

# DISCRIMINANTS AND DIFFERENTS

0DWH

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## 1. Introduction

0DWI In this chapter we study the different and discriminant of locally quasi-finite morphisms of schemes. A good reference for some of this material is [Kun86].

Given a quasi-finite morphism  $f : Y \rightarrow X$  of Noetherian schemes there is a relative dualizing module  $\omega_{Y/X}$ . In Section 2 we construct this module from scratch, using Zariski's main theorem and étale localization methods. The key property is that given a diagram

$$\begin{array}{ccc}
 Y' & \xrightarrow{g'} & Y \\
 f' \downarrow & & \downarrow f \\
 X' & \xrightarrow{g} & X
 \end{array}$$

with  $g : X' \rightarrow X$  flat,  $Y' \subset X' \times_X Y$  open, and  $f' : Y' \rightarrow X'$  finite, then there is a canonical isomorphism

$$f'_*(g')^*\omega_{Y/X} = \mathcal{H}om_{\mathcal{O}_{X'}}(f'_*\mathcal{O}_{Y'}, \mathcal{O}_{X'})$$

as sheaves of  $f'_*\mathcal{O}_{Y'}$ -modules. In Section 4 we prove that if  $f$  is flat, then there is a canonical global section  $\tau_{Y/X} \in H^0(Y, \omega_{Y/X})$  which for every commutative diagram as above maps  $(g')^*\tau_{Y/X}$  to the trace map of Section 3 for the finite locally free

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morphism  $f'$ . In Section 8 we define the different for a flat quasi-finite morphism of Noetherian schemes as the annihilator of the cokernel of  $\tau_{Y/X} : \mathcal{O}_X \rightarrow \omega_{Y/X}$ .

The main goal of this chapter is to prove that for quasi-finite syntomic<sup>1</sup>  $f$  the different agrees with the Kähler different. The Kähler different is the zeroth fitting ideal of  $\Omega_{Y/X}$ , see Section 6. This agreement is not obvious; we use a slick argument due to Tate, see Section 10. On the way we also discuss the Noether different and the Dedekind different.

Only in the end of this chapter, see Sections 12 and 13, do we make the link with the more advanced material on duality for schemes.

## 2. Dualizing modules for quasi-finite ring maps

0BUK Let  $A \rightarrow B$  be a quasi-finite homomorphism of Noetherian rings. By Zariski's main theorem (Algebra, Lemma 122.15) there exists a factorization  $A \rightarrow B' \rightarrow B$  with  $A \rightarrow B'$  finite and  $B' \rightarrow B$  inducing an open immersion of spectra. We set

$$0BSZ \quad (2.0.1) \quad \omega_{B/A} = \mathrm{Hom}_A(B', A) \otimes_{B'} B$$

in this situation. The reader can think of this as a kind of relative dualizing module, see Lemmas 12.1 and 2.12. In this section we will show by elementary commutative algebra methods that  $\omega_{B/A}$  is independent of the choice of the factorization and that formation of  $\omega_{B/A}$  commutes with flat base change. To help prove the independence of factorizations we compare two given factorizations.

0BT0 **Lemma 2.1.** *Let  $A \rightarrow B$  be a quasi-finite ring map. Given two factorizations  $A \rightarrow B' \rightarrow B$  and  $A \rightarrow B'' \rightarrow B$  with  $A \rightarrow B'$  and  $A \rightarrow B''$  finite and  $\mathrm{Spec}(B) \rightarrow \mathrm{Spec}(B')$  and  $\mathrm{Spec}(B) \rightarrow \mathrm{Spec}(B'')$  open immersions, there exists an  $A$ -subalgebra  $B''' \subset B$  finite over  $A$  such that  $\mathrm{Spec}(B) \rightarrow \mathrm{Spec}(B''')$  an open immersion and  $B' \rightarrow B$  and  $B'' \rightarrow B$  factor through  $B'''$ .*

**Proof.** Let  $B''' \subset B$  be the  $A$ -subalgebra generated by the images of  $B' \rightarrow B$  and  $B'' \rightarrow B$ . As  $B'$  and  $B''$  are each generated by finitely many elements integral over  $A$ , we see that  $B'''$  is generated by finitely many elements integral over  $A$  and we conclude that  $B'''$  is finite over  $A$  (Algebra, Lemma 35.5). Consider the maps

$$B = B' \otimes_{B'} B \rightarrow B''' \otimes_{B'} B \rightarrow B \otimes_{B'} B = B$$

The final equality holds because  $\mathrm{Spec}(B) \rightarrow \mathrm{Spec}(B')$  is an open immersion (and hence a monomorphism). The second arrow is injective as  $B' \rightarrow B$  is flat. Hence both arrows are isomorphisms. This means that

$$\begin{array}{ccc} \mathrm{Spec}(B''') & \longleftarrow & \mathrm{Spec}(B) \\ \downarrow & & \downarrow \\ \mathrm{Spec}(B') & \longleftarrow & \mathrm{Spec}(B) \end{array}$$

is cartesian. Since the base change of an open immersion is an open immersion we conclude.  $\square$

0BT1 **Lemma 2.2.** *The module (2.0.1) is well defined, i.e., independent of the choice of the factorization.*

<sup>1</sup>AKA flat and lci.

**Proof.** Let  $B', B'', B'''$  be as in Lemma 2.1. We obtain a canonical map

$$\omega''' = \text{Hom}_A(B''', A) \otimes_{B'''} B \longrightarrow \text{Hom}_A(B', A) \otimes_{B'} B = \omega'$$

and a similar one involving  $B''$ . If we show these maps are isomorphisms then the lemma is proved. Let  $g \in B'$  be an element such that  $B'_g \rightarrow B_g$  is an isomorphism and hence  $B'_g \rightarrow (B''')_g \rightarrow B_g$  are isomorphisms. It suffices to show that  $(\omega''')_g \rightarrow \omega'_g$  is an isomorphism. The kernel and cokernel of the ring map  $B' \rightarrow B'''$  are finite  $A$ -modules and  $g$ -power torsion. Hence they are annihilated by a power of  $g$ . This easily implies the result.  $\square$

0BT2 **Lemma 2.3.** *Let  $A \rightarrow B$  be a quasi-finite map of Noetherian rings.*

- (1) *If  $A \rightarrow B$  factors as  $A \rightarrow A_f \rightarrow B$  for some  $f \in A$ , then  $\omega_{B/A} = \omega_{B/A_f}$ .*
- (2) *If  $g \in B$ , then  $(\omega_{B/A})_g = \omega_{B_g/A}$ .*
- (3) *If  $f \in A$ , then  $\omega_{B_f/A_f} = (\omega_{B/A})_f$ .*

**Proof.** Say  $A \rightarrow B' \rightarrow B$  is a factorization with  $A \rightarrow B'$  finite and  $\text{Spec}(B) \rightarrow \text{Spec}(B')$  an open immersion. In case (1) we may use the factorization  $A_f \rightarrow B'_f \rightarrow B$  to compute  $\omega_{B/A_f}$  and use Algebra, Lemma 10.2. In case (2) use the factorization  $A \rightarrow B' \rightarrow B_g$  to see the result. Part (3) follows from a combination of (1) and (2).  $\square$

Let  $A \rightarrow B$  be a quasi-finite ring map of Noetherian rings, let  $A \rightarrow A_1$  be an arbitrary ring map of Noetherian rings, and set  $B_1 = B \otimes_A A_1$ . We obtain a cocartesian diagram

$$\begin{array}{ccc} B & \longrightarrow & B_1 \\ \uparrow & & \uparrow \\ A & \longrightarrow & A_1 \end{array}$$

Observe that  $A_1 \rightarrow B_1$  is quasi-finite as well (Algebra, Lemma 121.8). In this situation we will define a canonical  $B$ -linear base change map

0BV B (2.3.1) 
$$\omega_{B/A} \longrightarrow \omega_{B_1/A_1}$$

Namely, we choose a factorization  $A \rightarrow B' \rightarrow B$  as in the construction of  $\omega_{B/A}$ . Then  $B'_1 = B' \otimes_A A_1$  is finite over  $A_1$  and we can use the factorization  $A_1 \rightarrow B'_1 \rightarrow B_1$  in the construction of  $\omega_{B_1/A_1}$ . Thus we have to construct a map

$$\text{Hom}_A(B', A) \otimes_{B'} B \longrightarrow \text{Hom}_{A_1}(B' \otimes_A A_1, A_1) \otimes_{B'_1} B_1$$

Thus it suffices to construct a  $B'$ -linear map  $\text{Hom}_A(B', A) \rightarrow \text{Hom}_{A_1}(B' \otimes_A A_1, A_1)$  which we will denote  $\varphi \mapsto \varphi_1$ . Namely, given an  $A$ -linear map  $\varphi : B' \rightarrow A$  we let  $\varphi_1$  be the map such that  $\varphi_1(b' \otimes a_1) = \varphi(b')a_1$ . This is clearly  $A_1$ -linear and the construction is complete.

0BVC **Lemma 2.4.** *The base change map (2.3.1) is independent of the choice of the factorization  $A \rightarrow B' \rightarrow B$ . Given ring maps  $A \rightarrow A_1 \rightarrow A_2$  the composition of the base change maps for  $A \rightarrow A_1$  and  $A_1 \rightarrow A_2$  is the base change map for  $A \rightarrow A_2$ .*

**Proof.** Omitted. Hint: argue in exactly the same way as in Lemma 2.2 using Lemma 2.1.  $\square$

0BT3 **Lemma 2.5.** *If  $A \rightarrow A_1$  is flat, then the base change map (2.3.1) induces an isomorphism  $\omega_{B/A} \otimes_B B_1 \rightarrow \omega_{B_1/A_1}$ .*

**Proof.** Assume that  $A \rightarrow A_1$  is flat. By construction of  $\omega_{B/A}$  we may assume that  $A \rightarrow B$  is finite. Then  $\omega_{B/A} = \text{Hom}_A(B, A)$  and  $\omega_{B_1/A_1} = \text{Hom}_{A_1}(B_1, A_1)$ . Since  $B_1 = B \otimes_A A_1$  the result follows from More on Algebra, Remark 60.20.  $\square$

0BT4 **Lemma 2.6.** *Let  $A \rightarrow B \rightarrow C$  be quasi-finite homomorphisms of Noetherian rings. There is a canonical map  $\omega_{B/A} \otimes_B \omega_{C/B} \rightarrow \omega_{C/A}$ .*

**Proof.** Choose  $A \rightarrow B' \rightarrow B$  with  $A \rightarrow B'$  finite such that  $\text{Spec}(B) \rightarrow \text{Spec}(B')$  is an open immersion. Then  $B' \rightarrow C$  is quasi-finite too. Choose  $B' \rightarrow C' \rightarrow C$  with  $B' \rightarrow C'$  finite and  $\text{Spec}(C) \rightarrow \text{Spec}(C')$  an open immersion. Then the source of the arrow is

$$\text{Hom}_A(B', A) \otimes_{B'} B \otimes_B \text{Hom}_B(B \otimes_{B'} C', B) \otimes_{B \otimes_{B'} C'} C$$

which is equal to

$$\text{Hom}_A(B', A) \otimes_{B'} \text{Hom}_{B'}(C', B) \otimes_{C'} C$$

This indeed comes with a canonical map to  $\text{Hom}_A(C', A) \otimes_{C'} C = \omega_{C/A}$  coming from composition  $\text{Hom}_A(B', A) \times \text{Hom}_{B'}(C', B) \rightarrow \text{Hom}_A(C', A)$ .  $\square$

0BT5 **Lemma 2.7.** *Let  $A \rightarrow B$  and  $A \rightarrow C$  be quasi-finite maps of Noetherian rings. Then  $\omega_{B \times C/A} = \omega_{B/A} \times \omega_{C/A}$  as modules over  $B \times C$ .*

**Proof.** Choose factorizations  $A \rightarrow B' \rightarrow B$  and  $A \rightarrow C' \rightarrow C$  such that  $A \rightarrow B'$  and  $A \rightarrow C'$  are finite and such that  $\text{Spec}(B) \rightarrow \text{Spec}(B')$  and  $\text{Spec}(C) \rightarrow \text{Spec}(C')$  are open immersions. Then  $A \rightarrow B' \times C' \rightarrow B \times C$  is a similar factorization. Using this factorization to compute  $\omega_{B \times C/A}$  gives the lemma.  $\square$

0BVD **Lemma 2.8.** *Let  $A \rightarrow B$  be a quasi-finite homomorphism of Noetherian rings. Then  $\text{Ass}_B(\omega_{B/A})$  is the set of primes of  $B$  lying over associated primes of  $A$ .*

**Proof.** Choose a factorization  $A \rightarrow B' \rightarrow B$  with  $A \rightarrow B'$  finite and  $B' \rightarrow B$  inducing an open immersion on spectra. As  $\omega_{B/A} = \omega_{B'/A} \otimes_{B'} B$  it suffices to prove the statement for  $\omega_{B'/A}$ . Thus we may assume  $A \rightarrow B$  is finite.

Assume  $\mathfrak{p} \in \text{Ass}(A)$  and  $\mathfrak{q}$  is a prime of  $B$  lying over  $\mathfrak{p}$ . Let  $x \in A$  be an element whose annihilator is  $\mathfrak{p}$ . Choose a nonzero  $\kappa(\mathfrak{p})$  linear map  $\lambda : \kappa(\mathfrak{q}) \rightarrow \kappa(\mathfrak{p})$ . Since  $A/\mathfrak{p} \subset B/\mathfrak{q}$  is a finite extension of rings, there is an  $f \in A$ ,  $f \notin \mathfrak{p}$  such that  $f\lambda$  maps  $B/\mathfrak{q}$  into  $A/\mathfrak{p}$ . Hence we obtain a nonzero  $A$ -linear map

$$B \rightarrow B/\mathfrak{q} \rightarrow A/\mathfrak{p} \rightarrow A, \quad b \mapsto f\lambda(b)x$$

An easy computation shows that this element of  $\omega_{B/A}$  has annihilator  $\mathfrak{q}$ , whence  $\mathfrak{q} \in \text{Ass}(\omega_{B/A})$ .

