DUALITY FOR SCHEMES

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Contents

1. Introduction

0DWF This chapter studies relative duality for morphisms of schemes and the dualizing complex on a scheme. A reference is [Har66].

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Dualizing complexes for Noetherian rings were defined and studied in Dualizing Complexes, Section 15 ff. In this chapter we continue this by studying dualizing complexes on schemes, see Section 2.

The bulk of this chapter is devoted to studying the right adjoint of pushforward in the setting of derived categories of sheaves of modules with quasi-coherent cohomology sheaves. See Sections 3, 4, 5, 6, 7, 8, 9, 11, 13, 14, and 15. Here we follow the papers [Nee96], [LN07], [Lip09], and [Nee14].

After a brief discussion of compactifications in Section 16 we discuss the upper shriek functors $f^!$ for compactifyable morphisms in Sections 17, 18, and 19 culminating in the overview Section 20.

In Section 21 we explain alternative theory of duality and dualizing complexes when working over a fixed locally Noetherian base endowed with a dualizing complex (this section corresponds to a remark in Hartshorne's book).

In the remaining sections we give a few applications.

This chapter is continued by the chapter on duality on algebraic spaces, see Duality for Spaces, Section 1.

2. Dualizing complexes on schemes

0A85 We define a dualizing complex on a locally Noetherian scheme to be a complex which affine locally comes from a dualizing complex on the corresponding ring. This is not completely standard but agrees with all definitions in the literature on Noetherian schemes of finite dimension.

0A86 **Lemma 2.1.** *Let X be a locally Noetherian scheme. Let K be an object of $D(\mathcal{O}_X)$. The following are equivalent*

1. *For every affine open $U = \mathrm{Spec}(A) \subset X$ there exists a dualizing complex $\omega_A^\bullet$ for A such that $K|_U$ is isomorphic to the image of $\omega_A^\bullet$ by the functor $\widetilde{\ }: D(A) \to D(\mathcal{O}_U)$.*
2. *There is an affine open covering $X = \bigcup U_i$, $U_i = \mathrm{Spec}(A_i)$ such that for each i there exists a dualizing complex $\omega_i^\bullet$ for A_i such that $K|_U$ is isomorphic to the image of $\omega_i^\bullet$ by the functor $\widetilde{\ }: D(A_i) \to D(\mathcal{O}_{U_i})$.*

Proof. Assume (2) and let $U = \mathrm{Spec}(A)$ be an affine open of X. Since condition (2) implies that K is in $D_{QCoh}(\mathcal{O}_X)$ we find an object $\omega_A^\bullet$ in $D(A)$ whose associated complex of quasi-coherent sheaves is isomorphic to $K|_U$, see Derived Categories of Schemes, Lemma 3.5. We will show that $\omega_A^\bullet$ is a dualizing complex for A which will finish the proof.

Since $X = \bigcup U_i$ is an open covering, we can find a standard open covering $U = D(f_1) \cup \ldots \cup D(f_m)$ such that each $D(f_j)$ is a standard open in one of the affine opens U_i, see Schemes, Lemma 11.5. Say $D(f_j) = D(g_j)$ for $g_j \in A_{i_j}$. Then $A_{f_j} \cong (A_{i_j})_{g_j}$ and we have

$$(\omega_A^\bullet)_{f_j} \cong (\omega_i^\bullet)_{g_j}$$

in the derived category by Derived Categories of Schemes, Lemma 3.5. By Dualizing Complexes, Lemma 15.6 we find that the complex $(\omega_A^\bullet)_{f_j}$ is a dualizing complex over A_{f_j} for $j = 1, \ldots, m$. This implies that $\omega_A^\bullet$ is dualizing by Dualizing Complexes, Lemma 15.7. $\square$

0A87 **Definition 2.2.** Let X be a locally Noetherian scheme. An object K of $D(\mathcal{O}_X)$ is called a *dualizing complex* if K satisfies the equivalent conditions of Lemma 2.1.

Please see remarks made at the beginning of this section.

0A88 **Lemma 2.3.** *Let A be a Noetherian ring and let $X = \mathrm{Spec}(A)$. Let K, L be objects of $D(A)$. If $K \in D_{Coh}(A)$ and L has finite injective dimension, then*

$$R\mathcal{H}om_{\mathcal{O}_X}(\widetilde{K}, \widetilde{L}) = \widetilde{R\,\mathrm{Hom}_A(K, L)}$$

in $D(\mathcal{O}_X)$.

Proof. We may assume that L is given by a finite complex $I^\bullet$ of injective A-modules. By induction on the length of $I^\bullet$ and compatibility of the constructions with distinguished triangles, we reduce to the case that $L = I[0]$ where I is an injective A-module. In this case, Derived Categories of Schemes, Lemma 9.8, tells us that the nth cohomology sheaf of $R\mathcal{H}om_{\mathcal{O}_X}(\widetilde{K}, \widetilde{L})$ is the sheaf associated to the presheaf

$$D(f) \longmapsto \mathrm{Ext}^n_{A_f}(K \otimes_A A_f, I \otimes_A A_f)$$

Since A is Noetherian, the A_f-module $I \otimes_A A_f$ is injective (Dualizing Complexes, Lemma 3.8). Hence we see that

$$\begin{aligned} \mathrm{Ext}^n_{A_f}(K \otimes_A A_f, I \otimes_A A_f) &= \mathrm{Hom}_{A_f}(H^{-n}(K \otimes_A A_f), I \otimes_A A_f) \\ &= \mathrm{Hom}_{A_f}(H^{-n}(K) \otimes_A A_f, I \otimes_A A_f) \\ &= \mathrm{Hom}_A(H^{-n}(K), I) \otimes_A A_f \end{aligned}$$

The last equality because $H^{-n}(K)$ is a finite A-module, see Algebra, Lemma 10.2. This proves that the canonical map

$$\widetilde{R\,\mathrm{Hom}_A(K, L)} \longrightarrow R\mathcal{H}om_{\mathcal{O}_X}(\widetilde{K}, \widetilde{L})$$

is a quasi-isomorphism in this case and the proof is done. $\square$

0A89 **Lemma 2.4.** *Let K be a dualizing complex on a locally Noetherian scheme X. Then K is an object of $D_{Coh}(\mathcal{O}_X)$ and $D = R\mathcal{H}om_{\mathcal{O}_X}(-, K)$ induces an anti-equivalence*

$$D : D_{Coh}(\mathcal{O}_X) \longrightarrow D_{Coh}(\mathcal{O}_X)$$

which comes equipped with a canonical isomorphism $id \to D \circ D$. If X is quasi-compact, then D exchanges $D^+_{Coh}(\mathcal{O}_X)$ and $D^-_{Coh}(\mathcal{O}_X)$ and induces an equivalence $D^b_{Coh}(\mathcal{O}_X) \to D^b_{Coh}(\mathcal{O}_X)$.

Proof. Let $U \subset X$ be an affine open. Say $U = \mathrm{Spec}(A)$ and let $\omega_A^\bullet$ be a dualizing complex for A corresponding to $K|_U$ as in Lemma 2.1. By Lemma 2.3 the diagram

$$\begin{tikzcd} D_{Coh}(A) \arrow[r] \arrow[d, "R\,\mathrm{Hom}_A(-,\omega_A^\bullet)"'] & D_{Coh}(\mathcal{O}_U) \arrow[d, "R\mathcal{H}om_{\mathcal{O}_X}(-,K|_U)"] \\ D_{Coh}(A) \arrow[r] & D(\mathcal{O}_U) \end{tikzcd}$$

commutes. We conclude that D sends $D_{Coh}(\mathcal{O}_X)$ into $D_{Coh}(\mathcal{O}_X)$. Moreover, the canonical map

$$L \longrightarrow R\mathcal{H}om_{\mathcal{O}_X}(R\mathcal{H}om_{\mathcal{O}_X}(L, K), K)$$

(Cohomology on Sites, Lemma 34.5) is an isomorphism for all L because this is true on affines by Dualizing Complexes, Lemma 15.2. The statement on boundedness properties of the functor D in the quasi-compact case also follows from the corresponding statements of Dualizing Complexes, Lemma 15.2. $\square$

Let X be a locally ringed space. We will say that an object L of $D(\mathcal{O}_X)$ is *invertible* if there is an open covering $X = \bigcup U_i$ such that $L|_{U_i} \cong \mathcal{O}_{U_i}[-n_i]$ for some integers n_i. In this case, the function

$$x \mapsto n_x, \quad \text{where } n_x \text{ is the unique integer such that } H^{n_x}(L_x) \neq 0$$

is locally constant on X. In particular, it follows that $L = \bigoplus H^n(L)[-n]$ which gives a well defined complex of $\mathcal{O}_X$-modules (with zero differentials) representing L. In particular L is a perfect object of $D(\mathcal{O}_X)$.

0ATP **Lemma 2.5.** *Let X be a locally Noetherian scheme. If K and K' are dualizing complexes on X, then K' is isomorphic to $K \otimes^{\mathbf{L}}_{\mathcal{O}_X} L$ for some invertible object L of $D(\mathcal{O}_X)$.*

Proof. Set

$$L = R\mathcal{H}om_{\mathcal{O}_X}(K, K')$$

This is an invertible object of $D(\mathcal{O}_X)$, because affine locally this is true, see Dualizing Complexes, Lemma 15.5 and its proof. The evaluation map $L \otimes^{\mathbf{L}}_{\mathcal{O}_X} K \to K'$ is an isomorphism for the same reason. $\square$

0AWF **Lemma 2.6.** *Let X be a locally Noetherian scheme. Let $\omega_X^\bullet$ be a dualizing complex on X. Then X is universally catenary and the function $X \to \mathbf{Z}$ defined by*

$$x \longmapsto \delta(x) \text{ such that } \omega_{X,x}^\bullet[-\delta(x)] \text{ is a normalized dualizing complex over } \mathcal{O}_{X,x}$$

is a dimension function.

Proof. Immediate from the affine case Dualizing Complexes, Lemma 17.3 and the definitions. $\square$

0ECM **Lemma 2.7.** *Let X be a locally Noetherian scheme. Let $\omega_X^\bullet$ be a dualizing complex on X with associated dimension function δ. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Set $\mathcal{E}^i = \mathcal{E}xt^{-i}_{\mathcal{O}_X}(\mathcal{F}, \omega_X^\bullet)$. Then $\mathcal{E}^i$ is a coherent $\mathcal{O}_X$-module and for $x \in X$ we have*

1. *$\mathcal{E}^i_x$ is nonzero only for $\delta(x) \leq i \leq \delta(x) + \dim(Supp(\mathcal{F}_x))$,*
2. *$\dim(Supp(\mathcal{E}_x^{i+\delta(x)})) \leq i$,*
3. *$depth(\mathcal{F}_x)$ is the smallest integer $i \geq 0$ such that $\mathcal{E}^{i+\delta(x)} \neq 0$, and*
4. *we have $x \in Supp(\bigoplus_{j \leq i} \mathcal{E}^j) \Leftrightarrow depth_{\mathcal{O}_{X,x}}(\mathcal{F}_x) + \delta(x) \leq i$.*

Proof. Lemma 2.4 tells us that $\mathcal{E}^i$ is coherent. Choosing an affine neighbourhood of x and using Derived Categories of Schemes, Lemma 9.8 and More on Algebra, Lemma 87.2 part (3) we have

$$\mathcal{E}^i_x = \mathcal{E}xt^{-i}_{\mathcal{O}_X}(\mathcal{F}, \omega_X^\bullet)_x = \mathrm{Ext}^{-i}_{\mathcal{O}_{X,x}}(\mathcal{F}_x, \omega_{X,x}^\bullet) = \mathrm{Ext}^{\delta(x)-i}_{\mathcal{O}_{X,x}}(\mathcal{F}_x, \omega_{X,x}^\bullet[-\delta(x)])$$

By construction of δ in Lemma 2.6 this reduces parts (1), (2), and (3) to Dualizing Complexes, Lemma 16.5. Part (4) is a formal consequence of (3) and (1). $\square$

3. Right adjoint of pushforward

0A9D References for this section and the following are [Nee96], [LN07], [Lip09], and [Nee14].

Let $f : X \to Y$ be a morphism of schemes. In this section we consider the right adjoint to the functor $Rf_* : D_{QCoh}(\mathcal{O}_X) \to D_{QCoh}(\mathcal{O}_Y)$. In the literature, if this functor exists, then it is sometimes denoted $f^\times$. This notation is not universally accepted and we refrain from using it. We will not use the notation $f^!$ for such a functor, as this would clash (for general morphisms f) with the notation in [Har66].

0A9E **Lemma 3.1.** *Let $f : X \to Y$ be a morphism between quasi-separated and quasi-compact schemes. The functor $Rf_* : D_{QCoh}(X) \to D_{QCoh}(Y)$ has a right adjoint.*

This is almost the same as [Nee96, Example 4.2].

Proof. We will prove a right adjoint exists by verifying the hypotheses of Derived Categories, Proposition 35.2. First off, the category $D_{QCoh}(\mathcal{O}_X)$ has direct sums, see Derived Categories of Schemes, Lemma 3.1. The category $D_{QCoh}(\mathcal{O}_X)$ is compactly generated by Derived Categories of Schemes, Theorem 14.3. Since X and Y are quasi-compact and quasi-separated, so is f, see Schemes, Lemmas 21.13 and 21.14. Hence the functor Rf_* commutes with direct sums, see Derived Categories of Schemes, Lemma 4.2. This finishes the proof. $\square$

0A9F **Example 3.2.** Let $A \to B$ be a ring map. Let $Y = \text{Spec}(A)$ and $X = \text{Spec}(B)$ and $f : X \to Y$ the morphism corresponding to $A \to B$. Then $Rf_* : D_{QCoh}(\mathcal{O}_X) \to D_{QCoh}(\mathcal{O}_Y)$ corresponds to restriction $D(B) \to D(A)$ via the equivalences $D(B) \to D_{QCoh}(\mathcal{O}_X)$ and $D(A) \to D_{QCoh}(\mathcal{O}_Y)$. Hence the right adjoint corresponds to the functor $K \longmapsto R\operatorname{Hom}(B, K)$ of Dualizing Complexes, Section 13.

0A9G **Example 3.3.** If $f : X \to Y$ is a separated finite type morphism of Noetherian schemes, then the right adjoint of $Rf_* : D_{QCoh}(\mathcal{O}_X) \to D_{QCoh}(\mathcal{O}_Y)$ does not map $D_{Coh}(\mathcal{O}_Y)$ into $D_{Coh}(\mathcal{O}_X)$. Namely, let k be a field and consider the morphism $f : \mathbf{A}^1_k \to \text{Spec}(k)$. By Example 3.2 this corresponds to the question of whether $R\operatorname{Hom}(B, -)$ maps $D_{Coh}(A)$ into $D_{Coh}(B)$ where $A = k$ and $B = k[x]$. This is not true because

$$R\operatorname{Hom}(k[x], k) = \left(\prod\nolimits_{n \geq 0} k\right)[0]$$

which is not a finite $k[x]$-module. Hence $a(\mathcal{O}_Y)$ does not have coherent cohomology sheaves.

0A9H **Example 3.4.** If $f : X \to Y$ is a proper or even finite morphism of Noetherian schemes, then the right adjoint of $Rf_* : D_{QCoh}(\mathcal{O}_X) \to D_{QCoh}(\mathcal{O}_Y)$ does not map $D^-_{QCoh}(\mathcal{O}_Y)$ into $D^-_{QCoh}(\mathcal{O}_X)$. Namely, let k be a field, let $k[\epsilon]$ be the dual numbers over k, let $X = \text{Spec}(k)$, and let $Y = \text{Spec}(k[\epsilon])$. Then $\text{Ext}^i_{k[\epsilon]}(k, k)$ is nonzero for all $i \geq 0$. Hence $a(\mathcal{O}_Y)$ is not bounded above by Example 3.2.

0A9I **Lemma 3.5.** *Let $f : X \to Y$ be a morphism of quasi-compact and quasi-separated schemes. Let $a : D_{QCoh}(\mathcal{O}_Y) \to D_{QCoh}(\mathcal{O}_X)$ be the right adjoint to Rf_* of Lemma 3.1. Then a maps $D^+_{QCoh}(\mathcal{O}_Y)$ into $D^+_{QCoh}(\mathcal{O}_X)$. In fact, there exists an integer N such that $H^i(K) = 0$ for $i \leq c$ implies $H^i(a(K)) = 0$ for $i \leq c - N$.*

Proof. By Derived Categories of Schemes, Lemma 4.1 the functor Rf_* has finite cohomological dimension. In other words, there exist an integer N such that

$H^i(Rf_*L) = 0$ for $i \geq N + c$ if $H^i(L) = 0$ for $i \geq c$. Say $K \in D^+_{QCoh}(\mathcal{O}_Y)$ has $H^i(K) = 0$ for $i \leq c$. Then

$$\mathrm{Hom}_{D(\mathcal{O}_X)}(\tau_{\leq c-N}a(K), a(K)) = \mathrm{Hom}_{D(\mathcal{O}_Y)}(Rf_*\tau_{\leq c-N}a(K), K) = 0$$

by what we said above. Clearly, this implies that $H^i(a(K)) = 0$ for $i \leq c - N$. □

Let $f : X \to Y$ be a morphism of quasi-separated and quasi-compact schemes. Let a denote the right adjoint to $Rf_* : D_{QCoh}(\mathcal{O}_X) \to D_{QCoh}(\mathcal{O}_Y)$. For every $K \in D_{QCoh}(\mathcal{O}_Y)$ and $L \in D_{QCoh}(\mathcal{O}_X)$ we obtain a canonical map

0B6H (3.5.1) $$Rf_*R\mathcal{H}om_{\mathcal{O}_X}(L, a(K)) \longrightarrow R\mathcal{H}om_{\mathcal{O}_Y}(Rf_*L, K)$$

Namely, this map is constructed as the composition

$$Rf_*R\mathcal{H}om_{\mathcal{O}_X}(L, a(K)) \to R\mathcal{H}om_{\mathcal{O}_Y}(Rf_*L, Rf_*a(K)) \to R\mathcal{H}om_{\mathcal{O}_Y}(Rf_*L, K)$$

where the first arrow is Cohomology, Remark 36.10 and the second arrow is the counit $Rf_*a(K) \to K$ of the adjunction.

0A9Q **Lemma 3.6.** *Let $f : X \to Y$ be a morphism of quasi-compact and quasi-separated schemes. Let a be the right adjoint to $Rf_* : D_{QCoh}(\mathcal{O}_X) \to D_{QCoh}(\mathcal{O}_Y)$. Then (3.5.1)*

$$Rf_*R\mathcal{H}om_{\mathcal{O}_X}(L, a(K)) \longrightarrow R\mathcal{H}om_{\mathcal{O}_Y}(Rf_*L, K)$$

is an isomorphism for all $L \in D_{QCoh}(\mathcal{O}_X)$ and $K \in D_{QCoh}(\mathcal{O}_Y)$.

Proof. Let $M \in D_{QCoh}(\mathcal{O}_Y)$. Then we have the following

$$\begin{aligned}
\mathrm{Hom}_Y(M, Rf_*R\mathcal{H}om_{\mathcal{O}_X}(L, a(K))) &= \mathrm{Hom}_X(Lf^*M, R\mathcal{H}om_{\mathcal{O}_X}(L, a(K))) \\
&= \mathrm{Hom}_X(Lf^*M \otimes^{\mathbf{L}}_{\mathcal{O}_X} L, a(K)) \\
&= \mathrm{Hom}_Y(Rf_*(Lf^*M \otimes^{\mathbf{L}}_{\mathcal{O}_X} L), K) \\
&= \mathrm{Hom}_Y(M \otimes^{\mathbf{L}}_{\mathcal{O}_Y} Rf_*L, K) \\
&= \mathrm{Hom}_Y(M, R\mathcal{H}om_{\mathcal{O}_Y}(Rf_*L, K))
\end{aligned}$$

The first equality holds by Cohomology, Lemma 29.1. The second equality by Cohomology, Lemma 36.2. The third equality by construction of a. The fourth equality by Derived Categories of Schemes, Lemma 21.1 (this is the important step). The fifth by Cohomology, Lemma 36.2. Thus the result holds by the Yoneda lemma. □

0B6I **Lemma 3.7.** *Let $f : X \to Y$ be a morphism of quasi-separated and quasi-compact schemes. For all $L \in D_{QCoh}(\mathcal{O}_X)$ and $K \in D_{QCoh}(\mathcal{O}_Y)$ (3.5.1) induces an isomorphism $R\mathrm{Hom}_X(L, a(K)) \to R\mathrm{Hom}_Y(Rf_*L, K)$ of global derived homs.*

Proof. By the construction in Cohomology, Section 38 we have

$$R\mathrm{Hom}_X(L, a(K)) = R\Gamma(X, R\mathcal{H}om_{\mathcal{O}_X}(L, a(K))) = R\Gamma(Y, Rf_*R\mathcal{H}om_{\mathcal{O}_X}(L, a(K)))$$

and

$$R\mathrm{Hom}_Y(Rf_*L, K) = R\Gamma(Y, R\mathcal{H}om_{\mathcal{O}_X}(Rf_*L, a(K)))$$

Thus the lemma is a consequence of Lemma 3.6. □

4. Right adjoint of pushforward and restriction to opens

0E4G In this section we study the question to what extend the right adjoint of pushforward commutes with restriction to open subschemes. This is a base change question, so let's first discuss this more generally.

We often want to know whether the right adjoints to pushforward commutes with base change. Thus we consider a cartesian square

0A9J (4.0.1)
$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

of quasi-compact and quasi-separated schemes. Denote

$$a : D_{QCoh}(\mathcal{O}_Y) \to D_{QCoh}(\mathcal{O}_X) \quad \text{and} \quad a' : D_{QCoh}(\mathcal{O}_{Y'}) \to D_{QCoh}(\mathcal{O}_{X'})$$

the right adjoints to Rf_* and Rf'_* (Lemma 3.1). Consider the base change map of Cohomology, Remark 29.3. It induces a transformation of functors

$$Lg^* \circ Rf_* \longrightarrow Rf'_* \circ L(g')^*$$

on derived categories of sheaves with quasi-coherent cohomology. Hence a transformation between the right adjoints in the opposite direction

$$a \circ Rg_* \longleftarrow Rg'_* \circ a'$$

0A9K **Lemma 4.1.** *In diagram (4.0.1) assume that g is flat or more generally that f and g are Tor independent. Then $a \circ Rg_* \leftarrow Rg'_* \circ a'$ is an isomorphism.*

Proof. In this case the base change map $Lg^* \circ Rf_*K \longrightarrow Rf'_* \circ L(g')^*K$ is an isomorphism for every K in $D_{QCoh}(\mathcal{O}_X)$ by Derived Categories of Schemes, Lemma 21.3. Thus the corresponding transformation between adjoint functors is an isomorphism as well. $\square$

Let $f : X \to Y$ be a morphism of quasi-compact and quasi-separated schemes. Let $V \subset Y$ be a quasi-compact open subscheme and set $U = f^{-1}(V)$. This gives a cartesian square

$$\begin{tikzcd} U \arrow[r, "j'"] \arrow[d, "f|_U"'] & X \arrow[d, "f"] \\ V \arrow[r, "j"] & Y \end{tikzcd}$$

as in (4.0.1). By Lemma 4.1 the map $\xi : a \circ Rj_* \leftarrow Rj'_* \circ a'$ is an isomorphism where a and a' are the right adjoints to Rf_* and $R(f|_U)_*$. We obtain a transformation of functors $D_{QCoh}(\mathcal{O}_Y) \to D_{QCoh}(\mathcal{O}_U)$

0A9L (4.1.1)
$$(j')^* \circ a \to (j')^* \circ a \circ Rj_* \circ j^* \xrightarrow{\xi^{-1}} (j')^* \circ Rj'_* \circ a' \circ j^* \to a' \circ j^*$$

where the first arrow comes from $\text{id} \to Rj_* \circ j^*$ and the final arrow from the isomorphism $(j')^* \circ Rj'_* \to \text{id}$. In particular, we see that (4.1.1) is an isomorphism when evaluated on K if and only if $a(K)|_U \to a(Rj_*(K|_V))|_U$ is an isomorphism.

0A9M **Example 4.2.** There is a finite morphism $f : X \to Y$ of Noetherian schemes such that (4.1.1) is not an isomorphism when evaluated on some $K \in D_{Coh}(\mathcal{O}_Y)$.

Namely, let $X = \mathrm{Spec}(B) \to Y = \mathrm{Spec}(A)$ with $A = k[x, \epsilon]$ where k is a field and $\epsilon^2 = 0$ and $B = k[x] = A/(\epsilon)$. For $n \in \mathbf{N}$ set $M_n = A/(\epsilon, x^n)$. Observe that

$$\mathrm{Ext}^i_A(B, M_n) = M_n, \quad i \geq 0$$

because B has the free periodic resolution $\ldots \to A \to A \to A$ with maps given by multiplication by ϵ. Consider the object $K = \bigoplus M_n[n] = \prod M_n[n]$ of $D_{Coh}(A)$ (equality in $D(A)$ by Derived Categories, Lemmas 31.5 and 32.2). Then we see that $a(K)$ corresponds to $R\mathrm{Hom}(B, K)$ by Example 3.2 and

$$H^0(R\mathrm{Hom}(B, K)) = \mathrm{Ext}^0_A(B, K) = \prod\nolimits_{n \geq 1} \mathrm{Ext}^n_A(B.M_n) = \prod\nolimits_{n \geq 1} M_n$$

by the above. But this module has elements which are not annihilated by any power of x, whereas the complex K does have every element of its cohomology annihilated by a power of x. In other words, for the map (4.1.1) with $V = D(x)$ and $U = D(x)$ and the complex K cannot be an isomorphism because $(j')^*(a(K))$ is nonzero and $a'(j^*K)$ is zero.

0A9N **Lemma 4.3.** *Let $f : X \to Y$ be a morphism of quasi-compact and quasi-separated schemes. Let a be the right adjoint to $Rf_* : D_{QCoh}(\mathcal{O}_X) \to D_{QCoh}(\mathcal{O}_Y)$. Let $V \subset Y$ be quasi-compact open with inverse image $U \subset X$.*

1. *For every $Q \in D^+_{QCoh}(\mathcal{O}_Y)$ supported on $Y \setminus V$ the image $a(Q)$ is supported on $X \setminus U$ if and only if (4.1.1) is an isomorphism on all K in $D^+_{QCoh}(\mathcal{O}_Y)$.*
2. *For every $Q \in D_{QCoh}(\mathcal{O}_Y)$ supported on $Y \setminus V$ the image $a(Q)$ is supported on $X \setminus U$ if and only if (4.1.1) is an isomorphism on all K in $D_{QCoh}(\mathcal{O}_Y)$.*
3. *If a commutes with direct sums, then the equivalent conditions of (1) imply the equivalent conditions of (2).*

Proof. Proof of (1). Let $K \in D^+_{QCoh}(\mathcal{O}_Y)$. Choose a distinguished triangle

$$K \to Rj_*K|_V \to Q \to K[1]$$

Observe that Q is in $D^+_{QCoh}(\mathcal{O}_Y)$ (Derived Categories of Schemes, Lemma 4.1) and is supported on $Y \setminus V$ (Derived Categories of Schemes, Definition 6.4). Applying a we obtain a distinguished triangle

$$a(K) \to a(Rj_*K|_V) \to a(Q) \to a(K)[1]$$

on X. If $a(Q)$ is supported on $X \setminus U$, then restricting to U the map $a(K)|_U \to a(Rj_*K|_V)|_U$ is an isomorphism, i.e., (4.1.1) is an isomorphism on K. The converse is immediate.

The proof of (2) is exactly the same as the proof of (1).

Proof of (3). Assume the equivalent conditions of (1) hold. Set $T = Y \setminus V$. We will use the notation $D_{QCoh,T}(\mathcal{O}_Y)$ and $D_{QCoh,f^{-1}(T)}(\mathcal{O}_X)$ to denote complexes whose cohomology sheaves are supported on T and $f^{-1}(T)$. Since a commutes with direct sums, the strictly full, saturated, triangulated subcategory $\mathcal{D}$ with objects

$$\{Q \in D_{QCoh,T}(\mathcal{O}_Y) \mid a(Q) \in D_{QCoh,f^{-1}(T)}(\mathcal{O}_X)\}$$

is preserved by direct sums and hence derived colimits. On the other hand, the category $D_{QCoh,T}(\mathcal{O}_Y)$ is generated by a perfect object E (see Derived Categories of Schemes, Lemma 14.4). By assumption we see that $E \in \mathcal{D}$. By Derived Categories, Lemma 34.3 every object Q of $D_{QCoh,T}(\mathcal{O}_Y)$ is a derived colimit of a system $Q_1 \to Q_2 \to Q_3 \to \ldots$ such that the cones of the transition maps are direct sums of shifts of E. Arguing by induction we see that $Q_n \in \mathcal{D}$ for all n and finally that Q is in $\mathcal{D}$. Thus the equivalent conditions of (2) hold. $\square$

0A9P **Lemma 4.4.** *Let Y be a quasi-compact and quasi-separated scheme. Let $f : X \to Y$ be a proper morphism. If*[1]

1. *f is flat and of finite presentation, or*
2. *Y is Noetherian*

then the equivalent conditions of Lemma 4.3 part (1) hold for all quasi-compact opens V of Y.

Proof. Let $Q \in D^+_{QCoh}(\mathcal{O}_Y)$ be supported on $Y \setminus V$. To get a contradiction, assume that $a(Q)$ is not supported on $X \setminus U$. Then we can find a perfect complex P_U on U and a nonzero map $P_U \to a(Q)|_U$ (follows from Derived Categories of Schemes, Theorem 14.3). Then using Derived Categories of Schemes, Lemma 12.9 we may assume there is a perfect complex P on X and a map $P \to a(Q)$ whose restriction to U is nonzero. By definition of a this map is adjoint to a map $Rf_*P \to Q$.

The complex Rf_*P is pseudo-coherent. In case (1) this follows from Derived Categories of Schemes, Lemma 26.5. In case (2) this follows from Derived Categories of Schemes, Lemmas 10.1 and 9.3. Thus we may apply Derived Categories of Schemes, Lemma 16.3 and get a map $I \to \mathcal{O}_Y$ of perfect complexes whose restriction to V is an isomorphism such that the composition $I \otimes^{\mathbf{L}}_{\mathcal{O}_Y} Rf_*P \to Rf_*P \to K$ is zero. By Derived Categories of Schemes, Lemma 21.1 we have $I \otimes^{\mathbf{L}}_{\mathcal{O}_Y} Rf_*P = Rf_*(Lf^*I \otimes^{\mathbf{L}}_{\mathcal{O}_X} P)$. We conclude that the composition

$$Lf^*I \otimes^{\mathbf{L}}_{\mathcal{O}_X} P \to P \to a(K)$$

is zero. However, the restriction to U is the map $P|_U \to a(K)|_U$ which we assumed to be nonzero. This contradiction finishes the proof. $\square$

5. Right adjoint of pushforward and base change, I

0AA5 The map (4.1.1) is a special case of a base change map. Namely, suppose that we have a cartesian diagram

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

of quasi-compact and quasi-separated schemes, i.e., a diagram as in (4.0.1). Assume f and g are **Tor independent**. Then we can consider the morphism of functors $D_{QCoh}(\mathcal{O}_Y) \to D_{QCoh}(\mathcal{O}_{X'})$ given by the composition

0AA6 (5.0.1) $$L(g')^* \circ a \to L(g')^* \circ a \circ Rg_* \circ Lg^* \leftarrow L(g')^* \circ Rg'_* \circ a' \circ Lg^* \to a' \circ Lg^*$$

The first arrow comes from the adjunction map $\text{id} \to Rg_*Lg^*$ and the last arrow from the adjunction map $L(g')^*Rg'_* \to \text{id}$. We need the assumption on Tor independence to invert the arrow in the middle, see Lemma 4.1. Alternatively, we can

[1]This proof works for those morphisms of quasi-compact and quasi-separated schemes such that Rf_*P is pseudo-coherent for all P perfect on X. It follows easily from a theorem of Kiehl [Kie72] that this holds if f is proper and pseudo-coherent. This is the correct generality for this lemma and some of the other results in this chapter.

think of (5.0.1) by adjointness of $L(g')^*$ and $R(g')_*$ as a natural transformation

$$a \to a \circ Rg_* \circ Lg^* \leftarrow Rg'_* \circ a' \circ Lg^*$$

were again the second arrow is invertible. If $M \in D_{QCoh}(\mathcal{O}_X)$ and $K \in D_{QCoh}(\mathcal{O}_Y)$ then on Yoneda functors this map is given by

$$\begin{aligned} \mathrm{Hom}_X(M, a(K)) &= \mathrm{Hom}_Y(Rf_*M, K) \\ &\to \mathrm{Hom}_Y(Rf_*M, Rg_*Lg^*K) \\ &= \mathrm{Hom}_{Y'}(Lg^*Rf_*M, Lg^*K) \\ &\leftarrow \mathrm{Hom}_{Y'}(Rf'_*L(g')^*M, Lg^*K) \\ &= \mathrm{Hom}_{X'}(L(g')^*M, a'(Lg^*K)) \\ &= \mathrm{Hom}_X(M, Rg'_*a'(Lg^*K)) \end{aligned}$$

(were the arrow pointing left is invertible by the base change theorem given in Derived Categories of Schemes, Lemma 21.3) which makes things a little bit more explicit.

In this section we first prove that the base change map satisfies some natural compatibilities with regards to stacking squares as in Cohomology, Remarks 29.4 and 29.5 for the usual base change map. We suggest the reader skip the rest of this section on a first reading.

0ATQ **Lemma 5.1.** *Consider a commutative diagram*

$$\begin{tikzcd} X' \arrow[r, "k"'] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "l"] \arrow[d, "g'"'] & Y \arrow[d, "g"] \\ Z' \arrow[r, "m"] & Z \end{tikzcd}$$

of quasi-compact and quasi-separated schemes where both diagrams are cartesian and where f and l as well as g and m are Tor independent. Then the maps (5.0.1) for the two squares compose to give the base change map for the outer rectangle (see proof for a precise statement).

