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### 1. Introduction

In this chapter we study the different and discriminant of locally quasi-finite morphisms of schemes. A good reference for some of this material is \[\text{[Kun86]}\].

Given a quasi-finite morphism \(f : Y \rightarrow X\) of Noetherian schemes there is a relative dualizing module \(\omega_{Y/X}\). In Section 2 we construct this module from scratch, using Zariski’s main theorem and étale localization methods. The key property is that given a diagram

\[
\begin{array}{ccc}
Y' & \xrightarrow{g'} & Y \\
\downarrow{f'} & & \downarrow{f} \\
X' & \xrightarrow{g} & X
\end{array}
\]

with \(g : X' \rightarrow X\) flat, \(Y' \subset X' \times_X Y\) open, and \(f' : Y' \rightarrow X'\) finite, then there is a canonical isomorphism

\[f'_*(g')^*\omega_{Y/X} = \mathcal{H}om_{\mathcal{O}_{Y'}}(f'_!\mathcal{O}_{Y'}, \mathcal{O}_{X'})\]
as sheaves of $f'_*\mathcal{O}_{Y'}$-modules. In Section 3 we prove that if $f$ is flat, then there is a canonical global section $\tau_{Y/X} \in H^0(Y, \omega_{Y/X})$ which for every commutative diagram as above maps $(g')^*\tau_{Y/X}$ to the trace map of Section 3 for the finite locally free morphism $f'$. In Section 9 we define the different for a flat quasi-finite morphism of Noetherian schemes as the annihilator of the cokernel of $\tau_{Y/X} : \mathcal{O}_X \to \omega_{Y/X}$.

The main goal of this chapter is to prove that for quasi-finite syntomic $^1f$ the different agrees with the Kähler different. The Kähler different is the zeroth fitting ideal of $\Omega_{Y/X}$, see Section 7. This agreement is not obvious; we use a slick argument due to Tate, see Section 12. On the way we also discuss the Noether different and the Dedekind different.

Only in the end of this chapter, see Sections 15 and 16, do we make the link with the more advanced material on duality for schemes.

2. Dualizing modules for quasi-finite ring maps

Let $A \to B$ be a quasi-finite homomorphism of Noetherian rings. By Zariski’s main theorem (Algebra, Lemma 122.14) there exists a factorization $A \to B' \to B$ with $A \to B'$ finite and $B' \to B$ inducing an open immersion of spectra. We set

$$\omega_{B/A} = \text{Hom}_A(B', A) \otimes_B B$$

in this situation. The reader can think of this as a kind of relative dualizing module, see Lemmas 15.1 and 2.12. In this section we will show by elementary commutative algebra methods that $\omega_{B/A}$ is independent of the choice of the factorization and that formation of $\omega_{B/A}$ commutes with flat base change. To help prove the independence of factorizations we compare two given factorizations.

**Lemma 2.1.** Let $A \to B$ be a quasi-finite ring map. Given two factorizations $A \to B'' \to B$ and $A \to B'' \to B$ with $A \to B'$ and $A \to B''$ finite and $\text{Spec}(B') \to \text{Spec}(B)$ open immersions, there exists an $A$-subalgebra $B'' \subset B$ finite over $A$ such that $\text{Spec}(B) \to \text{Spec}(B'')$ an open immersion and $B' \to B$ and $B'' \to B$ factor through $B''$.

**Proof.** Let $B'' \subset B$ be the $A$-subalgebra generated by the images of $B' \to B$ and $B'' \to B$. As $B'$ and $B''$ are each generated by finitely many elements integral over $A$, we see that $B''$ is generated by finitely many elements integral over $A$ and we conclude that $B''$ is finite over $A$ (Algebra, Lemma 35.5). Consider the maps

$$B = B' \otimes_{B'} B \to B'' \otimes_{B'} B \to B \otimes_{B'} B = B$$

The final equality holds because $\text{Spec}(B) \to \text{Spec}(B')$ is an open immersion (and hence a monomorphism). The second arrow is injective as $B' \to B$ is flat. Hence both arrows are isomorphisms. This means that

$$\text{Spec}(B'') \leftarrow \text{Spec}(B)$$

$$\downarrow \downarrow$$

$$\text{Spec}(B') \leftarrow \text{Spec}(B)$$

is cartesian. Since the base change of an open immersion is an open immersion we conclude. \qed

---

^1AKA flat and lci.
Lemma 2.2. The module \((2.0.1)\) is well defined, i.e., independent of the choice of the factorization.

Proof. Let \(B', B'', B'''\) be as in Lemma \(2.1\). We obtain a canonical map
\[
\omega''' = \text{Hom}_A(B''', A) \otimes_{B'''} B \rightarrow \text{Hom}_A(B', A) \otimes_{B'} B = \omega'
\]
and a similar one involving \(B''\). If we show these maps are isomorphisms then the lemma is proved. Let \(g \in B'\) be an element such that \(B'_g \rightarrow B_g\) is an isomorphism and hence \(B'_g \rightarrow (B''')_g \rightarrow B_g\) are isomorphisms. It suffices to show that \((\omega''')_g \rightarrow \omega'_g\) is an isomorphism. The kernel and cokernel of the ring map \(B' \rightarrow B'''\) are finite \(A\)-modules and \(g\)-power torsion. Hence they are annihilated by a power of \(g\). This easily implies the result. □

Lemma 2.3. Let \(A \rightarrow B\) be a quasi-finite map of Noetherian rings.

1. If \(A \rightarrow B\) factors as \(A \rightarrow A_f \rightarrow B\) for some \(f \in A\), then \(\omega_{B/A} = \omega_{B/A_f}\).
2. If \(g \in B\), then \((\omega_{B/A})_g = \omega_{B_g/A}\).
3. If \(f \in A\), then \(\omega_{B_f/A_f} = (\omega_{B/A})_f\).

Proof. Say \(A \rightarrow B' \rightarrow B\) is a factorization with \(A \rightarrow B'\) finite and \(\text{Spec}(B) \rightarrow \text{Spec}(B')\) an open immersion. In case (1) we may use the factorization \(A_f \rightarrow B'_f \rightarrow B\) to compute \(\omega_{B/A_f}\) and use Algebra, Lemma \(10.2\). In case (2) use the factorization \(A \rightarrow B' \rightarrow B_g\) to see the result. Part (3) follows from a combination of (1) and (2). □

Let \(A \rightarrow B\) be a quasi-finite ring map of Noetherian rings, let \(A \rightarrow A_1\) be an arbitrary ring map of Noetherian rings, and set \(B_1 = B \otimes_A A_1\). We obtain a cocartesian diagram

\[
\begin{array}{ccc}
B & \longrightarrow & B_1 \\
\uparrow & & \uparrow \\
A & \longrightarrow & A_1
\end{array}
\]

Observe that \(A_1 \rightarrow B_1\) is quasi-finite as well (Algebra, Lemma \(121.8\)). In this situation we will define a canonical \(B\)-linear base change map

\[
(2.3.1) \quad \omega_{B/A} \longrightarrow \omega_{B_1/A_1}
\]

Namely, we choose a factorization \(A \rightarrow B' \rightarrow B\) as in the construction of \(\omega_{B/A}\). Then \(B'_1 = B' \otimes_A A_1\) is finite over \(A_1\) and we can use the factorization \(A_1 \rightarrow B'_1 \rightarrow B_1\) in the construction of \(\omega_{B_1/A_1}\). Thus we have to construct a map
\[
\text{Hom}_A(B', A) \otimes_{B'} B \longrightarrow \text{Hom}_{A_1}(B'_1 \otimes_A A_1, A_1) \otimes_{B'_1} B_1
\]

Thus it suffices to construct a \(B'\)-linear map \(\text{Hom}_A(B', A) \rightarrow \text{Hom}_{A_1}(B'_1 \otimes_A A_1, A_1)\) which we will denote \(\varphi \mapsto \varphi_1\). Namely, given an \(A\)-linear map \(\varphi : B' \rightarrow A\) we let \(\varphi_1\) be the map such that \(\varphi_1(b' \otimes a_1) = \varphi(b')a_1\). This is clearly \(A_1\)-linear and the construction is complete.

Lemma 2.4. The base change map \((2.3.1)\) is independent of the choice of the factorization \(A \rightarrow B' \rightarrow B\). Given ring maps \(A \rightarrow A_1 \rightarrow A_2\) the composition of the base change maps for \(A \rightarrow A_1\) and \(A_1 \rightarrow A_2\) is the base change map for \(A \rightarrow A_2\).

Proof. Omitted. Hint: argue in exactly the same way as in Lemma 2.2 using Lemma 2.1. □
Lemma 2.5. If \( A \to A_1 \) is flat, then the base change map \([2.3.4]\) induces an isomorphism \( \omega_{B/A} \otimes_B B_1 \to \omega_{B_1/A_1} \).

**Proof.** Assume that \( A \to A_1 \) is flat. By construction of \( \omega_{B/A} \) we may assume that \( A \to B \) is finite. Then \( \omega_{B/A} = \text{Hom}_A(B, A) \) and \( \omega_{B_1/A_1} = \text{Hom}_{A_1}(B_1, A_1) \). Since \( B_1 = B \otimes_A A_1 \) the result follows from More on Algebra, Remark \([62.21]\). \( \square \)

Lemma 2.6. Let \( A \to B \to C \) be quasi-finite homomorphisms of Noetherian rings. Then there is a canonical map \( \omega_{B/A} \otimes_B \omega_{C/B} \to \omega_{C/A} \).

**Proof.** Choose \( A \to B' \to B \) with \( A \to B' \) finite such that \( \text{Spec}(B) \to \text{Spec}(B') \) is an open immersion. Then \( B' \to C \) is quasi-finite too. Choose \( B' \to C' \to C \) with \( B' \to C' \) finite and \( \text{Spec}(C) \to \text{Spec}(C') \) an open immersion. Then the source of the arrow is

\[
\text{Hom}_A(B', A) \otimes_B B \otimes_B \text{Hom}_B(B \otimes_B C', B) \otimes_B \text{Hom}_B(C', C)
\]

which is equal to

\[
\text{Hom}_A(B', A) \otimes_B \text{Hom}_{B'}(C', B) \otimes_{C'} C
\]

This indeed comes with a canonical map to \( \text{Hom}_A(C', A) \otimes_{C'} C = \omega_{C/A} \) coming from composition \( \text{Hom}_A(B', A) \times \text{Hom}_{B'}(C', B) \to \text{Hom}_A(C', A) \). \( \square \)

Lemma 2.7. Let \( A \to B \) and \( A \to C \) be quasi-finite maps of Noetherian rings. Then \( \omega_{B \times C/A} = \omega_{B/A} \times \omega_{C/A} \) as modules over \( B \times C \).

**Proof.** Choose factorizations \( A \to B' \to B \) and \( A \to C' \to C \) such that \( A \to B' \) and \( A \to C' \) are finite and such that \( \text{Spec}(B) \to \text{Spec}(B') \) and \( \text{Spec}(C) \to \text{Spec}(C') \) are open immersions. Then \( A \to B' \times C' \to B \times C \) is a similar factorization. Using this factorization to compute \( \omega_{B \times C/A} \) gives the lemma. \( \square \)

Lemma 2.8. Let \( A \to B \) be a quasi-finite homomorphism of Noetherian rings. Then \( \text{Ass}_B(\omega_{B/A}) \) is the set of primes of \( B \) lying over associated primes of \( A \).

**Proof.** Choose a factorization \( A \to B' \to B \) with \( A \to B' \) finite and \( B' \to B \) inducing an open immersion on spectra. As \( \omega_{B/A} = \omega_{B'/A} \otimes_{B'} B \) it suffices to prove the statement for \( \omega_{B'/A} \). Thus we may assume \( A \to B \) is finite.

Assume \( p \in \text{Ass}(A) \) and \( q \) is a prime of \( B \) lying over \( p \). Let \( x \in A \) be an element whose annihilator is \( p \). Choose a nonzero \( \kappa(p) \) linear map \( \lambda : \kappa(q) \to \kappa(p) \). Since \( A/p \subset B/q \) is a finite extension of rings, there is an \( f \in A \), \( f \notin p \) such that \( f\lambda \) maps \( B/q \) into \( A/p \). Hence we obtain a nonzero \( A \)-linear map

\[
B \to B/q \to A/p \to A, \quad b \mapsto f\lambda(b)x
\]

An easy computation shows that this element of \( \omega_{B/A} \) has annihilator \( q \), whence \( q \in \text{Ass}(\omega_{B/A}) \).

Conversely, suppose that \( q \subset B \) is a prime ideal lying over a prime \( p \subset A \) which is not an associated prime of \( A \). We have to show that \( q \notin \text{Ass}_B(\omega_{B/A}) \). After replacing \( A \) by \( A_p \) and \( B \) by \( B_p \) we may assume that \( p \) is a maximal ideal of \( A \).

This is allowed by Lemma \([2.5]\) and Algebra, Lemma \([62.16]\). Then there exists an \( f \in m \) which is a nonzerodivisor on \( A \). Then \( f \) is a nonzerodivisor on \( \omega_{B/A} \) and hence \( q \) is not an associated prime of this module. \( \square \)

Lemma 2.9. Let \( A \to B \) be a flat quasi-finite homomorphism of Noetherian rings. Then \( \omega_{B/A} \) is a flat \( A \)-module.
**Proof.** Let \( q \subset B \) be a prime lying over \( p \subset A \). We will show that the localization \( \omega_{B/A,B} \) is flat over \( A_{\bar{q}} \). This suffices by Algebra, Lemma \[142.21\] we can find an étale ring map \( A \to A' \) and a prime ideal \( p' \subset A' \) lying over \( p \) such that \( \kappa(p') = \kappa(p) \) and such that

\[
B' = B \otimes_A A' = C \times D
\]

with \( A' \to C \) finite and such that the unique prime \( q' \) of \( B \otimes_A A' \) lying over \( q \) and \( p' \) corresponds to a prime of \( C \). By Lemma \[2.5\] and Algebra, Lemma \[99.1\] it suffices to show \( \omega_{B'/A',q'} \) is flat over \( A'_{q'} \). Since \( \omega_{B'/A'} = \omega_{C/A'} \times \omega_{D/A'} \) by Lemma \[2.7\] this reduces us to the case where \( B \) is finite flat over \( A \). In this case \( B \) is finite locally free as an \( A \)-module and \( \omega_{B/A} = \text{Hom}_A(B, A) \) is the dual finite locally free \( A \)-module.

\[\Box\]

**Lemma 2.10.** If \( A \to B \) is flat, then the base change map \[(2.3.1)\] induces an isomorphism \( \omega_{B/A} \otimes_B B_1 \to \omega_{B_1/A_1} \).

**Proof.** If \( A \to B \) is finite flat, then \( B \) is finite locally free as an \( A \)-module. In this case \( \omega_{B/A} = \text{Hom}_A(B, A) \) is the dual finite locally free \( A \)-module and formation of this module commutes with arbitrary base change which proves the lemma in this case. In the next paragraph we reduce the general (quasi-finite flat) case to the finite flat case just discussed.

Let \( q_1 \subset B_1 \) be a prime. We will show that the localization of the map at the prime \( q_1 \) is an isomorphism, which suffices by Algebra, Lemma \[22.1\]. Let \( q \subset B \) and \( p \subset A \) be the prime ideals lying under \( q_1 \). By Algebra, Lemma \[142.21\] we can find an étale ring map \( A \to A' \) and a prime ideal \( p' \subset A' \) lying over \( p \) such that \( \kappa(p') = \kappa(p) \) and such that

\[
B' = B \otimes_A A' = C \times D
\]

with \( A' \to C \) finite and such that the unique prime \( q' \) of \( B \otimes_A A' \) lying over \( q \) and \( p' \) corresponds to a prime of \( C \). Set \( A'_1 = A' \otimes_A A_1 \) and consider the base change maps \[(2.3.1)\] for the ring maps \( A \to A' \to A'_1 \) and \( A \to A_1 \to A'_1 \) as in the diagram

\[
\begin{array}{ccc}
\omega_{B'/A'} \otimes_{B'} B_1' \\
\downarrow & & \downarrow \\
\omega_{B'/A_1'} \otimes_{B_1'} B_1' \\
\omega_{B/A} \otimes_B B_1' \\
\downarrow & & \downarrow \\
\omega_{B_1/A_1} \otimes_{B_1} B_1'
\end{array}
\]

where \( B' = B \otimes_A A' \), \( B_1 = B \otimes_A A_1 \), and \( B'_1 = B \otimes_A (A' \otimes_A A_1) \). By Lemma \[2.4\] the diagram commutes. By Lemma \[2.5\] the vertical arrows are isomorphisms. As \( B_1 \to B'_1 \) is étale and hence flat it suffices to prove the top horizontal arrow is an isomorphism after localizing at a prime \( q_1' \) of \( B'_1 \) lying over \( q \) (there is such a prime and use Algebra, Lemma \[38.17\]). Thus we may assume that \( B = C \times D \) with \( A \to C \) finite and \( q \) corresponding to a prime of \( C \). In this case the dualizing module \( \omega_{B/A} \) decomposes in a similar fashion (Lemma \[2.7\]) which reduces the question to the finite flat case \( A \to C \) handled above.

\[\Box\]

**Remark 2.11.** Let \( f : Y \to X \) be a locally quasi-finite morphism of locally Noetherian schemes. It is clear from Lemma \[2.8\] that there is a unique coherent
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\( \mathcal{O}_Y \)-module \( \omega_{Y/X} \) on \( Y \) such that for every pair of affine opens \( \text{Spec}(B) = V \subset Y \), \( \text{Spec}(A) = U \subset X \) with \( f(V) \subset U \) there is a canonical isomorphism

\[
H^0(V, \omega_{Y/X}) = \omega_{B/A}
\]

and where these isomorphisms are compatible with restriction maps.

**Lemma 2.12.** Let \( A \to B \) be a quasi-finite homomorphism of Noetherian rings. Let \( \omega^{\bullet}_{B/A} \in D(B) \) be the algebraic relative dualizing complex discussed in Dualizing Complexes, Section 25. Then there is a (nonunique) isomorphism \( \omega_{B/A} = H^0(\omega^{\bullet}_{B/A}) \).

