1. Introduction

In this chapter we start with a discussion of the de Rham complex of a ring map and we end with a proof that de Rham cohomology defines a Weil cohomology theory when the base field has characteristic zero.

2. The de Rham complex

Let \( p : X \rightarrow S \) be a morphism of schemes. There is a complex

\[
\Omega^\bullet_{X/S} = \mathcal{O}_{X/S} \rightarrow \Omega^1_{X/S} \rightarrow \Omega^2_{X/S} \rightarrow \ldots
\]

of \( p^{-1}\mathcal{O}_S \)-modules with \( \Omega^i_{X/S} = \wedge^i(\Omega_{X/S}) \) placed in degree \( i \) and differential determined by the rule \( d(g_0 \wedge g_1 \wedge \ldots \wedge g_p) = dg_0 \wedge g_1 \wedge \ldots \wedge g_p \) on local sections. See Modules, Section 26.

Given a commutative diagram

\[
\begin{array}{ccc}
X' & \rightarrow & X \\
\downarrow & & \downarrow \\
S' & \rightarrow & S
\end{array}
\]

of schemes, there are canonical maps of complexes \( f^\ast \Omega^\bullet_{X/S} \rightarrow \Omega^\bullet_{X'/S'} \) and \( \Omega^\bullet_{X/S} \rightarrow f_\ast \Omega^\bullet_{X'/S'} \). See Modules, Section 26. Linearizing, for every \( p \) we obtain a linear map \( f^\ast \Omega^p_{X/S} \rightarrow \Omega^p_{X'/S'} \).

In particular, if \( f : Y \rightarrow X \) be a morphism of schemes over a base scheme \( S \), then there is a map of complexes

\[
\Omega^\bullet_{X/S} \rightarrow f_\ast \Omega^\bullet_{Y/S}
\]
Linearizing, we see that for every $p \geq 0$ we obtain a canonical map
$$\Omega^p_{X/S} \otimes \mathcal{O}_X \xrightarrow{f_*} f_*\Omega^p_{Y/S}$$

**Lemma 2.1.** Let

\[
\begin{array}{ccc}
X' & \xrightarrow{f} & X \\
\downarrow & & \downarrow \\
S' & \xrightarrow{f} & S
\end{array}
\]

be a cartesian diagram of schemes. Then the maps discussed above induce isomorphisms $f^*\Omega^p_{Y/X} \to \Omega^p_{Y'/X'}$.

**Proof.** Combine Morphisms, Lemma 31.10 with the fact that formation of exterior power commutes with base change. \(\square\)

### 3. de Rham cohomology

Let $f : X \to S$ be a morphism of schemes. We define the de Rham cohomology of $X$ over $S$ to be the cohomology groups
$$H^i_{dR}(X/S) = H^i(R\Gamma(X, \Omega^\bullet_{X/S}))$$

Since $\Omega^\bullet_{X/S}$ is a complex of $f^{-1}\mathcal{O}_S$-modules, these cohomology groups are naturally modules over $H^0(S, \mathcal{O}_S)$.

### 4. Comparing sheaves of differential forms

The goal of this section is to compare the sheaves $\Omega^p_{X/Z}$ and $\Omega^p_{Y/Z}$ when given a locally quasi-finite syntomic morphism of schemes $f : Y \to X$. The result will be applied in Section 5 to the construction of the trace map on de Rham complexes if $f$ is finite.

**Lemma 4.1.** Let $R$ be a ring and consider a commutative diagram

\[
\begin{array}{cccccc}
0 & \longrightarrow & K^0 & \longrightarrow & L^0 & \longrightarrow & M^0 & \longrightarrow & 0 \\
& & \uparrow \sigma & & \uparrow & & \uparrow & & \\
& & L^{-1} & \longrightarrow & M^{-1} & & & & \\
\end{array}
\]

of $R$-modules with exact top row and $M^0$ and $M^{-1}$ finite free of the same rank. Then there are canonical maps
$$\wedge^i(H^0(L^\bullet)) \longrightarrow \wedge^i(K^0) \otimes_R \det(M^\bullet)$$
whose composition with $\wedge^i(K^0) \to \wedge^i(H^0(L^\bullet))$ is equal to multiplication with $\delta(M^\bullet)$.