Proof. It follows from the assumptions that $g \circ f$ and m are Tor independent (details omitted), hence the statement makes sense. In this proof we write k^* in place of Lk^* and f_* instead of Rf_*. Let a, b, and c be the right adjoints of Lemma 3.1 for f, g, and $g \circ f$ and similarly for the primed versions. The arrow corresponding to the top square is the composition

$$\gamma_{top} : k^* \circ a \to k^* \circ a \circ l_* \circ l^* \xleftarrow{\xi_{top}} k^* \circ k_* \circ a' \circ l^* \to a' \circ l^*$$

where $\xi_{top} : k_* \circ a' \to a \circ l_*$ is an isomorphism (hence can be inverted) and is the arrow "dual" to the base change map $l^* \circ f_* \to f'_* \circ k^*$. The outer arrows come from the canonical maps $1 \to l_* \circ l^*$ and $k^* \circ k_* \to 1$. Similarly for the second square we have

$$\gamma_{bot} : l^* \circ b \to l^* \circ b \circ m_* \circ m^* \xleftarrow{\xi_{bot}} l^* \circ l_* \circ b' \circ m^* \to b' \circ m^*$$

For the outer rectangle we get

$$\gamma_{rect} : k^* \circ c \to k^* \circ c \circ m_* \circ m^* \xleftarrow{\xi_{rect}} k^* \circ k_* \circ c' \circ m^* \to c' \circ m^*$$

We have $(g \circ f)_* = g_* \circ f_*$ and hence $c = a \circ b$ and similarly $c' = a' \circ b'$. The statement of the lemma is that γ_{rect} is equal to the composition

$$k^* \circ c = k^* \circ a \circ b \xrightarrow{\gamma_{top}} a' \circ l^* \circ b \xrightarrow{\gamma_{bot}} a' \circ b' \circ m^* = c' \circ m^*$$

To see this we contemplate the following diagram:

$$\begin{tikzcd}
 & & k^* \circ a \circ b \arrow[d] \arrow[ddll] \\
 & & k^* \circ a \circ l_* \circ l^* \circ b \arrow[dl] \\
k^* \circ a \circ b \circ m_* \circ m^* \arrow[r] & k^* \circ a \circ l_* \circ l^* \circ b \circ m_* \circ m^* & k^* \circ k_* \circ a' \circ l^* \circ b \arrow[u, "\xi_{top}"'] \arrow[dl] \arrow[d] \\
 & k^* \circ k_* \circ a' \circ l^* \circ b \circ m_* \circ m^* \arrow[u, "\xi_{top}"'] \arrow[dr] & a' \circ l^* \circ b \arrow[d] \\
k^* \circ k_* \circ a' \circ b' \circ m^* \arrow[uu, "\xi_{rect}"'] \arrow[ddrr] & k^* \circ k_* \circ a' \circ l^* \circ l_* \circ b' \circ m^* \arrow[l] \arrow[u, "\xi_{bot}"'] \arrow[dr] & a' \circ l^* \circ b \circ m_* \circ m^* \\
 & & a' \circ l^* \circ l_* \circ b' \circ m^* \arrow[u, "\xi_{bot}"'] \arrow[d] \\
 & & a' \circ b' \circ m^*
\end{tikzcd}$$

Going down the right hand side we have the composition and going down the left hand side we have γ_{rect}. All the quadrilaterals on the right hand side of this diagram commute by Categories, Lemma 27.2 or more simply the discussion preceding Categories, Definition 27.1. Hence we see that it suffices to show the diagram

$$\begin{tikzcd}
a \circ l_* \circ l^* \circ b \circ m_* & a \circ b \circ m_* \arrow[l] \\
k_* \circ a' \circ l^* \circ b \circ m_* \arrow[u, "\xi_{top}"'] & \\
k_* \circ a' \circ l^* \circ l_* \circ b' \arrow[u, "\xi_{bot}"'] \arrow[r] & k_* \circ a' \circ b' \arrow[uu, "\xi_{rect}"']
\end{tikzcd}$$

becomes commutative if we invert the arrows ξ_{top}, ξ_{bot}, and ξ_{rect} (note that this is different from asking the diagram to be commutative). However, the diagram

$$\begin{tikzcd}
 & a \circ l_* \circ l^* \circ b \circ m_* & \\
a \circ l_* \circ l^* \circ l_* \circ b' \arrow[ur, "\xi_{bot}"] & & k_* \circ a' \circ l^* \circ b \circ m_* \arrow[ul, "\xi_{top}"'] \\
 & k_* \circ a' \circ l^* \circ l_* \circ b' \arrow[ul, "\xi_{top}"] \arrow[ur, "\xi_{bot}"'] &
\end{tikzcd}$$

commutes by Categories, Lemma 27.2. Since the diagrams

$$\begin{tikzcd}
a \circ l_* \circ l^* \circ b \circ m_* & a \circ b \circ m \arrow[l] \\
a \circ l_* \circ l^* \circ l_* \circ b' \arrow[u] & a \circ l_* \circ b' \arrow[l] \arrow[u]
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
a \circ l_* \circ l^* \circ l_* \circ b' \arrow[r] & a \circ l_* \circ b' \\
k_* \circ a' \circ l^* \circ l_* \circ b' \arrow[r] \arrow[u] & k_* \circ a' \circ b' \arrow[u]
\end{tikzcd}$$

commute (see references cited) and since the composition of $l_* \to l_* \circ l^* \circ l_* \to l_*$ is the identity, we find that it suffices to prove that

$$k \circ a' \circ b' \xrightarrow{\xi_{bot}} a \circ l_* \circ b \xrightarrow{\xi_{top}} a \circ b \circ m_*$$

is equal to ξ_{rect} (via the identifications $a \circ b = c$ and $a' \circ b' = c'$). This is the statement dual to Cohomology, Remark 29.4 and the proof is complete. □

0ATR **Lemma 5.2.** *Consider a commutative diagram*

$$\begin{tikzcd}
X'' \arrow[r, "g'"'] \arrow[d, "f''"'] & X' \arrow[r, "g"'] \arrow[d, "f'"'] & X \arrow[d, "f"] \\
Y'' \arrow[r, "h'"] & Y' \arrow[r, "h"] & Y
\end{tikzcd}$$

of quasi-compact and quasi-separated schemes where both diagrams are cartesian and where f and h as well as f' and h' are Tor independent. Then the maps (5.0.1) for the two squares compose to give the base change map for the outer rectangle (see proof for a precise statement).

Proof. It follows from the assumptions that f and $h \circ h'$ are Tor independent (details omitted), hence the statement makes sense. In this proof we write g^* in place of Lg^* and f_* instead of Rf_*. Let a, a', and a'' be the right adjoints of Lemma 3.1 for f, f', and f''. The arrow corresponding to the right square is the composition

$$\gamma_{right} : g^* \circ a \to g^* \circ a \circ h_* \circ h^* \xleftarrow{\xi_{right}} g^* \circ g_* \circ a' \circ h^* \to a' \circ h^*$$

where $\xi_{right} : g_* \circ a' \to a \circ h_*$ is an isomorphism (hence can be inverted) and is the arrow "dual" to the base change map $h^* \circ f_* \to f'_* \circ g^*$. The outer arrows come from the canonical maps $1 \to h_* \circ h^*$ and $g^* \circ g_* \to 1$. Similarly for the left square we have

$$\gamma_{left} : (g')^* \circ a' \to (g')^* \circ a' \circ (h')_* \circ (h')^* \xleftarrow{\xi_{left}} (g')^* \circ (g')_* \circ a'' \circ (h')^* \to a'' \circ (h')^*$$

For the outer rectangle we get

$$\gamma_{rect} : k^* \circ a \to k^* \circ a \circ m_* \circ m^* \xleftarrow{\xi_{rect}} k^* \circ k_* \circ a'' \circ m^* \to a'' \circ m^*$$

where $k = g \circ g'$ and $m = h \circ h'$. We have $k^* = (g')^* \circ g^*$ and $m^* = (h')^* \circ h^*$. The statement of the lemma is that γ_{rect} is equal to the composition

$$k^* \circ a = (g')^* \circ g^* \circ a \xrightarrow{\gamma_{right}} (g')^* \circ a' \circ h^* \xrightarrow{\gamma_{left}} a'' \circ (h')^* \circ h^* = a'' \circ m^*$$

To see this we contemplate the following diagram

$$\begin{tikzcd}
& (g')^* \circ g^* \circ a \arrow[d] \arrow[ddl] \\
& (g')^* \circ g^* \circ a \circ h_* \circ h^* \arrow[dl] \\
(g')^* \circ g^* \circ a \circ h_* \circ (h')_* \circ (h')^* \circ h^* & (g')^* \circ g^* \circ g_* \circ a' \circ h^* \arrow[u, "\xi_{right}"'] \arrow[dl] \arrow[d] \\
(g')^* \circ g^* \circ g_* \circ a' \circ (h')_* \circ (h')^* \circ h^* \arrow[u, "\xi_{right}"'] \arrow[dr] & (g')^* \circ a' \circ h^* \arrow[d] \\
(g')^* \circ g^* \circ g_* \circ (g')_* \circ a'' \circ (h')^* \circ h^* \arrow[u, "\xi_{left}"'] \arrow[dr] \arrow[ddr] & (g')^* \circ a' \circ (h')_* \circ (h')^* \circ h^* \\
& (g')^* \circ (g')_* \circ a'' \circ (h')^* \circ h^* \arrow[u, "\xi_{left}"'] \arrow[d] \\
& a'' \circ (h')^* \circ h^*
\end{tikzcd}$$

Going down the right hand side we have the composition and going down the left hand side we have γ_{rect}. All the quadrilaterals on the right hand side of this diagram commute by Categories, Lemma 27.2 or more simply the discussion preceding Categories, Definition 27.1. Hence we see that it suffices to show that

$$g_* \circ (g')_* \circ a'' \xrightarrow{\xi_{left}} g_* \circ a' \circ (h')_* \xrightarrow{\xi_{right}} a \circ h_* \circ (h')_*$$

is equal to ξ_{rect}. This is the statement dual to Cohomology, Remark 29.5 and the proof is complete. □

0ATS **Remark 5.3.** Consider a commutative diagram

$$\begin{tikzcd}
X'' \arrow[r, "k'"'] \arrow[d, "f''"'] & X' \arrow[r, "k"'] \arrow[d, "f'"'] & X \arrow[d, "f"] \\
Y'' \arrow[r, "l'"] \arrow[d, "g''"'] & Y' \arrow[r, "l"] \arrow[d, "g'"'] & Y \arrow[d, "g"] \\
Z'' \arrow[r, "m'"] & Z' \arrow[r, "m"] & Z
\end{tikzcd}$$

of quasi-compact and quasi-separated schemes where all squares are cartesian and where (f, l), (g, m), (f', l'), (g', m') are Tor independent pairs of maps. Let a, a',

a'', b, b', b'' be the right adjoints of Lemma 3.1 for f, f', f'', g, g', g''. Let us label the squares of the diagram A, B, C, D as follows

$$\begin{matrix} A & B \\ C & D \end{matrix}$$

Then the maps (5.0.1) for the squares are (where we use $k^* = Lk^*$, etc)

$$\begin{matrix} \gamma_A : (k')^* \circ a' \to a'' \circ (l')^* & \gamma_B : k^* \circ a \to a' \circ l^* \\ \gamma_C : (l')^* \circ b' \to b'' \circ (m')^* & \gamma_D : l^* \circ b \to b' \circ m^* \end{matrix}$$

For the 2×1 and 1×2 rectangles we have four further base change maps

$$\begin{matrix} \gamma_{A+B} : (k \circ k')^* \circ a \to a'' \circ (l \circ l')^* \\ \gamma_{C+D} : (l \circ l')^* \circ b \to b'' \circ (m \circ m')^* \\ \gamma_{A+C} : (k')^* \circ (a' \circ b') \to (a'' \circ b'') \circ (m')^* \\ \gamma_{A+C} : k^* \circ (a \circ b) \to (a' \circ b') \circ m^* \end{matrix}$$

By Lemma 5.2 we have

$$\gamma_{A+B} = \gamma_A \circ \gamma_B, \quad \gamma_{C+D} = \gamma_C \circ \gamma_D$$

and by Lemma 5.1 we have

$$\gamma_{A+C} = \gamma_C \circ \gamma_A, \quad \gamma_{B+D} = \gamma_D \circ \gamma_B$$

Here it would be more correct to write $\gamma_{A+B} = (\gamma_A \star \mathrm{id}_{l^*}) \circ (\mathrm{id}_{(k')^*} \star \gamma_B)$ with notation as in Categories, Section 27 and similarly for the others. However, we continue the abuse of notation used in the proofs of Lemmas 5.1 and 5.2 of dropping $\star$ products with identities as one can figure out which ones to add as long as the source and target of the transformation is known. Having said all of this we find (a priori) two transformations

$$(k')^* \circ k^* \circ a \circ b \longrightarrow a'' \circ b'' \circ (m')^* \circ m^*$$

namely

$$\gamma_C \circ \gamma_A \circ \gamma_D \circ \gamma_B = \gamma_{A+C} \circ \gamma_{B+D}$$

and

$$\gamma_C \circ \gamma_D \circ \gamma_A \circ \gamma_B = \gamma_{C+D} \circ \gamma_{A+B}$$

The point of this remark is to point out that these transformations are equal. Namely, to see this it suffices to show that

$$\begin{tikzcd} (k')^* \circ a' \circ l^* \circ b \arrow[r, "\gamma_D"'] \arrow[d, "\gamma_A"'] & (k')^* \circ a' \circ b' \circ m^* \arrow[d, "\gamma_A"] \\ a'' \circ (l')^* \circ l^* \circ b \arrow[r, "\gamma_D"] & a'' \circ (l')^* \circ b' \circ m^* \end{tikzcd}$$

commutes. This is true by Categories, Lemma 27.2 or more simply the discussion preceding Categories, Definition 27.1.

6. Right adjoint of pushforward and base change, II

0BZF In this section we prove that the base change map of Section 5 is an isomorphism in some cases. We first observe that it suffices to check over affine opens, provided formation of the right adjoint of pushforward commutes with restriction to opens.

0E9S **Remark 6.1.** Consider a cartesian diagram

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

of quasi-compact and quasi-separated schemes with (g, f) Tor independent. Let $V \subset Y$ and $V' \subset Y'$ be affine opens with $g(V') \subset V$. Form the cartesian diagrams

$$\begin{tikzcd} U \arrow[r] \arrow[d] & X \arrow[d] \\ V \arrow[r] & Y \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} U' \arrow[r] \arrow[d] & X' \arrow[d] \\ V' \arrow[r] & Y' \end{tikzcd}$$

Assume (4.1.1) with respect to K and the first diagram and (4.1.1) with respect to Lg^*K and the second diagram are isomorphisms. Then the restriction of the base change map (5.0.1)

$$L(g')^*a(K) \longrightarrow a'(Lg^*K)$$

to U' is isomorphic to the base change map (5.0.1) for $K|_V$ and the cartesian diagram

$$\begin{tikzcd} U' \arrow[r] \arrow[d] & U \arrow[d] \\ V' \arrow[r] & V \end{tikzcd}$$

This follows from the fact that (4.1.1) is a special case of the base change map (5.0.1) and that the base change maps compose correctly if we stack squares horizontally, see Lemma 5.2. Thus in order to check the base change map restricted to U' is an isomorphism it suffices to work with the last diagram.

0AA8 **Lemma 6.2.** *In diagram (4.0.1) assume*

1. $g : Y' \to Y$ *is a morphism of affine schemes,*
2. $f : X \to Y$ *is proper, and*
3. f *and* g *are Tor independent.*

Then the base change map (5.0.1) induces an isomorphism

$$L(g')^*a(K) \longrightarrow a'(Lg^*K)$$

in the following cases

1. *for all* $K \in D_{QCoh}(\mathcal{O}_X)$ *if* f *is flat of finite presentation,*
2. *for all* $K \in D_{QCoh}(\mathcal{O}_X)$ *if* f *is perfect and* Y *Noetherian,*
3. *for* $K \in D^+_{QCoh}(\mathcal{O}_X)$ *if* g *has finite Tor dimension and* Y *Noetherian.*

Proof. Write $Y = \mathrm{Spec}(A)$ and $Y' = \mathrm{Spec}(A')$. As a base change of an affine morphism, the morphism g' is affine. Let M be a perfect generator for $D_{QCoh}(\mathcal{O}_X)$, see Derived Categories of Schemes, Theorem 14.3. Then $L(g')^*M$ is a generator for $D_{QCoh}(\mathcal{O}_{X'})$, see Derived Categories of Schemes, Remark 15.4. Hence it suffices to show that (5.0.1) induces an isomorphism

0E45 (6.2.1) $$R\,\mathrm{Hom}_{X'}(L(g')^*M, L(g')^*a(K)) \longrightarrow R\,\mathrm{Hom}_{X'}(L(g')^*M, a'(Lg^*K))$$

of global hom complexes, see Cohomology, Section 38, as this will imply the cone of $L(g')^*a(K) \to a'(Lg^*K)$ is zero. The structure of the proof is as follows: we will first show that these Hom complexes are isomorphic and in the last part of the proof we will show that the isomorphism is induced by (6.2.1).

The left hand side. Because M is perfect, the canonical map

$$R\operatorname{Hom}_X(M, a(K)) \otimes_A^{\mathbf{L}} A' \longrightarrow R\operatorname{Hom}_{X'}(L(g')^*M, L(g')^*a(K))$$

is an isomorphism by Derived Categories of Schemes, Lemma 21.4. We can combine this with the isomorphism $R\operatorname{Hom}_Y(Rf_*M, K) = R\operatorname{Hom}_X(M, a(K))$ of Lemma 3.7 to get that the left hand side equals $R\operatorname{Hom}_Y(Rf_*M, K) \otimes_A^{\mathbf{L}} A'$.

The right hand side. Here we first use the isomorphism

$$R\operatorname{Hom}_{X'}(L(g')^*M, a'(Lg^*K)) = R\operatorname{Hom}_{Y'}(Rf'_*L(g')^*M, Lg^*K)$$

of Lemma 3.7. Then we use the base change map $Lg^*Rf_*M \to Rf'_*L(g')^*M$ is an isomorphism by Derived Categories of Schemes, Lemma 21.3. Hence we may rewrite this as $R\operatorname{Hom}_{Y'}(Lg^*Rf_*M, Lg^*K)$. Since Y, Y' are affine and K, Rf_*M are in $D_{QCoh}(\mathcal{O}_Y)$ (Derived Categories of Schemes, Lemma 4.1) we have a canonical map

$$\beta : R\operatorname{Hom}_Y(Rf_*M, K) \otimes_A^{\mathbf{L}} A' \longrightarrow R\operatorname{Hom}_{Y'}(Lg^*Rf_*M, Lg^*K)$$

in $D(A')$. This is the arrow More on Algebra, Equation (87.1.1) where we have used Derived Categories of Schemes, Lemmas 3.5 and 9.8 to translate back and forth into algebra.

(1) If f is flat and of finite presentation, the complex Rf_*M is perfect on Y by Derived Categories of Schemes, Lemma 26.4 and β is an isomorphism by More on Algebra, Lemma 87.2 part (1).
(2) If f is perfect and Y Noetherian, the complex Rf_*M is perfect on Y by More on Morphisms, Lemma 51.12 and β is an isomorphism as before.
(3) If g has finite tor dimension and Y is Noetherian, the complex Rf_*M is pseudo-coherent on Y (Derived Categories of Schemes, Lemmas 10.1 and 9.3) and β is an isomorphism by More on Algebra, Lemma 87.2 part (4).

We conclude that we obtain the same answer as in the previous paragraph.

In the rest of the proof we show that the identifications of the left and right hand side of (6.2.1) given in the second and third paragraph are in fact given by (6.2.1). To make our formulas manageable we will use $(-,-)_X = R\operatorname{Hom}_X(-,-)$, use $- \otimes A'$ in stead of $- \otimes_A^{\mathbf{L}} A'$, and we will abbreviate $g^* = Lg^*$ and $f_* = Rf_*$. Consider the following commutative diagram

$$\begin{tikzcd}
((g')^*M, (g')^*a(K))_{X'} \arrow[d] & (M, a(K))_X \otimes A' \arrow[l, "\alpha"'] \arrow[r, equal] \arrow[d] & (f_*M, K)_Y \otimes A' \arrow[d] \\
((g')^*M, (g')^*a(g_*g^*K))_{X'} & (M, a(g_*g^*K))_X \otimes A' \arrow[l, "\alpha"'] \arrow[r, equal] & (f_*M, g_*g^*K)_Y \otimes A' \arrow[dd, bend right=60, "\mu'"'] \\
((g')^*M, (g')^*g'_*a'(g^*K))_{X'} \arrow[u] \arrow[d] & (M, g'_*a'(g^*K))_X \otimes A' \arrow[l, "\alpha"'] \arrow[u] \arrow[dl, "\mu"] & (f_*M, K) \otimes A' \arrow[d, "\beta"] \\
((g')^*M, a'(g^*K))_{X'} \arrow[r, equal] & (f'_*(g')^*M, g^*K)_{Y'} \arrow[r] & (g^*f_*M, g^*K)_{Y'}
\end{tikzcd}$$

The arrows labeled α are the maps from Derived Categories of Schemes, Lemma 21.4 for the diagram with corners X', X, Y', Y. The upper part of the diagram is commutative as the horizontal arrows are functorial in the entries. The middle vertical arrows come from the invertible transformation $g'_* \circ a' \to a \circ g_*$ of Lemma 4.1 and therefore the middle square is commutative. Going down the left hand side is (6.2.1). The upper horizontal arrows provide the identifications used in the second paragraph of the proof. The lower horizontal arrows including β provide the identifications used in the third paragraph of the proof. Given $E \in D(A)$, $E' \in D(A')$, and $c : E \to E'$ in $D(A)$ we will denote $\mu_c : E \otimes A' \to E'$ the map induced by c and the adjointness of restriction and base change; if c is clear we write $\mu = \mu_c$, i.e., we drop c from the notation. The map μ in the diagram is of this form with c given by the identification $(M, g'_*a(g^*K))_X = ((g')^*M, a'(g^*K))_{X'}$; the triangle involving μ is commutative by Derived Categories of Schemes, Remark 21.5.

Observe that

$$\begin{tikzcd}
(M, a(g_*g^*K))_X \arrow[r, equal] & (f_*M, g_*g^*K)_Y \arrow[r, equal] & (g^*f_*M, g^*K)_{Y'} \\
(M, g'_*a'(g^*K))_X \arrow[u] \arrow[r, equal] & ((g')^*M, a'(g^*K))_{X'} \arrow[r, equal] & (f'_*(g')^*M, g^*K)_{Y'} \arrow[u]
\end{tikzcd}$$

is commutative by the very definition of the transformation $g'_* \circ a' \to a \circ g_*$. Letting μ' be as above corresponding to the identification $(f_*M, g_*g^*K)_X = (g^*f_*M, g^*K)_{Y'}$, then the hexagon commutes as well. Thus it suffices to show that β is equal to the composition of $(f_*M, K)_Y \otimes A' \to (f_*M, g_*g^*K)_X \otimes A'$ and μ'. To do this, it suffices to prove the two induced maps $(f_*M, K)_Y \to (g^*f_*M, g^*K)_{Y'}$ are the same. In other words, it suffices to show the diagram

$$\begin{tikzcd}
R\operatorname{Hom}_A(E, K) \arrow[rr, "\text{induced by } \beta"'] \arrow[dr] & & R\operatorname{Hom}_{A'}(E \otimes_A^{\mathbf{L}} A', K \otimes_A^{\mathbf{L}} A') \\
& R\operatorname{Hom}_A(E, K \otimes_A^{\mathbf{L}} A') \arrow[ur] &
\end{tikzcd}$$

commutes for all $E, K \in D(A)$. Since this is how β is constructed in More on Algebra, Section 87 the proof is complete. □

7. Right adjoint of pushforward and trace maps

0AWG Let $f : X \to Y$ be a morphism of quasi-compact and quasi-separated schemes. Let $a : D_{QCoh}(\mathcal{O}_Y) \to D_{QCoh}(\mathcal{O}_X)$ be the right adjoint as in Lemma 3.1. By Categories, Section 24 we obtain a transformation of functors

$$\operatorname{Tr}_f : Rf_* \circ a \longrightarrow \operatorname{id}$$

The corresponding map $\operatorname{Tr}_{f,K} : Rf_*a(K) \longrightarrow K$ for $K \in D_{QCoh}(\mathcal{O}_Y)$ is sometimes called the *trace map*. This is the map which has the property that the bijection

$$\operatorname{Hom}_X(L, a(K)) \longrightarrow \operatorname{Hom}_Y(Rf_*L, K)$$

for $L \in D_{QCoh}(\mathcal{O}_X)$ which characterizes the right adjoint is given by

$$\varphi \longmapsto \operatorname{Tr}_{f,K} \circ Rf_*\varphi$$

The isomorphism

$$Rf_*R\mathcal{H}om_{\mathcal{O}_X}(L, a(K)) \longrightarrow R\mathcal{H}om_{\mathcal{O}_Y}(Rf_*L, K)$$

of Lemma 3.6 comes about by composition with $\text{Tr}_{f,K}$. Every trace map we are going to consider in this section will be a special case of this trace map. Before we discuss some special cases we show that formation of the trace map commutes with base change.

0B6J **Lemma 7.1** (Trace map and base change)**.** *Suppose we have a diagram (4.0.1) where f and g are tor independent. Then the maps $1 \star Tr_f : Lg^* \circ Rf_* \circ a \to Lg^*$ and $Tr_{f'} \star 1 : Rf'_* \circ a' \circ Lg^* \to Lg^*$ agree via the base change maps $\beta : Lg^* \circ Rf_* \to Rf'_* \circ L(g')^*$ (Cohomology, Remark 29.3) and $\alpha : L(g')^* \circ a \to a' \circ Lg^*$ (5.0.1). More precisely, the diagram*

$$\begin{tikzcd}
Lg^* \circ Rf_* \circ a \arrow[r, "1 \star Tr_f"'] \arrow[d, "\beta \star 1"'] & Lg^* \\
Rf'_* \circ L(g')^* \circ a \arrow[r, "1 \star \alpha"] & Rf'_* \circ a' \circ Lg^* \arrow[u, "Tr_{f'} \star 1"']
\end{tikzcd}$$

of transformations of functors commutes.

Proof. In this proof we write f_* for Rf_* and g^* for Lg^* and we drop $\star$ products with identities as one can figure out which ones to add as long as the source and target of the transformation is known. Recall that $\beta : g^* \circ f_* \to f'_* \circ (g')^*$ is an isomorphism and that α is defined using the isomorphism $\beta^\vee : g'_* \circ a' \to a \circ g_*$ which is the adjoint of β, see Lemma 4.1 and its proof. First we note that the top horizontal arrow of the diagram in the lemma is equal to the composition

$$g^* \circ f_* \circ a \to g^* \circ f_* \circ a \circ g_* \circ g^* \to g^* \circ g_* \circ g^* \to g^*$$

where the first arrow is the unit for (g^*, g_*), the second arrow is Tr_f, and the third arrow is the counit for (g^*, g_*). This is a simple consequence of the fact that the composition $g^* \to g^* \circ g_* \circ g^* \to g^*$ of unit and counit is the identity. Consider the diagram

$$\begin{tikzcd}
 & g^* \circ f_* \circ a \arrow[r, "\text{Tr}_f"'] \arrow[dl, "\beta"'] \arrow[d] & g^* & \\
f'_* \circ (g')^* \circ a \arrow[dr] & g^* \circ f_* \circ a \circ g_* \circ g^* \arrow[ur] \arrow[d, "\beta"'] & g^* \circ f_* \circ g'_* \circ a' \circ g^* \arrow[l, "\beta^\vee"'] \arrow[d, "\beta"'] & f'_* \circ a' \circ g^* \arrow[ul, "\text{Tr}_{f'}"'] \\
 & f'_* \circ (g')^* \circ a \circ g_* \circ g^* & f'_* \circ (g')^* \circ g'_* \circ a' \circ g^* \arrow[l, "\beta^\vee"'] \arrow[ur] &
\end{tikzcd}$$

In this diagram the two squares commute Categories, Lemma 27.2 or more simply the discussion preceding Categories, Definition 27.1. The triangle commutes by the discussion above. By Categories, Lemma 24.7 the square

$$\begin{tikzcd}
g^* \circ f_* \circ g'_* \circ a' \arrow[r, "\beta"'] \arrow[d, "\beta^\vee"'] & f'_* \circ (g')^* \circ g'_* \circ a' \arrow[d] \\
g^* \circ f_* \circ a \circ g_* \arrow[r] & \text{id}
\end{tikzcd}$$

commutes which implies the pentagon in the big diagram commutes. Since β and $\beta^\vee$ are isomorphisms, and since going on the outside of the big diagram equals $\text{Tr}_f \circ \alpha \circ \beta$ by definition this proves the lemma. $\square$

Let $f : X \to Y$ be a morphism of quasi-compact and quasi-separated schemes. Let $a : D_{QCoh}(\mathcal{O}_Y) \to D_{QCoh}(\mathcal{O}_X)$ be the right adjoint of Rf_* as in Lemma 3.1. By Categories, Section 24 we obtain a transformation of functors

$$\eta_f : \text{id} \to a \circ Rf_*$$

which is called the unit of the adjunction.

0B6K **Lemma 7.2.** *Suppose we have a diagram (4.0.1) where f and g are tor independent. Then the maps $1 \star \eta_f : L(g')^* \to L(g')^* \circ a \circ Rf_*$ and $\eta_{f'} \star 1 : L(g')^* \to a' \circ Rf'_* \circ L(g')^*$ agree via the base change maps $\beta : Lg^* \circ Rf_* \to Rf'_* \circ L(g')^*$ (Cohomology, Remark 29.3) and $\alpha : L(g')^* \circ a \to a' \circ Lg^*$ (5.0.1). More precisely, the diagram*

$$\begin{tikzcd} L(g')^* \arrow[r, "1\star\eta_f"'] \arrow[d, "\eta_{f'}\star 1"'] & L(g')^* \circ a \circ Rf_* \arrow[d, "\alpha"] \\ a' \circ Rf'_* \circ L(g')^* & a' \circ Lg^* \circ Rf_* \arrow[l, "\beta"'] \end{tikzcd}$$

of transformations of functors commutes.

Proof. This proof is dual to the proof of Lemma 7.1. In this proof we write f_* for Rf_* and g^* for Lg^* and we drop $\star$ products with identities as one can figure out which ones to add as long as the source and target of the transformation is known. Recall that $\beta : g^* \circ f_* \to f'_* \circ (g')^*$ is an isomorphism and that α is defined using the isomorphism $\beta^\vee : g'_* \circ a' \to a \circ g_*$ which is the adjoint of β, see Lemma 4.1 and its proof. First we note that the left vertical arrow of the diagram in the lemma is equal to the composition

$$(g')^* \to (g')^* \circ g'_* \circ (g')^* \to (g')^* \circ g'_* \circ a' \circ f'_* \circ (g')^* \to a' \circ f'_* \circ (g')^*$$

where the first arrow is the unit for $((g')^*, g'_*)$, the second arrow is $\eta_{f'}$, and the third arrow is the counit for $((g')^*, g'_*)$. This is a simple consequence of the fact that the composition $(g')^* \to (g')^* \circ (g')_* \circ (g')^* \to (g')^*$ of unit and counit is the identity. Consider the diagram

$$\begin{tikzcd} & (g')^* \circ a \circ f_* \arrow[r] & (g')^* \circ a \circ g_* \circ g^* \circ f_* \arrow[dl, "\beta"'] \\ (g')^* \arrow[ur, "\eta_f"] \arrow[dr] \arrow[dd, "\eta_{f'}"'] & (g')^* \circ a \circ g_* \circ f'_* \circ (g')^* & (g')^* \circ g'_* \circ a' \circ g^* \circ f_* \arrow[u, "\beta^\vee"'] \arrow[dl, "\beta"'] \arrow[d] \\ & (g')^* \circ g'_* \circ a' \circ f'_* \circ (g')^* \arrow[u, "\beta^\vee"'] \arrow[dl] & a' \circ g^* \circ f_* \arrow[dll, "\beta"] \\ a' \circ f'_* \circ (g')^* & & \end{tikzcd}$$

In this diagram the two squares commute Categories, Lemma 27.2 or more simply the discussion preceding Categories, Definition 27.1. The triangle commutes by the discussion above. By the dual of Categories, Lemma 24.7 the square

$$\begin{tikzcd} \mathrm{id} \arrow[r] \arrow[d] & g'_* \circ a' \circ g^* \circ f_* \arrow[d, "\beta"] \\ g'_* \circ a' \circ g^* \circ f_* \arrow[r, "\beta^\vee"] & a \circ g_* \circ f'_* \circ (g')^* \end{tikzcd}$$

commutes which implies the pentagon in the big diagram commutes. Since β and $\beta^\vee$ are isomorphisms, and since going on the outside of the big diagram equals $\beta \circ \alpha \circ \eta_f$ by definition this proves the lemma. □

0B6L **Example 7.3.** Let $A \to B$ be a ring map. Let $Y = \mathrm{Spec}(A)$ and $X = \mathrm{Spec}(B)$ and $f : X \to Y$ the morphism corresponding to $A \to B$. As seen in Example 3.2 the right adjoint of $Rf_* : D_{QCoh}(\mathcal{O}_X) \to D_{QCoh}(\mathcal{O}_Y)$ sends an object K of $D(A) = D_{QCoh}(\mathcal{O}_Y)$ to $R\mathrm{Hom}(B, K)$ in $D(B) = D_{QCoh}(\mathcal{O}_X)$. The trace map is the map

$$\mathrm{Tr}_{f,K} : R\mathrm{Hom}(B, K) \longrightarrow R\mathrm{Hom}(A, K) = K$$

induced by the A-module map $A \to B$.

8. Right adjoint of pushforward and pullback

0B6N Let $f : X \to Y$ be a morphism of quasi-compact and quasi-separated schemes. Let a be the right adjoint of pushforward as in Lemma 3.1. For $K, L \in D_{QCoh}(\mathcal{O}_Y)$ there is a canonical map

$$Lf^*K \otimes^{\mathbf{L}}_{\mathcal{O}_X} a(L) \longrightarrow a(K \otimes^{\mathbf{L}}_{\mathcal{O}_Y} L)$$

Namely, this map is adjoint to a map

$$Rf_*(Lf^*K \otimes^{\mathbf{L}}_{\mathcal{O}_X} a(L)) = K \otimes^{\mathbf{L}}_{\mathcal{O}_Y} Rf_*(a(L)) \longrightarrow K \otimes^{\mathbf{L}}_{\mathcal{O}_Y} L$$

(equality by Derived Categories of Schemes, Lemma 21.1) for which we use the trace map $Rf_*a(L) \to L$. When $L = \mathcal{O}_Y$ we obtain a map

0A9S (8.0.1)
$$Lf^*K \otimes^{\mathbf{L}}_{\mathcal{O}_X} a(\mathcal{O}_Y) \longrightarrow a(K)$$

functorial in K and compatible with distinguished triangles.

0A9T **Lemma 8.1.** *Let $f : X \to Y$ be a morphism of quasi-compact and quasi-separated schemes. The map $Lf^*K \otimes^{\mathbf{L}}_{\mathcal{O}_X} a(L) \to a(K \otimes^{\mathbf{L}}_{\mathcal{O}_Y} L)$ defined above for $K, L \in D_{QCoh}(\mathcal{O}_Y)$ is an isomorphism if K is perfect. In particular, (8.0.1) is an isomorphism if K is perfect.*

Proof. Let $K^\vee$ be the "dual" to K, see Cohomology, Lemma 43.11. For $M \in D_{QCoh}(\mathcal{O}_X)$ we have

$$\begin{aligned} \mathrm{Hom}_{D(\mathcal{O}_Y)}(Rf_*M, K \otimes^{\mathbf{L}}_{\mathcal{O}_Y} L) &= \mathrm{Hom}_{D(\mathcal{O}_Y)}(Rf_*M \otimes^{\mathbf{L}}_{\mathcal{O}_Y} K^\vee, L) \\ &= \mathrm{Hom}_{D(\mathcal{O}_X)}(M \otimes^{\mathbf{L}}_{\mathcal{O}_X} Lf^*K^\vee, a(L)) \\ &= \mathrm{Hom}_{D(\mathcal{O}_X)}(M, Lf^*K \otimes^{\mathbf{L}}_{\mathcal{O}_X} a(L)) \end{aligned}$$

Second equality by the definition of a and the projection formula (Cohomology, Lemma 45.3) or the more general Derived Categories of Schemes, Lemma 21.1. Hence the result by the Yoneda lemma. □

0B6P **Lemma 8.2.** *Suppose we have a diagram (4.0.1) where f and g are tor independent. Let $K \in D_{QCoh}(\mathcal{O}_Y)$. The diagram*

$$\begin{tikzcd} L(g')^*(Lf^*K \otimes^{\mathbf{L}}_{\mathcal{O}_X} a(\mathcal{O}_Y)) \arrow[r] \arrow[d] & L(g')^*a(K) \arrow[d] \\ L(f')^*Lg^*K \otimes^{\mathbf{L}}_{\mathcal{O}_{X'}} a'(\mathcal{O}_{Y'}) \arrow[r] & a'(Lg^*K) \end{tikzcd}$$

commutes where the horizontal arrows are the maps (8.0.1) for K and Lg^*K and the vertical maps are constructed using Cohomology, Remark 29.3 and (5.0.1).

Proof. In this proof we will write f_* for Rf_* and f^* for Lf^*, etc, and we will write $\otimes$ for $\otimes^{\mathbf{L}}_{\mathcal{O}_X}$, etc. Let us write (8.0.1) as the composition

$$\begin{aligned} f^*K \otimes a(\mathcal{O}_Y) &\to a(f_*(f^*K \otimes a(\mathcal{O}_Y))) \\ &\leftarrow a(K \otimes f_*a(\mathcal{O}_K)) \\ &\to a(K \otimes \mathcal{O}_Y) \\ &\to a(K) \end{aligned}$$

Here the first arrow is the unit η_f, the second arrow is a applied to Cohomology, Equation (45.2.1) which is an isomorphism by Derived Categories of Schemes, Lemma 21.1, the third arrow is a applied to $\text{id}_K \otimes \text{Tr}_f$, and the fourth arrow is a applied to the isomorphism $K \otimes \mathcal{O}_Y = K$. The proof of the lemma consists in showing that each of these maps gives rise to a commutative square as in the statement of the lemma. For η_f and Tr_f this is Lemmas 7.2 and 7.1. For the arrow using Cohomology, Equation (45.2.1) this is Cohomology, Remark 45.5. For the multiplication map it is clear. This finishes the proof. ☐

0B6Q **Lemma 8.3.** *Let $f : X \to Y$ be a proper morphism of Noetherian schemes. Let $V \subset Y$ be an open such that $f^{-1}(V) \to V$ is an isomorphism. Then for $K \in D^+_{QCoh}(\mathcal{O}_Y)$ the map (8.0.1) restricts to an isomorphism over $f^{-1}(V)$.*

Proof. By Lemma 4.4 the map (4.1.1) is an isomorphism for objects of $D^+_{QCoh}(\mathcal{O}_Y)$. Hence Lemma 8.2 tells us the restriction of (8.0.1) for K to $f^{-1}(V)$ is the map (8.0.1) for $K|_V$ and $f^{-1}(V) \to V$. Thus it suffices to show that the map is an isomorphism when f is the identity morphism. This is clear. ☐

0B6R **Lemma 8.4.** *Let $f : X \to Y$ and $g : Y \to Z$ be composable morphisms of quasi-compact and quasi-separated schemes and set $h = g \circ f$. Let a, b, c be the adjoints of Lemma 3.1 for f, g, h. For any $K \in D_{QCoh}(\mathcal{O}_Z)$ the diagram*

$$\begin{tikzcd} Lf^*(Lg^*K \otimes^{\mathbf{L}}_{\mathcal{O}_Y} b(\mathcal{O}_Z)) \otimes^{\mathbf{L}}_{\mathcal{O}_X} a(\mathcal{O}_Y) \arrow[r] \arrow[d, equal] & a(Lg^*K \otimes^{\mathbf{L}}_{\mathcal{O}_Y} b(\mathcal{O}_Z)) \arrow[r] & a(b(K)) \arrow[d, equal] \\ Lh^*K \otimes^{\mathbf{L}}_{\mathcal{O}_X} Lf^*b(\mathcal{O}_Z) \otimes^{\mathbf{L}}_{\mathcal{O}_X} a(\mathcal{O}_Y) \arrow[r] & Lh^*K \otimes^{\mathbf{L}}_{\mathcal{O}_X} c(\mathcal{O}_Z) \arrow[r] & c(K) \end{tikzcd}$$

is commutative where the arrows are (8.0.1) and we have used $Lh^* = Lf^* \circ Lg^*$ and $c = a \circ b$.