Conversely, suppose that  $\mathfrak{q} \subset B$  is a prime ideal lying over a prime  $\mathfrak{p} \subset A$  which is not an associated prime of  $A$ . We have to show that  $\mathfrak{q} \notin \text{Ass}_B(\omega_{B/A})$ . After replacing  $A$  by  $A_{\mathfrak{p}}$  and  $B$  by  $B_{\mathfrak{p}}$  we may assume that  $\mathfrak{p}$  is a maximal ideal of  $A$ . This is allowed by Lemma 2.5 and Algebra, Lemma 62.16. Then there exists an  $f \in \mathfrak{m}$  which is a nonzerodivisor on  $A$ . Then  $f$  is a nonzerodivisor on  $\omega_{B/A}$  and hence  $\mathfrak{q}$  is not an associated prime of this module.  $\square$

0BVE **Lemma 2.9.** *Let  $A \rightarrow B$  be a flat quasi-finite homomorphism of Noetherian rings. Then  $\omega_{B/A}$  is a flat  $A$ -module.*

**Proof.** Let  $\mathfrak{q} \subset B$  be a prime lying over  $\mathfrak{p} \subset A$ . We will show that the localization  $\omega_{B/A, \mathfrak{q}}$  is flat over  $A_{\mathfrak{p}}$ . This suffices by Algebra, Lemma 38.19. By Algebra, Lemma 141.21 we can find an étale ring map  $A \rightarrow A'$  and a prime ideal  $\mathfrak{p}' \subset A'$  lying over  $\mathfrak{p}$  such that  $\kappa(\mathfrak{p}') = \kappa(\mathfrak{p})$  and such that

$$B' = B \otimes_A A' = C \times D$$

with  $A' \rightarrow C$  finite and such that the unique prime  $\mathfrak{q}'$  of  $B \otimes_A A'$  lying over  $\mathfrak{q}$  and  $\mathfrak{p}'$  corresponds to a prime of  $C$ . By Lemma 2.5 and Algebra, Lemma 99.1 it suffices to show  $\omega_{B'/A', \mathfrak{q}'}$  is flat over  $A'_{\mathfrak{p}'}$ . Since  $\omega_{B'/A'} = \omega_{C/A'} \times \omega_{D/A'}$  by Lemma 2.7 this reduces us to the case where  $B$  is finite flat over  $A$ . In this case  $B$  is finite locally free as an  $A$ -module and  $\omega_{B/A} = \text{Hom}_A(B, A)$  is the dual finite locally free  $A$ -module.  $\square$

0BVF **Lemma 2.10.** *If  $A \rightarrow B$  is flat, then the base change map (2.3.1) induces an isomorphism  $\omega_{B/A} \otimes_B B_1 \rightarrow \omega_{B_1/A_1}$ .*

**Proof.** If  $A \rightarrow B$  is finite flat, then  $B$  is finite locally free as an  $A$ -module. In this case  $\omega_{B/A} = \text{Hom}_A(B, A)$  is the dual finite locally free  $A$ -module and formation of this module commutes with arbitrary base change which proves the lemma in this case. In the next paragraph we reduce the general (quasi-finite flat) case to the finite flat case just discussed.

Let  $\mathfrak{q}_1 \subset B_1$  be a prime. We will show that the localization of the map at the prime  $\mathfrak{q}_1$  is an isomorphism, which suffices by Algebra, Lemma 23.1. Let  $\mathfrak{q} \subset B$  and  $\mathfrak{p} \subset A$  be the prime ideals lying under  $\mathfrak{q}_1$ . By Algebra, Lemma 141.21 we can find an étale ring map  $A \rightarrow A'$  and a prime ideal  $\mathfrak{p}' \subset A'$  lying over  $\mathfrak{p}$  such that  $\kappa(\mathfrak{p}') = \kappa(\mathfrak{p})$  and such that

$$B' = B \otimes_A A' = C \times D$$

with  $A' \rightarrow C$  finite and such that the unique prime  $\mathfrak{q}'$  of  $B \otimes_A A'$  lying over  $\mathfrak{q}$  and  $\mathfrak{p}'$  corresponds to a prime of  $C$ . Set  $A'_1 = A' \otimes_A A_1$  and consider the base change maps (2.3.1) for the ring maps  $A \rightarrow A' \rightarrow A'_1$  and  $A \rightarrow A_1 \rightarrow A'_1$  as in the diagram

$$\begin{array}{ccc} \omega_{B'/A'} \otimes_{B'} B'_1 & \longrightarrow & \omega_{B'_1/A'_1} \\ \uparrow & & \uparrow \\ \omega_{B/A} \otimes_B B'_1 & \longrightarrow & \omega_{B_1/A_1} \otimes_{B_1} B'_1 \end{array}$$

where  $B' = B \otimes_A A'$ ,  $B_1 = B \otimes_A A_1$ , and  $B'_1 = B \otimes_A (A' \otimes_A A_1)$ . By Lemma 2.4 the diagram commutes. By Lemma 2.5 the vertical arrows are isomorphisms. As  $B_1 \rightarrow B'_1$  is étale and hence flat it suffices to prove the top horizontal arrow is an isomorphism after localizing at a prime  $\mathfrak{q}'_1$  of  $B'_1$  lying over  $\mathfrak{q}$  (there is such a prime and use Algebra, Lemma 38.17). Thus we may assume that  $B = C \times D$  with  $A \rightarrow C$  finite and  $\mathfrak{q}$  corresponding to a prime of  $C$ . In this case the dualizing module  $\omega_{B/A}$  decomposes in a similar fashion (Lemma 2.7) which reduces the question to the finite flat case  $A \rightarrow C$  handled above.  $\square$

0BVG **Remark 2.11.** Let  $f : Y \rightarrow X$  be a quasi-finite morphism of Noetherian schemes. It is clear from Lemma 2.3 that there is a unique coherent  $\mathcal{O}_Y$ -module  $\omega_{Y/X}$  on  $Y$

such that for every pair of affine opens  $\text{Spec}(B) = V \subset Y$ ,  $\text{Spec}(A) = U \subset X$  with  $f(V) \subset U$  there is a canonical isomorphism

$$H^0(V, \omega_{Y/X}) = \omega_{B/A}$$

and where these isomorphisms are compatible with restriction maps.

0C0I **Lemma 2.12.** *Let  $A \rightarrow B$  be a quasi-finite homomorphism of Noetherian rings. Let  $\omega_{B/A}^\bullet \in D(B)$  be the algebraic relative dualizing complex discussed in Dualizing Complexes, Section 25. Then there is a (nonunique) isomorphism  $\omega_{B/A} = H^0(\omega_{B/A}^\bullet)$ .*

**Proof.** Choose a factorization  $A \rightarrow B' \rightarrow B$  where  $A \rightarrow B'$  is finite and  $\text{Spec}(B') \rightarrow \text{Spec}(B)$  is an open immersion. Then  $\omega_{B/A}^\bullet = \omega_{B'/A}^\bullet \otimes_B^L B'$  by Dualizing Complexes, Lemmas 24.7 and 24.9 and the definition of  $\omega_{B/A}^\bullet$ . Hence it suffices to show there is an isomorphism when  $A \rightarrow B$  is finite. In this case we can use Dualizing Complexes, Lemma 24.8 to see that  $\omega_{B/A}^\bullet = R\text{Hom}(B, A)$  and hence  $H^0(\omega_{B/A}^\bullet) = \text{Hom}_A(B, A)$  as desired.  $\square$

### 3. Discriminant of a finite locally free morphism

0BVH Let  $X$  be a scheme and let  $\mathcal{F}$  be a finite locally free  $\mathcal{O}_X$ -module. Then there is a canonical *trace* map

$$\text{Trace} : \text{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F}) \longrightarrow \mathcal{O}_X$$

See Exercises, Exercise 21.6. This map has the property that  $\text{Trace}(\text{id})$  is the locally constant function on  $\mathcal{O}_X$  corresponding to the rank of  $\mathcal{F}$ .

Let  $\pi : X \rightarrow Y$  be a morphism of schemes which is finite locally free. Then there exists a canonical *trace for  $\pi$*  which is an  $\mathcal{O}_Y$ -linear map

$$\text{Trace}_\pi : \pi_* \mathcal{O}_X \longrightarrow \mathcal{O}_Y$$

sending a local section  $f$  of  $\pi_* \mathcal{O}_X$  to the trace of multiplication by  $f$  on  $\pi_* \mathcal{O}_X$ . Over affine opens this recovers the construction in Exercises, Exercise 21.7. The composition

$$\mathcal{O}_Y \xrightarrow{\pi^\sharp} \pi_* \mathcal{O}_X \xrightarrow{\text{Trace}_\pi} \mathcal{O}_Y$$

equals multiplication by the degree of  $\pi$  (which is a locally constant function on  $Y$ ). In analogy with Fields, Section 20 we can define the trace pairing

$$Q_\pi : \pi_* \mathcal{O}_X \times \pi_* \mathcal{O}_X \longrightarrow \mathcal{O}_Y$$

by the rule  $(f, g) \mapsto \text{Trace}_\pi(fg)$ . We can think of  $Q_\pi$  as a linear map  $\pi_* \mathcal{O}_X \rightarrow \text{Hom}_{\mathcal{O}_Y}(\pi_* \mathcal{O}_X, \mathcal{O}_Y)$  between locally free modules of the same rank, and hence obtain a determinant

$$\det(Q_\pi) : \wedge^{\text{top}}(\pi_* \mathcal{O}_X) \longrightarrow \wedge^{\text{top}}(\pi_* \mathcal{O}_X)^{\otimes -1}$$

or in other words a global section

$$\det(Q_\pi) \in \Gamma(Y, \wedge^{\text{top}}(\pi_* \mathcal{O}_X)^{\otimes -2})$$

The *discriminant* of  $\pi$  is by definition the closed subscheme  $D_\pi \subset Y$  cut out by this global section. Clearly,  $D_\pi$  is a locally principal closed subscheme of  $Y$ .

0BJF **Lemma 3.1.** *Let  $\pi : X \rightarrow Y$  be a morphism of schemes which is finite locally free. Then  $\pi$  is étale if and only if its discriminant is empty.*

**Proof.** By Morphisms, Lemma 34.8 it suffices to check that the fibres of  $\pi$  are étale. Since the construction of the trace pairing commutes with base change we reduce to the following question: Let  $k$  be a field and let  $A$  be a finite dimensional  $k$ -algebra. Show that  $A$  is étale over  $k$  if and only if the trace pairing  $Q_{A/k} : A \times A \rightarrow k$ ,  $(a, b) \mapsto \text{Trace}_{A/k}(ab)$  is nondegenerate.

Assume  $Q_{A/k}$  is nondegenerate. If  $a \in A$  is a nilpotent element, then  $ab$  is nilpotent for all  $b \in A$  and we conclude that  $Q_{A/k}(a, -)$  is identically zero. Hence  $A$  is reduced. Then we can write  $A = K_1 \times \dots \times K_n$  as a product where each  $K_i$  is a field (see Algebra, Lemmas 52.2, 52.6, and 24.1). In this case the quadratic space  $(A, Q_{A/k})$  is the orthogonal direct sum of the spaces  $(K_i, Q_{K_i/k})$ . It follows from Fields, Lemma 20.7 that each  $K_i$  is separable over  $k$ . This means that  $A$  is étale over  $k$  by Algebra, Lemma 141.4. The converse is proved by reading the argument backwards.  $\square$

#### 4. Traces for flat quasi-finite ring maps

0BSY The trace referred to in the title of this section is of a completely different nature than the trace discussed in Duality for Schemes, Section 7. Namely, it is the trace as discussed in Fields, Section 20 and generalized in Exercises, Exercises 21.6 and 21.7.

Let  $A \rightarrow B$  be a finite flat map of Noetherian rings. Then  $B$  is finite flat as an  $A$ -module and hence finite locally free (Algebra, Lemma 77.2). Given  $b \in B$  we can consider the *trace*  $\text{Trace}_{B/A}(b)$  of the  $A$ -linear map  $B \rightarrow B$  given by multiplication by  $b$  on  $B$ . By the references above this defines an  $A$ -linear map  $\text{Trace}_{B/A} : B \rightarrow A$ . Since  $\omega_{B/A} = \text{Hom}_A(B, A)$  as  $A \rightarrow B$  is finite, we see that  $\text{Trace}_{B/A} \in \omega_{B/A}$ .

For a general flat quasi-finite ring map we define the notion of a trace as follows.

0BT6 **Definition 4.1.** Let  $A \rightarrow B$  be a flat quasi-finite map of Noetherian rings. The *trace element* is the unique<sup>2</sup> element  $\tau_{B/A} \in \omega_{B/A}$  with the following property: for any Noetherian  $A$ -algebra  $A_1$  such that  $B_1 = B \otimes_A A_1$  comes with a product decomposition  $B_1 = C \times D$  with  $A_1 \rightarrow C$  finite the image of  $\tau_{B/A}$  in  $\omega_{C/A_1}$  is  $\text{Trace}_{C/A_1}$ . Here we use the base change map (2.3.1) and Lemma 2.7 to get  $\omega_{B/A} \rightarrow \omega_{B_1/A_1} \rightarrow \omega_{C/A_1}$ .

We first prove that trace elements are unique and then we prove that they exist.

