence L is the Proj of the Rees algebra

$$A \oplus A_+ \oplus (A_+)^2 \oplus \ldots = \bigoplus\nolimits_{d \geq 0} A_{\geq d}$$

Hence L is an example of the phenomenon studied in more generality in More on Morphisms, Section 46; we will use the observations we made there without further mention. In particular, we can consider the diagram

$$\begin{tikzcd}
P \arrow[r, "0"'] \arrow[d, "p"'] & L \arrow[r, "\pi"'] \arrow[d, "b"] & P \arrow[d] \\
\mathbf{A}^m \arrow[r, "i"] & \mathbf{A}^{m+n} \arrow[r] & \mathbf{A}^m
\end{tikzcd}$$

Thus we see that L is a line bundle over $P = \mathbf{A}^m \times \mathbf{P}^{n-1}$ whose zero section is the exceptional divisor of the blowup b.

0FUR **Lemma 16.1.** *Taking differentials over* $\text{Spec}(R)$ *for* $a \geq 0$ *we have*

1. *the map* $\Omega^a_{\mathbf{A}^{m+n}} \to b_*\Omega^a_L$ *is an isomorphism,*
2. *the map* $\Omega^a_{\mathbf{A}^m} \to p_*\Omega^a_P$ *is an isomorphism, and*
3. *the map* $Rb_*\Omega^a_L \to i_*Rp_*\Omega^a_P$ *is an isomorphism on cohomology sheaves in degree* ≥ 1.

Proof. Let us first prove part (2). Since $P = \mathbf{A}^m \times \mathbf{P}^{n-1}$ we see that

$$\Omega^a_P = \bigoplus\nolimits_{a=r+s} \text{pr}_1^*\Omega^r_{\mathbf{A}^m} \otimes \text{pr}_2^*\Omega^s_{\mathbf{P}^{n-1}}$$

Recalling that $p = \text{pr}_1$ by the projection formula (Cohomology, Lemma 49.2) we obtain

$$p_*\Omega^a_P = \bigoplus\nolimits_{a=r+s} \Omega^r_{\mathbf{A}^m} \otimes \text{pr}_{1,*}\text{pr}_2^*\Omega^s_{\mathbf{P}^{n-1}}$$

By the calculations in Section 11 and in particular in the proof of Lemma 11.3 we have $\text{pr}_{1,*}\text{pr}_2^*\Omega^s_{\mathbf{P}^{n-1}} = 0$ except if $s = 0$ in which case we get $\text{pr}_{1,*}\mathcal{O}_P = \mathcal{O}_{\mathbf{A}^m}$. This proves (2).

By the material in Section 10 and in particular Lemma 10.4 we have $\pi_*\Omega^a_L = \Omega^a_P \oplus \bigoplus_{k\geq 1} \Omega^a_{L,k}$ Since the composition $\pi \circ 0$ in the diagram above is the identity morphism on P to prove part (3) it suffices to show that $\Omega^a_{L,k}$ has vanishing higher cohomology for $k > 0$. By Lemmas 10.2 and 10.4 there are short exact sequences

$$0 \to \Omega^a_P \otimes \mathcal{O}_P(k) \to \Omega^a_{L,k} \to \Omega^{a-1}_P \otimes \mathcal{O}_P(k) \to 0$$

where $\Omega^{a-1} = 0$ if $a = 0$. By Lemma 11.2 we obtain the desired vanishing of higher cohomology.

We still have to prove (1). If $n = 1$, then we are blowing up an effective Cartier divisor and b is an isomorphism and we have (1). If $n > 1$, then the composition

$$\Gamma(\mathbf{A}^{n+m}, \Omega^a_{\mathbf{A}^{n+m}}) \to \Gamma(L, \Omega^a_L) \to \Gamma(L \setminus 0(P), \Omega^a_L) = \Gamma(\mathbf{A}^{n+m} \setminus \mathbf{A}^m, \Omega^a_{\mathbf{A}^{n+m}})$$

is an isomorphism as $\Omega^a_{\mathbf{A}^{m+n}}$ is finite free (small detail omitted). Thus the only way (1) can fail is if there are nonzero elements of $\Gamma(L, \Omega^a_L)$ which vanish outside of $0(P)$. Since L is a line bundle over P with zero section $0 : P \to L$, it suffices to show that on a line bundle there are no nonzero sections of a sheaf of differentials which vanish outside the zero section. The reader sees this is true either (preferably) by a local caculation or by using that $\Omega_{L,k} \subset \Omega_{L^\star,k}$ (see references above). $\square$

17. Blowing up and de Rham cohomology

0FUC Fix a base scheme S, a smooth morphism $X \to S$, and a closed subscheme $Z \subset X$ which is also smooth over S. Denote $b : X' \to X$ the blowing up of X along Z. Denote $E \subset X'$ the exceptional divisor. Picture

0FUS (17.0.1)
$$\begin{tikzcd} E \arrow[r, "j"] \arrow[d, "p"] & X' \arrow[d, "b"] \\ Z \arrow[r, "i"] & X \end{tikzcd}$$

Our goal in this section is to prove that the map $b^* : H^*_{dR}(X/S) \longrightarrow H^*_{dR}(X'/S)$ is injective (although a lot more can be said).

0FUT **Lemma 17.1.** *Let S be a scheme. Let $Z \to X$ be a closed immersion of schemes smooth over S. Let $b : X' \to X$ be the blowing up of Z with exceptional divisor $E \subset X'$. Then X' and E are smooth over S. The morphism $p : E \to Z$ is canonically isomorphic to the projective space bundle*

$$\mathbf{P}(\mathcal{I}/\mathcal{I}^2) \longrightarrow Z$$

where $\mathcal{I} \subset \mathcal{O}_X$ is the ideal sheaf of Z. The relative $\mathcal{O}_E(1)$ coming from the projective space bundle structure is isomorphic to the restriction of $\mathcal{O}_{X'}(-E)$ to E.

Proof. By Divisors, Lemma 22.11 the immersion $Z \to X$ is a regular immmersion, hence the ideal sheaf $\mathcal{I}$ is of finite type, hence b is a projective morphism with relatively ample invertible sheaf $\mathcal{O}_{X'}(1) = \mathcal{O}_{X'}(-E)$, see Divisors, Lemmas 32.4 and 32.13. The canonical map $\mathcal{I} \to b_*\mathcal{O}_{X'}(1)$ gives a closed immersion

$$X' \longrightarrow \mathbf{P}\left(\bigoplus\nolimits_{n \geq 0} \text{Sym}^n_{\mathcal{O}_X}(\mathcal{I})\right)$$

by the very construction of the blowup. The restriction of this morphism to E gives a canonical map

$$E \longrightarrow \mathbf{P}\left(\bigoplus\nolimits_{n \geq 0} \text{Sym}^n_{\mathcal{O}_Z}(\mathcal{I}/\mathcal{I}^2)\right)$$

over Z. Since $\mathcal{I}/\mathcal{I}^2$ is finite locally free if this canonical map is an isomorphism, then the final part of the lemma holds. Having said all of this, now the question is étale local on X. Namely, blowing up commutes with flat base change by Divisors, Lemma 32.3 and we can check smoothness after precomposing with a surjective étale morphism. Thus by the étale local structure of a closed immersion of schemes over S given in More on Morphisms, Lemma 34.9 this reduces to the situation discussed in Section 16. □

0FUU **Lemma 17.2.** *With notation as in Lemma 17.1 for $a \geq 0$ we have*

(1) *the map $\Omega^a_{X/S} \to b_*\Omega^a_{X'/S}$ is an isomorphism,*
(2) *the map $\Omega^a_{Z/S} \to p_*\Omega^a_{E/S}$ is an isomorphism,*
(3) *the map $Rb_*\Omega^a_{X'/S} \to i_*Rp_*\Omega^a_{E/S}$ is an isomorphism on cohomology sheaves in degree ≥ 1.*

Proof. Let $\epsilon : X_1 \to X$ be a surjective étale morphism. Denote $i_1 : Z_1 \to X_1$, $b_1 : X'_1 \to X_1$, $E_1 \subset X'_1$, and $p_1 : E_1 \to Z_1$ the base changes of the objects considered in Lemma 17.1. Observe that i_1 is a closed immersion of schemes smooth over S and that b_1 is the blowing up with center Z_1 by Divisors, Lemma 32.3. Suppose that we can prove (1), (2), and (3) for the morphisms b_1, p_1, and i_1. Then by Lemma 2.2 we obtain that the pullback by ϵ of the maps in (1), (2), and (3) are isomorphisms. As ϵ is a surjective flat morphism we conclude. Thus working étale locally, by More on Morphisms, Lemma 34.9, we may assume we are in the situation discussed in Section 16. In this case the lemma is the same as Lemma 16.1. □

0FUV **Lemma 17.3.** *With notation as in Lemma 17.1 and denoting $f : X \to S$ the structure morphism there is a canonical distinguished triangle*

$$\Omega^\bullet_{X/S} \to Rb_*(\Omega^\bullet_{X'/S}) \oplus i_*\Omega^\bullet_{Z/S} \to i_*Rp_*(\Omega^\bullet_{E/S}) \to \Omega^\bullet_{X/S}[1]$$

in $D(X, f^{-1}\mathcal{O}_S)$ where the four maps

$$\begin{matrix} \Omega^\bullet_{X/S} & \to & Rb_*(\Omega^\bullet_{X'/S}), \\ \Omega^\bullet_{X/S} & \to & i_*\Omega^\bullet_{Z/S}, \\ Rb_*(\Omega^\bullet_{X'/S}) & \to & i_*Rp_*(\Omega^\bullet_{E/S}), \\ i_*\Omega^\bullet_{Z/S} & \to & i_*Rp_*(\Omega^\bullet_{E/S}) \end{matrix}$$

are the canonical ones (Section 2), except with sign reversed for one of them.

Proof. Choose a distinguished triangle

$$C \to Rb_*\Omega^\bullet_{X'/S} \oplus i_*\Omega^\bullet_{Z/S} \to i_*Rp_*\Omega^\bullet_{E/S} \to C[1]$$

in $D(X, f^{-1}\mathcal{O}_S)$. It suffices to show that $\Omega^\bullet_{X/S}$ is isomorphic to C in a manner compatible with the canonical maps. By the axioms of triangulated categories there exists a map of distinguished triangles

$$\begin{tikzcd} C' \arrow[r] \arrow[d] & b_*\Omega^\bullet_{X'/S} \oplus i_*\Omega^\bullet_{Z/S} \arrow[r] \arrow[d] & i_*p_*\Omega^\bullet_{E/S} \arrow[r] \arrow[d] & C'[1] \arrow[d] \\ C \arrow[r] & Rb_*\Omega^\bullet_{X'/S} \oplus i_*\Omega^\bullet_{Z/S} \arrow[r] & i_*Rp_*\Omega^\bullet_{E/S} \arrow[r] & C[1] \end{tikzcd}$$

By Lemma 17.2 part (3) and Derived Categories, Proposition 4.22 we conclude that $C' \to C$ is an isomorphism. By Lemma 17.2 part (2) the map $i_*\Omega^\bullet_{Z/S} \to i_*p_*\Omega^\bullet_{E/S}$ is an isomorphism. Thus $C' = b_*\Omega^\bullet_{X'/S}$ in the derived category. Finally we use Lemma 17.2 part (1) tells us this is equal to $\Omega^\bullet_{X/S}$. We omit the verification this is compatible with the canonical maps. □

0FUW **Proposition 17.4.** *With notation as in Lemma 17.1 the map $\Omega^\bullet_{X/S} \to Rb_*\Omega^\bullet_{X'/S}$ has a splitting in $D(X, (X \to S)^{-1}\mathcal{O}_S)$.*

Proof. Consider the triangle constructed in Lemma 17.3. We claim that the map

$$Rb_*(\Omega^\bullet_{X'/S}) \oplus i_*\Omega^\bullet_{Z/S} \to i_*Rp_*(\Omega^\bullet_{E/S})$$

has a splitting whose image contains the summand $i_*\Omega^\bullet_{Z/S}$. By Derived Categories, Lemma 4.10 this will show that the first arrow of the triangle has a splitting which vanishes on the summand $i_*\Omega^\bullet_{Z/S}$ which proves the lemma. We will prove the claim by decomposing $Rp_*\Omega^\bullet_{E/S}$ into a direct sum where the first piece corresponds to $\Omega^\bullet_{Z/S}$ and the second piece can be lifted through $Rb_*\Omega^\bullet_{X'/S}$.

