1. Introduction

In this chapter we start with a discussion of the de Rham complex of a morphism of schemes and we end with a proof that de Rham cohomology defines a Weil cohomology theory when the base field has characteristic zero.

2. The de Rham complex

Let $p : X \to S$ be a morphism of schemes. There is a complex

$$\Omega^\bullet_{X/S} = \mathcal{O}_{X/S} \to \Omega^1_{X/S} \to \Omega^2_{X/S} \to \ldots$$

of $p^{-1}\mathcal{O}_S$-modules with $\Omega^i_{X/S} = \wedge^i(\Omega_X/S)$ placed in degree $i$ and differential determined by the rule $d(\omega_0 d\omega_1 \wedge \ldots \wedge d\omega_p) = d\omega_0 \wedge d\omega_1 \wedge \ldots \wedge d\omega_p$ on local sections. See Modules, Section 26.

Given a commutative diagram

$$\begin{array}{ccc}
X' & \xrightarrow{f} & X \\
\downarrow & & \downarrow \\
S' & \xrightarrow{} & S
\end{array}$$

This is a chapter of the Stacks Project, version 19ba3554, compiled on Oct 13, 2019.
of schemes, there are canonical maps of complexes \( f^{-1}\Omega^\bullet_{X/S} \to \Omega^\bullet_{X'/S'} \) and \( \Omega^\bullet_{X/S} \to f_*\Omega^\bullet_{X'/S'} \). See Modules, Section [26]. Linearizing, for every \( p \) we obtain a linear map \( f^*\Omega^p_{X/S} \to \Omega^p_{X'/S'} \).

In particular, if \( f : Y \to X \) be a morphism of schemes over a base scheme \( S \), then there is a map of complexes

\[
\Omega^\bullet_{X/S} \to f_*\Omega^\bullet_{Y/S}
\]

Linearizing, we see that for every \( p \geq 0 \) we obtain a canonical map

\[
\Omega^p_{X/S} \otimes_O f_*\mathcal{O}_Y \to f_*\Omega^p_{Y/S}
\]

**Lemma 2.1.** Let

\[
\begin{array}{ccc}
X' & \to & X \\
\downarrow & & \downarrow \\
S' & \to & S
\end{array}
\]

be a cartesian diagram of schemes. Then the maps discussed above induce isomorphisms \( f^*\Omega^p_{X/S} \to \Omega^p_{X'/S'} \).

**Proof.** Combine Morphisms, Lemma [31.10] with the fact that formation of exterior power commutes with base change. \(\square\)

**Lemma 2.2.** Consider a commutative diagram of schemes

\[
\begin{array}{ccc}
X' & \to & X \\
\downarrow & & \downarrow \\
S' & \to & S
\end{array}
\]

If \( X' \to X \) and \( S' \to S \) are étale, then the maps discussed above induce isomorphisms \( f^*\Omega^p_{X/S} \to \Omega^p_{X'/S'} \).

**Proof.** We have \( \Omega_{S'/S} = 0 \) and \( \Omega_{X'/X} = 0 \), see for example Morphisms, Lemma [34.15]. Then by the short exact sequences of Morphisms, Lemmas [31.9] and [32.16] we see that \( \Omega_{X'/S'} = \Omega_{X'/S} = f^*\Omega_{X/S} \). Taking exterior powers we conclude. \(\square\)

### 3. de Rham cohomology

Let \( p : X \to S \) be a morphism of schemes. We define the *de Rham cohomology of \( X \) over \( S \) to be the cohomology groups

\[
H^i_{dR}(X/S) = H^i(R\Gamma(X, \Omega^\bullet_{X/S}))
\]

Since \( \Omega^\bullet_{X/S} \) is a complex of \( p^{-1}\mathcal{O}_S \)-modules, these cohomology groups are naturally modules over \( H^0(S, \mathcal{O}_S) \).

Given a commutative diagram

\[
\begin{array}{ccc}
X' & \to & X \\
\downarrow & & \downarrow \\
S' & \to & S
\end{array}
\]
of schemes, using the canonical maps of Section 2 we obtain pullback maps
\[ f^* : R\Gamma(X, \Omega^\bullet_{X/S}) \longrightarrow R\Gamma(X', \Omega^\bullet_{X'/S'}) \]
and
\[ f^* : H^i_{DR}(X/S) \longrightarrow H^i_{DR}(X'/S') \]
These pullbacks satisfy an obvious composition law. In particular, if we work over a fixed base scheme \( S \), then de Rham cohomology is a contravariant functor on the category of schemes over \( S \).

Lemma 3.1. Let \( X \rightarrow S \) be a morphism of affine schemes given by the ring map \( R \rightarrow A \). Then \( R\Gamma(X, \Omega^\bullet_{X/S}) = \Omega^\bullet_{A/R} \) in \( D(R) \) and \( H^i_{dR}(X/S) = H^i(\Omega^\bullet_{A/R}) \).

Proof. This follows from Cohomology of Schemes, Lemma 2.2 and Leray’s acyclicity lemma (Derived Categories, Lemma 16.7).

Lemma 3.2. Let \( p : X \rightarrow S \) be a morphism of schemes. If \( p \) is quasi-compact and quasi-separated, then \( Rp_\ast \Omega^\bullet_{X/S} \) is an object of \( D_{QCoh}(\mathcal{O}_S) \).

Proof. There is a spectral sequence with first page \( E_1^{a,b} = \mathcal{R}^a \sigma_{b} \Omega^\bullet_{X/S} \) converging to \( Rp_\ast \Omega^\bullet_{X/S} \) (see Derived Categories, Lemma 21.3). Hence by Homology, Lemma 23.6 it suffices to show that \( \mathcal{R}^a \sigma_{b} \Omega^\bullet_{X/S} \) is quasi-coherent. This follows from Cohomology of Schemes, Lemma 4.5.

Lemma 3.3. Let \( p : X \rightarrow S \) be a proper morphism of schemes with \( S \) locally Noetherian. Then \( Rp_\ast \Omega^\bullet_{X/S} \) is an object of \( D_{Coh}(\mathcal{O}_S) \).

Proof. In this case by Morphisms, Lemma 31.12 the modules \( \Omega^\bullet_{X/S} \) are coherent. Hence we can use exactly the same argument as in the proof of Lemma 3.2 using Cohomology of Schemes, Proposition 19.1.

Lemma 3.4. Let \( A \) be a Noetherian ring. Let \( X \) be a proper scheme over \( S = \text{Spec}(A) \). Then \( H^i_{dR}(X/S) \) is a finite \( A \)-module for all \( i \).

Proof. This is a special case of Lemma 3.3.

Lemma 3.5. Let \( f : X \rightarrow S \) be a proper smooth morphism of schemes. Then \( Rf_\ast \Omega^p_{X/S}, p \geq 0 \) and \( Rf_\ast \Omega^\bullet_{X/S} \) are perfect objects of \( D(\mathcal{O}_S) \) whose formation commutes with arbitrary change of base.

Proof. Since \( f \) is smooth the modules \( \Omega^p_{X/S} \) are finite locally free \( \mathcal{O}_X \)-modules, see Morphisms, Lemma 32.12. Their formation commutes with arbitrary change of base by Lemma 2.1. Hence \( Rf_\ast \Omega^p_{X/S} \) is a perfect object of \( D(\mathcal{O}_S) \) whose formation commutes with arbitrary base change, see Derived Categories of Schemes, Lemma 27.4. This proves the first assertion of the lemma.

To prove that \( Rf_\ast \Omega^p_{X/S} \) is perfect on \( S \) we may work locally on \( S \). Thus we may assume \( S \) is quasi-compact. This means we may assume that \( \Omega^p_{X/S} \) is zero for \( n \) large enough. For every \( p \geq 0 \) we claim that \( Rf_\ast \sigma_{\geq p} \Omega^\bullet_{X/S} \) is a perfect object of \( D(\mathcal{O}_S) \) whose formation commutes with arbitrary change of base. By the above we see that this is true for \( p \gg 0 \). Suppose the claim holds for \( p \) and consider the distinguished triangle
\[ \sigma_{\geq p} \Omega^\bullet_{X/S} \rightarrow \sigma_{\geq p-1} \Omega^\bullet_{X/S} \rightarrow \Omega^{p-1}_{X/S}[-p+1] \rightarrow (\sigma_{\geq p} \Omega^\bullet_{X/S})[1] \]
in $D(f^{-1}\mathcal{O}_S)$. Applying the exact functor $Rf_*$ we obtain a distinguished triangle in $D(\mathcal{O}_S)$. Since we have the 2-out-of-3 property for being perfect (Cohomology, Lemma 45.7) we conclude $Rf_\sigma_{\geq p-1}\Omega^\bullet_{X/S}$ is a perfect object of $D(\mathcal{O}_S)$. Similarly for the commutation with arbitrary base change. □

4. Cup product

0FM1 Consider the maps $\Omega^p_{X/S} \times \Omega^q_{X/S} \to \Omega^{p+q}_{X/S}$ given by $\omega, \eta \mapsto \omega \wedge \eta$. Using the formula for $d$ given in Section 2 and the Leibniz rule for $d : \mathcal{O}_X \to \Omega^1_{X/S}$ we see that $d(\omega \wedge \eta) = d(\omega) \wedge \eta + (-1)^{\deg_\omega}\omega \wedge d(\eta)$. This means that $\wedge$ defines a morphism

$$\wedge : \text{Tot}(\Omega^\bullet_{X/S} \otimes_{p-1}\mathcal{O}_S \Omega^\bullet_{X/S}) \to \Omega^\bullet_{X/S}$$

of complexes of $p^{-1}\mathcal{O}_S$-modules.

Combining the cup product of Cohomology, Section 31 with (4.0.1) we find a $H^0(S, \mathcal{O}_S)$-bilinear cup product map

$$\cup : H^i_{dR}(X/S) \times H^j_{dR}(X/S) \to H^{i+j}_{dR}(X/S)$$

For example, if $\omega \in \Gamma(X, \Omega^i_{X/S})$ and $\eta \in \Gamma(X, \Omega^j_{X/S})$ are closed, then the cup product of the de Rham cohomology classes of $\omega$ and $\eta$ is the de Rham cohomology class of $\omega \wedge \eta$, see discussion in Cohomology, Section 31.

Given a commutative diagram

$$\begin{array}{ccc}
X' & \to & X \\
\downarrow & & \downarrow \\
S' & \to & S
\end{array}$$

of schemes, the pullback maps $f^* : R\Gamma(X, \Omega^\bullet_{X/S}) \to R\Gamma(X', \Omega^\bullet_{X'/S'})$ and $f^* : H^i_{dR}(X/S) \to H^i_{dR}(X'/S')$ are compatible with the cup product defined above.

0FM2 Lemma 4.1. Let $p : X \to S$ be a morphism of schemes. If the diagonal of $X$ is affine, then the cup product on $H^*_{dR}(X/S)$ is associative and graded commutative.

This lemma is true without the assumption on the diagonal of $X$. It can be proven by replacing open coverings in the proof below by hypercoverings.

Proof. Choose an affine open covering $U : X = \bigcup_{i \in I} U_i$. As the diagonal of $X$ is affine, for $i_0, \ldots, i_p \in I$ the intersection $U_{i_0 \ldots i_p} = U_{i_0} \cap \ldots \cap U_{i_p}$ is affine. Since the sheaves $\Omega^n_{X/S}$ are quasi-coherent we see that $H^i(U_{i_0 \ldots i_p}, \Omega^n_{X/S}) = 0$ for all $i > 0$ and $p, i_0, \ldots, i_p, n$, see Cohomology of Schemes, Lemma 2.2. Thus the canonical map

$$\text{Tot}((\mathcal{C}^\bullet(U, \Omega^\bullet_{X/S})) \to R\Gamma(X, \Omega^\bullet_{X/S})$$

is an isomorphism, see Cohomology, Lemma 25.2. By Cohomology, Lemma 31.1 the cup product on de Rham cohomology is given by the construction of Cohomology, Equation (25.3.2) on $\text{Tot}((\mathcal{C}^\bullet(U, \Omega^\bullet_{X/S})))$. This construction was shown to be associative (using that $\wedge$ is associative) and graded commutative (using that $\wedge$ is graded commutative) in Cohomology, Section 25. □
5. Hodge cohomology

Let \( p : X \to S \) be a morphism of schemes. We define the de Hodge cohomology of \( X \) over \( S \) to be the cohomology groups

\[
H_{\text{Hodge}}^n(X/S) = \bigoplus_{n=p+q} \Omega^p_X(S)
\]

viewed as a graded \( H^0(X, O_X) \)-module. The wedge product of forms combined with the cup product of Cohomology, Section 31 defines a \( H^0(X, O_X) \)-bilinear cup product

\[
\cup : H_{\text{Hodge}}^i(X/S) \times H_{\text{Hodge}}^j(X/S) \to H_{\text{Hodge}}^{i+j}(X/S)
\]

Of course if \( \xi \in H^q(X, \Omega^p_X/S) \) and \( \xi' \in H^q'(X, \Omega^{p'}_X/S) \) then \( \xi \cup \xi' \in H^{q+q'}(X, \Omega^{p+p'}_X/S) \).

**Lemma 5.1.** Let \( p : X \to S \) be a morphism of schemes. If the diagonal of \( X \) is affine, then the cup product on \( H_{\text{Hodge}}^*(X/S) \) is associative and graded commutative.

This lemma is true without the assumption on the diagonal of \( X \).

**Proof.** The proof is identical to the proof of Lemma 4.1

Given a commutative diagram

\[
\begin{array}{ccc}
X' & \xrightarrow{f} & X \\
\downarrow & & \downarrow \\
S' & \xrightarrow{g} & S
\end{array}
\]

of schemes, there are pullback maps \( f^* : H_{\text{Hodge}}^i(X/S) \to H_{\text{Hodge}}^i(X'/S') \) compatible with gradings and with the cup product defined above.

6. Two spectral sequences

Let \( p : X \to S \) be a morphism of schemes. Since the category of \( p^{-1}O_S \)-modules on \( X \) has enough injectives there exist a Cartan-Eilenberg resolution for \( \Omega^*_X/S \). See Derived Categories, Lemma [21.2] Hence we can apply Derived Categories, Lemma [21.3] to get two spectral sequences both converging to the de Rham cohomology of \( X \) over \( S \).