0BT7 **Lemma 4.2.** *Let  $A \rightarrow B$  be a flat quasi-finite map of Noetherian rings. Then there is at most one trace element in  $\omega_{B/A}$ .*

**Proof.** Let  $\mathfrak{q} \subset B$  be a prime ideal lying over the prime  $\mathfrak{p} \subset A$ . By Algebra, Lemma 141.21 we can find an étale ring map  $A \rightarrow A_1$  and a prime ideal  $\mathfrak{p}_1 \subset A_1$  lying over  $\mathfrak{p}$  such that  $\kappa(\mathfrak{p}_1) = \kappa(\mathfrak{p})$  and such that

$$B_1 = B \otimes_A A_1 = C \times D$$

with  $A_1 \rightarrow C$  finite and such that the unique prime  $\mathfrak{q}_1$  of  $B \otimes_A A_1$  lying over  $\mathfrak{q}$  and  $\mathfrak{p}_1$  corresponds to a prime of  $C$ . Observe that  $\omega_{C/A_1} = \omega_{B/A} \otimes_B C$  (combine Lemmas 2.5 and 2.7). Since the collection of ring maps  $B \rightarrow C$  obtained in this manner is a jointly injective family of flat maps and since the image of  $\tau_{B/A}$  in  $\omega_{C/A_1}$  is prescribed the uniqueness follows.  $\square$

<sup>2</sup>Uniqueness and existence will be justified in Lemmas 4.2 and 4.6.

Here is a sanity check.

0BT8 **Lemma 4.3.** *Let  $A \rightarrow B$  be a finite flat map of Noetherian rings. Then  $\text{Trace}_{B/A} \in \omega_{B/A}$  is the trace element.*

**Proof.** Suppose we have  $A \rightarrow A_1$  with  $A_1$  Noetherian and a product decomposition  $B \otimes_A A_1 = C \times D$  with  $A_1 \rightarrow C$  finite. Of course in this case  $A_1 \rightarrow D$  is also finite. Set  $B_1 = B \otimes_A A_1$ . Since the construction of traces commutes with base change we see that  $\text{Trace}_{B/A}$  maps to  $\text{Trace}_{B_1/A_1}$ . Thus the proof is finished by noticing that  $\text{Trace}_{B_1/A_1} = (\text{Trace}_{C/A_1}, \text{Trace}_{D/A_1})$  under the isomorphism  $\omega_{B_1/A_1} = \omega_{C/A_1} \times \omega_{D/A_1}$  of Lemma 2.7.  $\square$

0BT9 **Lemma 4.4.** *Let  $A \rightarrow B$  be a flat quasi-finite map of Noetherian rings. Let  $\tau \in \omega_{B/A}$  be a trace element.*

- (1) *If  $A \rightarrow A_1$  is a map with  $A_1$  Noetherian, then with  $B_1 = A_1 \otimes_A B$  the image of  $\tau$  in  $\omega_{B_1/A_1}$  is a trace element.*
- (2) *If  $A = R_f$ , then  $\tau$  is a trace element in  $\omega_{B/R}$ .*
- (3) *If  $g \in B$ , then the image of  $\tau$  in  $\omega_{B_g/A}$  is a trace element.*
- (4) *If  $B = B_1 \times B_2$ , then  $\tau$  maps to a trace element in both  $\omega_{B_1/A}$  and  $\omega_{B_2/A}$ .*

**Proof.** Part (1) is a formal consequence of the definition.

Statement (2) makes sense because  $\omega_{B/R} = \omega_{B/A}$  by Lemma 2.3. Denote  $\tau'$  the element  $\tau$  but viewed as an element of  $\omega_{B/R}$ . To see that (2) is true suppose that we have  $R \rightarrow R_1$  with  $R_1$  Noetherian and a product decomposition  $B \otimes_R R_1 = C \times D$  with  $R_1 \rightarrow C$  finite. Then with  $A_1 = (R_1)_f$  we see that  $B \otimes_A A_1 = C \times D$ . Since  $R_1 \rightarrow C$  is finite, a fortiori  $A_1 \rightarrow C$  is finite. Hence we can use the defining property of  $\tau$  to get the corresponding property of  $\tau'$ .

Statement (3) makes sense because  $\omega_{B_g/A} = (\omega_{B/A})_g$  by Lemma 2.3. The proof is similar to the proof of (2). Suppose we have  $A \rightarrow A_1$  with  $A_1$  Noetherian and a product decomposition  $B_g \otimes_A A_1 = C \times D$  with  $A_1 \rightarrow C$  finite. Set  $B_1 = B \otimes_A A_1$ . Then  $\text{Spec}(C) \rightarrow \text{Spec}(B_1)$  is an open immersion as  $B_g \otimes_A A_1 = (B_1)_g$  and the image is closed because  $B_1 \rightarrow C$  is finite (as  $A_1 \rightarrow C$  is finite). Thus we see that  $B_1 = C \times D_1$  and  $D = (D_1)_g$ . Then we can use the defining property of  $\tau$  to get the corresponding property for the image of  $\tau$  in  $\omega_{B_g/A}$ .

Statement (4) makes sense because  $\omega_{B/A} = \omega_{B_1/A} \times \omega_{B_2/A}$  by Lemma 2.7. Suppose we have  $A \rightarrow A'$  with  $A'$  Noetherian and a product decomposition  $B \otimes_A A' = C \times D$  with  $A' \rightarrow C$  finite. Then it is clear that we can refine this product decomposition into  $B \otimes_A A' = C_1 \times C_2 \times D_1 \times D_2$  with  $A' \rightarrow C_i$  finite such that  $B_i \otimes_A A' = C_i \times D_i$ . Then we can use the defining property of  $\tau$  to get the corresponding property for the image of  $\tau$  in  $\omega_{B_i/A}$ . This uses the obvious fact that  $\text{Trace}_{C/A'} = (\text{Trace}_{C_1/A'}, \text{Trace}_{C_2/A'})$  under the decomposition  $\omega_{C/A'} = \omega_{C_1/A'} \times \omega_{C_2/A'}$ .  $\square$

0BTA **Lemma 4.5.** *Let  $A \rightarrow B$  be a flat quasi-finite map of Noetherian rings. Let  $g_1, \dots, g_m \in B$  be elements generating the unit ideal. Let  $\tau \in \omega_{B/A}$  be an element whose image in  $\omega_{B_{g_i}/A}$  is a trace element for  $A \rightarrow B_{g_i}$ . Then  $\tau$  is a trace element.*

**Proof.** Suppose we have  $A \rightarrow A_1$  with  $A_1$  Noetherian and a product decomposition  $B \otimes_A A_1 = C \times D$  with  $A_1 \rightarrow C$  finite. We have to show that the image of  $\tau$  in  $\omega_{C/A_1}$  is  $\text{Trace}_{C/A_1}$ . Observe that  $g_1, \dots, g_m$  generate the unit ideal in  $B_1 = B \otimes_A A_1$  and



that  $\tau$  maps to a trace element in  $\omega_{(B_1)_{g_i}/A_1}$  by Lemma 4.4. Hence we may replace  $A$  by  $A_1$  and  $B$  by  $B_1$  to get to the situation as described in the next paragraph.

Here we assume that  $B = C \times D$  with  $A \rightarrow C$  finite. Let  $\tau_C$  be the image of  $\tau$  in  $\omega_{C/A}$ . We have to prove that  $\tau_C = \text{Trace}_{C/A}$  in  $\omega_{C/A}$ . By the compatibility of trace elements with products (Lemma 4.4) we see that  $\tau_C$  maps to a trace element in  $\omega_{C_{g_i}/A}$ . Hence, after replacing  $B$  by  $C$  we may assume that  $A \rightarrow B$  is finite flat.

Assume  $A \rightarrow B$  is finite flat. In this case  $\text{Trace}_{B/A}$  is a trace element by Lemma 4.3. Hence  $\text{Trace}_{B/A}$  maps to a trace element in  $\omega_{B_{g_i}/A}$  by Lemma 4.4. Since trace elements are unique (Lemma 4.2) we find that  $\text{Trace}_{B/A}$  and  $\tau$  map to the same elements in  $\omega_{B_{g_i}/A} = (\omega_{B/A})_{g_i}$ . As  $g_1, \dots, g_m$  generate the unit ideal of  $B$  the map  $\omega_{B/A} \rightarrow \prod \omega_{B_{g_i}/A}$  is injective and we conclude that  $\tau_C = \text{Trace}_{B/A}$  as desired.  $\square$

0BTB **Lemma 4.6.** *Let  $A \rightarrow B$  be a flat quasi-finite map of Noetherian rings. There exists a trace element  $\tau \in \omega_{B/A}$ .*

**Proof.** Choose a factorization  $A \rightarrow B' \rightarrow B$  with  $A \rightarrow B'$  finite and  $\text{Spec}(B) \rightarrow \text{Spec}(B')$  an open immersion. Let  $g_1, \dots, g_n \in B'$  be elements such that  $\text{Spec}(B) = \bigcup D(g_i)$  as opens of  $\text{Spec}(B')$ . Suppose that we can prove the existence of trace elements  $\tau_i$  for the quasi-finite flat ring maps  $A \rightarrow B_{g_i}$ . Then for all  $i, j$  the elements  $\tau_i$  and  $\tau_j$  map to trace elements of  $\omega_{B_{g_i g_j}/A}$  by Lemma 4.4. By uniqueness of trace elements (Lemma 4.2) they map to the same element. Hence the sheaf condition for the quasi-coherent module associated to  $\omega_{B/A}$  (see Algebra, Lemma 22.2) produces an element  $\tau \in \omega_{B/A}$ . Then  $\tau$  is a trace element by Lemma 4.5. In this way we reduce to the case treated in the next paragraph.

Assume we have  $A \rightarrow B'$  finite and  $g \in B'$  with  $B = B'_g$  flat over  $A$ . It is our task to construct a trace element in  $\omega_{B/A} = \text{Hom}_A(B', A) \otimes_{B'} B$ . Choose a resolution  $F_1 \rightarrow F_0 \rightarrow B' \rightarrow 0$  of  $B'$  by finite free  $A$ -modules  $F_0$  and  $F_1$ . Then we have an exact sequence

$$0 \rightarrow \text{Hom}_A(B', A) \rightarrow F_0^\vee \rightarrow F_1^\vee$$

where  $F_i^\vee = \text{Hom}_A(F_i, A)$  is the dual finite free module. Similarly we have the exact sequence

$$0 \rightarrow \text{Hom}_A(B', B') \rightarrow F_0^\vee \otimes_A B' \rightarrow F_1^\vee \otimes_A B'$$

The idea of the construction of  $\tau$  is to use the diagram

$$B' \xrightarrow{\mu} \text{Hom}_A(B', B') \leftarrow \text{Hom}_A(B', A) \otimes_A B' \xrightarrow{ev} A$$

where the first arrow sends  $b' \in B'$  to the  $A$ -linear operator given by multiplication by  $b'$  and the last arrow is the evaluation map. The problem is that the middle arrow, which sends  $\lambda' \otimes b'$  to the map  $b'' \mapsto \lambda'(b'')b'$ , is not an isomorphism. If  $B'$  is flat over  $A$ , the exact sequences above show that it is an isomorphism and the composition from left to right is the usual trace  $\text{Trace}_{B'/A}$ . In the general case, we consider the diagram

$$\begin{array}{ccc} \text{Hom}_A(B', A) \otimes_A B' & \longrightarrow & \text{Hom}_A(B', A) \otimes_A B'_g \\ \psi \nearrow & \downarrow & \downarrow \\ B' & \xrightarrow{\mu} \text{Hom}_A(B', B') \longrightarrow & \text{Ker}(F_0^\vee \otimes_A B'_g \rightarrow F_1^\vee \otimes_A B'_g) \end{array}$$

By flatness of  $A \rightarrow B'_g$  we see that the right vertical arrow is an isomorphism. Hence we obtain the unadorned dotted arrow. Since  $B'_g = \text{colim}_{g^n} \frac{1}{g^n} B'$ , since colimits commute with tensor products, and since  $B'$  is a finitely presented  $A$ -module we can find an  $n \geq 0$  and a  $B'$ -linear (for right  $B'$ -module structure) map  $\psi : B' \rightarrow \text{Hom}_A(B', A) \otimes_A B'$  whose composition with the left vertical arrow is  $g^n \mu$ . Composing with  $ev$  we obtain an element  $ev \circ \psi \in \text{Hom}_A(B', A)$ . Then we set

$$\tau = (ev \circ \psi) \otimes g^{-n} \in \text{Hom}_A(B', A) \otimes_{B'} B'_g = \omega_{B'_g/A} = \omega_{B/A}$$

We omit the easy verification that this element does not depend on the choice of  $n$  and  $\psi$  above.

Let us prove that  $\tau$  as constructed in the previous paragraph has the desired property in a special case. Namely, say  $B' = C' \times D'$  and  $g = (f, h)$  where  $A \rightarrow C'$  flat,  $D'_h$  is flat, and  $f$  is a unit in  $C'$ . To show:  $\tau$  maps to  $\text{Trace}_{C'/A}$  in  $\omega_{C'/A}$ . In this case we first choose  $n_D$  and  $\psi_D : D' \rightarrow \text{Hom}_A(D', A) \otimes_A D'$  as above for the pair  $(D', h)$  and we can let  $\psi_C : C' \rightarrow \text{Hom}_A(C', A) \otimes_A C' = \text{Hom}_A(C', C')$  be the map second-  
ing  $c' \in C'$  to multiplication by  $c'$ . Then we take  $n = n_D$  and  $\psi = (f^{n_D} \psi_C, \psi_D)$  and the desired compatibility is clear because  $\text{Trace}_{C'/A} = ev \circ \psi_C$  as remarked above.