Proof of the claim. We may decompose X into open and closed subschemes having fixed relative dimension to S, see Morphisms, Lemma 32.12. Since the derived category $D(X, f^{-1}\mathcal{O})_S)$ correspondingly decomposes as a product of categories, we may assume X has fixed relative dimension N over S. We may decompose $Z = \coprod Z_m$ into open and closed subschemes of relative dimension $m \geq 0$ over S. The restriction $i_m : Z_m \to X$ of i to Z_m is a regular immersion of codimension $N - m$, see Divisors, Lemma 22.11. Let $E = \coprod E_m$ be the corresponding decomposition, i.e., we set $E_m = p^{-1}(Z_m)$. If $p_m : E_m \to Z_m$ denotes the restriction of p to E_m, then we have a canonical isomorphism

$$\tilde{\xi}_m : \bigoplus\nolimits_{t=0,\ldots,N-m-1} \Omega^\bullet_{Z_m/S}[-2t] \longrightarrow Rp_{m,*}\Omega^\bullet_{E_m/S}$$

in $D(Z_m, (Z_m \to S)^{-1}\mathcal{O}_S)$ where in degree 0 we have the canonical map $\Omega^\bullet_{Z_m/S} \to Rp_{m,*}\Omega^\bullet_{E_m/S}$. See Remark 14.2. Thus we have an isomorphism

$$\tilde{\xi} : \bigoplus\nolimits_m \bigoplus\nolimits_{t=0,\ldots,N-m-1} \Omega^\bullet_{Z_m/S}[-2t] \longrightarrow Rp_*(\Omega^\bullet_{E/S})$$

in $D(Z, (Z \to S)^{-1}\mathcal{O}_S)$ whose restriction to the summand $\Omega^\bullet_{Z/S} = \bigoplus \Omega^\bullet_{Z_m/S}$ of the source is the canonical map $\Omega^\bullet_{Z/S} \to Rp_*(\Omega^\bullet_{E/S})$. Consider the subcomplexes M_m and K_m of the complex $\bigoplus_{t=0,\ldots,N-m-1} \Omega^\bullet_{Z_m/S}[-2t]$ introduced in Remark 14.2. We set

$$M = \bigoplus M_m \quad \text{and} \quad K = \bigoplus K_m$$

We have $M = K[-2]$ and by construction the map

$$c_{E/Z} \oplus \tilde{\xi}|_M : \Omega^\bullet_{Z/S} \oplus M \longrightarrow Rp_*(\Omega^\bullet_{E/S})$$

is an isomorphism (see remark referenced above).

Consider the map

$$\delta : \Omega^\bullet_{E/S}[-2] \longrightarrow \Omega^\bullet_{X'/S}$$

in $D(X', (X' \to S)^{-1}\mathcal{O}_S)$ of Lemma 15.5 with the property that the composition

$$\Omega^\bullet_{E/S}[-2] \longrightarrow \Omega^\bullet_{X'/S} \longrightarrow \Omega^\bullet_{E/S}$$

is the map θ' of Remark 4.3 for $c_1^{dR}(\mathcal{O}_{X'}(-E))|_E) = c_1^{dR}(\mathcal{O}_E(1))$. The final assertion of Remark 14.2 tells us that the diagram

$$\begin{tikzcd} K[-2] \arrow[r, "\mathrm{id}"'] \arrow[d, "(\tilde{\xi}|_K)[-2]"'] & M \arrow[d, "\tilde{x}|_M"] \\ Rp_*\Omega^\bullet_{E/S}[-2] \arrow[r, "Rp_*\theta'"] & Rp_*\Omega^\bullet_{E/S} \end{tikzcd}$$

commutes. Thus we see that we can obtain the desired splitting of the claim as the map

$$\begin{aligned} Rp_*(\Omega^\bullet_{E/S}) &\xrightarrow{(c_{E/Z}\oplus\tilde{\xi}|_M)^{-1}} \Omega^\bullet_{Z/S} \oplus M \\ &\xrightarrow{\mathrm{id}\oplus\mathrm{id}^{-1}} \Omega^\bullet_{Z/S} \oplus K[-2] \\ &\xrightarrow{\mathrm{id}\oplus(\tilde{\xi}|_K)[-2]} \Omega^\bullet_{Z/S} \oplus Rp_*\Omega^\bullet_{E/S}[-2] \\ &\xrightarrow{\mathrm{id}\oplus Rb_*\delta} \Omega^\bullet_{Z/S} \oplus Rb_*\Omega^\bullet_{X'/S} \end{aligned}$$

The relationship between θ' and δ stated above together with the commutative diagram involving θ', $\tilde{\xi}|_K$, and $\tilde{\xi}|_M$ above are exactly what's needed to show that this is a section to the canonical map $\Omega^\bullet_{Z/S} \oplus Rb_*(\Omega^\bullet_{X'/S}) \to Rp_*(\Omega^\bullet_{E/S})$ and the proof of the claim is complete. $\square$

18. Comparing sheaves of differential forms

0FL7 The goal of this section is to compare the sheaves $\Omega^p_{X/\mathbf{Z}}$ and $\Omega^p_{Y/\mathbf{Z}}$ when given a locally quasi-finite syntomic morphism of schemes $f : Y \to X$. The result will be applied in Section 19 to the construction of the trace map on de Rham complexes if f is finite.

0FL8 **Lemma 18.1.** *Let R be a ring and consider a commutative diagram*

$$\begin{tikzcd} 0 \arrow[r] & K^0 \arrow[r] & L^0 \arrow[r] & M^0 \arrow[r] & 0 \\ & & L^{-1} \arrow[u, "\partial"] \arrow[r, equal] & M^{-1} \arrow[u] & \end{tikzcd}$$

of R-modules with exact top row and M^0 and M^{-1} finite free of the same rank. Then there are canonical maps

$$\wedge^i(H^0(L^\bullet)) \longrightarrow \wedge^i(K^0) \otimes_R \det(M^\bullet)$$

whose composition with $\wedge^i(K^0) \to \wedge^i(H^0(L^\bullet))$ is equal to multiplication with $\delta(M^\bullet)$.

Proof. Say M^0 and M^{-1} are free of rank n. For every $i \geq 0$ there is a canonical surjection

$$\pi_i : \wedge^{n+i}(L^0) \longrightarrow \wedge^i(K^0) \otimes \wedge^n(M^0)$$

whose kernel is the submodule generated by wedges $l_1 \wedge \ldots \wedge l_{n+i}$ such that $> i$ of the l_j are in K^0. On the other hand, the exact sequence

$$L^{-1} \to L^0 \to H^0(L^\bullet) \to 0$$

similarly produces canonical maps

$$\wedge^i(H^0(L^\bullet)) \otimes \wedge^n(L^{-1}) \longrightarrow \wedge^{n+i}(L^0)$$

by sending $\eta \otimes \theta$ to $\tilde{\eta} \wedge \partial(\theta)$ where $\tilde{\eta} \in \wedge^i(L^0)$ is a lift of η. The composition of these two maps, combined with the identification $\wedge^n(L^{-1}) = \wedge^n(M^{-1})$ gives a map

$$\wedge^i(H^0(L^\bullet)) \otimes \wedge^n(M^{-1}) \longrightarrow \wedge^i(K^0) \otimes \wedge^n(M^0)$$

Since $\det(M^\bullet) = \wedge^n(M^0) \otimes (\wedge^n(M^{-1}))^{\otimes -1}$ this produces a map as in the statement of the lemma. If η is the image of $\omega \in \wedge^i(K^0)$, then we see that $\theta \otimes \eta$ is mapped to $\pi_i(\omega \wedge \partial(\theta)) = \omega \otimes \overline{\theta}$ in $\wedge^i(K^0) \otimes \wedge^n(M^0)$ where $\overline{\theta}$ is the image of θ in $\wedge^n(M^0)$. Since $\delta(M^\bullet)$ is simply the determinant of the map $M^{-1} \to M^0$ this proves the last statement. $\square$

0FL9 **Remark 18.2.** Let A be a ring. Let $P = A[x_1, \ldots, x_n]$. Let $f_1, \ldots, f_n \in P$ and set $B = P/(f_1, \ldots, f_n)$. Assume $A \to B$ is quasi-finite. Then B is a relative global complete intersection over A (Algebra, Definition 135.5) and $(f_1, \ldots, f_n)/(f_1, \ldots, f_n)^2$ is free with generators the classes $\overline{f}_i$ by Algebra, Lemma 135.13. Consider the following diagram

$$\begin{tikzcd}
\Omega_{A/\mathbf{Z}} \otimes_A B \arrow[r] & \Omega_{P/\mathbf{Z}} \otimes_P B \arrow[r] & \Omega_{P/A} \otimes_P B \\
& (f_1, \ldots, f_n)/(f_1, \ldots, f_n)^2 \arrow[u] \arrow[r, equal] & (f_1, \ldots, f_n)/(f_1, \ldots, f_n)^2 \arrow[u]
\end{tikzcd}$$

The right column represents $NL_{B/A}$ in $D(B)$ hence has cohomology $\Omega_{B/A}$ in degree 0. The top row is the split short exact sequence $0 \to \Omega_{A/\mathbf{Z}} \otimes_A B \to \Omega_{P/\mathbf{Z}} \otimes_P B \to \Omega_{P/A} \otimes_P B \to 0$. The middle column has cohomology $\Omega_{B/\mathbf{Z}}$ in degree 0 by Algebra, Lemma 130.9. Thus by Lemma 18.1 we obtain canonical B-module maps

$$\Omega^p_{B/\mathbf{Z}} \longrightarrow \Omega^p_{A/\mathbf{Z}} \otimes_A \det(NL_{B/A})$$

whose composition with $\Omega^p_{A/\mathbf{Z}} \to \Omega^p_{B/\mathbf{Z}}$ is multiplication by $\delta(NL_{B/A})$.

0FLA **Lemma 18.3.** *There exists a unique rule that to every locally quasi-finite syntomic morphism of schemes $f : Y \to X$ assigns $\mathcal{O}_Y$-module maps*

$$c^p_{Y/X} : \Omega^p_{Y/\mathbf{Z}} \longrightarrow f^*\Omega^p_{X/\mathbf{Z}} \otimes_{\mathcal{O}_Y} \det(NL_{Y/X})$$

satisfying the following two properties

1. *the composition with $f^*\Omega^p_{X/\mathbf{Z}} \to \Omega^p_{Y/\mathbf{Z}}$ is multiplication by $\delta(NL_{Y/X})$, and*
2. *the rule is compatible with restriction to opens and with base change.*

Proof. This proof is very similar to the proof of Discriminants, Proposition 13.2 and we suggest the reader look at that proof first. We fix $p \geq 0$ throughout the proof.