**Proof.** Say $M^0$ and $M^{-1}$ are free of rank $n$. For every $i \geq 0$ there is a canonical surjection
$$\pi_i : \wedge^{n+i}(L^0) \longrightarrow \wedge^i(K^0) \otimes \wedge^n(M^0)$$
whose kernel is the submodule generated by wedges $l_1 \wedge \ldots \wedge l_{n+i}$ such that $>i$ of the $l_j$ are in $K^0$. On the other hand, the exact sequence
$$L^{-1} \longrightarrow L^0 \longrightarrow H^0(L^\bullet) \longrightarrow 0$$
similarly produces canonical maps
$$\wedge^i(H^0(L^\bullet)) \otimes \wedge^n(L^{-1}) \longrightarrow \wedge^{n+i}(L^0)$$
There exists a unique rule that to every locally quasi-finite syntomic morphism of schemes $f : Y \to X$ assigns $\mathcal{O}_Y$-module maps

$$c^p_{Y/X} : \Omega^p_{Y/Z} \to f^*\Omega^p_{X/Z} \otimes_{\mathcal{O}_Y} \det(\mathcal{N}L_{Y/X})$$

satisfying the following two properties

1. the composition with $f^*\Omega^p_{X/Z} \to \Omega^p_{Y/Z}$ is multiplication by $\delta(\mathcal{N}L_{Y/X})$, and
2. the rule is compatible with restriction to opens and with base change.

Proof. This proof is very similar to the proof of Discriminants, Proposition 13.2 and we suggest the reader look at that proof first. We fix $p \geq 0$ throughout the proof.

Let us reformulate the statement. Consider the category $\mathcal{C}$ whose objects, denoted $Y/X$, are locally quasi-finite syntomic morphism $f : Y \to X$ of schemes and whose morphisms $b/a : Y'/X' \to Y/X$ are commutative diagrams

$$
\begin{array}{ccc}
Y' & \xrightarrow{b} & Y \\
\downarrow{f'} & & \downarrow{f} \\
X' & \xrightarrow{a} & X
\end{array}
$$

which induce an isomorphism of $Y'$ with an open subscheme of $X' \times_X Y$. The lemma means that for every object $Y/X$ of $\mathcal{C}$ we have maps $c^p_{Y/X}$ with property
(1) and for every morphism $b/a : Y'/X' \to Y/X$ of $\mathcal{C}$ we have $b^*c^p_{Y'/X'} = c^p_{Y/X'}$ via the identifications $b^*\det(NL_{Y'/X'}) = \det(NL_{Y'/X'}).$ (Discriminants, Section 13) and $b^*\Omega^p_{Y'/X'} = \Omega^p_{Y'/X'}.$ (Lemma 2.1).

Given $Y/X$ in $\mathcal{C}$ and $y \in Y$ we can find an affine open $V \subset Y$ and $U \subset X$ with $f(V) \subset U$ such that there exists some maps

$$\Omega^p_{Y/Z}|_V \longrightarrow \left(f^*\Omega^p_{X/Z} \otimes_{\mathcal{O}_V} \det(NL_{Y/X})\right)|_V$$

with property (1). This follows from picking affine opens as in Discriminants, Lemma 10.1 part (5) and Remark 4.2. If $\Omega^p_{X/Z}$ is finite locally free and annihilator of the section $\delta(NL_{Y/X})$ is zero, then these local maps are unique and automatically glue!

Let $\mathcal{C}_{nice} \subset \mathcal{C}$ denote the full subcategory of $Y/X$ such that

1. $X$ is of finite type over $Z$,
2. $\Omega_{X/Z}$ is locally free, and
3. the annihilator of $\delta(NL_{Y/X})$ is zero.

By the remarks in the previous paragraph, we see that for any object $Y/X$ of $\mathcal{C}_{nice}$ we have a unique map $c^p_{Y/X}$ satisfying condition (1). If $b/a : Y'/X' \to Y/X$ is a morphism of $\mathcal{C}_{nice}$, then $b^*c^p_{Y/X}$ is equal to $c^p_{Y'/X'}$, because $b^*\delta(NL_{Y/X}) = \delta(NL_{Y'/X'})$ (see Discriminants, Section 13). In other words, we have solved the problem on the full subcategory $\mathcal{C}_{nice}$. For $Y/X$ in $\mathcal{C}_{nice}$ we continue to denote $c^p_{Y/X}$ the solution we’ve just found.

Consider morphisms

$$Y_1/X_1 \leftarrow b_{1/a_1} Y/X \xrightarrow{b_{2/a_2}} Y_2/X_2$$

in $\mathcal{C}$ such that $Y_1/X_1$ and $Y_2/X_2$ are objects of $\mathcal{C}_{nice}$. Claim. $b_{1/a_1}c^p_{Y_1/X_1} = b_{2/a_2}c^p_{Y_2/X_2}$.