Proof. In this proof we will write f_* for Rf_* and f^* for Lf^*, etc, and we will write $\otimes$ for $\otimes^{\mathbf{L}}_{\mathcal{O}_X}$, etc. The composition of the top arrows is adjoint to a map

$$g_*f_*(f^*(g^*K \otimes b(\mathcal{O}_Z)) \otimes a(\mathcal{O}_Y)) \to K$$

The left hand side is equal to $K \otimes g_*f_*(f^*b(\mathcal{O}_Z) \otimes a(\mathcal{O}_Y))$ by Derived Categories of Schemes, Lemma 21.1 and inspection of the definitions shows the map comes from the map

$$g_*f_*(f^*b(\mathcal{O}_Z) \otimes a(\mathcal{O}_Y)) \xleftarrow{g_*\epsilon} g_*(b(\mathcal{O}_Z) \otimes f_*a(\mathcal{O}_Y)) \xrightarrow{g_*\alpha} g_*(b(\mathcal{O}_Z)) \xrightarrow{\beta} \mathcal{O}_Z$$

tensored with id_K. Here ϵ is the isomorphism from Derived Categories of Schemes, Lemma 21.1 and β comes from the counit map $g_*b \to \text{id}$. Similarly, the composition of the lower horizontal arrows is adjoint to id_K tensored with the composition

$$g_*f_*(f^*b(\mathcal{O}_Z) \otimes a(\mathcal{O}_Y)) \xrightarrow{g_*f_*\delta} g_*f_*(ab(\mathcal{O}_Z)) \xrightarrow{g_*\gamma} g_*(b(\mathcal{O}_Z)) \xrightarrow{\beta} \mathcal{O}_Z$$

where γ comes from the counit map $f_*a \to \text{id}$ and δ is the map whose adjoint is the composition

$$f_*(f^*b(\mathcal{O}_Z) \otimes a(\mathcal{O}_Y)) \xleftarrow{\epsilon} b(\mathcal{O}_Z) \otimes f_*a(\mathcal{O}_Y) \xrightarrow{\alpha} b(\mathcal{O}_Z)$$

By general properties of adjoint functors, adjoint maps, and counits (see Categories, Section 24) we have $\gamma \circ f_*\delta = \alpha \circ \epsilon^{-1}$ as desired. $\square$

9. Right adjoint of pushforward for closed immersions

0A74 Let $i : (Z, \mathcal{O}_Z) \to (X, \mathcal{O}_X)$ be a morphism of ringed spaces such that i is a homomorphism onto a closed subset and such that $i^\sharp : \mathcal{O}_X \to i_*\mathcal{O}_Z$ is surjective. (For example a closed immersion of schemes.) Let $\mathcal{I} = \text{Ker}(i^\sharp)$. For a sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ the sheaf

$$\mathcal{H}om_{\mathcal{O}_X}(i_*\mathcal{O}_Z, \mathcal{F})$$

a sheaf of $\mathcal{O}_X$-modules annihilated by $\mathcal{I}$. Hence by Modules, Lemma 13.4 there is a sheaf of $\mathcal{O}_Z$-modules, which we will denote $\mathcal{H}om(\mathcal{O}_Z, \mathcal{F})$, such that

$$i_*\,\mathcal{H}om(\mathcal{O}_Z, \mathcal{F}) = \mathcal{H}om_{\mathcal{O}_X}(i_*\mathcal{O}_Z, \mathcal{F})$$

as $\mathcal{O}_X$-modules. We spell out what this means.

0A75 **Lemma 9.1.** *With notation as above. The functor $\mathcal{H}om(\mathcal{O}_Z, -)$ is a right adjoint to the functor $i_* : Mod(\mathcal{O}_Z) \to Mod(\mathcal{O}_X)$. For $V \subset Z$ open we have*

$$\Gamma(V, \mathcal{H}om(\mathcal{O}_Z, \mathcal{F})) = \{s \in \Gamma(U, \mathcal{F}) \mid \mathcal{I}s = 0\}$$

where $U \subset X$ is an open whose intersection with Z is V.

Proof. Let $\mathcal{G}$ be a sheaf of $\mathcal{O}_Z$-modules. Then

$$\text{Hom}_{\mathcal{O}_X}(i_*\mathcal{G}, \mathcal{F}) = \text{Hom}_{i_*\mathcal{O}_Z}(i_*\mathcal{G}, \mathcal{H}om_{\mathcal{O}_X}(i_*\mathcal{O}_Z, \mathcal{F})) = \text{Hom}_{\mathcal{O}_Z}(\mathcal{G}, \mathcal{H}om(\mathcal{O}_Z, \mathcal{F}))$$

The first equality by Modules, Lemma 20.6 and the second by the fully faithfulness of i_*, see Modules, Lemma 13.4. The description of sections is left to the reader. $\square$

The functor

$$Mod(\mathcal{O}_X) \longrightarrow Mod(\mathcal{O}_Z), \quad \mathcal{F} \longmapsto \mathcal{H}om(\mathcal{O}_Z, \mathcal{F})$$

is left exact and has a derived extension

$$R\,\mathcal{H}om(\mathcal{O}_Z, -) : D(\mathcal{O}_X) \to D(\mathcal{O}_Z).$$

0A76 **Lemma 9.2.** *With notation as above. The functor $R\mathcal{H}om(\mathcal{O}_Z, -)$ is the right adjoint of the functor $Ri_* : D(\mathcal{O}_Z) \to D(\mathcal{O}_X)$.*

Proof. This is a consequence of the fact that i_* and $\mathcal{H}om(\mathcal{O}_Z, -)$ are adjoint functors by Lemma 9.1. See Derived Categories, Lemma 28.5. $\square$

0A77 **Lemma 9.3.** *With notation as above. We have*

$$Ri_* R\mathcal{H}om(\mathcal{O}_Z, K) = R\mathcal{H}om_{\mathcal{O}_X}(i_*\mathcal{O}_Z, K)$$

in $D(\mathcal{O}_X)$ for all K in $D(\mathcal{O}_X)$.

Proof. This is immediate from the construction of the functor $R\mathcal{H}om(\mathcal{O}_Z, -)$. $\square$

0E2I **Lemma 9.4.** *With notation as above. For $M \in D(\mathcal{O}_Z)$ we have*

$$R\mathcal{H}om_{\mathcal{O}_X}(Ri_*M, K) = Ri_* R\mathcal{H}om_{\mathcal{O}_Z}(M, R\mathcal{H}om(\mathcal{O}_Z, K))$$

in $D(\mathcal{O}_Z)$ for all K in $D(\mathcal{O}_X)$.

Proof. This is immediate from the construction of the functor $R\mathcal{H}om(\mathcal{O}_Z, -)$ and the fact that if $\mathcal{K}^\bullet$ is a K-injective complex of $\mathcal{O}_X$-modules, then $\mathcal{H}om(\mathcal{O}_Z, \mathcal{K}^\bullet)$ is a K-injective complex of $\mathcal{O}_Z$-modules, see Derived Categories, Lemma 29.9. $\square$

0A78 **Lemma 9.5.** *Let $i : Z \to X$ be a pseudo-coherent closed immersion of schemes (any closed immersion if X is locally Noetherian). Then*

1. *$R\mathcal{H}om(\mathcal{O}_Z, -)$ maps $D^+_{QCoh}(\mathcal{O}_X)$ into $D^+_{QCoh}(\mathcal{O}_Z)$, and*
2. *if $X = \mathrm{Spec}(A)$ and $Z = \mathrm{Spec}(B)$, then the diagram*

$$\begin{tikzcd} D^+(B) \arrow[r] & D^+_{QCoh}(\mathcal{O}_Z) \\ D^+(A) \arrow[r] \arrow[u, "R\mathrm{Hom}(B,-)"] & D^+_{QCoh}(\mathcal{O}_X) \arrow[u, "R\mathcal{H}om(\mathcal{O}_Z,-)"'] \end{tikzcd}$$

is commutative.

Proof. To explain the parenthetical remark, if X is locally Noetherian, then i is pseudo-coherent by More on Morphisms, Lemma 50.9.

Let K be an object of $D^+_{QCoh}(\mathcal{O}_X)$. To prove (1), by Morphisms, Lemma 4.1 it suffices to show that i_* applied to $H^n(R\mathcal{H}om(\mathcal{O}_Z, K))$ produces a quasi-coherent module on X. By Lemma 9.3 this means we have to show that $R\mathcal{H}om_{\mathcal{O}_X}(i_*\mathcal{O}_Z, K)$ is in $D_{QCoh}(\mathcal{O}_X)$. Since i is pseudo-coherent the sheaf $\mathcal{O}_Z$ is a pseudo-coherent $\mathcal{O}_X$-module. Hence the result follows from Derived Categories of Schemes, Lemma 9.8.

Assume $X = \mathrm{Spec}(A)$ and $Z = \mathrm{Spec}(B)$ as in (2). Let $I^\bullet$ be a bounded below complex of injective A-modules representing an object K of $D^+(A)$. Then we know that $R\mathrm{Hom}(B, K) = \mathrm{Hom}_A(B, I^\bullet)$ viewed as a complex of B-modules. Choose a quasi-isomorphism

$$\widetilde{I^\bullet} \longrightarrow \mathcal{I}^\bullet$$

where $\mathcal{I}^\bullet$ is a bounded below complex of injective $\mathcal{O}_X$-modules. It follows from the description of the functor $\mathcal{H}om(\mathcal{O}_Z, -)$ in Lemma 9.1 that there is a map

$$\mathrm{Hom}_A(B, I^\bullet) \longrightarrow \Gamma(Z, \mathcal{H}om(\mathcal{O}_Z, \mathcal{I}^\bullet))$$

Observe that $\mathcal{H}om(\mathcal{O}_Z, \mathcal{I}^\bullet)$ represents $R\mathcal{H}om(\mathcal{O}_Z, \widetilde{K})$. Applying the universal property of the $\widetilde{\ }$ functor we obtain a map

$$\widetilde{\mathrm{Hom}_A(B, I^\bullet)} \longrightarrow R\mathcal{H}om(\mathcal{O}_Z, \widetilde{K})$$

in $D(\mathcal{O}_Z)$. We may check that this map is an isomorphism in $D(\mathcal{O}_Z)$ after applying i_*. However, once we apply i_* we obtain the isomorphism of Derived Categories of Schemes, Lemma 9.8 via the identification of Lemma 9.3. $\square$

0A79 **Lemma 9.6.** *Let $i : Z \to X$ be a closed immersion of schemes. Assume X is a locally Noetherian. Then $R\mathcal{H}om(\mathcal{O}_Z, -)$ maps $D^+_{Coh}(\mathcal{O}_X)$ into $D^+_{Coh}(\mathcal{O}_Z)$.*

Proof. The question is local on X, hence we may assume that X is affine. Say $X = \mathrm{Spec}(A)$ and $Z = \mathrm{Spec}(B)$ with A Noetherian and $A \to B$ surjective. In this case, we can apply Lemma 9.5 to translate the question into algebra. The corresponding algebra result is a consequence of Dualizing Complexes, Lemma 13.4. $\square$

0A9X **Lemma 9.7.** *Let X be a quasi-compact and quasi-separated scheme. Let $i : Z \to X$ be a pseudo-coherent closed immersion (if X is Noetherian, then any closed immersion is pseudo-coherent). Let $a : D_{QCoh}(\mathcal{O}_X) \to D_{QCoh}(\mathcal{O}_Z)$ be the right adjoint to Ri_*. Then there is a functorial isomorphism*

$$a(K) = R\mathcal{H}om(\mathcal{O}_Z, K)$$

for $K \in D^+_{QCoh}(\mathcal{O}_X)$.

Proof. (The parenthetical statement follows from More on Morphisms, Lemma 50.9.) By Lemma 9.2 the functor $R\mathcal{H}om(\mathcal{O}_Z, -)$ is a right adjoint to $Ri_* : D(\mathcal{O}_Z) \to D(\mathcal{O}_X)$. Moreover, by Lemma 9.5 and Lemma 3.5 both $R\mathcal{H}om(\mathcal{O}_Z, -)$ and a map $D^+_{QCoh}(\mathcal{O}_X)$ into $D^+_{QCoh}(\mathcal{O}_Z)$. Hence we obtain the isomorphism by uniqueness of adjoint functors. $\square$

0B6M **Example 9.8.** If $i : Z \to X$ is closed immersion of Noetherian schemes, then the diagram

$$\begin{tikzcd}
i_*a(K) \arrow[rr, "\mathrm{Tr}_{i,K}"'] \arrow[d, equal] & & K \arrow[d, equal] \\
i_*R\mathcal{H}om(\mathcal{O}_Z, K) \arrow[r, equal] & R\mathcal{H}om_{\mathcal{O}_X}(i_*\mathcal{O}_Z, K) \arrow[r] & K
\end{tikzcd}$$

is commutative for $K \in D^+_{QCoh}(\mathcal{O}_X)$. Here the horizontal equality sign is Lemma 9.3 and the lower horizontal arrow is induced by the map $\mathcal{O}_X \to i_*\mathcal{O}_Z$. The commutativity of the diagram is a consequence of Lemma 9.7.

10. Right adjoint of pushforward for closed immersions and base change

0E2J Consider a cartesian diagram of schemes

$$\begin{tikzcd}
Z' \arrow[r, "i'"'] \arrow[d, "g"'] & X' \arrow[d, "f"] \\
Z \arrow[r, "i"] & X
\end{tikzcd}$$

where i is a closed immersion. If Z and X' are tor independent over X, then there is a canonical base change map

0E2K (10.0.1)
$$Lg^*R\mathcal{H}om(\mathcal{O}_Z, K) \longrightarrow R\mathcal{H}om(\mathcal{O}_{Z'}, Lf^*K)$$

in $D(\mathcal{O}_{Z'})$ functorial for K in $D(\mathcal{O}_X)$. Namely, by adjointness of Lemma 9.2 such an arrow is the same thing as a map

$$Ri'_*Lg^*R\mathcal{H}om(\mathcal{O}_Z, K) \longrightarrow Lf^*K$$

in $D(\mathcal{O}_{X'})$. By tor independence we have $Ri'_* \circ Lg^* = Lf^* \circ Ri_*$ (see Derived Categories of Schemes, Lemma 21.7). Thus this is the same thing as a map

$$Lf^*Ri_*R\mathcal{H}om(\mathcal{O}_Z, K) \longrightarrow Lf^*K$$

For this we can use $Lf^*(can)$ where $can : Ri_*R\mathcal{H}om(\mathcal{O}_Z, K) \to K$ is the counit of the adjunction.

0E2L **Lemma 10.1.** *In the situation above, the map (10.0.1) is an isomorphism if and only if the base change map*

$$Lf^*R\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_Z, K) \longrightarrow R\mathcal{H}om_{\mathcal{O}_{X'}}(\mathcal{O}_{Z'}, Lf^*K)$$

of Cohomology, Remark 36.11 is an isomorphism.

Proof. The statement makes sense because $\mathcal{O}_{Z'} = Lf^*\mathcal{O}_Z$ by the assumed tor independence. Since i'_* is exact and faithful we see that it suffices to show the map (10.0.1) is an isomorphism after applying Ri'_*. Since $Ri'_* \circ Lg^* = Lf^* \circ Ri_*$ by the assumed tor indepence and Derived Categories of Schemes, Lemma 21.7 we obtain a map

$$Lf^*Ri_*R\mathcal{H}om(\mathcal{O}_Z, K) \longrightarrow Ri'_*R\mathcal{H}om(\mathcal{O}_{Z'}, Lf^*K)$$

whose source and target are as in the statement of the lemma by Lemma 9.3. We omit the verification that this is the same map as the one constructed in Cohomology, Remark 36.11. $\square$

0E2M **Lemma 10.2.** *In the situation above, assume f is flat and i pseudo-coherent. Then (10.0.1) is an isomorphism for K in $D^+_{QCoh}(\mathcal{O}_X)$.*

Proof. First proof. To prove this map is an isomorphism, we may work locally. Hence we may assume X, X', Z, Z' are affine, say corresponding to the rings A, A', B, B'. Then B and A' are tor independent over A. By Lemma 10.1 it suffices to check that

$$R\operatorname{Hom}_A(B, K) \otimes_A^{\mathbf{L}} A' = R\operatorname{Hom}_{A'}(B', K \otimes_A^{\mathbf{L}} A')$$

in $D(A')$ for all $K \in D^+(A)$. Here we use Derived Categories of Schemes, Lemma 9.8 and the fact that B, resp. B' is pseudo-coherent as an A-module, resp. A'-module to compare derived hom on the level of rings and schemes. The displayed equality follows from More on Algebra, Lemma 86.3 part (3). See also the discussion in Dualizing Complexes, Section 14.

Second proof[2]. Let $z' \in Z'$ with image $z \in Z$. First show that (10.0.1) on stalks at z' induces the map

$$R\operatorname{Hom}(\mathcal{O}_{Z,z}, K_z) \otimes^{\mathbf{L}}_{\mathcal{O}_{Z,x}} \mathcal{O}_{Z',z'} \longrightarrow R\operatorname{Hom}(\mathcal{O}_{Z',z'}, K_z \otimes^{\mathbf{L}}_{\mathcal{O}_{X,z}} \mathcal{O}_{X',z'})$$

[2]This proof shows it suffices to assume K is in $D^+(\mathcal{O}_X)$.

from Dualizing Complexes, Equation (14.0.1). Namely, the constructions of these maps are identical. Then apply Dualizing Complexes, Lemma 14.2. $\square$

0E2N **Lemma 10.3.** *Let $i : Z \to X$ be a pseudo-coherent closed immersion of schemes. Let $M \in D_{QCoh}(\mathcal{O}_X)$ locally have tor-amplitude in $[a, \infty)$. Let $K \in D^+_{QCoh}(\mathcal{O}_X)$. Then there is a canonical isomorphism*

$$R\mathcal{H}om(\mathcal{O}_Z, K) \otimes^{\mathbf{L}}_{\mathcal{O}_Z} Li^*M = R\mathcal{H}om(\mathcal{O}_Z, K \otimes^{\mathbf{L}}_{\mathcal{O}_X} M)$$

in $D(\mathcal{O}_Z)$.

Proof. A map from LHS to RHS is the same thing as a map

$$Ri_*R\mathcal{H}om(\mathcal{O}_Z, K) \otimes^{\mathbf{L}}_{\mathcal{O}_X} M \longrightarrow K \otimes^{\mathbf{L}}_{\mathcal{O}_X} M$$

by Lemmas 9.2 and 9.3. For this map we take the counit $Ri_*R\mathcal{H}om(\mathcal{O}_Z, K) \to K$ tensored with id_M. To see this map is an isomorphism under the hypotheses given, translate into algebra using Lemma 9.5 and then for example use More on Algebra, Lemma 86.3 part (3). Instead of using Lemma 9.5 you can look at stalks as in the second proof of Lemma 10.2. $\square$

11. Right adjoint of pushforward for finite morphisms

0AWZ If $i : Z \to X$ is a closed immersion of schemes, then there is a right adjoint $\mathcal{H}om(\mathcal{O}_Z, -)$ to the functor $i_* : \mathit{Mod}(\mathcal{O}_Z) \to \mathit{Mod}(\mathcal{O}_X)$ whose derived extension $R\mathcal{H}om(\mathcal{O}_Z, -)$ is the right adjoint to $Ri_* : D(\mathcal{O}_Z) \to D(\mathcal{O}_X)$. See Section 9. In the case of a finite morphism $f : Y \to X$ this strategy cannot work, as the functor $f_* : \mathit{Mod}(\mathcal{O}_Y) \to \mathit{Mod}(\mathcal{O}_X)$ is not exact in general and hence does not have a right adjoint. A replacement is to consider the exact functor $\mathit{Mod}(f_*\mathcal{O}_Y) \to \mathit{Mod}(\mathcal{O}_X)$ and consider the corresponding right adjoint and its derived extension.

Let $f : Y \to X$ be an affine morphism of schemes. For a sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ the sheaf

$$\mathcal{H}om_{\mathcal{O}_X}(f_*\mathcal{O}_Y, \mathcal{F})$$

is a sheaf of $f_*\mathcal{O}_Y$-modules. We obtain a functor $\mathit{Mod}(\mathcal{O}_X) \to \mathit{Mod}(f_*\mathcal{O}_Y)$ which we will denote $\mathcal{H}om(f_*\mathcal{O}_Y, -)$.

0BUZ **Lemma 11.1.** *With notation as above. The functor $\mathcal{H}om(f_*\mathcal{O}_Y, -)$ is a right adjoint to the restriction functor $\mathit{Mod}(f_*\mathcal{O}_Y) \to \mathit{Mod}(\mathcal{O}_X)$. For an affine open $U \subset X$ we have*

$$\Gamma(U, \mathcal{H}om(f_*\mathcal{O}_Y, \mathcal{F})) = \text{Hom}_A(B, \mathcal{F}(U))$$

where $A = \mathcal{O}_X(U)$ and $B = \mathcal{O}_Y(f^{-1}(U))$.

Proof. Adjointness follows from Modules, Lemma 20.6. As f is affine we see that $f_*\mathcal{O}_Y$ is the quasi-coherent sheaf corresponding to B viewed as an A-module. Hence the description of sections over U follows from Schemes, Lemma 7.1. $\square$

The functor $\mathcal{H}om(f_*\mathcal{O}_Y, -)$ is left exact. Let

$$R\mathcal{H}om(f_*\mathcal{O}_Y, -) : D(\mathcal{O}_X) \longrightarrow D(f_*\mathcal{O}_Y)$$

be its derived extension.

0BV0 **Lemma 11.2.** *With notation as above. The functor $R\mathcal{H}om(f_*\mathcal{O}_Y, -)$ is the right adjoint of the functor $D(f_*\mathcal{O}_Y) \to D(\mathcal{O}_X)$.*

Proof. Follows from Lemma 11.1 and Derived Categories, Lemma 28.5. $\square$

0BV1 **Lemma 11.3.** *With notation as above. The composition*

$$D(\mathcal{O}_X) \xrightarrow{R\mathcal{H}om(f_*\mathcal{O}_Y,-)} D(f_*\mathcal{O}_Y) \to D(\mathcal{O}_X)$$

is the functor $K \mapsto R\mathcal{H}om_{\mathcal{O}_X}(f_*\mathcal{O}_Y, K)$.

Proof. This is immediate from the construction. □

0AX2 **Lemma 11.4.** *Let $f : Y \to X$ be a finite pseudo-coherent morphism of schemes (a finite morphism of Noetherian schemes is pseudo-coherent). The functor $R\mathcal{H}om(f_*\mathcal{O}_Y, -)$ maps $D^+_{QCoh}(\mathcal{O}_X)$ into $D^+_{QCoh}(f_*\mathcal{O}_Y)$. If X is quasi-compact and quasi-separated, then the diagram*

$$\begin{tikzcd} D^+_{QCoh}(\mathcal{O}_X) \arrow[rr, "a"'] \arrow[dr, "R\mathcal{H}om(f_*\mathcal{O}_Y,-)"'] & & D^+_{QCoh}(\mathcal{O}_Y) \arrow[dl, "\Phi"] \\ & D^+_{QCoh}(f_*\mathcal{O}_Y) & \end{tikzcd}$$

is commutative, where a is the right adjoint of Lemma 3.1 for f and Φ is the equivalence of Derived Categories of Schemes, Lemma 5.3.

Proof. (The parenthetical remark follows from More on Morphisms, Lemma 50.9.) Since f is pseudo-coherent, the $\mathcal{O}_X$-module $f_*\mathcal{O}_Y$ is pseudo-coherent, see More on Morphisms, Lemma 50.8. Thus $R\mathcal{H}om(f_*\mathcal{O}_Y, -)$ maps $D^+_{QCoh}(\mathcal{O}_X)$ into $D^+_{QCoh}(f_*\mathcal{O}_Y)$, see Derived Categories of Schemes, Lemma 9.8. Then $\Phi \circ a$ and $R\mathcal{H}om(f_*\mathcal{O}_Y, -)$ agree on $D^+_{QCoh}(\mathcal{O}_X)$ because these functors are both right adjoint to the restriction functor $D^+_{QCoh}(f_*\mathcal{O}_Y) \to D^+_{QCoh}(\mathcal{O}_X)$. To see this use Lemmas 3.5 and 11.2. □

0AX3 **Remark 11.5.** If $f : Y \to X$ is a finite morphism of Noetherian schemes, then the diagram

$$\begin{tikzcd} Rf_*a(K) \arrow[r, "\mathrm{Tr}_{f,K}"'] \arrow[d, equal] & K \arrow[d, equal] \\ R\mathcal{H}om_{\mathcal{O}_X}(f_*\mathcal{O}_Y, K) \arrow[r] & K \end{tikzcd}$$

is commutative for $K \in D^+_{QCoh}(\mathcal{O}_X)$. This follows from Lemma 11.4. The lower horizontal arrow is induced by the map $\mathcal{O}_X \to f_*\mathcal{O}_Y$ and the upper horizontal arrow is the trace map discussed in Section 7.

12. Right adjoint of pushforward for proper flat morphisms

0E4H For proper, flat, and finitely presented morphisms of quasi-compact and quasi-separated schemes the right adjoint of pushforward enjoys some remarkable properties.

0E4I **Lemma 12.1.** *Let Y be a quasi-compact and quasi-separated scheme. Let $f : X \to Y$ be a morphism of schemes which is proper, flat, and of finite presentation. Let a be the right adjoint for $Rf_* : D_{QCoh}(\mathcal{O}_X) \to D_{QCoh}(\mathcal{O}_Y)$ of Lemma 3.1. Then a commutes with direct sums.*

Proof. Let P be a perfect object of $D(\mathcal{O}_X)$. By Derived Categories of Schemes, Lemma 26.4 the complex Rf_*P is perfect on Y. Let K_i be a family of objects of $D_{QCoh}(\mathcal{O}_Y)$. Then

$$\begin{aligned}\mathrm{Hom}_{D(\mathcal{O}_X)}(P, a(\bigoplus K_i)) &= \mathrm{Hom}_{D(\mathcal{O}_Y)}(Rf_*P, \bigoplus K_i)\\ &= \bigoplus \mathrm{Hom}_{D(\mathcal{O}_Y)}(Rf_*P, K_i)\\ &= \bigoplus \mathrm{Hom}_{D(\mathcal{O}_X)}(P, a(K_i))\end{aligned}$$

because a perfect object is compact (Derived Categories of Schemes, Proposition 16.1). Since $D_{QCoh}(\mathcal{O}_X)$ has a perfect generator (Derived Categories of Schemes, Theorem 14.3) we conclude that the map $\bigoplus a(K_i) \to a(\bigoplus K_i)$ is an isomorphism, i.e., a commutes with direct sums. ☐

0E4J **Lemma 12.2.** *Let Y be a quasi-compact and quasi-separated scheme. Let $f : X \to Y$ be a morphism of schemes which is proper, flat, and of finite presentation. Let a be the right adjoint for $Rf_* : D_{QCoh}(\mathcal{O}_X) \to D_{QCoh}(\mathcal{O}_Y)$ of Lemma 3.1. Then*

1. *for every closed $T \subset Y$ if $Q \in D_{QCoh}(Y)$ is supported on T, then $a(Q)$ is supported on $f^{-1}(T)$,*
2. *for every open $V \subset Y$ and any $K \in D_{QCoh}(\mathcal{O}_Y)$ the map (4.1.1) is an isomorphism, and*

Proof. This follows from Lemmas 4.3, 4.4, and 12.1. ☐

0E4K **Lemma 12.3.** *Let Y be a quasi-compact and quasi-separated scheme. Let $f : X \to Y$ be a morphism of schemes which is proper, flat, and of finite presentation. The map (8.0.1) is an isomorphism for every object K of $D_{QCoh}(\mathcal{O}_Y)$.*

Proof. By Lemma 12.1 we know that a commutes with direct sums. Hence the collection of objects of $D_{QCoh}(\mathcal{O}_Y)$ for which (8.0.1) is an isomorphism is a strictly full, saturated, triangulated subcategory of $D_{QCoh}(\mathcal{O}_Y)$ which is moreover preserved under taking direct sums. Since $D_{QCoh}(\mathcal{O}_Y)$ is a module category (Derived Categories of Schemes, Theorem 17.3) generated by a single perfect object (Derived Categories of Schemes, Theorem 14.3) we can argue as in More on Algebra, Remark 57.13 to see that it suffices to prove (8.0.1) is an isomorphism for a single perfect object. However, the result holds for perfect objects, see Lemma 8.1. ☐

The following lemma shows that the base change map (5.0.1) is an isomorphism for proper, flat morphisms of finite presentation. We will see in Example 15.2 that this does not remain true for perfect proper morphisms; in that case one has to make a tor independence condition.

0AAB **Lemma 12.4.** *Let $g : Y' \to Y$ be a morphism of quasi-compact and quasi-separated schemes. Let $f : X \to Y$ be a proper, flat morphism of finite presentation. Then the base change map (5.0.1) is an isomorphism for all $K \in D_{QCoh}(\mathcal{O}_Y)$.*

Proof. By Lemma 12.2 formation of the functors a and a' commutes with restriction to opens of Y and Y'. Hence we may assume $Y' \to Y$ is a morphism of affine schemes, see Remark 6.1. In this case the statement follows from Lemma 6.2. ☐

0B6S **Remark 12.5.** Let Y be a quasi-compact and quasi-separated scheme. Let $f : X \to Y$ be a proper, flat morphism of finite presentation. Let a be the adjoint of Lemma 3.1 for f. In this situation, $\omega_{X/Y}^\bullet = a(\mathcal{O}_Y)$ is sometimes called the *relative dualizing complex*. By Lemma 12.3 there is a functorial isomorphism $a(K) = Lf^*K \otimes^{\mathbf{L}}_{\mathcal{O}_X} \omega^\bullet_{X/Y}$ for $K \in D_{QCoh}(\mathcal{O}_Y)$. Moreover, the trace map

$$\mathrm{Tr}_{f,\mathcal{O}_Y} : Rf_*\omega^\bullet_{X/Y} \to \mathcal{O}_Y$$

of Section 7 induces the trace map for all K in $D_{QCoh}(\mathcal{O}_Y)$. More precisely the diagram

$$\begin{tikzcd}
Rf_*a(K) \arrow[rr, "\mathrm{Tr}_{f,K}"'] \arrow[d, equal] & & K \arrow[d, equal] \\
Rf_*(Lf^*K \otimes^{\mathbf{L}}_{\mathcal{O}_X} \omega^\bullet_{X/Y}) \arrow[r, equal] & K \otimes^{\mathbf{L}}_{\mathcal{O}_Y} Rf_*\omega^\bullet_{X/Y} \arrow[r, "\mathrm{id}_K \otimes \mathrm{Tr}_{f,\mathcal{O}_Y}"] & K
\end{tikzcd}$$

where the equality on the lower right is Derived Categories of Schemes, Lemma 21.1. If $g : Y' \to Y$ is a morphism of quasi-compact and quasi-separated schemes and $X' = Y' \times_Y X$, then by Lemma 12.4 we have $\omega^\bullet_{X'/Y'} = L(g')^*\omega^\bullet_{X/Y}$ where $g' : X' \to X$ is the projection and by Lemma 7.1 the trace map

$$\mathrm{Tr}_{f',\mathcal{O}_{Y'}} : Rf'_*\omega^\bullet_{X'/Y'} \to \mathcal{O}_{Y'}$$

for $f' : X' \to Y'$ is the base change of $\mathrm{Tr}_{f,\mathcal{O}_Y}$ via the base change isomorphism.

0E4L **Lemma 12.6.** *Let Y be a quasi-compact and quasi-separated scheme. Let $f : X \to Y$ be a morphism of schemes which is proper, flat, and of finite presentation with relative dualizing complex $\omega^\bullet_{X/Y}$ (Remark 12.5). Then*

1. *$\omega^\bullet_{X/Y}$ is a Y-perfect object of $D(\mathcal{O}_X)$,*
2. *$Rf_*\omega^\bullet_{X/Y}$ has vanishing cohomology sheaves in positive degrees,*
3. *$\mathcal{O}_X \to R\mathcal{H}om_{\mathcal{O}_X}(\omega^\bullet_{X/Y}, \omega^\bullet_{X/Y})$ is an isomorphism.*

Proof. In view of the fact that formation of $\omega^\bullet_{X/Y}$ commutes with base change (see Remark 12.5), we may and do assume that Y is affine. We will repeatedly use that $Rf_*R\mathcal{H}om_{\mathcal{O}_X}(L, a(K)) = R\mathcal{H}om_{\mathcal{O}_Y}(Rf_*L, K)$, see Lemma 3.6. Let E be a perfect object of $D(\mathcal{O}_X)$ with dual $E^\vee$, see Cohomology, Lemma 43.11. Then

$$Rf_*(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} \omega^\bullet_{X/Y}) = Rf_*R\mathcal{H}om_{\mathcal{O}_X}(E^\vee, \omega^\bullet_{X/Y}) = R\mathcal{H}om_{\mathcal{O}_Y}(Rf_*E^\vee, \mathcal{O}_Y)$$

By Derived Categories of Schemes, Lemma 26.4 the complex $Rf_*E^\vee$ is perfect. Hence the dual $R\mathcal{H}om_{\mathcal{O}_Y}(Rf_*E^\vee, \mathcal{O}_Y)$ is perfect as well. We conclude that $\omega^\bullet_{X/Y}$ is pseudo-coherent by Derived Categories of Schemes, Lemma 30.3 amd More on Morphisms, Lemma 59.4.

Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_Y$-module. By Lemma 12.3 we have

$$a(\mathcal{F}) = Lf^*\mathcal{F} \otimes^{\mathbf{L}}_{\mathcal{O}_X} \omega^\bullet_{X/Y} = f^{-1}\mathcal{F} \otimes^{\mathbf{L}}_{f^{-1}\mathcal{O}_Y} \omega^\bullet_{X/Y}$$

Second equality by Cohomology, Lemma 28.4. By Lemma 3.5 there exists an integer N such that $H^i(a(\mathcal{F})) = 0$ for $i \leq -N$. Looking at stalks we conclude that $\omega^\bullet_{X/Y}$ has finite tor dimension (details omitted; hint: for $y \in Y$ any $\mathcal{O}_{Y,y}$-module occurs as $\mathcal{F}_y$ for some quasi-coherent module on the affine scheme Y).

Combining the results of the previous two paragraphs we find that $\omega^\bullet_{X/Y}$ is Y-perfect, see Derived Categories of Schemes, Definition 31.1. This proves (1).

Let M be an object of $D_{QCoh}(\mathcal{O}_Y)$. Then

$$\begin{aligned}\mathrm{Hom}_Y(M, Rf_*\omega_{X/Y}^\bullet) &= \mathrm{Hom}_X(Lf^*M, \omega_{X/Y}^\bullet) \\ &= \mathrm{Hom}_Y(Rf_*Lf^*M, \mathcal{O}_Y) \\ &= \mathrm{Hom}_Y(M \otimes_{\mathcal{O}_Y}^{\mathbf{L}} Rf_*\mathcal{O}_Y, \mathcal{O}_Y)\end{aligned}$$

The first equality holds by Cohomology, Lemma 29.1. The second equality by construction of a. The third equality by Derived Categories of Schemes, Lemma 21.1. Recall $Rf_*\mathcal{O}_X$ is perfect of tor amplitude in $[0, N]$ for some N, see Derived Categories of Schemes, Lemma 26.4. Thus we can represent $Rf_*\mathcal{O}_X$ by a complex of finite projective modules sitting in degrees $[0, N]$ (using More on Algebra, Lemma 69.2 and the fact that Y is affine). Hence if $M = \mathcal{O}_Y[-i]$ for some $i > 0$, then the last group is zero. Since Y is affine we conclude that $H^i(Rf_*\omega_{X/Y}^\bullet) = 0$ for $i > 0$. This proves (2).

