1. Introduction

This chapter studies relative duality for morphisms of schemes and the dualizing complex on a scheme. A reference is [Har66].

This is a chapter of the Stacks Project, version 37287dd8, compiled on Apr 13, 2019.
Duality for Noetherian rings were defined and studied in Dualizing Complexes, Section 15 ff. In this chapter we continue this by studying dualizing complexes on schemes, see Section 2.

The bulk of this chapter is devoted to studying the right adjoint of pushforward in the setting of derived categories of sheaves of modules with quasi-coherent cohomology sheaves. See Sections 3, 4, 5, 6, 7, 8, 9, 11, 13, 14, and 15. Here we follow the papers [Nee96], [LN07], [Lip09], and [Nee14].

We discuss the important and useful upper shriek functors \( f^! \) for separated morphisms of finite type between Noetherian schemes in Sections 16, 17, and 18 culminating in the overview Section 19.

In Section 20 we explain alternative theory of duality and dualizing complexes when working over a fixed locally Noetherian base endowed with a dualizing complex (this section corresponds to a remark in Hartshorne’s book).

In the remaining sections we give a few applications.

This chapter is continued by the chapter on duality on algebraic spaces, see Duality for Spaces, Section 1.

2. Dualizing complexes on schemes

We define a dualizing complex on a locally Noetherian scheme to be a complex which affine locally comes from a dualizing complex on the corresponding ring. This is not completely standard but agrees with all definitions in the literature on Noetherian schemes of finite dimension.

**Lemma 2.1.** Let \( X \) be a locally Noetherian scheme. Let \( K \) be an object of \( D(\mathcal{O}_X) \). The following are equivalent

1. For every affine open \( U = \text{Spec}(A) \subset X \) there exists a dualizing complex \( \omega_A^\bullet \) for \( A \) such that \( K|_U \) is isomorphic to the image of \( \omega_A^\bullet \) by the functor \( \tilde{\cdot} : D(A) \to D(\mathcal{O}_U) \).
2. There is an affine open covering \( X = \bigcup U_i, U_i = \text{Spec}(A_i) \) such that for each \( i \) there exists a dualizing complex \( \omega_i^\bullet \) for \( A_i \) such that \( K|_U \) is isomorphic to the image of \( \omega_i^\bullet \) by the functor \( \tilde{\cdot} : D(A_i) \to D(\mathcal{O}_{U_i}) \).

**Proof.** Assume (2) and let \( U = \text{Spec}(A) \) be an affine open of \( X \). Since condition (2) implies that \( K \) is in \( D_{QCoh}(\mathcal{O}_X) \) we find an object \( \omega_A^\bullet \) in \( D(A) \) whose associated complex of quasi-coherent sheaves is isomorphic to \( K|_U \), see Derived Categories of Schemes, Lemma 3.5. We will show that \( \omega_A^\bullet \) is a dualizing complex for \( A \) which will finish the proof.

Since \( X = \bigcup U_i \) is an open covering, we can find a standard open covering \( U = D(f_1) \cup \ldots \cup D(f_m) \) such that each \( D(f_j) \) is a standard open in one of the affine opens \( U_i \), see Schemes, Lemma 11.5. Say \( D(f_j) = D(g_j) \) for \( g_j \in A_i \). Then \( A_{f_j} \cong (A_i)_{g_j} \) and we have \( (\omega_A^\bullet)_{f_j} \cong (\omega_i^\bullet)_{g_j} \) in the derived category by Derived Categories of Schemes, Lemma 3.5. By Dualizing Complexes, Lemma 15.6 we find that the complex \( (\omega_A^\bullet)_{f_j} \) is a dualizing complex over \( A_{f_j} \) for \( j = 1, \ldots, m \). This implies that \( \omega_A^\bullet \) is dualizing by Dualizing Complexes, Lemma 15.7. \( \square \)
Let $X$ be a locally Noetherian scheme. An object $K$ of $D(\mathcal{O}_X)$ is called a dualizing complex if $K$ satisfies the equivalent conditions of Lemma 2.1.

Please see remarks made at the beginning of this section.

Let $A$ be a Noetherian ring and let $X = \text{Spec}(A)$. Let $K, L$ be objects of $D(A)$. If $K \in D_{\text{Coh}}(A)$ and $L$ has finite injective dimension, then

$$R\text{Hom}_{\mathcal{O}_X}(\bar{K}, \bar{L}) = R\text{Hom}_A(K, L)$$

in $D(\mathcal{O}_X)$.

**Proof.** We may assume that $L$ is given by a finite complex $I^\bullet$ of injective $A$-modules. By induction on the length of $I^\bullet$ and compatibility of the constructions with distinguished triangles, we reduce to the case that $L = I[0]$ where $I$ is an injective $A$-module. In this case, Derived Categories of Schemes, Lemma 9.8, tells us that the $n$th cohomology sheaf of $R\text{Hom}_{\mathcal{O}_X}(\bar{K}, \bar{L})$ is the sheaf associated to the presheaf

$$D(f) \mapsto \text{Ext}_{A_f}^n(K \otimes_A A_f, I \otimes_A A_f)$$

Since $A$ is Noetherian, the $A_f$-module $I \otimes_A A_f$ is injective (Dualizing Complexes, Lemma 3.8). Hence we see that

$$\text{Ext}_{A_f}^n(K \otimes_A A_f, I \otimes_A A_f) = \text{Hom}_{A_f}(H^{-n}(K \otimes_A A_f), I \otimes_A A_f)$$

$$= \text{Hom}_{A_f}(H^{-n}(K) \otimes A_f, I \otimes_A A_f)$$

$$= \text{Hom}_A(H^{-n}(K), I) \otimes A_f$$

The last equality because $H^{-n}(K)$ is a finite $A$-module, see Algebra, Lemma 10.2.

This proves that the canonical map

$$R\text{Hom}_A(K, L) \to R\text{Hom}_{\mathcal{O}_X}(\bar{K}, \bar{L})$$

is a quasi-isomorphism in this case and the proof is done. □

**Lemma 2.4.** Let $K$ be a dualizing complex on a locally Noetherian scheme $X$. Then $K$ is an object of $D_{\text{Coh}}(\mathcal{O}_X)$ and $D = R\text{Hom}_{\mathcal{O}_X}(-, K)$ induces an anti-equivalence

$$D : D_{\text{Coh}}(\mathcal{O}_X) \to D_{\text{Coh}}(\mathcal{O}_X)$$

which comes equipped with a canonical isomorphism $\text{id} \to D \circ D$. If $X$ is quasi-compact, then $D$ exchanges $D^b_{\text{Coh}}(\mathcal{O}_X)$ and $D^b_{\text{Coh}}(\mathcal{O}_X)$ and induces an equivalence $D^b_{\text{Coh}}(\mathcal{O}_X) \to D^b_{\text{Coh}}(\mathcal{O}_X)$.

**Proof.** Let $U \subset X$ be an affine open. Say $U = \text{Spec}(A)$ and let $\omega^*_A$ be a dualizing complex for $A$ corresponding to $K|_U$ as in Lemma 2.1. By Lemma 2.3 the diagram

$$D_{\text{Coh}}(A) \to D_{\text{Coh}}(\mathcal{O}_X)$$

$$R\text{Hom}_A(-, \omega^*_A) \quad R\text{Hom}_{\mathcal{O}_X}(-, K|_U)$$

$$D_{\text{Coh}}(\mathcal{O}_X) \to D(\mathcal{O}_U)$$

commutes. We conclude that $D$ sends $D_{\text{Coh}}(\mathcal{O}_X)$ into $D_{\text{Coh}}(\mathcal{O}_X)$. Moreover, the canonical map

$$L \to R\text{Hom}_{\mathcal{O}_X}(R\text{Hom}_{\mathcal{O}_X}(L, K), K)$$
(Cohomology on Sites, Lemma 34.5) is an isomorphism for all \( L \) because this is true on affines by Dualizing Complexes, Lemma 15.2. The statement on boundedness properties of the functor \( D \) in the quasi-compact case also follows from the corresponding statements of Dualizing Complexes, Lemma 15.2.

Let \( X \) be a locally ringed space. We will say that an object \( L \) of \( D(\mathcal{O}_X) \) is invertible if there is an open covering \( X = \bigcup U_i \) such that \( L|_{U_i} \cong \mathcal{O}_{U_i}[-n_i] \) for some integers \( n_i \). In this case, the function

\[
x \mapsto n_x, \quad \text{where } n_x \text{ is the unique integer such that } H^n(x)(L_x) \neq 0
\]

is locally constant on \( X \). In particular, it follows that \( L = \bigoplus H^n(L)[-n] \) which gives a well defined complex of \( \mathcal{O}_X \)-modules (with zero differentials) representing \( L \). In particular \( L \) is a perfect object of \( D(\mathcal{O}_X) \).

**Lemma 2.5.** Let \( X \) be a locally Noetherian scheme. If \( K \) and \( K' \) are dualizing complexes on \( X \), then \( K' \) is isomorphic to \( K \otimes_{\mathcal{O}_X} L \) for some invertible object \( L \) of \( D(\mathcal{O}_X) \).

**Proof.** Set

\[
L = R \mathcal{H}om_{\mathcal{O}_X}(K, K')
\]

This is an invertible object of \( D(\mathcal{O}_X) \), because affine locally this is true, see Dualizing Complexes, Lemma 15.3 and its proof. The evaluation map \( L \otimes_{\mathcal{O}_X} K \to K' \) is an isomorphism for the same reason.

**Lemma 2.6.** Let \( X \) be a locally Noetherian scheme. Let \( \omega_X^* \) be a dualizing complex on \( X \). Then \( X \) is universally catenary and the function \( X \to \mathbf{Z} \) defined by

\[
x \mapsto \delta(x) \quad \text{such that } \omega_{X,x}^*[-\delta(x)] \text{ is a normalized dualizing complex over } \mathcal{O}_{X,x}
\]

is a dimension function.

**Proof.** Immediate from the affine case Dualizing Complexes, Lemma 17.3 and the definitions.

**Lemma 2.7.** Let \( X \) be a locally Noetherian scheme. Let \( \omega_X^* \) be a dualizing complex on \( X \) with associated dimension function \( \delta \). Let \( \mathcal{F} \) be a coherent \( \mathcal{O}_X \)-module. Set \( \mathcal{E}^i = \mathcal{E}xt_{\mathcal{O}_X}^i(\mathcal{F}, \omega_X^*) \). Then \( \mathcal{E}^i \) is a coherent \( \mathcal{O}_X \)-module and for \( x \in X \) we have

1. \( \mathcal{E}^i_x \) is nonzero only for \( \delta(x) \leq i \leq \delta(x) + \dim(\text{Supp}(\mathcal{F}_x)) \),
2. \( \dim(\text{Supp}(\mathcal{E}^{i+\delta(x)}_x)) \leq i \),
3. \( \text{depth}(\mathcal{F}_x) \) is the smallest integer \( i \geq 0 \) such that \( \mathcal{E}^{i+\delta(x)} \neq 0 \), and
4. we have \( x \in \text{Supp}(\bigoplus_{j \leq i} \mathcal{E}^j) \leftrightarrow \text{depth}_{\mathcal{O}_{X,x}}(\mathcal{F}_x) + \delta(x) \leq i \).

**Proof.** Lemma 2.4 tells us that \( \mathcal{E}^i \) is coherent. Choosing an affine neighbourhood of \( x \) and using Derived Categories of Schemes, Lemma 9.8 and More on Algebra, Lemma 87.2 part (3) we have

\[
\mathcal{E}^i_x = \mathcal{E}xt_{\mathcal{O}_X}^{i-\delta(x)}(\mathcal{F}, \omega_X^*) = \mathcal{E}xt_{\mathcal{O}_{X,x}}^{i-\delta(x)}(\mathcal{F}_x, \omega_{X,x}^*) = \mathcal{E}xt_{\mathcal{O}_{X,x}}^{\delta(x)-i}(\mathcal{F}_x, \omega_{X,x}^*[-\delta(x)])
\]

By construction of \( \delta \) in Lemma 2.6 this reduces parts (1), (2), and (3) to Dualizing Complexes, Lemma 16.5 Part (4) is a formal consequence of (3) and (1). □
3. Right adjoint of pushforward

Let $f : X \to Y$ be a morphism of schemes. In this section we consider the right adjoint to the functor $Rf_* : D_{QCoh}(\mathcal{O}_X) \to D_{QCoh}(\mathcal{O}_Y)$. In the literature, if this functor exists, then it is sometimes denoted $f^\times$. This notation is not universally accepted and we refrain from using it. We will not use the notation $f^!$ for such a functor, as this would clash (for general morphisms $f$) with the notation in [Har66].

**Lemma 3.1.** Let $f : X \to Y$ be a morphism between quasi-separated and quasi-compact schemes. The functor $Rf_* : D_{QCoh}(X) \to D_{QCoh}(Y)$ has a right adjoint.

**Proof.** We will prove a right adjoint exists by verifying the hypotheses of Derived Categories, Proposition [35.2]. First off, the category $D_{QCoh}(\mathcal{O}_X)$ has direct sums, see Derived Categories of Schemes, Lemma [3.1]. The category $D_{QCoh}(\mathcal{O}_X)$ is compactly generated by Derived Categories of Schemes, Theorem [14.3]. Since $X$ and $Y$ are quasi-compact and quasi-separated, so is $f$, see Schemes, Lemmas [21.13] and [21.14]. Hence the functor $Rf_*$ commutes with direct sums, see Derived Categories of Schemes, Lemma [4.2]. This finishes the proof. $\square$

**Example 3.2.** Let $A \to B$ be a ring map. Let $Y = \text{Spec}(A)$ and $X = \text{Spec}(B)$ and $f : X \to Y$ the morphism corresponding to $A \to B$. Then $Rf_* : D_{QCoh}(\mathcal{O}_X) \to D_{QCoh}(\mathcal{O}_Y)$ corresponds to restriction $D(B) \to D(A)$ via the equivalences $D(B) \to D_{QCoh}(\mathcal{O}_X)$ and $D(A) \to D_{QCoh}(\mathcal{O}_Y)$. Hence the right adjoint corresponds to the functor $K \mapsto R\text{Hom}(B, K)$ of Dualizing Complexes, Section [13].

**Example 3.3.** If $f : X \to Y$ is a separated finite type morphism of Noetherian schemes, then the right adjoint of $Rf_* : D_{QCoh}(\mathcal{O}_X) \to D_{QCoh}(\mathcal{O}_Y)$ does not map $D_{QCoh}(\mathcal{O}_Y)$ into $D_{QCoh}(\mathcal{O}_X)$. Namely, let $k$ be a field and consider the morphism $f : \mathbb{A}^1_k \to \text{Spec}(k)$. By Example [3.2] this corresponds to the question of whether $R\text{Hom}(B, -)$ maps $D_{QCoh}(A)$ into $D_{QCoh}(B)$ where $A = k$ and $B = k[x]$. This is not true because

$$R\text{Hom}(k[x], k) = \left( \prod_{n \geq 0} k \right)[0]$$

which is not a finite $k[x]$-module. Hence $a(\mathcal{O}_Y)$ does not have coherent cohomology sheaves.

**Example 3.4.** If $f : X \to Y$ is a proper or even finite morphism of Noetherian schemes, then the right adjoint of $Rf_* : D_{QCoh}(\mathcal{O}_X) \to D_{QCoh}(\mathcal{O}_Y)$ does not map $D_{QCoh}(\mathcal{O}_Y)$ into $D_{QCoh}(\mathcal{O}_X)$ (see Example [3.2]). Namely, let $k$ be a field, let $[\epsilon]$ be the dual numbers over $k$, let $X = \text{Spec}(k)$, and let $Y = \text{Spec}(k[\epsilon])$. Then $\text{Ext}^i_{\mathcal{O}_Y}(k, k)$ is nonzero for all $i \geq 0$. Hence $a(\mathcal{O}_Y)$ is not bounded above by Example [3.2].

**Lemma 3.5.** Let $f : X \to Y$ be a morphism of quasi-compact and quasi-separated schemes. Let $a : D_{QCoh}(\mathcal{O}_Y) \to D_{QCoh}(\mathcal{O}_X)$ be the right adjoint to $Rf_*$ of Lemma [3.1]. Then a maps $D_{QCoh}(\mathcal{O}_Y)$ into $D_{QCoh}(\mathcal{O}_X)$. In fact, there exists an integer $N$ such that $H^i(K) = 0$ for $i \leq c$ implies $H^i(a(K)) = 0$ for $i \leq c - N$.