The first is customarily called the Hodge-to-de Rham spectral sequence. The first page of this spectral sequence has

\[
E_1^{p,q} = H^q(X, \Omega^p_X/S)
\]

which are the Hodge cohomology groups of \( X/S \) (whence the name). The differential \( d_1 \) on this page is given by the maps \( d_1^{p,q} : H^q(X, \Omega^p_X/S) \to H^q(X, \Omega^{p+1}_X/S) \)
induced by the differential \(d : \Omega^p_{X/S} \to \Omega^{p+1}_{X/S}\). Here is a picture

\[
\begin{align*}
H^2(X, \mathcal{O}_X) & \xrightarrow{d} H^2(X, \Omega^1_{X/S}) & H^2(X, \Omega^2_{X/S}) & \xrightarrow{d} H^2(X, \Omega^3_{X/S}) \\
H^1(X, \mathcal{O}_X) & \xrightarrow{d} H^1(X, \Omega^1_{X/S}) & H^1(X, \Omega^2_{X/S}) & \xrightarrow{d} H^1(X, \Omega^3_{X/S}) \\
H^0(X, \mathcal{O}_X) & \xrightarrow{d} H^0(X, \Omega^1_{X/S}) & H^0(X, \Omega^2_{X/S}) & \xrightarrow{d} H^0(X, \Omega^3_{X/S})
\end{align*}
\]

where we have drawn striped arrows to indicate the source and target of the differentials on the \(E_2\) page and a dotted arrow for a differential on the \(E_3\) page. Looking in degree 0 we conclude that

\[
H^0^dR(X/S) = \text{Ker}(d : H^0(X, \mathcal{O}_X) \to H^0(X, \Omega^1_{X/S}))
\]

Of course, this is also immediately clear from the fact that the de Rham complex starts in degree 0 with \(\mathcal{O}_X \to \Omega^1_{X/S}\).

The second spectral sequence is usually called the conjugate spectral sequence. The second page of this spectral sequence has

\[
E_2^{p,q} = H^p(X, H^q(\Omega^\bullet_{X/S})) = H^p(X, \mathcal{H}^q)
\]

where \(\mathcal{H}^q = H^q(\Omega^\bullet_{X/S})\) is the \(q\)th cohomology sheaf of the de Rham complex of \(X/S\). The differentials on this page are given by \(E_2^{p,q} \to E_2^{p+2,q-1}\). Here is a picture

\[
\begin{align*}
H^0(X, \mathcal{H}^2) & \xrightarrow{d} H^1(X, \mathcal{H}^2) & H^2(X, \mathcal{H}^2) & \xrightarrow{d} H^3(X, \mathcal{H}^2) \\
H^0(X, \mathcal{H}^1) & \xrightarrow{d} H^1(X, \mathcal{H}^1) & H^2(X, \mathcal{H}^1) & \xrightarrow{d} H^3(X, \mathcal{H}^1) \\
H^0(X, \mathcal{H}^0) & \xrightarrow{d} H^1(X, \mathcal{H}^0) & H^2(X, \mathcal{H}^0) & \xrightarrow{d} H^3(X, \mathcal{H}^0)
\end{align*}
\]

Looking in degree 0 we conclude that

\[
H^0^dR(X/S) = H^0(X, \mathcal{H}^0)
\]

which is obvious if you think about it. In degree 1 we get an exact sequence

\[
0 \to H^1(X, \mathcal{H}^0) \to H^1^dR(X/S) \to H^0(X, \mathcal{H}^1) \to H^2(X, \mathcal{H}^0) \to H^2^dR(X/S)
\]

It turns out that if \(X \to S\) is smooth and \(S\) lives in characteristic \(p\), then the sheaves \(\mathcal{H}^q\) are computable (in terms of a certain sheaves of differentials) and the conjugate spectral sequence is a valuable tool (insert future reference here).

7. The Hodge filtration

Let \(X \to S\) be a morphism of schemes. The Hodge filtration on \(H^p^dR(X/S)\) is the filtration induced by the Hodge-to-de Rham spectral sequence (Homology, Definition 22.5). To avoid misunderstanding, we explicitly define it as follows.
Let $X \to S$ be a morphism of schemes. The Hodge filtration on $H^n_{dR}(X/S)$ is the filtration with terms

$$F^pH^n_{dR}(X/S) = \text{Im} \left( H^n(X, \sigma_{\geq p}^\bullet \Omega^\bullet_{X/S}) \to H^n_{dR}(X/S) \right)$$

where $\sigma_{\geq p}^\bullet \Omega^\bullet_{X/S}$ is as in Homology, Section 13.

Of course $\sigma_{\geq p}^\bullet \Omega^\bullet_{X/S}$ is a subcomplex of the relative de Rham complex and we obtain a filtration

$$\Omega^\bullet_{X/S} = \sigma_{\geq 0}^\bullet \Omega^\bullet_{X/S} \supset \sigma_{\geq 1}^\bullet \Omega^\bullet_{X/S} \supset \sigma_{\geq 2}^\bullet \Omega^\bullet_{X/S} \supset \sigma_{\geq 3}^\bullet \Omega^\bullet_{X/S} \supset \ldots$$

of the relative de Rham complex with $\text{gr}^p(\Omega^\bullet_{X/S}) = \Omega^\bullet_{X/S}[-p]$. The spectral sequence constructed in Cohomology, Lemma 29.1 for $\Omega^\bullet_{X/S}$ viewed as a filtered complex of sheaves is the same as the Hodge-to-de Rham spectral sequence constructed in Section 12. Further the wedge product (4.0.1) sends $\text{Tot}(\sigma_{\geq j}^\bullet \Omega^\bullet_{X/S} \otimes \sigma_{\geq j}^\bullet \Omega^\bullet_{Y/S})$ into $\sigma_{\geq j}^\bullet \Omega^\bullet_{X/S}$, hence we get commutative diagrams

$$H^n(X, \sigma_{\geq j}^\bullet \Omega^\bullet_{X/S})) \times H^m(Y, \sigma_{\geq j}^\bullet \Omega^\bullet_{Y/S})) \to H^{n+m}(X \times S, \sigma_{\geq j}^\bullet \Omega^\bullet_{X/S})$$

In particular we find that

$$F^iH^n_{dR}(X/S) \cup F^jH^n_{dR}(X/S) \subset F^{i+j}H^n_{dR}(X/S)$$

8. Küneth formula

An important feature of de Rham cohomology is that there is a Künneth formula.

Let $a : X \to S$ and $b : Y \to S$ be morphisms of schemes with the same target. Let $p : X \times_S Y \to X$ and $q : X \times_S Y \to Y$ be the projection morphisms and $f = a \circ p = b \circ q$. Here is a picture

```
X \times_S Y
  |  |
  |  |
p  q

X a

Y f

S b
```

Lemma 8.1. In the situation above there is a canonical isomorphism

$$\text{Tot}(p^{-1} \Omega^\bullet_{X/S} \otimes f^{-1} \Omega^\bullet_{Y/S}) \to \Omega^\bullet_{X \times_S Y/S}$$

of complexes of $f^{-1}\mathcal{O}_S$-modules.

Proof. By Derived Categories of Schemes, Remark 22.2 we have

$$p^{-1} \Omega^i_{X/S} \otimes f^{-1} \Omega^j_{Y/S} = p^* \Omega^i_{X/S} \otimes \sigma_{X \times_S Y} q^* \Omega^j_{Y/S}$$

for all $i, j$. On the other hand, we know that $\Omega^\bullet_{X \times_S Y/S} = p^* \Omega^\bullet_{X/S} \oplus q^* \Omega^\bullet_{Y/S}$ by Morphisms, Lemma 31.11. Taking exterior powers we obtain

$$\Omega^n_{X \times_S Y/S} = \bigoplus_{i+j=n} p^{-1} \Omega^i_{X/S} \otimes \sigma_{X \times_S Y} q^* \Omega^j_{Y/S} = \bigoplus_{i+j=n} p^{-1} \Omega^i_{X/S} \otimes f^{-1} \sigma_{X} q^{-1} \Omega^j_{Y/S}$$

0FM8 Definition 7.1. Let $X \to S$ be a morphism of schemes. The Hodge filtration on $H^n_{dR}(X/S)$ is the filtration with terms

$$F^pH^n_{dR}(X/S) = \text{Im} \left( H^n(X, \sigma_{\geq p}^\bullet \Omega^\bullet_{X/S}) \to H^n_{dR}(X/S) \right)$$

0FM9 An important feature of de Rham cohomology is that there is a Künneth formula.

0FMA Lemma 8.1. In the situation above there is a canonical isomorphism

$$\text{Tot}(p^{-1} \Omega^\bullet_{X/S} \otimes f^{-1} \Omega^\bullet_{Y/S}) \to \Omega^\bullet_{X \times_S Y/S}$$

of complexes of $f^{-1}\mathcal{O}_S$-modules.

Proof. By Derived Categories of Schemes, Remark 22.2 we have

$$p^{-1} \Omega^i_{X/S} \otimes f^{-1} \Omega^j_{Y/S} = p^* \Omega^i_{X/S} \otimes \sigma_{X \times_S Y} q^* \Omega^j_{Y/S}$$

for all $i, j$. On the other hand, we know that $\Omega^\bullet_{X \times_S Y/S} = p^* \Omega^\bullet_{X/S} \oplus q^* \Omega^\bullet_{Y/S}$ by Morphisms, Lemma 31.11. Taking exterior powers we obtain

$$\Omega^n_{X \times_S Y/S} = \bigoplus_{i+j=n} p^{-1} \Omega^i_{X/S} \otimes \sigma_{X \times_S Y} q^* \Omega^j_{Y/S} = \bigoplus_{i+j=n} p^{-1} \Omega^i_{X/S} \otimes f^{-1} \sigma_{X} q^{-1} \Omega^j_{Y/S}$$
by elementary properties of exterior powers. This finishes the proof.

If $S = \text{Spec}(A)$ is affine, then combining the result of Lemma 8.1 with the cup product map of Derived Categories of Schemes, Equation (22.1.1) we obtain a cup product

$$R\Gamma(X, \Omega^\bullet_{X/S}) \otimes^L_{A} R\Gamma(Y, \Omega^\bullet_{Y/S}) \to R\Gamma(X \times_S Y, \Omega^\bullet_{X \times_S Y/S})$$

On the level of cohomology, using the discussion in More on Algebra, Section 61, we obtain a canonical map

$$H^i_{dR}(X/S) \otimes A H^j_{dR}(Y/S) \to H^{i+j}_{dR}(X \times_S Y/S), \quad (\xi, \zeta) \mapsto p^*\xi \cup q^*\zeta$$

We note that the construction above indeed proceeds by first pulling back and then taking the cup product.

Lemma 8.2. Assume $X$ and $Y$ are smooth, quasi-compact, and quasi-separated over $S = \text{Spec}(A)$. Then the map

$$R\Gamma(X, \Omega^\bullet_{X/S}) \otimes^L_{A} R\Gamma(Y, \Omega^\bullet_{Y/S}) \to R\Gamma(X \times_S Y, \Omega^\bullet_{X \times_S Y/S})$$

is an isomorphism in $D(A)$.

Proof. By Morphisms, Lemma 32.12 the sheaves $\Omega^n_{X/S}$ and $\Omega^n_{Y/S}$ are finite locally free $\mathcal{O}_X$ and $\mathcal{O}_Y$-modules. On the other hand, $X$ and $Y$ are flat over $S$ (Morphisms, Lemma 32.9) and hence we find that $\Omega^n_{X/S}$ and $\Omega^n_{Y/S}$ are flat over $S$. Thus the result by Lemma 8.1 and Derived Categories of Schemes, Lemma 22.3.

Given a possibly non-affine base scheme $S$ we can do this construction over all affine opens and upon sheafification we obtain a relative cup product

$$R_a \Omega^\bullet_{X/S} \otimes^L_{\mathcal{O}_S} R_b \Omega^\bullet_{Y/S} \to R_f \Omega^\bullet_{X \times_S Y/S}$$

in $D(\mathcal{O}_S)$ and taking global sections we get

$$H^i_{dR}(X/S) \otimes_{\mathcal{O}^1(S, \mathcal{O}_S)} H^j_{dR}(Y/S) \to H^{i+j}_{dR}(X \times_S Y/S), \quad (\xi, \zeta) \mapsto p^*\xi \cup q^*\zeta$$

Lemma 8.3. Assume $X$ and $Y$ are smooth, quasi-compact, and quasi-separated over $S$. Then the map

$$R_a \Omega^\bullet_{X/S} \otimes^L_{\mathcal{O}_S} R_b \Omega^\bullet_{Y/S} \to R_f \Omega^\bullet_{X \times_S Y/S}$$

is an isomorphism in $D(\mathcal{O}_S)$.

Proof. Immediate consequence of Lemma 8.2.

9. First chern class in de Rham cohomology

Let $X \to S$ be a morphism of schemes. There is a map of complexes

$$d \log : \mathcal{O}_X^1[-1] \to \Omega^\bullet_{X/S}$$

which sends the section $g \in \mathcal{O}_X(U)$ to the section $d \log(g) = g^{-1}dg$ of $\Omega^1_{X/S}(U)$. Thus we can consider the map

$$\text{Pic}(X) = H^1(X, \mathcal{O}_X^*) = H^2(X, \mathcal{O}_X^1[-1]) \to H^2_{dR}(X/S)$$

where the first equality is Cohomology, Lemma 6.1. The image of the isomorphism class of the invertible module $\mathcal{L}$ is denoted $c_1^{dR}(\mathcal{L}) \in H^2_{dR}(X/S)$. 
We can also use the map \( d \log : \mathcal{O}_X^* \to \Omega^1_{X/S} \) to define a Chern class in Hodge cohomology

\[
\epsilon_1^{Hodge} : \text{Pic}(X) \to H^1(X, \Omega^1_{X/S}) \subset H^2_{dR}(X/S)
\]

These constructions are compatible with pullbacks.

Lemma 9.1. Given a commutative diagram

\[
\begin{array}{c}
X' \\
\downarrow f \\
S'
\end{array} \quad \begin{array}{c}
\Downarrow \quad \Downarrow \\
X \\
S
\end{array}
\]

of schemes the diagrams

\[
\begin{array}{ccc}
\text{Pic}(X') & \xleftarrow{f^*} & \text{Pic}(X) \\
\epsilon_1^{dR} & \downarrow & \epsilon_1^{dR} \\
H^2_{dR}(X'/S') & f^* & H^2_{dR}(X/S) \end{array} \quad \begin{array}{ccc}
\text{Pic}(X') & \xleftarrow{f^*} & \text{Pic}(X) \\
\epsilon_1^{Hodge} & \downarrow & \epsilon_1^{Hodge} \\
H^1(X', \Omega^1_{X'/S'}) & f^* & H^1(X, \Omega^1_{X/S})
\end{array}
\]

commute.

Proof. Omitted. \(\square\)

Let us “compute” the element \( \epsilon_1^{dR}(L) \) in Čech cohomology (with sign rules for Čech differentials as in Cohomology, Section 25). Namely, choose an open covering \( U : X = \bigcup_{i \in I} U_i \) such that we have a trivializing section \( s_i \) of \( L|_{U_i} \) for all \( i \).