Let us reformulate the statement. Consider the category $\mathcal{C}$ whose objects, denoted Y/X, are locally quasi-finite syntomic morphism $f : Y \to X$ of schemes and whose morphisms $b/a : Y'/X' \to Y/X$ are commutative diagrams

$$\begin{tikzcd} Y' \arrow[r, "b"] \arrow[d, "f'"'] & Y \arrow[d, "f"] \\ X' \arrow[r, "a"] & X \end{tikzcd}$$

which induce an isomorphism of Y' with an open subscheme of $X' \times_X Y$. The lemma means that for every object Y/X of $\mathcal{C}$ we have maps $c^p_{Y/X}$ with property (1) and for every morphism $b/a : Y'/X' \to Y/X$ of $\mathcal{C}$ we have $b^* c^p_{Y/X} = c^p_{Y'/X'}$ via the identifications $b^* \det(NL_{Y/X}) = \det(NL_{Y'/X'})$ (Discriminants, Section 13) and $b^*\Omega^p_{Y/X} = \Omega^p_{Y'/X'}$ (Lemma 2.1).

Given Y/X in $\mathcal{C}$ and $y \in Y$ we can find an affine open $V \subset Y$ and $U \subset X$ with $f(V) \subset U$ such that there exists some maps

$$\Omega^p_{Y/\mathbf{Z}}|_V \longrightarrow \Big(f^*\Omega^p_{X/\mathbf{Z}} \otimes_{\mathcal{O}_Y} \det(NL_{Y/X})\Big)|_V$$

with property (1). This follows from picking affine opens as in Discriminants, Lemma 10.1 part (5) and Remark 18.2. If $\Omega^p_{X/\mathbf{Z}}$ is finite locally free and annihilator of the section $\delta(NL_{Y/X})$ is zero, then these local maps are unique and automatically glue!

Let $\mathcal{C}_{nice} \subset \mathcal{C}$ denote the full subcategory of Y/X such that

(1) X is of finite type over $\mathbf{Z}$,
(2) $\Omega_{X/\mathbf{Z}}$ is locally free, and
(3) the annihilator of $\delta(NL_{Y/X})$ is zero.

By the remarks in the previous paragraph, we see that for any object Y/X of $\mathcal{C}_{nice}$ we have a unique map $c^p_{Y/X}$ satisfying condition (1). If $b/a : Y'/X' \to Y/X$ is a morphism of $\mathcal{C}_{nice}$, then $b^*c^p_{Y/X}$ is equal to $c^p_{Y'/X'}$ because $b^*\delta(NL_{Y/X}) = \delta(NL_{Y'/X'})$ (see Discriminants, Section 13). In other words, we have solved the problem on the full subcategory $\mathcal{C}_{nice}$. For Y/X in $\mathcal{C}_{nice}$ we continue to denote $c^p_{Y/X}$ the solution we've just found.

Consider morphisms

$$Y_1/X_1 \xleftarrow{b_1/a_1} Y/X \xrightarrow{b_2/a_2} Y_2/X_2$$

in $\mathcal{C}$ such that Y_1/X_1 and Y_2/X_2 are objects of $\mathcal{C}_{nice}$. **Claim.** $b_1^*c^p_{Y_1/X_1} = b_2^*c^p_{Y_2/X_2}$. We will first show that the claim implies the lemma and then we will prove the claim.

Let $d, n \geq 1$ and consider the locally quasi-finite syntomic morphism $Y_{n,d} \to X_{n,d}$ constructed in Discriminants, Example 10.5. Then $Y_{n,d}$ and $Y_{n,d}$ are irreducible schemes of finite type and smooth over $\mathbf{Z}$. Namely, $X_{n,d}$ is a spectrum of a polynomial ring over $\mathbf{Z}$ and $Y_{n,d}$ is an open subscheme of such. The morphism $Y_{n,d} \to X_{n,d}$ is locally quasi-finite syntomic and étale over a dense open, see Discriminants, Lemma 10.6. Thus $\delta(NL_{Y_{n,d}/X_{n,d}})$ is nonzero: for example we have the local description of $\delta(NL_{Y/X})$ in Discriminants, Remark 13.1 and we have the local description of étale morphisms in Morphisms, Lemma 34.15 part (8). Now a nonzero section of an invertible module over an irreducible regular scheme has vanishing annihilator. Thus $Y_{n,d}/X_{n,d}$ is an object of $\mathcal{C}_{nice}$.

Let Y/X be an arbitrary object of $\mathcal{C}$. Let $y \in Y$. By Discriminants, Lemma 10.7 we can find $n, d \geq 1$ and morphisms

$$Y/X \leftarrow V/U \xrightarrow{b/a} Y_{n,d}/X_{n,d}$$

of $\mathcal{C}$ such that $V \subset Y$ and $U \subset X$ are open. Thus we can pullback the canonical morphism $c^p_{Y_{n,d}/X_{n,d}}$ constructed above by b to V. The claim guarantees these local isomorphisms glue! Thus we get a well defined global maps $c^p_{Y/X}$ with property (1). If $b/a : Y'/X' \to Y/X$ is a morphism of $\mathcal{C}$, then the claim also implies that the similarly constructed map $c^p_{Y'/X'}$ is the pullback by b of the locally constructed map $c^p_{Y/X}$. Thus it remains to prove the claim.

In the rest of the proof we prove the claim. We may pick a point $y \in Y$ and prove the maps agree in an open neighbourhood of y. Thus we may replace Y_1, Y_2 by open neighbourhoods of the image of y in Y_1 and Y_2. Thus we may assume Y, X, Y_1, X_1, Y_2, X_2 are affine. We may write $X = \lim X_\lambda$ as a cofiltered limit of affine schemes of finite type over $X_1 \times X_2$. For each λ we get

$$Y_1 \times_{X_1} X_\lambda \quad \text{and} \quad X_\lambda \times_{X_2} Y_2$$

If we take limits we obtain

$$\lim Y_1 \times_{X_1} X_\lambda = Y_1 \times_{X_1} X \supset Y \subset X \times_{X_2} Y_2 = \lim X_\lambda \times_{X_2} Y_2$$

By Limits, Lemma 4.11 we can find a λ and opens $V_{1,\lambda} \subset Y_1 \times_{X_1} X_\lambda$ and $V_{2,\lambda} \subset X_\lambda \times_{X_2} Y_2$ whose base change to X recovers Y (on both sides). After increasing λ we may assume there is an isomorphism $V_{1,\lambda} \to V_{2,\lambda}$ whose base change to X is the identity on Y, see Limits, Lemma 10.1. Then we have the commutative diagram

$$\begin{tikzcd} & Y/X \arrow[dl, "b_1/a_1"'] \arrow[d] \arrow[dr, "b_2/a_2"] & \\ Y_1/X_1 & V_{1,\lambda}/X_\lambda \arrow[l] \arrow[r] & Y_2/X_2 \end{tikzcd}$$

Thus it suffices to prove the claim for the lower row of the diagram and we reduce to the case discussed in the next paragraph.

Assume Y, X, Y_1, X_1, Y_2, X_2 are affine of finite type over $\mathbf{Z}$. Write $X = \text{Spec}(A)$, $X_i = \text{Spec}(A_i)$. The ring map $A_1 \to A$ corresponding to $X \to X_1$ is of finite type and hence we may choose a surjection $A_1[x_1, \ldots, x_n] \to A$. Similarly, we may choose a surjection $A_2[y_1, \ldots, y_m] \to A$. Set $X_1' = \text{Spec}(A_1[x_1, \ldots, x_n])$ and $X_2' = \text{Spec}(A_2[y_1, \ldots, y_m])$. Observe that $\Omega_{X_1'/\mathbf{Z}}$ is the direct sum of the pullback of $\Omega_{X_1/\mathbf{Z}}$ and a finite free module. Similarly for X_2'. Set $Y_1' = Y_1 \times_{X_1} X_1'$ and $Y_2' = Y_2 \times_{X_2} X_2'$. We get the following diagram

$$Y_1/X_1 \leftarrow Y_1'/X_1' \leftarrow Y/X \rightarrow Y_2'/X_2' \rightarrow Y_2/X_2$$

Since $X_1' \to X_1$ and $X_2' \to X_2$ are flat, the same is true for $Y_1' \to Y_1$ and $Y_2' \to Y_2$. It follows easily that the annihilators of $\delta(NL_{Y_1'/X_1'})$ and $\delta(NL_{Y_2'/X_2'})$ are zero. Hence Y_1'/X_1' and Y_2'/X_2' are in $\mathcal{C}_{nice}$. Thus the outer morphisms in the displayed diagram are morphisms of $\mathcal{C}_{nice}$ for which we know the desired compatibilities.

Thus it suffices to prove the claim for $Y'_1/X'_1 \leftarrow Y/X \rightarrow Y'_2/X'_2$. This reduces us to the case discussed in the next paragraph.

Assume Y, X, Y_1, X_1, Y_2, X_2 are affine of finite type over $\mathbf{Z}$ and $X \to X_1$ and $X \to X_2$ are closed immersions. Consider the open embeddings $Y_1 \times_{X_1} X \supset Y \subset X \times_{X_2} Y_2$. There is an open neighbourhood $V \subset Y$ of y which is a standard open of both $Y_1 \times_{X_1} X$ and $X \times_{X_2} Y_2$. This follows from Schemes, Lemma 11.5 applied to the scheme obtained by glueing $Y_1 \times_{X_1} X$ and $X \times_{X_2} Y_2$ along Y; details omitted. Since $X \times_{X_2} Y_2$ is a closed subscheme of Y_2 we can find a standard open $V_2 \subset Y_2$ such that $V_2 \times_{X_2} X = V$. Similarly, we can find a standard open $V_1 \subset Y_1$ such that $V_1 \times_{X_1} X = V$. After replacing Y, Y_1, Y_2 by V, V_1, V_2 we reduce to the case discussed in the next paragraph.

Assume Y, X, Y_1, X_1, Y_2, X_2 are affine of finite type over $\mathbf{Z}$ and $X \to X_1$ and $X \to X_2$ are closed immersions and $Y_1 \times_{X_1} X = Y = X \times_{X_2} Y_2$. Write $X = \mathrm{Spec}(A)$, $X_i = \mathrm{Spec}(A_i)$, $Y = \mathrm{Spec}(B)$, $Y_i = \mathrm{Spec}(B_i)$. Then we can consider the affine schemes

$$X' = \mathrm{Spec}(A_1 \times_A A_2) = \mathrm{Spec}(A') \quad \text{and} \quad Y' = \mathrm{Spec}(B_1 \times_B B_2) = \mathrm{Spec}(B')$$

Observe that $X' = X_1 \amalg_X X_2$ and $Y' = Y_1 \amalg_Y Y_2$, see More on Morphisms, Lemma 14.1. By More on Algebra, Lemma 5.1 the rings A' and B' are of finite type over $\mathbf{Z}$. By More on Algebra, Lemma 6.4 we have $B' \otimes_A A_1 = B_1$ and $B' \times_A A_2 = B_2$. In particular a fibre of $Y' \to X'$ over a point of $X' = X_1 \amalg_X X_2$ is always equal to either a fibre of $Y_1 \to X_1$ or a fibre of $Y_2 \to X_2$. By More on Algebra, Lemma 6.8 the ring map $A' \to B'$ is flat. Thus by Discriminants, Lemma 10.1 part (3) we conclude that Y'/X' is an object of $\mathcal{C}$. Consider now the commutative diagram

$$\begin{tikzcd}
& Y/X \arrow[dl, "b_1/a_1"'] \arrow[dr, "b_2/a_2"] & \\
Y_1/X_1 \arrow[dr] & & Y_2/X_2 \arrow[dl] \\
& Y'/X' &
\end{tikzcd}$$