To prove the desired property in general, suppose given  $A \rightarrow A_1$  with  $A_1$  Noetherian and a product decomposition  $B'_g \otimes_A A_1 = C \times D$  with  $A_1 \rightarrow C$  finite. Set  $B'_1 = B' \otimes_A A_1$ . Then  $\text{Spec}(C) \rightarrow \text{Spec}(B'_1)$  is an open immersion as  $B'_g \otimes_A A_1 = (B'_1)_g$  and the image is closed as  $B'_1 \rightarrow C$  is finite (since  $A_1 \rightarrow C$  is finite). Thus  $B'_1 = C \times D'$  and  $D'_g = D$ . We conclude that  $B'_1 = C \times D'$  and  $g$  over  $A_1$  are as in the previous paragraph. Since formation of the displayed diagram above commutes with base change, the formation of  $\tau$  commutes with the base change  $A \rightarrow A_1$  (details omitted; use the resolution  $F_1 \otimes_A A_1 \rightarrow F_0 \otimes_A A_1 \rightarrow B'_1 \rightarrow 0$  to see this). Thus the desired compatibility follows from the result of the previous paragraph.  $\square$

0BVJ **Remark 4.7.** Let  $f : Y \rightarrow X$  be a flat quasi-finite morphism of Noetherian schemes. Let  $\omega_{Y/X}$  be as in Remark 2.11. It is clear from the uniqueness, existence, and compatibility with localization of trace elements (Lemmas 4.2, 4.6, and 4.4) that there exists a global section

$$\tau_{Y/X} \in \Gamma(Y, \omega_{Y/X})$$

such that for every pair of affine opens  $\text{Spec}(B) = V \subset Y$ ,  $\text{Spec}(A) = U \subset X$  with  $f(V) \subset U$  that element  $\tau_{Y/X}$  maps to  $\tau_{B/A}$  under the canonical isomorphism  $H^0(V, \omega_{Y/X}) = \omega_{B/A}$ .

0C13 **Lemma 4.8.** *Let  $k$  be a field and let  $A$  be a finite  $k$ -algebra. Assume  $A$  is local with residue field  $k'$ . The following are equivalent*

- (1)  $\text{Trace}_{A/k}$  is nonzero,
- (2)  $\tau_{A/k} \in \omega_{A/k}$  is nonzero, and
- (3)  $k'/k$  is separable and  $\text{length}_A(A)$  is prime to the characteristic of  $k$ .

**Proof.** Conditions (1) and (2) are equivalent by Lemma 4.3. Let  $\mathfrak{m} \subset A$ . Since  $\dim_k(A) < \infty$  it is clear that  $A$  has finite length over  $A$ . Choose a filtration

$$A = I_0 \supset \mathfrak{m} = I_1 \supset I_2 \supset \dots \supset I_n = 0$$

by ideals such that  $I_i/I_{i+1} \cong k'$  as  $A$ -modules. See Algebra, Lemma 51.11 which also shows that  $n = \text{length}_A(A)$ . If  $a \in \mathfrak{m}$  then  $aI_i \subset I_{i+1}$  and it is immediate that  $\text{Trace}_{A/k}(a) = 0$ . If  $a \notin \mathfrak{m}$  with image  $\lambda \in k'$ , then we conclude

$$\text{Trace}_{A/k}(a) = \sum_{i=0, \dots, n-1} \text{Trace}_k(a : I_i/I_{i-1} \rightarrow I_i/I_{i-1}) = n \text{Trace}_{k'/k}(\lambda)$$

The proof of the lemma is finished by applying Fields, Lemma 20.7.  $\square$

## 5. The Noether different

0BVK There are many different differentials available in the literature. We list some of them in this and the next sections; for more information we suggest the reader consult [Kun86].

Let  $A \rightarrow B$  be a ring map. Denote

$$\mu : B \otimes_A B \longrightarrow B, \quad b \otimes b' \longmapsto bb'$$

the multiplication map. Let  $I = \text{Ker}(\mu)$ . It is clear that  $I$  is generated by the elements  $b \otimes 1 - 1 \otimes b$  for  $b \in B$ . Hence the annihilator  $J \subset B \otimes_A B$  of  $I$  is a  $B$ -module in a canonical manner. The *Noether different* of  $B$  over  $A$  is the image of  $J$  under the map  $\mu : B \otimes_A B \rightarrow B$ . Equivalently, the Noether different is the image of the map

$$J = \text{Hom}_{B \otimes_A B}(B, B \otimes_A B) \longrightarrow B, \quad \varphi \longmapsto \mu(\varphi(1))$$

We begin with some obligatory lemmas.

0BVL **Lemma 5.1.** *Let  $A \rightarrow B_i$ ,  $i = 1, 2$  be ring maps. Set  $B = B_1 \times B_2$ .*

- (1) *The annihilator  $J$  of  $\text{Ker}(B \otimes_A B \rightarrow B)$  is  $J_1 \times J_2$  where  $J_i$  is the annihilator of  $\text{Ker}(B_i \otimes_A B_i \rightarrow B_i)$ .*
- (2) *The Noether different  $\mathfrak{D}$  of  $B$  over  $A$  is  $\mathfrak{D}_1 \times \mathfrak{D}_2$ , where  $\mathfrak{D}_i$  is the Noether different of  $B_i$  over  $A$ .*

**Proof.** Omitted.  $\square$

0BVM **Lemma 5.2.** *Let  $A \rightarrow B$  be a finite type ring map. Let  $A \rightarrow A'$  be a flat ring map. Set  $B' = B \otimes_A A'$ .*

- (1) *The annihilator  $J'$  of  $\text{Ker}(B' \otimes_{A'} B' \rightarrow B')$  is  $J \otimes_A A'$  where  $J$  is the annihilator of  $\text{Ker}(B \otimes_A B \rightarrow B)$ .*
- (2) *The Noether different  $\mathfrak{D}'$  of  $B'$  over  $A'$  is  $\mathfrak{D}B'$ , where  $\mathfrak{D}$  is the Noether different of  $B$  over  $A$ .*

**Proof.** Choose generators  $b_1, \dots, b_n$  of  $B$  as an  $A$ -algebra. Then

$$J = \text{Ker}(B \otimes_A B \xrightarrow{b_i \otimes 1 - 1 \otimes b_i} (B \otimes_A B)^{\oplus n})$$

Hence we see that the formation of  $J$  commutes with flat base change. The result on the Noether different follows immediately from this.  $\square$

0BVN **Lemma 5.3.** *Let  $A \rightarrow B' \rightarrow B$  be ring maps with  $A \rightarrow B'$  of finite type and  $B' \rightarrow B$  inducing an open immersion of spectra.*

- (1) *The annihilator  $J$  of  $\text{Ker}(B \otimes_A B \rightarrow B)$  is  $J' \otimes_{B'} B$  where  $J'$  is the annihilator of  $\text{Ker}(B' \otimes_A B' \rightarrow B')$ .*
- (2) *The Noether different  $\mathfrak{D}$  of  $B$  over  $A$  is  $\mathfrak{D}'B$ , where  $\mathfrak{D}'$  is the Noether different of  $B'$  over  $A$ .*

**Proof.** Write  $I = \text{Ker}(B \otimes_A B \rightarrow B)$  and  $I' = \text{Ker}(B' \otimes_A B' \rightarrow B')$ . As  $\text{Spec}(B) \rightarrow \text{Spec}(B')$  is an open immersion, it follows that  $B = (B \otimes_A B) \otimes_{B' \otimes_A B'} B'$ . Thus we see that  $I = I'(B \otimes_A B)$ . Since  $I'$  is finitely generated and  $B' \otimes_A B' \rightarrow B \otimes_A B$  is flat, we conclude that  $J = J'(B \otimes_A B)$ , see Algebra, Lemma 39.4. Since the  $B' \otimes_A B'$ -module structure of  $J'$  factors through  $B' \otimes_A B' \rightarrow B'$  we conclude that (1) is true. Part (2) is a consequence of (1).  $\square$

0BVP **Remark 5.4.** Let  $A \rightarrow B$  be a quasi-finite homomorphism of Noetherian rings. Let  $J$  be the annihilator of  $\text{Ker}(B \otimes_A B \rightarrow B)$ . There is a canonical  $B$ -bilinear pairing

0BVQ (5.4.1) 
$$\omega_{B/A} \times J \longrightarrow B$$

defined as follows. Choose a factorization  $A \rightarrow B' \rightarrow B$  with  $A \rightarrow B'$  finite and  $B' \rightarrow B$  inducing an open immersion of spectra. Let  $J'$  be the annihilator of  $\text{Ker}(B' \otimes_A B' \rightarrow B')$ . We first define

$$\text{Hom}_A(B', A) \times J' \longrightarrow B', \quad (\lambda, \sum b_i \otimes c_i) \mapsto \sum \lambda(b_i)c_i$$

This is  $B'$ -bilinear exactly because for  $\xi \in J'$  and  $b \in B'$  we have  $(b \otimes 1)\xi = (1 \otimes b)\xi$ . By Lemma 5.3 and the fact that  $\omega_{B/A} = \text{Hom}_A(B', A) \otimes_{B'} B$  we can extend this to a  $B$ -bilinear pairing as displayed above.

0BVR **Lemma 5.5.** Let  $A \rightarrow B$  be a quasi-finite homomorphism of Noetherian rings.

(1) If  $A \rightarrow A'$  is a flat map of Noetherian rings, then

$$\begin{array}{ccc} \omega_{B/A} \times J & \longrightarrow & B \\ \downarrow & & \downarrow \\ \omega_{B'/A'} \times J' & \longrightarrow & B' \end{array}$$

is commutative where notation as in Lemma 5.2 and horizontal arrows are given by (5.4.1).

(2) If  $B = B_1 \times B_2$ , then

$$\begin{array}{ccc} \omega_{B/A} \times J & \longrightarrow & B \\ \downarrow & & \downarrow \\ \omega_{B_i/A} \times J_i & \longrightarrow & B_i \end{array}$$

is commutative for  $i = 1, 2$  where notation as in Lemma 5.1 and horizontal arrows are given by (5.4.1).

**Proof.** Because of the construction of the pairing in Remark 5.4 both (1) and (2) reduce to the case where  $A \rightarrow B$  is finite. Then (1) follows from the fact that the contraction map  $\text{Hom}_A(M, A) \otimes_A M \otimes_A M \rightarrow M$ ,  $\lambda \otimes m \otimes m' \mapsto \lambda(m)m'$  commuted with base change. To see (2) use that  $J = J_1 \times J_2$  is contained in the summands  $B_1 \otimes_A B_1$  and  $B_2 \otimes_A B_2$  of  $B \otimes_A B$ .  $\square$

0BVS **Lemma 5.6.** Let  $A \rightarrow B$  be a flat quasi-finite homomorphism of Noetherian rings. The pairing of Remark 5.4 induces an isomorphism  $J \rightarrow \text{Hom}_B(\omega_{B/A}, B)$ .

**Proof.** We first prove this when  $A \rightarrow B$  is finite and flat. In this case we can localize on  $A$  and assume  $B$  is finite free as an  $A$ -module. Let  $b_1, \dots, b_n$  be a basis of  $B$  as an  $A$ -module and denote  $b_1^\vee, \dots, b_n^\vee$  the dual basis of  $\omega_{B/A}$ . Note that  $\sum b_i \otimes c_i \in J$  maps to the element of  $\text{Hom}_B(\omega_{B/A}, B)$  which sends  $b_i^\vee$  to  $c_i$ . Suppose  $\varphi : \omega_{B/A} \rightarrow B$  is  $B$ -linear. Then we claim that  $\xi = \sum b_i \otimes \varphi(b_i^\vee)$  is an element of  $J$ . Namely, the  $B$ -linearity of  $\varphi$  exactly implies that  $(b \otimes 1)\xi = (1 \otimes b)\xi$  for all  $b \in B$ . Thus our map has an inverse and it is an isomorphism.

Let  $\mathfrak{q} \subset B$  be a prime lying over  $\mathfrak{p} \subset A$ . We will show that the localization

$$J_{\mathfrak{q}} \longrightarrow \text{Hom}_B(\omega_{B/A}, B)_{\mathfrak{q}}$$

is an isomorphism. This suffices by Algebra, Lemma 23.1. By Algebra, Lemma 141.21 we can find an étale ring map  $A \rightarrow A'$  and a prime ideal  $\mathfrak{p}' \subset A'$  lying over  $\mathfrak{p}$  such that  $\kappa(\mathfrak{p}') = \kappa(\mathfrak{p})$  and such that

$$B' = B \otimes_A A' = C \times D$$

with  $A' \rightarrow C$  finite and such that the unique prime  $\mathfrak{q}'$  of  $B \otimes_A A'$  lying over  $\mathfrak{q}$  and  $\mathfrak{p}'$  corresponds to a prime of  $C$ . Let  $J'$  be the annihilator of  $\text{Ker}(B' \otimes_{A'} B' \rightarrow B')$ . By Lemmas 2.5, 5.2, and 5.5 the map  $J' \rightarrow \text{Hom}_{B'}(\omega_{B'/A'}, B')$  is gotten by applying the functor  $-\otimes_B B'$  to the map  $J \rightarrow \text{Hom}_B(\omega_{B/A}, B)$ . Since  $B_{\mathfrak{q}} \rightarrow B'_{\mathfrak{q}'}$  is faithfully flat it suffices to prove the result for  $(A' \rightarrow B', \mathfrak{q}')$ . By Lemmas 2.7, 5.1, and 5.5 this reduces us to the case proved in the first paragraph of the proof.  $\square$

0BVT **Lemma 5.7.** *Let  $A \rightarrow B$  be a flat quasi-finite homomorphism of Noetherian rings. The diagram*

$$\begin{array}{ccc} J & \xrightarrow{\quad} & \text{Hom}_B(\omega_{B/A}, B) \\ & \searrow \mu & \swarrow \varphi \mapsto \varphi(\tau_{B/A}) \\ & & B \end{array}$$

*commutes where the horizontal arrow is the isomorphism of Lemma 5.6. Hence the Noether different of  $B$  over  $A$  is the image of the map  $\text{Hom}_B(\omega_{B/A}, B) \rightarrow B$ .*

**Proof.** Exactly as in the proof of Lemma 5.6 this reduces to the case of a finite free map  $A \rightarrow B$ . In this case  $\tau_{B/A} = \text{Trace}_{B/A}$ . Choose a basis  $b_1, \dots, b_n$  of  $B$  as an  $A$ -module. Let  $\xi = \sum b_i \otimes c_i \in J$ . Then  $\mu(\xi) = \sum b_i c_i$ . On the other hand, the image of  $\xi$  in  $\text{Hom}_B(\omega_{B/A}, B)$  sends  $\text{Trace}_{B/A}$  to  $\sum \text{Trace}_{B/A}(b_i) c_i$ . Thus we have to show

$$\sum b_i c_i = \sum \text{Trace}_{B/A}(b_i) c_i$$

when  $\xi = \sum b_i \otimes c_i \in J$ . Write  $b_i b_j = \sum_k a_{ij}^k b_k$  for some  $a_{ij}^k \in A$ . Then the right hand side is  $\sum_{i,j} a_{ij}^j c_i$ . On the other hand,  $\xi \in J$  implies

$$(b_j \otimes 1) \left( \sum_i b_i \otimes c_i \right) = (1 \otimes b_j) \left( \sum_i b_i \otimes c_i \right)$$

which implies that  $b_j c_i = \sum_k a_{jk}^i c_k$ . Thus the left hand side is  $\sum_{i,j} a_{ij}^i c_j$ . Since  $a_{ij}^k = a_{ji}^k$  the equality holds.  $\square$

0BVU **Lemma 5.8.** *Let  $A \rightarrow B$  be a finite type ring map. Let  $\mathfrak{D} \subset B$  be the Noether different. Then  $V(\mathfrak{D})$  is the set of primes  $\mathfrak{q} \subset B$  such that  $A \rightarrow B$  is not unramified at  $\mathfrak{q}$ .*

**Proof.** Assume  $A \rightarrow B$  is unramified at  $\mathfrak{q}$ . After replacing  $B$  by  $B_g$  for some  $g \in B$ ,  $g \notin \mathfrak{q}$  we may assume  $A \rightarrow B$  is unramified (Algebra, Definition 147.1 and Lemma 5.3). In this case  $\Omega_{B/A} = 0$ . Hence if  $I = \text{Ker}(B \otimes_A B \rightarrow B)$ , then  $I/I^2 = 0$  by Algebra, Lemma 130.13. Since  $A \rightarrow B$  is of finite type, we see that  $I$  is finitely generated. Hence by Nakayama's lemma (Algebra, Lemma 19.1) there exists an element of the form  $1 + i$  annihilating  $I$ . It follows that  $\mathfrak{D} = B$ .