On the overlaps \( U_{i_0 i_1} = U_{i_0} \cap U_{i_1} \) we have an invertible function \( f_{i_0 i_1} \) such that \( f_{i_0 i_1} = s_{i_1}|_{U_{i_0 i_1}} s_{i_0}^{-1}|_{U_{i_0 i_1}} \). Of course we have

\[
f_{i_0 i_1} f_{i_0 i_1}^{-1} f_{i_0 i_1}^{-1} f_{i_0 i_1} = 1
\]

The cohomology class of \( L \) in \( H^1(X, \mathcal{O}_X^*) \) is the image of the Čech cohomology class of the cocycle \( \{ f_{i_0 i_1} \} \) in \( \check{C}^*(U, \mathcal{O}_X^*) \). Therefore we see that \( \epsilon_1^{dR}(L) \) is the image of the cohomology class associated to the Čech cocycle \( \{ \alpha_{i_0 \ldots i_p} \} \) in \( \text{Tot}(\check{C}^*(U, \Omega^1_{X/S})) \) of degree 2 given by

\[
\begin{array}{l}
(1) \alpha_{i_0} = 0 \text{ in } \Omega^2_X(U_{i_0}), \\
(2) \alpha_{i_0 i_1} = f_{i_0 i_1}^{-1} d f_{i_0 i_1} \text{ in } \Omega^3_X(U_{i_0 i_1}), \text{ and}
\end{array}
\]

\[
(3) \alpha_{i_0 i_1 i_2} = 0 \text{ in } \mathcal{O}_X(U_{i_0 i_1 i_2}).
\]

Suppose we have invertible modules \( \mathcal{L}_k, k = 1, \ldots, a \) each trivialized over \( U_i \) for all \( i \in I \) giving rise to cocycles \( f_{k, i_0 i_1} \) and \( \alpha_k = \{ \alpha_{k, i_0 \ldots i_p} \} \) as above. Using the rule in Cohomology, Section 25 we can compute

\[
\beta = \alpha_1 \cup \alpha_2 \cup \ldots \cup \alpha_a
\]

to be given by the cocycle \( \beta = \{ \beta_{i_0 \ldots i_p} \} \) described as follows

\[
\begin{array}{l}
(1) \beta_{i_0 \ldots i_p} = 0 \text{ in } \Omega^2_{X/S}(U_{i_0 \ldots i_p}) \text{ unless } p = a, \text{ and}
\end{array}
\]

\[
(2) \beta_{i_0 \ldots i_a} = (-1)^{a(a-1)/2} \alpha_{i_0 i_1} \wedge \alpha_{2 i_1 i_2} \wedge \ldots \wedge \alpha_{a i_a+1 i_a} \text{ in } \Omega^a_{X/S}(U_{i_0 \ldots i_a}).
\]

The Čech differential of a 0-cycle \( \{ a_{i_0} \} \) has \( a_{i_1} - a_{i_0} \) over \( U_{i_0 i_1} \).
Thus this is a cocycle representing $c_1^{dR}(\mathcal{L}_1) \cup \ldots \cup c_1^{dR}(\mathcal{L}_n)$ Of course, the same computation shows that the cocycle $\{\beta_{i_0, \ldots, i_n}\}$ in $\check{\mathcal{C}}^2(\mathcal{U}, \Omega^a_{X/S})$ represents the cohomology class $c_1^{Hodge}(\mathcal{L}_1) \cup \ldots \cup c_1^{Hodge}(\mathcal{L}_n)$

**Remark 9.2.** Here is a reformulation of the calculations above in more abstract terms. Let $p: X \to S$ be a morphism of schemes. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. If we view $d\log$ as a map

$$\mathcal{O}_X[-1] \to \sigma_{\geq 1} \Omega^\bullet_{X/S}$$

then using $\text{Pic}(X) = H^1(X, \mathcal{O}_X^\bullet)$ as above we find a cohomology class

$$\gamma_1(\mathcal{L}) \in H^2(X, \sigma_{\geq 1} \Omega^\bullet_{X/S})$$

The image of $\gamma_1(\mathcal{L})$ under the map $\sigma_{\geq 1} \Omega^\bullet_{X/S} \to \Omega^\bullet_{X/S}$ recovers $c_1^{dR}(\mathcal{L})$. In particular we see that $c_1^{dR}(\mathcal{L}) \in F^1 H^2_{dR}(X/S)$, see Section 7. The image of $\gamma_1(\mathcal{L})$ under the map $\sigma_{\geq 1} \Omega^\bullet_{X/S} \to \Omega^\bullet_{X/S}[-1]$ recovers $c_1^{Hodge}(\mathcal{L})$. Taking the cup product (see Section 7) we obtain

$$\xi = \gamma_1(\mathcal{L}_1) \cup \ldots \cup \gamma_1(\mathcal{L}_n) \in H^{2a}(X, \sigma_{\geq 1} \Omega^\bullet_{X/S})$$

The commutative diagrams in Section 7 show that $\xi$ is mapped to $c_1^{dR}(\mathcal{L}_1) \cup \ldots \cup c_1^{dR}(\mathcal{L}_n)$ in $H^2_{dR}(X/S)$ by the map $\sigma_{\geq 1} \Omega^\bullet_{X/S} \to \Omega^\bullet_{X/S}$. Also, it follows $c_1^{dR}(\mathcal{L}_1) \cup \ldots \cup c_1^{dR}(\mathcal{L}_n)$ is contained in $F^a H^2_{dR}(X/S)$. Similarly, the map $\sigma_{\geq a} \Omega^\bullet_{X/S} \to \Omega^\bullet_{X/S}[-a]$ sends $\xi$ to $c_1^{Hodge}(\mathcal{L}_1) \cup \ldots \cup c_1^{Hodge}(\mathcal{L}_n)$ in $H^a(X, \Omega^\bullet_{X/S})$.

**Remark 9.3.** Let $p: X \to S$ be a morphism of schemes. For $i > 0$ denote $\Omega^i_{X/S, \log} \subset \Omega^\bullet_{X/S}$ the abelian subsheaf generated by local sections of the form

$$d\log(u_1) \wedge \ldots \wedge d\log(u_i)$$

where $u_1, \ldots, u_n$ are invertible local sections of $\mathcal{O}_X$. For $i = 0$ the subsheaf $\Omega^0_{X/S, \log} \subset \mathcal{O}_X$ is the image of $\mathcal{Z} \to \mathcal{O}_X$. For every $i \geq 0$ we have a map of complexes

$$\Omega^i_{X/S, \log} \to \Omega^\bullet_{X/S}$$

because the derivative of a logarithmic form is zero. Moreover, wedging logarithmic forms gives another, hence we find bilinear maps

$$\wedge: \Omega^i_{X/S, \log} \times \Omega^j_{X/S, \log} \to \Omega^{i+j}_{X/S, \log}$$

compatible with (4.0.1) and the maps above. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Using the map of abelian sheaves $d\log: \mathcal{O}_X \to \Omega^1_{X/S, \log}$ and the identification $\text{Pic}(X) = H^1(X, \mathcal{O}_X^\bullet)$ we find a canonical cohomology class

$$\tilde{\gamma}_1(\mathcal{L}) \in H^1(X, \Omega^1_{X/S, \log})$$

These classes have the following properties

1. The image of $\tilde{\gamma}_1(\mathcal{L})$ under the canonical map $\Omega^1_{X/S, \log}[-1] \to \sigma_{\geq 1} \Omega^\bullet_{X/S}$ sends $\tilde{\gamma}_1(\mathcal{L})$ to the class $\gamma_1(\mathcal{L}) \in H^2(X, \sigma_{\geq 1} \Omega^\bullet_{X/S})$ of Remark 9.2.
2. The image of $\tilde{\gamma}_1(\mathcal{L})$ under the canonical map $\Omega^1_{X/S, \log}[-1] \to \Omega^\bullet_{X/S}$ sends $\tilde{\gamma}_1(\mathcal{L})$ to $c_1^{dR}(\mathcal{L})$ in $H^2_{dR}(X/S)$.
3. The image of $\tilde{\gamma}_1(\mathcal{L})$ under the canonical map $\Omega^1_{X/S, \log} \to \Omega^\bullet_{X/S}$ sends $\tilde{\gamma}_1(\mathcal{L})$ to $c_1^{Hodge}(\mathcal{L})$ in $H^1(X, \Omega^1_{X/S})$. 

(4) the construction of these classes is compatible with pullbacks,
(5) add more here.

10. de Rham cohomology of projective space

Let $A$ be a ring. Let $n \geq 1$. The structure morphism $\mathbf{P}_A^n \to \text{Spec}(A)$ is a proper smooth of relative dimension $n$. It is smooth of relative dimension $n$ and of finite type as $\mathbf{P}_A^n$ has a finite affine open covering by schemes each isomorphic to $\mathbf{A}^n_A$; see Constructions, Lemma 13.3. It is proper because it is also separated and universally closed by Constructions, Lemma 13.4. Let us denote $\mathcal{O}$ and $\mathcal{O}(d)$ the structure sheaf $\mathcal{O}_{\mathbf{P}_A^n}$ and the Serre twists $\mathcal{O}_{\mathbf{P}_A^n}(d)$. Let us denote $\Omega = \Omega_{\mathbf{P}_A^n/A}$ the sheaf of relative differentials and $\Omega^p$ its exterior powers.

**Lemma 10.1.** There exists a short exact sequence

$$0 \to \Omega \to \mathcal{O}(-1)^{\oplus n+1} \to \mathcal{O} \to 0$$

**Proof.** To explain this, we recall that $\mathbf{P}_A^n = \text{Proj}(A[T_0, \ldots, T_n])$, and we write symbolically

$$\mathcal{O}(-1)^{\oplus n+1} = \bigoplus_{j=0, \ldots, n} \mathcal{O}(-1)dT_j$$

The first arrow

$$\Omega \to \bigoplus_{j=0, \ldots, n} \mathcal{O}(-1)dT_j$$

in the short exact sequence above is given on each of the standard opens $D_+(T_i) = \text{Spec}(A[T_0/T_i, \ldots, T_n/T_i])$ mentioned above by the rule

$$\sum_{j \neq i} g_j d(T_j/T_i) \mapsto \sum_{j \neq i} g_jdT_j - \left(\sum_{j \neq i} g_j T_j/T_i^2\right)dT_i$$

This makes sense because $1/T_i$ is a section of $\mathcal{O}(-1)$ over $D_+(T_i)$. The map

$$\bigoplus_{j=0, \ldots, n} \mathcal{O}(-1)dT_j \to \mathcal{O}$$

is given by sending $dT_j$ to $T_j$, more precisely, on $D_+(T_i)$ we send the section $\sum g_jdT_j$ to $\sum T_jg_j$. We omit the verification that this produces a short exact sequence. □

**Lemma 10.2.** We have $H^q(\mathbf{P}_A^n, \Omega^p) = 0$ unless $0 \leq p = q \leq n$. For $0 \leq p \leq n$ the $A$-module $H^p(\mathbf{P}_A^n, \Omega^p)$ free of rank $1$ with basis element $c^\text{Hodge}_1(\mathcal{O}(1))^p$.

**Proof.** We are going to use the results of Cohomology of Schemes, Lemma 8.1 without further mention. In particular, the statement is true for $H^q(\mathbf{P}_A^n, \mathcal{O})$.

Proof for $p = 1$. Consider the short exact sequence

$$0 \to \Omega \to \mathcal{O}(-1)^{\oplus n+1} \to \mathcal{O} \to 0$$

of Lemma 10.1. It shows that $H^q(\mathbf{P}_A^n, \Omega) = 0$ unless $q = 1$ and for $q = 1$ it free of rank $1$ generated by the boundary $\xi$ of $1 \in H^0(\mathbf{P}_A^n, \mathcal{O})$. As in the proof of Lemma 10.1 we will identify $\mathcal{O}(-1)^{\oplus n+1}$ with $\bigoplus_{j=0, \ldots, n} \mathcal{O}(-1)dT_j$. Consider the open covering

$$U : \mathbf{P}_A^n = \bigcup_{i=0, \ldots, n} D_+(T_i)$$

We can lift the restriction of the global section $1$ of $\mathcal{O}$ to $U_i = D_+(T_i)$ by the section $T_i^{-1}dT_i$ of $\bigoplus \mathcal{O}(-1)dT_j$ over $U_i$. Thus the cocyle representing $\xi$ is given by

$$T_i^{-1}dT_{i_1} - T_i^{-1}dT_{i_0} = d \log(T_{i_1}/T_{i_0}) \in \Omega(U_{i_0i_1})$$
On the other hand, for each \( i \) the section \( T_i \) is a trivializing section of \( O(1) \) over \( U_i \). Hence we see that \( f_{0i} = T_1/T_{i_0} \in O(U_{i_0i_1}) \) is the cocycle representing \( O(1) \) in \( \text{Pic}(\mathbb{P}^n_A) \), see Section \( \square \). Hence \( c^1_{H^{\text{Hodge}}(O(1))} \) is given by the cocycle \( d \log(T_1/T_{i_0}) \) which agrees with what we got for \( \xi \) above.

Proof for general \( p \) by induction. The base cases \( p = 0, 1 \) were handled above. Assume \( p > 1 \). Let us think of the short exact sequence above as defining a 2 step filtration on \( O(-1)^{\oplus n+1} \). The induced filtration on \( \wedge^p O(-1)^{\oplus n+1} \) looks like this

\[
0 \to \Omega^p \to \wedge^p O(-1)^{\oplus n+1} \to \Omega^{p-1} \to 0
\]

Observe that \( \wedge^p O(-1)^{\oplus n+1} \) is isomorphic to a direct sum of \( n+1 \) choose \( p \) copies of \( O(-p) \) and hence has vanishing cohomology in all degrees. By induction hypothesis, this shows that \( H^q(P^A_n, \Omega^p) \) is zero unless \( q = p \) and \( H^p(P^A_n, \Omega^p) \) is free of rank 1 with generator the boundary of \( c^1_{H^{\text{Hodge}}(O(1))} \). By the calculation in Section \( \square \) the cohomology class \( c^1_{H^{\text{Hodge}}(O(1))} \) is, up to a sign, represented by the cocycle with terms

\[
\beta_{i_0 \ldots i_{p-1}} = d \log(T_{i_1}/T_{i_2}) \wedge d \log(T_{i_2}/T_{i_3}) \wedge \ldots \wedge d \log(T_{i_{p-1}}/T_{i_p})
\]

in \( \Omega^{p-1}(U_{i_0 \ldots i_{p-1}}) \). These \( \beta_{i_0 \ldots i_{p-1}} \) can be lifted to the sections \( \tilde{\beta}_{i_0 \ldots i_{p-1}} = T_{i_0}^{-1}dT_{i_1} \wedge T_{i_1}^{-1}dT_{i_2} \wedge \ldots \wedge T_{i_{p-1}}^{-1}d\beta_{i_1 \ldots i_p} \) of \( \wedge^p \bigoplus \bigwedge^p O(-1)^{\oplus n+1} \) over \( U_{i_0 \ldots i_{p-1}} \). We conclude that the generator of \( H^p(P^A_n, \Omega^p) \) is given by the cocycle components are

\[
\sum_{a=0}^p (-1)^a \beta_{a_0 \ldots a_p} = T_{i_1}^{-1}dT_{i_2} \wedge \beta_{i_2 \ldots i_p} + \sum_{a=1}^p (-1)^a T_{i_0}^{-1}dT_{i_1} \wedge \beta_{a_1 \ldots a_p}
\]

viewed as a section of \( \Omega^p \) over \( U_{i_0 \ldots i_p} \). This is up to sign the same as the cocycle representing \( c^1_{H^{\text{Hodge}}(O(1))} \) and the proof is complete.

\[\square\]

Lemma 10.3. For \( 0 \leq i \leq n \) the de Rham cohomology \( H^{2i}_{dR}(P^A_n/A) \) is a free \( A \)-module of rank 1 with basis element \( c^i_{dR}(O(1))^i \). In all other degrees the de Rham cohomology of \( P^A_n \) over \( A \) is zero.