We will first show that the claim implies the lemma and then we will prove the claim.

Let $d, n \geq 1$ and consider the locally quasi-finite syntomic morphism $Y_{n,d} \to X_{n,d}$ constructed in Discriminants, Example 10.5. Then $Y_{n,d}$ and $X_{n,d}$ are irreducible schemes of finite type and smooth over $Z$. Namely, $X_{n,d}$ is a spectrum of a polynomial ring over $\mathcal{O}$. The morphism $Y_{n,d} \to X_{n,d}$ is locally quasi-finite syntomic and étale over a dense open, see Discriminants, Lemma 10.1. Thus $\delta(NL_{Y_{n,d}/X_{n,d}})$ is nonzero: for example we have the local description of $\delta(NL_{Y/X})$ in Discriminants, Remark 13.1 and we have the local description of étale morphisms in Morhisms, Lemma 34.15 part (8). Now a nonzero section of an invertible module over an irreducible regular scheme has vanishing annihilator. Thus $Y_{n,d}/X_{n,d}$ is an object of $\mathcal{C}_{nice}$.

Let $Y/X$ be an arbitrary object of $\mathcal{C}$. By Discriminants, Lemma 10.7 we can find $n, d \geq 1$ and morphisms

$$Y/X \leftarrow V/U \xrightarrow{b/a} Y_{n,d}/X_{n,d}$$

of $\mathcal{C}$ such that $V \subset Y$ and $U \subset X$ are open. Thus we can pullback the canonical morphism $c^p_{Y_{n,d}/X_{n,d}}$ constructed above by $b$ to $V$. The claim guarantees these local isomorphisms glue! Thus we get a well defined global maps $c^p_{Y/X}$ with property (1). If $b/a : Y'/X' \to Y/X$ is a morphism of $\mathcal{C}$, then the claim also implies that...
Thus it suffices to prove the claim.

In the rest of the proof we prove the claim. We may pick a point \( y \in Y \) and prove the maps agree in an open neighbourhood of \( y \). Thus we may replace \( Y_1, Y_2 \) by open neighbourhoods of the image of \( y \) in \( Y_1 \) and \( Y_2 \). Thus we may assume \( Y, X, Y_1, Y_2, X_2 \) are affine. We may write \( X = \text{lim} X_\lambda \) as a cofiltered limit of affine schemes of finite type over \( X_1 \times X_2 \). For each \( \lambda \) we get
\[
Y_1 \times X_1 X_\lambda \quad \text{and} \quad X_\lambda \times X_2 Y_2
\]
If we take limits we obtain
\[
\text{lim} Y_1 \times X_1 X_\lambda = Y_1 \times X_1 X \supset Y \subset X \times X_2 Y_2 = \text{lim} X_\lambda \times X_2 Y_2
\]
By Limits, Lemma \([10.1]\) we can find a \( \lambda \) and opens \( V_{1,\lambda} \subset Y_1 \times X_1 X_\lambda \) and \( V_{2,\lambda} \subset X_\lambda \times X_2 Y_2 \) whose base change to \( X \) recovers \( Y \) (on both sides). After increasing \( \lambda \) we may assume there is an isomorphism \( V_{1,\lambda} \to V_{2,\lambda} \) whose base change to \( X \) is the identity on \( Y \), see Limits, Lemma \([10.1]\). Then we have the commutative diagram
\[
\begin{array}{ccc}
Y/X & \xrightarrow{b_1/a_1} & Y_1/X_1 \\
\downarrow & & \downarrow \\
V_{1,\lambda}/X_\lambda & \xrightarrow{b_2/a_2} & Y_2/X_2
\end{array}
\]
Thus it suffices to prove the claim for the lower row of the diagram and we reduce to the case discussed in the next paragraph.