Now we would be done if Y'/X' is an object of $\mathcal{C}_{nice}$, but this is almost never the case. Namely, then pulling back $c^p_{Y'/X'}$ around the two sides of the square, we would obtain the desired conclusion. To get around the problem that Y'/X' is not in $\mathcal{C}_{nice}$ we note the arguments above show that, after possibly shrinking all of the schemes $X, Y, X_1, Y_1, X_2, Y_2, X', Y'$ we can find some $n, d \geq 1$, and extend the diagram like so:

$$\begin{tikzcd} & Y/X \arrow[dl, "b_1/a_1"'] \arrow[dr, "b_2/a_2"] & \\ Y_1/X_1 \arrow[dr] & & Y_2/X_2 \arrow[dl] \\ & Y'/X' \arrow[d] & \\ & Y_{n,d}/X_{n,d} & \end{tikzcd}$$

and then we can use the already given argument by pulling back from $c^p_{Y_{n,d}/X_{n,d}}$. This finishes the proof. □

19. Trace maps on de Rham complexes

0FK6 A reference for some of the material in this section is [Gar84]. Let S be a scheme. Let $f : Y \to X$ be a finite locally free morphism of schemes over S. Then there is a trace map $\text{Trace}_f : f_*\mathcal{O}_Y \to \mathcal{O}_X$, see Discriminants, Section 3. In this situation a trace map on de Rham complexes is a map of complexes

$$\Theta_{Y/X} : f_*\Omega^\bullet_{Y/S} \longrightarrow \Omega^\bullet_{X/S}$$

such that $\Theta_{Y/X}$ is equal to Trace_f in degree 0 and satisfies

$$\Theta_{Y/X}(\omega \wedge \eta) = \omega \wedge \Theta_{Y/X}(\eta)$$

for local sections ω of $\Omega^\bullet_{X/S}$ and η of $f_*\Omega^\bullet_{Y/S}$. It is not clear to us whether such a trace map $\Theta_{Y/X}$ exists for every finite locally free morphism $Y \to X$; please email stacks.project@gmail.com if you have a counterexample or a proof.

0FK7 **Example 19.1.** Here is an example where we do not have a trace map on de Rham complexes. For example, consider the $\mathbf{C}$-algebra $B = \mathbf{C}[x, y]$ with action of $G = \{\pm 1\}$ given by $x \mapsto -x$ and $y \mapsto -y$. The invariants $A = B^G$ form a normal domain of finite type over $\mathbf{C}$ generated by x^2, xy, y^2. We claim that for the inclusion $A \subset B$ there is no reasonable trace map $\Omega_{B/\mathbf{C}} \to \Omega_{A/\mathbf{C}}$ on 1-forms. Namely, consider the element $\omega = x\text{d}y \in \Omega_{B/\mathbf{C}}$. Since ω is invariant under the action of G if a "reasonable" trace map exists, then 2ω should be in the image of $\Omega_{A/\mathbf{C}} \to \Omega_{B/\mathbf{C}}$. This is not the case: there is no way to write 2ω as a linear combination of $\text{d}(x^2)$, $\text{d}(xy)$, and $\text{d}(y^2)$ even with coefficients in B. This example contradicts the main theorem in [Zan99].

0FLB **Lemma 19.2.** *There exists a unique rule that to every finite syntomic morphism of schemes $f : Y \to X$ assigns $\mathcal{O}_X$-module maps*

$$\Theta^p_{Y/X} : f_*\Omega^p_{Y/\mathbf{Z}} \longrightarrow \Omega^p_{X/\mathbf{Z}}$$

satisfying the following properties

(1) *the composition with $\Omega^p_{X/\mathbf{Z}} \otimes_{\mathcal{O}_X} f_*\mathcal{O}_Y \to f_*\Omega^p_{Y/\mathbf{Z}}$ is equal to $id \otimes Trace_f$ where $Trace_f : f_*\mathcal{O}_Y \to \mathcal{O}_X$ is the map from Discriminants, Section 3,*
(2) *the rule is compatible with base change.*

Proof. First, assume that X is locally Noetherian. By Lemma 18.3 we have a canonical map
$$c^p_{Y/X} : \Omega^p_{Y/S} \longrightarrow f^*\Omega^p_{X/S} \otimes_{\mathcal{O}_Y} \det(NL_{Y/X})$$
By Discriminants, Proposition 13.2 we have a canonical isomorphism
$$c_{Y/X} : \det(NL_{Y/X}) \to \omega_{Y/X}$$
mapping $\delta(NL_{Y/X})$ to $\tau_{Y/X}$. Combined these maps give
$$c^p_{Y/X} \otimes c_{Y/X} : \Omega^p_{Y/S} \longrightarrow f^*\Omega^p_{X/S} \otimes_{\mathcal{O}_Y} \omega_{Y/X}$$
By Discriminants, Section 5 this is the same thing as a map
$$\Theta^p_{Y/X} : f_*\Omega^p_{Y/S} \longrightarrow \Omega^p_{X/S}$$
Recall that the relationship between $c^p_{Y/X} \otimes c_{Y/X}$ and $\Theta^p_{Y/X}$ uses the evaluation map $f_*\omega_{Y/X} \to \mathcal{O}_X$ which sends $\tau_{Y/X}$ to $\text{Trace}_f(1)$, see Discriminants, Section 5. Hence property (1) holds. Property (2) holds for base changes by $X' \to X$ with X' locally Noetherian because both $c^p_{Y/X}$ and $c_{Y/X}$ are compatible with such base changes. For $f : Y \to X$ finite syntomic and X locally Noetherian, we will continue to denote $\Theta^p_{Y/X}$ the solution we've just found.

Uniqueness. Suppose that we have a finite syntomic morphism $f : Y \to X$ such that X is smooth over $\text{Spec}(\mathbf{Z})$ and f is étale over a dense open of X. We claim that in this case $\Theta^p_{Y/X}$ is uniquely determined by property (1). Namely, consider the maps
$$\Omega^p_{X/\mathbf{Z}} \otimes_{\mathcal{O}_X} f_*\mathcal{O}_Y \to f_*\Omega^p_{Y/\mathbf{Z}} \to \Omega^p_{X/\mathbf{Z}}$$
The sheaf $\Omega^p_{X/\mathbf{Z}}$ is torsion free (by the assumed smoothness), hence it suffices to check that the restriction of $\Theta^p_{Y/X}$ is uniquely determined over the dense open over which f is étale, i.e., we may assume f is étale. However, if f is étale, then $f^*\Omega_{X/\mathbf{Z}} = \Omega_{Y/\mathbf{Z}}$ hence the first arrow in the displayed equation is an isomorphism. Since we've pinned down the composition, this guarantees uniqueness.

Let $f : Y \to X$ be a finite syntomic morphism of locally Noetherian schemes. Let $x \in X$. By Discriminants, Lemma 11.7 we can find $d \geq 1$ and a commutative diagram
$$\begin{tikzcd} Y \arrow[d] & V \arrow[l] \arrow[r] \arrow[d] & V_d \arrow[d] \\ X & U \arrow[l] \arrow[r] & U_d \end{tikzcd}$$
such that $x \in U \subset X$ is open, $V = f^{-1}(U)$ and $V = U \times_{U_d} V_d$. Thus $\Theta^p_{Y/X}|_V$ is the pullback of the map $\Theta^p_{V_d/U_d}$. However, by the discussion on uniqueness above and Discriminants, Lemmas 11.4 and 11.5 the map $\Theta^p_{V_d/U_d}$ is uniquely determined by the requirement (1). Hence uniqueness holds.

At this point we know that we have existence and uniqueness for all finite syntomic morphisms $Y \to X$ with X locally Noetherian. We could now give an argument similar to the proof of Lemma 18.3 to extend to general X. However, instead it possible to directly use absolute Noetherian approximation to finish the proof. Namely, to construct $\Theta^p_{Y/X}$ it suffices to do so Zariski locally on X (provided we also show the uniqueness). Hence we may assume X is affine (small detail omitted).

Then we can write $X = \lim_{i \in I} X_i$ as the limit over a directed set I of Noetherian affine schemes. By Algebra, Lemma 126.8 we can find $0 \in I$ and a finitely presented morphism of affines $f_0 : Y_0 \to X_0$ whose base change to X is $Y \to X$. After increasing 0 we may assume $Y_0 \to X_0$ is finite and syntomic, see Algebra, Lemma 163.9 and 163.3. For $i \geq 0$ also the base change $f_i : Y_i = Y_0 \times_{X_0} X_i \to X_i$ is finite syntomic. Then

$$\Gamma(X, f_*\Omega^p_{Y/\mathbf{Z}}) = \Gamma(Y, \Omega^p_{Y/\mathbf{Z}}) = \operatorname{colim}_{i \geq 0} \Gamma(Y_i, \Omega^p_{Y_i/\mathbf{Z}}) = \operatorname{colim}_{i \geq 0} \Gamma(X_i, f_{i,*}\Omega^p_{Y_i/\mathbf{Z}})$$

Hence we can (and are forced to) define $\Theta^p_{Y/X}$ as the colimit of the maps $\Theta^p_{Y_i/X_i}$. This map is compatible with any cartesian diagram

$$\begin{tikzcd} Y' \arrow[r] \arrow[d] & Y \arrow[d] \\ X' \arrow[r] & X \end{tikzcd}$$

with X' affine as we know this for the case of Noetherian affine schemes by the arguments given above (small detail omitted; hint: if we also write $X' = \lim_{j \in J} X'_j$ then for every $i \in I$ there is a $j \in J$ and a morphism $X'_j \to X_i$ compatible with the morphism $X' \to X$). This finishes the proof. □

0FLC **Proposition 19.3.** *Let $f : Y \to X$ be a finite syntomic morphism of schemes. The maps $\Theta^p_{Y/X}$ of Lemma 19.2 define a map of complexes* [Gar84]

$$\Theta_{Y/X} : f_*\Omega^\bullet_{Y/\mathbf{Z}} \longrightarrow \Omega^\bullet_{X/\mathbf{Z}}$$

with the following properties

(1) *in degree 0 we get $Trace_f : f_*\mathcal{O}_Y \to \mathcal{O}_X$, see Discriminants, Section 3,*
(2) *we have $\Theta_{Y/X}(\omega \wedge \eta) = \omega \wedge \Theta_{Y/X}(\eta)$ for ω in $\Omega^\bullet_{X/S}$ and η in $f_*\Omega^\bullet_{Y/S}$,*
(3) *if f is a morphism over a base scheme S, then $\Theta_{Y/X}$ induces a map of complexes $f_*\Omega^\bullet_{Y/S} \to \Omega^\bullet_{X/S}$.*

Proof. By Discriminants, Lemma 11.7 for every $x \in X$ we can find $d \geq 1$ and a commutative diagram

$$\begin{tikzcd} Y \arrow[d] & V \arrow[l] \arrow[r] \arrow[d] & V_d \arrow[r] \arrow[d] & Y_d = \operatorname{Spec}(B_d) \arrow[d] \\ X & U \arrow[l] \arrow[r] & U_d \arrow[r] & X_d = \operatorname{Spec}(A_d) \end{tikzcd}$$

such that $x \in U \subset X$ is affine open, $V = f^{-1}(U)$ and $V = U \times_{U_d} V_d$. Write $U = \operatorname{Spec}(A)$ and $V = \operatorname{Spec}(B)$ and observe that $B = A \otimes_{A_d} B_d$ and recall that $B_d = A_d e_1 \oplus \ldots \oplus A_d e_d$. Suppose we have $a_1, \ldots, a_r \in A$ and $b_1, \ldots, b_s \in B$. We may write $b_j = \sum a_{j,l} e_d$ with $a_{j,l} \in A$. Set $N = r + sd$ and consider the factorizations

$$\begin{tikzcd} V \arrow[r] \arrow[d] & V' = \mathbf{A}^N \times V_d \arrow[r] \arrow[d] & V_d \arrow[d] \\ U \arrow[r] & U' = \mathbf{A}^N \times U_d \arrow[r] & U_d \end{tikzcd}$$

Here the horizontal lower right arrow is given by the morphism $U \to U_d$ (from the earlier diagram) and the morphism $U \to \mathbf{A}^N$ given by $a_1, \ldots, a_r, a_{1,1}, \ldots, a_{s,d}$.