Conversely, assume that  $\mathfrak{D} \not\subset \mathfrak{q}$ . Then after replacing  $B$  by a principal localization as above we may assume  $\mathfrak{D} = B$ . This means there exists an element of the form  $1 + i$  in the annihilator of  $I$ . Conversely this implies that  $I/I^2 = \Omega_{B/A}$  is zero and we conclude.  $\square$

## 6. The Kähler different

0BVV Let  $A \rightarrow B$  be a finite type ring map. The *Kähler different* is the zeroth fitting ideal of  $\Omega_{B/A}$  as a  $B$ -module. We globalize the definition as follows.

0BVW **Definition 6.1.** Let  $f : Y \rightarrow X$  be a morphism of schemes which is locally of finite type. The *Kähler different* is the 0th fitting ideal of  $\Omega_{Y/X}$ .

The Kähler different is a quasi-coherent sheaf of ideals on  $Y$ .

0BVX **Lemma 6.2.** Consider a cartesian diagram of schemes

$$\begin{array}{ccc} Y' & \longrightarrow & Y \\ f' \downarrow & & \downarrow f \\ X' & \xrightarrow{g} & X \end{array}$$

with  $f$  locally of finite type. Let  $R \subset Y$ , resp.  $R' \subset Y'$  be the closed subscheme cut out by the Kähler different of  $f$ , resp.  $f'$ . Then  $Y' \rightarrow Y$  induces an isomorphism  $R' \rightarrow R \times_Y Y'$ .

**Proof.** This is true because  $\Omega_{Y'/X'}$  is the pullback of  $\Omega_{Y/X}$  (Morphisms, Lemma 31.10) and then we can apply More on Algebra, Lemma 8.4.  $\square$

0BVY **Lemma 6.3.** Let  $f : Y \rightarrow X$  be a morphism of schemes which is locally of finite type. Let  $R \subset Y$  be the closed subscheme defined by the Kähler different. Then  $R \subset Y$  is exactly the set of points where  $f$  is not unramified.

**Proof.** This is a copy of Divisors, Lemma 10.1.  $\square$

0BVZ **Lemma 6.4.** Let  $A$  be a ring. Let  $n \geq 1$  and  $f_1, \dots, f_n \in A[x_1, \dots, x_n]$ . Set  $B = A[x_1, \dots, x_n]/(f_1, \dots, f_n)$ . The Kähler different of  $B$  over  $A$  is the ideal of  $B$  generated by  $\det(\partial f_i / \partial x_j)$ .

**Proof.** This is true because  $\Omega_{B/A}$  has a presentation

$$\bigoplus_{i=1, \dots, n} B f_i \xrightarrow{d} \bigoplus_{j=1, \dots, n} B dx_j \rightarrow \Omega_{B/A} \rightarrow 0$$

by Algebra, Lemma 130.9.  $\square$

### 7. The Dedekind different

0BW0 Let  $A \rightarrow B$  be a ring map. We say *the Dedekind different is defined* if  $A$  is Noetherian,  $A \rightarrow B$  is finite, any nonzerodivisor on  $A$  is a nonzerodivisor on  $B$ , and  $K \rightarrow L$  is étale where  $K = Q(A)$  and  $L = B \otimes_A K$ . Then  $K \subset L$  is finite étale and

$$\mathcal{L}_{B/A} = \{x \in L \mid \text{Trace}_{L/K}(bx) \in A \text{ for all } b \in B\}$$

is the Dedekind complementary module. In this situation the *Dedekind different* is

$$\mathfrak{D}_{B/A} = \{x \in L \mid x\mathcal{L}_{B/A} \subset B\}$$

viewed as a  $B$ -submodule of  $L$ . By Lemma 7.1 the Dedekind different is an ideal of  $B$  either if  $A$  is normal or if  $B$  is flat over  $A$ .

0BW1 **Lemma 7.1.** *Assume the Dedekind different of  $A \rightarrow B$  is defined. Consider the statements*

- (1)  $A \rightarrow B$  is flat,
- (2)  $A$  is a normal ring,
- (3)  $\text{Trace}_{L/K}(B) \subset A$ ,
- (4)  $1 \in \mathcal{L}_{B/A}$ , and
- (5) the Dedekind different  $\mathfrak{D}_{B/A}$  is an ideal of  $B$ .

Then we have (1)  $\Rightarrow$  (3), (2)  $\Rightarrow$  (3), (3)  $\Leftrightarrow$  (4), and (4)  $\Rightarrow$  (5).

**Proof.** The equivalence of (3) and (4) and the implication (4)  $\Rightarrow$  (5) are immediate.

If  $A \rightarrow B$  is flat, then we see that  $\text{Trace}_{B/A} : B \rightarrow A$  is defined and that  $\text{Trace}_{L/K}$  is the base change. Hence (3) holds.

If  $A$  is normal, then  $A$  is a finite product of normal domains, hence we reduce to the case of a normal domain. Then  $K$  is the fraction field of  $A$  and  $L = \prod L_i$  is a finite product of finite separable field extensions of  $K$ . Then  $\text{Trace}_{L/K}(b) = \sum \text{Trace}_{L_i/K}(b_i)$  where  $b_i \in L_i$  is the image of  $b$ . Since  $b$  is integral over  $A$  as  $B$  is finite over  $A$ , these traces are in  $A$ . This is true because the minimal polynomial of  $b_i$  over  $K$  has coefficients in  $A$  (Algebra, Lemma 37.6) and because  $\text{Trace}_{L_i/K}(b_i)$  is an integer multiple of one of these coefficients (Fields, Lemma 20.3).  $\square$

0BW2 **Lemma 7.2.** *If the Dedekind different of  $A \rightarrow B$  is defined, then there is a canonical isomorphism  $\mathcal{L}_{B/A} \rightarrow \omega_{B/A}$ .*

**Proof.** Recall that  $\omega_{B/A} = \text{Hom}_A(B, A)$  as  $A \rightarrow B$  is finite. We send  $x \in \mathcal{L}_{B/A}$  to the map  $b \mapsto \text{Trace}_{L/K}(bx)$ . Conversely, given an  $A$ -linear map  $\varphi : B \rightarrow A$  we obtain a  $K$ -linear map  $\varphi_K : L \rightarrow K$ . Since  $K \rightarrow L$  is finite étale, we see that the trace pairing is nondegenerate (Lemma 3.1) and hence there exists a  $x \in L$  such that  $\varphi_K(y) = \text{Trace}_{L/K}(xy)$  for all  $y \in L$ . Then  $x \in \mathcal{L}_{B/A}$  maps to  $\varphi$  in  $\omega_{B/A}$ .  $\square$

0BW3 **Lemma 7.3.** *If the Dedekind different of  $A \rightarrow B$  is defined and  $A \rightarrow B$  is flat, then*

- (1) the canonical isomorphism  $\mathcal{L}_{B/A} \rightarrow \omega_{B/A}$  sends  $1 \in \mathcal{L}_{B/A}$  to the trace element  $\tau_{B/A} \in \omega_{B/A}$ , and
- (2) the Dedekind different is  $\mathfrak{D}_{B/A} = \{b \in B \mid b\omega_{B/A} \subset B\tau_{B/A}\}$ .

**Proof.** The first assertion follows from the proof of Lemma 7.1 and Lemma 4.3. The second assertion is immediate from the first and the definitions.  $\square$

### 8. The different

0BTC The motivation for the following definition is that it recovers the Dedekind different in the finite flat case as we will see below.

0BW4 **Definition 8.1.** Let  $f : Y \rightarrow X$  be a flat quasi-finite morphism of Noetherian schemes. Let  $\omega_{Y/X}$  be the relative dualizing module and let  $\tau_{Y/X} \in \Gamma(Y, \omega_{Y/X})$  be the trace element (Remarks 2.11 and 4.7). The annihilator of

$$\text{Coker}(\mathcal{O}_Y \xrightarrow{\tau_{Y/X}} \omega_{Y/X})$$

is the *different* of  $Y/X$ . It is a coherent ideal  $\mathfrak{D}_f \subset \mathcal{O}_Y$ .

We will generalize this in Remark 11.2 below. Observe that  $\mathfrak{D}_f$  is locally generated by one element if  $\omega_{Y/X}$  is an invertible  $\mathcal{O}_Y$ -module. We first state the agreement with the Dedekind different.

0BW5 **Lemma 8.2.** *Let  $f : Y \rightarrow X$  be a flat quasi-finite morphism of Noetherian schemes. Let  $V = \text{Spec}(B) \subset Y$ ,  $U = \text{Spec}(A) \subset X$  be affine open subschemes with  $f(V) \subset U$ . If the Dedekind different of  $A \rightarrow B$  is defined, then*

$$\mathfrak{D}_f|_V = \widetilde{\mathfrak{D}_{B/A}}$$

as coherent ideal sheaves on  $V$ .

**Proof.** This is clear from Lemmas 7.1 and 7.3. □

0BW6 **Lemma 8.3.** *Let  $f : Y \rightarrow X$  be a flat quasi-finite morphism of Noetherian schemes. Let  $V = \text{Spec}(B) \subset Y$ ,  $U = \text{Spec}(A) \subset X$  be affine open subschemes with  $f(V) \subset U$ . If  $\omega_{Y/X}|_V$  is invertible, i.e., if  $\omega_{B/A}$  is an invertible  $B$ -module, then*

$$\mathfrak{D}_f|_V = \widetilde{\mathfrak{D}}$$

as coherent ideal sheaves on  $V$  where  $\mathfrak{D} \subset B$  is the Noether different of  $B$  over  $A$ .

**Proof.** Consider the map

$$\text{Hom}_{\mathcal{O}_Y}(\omega_{Y/X}, \mathcal{O}_Y) \longrightarrow \mathcal{O}_Y, \quad \varphi \longmapsto \varphi(\tau_{Y/X})$$

The image of this map corresponds to the Noether different on affine opens, see Lemma 5.7. Hence the result follows from the elementary fact that given an invertible module  $\omega$  and a global section  $\tau$  the image of  $\tau : \text{Hom}(\omega, \mathcal{O}) = \omega^{\otimes -1} \rightarrow \mathcal{O}$  is the same as the annihilator of  $\text{Coker}(\tau : \mathcal{O} \rightarrow \omega)$ . □

0BW7 **Lemma 8.4.** *Consider a cartesian diagram of Noetherian schemes*

$$\begin{array}{ccc} Y' & \longrightarrow & Y \\ f' \downarrow & & \downarrow f \\ X' & \xrightarrow{g} & X \end{array}$$

with  $f$  flat and quasi-finite. Let  $R \subset Y$ , resp.  $R' \subset Y'$  be the closed subscheme cut out by the different  $\mathfrak{D}_f$ , resp.  $\mathfrak{D}_{f'}$ . Then  $Y' \rightarrow Y$  induces a bijective closed immersion  $R' \rightarrow R \times_Y Y'$ . If  $g$  is flat or if  $\omega_{Y/X}$  is invertible, then  $R' = R \times_Y Y'$ .



