1. Introduction

This is basically just a list of things that we want to put in the stacks project. As we add material to the Stacks project continuously this is always somewhat behind the current state of the Stacks project. In fact, it may have been a mistake to try and list things we should add, because it seems impossible to keep it up to date.

Last updated: Thursday, August 31, 2017.

2. Conventions

We should have a chapter with a short list of conventions used in the document. This chapter already exists, see Conventions, Section 1, but a lot more could be added there. Especially useful would be to find “hidden” conventions and tacit assumptions and put those there.

This is a chapter of the Stacks Project, version 37287dd8, compiled on Apr 13, 2019.
3. Sites and Topoi

02BA We have a chapter on sites and sheaves, see Sites, Section 1. We have a chapter on ringed sites (and topoi) and modules on them, see Modules on Sites, Section 1. We have a chapter on cohomology in this setting, see Cohomology on Sites, Section 1. But a lot more could be added, especially in the chapter on cohomology.

4. Stacks

02BB We have a chapter on (abstract) stacks, see Stacks, Section 1. It would be nice if

1. improve the discussion on “stackyfication”,
2. give examples of stackyfication,
3. more examples in general,
4. improve the discussion of gerbes.

Example result which has not been added yet: Given a sheaf of abelian groups \( \mathcal{F} \) over \( \mathcal{C} \) the set of equivalence classes of gerbes banded by \( \mathcal{F} \) is bijective to \( H^2(\mathcal{C}, \mathcal{F}) \).

5. Simplicial methods

03MZ We have a chapter on simplicial methods, see Simplicial, Section 1. This has to be reviewed and improved. The discussion of the relationship between simplicial homotopy (also known as combinatorial homotopy) and Kan complexes should be improved upon. There is a chapter on simplicial spaces, see Simplicial Spaces, Section 1. This chapter briefly discusses simplicial topological spaces, simplicial sites, and simplicial topoi. We can further develop “simplicial algebraic geometry” to discuss simplicial schemes (or simplicial algebraic spaces, or simplicial algebraic stacks) and treat geometric questions, their cohomology, etc.

6. Cohomology of schemes

02BE There is already a chapter on cohomology of quasi-coherent sheaves, see Cohomology of Schemes, Section 1. We have a chapter discussing the derived category of quasi-coherent sheaves on a scheme, see Derived Categories of Schemes, Section 1. We have a chapter discussing duality for Noetherian schemes and relative duality for morphisms of schemes, see Duality for Schemes, Section 1. We also have chapters on étale cohomology of schemes and on crystalline cohomology of schemes. But most of the material in these chapters is very basic and a lot more could/should be added there.

7. Deformation theory à la Schlessinger

02BF We have a chapter on this material, see Formal Deformation Theory, Section 1. We have a chapter discussing examples of the general theory, see Deformation Problems, Section 1. We have a chapter, see Deformation Theory, Section 1 which discusses deformations of rings (and modules), deformations of ringed spaces (and sheaves of modules), deformations of ringed topoi (and sheaves of modules). In this chapter we use the naïve cotangent complex to describe obstructions, first order deformations, and infinitesimal automorphisms. This material has found some applications to algebraicity of moduli stacks in later chapters. There is also a chapter discussing the full cotangent complex, see Cotangent, Section 1.
8. Definition of algebraic stacks

An algebraic stack is a stack in groupoids over the category of schemes with the fppf topology that has a diagonal representable by algebraic spaces and is the target of a surjective smooth morphism from a scheme. See Algebraic Stacks, Section 12. A “Deligne-Mumford stack” is an algebraic stack for which there exists a scheme and a surjective étale morphism from that scheme to it as in the paper [DM69] of Deligne and Mumford, see Algebraic Stacks, Definition 12.2. We will reserve the term “Artin stack” for a stack such as in the papers by Artin, see [Art69], [Art70], and [Art74]. A possible definition is that an Artin stack is an algebraic stack \( \mathcal{X} \) over a locally Noetherian scheme \( S \) such that \( \mathcal{X} \to S \) is locally of finite type.\footnote{Namely, these are exactly the algebraic stacks over \( S \) satisfying Artin’s axioms [-1], [0], [1], [2], [3], [4], [5] of Artin’s Axioms, Section 14.}

9. Examples of schemes, algebraic spaces, algebraic stacks

The Stacks project currently contains two chapters discussing moduli stacks and their properties, see Moduli Stacks, Section 1 and Moduli of Curves, Section 1. Over time we intend to add more, for example:

1. \( A_g \), i.e., principally polarized abelian schemes of genus \( g \),
2. \( \mathcal{M}_{1,1} \), i.e., 1-pointed smooth projective genus 1 curves,
3. \( \mathcal{M}_{g,n} \), i.e., smooth projective genus \( g \)-curves with \( n \) pairwise distinct labeled points,
4. \( \overline{\mathcal{M}}_{g,n} \), i.e., stable \( n \)-pointed nodal projective genus \( g \)-curves,
5. \( \mathcal{H}om_S(\mathcal{X}, \mathcal{Y}) \), moduli of morphisms (with suitable conditions on the stacks \( \mathcal{X}, \mathcal{Y} \) and the base scheme \( S \)),
6. \( \mathcal{B}un_G(X) = \mathcal{H}om_S(X, BG) \), the stack of \( G \)-bundles of the geometric Langlands programme (with suitable conditions on the scheme \( X \), the group scheme \( G \), and the base scheme \( S \)),
7. \( \mathcal{P}ic_{X/S} \), i.e., the Picard stack associated to an algebraic stack over a base scheme (or space).

More generally, the Stacks project is somewhat lacking in geometrically meaningful examples.