Proof. Consider the Hodge-to-de Rham spectral sequence of Section \( \square \). By the computation of the Hodge cohomology of \( P^A_n \) over \( A \) done in Lemma \( \square \) we see that the spectral sequence degenerates on the \( E_1 \) page. In this way we see that \( H^{2i}_{dR}(P^A_n/A) \) is a free \( A \)-module of rank 1 for \( 0 \leq i \leq n \) and zero else. Observe that \( c^i_{dR}(O(1))^i \in H^2_{dR}(P^A_n/A) \) for \( i = 0, \ldots, n \) and that for \( i = n \) this element is the image of \( c^1_{H^{\text{Hodge}}(\mathcal{L})} \) by the map of complexes

\[
\Omega^0_{P^A_n/A}[n] \to \Omega^0_{P^A_n/A}
\]

This follows for example from the discussion in Remark \( \square \) or from the explicit description of cocycles representing these classes in Section \( \square \). The spectral sequence shows that the induced map

\[
H^n(P^A_n, \Omega^0_{P^A_n/A}) \to H^{2n}_{dR}(P^A_n/A)
\]

is an isomorphism and since \( c^1_{H^{\text{Hodge}}(\mathcal{L})} \) is a generator of the source (Lemma \( \square \)), we conclude that \( c^i_{dR}(\mathcal{L})^i \) is a generator of the target. By the \( A \)-bilinearity of the cup products, it follows that also \( c^i_{dR}(\mathcal{L})^i \) is a generator of \( H^{2i}_{dR}(P^A_n/A) \) for \( 0 \leq i \leq n \). \[\square\]
11. The spectral sequence for a smooth morphism

0FMK Consider a commutative diagram of schemes

\[ \begin{align*}
X & \xrightarrow{f} Y \\
\downarrow p & \quad & \downarrow q \\
S & \quad & \quad 
\end{align*} \]

where \( f \) is a smooth morphism. Then we obtain a locally split short exact sequence

\[ 0 \to f^* \Omega_{Y/S} \to \Omega_{X/S} \to \Omega_{X/Y} \to 0 \]

by Morphisms, Lemma \[\text{32.16}\]. Let us think of this as a descending filtration \( F \) on \( \Omega_{X/S} \) with \( F^0 \Omega_{X/S} = \Omega_{X/S}, \ F^1 \Omega_{X/S} = f^* \Omega_{Y/S}, \) and \( F^2 \Omega_{X/S} = 0 \). Applying the functor \( \wedge^p \) we obtain for every \( p \) an induced filtration

\[ \Omega_{X/S}^p = F^0 \Omega_{X/S}^p \supset F^1 \Omega_{X/S}^p \supset F^2 \Omega_{X/S}^p \supset \cdots \supset F^{p+1} \Omega_{X/S}^p = 0 \]

whose successive quotients are

\[ \text{gr}^k \Omega_{X/S}^p = F^k \Omega_{X/S}^p / F^{k+1} \Omega_{X/S}^p = f^* \Omega_{Y/S}^k \otimes_{O_Y} \Omega_{X/Y}^p = f^{-1} \Omega_{Y/S}^k \otimes f^{-1} O_Y \Omega_{X/Y}^p \]

for \( k = 0, \ldots, p \). In fact, the reader can check using the Leibniz rule that \( F^k \Omega_{X/S}^p \) is a subcomplex of \( \Omega_{X/S}^p \). In this way \( \Omega_{X/S}^p \) has the structure of a filtered complex.

We can also see this by observing that

\[ F^k \Omega_{X/S}^p = \text{Im} \left( \wedge : \text{Tot}(f^{-1} \Omega_{Y/S}^p \otimes_{O_Y} \Omega_{X/S}^p) \to \Omega_{X/S}^p \right) \]

is the image of a map of complexes on \( X \). The filtered complex

\[ \Omega_{X/S} = F^0 \Omega_{X/S}^p \supset F^1 \Omega_{X/S}^p \supset F^2 \Omega_{X/S}^p \supset \cdots \]

has the following associated graded parts

\[ \text{gr}^k \Omega_{X/S}^p = f^{-1} \Omega_{Y/S}^k[-k] \otimes f^{-1} O_Y \Omega_{X/Y}^p \]

by what was said above. We interrupt the discussion for a technical result.

0FML \[\text{Lemma 11.1.}\] Let \( f : X \to Y \) be a quasi-compact and quasi-separated morphism of schemes. Let \( F^\bullet \) be a locally bounded complex of \( f^{-1} O_Y \)-modules. Assume for all \( n \in \mathbb{Z} \) the sheaf \( F^n \) is a flat \( f^{-1} O_Y \)-module and \( F^n \) has the structure of a quasi-coherent \( O_X \)-module compatible with the given \( p^{-1} O_Y \)-module structure (but the differentials in the complex \( F^\bullet \) need not be \( O_X \)-linear). Then

\[ \mathcal{G} \otimes_{O_Y} Rf_* F^\bullet = Rf_*(f^{-1} \mathcal{G} \otimes_{f^{-1} O_Y} F^\bullet) \]

for any quasi-coherent \( O_Y \)-module \( \mathcal{G} \).

**Proof.** This lemma is a variant of Derived Categories of Schemes, Lemma \[\text{21.1}\] and we urge the reader to read the proof of that lemma first. Denote \( f^! : (X, f^{-1} O_Y) \to (Y, O_Y) \) the obvious flat morphism of ringed spaces. We view \( F^\bullet \) as a complex of modules on the ringed space \( (X, f^{-1} O_Y) \). Since \( F^\bullet \) is a locally bounded complex of flat \( f^{-1} O_Y \)-modules we see that the complex \( f^{-1} \mathcal{G} \otimes_{f^{-1} O_Y} F^\bullet \) represents \( L(f^!) \mathcal{G} \otimes_{f^{-1} O_Y} F^\bullet \) in \( D(f^{-1} O_Y) \). Hence the statement of the lemma is really that

\[ \mathcal{G} \otimes_{O_Y} Rf^! F^\bullet = Rf^!(L(f^!) \mathcal{G} \otimes_{f^{-1} O_Y} F^\bullet) \]
Formulated in this manner we can generalize the statement to the statement that the canonical, functorial arrow Cohomology, Equation (47.2.1)

\[ G \otimes_{\mathcal{O}_Y} Lf'_* F^\bullet \to Rf'_*(L(f')^*G \otimes_{\mathcal{O}_Y} F^\bullet) \]

is an isomorphism for all \( G \) in \( D_{QCoh}(\mathcal{O}_Y) \). Formulated in this manner the problem is local on \( Y \) and we may assume \( Y \) is affine. Moreover, the source and target of the arrow are exact functors \( D_{QCoh}(\mathcal{O}_Y) \to D(\mathcal{O}_Y) \) of triangulated categories. Hence if \( G_1 \to G_2 \to G_3 \to G_1[1] \) is a distinguished triangle in \( D_{QCoh}(\mathcal{O}_Y) \) and the result holds for two out of \( G_1, G_2, G_3 \), then the result holds for the third. Finally, both sides of the arrow commute with arbitrary direct sums (see below for the right hand side). Thus, exactly as in the proof of Derived Categories of Schemes, Lemma 21.1 it suffices to prove that

\[ \mathcal{O}_Y \otimes_{\mathcal{O}_Y} Lf'_* F^\bullet \to Rf'_*(L(f')^*\mathcal{O}_Y \otimes_{\mathcal{O}_Y} F^\bullet) \]

which is obvious.

We will have to show that \( G \mapsto Rf'_*(L(f')^*G \otimes_{\mathcal{O}_Y} F^\bullet) \) commutes with direct sums on \( D_{QCoh}(\mathcal{O}_Y) \). This is where we will use \( F^n \) has the structure of a quasi-coherent \( \mathcal{O}_X \)-module. First, observe that \( G \mapsto L(f')^*G \otimes_{\mathcal{O}_Y} F^\bullet \) commutes with arbitrary direct sums. Next, if \( F^\bullet \) consists of a single quasi-coherent \( \mathcal{O}_X \)-module \( F^\bullet = F^n[-n] \) then we have \( L(f')^*G \otimes_{\mathcal{O}_Y} F^\bullet = Lf^*G \otimes_{\mathcal{O}_X} F^n[-n] \), see Cohomology, Lemma 27.4. Hence in this case the commutation with direct sums follows from Derived Categories of Schemes, Lemma 4.2. Now, in general, since \( Y \) is affine and \( F^\bullet \) is locally bounded, we see that

\[ F^\bullet = (F^a \to \ldots \to F^b) \]

is bounded. Arguing by induction on \( b-a \) and considering the distinguished triangle

\[ F^b[-b] \to (F^a \to \ldots \to F^b) \to (F^a \to \ldots \to F^{b-1}) \to F^b[-b+1] \]

the proof is finished. Some details omitted.

**0FMN Lemma 11.2.** Let \( f : X \to Y \) be a quasi-compact, quasi-separated, and smooth morphism of schemes over a base scheme \( S \). There is a bounded spectral sequence with first page

\[ E_1^{p,q} = H^q(\Omega^p_{X/S} \otimes_{\mathcal{O}_Y} Rf_* \Omega^\bullet_{X/Y}) \]

converging to \( R^{p+q}f_* \Omega^\bullet_{X/S} \).

**Proof.** Consider \( \Omega^\bullet_{X/S} \) as a filtered complex with the filtration introduced above. The spectral sequence is the spectral sequence of Cohomology, Lemma 29.5. By Lemma 11.1 we have

\[ Rf_* g^k \Omega^\bullet_{X/S} = \Omega^L_{Y/S}[-k] \otimes_{\mathcal{O}_Y} Rf_* \Omega^\bullet_{X/Y} \]

and thus we conclude. \( \square \)

**0FMN Remark 11.3.** In Lemma 11.2 consider the cohomology sheaves

\[ \mathcal{H}^q_{dR}(X/Y) = H^q(Rf_* \Omega^\bullet_{X/Y}) \]

If \( f \) is proper in addition to being smooth and \( S \) is a scheme over \( Q \) then \( \mathcal{H}^q_{dR}(X/Y) \) is finite locally free (insert future reference here). If we only assume \( \mathcal{H}^q_{dR}(X/Y) \) are flat \( \mathcal{O}_Y \)-modules, then we obtain (tiny argument omitted)

\[ E_1^{p,q} = \Omega^p_{Y/S} \otimes_{\mathcal{O}_Y} \mathcal{H}^q_{dR}(X/Y) \]
and the differentials in the spectral sequence are maps
\[ d_{i}^{q} : \Omega_{Y/S}^{i} \otimes \mathcal{O}_{Y} \to \Omega_{Y/S}^{i+1} \otimes \mathcal{O}_{Y} \]
In particular, for \( p = 0 \) we obtain a map \( d_{i}^{0,q} : \mathcal{H}_{\text{dR}}^{q}(X/Y) \to \Omega_{Y/S}^{i} \otimes \mathcal{O}_{Y} \mathcal{H}_{\text{dR}}^{q}(X/Y) \)
which turns out to be an integrable connection \( \nabla \) (insert future reference here) and the complex
\[ \mathcal{H}_{\text{dR}}^{q}(X/Y) \to \Omega_{Y/S}^{1} \otimes \mathcal{O}_{Y} \mathcal{H}_{\text{dR}}^{q}(X/Y) \to \Omega_{Y/S}^{2} \otimes \mathcal{O}_{Y} \mathcal{H}_{\text{dR}}^{q}(X/Y) \to \ldots \]
with differentials given by \( d_{i}^{0,q} \) is the de Rham complex of \( \nabla \). The connection \( \nabla \) is known as the Gauss-Manin connection.

**Lemma 11.4.** Let \( f : X \to Y \) be a smooth proper morphism of schemes. Let \( N \) and \( n_{1}, \ldots, n_{N} \geq 0 \) be integers and let \( \xi_{i} \in H_{\text{dR}}^{n_{i}}(X/Y), 1 \leq i \leq N \). Assume for all points \( y \in Y \) the images of \( \xi_{1}, \ldots, \xi_{N} \) in \( H_{\text{dR}}^{q}(X_{y}/y) \) form a basis over \( \kappa(y) \). Then the map
\[ \bigoplus_{i=1}^{N} \mathcal{O}_{Y}[-n_{i}] \to Rf_{*}\Omega_{X/Y}^{\bullet} \]
associated to \( \xi_{1}, \ldots, \xi_{N} \) is an isomorphism.

**Proof.** By Lemma 3.5 \( Rf_{*}\Omega_{X/Y}^{\bullet} \) is a perfect object of \( D(\mathcal{O}_{Y}) \) whose formation commutes with arbitrary base change. Thus the map of the lemma is a map \( a : K \to L \) between perfect objects of \( D(\mathcal{O}_{Y}) \) whose derived restriction to any point is an isomorphism by our assumption on fibres. Then the cone \( C \) on \( a \) is a perfect object of \( D(\mathcal{O}_{Y}) \) (Cohomology, Lemma 45.7) whose derived restriction to any point is zero. It follows that \( C \) is zero by More on Algebra, Lemma 70.6 and \( a \) is an isomorphism. (This also uses Derived Categories of Schemes, Lemmas 3.5 and 9.7 to translate into algebra.)

**Lemma 11.5.** Let \( f : X \to Y \) be a morphism of schemes with \( Y = \text{Spec}(A) \) affine. Let \( \mathcal{U} : X = \bigcup_{i \in I} U_{i} \) be a finite affine open covering such that all the finite intersections \( U_{i_{0}} \cap \ldots \cap U_{i_{p}} \) are affine. Let \( \mathcal{F}^{\bullet} \) be a bounded complex of \( f^{-1}\mathcal{O}_{Y} \)-modules. Assume for all \( n \in \mathbb{Z} \) the sheaf \( \mathcal{F}^{n} \) is a flat \( f^{-1}\mathcal{O}_{Y} \)-module and \( \mathcal{F}^{n} \) has the structure of a quasi-coherent \( \mathcal{O}_{X} \)-module compatible with the given \( p^{-1}\mathcal{O}_{Y} \)-module structure (but the differentials in the complex \( \mathcal{F}^{\bullet} \) need not be \( \mathcal{O}_{X} \)-linear). Then the complex \( \text{Tot}((\mathcal{C}^{\bullet}(\mathcal{U}, \mathcal{F}^{\bullet})) \) is K-flat as a complex of \( A \)-modules.