Assume \( Y, X, Y_1, X_1, Y_2, X_2 \) are affine of finite type over \( \mathbf{Z} \). Write \( X = \text{Spec}(A), \ X_i = \text{Spec}(A_i) \). The ring map \( A_1 \to A \) corresponding to \( X \to X_1 \) is of finite type and hence we may choose a surjection \( A_1[x_1,\ldots,x_n] \to A \). Similarly, we may choose a surjection \( A_2[y_1,\ldots,y_m] \to A \). Set \( X'_1 = \text{Spec}(A_1[x_1,\ldots,x_n]) \) and \( X'_2 = \text{Spec}(A_2[y_1,\ldots,y_m]) \). Observe that \( \Omega_{X'_1}/\mathbf{Z} \) is the direct sum of the pullback of \( \Omega_{X_1}/\mathbf{Z} \) and a finite free module. Similarly for \( X'_2 \). Set \( Y'_1 = Y_1 \times X_1 X'_1 \) and \( Y'_2 = Y_2 \times X_2 X'_2 \). We get the following diagram
\[
Y_1/X_1 \leftarrow Y'_1/X'_1 \leftarrow Y/X \rightarrow Y'_2/X'_2 \rightarrow Y_2/X_2
\]
Since \( X'_1 \to X_1 \) and \( X'_2 \to X_2 \) are flat, the same is true for \( Y'_1 \to Y_1 \) and \( Y'_2 \to Y_2 \). It follows easily that the annihilators of \( \delta(\text{NL}_{Y'_1/X'_1}) \) and \( \delta(\text{NL}_{Y'_2/X'_2}) \) are zero.
Hence \( Y'_1/X'_1 \) and \( Y'_2/X'_2 \) are in \( \mathcal{C}_{\text{nice}} \). Thus the outer morphisms in the displayed diagram are morphisms of \( \mathcal{C}_{\text{nice}} \) for which we know the desired compatibilities. Thus it suffices to prove the claim for \( Y'_1/X'_1 \leftarrow Y/X \rightarrow Y'_2/X'_2 \). This reduces us to the case discussed in the next paragraph.

Assume \( Y, X, Y_1, X_1, Y_2, X_2 \) are affine of finite type over \( \mathbf{Z} \) and \( X \to X_1 \) and \( X \to X_2 \) are closed immersions. Consider the open embeddings \( Y_1 \times X_1 X \supset Y \subset X \times X_2 Y_2 \). There is an open neighbourhood \( V \subset Y \) of \( y \) which is a standard open of both \( Y_1 \times X_1 X \) and \( X \times X_2 Y_2 \). This follows from Schemes, Lemma \([11.5]\) applied to the scheme obtained by glueing \( Y_1 \times X_1 X \) and \( X \times X_2 Y_2 \) along \( Y \); details omitted. Since \( X \times X_2 Y_2 \) is a closed subscheme of \( Y_2 \) we can find a standard open \( V_2 \subset Y_2 \) such that \( V_2 \times X_2 X = V \). Similarly, we can find a standard open \( V_1 \subset Y_1 \) such that \( V_1 \times X_1 X = V \). After replacing \( Y, Y_1, Y_2 \) by \( V, V_1, V_2 \) we reduce to the case discussed in the next paragraph.
Assume $Y, X, Y_1, X_1, Y_2, X_2$ are affine of finite type over $\mathbb{Z}$ and $X \to X_1$ and $X \to X_2$ are closed immersions and $Y_1 \times X_1 = X = X_2 \times X_2$. Write $X = \text{Spec}(A)$, $X_i = \text{Spec}(A_i)$, $Y = \text{Spec}(B)$, $Y_i = \text{Spec}(B_i)$. Then we can consider the affine schemes

$$X' = \text{Spec}(A_1 \times_A A_2) = \text{Spec}(A') \quad \text{and} \quad Y' = \text{Spec}(B_1 \times_B B_2) = \text{Spec}(B')$$

Observe that $X' = X_1 \sqcup X_2$ and $Y' = Y_1 \sqcup Y_2$, see More on Morphisms, Lemma 14.1. By More on Algebra, Lemma 5.1 the rings $A'$ and $B'$ are of finite type over $\mathbb{Z}$. By More on Algebra, Lemma 6.4 we have $B' \otimes_A A_1 = B_1$ and $B' \times_A A_2 = B_2$. In particular a fibre of $Y' \to X'$ over a point of $X' = X_1 \sqcup X_2$ is always equal to either a fibre of $Y_1 \to X_1$ or a fibre of $Y_2 \to X_2$. By More on Algebra, Lemma 6.8 the ring map $A' \to B'$ is flat. Thus by Discriminants, Lemma 10.1 part (3) we conclude that $Y'/X'$ is an object of $C$. Consider now the commutative diagram

$$
\begin{array}{ccc}
Y/X & \xrightarrow{b_1/a_1} & Y_1/X_1 \\
& \searrow & \downarrow \\
& & Y''/X'
\end{array}
\quad
\begin{array}{ccc}
Y/X & \xrightarrow{b_2/a_2} & Y_2/X_2 \\
& \searrow & \downarrow \\
& & Y''/X'
\end{array}
$$