Let E be a perfect object of $D_{QCoh}(\mathcal{O}_X)$. Then we have

$$\begin{aligned}\mathrm{Hom}_X(E, R\mathcal{H}om_{\mathcal{O}_X}(\omega_{X/Y}^\bullet, \omega_{X/Y}^\bullet) &= \mathrm{Hom}_X(E \otimes_{\mathcal{O}_X}^{\mathbf{L}} \omega_{X/Y}^\bullet, \omega_{X/Y}^\bullet) \\ &= \mathrm{Hom}_Y(Rf_*(E \otimes_{\mathcal{O}_X}^{\mathbf{L}} \omega_{X/Y}^\bullet), \mathcal{O}_Y) \\ &= \mathrm{Hom}_Y(Rf_*(R\mathcal{H}om_{\mathcal{O}_X}(E^\vee, \omega_{X/Y}^\bullet)), \mathcal{O}_Y) \\ &= \mathrm{Hom}_Y(R\mathcal{H}om_{\mathcal{O}_Y}(Rf_*E^\vee, \mathcal{O}_Y), \mathcal{O}_Y) \\ &= R\Gamma(Y, Rf_*E^\vee) \\ &= \mathrm{Hom}_X(E, \mathcal{O}_X)\end{aligned}$$

The first equality holds by Cohomology, Lemma 36.2. The second equality is the definition of $\omega_{X/Y}^\bullet$. The third equality comes from the construction of the dual perfect complex $E^\vee$, see Cohomology, Lemma 43.11. The fourth equality is given in the first paragraph of the proof. The fifth equality holds by double duality for perfect complexes (Cohomology, Lemma 43.11) and the fact that Rf_*E is perfect by Derived Categories of Schemes, Lemma 26.4. The last equality is Leray for f. This string of equalities essentially shows (3) holds by the Yoneda lemma. Namely, the object $R\mathcal{H}om(\omega_{X/Y}^\bullet, \omega_{X/Y}^\bullet)$ is in $D_{QCoh}(\mathcal{O}_X)$ by Derived Categories of Schemes, Lemma 9.8. Taking $E = \mathcal{O}_X$ in the above we get a map $\alpha : \mathcal{O}_X \to R\mathcal{H}om_{\mathcal{O}_X}(\omega_{X/Y}^\bullet, \omega_{X/Y}^\bullet)$ corresponding to $\mathrm{id}_{\mathcal{O}_X} \in \mathrm{Hom}_X(\mathcal{O}_X, \mathcal{O}_X)$. Since all the isomorphisms above are functorial in E we see that the cone on α is an object C of $D_{QCoh}(\mathcal{O}_X)$ such that $\mathrm{Hom}(E, C) = 0$ for all perfect E. Since the perfect objects generate (Derived Categories of Schemes, Theorem 14.3) we conclude that α is an isomorphism. $\square$

0E2P **Lemma 12.7** (Rigidity)**.** *Let Y be a quasi-compact and quasi-separated scheme. Let $f : X \to Y$ be a proper, flat morphism of finite presentation with relative dualizing complex $\omega_{X/Y}^\bullet$ (Remark 12.5). There is a canonical isomorphism*

0E2Q (12.7.1) $$\mathcal{O}_X = c(Lpr_1^*\omega_{X/Y}^\bullet) = c(Lpr_2^*\omega_{X/Y}^\bullet)$$

and a canonical isomorphism

0E2R (12.7.2) $$\omega_{X/Y}^\bullet = c\left(Lpr_1^*\omega_{X/Y}^\bullet \otimes_{\mathcal{O}_{X\times_Y X}}^{\mathbf{L}} Lpr_2^*\omega_{X/Y}^\bullet\right)$$

where c is the right adjoint of Lemma 3.1 for the diagonal $\Delta : X \to X \times_Y X$.

Proof. Let a be the right adjoint for f as in Lemma 3.1. Consider the cartesian square

$$\begin{tikzcd} X \times_Y X \arrow[r, "q"] \arrow[d, "p"] & X \arrow[d, "f"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

Let b be the right adjoint for p as in Lemma 3.1. Then

$$\begin{aligned} \omega_{X/Y}^\bullet &= c(b(\omega_{X/Y}^\bullet)) \\ &= c(Lp^*\omega_{X/Y}^\bullet \otimes_{\mathcal{O}_{X\times_Y X}}^{\mathbf{L}} b(\mathcal{O}_X)) \\ &= c(Lp^*\omega_{X/Y}^\bullet \otimes_{\mathcal{O}_{X\times_Y X}}^{\mathbf{L}} Lq^*a(\mathcal{O}_Y)) \\ &= c(Lp^*\omega_{X/Y}^\bullet \otimes_{\mathcal{O}_{X\times_Y X}}^{\mathbf{L}} Lq^*\omega_{X/Y}^\bullet) \end{aligned}$$

as in (12.7.2). Explanation as follows:

(1) The first equality holds as $\text{id} = c \circ b$ because $\text{id}_X = p \circ \Delta$.
(2) The second equality holds by Lemma 12.3.
(3) The third holds by Lemma 12.4 and the fact that $\mathcal{O}_X = Lf^*\mathcal{O}_Y$.
(4) The fourth holds because $\omega_{X/Y}^\bullet = a(\mathcal{O}_Y)$.

Equation (12.7.1) is proved in exactly the same way. □

0BRU **Remark 12.8.** Lemma 12.7 means our relative dualizing complex is *rigid* in a sense analogous to the notion introduced in [vdB97]. Namely, since the functor on the right of (12.7.2) is "quadratic" in $\omega_{X/Y}^\bullet$ and the functor on the left of (12.7.2) is "linear" this "pins down" the complex $\omega_{X/Y}^\bullet$ to some extent. There is an approach to duality theory using "rigid" (relative) dualizing complexes, see for example [Nee11], [Yek10], and [YZ09]. We will return to this in Section 28.

13. Right adjoint of pushforward for perfect proper morphisms

0AA9 The correct generality for this section would be to consider perfect proper morphisms of quasi-compact and quasi-separated schemes, see [LN07].

0A9R **Lemma 13.1.** *Let $f : X \to Y$ be a perfect proper morphism of Noetherian schemes. Let a be the right adjoint for $Rf_* : D_{QCoh}(\mathcal{O}_X) \to D_{QCoh}(\mathcal{O}_Y)$ of Lemma 3.1. Then a commutes with direct sums.*

Proof. Let P be a perfect object of $D(\mathcal{O}_X)$. By More on Morphisms, Lemma 51.12 the complex Rf_*P is perfect on Y. Let K_i be a family of objects of $D_{QCoh}(\mathcal{O}_Y)$. Then

$$\begin{aligned} \text{Hom}_{D(\mathcal{O}_X)}(P, a(\bigoplus K_i)) &= \text{Hom}_{D(\mathcal{O}_Y)}(Rf_*P, \bigoplus K_i) \\ &= \bigoplus \text{Hom}_{D(\mathcal{O}_Y)}(Rf_*P, K_i) \\ &= \bigoplus \text{Hom}_{D(\mathcal{O}_X)}(P, a(K_i)) \end{aligned}$$

because a perfect object is compact (Derived Categories of Schemes, Proposition 16.1). Since $D_{QCoh}(\mathcal{O}_X)$ has a perfect generator (Derived Categories of Schemes, Theorem 14.3) we conclude that the map $\bigoplus a(K_i) \to a(\bigoplus K_i)$ is an isomorphism, i.e., a commutes with direct sums. □

0AAA **Lemma 13.2.** *Let $f : X \to Y$ be a perfect proper morphism of Noetherian schemes. Let a be the right adjoint for $Rf_* : D_{QCoh}(\mathcal{O}_X) \to D_{QCoh}(\mathcal{O}_Y)$ of Lemma 3.1. Then*

1. *for every closed $T \subset Y$ if $Q \in D_{QCoh}(Y)$ is supported on T, then $a(Q)$ is supported on $f^{-1}(T)$,*
2. *for every open $V \subset Y$ and any $K \in D_{QCoh}(\mathcal{O}_Y)$ the map (4.1.1) is an isomorphism, and*

Proof. This follows from Lemmas 4.3, 4.4, and 13.1. □

0A9U **Lemma 13.3.** *Let $f : X \to Y$ be a perfect proper morphism of Noetherian schemes. The map (8.0.1) is an isomorphism for every object K of $D_{QCoh}(\mathcal{O}_Y)$.*

Proof. By Lemma 13.1 we know that a commutes with direct sums. Hence the collection of objects of $D_{QCoh}(\mathcal{O}_Y)$ for which (8.0.1) is an isomorphism is a strictly full, saturated, triangulated subcategory of $D_{QCoh}(\mathcal{O}_Y)$ which is moreover preserved under taking direct sums. Since $D_{QCoh}(\mathcal{O}_Y)$ is a module category (Derived Categories of Schemes, Theorem 17.3) generated by a single perfect object (Derived Categories of Schemes, Theorem 14.3) we can argue as in More on Algebra, Remark 57.13 to see that it suffices to prove (8.0.1) is an isomorphism for a single perfect object. However, the result holds for perfect objects, see Lemma 8.1. □

0BZG **Lemma 13.4.** *Let $f : X \to Y$ be a perfect proper morphism of Noetherian schemes. Let $g : Y' \to Y$ be a morphism with Y' Noetherian. If X and Y' are tor independent over Y, then the base change map (5.0.1) is an isomorphism for all $K \in D_{QCoh}(\mathcal{O}_Y)$.*

Proof. By Lemma 13.2 formation of the functors a and a' commutes with restriction to opens of Y and Y'. Hence we may assume $Y' \to Y$ is a morphism of affine schemes, see Remark 6.1. In this case the statement follows from Lemma 6.2. □

14. Right adjoint of pushforward for effective Cartier divisors

0B4A Let X be a scheme and let $i : D \to X$ be the inclusion of an effective Cartier divisor. Denote $\mathcal{N} = i^*\mathcal{O}_X(D)$ the normal sheaf of i, see Morphisms, Section 30 and Divisors, Section 13. Recall that $R\mathcal{H}om(\mathcal{O}_D, -)$ denotes the right adjoint to $i_* : D(\mathcal{O}_D) \to D(\mathcal{O}_X)$ and has the property $i_*R\mathcal{H}om(\mathcal{O}_D, -) = R\mathcal{H}om_{\mathcal{O}_X}(i_*\mathcal{O}_D, -)$, see Section 9.

0B4B **Lemma 14.1.** *As above, let X be a scheme and let $D \subset X$ be an effective Cartier divisor. There is a canonical isomorphism $R\mathcal{H}om(\mathcal{O}_D, \mathcal{O}_X) = \mathcal{N}[-1]$ in $D(\mathcal{O}_D)$.*

Proof. Equivalently, we are saying that $R\mathcal{H}om(\mathcal{O}_D, \mathcal{O}_X)$ has a unique nonzero cohomology sheaf in degree 1 and that this sheaf is isomorphic to $\mathcal{N}$. Since i_* is exact and fully faithful, it suffices to prove that $i_*R\mathcal{H}om(\mathcal{O}_D, \mathcal{O}_X)$ is isomorphic to $i_*\mathcal{N}[-1]$. We have $i_*R\mathcal{H}om(\mathcal{O}_D, \mathcal{O}_X) = R\mathcal{H}om_{\mathcal{O}_X}(i_*\mathcal{O}_D, \mathcal{O}_X)$ by Lemma 9.3. We have a resolution

$$0 \to \mathcal{I} \to \mathcal{O}_X \to i_*\mathcal{O}_D \to 0$$

where $\mathcal{I}$ is the ideal sheaf of D which we can use to compute. Since $R\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{O}_X) = \mathcal{O}_X$ and $R\mathcal{H}om_{\mathcal{O}_X}(\mathcal{I}, \mathcal{O}_X) = \mathcal{O}_X(D)$ by a local computation, we see that

$$R\mathcal{H}om_{\mathcal{O}_X}(i_*\mathcal{O}_D, \mathcal{O}_X) = (\mathcal{O}_X \to \mathcal{O}_X(D))$$

where on the right hand side we have $\mathcal{O}_X$ in degree 0 and $\mathcal{O}_X(D)$ in degree 1. The result follows from the short exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(D) \to i_*\mathcal{N} \to 0$$

coming from the fact that D is the zero scheme of the canonical section of $\mathcal{O}_X(D)$ and from the fact that $\mathcal{N} = i^*\mathcal{O}_X(D)$. $\square$

For every object K of $D(\mathcal{O}_X)$ there is a canonical map

0B4C (14.1.1) $$Li^*K \otimes^{\mathbf{L}}_{\mathcal{O}_D} R\mathcal{H}om(\mathcal{O}_D, \mathcal{O}_X) \longrightarrow R\mathcal{H}om(\mathcal{O}_D, K)$$

in $D(\mathcal{O}_D)$ functorial in K and compatible with distinguished triangles. Namely, this map is adjoint to a map

$$i_*(Li^*K \otimes^{\mathbf{L}}_{\mathcal{O}_D} R\mathcal{H}om(\mathcal{O}_D, \mathcal{O}_X)) = K \otimes^{\mathbf{L}}_{\mathcal{O}_X} R\mathcal{H}om_{\mathcal{O}_X}(i_*\mathcal{O}_D, \mathcal{O}_X) \longrightarrow K$$

where the equality is Cohomology, Lemma 45.4 and the arrow comes from the canonical map $R\mathcal{H}om_{\mathcal{O}_X}(i_*\mathcal{O}_D, \mathcal{O}_X) \to \mathcal{O}_X$ induced by $\mathcal{O}_X \to i_*\mathcal{O}_D$.

If $K \in D_{QCoh}(\mathcal{O}_X)$, then (14.1.1) is equal to (8.0.1) via the identification $a(K) = R\mathcal{H}om(\mathcal{O}_D, K)$ of Lemma 9.7. If $K \in D_{QCoh}(\mathcal{O}_X)$ and X is Noetherian, then the following lemma is a special case of Lemma 13.3.

0AA4 **Lemma 14.2.** *As above, let X be a scheme and let $D \subset X$ be an effective Cartier divisor. Then (14.1.1) combined with Lemma 14.1 defines an isomorphism*

$$Li^*K \otimes^{\mathbf{L}}_{\mathcal{O}_D} \mathcal{N}[-1] \longrightarrow R\mathcal{H}om(\mathcal{O}_D, K)$$

functorial in K in $D(\mathcal{O}_X)$.

Proof. Since i_* is exact and fully faithful on modules, to prove the map is an isomorphism, it suffices to show that it is an isomorphism after applying i_*. We will use the short exact sequences $0 \to \mathcal{I} \to \mathcal{O}_X \to i_*\mathcal{O}_D \to 0$ and $0 \to \mathcal{O}_X \to \mathcal{O}_X(D) \to i_*\mathcal{N} \to 0$ used in the proof of Lemma 14.1 without further mention. By Cohomology, Lemma 45.4 which was used to define the map (14.1.1) the left hand side becomes

$$K \otimes^{\mathbf{L}}_{\mathcal{O}_X} i_*\mathcal{N}[-1] = K \otimes^{\mathbf{L}}_{\mathcal{O}_X} (\mathcal{O}_X \to \mathcal{O}_X(D))$$

The right hand side becomes

$$\begin{aligned} R\mathcal{H}om_{\mathcal{O}_X}(i_*\mathcal{O}_D, K) &= R\mathcal{H}om_{\mathcal{O}_X}((\mathcal{I} \to \mathcal{O}_X), K) \\ &= R\mathcal{H}om_{\mathcal{O}_X}((\mathcal{I} \to \mathcal{O}_X), \mathcal{O}_X) \otimes^{\mathbf{L}}_{\mathcal{O}_X} K \end{aligned}$$

the final equality by Cohomology, Lemma 43.11. Since the map comes from the isomorphism

$$R\mathcal{H}om_{\mathcal{O}_X}((\mathcal{I} \to \mathcal{O}_X), \mathcal{O}_X) = (\mathcal{O}_X \to \mathcal{O}_X(D))$$

the lemma is clear. $\square$

15. Right adjoint of pushforward in examples

0BQV In this section we compute the right adjoint to pushforward in some examples. The isomorphisms are canonical but only in the weakest possible sense, i.e., we do not prove or claim that these isomorphisms are compatible with various operations such as base change and compositions of morphisms. There is a huge literature on these types of issues; the reader can start with the material in [Har66], [Con00] (these citations use a different starting point for duality but address the issue of constructing canonical representatives for relative dualizing complexes) and then continue looking at works by Joseph Lipman and collaborators.

0A9W **Lemma 15.1.** *Let Y be a Noetherian scheme. Let $\mathcal{E}$ be a finite locally free $\mathcal{O}_Y$-module of rank $n+1$ with determinant $\mathcal{L} = \wedge^{n+1}(\mathcal{E})$. Let $f : X = \mathbf{P}(\mathcal{E}) \to Y$ be the projection. Let a be the right adjoint for $Rf_* : D_{QCoh}(\mathcal{O}_X) \to D_{QCoh}(\mathcal{O}_Y)$ of Lemma 3.1. Then there is an isomorphism*

$$c : f^*\mathcal{L}(-n-1)[n] \longrightarrow a(\mathcal{O}_Y)$$

In particular, if $\mathcal{E} = \mathcal{O}_Y^{\oplus n+1}$, then $X = \mathbf{P}^n_Y$ and we obtain $a(\mathcal{O}_Y) = \mathcal{O}_X(-n-1)[n]$.

Proof. In (the proof of) Cohomology of Schemes, Lemma 8.4 we constructed a canonical isomorphism

$$R^nf_*(f^*\mathcal{L}(-n-1)) \longrightarrow \mathcal{O}_Y$$

Moreover, $Rf_*(f^*\mathcal{L}(-n-1))[n] = R^nf_*(f^*\mathcal{L}(-n-1))$, i.e., the other higher direct images are zero. Thus we find an isomorphism

$$Rf_*(f^*\mathcal{L}(-n-1)[n]) \longrightarrow \mathcal{O}_Y$$

This isomorphism determines c as in the statement of the lemma because a is the right adjoint of Rf_*. By Lemma 4.4 construction of the a is local on the base. In particular, to check that c is an isomorphism, we may work locally on Y. In other words, we may assume Y is affine and $\mathcal{E} = \mathcal{O}_Y^{\oplus n+1}$. In this case the sheaves $\mathcal{O}_X, \mathcal{O}_X(-1), \ldots, \mathcal{O}_X(-n)$ generate $D_{QCoh}(X)$, see Derived Categories of Schemes, Lemma 15.3. Hence it suffices to show that $c : \mathcal{O}_X(-n-1)[n] \to a(\mathcal{O}_Y)$ is transformed into an isomorphism under the functors

$$F_{i,p}(-) = \mathrm{Hom}_{D(\mathcal{O}_X)}(\mathcal{O}_X(i), (-)[p])$$

for $i \in \{-n, \ldots, 0\}$ and $p \in \mathbf{Z}$. For $F_{0,p}$ this holds by construction of the arrow c! For $i \in \{-n, \ldots, -1\}$ we have

$$\mathrm{Hom}_{D(\mathcal{O}_X)}(\mathcal{O}_X(i), \mathcal{O}_X(-n-1)[n+p]) = H^p(X, \mathcal{O}_X(-n-1-i)) = 0$$

by the computation of cohomology of projective space (Cohomology of Schemes, Lemma 8.1) and we have

$$\mathrm{Hom}_{D(\mathcal{O}_X)}(\mathcal{O}_X(i), a(\mathcal{O}_Y)[p]) = \mathrm{Hom}_{D(\mathcal{O}_Y)}(Rf_*\mathcal{O}_X(i), \mathcal{O}_Y[p]) = 0$$

because $Rf_*\mathcal{O}_X(i) = 0$ by the same lemma. Hence the source and the target of $F_{i,p}(c)$ vanish and $F_{i,p}(c)$ is necessarily an isomorphism. This finishes the proof. □

0AAC **Example 15.2.** The base change map (5.0.1) is not an isomorphism if f is perfect proper and g is perfect. Let k be a field. Let $Y = \mathbf{A}^2_k$ and let $f : X \to Y$ be the blowup of Y in the origin. Denote $E \subset X$ the exceptional divisor. Then we can factor f as

$$X \xrightarrow{i} \mathbf{P}^1_Y \xrightarrow{p} Y$$

This gives a factorization $a = c \circ b$ where a, b, and c are the right adjoints of Lemma 3.1 of Rf_*, Rp_*, and Ri_*. Denote $\mathcal{O}(n)$ the Serre twist of the structure sheaf on $\mathbf{P}^1_Y$ and denote $\mathcal{O}_X(n)$ its restriction to X. Note that $X \subset \mathbf{P}^1_Y$ is cut out by a degree one equation, hence $\mathcal{O}(X) = \mathcal{O}(1)$. By Lemma 15.1 we have $b(\mathcal{O}_Y) = \mathcal{O}(-2)[1]$. By Lemma 9.7 we have

$$a(\mathcal{O}_Y) = c(b(\mathcal{O}_Y)) = c(\mathcal{O}(-2)[1]) = R\mathcal{H}om(\mathcal{O}_X, \mathcal{O}(-2)[1]) = \mathcal{O}_X(-1)$$

Last equality by Lemma 14.2. Let $Y' = \mathrm{Spec}(k)$ be the origin in Y. The restriction of $a(\mathcal{O}_Y)$ to $X' = E = \mathbf{P}^1_k$ is an invertible sheaf of degree -1 placed in cohomological degree 0. But on the other hand, $a'(\mathcal{O}_{\mathrm{Spec}(k)}) = \mathcal{O}_E(-2)[1]$ which is an invertible sheaf of degree -2 placed in cohomological degree -1, so different. In this example (4) is the only hypothesis of Lemma 6.2 which is violated.

0BQW **Lemma 15.3.** *Let Y be a ringed space. Let $\mathcal{I} \subset \mathcal{O}_Y$ be a sheaf of ideals. Set $\mathcal{O}_X = \mathcal{O}_Y/\mathcal{I}$ and $\mathcal{N} = \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_X)$. There is a canonical isomorphism $c : \mathcal{N} \to \mathcal{E}xt^1_{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{O}_X)$.*

Proof. Consider the canonical short exact sequence

0BQX (15.3.1)
$$0 \to \mathcal{I}/\mathcal{I}^2 \to \mathcal{O}_Y/\mathcal{I}^2 \to \mathcal{O}_X \to 0$$

Let $U \subset X$ be open and let $s \in \mathcal{N}(U)$. Then we can pushout (15.3.1) via s to get an extension E_s of $\mathcal{O}_X|_U$ by $\mathcal{O}_X|_U$. This in turn defines a section $c(s)$ of $\mathcal{E}xt^1_{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{O}_X)$ over U. See Cohomology, Lemma 36.1 and Derived Categories, Lemma 27.6. Conversely, given an extension

$$0 \to \mathcal{O}_X|_U \to \mathcal{E} \to \mathcal{O}_X|_U \to 0$$

of $\mathcal{O}_U$-modules, we can find an open covering $U = \bigcup U_i$ and sections $e_i \in \mathcal{E}(U_i)$ mapping to $1 \in \mathcal{O}_X(U_i)$. Then e_i defines a map $\mathcal{O}_Y|_{U_i} \to \mathcal{E}|_{U_i}$ whose kernel contains $\mathcal{I}^2$. In this way we see that $\mathcal{E}|_{U_i}$ comes from a pushout as above. This shows that c is surjective. We omit the proof of injectivity. □

0BQY **Lemma 15.4.** *Let Y be a ringed space. Let $\mathcal{I} \subset \mathcal{O}_Y$ be a sheaf of ideals. Set $\mathcal{O}_X = \mathcal{O}_Y/\mathcal{I}$. If $\mathcal{I}$ is Koszul-regular (Divisors, Definition 20.2) then composition on $R\mathcal{H}om_{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{O}_X)$ defines isomorphisms*

$$\wedge^i(\mathcal{E}xt^1_{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{O}_X)) \longrightarrow \mathcal{E}xt^i_{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{O}_X)$$

for all i.

Proof. By composition we mean the map

$$R\mathcal{H}om_{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{O}_X) \otimes^{\mathbf{L}}_{\mathcal{O}_Y} R\mathcal{H}om_{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{O}_X) \longrightarrow R\mathcal{H}om_{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{O}_X)$$

of Cohomology, Lemma 36.6. This induces multiplication maps

$$\mathcal{E}xt^a_{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{O}_X) \otimes_{\mathcal{O}_Y} \mathcal{E}xt^b_{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{O}_X) \longrightarrow \mathcal{E}xt^{a+b}_{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{O}_X)$$

Please compare with More on Algebra, Equation (61.0.1). The statement of the lemma means that the induced map

$$\mathcal{E}xt^1_{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{O}_X) \otimes \ldots \otimes \mathcal{E}xt^1_{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{O}_X) \longrightarrow \mathcal{E}xt^i_{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{O}_X)$$

factors through the wedge product and then induces an isomorphism. To see this is true we may work locally on Y. Hence we may assume that we have global sections $f_1, \ldots, f_r$ of $\mathcal{O}_Y$ which generate $\mathcal{I}$ and which form a Koszul regular sequence. Denote

$$\mathcal{A} = \mathcal{O}_Y\langle \xi_1, \ldots, \xi_r \rangle$$

the sheaf of strictly commutative differential graded $\mathcal{O}_Y$-algebras which is a (divided power) polynomial algebra on $\xi_1, \ldots, \xi_r$ in degree -1 over $\mathcal{O}_Y$ with differential d given by the rule $\mathrm{d}\xi_i = f_i$. Let us denote $\mathcal{A}^\bullet$ the underlying complex of $\mathcal{O}_Y$-modules which is the Koszul complex mentioned above. Thus the canonical map $\mathcal{A}^\bullet \to \mathcal{O}_X$ is a quasi-isomorphism. We obtain quasi-isomorphisms

$$R\mathcal{H}om_{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{O}_X) \to \mathcal{H}om^\bullet(\mathcal{A}^\bullet, \mathcal{A}^\bullet) \to \mathcal{H}om^\bullet(\mathcal{A}^\bullet, \mathcal{O}_X)$$

by Cohomology, Lemma 40.9. The differentials of the latter complex are zero, and hence

$$\mathcal{E}xt^i_{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{O}_X) \cong \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{A}^{-i}, \mathcal{O}_X)$$

For $j \in \{1, \ldots, r\}$ let $\delta_j : \mathcal{A} \to \mathcal{A}$ be the derivation of degree 1 with $\delta_j(\xi_i) = \delta_{ij}$ (Kronecker delta). A computation shows that $\delta_j \circ \mathrm{d} = -\mathrm{d} \circ \delta_j$ which shows that we get a morphism of complexes.

$$\delta_j : \mathcal{A}^\bullet \to \mathcal{A}^\bullet[1].$$

Whence δ_j defines a section of the corresponding $\mathcal{E}xt$-sheaf. Another computation shows that $\delta_1, \ldots, \delta_r$ map to a basis for $\mathcal{H}om_{\mathcal{O}_Y}(\mathcal{A}^{-1}, \mathcal{O}_X)$ over $\mathcal{O}_X$. Since it is clear that $\delta_j \circ \delta_j = 0$ and $\delta_j \circ \delta_{j'} = -\delta_{j'} \circ \delta_j$ as endomorphisms of $\mathcal{A}$ and hence in the $\mathcal{E}xt$-sheaves we obtain the statement that our map above factors through the exterior power. To see we get the desired isomorphism the reader checks that the elements

$$\delta_{j_1} \circ \ldots \circ \delta_{j_i}$$

for $j_1 < \ldots < j_i$ map to a basis of the sheaf $\mathcal{H}om_{\mathcal{O}_Y}(\mathcal{A}^{-i}, \mathcal{O}_X)$ over $\mathcal{O}_X$. □

0BQZ **Lemma 15.5.** *Let Y be a ringed space. Let $\mathcal{I} \subset \mathcal{O}_Y$ be a sheaf of ideals. Set $\mathcal{O}_X = \mathcal{O}_Y/\mathcal{I}$ and $\mathcal{N} = \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_X)$. If $\mathcal{I}$ is Koszul-regular (Divisors, Definition 20.2) then*

$$R\mathcal{H}om_{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{O}_Y) = \wedge^r \mathcal{N}[r]$$

where $r : Y \to \{1, 2, 3, \ldots\}$ sends y to the minimal number of generators of $\mathcal{I}$ needed in a neighbourhood of y.

Proof. We can use Lemmas 15.3 and 15.4 to see that we have isomorphisms $\wedge^i \mathcal{N} \to \mathcal{E}xt^i_{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{O}_X)$ for $i \geq 0$. Thus it suffices to show that the map $\mathcal{O}_Y \to \mathcal{O}_X$ induces an isomorphism

$$\mathcal{E}xt^r_{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{O}_Y) \longrightarrow \mathcal{E}xt^r_{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{O}_X)$$

and that $\mathcal{E}xt^i_{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{O}_Y)$ is zero for $i \neq r$. These statements are local on Y. Thus we may assume that we have global sections $f_1, \ldots, f_r$ of $\mathcal{O}_Y$ which generate $\mathcal{I}$ and which form a Koszul regular sequence. Let $\mathcal{A}^\bullet$ be the Koszul complex on $f_1, \ldots, f_r$ as introduced in the proof of Lemma 15.4. Then

$$R\mathcal{H}om_{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{O}_Y) = \mathcal{H}om^\bullet(\mathcal{A}^\bullet, \mathcal{O}_Y)$$

by Cohomology, Lemma 40.9. Denote $1 \in H^0(\mathcal{H}om^\bullet(\mathcal{A}^\bullet, \mathcal{O}_Y))$ the identity map of $\mathcal{A}^0 = \mathcal{O}_Y \to \mathcal{O}_Y$. With δ_j as in the proof of Lemma 15.4 we get an isomorphism of graded $\mathcal{O}_Y$-modules

$$\mathcal{O}_Y\langle \delta_1, \ldots, \delta_r \rangle \longrightarrow \mathcal{H}om^\bullet(\mathcal{A}^\bullet, \mathcal{O}_Y)$$

by mapping $\delta_{j_1} \ldots \delta_{j_i}$ to $1 \circ \delta_{j_1} \circ \ldots \circ \delta_{j_i}$ in degree i. Via this isomorphism the differential on the right hand side induces a differential d on the left hand side. By our sign rules we have $\mathrm{d}(1) = -\sum f_j \delta_j$. Since $\delta_j : \mathcal{A}^\bullet \to \mathcal{A}^\bullet[1]$ is a morphism of complexes, it follows that

$$\mathrm{d}(\delta_{j_1} \ldots \delta_{j_i}) = (-\sum f_j \delta_j)\delta_{j_1} \ldots \delta_{j_i}$$

Observe that we have $\mathrm{d} = \sum f_j \delta_j$ on the differential graded algebra $\mathcal{A}$. Therefore the map defined by the rule

$$1 \circ \delta_{j_1} \ldots \delta_{j_i} \longmapsto (\delta_{j_1} \circ \ldots \circ \delta_{j_i})(\xi_1 \ldots \xi_r)$$

will define an isomorphism of complexes

$$\mathcal{H}om^\bullet(\mathcal{A}^\bullet, \mathcal{O}_Y) \longrightarrow \mathcal{A}^\bullet[-r]$$

if r is odd and commuting with differentials up to sign if r is even. In any case these complexes have isomorphic cohomology, which shows the desired vanishing. The isomorphism on cohomology in degree r under the map

$$\mathcal{H}om^\bullet(\mathcal{A}^\bullet, \mathcal{O}_Y) \longrightarrow \mathcal{H}om^\bullet(\mathcal{A}^\bullet, \mathcal{O}_X)$$

also follows in a straightforward manner from this. (We observe that our choice of conventions regarding Koszul complexes does intervene in the definition of the isomorphism $R\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{O}_Y) = \wedge^r\mathcal{N}[r]$.) □

0BR0 **Lemma 15.6.** *Let Y be a quasi-compact and quasi-separated scheme. Let $i : X \to Y$ be a Koszul-regular closed immersion. Let a be the right adjoint of $Ri_* : D_{QCoh}(\mathcal{O}_X) \to D_{QCoh}(\mathcal{O}_Y)$ of Lemma 3.1. Then there is an isomorphism*

$$\wedge^r\mathcal{N}[-r] \longrightarrow a(\mathcal{O}_Y)$$

where $\mathcal{N} = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{C}_{X/Y}, \mathcal{O}_X)$ is the normal sheaf of i (Morphisms, Section 30) and r is its rank viewed as a locally constant function on X.

Proof. Recall, from Lemmas 9.7 and 9.3, that $a(\mathcal{O}_Y)$ is an object of $D_{QCoh}(\mathcal{O}_X)$ whose pushforward to Y is $R\mathcal{H}om_{\mathcal{O}_Y}(i_*\mathcal{O}_X, \mathcal{O}_Y)$. Thus the result follows from Lemma 15.5. □

0BRT **Lemma 15.7.** *Let S be a Noetherian scheme. Let $f : X \to S$ be a smooth proper morphism of relative dimension d. Let a be the right adjoint of $Rf_* : D_{QCoh}(\mathcal{O}_X) \to D_{QCoh}(\mathcal{O}_S)$ as in Lemma 3.1. Then there is an isomorphism*

$$\wedge^d\Omega_{X/S}[d] \longrightarrow a(\mathcal{O}_S)$$

in $D(\mathcal{O}_X)$.

Proof. Set $\omega_{X/S}^\bullet = a(\mathcal{O}_S)$ as in Remark 12.5. Let c be the right adjoint of Lemma 3.1 for $\Delta : X \to X \times_S X$. Because Δ is the diagonal of a smooth morphism it is a Koszul-regular immersion, see Divisors, Lemma 22.10. In particular, Δ is a perfect proper morphism (More on Morphisms, Lemma 51.7) and we obtain

$$\begin{aligned}
\mathcal{O}_X &= c(L\mathrm{pr}_1^*\omega_{X/S}^\bullet) \\
&= L\Delta^*(L\mathrm{pr}_1^*\omega_{X/S}^\bullet) \otimes_{\mathcal{O}_X}^{\mathbf{L}} c(\mathcal{O}_{X\times_S X}) \\
&= \omega_{X/S}^\bullet \otimes_{\mathcal{O}_X}^{\mathbf{L}} c(\mathcal{O}_{X\times_S X}) \\
&= \omega_{X/S}^\bullet \otimes_{\mathcal{O}_X}^{\mathbf{L}} \wedge^d(\mathcal{N}_\Delta)[-d]
\end{aligned}$$

The first equality is (12.7.1) because $\omega_{X/S}^\bullet = a(\mathcal{O}_S)$. The second equality by Lemma 13.3. The third equality because $p \circ \Delta = \mathrm{id}_X$. The fourth equality by Lemma 15.6. Observe that $\wedge^d(\mathcal{N}_\Delta)$ is an invertible $\mathcal{O}_X$-module. Hence $\wedge^d(\mathcal{N}_\Delta)[-d]$ is an invertible object of $D(\mathcal{O}_X)$ and we conclude that $a(\mathcal{O}_S) = \omega_{X/S}^\bullet = \wedge^d(\mathcal{C}_\Delta)[d]$. Since the conormal sheaf $\mathcal{C}_\Delta$ of Δ is $\Omega_{X/S}$ by Morphisms, Lemma 31.7 the proof is complete. □

16. Compactifications

0ATT We interrupt the flow of the arguments for a little bit of geometry.

Let S be a quasi-compact and quasi-separated scheme. We will say a scheme X over S *has a compactification over S* or *is compactifyable over S* if there exists a quasi-compact open immersion $X \to \overline{X}$ into a scheme $\overline{X}$ proper over S. If X has a compactification over S, then $X \to S$ is separated and of finite type. It is a theorem of Nagata (see [Lüt93], [Con07], [Nag56], [Nag57], [Nag62], and [Nag63]) that the converse is true as well (we will give a precise statement and a proof if we ever need this result).

Let S be a quasi-compact and quasi-separated scheme. Let X be a scheme over S. The category of *compactifications of X over S* is the category whose objects are open immersions $j : X \to \overline{X}$ over S with $\overline{X} \to S$ proper and whose morphisms $(j' : X' \to \overline{X}') \to (j : X \to \overline{X})$ are morphisms $f : \overline{X}' \to \overline{X}$ of schemes over S such that $f \circ j' = j$.