**Proof.** We may write
\[ \mathcal{F}^{\bullet} = (\mathcal{F}^{a} \to \ldots \to \mathcal{F}^{b}) \]
Arguing by induction on \( b - a \) and considering the distinguished triangle
\[ \mathcal{F}^{b}[-b] \to (\mathcal{F}^{a} \to \ldots \to \mathcal{F}^{b}) \to (\mathcal{F}^{a} \to \ldots \to \mathcal{F}^{b-1}) \to \mathcal{F}^{b}[-b + 1] \]
and using More on Algebra, Lemma 57.7 we reduce to the case where \( \mathcal{F}^{\bullet} \) consists of a single quasi-coherent \( \mathcal{O}_{X} \)-module \( \mathcal{F} \) placed in degree 0. In this case the Čech complex for \( \mathcal{F} \) and \( \mathcal{U} \) is homotopy equivalent to the alternation Čech complex, see Cohomology, Lemma 23.6. Since \( U_{i_{0}} \cap \ldots \cap U_{i_{p}} \) is always affine, we see that \( \mathcal{F}(U_{i_{0}} \cap \ldots \cap U_{i_{p}}) \) is \( A \)-flat. Hence \( \mathcal{C}^{\bullet}_{\text{alt}}(\mathcal{U}, \mathcal{F}) \) is a bounded complex of flat \( A \)-modules and hence K-flat by More on Algebra, Lemma 57.9.

**Proposition 11.6.** Let \( f : X \to Y \) be a smooth proper morphism of schemes over a base \( S \). Let \( N \) and \( n_{1}, \ldots, n_{N} \geq 0 \) be integers and let \( \xi_{i} \in H_{\text{dR}}^{n_{i}}(X/S), 1 \leq i \leq N \).
Assume for all points $y \in Y$ the images of $\xi_1, \ldots, \xi_N$ in $H^*_dR(X_{/y})$ form a basis over $\kappa(y)$. Then the map
\[
\bigoplus_{i=1}^N H^*_dR(Y/S) \longrightarrow H^*_dR(X/S), \quad (a_1, \ldots, a_N) \longmapsto \sum f^*a_i \cup \xi_i
\]
is an isomorphism.

**Proof.** Denote $p : X \to S$ and $q : Y \to S$ be the structure morphisms. We can think of $\xi_i$ as a map $p^{-1}O_S[-n_i] \to \Omega^*_X/S$. Thus we can consider the map
\[
f^{-1}\Omega^*_Y/S[-n_i] = f^{-1}\Omega^*_Y/S \otimes_{p^{-1}O_S} p^{-1}O_S[-n_i]
\]
\[
\xrightarrow{id \otimes \xi_i} f^{-1}\Omega^*_Y/S \otimes_{p^{-1}O_S} \Omega^*_X/S \to \text{Tot}(f^{-1}\Omega^*_Y/S \otimes_{p^{-1}O_S} \Omega^*_X/S)
\]
\[
\xrightarrow{\cup} \Omega^*_X/S
\]
in $D(X, p^{-1}O_S)$. The adjoint of this is a map
\[
\tilde{\xi}_i : \Omega^*_Y/S[-n_i] \longrightarrow Rf_*\Omega^*_X/S
\]
in $D(Y, q^{-1}O_S)$. By the discussion in Cohomology, Section 31 on cohomology $\tilde{\xi}_i$ gives the map $a \mapsto f^*a \cup \xi_i$. Thus it suffices to show that the map
\[
\xi = \bigoplus \tilde{\xi}_i : \bigoplus \Omega^*_Y/S[-n_i] \longrightarrow Rf_*\Omega^*_X/S
\]
is an isomorphism in $D(Y, q^{-1}O_S)$. If $Y' \subset Y$ is open with inverse image $X' \subset X$, then $\xi_{|X'}$ induces the map
\[
\bigoplus_{i=1}^N H^*_dR(Y'/S) \longrightarrow H^*_dR(X'/S), \quad (a_1, \ldots, a_N) \longmapsto \sum f^*a_i \cup \xi_i|_{X'}
\]
on cohomology over $Y'$. Thus it suffices to find a basis for the topology on $Y$ such that the proposition holds for the members of the basis. This reduces us to the case discussed in the next paragraph.

Assume $Y$ and $S$ are affine. Say $Y = \text{Spec}(A)$ and $S = \text{Spec}(R)$. In this case $\Omega^*_A/R$ computes $R\Gamma(Y, \Omega^*_Y/S)$ by Lemma 3.1. Choose a finite affine open covering $U : X = \bigcup_{i \in I} U_i$. Consider the complex
\[
K^* = \text{Tot}(\tilde{\xi}^*(U, \Omega^*_X/S))
\]
as in Cohomology, Section 25. Let us collect some facts about this complex most of which can be found in the reference just given:

1. $K^*$ is a complex of $R$-modules whose terms are $A$-modules,
2. $K^*$ represents $R\Gamma(X, \Omega^*_X/S)$ in $D(R)$ (Cohomology of Schemes, Lemma 2.2 and Cohomology, Lemma 25.2),
3. there is a natural map $\Omega^*_A/R \to K^*$ of complexes of $R$-modules which is $A$-linear on terms and induces the pullback map $H^*_dR(Y/S) \to H^*_dR(X/S)$ on cohomology,
4. $K^*$ has a multiplication denoted $\wedge$ which turns it into a differential graded $R$-algebra,
5. the multiplication on $K^*$ induces the cup product on $H^*_dR(X/S)$ (Cohomology, Section 31).
Let the map \( \bigoplus \) and similarly for \( \square \) are finite in any given degree. is an isomorphism in \( A \) is an isomorphism in \( \text{gr} \) the derived tensor product derived tensor product as is true by inspection on the left hand side. Finally, taking isomorphisms to isomorphisms. Arguing by induction on \( i \) which sends \( \omega \) in the \( i \)th summand to \( \omega \wedge x_i \). All that remains is to show that this map is a quasi-isomorphism. We endow \( M^* \) with the structure of a filtered complex by the rule

\[
F^k M^* = \bigoplus_{i=1, \ldots, N} \Omega^*_{A/R}[-n_i] \rightarrow K^*
\]

With this choice the map \( \tilde{x} \) is a morphism of filtered complexes. Observe that \( \text{gr}^0 M^* = \bigoplus A[-n_i] \) and multiplication induces an isomorphism \( \text{gr}^k \Omega^k_{A/R}[-k] \otimes_A \text{gr}^0 M^* \rightarrow \text{gr}^k M^* \). By construction and Lemma 11.4 we see that \( \text{gr}^k \tilde{x} : \text{gr}^k M^* \rightarrow \text{gr}^k K^* \) is an isomorphism in \( D(A) \). It follows that for all \( k \geq 0 \) we obtain isomorphisms

\[
\text{gr}^k \tilde{x} : \text{gr}^k M^* = \Omega^k_{A/R}[-k] \otimes_A \text{gr}^0 M^* \rightarrow \Omega^k_{A/R}[-k] \otimes_A \text{gr}^0 K^* = \text{gr}^k K^*
\]

in \( D(A) \). Namely, the complex \( \text{gr}^0 K^* = \text{Tot}(\hat{\mathcal{C}}^* \langle U, \Omega^*_{X/Y} \rangle) \) is \( K \)-flat as a complex of \( A \)-modules by Lemma 11.5. Hence the tensor product on the right hand side is the derived tensor product as is true by inspection on the left hand side. Finally, taking the derived tensor product \( \Omega^k_{A/R}[-k] \otimes_A \) is a functor on \( D(A) \) and therefore sends isomorphisms to isomorphisms. Arguing by induction on \( k \) we deduce that \( \tilde{x} : M^*/F^k M^* \rightarrow K^*/F^k K^* \) is an isomorphism in \( D(R) \) since we have the short exact sequences

\[
0 \rightarrow F^k M^*/F^{k+1} M^* \rightarrow M^*/F^{k+1} M^* \rightarrow \text{gr}^k M^* \rightarrow 0
\]

and similarly for \( K^* \). This proves that \( \tilde{x} \) is a quasi-isomorphism as the filtrations are finite in any given degree.

\[ \square \]

12. Projective space bundle formula

\[ 0FMS \quad \text{The title says it all.} \]

\[ 0FMT \quad \text{Proposition 12.1. Let } X \rightarrow S \text{ be a morphism of schemes. Let } \mathcal{E} \text{ be a locally free } \mathcal{O}_X \text{-module of constant rank } r. \text{ Consider the morphism } p : P = \mathcal{P}(\mathcal{E}) \rightarrow X. \text{ Then the map} \]

\[
\bigoplus_{i=0, \ldots, r-1} H^i_{dR}(X/S) \longrightarrow H^i_{dR}(P/S), \quad (a_0, \ldots, a_{r-1}) \longmapsto \sum p^*(a_i) \cup c_{1}^{dR}(\mathcal{O}_P(1))^i
\]
is an isomorphism.

**Proof.** Choose an affine open Spec$(A) \subset X$ such that $\mathcal{E}$ restricts to the trivial locally free module $\mathcal{O}_{\text{Spec}(A)}$. Then $P \times X \text{Spec}(A) = P^r_A$. Thus we see that $p$ is proper and smooth, see Section [10]. Moreover, the classes $c_i^{dR}(\mathcal{O}_P(1))^i$, $i = 0,1,\ldots,r-1$ restricted to a fibre $X_y = P^r_{y}$ freely generate the de Rham cohomology $H^j_{dR}(X,y)$ over $\kappa(y)$, see Lemma [10.3]. Thus we’ve verified the conditions of Proposition [11.6] and we win. □

13. Log poles along a divisor

0FMU Let $X \to S$ be a morphism of schemes. Let $Y \subset X$ be an effective Cartier divisor. If $X$ étale locally along $Y$ looks like $Y \times \mathbb{A}^1$, then there is a canonical short exact sequence of complexes

$$0 \to \Omega^\bullet_{X/S} \to \Omega^\bullet_{X/S}(\log Y) \to \Omega^\bullet_{Y/S}[-1] \to 0$$

having many good properties we will discuss in this section.

0FMV **Definition 13.1.** Let $X \to S$ be a morphism of schemes. Let $Y \subset X$ be an effective Cartier divisor. We say the de Rham complex of log poles is defined for $Y \subset X$ over $S$ if for all $y \in Y$ and local equation $f \in \mathcal{O}_{X,y}$ of $Y$ we have

1. $\mathcal{O}_{X,x} \to \mathcal{O}_{X,S,x}$, $g \mapsto gdf$ is a split injection, and
2. $\Omega^p_{X,S,y}$ is $f$-torsion free for all $p$.

An easy local calculation shows that it suffices to find one local equation $f$ for which condition (1) holds.

0FMW **Lemma 13.2.** Let $X \to S$ be a morphism of schemes. Let $Y \subset X$ be an effective Cartier divisor. Assume the de Rham complex of log poles is defined for $Y \subset X$ over $S$. There is a canonical short exact sequence of complexes

$$0 \to \Omega^\bullet_{X/S} \to \Omega^\bullet_{X/S}(\log Y) \to \Omega^\bullet_{Y/S}[-1] \to 0$$

**Proof.** Our assumption is that for every $y \in Y$ and local equation $f \in \mathcal{O}_{X,y}$ of $Y$ we have

$$\Omega^p_{X,S,y} = \mathcal{O}_{X,y} df \oplus M$$

and

$$\Omega^p_{X,S,y} = \land^{p-1}(M) df \oplus \land^p(M)$$

for some module $M$ with $f$-torsion free exterior powers $\land^p(M)$. It follows that

$$\Omega^p_{Y,S,y} = \land^p(M/fM) = \land^p(M)/f \land^p(M)$$

Below we will tacitly use these facts. In particular the sheaves $\Omega^p_{X/S}$ have no nonzero torsion sections supported on $Y$ and we have a canonical inclusion

$$\Omega^p_{X/S} \subset \Omega^p_{X/S}(Y)$$

see More on Flatness, Section [42]. Let $U = \text{Spec}(A)$ be an affine open subscheme such that $Y \cap U = V(f)$ for some nonzerodivisor $f \in A$. Let us consider the $\mathcal{O}_U$-submodule of $\Omega^p_{X/S}(Y)|_U$ generated by $\Omega^p_{X/S}|_U$ and $d\log(f) \land \Omega^p_{X/S}|_U$ where $d\log(f) = f^{-1}d(f)$. This is independent of the choice of $f$ as another generator of the ideal of $Y$ on $U$ is equal to $uf$ for a unit $u \in A$ and we get

$$d\log(uf) - d\log(f) = d\log(u) = u^{-1}du$$
which is a section of $\Omega_{X/S}$ over $U$. Obviously, these local sheaves glue to give a quasi-coherent submodule

$$\Omega^p_{X/S} \subset \Omega^p_{X/S}(\log Y) \subset \Omega^p_{Y/S}(Y)$$

Let us agree to think of $\Omega^p_{Y/S}$ as a quasi-coherent $\mathcal{O}_X$-module. There is a unique surjective $\mathcal{O}_X$-linear map

$$\Omega^p_{X/S}(\log Y) \to \Omega^p_{Y/S}$$

which over $U$ sends a local section $d\log(f) \wedge \eta$ to the image $\eta$ in $\Omega^p_{Y/S}$ and annihilates the submodule $\Omega^p_{X/S}$. If a form $\eta$ over $U$ restricts to zero on $\Omega_{Y/S}$, then $\eta = df \wedge \eta' + f\eta''$ for some forms $\eta'$ and $\eta''$ over $U$. We conclude that we have a short exact sequence

$$0 \to \Omega^p_{X/S} \to \Omega^p_{X/S}(\log Y) \to \Omega^p_{Y/S} \to 0$$

for all $p$.

We still have to define the differentials $\Omega^p_{X/S}(\log Y) \to \Omega^{p+1}_{X/S}(\log Y)$. On the subsheaf $\Omega^p_{X/S}$ we use the differential of the de Rham complex of $X$ over $S$. Finally, we define $d(d\log(f) \wedge \eta) = -d\log(f) \wedge d\eta$. It is a pleasant exercise to show that we obtain a short exact sequence of complexes as stated in the lemma. □

**Lemma 13.3.** Let $X \to S$ be a morphism of schemes. Let $Y \subset X$ be an effective Cartier divisor. Assume

1. the diagonal of $X$ is affine, and
2. the de Rham complex of log poles is defined for $Y \subset X$ over $S$.

Let $b \in H^{m_1}_{dR}(X/S)$ be a de Rham cohomology class whose restriction to $Y$ is zero. Then $c^1_{dR}(\mathcal{O}_X(Y)) \cup b = 0$ in $H^{m_2+2}_{dR}(X/S)$.

This lemma is true without the assumption on the diagonal of $X$. It can be proven by replacing open coverings in the proof below by hypercoverings.

**Proof.** The short exact sequence of complexes of Lemma 13.2 gives a boundary map

$$H^{m_1}_{dR}(Y/S) = H^{m_1+1}(X, \Omega_{X/S}^\bullet[-1]) \to H^{m_2+2}_{dR}(X, \Omega_{X/S}^\bullet) = H^{m_2+2}_{dR}(X/S)$$

We claim that $b \cup c^1_{dR}(\mathcal{O}_X(-Y))$ is equal to the boundary of the restriction of $b$ to $Y$. The claim proves the lemma.