Now we would be done if $Y''/X'$ is an object of $C\text{nice}$, but this is almost never the case. Namely, then pulling back $c^p_{Y'/X'}$, around the two sides of the square, we would obtain the desired conclusion. To get around the problem that $Y''/X'$ is not in $C\text{nice}$ we note the arguments above show that, after possibly shrinking all of the schemes $X, Y, X_1, Y_1, X_2, Y_2, X', Y'$ we can find some $n, d \geq 1$, and extend the diagram like so:

$$
\begin{array}{ccc}
Y/X & \xrightarrow{b_1/a_1} & Y_1/X_1 \\
& \searrow & \downarrow \\
& & Y''/X'
\end{array}
\quad
\begin{array}{ccc}
Y/X & \xrightarrow{b_2/a_2} & Y_2/X_2 \\
& \searrow & \downarrow \\
& & Y''/X'
\end{array}
\quad
\begin{array}{c}
Y_{n,d}/X_{n,d}
\end{array}
$$

and then we can use the already given argument by pulling back from $c^p_{Y_{n,d}/X_{n,d}}$. This finishes the proof.

5. Trace maps on de Rham complexes

A reference for some of the material in this section is [Gar84]. Let $S$ be a scheme. Let $f : Y \to X$ be a finite locally free morphism of schemes over $S$. Then there is a trace map $\text{Trace}_f : f_* \mathcal{O}_Y \to \mathcal{O}_X$, see Discriminants, Section 3. In this situation a trace map on de Rham complexes is a map of complexes

$$\Theta_{Y/X} : f_* \Omega^\bullet_{Y/S} \longrightarrow \Omega^\bullet_{X/S}$$
such that $\Theta_{Y/X}$ is equal to $\text{Trace}_f$ in degree 0 and satisfies

$$\Theta_{Y/X}(\omega \wedge \eta) = \omega \wedge \Theta_{Y/X}(\eta)$$

for local sections $\omega$ of $\Omega_{X/S}^+$ and $\eta$ of $f^*\Omega_{Y/S}^+$. It is not clear to us whether such a trace map $\Theta_{Y/X}$ exists for every finite locally free morphism $Y \to X$; please email stacks.project@gmail.com if you have a counterexample or a proof.

**Example 5.1.** Here is an example where we do not have a trace map on de Rham complexes. For example, consider the $\mathbb{C}$-algebra $B = \mathbb{C}[x,y]$ with action of $G = \{±1\}$ given by $x \mapsto -x$ and $y \mapsto -y$. The invariants $A = B^G$ form a normal domain of finite type over $\mathbb{C}$ generated by $x^2, xy, y^2$. We claim that for the inclusion $A \subset B$ there is no reasonable trace map $\Omega_{B/\mathbb{C}} \to \Omega_{A/\mathbb{C}}$ on 1-forms. Namely, consider the element $\omega = xy \in \Omega_{B/\mathbb{C}}$. Since $\omega$ is invariant under the action of $G$ if a “reasonable” trace map exists, then $2\omega$ should be in the image of $\Omega_{A/\mathbb{C}} \to \Omega_{B/\mathbb{C}}$. This is not the case: there is no way to write $2\omega$ as a linear combination of $d(x^2)$, $d(xy)$, and $d(y^2)$ even with coefficients in $B$. This example contradicts the main theorem in [Zan99].

**Lemma 5.2.** There exists a unique rule that to every finite syntomic morphism of schemes $f : Y \to X$ assigns $\mathcal{O}_X$-module maps

$$\Theta^p_{Y/X} : f_*\Omega^p_{Y/\mathbb{Z}} \to \Omega^p_{X/\mathbb{Z}}$$

satisfying the following properties

1. the composition with $\Omega^p_{X/\mathbb{Z}} \otimes_{\mathcal{O}_X} f_*\mathcal{O}_Y \to f_*\Omega^p_{Y/\mathbb{Z}}$ is equal to $\text{id} \otimes \text{Trace}_f$, where $\text{Trace}_f : f_*\mathcal{O}_Y \to \mathcal{O}_X$ is the map from Discriminants, Section 5
2. the rule is compatible with base change.