0ATU **Lemma 16.1.** *Let S be a quasi-compact and quasi-separated scheme. Let X be a compactifyable scheme over S. The category of compactifications of X over S is cofiltered.*

Proof. We have to check conditions (1), (2), (3) of Categories, Definition 20.1. Condition (1) holds exactly because we assumed that X is compactifyable. Let $j_i : X \to \overline{X}_i$, $i = 1, 2$ be two compactifications. Then we can consider the scheme theoretic closure $\overline{X}$ of $(j_1, j_2) : X \to \overline{X}_1 \times_S \overline{X}_2$. This determines a third compactification $j : X \to \overline{X}$ which dominates both j_i:

$$(X, \overline{X}_1) \longleftarrow (X, \overline{X}) \longrightarrow (X, \overline{X}_2)$$

Thus (2) holds. Let $f_1, f_2 : \overline{X}_1 \to \overline{X}_2$ be two morphisms between compactifications $j_i : X \to \overline{X}_i$, $i = 1, 2$. Let $\overline{X} \subset \overline{X}_1$ be the equalizer of f_1 and f_2. As $\overline{X}_2 \to S$ is separated, we see that $\overline{X}$ is a closed subscheme of $\overline{X}_1$ and hence proper over S. Moreover, we obtain an open immersion $X \to \overline{X}$ because $f_1|_X = f_2|_X = \text{id}_X$. The morphism $(X \to \overline{X}) \to (j_1 : X \to \overline{X}_1)$ given by the closed immersion $\overline{X} \to \overline{X}_1$ equalizes f_1 and f_2 which proves condition (3) and finishes the proof. $\square$

We can also consider the category of all compactifications (for varying X). It turns out that this category, localized at the set of morphisms which induce an isomorphism on the interior is equivalent to the category of compactifyable schemes over S.

0A9Z **Lemma 16.2.** *Let S be a quasi-compact and quasi-separated scheme. Let $f : X \to Y$ be a morphism of schemes over S with Y separated and of finite type over S and X compactifyable over S. Then X has a compactification over Y.*

Proof. Let $f : X \to Y$ be a morphism of schemes over S with Y separated and of finite type over S. Let $j : X \to \overline{X}$ be a compactification of X over S. Then we let $\overline{X}'$ be the scheme theoretic image of $(j, f) : X \to \overline{X} \times_S Y$. The morphism $\overline{X}' \to Y$ is proper because $\overline{X} \times_S Y \to Y$ is proper as a base change of $\overline{X} \to S$. On the other hand, since Y is separated over S, the morphism $(1, f) : X \to X \times_S Y$ is a closed immersion (Schemes, Lemma 21.10) and hence $X \to \overline{X}'$ is an open immersion. $\square$

Let S be a quasi-compact and quasi-separated scheme. We define the *category of compactifications* to be the category whose objects are pairs $(X, \overline{X})$ where $\overline{X}$ is a scheme proper over S and $X \subset \overline{X}$ is a quasi-compact open and whose morphisms are commutative diagrams

$$\begin{tikzcd} X \arrow[r, "f"'] \arrow[d] & Y \arrow[d] \\ \overline{X} \arrow[r, "\overline{f}"] & \overline{Y} \end{tikzcd}$$

of morphisms of schemes over S.

0ATV **Lemma 16.3.** *Let S be a quasi-compact and quasi-separated scheme. The collection of morphisms $(u, \overline{u}) : (X', \overline{X}') \to (X, \overline{X})$ such that u is an isomorphism forms a right multiplicative system (Categories, Definition 26.1) of arrows in the category of compactifications.*

Proof. Axiom RMS1 is trivial to verify. Let us check RMS2 holds. Suppose given a diagram

$$\begin{tikzcd} & (X', \overline{X}') \arrow[d, "(u,\overline{u})"'] \\ (Y, \overline{Y}) \arrow[r, "(f,\overline{f})"] & (X, \overline{X}) \end{tikzcd}$$

with $u : X' \to X$ an isomorphism. Then we let $Y' = Y \times_X X'$ with the projection map $v : Y' \to Y$ (an isomorphism). We also set $\overline{Y}' = \overline{Y} \times_{\overline{X}} \overline{X}'$ with the projection map $\overline{v} : \overline{Y}' \to \overline{Y}$ It is clear that $Y' \to \overline{Y}'$ is an open immersion. The diagram

$$\begin{tikzcd} (Y', \overline{Y}') \arrow[r, "(g,\overline{g})"'] \arrow[d, "(v,\overline{v})"'] & (X', \overline{X}') \arrow[d, "(u,\overline{u})"'] \\ (Y, \overline{Y}) \arrow[r, "(f,\overline{f})"] & (X, \overline{X}) \end{tikzcd}$$

shows that axiom RMS2 holds.

Let us check RMS3 holds. Suppose given a pair of morphisms $(f, \overline{f}), (g, \overline{g}) : (X, \overline{X}) \to (Y, \overline{Y})$ of compactifications and a morphism $(v, \overline{v}) : (Y, \overline{Y}) \to (Y', \overline{Y}')$ such that v is an isomorphism and such that $(v, \overline{v}) \circ (f, \overline{f}) = (v, \overline{v}) \circ (g, \overline{g})$. Then $f = g$. Hence if we let $\overline{X}' \subset \overline{X}$ be the equalizer of $\overline{f}$ and $\overline{g}$, then $(u, \overline{u}) : (X, \overline{X}') \to (X, \overline{X})$ will be a morphism of the category of compactifications such that $(f, \overline{f}) \circ (u, \overline{u}) = (g, \overline{g}) \circ (u, \overline{u})$ as desired. $\square$

0ATW **Lemma 16.4.** *Let S be a quasi-compact and quasi-separated scheme. The functor $(X, \overline{X}) \mapsto X$ defines an equivalence from the category of compactifications localized (Categories, Lemma 26.11) at the right multiplicative system of Lemma 16.3 to the category of compactifyable schemes over S.*

Proof. Denote $\mathcal{C}$ the category of compactifications and denote $Q : \mathcal{C} \to \mathcal{C}'$ the localization functor of Categories, Lemma 26.16. Denote $\mathcal{D}$ the category of compactifyable schemes over S. It is clear from the lemma just cited and our choice of multiplicative system that we obtain a functor $\mathcal{C}' \to \mathcal{D}$. This functor is clearly essentially surjective. If $f : X \to Y$ is a morphism of compactifyable schemes, then we choose an open immersion $Y \to \overline{Y}$ into a scheme proper over S, and then we choose an embedding $X \to \overline{X}$ into a scheme $\overline{X}$ proper over $\overline{Y}$ (possible by Lemma 16.2 applied to $X \to \overline{Y}$). This gives a morphism $(X, \overline{X}) \to (Y, \overline{Y})$ of compactifications which produces our given morphism $X \to Y$. Finally, suppose given a pair of morphisms in the localized category with the same source and target: say

$$a = ((f, \overline{f}) : (X', \overline{X}') \to (Y, \overline{Y}), (u, \overline{u}) : (X', \overline{X}') \to (X, \overline{X}))$$

and

$$b = ((g, \overline{g}) : (X'', \overline{X}'') \to (Y, \overline{Y}), (v, \overline{v}) : (X'', \overline{X}'') \to (X, \overline{X}))$$

which produce the same morphism $X \to Y$ over S, in other words $f \circ u^{-1} = g \circ v^{-1}$. By Categories, Lemma 26.13 we may assume that $(X', \overline{X}') = (X'', \overline{X}'')$ and $(u, \overline{u}) = (v, \overline{v})$. In this case we can consider the equalizer $\overline{X}''' \subset \overline{X}'$ of $\overline{f}$ and $\overline{g}$. The morphism $(w, \overline{w}) : (X', \overline{X}''') \to (X', \overline{X}')$ is in the multiplicative subset and we see that $a = b$ in the localized category by precomposing with $(w, \overline{w})$. $\square$

17. Upper shriek functors

0A9Y In this section, we construct the functors $f^!$ for morphisms between compactifyable schemes over a fixed Noetherian base. As is customary in coherent duality, there are a number of diagrams that have to be shown to be commutative. We suggest the reader, after reading the construction, skips the verification of the lemmas and continues to the next section where we discuss properties of the upper shriek functors.

Given a morphism $f : X \to Y$ of compactifyable schemes over a Noetherian base scheme S, we will define an exact functor

$$f^! : D^+_{QCoh}(\mathcal{O}_Y) \to D^+_{QCoh}(\mathcal{O}_X)$$

of triangulated categories. Namely, we choose a compactification $X \to \overline{X}$ over Y which is possible by Lemma 16.2. Denote $\overline{f} : \overline{X} \to Y$ the structure morphism. Let $\overline{a} : D_{QCoh}(\mathcal{O}_Y) \to D_{QCoh}(\mathcal{O}_{\overline{X}})$ be the right adjoint of $R\overline{f}_*$ constructed in Lemma 3.1. Then we set

$$f^! K = \overline{a}(K)|_X$$

for $K \in D^+_{QCoh}(\mathcal{O}_Y)$. The result is an object of $D^+_{QCoh}(\mathcal{O}_X)$ by Lemma 3.5.

0AA0 **Lemma 17.1.** *Let $f : X \to Y$ be a morphism between compactifyable schemes over a Noetherian scheme S. The functor $f^!$ is, up to canonical isomorphism, independent of the choice of the compactification.*

Proof. Consider the category of compactifications of X over Y, which is cofiltered according to Lemmas 16.1 and 16.2. To every choice of a compactification

$$j : X \to \overline{X}, \quad \overline{f} : \overline{X} \to Y$$

the construction above associates the functor $j^* \circ \overline{a} : D^+_{QCoh}(\mathcal{O}_Y) \to D^+_{QCoh}(\mathcal{O}_X)$ where $\overline{a}$ is the right adjoint of $R\overline{f}_*$ constructed in Lemma 3.1.

Suppose given a morphism $g : \overline{X}_1 \to \overline{X}_2$ between compactifications $j_i : X \to \overline{X}_i$ over Y. Namely, let $\overline{c}$ be the right adjoint of Lemma 3.1 for g. Then $\overline{c} \circ \overline{a}_2 = \overline{a}_1$ because these functors are adjoint to $R\overline{f}_{2,*} \circ Rg_* = R(\overline{f}_2 \circ g)_*$. By (4.1.1) we have a canonical transformation

$$j_1^* \circ \overline{c} \longrightarrow j_2^*$$

of functors $D^+_{QCoh}(\mathcal{O}_{\overline{X}_2}) \to D^+_{QCoh}(\mathcal{O}_X)$ which is an isomorphism by Lemma 4.4. The composition

$$j_1^* \circ \overline{a}_1 \longrightarrow j_1^* \circ \overline{c} \circ \overline{a}_2 \longrightarrow j_2^* \circ \overline{a}_2$$

is an isomorphism of functors which we will denote by α_g.

To finish the proof, since the category of compactifications of X over Y is cofiltered, it suffices to show compositions of morphisms of compactifications of X over Y are turned into compositions of isomorphisms of functors[3]. To do this, suppose that $j_3 : X \to \overline{X}_3$ is a third compactification and that $h : \overline{X}_2 \to \overline{X}_3$ is a morphism of compactifications. Let $\overline{d}$ be the right adjoint of Lemma 3.1 for h. Then $\overline{d} \circ \overline{a}_3 = \overline{a}_2$ and there is a canonical transformation

$$j_2^* \circ \overline{d} \longrightarrow j_3^*$$

of functors $D^+_{QCoh}(\mathcal{O}_{\overline{X}_3}) \to D^+_{QCoh}(\mathcal{O}_X)$ for the same reasons as above. Denote $\overline{e}$ the right adjoint of Lemma 3.1 for $h \circ g$. There is a canonical transformation

$$j_1^* \circ \overline{e} \longrightarrow j_3^*$$

of functors $D^+_{QCoh}(\mathcal{O}_{\overline{X}_3}) \to D^+_{QCoh}(\mathcal{O}_X)$ given by (4.1.1). Spelling things out we have to show that the composition

$$\alpha_h \circ \alpha_g : j_1^* \circ \overline{a}_1 \to j_1^* \circ \overline{c} \circ \overline{a}_2 \to j_2^* \circ \overline{a}_2 \to j_2^* \circ \overline{d} \circ \overline{a}_3 \to j_3^* \circ \overline{a}_3$$

is the same as the composition

$$\alpha_{h \circ g} : j_1^* \circ \overline{a}_1 \to j_1^* \circ \overline{e} \circ \overline{a}_3 \to j_3^* \circ \overline{a}_3$$

We split this into two parts. The first is to show that the diagram

$$\begin{tikzcd}
\overline{a}_1 \arrow[r] \arrow[d] & \overline{c} \circ \overline{a}_2 \arrow[d] \\
\overline{e} \circ \overline{a}_3 \arrow[r] & \overline{c} \circ \overline{d} \circ \overline{a}_3
\end{tikzcd}$$

commutes where the lower horizontal arrow comes from the identification $\overline{e} = \overline{c} \circ \overline{d}$. This is true because the corresponding diagram of total direct image functors

$$\begin{tikzcd}
R\overline{f}_{1,*} \arrow[r] \arrow[d] & Rg_* \circ R\overline{f}_{2,*} \arrow[d] \\
R(h \circ g)_* \circ R\overline{f}_{3,*} \arrow[r] & Rg_* \circ Rh_* \circ R\overline{f}_{3,*}
\end{tikzcd}$$

is commutative (insert future reference here). The second part is to show that the composition

$$j_1^* \circ \overline{c} \circ \overline{d} \to j_2^* \circ \overline{d} \to j_3^*$$

is equal to the map

$$j_1^* \circ \overline{e} \to j_3^*$$

via the identification $\overline{e} = \overline{c} \circ \overline{d}$. This was proven in Lemma 5.1 (note that in the current case the morphisms f', g' of that lemma are equal to id_X). $\square$

[3] Namely, if $\alpha, \beta : F \to G$ are morphisms of functors and $\gamma : G \to H$ is an isomorphism of functors such that $\gamma \circ \alpha = \gamma \circ \beta$, then we conclude $\alpha = \beta$.

0ATX **Lemma 17.2.** *Let $f : X \to Y$ and $g : Y \to Z$ be composable morphisms between compactifyable schemes over a Noetherian scheme S. Then there is a canonical isomorphism $(g \circ f)^! \to f^! \circ g^!$.*

Proof. Choose a compactification $i : Y \to \overline{Y}$ of Y over Z. Choose a compactification $X \to \overline{X}$ of X over $\overline{Y}$. This uses Lemma 16.2 twice. Let $\overline{a}$ be the right adjoint of Lemma 3.1 for $\overline{X} \to \overline{Y}$ and let $\overline{b}$ be the right adjoint of Lemma 3.1 for $\overline{Y} \to Z$. Then $\overline{a} \circ \overline{b}$ is the right adjoint of Lemma 3.1 for the composition $\overline{X} \to Z$. Hence $g^! = j_Y^* \circ \overline{b}$ and $(g \circ f)^! = (X \to \overline{X})^* \circ \overline{a} \circ \overline{b}$. Let U be the inverse image of Y in $\overline{X}$ so that we get the commutative diagram

$$\begin{tikzcd} X \arrow[r, "j"'] \arrow[d] & U \arrow[r, "j'"'] \arrow[dl] & \overline{X} \arrow[dl] \\ Y \arrow[r, "i"'] \arrow[d] & \overline{Y} \arrow[dl] & \\ Z & & \end{tikzcd}$$

Let $\overline{a}'$ be the right adjoint of Lemma 3.1 for $U \to Y$. Then $f^! = j^* \circ \overline{a}'$. We obtain

$$\gamma : (j')^* \circ \overline{a} \to \overline{a}' \circ j_Y^*$$

by (4.1.1) and we can use it to define

$$(g \circ f)^! = j_X^* \circ \overline{a} \circ \overline{b} = j^* \circ (j')^* \circ \overline{a} \circ \overline{b} \to j^* \circ \overline{a}' \circ j_Y^* \circ \overline{b} = f^! \circ g^!$$

which is an isomorphism on objects of $D^+_{QCoh}(\mathcal{O}_Z)$ by Lemma 4.4. To finish the proof we show that this isomorphism is independent of choices made.

Suppose we have two diagrams

$$\begin{tikzcd} X \arrow[r, "j_1"'] \arrow[d] & U_1 \arrow[r, "j_1'"'] \arrow[dl] & \overline{X}_1 \arrow[dl] \\ Y \arrow[r, "i_1"'] \arrow[d] & \overline{Y}_1 \arrow[dl] & \\ Z & & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} X \arrow[r, "j_2"'] \arrow[d] & U_2 \arrow[r, "j_2'"'] \arrow[dl] & \overline{X}_2 \arrow[dl] \\ Y \arrow[r, "i_2"'] \arrow[d] & \overline{Y}_2 \arrow[dl] & \\ Z & & \end{tikzcd}$$

We can first choose a compactification $i : Y \to \overline{Y}$ of Y over Z which dominates both $\overline{Y}_1$ and $\overline{Y}_2$, see Lemma 16.1. By Lemma 16.3 and Categories, Lemmas 26.13 and 26.14 we can choose a compactification $X \to \overline{X}$ of X over $\overline{Y}$ with morphisms $\overline{X} \to \overline{X}_1$ and $\overline{X} \to \overline{X}_2$ and such that the composition $\overline{X} \to \overline{Y} \to \overline{Y}_1$ is equal to the composition $\overline{X} \to \overline{X}_1 \to \overline{Y}_1$ and such that the composition $\overline{X} \to \overline{Y} \to \overline{Y}_2$ is equal to the composition $\overline{X} \to \overline{X}_2 \to \overline{Y}_2$. Thus we see that it suffices to compare the maps determined by our diagrams when we have a commutative diagram as follows

$$
\begin{tikzcd}
X \arrow[r, "j_1"'] \arrow[d, equal] & U_1 \arrow[r, "j_1'"'] \arrow[d] \arrow[ddl] & \overline{X}_1 \arrow[d] \arrow[ddl] \\
X \arrow[r, "j_2"] \arrow[d] & U_2 \arrow[r, "j_2'"] \arrow[ddl] & \overline{X}_2 \arrow[ddl] \\
Y \arrow[r, "i_1"] \arrow[d, equal] & \overline{Y}_1 \arrow[d] & \\
Y \arrow[r, "i_2"] \arrow[d] & \overline{Y}_2 \arrow[dl] & \\
Z & &
\end{tikzcd}
$$

We use $\overline{a}_i$, $\overline{a}_i'$, $\overline{c}$, and $\overline{c}'$ for the right adjoint of Lemma 3.1 for $\overline{X}_i \to \overline{Y}_i$, $U_i \to Y$, $\overline{X}_1 \to \overline{X}_2$, and $U_1 \to U_2$. Each of the squares

$$
\begin{tikzcd}
X \arrow[r] \arrow[d] \arrow[dr, phantom, "A"] & U_1 \arrow[d] \\
X \arrow[r] & U_2
\end{tikzcd}
\quad
\begin{tikzcd}
U_2 \arrow[r] \arrow[d] \arrow[dr, phantom, "B"] & \overline{X}_2 \arrow[d] \\
Y \arrow[r] & \overline{Y}_2
\end{tikzcd}
\quad
\begin{tikzcd}
U_1 \arrow[r] \arrow[d] \arrow[dr, phantom, "C"] & \overline{X}_1 \arrow[d] \\
Y \arrow[r] & \overline{Y}_1
\end{tikzcd}
\quad
\begin{tikzcd}
Y \arrow[r] \arrow[d] \arrow[dr, phantom, "D"] & \overline{Y}_1 \arrow[d] \\
Y \arrow[r] & \overline{Y}_2
\end{tikzcd}
\quad
\begin{tikzcd}
X \arrow[r] \arrow[d] \arrow[dr, phantom, "E"] & \overline{X}_1 \arrow[d] \\
X \arrow[r] & \overline{X}_2
\end{tikzcd}
$$

gives rise to a base change map (4.1.1) as follows

$$
\gamma_A : j_1^* \circ \overline{c}' \to j_2^* \qquad \gamma_B : (j_2')^* \circ \overline{a}_2 \to \overline{a}_2' \circ i_2^* \qquad \gamma_C : (j_1')^* \circ \overline{a}_1 \to \overline{a}_1' \circ i_1^*
$$
$$
\gamma_D : i_1^* \circ \overline{d} \to i_2^* \qquad \gamma_E : (j_1' \circ j_1)^* \circ \overline{c} \to (j_2' \circ j_2)^*
$$

Denote $f_1^! = j_1^* \circ \overline{a}_1'$, $f_2^! = j_2^* \circ \overline{a}_2'$, $g_1^! = i_1^* \circ \overline{b}_1$, $g_2^! = i_2^* \circ \overline{b}_2$, $(g \circ f)_1^! = (j_1' \circ j_1)^* \circ \overline{a}_1 \circ \overline{b}_1$, and $(g \circ f)_2^! = (j_2' \circ j_2)^* \circ \overline{a}_2 \circ \overline{b}_2$. The construction given in the first paragraph of the proof and in Lemma 17.1 uses

(1) γ_C for the map $(g \circ f)_1^! \to f_1^! \circ g_1^!$,
(2) γ_B for the map $(g \circ f)_2^! \to f_2^! \circ g_2^!$,
(3) γ_A for the map $f_1^! \to f_2^!$,
(4) γ_D for the map $g_1^! \to g_2^!$, and
(5) γ_E for the map $(g \circ f)_1^! \to (g \circ f)_2^!$.

We have to show that the diagram

$$
\begin{tikzcd}
(g \circ f)_1^! \arrow[r, "\gamma_E"'] \arrow[d, "\gamma_C"'] & (g \circ f)_2^! \arrow[d, "\gamma_B"'] \\
f_1^! \circ g_1^! \arrow[r, "\gamma_A \circ \gamma_D"] & f_2^! \circ g_2^!
\end{tikzcd}
$$

is commutative. We will use Lemmas 5.1 and 5.2 and with (abuse of) notation as in Remark 5.3 (in particular dropping $\star$ products with identity transformations from the notation). We can write $\gamma_E = \gamma_A \circ \gamma_F$ where

$$\begin{tikzcd} U_1 \arrow[r] \arrow[d] \arrow[dr, phantom, "F"] & \overline{X}_1 \arrow[d] \\ U_2 \arrow[r] & \overline{X}_2 \end{tikzcd}$$

Thus we see that

$$\gamma_B \circ \gamma_E = \gamma_B \circ \gamma_A \circ \gamma_F = \gamma_A \circ \gamma_B \circ \gamma_F$$

the last equality because the two squares A and B only intersect in one point (similar to the last argument in Remark 5.3). Thus it suffices to prove that $\gamma_D \circ \gamma_C = \gamma_B \circ \gamma_F$. Since both of these are equal to the map (4.1.1) for the square

$$\begin{tikzcd} U_1 \arrow[r] \arrow[d] & \overline{X}_1 \arrow[d] \\ Y \arrow[r] & \overline{Y}_2 \end{tikzcd}$$

we conclude. □

0ATY **Lemma 17.3.** *Let S be a Noetherian scheme. The constructions of Lemmas 17.1 and 17.2 define a pseudo functor from the category of compactifyable schemes over S into the 2-category of categories (see Categories, Definition 28.5).*

Proof. To show this we have to prove given morphisms $f : X \to Y$, $g : Y \to Z$, $h : Z \to T$ that

$$\begin{tikzcd} (h \circ g \circ f)^! \arrow[r, "\gamma_{A+B}"'] \arrow[d, "\gamma_{B+C}"'] & f^! \circ (h \circ g)^! \arrow[d, "\gamma_C"] \\ (g \circ f)^! \circ h^! \arrow[r, "\gamma_A"] & f^! \circ g^! \circ h^! \end{tikzcd}$$

is commutative (for the meaning of the γ's, see below). To do this we choose a compactification $\overline{Z}$ of Z over T, then a compactification $\overline{Y}$ of Y over $\overline{Z}$, and then a compactification $\overline{X}$ of X over $\overline{Y}$. This uses Lemma 16.2 thrice. Let $W \subset \overline{Y}$ be the inverse image of Z under $\overline{Y} \to \overline{Z}$ and let $U \subset V \subset \overline{X}$ be the inverse images of $Y \subset W$ under $\overline{X} \to \overline{Y}$. This produces the following diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & U \arrow[r] \arrow[d] \arrow[dr, phantom, "A"] & V \arrow[r] \arrow[d] \arrow[dr, phantom, "B"] & \overline{X} \arrow[d] \\ Y \arrow[r] \arrow[d, "g"'] & Y \arrow[r] \arrow[d] & W \arrow[r] \arrow[d] \arrow[dr, phantom, "C"] & \overline{Y} \arrow[d] \\ Z \arrow[r] \arrow[d, "h"'] & Z \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d] & \overline{Z} \arrow[d] \\ T \arrow[r] & T \arrow[r] & T \arrow[r] & T \end{tikzcd}$$

Without introducing tons of notation but arguing exactly as in the proof of Lemma 17.2 we see that the maps in the first displayed diagram use the maps (4.1.1) for the rectangles $A + B$, $B + C$, A, and C as indicated. Since by Lemmas 5.1 and 5.2

we have $\gamma_{A+B} = \gamma_A \circ \gamma_B$ and $\gamma_{B+C} = \gamma_C \circ \gamma_B$ we conclude that the desired equality holds provided $\gamma_A \circ \gamma_C = \gamma_C \circ \gamma_A$. This is true because the two squares A and C only intersect in one point (similar to the last argument in Remark 5.3). $\square$

0B6T **Lemma 17.4.** *Let $f : X \to Y$ be a morphism between compactifyable schemes over a Noetherian scheme S. There are canonical maps*

$$\mu_{f,K} : Lf^*K \otimes^{\mathbf{L}}_{\mathcal{O}_X} f^!\mathcal{O}_Y \longrightarrow f^!K$$

functorial in K in $D^+_{QCoh}(\mathcal{O}_Y)$. If $g : Y \to Z$ is another morphism between compactifyable schemes, then the diagram

$$\begin{tikzcd}
Lf^*(Lg^*K \otimes^{\mathbf{L}}_{\mathcal{O}_Y} g^!\mathcal{O}_Z) \otimes^{\mathbf{L}}_{\mathcal{O}_X} f^!\mathcal{O}_Y \arrow[r, "\mu_f"'] \arrow[d, equal] & f^!(Lg^*K \otimes^{\mathbf{L}}_{\mathcal{O}_Y} g^!\mathcal{O}_Z) \arrow[r, "f^!\mu_g"'] & f^!g^!K \arrow[d, equal] \\
Lf^*Lg^*K \otimes^{\mathbf{L}}_{\mathcal{O}_X} Lf^*g^!\mathcal{O}_Z \otimes^{\mathbf{L}}_{\mathcal{O}_X} f^!\mathcal{O}_Y \arrow[r, "\mu_f"] & Lf^*Lg^*K \otimes^{\mathbf{L}}_{\mathcal{O}_X} f^!g^!\mathcal{O}_Z \arrow[r, "\mu_{g\circ f}"] & f^!g^!K
\end{tikzcd}$$

commutes for all $K \in D^+_{QCoh}(\mathcal{O}_Z)$.

Proof. If f is proper, then $f^! = a$ and we can use (8.0.1) and if g is also proper, then Lemma 8.4 proves the commutativity of the diagram (in greater generality).

In general, choose a compactification $j : X \to \overline{X}$ of X over Y. Since $f^!$ is defined as $j^* \circ \overline{a}$ we obtain μ_f as the restriction of the map (8.0.1)

$$L\overline{f}^*K \otimes^{\mathbf{L}}_{\mathcal{O}_{\overline{X}}} \overline{a}(\mathcal{O}_Y) \longrightarrow \overline{a}(K)$$

to X. To see this is independent of the choice of the compactification, we may assume given a morphism $g : \overline{X}_1 \to \overline{X}_2$ between compactifications $j_i : X \to \overline{X}_i$ over Y. But now we know that the maps

$$L\overline{f}_1^*K \otimes^{\mathbf{L}}_{\mathcal{O}_{\overline{X}}} \overline{a}_1(\mathcal{O}_Y) \longrightarrow \overline{a}_1(K) \quad \text{and} \quad L\overline{f}_2^*K \otimes^{\mathbf{L}}_{\mathcal{O}_{\overline{X}}} \overline{a}_2(\mathcal{O}_Y) \longrightarrow \overline{a}_2(K)$$

fit into a commutative diagram by Lemma 8.4 with two other maps given by μ_g which restrict to an isomorphism on X by Lemma 8.3. This implies the two displayed maps above restrict to the same map on the open, via the identification $\overline{a}_1(K)|_X = \overline{a}_2(K)|_X$ used in the definition of $f^!$. Having said this, the commutativity of the diagram follows from the construction of the isomorphism $(g\circ f)^! \to f^! \circ g^!$ (first part of the proof of Lemma 17.2 using $\overline{X} \to \overline{Y} \to Z$) and the result of Lemma 8.4 for $\overline{X} \to \overline{Y} \to Z$. $\square$

18. Properties of upper shriek functors

0ATZ Here are some properties of the upper shriek functors.

0AU0 **Lemma 18.1.** *Let S be a Noetherian scheme. Let Y be a compactifyable scheme over S and let $j : X \to Y$ be an open immersion. Then there is a canonical isomorphism $j^! = j^*$ of functors.*

Proof. In this case we may choose $\overline{X} = Y$ as our compactification. Then the right adjoint of Lemma 3.1 for $\mathrm{id} : Y \to Y$ is the identity functor and hence $j^! = j^*$ by definition. $\square$

0AA1 **Lemma 18.2.** *Let S be a Noetherian scheme. Let Y be a compactifyable scheme over S and let $f : X = \mathbf{A}^1_Y \to Y$ be the projection. Then there is a (noncanonical) isomorphism $f^!(-) \cong Lf^*(-)[1]$ of functors.*

Proof. Since $X = \mathbf{A}^1_Y \subset \mathbf{P}^1_Y$ and since $\mathcal{O}_{\mathbf{P}^1_Y}(-2)|_X \cong \mathcal{O}_X$ this follows from Lemmas 15.1 and 13.3. $\square$

0AA2 **Lemma 18.3.** *Let S be a Noetherian scheme. Let Y be a compactifyable scheme over S and let $i : X \to Y$ be a closed immersion. Then there is a canonical isomorphism $i^!(-) = R\mathcal{H}om(\mathcal{O}_X, -)$ of functors.*

Proof. This is a restatement of Lemma 9.7. $\square$

0BV2 **Remark 18.4** (Local description upper shriek)**.** Let S be a Noetherian scheme. Let $f : X \to Y$ be a morphism of compactifyable schemes over S. Using the lemmas above we can compute $f^!$ locally as follows. Suppose that we are given affine opens

$$\begin{tikzcd} U \arrow[r, "j"'] \arrow[d, "g"'] & X \arrow[d, "f"] \\ V \arrow[r, "i"] & Y \end{tikzcd}$$

Since $j^! \circ f^! = g^! \circ i^!$ (Lemma 17.2) and since $j^!$ and $i^!$ are given by restriction (Lemma 18.1) we see that

$$(f^!E)|_U = g^!(E|_V)$$

for any $E \in D^+_{QCoh}(\mathcal{O}_X)$. Write $U = \mathrm{Spec}(A)$ and $V = \mathrm{Spec}(R)$ and let $\varphi : R \to A$ be the finite type ring map corresponding to g. Choose a presentation $A = P/I$ where $P = R[x_1, \ldots, x_n]$ is a polynomial algebra in n variables over R. Choose an object $K \in D^+(R)$ corresponding to $E|_V$ (Derived Categories of Schemes, Lemma 3.5). Then we claim that $f^!E|_U$ corresponds to

$$\varphi^!(K) = R\,\mathrm{Hom}(A, K \otimes^{\mathbf{L}}_R P)[n]$$

where $R\,\mathrm{Hom}(A, -) : D(P) \to D(A)$ is the functor of Dualizing Complexes, Section 13 and where $\varphi^! : D(R) \to D(A)$ is the functor of Dualizing Complexes, Section 24. Namely, the choice of presentation gives a factorization

$$U \to \mathbf{A}^n_V \to \mathbf{A}^{n-1}_V \to \ldots \to \mathbf{A}^1_V \to V$$

Applying Lemma 18.2 exactly n times we see that $(\mathbf{A}^n_V \to V)^!(E|_V)$ corresponds to $K \otimes^{\mathbf{L}}_R P[n]$. By Lemmas 9.5 and 18.3 the last step corresponds to applying $R\,\mathrm{Hom}(A, -)$.

0AU1 **Lemma 18.5.** *Let S be a Noetherian scheme. Let $f : X \to Y$ be a morphism of compactifyable schemes over S. Then $f^!$ maps $D^+_{Coh}(\mathcal{O}_Y)$ into $D^+_{Coh}(\mathcal{O}_X)$.*

Proof. The question is local on X hence we may assume that X and Y are affine schemes. In this case we can factor $f : X \to Y$ as

$$X \xrightarrow{i} \mathbf{A}^n_Y \to \mathbf{A}^{n-1}_Y \to \ldots \to \mathbf{A}^1_Y \to Y$$

where i is a closed immersion. The lemma follows from By Lemmas 18.2 and 9.6 and Dualizing Complexes, Lemma 15.10 and induction. $\square$

0AA3 **Lemma 18.6.** *Let S be a Noetherian scheme. Let $f : X \to Y$ be a morphism of compactifyable schemes over S. If K is a dualizing complex for Y, then $f^!K$ is a dualizing complex for X.*

Proof. The question is local on X hence we may assume that X and Y are affine schemes. In this case we can factor $f : X \to Y$ as

$$X \xrightarrow{i} \mathbf{A}^n_Y \to \mathbf{A}^{n-1}_Y \to \ldots \to \mathbf{A}^1_Y \to Y$$

where i is a closed immersion. By Lemma 18.2 and Dualizing Complexes, Lemma 15.10 and induction we see that the $p^!K$ is a dualizing complex on $\mathbf{A}^n_Y$ where $p : \mathbf{A}^n_Y \to Y$ is the projection. Similarly, by Dualizing Complexes, Lemma 15.9 and Lemmas 9.5 and 18.3 we see that $i^!$ transforms dualizing complexes into dualizing complexes. □

0AU2 **Lemma 18.7.** *Let S be a Noetherian scheme. Let $f : X \to Y$ be a morphism of compactifyable schemes over S. Let K be a dualizing complex on Y. Set $D_Y(M) = R\mathcal{H}om_{\mathcal{O}_Y}(M, K)$ for $M \in D_{Coh}(\mathcal{O}_Y)$ and $D_X(E) = R\mathcal{H}om_{\mathcal{O}_X}(E, f^!K)$ for $E \in D_{Coh}(\mathcal{O}_X)$. Then there is a canonical isomorphism*

$$f^!M \longrightarrow D_X(Lf^*D_Y(M))$$

for $M \in D^+_{Coh}(\mathcal{O}_Y)$.

Proof. Choose compactification $j : X \subset \overline{X}$ of X over Y (Lemma 16.2). Let a be the right adjoint of Lemma 3.1 for $\overline{X} \to Y$. Set $D_{\overline{X}}(E) = R\mathcal{H}om_{\mathcal{O}_{\overline{X}}}(E, a(K))$ for $E \in D_{Coh}(\mathcal{O}_{\overline{X}})$. Since formation of $R\mathcal{H}om$ commutes with restriction to opens and since $f^! = j^* \circ a$ we see that it suffices to prove that there is a canonical isomorphism

$$a(M) \longrightarrow D_{\overline{X}}(L\overline{f}^*D_Y(M))$$

for $M \in D_{Coh}(\mathcal{O}_Y)$. For $F \in D_{QCoh}(\mathcal{O}_X)$ we have

$$\begin{aligned}
\mathrm{Hom}_{\overline{X}}(F, D_{\overline{X}}(L\overline{f}^*D_Y(M))) &= \mathrm{Hom}_{\overline{X}}(F \otimes^{\mathbf{L}}_{\mathcal{O}_X} L\overline{f}^*D_Y(M), a(K)) \\
&= \mathrm{Hom}_Y(R\overline{f}_*(F \otimes^{\mathbf{L}}_{\mathcal{O}_X} L\overline{f}^*D_Y(M)), K) \\
&= \mathrm{Hom}_Y(R\overline{f}_*(F) \otimes^{\mathbf{L}}_{\mathcal{O}_Y} D_Y(M), K) \\
&= \mathrm{Hom}_Y(R\overline{f}_*(F), D_Y(D_Y(M))) \\
&= \mathrm{Hom}_Y(R\overline{f}_*(F), M) \\
&= \mathrm{Hom}_{\overline{X}}(F, a(M))
\end{aligned}$$

The first equality by Cohomology, Lemma 36.2. The second by definition of a. The third by Derived Categories of Schemes, Lemma 21.1. The fourth equality by Cohomology, Lemma 36.2 and the definition of D_Y. The fifth equality by Lemma 2.4. The final equality by definition of a. Hence we see that $a(M) = D_{\overline{X}}(L\overline{f}^*D_Y(M))$ by Yoneda's lemma. □

0B6U **Lemma 18.8.** *Let S be a Noetherian scheme. Let $f : X \to Y$ be a perfect (e.g., flat) morphism of compactifyable schemes over S. Then*

(a) *$f^!$ maps $D^b_{Coh}(\mathcal{O}_Y)$ into $D^b_{Coh}(\mathcal{O}_X)$,*
(b) *the map $\mu_{f,K} : Lf^*K \otimes^{\mathbf{L}}_{\mathcal{O}_X} f^!\mathcal{O}_Y \to f^!K$ of Lemma 17.4 is an isomorphism for all $K \in D^+_{QCoh}(\mathcal{O}_Y)$.*

Proof. (A flat morphism of finite presentation is perfect, see More on Morphisms, Lemma 51.5.) We begin with a series of preliminary remarks.