**Proof.** Choose a factorization \( A \to B' \to B \) where \( A \to B' \) is finite and \( \text{Spec}(B') \to \text{Spec}(B) \) is an open immersion. Then \( \omega^{\bullet}_{B/A} = \omega^{\bullet}_{B'/A} \otimes_B B' \) by Dualizing Complexes, Lemmas 24.7 and 24.9 and the definition of \( \omega^{\bullet}_{B/A} \). Hence it suffices to show there is an isomorphism when \( A \to B \) is finite. In this case we can use Dualizing Complexes, Lemma 24.8 to see that \( \omega^{\bullet}_{B/A} = \mathcal{R}\text{Hom}(B, A) \) and hence \( H^0(\omega^{\bullet}_{B/A}) = \text{Hom}_A(B, A) \) as desired. \( \square \)

3. Discriminant of a finite locally free morphism

Let \( X \) be a scheme and let \( \mathcal{F} \) be a finite locally free \( \mathcal{O}_X \)-module. Then there is a canonical trace map

\[
\text{Trace} : \mathcal{H}\text{om}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F}) \to \mathcal{O}_X
\]

See Exercises, Exercise 22.6. This map has the property that \( \text{Trace}(\text{id}) \) is the locally constant function on \( \mathcal{O}_X \) corresponding to the rank of \( \mathcal{F} \).

Let \( \pi : X \to Y \) be a morphism of schemes which is finite locally free. Then there exists a canonical trace for \( \pi \) which is an \( \mathcal{O}_Y \)-linear map

\[
\text{Trace}_\pi : \pi_* \mathcal{O}_X \to \mathcal{O}_Y
\]

sending a local section \( f \) of \( \pi_* \mathcal{O}_X \) to the trace of multiplication by \( f \) on \( \pi_* \mathcal{O}_X \).

Over affine opens this recovers the construction in Exercises, Exercise 22.7. The composition

\[
\mathcal{O}_Y \to \pi_* \mathcal{O}_X \to \pi_* \mathcal{O}_X \to \mathcal{O}_Y
\]

equals multiplication by the degree of \( \pi \) (which is a locally constant function on \( Y \)). In analogy with Fields, Section 20 we can define the trace pairing

\[
Q_\pi : \pi_* \mathcal{O}_X \times \pi_* \mathcal{O}_X \to \mathcal{O}_Y
\]

by the rule \( (f, g) \mapsto \text{Trace}_\pi(fg) \). We can think of \( Q_\pi \) as a linear map \( \pi_* \mathcal{O}_X \to \mathcal{H}\text{om}_{\mathcal{O}_Y}(\pi_* \mathcal{O}_X, \mathcal{O}_Y) \) between locally free modules of the same rank, and hence obtain a determinant

\[
\det(Q_\pi) : \wedge^{\text{top}}(\pi_* \mathcal{O}_X) \to \wedge^{\text{top}}(\pi_* \mathcal{O}_X)^{\otimes -1}
\]

or in other words a global section

\[
\det(Q_\pi) \in \Gamma(Y, \wedge^{\text{top}}(\pi_* \mathcal{O}_X)^{\otimes -2})
\]

The discriminant of \( \pi \) is by definition the closed subscheme \( D_\pi \subset Y \) cut out by this global section. Clearly, \( D_\pi \) is a locally principal closed subscheme of \( Y \).

**Lemma 3.1.** Let \( \pi : X \to Y \) be a morphism of schemes which is finite locally free. Then \( \pi \) is étale if and only if its discriminant is empty.
Proof. By Morphisms, Lemma 34.8 it suffices to check that the fibres of $\pi$ are étale. Since the construction of the trace pairing commutes with base change we reduce to the following question: Let $k$ be a field and let $A$ be a finite dimensional $k$-algebra. Show that $A$ is étale over $k$ if and only if the trace pairing $Q_A/k : A \times A \to k$, $(a, b) \mapsto \text{Trace}_{A/k}(ab)$ is nondegenerate.

Assume $Q_A/k$ is nondegenerate. If $a \in A$ is a nilpotent element, then $ab$ is nilpotent for all $b \in A$ and we conclude that $Q_A/k(a, -)$ is identically zero. Hence $A$ is reduced. Then we can write $A = K_1 \times \ldots \times K_n$ as a product where each $K_i$ is a field (see Algebra, Lemmas 52.2, 52.6, and 24.1). In this case the quadratic space $(A, Q_{A/k})$ is the orthogonal direct sum of the spaces $(K_i, Q_{K_i/k})$. It follows from Fields, Lemma 20.7 that each $K_i$ is separable over $k$. This means that $A$ is étale over $k$ by Algebra, Lemma 142.4 The converse is proved by reading the argument backwards. □

4. Traces for flat quasi-finite ring maps

Let $A \to B$ be a finite flat map of Noetherian rings. Then $B$ is finite flat as an $A$-module and hence finite locally free (Algebra, Lemma 77.2). Given $b \in B$ we can consider the trace $\text{Trace}_{B/A}(b)$ of the $A$-linear map $B \to B$ given by multiplication by $b$ on $B$. By the references above this defines an $A$-linear map $\text{Trace}_{B/A} : B \to A$. Since $\omega_{B/A} = \text{Hom}_A(B, A)$ as $A \to B$ is finite, we see that $\text{Trace}_{B/A} \in \omega_{B/A}$.

For a general flat quasi-finite ring map we define the notion of a trace as follows.

Definition 4.1. Let $A \to B$ be a flat quasi-finite map of Noetherian rings. The trace element $\tau_{B/A} \in \omega_{B/A}$ with the following property: for any Noetherian $A$-algebra $A_1$ such that $B_1 = B \otimes_A A_1$ comes with a product decomposition $B_1 = C \times D$ with $A_1 \to C$ finite the image of $\tau_{B/A}$ in $\omega_{C/A_1}$ is $\text{Trace}_{C/A_1}$. Here we use the base change map 2.3.1 and Lemma 2.7 to get $\omega_{B/A} \to \omega_{B/A_1} \to \omega_{C/A_1}$.

We first prove that trace elements are unique and then we prove that they exist.

Lemma 4.2. Let $A \to B$ be a flat quasi-finite map of Noetherian rings. Then there is at most one trace element in $\omega_{B/A}$.

Proof. Let $q \subset B$ be a prime ideal lying over the prime $p \subset A$. By Algebra, Lemma 142.21 we can find an étale ring map $A \to A_1$ and a prime ideal $p_1 \subset A_1$ lying over $p$ such that $\kappa(p_1) = \kappa(p)$ and such that $B_1 = B \otimes_A A_1 = C \times D$

with $A_1 \to C$ finite and such that the unique prime $q_1$ of $B \otimes_A A_1$ lying over $q$ and $p_1$ corresponds to a prime of $C$. Observe that $\omega_{C/A_1} = \omega_{B/A} \otimes_B C$ (combine Lemmas 2.5 and 2.7). Since the collection of ring maps $B \to C$ obtained in this manner is a jointly injective family of flat maps and since the image of $\tau_{B/A}$ in $\omega_{C/A_1}$ is prescribed the uniqueness follows. □

2Uniqueness and existence will be justified in Lemmas 4.2 and 4.6
Here is a sanity check.

**Lemma 4.3.** Let $A \to B$ be a finite flat map of Noetherian rings. Then $\text{Trace}_{B/A} \in \omega_{B/A}$ is the trace element.

**Proof.** Suppose we have $A \to A_1$ with $A_1$ Noetherian and a product decomposition $B \otimes_A A_1 = C \times D$ with $A_1 \to C$ finite. Of course in this case $A_1 \to D$ is also finite. Set $B_1 = B \otimes_A A_1$. Since the construction of traces commutes with base change we see that $\text{Trace}_{B_1/A_1}$ maps to $\text{Trace}_{B_1/A_1}$. Thus the proof is finished by noticing that $\text{Trace}_{B_1/A_1} = (\text{Trace}_{C/A_1}, \text{Trace}_{D/A_1})$ under the isomorphism $\omega_{B_1/A_1} = \omega_{C/A_1} \times \omega_{D/A_1}$ of Lemma 2.7.

**Lemma 4.4.** Let $A \to B$ be a flat quasi-finite map of Noetherian rings. Let $\tau \in \omega_{B/A}$ be a trace element.

1. If $A \to A_1$ is a map with $A_1$ Noetherian, then with $B_1 = A_1 \otimes_A B$ the image of $\tau$ in $\omega_{B_1/A_1}$ is a trace element.
2. If $A = R_1$, then $\tau$ is a trace element in $\omega_{B/R}$.
3. If $g \in B$, then the image of $\tau$ in $\omega_{B_g/A}$ is a trace element.
4. If $B = B_1 \times B_2$, then $\tau$ maps to a trace element in both $\omega_{B_1/A}$ and $\omega_{B_2/A}$.

**Proof.** Part (1) is a formal consequence of the definition. Statement (2) makes sense because $\omega_{B/R} = \omega_{B/A}$ by Lemma 2.3. Denote $\tau'$ the element $\tau$ but viewed as an element of $\omega_{B/R}$. To see that (2) is true suppose that we have $R \to R_1$ with $R_1$ Noetherian and a product decomposition $B \otimes_R R_1 = C \times D$ with $R_1 \to C$ finite. Then with $A_1 = (R_1)_g$ we see that $B \otimes_A A_1 = C \times D$. Since $R_1 \to C$ is finite, a fortiori $A_1 \to C$ is finite. Hence we can use the defining property of $\tau$ to get the corresponding property of $\tau'$.

Statement (3) makes sense because $\omega_{B_g/A} = (\omega_{B/A})_g$ by Lemma 2.3. The proof is similar to the proof of (2). Suppose we have $A \to A_1$ with $A_1$ Noetherian and a product decomposition $B_g \otimes_A A_1 = C \times D$ with $A_1 \to C$ finite. Set $B_1 = B \otimes_A A_1$. Then $\text{Spec}(C) \to \text{Spec}(B_1)$ is an open immersion as $B_g \otimes_A A_1 = (B_1)_g$ and the image is closed because $B_1 \to C$ is finite (as $A_1 \to C$ is finite). Thus we see that $B_1 = C \times D_1$ and $D = (D_1)_g$. Then we can use the defining property of $\tau$ to get the corresponding property for the image of $\tau$ in $\omega_{B_g/A}$.

Statement (4) makes sense because $\omega_{B/A} = \omega_{B_1/A} \times \omega_{B_2/A}$ by Lemma 2.7. Suppose we have $A \to A'$ with $A'$ Noetherian and a product decomposition $B \otimes_A A' = C \times D$ with $A' \to C$ finite. Then it is clear that we can refine this product decomposition into $B \otimes_A A' = C_1 \times C_2 \times D_1 \times D_2$ with $A' \to C_i$ finite such that $B_i \otimes_A A' = C_i \times D_i$. Then we can use the defining property of $\tau$ to get the corresponding property for the image of $\tau$ in $\omega_{B_i/A}$. This uses the obvious fact that $\text{Trace}_{C/A'} = (\text{Trace}_{C_1/A'}, \text{Trace}_{C_2/A'})$ under the decomposition $\omega_{C/A'} = \omega_{C_1/A'} \times \omega_{C_2/A'}$.

**Lemma 4.5.** Let $A \to B$ be a flat quasi-finite map of Noetherian rings. Let $g_1, \ldots, g_m \in B$ be elements generating the unit ideal. Let $\tau \in \omega_{B/A}$ be an element whose image in $\omega_{B_{g_i}/A}$ is a trace element for $A \to B_{g_i}$. Then $\tau$ is a trace element.

**Proof.** Suppose we have $A \to A_1$ with $A_1$ Noetherian and a product decomposition $B \otimes_A A_1 = C \times D$ with $A_1 \to C$ finite. We have to show that the image of $\tau$ in $\omega_{C/A_1}$ is $\text{Trace}_{C/A_1}$. Observe that $g_1, \ldots, g_m$ generate the unit ideal in $B_1 = B \otimes_A A_1$ and
that \( \tau \) maps to a trace element in \( \omega_{(B_1)_{\eta}/A_1} \) by Lemma 4.4. Hence we may replace \( A \) by \( A_1 \) and \( B \) by \( B_1 \) to get to the situation as described in the next paragraph.

Here we assume that \( B = C \times D \) with \( A \to C \) is finite. Let \( \tau_C \) be the image of \( \tau \) in \( \omega_{C/A} \). We have to prove that \( \tau_C = \text{Trace}_{C/A} \) in \( \omega_{C/A} \). By the compatibility of trace elements with products (Lemma 4.4) we see that \( \tau_C \) maps to a trace element in \( \omega_{C_\eta/A} \). Hence, after replacing \( B \) by \( C \) we may assume that \( A \to B \) is finite flat.

Assume \( A \to B \) is finite flat. In this case \( \text{Trace}_{B/A} \) is a trace element by Lemma 4.3. Hence \( \text{Trace}_{B/A} \) maps to a trace element in \( \omega_{B_\eta/A} \) by Lemma 4.4. Since trace elements are unique (Lemma 4.2) we find that \( \text{Trace}_{B/A} \) and \( \tau \) map to the same elements in \( \omega_{B_\eta/A} = (\omega_{B/A})_\eta \). As \( g_1, \ldots, g_m \) generate the unit ideal of \( B \) the map \( \omega_{B/A} \to \prod \omega_{B_\eta/A} \) is injective and we conclude that \( \tau_C = \text{Trace}_{B/A} \) as desired. \( \square \)

**Lemma 4.6.** Let \( A \to B \) be a flat quasi-finite map of Noetherian rings. There exists a trace element \( \tau \in \omega_{B/A} \).

**Proof.** Choose a factorization \( A \to B' \to B \) with \( A \to B' \) finite and \( \text{Spec}(B) \to \text{Spec}(B') \) an open immersion. Let \( g_1, \ldots, g_n \in B' \) be elements such that \( \text{Spec}(B) = \bigcup D(g_i) \) as opens of \( \text{Spec}(B') \). Suppose that we can prove the existence of trace elements \( \tau_i \) for the quasi-finite flat ring maps \( A \to B_{g_i} \). Then for all \( i, j \) the elements \( \tau_i \) and \( \tau_j \) map to trace elements of \( \omega_{B_{g_i}g_j}/A \) by Lemma 4.4. By uniqueness of trace elements (Lemma 4.2) they map to the same element. Hence the sheaf condition for the quasi-coherent module associated to \( \omega_{B/A} \) (see Algebra, Lemma 23.1) produces an element \( \tau \in \omega_{B/A} \). Then \( \tau \) is a trace element by Lemma 4.5. In this way we reduce to the case treated in the next paragraph.

Assume we have \( A \to B' \) finite and \( g \in B' \) with \( B = B'_{g} \) flat over \( A \). It is our task to construct a trace element in \( \omega_{B/A} = \text{Hom}_A(B',A) \otimes_{B'} B \). Choose a resolution \( F_1 \to F_0 \to B' \to 0 \) of \( B' \) by finite free \( A \)-modules \( F_0 \) and \( F_1 \). Then we have an exact sequence

\[
0 \to \text{Hom}_A(B',A) \to F_0^\vee \to F_1^\vee
\]

where \( F_i^\vee = \text{Hom}_A(F_i,A) \) is the dual finite free module. Similarly we have the exact sequence

\[
0 \to \text{Hom}_A(B',B') \to F_0^\vee \otimes_A B' \to F_1^\vee \otimes_A B'
\]

The idea of the construction of \( \tau \) is to use the diagram\[
B' \xrightarrow{\mu} \text{Hom}_A(B',B') \leftrightarrow \text{Hom}_A(B',A) \otimes_A B' \xrightarrow{ev} A
\]

where the first arrow sends \( b' \in B' \) to the \( A \)-linear operator given by multiplication by \( b' \) and the last arrow is the evaluation map. The problem is that the middle arrow, which sends \( \lambda' \otimes b' \) to the map \( b'' \mapsto \lambda'(b'')b' \), is not an isomorphism. If \( B' \) is flat over \( A \), the exact sequences above show that it is an isomorphism and the composition from left to right is the usual trace \( \text{Trace}_{B'/A} \). In the general case, we consider the diagram

\[
\begin{array}{ccc}
\text{Hom}_A(B',A) \otimes_A B' & \overset{\psi}{\longrightarrow} & \text{Hom}_A(B',A) \otimes_A B' \\
\downarrow & & \downarrow \\
B' & \overset{\mu}{\longrightarrow} & \text{Hom}_A(B',B') \\
\end{array}
\]

\[
\begin{array}{ccc}
B' & \longrightarrow & \text{Ker}(F_0^\vee \otimes_A B'_{g} \to F_1^\vee \otimes_A B'_{g}) \\
\end{array}
\]
By flatness of $A \to B'_g$ we see that the right vertical arrow is an isomorphism. Hence we obtain the unadorned dotted arrow. Since $B'_g = \colim B'$, since colimits commute with tensor products, and since $B'$ is a finitely presented $A$-module we can find an $n \geq 0$ and a $B'$-linear (for right $B'$-module structure) map $\psi : B' \to \Hom_A(B', A) \otimes_A B'$ whose composition with the left vertical arrow is $g^n\mu$. Composing with $ev$ we obtain an element $ev \circ \psi \in \Hom_A(B', A)$. Then we set

$$\tau = (ev \circ \psi) \otimes g^{-n} \in \Hom_A(B', A) \otimes_{B'} B'_g = \omega_{B'/A} = \omega_B/A.$$  

We omit the easy verification that this element does not depend on the choice of $n$ and $\psi$ above.

Let us prove that $\tau$ as constructed in the previous paragraph has the desired property in a special case. Namely, say $B' = C' \times D'$ and $g = (f, h)$ where $A \to C'$ flat, $D'_h$ is flat, and $f$ is a unit in $C'$. To show: $\tau$ maps to $\Trace_{C'/A}$ in $\omega_{C'/A}$. In this case we first choose $\nu_D$ and $\psi_D : B' \to \Hom_A(D', A) \otimes_A D'$ as above for the pair $(D', h)$ and we can let $\psi_C : C' \to \Hom_A(C', A) \otimes_A C' = \Hom_A(C', C')$ be the map sending $c' \in C'$ to multiplication by $c'$. Then we take $n = \nu_D$ and $\psi = (f^n \nu_D \psi_C, \psi_D)$ and the desired compatibility is clear because $\Trace_{C'/A} = ev \circ \psi_C$ as remarked above.