**Proof.** First, assume that $X$ is locally Noetherian. By Lemma 4.3 we have a canonical map

$$c^p_{Y/X} : \Omega^p_{Y/\mathbb{Z}} \to f^*\Omega^p_{X/\mathbb{Z}} \otimes_{\mathcal{O}_Y} \text{det}(\mathcal{N}L_{Y/X})$$

By Discriminants, Proposition 13.2 we have a canonical isomorphism

$$c_{Y/X} : \text{det}(\mathcal{N}L_{Y/X}) \to \omega_{Y/X}$$

mapping $\delta(\mathcal{N}L_{Y/X})$ to $\tau_{Y/X}$. Combined these maps give

$$c^p_{Y/X} \otimes c_{Y/X} : \Omega^p_{Y/\mathbb{Z}} \to f^*\Omega^p_{X/\mathbb{Z}} \otimes_{\mathcal{O}_Y} \omega_{Y/X}$$

By Discriminants, Section 5 this is the same thing as a map

$$\Theta^p_{Y/X} : f_*\Omega^p_{Y/\mathbb{Z}} \to \Omega^p_{X/\mathbb{Z}}$$

Recall that the relationship between $c^p_{Y/X} \otimes c_{Y/X}$ and $\Theta^p_{Y/X}$ uses the evaluation map $f_*\omega_{Y/X} \to \mathcal{O}_X$ which sends $\tau_{Y/X}$ to $\text{Trace}_f(1)$, see Discriminants, Section 5. Hence property (1) holds. Property (2) holds for base changes by $X' \to X$ with $X'$ locally Noetherian because both $c^p_{Y/X}$ and $c_{Y/X}$ are compatible with such base changes. For $f : Y \to X$ finite syntomic and $X$ locally Noetherian, we will continue to denote $\Theta^p_{Y/X}$ the solution we’ve just found.

Uniqueness. Suppose that we have a finite syntomic morphism $f : Y \to X$ such that $X$ is smooth over Spec($\mathbb{Z}$) and $f$ is étale over a dense open of $X$. We claim that in this case $\Theta^p_{Y/X}$ is uniquely determined by property (1). Namely, consider the maps

$$\Omega^p_{X/\mathbb{Z}} \otimes_{\mathcal{O}_X} f_*\mathcal{O}_Y \to f_*\Omega^p_{Y/\mathbb{Z}} \to \Omega^p_{X/\mathbb{Z}}$$
The sheaf $\Omega^p_{X/Z}$ is torsion free (by the assumed smoothness), hence it suffices to check that the restriction of $\Theta^p_{Y/X}$ is uniquely determined over the dense open over which $f$ is étale, i.e., we may assume $f$ is étale. However, if $f$ is étale, then $f^*\Omega^p_{X/Z} = \Omega^p_{Y/Z}$ hence the first arrow in the displayed equation is an isomorphism. Since we’ve pinned down the composition, this guarantees uniqueness.

Let $f : Y \to X$ be a finite syntomic morphism of locally Noetherian schemes. Let $x \in X$. By Discriminants, Lemma [11.7] we can find $d \geq 1$ and a commutative diagram

$$
\begin{array}{ccc}
Y & \to & V \\
\downarrow & & \downarrow \\
X & \to & U \\
\end{array}
$$

such that $x \in U \subset X$ is open, $V = f^{-1}(U)$ and $V = U \times_U V_d$. Thus $\Theta^p_{Y/X}|_V$ is the pullback of the map $\Theta^p_{V_u/U_u}$. However, by the discussion on uniqueness above and Discriminants, Lemmas [11.4] and [11.5] the map $\Theta^p_{V_d/U_d}$ is uniquely determined by the requirement (1). Hence uniqueness holds.

At this point we know that we have existence and uniqueness for all finite syntomic morphisms $Y \to X$ with $X$ locally Noetherian. We could now give an argument similar to the proof of Lemma 4.3 to extend to general $X$. However, instead it possible to directly use absolute Noetherian approximation to finish the proof. Namely, to construct $\Theta^p_{Y/X}$ it suffices to do so Zariski locally on $X$ (provided we also show the uniqueness). Hence we may assume $X$ is affine (small detail omitted). Then we can write $X = \lim_{i \in I} X_i$ as the limit over a directed set $I$ of Noetherian affine schemes. By Algebra, Lemma [126.8] we can find $0 \in I$ and a finitely presented morphism of affines $f_0 : Y_0 \to X_0$ whose base change to $X$ is $Y \to X$. After increasing $0$ we may assume $Y_0 \to X_0$ is finite and syntomic, see Algebra, Lemma [163.9] and [163.3].