**Proof.** There is an immediate reduction to the case where  $X, X', Y, Y'$  are affine. In other words, we have a cocartesian diagram of Noetherian rings

$$\begin{array}{ccc} B' & \longleftarrow & B \\ \uparrow & & \uparrow \\ A' & \longleftarrow & A \end{array}$$

with  $A \rightarrow B$  flat and quasi-finite. The base change map  $\omega_{B/A} \otimes_B B' \rightarrow \omega_{B'/A'}$  is an isomorphism (Lemma 2.10) and maps the trace element  $\tau_{B/A}$  to the trace element  $\tau_{B'/A'}$  (Lemma 4.4). Hence the finite  $B$ -module  $Q = \text{Coker}(\tau_{B/A} : B \rightarrow \omega_{B/A})$  satisfies  $Q \otimes_B B' = \text{Coker}(\tau_{B'/A'} : B' \rightarrow \omega_{B'/A'})$ . Thus  $\mathfrak{D}_{B/A} B' \subset \mathfrak{D}_{B'/A'}$  which means we obtain the closed immersion  $R' \rightarrow R \times_Y Y'$ . Since  $R = \text{Supp}(Q)$  and  $R' = \text{Supp}(Q \otimes_B B')$  (Algebra, Lemma 39.5) we see that  $R' \rightarrow R \times_Y Y'$  is bijective by Algebra, Lemma 39.6. The equality  $\mathfrak{D}_{B/A} B' = \mathfrak{D}_{B'/A'}$  holds if  $B \rightarrow B'$  is flat, e.g., if  $A \rightarrow A'$  is flat, see Algebra, Lemma 39.4. Finally, if  $\omega_{B/A}$  is invertible, then we can localize and assume  $\omega_{B/A} = B\lambda$ . Writing  $\tau_{B/A} = b\lambda$  we see that  $Q = B/bB$  and  $\mathfrak{D}_{B/A} = bB$ . The same reasoning over  $B'$  gives  $\mathfrak{D}_{B'/A'} = bB'$  and the lemma is proved.  $\square$

0BW8 **Lemma 8.5.** *Let  $f : Y \rightarrow X$  be a finite flat morphism of Noetherian schemes. Then  $\text{Norm}_f : f_* \mathcal{O}_Y \rightarrow \mathcal{O}_X$  maps  $f_* \mathfrak{D}_f$  into the ideal sheaf of the discriminant  $D_f$ .*

**Proof.** The norm map is constructed in Divisors, Lemma 17.6 and the discriminant of  $f$  in Section 3. The question is affine local, hence we may assume  $X = \text{Spec}(A)$ ,  $Y = \text{Spec}(B)$  and  $f$  given by a finite locally free ring map  $A \rightarrow B$ . Localizing further we may assume  $B$  is finite free as an  $A$ -module. Choose a basis  $b_1, \dots, b_n \in B$  for  $B$  as an  $A$ -module. Denote  $b_1^\vee, \dots, b_n^\vee$  the dual basis of  $\omega_{B/A} = \text{Hom}_A(B, A)$  as an  $A$ -module. Since the norm of  $b$  is the determinant of  $b : B \rightarrow B$  as an  $A$ -linear map, we see that  $\text{Norm}_{B/A}(b) = \det(b_i^\vee(bb_j))$ . The discriminant is the principal closed subscheme of  $\text{Spec}(A)$  defined by  $\det(\text{Trace}_{B/A}(b_i b_j))$ . If  $b \in \mathfrak{D}_{B/A}$  then there exist  $c_i \in B$  such that  $b \cdot b_i^\vee = c_i \cdot \text{Trace}_{B/A}$  where we use a dot to indicate the  $B$ -module structure on  $\omega_{B/A}$ . Write  $c_i = \sum a_{il} b_l$ . We have

$$\begin{aligned} \text{Norm}_{B/A}(b) &= \det(b_i^\vee(bb_j)) \\ &= \det((b \cdot b_i^\vee)(b_j)) \\ &= \det((c_i \cdot \text{Trace}_{B/A})(b_j)) \\ &= \det(\text{Trace}_{B/A}(c_i b_j)) \\ &= \det(a_{il}) \det(\text{Trace}_{B/A}(b_l b_j)) \end{aligned}$$

which proves the lemma.  $\square$

0BW9 **Lemma 8.6.** *Let  $f : Y \rightarrow X$  be a flat quasi-finite morphism of Noetherian schemes. The closed subscheme  $R \subset Y$  defined by the different  $\mathfrak{D}_f$  is exactly the set of points where  $f$  is not étale (equivalently not unramified).*

**Proof.** Since  $f$  is of finite presentation and flat, we see that it is étale at a point if and only if it is unramified at that point. Moreover, the formation of the locus of ramified points commutes with base change. See Morphisms, Section 34 and especially Morphisms, Lemma 34.17. By Lemma 8.4 the formation of  $R$  commutes set theoretically with base change. Hence it suffices to prove the lemma when  $X$  is the spectrum of a field. On the other hand, the construction of  $(\omega_{Y/X}, \tau_{Y/X})$  is

local on  $Y$ . Since  $Y$  is a finite discrete space (being quasi-finite over a field), we may assume  $Y$  has a unique point.

Say  $X = \text{Spec}(k)$  and  $Y = \text{Spec}(B)$  where  $k$  is a field and  $B$  is a finite local  $k$ -algebra. If  $Y \rightarrow X$  is étale, then  $B$  is a finite separable extension of  $k$ , and the trace element  $\text{Trace}_{B/k}$  is a basis element of  $\omega_{B/k}$  by Fields, Lemma 20.7. Thus  $\mathfrak{D}_{B/k} = B$  in this case. Conversely, if  $\mathfrak{D}_{B/k} = B$ , then we see from Lemma 8.5 and the fact that the norm of 1 equals 1 that the discriminant is empty. Hence  $Y \rightarrow X$  is étale by Lemma 3.1.  $\square$

0BWA **Lemma 8.7.** *Let  $f : Y \rightarrow X$  be a flat quasi-finite morphism of Noetherian schemes. Let  $R \subset Y$  be the closed subscheme defined by  $\mathfrak{D}_f$ .*

- (1) *If  $\omega_{Y/X}$  is invertible, then  $R$  is a locally principal closed subscheme of  $Y$ .*
- (2) *If  $\omega_{Y/X}$  is invertible and  $f$  is finite, then the norm of  $R$  is the discriminant  $D_f$  of  $f$ .*
- (3) *If  $\omega_{Y/X}$  is invertible and  $f$  is étale at the associated points of  $Y$ , then  $R$  is an effective Cartier divisor and there is an isomorphism  $\mathcal{O}_Y(R) = \omega_{Y/X}$ .*

**Proof.** Proof of (1). We may work locally on  $Y$ , hence we may assume  $\omega_{Y/X}$  is free of rank 1. Say  $\omega_{Y/X} = \mathcal{O}_Y \lambda$ . Then we can write  $\tau_{Y/X} = h\lambda$  and then we see that  $R$  is defined by  $h$ , i.e.,  $R$  is locally principal.

Proof of (2). We may assume  $Y \rightarrow X$  is given by a finite free ring map  $A \rightarrow B$  and that  $\omega_{B/A}$  is free of rank 1 as  $B$ -module. Choose a  $B$ -basis element  $\lambda$  for  $\omega_{B/A}$  and write  $\text{Trace}_{B/A} = b \cdot \lambda$  for some  $b \in B$ . Then  $\mathfrak{D}_{B/A} = (b)$  and  $D_f$  is cut out by  $\det(\text{Trace}_{B/A}(b_i b_j))$  where  $b_1, \dots, b_n$  is a basis of  $B$  as an  $A$ -module. Let  $b_1^\vee, \dots, b_n^\vee$  be the dual basis. Writing  $b_i^\vee = c_i \cdot \lambda$  we see that  $c_1, \dots, c_n$  is a basis of  $B$  as well. Hence with  $c_i = \sum a_{il} b_l$  we see that  $\det(a_{il})$  is a unit in  $A$ . Clearly,  $b \cdot b_i^\vee = c_i \cdot \text{Trace}_{B/A}$  hence we conclude from the computation in the proof of Lemma 8.5 that  $\text{Norm}_{B/A}(b)$  is a unit times  $\det(\text{Trace}_{B/A}(b_i b_j))$ .

Proof of (3). In the notation above we see from Lemma 8.6 and the assumption that  $h$  does not vanish in the associated points of  $Y$ , which implies that  $h$  is a nonzerodivisor. The canonical isomorphism sends 1 to  $\tau_{Y/X}$ , see Divisors, Lemma 14.10.  $\square$

### 9. Quasi-finite syntomic morphisms

0DWJ This section discusses the fact that a quasi-finite syntomic morphism has an invertible relative dualizing module.

0BWE **Lemma 9.1.** *Let  $f : Y \rightarrow X$  be a morphism of schemes. The following are equivalent*

- (1)  *$f$  is locally quasi-finite and syntomic,*
- (2)  *$f$  is locally quasi-finite, flat, and a local complete intersection morphism,*
- (3)  *$f$  is locally quasi-finite, flat, locally of finite presentation, and the fibres of  $f$  are local complete intersections.*
- (4)  *$f$  is locally quasi-finite and for every  $y \in Y$  there are affine opens  $y \in V = \text{Spec}(B) \subset Y$ ,  $U = \text{Spec}(A) \subset X$  with  $f(V) \subset U$  an integer  $n$  and  $h, f_1, \dots, f_n \in A[x_1, \dots, x_n]$  such that  $B = A[x_1, \dots, x_n, 1/h]/(f_1, \dots, f_n)$ ,*
- (5) *for every  $y \in Y$  there are affine opens  $y \in V = \text{Spec}(B) \subset Y$ ,  $U = \text{Spec}(A) \subset X$  with  $f(V) \subset U$  such that  $A \rightarrow B$  is a relative global complete intersection of the form  $B = A[x_1, \dots, x_n]/(f_1, \dots, f_n)$ .*

**Proof.** The equivalence of (1) and (2) is More on Morphisms, Lemma 51.8. The equivalence of (1) and (3) is Morphisms, Lemma 29.11.

If  $A \rightarrow B$  is as in (4), then  $B = A[x, x_1, \dots, x_n]/(xh - 1, f_1, \dots, f_n)$  is a relative global complete intersection by see Algebra, Definition 134.5. Thus (4) implies (5). It is clear that (5) implies (4).

Condition (5) implies (1): by Algebra, Lemma 134.14 a relative global complete intersection is syntomic and the definition of a relative global complete intersection guarantees that a relative global complete intersection on  $n$  variables with  $n$  equations is quasi-finite, see Algebra, Definition 134.5 and Lemma 121.2.

Finally, either Algebra, Lemma 134.15 or Morphisms, Lemma 29.10 shows that (1) implies (5).  $\square$

0DWK **Lemma 9.2.** *Invertibility of the relative dualizing module.*

- (1) *If  $A \rightarrow B$  is a quasi-finite flat homomorphism of Noetherian rings, then  $\omega_{B/A}$  is an invertible  $B$ -module if and only if  $\omega_{B \otimes_A \kappa(\mathfrak{p})/\kappa(\mathfrak{p})}$  is an invertible  $B \otimes_A \kappa(\mathfrak{p})$ -module for all primes  $\mathfrak{p} \subset A$ .*
- (2) *If  $Y \rightarrow X$  is a quasi-finite flat morphism of Noetherian schemes, then  $\omega_{Y/X}$  is invertible if and only if  $\omega_{Y_x/x}$  is invertible for all  $x \in X$ .*

**Proof.** Proof of (1). As  $A \rightarrow B$  is flat, the module  $\omega_{B/A}$  is  $A$ -flat, see Lemma 2.9. Thus  $\omega_{B/A}$  is an invertible  $B$ -module if and only if  $\omega_{B/A} \otimes_A \kappa(\mathfrak{p})$  is an invertible  $B \otimes_A \kappa(\mathfrak{p})$ -module for every prime  $\mathfrak{p} \subset A$ , see More on Morphisms, Lemma 16.7. Still using that  $A \rightarrow B$  is flat, we have that formation of  $\omega_{B/A}$  commutes with base change, see Lemma 2.10. Thus we see that invertibility of the relative dualizing module, in the presence of flatness, is equivalent to invertibility of the relative dualizing module for the maps  $\kappa(\mathfrak{p}) \rightarrow B \otimes_A \kappa(\mathfrak{p})$ .

Part (2) follows from (1) and the fact that affine locally the dualizing modules are given by their algebraic counterparts, see Remark 2.11.  $\square$

0DWL **Lemma 9.3.** *Let  $k$  be a field. Let  $B = k[x_1, \dots, x_n]/(f_1, \dots, f_n)$  be a global complete intersection over  $k$  of dimension 0. Then  $\omega_{B/k}$  is invertible.*

**Proof.** By Noether normalization, see Algebra, Lemma 114.4 we see that there exists a finite injection  $k \rightarrow B$ , i.e.,  $\dim_k(B) < \infty$ . Hence  $\omega_{B/k} = \text{Hom}_k(B, k)$  as a  $B$ -module. By Dualizing Complexes, Lemma 15.8 we see that  $R\text{Hom}(B, k)$  is a dualizing complex for  $B$  and by Dualizing Complexes, Lemma 13.3 we see that  $R\text{Hom}(B, k)$  is equal to  $\omega_{B/k}$  placed in degree 0. Thus it suffices to show that  $B$  is Gorenstein (Dualizing Complexes, Lemma 21.4). This is true by Dualizing Complexes, Lemma 21.7.  $\square$

0BWF **Lemma 9.4.** *Let  $f : Y \rightarrow X$  be a morphism of Noetherian schemes. If  $f$  satisfies the equivalent conditions of Lemma 9.1 then  $\omega_{Y/X}$  is an invertible  $\mathcal{O}_Y$ -module.*

**Proof.** We may assume  $A \rightarrow B$  is a relative global complete intersection of the form  $B = A[x_1, \dots, x_n]/(f_1, \dots, f_n)$  and we have to show  $\omega_{B/A}$  is invertible. This follows in combining Lemmas 9.2 and 9.3.  $\square$

### 10. A formula for the different

0BWB In this section we discuss the material in [MR70, Appendix A] due to Tate. In our language, this will show that the different is equal to the Kähler different in the case of a flat, quasi-finite, local complete intersection morphism. First we compute the Noether different in a special case.