To prove the desired property in general, suppose given $A \to A_1$ with $A_1$ Noetherian and a product decomposition $B'_g \otimes_A A_1 = C \times D$ with $A_1 \to C$ finite. Set $B'_1 = B'_g \otimes_A A_1$. Then $\Spec(C) \to \Spec(B'_1)$ is an open immersion as $B'_g \otimes_A A_1 = (B'_g)_g$ and the image is closed as $B'_1 \to C$ is finite (since $A_1 \to C$ is finite). Thus $B'_1 = C \times D'$ and $D'_g = D$. We conclude that $B'_g = C \times D'$ and $A_1 \to B'_1 \to 0$ to see this). Thus the desired compatibility follows from the result of the previous paragraph. □

0BVJ **Remark 4.7.** Let $f : Y \to X$ be a flat locally quasi-finite morphism of locally Noetherian schemes. Let $\omega_{Y/X}$ be as in Remark 2.11. It is clear from the uniqueness, existence, and compatibility with localization of trace elements (Lemmas 4.2, 4.6, and 4.4) that there exists a global section

$$\tau_{Y/X} \in \Gamma(Y, \omega_{Y/X})$$

such that for every pair of affine opens $\Spec(B) = V \subset Y$, $\Spec(A) = U \subset X$ with $f(V) \subset U$ that element $\tau_{Y/X}$ maps to $\tau_{B/A}$ under the canonical isomorphism $\Res^0(V, \omega_{Y/X}) = \omega_{B/A}$.

0C13 **Lemma 4.8.** Let $k$ be a field and let $A$ be a finite $k$-algebra. Assume $A$ is local with residue field $k'$. The following are equivalent

1. $\Trace_{A/k}$ is nonzero,
2. $\tau_{A/k} \in \omega_{A/k}$ is nonzero, and
3. $k'/k$ is separable and $\dim_A(A)$ is prime to the characteristic of $k$.

**Proof.** Conditions (1) and (2) are equivalent by Lemma 4.3. Let $m \subset A$. Since $\dim_k(A) < \infty$ it is clear that $A$ has finite length over $A$. Choose a filtration

$$A = I_0 \supset m = I_1 \supset I_2 \supset \ldots \supset I_n = 0$$

such that $I_{i+1}/I_i$ is a finite $A/I_i$-module for $0 \leq i < n$. Since $I_{i+1}/I_i$ is annihilated by $\tau_{A/k}$ for $0 \leq i < n$, we have

$$I_{i+1}/I_i \subset \Hom_A(I_{i+1}/I_i, A) \otimes_{I_{i+1}/I_i} I_{i+1}/I_i.$$
by ideals such that $I_i/I_{i+1} \cong k'$ as $A$-modules. See Algebra, Lemma 31.11 which also shows that $n = \text{length}_A(a)$. If $a \in \mathfrak{m}$ then $aI_i \subset I_{i+1}$ and it is immediate that $\text{Trace}_{A/k}(a) = 0$. If $a \notin \mathfrak{m}$ with image $\lambda \in k'$, then we conclude

$$\text{Trace}_{A/k}(a) = \sum_{i=0}^{n-1} \text{Trace}(a : I_i/I_{i-1} \to I_i/I_{i-1}) = n\text{Trace}_{k'/k}(\lambda)$$

The proof of the lemma is finished by applying Fields, Lemma 20.7.

5. Finite morphisms

In this section we collect some observations about the constructions in the previous sections for finite morphisms. Let $f : Y \to X$ be a finite morphism of locally Noetherian schemes. Let $\omega_{Y/X}$ be as in Remark 2.11

The first remark is that

$$f_* \omega_{Y/X} = \mathcal{H}om_{\mathcal{O}_X}(f_* \mathcal{O}_Y, \mathcal{O}_X)$$

as sheaves of $f_* \mathcal{O}_Y$-modules. Since $f$ is affine, this formula uniquely characterizes $\omega_{Y/X}$, see Morphisms, Lemma 11.6. The formula holds because for $\text{Spec}(A) = U \subset X$ affine open, the inverse image $V = f^{-1}(U)$ is the spectrum of a finite $A$-algebra $B$ and hence

$$H^0(U, f_* \omega_{Y/X}) = H^0(V, \omega_{Y/X}) = \omega_{B/A} = \text{Hom}_A(B, A) = H^0(U, \text{Hom}_{\mathcal{O}_X}(f_* \mathcal{O}_Y, \mathcal{O}_X))$$

by construction. In particular, we obtain a canonical evaluation map $f_* \omega_{Y/X} \to \mathcal{O}_X$

which is given by evaluation at 1 if we think of $f_* \omega_{Y/X}$ as the sheaf $\text{Hom}_{\mathcal{O}_X}(f_* \mathcal{O}_Y, \mathcal{O}_X)$. The second remark is that using the evaluation map we obtain canonical identifications

$$\text{Hom}_Y(F, f^* \mathcal{G} \otimes_{\mathcal{O}_Y} \omega_{Y/X}) = \text{Hom}_X(f_* F, \mathcal{G})$$

functorially in the quasi-coherent module $F$ on $Y$ and the finite locally free module $\mathcal{G}$ on $X$. If $\mathcal{G} = \mathcal{O}_X$ this follows immediately from the above and Algebra, Lemma 13.4. For general $\mathcal{G}$ we can use the same lemma and the isomorphisms

$$f_*(f^* \mathcal{G} \otimes_{\mathcal{O}_Y} \omega_{Y/X}) = \mathcal{G} \otimes_{\mathcal{O}_X} \text{Hom}_{\mathcal{O}_X}(f_* \mathcal{O}_Y, \mathcal{O}_X) = \text{Hom}_{\mathcal{O}_X}(f_* \mathcal{O}_Y, \mathcal{G})$$

of $f_* \mathcal{O}_Y$-modules where the first equality is the projection formula (Cohomology, Lemma 47.2). An alternative is to prove the formula affine locally by direct computation.

The third remark is that if $f$ is in addition flat, then the composition

$$f_* \mathcal{O}_Y \xrightarrow{f_{\tau Y/X}} f_* \omega_{Y/X} \to \mathcal{O}_X$$

is equal to the trace map $\text{Trace}_f$ discussed in Section 3. This follows immediately by looking over affine opens.

The fourth remark is that if $f$ is flat and $X$ Noetherian, then we obtain

$$\text{Hom}_Y(K, Lf^* M \otimes_{\mathcal{O}_Y} \omega_{Y/X}) = \text{Hom}_X(Rf_* K, M)$$

for any $K$ in $D_{QCoh}(\mathcal{O}_Y)$ and $M$ in $D_{QCoh}(\mathcal{O}_X)$. This follows from the material in Duality for Schemes, Section 12, but can be proven directly in this case as follows. First, if $X$ is affine, then it holds by Dualizing Complexes, Lemmas 13.1 and 13.4.

\[\text{There is a simpler proof of this lemma in our case.}\]
and Derived Categories of Schemes, Lemma \[3.5\]. Then we can use the induction principle (Cohomology of Schemes, Lemma \[4.1\]) and Mayer-Vietoris (in the form of Cohomology, Lemma \[33.3\]) to finish the proof.

6. The Noether different

There are many different differents available in the literature. We list some of them in this and the next sections; for more information we suggest the reader consult [Kun86].

Let \( A \rightarrow B \) be a ring map. Denote

\[
\mu : B \otimes_A B \rightarrow B, \quad b \otimes b' \mapsto(bb')
\]

the multiplication map. Let \( I = \text{Ker}(\mu) \). It is clear that \( I \) is generated by the elements \( b \otimes 1 - 1 \otimes b \) for \( b \in B \). Hence the annihilator \( J \subset B \otimes_A B \) of \( I \) is a \( B \)-module in a canonical manner. The Noether different of \( B \) over \( A \) is the image of \( J \) under the map \( \mu : B \otimes_A B \rightarrow B \). Equivalently, the Noether different is the image of the map

\[
J = \text{Hom}_{B \otimes A B}(B, B \otimes_A B) \rightarrow B, \quad \varphi \mapsto \mu(\varphi(1))
\]

We begin with some obligatory lemmas.

**Lemma 6.1.** Let \( A \rightarrow B_i, i = 1, 2 \) be ring maps. Set \( B = B_1 \times B_2 \).

1. The annihilator \( J \) of \( \text{Ker}(B \otimes_A B \rightarrow B) \) is \( J_1 \times J_2 \) where \( J_i \) is the annihilator of \( \text{Ker}(B_i \otimes_A B_i \rightarrow B_i) \).

2. The Noether different \( \mathcal{D} \) of \( B \) over \( A \) is \( \mathcal{D}_1 \times \mathcal{D}_2 \), where \( \mathcal{D}_i \) is the Noether different of \( B_i \) over \( A \).

**Proof.** Omitted.

**Lemma 6.2.** Let \( A \rightarrow B \) be a finite type ring map. Let \( A \rightarrow A' \) be a flat ring map. Set \( B' = B \otimes_A A' \).

1. The annihilator \( J' \) of \( \text{Ker}(B' \otimes_{A'} B' \rightarrow B') \) is \( J \otimes_A A' \) where \( J \) is the annihilator of \( \text{Ker}(B \otimes_A B \rightarrow B) \).

2. The Noether different \( \mathcal{D}' \) of \( B' \) over \( A' \) is \( \mathcal{D}B' \), where \( \mathcal{D} \) is the Noether different of \( B \) over \( A \).

**Proof.** Choose generators \( b_1, \ldots, b_n \) of \( B \) as an \( A \)-algebra. Then

\[
J = \text{Ker}(B \otimes_A B \xrightarrow{b_1 \otimes 1 - 1 \otimes b_1} (B \otimes_A B)^{\oplus n})
\]

Hence we see that the formation of \( J \) commutes with flat base change. The result on the Noether different follows immediately from this.

**Lemma 6.3.** Let \( A \rightarrow B' \rightarrow B \) be ring maps with \( A \rightarrow B' \) of finite type and \( B' \rightarrow B \) inducing an open immersion of spectra.

1. The annihilator \( J' \) of \( \text{Ker}(B \otimes_A B \rightarrow B) \) is \( J' \otimes_{B'} B \) where \( J' \) is the annihilator of \( \text{Ker}(B' \otimes_A B' \rightarrow B') \).

2. The Noether different \( \mathcal{D}' \) of \( B \) over \( A \) is \( \mathcal{D}'B \), where \( \mathcal{D}' \) is the Noether different of \( B' \) over \( A \).
Proof. Write \( I = \text{Ker}(B \otimes_A B \to B) \) and \( I' = \text{Ker}(B' \otimes_A B' \to B') \). As \( \text{Spec}(B) \to \text{Spec}(B') \) is an open immersion, it follows that \( B = (B \otimes_A B) \otimes_B B' \). Thus we see that \( I = I'(B \otimes_A B) \). Since \( I' \) is finitely generated and \( B' \otimes_A B' \to B \otimes_A B \) is flat, we conclude that \( J = J'(B \otimes_A B) \), see Algebra, Lemma 39.4. Since the \( B' \otimes_A B' \)-module structure of \( J' \) factors through \( B' \otimes_A B' \to B' \) we conclude that (1) is true. Part (2) is a consequence of (1).

\[ \square \]

**Remark 6.4.** Let \( A \to B \) be a quasi-finite homomorphism of Noetherian rings. Let \( J \) be the annihilator of \( \text{Ker}(B \otimes_A B \to B) \). There is a canonical \( B \)-bilinear pairing

\[ \omega_{B/A} \times J \to B \]

defined as follows. Choose a factorization \( A \to B' \to B \) with \( A \to B' \) finite and \( B' \to B \) inducing an open immersion of spectra. Let \( J' \) be the annihilator of \( \text{Ker}(B' \otimes_A B' \to B') \). We first define

\[ \text{Hom}_A(B', A) \times J' \to B' , \quad (\lambda, \sum b_i \otimes c_i) \mapsto \sum \lambda(b_i)c_i \]

This is \( B' \)-bilinear exactly because for \( \xi \in J' \) and \( b \in B' \) we have \( (b \otimes 1)\xi = (1 \otimes b)\xi \). By Lemma 6.3 and the fact that \( \omega_{B/A} = \text{Hom}_A(B', A) \otimes_{B'} B \) we can extend this to a \( B \)-bilinear pairing as displayed above.

**Lemma 6.5.** Let \( A \to B \) be a quasi-finite homomorphism of Noetherian rings.

1. If \( A \to A' \) is a flat map of Noetherian rings, then

\[ \omega_{B/A} \times J \to B \]

is commutative where notation as in Lemma 6.3 and horizontal arrows are given by \( 6.4.1 \).

2. If \( B = B_1 \times B_2 \), then

\[ \omega_{B/A} \times J \to B \]

is commutative for \( i = 1, 2 \) where notation as in Lemma 6.4 and horizontal arrows are given by \( 6.4.1 \).

**Proof.** Because of the construction of the pairing in Remark 6.4 both (1) and (2) reduce to the case where \( A \to B \) is finite. Then (1) follows from the fact that the contraction map \( \text{Hom}_A(M, A) \otimes_A M \otimes_A M \to M, \lambda \otimes m \otimes m' \mapsto \lambda(m)m' \) commuted with base change. To see (2) use that \( J = J_1 \times J_2 \) is contained in the summands \( B_1 \otimes_A B_1 \) and \( B_2 \otimes_A B_2 \) of \( B \otimes_A B \).

\[ \square \]

**Lemma 6.6.** Let \( A \to B \) be a flat quasi-finite homomorphism of Noetherian rings. The pairing of Remark 6.4 induces an isomorphism \( J \to \text{Hom}_B(\omega_{B/A}, B) \).
Proof. We first prove this when $A \to B$ is finite and flat. In this case we can localize on $A$ and assume $B$ is finite free as an $A$-module. Let $b_1, \ldots, b_n$ be a basis of $B$ as an $A$-module and denote $b_1', \ldots, b_n'$ the dual basis of $\omega_{B/A}$. Note that $\sum b_i \otimes c_i \in J$ maps to the element of $\text{Hom}_B(\omega_{B/A}, B)$ which sends $b_i'$ to $c_i$. Suppose $\varphi : \omega_{B/A} \to B$ is $B$-linear. Then we claim that $\xi = \sum b_i \otimes \varphi(b_i')$ is an element of $J$. Namely, the $B$-linearity of $\varphi$ exactly implies that $(b \otimes 1)\xi = (1 \otimes b)\xi$ for all $b \in B$. Thus our map has an inverse and it is an isomorphism.

Let $q \subset B$ be a prime lying over $p \subset A$. We will show that the localization

$$J_q \to \text{Hom}_B(\omega_{B/A}, B)_q$$

is an isomorphism. This suffices by Algebra, Lemma 22.1. By Algebra, Lemma 142.21 we can find an étale ring map $A \to A'$ and a prime ideal $p' \subset A'$ lying over $p$ such that $\kappa(p') = \kappa(p)$ and such that

$$B' = B \otimes_A A' = C \times D$$

with $A' \to C$ finite and such that the unique prime $q'$ of $B \otimes_A A'$ lying over $q$ and $p'$ corresponds to a prime of $C$. Let $J'$ be the annihilator of $\text{Ker}(B' \otimes_{A'} B' \to B')$. By Lemmas 2.5, 6.1, and 6.5 the map $J' \to \text{Hom}_{B'}(\omega_{B'/A'}, B')$ is gotten by applying the functor $- \otimes_{A'} B'$ to the map $J \to \text{Hom}_B(\omega_{B/A}, B)$. Since $B_q \to B'_q$ is faithfully flat it suffices to prove the result for $(A' \to B', q')$. By Lemmas 2.7, 6.1, and 6.5 this reduces us to the case proved in the first paragraph of the proof. 

0BVU [Lemma 6.7. Let $A \to B$ be a flat quasi-finite homomorphism of Noetherian rings. The diagram

$$\begin{array}{ccc}
J & \to & \text{Hom}_B(\omega_{B/A}, B) \\
\mu & & \downarrow \varphi \\
B & \to & \varphi(\tau_{B/A})
\end{array}$$

commutes where the horizontal arrow is the isomorphism of Lemma 6.6. Hence the Noether different of $B$ over $A$ is the image of the map $\text{Hom}_B(\omega_{B/A}, B) \to B$.

Proof. Exactly as in the proof of Lemma 6.6 this reduces to the case of a finite free map $A \to B$. In this case $\tau_{B/A} = \text{Trace}_{B/A}$. Choose a basis $b_1, \ldots, b_n$ of $B$ as an $A$-module. Let $\xi = \sum b_i \otimes c_i \in J$. Then $\mu(\xi) = \sum b_i c_i$. On the other hand, the image of $\xi$ in $\text{Hom}_B(\omega_{B/A}, B)$ sends $\text{Trace}_{B/A}$ to $\sum \text{Trace}_{B/A}(b_i) c_i$. Thus we have to show

$$\sum b_i c_i = \sum \text{Trace}_{B/A}(b_i) c_i$$

when $\xi = \sum b_i \otimes c_i \in J$. Write $b_i b_j = \sum_k a_{ij}^k b_k$ for some $a_{ij}^k \in A$. Then the right hand side is $\sum \text{Trace}_{B/A}(b_i) c_i$ and on the other hand, $\xi \in J$ implies

$$(b_j \otimes 1)(\sum_i b_i \otimes c_i) = (1 \otimes b_j)(\sum_i b_i \otimes c_i)$$

which implies that $b_j c_i = \sum_k a_{ij}^k c_k$. Thus the left hand side is $\sum \text{Trace}_{B/A}(b_i) c_i$ and on the other hand, $\xi \in J$ implies

$$a_{ij}^k = a_{ji}^k$$

the equality holds. 

0BVU [Lemma 6.8. Let $A \to B$ be a finite type ring map. Let $\mathfrak{D} \subset B$ be the Noether different. Then $V(\mathfrak{D})$ is the set of primes $q \subset B$ such that $A \to B$ is not unramified at $q$.}
Proof. Assume $A \to B$ is unramified at $q$. After replacing $B$ by $B_g$ for some $g \in B$, $g \notin q$ we may assume $A \to B$ is unramified (Algebra, Definition 148.1 and Lemma 6.3). In this case $\Omega_{B/A} = 0$. Hence if $I = \text{Ker}(B \otimes_A B \to B)$, then $I/I^2 = 0$ by Algebra, Lemma 130.13. Since $A \to B$ is of finite type, we see that $I$ is finitely generated. Hence by Nakayama’s lemma (Algebra, Lemma 19.1) there exists an element of the form $1 + i$ annihilating $I$. It follows that $D = B$.