To prove this, choose an affine open covering $U : X = \bigcup_{i \in I} U_i$ such that $Y \cap U_i$ is the vanishing scheme of a nonzerodivisor $f_i \in \mathcal{O}_X(U_i)$. Then $U_{i_0, \ldots, i_p} = U_{i_0} \cap \cdots \cap U_{i_p}$ is affine for all $i_0, \ldots, i_p \in I$ by our assumption on the diagonal of $X$. Now the short exact sequence of complexes of Lemma 13.2 is termwise given by a short exact sequence of quasi-coherent modules. Hence taking sections over any affine open produces a short exact sequence. Thus we obtain a short exact sequence

$$0 \to \text{Tot}(\mathcal{C}^\bullet(\mathcal{U}, \Omega_{X/S}^\bullet)) \to \text{Tot}(\mathcal{C}^\bullet(\mathcal{U}, \Omega_{Y/S}^\bullet(\log Y))) \to \text{Tot}(\mathcal{C}^\bullet(\mathcal{U}, \Omega_{Y/S}^\bullet[-1])) \to 0$$

of Čech complexes. By Cohomology of Schemes, Lemma 2.2 and Cohomology, Lemma 25.2 the associated long exact sequence of cohomology groups is the long exact sequence of cohomology associated to the short exact sequence of complexes in Lemma 13.2 (small detail omitted).
Using notation as in Cohomology, Section 25 let \( \beta = \{ \beta_{i_0...i_p} \} \) be a cocycle representing \( b \) which has degree \( m \). Let us compute the de Rham chern class of the invertible \( \mathcal{O}_X \)-module \( \mathcal{O}_X(-Y) \). On \( U_i \) the section \( f_i \) is a trivialization. On \( U_{i_0i_1} \) the regular functions \( f_{i_0} \) and \( f_{i_1} \) differ by a unit \( f_{i_0i_1} \), like so
\[
\frac{f_{i_1}}{f_{i_0}}|_{U_{i_0i_1}} = f_{i_0i_1} \cdot \frac{f_{i_0}}{f_{i_0i_1}}
\]
By the discussion in Section 9 the de Rham cohomology class \( c^dR_1(\mathcal{O}_X(-Y)) \) is the class of the cocycle \( \alpha = \{ \alpha_{i_0...i_p} \} = f_{i_0i_1}^{-1} df_{i_0i_1} \) and zero in other degrees. The cup product is given by
\[
(\alpha \cup \beta)_{i_0...i_p} = \sum_{r=0}^p \epsilon(2, m, p, r) \alpha_{i_0...i_r} \wedge \beta_{i_r...i_p} = (-1)^{p+1} \alpha_{i_0i_1} \wedge \beta_{i_1...i_p}
\]
because \( \epsilon(n, m, p, r) \) is \(-1\) to the power \((p + r)n + rp + r\).

On the other hand, the restriction of \( b \) to \( Y \) is represented by the cocycle \( \beta|_Y = \{ \beta_{i_0...i_p}|_{U_{i_0...i_p} \cap Y} \} \) of the complex \( \text{Tot}(\mathcal{C}^\bullet(\mathcal{U}, \Omega^\bullet_{X/S}^*)) \). This means that the cocycle in \( \text{Tot}(\mathcal{C}^\bullet(\mathcal{U}, \Omega^\bullet_{Y/S}^*|[-1])) \) corresponding to \( \beta|_Y \) has is given by \( \{ (-1)^p \beta_{i_0...i_p}|_{U_{i_0...i_p} \cap Y} \} \) by the commutative diagram in Cohomology, Remark 25.3 and the construction of the map
\[
\gamma : \text{Tot}(\mathcal{C}^\bullet(\mathcal{U}, \Omega^\bullet_{X/S}^*))[-1] \longrightarrow \text{Tot}(\mathcal{C}^\bullet(\mathcal{U}, \Omega^\bullet_{Y/S}^*)[-1])
\]
in Homology, Remark 23.9. By our construction of the complex with log poles in the proof of Lemma 13.2 this cocycle lifts to the cycle
\[
\gamma = \{ (-1)^p d \log(f_{i_0}) \wedge \beta_{i_0...i_p} \}
\]
in the complex \( \text{Tot}(\mathcal{C}^\bullet(\mathcal{U}, \Omega^\bullet_{X/S}^*(\log Y))) \). We have
\[
d(\gamma)_{i_0...i_p} = (-1)^{p-1} d \log(f_{i_1}) \wedge \beta_{i_1...i_p}
+ \sum_{j=1}^p (-1)^{j+p-1} d \log(f_{i_0}) \wedge \beta_{i_0...i_j...i_p}
+ (-1)^p p d \log(f_{i_0}) \wedge \beta_{i_0...i_p}
= (-1)^{p-1} (d \log(f_{i_1}) - d \log(f_{i_0})) \wedge \beta_{i_1...i_p}
+ \sum_{j=0}^p (-1)^{j+p-1} d \log(f_{i_0}) \wedge \beta_{i_0...i_j...i_p}
- d \log(f_{i_0}) \wedge d(\beta_{i_0...i_p})
= (-1)^{p-1} f_{i_0i_1}^{-1} df_{i_0i_1} \wedge \beta_{i_1...i_p}
\]
The last equality because the remaining terms sum up to \((-1)^{p-1} d \log(f_{i_0}) \wedge d(\beta)_{i_0...i_p}\), which is zero as \( \beta \) is a cocycle. This is the same result we got above and the proof is complete. \( \square \)

**Lemma 13.4.** Let \( X \to S \) be a morphism of schemes. Let \( Y \subset X \) be an effective Cartier divisor. If both \( X \to S \) and \( Y \to S \) are smooth, then the de Rham complex of log poles is defined for \( Y \subset X \) over \( S \).

**Proof.** In this case the modules \( \Omega^p_{X/S} \) are locally free, see Morphisms, Lemma 32.14 and hence the second condition of Definition 33.1 holds. On the other hand, for \( s \in S \) the fibre \( Y_s \subset X_s \) is an effective Cartier divisor (Divisors, Lemma 18.1). Hence if \( y \in Y \) maps to \( s \in S \) we have \( \dim \mathcal{O}_{Y_s,y} = \dim \mathcal{O}_{X_s,y} - 1 \) for example by Algebra, Lemma 59.12. Thus \( \dim_y(Y_s) = \dim_y(X_s) - 1 \) by Morphisms, Lemma
Now $\Omega_{X/S,x}$ is free of rank $\dim_y(X_s)$ and $\Omega_{Y/S,y}$ is free of rank $\dim_y(Y_s) = \dim_y(X_s) - 1$ by the already used Morphisms, Lemma 32.14. Since $\Omega_{Y/S,y}$ is the quotient of $\Omega_{X/S,x}$ by the submodule $f\Omega_{X/S,x}$ and $\mathcal{O}_{X,x}$ we conclude that $df$ must map to a nonzero element of $\Omega_{X/S,x} \otimes \kappa(y)$. Hence $df$ generates a direct summand and the proof is complete.

**Remark 13.5.** Let $S$ be a locally Noetherian scheme. Let $X$ be locally of finite type over $S$. Let $Y \subset X$ be an effective Cartier divisor. If the map $\mathcal{O}_{X,Y} \to \mathcal{O}_{Y,Y}$ has a section for all $y \in Y$, then the de Rham complex of log poles is defined for $Y \subset X$ over $S$. If we ever need this result we will formulate a precise statement and add a proof here.

**Remark 13.6.** Let $S$ be a locally Noetherian scheme. Let $X$ be locally of finite type over $S$. Let $Y \subset X$ be an effective Cartier divisor. If for every $y \in Y$ we can find a diagram of schemes over $S$

$$X \xleftarrow{\varphi} U \xrightarrow{\psi} V$$

with $\varphi$ étale and $\psi|_{\varphi^{-1}(Y)} : \varphi^{-1}(Y) \to V$ étale, then the de Rham complex of log poles is defined for $Y \subset X$ over $S$. A special case is when the pair $(X,Y)$ étale locally looks like $(V \times \mathbb{A}^1, V \times \{0\})$. If we ever need this result we will formulate a precise statement and add a proof here.

### 14. Comparing sheaves of differential forms

The goal of this section is to compare the sheaves $\Omega^p_{X/Z}$ and $\Omega^p_{Y/Z}$ when given a locally quasi-finite syntomic morphism of schemes $f : Y \to X$. The result will be applied in Section 15 to the construction of the trace map on de Rham complexes if $f$ is finite.

**Lemma 14.1.** Let $R$ be a ring and consider a commutative diagram

$$
\begin{array}{cccccc}
0 & \longrightarrow & K^0 & \longrightarrow & L^0 & \longrightarrow & M^0 & \longrightarrow & 0 \\
& & & \downarrow \theta & \downarrow & \downarrow & \downarrow & \downarrow & \\
& & L^{-1} & \longrightarrow & M^{-1} & \longrightarrow & 0
\end{array}
$$

of $R$-modules with exact top row and $M^0$ and $M^{-1}$ finite free of the same rank. Then there are canonical maps

$$\wedge^i(H^0(L^\bullet)) \longrightarrow \wedge^i(K^0) \otimes_R \det(M^\bullet)$$

whose composition with $\wedge^i(K^0) \to \wedge^i(H^0(L^\bullet))$ is equal to multiplication with $\delta(M^\bullet)$.

**Proof.** Say $M^0$ and $M^{-1}$ are free of rank $n$. For every $i \geq 0$ there is a canonical surjection

$$\pi_i : \wedge^{n+i}(L^0) \longrightarrow \wedge^i(K^0) \otimes \wedge^n(M^0)$$

whose kernel is the submodule generated by wedges $l_1 \wedge \ldots \wedge l_{n+i}$ such that $> i$ of the $l_j$ are in $K^0$. On the other hand, the exact sequence

$$L^{-1} \to L^0 \to H^0(L^\bullet) \to 0$$
similarly produces canonical maps
\[ \wedge^i(H^0(L^\bullet)) \otimes \wedge^n(L^{-1}) \longrightarrow \wedge^{n+i}(L^0) \]
by sending \( \eta \otimes \theta \) to \( \tilde{\eta} \wedge \partial(\theta) \) where \( \tilde{\eta} \in \wedge^i(L^0) \) is a lift of \( \eta \). The composition of these two maps, combined with the identification \( \wedge^n(L^{-1}) = \wedge^n(M^{-1}) \) gives a map
\[ \wedge^i(H^0(L^\bullet)) \otimes \wedge^n(M^{-1}) \longrightarrow \wedge^i(K^0) \otimes \wedge^n(M^0) \]
Since \( \det(M^\bullet) = \wedge^n(M^0) \otimes (\wedge^n(M^{-1}))^{\otimes -1} \) this produces a map as in the statement of the lemma. If \( \eta \) is the image of \( \omega \in \wedge^i(K^0) \), then we see that \( \theta \otimes \eta \) is mapped to \( \pi_1(\omega \wedge \partial(\theta)) = \omega \otimes \tilde{\theta} \) in \( \wedge^i(K^0) \otimes \wedge^n(M^0) \) where \( \tilde{\theta} \) is the image of \( \theta \) in \( \wedge^n(M^0) \). Since \( \delta(M^\bullet) \) is simply the determinant of the map \( M^{-1} \rightarrow M^0 \) this proves the last statement.

\[ \square \]

**Remark 14.2.** Let \( A \) be a ring. Let \( P = A[x_1, \ldots, x_n] \). Let \( f_1, \ldots, f_n \in P \) and set \( B = P/(f_1, \ldots, f_n) \). Assume \( A \rightarrow B \) is quasi-finite. Then \( B \) is a relative global complete intersection over \( A \) (Algebra, Definition 135.5) and \( (f_1, \ldots, f_n)/(f_1, \ldots, f_n)^2 \) is free with generators the classes \( \tilde{f}_i \) by Algebra, Lemma 135.13. Consider the following diagram
\[
\begin{array}{c}
\Omega_{A/Z} \otimes_A B & \longrightarrow & \Omega_{P/Z} \otimes_P B & \longrightarrow & \Omega_{P/A} \otimes_P B \\
(f_1, \ldots, f_n)/(f_1, \ldots, f_n)^2 & \quad & (f_1, \ldots, f_n)/(f_1, \ldots, f_n)^2 & \quad & \\
\end{array}
\]
The left column represents \( \text{NL}_{B/A} \) in \( D(B) \). The middle column represents \( \text{NL}_{B/Z} \) and hence has cohomology \( \Omega_{B/Z} \) in degree 0. The top row is the split short exact sequence \( 0 \rightarrow \Omega_{A/Z} \otimes_A B \rightarrow \Omega_{P/Z} \otimes_P B \rightarrow \Omega_{P/A} \otimes_P B \rightarrow 0 \). Thus by Lemma 14.1 we obtain canonical \( B \)-module maps
\[ \Omega_{B/Z}^P \longrightarrow \Omega_{A/Z}^P \otimes_A \det(\text{NL}_{B/A}) \]
whose composition with \( \Omega_{A/Z}^P \rightarrow \Omega_{B/Z}^P \) is multiplication by \( \delta(\text{NL}_{B/A}) \).

**Lemma 14.3.** There exists a unique rule that to every locally quasi-finite syntomic morphism of schemes \( f : Y \rightarrow X \) assigns \( \mathcal{O}_Y \)-module maps
\[ c^p_{Y/X} : \Omega_{Y/Z}^P \longrightarrow f^*\Omega_{X/Z}^P \otimes_{\mathcal{O}_Y} \det(\text{NL}_{Y/X}) \]
satisfying the following two properties

1. the composition with \( f^*\Omega_{X/Z}^P \rightarrow \Omega_{Y/Z}^P \) is multiplication by \( \delta(\text{NL}_{Y/X}) \), and
2. the rule is compatible with restriction to opens and with base change.

**Proof.** This proof is very similar to the proof of Discriminants, Proposition 13.2 and we suggest the reader look at that proof first. We fix \( p \geq 0 \) throughout the proof.

Let us reformulate the statement. Consider the category \( \mathcal{C} \) whose objects, denoted \( Y/X \), are locally quasi-finite syntomic morphism \( f : Y \rightarrow X \) of schemes and whose morphisms \( b/a : Y'/X' \rightarrow Y/X \) are commutative diagrams
\[
\begin{array}{ccc}
Y' & \longrightarrow & Y \\
\downarrow f' & & \downarrow f \\
X' & \longrightarrow & X \\
\end{array}
\]
which induce an isomorphism of $Y'$ with an open subscheme of $X' \times_X Y$. The lemma means that for every object $Y/X$ of $\mathcal{C}$ we have maps $c_{Y/X}^p$ with property (1) and for every morphism $b/a : Y'/X' \to Y/X$ of $\mathcal{C}$ we have $b^*c_{Y/X}^p = c_{Y'/X'}^p$, via the identifications $b^*\det(NL_{Y/X}) = \det(NL_{Y'/X'})$ (Discriminants, Section 13) and $b^*\Omega_{Y/X}^p = \Omega_{Y'/X'}^p$ (Lemma 2.1).