**Proof.** By Derived Categories of Schemes, Lemma [4.1] the functor $Rf_*$ has finite cohomological dimension. In other words, there exist an integer $N$ such that
$H^i(Rf_*L) = 0$ for $i \geq N + c$ if $H^i(L) = 0$ for $i \geq c$. Say $K \in D^+_\QCoh(\mathcal{O}_Y)$ has $H^i(K) = 0$ for $i \leq c$. Then

$$\Hom_{D(\mathcal{O}_X)}(\tau_{\leq c-N}a(K), a(K)) = \Hom_{D(\mathcal{O}_Y)}(Rf_*\tau_{\leq c-N}a(K), K) = 0$$

by what we said above. Clearly, this implies that $H^i(a(K)) = 0$ for $i \leq c - N$. □

Let $f : X \to Y$ be a morphism of quasi-separated and quasi-compact schemes. Let $\alpha$ denote the right adjoint to $Rf_* : D^+_{\QCoh}(\mathcal{O}_X) \to D^+_{\QCoh}(\mathcal{O}_Y)$. For every $K \in D^+_{\QCoh}(\mathcal{O}_Y)$ and $L \in D^+_{\QCoh}(\mathcal{O}_X)$ we obtain a canonical map

\[ (3.5.1) \quad Rf_* R\mathcal{H}om_{\mathcal{O}_X}(L, a(K)) \longrightarrow R\mathcal{H}om_{\mathcal{O}_Y}(Rf_*L, K) \]

Namely, this map is constructed as the composition

$$Rf_* R\mathcal{H}om_{\mathcal{O}_X}(L, a(K)) \to R\mathcal{H}om_{\mathcal{O}_Y}(Rf_*L, Rf_*a(K)) \to R\mathcal{H}om_{\mathcal{O}_Y}(Rf_*L, K)$$

where the first arrow is Cohomology, Remark 36.10 and the second arrow is the counit $Rf_*a(K) \to K$ of the adjunction.

\[ \textbf{Lemma 3.6.} \quad \text{Let } f : X \to Y \text{ be a morphism of quasi-compact and quasi-separated schemes. Let } \alpha \text{ be the right adjoint to } Rf_* : D^+_{\QCoh}(\mathcal{O}_X) \to D^+_{\QCoh}(\mathcal{O}_Y). \text{ Then} \]

\[ (3.5.1) \quad Rf_* R\mathcal{H}om_{\mathcal{O}_X}(L, a(K)) \longrightarrow R\mathcal{H}om_{\mathcal{O}_Y}(Rf_*L, K) \]

\text{is an isomorphism for all } L \in D^+_{\QCoh}(\mathcal{O}_X) \text{ and } K \in D^+_{\QCoh}(\mathcal{O}_Y). \]

\[ \textbf{Proof.} \quad \text{Let } M \in D^+_{\QCoh}(\mathcal{O}_Y). \text{ Then we have the following} \]

$$\Hom_Y(M, Rf_* R\mathcal{H}om_{\mathcal{O}_X}(L, a(K))) = \Hom_X(Lf^*M, R\mathcal{H}om_{\mathcal{O}_X}(L, a(K)))$$

$$= \Hom_X(Lf^*M \otimes_{\mathcal{O}_X} L, a(K))$$

$$= \Hom_Y(Rf_*(Lf^*M \otimes_{\mathcal{O}_X} L), K)$$

$$= \Hom_Y(M \otimes_{\mathcal{O}_Y} Rf_*L, K)$$

$$= \Hom_Y(M, R\mathcal{H}om_{\mathcal{O}_Y}(Rf_*L, K))$$

The first equality holds by Cohomology, Lemma 29.1. The second equality by Cohomology, Lemma 36.2. The third equality by construction of $\alpha$. The fourth equality by Derived Categories of Schemes, Lemma 21.1 (this is the important step). The fifth by Cohomology, Lemma 36.2 Thus the result holds by the Yoneda lemma. □

\[ \textbf{Lemma 3.7.} \quad \text{Let } f : X \to Y \text{ be a morphism of quasi-separated and quasi-compact schemes. For all } L \in D^+_{\QCoh}(\mathcal{O}_X) \text{ and } K \in D^+_{\QCoh}(\mathcal{O}_Y) \text{ (3.5.1) induces an isomorphism} \]

$$R\mathcal{H}om_X(L, a(K)) \to R\mathcal{H}om_Y(Rf_*L, K) \text{ of global derived homs.}$$

\[ \textbf{Proof.} \quad \text{By the construction in Cohomology, Section 38 we have} \]

$$R\mathcal{H}om_X(L, a(K)) = \Gamma(X, R\mathcal{H}om_{\mathcal{O}_X}(L, a(K))) = \Gamma(Y, Rf_* R\mathcal{H}om_{\mathcal{O}_X}(L, a(K)))$$

and

$$R\mathcal{H}om_Y(Rf_*L, K) = \Gamma(Y, R\mathcal{H}om_{\mathcal{O}_Y}(Rf_*L, a(K)))$$

Thus the lemma is a consequence of Lemma 3.6. □
4. Right adjoint of pushforward and restriction to opens

In this section we study the question to what extend the right adjoint of pushforward commutes with restriction to open subschemes. This is a base change question, so let’s first discuss this more generally.

We often want to know whether the right adjoints to pushforward commutes with base change. Thus we consider a cartesian square

\begin{equation}
\begin{array}{c}
X' \\
\downarrow g' \\
Y'
\end{array}
\begin{array}{c}
X \\
\downarrow f \\
Y
\end{array}
\end{equation}

of quasi-compact and quasi-separated schemes. Denote

\[ a : \mathcal{D}_{QCoh}(\mathcal{O}_Y) \to \mathcal{D}_{QCoh}(\mathcal{O}_X) \quad \text{and} \quad a' : \mathcal{D}_{QCoh}(\mathcal{O}_Y') \to \mathcal{D}_{QCoh}(\mathcal{O}_X') \]

the right adjoints to \( Rf_* \) and \( Rf'_* \) (Lemma 3.1). Consider the base change map of Cohomology, Remark 29.3. It induces a transformation of functors

\[ Lg^* \circ Rf_* \to Rf'_* \circ L(g')^* \]

on derived categories of sheaves with quasi-coherent cohomology. Hence a transformation between the right adjoints in the opposite direction

\[ a \circ Rg_* \leftarrow Rg'_* \circ a' \]

Lemma 4.1. In diagram (4.0.1) assume that \( g \) is flat or more generally that \( f \) and \( g \) are Tor independent. Then \( a \circ Rg_* \leftarrow Rg'_* \circ a' \) is an isomorphism.

Proof. In this case the base change map \( Lg^* \circ Rf_* K \to Rf'_* \circ L(g')^* K \) is an isomorphism for every \( K \) in \( \mathcal{D}_{QCoh}(\mathcal{O}_X) \) by Derived Categories of Schemes, Lemma 21.3. Thus the corresponding transformation between adjoint functors is an isomorphism as well. \( \square \)

Let \( f : X \to Y \) be a morphism of quasi-compact and quasi-separated schemes. Let \( V \subseteq Y \) be a quasi-compact open subscheme and set \( U = f^{-1}(V) \). This gives a cartesian square

\begin{equation}
\begin{array}{c}
U \\
\downarrow j' \\
V
\end{array}
\begin{array}{c}
X \\
\downarrow f \\
Y
\end{array}
\end{equation}

as in (4.0.1). By Lemma 4.1 the map \( \xi : a \circ Rj_* \leftarrow Rj'_* \circ a' \) is an isomorphism where \( a \) and \( a' \) are the right adjoints to \( Rf_* \) and \( R(f|_U)_* \). We obtain a transformation of functors \( \mathcal{D}_{QCoh}(\mathcal{O}_Y) \to \mathcal{D}_{QCoh}(\mathcal{O}_U) \)

\begin{equation}
(j')^* \circ a \to (j')^* \circ a \circ Rj_* \circ j^* \xrightarrow{\xi^{-1}} (j')^* \circ Rj'_* \circ a' \circ j^* \to a' \circ j^* \end{equation}

where the first arrow comes from \( id \to Rj_* \circ j^* \) and the final arrow from the isomorphism \( (j')^* \circ Rj'_* \to id \). In particular, we see that \( (4.1.1) \) is an isomorphism when evaluated on \( K \) if and only if \( a(K)|_U \to a(Rj_*(K|_Y)|_U) \) is an isomorphism.

Example 4.2. There is a finite morphism \( f : X \to Y \) of Noetherian schemes such that \( (4.1.1) \) is not an isomorphism when evaluated on some \( K \in \mathcal{D}_{Coh}(\mathcal{O}_Y) \).
Namely, let \( X = \text{Spec}(B) \to Y = \text{Spec}(A) \) with \( A = k[x, \epsilon] \) where \( k \) is a field and \( \epsilon^2 = 0 \) and \( B = k[x] = A/(\epsilon) \). For \( n \in \mathbb{N} \) set \( M_n = A/(\epsilon, x^n) \). Observe that
\[
\text{Ext}_A^i(B, M_n) = M_n, \quad i \geq 0
\]
because \( B \) has the free periodic resolution \( \ldots \to A \to A \to A \) with maps given by multiplication by \( \epsilon \). Consider the object \( K = \bigoplus M_n[n] = \prod M_n[n] \) of \( \mathcal{D}_{\text{Coh}}(A) \) (equality in \( \mathcal{D}(A) \) by Derived Categories, Lemmas \[31.5\] and \[32.2\]). Then we see that \( a(K) \) corresponds to \( R \text{Hom}(B, K) \) by Example \[3.2\] and
\[
H^0(R \text{Hom}(B, K)) = \text{Ext}^0_A(B, K) = \prod_{n \geq 1} \text{Ext}^n_A(B, M_n) = \prod_{n \geq 1} M_n
\]
by the above. But this module has elements which are not annihilated by any power of \( x \), whereas the complex \( K \) does have every element of its cohomology annihilated by a power of \( x \). In other words, for the map \( (4.1.1) \) with \( V = D(x) \) and \( U = D(x) \) and the complex \( K \) cannot be an isomorphism because \( (j')^*(a(K)) \) is nonzero and \( a^*(j^*K) \) is zero.

**Lemma 4.3.** Let \( f : X \to Y \) be a morphism of quasi-compact and quasi-separated schemes. Let \( a \) be the right adjoint to \( Rf_* : \mathcal{D}_{\text{Coh}}(\mathcal{O}_X) \to \mathcal{D}_{\text{Coh}}(\mathcal{O}_Y) \). Let \( V \subset Y \) be quasi-compact open with inverse image \( U \subset X \).

1. For every \( Q \in \mathcal{D}_{\text{Coh}}(\mathcal{O}_Y) \) supported on \( Y \setminus V \) the image \( a(Q) \) is supported on \( X \setminus U \) if and only if \( (4.1.1) \) is an isomorphism on all \( K \) in \( \mathcal{D}_{\text{Coh}}(\mathcal{O}_Y) \).
2. For every \( Q \in \mathcal{D}_{\text{Coh}}(\mathcal{O}_Y) \) supported on \( Y \setminus V \) the image \( a(Q) \) is supported on \( X \setminus U \) if and only if \( (4.1.1) \) is an isomorphism on all \( K \) in \( \mathcal{D}_{\text{Coh}}(\mathcal{O}_Y) \).
3. If \( a \) commutes with direct sums, then the equivalent conditions of (1) imply the equivalent conditions of (2).

**Proof.** Proof of (1). Let \( K \in \mathcal{D}_{\text{Coh}}^+(\mathcal{O}_Y) \). Choose a distinguished triangle
\[
K \to Rj_*K|_V \to Q \to K[1]
\]
Observe that \( Q \) is in \( \mathcal{D}_{\text{Coh}}^+(\mathcal{O}_Y) \) (Derived Categories of Schemes, Lemma \[4.1\]) and is supported on \( Y \setminus V \) (Derived Categories of Schemes, Definition \[6.4\]). Applying \( a \) we obtain a distinguished triangle
\[
a(K) \to a(Rj_*K|_V) \to a(Q) \to a(K)[1]
\]
on \( X \). If \( a(Q) \) is supported on \( X \setminus U \), then restricting to \( U \) the map \( a(K)|_U \to a(Rj_*K|_V)|_U \) is an isomorphism, i.e., \( (4.1.1) \) is an isomorphism on \( K \). The converse is immediate.

The proof of (2) is exactly the same as the proof of (1).

Proof of (3). Assume the equivalent conditions of (1) hold. Set \( T = Y \setminus V \). We will use the notation \( \mathcal{D}_{\text{Coh}, T}(\mathcal{O}_Y) \) and \( \mathcal{D}_{\text{Coh}, f^{-1}(T)}(\mathcal{O}_X) \) to denote complexes whose cohomology sheaves are supported on \( T \) and \( f^{-1}(T) \). Since \( a \) commutes with direct sums, the strictly full, saturated, triangulated subcategory \( \mathcal{D} \) with objects
\[
\{ Q \in \mathcal{D}_{\text{Coh}, T}(\mathcal{O}_Y) \mid a(Q) \in \mathcal{D}_{\text{Coh}, f^{-1}(T)}(\mathcal{O}_X) \}
\]
is preserved by direct sums and hence derived colimits. On the other hand, the category \( \mathcal{D}_{\text{Coh}, T}(\mathcal{O}_Y) \) is generated by a perfect object \( E \) (see Derived Categories of Schemes, Lemma \[4.4\]). By assumption we see that \( E \in \mathcal{D} \). By Derived Categories, Lemma \[34.3\] every object \( Q \) of \( \mathcal{D}_{\text{Coh}, T}(\mathcal{O}_Y) \) is a derived colimit of a system \( Q_1 \to Q_2 \to Q_3 \to \ldots \) such that the cones of the transition maps are direct sums of shifts.
of $E$. Arguing by induction we see that $Q_n \in \mathcal{D}$ for all $n$ and finally that $Q \in \mathcal{D}$. Thus the equivalent conditions of (2) hold.

\[ \square \]

**Lemma 4.4.** Let $Y$ be a quasi-compact and quasi-separated scheme. Let $f : X \to Y$ be a proper morphism. If

1. $f$ is flat and of finite presentation, or
2. $Y$ is Noetherian

then the equivalent conditions of Lemma 4.3 part (1) hold for all quasi-compact opens $V$ of $Y$.

**Proof.** Let $Q \in D_{QCoh}^+(\mathcal{O}_Y)$ be supported on $Y \setminus V$. To get a contradiction, assume that $a(Q)$ is not supported on $X \setminus U$. Then we can find a perfect complex $P_U$ on $U$ and a nonzero map $P_U \to a(Q)|_U$ (follows from Derived Categories of Schemes, Theorem 14.3). Then using Derived Categories of Schemes, Lemma 12.9 we may assume there is a perfect complex $P$ on $X$ and a map $P \to a(Q)$ whose restriction to $U$ is nonzero. By definition of $a$ this map is adjoint to a map $Rf_*P \to Q$.

The complex $Rf_*P$ is pseudo-coherent. In case (1) this follows from Derived Categories of Schemes, Lemma 26.5. In case (2) this follows from Derived Categories of Schemes, Lemmas 10.1 and 9.3. Thus we may apply Derived Categories of Schemes, Lemma 16.3 and get a map $I \to \mathcal{O}_Y$ of perfect complexes whose restriction to $V$ is an isomorphism such that the composition $I \otimes \mathcal{O}_Y \to Rf_*P \to Rf_*P \to K$ is zero. By Derived Categories of Schemes, Lemma 21.1 we have $I \otimes \mathcal{O}_Y \to Rf_*P = Rf_*(Lf^*I \otimes \mathcal{O}_X P)$. We conclude that the composition

\[ Lf^*I \otimes \mathcal{O}_X P \to P \to a(K) \]

is zero. However, the restriction to $U$ is the map $P|_U \to a(K)|_U$ which we assumed to be nonzero. This contradiction finishes the proof. \[ \square \]

### 5. Right adjoint of pushforward and base change, I

The map (4.1.1) is a special case of a base change map. Namely, suppose that we have a cartesian diagram

\[
\begin{array}{ccc}
X' & \xrightarrow{g'} & X \\
\downarrow f' & & \downarrow f \\
Y' & \xrightarrow{g} & Y
\end{array}
\]

of quasi-compact and quasi-separated schemes, i.e., a diagram as in (4.0.1). Assume $f$ and $g$ are Tor independent. Then we can consider the morphism of functors $D_{QCoh}(\mathcal{O}_Y) \to D_{QCoh}(\mathcal{O}_{X'})$ given by the composition

\[ (5.0.1) \quad L(g')^* \circ a \to L(g)^* \circ a \circ Rg_* \circ Lg^* \leftarrow L(g'^*) \circ Rg'_* \circ a' \circ Lg^* \to a' \circ Lg^* \]

The first arrow comes from the adjunction map $id \to Rg_*Lg^*$ and the last arrow from the adjunction map $L(g'^*)Rg'_* \to id$. We need the assumption on Tor independence to invert the arrow in the middle, see Lemma 4.1. Alternatively, we can

\[ ^1 \text{This proof works for those morphisms of quasi-compact and quasi-separated schemes such that } Rf_*P \text{ is pseudo-coherent for all } P \text{ perfect on } X \text{. It follows easily from a theorem of Kiehl [Kie72] that this holds if } f \text{ is proper and pseudo-coherent. This is the correct generality for this lemma and some of the other results in this chapter.} \]
think of \((5.0.1)\) by adjointness of \(L(g')^*\) and \(R(g)_*\) as a natural transformation
\[ a \rightarrow a \circ Rg_\circ Lg^* \leftarrow Rg'_* \circ a' \circ Lg^* \]
were again the second arrow is invertible. If \(M \in D_{QCoh}(\mathcal{O}_X)\) and \(K \in D_{QCoh}(\mathcal{O}_Y)\) then on Yoneda functors this map is given by
\[ \text{Hom}_X(M, a(K)) = \text{Hom}_Y(Rf_* M, K) \]
\[ \rightarrow \text{Hom}_Y(Rf_* M, Rg_* Lg^* K) \]
\[ = \text{Hom}_Y(Lg^* Rf_* M, Lg^* K) \]
\[ \leftarrow \text{Hom}_Y(Rf'_* L(g')^* M, Lg^* K) \]
\[ = \text{Hom}_X(L(g')^* M, a'(Lg^* K)) \]
\[ = \text{Hom}_X(M, Rg'_* a'(Lg^* K)) \]
(were the arrow pointing left is invertible by the base change theorem given in Derived Categories of Schemes, Lemma \[21.3\]) which makes things a little bit more explicit.

In this section we first prove that the base change map satisfies some natural compatibilities with regards to stacking squares as in Cohomology, Remarks \[29.4\] and \[29.5\] for the usual base change map. We suggest the reader skip the rest of this section on a first reading.

\textbf{Lemma 5.1.} Consider a commutative diagram

\[
\begin{array}{ccc}
X' & \xrightarrow{k} & X \\
\downarrow f' & & \downarrow f \\
Y' & \xrightarrow{l} & Y \\
\downarrow g' & & \downarrow g \\
Z' & \xrightarrow{m} & Z
\end{array}
\]

of quasi-compact and quasi-separated schemes where both diagrams are cartesian and where \(f\) and \(l\) as well as \(g\) and \(m\) are Tor independent. Then the maps \((5.0.1)\) for the two squares compose to give the base change map for the outer rectangle (see proof for a precise statement).