Conversely, assume that $D \not\subset q$. Then after replacing $B$ by a principal localization as above we may assume $D = B$. This means there exists an element of the form $1 + i$ in the annihilator of $I$. Conversely this implies that $I/I^2 = \Omega_{B/A}$ is zero and we conclude. □

7. The Kähler different

0BVV Let $A \to B$ be a finite type ring map. The Kähler different is the zeroth fitting ideal of $\Omega_{B/A}$ as a $B$-module. We globalize the definition as follows.

0BVW Definition 7.1. Let $f : Y \to X$ be a morphism of schemes which is locally of finite type. The Kähler different is the 0th fitting ideal of $\Omega_{Y/X}$.

The Kähler different is a quasi-coherent sheaf of ideals on $Y$.

0BVX Lemma 7.2. Consider a cartesian diagram of schemes

$$
\begin{array}{ccc}
Y' & \longrightarrow & Y \\
\downarrow f' & & \downarrow f \\
X' & \longrightarrow & X
\end{array}
$$

with $f$ locally of finite type. Let $R \subset Y$, resp. $R' \subset Y'$ be the closed subscheme cut out by the Kähler different of $f$, resp. $f'$. Then $Y' \to Y$ induces an isomorphism $R' \to R \times_Y Y'$.

Proof. This is true because $\Omega_{Y'/X'}$ is the pullback of $\Omega_{Y/X}$ (Morphisms, Lemma 31.10) and then we can apply More on Algebra, Lemma 8.4. □

0BVY Lemma 7.3. Let $f : Y \to X$ be a morphism of schemes which is locally of finite type. Let $R \subset Y$ be the closed subscheme defined by the Kähler different. Then $R \subset Y$ is exactly the set of points where $f$ is not unramified.

Proof. This is a copy of Divisors, Lemma 10.3. □

0BVZ Lemma 7.4. Let $A$ be a ring. Let $n \geq 1$ and $f_1, \ldots, f_n \in A[x_1, \ldots, x_n]$. Set $B = A[x_1, \ldots, x_n]/(f_1, \ldots, f_n)$. The Kähler different of $B$ over $A$ is the ideal of $B$ generated by $\det(\partial f_i/\partial x_j)$.

Proof. This is true because $\Omega_{B/A}$ has a presentation

$$
\bigoplus_{i=1}^n Bf_i \overset{d}{\longrightarrow} \bigoplus_{j=1}^n Bdx_j \rightarrow \Omega_{B/A} \rightarrow 0
$$

by Algebra, Lemma 130.9. □
8. The Dedekind different

Let $A \to B$ be a ring map. We say the Dedekind different is defined if $A$ is Noetherian, $A \to B$ is finite, any nonzerodivisor on $A$ is a nonzerodivisor on $B$, and $K \to L$ is étale where $K = Q(A)$ and $L = B \otimes_A K$. Then $K \subset L$ is finite étale and 

$$L_{B/A} = \{ x \in L \mid \text{Trace}_{L/K}(bx) \in A \text{ for all } b \in B \}$$

is the Dedekind complementary module. In this situation the Dedekind different is 

$$D_{B/A} = \{ x \in L \mid xL_{B/A} \subset B \}$$

viewed as a $B$-submodule of $L$. By Lemma 8.1 the Dedekind different is an ideal of $B$ either if $A$ is normal or if $B$ is flat over $A$.

**Lemma 8.1.** Assume the Dedekind different of $A \to B$ is defined. Consider the statements

1. $A \to B$ is flat,
2. $A$ is a normal ring,
3. $\text{Trace}_{L/K}(B) \subset A$,
4. $1 \in L_{B/A}$, and
5. the Dedekind different $D_{B/A}$ is an ideal of $B$.

Then we have (1) $\Rightarrow$ (3), (2) $\Rightarrow$ (3), (3) $\iff$ (4), and (4) $\Rightarrow$ (5).

**Proof.** The equivalence of (3) and (4) and the implication (4) $\Rightarrow$ (5) are immediate.

If $A \to B$ is flat, then we see that $\text{Trace}_{B/A} : B \to A$ is defined and that $\text{Trace}_{L/K}$ is the base change. Hence (3) holds.

If $A$ is normal, then $A$ is a finite product of normal domains, hence we reduce to the case of a normal domain. Then $K$ is the fraction field of $A$ and $L = \prod L_i$ is a finite product of finite separable field extensions of $K$. Then $\text{Trace}_{L/K}(b) = \sum \text{Trace}_{L_i/K}(b_i)$ where $b_i \in L_i$ is the image of $b$. Since $b$ is integral over $A$ as $B$ is finite over $A$, these traces are in $A$. This is true because the minimal polynomial of $b_i$ over $K$ has coefficients in $A$ (Algebra, Lemma 37.6) and because $\text{Trace}_{L_i/K}(b_i)$ is an integer multiple of one of these coefficients (Fields, Lemma 20.3).

**Lemma 8.2.** If the Dedekind different of $A \to B$ is defined, then there is a canonical isomorphism $L_{B/A} \to \omega_{B/A}$.

**Proof.** Recall that $\omega_{B/A} = \text{Hom}_A(B, A)$ as $A \to B$ is finite. We send $x \in L_{B/A}$ to the map $b \mapsto \text{Trace}_{L/K}(bx)$. Conversely, given an $A$-linear map $\varphi : B \to A$ we obtain a $K$-linear map $\varphi_K : L \to K$. Since $K \to L$ is finite étale, we see that the trace pairing is nondegenerate (Lemma 3.1) and hence there exists a $x \in L$ such that $\varphi_K(y) = \text{Trace}_{L/K}(xy)$ for all $y \in L$. Then $x \in L_{B/A}$ maps to $\varphi$ in $\omega_{B/A}$.

**Lemma 8.3.** If the Dedekind different of $A \to B$ is defined and $A \to B$ is flat, then

1. the canonical isomorphism $L_{B/A} \to \omega_{B/A}$ sends $1 \in L_{B/A}$ to the trace element $\tau_{B/A} \in \omega_{B/A}$, and
2. the Dedekind different is $D_{B/A} = \{ b \in B \mid b\omega_{B/A} \subset B\tau_{B/A} \}$.

**Proof.** The first assertion follows from the proof of Lemma 8.1 and Lemma 4.3. The second assertion is immediate from the first and the definitions.
9. The different

The motivation for the following definition is that it recovers the Dedekind different in the finite flat case as we will see below.

Definition 9.1. Let \( f : Y \rightarrow X \) be a flat quasi-finite morphism of Noetherian schemes. Let \( \omega_{Y/X} \) be the relative dualizing module and let \( \tau_{Y/X} \in \Gamma(Y, \omega_{Y/X}) \) be the trace element (Remarks 2.11 and 4.7). The annihilator of

\[
\text{Coker}(O_Y \xrightarrow{\tau_{Y/X}} \omega_{Y/X})
\]

is the different of \( Y/X \). It is a coherent ideal \( \mathcal{D}_f \subset O_Y \).

We will generalize this in Remark 14.2 below. Observe that \( \mathcal{D}_f \) is locally generated by one element if \( \omega_{Y/X} \) is an invertible \( O_Y \)-module. We first state the agreement with the Dedekind different.

Lemma 9.2. Let \( f : Y \rightarrow X \) be a flat quasi-finite morphism of Noetherian schemes. Let \( V = \text{Spec}(B) \subset Y, \ U = \text{Spec}(A) \subset X \) be affine open subschemes with \( f(V) \subset U \). If the Dedekind different of \( A \rightarrow B \) is defined, then

\[
\mathcal{D}_f|_V = \widetilde{\mathcal{D}_{B/A}}
\]

as coherent ideal sheaves on \( V \).

Proof. This is clear from Lemmas 8.1 and 8.3. \( \square \)

Lemma 9.3. Let \( f : Y \rightarrow X \) be a flat quasi-finite morphism of Noetherian schemes. Let \( V = \text{Spec}(B) \subset Y, \ U = \text{Spec}(A) \subset X \) be affine open subschemes with \( f(V) \subset U \). If \( \omega_{Y/X}|_V \) is invertible, i.e., if \( \omega_{B/A} \) is an invertible \( B \)-module, then

\[
\mathcal{D}_f|_V = \mathcal{D}
\]

as coherent ideal sheaves on \( V \) where \( \mathcal{D} \subset B \) is the Noether different of \( B \) over \( A \).

Proof. Consider the map

\[
\text{Hom}_{O_Y}(\omega_{Y/X}, O_Y) \longrightarrow O_Y, \ \varphi \mapsto \varphi(\tau_{Y/X})
\]

The image of this map corresponds to the Noether different on affine opens, see Lemma 6.7. Hence the result follows from the elementary fact that given an invertible module \( \omega \) and a global section \( \tau \) the image of \( \tau : \text{Hom}(\omega, O) = \omega^{\otimes -1} \rightarrow O \) is the same as the annihilator of \( \text{Coker}(\tau : O \rightarrow \omega) \). \( \square \)

Lemma 9.4. Consider a cartesian diagram of Noetherian schemes

\[
\begin{array}{ccc}
Y' & \rightarrow & Y \\
\downarrow f' & & \downarrow f \\
X' & \rightarrow & X
\end{array}
\]

with \( f \) flat and quasi-finite. Let \( R \subset Y \), resp. \( R' \subset Y' \) be the closed subscheme cut out by the different \( \mathcal{D}_f \), resp. \( \mathcal{D}_{f'} \). Then \( Y' \rightarrow Y \) induces a bijective closed immersion \( R' \rightarrow R \times_Y Y' \). If \( g \) is flat or if \( \omega_{Y/X} \) is invertible, then \( R' = R \times_Y Y' \).
**Proof.** There is an immediate reduction to the case where $X$, $X'$, $Y$, $Y'$ are affine. In other words, we have a cocartesian diagram of Noetherian rings

$$
\begin{array}{ccc}
B' & \leftarrow & B \\
\uparrow & & \uparrow \\
A' & \leftarrow & A
\end{array}
$$

with $A \to B$ flat and quasi-finite. The base change map $\omega_{B'/A} \otimes_B B' \to \omega_{B'/A'}$ is an isomorphism (Lemma [2.10]) and maps the trace element $\tau_{B'/A}$ to the trace element $\tau_{B'/A'}$ (Lemma [4.4]). Hence the finite $B$-module $Q = \text{Coker}(\tau_{B'/A} : B \to \omega_{B/A})$ satisfies $Q \otimes_B B' = \text{Coker}(\tau_{B'/A'} : B' \to \omega_{B'/A'})$. Thus $\mathfrak{D}_{B'/A} \subset \mathfrak{D}_{B'/A'}$ which means we obtain the closed immersion $R' \to R \times_Y Y'$. Since $R = \text{Supp}(Q)$ and $R' = \text{Supp}(Q \otimes_B B')$ (Algebra, Lemma [39.5]) we see that $R' \to R \times_Y Y'$ is bijective by Algebra, Lemma [39.6]. The equality $\mathfrak{D}_{B/A} B' = \mathfrak{D}_{B'/A'}$ holds if $B \to B'$ is flat, e.g., if $A \to A'$ is flat, see Algebra, Lemma [39.4]. Finally, if $\omega_{B/A}$ is invertible, then we can localize and assume $\omega_{B/A} = B \lambda$. Writing $\tau_{B/A} = b \lambda$ we see that $Q = B/bB$ and $\mathfrak{D}_{B/A} = bB$. The same reasoning over $B'$ gives $\mathfrak{D}_{B'/A'} = bB'$ and the lemma is proved. \hfill \Box

**0BW8 Lemma 9.5.** Let $f : Y \to X$ be a finite flat morphism of Noetherian schemes. Then $\text{Norm}_f : f_* \mathcal{O}_Y \to \mathcal{O}_X$ maps $f_* \mathcal{D}_f$ into the ideal sheaf of the discriminant $\mathcal{D}_f$.

**Proof.** The norm map is constructed in Divisors, Lemma [17.6] and the discriminant of $f$ in Section [3]. The question is affine local, hence we may assume $X = \text{Spec}(A)$, $Y = \text{Spec}(B)$ and $f$ given by a finite locally free ring map $A \to B$. Localizing further we may assume $B$ is finite free as an $A$-module. Choose a basis $b_1, \ldots, b_n \in B$ for $B$ as an $A$-module. Denote $b_1^\gamma, \ldots, b_n^\gamma$ the dual basis of $\omega_{B/A} = \text{Hom}_A(B, A)$ as an $A$-module. Since the norm of $b$ is the determinant of $b : B \to B$ as an $A$-linear map, we see that $\text{Norm}_{B/A}(b) = \det(b_1^\gamma(b))$. The discriminant is the principal closed subscheme of $\text{Spec}(A)$ defined by $\det(\text{Trace}_{B/A}(b_1^\gamma))$. If $b \in \mathfrak{D}_{B/A}$ then there exist $c_i \in B$ such that $b \cdot b_1^\gamma = c_i \cdot \text{Trace}_{B/A}$ where we use a dot to indicate the $B$-module structure on $\omega_{B/A}$. Write $c_i = \sum a_{i,j} b_j$. We have

$$
\text{Norm}_{B/A}(b) = \det(b_1^\gamma(b))
= \det((b \cdot b_1^\gamma)(b))
= \det((c_i \cdot \text{Trace}_{B/A})(b))
= \det(\text{Trace}_{B/A}(c_i b_j))
= \det(a_{i,j}) \det(\text{Trace}_{B/A}(b_i b_j))
$$

which proves the lemma. \hfill \Box

**0BW9 Lemma 9.6.** Let $f : Y \to X$ be a flat quasi-finite morphism of Noetherian schemes. The closed subscheme $R \subset Y$ defined by the different $\mathcal{D}_f$ is exactly the set of points where $f$ is not étale (equivalently not unramified).

**Proof.** Since $f$ is of finite presentation and flat, we see that it is étale at a point if and only if it is unramified at that point. Moreover, the formation of the locus of ramified points commutes with base change. See Morphisms, Section [24] and especially Morphisms, Lemma [34.17] By Lemma [9.4] the formation of $R$ commutes set theoretically with base change. Hence it suffices to prove the lemma when $X$ is the spectrum of a field. On the other hand, the construction of $(\omega_Y/X, \tau_Y/X)$ is
local on $Y$. Since $Y$ is a finite discrete space (being quasi-finite over a field), we may assume $Y$ has a unique point.

Say $X = \text{Spec}(k)$ and $Y = \text{Spec}(B)$ where $k$ is a field and $B$ is a finite local $k$-algebra. If $Y \to X$ is étale, then $B$ is a finite separable extension of $k$, and the trace element $\text{Trace}_{B/k}$ is a basis element of $\omega_{B/k}$ by Fields, Lemma 20.7. Thus $\mathcal{O}_{B/k} = B$ in this case. Conversely, if $\mathcal{O}_{B/k} = B$, then we see from Lemma 9.5 and the fact that the norm of 1 equals 1 that the discriminant is empty. Hence $Y \to X$ is étale by Lemma 3.1.

\textbf{Proof.} Proof of (1). We may work locally on $Y$, hence we may assume $\omega_{Y/X}$ is free of rank 1. Say $\omega_{Y/X} = \mathcal{O}_Y \cdot \lambda$. Then we can write $\tau_{Y/X} = h\lambda$ and then we see that $R$ is defined by $h$, i.e., $R$ is locally principal.

Proof of (2). We may assume $Y \to X$ is given by a finite free ring map $A \to B$ and that $\omega_{B/A}$ is free of rank 1 as $B$-module. Choose a $B$-basis element $\lambda$ for $\omega_{B/A}$ and write $\text{Trace}_{B/A} = b \cdot \lambda$ for some $b \in B$. Then $\mathcal{O}_{B/A} = (b)$ and $D_f$ is cut out by $\det(\text{Trace}_{B/A}(b_i b_j))$ where $b_1, \ldots, b_n$ is a basis of $B$ as an $A$-module. Let $b_1^\vee, \ldots, b_n^\vee$ be the dual basis. Writing $b_i^\vee = c_i \cdot \lambda$ we see that $c_1, \ldots, c_n$ is a basis of $B$ as well. Hence with $c_i = \sum a_i b_i$ we see that $\det(a_i)$ is a unit in $A$. Clearly, $b \cdot b_i^\vee = c_i \cdot \text{Trace}_{B/A}$ hence we conclude from the computation in the proof of Lemma 9.5 that $\text{Norm}_{B/A}(b)$ is a unit times $\det(\text{Trace}_{B/A}(b_i b_j))$.

Proof of (3). In the notation above we see from Lemma 9.6 and the assumption that $h$ does not vanish in the associated points of $Y$, which implies that $h$ is a nonzerodivisor. The canonical isomorphism sends 1 to $\tau_{Y/X}$, see Divisors, Lemma 14.10.

\begin{lemma} 
Let $f : Y \to X$ be a flat quasi-finite morphism of Noetherian schemes. Let $R \subset Y$ be the closed subscheme defined by $\mathcal{Q}_f$.

\begin{enumerate}
\item If $\omega_{Y/X}$ is invertible, then $R$ is a locally principal closed subscheme of $Y$.
\item If $\omega_{Y/X}$ is invertible and $f$ is finite, then the norm of $R$ is the discriminant $D_f$ of $f$.
\item If $\omega_{Y/X}$ is invertible and $f$ is étale at the associated points of $Y$, then $R$ is an effective Cartier divisor and there is an isomorphism $\mathcal{O}_Y(R) = \omega_{Y/X}$.
\end{enumerate}
\end{lemma}

\textbf{Proof.} Proof of (1). We may work locally on $Y$, hence we may assume $\omega_{Y/X}$ is free of rank 1. Say $\omega_{Y/X} = \mathcal{O}_Y \cdot \lambda$. Then we can write $\tau_{Y/X} = h\lambda$ and then we see that $R$ is defined by $h$, i.e., $R$ is locally principal.