For $i \geq 0$ also the base change $f_i : Y_i = Y_0 \times_{X_0} X_i \to X_i$ is finite syntomic. Then

$$
\Gamma(X, f_\ast \Omega^p_{Y/Z}) = \Gamma(Y, \Omega^p_{Y/Z}) = \colim_{i \geq 0} \Gamma(Y_i, \Omega^p_{Y_i/Z}) = \colim_{i \geq 0} \Gamma(X_i, f_i\ast \Omega^p_{Y_i/Z})
$$

Hence we can (and are forced to) define $\Theta^p_{Y/X}$ as the colimit of the maps $\Theta^p_{Y_i/X_i}$. This map is compatible with any cartesian diagram

$$
\begin{array}{ccc}
Y' & \to & Y \\
\downarrow & & \downarrow \\
X' & \to & X \\
\end{array}
$$

with $X'$ affine as we know this for the case of Noetherian affine schemes by the arguments given above (small detail omitted; hint: if we also write $X' = \lim_{j \in J} X'_j$ then for every $i \in I$ there is a $j \in J$ and a morphism $X'_j \to X_i$ compatible with the morphism $X' \to X$). This finishes the proof. □

**Proposition 5.3.** Let $f : Y \to X$ be a finite syntomic morphism of schemes. The maps $\Theta^p_{Y/X}$ of Lemma 5.2 define a map of complexes

$$
\Theta^p_{Y/X} : f_\ast \Omega^p_{Y/Z} \to \Omega^p_{X/Z}
$$

with the following properties
(1) in degree 0 we get \( \text{Trace}_f : f_!\mathcal{O}_Y \to \mathcal{O}_X \), see Discriminants, Section 11.4.

(2) we have \( \Theta_{Y/X}(\omega \wedge \eta) = \omega \wedge \Theta_{Y/X}(\eta) \) for \( \omega \) in \( \Omega^*_X/S \) and \( \eta \) in \( f_!\Omega^*_Y/S \).

(3) if \( f \) is a morphism over a base scheme \( S \), then \( \Theta_{Y/X} \) induces a map of complexes \( f_*\Omega^*_Y/S \to \Omega^*_X/S \).

**Proof.** By Discriminants, Lemma 11.7 for every \( x \in X \) we can find \( d \geq 1 \) and a commutative diagram

\[
\begin{array}{ccc}
Y & \to & V_d \\
\downarrow & & \downarrow \\
X & \to & U_d \\
\end{array}
\]

such that \( x \in U \subset X \) is affine open, \( V = f^{-1}(U) \) and \( V = U \times_{U_d} V_d \). Write \( U = \text{Spec}(A) \) and \( V = \text{Spec}(B) \) and observe that \( B = A \otimes_{A_d} B_d \) and recall that \( B_d = A_d e_1 \oplus \cdots \oplus A_d e_d \). Suppose we have \( a_1, \ldots, a_r, a_{1,1}, \ldots, a_{s,d} \) then we see that the functions \( a_1, \ldots, a_r \) are in the image of \( \Gamma(U', \mathcal{O}_{U'}) \to \Gamma(U, \mathcal{O}_U) \) and the functions \( b_1, \ldots, b_s \) are in the image of \( \Gamma(V', \mathcal{O}_{V'}) \to \Gamma(V, \mathcal{O}_V) \). In this way we see that for any finite collection of elements\(^1\) of the groups

\[
\Gamma(V, \Omega^i_{Y/Z}), \quad i = 0, 1, 2, \ldots \quad \text{and} \quad \Gamma(U, \Omega^j_{X/Z}), \quad j = 0, 1, 2, \ldots
\]

we can find a factorizations \( V \to V' \to V_d \) and \( U \to U' \to U_d \) with \( V' = A^N \times V_d \) and \( U' = A^N \times U_d \) as above such that these sections are the pullbacks of sections from

\[
\Gamma(V', \Omega^i_{V'/Z}), \quad i = 0, 1, 2, \ldots \quad \text{and} \quad \Gamma(U', \Omega^j_{U'/Z}), \quad j = 0, 1, 2, \ldots
\]

The upshot of this is that to check \( d \circ \Theta_{Y/X} = \Theta_{Y/X} \circ d \) it suffices to check this is true for \( \Theta_{V'/U'} \). Similarly, for property (2) of the lemma.