\textbf{Proof.} It follows from the assumptions that \(g \circ f\) and \(m\) are Tor independent (details omitted), hence the statement makes sense. In this proof we write \(k^*\) in place of \(Lk^*\) and \(f_*\) instead of \(Rf_*\). Let \(a, b, c\) be the right adjoints of Lemma \[3.1\] for \(f, g,\) and \(g \circ f\) and similarly for the primed versions. The arrow corresponding to the top square is the composition
\[ \gamma_{\text{top}} : k^* \circ a \rightarrow k^* \circ a \circ l_* \circ l^* \xrightarrow{\xi_{\text{top}}} k^* \circ k_* \circ a' \circ l^* \rightarrow a' \circ l^* \]
where \(\xi_{\text{top}} : k_* \circ a' \rightarrow a \circ l_*\) is an isomorphism (hence can be inverted) and is the arrow “dual” to the base change map \(l^* \circ f_* \rightarrow f'_* \circ k^*\). The outer arrows come from the canonical maps \(1 \rightarrow l_* \circ l^*\) and \(k^* \circ k_* \rightarrow 1\). Similarly for the second square we have
\[ \gamma_{\text{bot}} : l^* \circ b \rightarrow l^* \circ b \circ m_* \circ m^* \xrightarrow{\xi_{\text{bot}}} l^* \circ l_* \circ b' \circ m^* \rightarrow b' \circ m^* \]
For the outer rectangle we get

$$\gamma_{\text{rect}} : k^* \circ c \rightarrow k^* \circ c \circ m_* \circ m^* \xrightarrow{\xi_{\text{rect}}} k^* \circ k_* \circ c' \circ m^* \rightarrow c' \circ m^*$$

We have \((g \circ f)_* = g_* \circ f_*\) and hence \(c = a \circ b\) and similarly \(c' = a' \circ b'\). The statement of the lemma is that \(\gamma_{\text{rect}}\) is equal to the composition

$$k^* \circ c = k^* \circ a \circ b \xrightarrow{\gamma_{\text{top}}} a' \circ l^* \circ b \xrightarrow{\gamma_{\text{bot}}} a' \circ b' \circ m^* = c' \circ m^*$$

To see this we contemplate the following diagram:

Going down the right hand side we have the composition and going down the left hand side we have \(\gamma_{\text{rect}}\). All the quadrilaterals on the right hand side of this diagram commute by Categories, Lemma 27.2 or more simply the discussion preceding Categories, Definition 27.1. Hence we see that it suffices to show the diagram
becomes commutative if we invert the arrows \( \xi_{\text{top}}, \xi_{\text{bot}}, \) and \( \xi_{\text{rect}} \) (note that this is different from asking the diagram to be commutative). However, the diagram

\[
\begin{array}{ccc}
  a \circ l_\ast \circ l^\ast \circ b \circ m_\ast & \xrightarrow{\xi_{\text{top}}} & k_\ast \circ a' \circ l^\ast \circ b \circ m_\ast \\
  a \circ l_\ast \circ l^\ast \circ l_\ast \circ b' & \xrightarrow{\xi_{\text{bot}}} & k_\ast \circ a' \circ l^\ast \circ l_\ast \circ b' \\
\end{array}
\]

commutes by Categories, Lemma 27.2. Since the diagrams

\[
\begin{array}{ccc}
a \circ l_\ast \circ l^\ast \circ b \circ m_\ast & \xrightarrow{a \circ b \circ m} & a \circ l_\ast \circ l^\ast \circ l_\ast \circ b' & \xrightarrow{a \circ l_\ast \circ b'} \\
\end{array}
\]

and

\[
\begin{array}{ccc}
a \circ l_\ast \circ l^\ast \circ l_\ast \circ b' & \xrightarrow{a \circ l_\ast \circ b'} & k_\ast \circ a' \circ l^\ast \circ l_\ast \circ b' & \xrightarrow{k_\ast \circ a' \circ b'} \\
\end{array}
\]

commute (see references cited) and since the composition of \( l_\ast \rightarrow l_\ast \circ l^\ast \rightarrow l_\ast \) is the identity, we find that it suffices to prove that

\[
k \circ a' \circ b' \xrightarrow{\xi_{\text{bot}}} a \circ l_\ast \circ b \xrightarrow{\xi_{\text{top}}} a \circ b \circ m_\ast
\]

is equal to \( \xi_{\text{rect}} \) (via the identifications \( a \circ b = c \) and \( a' \circ b' = c' \)). This is the statement dual to Cohomology, Remark 29.4 and the proof is complete. \( \square \)

**0ATR Lemma 5.2.** Consider a commutative diagram

\[
\begin{array}{ccc}
  X'' & \xrightarrow{g''} & X' & \xrightarrow{g} & X \\
  Y'' & \xrightarrow{h''} & Y' & \xrightarrow{h} & Y \\
  \downarrow{f''} & & \downarrow{f'} & & \downarrow{f} \\
  \end{array}
\]

of quasi-compact and quasi-separated schemes where both diagrams are cartesian and where \( f \) and \( h \) as well as \( f' \) and \( h' \) are Tor independent. Then the maps \( \xi_{\text{left}} \) for the two squares compose to give the base change map for the outer rectangle (see proof for a precise statement).

**Proof.** It follows from the assumptions that \( f \) and \( h \circ h' \) are Tor independent (details omitted), hence the statement makes sense. In this proof we write \( g' \) in place of \( Lg \) and \( f_\ast \) instead of \( Rf_\ast \). Let \( a, a', \) and \( a'' \) be the right adjoints of Lemma 3.1 for \( f, f', \) and \( f'' \). The arrow corresponding to the right square is the composition

\[
\gamma_{\text{right}} : g^* \circ a \rightarrow g^* \circ a \circ h_\ast \circ h^* \xrightarrow{\xi_{\text{right}}} g^* \circ g_\ast \circ a' \circ h^* \rightarrow a' \circ h^*
\]

where \( \xi_{\text{right}} : g_\ast \circ a' \rightarrow a \circ h_\ast \) is an isomorphism (hence can be inverted) and is the arrow “dual” to the base change map \( h^* \circ f_\ast \rightarrow f'_\ast \circ g^* \). The outer arrows come from the canonical maps \( 1 \rightarrow h_\ast \circ h^* \) and \( g^* \circ g_\ast \rightarrow 1 \). Similarly for the left square we have

\[
\gamma_{\text{left}} : (g')^* \circ a' \rightarrow (g')^* \circ a' \circ (h')_\ast \circ (h')^* \xrightarrow{\xi_{\text{left}}} (g')^* \circ (g')_\ast \circ a'' \circ (h')^* \rightarrow a'' \circ (h')^*
\]

For the outer rectangle we get

\[
\gamma_{\text{rect}} : k^* \circ a \rightarrow k^* \circ a \circ m_\ast \circ m^* \xrightarrow{\xi_{\text{rect}}} k^* \circ k_\ast \circ a'' \circ m^* \rightarrow a'' \circ m^*
\]
where \( k = g \circ g' \) and \( m = h \circ h' \). We have \( k^* = (g')^* \circ g^* \) and \( m^* = (h')^* \circ h^* \). The statement of the lemma is that \( \gamma_{\text{rect}} \) is equal to the composition

\[
k^* \circ a = (g')^* \circ g^* \circ a \xrightarrow{\gamma_{\text{right}}} (g')^* \circ a' \circ h^* \xrightarrow{\gamma_{\text{left}}} a'' \circ (h')^* \circ h^* = a'' \circ m^*
\]

To see this we contemplate the following diagram

\[
\begin{array}{c}
(g')^* \circ g^* \circ a \\
\downarrow \\
(g')^* \circ g^* \circ a \circ h_\ast \circ h^* \\
\downarrow \xi_{\text{right}} \\
(g')^* \circ g^* \circ a \circ (h')_\ast \circ (h')^* \circ h^* \\
\downarrow \xi_{\text{left}} \\
(g')^* \circ g_\ast \circ a' \circ (h')^* \circ h^* \\
\downarrow \xi_{\text{left}} \\
(g')^* \circ g_\ast \circ (g')_\ast \circ a'' \circ (h')^* \circ h^* \\
\downarrow \xi_{\text{left}} \\
(g')^* \circ (g')_\ast \circ a'' \circ (h')^* \circ h^* \\
\rightarrow a'' \circ (h')^* \circ h^*
\end{array}
\]

Going down the right hand side we have the composition and going down the left hand side we have \( \gamma_{\text{rect}} \). All the quadrilaterals on the right hand side of this diagram commute by Categories, Lemma 27.2 or more simply the discussion preceding Categories, Definition 27.1. Hence we see that it suffices to show that

\[
g_\ast \circ (g')_\ast \circ a'' \xrightarrow{\xi_{\text{left}}} g_\ast \circ a' \circ (h')_\ast \xrightarrow{\xi_{\text{right}}} a \circ h_\ast \circ (h')_\ast
\]

is equal to \( \xi_{\text{rect}} \). This is the statement dual to Cohomology, Remark 29.5 and the proof is complete. \( \square \)

**Remark 5.3.** Consider a commutative diagram

\[
\begin{array}{cccccccc}
X'' & \xrightarrow{k'} & X' & \xrightarrow{k} & X \\
\downarrow f'' & \quad & \downarrow f' & \quad & \downarrow f \\
Y'' & \xrightarrow{l'} & Y' & \xrightarrow{l} & Y \\
\downarrow g'' & \quad & \downarrow g' & \quad & \downarrow g \\
Z'' & \xrightarrow{m'} & Z' & \xrightarrow{m} & Z
\end{array}
\]

of quasi-compact and quasi-separated schemes where all squares are cartesian and where \((f, l), (g, m), (f', l'), (g', m')\) are Tor independent pairs of maps. Let \( a, a' \),
In this section we prove that the base change map of Section 5 is an isomorphism
for $f, f', f'', g, g', g''$. Let us label the squares of the diagram $A, B, C, D$ as follows
\[
\begin{array}{ccc}
\ & A & B \\
\ & C & D
\end{array}
\]
Then the maps \[5.0.1\] for the squares are (where we use $k^* = Lk^*$, etc)
\[
\begin{align*}
\gamma_A : (k')^* \circ a' & \to a'' \circ (l')^* \\
\gamma_C : (l')^* \circ b' & \to b'' \circ (m')^* \\
\gamma_B : k^* \circ a & \to a' \circ l^* \\
\gamma_D : l^* \circ b & \to b' \circ m^*
\end{align*}
\]
For the $2 \times 1$ and $1 \times 2$ rectangles we have four further base change maps
\[
\begin{align*}
\gamma_{A+D} : (k \circ k')^* \circ a & \to a'' \circ (l \circ l')^* \\
\gamma_{C+D} : (l \circ l')^* \circ b & \to b'' \circ (m \circ m')^* \\
\gamma_{A+C} : (k')^* \circ (a' \circ b') & \to (a'' \circ b'') \circ (m')^* \\
\gamma_{B+D} : k^* \circ (a \circ b) & \to (a' \circ b') \circ m^*
\end{align*}
\]
By Lemma \[5.2\] we have
\[
\gamma_{A+B} = \gamma_A \circ \gamma_B, \quad \gamma_{C+D} = \gamma_C \circ \gamma_D
\]
and by Lemma \[5.1\] we have
\[
\gamma_{A+C} = \gamma_C \circ \gamma_A, \quad \gamma_{B+D} = \gamma_D \circ \gamma_B
\]
Here it would be more correct to write $\gamma_{A+B} = (\gamma_A \star \text{id}_{k'}^*) \circ (\text{id}_{(k')}^* \star \gamma_B)$ with notation as in Categories, Section \[27\] and similarly for the others. However, we continue the abuse of notation used in the proofs of Lemmas \[5.1\] and \[5.2\] of dropping $\star$ products with identities as one can figure out which ones to add as long as the source and target of the transformation is known. Having said all of this we find (a priori) two transformations
\[
(k')^* \circ k^* \circ a \circ b \longrightarrow a'' \circ b'' \circ (m')^* \circ m^*
\]
namely
\[
\gamma_C \circ \gamma_A \circ \gamma_D \circ \gamma_B = \gamma_{A+C} \circ \gamma_{B+D}
\]
and
\[
\gamma_C \circ \gamma_D \circ \gamma_A \circ \gamma_B = \gamma_{C+D} \circ \gamma_{A+B}
\]
The point of this remark is to point out that these transformations are equal. Namely, to see this it suffices to show that
\[
\begin{array}{c}
(k')^* \circ a' \circ l^* \circ b \\
(\gamma_A \downarrow) \quad (\gamma_D \downarrow)
\end{array}
\]
\[
\begin{array}{c}
\gamma_A \\
\gamma_A
\end{array}
\]
\[
\begin{array}{c}
a'' \circ (l')^* \circ l^* \circ b \\
(\gamma_D \downarrow) \quad (\gamma_B \downarrow)
\end{array}
\]
\[
\begin{array}{c}
\gamma_B \\
\gamma_B
\end{array}
\]
commutes. This is true by Categories, Lemma \[27.2\] or more simply the discussion preceding Categories, Definition \[27.1\]

6. Right adjoint of pushforward and base change, II

In this section we prove that the base change map of Section 3 is an isomorphism
in some cases. We first observe that it suffices to check over affine opens, provided
formation of the right adjoint of pushforward commutes with restriction to opens.
Remark 6.1. Consider a cartesian diagram

\[
\begin{array}{ccc}
X' & \longrightarrow & X \\
\downarrow g' & & \downarrow f \\
Y' & \longrightarrow & Y
\end{array}
\]

of quasi-compact and quasi-separated schemes with \((g, f)\) Tor independent. Let \(V \subset Y\) and \(V' \subset Y'\) be affine opens with \(g(V') \subset V\). Form the cartesian diagrams

\[
\begin{array}{ccc}
U & \longrightarrow & X \\
\downarrow & & \downarrow \\
V & \longrightarrow & Y
\end{array} \quad \text{and} \quad \begin{array}{ccc}
U' & \longrightarrow & X' \\
\downarrow & & \downarrow \\
V' & \longrightarrow & Y'
\end{array}
\]

Assume (4.1.1) with respect to \(K\) and the first diagram and (4.1.1) with respect to \(Lg^*K\) and the second diagram are isomorphisms. Then the restriction of the base change map (5.0.1)

\[
L(g')^*a(K) \longrightarrow a'(Lg^*K)
\]

to \(U'\) is isomorphic to the base change map (5.0.1) for \(K|_V\) and the cartesian diagram

\[
\begin{array}{ccc}
U' & \longrightarrow & U \\
\downarrow & & \downarrow \\
V' & \longrightarrow & V
\end{array}
\]

This follows from the fact that (4.1.1) is a special case of the base change map (5.0.1) and that the base change maps compose correctly if we stack squares horizontally, see Lemma 5.2. Thus in order to check the base change map restricted to \(U'\) is an isomorphism it suffices to work with the last diagram.

Lemma 6.2. In diagram (4.0.1) assume

1. \(g : Y' \rightarrow Y\) is a morphism of affine schemes,
2. \(f : X \rightarrow Y\) is proper, and
3. \(f\) and \(g\) are Tor independent.

Then the base change map (5.0.1) induces an isomorphism

\[
L(g')^*a(K) \longrightarrow a'(Lg^*K)
\]

in the following cases

1. for all \(K \in D_{QCoh}(O_X)\) if \(f\) is flat of finite presentation,
2. for all \(K \in D_{QCoh}(O_X)\) if \(f\) is perfect and \(Y\) Noetherian,
3. for \(K \in D^{+}_{QCoh}(O_X)\) if \(g\) has finite Tor dimension and \(Y\) Noetherian.

Proof. Write \(Y = \text{Spec}(A)\) and \(Y' = \text{Spec}(A')\). As a base change of an affine morphism, the morphism \(g'\) is affine. Let \(M\) be a perfect generator for \(D_{QCoh}(O_X)\), see Derived Categories of Schemes, Remark 14.3. Then \(L(g')^*M\) is a generator for \(D_{QCoh}(O_X)\), see Derived Categories of Schemes, Remark 15.4. Hence it suffices to show that (5.0.1) induces an isomorphism

\[
R\text{Hom}_X(L(g')^*M, L(g')^*a(K)) \longrightarrow R\text{Hom}_X(L(g')^*M, a'(Lg^*K))
\]

of global hom complexes, see Cohomology, Section 38 as this will imply the cone of \(L(g')^*a(K) \rightarrow a'(Lg^*K)\) is zero. The structure of the proof is as follows: we
will first show that these Hom complexes are isomorphic and in the last part of the proof we will show that the isomorphism is induced by (6.2.1).

The left hand side. Because $M$ is perfect, the canonical map

$$R\text{Hom}_X(M, a(K)) \otimes^L_A A' \longrightarrow R\text{Hom}_{X'}(L(g')^*M, L(g')^*a(K))$$

is an isomorphism by Derived Categories of Schemes, Lemma [21.4]. We can combine this with the isomorphism $R\text{Hom}_Y(Rf_*M, K) = R\text{Hom}_X(M, a(K))$ of Lemma [3.7] to get that the left hand side equals $R\text{Hom}_Y(Rf_*M, K) \otimes^L_A A'$.

The right hand side. Here we first use the isomorphism

$$R\text{Hom}_{X'}(L(g')^*M, a'(Lg^*K)) = R\text{Hom}_{Y'}(Rf'_*L(g')^*M, Lg^*K)$$

of Lemma [3.7]. Then we use the base change map $Lg^*Rf_*M \to Rf'_*L(g')^*M$ is an isomorphism by Derived Categories of Schemes, Lemma [21.3]. Hence we may rewrite this as $R\text{Hom}_{Y'}(Lg^*Rf_*M, Lg^*K)$. Since $Y, Y'$ are affine and $K, Rf_*M$ are in $D_{QCoh}(\mathcal{O}_Y)$ (Derived Categories of Schemes, Lemma [4.1]) we have a canonical map

$$\beta : R\text{Hom}_{Y'}(Rf_*M, K) \otimes^L_A A' \longrightarrow R\text{Hom}_{Y'}(Lg^*Rf_*M, Lg^*K)$$

in $D(A')$. This is the arrow More on Algebra, Equation (87.1.1) where we have used Derived Categories of Schemes, Lemmas [3.5] and [9.8] to translate back and forth into algebra.

1. If $f$ is flat and of finite presentation, the complex $Rf_*M$ is perfect on $Y$ by Derived Categories of Schemes, Lemma [26.4] and $\beta$ is an isomorphism by More on Algebra, Lemma [87.2] part (1).

2. If $f$ is perfect and $Y$ Noetherian, the complex $Rf_*M$ is perfect on $Y$ by More on Morphisms, Lemma [53.12] and $\beta$ is an isomorphism as before.

3. If $g$ has finite tor dimension and $Y$ is Noetherian, the complex $Rf_*M$ is pseudo-coherent on $Y$ (Derived Categories of Schemes, Lemmas [10.1] and [9.3]) and $\beta$ is an isomorphism by More on Algebra, Lemma [87.2] part (4).

We conclude that we obtain the same answer as in the previous paragraph.

In the rest of the proof we show that the identifications of the left and right hand side of (6.2.1) given in the second and third paragraph are in fact given by (6.2.1). To make our formulas manageable we will use $(-, -)_X = R\text{Hom}_X(-, -)$, use $- \otimes A'$ in stead of $- \otimes^L_A A'$, and we will abbreviate $g^* = Lg^*$ and $f_* = Rf_*$. Consider the following commutative diagram

$$
\begin{array}{cccccc}
((g')^*M, (g')^*a(K))_X & \xleftarrow{\alpha} & (M, a(K))_X \otimes A' & \longrightarrow & (f_*M, K)_Y \otimes A' \\
\downarrow & & & & \downarrow \\
((g')^*M, (g')^*a(g^*K))_X & \xleftarrow{\alpha} & (M, a(g^*K))_X \otimes A' & \longrightarrow & (f_*M, g^*K)_Y \otimes A' \\
\downarrow & & & & \downarrow \\
((g')^*M, (g')^*a'(g^*K))_X & \xleftarrow{\alpha} & (M, g^*a'(g^*K))_X \otimes A' & \longrightarrow & (f_*M, K) \otimes A' \\
\downarrow & & & & \downarrow \beta \\
((g')^*M, a'(g^*K))_X & \xleftarrow{\mu} & (f'_*(g')^*M, g^*K)_Y & \longrightarrow & (g^*f_*M, g^*K)_Y
\end{array}
$$
The arrows labeled $\alpha$ are the maps from Derived Categories of Schemes, Lemma 21.4 for the diagram with corners $X', X, Y', Y$. The upper part of the diagram is commutative as the horizontal arrows are functorial in the entries. The middle vertical arrows come from the invertible transformation $g'_* \circ \alpha \rightarrow a \circ g_*$ of Lemma 4.1 and therefore the middle square is commutative. Going down the left hand side is (6.2.1). The upper horizontal arrows provide the identifications used in the second paragraph of the proof. The lower horizontal arrows including $\beta$ provide the identifications used in the third paragraph of the proof. Given $E \in D(A)$, $E' \in D(A')$, and $c : E \rightarrow E'$ in $D(A)$ we will denote $\mu_c : E \otimes A' \rightarrow E'$ the map induced by $c$ and the adjointness of restriction and base change; if $c$ is clear we write $\mu = \mu_c$, i.e., we drop $c$ from the notation. The map $\mu$ in the diagram is of this form with $c$ given by the identification $(E, g'_* a(g^* K))_X = ((g^')^* E, a'(g^* K))_{Y'}$; the triangle involving $\mu$ is commutative by Derived Categories of Schemes, Remark 21.5.