Proof of (2). We may assume $Y \to X$ is given by a finite free ring map $A \to B$ and that $\omega_{B/A}$ is free of rank 1 as $B$-module. Choose a $B$-basis element $\lambda$ for $\omega_{B/A}$ and write $\text{Trace}_{B/A} = b \cdot \lambda$ for some $b \in B$. Then $\mathcal{O}_{B/A} = (b)$ and $D_f$ is cut out by $\det(\text{Trace}_{B/A}(b_i b_j))$ where $b_1, \ldots, b_n$ is a basis of $B$ as an $A$-module. Let $b_1^\vee, \ldots, b_n^\vee$ be the dual basis. Writing $b_i^\vee = c_i \cdot \lambda$ we see that $c_1, \ldots, c_n$ is a basis of $B$ as well. Hence with $c_i = \sum a_i b_i$ we see that $\det(a_i)$ is a unit in $A$. Clearly, $b \cdot b_i^\vee = c_i \cdot \text{Trace}_{B/A}$ hence we conclude from the computation in the proof of Lemma 9.5 that $\text{Norm}_{B/A}(b)$ is a unit times $\det(\text{Trace}_{B/A}(b_i b_j))$.

Proof of (3). In the notation above we see from Lemma 9.6 and the assumption that $h$ does not vanish in the associated points of $Y$, which implies that $h$ is a nonzerodivisor. The canonical isomorphism sends 1 to $\tau_{Y/X}$, see Divisors, Lemma 14.10.

10. Quasi-finite syntomic morphisms

\begin{lemma} 
Let $f : Y \to X$ be a morphism of schemes. The following are equivalent.

\begin{enumerate}
\item $f$ is locally quasi-finite and syntomic,
\item $f$ is locally quasi-finite, flat, and a local complete intersection morphism,
\item $f$ is locally quasi-finite, flat, locally of finite presentation, and the fibres of $f$ are local complete intersections,
\item $f$ is locally quasi-finite and for every $y \in Y$ there are affine opens $y \in V = \text{Spec}(B) \subset U = \text{Spec}(A) \subset X$ with $f(V) \subset U$ an integer $n$ and $h, f_1, \ldots, f_n \in A[x_1, \ldots, x_n]$ such that $B = A[x_1, \ldots, x_n, 1/h]/(f_1, \ldots, f_n)$,
\item for every $y \in Y$ there are affine opens $y \in V = \text{Spec}(B) \subset U = \text{Spec}(A) \subset X$ with $f(V) \subset U$ such that $A \to B$ is a relative global complete intersection of the form $B = A[x_1, \ldots, x_n]/(f_1, \ldots, f_n)$.
\end{enumerate}
\end{lemma}
(6) \( f \) is locally quasi-finite, flat, locally of finite presentation, and \( \text{NL}_{X/Y} \) has tor-amplitude in \([-1, 0]\), and
(7) \( f \) is flat, locally of finite presentation, \( \text{NL}_{X/Y} \) is perfect of rank 0 with tor-amplitude in \([-1, 0]\),

**Proof.** The equivalence of (1) and (2) is More on Morphisms, Lemma \[54.8\]. The equivalence of (1) and (3) is Morphisms, Lemma \[29.11\].

If \( A \to B \) is as in (4), then \( B = A[x_1, \ldots, x_n]/(xh - 1, f_1, \ldots, f_n) \) is a relative global complete intersection by see Algebra, Definition \[135.5\]. Thus (4) implies (5). It is clear that (5) implies (4).

Condition (5) implies (1): by Algebra, Lemma \[135.14\] a relative global complete intersection is syntomic and the definition of a relative global complete intersection guarantees that a relative global complete intersection on \( n \) variables with \( n \) equations is quasi-finite, see Algebra, Definition \[135.5\] and Lemma \[121.2\].

Either Algebra, Lemma \[135.15\] or Morphisms, Lemma \[29.10\] shows that (1) implies (5).

More on Morphisms, Lemma \[54.16\] shows that (6) is equivalent to (1). If the equivalent conditions (1) – (6) hold, then we see that affine locally \( Y \to X \) is given by a relative global complete intersection \( B = A[x_1, \ldots, x_n]/(f_1, \ldots, f_n) \) with the same number of variables as the number of equations. Using this presentation we see that

\[
\text{NL}_{B/A} = \left( (f_1, \ldots, f_n)/(f_1, \ldots, f_n)^2 \right) \to \bigoplus_{i=1, \ldots, n} Bdx_i
\]

By Algebra, Lemma \[135.13\] the module \( (f_1, \ldots, f_n)/(f_1, \ldots, f_n)^2 \) is free with generators the congruence classes of the elements \( f_1, \ldots, f_n \). Thus \( \text{NL}_{B/A} \) has rank 0 and so does \( \text{NL}_{Y/X} \). In this way we see that (1) – (6) imply (7).

Finally, assume (7). By More on Morphisms, Lemma \[54.16\] we see that \( f \) is syntomic. Thus on suitable affine opens \( f \) is given by a relative global complete intersection \( A \to B = A[x_1, \ldots, x_n]/(f_1, \ldots, f_m) \), see Morphisms, Lemma \[29.10\].

Exactly as above we see that \( \text{NL}_{B/A} \) is a perfect complex of rank \( n - m \). Thus \( n = m \) and we see that (5) holds. This finishes the proof. \( \square \)

**Lemma 10.2.** Invertibility of the relative dualizing module.

(1) If \( A \to B \) is a quasi-finite flat homomorphism of Noetherian rings, then \( \omega_{B/A} \) is an invertible \( B \)-module if and only if \( \omega_B \otimes_A \kappa(p) \) is an invertible \( B \otimes_A \kappa(p) \)-module for all primes \( p \subset A \).

(2) If \( Y \to X \) is a quasi-finite flat morphism of Noetherian schemes, then \( \omega_{Y/X} \) is invertible if and only if \( \omega_{Y/x} \) is invertible for all \( x \in X \).

**Proof.** Proof of (1). As \( A \to B \) is flat, the module \( \omega_{B/A} \) is \( A \)-flat, see Lemma \[2.9\]. Thus \( \omega_{B/A} \) is an invertible \( B \)-module if and only if \( \omega_{B_A} \otimes_A \kappa(p) \) is an invertible \( B \otimes_A \kappa(p) \)-module for every prime \( p \subset A \), see More on Morphisms, Lemma \[16.7\].

Still using that \( A \to B \) is flat, we have that formation of \( \omega_{B/A} \) commutes with base change, see Lemma \[2.10\]. Thus we see that invertibility of the relative dualizing module, in the presence of flatness, is equivalent to invertibility of the relative dualizing module for the maps \( \kappa(p) \to B \otimes_A \kappa(p) \).

Part (2) follows from (1) and the fact that affine locally the dualizing modules are given by their algebraic counterparts, see Remark \[2.11\]. \( \square \)
0DWL **Lemma 10.3.** Let \( k \) be a field. Let \( B = k[x_1, \ldots, x_n]/(f_1, \ldots, f_n) \) be a global complete intersection over \( k \) of dimension 0. Then \( \omega_{B/k} \) is invertible.

**Proof.** By Noether normalization, see Algebra, Lemma 14.4 we see that there exists a finite injection \( k \to B \), i.e., \( \dim_k(B) < \infty \). Hence \( \omega_{B/k} = \text{Hom}_k(B, k) \) as a \( B \)-module. By Dualizing Complexes, Lemma 15.8 we see that \( \text{RHom}(B, k) \) is a dualizing complex for \( B \) and by Dualizing Complexes, Lemma 13.3 we see that \( \text{RHom}(B, k) \) is equal to \( \omega_{B/k} \) placed in degree 0. Thus it suffices to show that \( B \) is Gorenstein (Dualizing Complexes, Lemma 21.4). This is true by Dualizing Complexes, Lemma 21.7. \( \square \)

0BWF **Lemma 10.4.** Let \( f : Y \to X \) be a morphism of locally Noetherian schemes. If \( f \) satisfies the equivalent conditions of Lemma 10.3 then \( \omega_{Y/X} \) is an invertible \( \mathcal{O}_Y \)-module.

**Proof.** We may assume \( A \to B \) is a relative global complete intersection of the form \( B = A[x_1, \ldots, x_n]/(f_1, \ldots, f_n) \) and we have to show \( \omega_{B/A} \) is invertible. This follows in combining Lemmas 10.2 and 10.3. \( \square \)

0FK8 **Example 10.5.** Let \( n \geq 1 \) and \( d \geq 1 \) be integers. Let \( T \) be the set of multi-indices \( E = (e_1, \ldots, e_n) \) with \( e_i \geq 0 \) and \( \sum e_i \leq d \). Consider the ring

\[
A = \mathbb{Z}[a_{i,E}; 1 \leq i \leq n, E \in T]
\]

In \( A[x_1, \ldots, x_n] \) consider the elements \( f_i = \sum_{E \in T} a_{i,E} x^E \) where \( x^E = x_1^{e_1} \cdots x_n^{e_n} \) as is customary. Consider the \( A \)-algebra

\[
B = A[x_1, \ldots, x_n]/(f_1, \ldots, f_n)
\]

Denote \( X_{n,d} = \text{Spec}(A) \) and let \( Y_{n,d} \subset \text{Spec}(B) \) be the maximal open subscheme such that the restriction of the morphism \( \text{Spec}(B) \to \text{Spec}(A) = X_{n,d} \) is quasi-finite, see Algebra, Lemma 122.13.

0FK9 **Lemma 10.6.** With notation as in Example 10.5 the schemes \( X_{n,d} \) and \( Y_{n,d} \) are regular and irreducible, the morphism \( Y_{n,d} \to X_{n,d} \) is locally quasi-finite and syntomic, and there is a dense open subscheme \( V \subset Y_{n,d} \) such that \( Y_{n,d} \to X_{n,d} \) restricts to an étale morphism \( V \to X_{n,d} \).

**Proof.** The scheme \( X_{n,d} \) is the spectrum of the polynomial ring \( A \). Hence \( X_{n,d} \) is regular and irreducible. Since we can write

\[
f_i = a_{i,(0,\ldots,0)} + \sum_{E \in T, E \neq (0,\ldots,0)} a_{i,E} x^E
\]

we see that the ring \( B \) is isomorphic to the polynomial ring on \( x_1, \ldots, x_n \) and the elements \( a_{i,E} \) with \( E \neq (0, \ldots, 0) \). Hence \( \text{Spec}(B) \) is an irreducible and regular scheme and so is the open \( Y_{n,d} \). The morphism \( Y_{n,d} \to X_{n,d} \) is locally quasi-finite and syntomic by Lemma 10.1. To find \( V \) it suffices to find a single point where \( Y_{n,d} \to X_{n,d} \) is étale (the locus of points where a morphism is étale is open by definition). Thus it suffices to find a point of \( X_{n,d} \) where the fibre of \( Y_{n,d} \to X_{n,d} \) is nonempty and étale, see Morphisms, Lemma 34.15. We choose the point corresponding to the ring map \( \chi : A \to \mathbb{Q} \) sending \( f_i \) to \( 1 + x_i^d \). Then

\[
B \otimes_{A, \chi} \mathbb{Q} = \mathbb{Q}[x_1, \ldots, x_n]/(x_1^d - 1, \ldots, x_n^d - 1)
\]

which is a nonzero étale algebra over \( \mathbb{Q} \). \( \square \)
0FKA **Lemma 10.7.** Let \( f : Y \to X \) be a morphism of schemes. If \( f \) satisfies the equivalent conditions of Lemma 10.1 then for every \( y \in Y \) there exist \( n,d \) and a commutative diagram

\[
\begin{array}{ccc}
Y & \leftarrow & V \\
\downarrow & & \downarrow \\
X & \leftarrow & U \\
& & \downarrow \\
& & X_{n,d}
\end{array}
\]

where \( U \subset X \) and \( V \subset Y \) are open, where \( Y_{n,d} \to X_{n,d} \) is as in Example 10.5, and where the square on the right hand side is cartesian.

**Proof.** By Lemma 10.1 we can choose \( U \) and \( V \) affine so that \( U = \text{Spec}(R) \) and \( V = \text{Spec}(S) \) with \( S = R[y_1, \ldots, y_n]/(g_1, \ldots, g_n) \). With notation as in Example 10.5 if we pick \( d \) large enough, then we can write each \( g_i \) as \( g_i = \sum_{E \in T} g_{i,E} y^E \) with \( g_{i,E} \in R \). Then the map \( A \to R \) sending \( a_{i,E} \) to \( g_{i,E} \) and the map \( B \to S \) sending \( x_i \to y_i \) give a cocartesian diagram of rings

\[
\begin{array}{ccc}
S & \leftarrow & B \\
\uparrow & & \uparrow \\
R & \leftarrow & A
\end{array}
\]

which proves the lemma. \( \square \)

11. Finite syntomic morphisms

0FKX This section is the analogue of Section 10 for finite syntomic morphisms.

0FKY **Lemma 11.1.** Let \( f : Y \to X \) be a morphism of schemes. The following are equivalent

(1) \( f \) is finite and syntomic,
(2) \( f \) is finite, flat, and a local complete intersection morphism,
(3) \( f \) is finite, flat, locally of finite presentation, and the fibres of \( f \) are local complete intersections,
(4) \( f \) is finite and for every \( x \in X \) there is an affine open \( x \in U = \text{Spec}(A) \subset X \) an integer \( n \) and \( f_1, \ldots, f_n \in A[x_1, \ldots, x_n] \) such that \( f^{-1}(U) \) is isomorphic to the spectrum of \( A[x_1, \ldots, x_n]/(f_1, \ldots, f_n) \),
(5) \( f \) is finite, flat, locally of finite presentation, and \( NL_{X/Y} \) has tor-amplitude in \([-1,0]\), and
(6) \( f \) is finite, flat, locally of finite presentation, and \( NL_{X/Y} \) is perfect of rank 0 with tor-amplitude in \([-1,0]\).

**Proof.** The equivalence of (1), (2), (3), (5), and (6) and the implication (4) \( \Rightarrow \) (1) follow immediately from Lemma 10.1. Assume the equivalent conditions (1), (2), (3), (5), (6) hold. Choose a point \( x \in X \) and an affine open \( U = \text{Spec}(A) \) of \( x \) in \( X \) and say \( x \) corresponds to the prime ideal \( p \subset A \). Write \( f^{-1}(U) = \text{Spec}(B) \). Write \( B = A[x_1, \ldots, x_n]/I \). Since \( NL_{B/A} \) is perfect of tor-amplitude in \([-1,0]\) by (6) we see that \( I/I^2 \) is a finite locally free \( B \)-module of rank \( n \). Since \( B_p \) is semi-local we see that \( (I/I^2)_p \) is free of rank \( n \), see Algebra, Lemma 77.6. Thus after replacing \( A \) by a principal localization at an element not in \( p \) we may assume \( I/I^2 \) is a free \( B \)-module of rank \( n \). Thus by Algebra, Lemma 135.6 we can find a presentation of
Let $d \geq 1$ be an integer. Consider variables $a_{ij}^l$ for $1 \leq i, j, l \leq d$ and denote

$$A_d = \mathbb{Z}[a_{ij}^l]/J$$

where $J$ is the ideal generated by the elements

$$\begin{align*}
\sum_l a_{ij}^l a_{ik}^m - \sum_l a_{ij}^m a_{ik}^l & \quad \forall i, j, k, m \\
\alpha_{ij}^k - a_{ij}^k & \quad \forall i, j, k \\
\delta_{ij} & \quad \forall i, j
\end{align*}$$

where $\delta_{ij}$ indices the Kronecker delta function. We define an $A_d$-algebra $B_d$ as follows: as an $A_d$-module we set

$$B_d = A_d e_1 \oplus \ldots \oplus A_d e_d$$

The algebra structure is given by $A_d \to B_d$ mapping $1$ to $e_1$. The multiplication on $B_d$ is the $A_d$-bilinear map

$$m : B_d \times B_d \to B_d, \quad m(e_i, e_j) = \sum a_{ij}^k e_k$$

It is straightforward to check that the relations given above exactly force this to be an $A_d$-algebra structure. The morphism

$$\pi_d : Y_d = \text{Spec}(B_d) \to \text{Spec}(A_d) = X_d$$

is the “universal” finite free morphism of rank $d$.

**Example 11.2.** Let $d \geq 1$ be an integer. Consider variables $a_{ij}^l$ for $1 \leq i, j, l \leq d$ and denote

$$A_d = \mathbb{Z}[a_{ij}^l]/J$$

where $J$ is the ideal generated by the elements

$$\begin{align*}
\sum_l a_{ij}^l a_{ik}^m - \sum_l a_{ij}^m a_{ik}^l & \quad \forall i, j, k, m \\
\alpha_{ij}^k - a_{ij}^k & \quad \forall i, j, k \\
\delta_{ij} & \quad \forall i, j
\end{align*}$$

where $\delta_{ij}$ indices the Kronecker delta function. We define an $A_d$-algebra $B_d$ as follows: as an $A_d$-module we set

$$B_d = A_d e_1 \oplus \ldots \oplus A_d e_d$$

The algebra structure is given by $A_d \to B_d$ mapping $1$ to $e_1$. The multiplication on $B_d$ is the $A_d$-bilinear map

$$m : B_d \times B_d \to B_d, \quad m(e_i, e_j) = \sum a_{ij}^k e_k$$

It is straightforward to check that the relations given above exactly force this to be an $A_d$-algebra structure. The morphism

$$\pi_d : Y_d = \text{Spec}(B_d) \to \text{Spec}(A_d) = X_d$$

is the “universal” finite free morphism of rank $d$.

**Lemma 11.3.** With notation as in Example 11.2 there is an open subscheme $U_d \subset X_d$ with the following property: a morphism of schemes $X \to X_d$ factors through $U_d$ if and only if $Y_d \times_{X_d} X \to X$ is syntomic.

**Proof.** Recall that being syntomic is the same thing as being flat and a local complete intersection morphism, see More on Morphisms, Lemma 54.1.8. The set $W_d \subset Y_d$ of points where $\pi_d$ is Koszul is open in $Y_d$ and its formation commutes with arbitrary base change, see More on Morphisms, Lemma 54.1.20. Since $\pi_d$ is finite and hence closed, we see that $Z = \pi_d(Y_d\setminus W_d)$ is closed. Since clearly $U_d = X_d \setminus Z$ and since its formation commutes with base change we find that the lemma is true. \hfill $\square$

**Lemma 11.4.** With notation as in Example 11.2 and $U_d$ as in Lemma 11.3 then $U_d$ is smooth over $\text{Spec}(\mathbb{Z})$.