0BWC **Lemma 10.1.** *Let  $A \rightarrow P$  be a ring map. Let  $f_1, \dots, f_n \in P$  be a Koszul regular sequence. Assume  $B = P/(f_1, \dots, f_n)$  is flat over  $A$ . Let  $g_1, \dots, g_n \in P \otimes_A B$  be a Koszul regular sequence generating the kernel of the multiplication map  $P \otimes_A B \rightarrow B$ . Write  $f_i \otimes 1 = \sum g_{ij} g_j$ . Then the annihilator of  $\text{Ker}(B \otimes_A B \rightarrow B)$  is a principal ideal generated by the image of  $\det(g_{ij})$ .* [MR70, Appendix]

**Proof.** The Koszul complex  $K_\bullet = K(P, f_1, \dots, f_n)$  is a resolution of  $B$  by finite free  $P$ -modules. The Koszul complex  $M_\bullet = K(P \otimes_A B, g_1, \dots, g_n)$  is a resolution of  $B$  by finite free  $P \otimes_A B$ -modules. There is a map of complexes

$$K_\bullet \longrightarrow M_\bullet$$

which in degree 1 is given by the matrix  $(g_{ij})$  and in degree  $n$  by  $\det(g_{ij})$ . See More on Algebra, Lemma 26.3. As  $B$  is a flat  $A$ -module, we can view  $M_\bullet$  as a complex of flat  $P$ -modules (via  $P \rightarrow P \otimes_A B, p \mapsto p \otimes 1$ ). Thus we may use both complexes to compute  $\text{Tor}_*^P(B, B)$  and it follows that the displayed map defines a quasi-isomorphism after tensoring with  $B$ . It is clear that  $H_n(K_\bullet \otimes_P B) = B$ . On the other hand,  $H_n(M_\bullet \otimes_P B)$  is the kernel of

$$B \otimes_A B \xrightarrow{g_1, \dots, g_n} (B \otimes_A B)^{\oplus n}$$

Since  $g_1, \dots, g_n$  generate the kernel of  $B \otimes_A B \rightarrow B$  this proves the lemma.  $\square$

0BWD **Lemma 10.2.** *Let  $A$  be a ring. Let  $n \geq 1$  and  $h, f_1, \dots, f_n \in A[x_1, \dots, x_n]$ . Set  $B = A[x_1, \dots, x_n, 1/h]/(f_1, \dots, f_n)$ . Assume that  $B$  is quasi-finite over  $A$ . Then*

- (1)  $B$  is flat over  $A$  and  $A \rightarrow B$  is a relative local complete intersection,
- (2) the annihilator  $J$  of  $I = \text{Ker}(B \otimes_A B \rightarrow B)$  is free of rank 1 over  $B$ ,
- (3) the Noether different of  $B$  over  $A$  is generated by  $\det(\partial f_i / \partial x_j)$  in  $B$ .

**Proof.** Note that  $B = A[x, x_1, \dots, x_n]/(xh - 1, f_1, \dots, f_n)$  is a relative global complete intersection over  $A$ , see Algebra, Definition 134.5. By Algebra, Lemma 134.14 we see that  $B$  is flat over  $A$ .

Write  $P' = A[x, x_1, \dots, x_n]$  and  $P = P'/(xh - 1) = A[x_1, \dots, x_n, 1/g]$ . Then we have  $P' \rightarrow P \rightarrow B$ . By More on Algebra, Lemma 30.4 we see that  $xh - 1, f_1, \dots, f_n$  is a Koszul regular sequence in  $P'$ . Since  $xh - 1$  is a Koszul regular sequence of length one in  $P'$  (by the same lemma for example) we conclude that  $f_1, \dots, f_n$  is a Koszul regular sequence in  $P$  by More on Algebra, Lemma 27.14.

Let  $g_i \in P \otimes_A B$  be the image of  $x_i \otimes 1 - 1 \otimes x_i$ . Let us use the short hand  $y_i = x_i \otimes 1$  and  $z_i = 1 \otimes x_i$  in  $A[x_1, \dots, x_n] \otimes_A A[x_1, \dots, x_n]$  so that  $g_i$  is the image of  $y_i - z_i$ . For a polynomial  $f \in A[x_1, \dots, x_n]$  we write  $f(y) = f \otimes 1$  and  $f(z) = 1 \otimes f$  in the above tensor product. Then we have

$$P \otimes_A B / (g_1, \dots, g_n) = \frac{A[y_1, \dots, y_n, z_1, \dots, z_n, \frac{1}{h(y)h(z)}]}{(f_1(z), \dots, f_n(z), y_1 - z_1, \dots, y_n - z_n)}$$

which is clearly isomorphic to  $B$ . Hence by the same arguments as above we find that  $f_1(z), \dots, f_n(z), y_1 - z_1, \dots, y_n - z_n$  is a Koszul regular sequence in  $A[y_1, \dots, y_n, z_1, \dots, z_n, \frac{1}{h(y)h(z)}]$ .

The sequence  $f_1(z), \dots, f_n(z)$  is a Koszul regular in  $A[y_1, \dots, y_n, z_1, \dots, z_n, \frac{1}{h(y)h(z)}]$  by flatness of the map

$$P \longrightarrow A[y_1, \dots, y_n, z_1, \dots, z_n, \frac{1}{h(y)h(z)}], \quad x_i \longmapsto z_i$$

and More on Algebra, Lemma 27.5. By More on Algebra, Lemma 27.14 we conclude that  $g_1, \dots, g_n$  is a regular sequence in  $P \otimes_A B$ .

At this point we have verified all the assumptions of Lemma 10.1 above with  $P$ ,  $f_1, \dots, f_n$ , and  $g_i \in P \otimes_A B$  as above. In particular the annihilator  $J$  of  $I$  is freely generated by one element  $\delta$  over  $B$ . Set  $f_{ij} = \partial f_i / \partial x_j \in A[x_1, \dots, x_n]$ . An elementary computation shows that we can write

$$f_i(y) = f_i(z_1 + g_1, \dots, z_n + g_n) = f_i(z) + \sum_j f_{ij}(z)g_j + \sum_{j,j'} F_{ijj'}g_jg_{j'}$$

for some  $F_{ijj'} \in A[y_1, \dots, y_n, z_1, \dots, z_n]$ . Taking the image in  $P \otimes_A B$  the terms  $f_i(z)$  map to zero and we obtain

$$f_i \otimes 1 = \sum_j \left( 1 \otimes f_{ij} + \sum_{j'} F_{ijj'}g_{j'} \right) g_j$$

Thus we conclude from Lemma 10.1 that  $\delta = \det(g_{ij})$  with  $g_{ij} = 1 \otimes f_{ij} + \sum_{j'} F_{ijj'}g_{j'}$ . Since  $g_{j'}$  maps to zero in  $B$ , we conclude that the image of  $\det(\partial f_i / \partial x_j)$  in  $B$  generates the Noether different of  $B$  over  $A$ .  $\square$

0BWG **Lemma 10.3.** *Let  $f : Y \rightarrow X$  be a morphism of Noetherian schemes. If  $f$  satisfies the equivalent conditions of Lemma 9.1 then the different  $\mathfrak{D}_f$  of  $f$  is the Kähler different of  $f$ .*

**Proof.** By Lemmas 8.3 and 9.4 the different of  $f$  affine locally is the same as the Noether different. Then the lemma follows from the computation of the Noether different and the Kähler different on standard affine pieces done in Lemmas 6.4 and 10.2.  $\square$

0BWH **Lemma 10.4.** *Let  $A$  be a ring. Let  $n \geq 1$  and  $h, f_1, \dots, f_n \in A[x_1, \dots, x_n]$ . Set  $B = A[x_1, \dots, x_n, 1/h]/(f_1, \dots, f_n)$ . Assume that  $B$  is quasi-finite over  $A$ . Then there is an isomorphism  $B \rightarrow \omega_{B/A}$  mapping  $\det(\partial f_i / \partial x_j)$  to  $\tau_{B/A}$ .*

**Proof.** Let  $J$  be the annihilator of  $\text{Ker}(B \otimes_A B \rightarrow B)$ . By Lemma 10.2 the map  $A \rightarrow B$  is flat and  $J$  is a free  $B$ -module with generator  $\xi$  mapping to  $\det(\partial f_i / \partial x_j)$  in  $B$ . Thus the lemma follows from Lemma 5.7 and the fact (Lemma 9.4) that  $\omega_{B/A}$  is an invertible  $B$ -module. (Warning: it is necessary to prove  $\omega_{B/A}$  is invertible because a finite  $B$ -module  $M$  such that  $\text{Hom}_B(M, B) \cong B$  need not be free.)  $\square$

0BWI **Example 10.5.** Let  $A$  be a Noetherian ring. Let  $f, h \in A[x]$  such that

$$B = (A[x]/(f))_h = A[x, 1/h]/(f)$$

is quasi-finite over  $A$ . Let  $f' \in A[x]$  be the derivative of  $f$  with respect to  $x$ . The ideal  $\mathfrak{D} = (f') \subset B$  is the Noether different of  $B$  over  $A$ , is the Kähler different of  $B$  over  $A$ , and is the ideal whose associated quasi-coherent sheaf of ideals is the different of  $\text{Spec}(B)$  over  $\text{Spec}(A)$ .

0BWJ **Lemma 10.6.** *Let  $S$  be a Noetherian scheme. Let  $X, Y$  be smooth schemes of relative dimension  $n$  over  $S$ . Let  $f : Y \rightarrow X$  be a quasi-finite morphism over  $S$ .*

Then  $f$  is flat and the closed subscheme  $R \subset Y$  cut out by the different of  $f$  is the locally principal closed subscheme cut out by

$$\wedge^n(df) \in \Gamma(Y, (f^*\Omega_{X/S}^n)^{\otimes -1} \otimes_{\mathcal{O}_Y} \Omega_{Y/S}^n)$$

If  $f$  is étale at the associated points of  $Y$ , then  $R$  is an effective Cartier divisor and

$$f^*\Omega_{X/S}^n \otimes_{\mathcal{O}_Y} \mathcal{O}(R) = \Omega_{Y/S}^n$$

as invertible sheaves on  $Y$ .

**Proof.** To prove that  $f$  is flat, it suffices to prove  $Y_s \rightarrow X_s$  is flat for all  $s \in S$  (More on Morphisms, Lemma 16.3). Flatness of  $Y_s \rightarrow X_s$  follows from Algebra, Lemma 127.1. By More on Morphisms, Lemma 51.10 the morphism  $f$  is a local complete intersection morphism. Thus the statement on the different follows from the corresponding statement on the Kähler different by Lemma 10.3. Finally, since we have the exact sequence

$$f^*\Omega_{X/S} \xrightarrow{df} \Omega_{X/S} \rightarrow \Omega_{Y/X} \rightarrow 0$$

by Morphisms, Lemma 31.9 and since  $\Omega_{X/S}$  and  $\Omega_{Y/S}$  are finite locally free of rank  $n$  (Morphisms, Lemma 32.12), the statement for the Kähler different is clear from the definition of the zeroth fitting ideal. If  $f$  is étale at the associated points of  $Y$ , then  $\wedge^n df$  does not vanish in the associated points of  $Y$ , which implies that the local equation of  $R$  is a nonzerodivisor. Hence  $R$  is an effective Cartier divisor. The canonical isomorphism sends 1 to  $\wedge^n df$ , see Divisors, Lemma 14.10.  $\square$

## 11. A generalization of the different

0BWK In this section we generalize Definition 8.1 to take into account all cases of ring maps  $A \rightarrow B$  where the Dedekind different is defined and  $1 \in \mathcal{L}_{B/A}$ . First we explain the condition “ $A \rightarrow B$  maps nonzerodivisors to nonzerodivisors and induces a flat map  $Q(A) \rightarrow Q(A) \otimes_A B$ ”.

0BWL **Lemma 11.1.** *Let  $A \rightarrow B$  be a map of Noetherian rings. Consider the conditions*

- (1) *nonzerodivisors of  $A$  map to nonzerodivisors of  $B$ ,*
- (2) *(1) holds and  $Q(A) \rightarrow Q(A) \otimes_A B$  is flat,*
- (3)  *$A \rightarrow B_{\mathfrak{q}}$  is flat for every  $\mathfrak{q} \in \text{Ass}(B)$ ,*
- (4) *(3) holds and  $A \rightarrow B_{\mathfrak{q}}$  is flat for every  $\mathfrak{q}$  lying over an element in  $\text{Ass}(A)$ .*

*Then we have the following implications*

$$\begin{array}{ccc} (1) & \longleftarrow & (2) \\ \uparrow & & \downarrow \\ (3) & \longleftarrow & (4) \end{array}$$

*If going up holds for  $A \rightarrow B$  then (2) and (4) are equivalent.*

**Proof.** The horizontal implications in the diagram are trivial. Let  $S \subset A$  be the set of nonzerodivisors so that  $Q(A) = S^{-1}A$  and  $Q(A) \otimes_A B = S^{-1}B$ . Recall that  $S = A \setminus \bigcup_{\mathfrak{p} \in \text{Ass}(A)} \mathfrak{p}$  by Algebra, Lemma 62.9. Let  $\mathfrak{q} \subset B$  be a prime lying over  $\mathfrak{p} \subset A$ .

Assume (2). If  $\mathfrak{q} \in \text{Ass}(B)$  then  $\mathfrak{q}$  consists of zerodivisors, hence (1) implies the same is true for  $\mathfrak{p}$ . Hence  $\mathfrak{p}$  corresponds to a prime of  $S^{-1}A$ . Hence  $A \rightarrow B_{\mathfrak{q}}$  is

flat by our assumption (2). If  $\mathfrak{q}$  lies over an associated prime  $\mathfrak{p}$  of  $A$ , then certainly  $\mathfrak{p} \in \text{Spec}(S^{-1}A)$  and the same argument works.

Assume (3). Let  $f \in A$  be a nonzerodivisor. If  $f$  were a zerodivisor on  $B$ , then  $f$  is contained in an associated prime  $\mathfrak{q}$  of  $B$ . Since  $A \rightarrow B_{\mathfrak{q}}$  is flat by assumption, we conclude that  $\mathfrak{p}$  is an associated prime of  $A$  by Algebra, Lemma 64.3. This would imply that  $f$  is a zerodivisor on  $A$ , a contradiction.