10. Properties of algebraic stacks

This is perhaps one of the easier projects to work on, as most of the basic theory is there now. Of course these things are really properties of morphisms of stacks. We can define singularities (up to smooth factors) etc. Prove that a connected normal stack is irreducible, etc.

11. Lisse étale site of an algebraic stack

This has been introduced in Cohomology of Stacks, Section 11. An example to show that it is not functorial with respect to 1-morphisms of algebraic stacks is discussed in Examples, Section 52. Of course a lot more could be said about this, but it turns out to be very useful to prove things using the “big” étale site as much as possible.
12. Things you always wanted to know but were afraid to ask

There are going to be lots of lemmas that you use over and over again that are useful but aren’t really mentioned specifically in the literature, or it isn’t easy to find references for. Bag of tricks.

Example: Given two groupoids in schemes $R \Rightarrow U$ and $R' \Rightarrow U'$ what does it mean to have a 1-morphism $[U/R] \to [U'/R']$ purely in terms of groupoids in schemes.

13. Quasi-coherent sheaves on stacks

These are defined and discussed in the chapter Cohomology of Stacks, Section [1]. Derived categories of modules are discussed in the chapter Derived Categories of Stacks, Section [1]. A lot more could be added to these chapters.

14. Flat and smooth

Artin’s theorem that having a flat surjection from a scheme is a replacement for the smooth surjective condition. This is now available as Criteria for Representability, Theorem [16.1].

15. Artin’s representability theorem

This is discussed in the chapter Artin’s Axioms, Section [1]. We also have an application, see Quot, Theorem [5.12]. There should be a lot more applications and the chapter itself has to be cleaned up as well.

16. DM stacks are finitely covered by schemes

We already have the corresponding result for algebraic spaces, see Limits of Spaces, Section [16]. What is missing is the result for DM and quasi-DM stacks.

17. Martin Olsson’s paper on properness

This proves two notions of proper are the same. The first part of this is now available in the form of Chow’s lemma for algebraic stacks, see More on Morphisms of Stacks, Theorem [10.3]. As a consequence we show that it suffices to use DVR’s in checking the valuative criterion for properness for algebraic stacks in certain cases, see More on Morphisms of Stacks, Section [11].

18. Proper pushforward of coherent sheaves

We can start working on this now that we have Chow’s lemma for algebraic stacks, see previous section.

19. Keel and Mori

See [KM97]. Their result has been added in More on Morphisms of Stacks, Section [13].

20. Add more here

Actually, no we should never have started this list as part of the Stacks project itself! There is a todo list somewhere else which is much easier to update.
21. Other chapters

Preliminaries

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2. Conventions
3. Set Theory
4. Categories
5. Topology
6. Sheaves on Spaces
7. Sites and Sheaves
8. Stacks
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10. Commutative Algebra
11. Brauer Groups
12. Homological Algebra
13. Derived Categories
14. Simplicial Methods
15. More on Algebra
16. Smoothing Ring Maps
17. Sheaves of Modules
18. Modules on Sites
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20. Cohomology of Sheaves
21. Cohomology on Sites
22. Differential Graded Algebra
23. Divided Power Algebra
24. Hypercoverings

Schemes

25. Schemes
26. Constructions of Schemes
27. Properties of Schemes
28. Morphisms of Schemes
29. Cohomology of Schemes
30. Divisors
31. Limits of Schemes
32. Varieties
33. Topologies on Schemes
34. Descent
35. Derived Categories of Schemes
36. More on Morphisms
37. More on Flatness
38. Groupoid Schemes
39. More on Groupoid Schemes
40. Étale Morphisms of Schemes

Topics in Scheme Theory

41. Chow Homology
42. Intersection Theory
43. Picard Schemes of Curves
44. Adequate Modules
45. Dualizing Complexes
46. Duality for Schemes
47. Discriminants and Differents
48. Local Cohomology
49. Algebraic and Formal Geometry
50. Algebraic Curves
51. Resolution of Surfaces
52. Semistable Reduction
53. Fundamental Groups of Schemes
54. Étale Cohomology
55. Crystalline Cohomology
56. Pro-étale Cohomology
57. More Étale Cohomology
58. The Trace Formula

Algebraic Spaces

59. Algebraic Spaces
60. Properties of Algebraic Spaces
61. Morphisms of Algebraic Spaces
62. Decent Algebraic Spaces
63. Cohomology of Algebraic Spaces
64. Limits of Algebraic Spaces
65. Divisors on Algebraic Spaces
66. Algebraic Spaces over Fields
67. Topologies on Algebraic Spaces
68. Descent and Algebraic Spaces
69. Derived Categories of Spaces
70. More on Morphisms of Spaces
71. Flatness on Algebraic Spaces
72. Groupoids in Algebraic Spaces
73. More on Groupoids in Spaces
74. Bootstrap
75. Pushouts of Algebraic Spaces

Topics in Geometry

76. Chow Groups of Spaces
77. Quotients of Groupoids
78. More on Cohomology of Spaces
79. Simplicial Spaces
80. Duality for Spaces
81. Formal Algebraic Spaces
82. Restricted Power Series
83. Resolution of Surfaces Revisited

Deformation Theory

84. Formal Deformation Theory
Deformation Theory  
The Cotangent Complex 
Deformation Problems

Algebraic Stacks
Algebraic Stacks  
Examples of Stacks 
Sheaves on Algebraic Stacks 
Criteria for Representability 
Artin’s Axioms 
Quot and Hilbert Spaces 
Properties of Algebraic Stacks 
Morphisms of Algebraic Stacks 
Limits of Algebraic Stacks 
Cohomology of Algebraic Stacks 
Derived Categories of Stacks 
Introducing Algebraic Stacks

More on Morphisms of Stacks 
The Geometry of Stacks 
Topics in Moduli Theory 
Moduli Stacks  
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References