**Proof.** Let us use More on Morphisms, Lemma 12.1 to show that $U_d \to \text{Spec}(\mathbb{Z})$ is smooth. Namely, suppose that $\text{Spec}(A) \to U_d$ is a morphism and $A' \to A$ is a small extension. Then $B = A \otimes_A B_d$ is a finite free $A$-algebra which is syntomic over $A$ (by construction of $U_d$). By Smoothing Ring Maps, Proposition 3.2 there exists a syntomic ring map $A' \to B'$ such that $B \cong B' \otimes_{A'} A$. Set $e_i' = 1 \in B'$. For $1 \leq i \leq d$ choose lifts $e_i' \in B'$ of the elements $1 \otimes e_i \in A \otimes_A B_d = B$. Then $e_1', \ldots, e_d'$ is a basis for $B'$ over $A'$ (for example see Algebra, Lemma 100.1). Thus we can write $e_i' e_j' = \sum \alpha_{ij}^l e_l'$ for unique elements $\alpha_{ij}^l \in A'$ which satisfy the relations $\sum \alpha_{ij}^l \alpha_{il}^m = \sum \alpha_{ij}^m \alpha_{il}^l$ and $\alpha_{ij}^l = \alpha_{ji}^l = \delta_{ij} \alpha_{ii}^l$ in $A$. This determines a morphism $\text{Spec}(A') \to X_d$ by sending $a_{ij}^l \in A_d$ to $\alpha_{ij}^l \in A'$. This morphism agrees with the given morphism $\text{Spec}(A) \to U_d$. Since $\text{Spec}(A')$ and $\text{Spec}(A)$ have the same
underlying topological space, we see that we obtain the desired lift \( \text{Spec}(A') \to U_d \)
and we conclude that \( U_d \) is smooth over \( \mathbb{Z} \).

\[ \text{0FL2 Lemma 11.5.} \] With notation as in Example 11.2 consider the open subscheme
\( U'_d \subseteq X_d \) over which \( \pi_d \) is étale. Then \( U'_d \) is a dense subset of the open \( U_d \) of
Lemma 11.3

**Proof.** By exactly the same reasoning as in the proof of Lemma 11.3 using Morphisms, Lemma 34.17 there is a maximal open \( U'_d \subseteq X_d \) over which \( \pi_d \) is étale. Moreover, since an étale morphism is syntomic, we see that \( U'_d \subseteq U_d \). To finish the proof we have to show that \( U'_d \subseteq U_d \) is dense. Let \( u : \text{Spec}(k) \to U_d \) be a morphism where \( k \) is a field. Let \( B = k \otimes_{A_d} B_d \) as in the proof of Lemma 11.4

We will show there is a local domain \( A' \) with residue field \( k \) and a finite syntomic \( A' \) algebra \( B' \) with \( B = k \otimes_{A'} B' \) whose generic fibre is étale. Exactly as in the previous paragraph this will determine a morphism \( \text{Spec}(A') \to U_d \) which will map the generic point into \( U'_d \) and the closed point to \( u \), thereby finishing the proof.

By Lemma 11.1 part (4) we can choose a presentation \( B = k[x_1, \ldots, x_n]/(f_1, \ldots, f_n) \).

Let \( d' \) be the maximum total degree of the polynomials \( f_1, \ldots, f_n \). Let \( Y_{n,d'} \to X_{n,d'} \)
be as in Example 10.5 By construction there is a morphism \( u' : \text{Spec}(k) \to X_{n,d'} \)
such that

\[ \text{Spec}(B) \cong Y_{n,d'} \times_{X_{n,d'},u'} \text{Spec}(k) \]

Denote \( A = \mathcal{O}_{X_{n,d'},u'} \) the henselization of the local ring of \( X_{n,d'} \) at the image of \( u' \).

Then we can write

\[ Y_{n,d'} \times_{X_{n,d'}} \text{Spec}(A) = Z \amalg W \]

with \( Z \to \text{Spec}(A) \) finite and \( W \to \text{Spec}(A) \) having empty closed fibre, see Algebra, Lemma 149.3 part (13) or the discussion in More on Morphisms, Section 36. By Lemma 10.6 the local ring \( A \) is regular (here we also use More on Algebra, Lemma 44.10) and the morphism \( Z \to \text{Spec}(A) \) is étale over the generic point of \( \text{Spec}(A) \)
(because it is mapped to the generic point of \( X_{d,n'} \)). By construction \( Z \times_{\text{Spec}(A)} \text{Spec}(k) \cong \text{Spec}(B) \). This proves what we want except that the map from residue field of \( A \) to \( k \) may not be an isomorphism. By Algebra, Lemma 15.1 there exists a flat local ring map \( A \to A' \) such that the residue field of \( A' \) is \( k \). If \( A' \) isn’t a domain, then we choose a minimal prime \( p \subset A' \) (which lies over the unique minimal prime of \( A \) by flatness) and we replace \( A' \) by \( A' / p \). Set \( B' \) equal to the unique \( A' \)-algebra such that \( Z \times_{\text{Spec}(A')} \text{Spec}(A') = \text{Spec}(B') \). This finishes the proof.

\[ \text{0FL3 Remark 11.6.} \] Let \( \pi_d : Y_d \to X_d \) be as in Example 11.2. Let \( U_d \subseteq X_d \) be the maximal open over which \( V_d = \pi_d^{-1}(U_d) \) is finite syntomic as in Lemma 11.3 Then it is also true that \( V_d \) is smooth over \( \mathbb{Z} \). (Of course the morphism \( V_d \to U_d \) is not smooth when \( d \geq 2 \).) Arguing as in the proof of Lemma 11.4 this corresponds to the following deformation problem: given a small extension \( C' \to C \) and a finite syntomic \( C \)-algebra \( B \) with a section \( B \to C \), find a finite syntomic \( C' \)-algebra \( B' \) and a section \( B' \to C' \) whose tensor product with \( C \) recovers \( B \to C \). By Lemma 11.1 we may write \( B = C[x_1, \ldots, x_n]/(f_1, \ldots, f_n) \) as a relative global complete intersection. After a change of coordinates with may assume \( x_1, \ldots, x_n \) are in the kernel of \( B \to C \). Then the polynomials \( f_i \) have vanishing constant terms. Choose any lifts \( f_i' \in C'[x_1, \ldots, x_n] \) of \( f_i \) with vanishing constant terms. Then \( B' = C'[x_1, \ldots, x_n]/(f_1', \ldots, f_n') \) with section \( B' \to C' \) sending \( x_i \) to zero works.
Lemma 11.7. Let \( f : Y \to X \) be a morphism of schemes. If \( f \) satisfies the equivalent conditions of Lemma [11.1] then for every \( x \in X \) there exist a \( d \) and a commutative diagram

\[
\begin{array}{ccc}
Y & \xrightarrow{V} & Y_d \\
\downarrow & & \downarrow \\
X & \xrightarrow{U} & X_d \\
\end{array}
\]

with the following properties

1. \( U \subset X \) is open and \( V = f^{-1}(U) \).
2. \( \pi_d : Y_d \to X_d \) is as in Example [11.2].
3. \( U_d \subset X_d \) is as in Lemma [11.3] and \( V_d = \pi_d^{-1}(U_d) \subset Y_d \).
4. where the middle square is cartesian.

Proof. Choose an affine open neighbourhood \( U = \text{Spec}(A) \subset X \) of \( x \). Write \( V = f^{-1}(U) = \text{Spec}(B) \). Then \( B \) is a finite locally free \( A \)-module and the inclusion \( A \subset B \) is a locally direct summand. Thus after shrinking \( U \) we can choose a basis \( 1 = e_1, e_2, \ldots, e_d \) of \( B \) as an \( A \)-module. Write \( e_i e_j = \sum \alpha_{ij}^k e_k \) for unique elements \( \alpha_{ij}^k \in A \) which satisfy the relations \( \sum_i \alpha_{ij}^i \alpha_{ik}^m = \sum_j \alpha_{ij}^j \alpha_{jk}^m \) and \( \alpha_{ij}^k = \alpha_{ji}^k \) and \( \alpha_{ij}^l - \delta_{ij} \) in \( A \). This determines a morphism \( \text{Spec}(A) \to X_d \) by sending \( a_{ij}^1 \in A_d \) to \( a_{ij}^l \in A \). By construction \( V \cong \text{Spec}(A) \times_{X_d} Y_d \). By the definition of \( U_d \) we see that \( \text{Spec}(A) \to X_d \) factors through \( U_d \). This finishes the proof. \( \square \)

12. A formula for the different

In this section we discuss the material in [MR70 Appendix A] due to Tate. In our language, this will show that the different is equal to the Kähler different in the case of a flat, quasi-finite, local complete intersection morphism. First we compute the Noether different in a special case.

Lemma 12.1. Let \( A \to P \) be a ring map. Let \( f_1, \ldots, f_n \in P \) be a Koszul regular sequence. Assume \( B = P/(f_1, \ldots, f_n) \) is flat over \( A \). Let \( g_1, \ldots, g_n \in P \otimes_A B \) be a Koszul regular sequence generating the kernel of the multiplication map \( P \otimes_A B \to B \). Write \( f \otimes 1 = \sum g_{ij} g_j \). Then the annihilator of \( \ker(B \otimes_A B \to B) \) is a principal ideal generated by the image of \( \text{det}(g_{ij}) \).

Proof. The Koszul complex \( K_* = K(P, f_1, \ldots, f_n) \) is a resolution of \( B \) by finite free \( P \)-modules. The Koszul complex \( M_* = K(P \otimes_A B, g_1, \ldots, g_n) \) is a resolution of \( B \) by finite free \( P \otimes_A B \)-modules. There is a map of complexes

\[ K_* \to M_* \]

which in degree 1 is given by the matrix \( (g_{ij}) \) and in degree \( n \) by \( \text{det}(g_{ij}) \). See More on Algebra, Lemma [28.3] As \( B \) is a flat \( A \)-module, we can view \( M_* \) as a complex of flat \( \text{free} \) \( P \)-modules (via \( P \to P \otimes_A B, p \mapsto p \otimes 1 \)). Thus we may use both complexes to compute \( \text{Tor}_n^P(B, B) \) and it follows that the displayed map defines a quasi-isomorphism after tensoring with \( B \). It is clear that \( H_n(K_* \otimes_P B) = B \). On the other hand, \( H_n(M_* \otimes_P B) \) is the kernel of \( B \otimes_A B \to B \) since \( g_1, \ldots, g_n \) generate the kernel of \( B \otimes_A B \to B \) this proves the lemma. \( \square \)
Lemma 12.2. Let $A$ be a ring. Let $n \geq 1$ and $f_1, \ldots, f_n \in A[x_1, \ldots, x_n]$. Set $B = A[x_1, \ldots, x_n]/(h, f_1, \ldots, f_n)$. Assume that $B$ is quasi-finite over $A$. Then

1. $B$ is flat over $A$ and $A \to B$ is a relative local complete intersection,
2. the annihilator $J$ of $I = \text{Ker}(B \otimes_A B \to B)$ is free of rank 1 over $B$.
3. the Noether different of $B$ over $A$ is generated by $\det(\partial f_i/\partial x_j)$ in $B$.

Proof. Note that $B = A[x_1, x_2, \ldots, x_n]/(h, f_1, \ldots, f_n)$ is a relative global complete intersection over $A$, see Algebra, Definition 135.5. By Algebra, Lemma 135.14 we see that $B$ is flat over $A$.

Write $P' = A[x_1, x_2, \ldots, x_n]$ and $P = P'/\langle xh - 1, f_1, \ldots, f_n \rangle$. Then we have $P' \to P \to B$. By More on Algebra, Lemma 32.4 we see that $xh - 1, f_1, \ldots, f_n$ is a Koszul regular sequence in $P'$. Since $xh - 1$ is a Koszul regular sequence of length one in $P'$ (by the same lemma for example) we conclude that $f_1, \ldots, f_n$ is a Koszul regular sequence in $P$ by More on Algebra, Lemma 29.14.

Let $g_i \in P \otimes_A B$ be the image of $x_i \otimes 1 - 1 \otimes x_i$. Let us use the short hand $y_i = x_i \otimes 1$ and $z_i = 1 \otimes x_i$ in $A[x_1, \ldots, x_n] \otimes_A A[x_1, \ldots, x_n]$ so that $g_i$ is the image of $y_i - z_i$.

For a polynomial $f \in A[x_1, \ldots, x_n]$ we write $f(y) = f \otimes 1$ and $f(z) = 1 \otimes f$ in the above tensor product. Then we have

$$P \otimes_A B/(g_1, \ldots, g_n) = \frac{A[y_1, \ldots, y_n, z_1, \ldots, z_n, 1/(y_i h(z_i))]}{(f_1(z), \ldots, f_n(z), y_1 - z_1, \ldots, y_n - z_n)}$$

which is clearly isomorphic to $B$. Hence by the same arguments as above we find that $f_1(z), \ldots, f_n(z), y_1 - z_1, \ldots, y_n - z_n$ is a Koszul regular sequence in $A[y_1, \ldots, y_n, z_1, \ldots, z_n, 1/(y_i h(z_i))]$.

The sequence $f_1(z), \ldots, f_n(z)$ is a Koszul regular in $A[y_1, \ldots, y_n, z_1, \ldots, z_n, 1/(y_i h(z_i))]$ by flatness of the map

$$P \longrightarrow A[y_1, \ldots, y_n, z_1, \ldots, z_n, 1/(y_i h(z_i))], \quad x_i \longmapsto z_i$$

and More on Algebra, Lemma 29.5. By More on Algebra, Lemma 29.14 we conclude that $g_1, \ldots, g_n$ is a regular sequence in $P \otimes_A B$.

At this point we have verified all the assumptions of Lemma 12.1 above with $P$, $f_1, \ldots, f_n$, and $g_i \in P \otimes_A B$ as above. In particular the annihilator $J$ of $I$ is freely generated by one element $\delta$ over $B$. Set $f_{ij} = \partial f_i/\partial x_j \in A[x_1, \ldots, x_n]$. An elementary computation shows that we can write

$$f_i(y) = f_i(z_1 + g_1, \ldots, z_n + g_n) = f_i(z) + \sum_j f_{ij}(z)g_j + \sum_{j,j'} F_{ijj'} g_j g_{j'}$$

for some $F_{ijj'} \in A[y_1, \ldots, y_n, z_1, \ldots, z_n]$. Taking the image in $P \otimes_A B$ the terms $f_i(z)$ map to zero and we obtain

$$f_i \otimes 1 = \sum_j \left( 1 \otimes f_{ij} + \sum_{j'} F_{ijj'} g_{j'} \right) g_j$$

Thus we conclude from Lemma 12.1 that $\delta = \det(g_{ij})$ with $g_{ij} = 1 \otimes f_{ij} + \sum_{j'} F_{ijj'} g_{j'}$. Since $g_{ij}$ maps to zero in $B$, we conclude that the image of $\det(\partial f_i/\partial x_j)$ in $B$ generates the Noether different of $B$ over $A$. \qed
Proof. By Lemmas \[9.3\] and \[10.4\] the different of \(f\) affine locally is the same as the Noether different. Then the lemma follows from the computation of the Noether different and the Kähler different on standard affine pieces done in Lemmas \[7.4\] and \[12.2\].

0BWH Lemma 12.4. Let \(A\) be a ring. Let \(n \geq 1\) and \(h, f_1, \ldots, f_n \in A[x_1, \ldots, x_n]\). Set \(B = A[x_1, \ldots, x_n, 1/h]/(f_1, \ldots, f_n)\). Assume that \(B\) is quasi-finite over \(A\). Then there is an isomorphism \(B \to \omega_{B/A}\) mapping \(\det(\partial f_i/\partial x_j)\) to \(\tau_{B/A}\).

Proof. Let \(J\) be the annihilator of \(\text{Ker}(B \otimes_A B \to B)\). By Lemma \[12.2\] the map \(A \to B\) is flat and \(J\) is a free \(B\)-module with generator \(\xi\) mapping to \(\det(\partial f_i/\partial x_j)\) in \(B\). Thus the lemma follows from Lemma \[6.7\] and the fact (Lemma \[10.4\]) that \(\omega_{B/A}\) is an invertible \(B\)-module. (Warning: it is necessary to prove \(\omega_{B/A}\) is invertible because a finite \(B\)-module \(M\) such that \(\text{Hom}_B(M, B) \cong B\) need not be free.)

0BWJ Example 12.5. Let \(A\) be a Noetherian ring. Let \(f, h \in A[x]\) such that

\[
B = (A[x]/(f))_h = A[x, 1/h]/(f)
\]

is quasi-finite over \(A\). Let \(f' \in A[x]\) be the derivative of \(f\) with respect to \(x\). The ideal \(\mathcal{O} = (f') \subset B\) is the Noether different of \(B\) over \(A\), is the Kähler different of \(B\) over \(A\), and is the ideal whose associated quasi-coherent sheaf of ideals is the different of \(\text{Spec}(B)\) over \(\text{Spec}(A)\).

0BWJ Lemma 12.6. Let \(S\) be a Noetherian scheme. Let \(X, Y\) be smooth schemes of relative dimension \(n\) over \(S\). Let \(f : Y \to X\) be a quasi-finite morphism over \(S\). Then \(f\) is flat and the closed subscheme \(R \subset Y\) cut out by the different of \(f\) is the locally principal closed subscheme cut out by

\[
\wedge^n(df) \in \Gamma(Y, (f^*\Omega^n_{X/S})^\otimes -1 \otimes_{\mathcal{O}_Y} \Omega^n_{Y/S})
\]

If \(f\) is étale at the associated points of \(Y\), then \(R\) is an effective Cartier divisor and

\[
f^*\Omega^n_{X/S} \otimes_{\mathcal{O}_Y} \mathcal{O}(R) = \Omega^n_{Y/S}
\]
as invertible sheaves on \(Y\).