(1) We already know that $f^!$ sends $D^+_{Coh}(\mathcal{O}_Y)$ into $D^+_{Coh}(\mathcal{O}_X)$, see Lemma 18.5.

(2) If f is an open immersion, then (a) and (b) are true because we can take $\overline{X} = Y$ in the construction of $f^!$ and μ_f. See also Lemma 18.1.
(3) If f is a perfect proper morphism, then (b) is true by Lemma 13.3.
(4) If there exists an open covering $X = \bigcup U_i$ and (a) is true for $U_i \to Y$, then (a) is true for $X \to Y$. Same for (b). This holds because the construction of $f^!$ and μ_f commutes with passing to open subschemes.
(5) If $g : Y \to Z$ is a second perfect morphism of compactifyable schemes over S and (b) holds for f and g, then $f^!g^!\mathcal{O}_Z = Lf^*g^!\mathcal{O}_Z \otimes^{\mathbf{L}}_{\mathcal{O}_X} f^!\mathcal{O}_Y$ and (b) holds for $g \circ f$ by the commutative diagram of Lemma 17.4.
(6) If (a) and (b) hold for both f and g, then (a) and (b) hold for $g \circ f$. Namely, then $f^!g^!\mathcal{O}_Z$ is bounded above (by the previous point) and $L(g \circ f)^*$ has finite cohomological dimension and (a) follows from (b) which we saw above.

From these points we see it suffices to prove the result in case X is affine. Choose an immersion $X \to \mathbf{A}^n_Y$ (Morphisms, Lemma 37.2) which we factor as $X \to U \to \mathbf{A}^n_Y \to Y$ where $X \to U$ is a closed immersion and $U \subset \mathbf{A}^n_Y$ is open. Note that $X \to U$ is a perfect closed immersion by More on Morphisms, Lemma 51.8. Thus it suffices to prove the lemma for a perfect closed immersion and for the projection $\mathbf{A}^n_Y \to Y$.

Let $f : X \to Y$ be a perfect closed immersion. We already know (b) holds. Let $K \in D^b_{Coh}(\mathcal{O}_Y)$. Then $f^!K = R\mathcal{H}om(\mathcal{O}_X, K)$ (Lemma 18.3) and $f_*f^!K = R\mathcal{H}om_{\mathcal{O}_Y}(f_*\mathcal{O}_X, K)$. Since f is perfect, the complex $f_*\mathcal{O}_X$ is perfect and hence $R\mathcal{H}om_{\mathcal{O}_Y}(f_*\mathcal{O}_X, K)$ is bounded above. This proves that (a) holds. Some details omitted.

Let $f : \mathbf{A}^n_Y \to Y$ be the projection. Then (a) holds by repeated application of Lemma 18.2. Finally, (b) is true because it holds for $\mathbf{P}^n_Y \to Y$ (flat and proper) and because $\mathbf{A}^n_Y \subset \mathbf{P}^n_Y$ is an open. □

0E9T **Lemma 18.9.** *Let S be a Noetherian scheme. Let $f : X \to Y$ be a flat morphism of compactifyable schemes over S. Then $f^!\mathcal{O}_Y$ is a Y-perfect object of $D(\mathcal{O}_X)$ and $\mathcal{O}_X \to R\mathcal{H}om_{\mathcal{O}_X}(f^!\mathcal{O}_Y, f^!\mathcal{O}_Y)$ is an isomorphism.*

Proof. Both assertions are local on X. Thus we may assume X and Y are affine. Then Remark 18.4 turns the lemma into an algebra lemma, namely Dualizing Complexes, Lemma 25.2. (Use Derived Categories of Schemes, Lemma 31.3 to match the languages.) □

0B6V **Lemma 18.10.** *Let S be a Noetherian scheme. Let $f : X \to Y$ be a local complete intersection morphism of compactifyable schemes over S. Then*
(1) *$f^!\mathcal{O}_Y$ is an invertible object of $D(\mathcal{O}_X)$, and*
(2) *$f^!$ maps perfect complexes to perfect complexes.*

Proof. Recall that a local complete intersection morphism is perfect, see More on Morphisms, Lemma 52.4. By Lemma 18.8 it suffices to show that $f^!\mathcal{O}_Y$ is an invertible object in $D(\mathcal{O}_X)$. This question is local on X and Y. Hence we may assume that $X \to Y$ factors as $X \to \mathbf{A}^n_Y \to Y$ where the first arrow is a Koszul regular immersion. See More on Morphisms, Section 52. The result holds for $\mathbf{A}^n_Y \to Y$ by Lemma 18.2. Thus it suffices to prove the lemma when f is a Koszul regular immersion. Working locally once again we reduce to the case $X = \mathrm{Spec}(A)$ and $Y = \mathrm{Spec}(B)$, where $A = B/(f_1, \ldots, f_r)$ for some regular sequence $f_1, \ldots, f_r \in B$ (use that for Noetherian local rings the notion of Koszul regular and regular are the same, see More on Algebra, Lemma 29.7). Thus $X \to Y$ is a composition

$$X = X_r \to X_{r-1} \to \ldots \to X_1 \to X_0 = Y$$

where each arrow is the inclusion of an effective Cartier divisor. In this way we reduce to the case of an inclusion of an effective Cartier divisor $i : D \to X$. In this case $i^!\mathcal{O}_X = \mathcal{N}[1]$ by Lemma 14.1 and the proof is complete. $\square$

19. Base change for upper shriek

0BZX Let S be a Noetherian scheme. Let

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

be a cartesian diagram of compactifyable schemes over S such that X and Y' are Tor independent over Y. Our setup is currently not sufficient to construct a base change map $L(g')^* \circ f^! \to (f')^! \circ Lg^*$ in this generality. The reason is that in general it will not be possible to choose a compactification $j : X \to \overline{X}$ over Y such that $\overline{X}$ and Y' are tor independent over Y and hence our construction of the base change map in Section 5 does not apply[4].

0E9U **Lemma 19.1.** *Let S be a Noetherian scheme. Let*

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

be a cartesian diagram of compactifyable schemes over S with g flat. Then there is an isomorphism $L(g')^* \circ f^! \to (f')^! \circ Lg^*$ on $D^+_{QCoh}(\mathcal{O}_Y)$.

Proof. Namely, because g is flat, for every choice of compactification $j : X \to \overline{X}$ of X over Y the scheme $\overline{X}$ is Tor independent of Y'. Denote $j' : X' \to \overline{X}'$ the base change of j and $\overline{g}' : \overline{X}' \to \overline{X}$ the projection. We define the base change map as the composition

$$L(g')^* \circ f^! = L(g')^* \circ j^* \circ a = (j')^* \circ L(\overline{g}')^* \circ a \longrightarrow (j')^* \circ a' \circ Lg^* = (f')^! \circ Lg^*$$

where the middle arrow is the base change map (5.0.1) and a and a' are the right adjoints to pushforward of Lemma 3.1 for $\overline{X} \to Y$ and $\overline{X}' \to Y'$. This construction is independent of the choice of compactification (we will formulate a precise lemma and prove it, if we ever need this result).

To finish the proof it suffices to show that the base change map $L(g')^* \circ a \to a' \circ Lg^*$ is an isomorphism on $D^+_{QCoh}(\mathcal{O}_Y)$. By Lemma 4.4 formation of a and a' commutes

[4] The reader who is well versed with derived algebraic geometry will realize this is not a "real" problem. Namely, taking $\overline{X}'$ to be the derived fibre product of $\overline{X}$ and Y' over Y, one can argue exactly as in the next paragraph to define this map. After all, the Tor independence of X and Y' guarantees that X' will be an open subscheme of the derived scheme $\overline{X}'$.

with restriction to affine opens of Y and Y'. Thus by Remark 6.1 we may assume that Y and Y' are affine. Thus the result by Lemma 6.2. □

In the rest of this section, we formulate some easy to prove results which would be consequences of a good theory of the base change map.

0BZY **Lemma 19.2** (Poor man's base change). *Let S be a Noetherian scheme. Let*

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

be a cartesian diagram of compactifyable schemes over S. Let $E \in D^+_{QCoh}(\mathcal{O}_Y)$. If f is flat, then $L(g')^*f^!E$ and $(f')^!Lg^*E$ restrict to isomorphic objects of $D(\mathcal{O}_{U'})$ for $U' \subset X'$ affine open mapping into affine opens of Y, Y', and X.

Proof. By our assumptions we immediately reduce to the case where X, Y, Y', and X' are affine. Say $Y = \text{Spec}(R)$, $Y' = \text{Spec}(R')$, $X = \text{Spec}(A)$, and $X' = \text{Spec}(A')$. Then $A' = A \otimes_R R'$. Let E correspond to $K \in D^+(R)$. Denoting $\varphi : R \to A$ and $\varphi' : R' \to A'$ the given maps we see from Remark 18.4 that $L(g')^*f^!E$ and $(f')^!Lg^*E$ correspond to $\varphi^!(K) \otimes_A^{\mathbf{L}} A'$ and $(\varphi')^!(K \otimes_R^{\mathbf{L}} R')$ where $\varphi^!$ and $(\varphi')^!$ are the functors from Dualizing Complexes, Section 24. The result follows from Dualizing Complexes, Lemma 24.6. □

0BZZ **Lemma 19.3.** *Let $f : X \to Y$ be a flat morphism of compactifyable schemes over a Noetherian scheme S. Set $\omega^\bullet_{X/Y} = f^!\mathcal{O}_Y$ in $D^b_{Coh}(X)$. Let $y \in Y$ and $h : X_y \to X$ the projection. Then $Lh^*\omega^\bullet_{X/Y}$ is a dualizing complex on X_y.*

Proof. The complex $\omega^\bullet_{X/Y}$ is in D^b_{Coh} by Lemma 18.8. Being a dualizing complex is a local property. Hence by Lemma 19.2 it suffices to show that $(X_y \to y)^!\mathcal{O}_y$ is a dualizing complex on X_y. This follows from Lemma 18.6. □

20. A duality theory

0AU3 In this section we spell out what kind of a duality theory our very general results above give for compactifyable schemes over a fixed Noetherian base scheme.

Recall that a dualizing complex on a Noetherian scheme X, is an object of $D(\mathcal{O}_X)$ which affine locally gives a dualizing complex for the corresponding rings, see Definition 2.2.

Let S be a Noetherian base scheme. We summarize the most important points of the results obtained above:

(1) the functors $f^!$ turn D^+_{QCoh} into a pseudo functor on the category of compactifyable schemes over S,
(2) if $f : X \to Y$ is a proper morphism between compactifyable schemes over S, then $f^!$ is the restriction of the right adjoint of $Rf_* : D_{QCoh}(\mathcal{O}_X) \to D_{QCoh}(\mathcal{O}_Y)$ to $D^+_{QCoh}(\mathcal{O}_Y)$ and there is a canonical isomorphism

$$Rf_*R\mathcal{H}om_{\mathcal{O}_X}(K, f^!M) \to R\mathcal{H}om_{\mathcal{O}_Y}(Rf_*K, M)$$

for all $K \in D_{QCoh}(\mathcal{O}_X)$ and $M \in D^+_{QCoh}(\mathcal{O}_Y)$,

(3) if X has a dualizing complex $\omega_X^\bullet$, then the functor $D_X = R\mathcal{H}om_{\mathcal{O}_X}(-, \omega_X^\bullet)$ defines an involution of $D_{Coh}(\mathcal{O}_X)$ switching $D^+_{Coh}(\mathcal{O}_X)$ and $D^-_{Coh}(\mathcal{O}_X)$ and fixing $D^b_{Coh}(\mathcal{O}_X)$,
(4) if $f : X \to Y$ is a morphism of compactifyable schemes over S and $\omega_Y^\bullet$ is a dualizing complex on Y, then
  (a) $\omega_X^\bullet = f^!\omega_Y^\bullet$ is a dualizing complex for X,
  (b) $f^!M = D_X(Lf^*D_Y(M))$ canonically for $M \in D^+_{Coh}(\mathcal{O}_Y)$, and
  (c) if in addition f is proper then
$$Rf_*R\mathcal{H}om_{\mathcal{O}_X}(K, \omega_X^\bullet) = R\mathcal{H}om_{\mathcal{O}_Y}(Rf_*K, \omega_Y^\bullet)$$
(5) if $f : X \to Y$ is a closed immersion of compactifyable schemes over S, then $f^!(-) = R\mathcal{H}om(\mathcal{O}_X, -)$,
(6) if $f : Y \to X$ is a finite morphism of compactifyable schemes over S, then $f_*f^!(-) = R\mathcal{H}om_{\mathcal{O}_X}(f_*\mathcal{O}_X, -)$,
(7) if $f : X \to Y$ is the inclusion of an effective Cartier divisor into a compactifyable scheme over S, then $f^!(-) = Lf^*(-) \otimes_{\mathcal{O}_X} \mathcal{O}_Y(-X)[-1]$,
(8) if $f : X \to Y$ is a Koszul regular immersion of codimension c into a compactifyable scheme over S, then $f^!(-) \cong Lf^*(-) \otimes_{\mathcal{O}_X} \wedge^c\mathcal{N}[-c]$, and
(9) if $f : X \to Y$ is a smooth proper morphism of relative dimension d of compactifyable schemes over S, then $f^!(-) \cong Lf^* \otimes_{\mathcal{O}_X} \Omega^d_{X/Y}[d]$.

See Lemmas 2.4, 3.6, 9.7, 11.4, 14.2, 15.6, 15.7, 17.2, 17.3, 18.3, 18.6, 18.7, and 18.8. We have obtained our functors by a very abstract procedure which finally rests on invoking an existence theorem (Derived Categories, Proposition 35.2). This means we have, in general, no explicit description of the functors $f^!$. This can sometimes be a problem. But in fact, it is often enough to know the existence of a dualizing complex and the duality isomorphism to pin down $f^!$.

21. Glueing dualizing complexes

0AU5 We will now use glueing of dualizing complexes to get a theory which works for all finite type schemes over S given a pair $(S, \omega_S^\bullet)$ as in Situation 21.1. This is similar to [Har66, Remark on page 310].

0AU4 **Situation 21.1.** Here S is a Noetherian scheme and $\omega_S^\bullet$ is a dualizing complex.

Let X be a scheme of finite type over S. Let $\mathcal{U} : X = \bigcup_{i=1,\ldots,n} U_i$ be a finite open covering of X by quasi-compact compactifyable schemes over S. Every affine scheme of finite type over S is compactifyable over S by Morphisms, Lemma 37.3 hence such open coverings certainly exist. For each $i, j, k \in \{1, \ldots, n\}$ the schemes $p_i : U_i \to S$, $p_{ij} : U_i \cap U_j \to S$, and $p_{ijk} : U_i \cap U_j \cap U_k \to S$ are compactifyable. From such an open covering we obtain

(1) $\omega_i^\bullet = p_i^!\omega_S^\bullet$ a dualizing complex on U_i, see Section 20,
(2) for each i, j a canonical isomorphism $\varphi_{ij} : \omega_i^\bullet|_{U_i \cap U_j} \to \omega_j^\bullet|_{U_i\cap U_j}$, and
0AU6 (3) for each i, j, k we have
$$\varphi_{ik}|_{U_i\cap U_j\cap U_k} = \varphi_{jk}|_{U_i\cap U_j\cap U_k} \circ \varphi_{ij}|_{U_i\cap U_j\cap U_k}$$
in $D(\mathcal{O}_{U_i\cap U_j\cap U_k})$.

Here, in (2) we use that $(U_i \cap U_j \to U_i)^!$ is given by restriction (Lemma 18.1) and that we have canonical isomorphisms
$$(U_i \cap U_j \to U_i)^! \circ p_i^! = p_{ij}^! = (U_i \cap U_j \to U_j)^! \circ p_j^!$$

by Lemma 17.2 and to get (3) we use that the upper shriek functors form a pseudo functor by Lemma 17.3.

In the situation just described a *dualizing complex normalized relative to* $\omega_S^\bullet$ *and* $\mathcal{U}$ is a pair (K, α_i) where $K \in D(\mathcal{O}_X)$ and $\alpha_i : K|_{U_i} \to \omega_i^\bullet$ are isomorphisms such that φ_{ij} is given by $\alpha_j|_{U_i \cap U_j} \circ \alpha_i^{-1}|_{U_i \cap U_j}$. Since being a dualizing complex on a scheme is a local property we see that dualizing complexes normalized relative to $\omega_S^\bullet$ and $\mathcal{U}$ are indeed dualizing complexes.

0AU7 **Lemma 21.2.** *In Situation 21.1 let X be a scheme of finite type over S and let $\mathcal{U}$ be a finite open covering of X by compactifyable schemes. If there exists a dualizing complex normalized relative to $\omega_S^\bullet$ and $\mathcal{U}$, then it is unique up to unique isomorphism.*

Proof. If (K, α_i) and (K', α'_i) are two, then we consider $L = R\mathcal{H}om_{\mathcal{O}_X}(K, K')$. By Lemma 2.5 and its proof, this is an invertible object of $D(\mathcal{O}_X)$. Using α_i and α'_i we obtain an isomorphism

$$\alpha_i^t \otimes \alpha'_i : L|_{U_i} \longrightarrow R\mathcal{H}om_{\mathcal{O}_X}(\omega_i^\bullet, \omega_i^\bullet) = \mathcal{O}_{U_i}[0]$$

This already implies that $L = H^0(L)[0]$ in $D(\mathcal{O}_X)$. Moreover, $H^0(L)$ is an invertible sheaf with given trivializations on the opens U_i of X. Finally, the condition that $\alpha_j|_{U_i \cap U_j} \circ \alpha_i^{-1}|_{U_i \cap U_j}$ and $\alpha'_j|_{U_i \cap U_j} \circ (\alpha'_i)^{-1}|_{U_i \cap U_j}$ both give φ_{ij} implies that the transition maps are 1 and we get an isomorphism $H^0(L) = \mathcal{O}_X$. □

0AU8 **Lemma 21.3.** *In Situation 21.1 let X be a scheme of finite type over S and let $\mathcal{U}$, $\mathcal{V}$ be two finite open coverings of X by compactifyable schemes. If there exists a dualizing complex normalized relative to $\omega_S^\bullet$ and $\mathcal{U}$, then there exists a dualizing complex normalized relative to $\omega_S^\bullet$ and $\mathcal{V}$ and these complexes are canonically isomorphic.*

Proof. It suffices to prove this when $\mathcal{U}$ is given by the opens $U_1, \ldots, U_n$ and $\mathcal{V}$ by the opens $U_1, \ldots, U_{n+m}$. In fact, we may and do even assume $m = 1$. To go from a dualizing complex (K, α_i) normalized relative to $\omega_S^\bullet$ and $\mathcal{V}$ to a dualizing complex normalized relative to $\omega_S^\bullet$ and $\mathcal{U}$ is achieved by forgetting about α_i for $i = n+1$. Conversely, let (K, α_i) be a dualizing complex normalized relative to $\omega_S^\bullet$ and $\mathcal{U}$. To finish the proof we need to construct a map $\alpha_{n+1} : K|_{U_{n+1}} \to \omega_{n+1}^\bullet$ satisfying the desired conditions. To do this we observe that $U_{n+1} = \bigcup U_i \cap U_{n+1}$ is an open covering. It is clear that $(K|_{U_{n+1}}, \alpha_i|_{U_i \cap U_{n+1}})$ is a dualizing complex normalized relative to $\omega_S^\bullet$ and the covering $U_{n+1} = \bigcup U_i \cap U_{n+1}$. On the other hand, by condition (3) the pair $(\omega_{n+1}^\bullet|_{U_{n+1}}, \varphi_{n+1i})$ is another dualizing complex normalized relative to $\omega_S^\bullet$ and the covering $U_{n+1} = \bigcup U_i \cap U_{n+1}$. By Lemma 21.2 we obtain a unique isomorphism

$$\alpha_{n+1} : K|_{U_{n+1}} \longrightarrow \omega_{n+1}^\bullet$$

compatible with the given local isomorphisms. It is a pleasant exercise to show that this means it satisfies the required property. □

0AU9 **Lemma 21.4.** *In Situation 21.1 let X be a scheme of finite type over S and let $\mathcal{U}$ be a finite open covering of X by compactifyable schemes. Then there exists a dualizing complex normalized relative to $\omega_S^\bullet$ and $\mathcal{U}$.*

Proof. Say $\mathcal{U} : X = \bigcup_{i=1,\ldots,n} U_i$. We prove the lemma by induction on n. The base case $n = 1$ is immediate. Assume $n > 1$. Set $X' = U_1 \cup \ldots \cup U_{n-1}$ and let $(K', \{\alpha'_i\}_{i=1,\ldots,n-1})$ be a dualizing complex normalized relative to $\omega_S^\bullet$ and $\mathcal{U}' : X' = \bigcup_{i=1,\ldots,n-1} U_i$. It is clear that $(K'|_{X' \cap U_n}, \alpha'_i|_{U_i \cap U_n})$ is a dualizing complex normalized relative to $\omega_S^\bullet$ and the covering $X' \cap U_n = \bigcup_{i=1,\ldots,n-1} U_i \cap U_n$. On the other hand, by condition (3) the pair $(\omega_n^\bullet|_{X' \cap U_n}, \varphi_{ni})$ is another dualizing complex normalized relative to $\omega_S^\bullet$ and the covering $X' \cap U_n = \bigcup_{i=1,\ldots,n-1} U_i \cap U_n$. By Lemma 21.2 we obtain a unique isomorphism

$$\epsilon : K'|_{X' \cap U_n} \longrightarrow \omega_i^\bullet|_{X' \cap U_n}$$

compatible with the given local isomorphisms. By Cohomology, Lemma 39.1 we obtain $K \in D(\mathcal{O}_X)$ together with isomorphisms $\beta : K|_{X'} \to K'$ and $\gamma : K|_{U_n} \to \omega_n^\bullet$ such that $\epsilon = \gamma|_{X' \cap U_n} \circ \beta|_{X' \cap U_n}^{-1}$. Then we define

$$\alpha_i = \alpha'_i \circ \beta|_{U_i}, i = 1, \ldots, n-1, \text{ and } \alpha_n = \gamma$$

We still need to verify that φ_{ij} is given by $\alpha_j|_{U_i \cap U_j} \circ \alpha_i^{-1}|_{U_i \cap U_j}$. For $i, j \leq n-1$ this follows from the corresponding condition for α'_i. For $i = j = n$ it is clear as well. If $i < j = n$, then we get

$$\alpha_n|_{U_i \cap U_n} \circ \alpha_i^{-1}|_{U_i \cap U_n} = \gamma|_{U_i \cap U_n} \circ \beta^{-1}|_{U_i \cap U_n} \circ (\alpha'_i)^{-1}|_{U_i \cap U_n} = \epsilon|_{U_i \cap U_n} \circ (\alpha'_i)^{-1}|_{U_i \cap U_n}$$

This is equal to α_{in} exactly because ϵ is the unique map compatible with the maps α'_i and α_{ni}. $\square$

Let $(S, \omega_S^\bullet)$ be as in Situation 21.1. The upshot of the lemmas above is that given any scheme X of finite type over S, there is a pair (K, α_U) given up to unique isomorphism, consisting of an object $K \in D(\mathcal{O}_X)$ and isomorphisms $\alpha_U : K|_U \to \omega_U^\bullet$ for every open subscheme $U \subset X$ which has a compactification over S. Here $\omega_U^\bullet = (U \to S)^! \omega_S^\bullet$ is a dualizing complex on U, see Section 20. Moreover, if $\mathcal{U} : X = \bigcup U_i$ is a finite open covering by opens which are compactifyable over S, then (K, α_{U_i}) is a dualizing complex normalized relative to $\omega_S^\bullet$ and $\mathcal{U}$. Namely, uniqueness up to unique isomorphism by Lemma 21.2, existence for one open covering by Lemma 21.4, and the fact that K then works for all open coverings is Lemma 21.3.

0AUA **Definition 21.5.** Let S be a Noetherian scheme and let $\omega_S^\bullet$ be a dualizing complex on S. Let X be a scheme of finite type over S. The complex K constructed above is called the *dualizing complex normalized relative to* $\omega_S^\bullet$ and is denoted $\omega_X^\bullet$.

As the terminology suggest, a dualizing complex normalized relative to $\omega_S^\bullet$ is not just an object of the derived category of X but comes equipped with the local isomorphisms described above. This does not conflict with setting $\omega_X^\bullet = p^! \omega_S^\bullet$ where $p : X \to S$ is the structure morphism if X has a compactification over S (see Dualizing Complexes, Section 15). More generally we have the following sanity check.

0AUB **Lemma 21.6.** *Let $(S, \omega_S^\bullet)$ be as in Situation 21.1. Let $f : X \to Y$ be a morphism of finite type schemes over S. Let $\omega_X^\bullet$ and $\omega_Y^\bullet$ be dualizing complexes normalized relative to $\omega_S^\bullet$. Then $\omega_X^\bullet$ is a dualizing complex normalized relative to $\omega_Y^\bullet$.*

Proof. This is just a matter of bookkeeping. Choose a finite affine open covering $\mathcal{V} : Y = \bigcup V_j$. For each j choose a finite affine open covering $f^{-1}(V_j) = U_{ji}$. Set $\mathcal{U} : X = \bigcup U_{ji}$. The schemes V_j and U_{ji} are compactifyable over S, hence we have the upper shriek functors for $q_j : V_j \to S$, $p_{ji} : U_{ji} \to S$ and $f_{ji} : U_{ji} \to V_j$ and $f'_{ji} : U_{ji} \to Y$. Let (L, β_j) be a dualizing complex normalized relative to $\omega_S^\bullet$ and $\mathcal{V}$. Let (K, γ_{ji}) be a dualizing complex normalized relative to $\omega_S^\bullet$ and $\mathcal{U}$. (In other words, $L = \omega_Y^\bullet$ and $K = \omega_X^\bullet$.) We can define

$$\alpha_{ji} : K|_{U_{ji}} \xrightarrow{\gamma_{ji}} p_{ji}^!\omega_S^\bullet = f_{ji}^! q_j^! \omega_S^\bullet \xrightarrow{f_{ji}^!\beta_j^{-1}} f_{ji}^!(L|_{V_j}) = (f'_{ji})^!(L)$$

To finish the proof we have to show that $\alpha_{ji}|_{U_{ji} \cap U_{j'i'}} \circ \alpha_{j'i'}^{-1}|_{U_{ji} \cap U_{j'i'}}$ is the canonical isomorphism $(f'_{ji})^!(L)|_{U_{ji} \cap U_{j'i'}} \to (f'_{j'i'})^!(L)|_{U_{ji} \cap U_{j'i'}}$. This is formal and we omit the details. $\square$

0AUC **Lemma 21.7.** *Let $(S, \omega_S^\bullet)$ be as in Situation 21.1. Let $j : X \to Y$ be an open immersion of schemes of finite type over S. Let $\omega_X^\bullet$ and $\omega_Y^\bullet$ be dualizing complexes normalized relative to $\omega_S^\bullet$. Then there is a canonical isomorphism $\omega_X^\bullet = \omega_Y^\bullet|_X$.*

Proof. Immediate from the construction of normalized dualizing complexes given just above Definition 21.5. $\square$

0AUD **Lemma 21.8.** *Let $(S, \omega_S^\bullet)$ be as in Situation 21.1. Let $f : X \to Y$ be a proper morphism of schemes of finite type over S. Let $\omega_X^\bullet$ and $\omega_Y^\bullet$ be dualizing complexes normalized relative to $\omega_S^\bullet$. Let a be the right adjoint of Lemma 3.1 for f. Then there is a canonical isomorphism $a(\omega_Y^\bullet) = \omega_X^\bullet$.*

Proof. Let $p : X \to S$ and $q : Y \to S$ be the structure morphisms. If X and Y are compactifyable over S, then this follows from the fact that $\omega_X^\bullet = p^!\omega_S^\bullet$, $\omega_Y^\bullet = q^!\omega_S^\bullet$, $f^! = a$, and $f^! \circ q^! = p^!$ (Lemma 17.2). In the general case we first use Lemma 21.6 to reduce to the case $Y = S$. In this case X and Y are compactifyable over S and we've just seen the result. $\square$

Let $(S, \omega_S^\bullet)$ be as in Situation 21.1. For a scheme X of finite type over S denote $\omega_X^\bullet$ the dualizing complex for X normalized relative to $\omega_S^\bullet$. Define $D_X(-) = R\mathcal{H}om_{\mathcal{O}_X}(-, \omega_X^\bullet)$ as in Lemma 2.4. Let $f : X \to Y$ be a morphism of finite type schemes over S. Define

$$f_{new}^! = D_X \circ Lf^* \circ D_Y : D_{Coh}^+(\mathcal{O}_Y) \to D_{Coh}^+(\mathcal{O}_X)$$

If $f : X \to Y$ and $g : Y \to Z$ are composable morphisms between schemes of finite type over S, define

$$\begin{aligned} (g \circ f)_{new}^! &= D_X \circ L(g \circ f)^* \circ D_Z \\ &= D_X \circ Lf^* \circ Lg^* \circ D_Z \\ &\to D_X \circ Lf^* \circ D_Y \circ D_Y \circ Lg^* \circ D_Z \\ &= f_{new}^! \circ g_{new}^! \end{aligned}$$

where the arrow is defined in Lemma 2.4. We collect the results together in the following lemma.

0AUE **Lemma 21.9.** *Let $(S, \omega_S^\bullet)$ be as in Situation 21.1. With $f_{new}^!$ and $\omega_X^\bullet$ defined for all (morphisms of) schemes of finite type over S as above:*

1. *the functors $f_{new}^!$ and the arrows $(g \circ f)_{new}^! \to f_{new}^! \circ g_{new}^!$ turn D_{Coh}^+ into a pseudo functor from the category of schemes of finite type over S into the 2-category of categories,*
2. *$\omega_X^\bullet = (X \to S)_{new}^! \omega_S^\bullet$,*

(3) *the functor* D_X *defines an involution of* $D_{Coh}(\mathcal{O}_X)$ *switching* $D^+_{Coh}(\mathcal{O}_X)$ *and* $D^-_{Coh}(\mathcal{O}_X)$ *and fixing* $D^b_{Coh}(\mathcal{O}_X)$*,*
(4) $\omega_X^\bullet = f^!_{new}\omega_Y^\bullet$ *for* $f : X \to Y$ *a morphism of finite type schemes over* S*,*
(5) $f^!_{new}M = D_X(Lf^*D_Y(M))$ *for* $M \in D^+_{Coh}(\mathcal{O}_Y)$*, and*
(6) *if in addition* f *is proper, then* $f^!_{new}$ *is isomorphic to the restriction of the right adjoint of* $Rf_* : D_{QCoh}(\mathcal{O}_X) \to D_{QCoh}(\mathcal{O}_Y)$ *to* $D^+_{Coh}(\mathcal{O}_Y)$ *and there is a canonical isomorphism*
$$Rf_*R\mathcal{H}om_{\mathcal{O}_X}(K, f^!_{new}M) \to R\mathcal{H}om_{\mathcal{O}_Y}(Rf_*K, M)$$
for all $K \in D_{QCoh}(\mathcal{O}_X)$ *and* $M \in D^+_{Coh}(\mathcal{O}_Y)$*, and most importantly*
$$Rf_*R\mathcal{H}om_{\mathcal{O}_X}(K, \omega_X^\bullet) = R\mathcal{H}om_{\mathcal{O}_Y}(Rf_*K, \omega_Y^\bullet)$$

If X *is compactifyable over* S*, then* $\omega_X^\bullet$ *is canonically isomorphic to* $(X \to S)^!\omega_S^\bullet$ *and if* f *is a morphism between compactifyable schemes over* S*, then there is a canonical isomorphism*[5] $f^!_{new}K = f^!K$ *for* K *in* D^+_{Coh}*.*

Proof. Let $f : X \to Y$, $g : Y \to Z$, $h : Z \to T$ be morphisms of schemes of finite type over S. We have to show that

$$\begin{tikzcd}
(h \circ g \circ f)^!_{new} \arrow[r] \arrow[d] & f^!_{new} \circ (h \circ g)^!_{new} \arrow[d] \\
(g \circ f)^!_{new} \circ h^!_{new} \arrow[r] & f^!_{new} \circ g^!_{new} \circ h^!_{new}
\end{tikzcd}$$

is commutative. Let $\eta_Y : \text{id} \to D_Y^2$ and $\eta_Z : \text{id} \to D_Z^2$ be the canonical isomorphisms of Lemma 2.4. Then, using Categories, Lemma 27.2, a computation (omitted) shows that both arrows $(h \circ g \circ f)^!_{new} \to f^!_{new} \circ g^!_{new} \circ h^!_{new}$ are given by

$$1 \star \eta_Y \star 1 \star \eta_Z \star 1 : D_X \circ Lf^* \circ Lg^* \circ Lh^* \circ D_T \longrightarrow D_X \circ Lf^* \circ D_Y^2 \circ Lg^* \circ D_Z^2 \circ Lh^* \circ D_T$$

This proves (1). Part (2) is immediate from the definition of $(X \to S)^!_{new}$ and the fact that $D_S(\omega_S^\bullet) = \mathcal{O}_S$. Part (3) is Lemma 2.4. Part (4) follows by the same argument as part (2). Part (5) is the definition of $f^!_{new}$.

Proof of (6). Let a be the right adjoint of Lemma 3.1 for the proper morphism $f : X \to Y$ of schemes of finite type over S. The issue is that we do not know X or Y is compactifyable over S (and in general this won't be true) hence we cannot immediately apply Lemma 18.7 to f over S. To get around this we use the canonical identification $\omega_X^\bullet = a(\omega_Y^\bullet)$ of Lemma 21.8. Hence $f^!_{new}$ is the restriction of a to $D^+_{Coh}(\mathcal{O}_Y)$ by Lemma 18.7 applied to $f : X \to Y$ over the base scheme Y! Thus the result is true by Lemma 3.6.

The final assertions follow from the construction of normalized dualizing complexes and the already used Lemma 18.7. □

0BV3 **Remark 21.10.** Let S be a Noetherian scheme which has a dualizing complex. Let $f : X \to Y$ be a morphism of schemes of finite type over S. Then the functor
$$f^!_{new} : D^+_{Coh}(\mathcal{O}_Y) \to D^+_{Coh}(\mathcal{O}_X)$$
is independent of the choice of the dualizing complex $\omega_S^\bullet$ up to canonical isomorphism. We sketch the proof. Any second dualizing complex is of the form $\omega_S^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_S} \mathcal{L}$

[5] We haven't checked that these are compatible with the isomorphisms $(g \circ f)^! \to f^! \circ g^!$ and $(g \circ f)^!_{new} \to f^!_{new} \circ g^!_{new}$. We will do this here if we need this later.

where $\mathcal{L}$ is an invertible object of $D(\mathcal{O}_S)$, see Lemma 2.5. For any compactifyable $p : U \to S$ we have $p^!(\omega_S^\bullet \otimes_{\mathcal{O}_S}^{\mathbf{L}} \mathcal{L}) = p^!(\omega_S^\bullet) \otimes_{\mathcal{O}_U}^{\mathbf{L}} Lp^*\mathcal{L}$ by Lemma 8.1. Hence, if $\omega_X^\bullet$ and $\omega_Y^\bullet$ are the dualizing complexes normalized relative to $\omega_S^\bullet$ we see that $\omega_X^\bullet \otimes_{\mathcal{O}_X}^{\mathbf{L}} La^*\mathcal{L}$ and $\omega_Y^\bullet \otimes_{\mathcal{O}_Y}^{\mathbf{L}} Lb^*\mathcal{L}$ are the dualizing complexes normalized relative to $\omega_S^\bullet \otimes_{\mathcal{O}_S}^{\mathbf{L}} \mathcal{L}$ (where $a : X \to S$ and $b : Y \to S$ are the structure morphisms). Then the result follows as

$$\begin{aligned}
& R\mathcal{H}om_{\mathcal{O}_X}(Lf^*R\mathcal{H}om_{\mathcal{O}_Y}(K, \omega_Y^\bullet \otimes_{\mathcal{O}_Y}^{\mathbf{L}} Lb^*\mathcal{L}), \omega_X^\bullet \otimes_{\mathcal{O}_X}^{\mathbf{L}} La^*\mathcal{L}) \\
& = R\mathcal{H}om_{\mathcal{O}_X}(Lf^*R(\mathcal{H}om_{\mathcal{O}_Y}(K, \omega_Y^\bullet) \otimes_{\mathcal{O}_Y}^{\mathbf{L}} Lb^*\mathcal{L}), \omega_X^\bullet \otimes_{\mathcal{O}_X}^{\mathbf{L}} La^*\mathcal{L}) \\
& = R\mathcal{H}om_{\mathcal{O}_X}(Lf^*R\mathcal{H}om_{\mathcal{O}_Y}(K, \omega_Y^\bullet) \otimes_{\mathcal{O}_X}^{\mathbf{L}} La^*\mathcal{L}, \omega_X^\bullet \otimes_{\mathcal{O}_X}^{\mathbf{L}} La^*\mathcal{L}) \\
& = R\mathcal{H}om_{\mathcal{O}_X}(Lf^*R\mathcal{H}om_{\mathcal{O}_Y}(K, \omega_Y^\bullet), \omega_X^\bullet)
\end{aligned}$$

for $K \in D^+_{Coh}(\mathcal{O}_Y)$. The last equality because $La^*\mathcal{L}$ is invertible in $D(\mathcal{O}_X)$.