Observe that

\[
(M, a(g^* K))_X \xrightarrow{(f_* M, g_* g^* K)} (g^* f_* M, g^* K)_{Y'}
\]

\[
(M, g'_* a'(g^* K))_X \xrightarrow{((g')^* M, a'(g^* K))} (f'_*(g')^* M, g^* K)_{Y'}
\]

is commutative by the very definition of the transformation $g'_* \circ \alpha \rightarrow a \circ g_*$. Letting $\mu'$ be as above corresponding to the identification $(f_* M, g_* g^* K)_X = (g^* f_* M, g^* K)_{Y'}$, then the hexagon commutes as well. Thus it suffices to show that $\beta$ is equal to the composition of $(f_* M, g_* g^* K)_Y \otimes A' \rightarrow (f_* M, g_* g^* K)_X \otimes A'$ and $\mu'$. To do this, it suffices to prove the two induced maps $(f_* M, K)_Y \rightarrow (g^* f_* M, g^* K)_{Y'}$ are the same. In other words, it suffices to show the diagram

\[
\begin{array}{ccc}
R \Hom_A(E, K) & \xrightarrow{\text{induced by } \beta} & R \Hom_{A'}(E \otimes_A^L A', K \otimes_A^L A') \\
\downarrow & & \downarrow \\
R \Hom_A(E, K \otimes_A^L A')
\end{array}
\]

commutes for all $E, K \in D(A)$. Since this is how $\beta$ is constructed in More on Algebra, Section 87 the proof is complete.

7. Right adjoint of pushforward and trace maps

Let $f : X \rightarrow Y$ be a morphism of quasi-compact and quasi-separated schemes. Let $a : D_{QCoh}(\mathcal{O}_Y) \rightarrow D_{QCoh}(\mathcal{O}_X)$ be the right adjoint as in Lemma 3.1. By Categories, Section 24 we obtain a transformation of functors

\[
\Tr_f : Rf_* \circ a \rightarrow \text{id}
\]

The corresponding map $\Tr_{f, K} : Rf_* a(K) \rightarrow K$ for $K \in D_{QCoh}(\mathcal{O}_Y)$ is sometimes called the trace map. This is the map which has the property that the bijection

\[
\Hom_X(L, a(K)) \rightarrow \Hom_Y(Rf_* L, K)
\]

for $L \in D_{QCoh}(\mathcal{O}_X)$ which characterizes the right adjoint is given by

\[
\varphi \mapsto \Tr_{f, K} \circ Rf_* \varphi
\]
The isomorphism
\[ Rf_* \mathcal{R} \text{Hom}_{\mathcal{O}_X}(L, a(K)) \to R\mathcal{R} \text{Hom}_{
abla}(Rf_* L, K) \]
of Lemma 3.6 comes about by composition with $\text{Tr}_{f,K}$. Every trace map we are going to consider in this section will be a special case of this trace map. Before we discuss some special cases we show that formation of the trace map commutes with base change.

**Lemma 7.1** (Trace map and base change). Suppose we have a diagram (4.0.1) where $f$ and $g$ are tor independent. Then the maps $1 \ast \text{Tr}_f : Lg^* \circ Rf_* \circ a \to Lg^*$ and $\text{Tr}_f \ast 1 : Rf'_* \circ a' \circ Lg^* \to Lg^*$ agree via the base change maps $\beta : Lg^* \circ Rf_* \to Rf'_* \circ L(g')^*$ (Cohomology, Remark 29.3) and $\alpha : L(g')^* \circ a \to a' \circ Lg^*$ (5.0.1).

More precisely, the diagram
\[
\begin{array}{ccc}
Lg^* \circ Rf_* \circ a & \xrightarrow{1 \ast \text{Tr}_f} & Lg^* \\
\downarrow{\beta \ast 1} & & \downarrow{\text{Tr}_f \ast 1} \\
Rf'_* \circ L(g')^* \circ a & \xrightarrow{1 \ast \alpha} & Rf'_* \circ a' \circ Lg^*
\end{array}
\]
of transformations of functors commutes.

**Proof.** In this proof we write $f_*$ for $Rf_*$ and $g^*$ for $Lg^*$ and we drop $\ast$ products with identities as one can figure out which ones to add as long as the source and target of the transformation is known. Recall that $\beta : g^* \circ f_* \to f'_* \circ (g')^*$ is an isomorphism and that $\alpha$ is defined using the isomorphism $\beta' : g'_* \circ a' \to a \circ g_*$ which is the adjoint of $\beta$, see Lemma 4.1 and its proof. First we note that the top horizontal arrow of the diagram in the lemma is equal to the composition
\[ g^* \circ f_* \circ a \to g^* \circ f_* \circ a \circ g_* \circ g^* \to g^* \circ g_* \circ g^* \to g^* \]
where the first arrow is the unit for $(g^*, g_*)$, the second arrow is $\text{Tr}_f$, and the third arrow is the counit for $(g^*, g_*)$. This is a simple consequence of the fact that the composition $g^* \to g^* \circ g_* \to g^*$ of unit and counit is the identity. Consider the diagram

In this diagram the two squares commute Categories, Lemma 27.2 or more simply the discussion preceding Categories, Definition 27.1. The triangle commutes by the discussion above. By Categories, Lemma 27.1 the square
\[
\begin{array}{ccc}
g^* \circ f_* \circ g_* \circ a' & \xrightarrow{\beta} & f'_* \circ (g')^* \circ g'_* \circ a' \\
\downarrow{\beta'} & & \downarrow{\beta'} \\
g^* \circ f_* \circ a \circ g_* & \xrightarrow{\beta} & f'_* \circ (g')^* \circ g'_* \circ a'
\end{array}
\]
and the triangle commutes by Categories, Lemma 27.1.
commutes which implies the pentagon in the big diagram commutes. Since $\beta$ and $\beta^\vee$ are isomorphisms, and since going on the outside of the big diagram equals $\text{Tr}_f \circ \alpha \circ \beta$ by definition this proves the lemma.

Let $f : X \to Y$ be a morphism of quasi-compact and quasi-separated schemes. Let $a : D_{QCoh}(\mathcal{O}_Y) \to D_{QCoh}(\mathcal{O}_X)$ be the right adjoint of $Rf_*$ as in Lemma 3.1. By Categories, Section 24 we obtain a transformation of functors

$$\eta_f : \text{id} \to a \circ Rf_*$$

which is called the unit of the adjunction.

**Lemma 7.2.** Suppose we have a diagram [4.0.1] where $f$ and $g$ are tor independent. Then the maps $1 \star \eta_f : L(g')^* \to L(g')^* \circ a \circ Rf_*$ and $\eta_f \star 1 : L(g')^* \to a' \circ Rf_* \circ L(g')^*$ agree via the base change maps $\beta : Lg^* \circ Rf_* \to Rf'_* \circ L(g')^*$ (Cohomology, Remark 29.3) and $\alpha : L(g')^* \circ a \to a' \circ Lg^*$ [5.0.1]. More precisely, the diagram

$$
\begin{array}{ccc}
L(g')^* & \xrightarrow{\eta_f \star 1} & L(g')^* \circ a \circ Rf_* \\
\downarrow{\eta_f} & & \downarrow{\alpha} \\
\eta_f & & \\
\end{array}
$$

of transformations of functors commutes.

**Proof.** This proof is dual to the proof of Lemma 7.1. In this proof we write $f_*$ for $Rf_*$ and $g^*$ for $Lg^*$ and we drop $\star$ products with identities as one can figure out which ones to add as long as the source and target of the transformation is known. Recall that $\beta : g^* \circ f_* \to f'_* \circ (g')^*$ is an isomorphism and that $\alpha$ is defined using the isomorphism $\beta^\vee : g'_* \circ a' \to a \circ g_*$, which is the adjoint of $\beta$, see Lemma 4.1 and its proof. First we note that the left vertical arrow of the diagram in the lemma is equal to the composition

$$(g')^* \to (g')^* \circ g'_* \circ (g')^* \to (g')^* \circ g'_* \circ a' \circ f'_* \circ (g')^* \to a' \circ f'_* \circ (g')^*$$

where the first arrow is the unit for $((g')^*, g'_*)$, the second arrow is $\eta_f \star 1$, and the third arrow is the counit for $((g')^*, g'_*)$. This is a simple consequence of the fact that the composition $(g')^* \to (g')^* \circ (g')_* \circ (g')^* \to (g')^*$ of unit and counit is the identity. Consider the diagram

In this diagram the two squares commute Categories, Lemma 27.2 or more simply the discussion preceding Categories, Definition 27.1. The triangle commutes by the
discussion above. By the dual of Categories, Lemma 24.7 the square
\[
\begin{array}{ccc}
\text{id} & \rightarrow & g'_* \circ a' \circ g^* \circ f_* \\
\downarrow & & \downarrow \beta \\
g'_* \circ a' \circ g^* \circ f_* & \rightarrow & a \circ g_* \circ f'_* \circ (g')^*
\end{array}
\]
commutes which implies the pentagon in the big diagram commutes. Since \(\beta\) and \(\beta^v\) are isomorphisms, and since going on the outside of the big diagram equals \(\beta \circ \alpha \circ \eta_f\) by definition this proves the lemma. \(\square\)

**Example 7.3.** Let \(A \rightarrow B\) be a ring map. Let \(Y = \text{Spec}(A)\) and \(X = \text{Spec}(B)\) and \(f : X \rightarrow Y\) the morphism corresponding to \(A \rightarrow B\). As seen in Example 3.2 the right adjoint of \(Rf_* : D\text{QCoh}(\mathcal{O}_X) \rightarrow D\text{QCoh}(\mathcal{O}_Y)\) sends an object \(K\) of \(D(A) = D\text{QCoh}(\mathcal{O}_Y)\) to \(R\text{Hom}(B,K)\) in \(D(B) = D\text{QCoh}(\mathcal{O}_X)\). The trace map is the map
\[
\text{Tr}_{f,K} : R\text{Hom}(B,K) \rightarrow R\text{Hom}(A,K) = K
\]
induced by the \(A\)-module map \(A \rightarrow B\).

8. Right adjoint of pushforward and pullback

Let \(f : X \rightarrow Y\) be a morphism of quasi-compact and quasi-separated schemes. The map \(L^f*K \otimes^L_\mathcal{O}_X a(\mathcal{O}_Y) \rightarrow a(K \otimes^L_\mathcal{O}_Y L)\) defined above for \(K,L \in D\text{QCoh}(\mathcal{O}_Y)\) is an isomorphism if \(K\) is perfect. In particular, (8.0.1) is an isomorphism if \(K\) is perfect.

**Proof.** Let \(K^v\) be the “dual” to \(K\), see Cohomology, Lemma 43.11. For \(M \in D\text{QCoh}(\mathcal{O}_X)\) we have
\[
\text{Hom}_{D(\mathcal{O}_Y)}(Rf_*M,K \otimes^L_\mathcal{O}_Y L) = \text{Hom}_{D(\mathcal{O}_Y)}(Rf_*M \otimes^L_\mathcal{O}_Y K^v,L)
= \text{Hom}_{D(\mathcal{O}_X)}(M \otimes^L_\mathcal{O}_X Lf^*K^v,a(L))
= \text{Hom}_{D(\mathcal{O}_X)}(M,Lf^*K \otimes^L_\mathcal{O}_X a(L))
\]
Second equality by the definition of \(a\) and the projection formula (Cohomology, Lemma 45.3) or the more general Derived Categories of Schemes, Lemma 21.1. Hence the result by the Yoneda lemma. \(\square\)
Lemma 8.2. Suppose we have a diagram (4.0.1) where \( f \) and \( g \) are tor independent. Let \( K \in D_{QCoh}(O_Y) \). The diagram

\[
\begin{array}{ccc}
L(g')^* (Lf^* K \otimes_{O_X} a(O_Y)) & \longrightarrow & L(g')^* a(K) \\
\downarrow & & \downarrow \\
L(f')^* Lg^* K \otimes_{O_X} a'(O_Y) & \longrightarrow & a'(Lg^* K)
\end{array}
\]

commutes where the horizontal arrows are the maps (8.0.1) for \( K \) and \( Lg^* K \) and the vertical maps are constructed using Cohomology, Remark 29.3 and (5.0.1).

Proof. In this proof we will write \( f_* \) for \( Rf_* \) and \( f^* \) for \( Lf^* \), etc, and we will write \( \otimes \) for \( \otimes_{O_X} \), etc. Let us write (8.0.1) as the composition

\[
f^* K \otimes a(O_Y) \to a(f_* (f^* K \otimes a(O_Y))) \\
\leftarrow a(K \otimes f_* a(O_K)) \\
\to a(K \otimes O_Y) \\
\to a(K)
\]

Here the first arrow is the unit \( \eta_f \), the second arrow is \( a \) applied to Cohomology, Equation (45.2.1) which is an isomorphism by Derived Categories of Schemes, Lemma 21.1 the third arrow is \( a \) applied to \( id_K \otimes Tr_f \) and the fourth arrow is \( a \) applied to the isomorphism \( K \otimes O_Y = K \). The proof of the lemma consists in showing that each of these maps gives rise to a commutative square as in the statement of the lemma. For \( \eta_f \) and \( Tr_f \) this is Lemmas 7.2 and 7.1. For the arrow using Cohomology, Equation (45.2.1) this is Cohomology, Remark 45.5. For the multiplication map it is clear. This finishes the proof. \( \square \)

Lemma 8.3. Let \( f : X \to Y \) be a proper morphism of Noetherian schemes. Let \( V \subset Y \) be an open such that \( f^{-1}(V) \to V \) is an isomorphism. Then for \( K \in D_{QCoh}^+(O_Y) \) the map (8.0.1) restricts to an isomorphism over \( f^{-1}(V) \).

Proof. By Lemma 4.4 the map (4.1.1) is an isomorphism for objects of \( D_{QCoh}^+(O_Y) \). Hence Lemma 8.2 tells us the restriction of (8.0.1) for \( K \) to \( f^{-1}(V) \) is the map (8.0.1) for \( K|_V \) and \( f^{-1}(V) \to V \). Thus it suffices to show that the map is an isomorphism when \( f \) is the identity morphism. This is clear. \( \square \)

Lemma 8.4. Let \( f : X \to Y \) and \( g : Y \to Z \) be composable morphisms of quasi-compact and quasi-separated schemes and set \( h = g \circ f \). Let \( a, b, c \) be the adjoints of Lemma 3.1 for \( f, g, h \). For any \( K \in D_{QCoh}(O_Z) \) the diagram

\[
\begin{array}{ccc}
Lf^* (Lg^* K \otimes_{O_Y} b(O_Z)) \otimes_{O_X} a(O_Y) & \longrightarrow & a(Lg^* K \otimes_{O_Y} b(O_Z)) \\
Lt^* K \otimes_{O_X} Lf^* b(O_Z) \otimes_{O_X} a(O_Y) & \longrightarrow & Lh^* K \otimes_{O_X} c(O_Z)
\end{array}
\]

is commutative where the arrows are (8.0.1) and we have used \( Lh^* = Lf^* \circ Lg^* \) and \( c = a \circ b \).
Proof. In this proof we will write $f_*$ for $Rf_*$ and $f^*$ for $Lf^*$, etc, and we will write $\otimes$ for $\otimes_{\mathcal{O}_X}$, etc. The composition of the top arrows is adjoint to a map

$g_*f_*(f^*(g^*K \otimes b(\mathcal{O}_Z)) \otimes a(\mathcal{O}_Y)) \to K$

The left hand side is equal to $K \otimes g_*f_*(f^*(b(\mathcal{O}_Z) \otimes a(\mathcal{O}_Y)))$ by Derived Categories of Schemes, Lemma 21.1 and inspection of the definitions shows the map comes from Lemma 21.1 and tensored with id).

The composition of the top arrows is adjoint to a map $\epsilon \circ f_*a \to id$. Similarly, the composition of the lower horizontal arrows is adjoint to id tensored with the composition

$g_*f_*((b(\mathcal{O}_Z) \otimes a(\mathcal{O}_Y))) \xrightarrow{g_*f_*\delta} g_*f_*(ab(\mathcal{O}_Z)) \xrightarrow{g_*f_*\gamma} g_*(b(\mathcal{O}_Z)) \to \mathcal{O}_Z$

where $\gamma$ comes from the counit map $g_*b \to id$. Similarly, the composition is the map whose adjoint is the composition

$f_*(f^*(b(\mathcal{O}_Z) \otimes a(\mathcal{O}_Y))) \xleftarrow{f_*f_*\delta} b(\mathcal{O}_Z) \otimes f_*a(\mathcal{O}_Y) \xrightarrow{\alpha} b(\mathcal{O}_Z)$

By general properties of adjoint functors, adjoint maps, and counits (see Categories, Section 24) we have $\gamma \circ f_*\delta = \alpha \circ \epsilon^{-1}$ as desired. \hfill $\square$

9. Right adjoint of pushforward for closed immersions

Let $i : (Z, \mathcal{O}_Z) \to (X, \mathcal{O}_X)$ be a morphism of ringed spaces such that $i$ is a homomorphism onto a closed subset and such that $i^\sharp : \mathcal{O}_X \to \mathcal{O}_Z$ is surjective. (For example a closed immersion of schemes.) Let $\mathcal{I} = \text{Ker}(i^\sharp)$. For a sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ the sheaf

$\mathcal{H}(\mathcal{O}_Z, \mathcal{F})$

is a sheaf of $\mathcal{O}_X$-modules annihilated by $\mathcal{I}$. Hence by Modules, Lemma 13.4 there is a sheaf of $\mathcal{O}_Z$-modules, which we will denote $\mathcal{H}(\mathcal{O}_Z, \mathcal{F})$, such that

$i_* \mathcal{H}(\mathcal{O}_Z, \mathcal{F}) = \mathcal{H}(\mathcal{O}_X, i_*\mathcal{O}_Z, \mathcal{F})$

as $\mathcal{O}_X$-modules. We spell out what this means.