Proof. To prove that \(f\) is flat, it suffices to prove \(Y_s \to X_s\) is flat for all \(s \in S\) (More on Morphisms, Lemma \[16.3\]). Flatness of \(Y_s \to X_s\) follows from Algebra, Lemma \[127.1\]. By More on Morphisms, Lemma \[54.10\] the morphism \(f\) is a local complete intersection morphism. Thus the statement on the different follows from the corresponding statement on the Kähler different by Lemma \[12.3\]. Finally, since we have the exact sequence

\[
f^*\Omega^n_{X/S} \overset{df}{\longrightarrow} \Omega^n_{X/S} \to \Omega^n_{Y/X} \to 0
\]

by Morphisms, Lemma \[31.9\] and since \(\Omega^n_{X/S}\) and \(\Omega^n_{Y/S}\) are finite locally free of rank \(n\) (Morphisms, Lemma \[32.12\]), the statement for the Kähler different is clear from the definition of the zeroth fitting ideal. If \(f\) is étale at the associated points of \(Y\), then \(\wedge^n df\) does not vanish in the associated points of \(Y\), which implies that the local equation of \(R\) is a nonzerodivisor. Hence \(R\) is an effective Cartier divisor. The canonical isomorphism sends 1 to \(\wedge^n df\), see Divisors, Lemma \[14.10\].

\[\square\]
13. The Tate map

In this section we produce an isomorphism between the determinant of the relative cotangent complex and the relative dualizing module for a locally quasi-finite syntomic morphism of locally Noetherian schemes. Following [Gar84, 1.4.4] we dub the isomorphism the Tate map. Our approach is to avoid doing local calculations as much as is possible.

Let $Y \to X$ be a locally quasi-finite syntomic morphism of schemes. We will use all the equivalent conditions for this notion given in Lemma 10.1 without further mention in this section. In particular, we see that $NL_{Y/X}$ is a perfect object of $D(O_Y)$ with tor-amplitude in $[-1, 0]$. Thus we have a canonical invertible module $\det(NL_{Y/X})$ on $Y$ and a global section $\delta(NL_{Y/X}) \in \Gamma(Y, \det(NL_{Y/X}))$.

Suppose given a commutative diagram of schemes

$$
\begin{array}{ccc}
Y' & \rightarrow & Y \\
\downarrow & & \downarrow \\
X' & \rightarrow & X
\end{array}
$$

whose vertical arrows are locally quasi-finite syntomic and which induces an isomorphism of $Y'$ with an open of $X' \times_X Y$. Then the canonical map $Lb^* NL_{Y/X} \longrightarrow NL_{Y'/X'}$ is a quasi-isomorphism by More on Morphisms, Lemma 13.10. Thus we get a canonical isomorphism $b^* \det(NL_{Y/X}) \rightarrow \det(NL_{Y'/X'})$ which sends the canonical section $\delta(NL_{Y/X})$ to $\delta(NL_{Y'/X'})$, see Derived Categories of Schemes, Remark 36.2.

**Remark 13.1.** Let $Y \to X$ be a locally quasi-finite syntomic morphism of schemes. What does the pair $(\det(NL_{Y/X}), \delta(NL_{Y/X}))$ look like locally? Choose affine opens $V = \text{Spec}(B) \subset Y$, $U = \text{Spec}(A) \subset X$ with $f(V) \subset U$ and an integer $n$ and $f_1, \ldots, f_n \in A[x_1, \ldots, x_n]$ such that $B = A[x_1, \ldots, x_n]/(f_1, \ldots, f_n)$. Then

$$
NL_{B/A} = \left( (f_1, \ldots, f_n)/(f_1, \ldots, f_n)^2 \right) \longrightarrow \bigoplus_{i=1, \ldots, n} B dx_i
$$

and $(f_1, \ldots, f_n)/(f_1, \ldots, f_n)^2$ is free with generators the classes $\bar{f}_i$. See proof of Lemma 10.1. Thus $\det(L_{B/A})$ is free on the generator $dx_1 \wedge \ldots \wedge dx_n \otimes (\bar{f}_1 \wedge \ldots \wedge \bar{f}_n)^{\otimes -1}$ and the section $\delta(NL_{B/A})$ is the element

$$
\delta(NL_{B/A}) = \det(\partial f_j/\partial x_i) \cdot dx_1 \wedge \ldots \wedge dx_n \otimes (\bar{f}_1 \wedge \ldots \wedge \bar{f}_n)^{\otimes -1}
$$

by definition.

Let $Y \to X$ be a locally quasi-finite syntomic morphism of locally Noetherian schemes. By Remarks 2.11 and 4.7 we have a coherent $O_Y$-module $\omega_{Y/X}$ and a canonical global section $\tau_{Y/X} \in \Gamma(Y, \omega_{Y/X})$. 

which affine locally recovers the pair \( \omega_{B/A}, \tau_{B/A} \). By Lemma \[10.4\] the module \( \omega_{Y/X} \) is invertible. Suppose given a commutative diagram of locally Noetherian schemes

\[
\begin{array}{ccc}
Y' & \xrightarrow{b} & Y \\
\downarrow & & \downarrow \\
X' & \xrightarrow{a} & X 
\end{array}
\]

whose vertical arrows are locally quasi-finite syntomic and which induces an isomorphism of \( Y' \) with an open of \( X' \times_X Y \). Then there is a canonical base change map

\[ b^*\omega_{Y/X} \to \omega_{Y'/X'} \]

which is an isomorphism mapping \( \tau_{Y/X} \) to \( \tau_{Y'/X'} \). Namely, the base change map in the affine setting is \( \text{(2.3.1)} \), it is an isomorphism by Lemma \[2.10\] and it maps \( \tau_{Y/X} \) to \( \tau_{Y'/X'} \) by Lemma \[4.4\] part (1).

0FKD Proposition 13.2. There exists a unique rule that to every locally quasi-finite syntomic morphism of locally Noetherian schemes \( Y \to X \) assigns an isomorphism

\[ c_{Y/X} : \det(NL_{Y/X}) \to \omega_{Y/X} \]

satisfying the following two properties

1. the section \( \delta(NL_{Y/X}) \) is mapped to \( \tau_{Y/X} \), and
2. the rule is compatible with restriction to opens and with base change.

Proof. Let us reformulate the statement of the proposition. Consider the category \( \mathcal{C} \) whose objects, denoted \( Y/X \), are locally quasi-finite syntomic morphism \( Y \to X \) of locally Noetherian schemes and whose morphisms \( b/a : Y'/X' \to Y/X \) are commutative diagrams

\[
\begin{array}{ccc}
Y' & \xrightarrow{b} & Y \\
\downarrow & & \downarrow \\
X' & \xrightarrow{a} & X 
\end{array}
\]

which induce an isomorphism of \( Y' \) with an open subscheme of \( X' \times_X Y \). The proposition means that for every object \( Y/X \) of \( \mathcal{C} \) we have an isomorphism \( c_{Y/X} : \det(NL_{Y/X}) \to \omega_{Y/X} \) with \( c_{Y/X}(\delta(NL_{Y/X})) = \tau_{Y/X} \) and for every morphism \( b/a : Y'/X' \to Y/X \) of \( \mathcal{C} \) we have \( b^*c_{Y/X} = c_{Y'/X'} \), through the identifications \( b^*\det(NL_{Y/X}) = \det(NL_{Y'/X'}) \) and \( b^*\omega_{Y/X} = \omega_{Y'/X'} \), described above.

Given \( Y/X \) in \( \mathcal{C} \) and \( y \in Y \) we can find an affine open \( V \subset Y \) and \( U \subset X \) with \( f(V) \subset U \) such that there exists some isomorphism

\[ \det(NL_{Y/X})|_V \to \omega_{Y/X}|_V \]

mapping \( \delta(NL_{Y/X})|_V \) to \( \tau_{Y/X}|_V \). This follows from picking affine opens as in Lemma \[10.1\] part (5), the affine local description of \( \delta(NL_{Y/X}) \) in Remark \[13.1\] and Lemma \[12.4\]. If the annihilator of the section \( \tau_{Y/X} \) is zero, then these local maps are unique and automatically glue. Hence if the annihilator of \( \tau_{Y/X} \) is zero, then there is a unique isomorphism \( c_{Y/X} : \det(NL_{Y/X}) \to \omega_{Y/X} \) with \( c_{Y/X}(\delta(NL_{Y/X})) = \tau_{Y/X} \). If \( b/a : Y'/X' \to Y/X \) is a morphism of \( \mathcal{C} \) and the annihilator of \( \tau_{Y'/X'} \) is zero as well, then \( b^*c_{Y/X} \) is the unique isomorphism \( c_{Y'/X'} : \det(NL_{Y'/X'}) \to \omega_{Y'/X'} \) with \( c_{Y'/X'}(\delta(NL_{Y'/X'})) = \tau_{Y'/X'} \). This follows formally from the fact that \( b^*\delta(NL_{Y/X}) = \delta(NL_{Y'/X'}) \) and \( b^*\tau_{Y/X} = \tau_{Y'/X'} \).
We can summarize the results of the previous paragraph as follows. Let $\mathcal{C}_{\text{nice}} \subset \mathcal{C}$ denote the full subcategory of $Y/X$ such that the annihilator of $\tau_{Y/X}$ is zero. Then we have solved the problem on $\mathcal{C}_{\text{nice}}$. For $Y/X$ in $\mathcal{C}_{\text{nice}}$ we continue to denote $c_{Y/X}$ the solution we’ve just found.

Consider morphisms

$$Y_1/X_1 \leftarrow b_1/a_1 \quad Y/X \quad b_2/a_2 \rightarrow Y_2/X_2$$

in $\mathcal{C}$ such that $Y_1/X_1$ and $Y_2/X_2$ are objects of $\mathcal{C}_{\text{nice}}$. \textbf{Claim.} $b_1^* c_{Y_1/X_1} = b_2^* c_{Y_2/X_2}$.

We will first show that the claim implies the proposition and then we will prove the claim.

Let $d, n \geq 1$ and consider the locally quasi-finite syntomic morphism $Y_{n,d} \rightarrow X_{n,d}$ constructed in Example 10.5. Then $Y_{n,d}$ is an irreducible regular scheme and the morphism $Y_{n,d} \rightarrow X_{n,d}$ is locally quasi-finite syntomic and étale over a dense open, see Lemma 10.6. Thus $\tau_{Y_{n,d}/X_{n,d}}$ is nonzero for example by Lemma 9.6. Now a nonzero section of an invertible module over an irreducible regular scheme has vanishing annihilator. Thus $Y_{n,d}/X_{n,d}$ is an object of $\mathcal{C}_{\text{nice}}$.

Let $Y/X$ be an arbitrary object of $\mathcal{C}$. Let $y \in Y$. By Lemma 10.7 we can find $n, d \geq 1$ and morphisms

$$Y/X \leftarrow V/U \quad b/a \rightarrow Y_{n,d}/X_{n,d}$$

of $\mathcal{C}$ such that $V \subset Y$ and $U \subset X$ are open. Thus we can pullback the canonical morphism $c_{Y_{n,d}/X_{n,d}}$ constructed above by $b$ to $V$. The claim guarantees these local isomorphisms glue! Thus we get a well defined global isomorphism $c_{Y/X} : \det(NL_{Y/X}) \rightarrow \omega_{Y/X}$ with $c_{Y/X}(\delta(NL_{Y/X})) = \tau_{Y/X}$. If $b/a : Y'/X' \rightarrow Y/X$ is a morphism of $\mathcal{C}$, then the claim also implies that the similarly constructed map $c_{Y'/X'}$ is the pullback by $b$ of the locally constructed map $c_{Y/X}$. Thus it remains to prove the claim.

In the rest of the proof we prove the claim. We may pick a point $y \in Y$ and prove the maps agree in an open neighbourhood of $y$. Thus we may replace $Y_1, Y_2$ by open neighbourhoods of the image of $y$ in $Y_1$ and $Y_2$. Thus we may assume there are morphisms

$$Y_{n_1,d_1}/X_{n_1,d_1} \leftarrow Y_1/X_1 \quad \text{and} \quad Y_2/X_2 \rightarrow Y_{n_2,d_2}/X_{n_2,d_2}$$

These are morphisms of $\mathcal{C}_{\text{nice}}$ for which we know the desired compatibilities. Thus we may replace $Y_1/X_1$ by $Y_{n_1,d_1}/X_{n_1,d_1}$ and $Y_2/X_2$ by $Y_{n_2,d_2}/X_{n_2,d_2}$. This reduces us to the case that $Y_1, Y_2, X_2$ are of finite type over $\mathbf{Z}$. (The astute reader will realize that this step wouldn’t have been necessary if we’d defined $\mathcal{C}_{\text{nice}}$ to consist only of those objects $Y/X$ with $Y$ and $X$ of finite type over $\mathbf{Z}$.)

Assume $Y_1, X_1, Y_2, X_2$ are of finite type over $\mathbf{Z}$. After replacing $Y, X, Y_1, X_1, Y_2, X_2$ by suitable open neighbourhoods of the image of $y$ we may assume $Y, X, Y_1, X_1, Y_2, X_2$ are affine. We may write $X = \lim X_\lambda$ as a cofiltered limit of affine schemes of finite type over $X_1 \times X_2$. For each $\lambda$ we get

$$Y_1 \times_{X_1} X_\lambda \quad \text{and} \quad X_\lambda \times_{X_2} Y_2$$

If we take limits we obtain

$$\lim Y_1 \times_{X_1} X_\lambda = Y_1 \times_{X_1} X \supset Y \subset X \times_{X_2} Y_2 = \lim X_\lambda \times_{X_2} Y_2$$
By Limits, Lemma \cite{Limits_4.11} we can find a $\lambda$ and opens $V_{i,\lambda} \subset Y_1 \times X_1, X_\lambda$ and $V_{2,\lambda} \subset X_\lambda \times X_2 Y_2$ whose base change to $X$ recovers $Y$ (on both sides). After increasing $\lambda$ we may assume there is an isomorphism $V_{i,\lambda} \rightarrow V_{2,\lambda}$ whose base change to $X$ is the identity on $Y$, see Limits, Lemma \cite{Limits_10.1}. Then we have the commutative diagram

\[
\begin{array}{ccc}
Y/X & \longrightarrow & Y_2/X_2 \\
\downarrow & & \downarrow \quad b_2/a_2 \quad b_1/a_1 \\
Y_1/X_1 & \leftarrow & V_{1,\lambda}/X_\lambda \longrightarrow Y_2/X_2
\end{array}
\]

Thus it suffices to prove the claim for the lower row of the diagram and we reduce to the case discussed in the next paragraph.

Assume $Y, X, Y_1, X_1, Y_2, X_2$ are affine of finite type over $\mathbb{Z}$. Write $X = \text{Spec}(A)$, $X_1 = \text{Spec}(A_1)$. The ring map $A_1 \rightarrow A$ corresponding to $X \rightarrow X_1$ is of finite type and hence we may choose a surjection $A_1[x_1, \ldots, x_n] \rightarrow A$. Similarly, we may choose a surjection $A_2[y_1, \ldots, y_m] \rightarrow A$. Set $X'_1 = \text{Spec}(A_1[x_1, \ldots, x_n])$ and $X'_2 = \text{Spec}(A_2[y_1, \ldots, y_m])$. Set $Y'_1 = Y_1 \times X_1, X'_1$ and $Y'_2 = Y_2 \times X_2, X'_2$. We get the following diagram

\[
Y_1/X_1 \leftarrow Y'_1/X'_1 \leftarrow Y/X \rightarrow Y'_2/X'_2 \rightarrow Y_2/X_2
\]

Since $X'_1 \rightarrow X_1$ and $X'_2 \rightarrow X_2$ are flat, the same is true for $Y'_1 \rightarrow Y_1$ and $Y'_2 \rightarrow Y_2$. It follows easily that the annihilators of $\tau_{Y'_1/X'_1}$ and $\tau_{Y'_2/X'_2}$ are zero. Hence $Y'_1/X'_1$ and $Y'_2/X'_2$ are in $C_{\text{nice}}$. Thus the outer morphisms in the displayed diagram are morphisms of $C_{\text{nice}}$ for which we know the desired compatibilities. Thus it suffices to prove the claim for $Y'_1/X'_1 \leftarrow Y/X \rightarrow Y'_2/X'_2$. This reduces us to the case discussed in the next paragraph.

Assume $Y, X, Y_1, X_1, Y_2, X_2$ are affine of finite type over $\mathbb{Z}$ and $X \rightarrow X_1$ and $X \rightarrow X_2$ are closed immersions. Consider the open embeddings $Y_1 \times X_1, X_1 \supset Y \subset X \times X_2, Y_2$. There is an open neighbourhood $V \subset Y$ which is a standard open of both $Y_1 \times X_1, X_1 \times X_2, Y_2$. This follows from Schemes, Lemma \cite{Schemes_11.5} applied to the scheme obtained by glueing $Y_1 \times X_1, X_1 \times X_2, Y_2$ along $Y$; details omitted. Since $X \times X_2, Y_2$ is a closed subscheme of $Y_2$ we can find a standard open $V_2 \subset Y_2$ such that $V_2 \times X_2, X_2 \times X_2, Y_2 = V$. Similarly, we can find a standard open $V_1 \subset Y_1$ such that $V_1 \times X_1, X_1 = V$. After replacing $Y, Y_1, Y_2$ by $V, V_1, V_2$ we reduce to the case discussed in the next paragraph.