By Discriminants, Lemmas 11.4 and 11.5 the scheme \( U_d \) is smooth and the morphism \( V_d \to U_d \) is étale over a dense open of \( U_d \). Hence the same is true for the morphism \( V' \to U' \). Since \( \Omega^i_{U'/Z} \) is locally free and hence \( \Omega^i_{U'/Z} \) is torsion free, it suffices to check the desired relations after restricting to the open over which \( V' \) is finite étale. Then we may check the relations after a surjective étale base change. Hence we may split the finite étale cover and assume we are looking at a morphism of the form

\[
\prod_{i=1, \ldots, d} W \to W
\]

with \( W \) smooth over \( \mathbf{Z} \). In this case any local properties of our construction are trivial to check (provided they are true). This finishes the proof of (1) and (2).

---

\(^1\)After all these elements will be finite sums of elements of the form \( a_0 a_1 \wedge \cdots \wedge a_i \wedge b_i \) with \( a_0, \ldots, a_i \in A \) or finite sums of elements of the form \( b_0 b_1 \wedge \cdots \wedge b_j \) with \( b_0, \ldots, b_j \in B \).
Finally, we observe that (3) follows from (2) because $\Omega_{Y/S}$ is the quotient of $\Omega_{Y/Z}$ by the submodule generated by pullbacks of local sections of $\Omega_{S/Z}$.

Example 5.4. Let $A$ be a ring. Let $f = x^d + \sum_{0 \leq i < d} a_{d-i}x^i \in A[x]$. Let $B = A[x]/(f)$. By Proposition 5.3 we have a morphism of complexes

$$\Theta_{B/A} : \Omega^\bullet_B \rightarrow \Omega^\bullet_A$$

In particular, if $t \in B$ denotes the image of $x \in A[x]$ we can consider the elements

$$\Theta_{B/A}(t^i dt) \in \Omega^1_A, \quad i = 0, \ldots, d - 1$$

What are these elements? By the same principle as used in the proof of Proposition 5.3 it suffices to compute this in the universal case, i.e., when $A = \mathbb{Z}[a_1, \ldots, a_d]$ or even when $A$ is replaced by the fraction field $\mathbb{Q}(a_1, \ldots, a_d)$. Writing symbolically

$$f = \prod_{i=1, \ldots, d} (x - \alpha_i)$$

we see that over $\mathbb{Q}(a_1, \ldots, a_d)$ the algebra $B$ becomes split:

$$\mathbb{Q}(a_0, \ldots, a_{d-1})[x]/(f) \rightarrow \prod_{i=1, \ldots, d} \mathbb{Q}(a_1, \ldots, a_d), \quad t \mapsto (a_1, \ldots, a_d)$$

Thus for example

$$\Theta(dt) = \sum d\alpha_i = -da_1$$

Next, we have

$$\Theta(tdt) = \sum \alpha_i d\alpha_i = a_1da_1 - da_2$$

Next, we have

$$\Theta(t^2 dt) = \sum \alpha_i^2 d\alpha_i = -a_1^2 da_1 + a_1 da_2 + a_2 da_1 - da_3$$

(modulo calculation error), and so on. This suggests that if $f(x) = x^d - a$ then

$$\Theta_{B/A}(t^i dt) = \begin{cases} 0 & \text{if } i = 0, \ldots, d - 2 \\ da & \text{if } i = d - 1 \end{cases}$$

in $\Omega_A$. This is true for in this particular case one can do the calculation for the extension $\mathbb{Q}(a)[x]/(x^d - a)$ to verify this directly.

6. First chern class in de Rham cohomology

Let $X \rightarrow S$ be a morphism of schemes. There is a map of complexes

$$d \log : \mathcal{O}_X^\star[-1] \rightarrow \Omega^\bullet_{X/S}$$

which sends the section $g \in \mathcal{O}_X^\star(U)$ to the section $d \log(g) = g^{-1}dg$ of $\Omega^1_{X/S}(U)$. Thus we can consider the map

$$\text{Pic}(X) = H^1(X, \mathcal{O}_X^\star) = H^2(X, \mathcal{O}_X^\star[-1]) \rightarrow H^2_{dR}(X/S)$$

where the first equality is Cohomology, Lemma 6.1. The image of the isomorphism class of the invertible module $\mathcal{L}$ is often denoted $c_1^{dR}(\mathcal{L}) \in H^2_{dR}(X/S)$.

7. Other chapters

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