0B6X **Example 21.11.** Let S be a Noetherian scheme and let $\omega_S^\bullet$ be a dualizing complex. Let $f : X \to Y$ be a proper morphism of finite type schemes over S. Let $\omega_X^\bullet$ and $\omega_Y^\bullet$ be dualizing complexes normalized relative to $\omega_S^\bullet$. In this situation we have $a(\omega_Y^\bullet) = \omega_X^\bullet$ (Lemma 21.8) and hence the trace map (Section 7) is a canonical arrow

$$\mathrm{Tr}_f : Rf_*\omega_X^\bullet \longrightarrow \omega_Y^\bullet$$

which produces the isomorphisms (Lemma 21.9)

$$\mathrm{Hom}_X(L, \omega_X^\bullet) = \mathrm{Hom}_Y(Rf_*L, \omega_Y^\bullet)$$

and

$$Rf_*R\mathcal{H}om_{\mathcal{O}_X}(L, \omega_X^\bullet) = R\mathcal{H}om_{\mathcal{O}_Y}(Rf_*L, \omega_Y^\bullet)$$

for L in $D_{QCoh}(\mathcal{O}_X)$.

0AX4 **Remark 21.12.** Let S be a Noetherian scheme and let $\omega_S^\bullet$ be a dualizing complex. Let $f : X \to Y$ be a finite morphism between schemes of finite type over S. Let $\omega_X^\bullet$ and $\omega_Y^\bullet$ be dualizing complexes normalized relative to $\omega_S^\bullet$. Then we have

$$f_*\omega_X^\bullet = R\mathcal{H}om(f_*\mathcal{O}_X, \omega_Y^\bullet)$$

in $D^+_{QCoh}(f_*\mathcal{O}_X)$ by Lemmas 11.4 and 21.8 and the trace map of Example 21.11 is the map

$$\mathrm{Tr}_f : Rf_*\omega_X^\bullet = f_*\omega_X^\bullet = R\mathcal{H}om(f_*\mathcal{O}_X, \omega_Y^\bullet) \longrightarrow \omega_Y^\bullet$$

which often goes under the name "evaluation at 1".

0B6W **Remark 21.13.** Let $f : X \to Y$ be a flat proper morphism of finite type schemes over a pair $(S, \omega_S^\bullet)$ as in Situation 21.1. The relative dualizing complex (Remark 12.5) is $\omega_{X/Y}^\bullet = a(\mathcal{O}_Y)$. By Lemma 21.8 we have the first canonical isomorphism in

$$\omega_X^\bullet = a(\omega_Y^\bullet) = Lf^*\omega_Y^\bullet \otimes_{\mathcal{O}_X}^{\mathbf{L}} \omega_{X/Y}^\bullet$$

in $D(\mathcal{O}_X)$. The second canonical isomorphism follows from the discussion in Remark 12.5.

22. Dimension functions

0BV4 We need a bit more information about how the dimension functions change when passing to a scheme of finite type over another.

0AWL **Lemma 22.1.** *Let S be a Noetherian scheme and let $\omega_S^\bullet$ be a dualizing complex. Let X be a scheme of finite type over S and let $\omega_X^\bullet$ be the dualizing complex normalized relative to $\omega_S^\bullet$. If $x \in X$ is a closed point lying over a closed point s of S, then $\omega_{X,x}^\bullet$ is a normalized dualizing complex over $\mathcal{O}_{X,x}$ provided that $\omega_{S,s}^\bullet$ is a normalized dualizing complex over $\mathcal{O}_{S,s}$.*

Proof. We may replace X by an affine neighbourhood of x, hence we may and do assume that $f : X \to S$ is compactifyable. Then $\omega_X^\bullet = f^!\omega_S^\bullet$. We have to show that $R\operatorname{Hom}_{\mathcal{O}_{X,x}}(\kappa(x), \omega_{X,x}^\bullet)$ is sitting in degree 0. Let $i_x : x \to X$ denote the inclusion morphism which is a closed immersion as x is a closed point. Hence $R\operatorname{Hom}_{\mathcal{O}_{X,x}}(\kappa(x), \omega_{X,x}^\bullet)$ represents $i_x^!\omega_X^\bullet$ by Lemma 18.3. Consider the commutative diagram

$$\begin{tikzcd} x \arrow[r, "i_x"'] \arrow[d, "\pi"'] & X \arrow[d, "f"] \\ s \arrow[r, "i_s"] & S \end{tikzcd}$$

By Morphisms, Lemma 19.3 the extension $\kappa(s) \subset \kappa(x)$ is finite and hence π is a finite morphism. We conclude that

$$i_x^!\omega_X^\bullet = i_x^! f^!\omega_S^\bullet = \pi^! i_s^!\omega_S^\bullet$$

Thus if $\omega_{S,s}^\bullet$ is a normalized dualizing complex over $\mathcal{O}_{S,s}$, then $i_s^!\omega_S^\bullet = \kappa(s)[0]$ by the same reasoning as above. We have

$$R\pi_*(\pi^!(\kappa(s)[0])) = R\,\mathcal{H}om_{\mathcal{O}_s}(R\pi_*(\kappa(x)[0]), \kappa(s)[0]) = \widetilde{\operatorname{Hom}_{\kappa(s)}(\kappa(x), \kappa(s))}$$

The first equality by Lemma 3.6 applied with $L = \kappa(x)[0]$. The second equality holds because π_* is exact. Thus $\pi^!(\kappa(s)[0])$ is supported in degree 0 and we win. $\square$

0AWM **Lemma 22.2.** *Let S be a Noetherian scheme and let $\omega_S^\bullet$ be a dualizing complex. Let $f : X \to S$ be of finite type and let $\omega_X^\bullet$ be the dualizing complex normalized relative to $\omega_S^\bullet$. For all $x \in X$ we have*

$$\delta_X(x) - \delta_S(f(x)) = trdeg_{\kappa(f(x))}(\kappa(x))$$

where δ_S, resp. δ_X is the dimension function of $\omega_S^\bullet$, resp. $\omega_X^\bullet$, see Lemma 2.6.

Proof. We may replace X by an affine neighbourhood of x. Hence we may and do assume there is a compactification $X \subset \overline{X}$ over S. Then we may replace X by $\overline{X}$ and assume that X is proper over S. We may also assume X is connected by replacing X by the connected component of X containing x. Next, recall that both δ_X and the function $x \mapsto \delta_S(f(x)) + \operatorname{trdeg}_{\kappa(f(x))}(\kappa(x))$ are dimension functions on X, see Morphisms, Lemma 50.3 (and the fact that S is universally catenary by Lemma 2.6). By Topology, Lemma 20.3 we see that the difference is locally constant, hence constant as X is connected. Thus it suffices to prove equality in any point of X. By Properties, Lemma 5.9 the scheme X has a closed point x. Since $X \to S$ is proper the image s of x is closed in S. Thus we may apply Lemma 22.1 to conclude. $\square$

0BV5 **Lemma 22.3.** *Let S be a Noetherian scheme. Let $f : X \to Y$ be a morphism of compactifyable schemes over S. Let $x \in X$ with image $y \in Y$. Then*

$$H^i(f^!\mathcal{O}_Y)_x \neq 0 \Rightarrow -\dim_x(X_y) \leq i.$$

Proof. Since the statement is local on X we may assume X and Y are affine schemes. Write $X = \text{Spec}(A)$ and $Y = \text{Spec}(R)$. Then $f^!\mathcal{O}_Y$ corresponds to the relative dualizing complex $\omega_{A/R}^\bullet$ of Dualizing Complexes, Section 25 by Remark 18.4. Thus the lemma follows from Dualizing Complexes, Lemma 25.7. $\square$

0BV6 **Lemma 22.4.** *Let S be a Noetherian scheme. Let $f : X \to Y$ be a flat morphism of compactifyable schemes over S. Let $x \in X$ with image $y \in Y$. Then*

$$H^i(f^!\mathcal{O}_Y)_x \neq 0 \Rightarrow -\dim_x(X_y) \leq i \leq 0.$$

In fact, if all fibres of f have dimension $\leq d$, then $f^!\mathcal{O}_Y$ has tor-amplitude in $[-d, 0]$ as an object of $D(X, f^{-1}\mathcal{O}_Y)$.

Proof. Arguing exactly as in the proof of Lemma 22.3 this follows from Dualizing Complexes, Lemma 25.8. $\square$

0E9V **Lemma 22.5.** *Let S be a Noetherian scheme. Let $f : X \to Y$ be a flat morphism of compactifyable schemes over S. Let $x \in X$ with image $y \in Y$. Assume*

1. *$\mathcal{O}_{Y,y}$ is Cohen-Macaulay, and*
2. *$trdeg_{\kappa(f(\xi))}(\kappa(\xi)) \leq r$ for any generic point ξ of an irreducible component of X containing x.*

Then

$$H^i(f^!\mathcal{O}_Y)_x \neq 0 \Rightarrow -r \leq i$$

and the stalk $H^{-r}(f^!\mathcal{O}_Y)_x$ is (S_2) as an $\mathcal{O}_{X,x}$-module.

Proof. After replacing X by an open neighbourhood of x, we may assume every irreducible component of X passes through x. Then arguing exactly as in the proof of Lemma 22.3 this follows from Dualizing Complexes, Lemma 25.9. $\square$

0BV7 **Lemma 22.6.** *Let S be a Noetherian scheme. Let $f : X \to Y$ be a flat quasi-finite morphism of schemes of compactifyable schemes over S. Then*

$$f^!\mathcal{O}_Y = \omega_{X/Y}[0]$$

for some coherent $\mathcal{O}_X$-module $\omega_{X/Y}$ flat over Y.

Proof. Consequence of Lemma 22.4 and the fact that the cohomology sheaves of $f^!\mathcal{O}_Y$ are coherent by Lemma 18.5. $\square$

0BV8 **Lemma 22.7.** *Let S be a Noetherian scheme. Let $f : X \to Y$ be a Cohen-Macaulay morphism (More on Morphisms, Definition 20.1) of compactifyable schemes over S. Then*

$$f^!\mathcal{O}_Y = \omega_{X/Y}[d]$$

for some coherent $\mathcal{O}_X$-module $\omega_{X/Y}$ flat over Y where d is the locally constant function on X which gives the relative dimension of X over Y.

Proof. The relative dimension d is well defined and locally constant by Morphisms, Lemma 28.4. The cohomology sheaves of $f^!\mathcal{O}_Y$ are coherent by Lemma 18.5. We will get flatness of $\omega_{X/Y}$ from Lemma 22.4 if we can show the other cohomology sheaves of $f^!\mathcal{O}_Y$ are zero.

The question is local on X, hence we may assume X and Y are affine and the morphism has relative dimension d. If $d = 0$, then the result follows directly from Lemma 22.6. If $d > 0$, then we may assume there is a factorization

$$X \xrightarrow{g} \mathbf{A}^d_Y \xrightarrow{p} Y$$

with g quasi-finite and flat, see More on Morphisms, Lemma 20.8. Then $f^! = g^! \circ p^!$. By Lemma 18.2 we see that $p^! \mathcal{O}_Y \cong \mathcal{O}_{\mathbf{A}^d_Y}[-d]$. We conclude by the case $d = 0$. □

0BV9 **Remark 22.8.** Let S be a Noetherian scheme endowed with a dualizing complex $\omega_S^\bullet$. In this case Lemmas 22.3, 22.4, 22.6, and 22.7 are true for any morphism $f : X \to Y$ of finite type schemes over S but with $f^!$ replaced by $f^!_{new}$. This is clear because in each case the proof reduces immediately to the affine case and then $f^! = f^!_{new}$ by Lemma 21.9.

23. Dualizing modules

0AWH This section is a continuation of Dualizing Complexes, Section 19.

Let X be a Noetherian scheme and let $\omega_X^\bullet$ be a dualizing complex. Let $n \in \mathbf{Z}$ be the smallest integer such that $H^n(\omega_X^\bullet)$ is nonzero. In other words, $-n$ is the maximal value of the dimension function associated to $\omega_X^\bullet$ (Lemma 2.6). Sometimes $H^n(\omega_X^\bullet)$ is called a *dualizing module* or *dualizing sheaf* for X and then it is often denoted by ω_X. We will say "let ω_X be a dualizing module" to indicate the above.

Care has to be taken when using dualizing modules ω_X on Noetherian schemes X:

(1) the integer n may change when passing from X to an open U of X and then it won't be true that $\omega_X|_U = \omega_U$,
(2) the dualizing complex isn't unique; the dualizing module is only unique up to tensoring by an invertible module.

The second problem will often be irrelevant because we will work with X of finite type over a base change S which is endowed with a fixed dualizing complex $\omega_S^\bullet$ and $\omega_X^\bullet$ will be the dualizing complex normalized relative to $\omega_S^\bullet$. The first problem will not occur if X is equidimensional, more precisely, if the dimension function associated to $\omega_X^\bullet$ (Lemma 2.6) maps every generic point of X to the same integer.

0AWI **Example 23.1.** Say $S = \mathrm{Spec}(A)$ with $(A, \mathfrak{m}, \kappa)$ a local Noetherian ring, and $\omega_S^\bullet$ corresponds to a normalized dualizing complex $\omega_A^\bullet$. Then if $f : X \to S$ is proper over S and $\omega_X^\bullet = f^! \omega_S^\bullet$ the coherent sheaf

$$\omega_X = H^{-\dim(X)}(\omega_X^\bullet)$$

is a dualizing module and is often called the dualizing module of X (with S and $\omega_S^\bullet$ being understood). We will see that this has good properties.

0AWJ **Example 23.2.** Say X is an equidimensional scheme of finite type over a field k. Then it is customary to take $\omega_X^\bullet$ the dualizing complex normalized relative to $k[0]$ and to refer to

$$\omega_X = H^{-\dim(X)}(\omega_X^\bullet)$$

as the dualizing module of X. If X is compactifyable over k, then this is a special case of Example 23.1, i.e., $\omega_X^\bullet = f^! \mathcal{O}_{\mathrm{Spec}(k)}$ where $f : X \to \mathrm{Spec}(k)$ is the structure morphism (follows from Lemma 21.9).

0AWK **Lemma 23.3.** *Let X be a connected Noetherian scheme and let ω_X be a dualizing module on X. The support of ω_X is the union of the irreducible components of maximal dimension with respect to any dimension function and ω_X is a coherent $\mathcal{O}_X$-module having property (S_2).*

Proof. By our conventions discussed above there exists a dualizing complex $\omega_X^\bullet$ such that ω_X is the leftmost nonvanishing cohomology sheaf. Since X is connected, any two dimension functions differ by a constant (Topology, Lemma 20.3). Hence we may use the dimension function associated to $\omega_X^\bullet$ (Lemma 2.6). With these remarks in place, the lemma now follows from Dualizing Complexes, Lemma 17.5 and the definitions (in particular Cohomology of Schemes, Definition 11.1). $\square$

0AWN **Lemma 23.4.** *Let X/A with $\omega_X^\bullet$ and ω_X be as in Example 23.1. Then*

(1) $H^i(\omega_X^\bullet) \neq 0 \Rightarrow i \in \{-\dim(X), \ldots, 0\}$,
(2) *the dimension of the support of $H^i(\omega_X^\bullet)$ is at most $-i$,*
(3) *Supp(ω_X) is the union of the components of dimension $\dim(X)$, and*
(4) *ω_X has property (S_2).*

Proof. Let δ_X and δ_S be the dimension functions associated to $\omega_X^\bullet$ and $\omega_S^\bullet$ as in Lemma 22.2. As X is proper over A, every closed subscheme of X contains a closed point x which maps to the closed point $s \in S$ and $\delta_X(x) = \delta_S(s) = 0$. Hence $\delta_X(\xi) = \dim(\overline{\{\xi\}})$ for any point $\xi \in X$. Hence we can check each of the statements of the lemma by looking at what happens over $\mathrm{Spec}(\mathcal{O}_{X,x})$ in which case the result follows from Dualizing Complexes, Lemmas 16.5 and 17.5. Some details omitted. The last two statements can also be deduced from Lemma 23.3. $\square$

0AWP **Lemma 23.5.** *Let X/A with dualizing module ω_X be as in Example 23.1. Let $d = \dim(X_s)$ be the dimension of the closed fibre. If $\dim(X) = d + \dim(A)$, then the dualizing module ω_X represents the functor*

$$\mathcal{F} \longmapsto \mathrm{Hom}_A(H^d(X, \mathcal{F}), \omega_A)$$

on the category of coherent $\mathcal{O}_X$-modules.

Proof. We have

$$\begin{aligned}
\mathrm{Hom}_X(\mathcal{F}, \omega_X) &= \mathrm{Ext}_X^{-\dim(X)}(\mathcal{F}, \omega_X^\bullet) \\
&= \mathrm{Hom}_X(\mathcal{F}[\dim(X)], \omega_X^\bullet) \\
&= \mathrm{Hom}_X(\mathcal{F}[\dim(X)], f^!(\omega_A^\bullet)) \\
&= \mathrm{Hom}_S(Rf_*\mathcal{F}[\dim(X)], \omega_A^\bullet) \\
&= \mathrm{Hom}_A(H^d(X, \mathcal{F}), \omega_A)
\end{aligned}$$

The first equality because $H^i(\omega_X^\bullet) = 0$ for $i < -\dim(X)$, see Lemma 23.4 and Derived Categories, Lemma 27.3. The second equality is follows from the definition of Ext groups. The third equality is our choice of $\omega_X^\bullet$. The fourth equality holds because $f^!$ is the right adjoint of Lemma 3.1 for f, see Section 20. The final equality holds because $R^if_*\mathcal{F}$ is zero for $i > d$ (Cohomology of Schemes, Lemma 20.9) and $H^j(\omega_A^\bullet)$ is zero for $j < -\dim(A)$. $\square$

24. Cohen-Macaulay schemes

0AWQ This section is the continuation of Dualizing Complexes, Section 20. Duality takes a particularly simple form for Cohen-Macaulay schemes.

0AWT **Lemma 24.1.** *Let X be a locally Noetherian scheme with dualizing complex $\omega_X^\bullet$.*

1. *X is Cohen-Macaulay $\Leftrightarrow$ $\omega_X^\bullet$ locally has a unique nonzero cohomology sheaf,*
2. *$\mathcal{O}_{X,x}$ is Cohen-Macaulay $\Leftrightarrow$ $\omega_{X,x}^\bullet$ has a unique nonzero cohomology,*
3. *$U = \{x \in X \mid \mathcal{O}_{X,x} \text{ is Cohen-Macaulay}\}$ is open and Cohen-Macaulay.*

If X is connected and Cohen-Macaulay, then there is an integer n and a coherent Cohen-Macaulay $\mathcal{O}_X$-module ω_X such that $\omega_X^\bullet = \omega_X[-n]$.

Proof. By definition and Dualizing Complexes, Lemma 15.6 for every $x \in X$ the complex $\omega_{X,x}^\bullet$ is a dualizing complex over $\mathcal{O}_{X,x}$. By Dualizing Complexes, Lemma 20.2 we see that (2) holds.

To see (3) assume that $\mathcal{O}_{X,x}$ is Cohen-Macaulay. Let n_x be the unique integer such that $H^{n_x}(\omega_{X,x}^\bullet)$ is nonzero. For an affine neighbourhood $V \subset X$ of x we have $\omega_X^\bullet|_V$ is in $D^b_{Coh}(\mathcal{O}_V)$ hence there are finitely many nonzero coherent modules $H^i(\omega_X^\bullet)|_V$. Thus after shrinking V we may assume only H^{n_x} is nonzero, see Modules, Lemma 9.5. In this way we see that $\mathcal{O}_{X,v}$ is Cohen-Macaulay for every $v \in V$. This proves that U is open as well as a Cohen-Macaulay scheme.

Proof of (1). The implication $\Leftarrow$ follows from (2). The implication $\Rightarrow$ follows from the discussion in the previous paragraph, where we showed that if $\mathcal{O}_{X,x}$ is Cohen-Macaulay, then in a neighbourhood of x the complex $\omega_X^\bullet$ has only one nonzero cohomology sheaf.

Assume X is connected and Cohen-Macaulay. The above shows that the map $x \mapsto n_x$ is locally constant. Since X is connected it is constant, say equal to n. Setting $\omega_X = H^n(\omega_X^\bullet)$ we see that the lemma holds because ω_X is Cohen-Macaulay by Dualizing Complexes, Lemma 20.2 (and Cohomology of Schemes, Definition 11.4). □

0AWU **Lemma 24.2.** *Let X be a locally Noetherian scheme. If there exists a coherent sheaf ω_X such that $\omega_X[0]$ is a dualizing complex on X, then X is a Cohen-Macaulay scheme.*

Proof. This follows immediately from Dualizing Complexes, Lemma 20.3 and our definitions. □

0C0Z **Lemma 24.3.** *Let S be a Noetherian scheme. Let $f : X \to Y$ be a flat morphism of compactifyable schemes over S. Let $x \in X$. The following are equivalent*

1. *f is Cohen-Macaulay at x,*
2. *$f^!\mathcal{O}_Y$ has a unique nonzero cohomology sheaf in a neighbourhood of x.*

Proof. One direction of the lemma follows from Lemma 22.7. To prove the converse, we may assume $f^!\mathcal{O}_Y$ has a unique nonzero cohomology sheaf. Let $y = f(x)$. Let $\xi_1, \ldots, \xi_n \in X_y$ be the generic points of the fibre X_y specializing to x. Let $d_1, \ldots, d_n$ be the dimensions of the corresponding irreducible components of X_y. The morphism $f : X \to Y$ is Cohen-Macaulay at η_i by More on Morphisms, Lemma 20.7. Hence by Lemma 22.7 we see that $d_1 = \ldots = d_n$. If d denotes the common value, then $d = \dim_x(X_y)$. After shrinking X we may assume all fibres have dimension at most d (Morphisms, Lemma 27.4). Then the only nonzero cohomology sheaf $\omega = H^{-d}(f^!\mathcal{O}_Y)$ is flat over Y by Lemma 22.4. Hence, if $h : X_y \to X$ denotes the canonical morphism, then $Lh^*(f^!\mathcal{O}_Y) = Lh^*(\omega[d]) = (h^*\omega)[d]$ by Derived Categories of Schemes, Lemma 21.6. Thus $h^*\omega[d]$ is the dualizing complex of X_y by Lemma 19.3. Hence X_y is Cohen-Macaulay by Lemma 24.1. This proves f is Cohen-Macaulay at x as desired. □

0C10 **Remark 24.4.** Let S be a Noetherian scheme. Let $f : X \to Y$ be a Cohen-Macaulay morphism of relative dimension d of compactifyable schemes over S. Let $\omega_{X/Y} = H^{-d}(f^!\mathcal{O}_Y)$ be the unique nonzero cohomology sheaf of $f^!\mathcal{O}_Y$, see Lemma 22.7. Then there is a canonical isomorphism

$$f^!K = Lf^*K \otimes^{\mathbf{L}}_{\mathcal{O}_X} \omega_{X/Y}[d]$$

for $K \in D^+_{QCoh}(\mathcal{O}_Y)$, see Lemma 18.8. In particular, if S has a dualizing complex $\omega_S^\bullet$, $\omega_Y^\bullet = (Y \to S)^!\omega_S^\bullet$, and $\omega_X^\bullet = (X \to S)^!\omega_S^\bullet$ then we have

$$\omega_X^\bullet = Lf^*\omega_Y^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_X} \omega_{X/Y}[d]$$

Thus if further X and Y are connected and Cohen-Macaulay and if ω_Y and ω_X denote the unique nonzero cohomology sheaves of $\omega_Y^\bullet$ and $\omega_X^\bullet$, then we have

$$\omega_X = f^*\omega_Y \otimes_{\mathcal{O}_X} \omega_{X/Y}.$$

Similar results hold for X and Y not necessarily compactifyable over S with dualizing complexes normalized with respect to $\omega_S^\bullet$ as in Section 21.

25. Gorenstein schemes

0AWV This section is the continuation of Dualizing Complexes, Section 21.

0AWW **Definition 25.1.** Let X be a scheme. We say X is *Gorenstein* if X is locally Noetherian and $\mathcal{O}_{X,x}$ is Gorenstein for all $x \in X$.

This definition makes sense because a Noetherian ring is said to be Gorenstein if and only if all of its local rings are Gorenstein, see Dualizing Complexes, Definition 21.1.

0C00 **Lemma 25.2.** *A Gorenstein scheme is Cohen-Macaulay.*

Proof. Looking affine locally this follows from the corresponding result in algebra, namely Dualizing Complexes, Lemma 21.2. □

0DWG **Lemma 25.3.** *A regular scheme is Gorenstein.*

Proof. Looking affine locally this follows from the corresponding result in algebra, namely Dualizing Complexes, Lemma 21.3. □

0BFQ **Lemma 25.4.** *Let X be a locally Noetherian scheme.*

(1) *If X has a dualizing complex $\omega_X^\bullet$, then*
   (a) *X is Gorenstein $\Leftrightarrow$ $\omega_X^\bullet$ is an invertible object of $D(\mathcal{O}_X)$,*
   (b) *$\mathcal{O}_{X,x}$ is Gorenstein $\Leftrightarrow$ $\omega_{X,x}^\bullet$ is an invertible object of $D(\mathcal{O}_{X,x})$,*
   (c) *$U = \{x \in X \mid \mathcal{O}_{X,x}$ is Gorenstein$\}$ is an open Gorenstein subscheme.*
(2) *If X is Gorenstein, then X has a dualizing complex if and only if $\mathcal{O}_X[0]$ is a dualizing complex.*

Proof. Looking affine locally this follows from the corresponding result in algebra, namely Dualizing Complexes, Lemma 21.4. $\square$

0BVA **Lemma 25.5.** *If $f : Y \to X$ is a local complete intersection morphism with X a Gorenstein scheme, then Y is Gorenstein.*

Proof. By More on Morphisms, Lemma 52.5 it suffices to prove the corresponding statement about ring maps. This is Dualizing Complexes, Lemma 21.7. $\square$

0C01 **Lemma 25.6.** *The property $\mathcal{P}(S) =$"S is Gorenstein" is local in the syntomic topology.*

Proof. Let $\{S_i \to S\}$ be a syntomic covering. The scheme S is locally Noetherian if and only if each S_i is Noetherian, see Descent, Lemma 13.1. Thus we may now assume S and S_i are locally Noetherian. If S is Gorenstein, then each S_i is Gorenstein by Lemma 25.5. Conversely, if each S_i is Gorenstein, then for each point $s \in S$ we can pick i and $t \in S_i$ mapping to s. Then $\mathcal{O}_{S,s} \to \mathcal{O}_{S_i,t}$ is a flat local ring homomorphism with $\mathcal{O}_{S_i,t}$ Gorenstein. Hence $\mathcal{O}_{S,s}$ is Gorenstein by Dualizing Complexes, Lemma 21.8. $\square$

26. Gorenstein morphisms

0C02 This section is one in a series. The corresponding sections for normal morphisms, regular morphisms, and Cohen-Macaulay morphisms can be found in More on Morphisms, Sections 18, 19, and 20.

The following lemma says that it does not make sense to define geometrically Gorenstein schemes, since these would be the same as Gorenstein schemes.

0C03 **Lemma 26.1.** *Let X be a locally Noetherian scheme over the field k. Let $k \subset k'$ be a finitely generated field extension. Let $x \in X$ be a point, and let $x' \in X_{k'}$ be a point lying over x. Then we have*

$$\mathcal{O}_{X,x} \text{ is Gorenstein} \Leftrightarrow \mathcal{O}_{X_{k'},x'} \text{ is Gorenstein}$$

If X is locally of finite type over k, the same holds for any field extension $k \subset k'$.

Proof. In both cases the ring map $\mathcal{O}_{X,x} \to \mathcal{O}_{X_{k'},x'}$ is a faithfully flat local homomorphism of Noetherian local rings. Thus if $\mathcal{O}_{X_{k'},x'}$ is Gorenstein, then so is $\mathcal{O}_{X,x}$ by Dualizing Complexes, Lemma 21.8. To go up, we use Dualizing Complexes, Lemma 21.8 as well. Thus we have to show that

$$\mathcal{O}_{X_{k'},x'}/\mathfrak{m}_x\mathcal{O}_{X_{k'},x'} = \kappa(x) \otimes_k k'$$

is Gorenstein. Note that in the first case $k \to k'$ is finitely generated and in the second case $k \to \kappa(x)$ is finitely generated. Hence this follows as property (A) holds for Gorenstein, see Dualizing Complexes, Lemma 23.1. $\square$

The lemma above guarantees that the following is the correct definition of Gorenstein morphisms.

0C04 **Definition 26.2.** Let $f : X \to Y$ be a morphism of schemes. Assume that all the fibres X_y are locally Noetherian schemes.

(1) Let $x \in X$, and $y = f(x)$. We say that f is *Gorenstein at x* if f is flat at x, and the local ring of the scheme X_y at x is Gorenstein.
(2) We say f is a *Gorenstein morphism* if f is Gorenstein at every point of X.

Here is a translation.

0C05 **Lemma 26.3.** *Let $f : X \to Y$ be a morphism of schemes. Assume all fibres of f are locally Noetherian. The following are equivalent*

(1) *f is Gorenstein, and*
(2) *f is flat and its fibres are Gorenstein schemes.*

Proof. This follows directly from the definitions. $\square$

0C06 **Lemma 26.4.** *A Gorenstein morphism is Cohen-Macaulay.*

Proof. Follows from Lemma 25.2 and the definitions. $\square$

0C15 **Lemma 26.5.** *A syntomic morphism is Gorenstein. Equivalently a flat local complete intersection morphism is Gorenstein.*

Proof. Recall that a syntomic morphism is flat and its fibres are local complete intersections over fields, see Morphisms, Lemma 29.11. Since a local complete intersection over a field is a Gorenstein scheme by Lemma 25.5 we conclude. The properties "syntomic" and "flat and local complete intersection morphism" are equivalent by More on Morphisms, Lemma 52.8. $\square$

0C11 **Lemma 26.6.** *Let $f : X \to Y$ and $g : Y \to Z$ be morphisms. Assume that the fibres X_y, Y_z and X_z of f, g, and $g \circ f$ are locally Noetherian.*

(1) *If f is Gorenstein at x and g is Gorenstein at $f(x)$, then $g \circ f$ is Gorenstein at x.*
(2) *If f and g are Gorenstein, then $g \circ f$ is Gorenstein.*
(3) *If $g \circ f$ is Gorenstein at x and f is flat at x, then f is Gorenstein at x and g is Gorenstein at $f(x)$.*
(4) *If $f \circ g$ is Gorenstein and f is flat, then f is Gorenstein and g is Gorenstein at every point in the image of f.*

Proof. After translating into algebra this follows from Dualizing Complexes, Lemma 21.8. $\square$

0C12 **Lemma 26.7.** *Let $f : X \to Y$ be a flat morphism of locally Noetherian schemes. If X is Gorenstein, then f is Gorenstein and $\mathcal{O}_{Y,f(x)}$ is Gorenstein for all $x \in X$.*

Proof. After translating into algebra this follows from Dualizing Complexes, Lemma 21.8. $\square$

0C07 **Lemma 26.8.** *Let $f : X \to Y$ be a morphism of schemes. Assume that all the fibres X_y are locally Noetherian schemes. Let $Y' \to Y$ be locally of finite type. Let $f' : X' \to Y'$ be the base change of f. Let $x' \in X'$ be a point with image $x \in X$.*

(1) *If f is Gorenstein at x, then $f' : X' \to Y'$ is Gorenstein at x'.*
(2) *If f is flat and x and f' is Gorenstein at x', then f is Gorenstein at x.*
(3) *If $Y' \to Y$ is flat at $f'(x')$ and f' is Gorenstein at x', then f is Gorenstein at x.*

Proof. Note that the assumption on $Y' \to Y$ implies that for $y' \in Y'$ mapping to $y \in Y$ the field extension $\kappa(y) \subset \kappa(y')$ is finitely generated. Hence also all the fibres $X'_{y'} = (X_y)_{\kappa(y')}$ are locally Noetherian, see Varieties, Lemma 11.1. Thus the lemma makes sense. Set $y' = f'(x')$ and $y = f(x)$. Hence we get the following commutative diagram of local rings

$$\begin{tikzcd} \mathcal{O}_{X',x'} & \mathcal{O}_{X,x} \arrow[l] \\ \mathcal{O}_{Y',y'} \arrow[u] & \mathcal{O}_{Y,y} \arrow[l] \arrow[u] \end{tikzcd}$$

where the upper left corner is a localization of the tensor product of the upper right and lower left corners over the lower right corner.

Assume f is Gorenstein at x. The flatness of $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ implies the flatness of $\mathcal{O}_{Y',y'} \to \mathcal{O}_{X',x'}$, see Algebra, Lemma 99.1. The fact that $\mathcal{O}_{X,x}/\mathfrak{m}_y\mathcal{O}_{X,x}$ is Gorenstein implies that $\mathcal{O}_{X',x'}/\mathfrak{m}_{y'}\mathcal{O}_{X',x'}$ is Gorenstein, see Lemma 26.1. Hence we see that f' is Gorenstein at x'.

Assume f is flat at x and f' is Gorenstein at x'. The fact that $\mathcal{O}_{X',x'}/\mathfrak{m}_{y'}\mathcal{O}_{X',x'}$ is Gorenstein implies that $\mathcal{O}_{X,x}/\mathfrak{m}_y\mathcal{O}_{X,x}$ is Gorenstein, see Lemma 26.1. Hence we see that f is Gorenstein at x.

Assume $Y' \to Y$ is flat at y' and f' is Gorenstein at x'. The flatness of $\mathcal{O}_{Y',y'} \to \mathcal{O}_{X',x'}$ and $\mathcal{O}_{Y,y} \to \mathcal{O}_{Y',y'}$ implies the flatness of $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$, see Algebra, Lemma 99.1. The fact that $\mathcal{O}_{X',x'}/\mathfrak{m}_{y'}\mathcal{O}_{X',x'}$ is Gorenstein implies that $\mathcal{O}_{X,x}/\mathfrak{m}_y\mathcal{O}_{X,x}$ is Gorenstein, see Lemma 26.1. Hence we see that f is Gorenstein at x. □

0E0Q **Lemma 26.9.** *Let $f : X \to Y$ be a morphism of schemes which is flat and locally of finite type. Then formation of the set $\{x \in X \mid f \text{ is Gorenstein at } x\}$ commutes with arbitrary base change.*

Proof. The assumption implies any fibre of f is locally of finite type over a field and hence locally Noetherian and the same is true for any base change. Thus the statement makes sense. Looking at fibres we reduce to the following problem: let X be a scheme locally of finite type over a field k, let K/k be a field extension, and let $x_K \in X_K$ be a point with image $x \in X$. Problem: show that $\mathcal{O}_{X_K,x_K}$ is Gorenstein if and only if $\mathcal{O}_{X,x}$ is Gorenstein.