Assume $Y, X, Y_1, X_1, Y_2, X_2$ are affine of finite type over $\mathbb{Z}$ and $X \rightarrow X_1$ and $X \rightarrow X_2$ are closed immersions and $Y_1 \times X_1, X_1 = Y = X \times X_2, Y_2$. Write $X = \text{Spec}(A)$, $X_1 = \text{Spec}(A_1)$, $Y = \text{Spec}(B)$, $Y_2 = \text{Spec}(B_2)$. Then we can consider the affine schemes

\[
X' = \text{Spec}(A_1 \times_A, A_2) = \text{Spec}(A') \quad \text{and} \quad Y' = \text{Spec}(B_1 \times_B, B_2) = \text{Spec}(B')
\]

Observe that $X' = X_1 \amalg X_2$ and $Y' = Y_1 \amalg Y_2$, see More on Morphisms, Lemma \cite{More_on_Morphisms_14.1}. By More on Algebra, Lemma \cite{More_on_Algebra_5.1} the rings $A'$ and $B'$ are of finite type over $\mathbb{Z}$. By More on Algebra, Lemma \cite{More_on_Algebra_6.4} we have $B' \otimes_A, A_1 = B_1$ and $B' \times_A, A_2 = B_2$. In particular a fibre of $Y' \rightarrow X'$ over a point of $X' = X_1 \amalg X_2$ is always equal to either a fibre of $Y_1 \rightarrow X_1$ or a fibre of $Y_2 \rightarrow X_2$. By More on Algebra, Lemma \cite{More_on_Algebra_6.8} the ring map $A' \rightarrow B'$ is flat. Thus by Lemma \cite{Limits_10.1} part (3) we conclude that
\[ Y'/X' \text{ is an object of } \mathcal{C}. \text{ Consider now the commutative diagram} \]

\[
\begin{array}{ccc}
Y/X & \xrightarrow{b_2/a_2} & Y_2/X_2 \\
\downarrow{b_1/a_1} & & \downarrow{b_1/a_1}\\
Y'/X' & \downarrow{b_2/a_2} & Y_2'/X_2 \\
& \downarrow & \downarrow
\end{array}
\]

Now we would be done if \( Y'/X' \) is an object of \( \mathcal{C}_{nice} \). Namely, then pulling back \( c_{Y'/X'} \) around the two sides of the square, we would obtain the desired conclusion. Now, in fact, it is true that \( Y'/X' \) is an object of \( \mathcal{C}_{nice} \). But it is amusing to note that we don’t even need this. Namely, the arguments above show that, after possibly shrinking all of the schemes \( X, Y, X_1, Y_1, X_2, Y_2, X', Y' \) we can find some \( n, d \geq 1 \), and extend the diagram like so:

\[
\begin{array}{ccc}
Y/X & \xrightarrow{b_2/a_2} & Y_2/X_2 \\
\downarrow{b_1/a_1} & & \downarrow{b_1/a_1}\\
Y'/X' & \downarrow{b_2/a_2} & Y_2'/X_2 \\
& \downarrow & \downarrow
\end{array}
\]

and then we can use the already given argument by pulling back from \( c_{Y_{n,d}/X_{n,d}} \). This finishes the proof. \( \square \)

14. A generalization of the different

In this section we generalize Definition 9.1 to take into account all cases of ring maps \( A \to B \) where the Dedekind different is defined and \( 1 \in \mathcal{L}_{B/A} \). First we explain the condition “\( A \to B \) maps nonzerodivisors to nonzerodivisors and induces a flat map \( Q(A) \to Q(A) \otimes_A B \)”.

Lemma 14.1. Let \( A \to B \) be a map of Noetherian rings. Consider the conditions

1. nonzerodivisors of \( A \to 0 \) map to nonzerodivisors of \( B \),
2. (1) holds and \( Q(A) \to Q(A) \otimes_A B \) is flat,
3. \( A \to B_q \) is flat for every \( q \in \text{Ass}(B) \),
4. (3) holds and \( A \to B_q \) is flat for every \( q \) lying over an element in \( \text{Ass}(A) \).

\[
\text{Namely, the structure sheaf } \mathcal{O}_{Y'} \text{ is a subsheaf of } (Y_1 \to Y')^* \mathcal{O}_{Y_1} \times (Y_2 \to Y')^* \mathcal{O}_{Y_2}.
\]
Then we have the following implications

\[
\begin{array}{c}
(1) \quad \iff \quad (2) \\
(3) \quad \iff \quad (4)
\end{array}
\]

If going up holds for \( A \to B \) then (2) and (4) are equivalent.

**Proof.** The horizontal implications in the diagram are trivial. Let \( S \subset A \) be the set of nonzerodivisors so that \( Q(A) = S^{-1}A \) and \( Q(A) \otimes_A B = S^{-1}B \). Recall that \( S = A \setminus \bigcup_{p \in \text{Ass}(A)} \mathfrak{p} \) by Algebra, Lemma [62.9]. Let \( q \subset B \) be a prime lying over \( p \subset A \).

Assume (2). If \( q \in \text{Ass}(B) \) then \( q \) consists of zerodivisors, hence (1) implies the same is true for \( p \). Hence \( p \) corresponds to a prime of \( S^{-1}A \). Hence \( A \to B_q \) is flat by our assumption (2). If \( q \) lies over an associated prime \( p \) of \( A \), then certainly \( p \in \text{Spec}(S^{-1}A) \) and the same argument works.

Assume (3). Let \( f \in A \) be a nonzerodivisor. If \( f \) were a zerodivisor on \( B \), then \( f \) is contained in an associated prime \( q \) of \( B \). Since \( A \to B_q \) is flat by assumption, we conclude that \( p \) is an associated prime of \( A \) by Algebra, Lemma [64.3]. This would imply that \( f \) is a zerodivisor on \( A \), a contradiction.

Assume (4) and going up for \( A \to B \). We already know (1) holds. If \( q \) corresponds to a prime of \( S^{-1}B \) then \( p \) is contained in an associated prime \( p' \) of \( A \). By going up there exists a prime \( q' \) containing \( q \) and lying over \( p \). Then \( A \to B_{q'} \) is flat by (4). Hence \( A \to B_q \) is flat as a localization. Thus \( A \to S^{-1}B \) is flat and so is \( S^{-1}A \to S^{-1}B \), see Algebra, Lemma [38.18].

**Remark 14.2.** We can generalize Definition 9.1. Suppose that \( f : Y \to X \) is a quasi-finite morphism of Noetherian schemes with the following properties

1. the open \( V \subset Y \) where \( f \) is flat contains \( \text{Ass}(\mathcal{O}_Y) \) and \( f^{-1}(\text{Ass}(\mathcal{O}_X)) \),
2. the trace element \( \tau_{Y/X} \) comes from a section \( \tau \in \Gamma(Y, \omega_{Y/X}) \).

Condition (1) implies that \( V \) contains the associated points of \( \omega_{Y/X} \) by Lemma [2.8]. In particular, \( \tau \) is unique if it exists (Divisors, Lemma [2.8]). Given \( \tau \) we can define the different \( D_f \) as the annihilator of \( \text{Coker}(\tau : \mathcal{O}_Y \to \omega_{Y/X}) \). This agrees with the Dedekind different in many cases (Lemma [14.3]). However, for non-flat maps between non-normal rings, this generalization no longer measures ramification of the morphism, see Example [14.4].

**Lemma 14.3.** Assume the Dedekind different is defined for \( A \to B \). Set \( X = \text{Spec}(A) \) and \( Y = \text{Spec}(B) \). The generalization of Remark 14.2 applies to the morphism \( f : Y \to X \) if and only if \( 1 \in \mathcal{L}_{B/A} \) (e.g., if \( A \) is normal, see Lemma [8.7]). In this case \( D_{B/A} \) is an ideal of \( B \) and we have

\[
\mathcal{D}_f = \widetilde{D}_{B/A}
\]

as coherent ideal sheaves on \( Y \).

**Proof.** As the Dedekind different for \( A \to B \) is defined we can apply Lemma [14.1] to see that \( Y \to X \) satisfies condition (1) of Remark 14.2. Recall that there is
a canonical isomorphism \( c : \mathcal{L}_{B/A} \to \omega_{B/A} \), see Lemma 8.2. Let \( K = Q(A) \) and \( L = K \otimes_A B \) as above. By construction the map \( c \) fits into a commutative diagram

\[
\begin{array}{ccc}
\mathcal{L}_{B/A} & \to & L \\
\downarrow c & & \downarrow \\
\omega_{B/A} & \to & \text{Hom}_K(L, K)
\end{array}
\]

where the right vertical arrow sends \( x \in L \) to the map \( y \mapsto \text{Trace}_{L/K}(xy) \) and the lower horizontal arrow is the base change map \( \text{Trace}_K \) for \( \omega_{B/A} \). We can factor the lower horizontal map as

\[
\omega_{B/A} = \Gamma(Y, \omega_{Y/X}) \to \Gamma(V, \omega_{V/X}) \to \text{Hom}_K(L, K)
\]

Since all associated points of \( \omega_{V/X} \) map to associated primes of \( A \) (Lemma 2.8) we see that the second map is injective. The element \( \tau_{V/X} \) maps to \( \text{Trace}_{L/K} \) in \( \text{Hom}_K(L, K) \) by the very definition of trace elements (Definition 4.1). Thus \( \tau \) as in condition (2) of Remark 14.2 exists if and only if \( 1 \in \mathcal{L}_{B/A} \) and then \( \tau = c(1) \).

In this case, by Lemma 8.1 we see that \( D_{B/A} \subset B \). Finally, the agreement of \( D_f \) with \( D_{B/A} \) is immediate from the definitions and the fact \( \tau = c(1) \) seen above.

**Example 14.4.** Let \( k \) be a field. Let \( A = k[x,y]/(xy) \) and \( B = k[u,v]/(uv) \) and let \( A \to B \) be given by \( x \mapsto u^n \) and \( y \mapsto v^m \) for some \( n, m \in \mathbb{N} \) prime to the characteristic of \( k \). Then \( A_{x+y} \to B_{x+y} \) is (finite) étale hence we are in the situation where the Dedekind different is defined. A computation shows that

\[
\text{Trace}_{L/K}(1) = (nx + my)/(x + y), \quad \text{Trace}_{L/K}(u^j) = 0, \quad \text{Trace}_{L/K}(v^j) = 0
\]

for \( 1 \leq i < n \) and \( 1 \leq j < m \). We conclude that \( 1 \in \mathcal{L}_{B/A} \) if and only if \( n = m \). Moreover, a computation shows that if \( n = m \), then \( \mathcal{L}_{B/A} = B \) and the Dedekind different is \( B \) as well. In other words, we find that the different of Remark 14.2 is defined for \( \text{Spec}(B) \to \text{Spec}(A) \) if and only if \( n = m \), and in this case the different is the unit ideal. Thus we see that in nonflat cases the nonvanishing of the different does not guarantee the morphism is étale or unramified.

### 15. Comparison with duality theory

**Lemma 15.1.** Let \( f : Y \to X \) be a quasi-finite separated morphism of Noetherian schemes. For every pair of affine opens \( \text{Spec}(B) = V \subset Y \), \( \text{Spec}(A) = U \subset X \) with \( f(V) \subset U \) there is an isomorphism

\[
H^0(V, f^!\mathcal{O}_Y) = \omega_{B/A}
\]

where \( f^! \) is as in Duality for Schemes, Section 16. These isomorphisms are compatible with restriction maps and define a canonical isomorphism \( H^0(f^!\mathcal{O}_X) = \omega_{Y/X} \) with \( \omega_{Y/X} \) as in Remark 2.17. Similarly, if \( f : Y \to X \) is a quasi-finite morphism of schemes of finite type over a Noetherian base \( S \) endowed with a dualizing complex \( \omega^S \), then \( H^0(f_{\text{new}}^!\mathcal{O}_X) = \omega_{Y/X} \).

**Proof.** By Zariski’s main theorem we can choose a factorization \( f = f' \circ j \) where \( j : Y \to Y' \) is an open immersion and \( f' : Y' \to X \) is a finite morphism, see More on Morphisms, Lemma 38.3. By our construction in Duality for Schemes, Lemma 16.2
we have $f^! = j^* \circ a'$ where $a' : D_{QCoh}(\mathcal{O}_X) \to D_{QCoh}(\mathcal{O}_Y^\prime)$ is the right adjoint to $Rf^!$ of Duality for Schemes, Lemma \[\text{[3.1]}\]. By Duality for Schemes, Lemma \[\text{[11.4]}\] we see that $\Phi(a'(\mathcal{O}_X)) = R\mathcal{H}om(f^!\mathcal{O}_Y^\prime, \mathcal{O}_X)$ in $D^{+}_{QCoh}(f^!\mathcal{O}_Y^\prime)$. In particular $a'(\mathcal{O}_X)$ has vanishing cohomology sheaves in degrees $< 0$. The zeroth cohomology sheaf is determined by the isomorphism

$$f^!H^0(a'(\mathcal{O}_X)) = \mathcal{H}om_{\mathcal{O}_X}(f^!\mathcal{O}_Y^\prime, \mathcal{O}_X)$$

as $f^!\mathcal{O}_Y^\prime$-modules via the equivalence of Morphisms, Lemma \[\text{[11.6]}\]. Writing $(f^!)^{-1}U = V' = \text{Spec}(B')$, we obtain

$$H^0(V', a'(\mathcal{O}_X)) = \text{Hom}_A(B', A).$$

As the zeroth cohomology sheaf of $a'(\mathcal{O}_X)$ is a quasi-coherent module we find that the restriction to $V$ is given by $\omega_{B/A} = \text{Hom}_A(B', A) \otimes_{B'} B$ as desired.

The statement about restriction maps signifies that the restriction mappings of the quasi-coherent $\mathcal{O}_Y^\prime$-module $H^0(a'(\mathcal{O}_X))$ for opens in $Y'$ agrees with the maps defined in Lemma \[\text{[2.3]}\] for the modules $\omega_{B/A}$ via the isomorphisms given above. This is clear.

Let $f : Y \to X$ be a quasi-finite morphism of schemes of finite type over a Noetherian base $S$ endowed with a dualizing complex $\omega_S^\bullet$. Consider opens $V \subset Y$ and $U \subset X$ with $f(V) \subset U$ and $V$ and $U$ separated over $S$. Denote $f|_V : V \to U$ the restriction of $f$. By the discussion above and Duality for Schemes, Lemma \[\text{[20.9]}\] there are canonical isomorphisms

$$H^0(f^!_{new}\mathcal{O}_X)|_V = H^0((f|_V)^!\mathcal{O}_U) = \omega_{V/U} = \omega_{Y/X}|_V$$

We omit the verification that these isomorphisms glue to a global isomorphism $H^0(f^!_{new}\mathcal{O}_X) \to \omega_{Y/X}$. \hfill \Box

0BVI \textbf{Lemma 15.2.} Let $f : Y \to X$ be a finite flat morphism of Noetherian schemes. The map

$$\text{Trace}_f : f_*\mathcal{O}_Y \to \mathcal{O}_X$$

of Section 3 corresponds to a map $\mathcal{O}_Y \to f^!\mathcal{O}_X$. Denote $\tau_{Y/X} \in H^0(Y, f^!\mathcal{O}_X)$ the image of $1$. Via the isomorphism $H^0(f^!\mathcal{O}_X) = \omega_{X/Y}$ of Lemma \[\text{[15.1]}\] this agrees with the construction in Remark \[\text{[4.7]}\].

\textbf{Proof.} Unwinding all the definitions, this is immediate from the fact that if $A \to B$ is finite flat, then $\tau_{B/A} = \text{Trace}_{B/A}$ (Lemma \[\text{[4.3]}\]) and the compatibility of traces with localizations (Lemma \[\text{[4.4]}\]). \hfill \Box

16. Quasi-finite Gorenstein morphisms

0C14 This section discusses quasi-finite Gorenstein morphisms.

0C16 \textbf{Lemma 16.1.} Let $f : Y \to X$ be a quasi-finite morphism of Noetherian schemes. The following are equivalent

1. $f$ is Gorenstein,
2. $f$ is flat and the fibres of $f$ are Gorenstein,
3. $f$ is flat and $\omega_{Y/X}$ is invertible (Remark \[\text{[2.14]}\]),
4. for every $y \in Y$ there are affine opens $y \in V = \text{Spec}(B) \subset Y$, $U = \text{Spec}(A) \subset X$ with $f(V) \subset U$ such that $A \to B$ is flat and $\omega_{B/A}$ is an invertible $B$-module.
Proof. Parts (1) and (2) are equivalent by definition. Parts (3) and (4) are equivalent by the construction of $\omega_{Y/X}$ in Remark 2.11. Thus we have to show that (1)-(2) is equivalent to (3)-(4).

First proof. Working affine locally we can assume $f$ is a separated morphism and apply Lemma 15.1 to see that $\omega_{Y/X}$ is the zeroth cohomology sheaf of $f^!\mathcal{O}_X$. Under both assumptions $f$ is flat and quasi-finite, hence $f^!\mathcal{O}_X$ is isomorphic to $\omega_{Y/X}[0]$, see Duality for Schemes, Lemma 21.6. Hence the equivalence follows from Duality for Schemes, Lemma 25.10.

Second proof. By Lemma 10.2 we see that it suffices to prove the equivalence of (2) and (3) when $X$ is the spectrum of a field $k$. Then $Y = \text{Spec}(B)$ where $B$ is a finite $k$-algebra. In this case $\omega_{B/A} = \omega_{B/k} = \text{Hom}_k(B,k)$ placed in degree 0 is a dualizing complex for $B$, see Dualizing Complexes, Lemma 15.8. Thus the equivalence follows from Dualizing Complexes, Lemma 21.4. □

Remark 16.2. Let $f : Y \to X$ be a quasi-finite Gorenstein morphism of Noetherian schemes. Let $D_f \subset \mathcal{O}_Y$ be the different and let $R \subset Y$ be the closed subscheme cut out by $D_f$. Then we have

1. $D_f$ is a locally principal ideal,
2. $R$ is a locally principal closed subscheme,
3. $D_f$ is affine locally the same as the Noether different,
4. formation of $R$ commutes with base change,
5. if $f$ is finite, then the norm of $R$ is the discriminant of $f$, and
6. if $f$ is étale in the associated points of $Y$, then $R$ is an effective Cartier divisor and $\omega_{Y/X} = \mathcal{O}_Y(R)$.

This follows from Lemmas 9.3, 9.4, and 9.7.

Remark 16.3. Let $S$ be a Noetherian scheme endowed with a dualizing complex $\omega_S$. Let $f : Y \to X$ be a quasi-finite Gorenstein morphism of compactifyable schemes over $S$. Assume moreover $Y$ and $X$ Cohen-Macaulay and $f$ étale at the generic points of $Y$. Then we can combine Duality for Schemes, Remark 23.4 and Remark 16.2 to see that we have a canonical isomorphism

$$\omega_Y = f^*\omega_X \otimes_{\mathcal{O}_Y} \omega_{Y/X} = f^*\omega_X \otimes_{\mathcal{O}_Y} \mathcal{O}_Y(R)$$

diagram of $\mathcal{O}_Y$-modules. If further $f$ is finite, then the isomorphism $\mathcal{O}_Y(R) = \omega_{Y/X}$ comes from the global section $\tau_{Y/X} \in H^0(Y, \omega_{Y/X})$ which corresponds via duality to the map $\text{Trace}_f : f_*\mathcal{O}_Y \to \mathcal{O}_X$, see Lemma 16.2.

17. Other chapters
References