Given $Y/X$ in $\mathcal{C}$ and $y \in Y$ we can find an affine open $V \subset Y$ and $U \subset X$ with $f(V) \subset U$ such that there exists some maps

$$\Omega_{Y/Z}^p |_V \to \left(f^*\Omega_{X/Z}^p \otimes_{O_Y} \det(NL_{Y/X})\right) |_V$$

with property (1). This follows from picking affine opens as in Discriminants, Lemma 10.1 part (5) and Remark 14.2. If $\Omega_{Y/Z}^p$ is finite locally free and annihilator of the section $\delta(NL_{Y/X})$ is zero, then these local maps are unique and automatically glue!

Let $\mathcal{C}_{nice} \subset \mathcal{C}$ denote the full subcategory of $Y/X$ such that

1. $X$ is of finite type over $\mathbb{Z}$,
2. $\Omega_{X/Z}$ is locally free, and
3. the annihilator of $\delta(NL_{Y/X})$ is zero.

By the remarks in the previous paragraph, we see that for any object $Y/X$ of $\mathcal{C}_{nice}$ we have a unique map $c_{Y/X}^p$ satisfying condition (1). If $b/a : Y'/X' \to Y/X$ is a morphism of $\mathcal{C}_{nice}$, then $b^*c_{Y/X}^p$ is equal to $c_{Y'/X'}^p$, because $b^*\delta(NL_{Y/X}) = \delta(NL_{Y'/X'})$ (see Discriminants, Section 13). In other words, we have solved the problem on the full subcategory $\mathcal{C}_{nice}$. For $Y/X$ in $\mathcal{C}_{nice}$ we continue to denote $c_{Y/X}^p$ the solution we’ve just found.

Consider morphisms

$$Y_1/X_1 \xleftarrow{b_1/a_1} Y/X \xrightarrow{b_2/a_2} Y_2/X_2$$

in $\mathcal{C}$ such that $Y_1/X_1$ and $Y_2/X_2$ are objects of $\mathcal{C}_{nice}$. Claim. $b_1^*c_{Y_1/X_1}^p = b_2^*c_{Y_2/X_2}^p$. We will first show that the claim implies the lemma and then we will prove the claim.

Let $d, n \geq 1$ and consider the locally quasi-finite syntomic morphism $Y_{n,d} \to X_{n,d}$ constructed in Discriminants, Example 10.5. Then $Y_{n,d}$ and $Y_{n,d}$ are irreducible schemes of finite type and smooth over $\mathbb{Z}$. Namely, $X_{n,d}$ is a spectrum of a polynomial ring over $\mathbb{Z}$ and $Y_{n,d}$ is an open subscheme of such. The morphism $Y_{n,d} \to X_{n,d}$ is locally quasi-finite syntomic and étale over a dense open, see Discriminants, Lemma 10.6. Thus $\delta(NL_{Y_{n,d}/X_{n,d}})$ is nonzero: for example we have the local description of $\delta(NL_{Y/X})$ in Discriminants, Remark 13.1 and we have the local description of étale morphisms in Morphisms, Lemma 34.15 part (8). Now a nonzero section of an invertible module over an irreducible regular scheme has vanishing annihilator. Thus $Y_{n,d}/X_{n,d}$ is an object of $\mathcal{C}_{nice}$.

Let $Y/X$ be an arbitrary object of $\mathcal{C}$. Let $y \in Y$. By Discriminants, Lemma 10.7 we can find $n, d \geq 1$ and morphisms

$$Y/X \leftarrow V/U \xrightarrow{b/a} Y_{n,d}/X_{n,d}$$

of $\mathcal{C}$ such that $V \subset Y$ and $U \subset X$ are open. Thus we can pullback the canonical morphism $c_{Y_{n,d}/X_{n,d}}^p$ constructed above by $b$ to $V$. The claim guarantees these local
isomorphisms glue! Thus we get a well defined global maps $c_{Y/X}^p$ with property (1). If $b/a : Y'/X' \to Y/X$ is a morphism of $\mathcal{C}$, then the claim also implies that the similarly constructed map $c_{Y'/X'}^p$ is the pullback by $b$ of the locally constructed map $c_{Y/X}^p$. Thus it remains to prove the claim.

In the rest of the proof we prove the claim. We may pick a point $y \in Y$ and prove the maps agree in an open neighbourhood of $y$. Thus we may replace $Y_1$, $Y_2$ by open neighbourhoods of the image of $y$ in $Y_1$ and $Y_2$. Thus we may assume $Y, X, Y_1, X_1, Y_2, X_2$ are affine. We may write $X = \lim X_\lambda$ as a cofiltered limit of affine schemes of finite type over $X_1 \times X_2$. For each $\lambda$ we get

$$Y_1 \times_{X_1} X_\lambda \text{ and } X_\lambda \times_{X_2} Y_2$$

If we take limits we obtain

$$\lim Y_1 \times_{X_1} X_\lambda = Y_1 \times_{X_1} X \supset Y \subset X \times_{X_2} Y_2 = \lim X_\lambda \times_{X_2} Y_2$$

By Limits, Lemma 4.11 we can find a $\lambda$ and opens $V_{1,\lambda} \subset Y_1 \times_{X_1} X_\lambda$ and $V_{2,\lambda} \subset X_\lambda \times_{X_2} Y_2$ whose base change to $X$ recovers $Y$ (on both sides). After increasing $\lambda$ we may assume there is an isomorphism $V_{1,\lambda} \to V_{2,\lambda}$ whose base change to $X$ is the identity on $Y$, see Limits, Lemma 10.1. Then we have the commutative diagram

$$
\begin{array}{ccc}
Y/X & & \\
\downarrow & & \downarrow \\
Y_1/X_1 & \leftarrow & V_{1,\lambda}/X_\lambda \\
& b_1/a_1 & \leftarrow b_2/a_2 \rightarrow & \rightarrow & V_{2,\lambda}/X_\lambda \\
& Y_2/X_2 & \\
\end{array}
$$

Thus it suffices to prove the claim for the lower row of the diagram and we reduce to the case discussed in the next paragraph.

Assume $Y, X, Y_1, X_1, Y_2, X_2$ are affine of finite type over $\mathbb{Z}$. Write $X = \Spec(A)$, $X_i = \Spec(A_i)$. The ring map $A_1 \to A$ corresponding to $X \to X_1$ is of finite type and hence we may choose a surjection $A_1[x_1, \ldots, x_n] \to A$. Similarly, we may choose a surjection $A_2[y_1, \ldots, y_m] \to A$. Set $X'_1 = \Spec(A_1[x_1, \ldots, x_n])$ and $X'_2 = \Spec(A_2[y_1, \ldots, y_m])$. Observe that $\Omega_{X'_1/X_1}$ is the direct sum of the pullback of $\Omega_{X_1/Y_1}$ and a finite free module. Similarly for $X'_2$. Set $Y'_1 = Y_1 \times_{X_1} X'_1$ and $Y'_2 = Y_2 \times_{X_2} X'_2$. We get the following diagram

$$
Y_1/X_1 \leftarrow Y'_1/X'_1 \leftarrow Y/X \to Y'_2/X'_2 \to Y_2/X_2
$$

Since $X'_1 \to X_1$ and $X'_2 \to X_2$ are flat, the same is true for $Y'_1 \to Y_1$ and $Y'_2 \to Y_2$. It follows easily that the annihilators of $\delta(\Omega_{X'_1/X'_1})$ and $\delta(\Omega_{X'_2/X'_2})$ are zero. Hence $Y'_1/X'_1$ and $Y'_2/X'_2$ are in $\mathcal{C}_{nice}$. Thus the outer morphisms in the displayed diagram are morphisms of $\mathcal{C}_{nice}$ for which we know the desired compatibilities. Thus it suffices to prove the claim for $Y'_1/X'_1 \leftarrow Y/X \to Y'_2/X'_2$. This reduces us to the case discussed in the next paragraph.

Assume $Y, X, Y_1, X_1, Y_2, X_2$ are affine of finite type over $\mathbb{Z}$ and $X \to X_1$ and $X \to X_2$ are closed immersions. Consider the open embeddings $Y_1 \times_{X_1} X \supset Y \subset X \times_{X_2} Y_2$. There is an open neighbourhood $V \subset Y$ of $y$ which is a standard open of both $Y_1 \times_{X_1} X$ and $X \times_{X_2} Y_2$. This follows from Schemes, Lemma 11.5 applied to the scheme obtained by gluing $Y_1 \times_{X_1} X$ and $X \times_{X_2} Y_2$ along $Y$; details omitted. Since $X \times_{X_1} Y_2$ is a closed subscheme of $Y_2$ we can find a standard open $V_2 \subset Y_2$ such that $V_2 \times_{X_2} X = V$. Similarly, we can find a standard open $V_1 \subset Y_1$ such that
V_1 \times_X X = V$. After replacing $Y, Y_1, Y_2$ by $V, V_1, V_2$ we reduce to the case discussed in the next paragraph.

Assume $Y, X, Y_1, X_1, Y_2, X_2$ are affine of finite type over $\mathbb{Z}$ and $X \to X_1$ and $X \to X_2$ are closed immersions and $Y \times_X Y_1 = Y = X \times_X Y_2$. Write $X = \text{Spec}(A)$, $X_i = \text{Spec}(A_i)$, $Y = \text{Spec}(B)$, $Y_i = \text{Spec}(B_i)$. Then we can consider the affine schemes

$$X' = \text{Spec}(A_1 \times_A A_2) = \text{Spec}(A') \quad \text{and} \quad Y' = \text{Spec}(B_1 \times_B B_2) = \text{Spec}(B')$$

Observe that $X' = X_1 \amalg X_2$ and $Y' = Y_1 \amalg Y_2$, see More on Morphisms, Lemma 14.1. By More on Algebra, Lemma 6.4 we have $B' \otimes_A A_1 = B_1$ and $B' \times_A A_2 = B_2$. In particular a fibre of $Y' \to X'$ over a point of $X' = X_1 \amalg X_2$ is always equal to either a fibre of $Y_1 \to X_1$ or a fibre of $Y_2 \to X_2$. By More on Algebra, Lemma 6.8 the ring map $A' \to B'$ is flat. Thus by Discriminants, Lemma 10.1 part (3) we conclude that $Y'/X'$ is an object of $\mathcal{C}$. Consider now the commutative diagram

Now we would be done if $Y'/X'$ is an object of $\mathcal{C}_{\text{nice}}$, but this is almost never the case. Namely, then pulling back $c_{Y'/X'}^\alpha$ around the two sides of the square, we would obtain the desired conclusion. To get around the problem that $Y'/X'$ is not in $\mathcal{C}_{\text{nice}}$ we note the arguments above show that, after possibly shrinking all of the schemes $X, Y, X_1, Y_1, X_2, Y_2, X', Y'$ we can find some $n, d \geq 1$, and extend the diagram like so:

and then we can use the already given argument by pulling back from $c_{Y_{n,d}/X_{n,d}}^\alpha$. This finishes the proof.

15. Trace maps on de Rham complexes

A reference for some of the material in this section is [Gar84]. Let $S$ be a scheme. Let $f : Y \to X$ be a finite locally free morphism of schemes over $S$. Then there is
a trace map $\text{Trace}_f : f_*\mathcal{O}_Y \to \mathcal{O}_X$, see Discriminants, Section 3. In this situation a trace map on de Rham complexes is a map of complexes

$$\Theta_{Y/X} : f_*\Omega^*_{Y/S} \to \Omega^*_{X/S}$$

such that $\Theta_{Y/X}$ is equal to $\text{Trace}_f$ in degree 0 and satisfies

$$\Theta_{Y/X}(\omega \wedge \eta) = \omega \wedge \Theta_{Y/X}(\eta)$$

for local sections $\omega$ of $\Omega^*_{X/S}$ and $\eta$ of $f_*\Omega^*_{Y/S}$. It is not clear to us whether such a trace map $\Theta_{Y/X}$ exists for every finite locally free morphism $Y \to X$; please email stacks.project@gmail.com if you have a counterexample or a proof.

**Example 15.1.** Here is an example where we do not have a trace map on de Rham complexes. For example, consider the $\mathbb{C}$-algebra $B = \mathbb{C}[x, y]$ with action of $G = \{\pm 1\}$ given by $x \mapsto -x$ and $y \mapsto -y$. The invariants $A = B^G$ form a normal domain of finite type over $\mathbb{C}$ generated by $x^2, xy, y^2$. We claim that for the inclusion $A \subset B$ there is no reasonable trace map $\Omega_B/\mathbb{C} \to \Omega_A/\mathbb{C}$ on 1-forms. Namely, consider the element $\omega = xdy \in \Omega_B/\mathbb{C}$. Since $\omega$ is invariant under the action of $G$ if a “reasonable” trace map exists, then $2\omega$ should be in the image of $\Omega_A/\mathbb{C} \to \Omega_B/\mathbb{C}$. This is not the case: there is no way to write $2\omega$ as a linear combination of $d(x^2)$, $d(xy)$, and $d(y^2)$ even with coefficients in $B$. This example contradicts the main theorem in [Zan99].

**Lemma 15.2.** There exists a unique rule that to every finite syntomic morphism of schemes $f : Y \to X$ assigns $\mathcal{O}_X$-module maps

$$\Theta^p_{Y/X} : f_*\Omega^p_{Y/S} \to \Omega^p_{X/S}$$

satisfying the following properties

1. the composition with $\Omega^p_{X/Z} \otimes_{\mathcal{O}_X} f_*\mathcal{O}_Y \to f_*\mathcal{O}^p_{Y/S}$ is equal to $\text{id} \otimes \text{Trace}_f$
2. the rule is compatible with base change.

**Proof.** First, assume that $X$ is locally Noetherian. By Lemma 14.3 we have a canonical map

$$c^p_{Y/X} : \Omega^p_{Y/S} \to f^*\Omega^p_{X/S} \otimes_{\mathcal{O}_Y} \det(NL_{Y/X})$$

By Discriminants, Proposition 13.2 we have a canonical isomorphism

$$c_{Y/X} : \det(NL_{Y/X}) \to \omega_{Y/X}$$

mapping $\delta(NL_{Y/X})$ to $\tau_{Y/X}$. Combined these maps give

$$c^p_{Y/X} \otimes c_{Y/X} : \Omega^p_{Y/S} \to f^*\Omega^p_{X/S} \otimes_{\mathcal{O}_Y} \omega_{Y/X}$$

By Discriminants, Section 5 this is the same thing as a map

$$\Theta^p_{Y/X} : f_*\Omega^p_{Y/S} \to \Omega^p_{X/S}$$

Recall that the relationship between $c^p_{Y/X} \otimes c_{Y/X}$ and $\Theta^p_{Y/X}$ uses the evaluation map $f_*\omega_{Y/X} \to \mathcal{O}_X$ which sends $\tau_{Y/X}$ to $\text{Trace}_f(1)$, see Discriminants, Section 5. Hence property (1) holds. Property (2) holds for base changes by $X' \to X$ with $X'$ locally Noetherian because both $c^p_{Y/X}$ and $c_{Y/X}$ are compatible with such base changes. For $f : Y \to X$ finite syntomic and $X$ locally Noetherian, we will continue to denote $\Theta^p_{Y/X}$ the solution we’ve just found.
Uniqueness. Suppose that we have a finite syntomic morphism $f : Y \to X$ such that $X$ is smooth over $\text{Spec}(\mathbb{Z})$ and $f$ is étale over a dense open of $X$. We claim that in this case $\Theta^p_{Y/X}$ is uniquely determined by property (1). Namely, consider the maps

$$\Omega^p_{X/\mathbb{Z}} \otimes_{\mathcal{O}_X} f_* \mathcal{O}_Y \to f_* \Omega^p_{Y/\mathbb{Z}} \to \Omega^p_{X/\mathbb{Z}}$$

The sheaf $\Omega^p_{X/\mathbb{Z}}$ is torsion free (by the assumed smoothness), hence it suffices to check that the restriction of $\Theta^p_{Y/X}$ is uniquely determined over the dense open over which $f$ is étale, i.e., we may assume $f$ is étale. However, if $f$ is étale, then $f^* \Omega^p_{X/\mathbb{Z}} = \Omega^p_{Y/\mathbb{Z}}$ hence the first arrow in the displayed equation is an isomorphism. Since we’ve pinned down the composition, this guarantees uniqueness.