Lemma 9.1. With notation as above. The functor $\mathcal{H}(\mathcal{O}_Z, -)$ is a right adjoint to the functor $i_* : \mathcal{M}(\mathcal{O}_Z) \to \mathcal{M}(\mathcal{O}_X)$. For $V \subset Z$ open we have

$\Gamma(V, \mathcal{H}(\mathcal{O}_Z, \mathcal{F})) = \{s \in \Gamma(U, \mathcal{F}) | \mathcal{I}s = 0\}$

where $U \subset X$ is an open whose intersection with $Z$ is $V$.

Proof. Let $\mathcal{G}$ be a sheaf of $\mathcal{O}_Z$-modules. Then

$\mathcal{H}(\mathcal{O}_X, i_*\mathcal{G}, \mathcal{F}) = \mathcal{H}(\mathcal{O}_X, \mathcal{H}(\mathcal{O}_Z, \mathcal{F}), i_*\mathcal{G}) = \mathcal{H}(\mathcal{O}_Z, \mathcal{G}, \mathcal{H}(\mathcal{O}_Z, \mathcal{F}))$

The first equality by Modules, Lemma 20.6 and the second by the fully faithfulness of $i_*$, see Modules, Lemma 13.4. The description of sections is left to the reader. \hfill $\square$

The functor

$\mathcal{M}(\mathcal{O}_X) \to \mathcal{M}(\mathcal{O}_Z), \quad \mathcal{F} \mapsto \mathcal{H}(\mathcal{O}_Z, \mathcal{F})$

is left exact and has a derived extension

$R\mathcal{H}(\mathcal{O}_Z, -) : D(\mathcal{O}_X) \to D(\mathcal{O}_Z)$. 

- 

-
Lemma 9.2. With notation as above. The functor $R\mathcal{H}om(\mathcal{O}_Z, -)$ is the right adjoint of the functor $Ri_*: D(\mathcal{O}_Z) \to D(\mathcal{O}_X)$.

Proof. This is a consequence of the fact that $i_*$ and $\mathcal{H}om(\mathcal{O}_Z, -)$ are adjoint functors by Lemma 9.1. See Derived Categories, Lemma 28.5.

Lemma 9.3. With notation as above. We have

$$Ri_* R\mathcal{H}om(\mathcal{O}_Z, K) = R\mathcal{H}om_{\mathcal{O}_X}(i_* \mathcal{O}_Z, K)$$

in $D(\mathcal{O}_X)$ for all $K$ in $D(\mathcal{O}_X)$.

Proof. This is immediate from the construction of the functor $R\mathcal{H}om(\mathcal{O}_Z, -)$.

Lemma 9.4. With notation as above. For $M \in D(\mathcal{O}_Z)$ we have

$$R\mathcal{H}om_{\mathcal{O}_X}(Ri_* M, K) = Ri_* R\mathcal{H}om_{\mathcal{O}_Z}(M, R\mathcal{H}om(\mathcal{O}_Z, K))$$

in $D(\mathcal{O}_X)$ for all $K$ in $D(\mathcal{O}_X)$.

Proof. This is immediate from the construction of the functor $R\mathcal{H}om(\mathcal{O}_Z, -)$ and the fact that if $K^\bullet$ is a $K$-injective complex of $\mathcal{O}_X$-modules, then $\mathcal{H}om(\mathcal{O}_Z, K^\bullet)$ is a $K$-injective complex of $\mathcal{O}_Z$-modules, see Derived Categories, Lemma 29.9.

Lemma 9.5. Let $i: Z \to X$ be a pseudo-coherent closed immersion of schemes (any closed immersion if $X$ is locally Noetherian). Then

(1) $R\mathcal{H}om(\mathcal{O}_Z, -)$ maps $D^+_{QCoh}(\mathcal{O}_X)$ into $D^+_{QCoh}(\mathcal{O}_Z)$, and

(2) if $X = \text{Spec}(A)$ and $Z = \text{Spec}(B)$, then the diagram

\[
\begin{array}{ccc}
D^+(B) & \longrightarrow & D^+_{QCoh}(\mathcal{O}_Z) \\
\uparrow R\mathcal{H}om(B, -) & & \uparrow R\mathcal{H}om(\mathcal{O}_Z, -) \\
D^+(A) & \longrightarrow & D^+_{QCoh}(\mathcal{O}_X)
\end{array}
\]

is commutative.

Proof. To explain the parenthetical remark, if $X$ is locally Noetherian, then $i$ is pseudo-coherent by More on Morphisms, Lemma 52.9.

Let $K$ be an object of $D^+_{QCoh}(\mathcal{O}_X)$. To prove (1), by Morphisms, Lemma 4.1 it suffices to show that $i_*$ applied to $H^n (R\mathcal{H}om(\mathcal{O}_Z, K))$ produces a quasi-coherent module on $X$. By Lemma 9.3 this means we have to show that $R\mathcal{H}om_{\mathcal{O}_X}(i_* \mathcal{O}_Z, K)$ is in $D_{QCoh}(\mathcal{O}_X)$. Since $i$ is pseudo-coherent the sheaf $\mathcal{O}_Z$ is a pseudo-coherent $\mathcal{O}_X$-module. Hence the result follows from Derived Categories of Schemes, Lemma 9.8.

Assume $X = \text{Spec}(A)$ and $Z = \text{Spec}(B)$ as in (2). Let $I^\bullet$ be a bounded below complex of injective $A$-modules representing an object $K$ of $D^+(A)$. Then we know that $R\mathcal{H}om(B, K) = \text{Hom}_A(B, I^\bullet)$ viewed as a complex of $B$-modules. Choose a quasi-isomorphism

$$\tilde{I}^\bullet \to I^\bullet$$

where $\tilde{I}^\bullet$ is a bounded below complex of injective $\mathcal{O}_X$-modules. It follows from the description of the functor $\mathcal{H}om(\mathcal{O}_Z, -)$ in Lemma 9.1 that there is a map

$$\text{Hom}_A(B, I^\bullet) \to \Gamma(Z, \mathcal{H}om(\mathcal{O}_Z, \tilde{I}^\bullet))$$
Observe that $\text{Hom}(O_Z, I^\bullet)$ represents $R\text{Hom}(O_Z, \bar{K})$. Applying the universal property of the $\sim$ functor we obtain a map

$$\text{Hom}_A(B, I^\bullet) \to R\text{Hom}(O_Z, \bar{K})$$

in $D(O_Z)$. We may check that this map is an isomorphism in $D(O_Z)$ after applying $i_*$. However, once we apply $i_*$ we obtain the isomorphism of Derived Categories of Schemes, Lemma 9.8 via the identification of Lemma 9.3.

□

Lemma 9.6. Let $i : Z \to X$ be a closed immersion of schemes. Assume $X$ is a locally Noetherian. Then $R\text{Hom}(O_Z, -)$ maps $D^{+}_{\text{Coh}}(O_X)$ into $D^{+}_{\text{Coh}}(O_Z)$.

Proof. The question is local on $X$, hence we may assume that $X$ is affine. Say $X = \text{Spec}(A)$ and $Z = \text{Spec}(B)$ with $A$ Noetherian and $A \to B$ surjective. In this case, we can apply Lemma 9.5 to translate the question into algebra. The corresponding algebra result is a consequence of Dualizing Complexes, Lemma 13.4. □

Lemma 9.7. Let $X$ be a quasi-compact and quasi-separated scheme. Let $i : Z \to X$ be a pseudo-coherent closed immersion (if $X$ is Noetherian, then any closed immersion is pseudo-coherent). Let $a : D^{+}_{\text{QCoh}}(O_X) \to D^{+}_{\text{QCoh}}(O_Z)$ be the right adjoint to $Ri_*$. Then there is a functorial isomorphism

$$a(K) = R\text{Hom}(O_Z, K)$$

for $K \in D^{+}_{\text{QCoh}}(O_X)$.

Proof. (The parenthetical statement follows from More on Morphisms, Lemma 52.9.) By Lemma 9.2 the functor $R\text{Hom}(O_Z, -)$ is a right adjoint to $Ri_* : D(O_Z) \to D(O_X)$. Moreover, by Lemma 9.5 and Lemma 3.5 both $R\text{Hom}(O_Z, -)$ and a map $D^{+}_{\text{QCoh}}(O_X)$ into $D^{+}_{\text{QCoh}}(O_Z)$. Hence we obtain the isomorphism by uniqueness of adjoint functors. □

Example 9.8. If $i : Z \to X$ is closed immersion of Noetherian schemes, then the diagram

$$\begin{array}{ccc}
i_*a(K) & \xrightarrow{\text{Tr}_{i, K}} & K \\
\| & & \\
i_*\text{Hom}(O_Z, K) & \xrightarrow{R\text{Hom}_{O_X}(i_*O_Z, K)} & K
\end{array}$$

is commutative for $K \in D^{+}_{\text{QCoh}}(O_X)$. Here the horizontal equality sign is Lemma 9.3 and the lower horizontal arrow is induced by the map $O_X \to i_*O_Z$. The commutativity of the diagram is a consequence of Lemma 9.7.

10. Right adjoint of pushforward for closed immersions and base change

Consider a cartesian diagram of schemes

$$\begin{array}{ccc}
Z' & \xrightarrow{i'} & X' \\
\downarrow g & & \downarrow f \\
Z & \xrightarrow{i} & X
\end{array}$$
In the situation above, assume
\[ (10.0.1) \quad Lg^* R\mathcal{H}om(O_Z, K) \to R\mathcal{H}om(O_{Z'}, Lf^* K) \]
in \( D(O_{Z'}) \) functorial for \( K \) in \( D(O_X) \). Namely, by adjointness of Lemma 9.2 such an arrow is the same thing as a map
\[ Ri'_* Lg^* R\mathcal{H}om(O_Z, K) \to Lf^* K \]
in \( D(O_X) \). By tor independence we have \( Ri'_* \circ Lg^* = Lf^* \circ Ri_* \) (see Derived Categories of Schemes, Lemma 21.7). Thus this is the same thing as a map
\[ Lf^* Ri_* R\mathcal{H}om(O_Z, K) \to Lf^* K \]
For this we can use \( Lf^*(can) \) where \( can : Ri_* R\mathcal{H}om(O_Z, K) \to K \) is the counit of the adjunction.

0E2L **Lemma 10.1.** In the situation above, the map \( (10.0.1) \) is an isomorphism if and only if the base change map
\[ Lf^* R\mathcal{H}om_{O_X}(O_Z, K) \to R\mathcal{H}om_{O_{X'}}(O_{Z'}, Lf^* K) \]
of Cohomology, Remark 36.11 is an isomorphism.

**Proof.** The statement makes sense because \( O_{Z'} = Lf^* O_Z \) by the assumed tor independence. Since \( i'_* \) is exact and faithful we see that it suffices to show the map \( (10.0.1) \) is an isomorphism after applying \( Ri'_* \). Since \( Ri'_* \circ Lg^* = Lf^* \circ Ri_* \) by the assumed tor independence and Derived Categories of Schemes, Lemma 21.7 we obtain a map
\[ Lf^* Ri_* R\mathcal{H}om(O_Z, K) \to Ri'_* R\mathcal{H}om(O_{Z'}, Lf^* K) \]
whose source and target are as in the statement of the lemma by Lemma 9.3. We omit the verification that this is the same map as the one constructed in Cohomology, Remark 36.11. \( \square \)

0E2M **Lemma 10.2.** In the situation above, assume \( f \) is flat and \( i \) pseudo-coherent. Then \( (10.0.1) \) is an isomorphism for \( K \) in \( D^{QCoh}_A(O_X) \).

**Proof.** First proof. To prove this map is an isomorphism, we may work locally. Hence we may assume \( X, X', Z, Z' \) are affine, say corresponding to the rings \( A, A', B, B' \). Then \( B \) and \( A' \) are tor independent over \( A \). By Lemma 10.1 it suffices to check that
\[ R\mathcal{H}om_A(B, K) \otimes^L_A A' = R\mathcal{H}om_{A'}(B', K \otimes^L_A A') \]
in \( D(A') \) for all \( K \in D^+(A) \). Here we use Derived Categories of Schemes, Lemma 9.8 and the fact that \( B, \) resp. \( B' \) is pseudo-coherent as an \( A \)-module, resp. \( A' \)-module to compare derived hom on the level of rings and schemes. The displayed equality follows from More on Algebra, Lemma 36.3 part (3). See also the discussion in Dualizing Complexes, Section 14.

Second proof. Let \( z' \in Z' \) with image \( z \in Z \). First show that \( (10.0.1) \) on stalks at \( z' \) induces the map
\[ R\mathcal{H}om(O_{Z,z}, K_z) \otimes^L_{O_{Z,z}} O_{Z',z'} \to R\mathcal{H}om(O_{Z',z'}, K_z \otimes^L_{O_{X,z}} O_{X',z'}) \]
\[ ^2 \text{This proof shows it suffices to assume } K \text{ is in } D^+(O_X). \]
from Dualizing Complexes, Equation (14.0.1). Namely, the constructions of these maps are identical. Then apply Dualizing Complexes, Lemma 14.2. □

**Lemma 10.3.** Let \( i : Z \to X \) be a pseudo-coherent closed immersion of schemes. Let \( M \in D_{QCoh}(\mathcal{O}_X) \) locally have tor-amplitude in \([a, \infty)\). Let \( K \in D^+_{QCoh}(\mathcal{O}_X) \). Then there is a canonical isomorphism

\[
R\mathcal{H}om(\mathcal{O}_Z, K) \otimes^L_{\mathcal{O}_Z} Li^* M = R\mathcal{H}om(\mathcal{O}_Z, K \otimes^L_{\mathcal{O}_X} M)
\]

in \( D(\mathcal{O}_Z) \).

**Proof.** A map from LHS to RHS is the same thing as a map

\[
Ri_* R\mathcal{H}om(\mathcal{O}_Z, K) \otimes^L_{\mathcal{O}_X} M \to K \otimes^L_{\mathcal{O}_X} M
\]

by Lemmas 9.2 and 9.3. For this map we take the counit \( Ri_* R\mathcal{H}om(\mathcal{O}_Z, K) \to K \) tensored with \( \text{id}_M \). To see this map is an isomorphism under the hypotheses given, translate into algebra using Lemma 9.5 and then for example use More on Algebra, Lemma 86.3 part (3). Instead of using Lemma 9.5 you can look at stalks as in the second proof of Lemma 10.2. □

### 11. Right adjoint of pushforward for finite morphisms

If \( i : Z \to X \) is a closed immersion of schemes, then there is a right adjoint \( \mathcal{H}om(\mathcal{O}_Z, -) \) to the functor \( i_* : \text{Mod}(\mathcal{O}_Z) \to \text{Mod}(\mathcal{O}_X) \) whose derived extension \( R\mathcal{H}om(\mathcal{O}_Z, -) \) is the right adjoint to \( Ri_* : D(\mathcal{O}_Z) \to D(\mathcal{O}_X) \). See Section 9. In the case of a finite morphism \( f : Y \to X \) this strategy cannot work, as the functor \( f_* : \text{Mod}(\mathcal{O}_Y) \to \text{Mod}(\mathcal{O}_X) \) is not exact in general and hence does not have a right adjoint. A replacement is to consider the exact functor \( \text{Mod}(f_* \mathcal{O}_Y) \to \text{Mod}(\mathcal{O}_X) \) and consider the corresponding right adjoint and its derived extension.

Let \( f : Y \to X \) be an affine morphism of schemes. For a sheaf of \( \mathcal{O}_X \)-modules \( \mathcal{F} \) the sheaf

\[
\mathcal{H}om_{\mathcal{O}_X}(f_* \mathcal{O}_Y, \mathcal{F})
\]

is a sheaf of \( f_* \mathcal{O}_Y \)-modules. We obtain a functor \( \text{Mod}(\mathcal{O}_X) \to \text{Mod}(f_* \mathcal{O}_Y) \) which we will denote \( \mathcal{H}om(f_* \mathcal{O}_Y, -) \).

**Lemma 11.1.** With notation as above. The functor \( \mathcal{H}om(f_* \mathcal{O}_Y, -) \) is a right adjoint to the restriction functor \( \text{Mod}(f_* \mathcal{O}_Y) \to \text{Mod}(\mathcal{O}_X) \). For an affine open \( U \subset X \) we have

\[
\Gamma(U, \mathcal{H}om(f_* \mathcal{O}_Y, \mathcal{F})) = \text{Hom}_A(B, \mathcal{F}(U))
\]

where \( A = \mathcal{O}_X(U) \) and \( B = \mathcal{O}_Y(f^{-1}(U)) \).

**Proof.** Adjointness follows from Modules, Lemma 20.6. As \( f \) is affine we see that \( f_* \mathcal{O}_Y \) is the quasi-coherent sheaf corresponding to \( B \) viewed as an \( A \)-module. Hence the description of sections over \( U \) follows from Schemes, Lemma 7.1. □

The functor \( \mathcal{H}om(f_* \mathcal{O}_Y, -) \) is left exact. Let

\[
R\mathcal{H}om(f_* \mathcal{O}_Y, -) : D(\mathcal{O}_X) \to D(f_* \mathcal{O}_Y)
\]

be its derived extension.

**Lemma 11.2.** With notation as above. The functor \( R\mathcal{H}om(f_* \mathcal{O}_Y, -) \) is the right adjoint of the functor \( D(f_* \mathcal{O}_Y) \to D(\mathcal{O}_X) \).

**Proof.** Follows from Lemma 11.1 and Derived Categories, Lemma 28.5. □
Lemma 11.3. With notation as above. The composition
\[ D(\mathcal{O}_X) \xrightarrow{R\text{Hom}(f_*\mathcal{O}_Y,-)} D(f_*\mathcal{O}_Y) \rightarrow D(\mathcal{O}_X) \]
is the functor \( K \mapsto R\text{Hom}_{\mathcal{O}_X}(f_*\mathcal{O}_Y, K) \).

Proof. This is immediate from the construction. \( \square \)

Lemma 11.4. Let \( f : Y \to X \) be a finite pseudo-coherent morphism of schemes (a finite morphism of Noetherian schemes is pseudo-coherent). The functor \( R\text{Hom}(f_*\mathcal{O}_Y, -) \) maps \( D^+_{\text{QCoh}}(\mathcal{O}_X) \) into \( D^+_{\text{QCoh}}(f_*\mathcal{O}_Y) \). If \( X \) is quasi-compact and quasi-separated, then the diagram
\[ \begin{array}{ccc}
D^+_{\text{QCoh}}(\mathcal{O}_X) & \xrightarrow{\alpha} & D^+_{\text{QCoh}}(\mathcal{O}_Y) \\
\downarrow \text{RHom}(f_*\mathcal{O}_Y,-) & & \downarrow \Phi \\
D^+_{\text{QCoh}}(f_*\mathcal{O}_Y) & & 
\end{array} \]
is commutative, where \( \alpha \) is the right adjoint of Lemma 3.1 for \( f \) and \( \Phi \) is the equivalence of Derived Categories of Schemes, Lemma 5.3.