Then we see that the functions $a_1, \ldots, a_r$ are in the image of $\Gamma(U', \mathcal{O}_{U'}) \to \Gamma(U, \mathcal{O}_U)$ and the functions $b_1, \ldots, b_s$ are in the image of $\Gamma(V', \mathcal{O}_{V'}) \to \Gamma(V, \mathcal{O}_V)$. In this way we see that for any finite collection of elements[4] of the groups

$$\Gamma(V, \Omega^i_{Y/\mathbf{Z}}), \quad i = 0, 1, 2, \ldots \quad \text{and} \quad \Gamma(U, \Omega^j_{X/\mathbf{Z}}), \quad j = 0, 1, 2, \ldots$$

we can find a factorizations $V \to V' \to V_d$ and $U \to U' \to U_d$ with $V' = \mathbf{A}^N \times V_d$ and $U' = \mathbf{A}^N \times U_d$ as above such that these sections are the pullbacks of sections from

$$\Gamma(V', \Omega^i_{V'/\mathbf{Z}}), \quad i = 0, 1, 2, \ldots \quad \text{and} \quad \Gamma(U', \Omega^j_{U'/\mathbf{Z}}), \quad j = 0, 1, 2, \ldots$$

The upshot of this is that to check $\mathrm{d} \circ \Theta_{Y/X} = \Theta_{Y/X} \circ \mathrm{d}$ it suffices to check this is true for $\Theta_{V'/U'}$. Similarly, for property (2) of the lemma.

By Discriminants, Lemmas 11.4 and 11.5 the scheme U_d is smooth and the morphism $V_d \to U_d$ is étale over a dense open of U_d. Hence the same is true for the morphism $V' \to U'$. Since $\Omega_{U'/\mathbf{Z}}$ is locally free and hence $\Omega^p_{U'/\mathbf{Z}}$ is torsion free, it suffices to check the desired relations after restricting to the open over which V' is finite étale. Then we may check the relations after a surjective étale base change. Hence we may split the finite étale cover and assume we are looking at a morphism of the form

$$\coprod\nolimits_{i=1,\ldots,d} W \longrightarrow W$$

with W smooth over $\mathbf{Z}$. In this case any local properties of our construction are trivial to check (provided they are true). This finishes the proof of (1) and (2).

Finally, we observe that (3) follows from (2) because $\Omega_{Y/S}$ is the quotient of $\Omega_{Y/\mathbf{Z}}$ by the submodule generated by pullbacks of local sections of $\Omega_{S/\mathbf{Z}}$. □

0FLD **Example 19.4.** Let A be a ring. Let $f = x^d + \sum_{0 \leq i < d} a_{d-i} x^i \in A[x]$. Let $B = A[x]/(f)$. By Proposition 19.3 we have a morphism of complexes

$$\Theta_{B/A} : \Omega^\bullet_B \longrightarrow \Omega^\bullet_A$$

In particular, if $t \in B$ denotes the image of $x \in A[x]$ we can consider the elements

$$\Theta_{B/A}(t^i \mathrm{d}t) \in \Omega^1_A, \quad i = 0, \ldots, d-1$$

What are these elements? By the same principle as used in the proof of Proposition 19.3 it suffices to compute this in the universal case, i.e., when $A = \mathbf{Z}[a_1, \ldots, a_d]$ or even when A is replaced by the fraction field $\mathbf{Q}(a_1, \ldots, a_d)$. Writing symbolically

$$f = \prod\nolimits_{i=1,\ldots,d} (x - \alpha_i)$$

we see that over $\mathbf{Q}(\alpha_1, \ldots, \alpha_d)$ the algebra B becomes split:

$$\mathbf{Q}(a_0, \ldots, a_{d-1})[x]/(f) \longrightarrow \prod\nolimits_{i=1,\ldots,d} \mathbf{Q}(\alpha_1, \ldots, \alpha_d), \quad t \longmapsto (\alpha_1, \ldots, \alpha_d)$$

Thus for example

$$\Theta(\mathrm{d}t) = \sum \mathrm{d}\alpha_i = -\mathrm{d}a_1$$

Next, we have

$$\Theta(t\mathrm{d}t) = \sum \alpha_i \mathrm{d}\alpha_i = a_1 \mathrm{d}a_1 - \mathrm{d}a_2$$

[4] After all these elements will be finite sums of elements of the form $a_0 \mathrm{d}a_1 \wedge \ldots \wedge \mathrm{d}a_i$ with $a_0, \ldots, a_i \in A$ or finite sums of elements of the form $b_0 \mathrm{d}b_1 \wedge \ldots \wedge \mathrm{d}b_j$ with $b_0, \ldots, b_j \in B$.

Next, we have

$$\Theta(t^2\mathrm{d}t) = \sum \alpha_i^2 \mathrm{d}\alpha_i = -a_1^2\mathrm{d}a_1 + a_1\mathrm{d}a_2 + a_2\mathrm{d}a_1 - \mathrm{d}a_3$$

(modulo calculation error), and so on. This suggests that if $f(x) = x^d - a$ then

$$\Theta_{B/A}(t^i\mathrm{d}t) = \begin{cases} 0 & \text{if} \quad i = 0, \ldots, d-2 \\ \mathrm{d}a & \text{if} \quad i = d-1 \end{cases}$$

in Ω_A. This is true for in this particular case one can do the calculation for the extension $\mathbf{Q}(a)[x]/(x^d - a)$ to verify this directly.

0FW2 **Lemma 19.5.** *Let p be a prime number. Let $X \to S$ be a smooth morphism of relative dimension d of schemes in characteristic p. The relative Frobenius $F_{X/S} : X \to X^{(p)}$ of X/S (Varieties, Definition 35.4) is finite syntomic and the corresponding map*

$$\Theta_{X/X^{(p)}} : F_{X/S,*}\Omega^\bullet_{X/S} \to \Omega^\bullet_{X^{(p)}/S}$$

is zero in all degrees except in degree d where it defines a surjection.

Proof. Observe that $F_{X/S}$ is a finite morphism by Varieties, Lemma 35.8. To prove that $F_{X/S}$ is flat, it suffices to show that the morphism $F_{X/S,s} : X_s \to X_s^{(p)}$ between fibres is flat for all $s \in S$, see More on Morphisms, Theorem 16.2. Flatness of $X_s \to X_s^{(p)}$ follows from Algebra, Lemma 127.1 (and the finiteness already shown). By More on Morphisms, Lemma 54.10 the morphism $F_{X/S}$ is a local complete intersection morphism. Hence $F_{X/S}$ is finite syntomic (see More on Morphisms, Lemma 54.8).

For every point $x \in X$ we may choose a commutative diagram

$$\begin{tikzcd} X \arrow[d] & U \arrow[l] \arrow[d, "\pi"'] \\ S & \mathbf{A}^d_S \arrow[l] \end{tikzcd}$$

where π is étale and $x \in U$ is open in X, see Morphisms, Lemma 34.20. Observe that $\mathbf{A}^d_S \to \mathbf{A}^d_S$, $(x_1, \ldots, x_d) \mapsto (x_1^p, \ldots, x_d^p)$ is the relative Frobenius for $\mathcal{A}^d_S$ over S. The commutative diagram

$$\begin{tikzcd} U \arrow[r, "F_{X/S}"'] \arrow[d, "\pi"'] & U^{(p)} \arrow[d, "\pi^{(p)}"] \\ \mathbf{A}^d_S \arrow[r, "x_i \mapsto x_i^p"] & \mathbf{A}^d_S \end{tikzcd}$$

of Varieties, Lemma 35.5 for $\pi : U \to \mathbf{A}^d_S$ is cartesian by Étale Morphisms, Lemma 14.3. Since the construction of Θ is compatible with base change and since $\Omega_{U/S} = \pi^*\Omega_{\mathbf{A}^d_S/S}$ (Lemma 2.2) we conclude that it suffices to show the lemma for $\mathbf{A}^d_S$.

Let A be a ring of characteristic p. Consider the unique A-algebra homomorphism $A[y_1, \ldots, y_d] \to A[x_1, \ldots, x_d]$ sending y_i to x_i^p. The arguments above reduce us to computing the map

$$\Theta^i : \Omega^i_{A[x_1,\ldots,x_d]/A} \to \Omega^i_{A[y_1,\ldots,y_d]/A}$$

We urge the reader to do the computation in this case for themselves. As in Example 19.4 we may reduce this to computing a formula for Θ^i in the universal case

$$\mathbf{Z}[y_1, \ldots, y_d] \to \mathbf{Z}[x_1, \ldots, x_d], \quad y_i \mapsto x_i^p$$

In turn, we can find the formula for Θ^i by computing in the complex case, i.e., for the $\mathbf{C}$-algebra map

$$\mathbf{C}[y_1, \ldots, y_d] \to \mathbf{C}[x_1, \ldots, x_d], \quad y_i \mapsto x_i^p$$

We may even invert $x_1, \ldots, x_d$ and $y_1, \ldots, y_d$. In this case, we have $\mathrm{d}x_i = p^{-1}x_i^{-p+1}\mathrm{d}y_i$. Hence we see that

$$\begin{aligned}\Theta^i(x_1^{e_1} \ldots x_d^{e_d}\mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_i) &= p^{-i}\Theta^i(x_1^{e_1-p+1} \ldots x_i^{e_i-p+1}x_{i+1}^{e_{i+1}} \ldots x_d^{e_d}\mathrm{d}y_1 \wedge \ldots \wedge \mathrm{d}y_i) \\ &= p^{-i}\mathrm{Trace}(x_1^{e_1-p+1} \ldots x_i^{e_i-p+1}x_{i+1}^{e_{i+1}} \ldots x_d^{e_d})\mathrm{d}y_1 \wedge \ldots \wedge \mathrm{d}y_i\end{aligned}$$

by the properties of Θ^i. An elementary computation shows that the trace in the expression above is zero unless $e_1, \ldots, e_i$ are congruent to -1 modulo p and $e_{i+1}, \ldots, e_d$ are divisible by p. Moreover, in this case we obtain

$$p^{d-i}y_1^{(e_1-p+1)/p} \ldots y_i^{(e_i-p+1)/p}y_{i+1}^{e_{i+1}/p} \ldots y_d^{e_d/p}\mathrm{d}y_1 \wedge \ldots \wedge \mathrm{d}y_i$$

We conclude that we get zero in characteristic p unless $d = i$ and in this case we get every possible d-form. □

20. Poincaré duality

0FW3 In this section we prove Poincar'e duality for the de Rham cohomology of a proper smooth scheme over a field. Let us first explain how this works for Hodge cohomology.

0FW4 **Lemma 20.1.** *Let k be a field. Let X be a nonempty smooth proper scheme over k equidimensional of dimension d. There exists a k-linear map*

$$t : H^d(X, \Omega^d_{X/k}) \longrightarrow k$$

unique up to precomposing by multiplication by a unit of $H^0(X, \mathcal{O}_X)$ with the following property: for all p, q the pairing

$$H^q(X, \Omega^p_{X/k}) \times H^{d-q}(X, \Omega^{d-p}_{X/k}) \longrightarrow k, \quad (\xi, \xi') \longmapsto t(\xi \cup \xi')$$

is perfect.