Assume (4) and going up for  $A \rightarrow B$ . We already know (1) holds. If  $\mathfrak{q}$  corresponds to a prime of  $S^{-1}B$  then  $\mathfrak{p}$  is contained in an associated prime  $\mathfrak{p}'$  of  $A$ . By going up there exists a prime  $\mathfrak{q}'$  containing  $\mathfrak{q}$  and lying over  $\mathfrak{p}$ . Then  $A \rightarrow B_{\mathfrak{q}'}$  is flat by (4). Hence  $A \rightarrow B_{\mathfrak{q}}$  is flat as a localization. Thus  $A \rightarrow S^{-1}B$  is flat and so is  $S^{-1}A \rightarrow S^{-1}B$ , see Algebra, Lemma 38.19.  $\square$

OBWM **Remark 11.2.** We can generalize Definition 8.1. Suppose that  $f : Y \rightarrow X$  is a quasi-finite morphism of Noetherian schemes with the following properties

- (1) the open  $V \subset Y$  where  $f$  is flat contains  $\text{Ass}(\mathcal{O}_Y)$  and  $f^{-1}(\text{Ass}(\mathcal{O}_X))$ ,
- (2) the trace element  $\tau_{V/X}$  comes from a section  $\tau \in \Gamma(Y, \omega_{Y/X})$ .

Condition (1) implies that  $V$  contains the associated points of  $\omega_{Y/X}$  by Lemma 2.8. In particular,  $\tau$  is unique if it exists (Divisors, Lemma 2.8). Given  $\tau$  we can define the different  $\mathfrak{D}_f$  as the annihilator of  $\text{Coker}(\tau : \mathcal{O}_Y \rightarrow \omega_{Y/X})$ . This agrees with the Dedekind different in many cases (Lemma 11.3). However, for non-flat maps between non-normal rings, this generalization no longer measures ramification of the morphism, see Example 11.4.

OBWN **Lemma 11.3.** *Assume the Dedekind different is defined for  $A \rightarrow B$ . Set  $X = \text{Spec}(A)$  and  $Y = \text{Spec}(B)$ . The generalization of Remark 11.2 applies to the morphism  $f : Y \rightarrow X$  if and only if  $1 \in \mathcal{L}_{B/A}$  (e.g., if  $A$  is normal, see Lemma 7.1). In this case  $\mathfrak{D}_{B/A}$  is an ideal of  $B$  and we have*

$$\mathfrak{D}_f = \widetilde{\mathfrak{D}_{B/A}}$$

as coherent ideal sheaves on  $Y$ .

**Proof.** As the Dedekind different for  $A \rightarrow B$  is defined we can apply Lemma 11.1 to see that  $Y \rightarrow X$  satisfies condition (1) of Remark 11.2. Recall that there is a canonical isomorphism  $c : \mathcal{L}_{B/A} \rightarrow \omega_{B/A}$ , see Lemma 7.2. Let  $K = Q(A)$  and  $L = K \otimes_A B$  as above. By construction the map  $c$  fits into a commutative diagram

$$\begin{array}{ccc} \mathcal{L}_{B/A} & \longrightarrow & L \\ c \downarrow & & \downarrow \\ \omega_{B/A} & \longrightarrow & \text{Hom}_K(L, K) \end{array}$$

where the right vertical arrow sends  $x \in L$  to the map  $y \mapsto \text{Trace}_{L/K}(xy)$  and the lower horizontal arrow is the base change map (2.3.1) for  $\omega_{B/A}$ . We can factor the lower horizontal map as

$$\omega_{B/A} = \Gamma(Y, \omega_{Y/X}) \rightarrow \Gamma(V, \omega_{V/X}) \rightarrow \text{Hom}_K(L, K)$$

Since all associated points of  $\omega_{V/X}$  map to associated primes of  $A$  (Lemma 2.8) we see that the second map is injective. The element  $\tau_{V/X}$  maps to  $\text{Trace}_{L/K}$  in  $\text{Hom}_K(L, K)$  by the very definition of trace elements (Definition 4.1). Thus  $\tau$  as in condition (2) of Remark 11.2 exists if and only if  $1 \in \mathcal{L}_{B/A}$  and then  $\tau = c(1)$ . In

this case, by Lemma 7.1 we see that  $\mathfrak{D}_{B/A} \subset B$ . Finally, the agreement of  $\mathfrak{D}_f$  with  $\mathfrak{D}_{B/A}$  is immediate from the definitions and the fact  $\tau = c(1)$  seen above.  $\square$

0BWP **Example 11.4.** Let  $k$  be a field. Let  $A = k[x, y]/(xy)$  and  $B = k[u, v]/(uv)$  and let  $A \rightarrow B$  be given by  $x \mapsto u^n$  and  $y \mapsto v^m$  for some  $n, m \in \mathbf{N}$  prime to the characteristic of  $k$ . Then  $A_{x+y} \rightarrow B_{x+y}$  is (finite) étale hence we are in the situation where the Dedekind different is defined. A computation shows that

$$\text{Trace}_{L/K}(1) = (nx + my)/(x + y), \quad \text{Trace}_{L/K}(u^i) = 0, \quad \text{Trace}_{L/K}(v^j) = 0$$

for  $1 \leq i < n$  and  $1 \leq j < m$ . We conclude that  $1 \in \mathcal{L}_{B/A}$  if and only if  $n = m$ . Moreover, a computation shows that if  $n = m$ , then  $\mathcal{L}_{B/A} = B$  and the Dedekind different is  $B$  as well. In other words, we find that the different of Remark 11.2 is defined for  $\text{Spec}(B) \rightarrow \text{Spec}(A)$  if and only if  $n = m$ , and in this case the different is the unit ideal. Thus we see that in nonflat cases the nonvanishing of the different does not guarantee the morphism is étale or unramified.

## 12. Comparison with duality theory

0DWM In this section we compare the elementary algebraic constructions above with the constructions in the chapter on duality theory for schemes.

0BUL **Lemma 12.1.** *Let  $f : Y \rightarrow X$  be a quasi-finite separated morphism of Noetherian schemes. For every pair of affine opens  $\text{Spec}(B) = V \subset Y$ ,  $\text{Spec}(A) = U \subset X$  with  $f(V) \subset U$  there is an isomorphism*

$$H^0(V, f^! \mathcal{O}_Y) = \omega_{B/A}$$

where  $f^!$  is as in Duality for Schemes, Section 17. These isomorphisms are compatible with restriction maps and define a canonical isomorphism  $H^0(f^! \mathcal{O}_X) = \omega_{Y/X}$  with  $\omega_{Y/X}$  as in Remark 2.11. Similarly, if  $f : Y \rightarrow X$  lives over a Noetherian base  $S$  endowed with a dualizing complex  $\omega_S^\bullet$ , then  $H^0(f_{new}^! \mathcal{O}_X) = \omega_{Y/X}$ .

**Proof.** By Zariski's main theorem we can choose a factorization  $f = f' \circ j$  where  $j : Y \rightarrow Y'$  is an open immersion and  $f' : Y' \rightarrow X$  is a finite morphism, see More on Morphisms, Lemma 38.3. Thus  $f$  is compactifiable and  $f^!$  is defined, see Duality for Schemes, Section 17. In fact, by our construction in Duality for Schemes, Lemma 17.1 we have  $f^! = j^* \circ a'$  where  $a' : D_{QCoh}(\mathcal{O}_X) \rightarrow D_{QCoh}(\mathcal{O}_{Y'})$  is the right adjoint to  $Rf'_*$  of Duality for Schemes, Lemma 3.1. By Duality for Schemes, Lemma 11.4 we see that  $\Phi(a'(\mathcal{O}_X)) = R\mathcal{H}om(f'_* \mathcal{O}_{Y'}, \mathcal{O}_X)$  in  $D_{QCoh}^+(f'_* \mathcal{O}_{Y'})$ . In particular  $a'(\mathcal{O}_X)$  has vanishing cohomology sheaves in degrees  $< 0$ . The zeroth cohomology sheaf is determined by the isomorphism

$$f'_* H^0(a'(\mathcal{O}_X)) = \mathcal{H}om_{\mathcal{O}_X}(f'_* \mathcal{O}_{Y'}, \mathcal{O}_X)$$

as  $f'_* \mathcal{O}_{Y'}$ -modules via the equivalence of Morphisms, Lemma 11.6. Writing  $(f')^{-1}U = V' = \text{Spec}(B')$ , we obtain

$$H^0(V', a'(\mathcal{O}_X)) = \text{Hom}_A(B', A).$$

As the zeroth cohomology sheaf of  $a'(\mathcal{O}_X)$  is a quasi-coherent module we find that the restriction to  $V$  is given by  $\omega_{B/A} = \text{Hom}_A(B', A) \otimes_{B'} B$  as desired.

The statement about restriction maps signifies that the restriction mappings of the quasi-coherent  $\mathcal{O}_{Y'}$ -module  $H^0(a'(\mathcal{O}_X))$  for opens in  $Y'$  agrees with the maps



defined in Lemma 2.3 for the modules  $\omega_{B/A}$  via the isomorphisms given above. This is clear.

The result for  $f_{new}^!$  follows from this result as we've seen that  $f_{new}^!$  is equal to the right adjoint to  $f_*$  by Duality for Schemes, Lemma 21.9.  $\square$

0BVI **Lemma 12.2.** *Let  $f : Y \rightarrow X$  be a finite flat morphism of Noetherian schemes. The map*

$$\text{Trace}_f : f_* \mathcal{O}_Y \longrightarrow \mathcal{O}_X$$

*of Section 3 corresponds to a map  $\mathcal{O}_Y \rightarrow f^! \mathcal{O}_X$ . Denote  $\tau_{Y/X} \in H^0(Y, f^! \mathcal{O}_X)$  the image of 1. Via the isomorphism  $H^0(f^! \mathcal{O}_X) = \omega_{X/Y}$  of Lemma 12.1 this agrees with the construction in Remark 4.7.*

**Proof.** Unwinding all the definitions, this is immediate from the fact that if  $A \rightarrow B$  is finite flat, then  $\tau_{B/A} = \text{Trace}_{B/A}$  (Lemma 4.3) and the compatibility of traces with localizations (Lemma 4.4).  $\square$

### 13. Quasi-finite Gorenstein morphisms

0C14 This section discusses quasi-finite Gorenstein morphisms.

0C16 **Lemma 13.1.** *Let  $f : Y \rightarrow X$  be a quasi-finite morphism of Noetherian schemes. The following are equivalent*

- (1)  *$f$  is Gorenstein,*
- (2)  *$f$  is flat and the fibres of  $f$  are Gorenstein,*
- (3)  *$f$  is flat and  $\omega_{Y/X}$  is invertible (Remark 2.11),*
- (4) *for every  $y \in Y$  there are affine opens  $y \in V = \text{Spec}(B) \subset Y$ ,  $U = \text{Spec}(A) \subset X$  with  $f(V) \subset U$  such that  $A \rightarrow B$  is flat and  $\omega_{B/A}$  is an invertible  $B$ -module.*

**Proof.** Parts (1) and (2) are equivalent by definition. Parts (3) and (4) are equivalent by the construction of  $\omega_{Y/X}$  in Remark 2.11. Thus we have to show that (1)-(2) is equivalent to (3)-(4).

First proof. Working affine locally we can assume  $f$  is a separated morphism and apply Lemma 12.1 to see that  $\omega_{Y/X}$  is the zeroth cohomology sheaf of  $f^! \mathcal{O}_X$ . Under both assumptions  $f$  is flat and quasi-finite, hence  $f^! \mathcal{O}_X$  is isomorphic to  $\omega_{Y/X}[0]$ , see Duality for Schemes, Lemma 22.6. Hence the equivalence follows from Duality for Schemes, Lemma 26.10.

Second proof. By Lemma 9.2, we see that it suffices to prove the equivalence of (2) and (3) when  $X$  is the spectrum of a field  $k$ . Then  $Y = \text{Spec}(B)$  where  $B$  is a finite  $k$ -algebra. In this case  $\omega_{B/A} = \omega_{B/k} = \text{Hom}_k(B, k)$  placed in degree 0 is a dualizing complex for  $B$ , see Dualizing Complexes, Lemma 15.8. Thus the equivalence follows from Dualizing Complexes, Lemma 21.4.  $\square$

0C17 **Remark 13.2.** *Let  $f : Y \rightarrow X$  be a quasi-finite Gorenstein morphism of Noetherian schemes. Let  $\mathfrak{D}_f \subset \mathcal{O}_Y$  be the different and let  $R \subset Y$  be the closed subscheme cut out by  $\mathfrak{D}_f$ . Then we have*

- (1)  *$\mathfrak{D}_f$  is a locally principal ideal,*
- (2)  *$R$  is a locally principal closed subscheme,*
- (3)  *$\mathfrak{D}_f$  is affine locally the same as the Noether different,*
- (4) *formation of  $R$  commutes with base change,*

- (5) if  $f$  is finite, then the norm of  $R$  is the discriminant of  $f$ , and
- (6) if  $f$  is étale in the associated points of  $Y$ , then  $R$  is an effective Cartier divisor and  $\omega_{Y/X} = \mathcal{O}_Y(R)$ .

This follows from Lemmas 8.3, 8.4, and 8.7.

0C18 **Remark 13.3.** Let  $S$  be a Noetherian scheme endowed with a dualizing complex  $\omega_S^\bullet$ . Let  $f : Y \rightarrow X$  be a quasi-finite Gorenstein morphism of compactifiable schemes over  $S$ . Assume moreover  $Y$  and  $X$  Cohen-Macaulay and  $f$  étale at the generic points of  $Y$ . Then we can combine Duality for Schemes, Remark 24.4 and Remark 13.2 to see that we have a canonical isomorphism

$$\omega_Y = f^*\omega_X \otimes_{\mathcal{O}_Y} \omega_{Y/X} = f^*\omega_X \otimes_{\mathcal{O}_Y} \mathcal{O}_Y(R)$$

of  $\mathcal{O}_Y$ -modules. If further  $f$  is finite, then the isomorphism  $\mathcal{O}_Y(R) = \omega_{Y/X}$  comes from the global section  $\tau_{Y/X} \in H^0(Y, \omega_{Y/X})$  which corresponds via duality to the map  $\text{Trace}_f : f_*\mathcal{O}_Y \rightarrow \mathcal{O}_X$ , see Lemma 12.2.

## 14. Other chapters

Preliminaries	(30) Divisors
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(2) Conventions	(32) Varieties
(3) Set Theory	(33) Topologies on Schemes
(4) Categories	(34) Descent
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