Proof. (The parenthetical remark follows from More on Morphisms, Lemma 52.9) Since \( f \) is pseudo-coherent, the \( \mathcal{O}_X \)-module \( f_*\mathcal{O}_Y \) is pseudo-coherent, see More on Morphisms, Lemma 52.8. Thus \( R\text{Hom}(f_*\mathcal{O}_Y,-) \) maps \( D^+_{\text{QCoh}}(\mathcal{O}_X) \) into \( D^+_{\text{QCoh}}(f_*\mathcal{O}_Y) \), see Derived Categories of Schemes, Lemma 9.8. Then \( \Phi \circ \alpha \) and \( R\text{Hom}(f_*\mathcal{O}_Y,-) \) agree on \( D^+_{\text{QCoh}}(\mathcal{O}_X) \) because these functors are both right adjoint to the restriction functor \( D^+_{\text{QCoh}}(f_*\mathcal{O}_Y) \to D^+_{\text{QCoh}}(\mathcal{O}_X) \). To see this use Lemmas 3.3 and 11.2. \( \square \)

Remark 11.5. If \( f : Y \to X \) is a finite morphism of Noetherian schemes, then the diagram
\[ \begin{array}{ccc}
Rf_*a(K) & \xrightarrow{\text{Tr}_{f,K}} & K \\
\downarrow \text{RHom}_{\mathcal{O}_X}(f_*\mathcal{O}_Y, K) & & \downarrow \\
Rf_*a(K) & \xrightarrow{\text{Tr}_{f,K}} & K 
\end{array} \]
is commutative for \( K \in D^+_{\text{QCoh}}(\mathcal{O}_X) \). This follows from Lemma 11.4. The lower horizontal arrow is induced by the map \( \mathcal{O}_X \to f_*\mathcal{O}_Y \) and the upper horizontal arrow is the trace map discussed in Section 7.

12. Right adjoint of pushforward for proper flat morphisms

For proper, flat, and finitely presented morphisms of quasi-compact and quasi-separated schemes the right adjoint of pushforward enjoys some remarkable properties.

Lemma 12.1. Let \( Y \) be a quasi-compact and quasi-separated scheme. Let \( f : X \to Y \) be a morphism of schemes which is proper, flat, and of finite presentation. Let \( a \) be the right adjoint for \( Rf_* : D_{\text{QCoh}}(\mathcal{O}_X) \to D_{\text{QCoh}}(\mathcal{O}_Y) \) of Lemma 3.1. Then \( a \) commutes with direct sums.
Proof. Let $P$ be a perfect object of $D(O_X)$. By Derived Categories of Schemes, Lemma 26.4 the complex $Rf_*P$ is perfect on $Y$. Let $K_i$ be a family of objects of $D_{QCoh}(O_Y)$. Then

$$\text{Hom}_{D(O_X)}(P, a(\bigoplus K_i)) = \text{Hom}_{D(O_Y)}(Rf_*P, \bigoplus K_i)$$

$$= \bigoplus \text{Hom}_{D(O_Y)}(Rf_*P, K_i)$$

$$= \bigoplus \text{Hom}_{D(O_X)}(P, a(K_i))$$

because a perfect object is compact (Derived Categories of Schemes, Proposition 16.1). Since $D_{QCoh}(O_X)$ has a perfect generator (Derived Categories of Schemes, Theorem 14.3) we conclude that the map $\bigoplus a(K_i) \to a(\bigoplus K_i)$ is an isomorphism, i.e., $a$ commutes with direct sums. □

Lemma 12.2. Let $Y$ be a quasi-compact and quasi-separated scheme. Let $f : X \to Y$ be a morphism of schemes which is proper, flat, and of finite presentation. Let $a$ be the right adjoint for $Rf_* : D_{QCoh}(O_X) \to D_{QCoh}(O_Y)$ of Lemma 3.1. Then

1. for every closed $T \subset Y$ if $Q \in D_{QCoh}(Y)$ is supported on $T$, then $a(Q)$ is supported on $f^{-1}(T)$,
2. for every open $V \subset Y$ and any $K \in D_{QCoh}(O_Y)$ the map (4.1.1) is an isomorphism, and

Proof. This follows from Lemmas 4.3, 4.4, and 12.1. □

Lemma 12.3. Let $Y$ be a quasi-compact and quasi-separated scheme. Let $f : X \to Y$ be a morphism of schemes which is proper, flat, and of finite presentation. The map (8.0.1) is an isomorphism for every object $K$ of $D_{QCoh}(O_Y)$.

Proof. By Lemma 12.2 formation of the functors $a$ and $a'$ commutes with restriction to opens of $Y$ and $Y'$. Hence we may assume $Y' \to Y$ is a morphism of affine schemes, see Remark 6.1. In this case the statement follows from Lemma 6.2. □

The following lemma shows that the base change map (5.0.1) is an isomorphism for proper, flat morphisms of finite presentation. We will see in Example 15.2 that this does not remain true for perfect proper morphisms; in that case one has to make a tor independence condition.

Lemma 12.4. Let $g : Y' \to Y$ be a morphism of quasi-compact and quasi-separated schemes. Let $f : X \to Y$ be a proper, flat morphism of finite presentation. Then the base change map (5.0.1) is an isomorphism for all $K \in D_{QCoh}(O_Y)$.

Proof. By Lemma 12.2 formation of the functors $a$ and $a'$ commutes with restriction to opens of $Y$ and $Y'$. Hence we may assume $Y' \to Y$ is a morphism of affine schemes, see Remark 6.1. In this case the statement follows from Lemma 6.2. □

Remark 12.5. Let $Y$ be a quasi-compact and quasi-separated scheme. Let $f : X \to Y$ be a proper, flat morphism of finite presentation. Let $a$ be the adjoint of Lemma 3.1 for $f$. In this situation, $\omega^\bullet_{X/Y} = a(O_Y)$ is sometimes called...
the relative dualizing complex. By Lemma 12.3 there is a functorial isomorphism $a(K) = Lf^*K \otimes_{\mathcal{O}_X} \omega_{X/Y}^\bullet$ for $K \in D_{QCoh}(\mathcal{O}_Y)$. Moreover, the trace map

$$\text{Tr}_{f,\mathcal{O}_Y} : Rf_*\omega_{X/Y}^\bullet \to \mathcal{O}_Y$$

of Section 7 induces the trace map for all $K$ in $D_{QCoh}(\mathcal{O}_Y)$. More precisely the diagram

$$\begin{array}{ccc}
Rf_*a(K) & \xrightarrow{\text{Tr}_{f,K}} & K \\
\downarrow & & \downarrow \\
Rf_*(Lf^*K \otimes_{\mathcal{O}_X} \omega_{X/Y}^\bullet) & \xrightarrow{\text{id}_K \otimes \text{Tr}_{f,\mathcal{O}_Y}} & K
\end{array}$$

where the equality on the lower right is Derived Categories of Schemes, Lemma 21.1. If $g : Y' \to Y$ is a morphism of quasi-compact and quasi-separated schemes and $X' = Y' \times_Y X$, then by Lemma 12.4 we have $\omega_{X'/Y'}^\bullet = L(g')^*\omega_{X/Y}^\bullet$ where $g' : X' \to X$ is the projection and by Lemma 7.1 the trace map

$$\text{Tr}_{g',\mathcal{O}_{Y'}} : Rf'_*\omega_{X'/Y'}^\bullet \to \mathcal{O}_{Y'}$$

for $f' : X' \to Y'$ is the base change of $\text{Tr}_{f,\mathcal{O}_Y}$ via the base change isomorphism.

Lemma 12.6. Let $Y$ be a quasi-compact and quasi-separated scheme. Let $f : X \to Y$ be a morphism of schemes which is proper, flat, and of finite presentation with relative dualizing complex $\omega_{X/Y}^\bullet$ (Remark 12.5). Then

1. $\omega_{X/Y}^\bullet$ is a $Y$-perfect object of $D(\mathcal{O}_X)$,
2. $Rf_*\omega_{X/Y}^\bullet$ has vanishing cohomology sheaves in positive degrees,
3. $\mathcal{O}_X \to R\mathcal{H}om_{\mathcal{O}_X}(\omega_{X/Y}^\bullet, \omega_{X/Y}^\bullet)$ is an isomorphism.

Proof. In view of the fact that formation of $\omega_{X/Y}^\bullet$ commutes with base change (see Remark 12.5), we may and do assume that $Y$ is affine. We will repeatedly use that $Rf_*R\mathcal{H}om_{\mathcal{O}_X}(L, a(K)) = R\mathcal{H}om_{\mathcal{O}_Y}(Rf_*L, K)$, see Lemma 3.6. Let $E$ be a perfect object of $D(\mathcal{O}_X)$ with dual $E'$, see Cohomology, Lemma 13.11. Then

$$Rf_*(E \otimes_{\mathcal{O}_X} L_X \omega_{X/Y}^\bullet) = Rf_*R\mathcal{H}om_{\mathcal{O}_X}(E', \omega_{X/Y}^\bullet) = R\mathcal{H}om_{\mathcal{O}_Y}(Rf_*E', \mathcal{O}_Y)$$

By Derived Categories of Schemes, Lemma 26.4 the complex $Rf_*E'$ is perfect. Hence the dual $R\mathcal{H}om_{\mathcal{O}_Y}(Rf_*E', \mathcal{O}_Y)$ is perfect as well. We conclude that $\omega_{X/Y}^\bullet$ is pseudo-coherent by Derived Categories of Schemes, Lemma 30.3 and More on Morphisms, Lemma 61.4.

Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_Y$-module. By Lemma 12.3 we have

$$a(\mathcal{F}) = Lf^*\mathcal{F} \otimes_{\mathcal{O}_X} \omega_{X/Y}^\bullet = f^{-1}\mathcal{F} \otimes_{f^{-1}\mathcal{O}_Y} \omega_{X/Y}^\bullet$$

Second equality by Cohomology, Lemma 28.4. By Lemma 3.5 there exists an integer $N$ such that $H^i(a(\mathcal{F})) = 0$ for $i \leq -N$. Looking at stalks we conclude that $\omega_{X/Y}^\bullet$ has finite tor dimension (details omitted; hint: for $y \in Y$ any $\mathcal{O}_{Y,y}$-module occurs as $\mathcal{F}_y$ for some quasi-coherent module on the affine scheme $Y$).

Combining the results of the previous two paragraphs we find that $\omega_{X/Y}^\bullet$ is $Y$-perfect, see Derived Categories of Schemes, Definition 31.1. This proves (1).
Let $M$ be an object of $D_{QCoh}(\mathcal{O}_Y)$. Then
\[
\Hom_Y(M, Rf_*\omega_{X/Y}^\bullet) = \Hom_X(Lf^*M, \omega_{X/Y}^\bullet)
= \Hom_Y(Rf_*Lf^*M, \mathcal{O}_Y)
= \Hom_Y(M \otimes_{\mathcal{O}_Y}^{L} Rf_*\mathcal{O}_Y, \mathcal{O}_Y)
\]
The first equality holds by Cohomology, Lemma 29.1. The second equality by construction of $a$. The third equality by Derived Categories of Schemes, Lemma 21.1. Recall $Rf_*\mathcal{O}_X$ is perfect of tor amplitude in $[0, N]$ for some $N$, see Derived Categories of Schemes, Lemma 26.4. Thus we can represent $Rf_*\mathcal{O}_X$ by a complex of finite projective modules sitting in degrees $[0, N]$ (using More on Algebra, Lemma 69.2 and the fact that $Y$ is affine). Hence if $M = \mathcal{O}_Y[-i]$ for some $i > 0$, then the last group is zero. Since $Y$ is affine we conclude that $H^i(Rf_*\omega_{X/Y}^\bullet) = 0$ for $i > 0$. This proves (2).

Let $E$ be a perfect object of $D_{QCoh}(\mathcal{O}_X)$. Then we have
\[
\Hom_X(E, R\Hom_{\mathcal{O}_X}(\omega_{X/Y}^\bullet, \omega_{X/Y}^\bullet)) = \Hom_X(E \otimes_{\mathcal{O}_X}^{L} \omega_{X/Y}^\bullet, \omega_{X/Y}^\bullet)
= \Hom_Y(Rf_*(E \otimes_{\mathcal{O}_X}^{L} \omega_{X/Y}^\bullet), \mathcal{O}_Y)
= \Hom_Y(Rf_*(R\Hom_{\mathcal{O}_X}(E^\vee, \omega_{X/Y}^\bullet)), \mathcal{O}_Y)
= \Hom_Y(R\Hom_{\mathcal{O}_Y}(Rf_*E^\vee, \mathcal{O}_Y), \mathcal{O}_Y)
= R\Gamma(Y, Rf_*E^\vee)
= \Hom_X(E, \mathcal{O}_X)
\]
The first equality holds by Cohomology, Lemma 36.2. The second equality is the definition of $\omega_{X/Y}^\bullet$. The third equality comes from the construction of the dual perfect complex $E^\vee$, see Cohomology, Lemma 43.11. The fourth equality is given in the first paragraph of the proof. The fifth equality holds by double duality for perfect complexes (Cohomology, Lemma 43.11) and the fact that $Rf_*E$ is perfect by Derived Categories of Schemes, Lemma 26.4. The last equality is Leray for $f$. This string of equalities essentially shows (3) holds by the Yoneda lemma. Namely, the object $R\Hom(\omega_{X/Y}^\bullet, \omega_{X/Y}^\bullet)$ is in $D_{QCoh}(\mathcal{O}_X)$ by Derived Categories of Schemes, Lemma 9.8. Taking $E = \mathcal{O}_X$ in the above we get a map $\alpha : \mathcal{O}_X \to R\Hom_{\mathcal{O}_X}(\omega_{X/Y}^\bullet, \omega_{X/Y}^\bullet)$ corresponding to $id_{\mathcal{O}_X} \in \Hom_X(\mathcal{O}_X, \mathcal{O}_X)$. Since all the isomorphisms above are functorial in $E$ we see that the cone on $\alpha$ is an object $C$ of $D_{QCoh}(\mathcal{O}_X)$ such that $\Hom(E, C) = 0$ for all perfect $E$. Since the perfect objects generate (Derived Categories of Schemes, Theorem 14.3) we conclude that $\alpha$ is an isomorphism. \hfill $\square$

\begin{lemma} \textbf{Lemma 12.7} \textbf{(Rigidity).} Let $Y$ be a quasi-compact and quasi-separated scheme. Let $f : X \to Y$ be a proper, flat morphism of finite presentation with relative dualizing complex $\omega_{X/Y}^\bullet$ (Remark 12.5). There is a canonical isomorphism
\end{lemma}

0E2P \hfill 12.7.1 \hfill $\mathcal{O}_X = c(Lpr_1^*\omega_{X/Y}^\bullet) = c(Lpr_2^*\omega_{X/Y}^\bullet)$

and a canonical isomorphism

0E2Q \hfill 12.7.2 \hfill $\omega_{X/Y}^\bullet = c \left( Lpr_1^*\omega_{X/Y}^\bullet \otimes_{\mathcal{O}_{X \times Y \times X}}^{L} Lpr_2^*\omega_{X/Y}^\bullet \right)$

where $c$ is the right adjoint of Lemma 3.1 for the diagonal $\Delta : X \to X \times_Y X$. 

**Proof.** Let \( a \) be the right adjoint for \( f \) as in Lemma 3.1. Consider the cartesian square

\[
\begin{array}{ccc}
X 
\times \ Y & \xrightarrow{q} & X \\
\downarrow^p & & \downarrow^f \\
X & \xrightarrow{f} & Y
\end{array}
\]

Let \( b \) be the right adjoint for \( p \) as in Lemma 3.1. Then

\[
\omega_{X/Y} = c(b(\omega_{X/Y}))
\]

\[
= c(Lp^*\omega_{X/Y} \otimes_{O_{X \times Y}} b(O_X))
\]

\[
= c(Lp^*\omega_{X/Y} \otimes_{O_{X \times Y}} Lq^*a(O_Y))
\]

\[
= c(Lp^*\omega_{X/Y} \otimes_{O_{X \times Y}} Lq^*\omega_{X/Y})
\]

as in (12.7.2). Explanation as follows:

1. The first equality holds as \( \text{id} = c \circ b \) because \( \text{id}_X = p \circ \Delta \).
2. The second equality holds by Lemma 12.3.
3. The third holds by Lemma 12.4 and the fact that \( O_X = Lf^*O_Y \).
4. The fourth holds because \( \omega_{X/Y} = a(O_Y) \).

Equation (12.7.1) is proved in exactly the same way. \( \square \)

0BRU **Remark 12.8.** Lemma 12.7 means our relative dualizing complex is *rigid* in a sense analogous to the notion introduced in [vdB97]. Namely, since the functor on the right of (12.7.2) is “quadratic” in \( \omega_{X/Y} \) and the functor on the left of (12.7.2) is “linear” this “pins down” the complex \( \omega_{X/Y} \) to some extent. There is an approach to duality theory using “rigid” (relative) dualizing complexes, see for example [Nec11], [Yek10], and [YZ09]. We will return to this in Section 27.

### 13. Right adjoint of pushforward for perfect proper morphisms

0AA9 The correct generality for this section would be to consider perfect proper morphisms of quasi-compact and quasi-separated schemes, see [LN07].

0A9R **Lemma 13.1.** Let \( f : X \to Y \) be a perfect proper morphism of Noetherian schemes. Let \( a \) be the right adjoint for \( Rf_* : D_{QCoh}(O_X) \to D_{QCoh}(O_Y) \) of Lemma 3.1 Then \( a \) commutes with direct sums.

**Proof.** Let \( P \) be a perfect object of \( D(O_X) \). By More on Morphisms, Lemma 53.12 the complex \( Rf_*P \) is perfect on \( Y \). Let \( K_i \) be a family of objects of \( D_{QCoh}(O_Y) \). Then

\[
\text{Hom}_{D(O_X)}(P, a(\bigoplus K_i)) = \text{Hom}_{D(O_Y)}(Rf_*P, \bigoplus K_i)
\]

\[
= \bigoplus \text{Hom}_{D(O_Y)}(Rf_*P, K_i)
\]

\[
= \bigoplus \text{Hom}_{D(O_X)}(P, a(K_i))
\]

because a perfect object is compact (Derived Categories of Schemes, Proposition 16.1). Since \( D_{QCoh}(O_X) \) has a perfect generator (Derived Categories of Schemes, Theorem 14.3) we conclude that the map \( \bigoplus a(K_i) \to a(\bigoplus K_i) \) is an isomorphism, i.e., \( a \) commutes with direct sums. \( \square \)
**Lemma 13.2.** Let $f : X \to Y$ be a perfect proper morphism of Noetherian schemes. Let $a$ be the right adjoint for $Rf_* : D_{QCoh}(\mathcal{O}_X) \to D_{QCoh}(\mathcal{O}_Y)$ of Lemma 7.1. Then

1. for every closed $T \subset Y$ if $Q \in D_{QCoh}(Y)$ is supported on $T$, then $a(Q)$ is supported on $f^{-1}(T)$,
2. for every open $V \subset Y$ and any $K \in D_{QCoh}(\mathcal{O}_Y)$ the map (4.1.1) is an isomorphism, and

**Proof.** This follows from Lemmas 4.3, 4.4, and 13.1. □

**Lemma 13.3.** Let $f : X \to Y$ be a perfect proper morphism of Noetherian schemes. The map (8.0.1) is an isomorphism for every object $K$ of $D_{QCoh}(\mathcal{O}_Y)$.