Proof. By Duality for Schemes, Lemma 27.1 we have $\omega_X^\bullet = \Omega^d_{X/k}[d]$. Since $\Omega_{X/k}$ is locally free of rank d (Morphisms, Lemma 32.12) we have

$$\Omega^d_{X/k} \otimes_{\mathcal{O}_X} (\Omega^p_{X/k})^\vee \cong \Omega^{d-p}_{X/k}$$

Thus we obtain a k-linear map $t : H^d(X, \Omega^d_{X/k}) \to k$ such that the statement is true by Duality for Schemes, Lemma 27.4. In particular the pairing $H^0(X, \mathcal{O}_X) \times H^d(X, \Omega^d_{X/k}) \to k$ is perfect, which implies that any k-linear map $t' : H^d(X, \Omega^d_{X/k}) \to k$ is of the form $\xi \mapsto t(g\xi)$ for some $g \in H^0(X, \mathcal{O}_X)$. Of course, in order for t' to still produce a duality between $H^0(X, \mathcal{O}_X)$ and $H^d(X, \Omega^d_{X/k})$ we need g to be a unit. Denote $\langle -, - \rangle_{p,q}$ the pairing constructed using t and denote $\langle -, - \rangle'_{p,q}$ the pairing constructed using t'. Clearly we have

$$\langle \xi, \xi' \rangle'_{p,q} = \langle g\xi, \xi' \rangle_{p,q}$$

for $\xi \in H^q(X, \Omega^p_{X/k})$ and $\xi' \in H^{d-q}(X, \Omega^{d-p}_{X/k})$. Since g is a unit, i.e., invertible, we see that using t' instead of t we still get perfect pairings for all p, q. $\square$

0FW5 **Lemma 20.2.** *Let k be a field. Let X be a smooth proper scheme over k. The map*

$$d : H^0(X, \mathcal{O}_X) \to H^0(X, \Omega^1_{X/k})$$

is zero.

Proof. Since X is smooth over k it is geometrically reduced over k, see Varieties, Lemma 25.4. Hence $H^0(X, \mathcal{O}_X) = \prod k_i$ is a finite product of finite separable field extensions k_i/k, see Varieties, Lemma 9.3. It follows that $\Omega_{H^0(X,\mathcal{O}_X)/k} = \prod \Omega_{k_i/k} = 0$ (see for example Algebra, Lemma 153.1). Since the map of the lemma factors as

$$H^0(X, \mathcal{O}_X) \to \Omega_{H^0(X,\mathcal{O}_X)/k} \to H^0(X, \Omega_{X/k})$$

by functoriality of the de Rham complex (see Section 2), we conclude. $\square$

0FW6 **Lemma 20.3.** *Let k be a field. Let X be a smooth proper scheme over k equidimensional of dimension d. The map*

$$d : H^d(X, \Omega^{d-1}_{X/k}) \to H^d(X, \Omega^d_{X/k})$$

is zero.

Proof. It is tempting to think this follows from a combination of Lemmas 20.2 and 20.1. However this doesn't work because the maps $\mathcal{O}_X \to \Omega^1_{X/k}$ and $\Omega^{d-1}_{X/k} \to \Omega^d_{X/k}$ are not $\mathcal{O}_X$-linear and hence we cannot use the functoriality discussed in Duality for Schemes, Remark 27.3 to conclude the map in Lemma 20.2 is dual to the one in this lemma.

We may replace X by a connected component of X. Hence we may assume X is irreducible. By Varieties, Lemmas 25.4 and 9.3 we see that $k' = H^0(X, \mathcal{O}_X)$ X is a finite separable extension k'/k. Since $\Omega_{k'/k} = 0$ (see for example Algebra, Lemma 153.1) we see that $\Omega_{X/k} = \Omega_{X/k'}$ (see Morphisms, Lemma 31.9). Thus we may replace k by k' and assume that $H^0(X, \mathcal{O}_X) = k$.

Assume $H^0(X, \mathcal{O}_X) = k$. We conclude that $\dim H^d(X, \Omega^d_{X/k}) = 1$ by Lemma 20.1. Assume first that the characteristic of k is a prime number p. Denote $F_{X/k} : X \to X^{(p)}$ the relative Frobenius of X over k; please keep in mind the facts proved about this morphism in Lemma 19.5. Consider the commutative diagram

$$\begin{tikzcd}
H^d(X, \Omega^{d-1}_{X/k}) \arrow[r] \arrow[d] & H^d(X^{(p)}, F_{X/k,*}\Omega^{d-1}_{X/k}) \arrow[r, "\Theta^{d-1}"'] \arrow[d] & H^d(X^{(p)}, \Omega^{d-1}_{X^{(p)}/k}) \arrow[d] \\
H^d(X, \Omega^d_{X/k}) \arrow[r] & H^d(X^{(p)}, F_{X/k,*}\Omega^d_{X/k}) \arrow[r, "\Theta^d"] & H^d(X^{(p)}, \Omega^d_{X^{(p)}/k})
\end{tikzcd}$$

The left two horizontal arrows are isomorphisms as $F_{X/k}$ is finite, see Cohomology of Schemes, Lemma 2.4. The right square commutes as $\Theta_{X^{(p)}/X}$ is a morphism of complexes and Θ^{d-1} is zero. Thus it suffices to show that Θ^d is nonzero (because the dimension of the source of the map Θ^d is 1 by the discussion above). However, we know that

$$\Theta^d : F_{X/k,*}\Omega^d_{X/k} \to \Omega^d_{X^{(p)}/k}$$

is surjective and hence surjective after applying the right exact functor $H^d(X^{(p)}, -)$ (right exactness by the vanishing of cohomology beyond d as follows from Cohomology, Proposition 20.7). Finally, $H^d(X^{(d)}, \Omega^d_{X^{(d)}/k})$ is nonzero for example because it is dual to $H^0(X^{(d)}, \mathcal{O}_{X^{(p)}})$ by Lemma 20.1 applied to $X^{(p)}$ over k. This finishes the proof in this case.

Finally, assume the characteristic of k is 0. We can write k as the filtered colimit of its finite type $\mathbf{Z}$-subalgebras R. For one of these we can find a cartesian diagram of schemes

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ \mathrm{Spec}(k) \arrow[r] & \mathrm{Spec}(R) \end{tikzcd}$$

such that $Y \to \mathrm{Spec}(R)$ is smooth of relative dimension d and proper. See Limits, Lemmas 10.1, 8.9, 16.3, and 13.1. The modules $M^{i,j} = H^j(Y, \Omega^i_{Y/R})$ are finite R-modules, see Cohomology of Schemes, Lemma 19.2. Thus after replacing R by a localization we may assume all of these modules are finite free. We have $M^{i,j} \otimes_R k = H^j(X, \Omega^i_{X/k})$ by flat base change (Cohomology of Schemes, Lemma 5.2). Thus it suffices to show that $M^{d-1,d} \to M^{d,d}$ is zero. This is a map of finite free modules over a domain, hence it suffices to find a dense set of primes $\mathfrak{p} \subset R$ such that after tensoring with $\kappa(\mathfrak{p})$ we get zero. Since R is of finite type over $\mathbf{Z}$, we can take the collection of primes $\mathfrak{p}$ whose residue field has positive characteristic (details omitted). Observe that

$$M^{d-1,d} \otimes_R \kappa(\mathfrak{p}) = H^d(Y_{\kappa(\mathfrak{p})}, \Omega^{d-1}_{Y_{\kappa(\mathfrak{p})}/\kappa(\mathfrak{p})})$$

for example by Limits, Lemma 17.2. Similarly for $M^{d,d}$. Thus we see that $M^{d-1,d} \otimes_R \kappa(\mathfrak{p}) \to M^{d,d} \otimes_R \kappa(\mathfrak{p})$ is zero by the case of positive characteristic handled above. $\square$

0FW7 **Proposition 20.4.** *Let k be a field. Let X be a nonempty smooth proper scheme over k equidimensional of dimension d. There exists a k-linear map*

$$t : H^d_{dR}(X/k) \longrightarrow k$$

unique up to precomposing by multiplication by a unit of $H^0(X, \mathcal{O}_X)$ with the following property: for all i the pairing

$$H^i_{dR}(X/k) \times H^{2d-i}_{dR}(X/k) \longrightarrow k, \quad (\xi, \xi') \longmapsto t(\xi \cup \xi')$$

is perfect.

Proof. By the Hodge-to-de Rham spectral sequence (Section 6), the vanishing of $\Omega^i_{X/k}$ for $i > d$, the vanishing in Cohomology, Proposition 20.7 and the results of Lemmas 20.2 and 20.3 we see that $H^0_{dR}(X/k) = H^0(X, \mathcal{O}_X)$ and $H^d(X, \Omega^d_{X/k}) = H^{2d}_{dR}(X/k)$. More precisely, these identifications come from the maps of complexes

$$\Omega^\bullet_{X/k} \to \mathcal{O}_X[0] \quad \text{and} \quad \Omega^d_{X/k}[-d] \to \Omega^\bullet_{X/k}$$

Let us choose $t : H^{2d}_{dR}(X/k) \to k$ which via this identification corresponds to a t as in Lemma 20.1. Then in any case we see that the pairing displayed in the lemma is perfect for $i = 0$.

Denote $\underline{k}$ the constant sheaf with value k on X. Let us abbreviate $\Omega^\bullet = \Omega^\bullet_{X/k}$. Consider the map (4.0.1) which in our situation reads

$$\wedge : \mathrm{Tot}(\Omega^\bullet \otimes_{\underline{k}} \Omega^\bullet) \longrightarrow \Omega^\bullet$$

For every integer $p = 0, 1, \ldots, d$ this map annihilates the subcomplex $\mathrm{Tot}(\sigma_{\leq p}\Omega^\bullet \otimes_{\underline{k}} \sigma_{<d-p}\Omega^\bullet)$ for degree reasons. Hence we find that the restriction of $\wedge$ to the subcomplex $\mathrm{Tot}(\sigma_{\geq p}\Omega^\bullet \otimes_{\underline{k}} \Omega^\bullet)$ factors through a map of complexes

$$\gamma_p : \mathrm{Tot}(\sigma_{\leq p}\Omega^\bullet \otimes_{\underline{k}} \sigma_{\geq d-p}\Omega^\bullet) \longrightarrow \Omega^\bullet$$

Using the same procedure as in Section 4 we obtain cup products

$$H^i(X, \sigma_{\leq p}\Omega^\bullet) \times H^{2d-i}(X, \sigma_{\geq d-p}\Omega^\bullet) \longrightarrow H^{2d}_{dR}(X, \Omega^\bullet)$$

We will prove by induction on p that these cup products via t induce perfect pairings between $H^i(X, \sigma_{\leq p}\Omega^\bullet)$ and $H^{2d-i}(X, \sigma_{\geq d-p}\Omega^\bullet)$. For $p = d$ this is the assertion of the proposition.

The base case is $p = 0$. In this case we simply obtain the pairing between $H^i(X, \mathcal{O}_X)$ and $H^{d-i}(X, \Omega^d)$ of Lemma 20.1 and the result is true.