Let $f : Y \to X$ be a finite syntomic morphism of locally Noetherian schemes. Let $x \in X$. By Discriminants, Lemma 11.7 we can find $d \geq 1$ and a commutative diagram

$$
\begin{array}{ccc}
Y & \to & V_d \\
\downarrow & & \downarrow \\
X & \to & U_d
\end{array}
$$

such that $x \in U \subset X$ is open, $V = f^{-1}(U)$ and $V = U \times_{U_d} V_d$. Thus $\Theta^p_{Y/X}|_{V_d}$ is the pullback of the map $\Theta^p_{V_d/U_d}$. However, by the discussion on uniqueness above and Discriminants, Lemmas 11.4 and 11.5 the map $\Theta^p_{V_d/U_d}$ is uniquely determined by the requirement (1). Hence uniqueness holds.

At this point we know that we have existence and uniqueness for all finite syntomic morphisms $Y \to X$ with $X$ locally Noetherian. We could now give an argument similar to the proof of Lemma 14.3 to extend to general $X$. However, instead it possible to directly use absolute Noetherian approximation to finish the proof. Namely, to construct $\Theta^p_{Y/X}$ it suffices to do so Zariski locally on $X$ (provided we also show the uniqueness). Hence we may assume $X$ is affine (small detail omitted). Then we can write $X = \lim_{i \in I} X_i$ as the limit over a directed set $I$ of Noetherian affine schemes. By Algebra, Lemma 126.8 we can find $0 \in I$ and a finitely presented morphism of affines $f_0 : Y_0 \to X_0$ whose base change to $X$ is $Y \to X$. After increasing $0$ we may assume $Y_0 \to X_0$ is finite and syntomic, see Algebra, Lemma 163.9 and 163.3. For $i \geq 0$ also the base change $f_i : Y_i = Y_0 \times_{X_0} X_i \to X_i$ is finite syntomic. Then

$$\Gamma(X, f_* \Omega^p_{Y/\mathbb{Z}}) = \Gamma(Y, \Omega^p_{Y_i/\mathbb{Z}}) = \text{colim}_{i \geq 0} \Gamma(Y_i, \Omega^p_{Y_i/\mathbb{Z}}) = \text{colim}_{i \geq 0} \Gamma(X_i, f_i_* \Omega^p_{Y_i/\mathbb{Z}})$$

Hence we can (and are forced to) define $\Theta^p_{Y_i/X_i}$ as the colimit of the maps $\Theta^p_{Y_i/X_i}$. This map is compatible with any cartesian diagram

$$
\begin{array}{ccc}
Y' & \to & Y \\
\downarrow & & \downarrow \\
X' & \to & X
\end{array}
$$

with $X'$ affine as we know this for the case of Noetherian affine schemes by the arguments given above (small detail omitted; hint: if we also write $X' = \lim_{j \in J} X'_j$ then for every $i \in I$ there is a $j \in J$ and a morphism $X'_j \to X_i$ compatible with the morphism $X' \to X$). This finishes the proof. □
Proposition 15.3. Let $f : Y \to X$ be a finite syntomic morphism of schemes. The maps $\Theta_{Y/X}$ of Lemma 15.2 define a map of complexes

$$\Theta_{Y/X} : f_*\Omega_{Y/Z}^\bullet \to \Omega_{X/Z}^\bullet$$

with the following properties

1. in degree 0 we get $\text{Trace}_f : f_*\mathcal{O}_Y \to \mathcal{O}_X$, see Discriminants, Section 3,
2. we have $\Theta_{Y/X}(\omega \wedge \eta) = \omega \wedge \Theta_{Y/X}(\eta)$ for $\omega$ in $\Omega_{X/S}^\bullet$ and $\eta$ in $f_*\Omega_{Y/S}^\bullet$,
3. if $f$ is a morphism over a base scheme $S$, then $\Theta_{Y/X}$ induces a map of complexes $f_*\Omega_{Y/S}^\bullet \to \Omega_{X/S}^\bullet$.

Proof. By Discriminants, Lemma 11.7 for every $x \in X$ we can find $d \geq 1$ and a commutative diagram

$$
\begin{array}{ccccccc}
Y & \leftarrow & V & \rightarrow & V_d & \rightarrow & Y_d = \Spec(B_d) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
X & \leftarrow & U & \rightarrow & U_d & \rightarrow & X_d = \Spec(A_d)
\end{array}
$$

such that $x \in U \subset X$ is affine open, $V = f^{-1}(U)$ and $V = U \times_{U_d} V_d$. Write $U = \Spec(A)$ and $V = \Spec(B)$ and observe that $B = A \otimes_{A_d} B_d$ and recall that $B_d = A_d e_1 \oplus \ldots \oplus A_d e_d$. Suppose we have $a_1, \ldots, a_r \in A$ and $b_1, \ldots, b_s \in B$. We may write $b_j = \sum a_{j,l} e_d$ with $a_{j,l} \in A$. Set $N = r + sd$ and consider the factorizations

$$
\begin{array}{ccccccc}
V & \rightarrow & V' = A^N \times V_d & \rightarrow & V_d \\
\downarrow & & \downarrow & & \downarrow \\
U & \rightarrow & U' = A^N \times U_d & \rightarrow & U_d
\end{array}
$$

Here the horizontal lower right arrow is given by the morphism $U \to U_d$ (from the earlier diagram) and the morphism $U \to A^N$ given by $a_1, \ldots, a_r, a_{1,1}, \ldots, a_{s,d}$. Then we see that the functions $a_1, \ldots, a_r$ are in the image of $\Gamma(U', \mathcal{O}_{U'}) \to \Gamma(U, \mathcal{O}_U)$ and the functions $b_1, \ldots, b_s$ are in the image of $\Gamma(V', \mathcal{O}_{V'}) \to \Gamma(V, \mathcal{O}_V)$. In this way we see that for any finite collection of elements $^{2}$ of the groups

$$
\Gamma(V, \Omega_{Y/Z}^i), \quad i = 0, 1, 2, \ldots \quad \text{and} \quad \Gamma(U, \Omega_{X/Z}^j), \quad j = 0, 1, 2, \ldots
$$

we can find a factorizations $V \to V' \to V_d$ and $U \to U' \to U_d$ with $V' = A^N \times V_d$ and $U' = A^N \times U_d$ as above such that these sections are the pullbacks of sections from

$$
\Gamma(V', \Omega_{Y'/Z}^i), \quad i = 0, 1, 2, \ldots \quad \text{and} \quad \Gamma(U', \Omega_{X'/Z}^j), \quad j = 0, 1, 2, \ldots
$$

The upshot of this is that to check $d \circ \Theta_{Y/X} = \Theta_{Y/X} \circ d$ it suffices to check this is true for $\Theta_{V'/U'}$. Similarly, for property (2) of the lemma.

By Discriminants, Lemmas 11.4 and 11.5 the scheme $U_d$ is smooth and the morphism $V_d \to U_d$ is étale over a dense open of $U_d$. Hence the same is true for the morphism $V' \to U'$. Since $\Omega_{U'/Z}$ is locally free and hence $\omega_{U'/Z}$ is torsion free, it suffices to check the desired relations after restricting to the open over which $V'$ is finite étale. Then we may check the relations after a surjective étale base change.

---

$^{2}$After all these elements will be finite sums of elements of the form $a_0da_1 \wedge \ldots \wedge da_1$ with $a_0, \ldots, a_1 \in A$ or finite sums of elements of the form $b_0db_1 \wedge \ldots \wedge db_1$ with $b_0, \ldots, b_f \in B$. 
Hence we may split the finite étale cover and assume we are looking at a morphism of the form

$$\prod_{i=1,\ldots,d} W \rightarrow W$$

with $W$ smooth over $\mathbb{Z}$. In this case any local properties of our construction are trivial to check (provided they are true). This finishes the proof of (1) and (2).

Finally, we observe that (3) follows from (2) because $\Omega_{Y/S}$ is the quotient of $\Omega_Y/\mathbb{Z}$ by the submodule generated by pullbacks of local sections of $\Omega_S/\mathbb{Z}$. 

Example 15.4. Let $A$ be a ring. Let $f = x^d + \sum_{0 \leq i < d} a_{d-i}x^i \in A[x]$. Let $B = A[x]/(f)$. By Proposition 15.3 we have a morphism of complexes

$$\Theta_{B/A} : \Omega^\bullet_B \rightarrow \Omega^\bullet_A$$

In particular, if $t \in B$ denotes the image of $x \in A[x]$ we can consider the elements

$$\Theta_{B/A}(t^i dt) \in \Omega^1_A, \quad i = 0, \ldots, d-1$$

What are these elements? By the same principle as used in the proof of Proposition 15.3 it suffices to compute this in the universal case, i.e., when $A = \mathbb{Z}[a_1, \ldots, a_d]$ or even when $A$ is replaced by the fraction field $\mathbb{Q}(a_1, \ldots, a_d)$. Writing symbolically

$$f = \prod_{i=1,\ldots,d} (x - \alpha_i)$$

we see that over $\mathbb{Q}(\alpha_1, \ldots, \alpha_d)$ the algebra $B$ becomes split:

$$\mathbb{Q}(a_0, \ldots, a_{d-1})[x]/(f) \rightarrow \prod_{i=1,\ldots,d} \mathbb{Q}(\alpha_1, \ldots, \alpha_d), \quad t \mapsto (\alpha_1, \ldots, \alpha_d)$$

Thus for example

$$\Theta(dt) = \sum \alpha_i = -da_1$$

Next, we have

$$\Theta(tdt) = \sum \alpha_i d\alpha_i = a_1 da_1 - da_2$$

Next, we have

$$\Theta(t^2 dt) = \sum \alpha_i^2 d\alpha_i = -a_1^2 da_1 + a_1 da_2 + a_2 da_1 - da_3$$

(modulo calculation error), and so on. This suggests that if $f(x) = x^d - a$ then

$$\Theta_{B/A}(t^i dt) = \begin{cases} 0 & \text{if } i = 0, \ldots, d-2 \\ da_i & \text{if } i = d-1 \end{cases}$$

in $\Omega_A$. This is true for in this particular case one can do the calculation for the extension $\mathbb{Q}(a)[x]/(x^d - a)$ to verify this directly.

16. Other chapters

Preliminaries

(1) Introduction
(2) Conventions
(3) Set Theory
(4) Categories
(5) Topology
(6) Sheaves on Spaces
(7) Sites and Sheaves
(8) Stacks

(9) Fields
(10) Commutative Algebra
(11) Brauer Groups
(12) Homological Algebra
(13) Derived Categories
(14) Simplicial Methods
(15) More on Algebra
(16) Smoothing Ring Maps
(17) Sheaves of Modules
(18) Modules on Sites
(19) Injectives
(20) Cohomology of Sheaves
(21) Cohomology on Sites
(22) Differential Graded Algebra
(23) Divided Power Algebra
(24) Hypercoverings

Schemes
(25) Schemes
(26) Constructions of Schemes
(27) Properties of Schemes
(28) Morphisms of Schemes
(29) Cohomology of Schemes
(30) Divisors
(31) Limits of Schemes
(32) Varieties
(33) Topologies on Schemes
(34) Descent
(35) Derived Categories of Schemes
(36) More on Morphisms
(37) More on Flatness
(38) Groupoid Schemes
(39) More on Groupoid Schemes
(40) Étale Morphisms of Schemes

Topics in Scheme Theory
(41) Chow Homology
(42) Intersection Theory
(43) Picard Schemes of Curves
(44) Weil Cohomology Theories
(45) Adequate Modules
(46) Dualizing Complexes
(47) Duality for Schemes
(48) Discriminants and Differents
(49) de Rham Cohomology
(50) Local Cohomology
(51) Algebraic and Formal Geometry
(52) Algebraic Curves
(53) Resolution of Surfaces
(54) Semistable Reduction
(55) Fundamental Groups of Schemes
(56) Étale Cohomology
(57) Crystalline Cohomology
(58) Pro-étale Cohomology
(59) More Étale Cohomology
(60) The Trace Formula

Algebraic Spaces
(61) Algebraic Spaces

(62) Properties of Algebraic Spaces
(63) Morphisms of Algebraic Spaces
(64) Decent Algebraic Spaces
(65) Cohomology of Algebraic Spaces
(66) Limits of Algebraic Spaces
(67) Divisors on Algebraic Spaces
(68) Algebraic Spaces over Fields
(69) Topologies on Algebraic Spaces
(70) Descent and Algebraic Spaces
(71) Derived Categories of Spaces
(72) More on Morphisms of Spaces
(73) Flatness on Algebraic Spaces
(74) Groupoids in Algebraic Spaces
(75) More on Groupoids in Spaces
(76) Bootstrap
(77) Pushouts of Algebraic Spaces

Topics in Geometry
(78) Chow Groups of Spaces
(79) Quotients of Groupoids
(80) More on Cohomology of Spaces
(81) Simplicial Spaces
(82) Duality for Spaces
(83) Formal Algebraic Spaces
(84) Restricted Power Series
(85) Resolution of Surfaces Revisited

Deformation Theory
(86) Formal Deformation Theory
(87) Deformation Theory
(88) The Cotangent Complex
(89) Deformation Problems

Algebraic Stacks
(90) Algebraic Stacks
(91) Examples of Stacks
(92) Sheaves on Algebraic Stacks
(93) Criteria for Representability
(94) Artin’s Axioms
(95) Quot and Hilbert Spaces
(96) Properties of Algebraic Stacks
(97) Morphisms of Algebraic Stacks
(98) Limits of Algebraic Stacks
(99) Cohomology of Algebraic Stacks
(100) Derived Categories of Stacks
(101) Introducing Algebraic Stacks
(102) More on Morphisms of Stacks
(103) The Geometry of Stacks

Topics in Moduli Theory
(104) Moduli Stacks
References