**Proof.** By Lemma 13.1 we know that $a$ commutes with direct sums. Hence the collection of objects of $D_{QCoh}(\mathcal{O}_Y)$ for which (8.0.1) is an isomorphism is a strictly full, saturated, triangulated subcategory of $D_{QCoh}(\mathcal{O}_Y)$ which is moreover preserved under taking direct sums. Since $D_{QCoh}(\mathcal{O}_Y)$ is a module category (Derived Categories of Schemes, Theorem 17.3) generated by a single perfect object (Derived Categories of Schemes, Theorem 14.3) we can argue as in More on Algebra, Remark 57.13 to see that it suffices to prove (8.0.1) is an isomorphism for a single perfect object. However, the result holds for perfect objects, see Lemma 8.1. □

**Lemma 13.4.** Let $f : X \to Y$ be a perfect proper morphism of Noetherian schemes. Let $g : Y' \to Y$ be a morphism with $Y'$ Noetherian. If $X$ and $Y'$ are tor independent over $Y$, then the base change map (5.0.1) is an isomorphism for all $K \in D_{QCoh}(\mathcal{O}_Y)$.

**Proof.** By Lemma 13.2 formation of the functors $a$ and $a'$ commutes with restriction to opens of $Y$ and $Y'$. Hence we may assume $Y' \to Y$ is a morphism of affine schemes, see Remark 6.1. In this case the statement follows from Lemma 6.2. □

**14. Right adjoint of pushforward for effective Cartier divisors**

**Lemma 14.1.** As above, let $X$ be a scheme and let $D \subset X$ be an effective Cartier divisor. There is a canonical isomorphism $R\mathcal{H}om(\mathcal{O}_D, \mathcal{O}_X) = \mathcal{N}[-1]$ in $D(\mathcal{O}_D)$.

**Proof.** Equivalently, we are saying that $R\mathcal{H}om(\mathcal{O}_D, \mathcal{O}_X)$ has a unique nonzero cohomology sheaf in degree 1 and that this sheaf is isomorphic to $\mathcal{N}$. Since $i_*$ is exact and fully faithful, it suffices to prove that $i_* R\mathcal{H}om(\mathcal{O}_D, \mathcal{O}_X)$ is isomorphic to $i_* \mathcal{N}[-1]$. We have $i_* R\mathcal{H}om(\mathcal{O}_D, \mathcal{O}_X) = R\mathcal{H}om_{\mathcal{O}_X}(i_* \mathcal{O}_D, \mathcal{O}_X)$ by Lemma 9.3. We have a resolution

$$0 \to \mathcal{I} \to \mathcal{O}_X \to i_* \mathcal{O}_D \to 0$$

where $\mathcal{I}$ is the ideal sheaf of $D$ which we can use to compute. Since $R\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{O}_X) = \mathcal{O}_X$ and $R\mathcal{H}om_{\mathcal{O}_X}(\mathcal{I}, \mathcal{O}_X) = \mathcal{O}_X(D)$ by a local computation, we see that

$$R\mathcal{H}om_{\mathcal{O}_X}(i_* \mathcal{O}_D, \mathcal{O}_X) = (\mathcal{O}_X \to \mathcal{O}_X(D))$$
where on the right hand side we have $\mathcal{O}_X$ in degree 0 and $\mathcal{O}_X(D)$ in degree 1. The result follows from the short exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(D) \to i_*\mathcal{N} \to 0$$

coming from the fact that $D$ is the zero scheme of the canonical section of $\mathcal{O}_X(D)$ and from the fact that $\mathcal{N} = i^*\mathcal{O}_X(D)$. \hfill $\square$

For every object $K$ of $D(\mathcal{O}_X)$ there is a canonical map

$$0 \to i^*K \otimes_{\mathcal{O}_X} R\mathcal{H}om(\mathcal{O}_D, \mathcal{O}_X) \to R\mathcal{H}om(\mathcal{O}_D, K)$$

in $D(\mathcal{O}_D)$ functorial in $K$ and compatible with distinguished triangles. Namely, this map is adjoint to a map

$$i_*(i^*K \otimes_{\mathcal{O}_D} R\mathcal{H}om(\mathcal{O}_D, \mathcal{O}_X)) = K \otimes_{\mathcal{O}_X} R\mathcal{H}om_{\mathcal{O}_X}(i_*\mathcal{O}_D, \mathcal{O}_X) \to K$$

where the equality is Cohomology, Lemma 45.4 and the arrow comes from the canonical map $R\mathcal{H}om_{\mathcal{O}_X}(i_*\mathcal{O}_D, \mathcal{O}_X) \to \mathcal{O}_X$ induced by $\mathcal{O}_X \to i_*\mathcal{O}_D$.

If $K \in D_{Q\mathcal{C}\mathcal{O}b}(\mathcal{O}_X)$, then (14.1.1) is equal to (8.0.1) via the identification $a(K) = R\mathcal{H}om(\mathcal{O}_D, K)$ of Lemma 9.7. If $K \in D_{Q\mathcal{C}\mathcal{O}b}(\mathcal{O}_X)$ and $X$ is Noetherian, then the following lemma is a special case of Lemma 13.3.

**Lemma 14.2.** As above, let $X$ be a scheme and let $D \subset X$ be an effective Cartier divisor. Then (14.1.1) combined with Lemma 14.1 defines an isomorphism

$$i^*K \otimes_{\mathcal{O}_D} \mathcal{N}[1] \to R\mathcal{H}om(\mathcal{O}_D, K)$$

functorial in $K$ in $D(\mathcal{O}_X)$.

**Proof.** Since $i_*$ is exact and fully faithful on modules, to prove the map is an isomorphism, it suffices to show that it is an isomorphism after applying $i_*$. We will use the short exact sequences $0 \to \mathcal{I} \to \mathcal{O}_X \to i_*\mathcal{O}_D \to 0$ and $0 \to \mathcal{O}_X \to \mathcal{O}_X(D) \to i_*\mathcal{N} \to 0$ used in the proof of Lemma 14.1 without further mention. By Cohomology, Lemma 45.4 which was used to define the map (14.1.1) the left hand side becomes

$$K \otimes_{\mathcal{O}_X} i_*\mathcal{N}[1] = K \otimes_{\mathcal{O}_X} (\mathcal{O}_X \to \mathcal{O}_X(D))$$

The right hand side becomes

$$R\mathcal{H}om_{\mathcal{O}_X}(i_*\mathcal{O}_D, K) = R\mathcal{H}om_{\mathcal{O}_X}((\mathcal{I} \to \mathcal{O}_X), K)$$

$$= R\mathcal{H}om_{\mathcal{O}_X}((\mathcal{I} \to \mathcal{O}_X), \mathcal{O}_X) \otimes_{\mathcal{O}_X} K$$

the final equality by Cohomology, Lemma 45.11. Since the map comes from the isomorphism

$$R\mathcal{H}om_{\mathcal{O}_X}((\mathcal{I} \to \mathcal{O}_X), \mathcal{O}_X) = (\mathcal{O}_X \to \mathcal{O}_X(D))$$

the lemma is clear. \hfill $\square$

**15. Right adjoint of pushforward in examples**

In this section we compute the right adjoint to pushforward in some examples. The isomorphisms are canonical but only in the weakest possible sense, i.e., we do not prove or claim that these isomorphisms are compatible with various operations such as base change and compositions of morphisms. There is a huge literature on these types of issues; the reader can start with the material in [Har66], [Con00] (these citations use a different starting point for duality but address the issue of
Lemma 15.1. Let $Y$ be a Noetherian scheme. Let $E$ be a finite locally free $\mathcal{O}_Y$-module of rank $n+1$ with determinant $L = \wedge^{n+1}(E)$. Let $f : X = \mathbb{P}(E) \to Y$ be the projection. Let $a$ be the right adjoint for $Rf_* : D_{QCoh}(\mathcal{O}_X) \to D_{QCoh}(\mathcal{O}_Y)$ of Lemma 8.4. Then there is an isomorphism

$$c : f^*L(-n-1)[n] \to a(\mathcal{O}_Y)$$

In particular, if $E = \mathcal{O}_Y^{\oplus n+1}$, then $X = \mathbb{P}_Y^n$ and we obtain $a(\mathcal{O}_Y) = \mathcal{O}_X(-n-1)[n]$.

Proof. In (the proof of) Cohomology of Schemes, Lemma 8.4 we constructed a canonical isomorphism

$$R^n f_*(f^*L(-n-1)) \to \mathcal{O}_Y$$

Moreover, $R^n f_*(f^*L(-n-1))[n] = R^n f_*(f^*L(-n-1))$, i.e., the other higher direct images are zero. Thus we find an isomorphism

$$R^n f_*(f^*L(-n-1)[n]) \to \mathcal{O}_Y$$

This isomorphism determines $c$ as in the statement of the lemma because $a$ is the right adjoint of $Rf_*$. By Lemma 14.4 construction of the $a$ is local on the base. In particular, to check that $c$ is an isomorphism, we may work locally on $Y$. In other words, we may assume $Y$ is affine and $E = \mathcal{O}_Y^{\oplus n+1}$. In this case the sheaves $\mathcal{O}_X, \mathcal{O}_X(-1), \ldots, \mathcal{O}_X(-n)$ generate $D_{QCoh}(X)$, see Derived Categories of Schemes, Lemma 15.3. Hence it suffices to show that $c : \mathcal{O}_X(-n-1)[n] \to a(\mathcal{O}_Y)$ is transformed into an isomorphism under the functors

$$F_{i,p}(-) = \text{Hom}_{D(\mathcal{O}_X)}(\mathcal{O}_X(i), (-)[p])$$

for $i \in \{-n, \ldots, 0\}$ and $p \in \mathbb{Z}$. For $F_{0,p}$ this holds by construction of the arrow $c$! For $i \in \{-n, \ldots, -1\}$ we have

$$\text{Hom}_{D(\mathcal{O}_X)}(\mathcal{O}_X(i), \mathcal{O}_X(-n-1)[n+p]) = H^p(X, \mathcal{O}_X(-n-1-i)) = 0$$

by the computation of cohomology of projective space (Cohomology of Schemes, Lemma 8.1) and we have

$$\text{Hom}_{D(\mathcal{O}_X)}(\mathcal{O}_X(i), a(\mathcal{O}_Y)[p]) = \text{Hom}_{D(\mathcal{O}_Y)}(Rf_*\mathcal{O}_X(i), \mathcal{O}_Y[p]) = 0$$

because $Rf_*\mathcal{O}_X(i) = 0$ by the same lemma. Hence the source and the target of $F_{i,p}(c)$ vanish and $F_{i,p}(c)$ is necessarily an isomorphism. This finishes the proof. □

Example 15.2. The base change map (5.0.1) is not an isomorphism if $f$ is perfect proper and $g$ is perfect. Let $k$ be a field. Let $Y = \mathbb{A}^2_k$ and let $f : X = \mathbb{P}_Y^1 \to Y$ be the blowup of $Y$ in the origin. Denote $E \subset X$ the exceptional divisor. Then we can factor $f$ as

$$X \overset{i}{\to} \mathbb{P}_Y^1 \overset{p}{\to} Y$$

This gives a factorization $a = c o b$ where $a$, $b$, and $c$ are the right adjoints of Lemma 9.1 of $Rf_*$, $Rp_*$, and $Ri_*$. Denote $\mathcal{O}(n)$ the Serre twist of the structure sheaf on $\mathbb{P}_Y^1$ and denote $\mathcal{O}_X(n)$ its restriction to $X$. Note that $X \subset \mathbb{P}_Y^1$ is cut out by a degree one equation, hence $\mathcal{O}(X) = \mathcal{O}(1)$. By Lemma 15.1 we have $b(\mathcal{O}_Y) = \mathcal{O}(-2)[1]$. By Lemma 9.7 we have

$$a(\mathcal{O}_Y) = c(b(\mathcal{O}_Y)) = c(\mathcal{O}(-2)[1]) = R\text{Hom}(\mathcal{O}_X, \mathcal{O}(-2)[1]) = \mathcal{O}_X(-1)$$
Last equality by Lemma \[14.2\] Let $Y' = \text{Spec}(k)$ be the origin in $Y$. The restriction of $a(O_Y)$ to $X' = E = \mathbb{P}_k^1$ is an invertible sheaf of degree $-1$ placed in cohomological degree $0$. But on the other hand, $a'(O_{\text{Spec}(k)}) = O_E(-2)[1]$ which is an invertible sheaf of degree $-2$ placed in cohomological degree $-1$, so different. In this example (4) is the only hypothesis of Lemma \[6.2\] which is violated.

**Lemma 15.3.** Let $Y$ be a ringed space. Let $\mathfrak{I} \subset O_Y$ be a sheaf of ideals. Set $O_X = O_Y/\mathfrak{I}$ and $\mathcal{N} = \mathcal{H}om_{O_Y}(\mathfrak{I}/\mathfrak{I}^2, O_X)$. There is a canonical isomorphism $c: \mathcal{N} \rightarrow \mathfrak{E}xt^1_{O_Y}(O_X, O_X)$.

**Proof.** Consider the canonical short exact sequence

\[ 0 \rightarrow \mathfrak{I}/\mathfrak{I}^2 \rightarrow O_Y/\mathfrak{I}^2 \rightarrow O_X \rightarrow 0 \]

Let $U \subset X$ be open and let $s \in \mathcal{N}(U)$. Then we can pushout \[15.3.1\] via $s$ to get an extension $E_s$ of $O_X|_U$ by $O_X|_U$. This in turn defines a section $c(s)$ of $\mathfrak{E}xt^1_{O_Y}(O_X, O_X)$ over $U$. See Cohomology, Lemma \[36.1\] and Derived Categories, Lemma \[27.6\]. Conversely, given an extension

\[ 0 \rightarrow O_X|_U \rightarrow \mathcal{E} \rightarrow O_X|_U \rightarrow 0 \]

of $O_U$-modules, we can find an open covering $U = \bigcup U_i$ and sections $e_i \in E(U_i)$ mapping to $1 \in O_X(U_i)$. Then $e_i$ defines a map $O_X|_{U_i} \rightarrow \mathcal{E}|_{U_i}$ whose kernel contains $\mathfrak{I}^2$. In this way we see that $\mathcal{E}|_{U_i}$ comes from a pushout as above. This shows that $c$ is surjective. We omit the proof of injectivity. \hfill \Box

**Lemma 15.4.** Let $Y$ be a ringed space. Let $\mathfrak{I} \subset O_Y$ be a sheaf of ideals. Set $O_X = O_Y/\mathfrak{I}$. If $\mathfrak{I}$ is Koszul-regular (Divisors, Definition \[20.2\]) then composition on $R\mathcal{H}om_{O_Y}(O_X, O_X)$ defines isomorphisms

\[ \wedge^i(\mathfrak{E}xt^1_{O_Y}(O_X, O_X)) \longrightarrow \mathfrak{E}xt^i_{O_Y}(O_X, O_X) \]

for all $i$.

**Proof.** By composition we mean the map

\[ R\mathcal{H}om_{O_Y}(O_X, O_X) \otimes^L_{O_Y} R\mathcal{H}om_{O_Y}(O_X, O_X) \rightarrow R\mathcal{H}om_{O_Y}(O_X, O_X) \]

of Cohomology, Lemma \[36.6\]. This induces multiplication maps

\[ \mathfrak{E}xt^1_{O_Y}(O_X, O_X) \otimes_{O_Y} \mathfrak{E}xt^1_{O_Y}(O_X, O_X) \rightarrow \mathfrak{E}xt^2_{O_Y}(O_X, O_X) \]

Please compare with More on Algebra, Equation \[61.0.1\]. The statement of the lemma means that the induced map

\[ \mathfrak{E}xt^1_{O_Y}(O_X, O_X) \otimes \cdots \otimes \mathfrak{E}xt^1_{O_Y}(O_X, O_X) \rightarrow \mathfrak{E}xt^i_{O_Y}(O_X, O_X) \]

factors through the wedge product and then induces an isomorphism. To see this is true we may work locally on $Y$. Hence we may assume that we have global sections $f_1, \ldots, f_r$ of $O_Y$ which generate $\mathfrak{I}$ and which form a Koszul regular sequence. Denote

\[ \mathcal{A} = O_Y(\xi_1, \ldots, \xi_r) \]

the sheaf of strictly commutative differential graded $O_Y$-algebras which is a (divided power) polynomial algebra on $\xi_1, \ldots, \xi_r$ in degree $-1$ over $O_Y$ with differential $d$ given by the rule $d\xi_i = f_i$. Let us denote $\mathcal{A}^\bullet$ the underlying complex of $O_Y$-modules which is the Koszul complex mentioned above. Thus the canonical map $\mathcal{A}^\bullet \rightarrow O_X$ is a quasi-isomorphism. We obtain quasi-isomorphisms

\[ R\mathcal{H}om_{O_Y}(O_X, O_X) \rightarrow \mathcal{H}om^\bullet(\mathcal{A}^\bullet, \mathcal{A}^\bullet) \rightarrow \mathcal{H}om^\bullet(\mathcal{A}^\bullet, O_X) \]
Let \( \delta_j : \mathcal{A} \to \mathcal{A} \) be the derivation of degree 1 with \( \delta_j(\xi_i) = \delta_i \) (Kronecker delta). A computation shows that \( \delta_j \circ d = -d \circ \delta_j \) which shows that we get a morphism of complexes.

\[
\delta_j : \mathcal{A}^* \to \mathcal{A}^*[1].
\]

Whence \( \delta_j \) defines a section of the corresponding \( \mathcal{E}xt \)-sheaf. Another computation shows that \( \delta_1, \ldots, \delta_r \) map to a basis for \( \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{A}^{-1}, \mathcal{O}_X) \) over \( \mathcal{O}_X \). Since it is clear that \( \delta_j \circ \delta_j = 0 \) and \( \delta_j \circ \delta_j' = -\delta_j' \circ \delta_j \) as endomorphisms of \( \mathcal{A} \) and hence in the \( \mathcal{E}xt \)-sheaves we obtain the statement that our map above factors through the exterior power. To see we get the desired isomorphism the reader checks that the elements

\[
\delta_j_1 \circ \ldots \circ \delta_j_i
\]

for \( j_1 < \ldots < j_i \) map to a basis of the sheaf \( \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{A}^{-i}, \mathcal{O}_X) \) over \( \mathcal{O}_X \).