Induction step. Say we know the result is true for p. Then we consider the distinguished triangle

$$\Omega^{p+1}[-p-1] \to \sigma_{\leq p+1}\Omega^\bullet \to \sigma_{\leq p}\Omega^\bullet \to \Omega^{p+1}[-p]$$

and the distinguished triangle

$$\sigma_{\geq d-p}\Omega^\bullet \to \sigma_{\geq d-p-1}\Omega^\bullet \to \Omega^{d-p-1}[-d+p+1] \to (\sigma_{\geq d-p}\Omega^\bullet)[1]$$

Observe that both are distinguished triangles in the homotopy category of complexes of sheaves of $\underline{k}$-modules; in particular the maps $\sigma_{\leq p}\Omega^\bullet \to \Omega^{p+1}[-p]$ and $\Omega^{d-p-1}[-d+p+1] \to (\sigma_{\geq d-p}\Omega^\bullet)[1]$ are given by actual maps of complexes, namely using the differential $\Omega^p \to \Omega^{p+1}$ and the differential $\Omega^{d-p-1} \to \Omega^{d-p}$. Consider the long exact cohomology sequences associated to these distinguished triangles

$$\begin{array}{ccc}
H^{i-1}(X, \sigma_{\leq p}\Omega^\bullet) & & H^{2d-i+1}(X, \sigma_{\geq d-p}\Omega^\bullet) \\
\downarrow a & & \uparrow a' \\
H^i(X, \Omega^{p+1}[-p-1]) & & H^{2d-i}(X, \Omega^{d-p-1}[-d+p+1]) \\
\downarrow b & & \uparrow b' \\
H^i(X, \sigma_{\leq p+1}\Omega^\bullet) & & H^{2d-i}(X, \sigma_{\geq d-p-1}\Omega^\bullet) \\
\downarrow c & & \uparrow c' \\
H^i(X, \sigma_{\leq p}\Omega^\bullet) & & H^{2d-i}(X, \sigma_{\geq d-p}\Omega^\bullet) \\
\downarrow d & & \uparrow d' \\
H^{i+1}(X, \Omega^{p+1}[-p-1]) & & H^{2d-i-1}(X, \Omega^{d-p-1}[-d+p+1])
\end{array}$$

By induction and Lemma 20.1 we know that the pairings constructed above between the k-vectorspaces on the first, second, fourth, and fifth rows are perfect. By the 5-lemma, in order to show that the pairing between the cohomology groups in the middle row is perfect, it suffices to show that the pairs (a, a'), (b, b'), (c, c'), and (d, d') are compatible with the given pairings (see below).

Let us prove this for the pair (c, c'). Here we observe simply that we have a commutative diagram

$$\begin{tikzcd}
\mathrm{Tot}(\sigma_{\leq p}\Omega^\bullet \otimes_{\underline{k}} \sigma_{\geq d-p}\Omega^\bullet) \arrow[d, "\gamma_p"'] & \mathrm{Tot}(\sigma_{\leq p+1}\Omega^\bullet \otimes_{\underline{k}} \sigma_{\geq d-p}\Omega^\bullet) \arrow[l] \arrow[d] \\
\Omega^\bullet & \mathrm{Tot}(\sigma_{\leq p+1}\Omega^\bullet \otimes_{\underline{k}} \sigma_{\geq d-p-1}\Omega^\bullet) \arrow[l, "\gamma_{p+1}"']
\end{tikzcd}$$

Hence if we have $\alpha \in H^i(X, \sigma_{\leq p+1}\Omega^\bullet)$ and $\beta \in H^{2d-i}(X, \sigma_{\geq d-p}\Omega^\bullet)$ then we get $\gamma_p(\alpha \cup c'(\beta)) = \gamma_{p+1}(c(\alpha) \cup \beta)$ by functoriality of the cup product.

Similarly for the pair (b, b') we use the commutative diagram

$$\begin{tikzcd}
\mathrm{Tot}(\sigma_{\leq p+1}\Omega^\bullet \otimes_{\underline{k}} \sigma_{\geq d-p-1}\Omega^\bullet) \arrow[d, "\gamma_{p+1}"'] & \mathrm{Tot}(\Omega^{p+1}[-p-1] \otimes_{\underline{k}} \sigma_{\geq d-p-1}\Omega^\bullet) \arrow[l] \arrow[d] \\
\Omega^\bullet & \Omega^{p+1}[-p-1] \otimes_{\underline{k}} \Omega^{d-p-1}[-d+p+1] \arrow[l, "\wedge"']
\end{tikzcd}$$

and argue in the same manner.

For the pair (d, d') we use the commutative diagram

$$\begin{tikzcd}
\Omega^{p+1}[-p] \otimes_{\underline{k}} \Omega^{d-p-1}[-d+p] \arrow[d] & \mathrm{Tot}(\sigma_{\leq p}\Omega^\bullet \otimes_{\underline{k}} \Omega^{d-p-1}[-d+p]) \arrow[l] \arrow[d] \\
\Omega^\bullet & \mathrm{Tot}(\sigma_{\leq p}\Omega^\bullet \otimes_{\underline{k}} \sigma_{\geq d-p}\Omega^\bullet) \arrow[l]
\end{tikzcd}$$

and we look at cohomology classes in $H^i(X, \sigma_{\leq p}\Omega^\bullet)$ and $H^{2d-i}(X, \Omega^{d-p-1}[-d+p])$. Changing i to $i-1$ we get the result for the pair (a, a') thereby finishing the proof that our pairings are perfect.

We omit the argument showing the uniqueness of t up to precomposing by multiplication by a unit in $H^0(X, \mathcal{O}_X)$. $\square$

21. Chern classes

0FW8 The results proved so far suffice to use the discussion in Weil Cohomology Theories, Section 12 to produce chern classes in de Rham cohomology.

0FW9 **Lemma 21.1.** *There is a unique rule which assigns to every quasi-compact and quasi-separated scheme X a total chern class*

$$c^{dR} : K_0(\mathit{Vect}(X)) \longrightarrow \prod\nolimits_{i \geq 0} H^{2i}_{dR}(X/\mathbf{Z})$$

with the following properties

1. *we have $c^{dR}(\alpha + \beta) = c^{dR}(\alpha)c^{dR}(\beta)$ for $\alpha, \beta \in K_0(\mathit{Vect}(X))$,*
2. *if $f : X \to X'$ is a morphism of quasi-compact and quasi-separated schemes, then $c^{dR}(f^*\alpha) = f^*c^{dR}(\alpha)$,*
3. *given $\mathcal{L} \in \mathrm{Pic}(X)$ we have $c^{dR}([\mathcal{L}]) = 1 + c_1^{dR}(\mathcal{L})$*

The construction can easily be extended to all schemes, but to do so one needs to slightly upgrade the discussion in Weil Cohomology Theories, Section 12.

Proof. We will apply Weil Cohomology Theories, Proposition 12.1 to get this.

Let $\mathcal{C}$ be the category of all quasi-compact and quasi-separated schemes. This certainly satisfies conditions (1), (2), and (3) (a), (b), and (c) of Weil Cohomology Theories, Section 12.

As our contravariant functor A from $\mathcal{C}$ to the category of graded algebras will send X to $A(X) = \bigoplus_{i \geq 0} H^{2i}_{dR}(X/\mathbf{Z})$ endowed with its cup product. Functoriality is discussed in Section 3 and the cup product in Section 4. For the additive maps c_1^A we take c_1^{dR} constructed in Section 9.

In fact, we obtain commutative algebras by Lemma 4.1 which shows we have axiom (1) for A.

To check axiom (2) for A it suffices to check that $H^*_{dR}(X \coprod Y/\mathbf{Z}) = H^*_{dR}(X/\mathbf{Z}) \times H^*_{dR}(Y/\mathbf{Z})$. This is a consequence of the fact that de Rham cohomology is constructed by taking the cohomology of a sheaf of differential graded algebras (in the Zariski topology).

Axiom (3) for A is just the statement that taking first chern classes of invertible modules is compatible with pullbacks. This follows from the more general Lemma 9.1.

Axiom (4) for A is the projective space bundle formula which we proved in Proposition 14.1.

Axiom (5). Let X be a quasi-compact and quasi-separated scheme and let $\mathcal{E} \to \mathcal{F}$ be a surjection of finite locally free $\mathcal{O}_X$-modules of ranks $r+1$ and r. Denote $i : P' = \mathbf{P}(\mathcal{F}) \to \mathbf{P}(\mathcal{E}) = P$ the corresponding incusion morphism. This is a morphism of smooth projective schemes over X which exhibits P' as an effective Cartier divisor on P. Thus by Lemma 15.7 the complex of log poles for $P' \subset P$ over $\mathbf{Z}$ is defined. Hence for $a \in A(P)$ with $i^*a = 0$ we have $a \cup c_1^A(\mathcal{O}_P(P')) = 0$ by Lemma 15.6. This finishes the proof. □

0FWA **Remark 21.2.** The analogues of Weil Cohomology Theories, Lemmas 12.2 (splitting principle) and 12.3 (chern classes of tensor products) hold for de Rham chern classes on quasi-compact and quasi-separated schemes. This is clear as we've shown in the proof of Lemma 21.1 that all the axioms of Weil Cohomology Theories, Section 12 are satisfied.

Working with schemes over $\mathbf{Q}$ we can construct a chern character.

0FWB **Lemma 21.3.** *There is a unique rule which assigns to every quasi-compact and quasi-separated scheme X over $\mathbf{Q}$ a "chern character"*

$$ch^{dR} : K_0(\mathit{Vect}(X)) \longrightarrow \prod\nolimits_{i \geq 0} H^{2i}_{dR}(X/\mathbf{Q})$$

with the following properties

1. *ch^{dR} is a ring map for all X,*
2. *if $f : X' \to X$ is a morphism of quasi-compact and quasi-separated schemes over $\mathbf{Q}$, then $f^* \circ ch^{dR} = ch^{dR} \circ f^*$, and*
3. *given $\mathcal{L} \in \mathrm{Pic}(X)$ we have $ch^{dR}([\mathcal{L}]) = \exp(c_1^{dR}(\mathcal{L}))$.*

The construction can easily be extended to all schemes over $\mathbf{Q}$, but to do so one needs to slightly upgrade the discussion in Weil Cohomology Theories, Section 12.

Proof. Exactly as in the proof of Lemma 21.1 one shows that the category of quasi-compact and quasi-separated schemes over $\mathbf{Q}$ together with the functor $A^*(X) = \bigoplus_{i \geq 0} H^{2i}_{dR}(X/\mathbf{Q})$ satisfy the axioms of Weil Cohomology Theories, Section 12. Moreover, in this case $A(X)$ is a $\mathbf{Q}$-algebra for all X. Hence the lemma follows from Weil Cohomology Theories, Proposition 12.4. $\square$

22. A Weil cohomology theory

0FWC Let k be a field of characteristic 0. In this section we prove that the functor

$$X \longmapsto H^*_{dR}(X/k)$$

defines a Weil cohomology theory over k with coefficients in k as defined in Weil Cohomology Theories, Definition 11.4. We will proceed by checking the constructions earlier in this chapter provide us with data (D0), (D1), and (D2') satisfying axioms (A1) – (A9) of Weil Cohomology Theories, Section 14.

Throughout the rest of this section we fix the field k of characteristic 0 and we set $F = k$. Next, we take the following data

- (D0) For our 1-dimensional F vector space $F(1)$ we take $F(1) = F = k$.
- (D1) For our functor H^* we take the functor sending a smooth projective scheme X over k to $H^*_{dR}(X/k)$. Functoriality is discussed in Section 3 and the cup product in Section 4. We obtain graded commutative F-algebras by Lemma 4.1.
- (D2') For the maps $c_1^H : \text{Pic}(X) \to H^2(X)(1)$ we use the de Rham first chern class introduced in Section 9.

We are going to show axioms (A1) – (A9) hold.

In this paragraph, we are going to reduce the checking of the axioms to the case where k is algebraically closed by using Weil Cohomology Theories, Lemma 14.18. Denote k' the algebraic closure of k. Set $F' = k'$. We obtain data (D0), (D1), (D2') over k' with coefficient field F' in exactly the same way as above. By Lemma 3.5 there are functorial isomorphisms

$$H^{2d}_{dR}(X/k) \otimes_k k' \longrightarrow H^{2d}_{dR}(X_{k'}/k')$$

for X smooth and projective over k. Moreover, the diagrams

$$\begin{tikzcd}
\text{Pic}(X) \arrow[r, "c_1^{dR}"'] \arrow[d] & H^2_{dR}(X/k) \arrow[d] \\
\text{Pic}(X_{k'}) \arrow[r, "c_1^{dR}"] & H^2_{dR}(X_{k'}/k')
\end{tikzcd}$$

commute by Lemma 9.1. This finishes the proof of the reduction.