The problem can be solved using a bit of algebra as follows. Choose an affine open $\text{Spec}(A) \subset X$ containing x. Say x corresponds to $\mathfrak{p} \subset A$. With $A_K = A \otimes_k K$ we see that $\text{Spec}(A_K) \subset X_K$ contains x_K. Say x_K corresponds to $\mathfrak{p}_K \subset A_K$. Let $\omega_A^\bullet$ be a dualizing complex for A. By Dualizing Complexes, Lemma 25.3 $\omega_A^\bullet \otimes_A A_K$ is a dualizing complex for A_K. Now we are done because $A_\mathfrak{p} \to (A_K)_{\mathfrak{p}_K}$ is a flat local homomorphism of Noetherian rings and hence $(\omega_A^\bullet)_\mathfrak{p}$ is an invertible object of $D(A_\mathfrak{p})$ if and only if $(\omega_A^\bullet)_\mathfrak{p} \otimes_{A_\mathfrak{p}} (A_K)_{\mathfrak{p}_K}$ is an invertible object of $D((A_K)_{\mathfrak{p}_K})$. Some details omitted; hint: look at cohomology modules. □

0C08 **Lemma 26.10.** *Let S be a Noetherian scheme. Let $f : X \to Y$ be a flat morphism of compactifyable schemes over S. Let $x \in X$. The following are equivalent*

1. *f is Gorenstein at x,*
2. *$f^!\mathcal{O}_X$ is isomorphic to an invertible object in a neighbourhood of x.*

In particular, the set of points where f is Gorenstein is open in X.

Proof. Set $\omega^\bullet = f^!\mathcal{O}_Y$. By Lemma 19.3 this is a bounded complex with coherent cohomology sheaves whose derived restriction $Lh^*\omega^\bullet$ to the fibre X_y is a dualizing complex on X_y. Denote $i : x \to X_y$ the inclusion of a point. Then the following are equivalent

(1) f is Gorenstein at x,
(2) $\mathcal{O}_{X_y,x}$ is Gorenstein,
(3) $Lh^*\omega^\bullet$ is invertible in a neighbourhood of x,
(4) $Li^*Lh^*\omega^\bullet$ has exactly one nonzero cohomology of dimension 1 over $\kappa(x)$,
(5) $L(h \circ i)^*\omega^\bullet$ has exactly one nonzero cohomology of dimension 1 over $\kappa(x)$,
(6) $\omega^\bullet$ is invertible in a neighbourhood of x.

The equivalence of (1) and (2) is by definition (as f is flat). The equivalence of (2) and (3) follows from Lemma 25.4. The equivalence of (3) and (4) follows from More on Algebra, Lemma 71.5. The equivalence of (4) and (5) holds because $Li^*Lh^* = L(h \circ i)^*$. The equivalence of (5) and (6) holds by More on Algebra, Lemma 71.5. Thus the lemma is clear. □

0C09 **Lemma 26.11.** *Let $f : X \to S$ be a morphism of schemes which is flat and locally of finite presentation. Let $x \in X$ with image $s \in S$. Set $d = \dim_x(X_s)$. The following are equivalent*

(1) f is Gorenstein at x,
(2) there exists an open neighbourhood $U \subset X$ of x and a locally quasi-finite morphism $U \to \mathbf{A}^d_S$ over S which is Gorenstein at x,
(3) there exists an open neighbourhood $U \subset X$ of x and a locally quasi-finite Gorenstein morphism $U \to \mathbf{A}^d_S$ over S,
(4) for any S-morphism $g : U \to \mathbf{A}^d_S$ of an open neighbourhood $U \subset X$ of x we have: g is quasi-finite at $x \Rightarrow g$ is Gorenstein at x.

In particular, the set of points where f is Gorenstein is open in X.

Proof. Choose affine open $U = \mathrm{Spec}(A) \subset X$ with $x \in U$ and $V = \mathrm{Spec}(R) \subset S$ with $f(U) \subset V$. Then $R \to A$ is a flat ring map of finite presentation. Let $\mathfrak{p} \subset A$ be the prime ideal corresponding to x. After replacing A by a principal localization we may assume there exists a quasi-finite map $R[x_1, \ldots, x_d] \to A$, see Algebra, Lemma 124.2. Thus there exists at least one pair (U, g) consisting of an open neighbourhood $U \subset X$ of x and a locally[6] quasi-finite morphism $g : U \to \mathbf{A}^d_S$.

Having said this, the lemma translates into the following algebra problem (translation omitted). Given $R \to A$ flat and of finite presentation, a prime $\mathfrak{p} \subset A$ and $\varphi : R[x_1, \ldots, x_d] \to A$ quasi-finite at $\mathfrak{p}$ the following are equivalent

(a) $\mathrm{Spec}(\varphi)$ is Gorenstein at $\mathfrak{p}$, and
(b) $\mathrm{Spec}(A) \to \mathrm{Spec}(R)$ is Gorenstein at $\mathfrak{p}$.
(c) $\mathrm{Spec}(A) \to \mathrm{Spec}(R)$ is Gorenstein in an open neighbourhood of $\mathfrak{p}$.

In each case $R[x_1, \ldots, x_n] \to A$ is flat at $\mathfrak{p}$ hence by openness of flatness (Algebra, Theorem 128.4), we may assume $R[x_1, \ldots, x_n] \to A$ is flat (replace A by a suitable principal localization). By Algebra, Lemma 162.1 there exists $R_0 \subset R$ and $R_0[x_1, \ldots, x_n] \to A_0$ such that R_0 is of finite type over $\mathbf{Z}$ and $R_0 \to A_0$ is of finite type and $R_0[x_1, \ldots, x_n] \to A_0$ is flat. Note that the set of points where a flat finite type morphism is Gorenstein commutes with base change by Lemma 26.8. In this way we reduce to the case where R is Noetherian.

[6] If S is quasi-separated, then g will be quasi-finite.

Thus we may assume X and S affine and that we have a factorization of f of the form

$$X \xrightarrow{g} \mathbf{A}^n_S \xrightarrow{p} S$$

with g flat and quasi-finite and S Noetherian. Then X and $\mathbf{A}^n_S$ are compactifyable over S and we have

$$f^!\mathcal{O}_S = g^!p^!\mathcal{O}_S = g^!\mathcal{O}_{\mathbf{A}^n_S}[n]$$

by know properties of upper shriek functors (Lemmas 17.2 and 18.2). Hence the equivalence of (a), (b), and (c) by Lemma 26.10. □

0C0A **Lemma 26.12.** *The property $\mathcal{P}(f) =$"the fibres of f are locally Noetherian and f is Gorenstein" is local in the fppf topology on the target and local in the syntomic topology on the source.*

Proof. We have $\mathcal{P}(f) = \mathcal{P}_1(f) \wedge \mathcal{P}_2(f)$ where $\mathcal{P}_1(f) =$"f is flat", and $\mathcal{P}_2(f) =$"the fibres of f are locally Noetherian and Gorenstein". We know that $\mathcal{P}_1$ is local in the fppf topology on the source and the target, see Descent, Lemmas 20.15 and 24.1. Thus we have to deal with $\mathcal{P}_2$.

Let $f : X \to Y$ be a morphism of schemes. Let $\{\varphi_i : Y_i \to Y\}_{i \in I}$ be an fppf covering of Y. Denote $f_i : X_i \to Y_i$ the base change of f by φ_i. Let $i \in I$ and let $y_i \in Y_i$ be a point. Set $y = \varphi_i(y_i)$. Note that

$$X_{i,y_i} = \mathrm{Spec}(\kappa(y_i)) \times_{\mathrm{Spec}(\kappa(y))} X_y.$$

and that $\kappa(y) \subset \kappa(y_i)$ is a finitely generated field extension. Hence if X_y is locally Noetherian, then X_{i,y_i} is locally Noetherian, see Varieties, Lemma 11.1. And if in addition X_y is Gorenstein, then X_{i,y_i} is Gorenstein, see Lemma 26.1. Thus $\mathcal{P}_2$ is fppf local on the target.

Let $\{X_i \to X\}$ be a syntomic covering of X. Let $y \in Y$. In this case $\{X_{i,y} \to X_y\}$ is a syntomic covering of the fibre. Hence the locality of $\mathcal{P}_2$ for the syntomic topology on the source follows from Lemma 25.6. □

27. More on dualizing complexes

0E4M Some lemmas which don't fit anywhere else very well.

0E4N **Lemma 27.1.** *Let $f : X \to Y$ be a morphism of locally Noetherian schemes. Assume*

1. *f is syntomic and surjective, or*
2. *f is a surjective flat local complete intersection morphism, or*
3. *f is a surjective Gorenstein morphism of finite type.*

Then $K \in D_{QCoh}(\mathcal{O}_Y)$ is a dualizing complex on Y if and only if Lf^*K is a dualizing complex on X.

Proof. Taking affine opens and using Derived Categories of Schemes, Lemma 3.5 this translates into Dualizing Complexes, Lemma 26.2. □

28. Relative dualizing complexes

0E2S For a proper, flat morphism of finite presentation we have a rigid relative dualizing complex, see Remark 12.5 and Lemma 12.7. For a compactifyable morphism $f : X \to Y$ of Noetherian schemes, we can consider $f^!\mathcal{O}_Y$. In this section we define relative dualizing complexes for morphisms which are flat and locally of finite presentation (but not necessarily quasi-separated or quasi-compact) between schemes (not necessarily locally Noetherian). We show such complexes exist, are unique up to unique isomorphism, and agree with the cases mentioned above. Before reading this section, please read Dualizing Complexes, Section 27.

0E2T **Definition 28.1.** Let $X \to S$ be a morphism of schemes which is flat and locally of finite presentation. Let $W \subset X \times_S X$ be any open such that the diagonal $\Delta_{X/S} : X \times_S X$ factors through a closed immersion $\Delta : X \to W$. A *relative dualizing complex* is a pair (K, ξ) consisting of an object $K \in D(\mathcal{O}_X)$ and a map

$$\xi : \Delta_*\mathcal{O}_X \longrightarrow L\text{pr}_1^*K|_W$$

in $D(\mathcal{O}_W)$ such that

(1) K is S-perfect (Derived Categories of Schemes, Definition 31.1), and
(2) ξ defines an isomorphism of $\Delta_*\mathcal{O}_X$ with $R\mathcal{H}om_{\mathcal{O}_W}(\Delta_*\mathcal{O}_X, L\text{pr}_1^*K|_W)$.

By Lemma 9.3 condition (2) is equivalent to the existence of an isomorphism

$$\mathcal{O}_X \longrightarrow R\mathcal{H}om(\mathcal{O}_X, L\text{pr}_1^*K|_W)$$

in $D(\mathcal{O}_X)$ whose pushforward via Δ is equal to ξ. Since $R\mathcal{H}om(\mathcal{O}_X, L\text{pr}_1^*K|_W)$ is independent of the choice of the open W, so is the category of pairs (K, ξ). If $X \to S$ is separated, then we can choose $W = X \times_S X$. We will reduce many of the arguments to the case of rings using the following lemma.

0E2U **Lemma 28.2.** *Let $X \to S$ be a morphism of schemes which is flat and locally of finite presentation. Let (K, ξ) be a relative dualizing complex. Then for any commutative diagram*

$$\begin{tikzcd} \text{Spec}(A) \arrow[r] \arrow[d] & X \arrow[d] \\ \text{Spec}(R) \arrow[r] & S \end{tikzcd}$$

whose horizontal arrows are open immersions, the restriction of K to $\text{Spec}(A)$ *corresponds via Derived Categories of Schemes, Lemma 3.5 to a relative dualizing complex for $R \to A$ in the sense of Dualizing Complexes, Definition 27.1.*

Proof. Since formation of $R\mathcal{H}om$ commutes with restrictions to opens we may as well assume $X = \text{Spec}(A)$ and $S = \text{Spec}(R)$. Observe that relatively perfect objects of $D(\mathcal{O}_X)$ are pseudo-coherent and hence are in $D_{QCoh}(\mathcal{O}_X)$ (Derived Categories of Schemes, Lemma 9.1). Thus the statement makes sense. Observe that taking Δ_*, $L\text{pr}_1^*$, and $R\mathcal{H}om$ is compatible with what happens on the algebraic side by Derived Categories of Schemes, Lemmas 3.7, 3.8, 9.8. For the last one we observe that $L\text{pr}_1^*K$ is S-perfect (hence bounded below) and that $\Delta_*\mathcal{O}_X$ is a pseudo-coherent object of $D(\mathcal{O}_W)$; translated into algebra this means that A is pseudo-coherent as an $A \otimes_R A$-module which follows from More on Algebra, Lemma 75.8 applied to $R \to A \otimes_R A \to A$. Thus we recover exactly the conditions in Dualizing Complexes, Definition 27.1. $\square$

0E2V **Lemma 28.3.** *Let $X \to S$ be a morphism of schemes which is flat and locally of finite presentation. Let (K, ξ) be a relative dualizing complex. Then $\mathcal{O}_X \to R\mathcal{H}om_{\mathcal{O}_X}(K, K)$ is an isomorphism.*

Proof. Looking affine locally this reduces using Lemma 28.2 to the algebraic case which is Dualizing Complexes, Lemma 27.5. $\square$

0E2W **Lemma 28.4.** *Let $X \to S$ be a morphism of schemes which is flat and locally of finite presentation. If (K, ξ) and (L, η) are two relative dualizing complexes on X/S, then there is a unique isomorphism $K \to L$ sending ξ to η.*

Proof. Let $U \subset X$ be an affine open mapping into an affine open of S. Then there is an isomorphism $K|_U \to L|_U$ by Lemma 28.2 and Dualizing Complexes, Lemma 27.2. The reader can reuse the argument of that lemma in the schemes case to obtain a proof in this case. We will instead use a glueing argument.

Suppose we have an isomorphism $\alpha : K \to L$. Then $\alpha(\xi) = u\eta$ for some invertible section $u \in H^0(W, \Delta_*\mathcal{O}_X) = H^0(X, \mathcal{O}_X)$. (Because both η and $\alpha(\xi)$ are generators of an invertible $\Delta_*\mathcal{O}_X$-module by assumption.) Hence after replacing α by $u^{-1}\alpha$ we see that $\alpha(\xi) = \eta$. Since the automorphism group of K is $H^0(X, \mathcal{O}_X^*)$ by Lemma 28.3 there is at most one such α.

Let $\mathcal{B}$ be the collection of affine opens of X which map into an affine open of S. For each $U \in \mathcal{B}$ we have a unique isomorphism $\alpha_U : K|_U \to L|_U$ mapping ξ to η by the discussion in the previous two paragraphs. Observe that $\text{Ext}^i(K|_U, K|_U) = 0$ for $i < 0$ and any open U of X by Lemma 28.3. By Cohomology, Lemma 39.2 applied to $\text{id} : X \to X$ we get a unique morphism $\alpha : K \to L$ agreeing with α_U for all $U \in \mathcal{B}$. Then α sends ξ to η as this is true locally. $\square$

0E2X **Lemma 28.5.** *Let $X \to S$ be a morphism of schemes which is flat and locally of finite presentation. There exists a relative dualizing complex (K, ξ).*

Proof. Let $\mathcal{B}$ be the collection of affine opens of X which map into an affine open of S. For each U we have a relative dualizing complex (K_U, ξ_U) for U over S. Namely, choose an affine open $V \subset S$ such that $U \to X \to S$ factors through V. Write $U = \text{Spec}(A)$ and $V = \text{Spec}(R)$. By Dualizing Complexes, Lemma 27.4 there exists a relative dualizing complex $K_A \in D(A)$ for $R \to A$. Arguing backwards through the proof of Lemma 28.2 this determines an V-perfect object $K_U \in D(\mathcal{O}_U)$ and a map

$$\xi : \Delta_*\mathcal{O}_U \to L\text{pr}_1^*K_U$$

in $D(\mathcal{O}_{U \times_V U})$. Since being V-perfect is the same as being S-perfect and since $U \times_V U = U \times_S U$ we find that (K_U, ξ_U) is as desired.

If $U' \subset U \subset X$ with $U', U \in \mathcal{B}$, then we have a unique isomorphism $\rho^U_{U'} : K_U|_{U'} \to K_{U'}$ in $D(\mathcal{O}_{U'})$ sending $\xi_U|_{U' \times_S U'}$ to $\xi_{U'}$ by Lemma 28.4 (note that trivially the restriction of a relative dualizing complex to an open is a relative dualizing complex). The uniqueness guarantees that $\rho^U_{U''} = \rho^V_{U''} \circ \rho^U_{U'}|_{U''}$ for $U'' \subset U' \subset U$ in $\mathcal{B}$. Observe that $\text{Ext}^i(K_U, K_U) = 0$ for $i < 0$ for $U \in \mathcal{B}$ by Lemma 28.3 applied to U/S and K_U. Thus the BBD glueing lemma (Cohomology, Theorem 39.8) tells us there is a unique solution, namely, an object $K \in D(\mathcal{O}_X)$ and isomorphisms $\rho_U : K|_U \to K_U$ such that we have $\rho^U_{U'} \circ \rho_U|_{U'} = \rho_{U'}$ for all $U' \subset U$, $U, U' \in \mathcal{B}$.

To finish the proof we have to construct the map

$$\xi : \Delta_*\mathcal{O}_X \longrightarrow L\text{pr}_1^*K|_W$$

in $D(\mathcal{O}_W)$ inducing an isomorphism from $\Delta_*\mathcal{O}_X$ to $R\mathcal{H}om_{\mathcal{O}_W}(\Delta_*\mathcal{O}_X, L\text{pr}_1^*K|_W)$. Since we may change W, we choose $W = \bigcup_{U \in \mathcal{B}} U \times_S U$. We can use ρ_U to get isomorphisms

$$R\mathcal{H}om_{\mathcal{O}_W}(\Delta_*\mathcal{O}_X, L\text{pr}_1^*K|_W)|_{U \times_S U} \xrightarrow{\rho_U} R\mathcal{H}om_{\mathcal{O}_{U \times_S U}}(\Delta_*\mathcal{O}_U, L\text{pr}_1^*K_U)$$

As W is covered by the opens $U \times_S U$ we conclude that the cohomology sheaves of $R\mathcal{H}om_{\mathcal{O}_W}(\Delta_*\mathcal{O}_X, L\text{pr}_1^*K|_W)$ are zero except in degree 0. Moreover, we obtain isomorphisms

$$H^0\left(U \times_S U, R\mathcal{H}om_{\mathcal{O}_W}(\Delta_*\mathcal{O}_X, L\text{pr}_1^*K|_W)\right) \xrightarrow{\rho_U} H^0\left((R\mathcal{H}om_{\mathcal{O}_{U \times_S U}}(\Delta_*\mathcal{O}_U, L\text{pr}_1^*K_U)\right)$$

Let τ_U in the LHS be an element mapping to ξ_U under this map. The compatibilities between $\rho^U_{U'}$, ξ_U, $\xi_{U'}$, ρ_U, and $\rho_{U'}$ for $U' \subset U \subset X$ open $U', U \in \mathcal{B}$ imply that $\tau_U|_{U' \times_S U'} = \tau_{U'}$. Thus we get a global section τ of the 0th cohomology sheaf $H^0(R\mathcal{H}om_{\mathcal{O}_W}(\Delta_*\mathcal{O}_X, L\text{pr}_1^*K|_W))$. Since the other cohomology sheaves of $R\mathcal{H}om_{\mathcal{O}_W}(\Delta_*\mathcal{O}_X, L\text{pr}_1^*K|_W)$ are zero, this global section τ determines a morphism ξ as desired. Since the restriction of ξ to $U \times_S U$ gives ξ_U, we see that it satisfies the final condition of Definition 28.1. □

0E2Y **Lemma 28.6.** *Consider a cartesian square*

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ S' \arrow[r, "g"] & S \end{tikzcd}$$

of schemes. Assume $X \to S$ is flat and locally of finite presentation. Let (K, ξ) be a relative dualizing complex for f. Set $K' = L(g')^*K$. Let ξ' be the derived base change of ξ (see proof). Then (K', ξ') is a relative dualizing complex for f'.

Proof. Consider the cartesian square

$$\begin{tikzcd} X' \arrow[rr] \arrow[d, "\Delta_{X'/S'}"] & & X \arrow[d, "\Delta_{X/S}"] \\ X' \times_{S'} X' \arrow[rr, "g' \times g'"] & & X \times_S X \end{tikzcd}$$

Choose $W \subset X \times_S X$ open such that $\Delta_{X/S}$ factors through a closed immersion $\Delta : X \to W$. Choose $W' \subset X' \times_{S'} X'$ open such that $\Delta_{X'/S'}$ factors through a closed immersion $\Delta' : X \to W'$ and such that $(g' \times g')(W') \subset W$. Let us still denote $g' \times g' : W' \to W$ the induced morphism. We have

$$L(g' \times g')^*\Delta_*\mathcal{O}_X = \Delta'_*\mathcal{O}_{X'} \quad \text{and} \quad L(g' \times g')^*L\text{pr}_1^*K|_W = L\text{pr}_1^*K'|_{W'}$$

The first equality holds because X and $X' \times_{S'} X'$ are tor independent over $X \times_S X$ (see for example More on Morphisms, Lemma 59.1). The second holds by transitivity of derived pullback (Cohomology, Lemma 28.2). Thus $\xi' = L(g' \times g')^*\xi$ can be viewed as a map

$$\xi' : \Delta'_*\mathcal{O}_{X'} \longrightarrow L\text{pr}_1^*K'|_{W'}$$

Having said this the proof of the lemma is straightforward. First, K' is S'-perfect by Derived Categories of Schemes, Lemma 31.6. To check that ξ' induces an isomorphism of $\Delta'_*\mathcal{O}_{X'}$ to $R\mathcal{H}om_{\mathcal{O}_{W'}}(\Delta'_*\mathcal{O}_{X'}, L\text{pr}_1^*K'|_{W'})$ we may work affine locally. By Lemma 28.2 we reduce to the corresponding statement in algebra which is proven in Dualizing Complexes, Lemma 27.4. ☐

0E2Z **Lemma 28.7.** *Let S be a quasi-compact and quasi-separated scheme. Let $f : X \to S$ be a proper, flat morphism of finite presentation. The relative dualizing complex $\omega_{X/S}^\bullet$ of Remark 12.5 together with (12.7.1) is a relative dualizing complex in the sense of Definition 28.1.*

Proof. In Lemma 12.6 we proved that $\omega_{X/S}^\bullet$ is S-perfect. Let c be the right adjoint of Lemma 3.1 for the diagonal $\Delta : X \to X \times_S X$. Then we can apply Δ_* to (12.7.1) to get an isomorphism

$$\Delta_*\mathcal{O}_X \to \Delta_*(c(L\text{pr}_1^*\omega_{X/S}^\bullet)) = R\mathcal{H}om_{\mathcal{O}_{X\times_S X}}(\Delta_*\mathcal{O}_X, L\text{pr}_1^*\omega_{X/S}^\bullet)$$

The equality holds by Lemmas 9.7 and 9.3. This finishes the proof. ☐

0E4P **Remark 28.8.** Let $X \to S$ be a morphism of schemes which is flat, proper, and of finite presentation. By Lemma 28.5 there exists a relative dualizing complex $(\omega_{X/S}^\bullet, \xi)$ in the sense of Definition 28.1. Consider any morphism $g : S' \to S$ where S' is quasi-compact and quasi-separated (for example an affine open of S). By Lemma 28.6 we see that $(L(g')^*\omega_{X/S}^\bullet, L(g')^*\xi)$ is a relative dualizing complex for the base change $f' : X' \to S'$ in the sense of Definition 28.1. Let $\omega_{X'/S'}^\bullet$ be the relative dualizing complex for $X' \to S'$ in the sense of Remark 12.5. Combining Lemmas 28.7 and 28.4 we see that there is a unique isomorphism

$$\omega_{X'/S'}^\bullet \longrightarrow L(g')^*\omega_{X/S}^\bullet$$

compatible with (12.7.1) and $L(g')^*\xi$. These isomorphisms are compatible with morphisms between quasi-compact and quasi-separated schemes over S and the base change isomorphisms of Lemma 12.4 (if we ever need this compatibility we will carefully state and prove it here).

0E9W **Lemma 28.9.** *Let S be a Noetherian scheme. Let $f : X \to Y$ be a flat morphism of compactifyable schemes over S. Then $f^!\mathcal{O}_Y$ is (the first component of) a relative dualizing complex for X over Y in the sense of Definition 28.1.*

Proof. By Lemma 18.9 we have that $f^!\mathcal{O}_Y$ is Y-perfect. As f is separated the diagonal $\Delta : X \to X \times_Y X$ is a closed immersion and $\Delta_*\Delta^!(-) = R\mathcal{H}om_{\mathcal{O}_{X\times_Y X}}(\mathcal{O}_X, -)$, see Lemmas 9.7 and 9.3. Hence to finish the proof it suffices to show $\Delta^!(L\text{pr}_1^*f^!(\mathcal{O}_Y)) \cong \mathcal{O}_X$ where $\text{pr}_1 : X \times_Y X \to X$ is the first projection. We have

$$\mathcal{O}_X = \Delta^!\text{pr}_1^!\mathcal{O}_X = \Delta^!\text{pr}_1^!L\text{pr}_2^*\mathcal{O}_Y = \Delta^!(L\text{pr}_1^*f^!\mathcal{O}_Y)$$

where $\text{pr}_2 : X \times_Y X \to X$ is the second projection and where we have used the base change isomorphism $\text{pr}_1^! \circ L\text{pr}_2^* = L\text{pr}_1^* \circ f^!$ of Lemma 19.1. ☐

0E30 **Lemma 28.10.** *Let $f : Y \to X$ and $X \to S$ be morphisms of schemes which are flat and of finite presentation. Let (K, ξ) and (M, η) be a relative dualizing complex for $X \to S$ and $Y \to X$. Set $E = M \otimes_{\mathcal{O}_Y}^{\mathbf{L}} Lf^*K$. Then (E, ζ) is a relative dualizing complex for $Y \to S$ for a suitable ζ.*

Proof. Using Lemma 28.2 and the algebraic version of this lemma (Dualizing Complexes, Lemma 27.6) we see that E is affine locally the first component of a relative dualizing complex. In particular we see that E is S-perfect since this may be checked affine locally, see Derived Categories of Schemes, Lemma 31.3.

Let us first prove the existence of ζ in case the morphisms $X \to S$ and $Y \to X$ are separated so that $\Delta_{X/S}$, $\Delta_{Y/X}$, and $\Delta_{Y/S}$ are closed immersions. Consider the following diagram

$$\begin{tikzcd} & & & Y \arrow[r, equal] & Y \arrow[d, "f"] \\ Y \arrow[r, "\Delta_{Y/X}"'] & Y \times_X Y \arrow[r, "\delta"] \arrow[d, "m"] \arrow[urr, "q"] & & Y \times_S Y \arrow[d, "f \times f"] \arrow[ur, "p"] & X \\ & X \arrow[rr, "\Delta_{X/S}"] & & X \times_S X \arrow[ur, "r"'] & \end{tikzcd}$$

where p, q, r are the first projections. By Lemma 9.4 we have

$$R\mathcal{H}om_{\mathcal{O}_{Y \times_S Y}}(\Delta_{Y/S,*}\mathcal{O}_Y, Lp^*E) = R\delta_*\left(R\mathcal{H}om_{\mathcal{O}_{Y \times_X Y}}(\Delta_{Y/X,*}\mathcal{O}_Y, R\mathcal{H}om(\mathcal{O}_{Y \times_X Y}, Lp^*E))\right)$$

By Lemma 10.3 we have

$$R\mathcal{H}om(\mathcal{O}_{Y \times_X Y}, Lp^*E) = R\mathcal{H}om(\mathcal{O}_{Y \times_X Y}, L(f \times f)^*Lr^*K) \otimes^{\mathbf{L}}_{\mathcal{O}_{Y \times_S Y}} Lq^*M$$

By Lemma 10.2 we have

$$R\mathcal{H}om(\mathcal{O}_{Y \times_X Y}, L(f \times f)^*Lr^*K) = Lm^*R\mathcal{H}om(\mathcal{O}_X, Lr^*K)$$

The last expression is isomorphic (via ξ) to $Lm^*\mathcal{O}_X = \mathcal{O}_{Y \times_X Y}$. Hence the expression preceding is isomorphic to Lq^*M. Hence

$$R\mathcal{H}om_{\mathcal{O}_{Y \times_S Y}}(\Delta_{Y/S,*}\mathcal{O}_Y, Lp^*E) = R\delta_*\left(R\mathcal{H}om_{\mathcal{O}_{Y \times_X Y}}(\Delta_{Y/X,*}\mathcal{O}_Y, Lq^*M)\right)$$

The material inside the parentheses is isomorphic to $\Delta_{Y/X,*} * \mathcal{O}_X$ via η. This finishes the proof in the separated case.

In the general case we choose an open $W \subset X \times_S X$ such that $\Delta_{X/S}$ factors through a closed immersion $\Delta : X \to W$ and we choose an open $V \subset Y \times_X Y$ such that $\Delta_{Y/X}$ factors through a closed immersion $\Delta' : Y \to V$. Finally, choose an open $W' \subset Y \times_S Y$ whose intersection with $Y \times_X Y$ gives V and which maps into W. Then we consider the diagram

$$\begin{tikzcd} & & Y \arrow[r, equal] & Y \arrow[d, "f"] \\ Y \arrow[r, "\Delta'"'] & V \arrow[r, "\delta"] \arrow[d, "m"] \arrow[ur, "q"] & W' \arrow[d, "f \times f"] \arrow[ur, "p"] & X \\ & X \arrow[r, "\Delta"] & W \arrow[ur, "r"'] & \end{tikzcd}$$

and we use exactly the same argument as before. $\square$

29. The fundamental class of an lci morphism

0E9X In this section we will use the computations made in Section 15. Thus our result will suffer from the same kind of non-uniqueness as we have in that section.

0E9Y **Lemma 29.1.** *Let X be a locally ringed space. Let*

$$\mathcal{E}_1 \xrightarrow{\alpha} \mathcal{E}_0 \to \mathcal{F} \to 0$$

be a short exact sequence of $\mathcal{O}_X$-modules. Assume $\mathcal{E}_1$ and $\mathcal{E}_0$ are locally free of ranks r_1, r_0. Then there is a canonical map

$$\wedge^{r_0 - r_1}\mathcal{F} \longrightarrow \wedge^{r_1}(\mathcal{E}_1^\vee) \otimes \wedge^{r_0}\mathcal{E}_0$$

which is an isomorphism on the stalk at $x \in X$ if and only if $\mathcal{F}$ is locally free of rank $r_0 - r_1$ in an open neighbourhood of x.

Proof. If $r_1 > r_0$ then $\wedge^{r_0 - r_1}\mathcal{F} = 0$ by convention and the unique map cannot be an isomorphism. Thus we may assume $r = r_0 - r_1 \geq 0$. Define the map by the formula

$$s_1 \wedge \ldots \wedge s_r \mapsto t_1^\vee \wedge \ldots \wedge t_{r_1}^\vee \otimes \alpha(t_1) \wedge \ldots \wedge \alpha(t_{r_1}) \wedge \tilde{s}_1 \wedge \ldots \wedge \tilde{s}_r$$

where $t_1, \ldots, t_{r_1}$ is a local basis for $\mathcal{E}_1$, correspondingly $t_1^\vee, \ldots, t_{r_1}^\vee$ is the dual basis for $\mathcal{E}_1^\vee$, and s_i' is a local lift of s_i to a section of $\mathcal{E}_0$. We omit the proof that this is well defined.

If $\mathcal{F}$ is locally free of rank r, then it is straightforward to verify that the map is an isomorphism. Conversely, assume the map is an isomorphism on stalks at x. Then $\wedge^r\mathcal{F}_x$ is invertible. This implies that $\mathcal{F}_x$ is generated by at most r elements. This can only happen if α has rank r modulo $\mathfrak{m}_x$, i.e., α has maximal rank modulo $\mathfrak{m}_x$. This implies that α has maximal rank in a neighbourhood of x and hence $\mathcal{F}$ is locally free of rank r in a neighbourhood as desired. □

0E9Z **Lemma 29.2.** *Let Y be a Noetherian scheme. Let $f : X \to Y$ be a local complete intersection morphism. Let r be the locally constant function on X such that $\omega_{Y/X} = H^{-r}(f^!\mathcal{O}_Y)$ is the unique nonzero cohomology sheaf of $f^!\mathcal{O}_Y$, see Lemma 18.10. Assume f factors as an immersion $X \to P$ followed by a proper smooth morphism $P \to Y$. Then there is a map*

$$\wedge^r\Omega_{X/Y} \longrightarrow \omega_{Y/X}$$

which is an isomorphism on the stalk at a point x if and only if f is smooth at x.

Proof. The assumption implies that X is compactifyable over Y hence $f^!$ is defined. Let $j : W \to P$ be an open subscheme such that $X \to P$ factors through a closed immersion $i : X \to W$. Moreover, we have $f^! = i^! \circ j^! \circ g^!$ where $g : P \to Y$ is the given morphism. We have $g^!\mathcal{O}_Y = \wedge^d\Omega_{P/Y}[d]$ by Lemma 15.7 where d is the locally constant function giving the relative dimension of P over Y. We have $j^! = j^*$. We have $i^!\mathcal{O}_W = \wedge^c\mathcal{N}[-c]$ where c is the codimension of X in W (a locally constant function on X) and where $\mathcal{N}$ is the normal sheaf of the Koszul-regular immersion i, see Lemma 15.6. Combining the above we find

$$f^!\mathcal{O}_Y = \left(\wedge^c\mathcal{N} \otimes_{\mathcal{O}_X} \wedge^d\Omega_{P/Y}|_X\right)[d - c]$$

where we have also used Lemma 18.8. Thus $r = d|_X - c$ as locally constant functions on X. The conormal sheaf of $X \to P$ is the module $\mathcal{I}/\mathcal{I}^2$ where $\mathcal{I} \subset \mathcal{O}_W$ is the ideal sheaf of i, see Morphisms, Section 30. Consider the canonical exact sequence

$$\mathcal{I}/\mathcal{I}^2 \to \Omega_{P/Y}|_X \to \Omega_{X/Y} \to 0$$

of Morphisms, Lemma 31.15. We obtain our map by an application of Lemma 29.1.

If f is smooth at x, then the map is an isomorphism by an application of Lemma 29.1 and the fact that $\Omega_{X/Y}$ is locally free at x of rank r. Conversely, assume that our map is an isomorphism on stalks at x. Then the lemma shows that $\Omega_{X/Y}$ is free of rank r after replacing X by an open neighbourhood of x. On the other hand, we may also assume that $X = \mathrm{Spec}(A)$ and $Y = \mathrm{Spec}(R)$ where $A = R[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$ and where $f_1, \ldots, f_m$ is a Koszul regular sequence (this follows from the definition of local complete intersection morphisms). Clearly this implies $r = n - m$. We conclude that the rank of the matrix of partials $\partial f_j/\partial x_i$ in the residue field at x is m. Thus after reordering the variables we may assume the determinant of $(\partial f_j/\partial x_i)_{1 \leq i,j \leq m}$ is invertible in an open neighbourhood of x. It follows that $R \to A$ is smooth at this point, see for example Algebra, Example 135.8. □

0EA0 **Lemma 29.3.** *Let $f : X \to Y$ be a morphism of schemes. Let $r \geq 0$. Assume*

1. *Y is Cohen-Macaulay (Properties, Definition 8.1),*
2. *f factors as $X \to P \to Y$ where the first morphism is an immersion and the second is smooth and proper,*
3. *if $x \in X$ and $\dim(\mathcal{O}_{X,x}) \leq 1$, then f is Koszul at x (More on Morphisms, Definition 52.2), and*
4. *if ξ is a generic point of an irreducible component of X, then we have $trdeg_{\kappa(f(\xi))}\kappa(\xi) = r$.*

Then with $\omega_{Y/X} = H^{-r}(f^!\mathcal{O}_Y)$ there is a map

$$\wedge^r \Omega_{X/Y} \longrightarrow \omega_{Y/X}$$

which is an isomorphism on the locus where f is smooth.

Proof. Let $U \subset X$ be the open subscheme over which f is a local complete intersection morphism. Since f has relative dimension r at all generic points by assumption (4) we see that the locally constant function of Lemma 29.2 is constant with value r and we obtain a map

$$\wedge^r \Omega_{X/Y}|_U = \wedge^r \Omega_{U/Y} \longrightarrow \omega_{U/Y} = \omega_{X/Y}|_U$$

which is an isomorphism in the smooth points of f (this locus is contained in U because a smooth morphism is a local complete intersection morphism). By Lemma 22.5 and the assumption that Y is Cohen-Macaulay the module $\omega_{X/Y}$ is (S_2). Since U contains all the points of codimension 1 by condition (3) and using Divisors, Lemma 5.11 we see that $j_*\omega_{U/Y} = \omega_{X/Y}$. Hence the map over U extends to X and the proof is complete. □

30. Other chapters

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