Assume k is algebraically closed field of characteristic zero. We will show axioms (A1) – (A9) for the data (D0), (D1), and (D2') given above.

Axiom (A1). Here we have to check that $H^*_{dR}(X \coprod Y/k) = H^*_{dR}(X/k) \times H^*_{dR}(Y/k)$. This is a consequence of the fact that de Rham cohomology is constructed by taking the cohomology of a sheaf of differential graded algebras (in the Zariski topology).

Axiom (A2). This is just the statement that taking first chern classes of invertible modules is compatible with pullbacks. This follows from the more general Lemma 9.1.

Axiom (A3). This follows from the more general Proposition 14.1.

Axiom (A4). This follows from the more general Lemma 15.6.

Already at this point, using Weil Cohomology Theories, Lemmas 14.1 and 14.2, we obtain a chern character and cycle class maps

$$\gamma : \mathrm{CH}^*(X) \longrightarrow \bigoplus\nolimits_{i \geq 0} H^{2i}_{dR}(X/k)$$

for X smooth projective over k which are graded ring homomorphisms compatible with pullbacks between morphisms $f : X \to Y$ of smooth projective schemes over k.

Axiom (A5). We have $H^*_{dR}(\mathrm{Spec}(k)/k) = k = F$ in degree 0. We have the Künneth formula for the product of two smooth projective k-schemes by Lemma 8.2 (observe that the derived tensor products in the statement are harmless as we are tensoring over the field k).

Axiom (A7). This follows from Proposition 17.4.

Axiom (A8). Let X be a smooth projective scheme over k. By the explanatory text to this axiom in Weil Cohomology Theories, Section 14 we see that $k' = H^0(X, \mathcal{O}_X)$ is a finite separable k-algebra. It follows that $H^*_{dR}(\mathrm{Spec}(k')/k) = k'$ sitting in degree 0 because $\Omega_{k'/k} = 0$. By Lemma 20.2 we also have $H^0_{dR}(X, \mathcal{O}_X) = k'$ and we get the axiom.

Axiom (A6). Let X be a nonempty smooth projective scheme over k which is equidimensional of dimension d. Denote $\Delta : X \to X \times_{\mathrm{Spec}(k)} X$ the diagonal morphism of X over k. We have to show that there exists a k-linear map

$$\lambda : H^{2d}_{dR}(X/k) \longrightarrow k$$

such that $(1 \otimes \lambda)\gamma([\Delta]) = 1$ in $H^0_{dR}(X/k)$. Let us write

$$\gamma = \gamma([\Delta]) = \gamma_0 + \ldots + \gamma_{2d}$$

with $\gamma_i \in H^i_{dR}(X/k) \otimes_k H^{2d-i}_{dR}(X/k)$ the Künneth components. Our problem is to show that there is a linear map $\lambda : H^{2d}_{dR}(X/k) \to k$ such that $(1 \otimes \lambda)\gamma_0 = 1$ in $H^0_{dR}(X/k)$.

Let $X = \coprod X_i$ be the decomposition of X into connected and hence irreducible components. Then we have correspondingly $\Delta = \coprod \Delta_i$ with $\Delta_i \subset X_i \times X_i$. It follows that

$$\gamma([\Delta]) = \sum \gamma([\Delta_i])$$

and moreover $\gamma([\Delta_i])$ corresponds to the class of $\Delta_i \subset X_i \times X_i$ via the decomposition

$$H^*_{dR}(X \times X) = \prod\nolimits_{i,j} H^*_{dR}(X_i \times X_j)$$

We omit the details; one way to show this is to use that in $\mathrm{CH}^0(X \times X)$ we have idempotents $e_{i,j}$ corresponding to the open and closed subschemes $X_i \times X_j$ and to use that γ is a ring map which sends $e_{i,j}$ to the corresponding idempotent in the displayed product decomposition of cohomology. If we can find $\lambda_i : H^{2d}_{dR}(X_i/k) \to k$ with $(1 \otimes \lambda_i)\gamma([\Delta_i]) = 1$ in $H^0_{dR}(X_i/k)$ then taking $\lambda = \sum \lambda_i$ will solve the problem for X. Thus we may and do assume X is irreducible.

Proof of Axiom (A6) for X irreducible. Since k is algebraically closed we have $H^0_{dR}(X/k) = k$ because $H^0(X, \mathcal{O}_X) = k$ as X is a projective variety over an algebraically closed field (see Varieties, Lemma 9.3 for example). Hence it clearly suffices to show that $\gamma([\Delta])$ is nonzero. Let $x \in X$ be any closed point. Consider the cartesian diagram

$$\begin{tikzcd} x \arrow[r] \arrow[d] & X \arrow[d, "\Delta"] \\ X \arrow[r, "x \times \mathrm{id}"] & X \times_{\mathrm{Spec}(k)} X \end{tikzcd}$$

Compatibility of γ with pullbacks implies that $\gamma([\Delta])$ maps to $\gamma([x])$ in $H^{2d}_{dR}(X/k)$. We conclude two things from this: (a) the class $\gamma([x])$ is indepdendent of x, and (b) it suffices to find any zero cycle α on X such that $\gamma(\alpha) \neq 0$. To do this we choose a finite morphism

$$f : X \longrightarrow \mathbf{P}^d_k$$

To see such a morphism exist, see Intersection Theory, Section 23 and in particular Lemma 23.1. Observe that f is finite syntomic (local complete intersection morphism by More on Morphisms, Lemma 54.10 and flat by Algebra, Lemma 127.1). By Proposition 19.3 we have a trace map

$$\Theta_f : f_*\Omega^\bullet_{X/k} \longrightarrow \Omega^\bullet_{\mathbf{P}^d_k/k}$$

whose composition with the canonical map

$$\Omega^\bullet_{\mathbf{P}^d_k/k} \longrightarrow f_*\Omega^\bullet_{X/k}$$

is multiplication by the degree of f. Hence we see that we get a map

$$\Theta : H^{2d}_{dR}(X/k) \to H^{2d}_{dR}(\mathbf{P}^d_k/k)$$

such that $\Theta \circ f^*$ is multiplication by a positive integer. Hence if we can find a zero cycle on $\mathbf{P}^d_k$ whose class is nonzero, then we conclude by the compatibility of γ with pullbacks. This is true by Lemma 11.4 and this finishes the proof of axiom (A6).

Axiom (A9). Let $Y \subset X$ be a nonempty smooth divisor on a nonempty smooth equidimensional projective scheme X over k of dimension d. We have to show that the diagram

$$\begin{tikzcd} H^{2d-2}_{dR}(X/k) \arrow[r, "c_1^{dR}(\mathcal{O}_X(Y)) \cap -"'] \arrow[d, "restriction"'] & H^{2d}_{dR}(X) \arrow[d, "\lambda_X"] \\ H^{2d-2}_{dR}(Y/k) \arrow[r, "\lambda_Y"] & k \end{tikzcd}$$

commutes where λ_X and λ_Y are as in axiom (A6). By Weil Cohomology Theories, Remark 14.6 the maps λ_X and λ_Y are unique. Above we have seen that if we decompose $X = \coprod X_i$ into connected (equivalently irreducible) components, then we have correspondingly $\lambda_X = \sum \lambda_{X_i}$. Similarly, if we decompoese $Y = \coprod Y_j$ into connected (equivalently irreducible) components, then we have $\lambda_Y = \sum \lambda_{Y_j}$. Moreover, in this case we have $\mathcal{O}_X(Y) = \otimes_j \mathcal{O}_X(Y_j)$ and hence

$$c_1^{dR}(\mathcal{O}_X(Y)) = \sum\nolimits_j c_1^{dR}(\mathcal{O}_X(Y_j))$$

in $H^2_{dR}(X/k)$. A straightforward diagram chase shows that it suffices to prove the commutativity of the diagram in case X and Y are both irreducible. In this case

$H^{2d-2}_{dR}(Y/k)$ is 1-dimensional (namely, we have Poincar'e duality for Y by Weil Cohomology Theories, Lemma 14.5). By axiom (A4) the kernel of restriction (left vertical arrow) is contained in the kernel of cupping with $c_1^{dR}(\mathcal{O}_X(Y))$. This means it suffices to find one cohomology class $a \in H^{2d-2}_{dR}(X)$ whose restriction to Y is nonzero such that we have commutativity in the diagram for a. Take any ample invertible module $\mathcal{L}$ and set

$$a = c_1^{dR}(\mathcal{L})^{d-1}$$

Then we know that $a|_Y = c_1^{dR}(\mathcal{L}|_Y)^{d-1}$ and hence

$$\lambda_Y(a|_Y) = \deg(c_1(\mathcal{L}|_Y)^{d-1} \cap [Y])$$

by our description of λ_Y above. This is a positive integer for by Chow Homology, Lemma 40.4 combined with Varieties, Lemma 44.9. Similarly, we find

$$\lambda_X(c_1^{dR}(\mathcal{O}_X(Y)) \cap a) = \deg(c_1(\mathcal{O}_X(Y)) \cap c_1(\mathcal{L})^{d-1} \cap [X])$$

Since we know that $c_1(\mathcal{O}_X(Y)) \cap [X] = [Y]$ more or less by definition we have an equality of zero cycles

$$(Y \to X)_* \left(c_1(\mathcal{L}|_Y)^{d-1} \cap [Y]\right) = c_1(\mathcal{O}_X(Y)) \cap c_1(\mathcal{L})^{d-1} \cap [X]$$

on X. Thus these cycles have the same degree and the proof is complete.

0FWD **Proposition 22.1.** *Let k be a field of characteristic zero. The functor that sends a smooth projective scheme X over k to $H^*_{dR}(X/k)$ is a Weil cohomology theory in the sense of Weil Cohomology Theories, Definition 11.4.*

Proof. In the discussion above we showed that our data (D0), (D1), (D2') satisfies axioms (A1) – (A9) of Weil Cohomology Theories, Section 14. Hence we conclude by Weil Cohomology Theories, Proposition 14.17.

Please don't read what follows. In the proof of the assertions we also used Lemmas 3.5, 9.1, 15.6, 8.2, 20.2, and 11.4, Propositions 14.1, 17.4, and 19.3, Weil Cohomology Theories, Lemmas 14.18, 14.1, 14.2, and 14.5, Weil Cohomology Theories, Remark 14.6, Varieties, Lemmas 9.3 and 44.9, Intersection Theory, Section 23 and Lemma 23.1, More on Morphisms, Lemma 54.10, Algebra, Lemma 127.1, and Chow Homology, Lemma 40.4. ☐

0FWE **Remark 22.2.** In exactly the same manner as above one can show that Hodge cohomology $X \mapsto H^*_{Hodge}(X/k)$ equipped with c_1^{Hodge} determines a Weil cohomology theory. If we ever need this, we will precisely formulate and prove this here. This leads to the following amusing consequence: If the betti numbers of a Weil cohomology theory are independent of the chosen Weil cohomology theory (over our field k of characteristic 0), then the Hodge-to-de Rham spectral sequence degenerates at E_1! Of course, the degeneration of the Hodge-to-de Rham spectral sequence is known (see for example [DI87] for a marvelous algebraic proof), but it is by no means an easy result! This suggests that proving the independence of betti numbers is a hard problem as well and as far as we know is still an open problem. See Weil Cohomology Theories, Remark 11.5 for a related question.

23. Other chapters

References

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