**Lemma 15.5.** Let \( Y \) be a ringed space. Let \( \mathcal{I} \subset \mathcal{O}_Y \) be a sheaf of ideals. Set \( \mathcal{O}_X = \mathcal{O}_Y/\mathcal{I} \) and \( \mathcal{N} = \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_X) \). If \( \mathcal{I} \) is Koszul-regular (Divisors, Definition 20.2) then

\[
R \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{O}_Y) = \wedge^r \mathcal{N}[r]
\]

where \( r : Y \to \{1, 2, 3, \ldots\} \) sends \( y \) to the minimal number of generators of \( \mathcal{I} \) needed in a neighbourhood of \( y \).

**Proof.** We can use Lemmas 15.3 and 15.4 to see that we have isomorphisms \( \wedge^i \mathcal{N} \to \mathcal{E}xt^i_{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{O}_X) \) for \( i \geq 0 \). Thus it suffices to show that the map \( \mathcal{O}_Y \to \mathcal{O}_X \) induces an isomorphism

\[
\mathcal{E}xt^r_{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{O}_Y) \to \mathcal{E}xt^r_{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{O}_X)
\]

and that \( \mathcal{E}xt^i_{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{O}_Y) \) is zero for \( i \neq r \). These statements are local on \( Y \). Thus we may assume that we have global sections \( f_1, \ldots, f_r \) of \( \mathcal{O}_Y \) which generate \( \mathcal{I} \) and which form a Koszul regular sequence. Let \( \mathcal{A}^* \) be the Koszul complex on \( f_1, \ldots, f_r \) as introduced in the proof of Lemma 15.4. Then

\[
R \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{O}_Y) = \mathcal{H}om^*(\mathcal{A}^*, \mathcal{O}_Y)
\]

by Cohomology, Lemma 40.9. Denote \( 1 \in H^0(\mathcal{H}om^*(\mathcal{A}^*, \mathcal{O}_Y)) \) the identity map of \( \mathcal{A}^0 = \mathcal{O}_Y \to \mathcal{O}_Y \). With \( \delta_j \) as in the proof of Lemma 15.4 we get an isomorphism of graded \( \mathcal{O}_Y \)-modules

\[
\mathcal{O}_Y(\delta_1, \ldots, \delta_r) \to \mathcal{H}om^*(\mathcal{A}^*, \mathcal{O}_Y)
\]

by mapping \( \delta_j_1 \ldots \delta_j_i \) to \( 1 \circ \delta_j_1 \circ \ldots \circ \delta_j_i \) in degree \( i \). Via this isomorphism the differential on the right hand side induces a differential \( d \) on the left hand side. By our sign rules we have \( d(1) = -\sum f_j \delta_j \). Since \( \delta_j : \mathcal{A}^* \to \mathcal{A}^*[1] \) is a morphism of complexes, it follows that

\[
d(\delta_j_1 \ldots \delta_j_i) = (-\sum f_j \delta_j) \delta_j_1 \ldots \delta_j_i
\]

Observe that we have \( d = \sum f_j \delta_j \) on the differential graded algebra \( \mathcal{A} \). Therefore the map defined by the rule

\[
1 \circ \delta_j_1 \ldots \delta_j_i \mapsto (\delta_j_1 \circ \ldots \circ \delta_j_i)(\xi_1 \ldots \xi_r)
\]
will define an isomorphism of complexes
\[ \text{Hom}^\bullet(A^\bullet, \mathcal{O}_Y) \to A^\bullet[-r] \]
if \( r \) is odd and commuting with differentials up to sign if \( r \) is even. In any case these complexes have isomorphic cohomology, which shows the desired vanishing. The isomorphism on cohomology in degree \( r \) under the map
\[ \text{Hom}^\bullet(A^\bullet, \mathcal{O}_Y) \to \text{Hom}^\bullet(A^\bullet, \mathcal{O}_X) \]
also follows in a straightforward manner from this. (We observe that our choice of conventions regarding Koszul complexes does intervene in the definition of the isomorphism \( R\text{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{O}_Y) = \wedge^r\mathcal{N}[r] \).) \[ \square \]

**Lemma 15.6.** Let \( Y \) be a quasi-compact and quasi-separated scheme. Let \( i : X \to Y \) be a Koszul-regular closed immersion. Let \( a \) be the right adjoint of \( R\text{f}_\ast : D_{\mathcal{QCoh}}(\mathcal{O}_X) \to D_{\mathcal{QCoh}}(\mathcal{O}_Y) \) of Lemma 3.1. Then there is an isomorphism
\[ \wedge^r\mathcal{N}[-r] \to a(\mathcal{O}_Y) \]
where \( \mathcal{N} = \text{Hom}_{\mathcal{O}_X}(\mathcal{C}_{X/Y}, \mathcal{O}_X) \) is the normal sheaf of \( i \) (Morphisms, Section 30) and \( r \) is its rank viewed as a locally constant function on \( X \).

**Proof.** Recall, from Lemmas 9.7 and 9.3, that \( a(\mathcal{O}_Y) \) is an object of \( D_{\mathcal{QCoh}}(\mathcal{O}_X) \). Thus the result follows from Lemma 15.5. \[ \square \]

**Lemma 15.7.** Let \( S \) be a Noetherian scheme. Let \( f : X \to S \) be a smooth proper morphism of relative dimension \( d \). Let \( a \) be the right adjoint of \( R\text{f}_\ast : D_{\mathcal{QCoh}}(\mathcal{O}_X) \to D_{\mathcal{QCoh}}(\mathcal{O}_S) \) as in Lemma 3.1. Then there is an isomorphism
\[ \wedge^d\Omega_{X/S}[d] \to a(\mathcal{O}_S) \]
in \( D(\mathcal{O}_X) \).

**Proof.** Set \( \omega_{X/S}^\bullet = a(\mathcal{O}_S) \) as in Remark 12.6. Let \( c \) be the right adjoint of Lemma 3.1 for \( \Delta : X \to X \times_X X \). Because \( \Delta \) is the diagonal of a smooth morphism it is a Koszul-regular immersion, see Divisors, Lemma 22.10. In particular, \( \Delta \) is a perfect proper morphism (More on Morphisms, Lemma 53.7) and we obtain
\[ \begin{align*}
\mathcal{O}_X &= c(L\text{pr}_1^\ast\omega_{X/S}) \\
&= L\Delta^\ast(L\text{pr}_1^\ast\omega_{X/S}) \otimes L^\Delta\mathcal{O}_X c(\mathcal{O}_{X \times_X X}) \\
&= \omega_{X/S}^\bullet \otimes L^\Delta\mathcal{O}_X c(\mathcal{O}_{X \times_X X}) \\
&= \omega_{X/S}^\bullet \otimes L^\Delta\mathcal{N}\wedge^d(\mathcal{N}_\Delta)[d] \\
\end{align*} \]
The first equality is (12.7.1) because \( \omega_{X/S}^\bullet = a(\mathcal{O}_S) \). The second equality by Lemma 13.3. The third equality because \( p \circ \Delta = \text{id}_X \). The fourth equality by Lemma 15.6. Observe that \( \wedge^d(\mathcal{N}_\Delta) \) is an invertible \( \mathcal{O}_X \)-module. Hence \( \wedge^d(\mathcal{N}_\Delta)[d] \) is an invertible object of \( D(\mathcal{O}_X) \) and we conclude that \( a(\mathcal{O}_S) = \omega_{X/S}^\bullet = \wedge^d(\mathcal{C}_\Delta)[d] \).
Since the conormal sheaf \( \mathcal{C}_\Delta \) of \( \Delta \) is \( \Omega_{X/S} \) by Morphisms, Lemma 31.7, the proof is complete. \[ \square \]
16. Upper shriek functors

In this section, we construct the functors $f^!$ for morphisms between schemes which are of finite type and separated over a fixed Noetherian base using compactifications. As is customary in coherent duality, there are a number of diagrams that have to be shown to be commutative. We suggest the reader, after reading the construction, skips the verification of the lemmas and continues to the next section where we discuss properties of the upper shriek functors.

**Situation 16.1.** Here $S$ is a Noetherian scheme and $\text{FTS}_S$ is the category whose

1. objects are schemes $X$ over $S$ such that the structure morphism $X \to S$ is both separated and of finite type, and
2. morphisms $f : X \to Y$ between objects are morphisms of schemes over $S$.

In Situation 16.1 given a morphism $f : X \to Y$ in $\text{FTS}_S$, we will define an exact functor $f^! : D^+_{\text{QCoh}}(\mathcal{O}_Y) \to D^+_{\text{QCoh}}(\mathcal{O}_X)$ of triangulated categories. Namely, we choose a compactification $X \to \overline{X}$ over $Y$ which is possible by More on Flatness, Theorem 33.8 and Lemma 32.2. Denote $\overline{f} : \overline{X} \to Y$ the structure morphism. Let $\overline{\pi} : D^+_{\text{QCoh}}(\mathcal{O}_Y) \to D^+_{\text{QCoh}}(\mathcal{O}_{\overline{X}})$ be the right adjoint of $R\overline{f}_*$ constructed in Lemma 3.1. Then we set

$$f^! K = \overline{\pi}(K)|_X$$

for $K \in D^+_{\text{QCoh}}(\mathcal{O}_Y)$. The result is an object of $D^+_{\text{QCoh}}(\mathcal{O}_X)$ by Lemma 3.5.

**Lemma 16.2.** In Situation 16.1 let $f : X \to Y$ be a morphism of $\text{FTS}_S$. The functor $f^!$ is, up to canonical isomorphism, independent of the choice of the compactification.

**Proof.** Consider the category of compactifications of $X$ over $Y$, which is cofiltered according to More on Flatness, Theorem 33.8 and Lemmas 32.1 and 32.2. To every choice of a compactification

$$j : X \to \overline{X}, \quad \overline{f} : \overline{X} \to Y$$

the construction above associates the functor $j^* \circ \overline{\pi} : D^+_{\text{QCoh}}(\mathcal{O}_Y) \to D^+_{\text{QCoh}}(\mathcal{O}_X)$ where $\overline{\pi}$ is the right adjoint of $R\overline{f}_*$ constructed in Lemma 3.1.

Suppose given a morphism $g : \overline{X}_1 \to \overline{X}_2$ between compactifications $j_i : X \to \overline{X}_i$ over $Y$. Namely, let $\overline{\pi}$ be the right adjoint of Lemma 3.1 for $g$. Then $\overline{\pi}_2 \circ \overline{\pi}_1 = \overline{\pi}_1$ because these functors are adjoint to $R\overline{f}_{2,*} \circ Rg_* = R(\overline{f}_2 \circ g)_*$. By (4.1.1) we have a canonical transformation

$$j_1^* \circ \overline{\pi} \to j_2^* \circ \overline{\pi}$$

of functors $D^+_{\text{QCoh}}(\mathcal{O}_{\overline{X}_2}) \to D^+_{\text{QCoh}}(\mathcal{O}_X)$ which is an isomorphism by Lemma 4.4. The composition

$$j_1^* \circ \overline{\pi}_1 \to j_1^* \circ \overline{\pi} \circ \overline{\pi}_2 \to j_2^* \circ \overline{\pi}_2$$

is an isomorphism of functors which we will denote by $\alpha_g$.

To finish the proof, since the category of compactifications of $X$ over $Y$ is cofiltered, it suffices to show compositions of morphisms of compactifications of $X$ over $Y$ are
Lemma 16.3. Let \( F \rightarrow G \) be a morphism of functors, and suppose that \( \alpha, \beta : F \rightarrow G \) are isomorphisms of functors such that \( \gamma : G \rightarrow H \) is an isomorphism of functors such that \( \gamma \circ \alpha = \gamma \circ \beta \). Then we conclude \( \alpha = \beta \).
for the composition $X \rightarrow Z$. Hence $g^! = j^*_{\gamma} \circ \overline{b}$ and $(g \circ f)^! = (X \rightarrow X)^* \circ \overline{b} \circ \overline{a}$.

Let $U$ be the inverse image of $Y$ in $X$ so that we get the commutative diagram

Let $\overline{a}'$ be the right adjoint of Lemma 3.1 for $U \rightarrow Y$. Then $f^! = j^* \circ \overline{a}'$. We obtain

$$\gamma : (j')^* \circ \pi \rightarrow \pi' \circ j_Y^*.$$ 

by (4.1.1) and we can use it to define

$$(g \circ f)^! = j^*_{\overline{X}} \circ \overline{\pi} \circ \overline{b} = j^* \circ (j')^* \circ \pi \circ \overline{b} \rightarrow j^* \circ \overline{\pi'} \circ j_Y^* \circ \overline{b} = f^! \circ g^!$$

which is an isomorphism on objects of $D^+_{QCoh}(O_Z)$ by Lemma 4.4. To finish the proof we show that this isomorphism is independent of choices made.

Suppose we have two diagrams

We can first choose a compactification $i : Y \rightarrow \overline{Y}$ of $Y$ over $Z$ which dominates both $\overline{Y}_1$ and $\overline{Y}_2$, see More on Flatness, Lemma 32.1. By More on Flatness, Lemma 32.3 and Categories, Lemmas 26.13 and 26.14 we can choose a compactification $X \rightarrow \overline{X}$ of $X$ over $Y$ with morphisms $\overline{X} \rightarrow \overline{X}_1$ and $\overline{X} \rightarrow \overline{X}_2$ and such that the composition $\overline{X} \rightarrow \overline{Y} \rightarrow \overline{Y}_1$ is equal to the composition $\overline{X} \rightarrow \overline{X}_1 \rightarrow \overline{Y}_1$ and such that the composition $\overline{X} \rightarrow \overline{Y} \rightarrow \overline{Y}_2$ is equal to the composition $\overline{X} \rightarrow \overline{X}_2 \rightarrow \overline{Y}_2$. Thus we see that it suffices to compare the maps determined by our diagrams when
we have a commutative diagram as follows

\[
\begin{array}{ccc}
X & \xrightarrow{j_1} & U_1 \\
X & \xrightarrow{j_2} & U_2 \\
Y & \xrightarrow{i_1} & Y_1 \\
Y & \xrightarrow{i_2} & Y_2 \\
Z & \xrightarrow{} & \end{array}
\]

We use \(a_i, a'_i, c, \) and \(c'\) for the right adjoint of Lemma 3.1 for \(X_i \to Y_i, U_i \to Y, \) \(X_1 \to X_2, \) and \(U_1 \to U_2.\) Each of the squares

\[
\begin{array}{ccc}
X & \xrightarrow{A} & U_1 \\
X & \xrightarrow{B} & U_2 \\
Y & \xrightarrow{C} & Y_1 \\
Y & \xrightarrow{D} & Y_2 \\
X & \xrightarrow{E} & X_1 \\
X & \xrightarrow{F} & X_2 \\
\end{array}
\]

gives rise to a base change map (4.1.1) as follows

\[
\begin{align*}
\gamma_A : j_1^* \circ \sigma' & \to j_2^* \\
\gamma_B : (j_2')^* \circ \sigma_2 & \to (j_2')^* \circ i_2^* \\
\gamma_C : (j_1')^* \circ \sigma_1 & \to (j_1')^* \circ i_1^* \\
\gamma_D : i_1^* \circ \sigma & \to i_2^* \\
\gamma_E : (j_1')^* \circ \sigma_1 & \circ \sigma_1 \circ \sigma_1 \\
\end{align*}
\]

Denote \(f_1' = f_1 \circ \sigma_1', f_2' = j_2 \circ \sigma_2', g_1' = i_1 \circ \sigma_1, g_2' = i_2 \circ \sigma_2, (g \circ f)_1' = (j_1')^* \circ \sigma_1 \circ \sigma_1,\) and \((g \circ f)_2' = (j_2')^* \circ \sigma_2 \circ \sigma_2.\) The construction given in the first paragraph of the proof and in Lemma [6.2] uses

1. \(\gamma_C\) for the map \((g \circ f)_1' \to f_1' \circ g_1',\)
2. \(\gamma_B\) for the map \((g \circ f)_2' \to f_2' \circ g_2',\)
3. \(\gamma_A\) for the map \(f_1' \to f_2',\)
4. \(\gamma_D\) for the map \(g_1' \to g_2',\)
5. \(\gamma_E\) for the map \((g \circ f)_1' \to (g \circ f)_2'.\)

We have to show that the diagram

\[
\begin{array}{ccc}
(g \circ f)_1' & \xrightarrow{\gamma_E} & (g \circ f)_2' \\
\gamma_C & \downarrow & \gamma_B \\
f_1' \circ g_1' & \xrightarrow{\gamma_A \circ \gamma_D} & f_2' \circ g_2' \\
\end{array}
\]

is commutative. We will use Lemmas [5.1] and [5.2] and with (abuse of) notation as in Remark [5.3] (in particular dropping \(\star\) products with identity transformations from
the notation). We can write \( \gamma_E = \gamma_A \circ \gamma_F \) where

\[
\begin{array}{ccc}
U_1 & \longrightarrow & \overline{X}_1 \\
\ \\ \downarrow F \\
U_2 & \longrightarrow & \overline{X}_2
\end{array}
\]

Thus we see that

\[
\gamma_B \circ \gamma_E = \gamma_B \circ \gamma_A \circ \gamma_F = \gamma_A \circ \gamma_B \circ \gamma_F
\]

the last equality because the two squares \( A \) and \( B \) only intersect in one point (similar to the last argument in Remark 5.3). Thus it suffices to prove that \( \gamma_D \circ \gamma_C = \gamma_B \circ \gamma_F \). Since both of these are equal to the map (4.1.1) for the square

\[
\begin{array}{ccc}
U_1 & \longrightarrow & \overline{X}_1 \\
\ \\ \downarrow \\
Y & \longrightarrow & \overline{Y}_2
\end{array}
\]

we conclude. \( \Box \)

0ATY **Lemma 16.4.** In Situation 16.1 the constructions of Lemmas 16.2 and 16.3 define a pseudo functor from the category \( FTTS_S \) into the 2-category of categories (see Categories, Definition 28.5).

**Proof.** To show this we have to prove given morphisms \( f : X \to Y, \ g : Y \to Z, \ h : Z \to T \) that

\[
(h \circ g \circ f)^! \xrightarrow{\gamma_{A+B}} f^! \circ (h \circ g)^!
\]

\[
\gamma_{B+C}
\]

\[
(g \circ f)^! \circ h^! \xrightarrow{\gamma_A} f^! \circ g^! \circ h^!
\]

is commutative (for the meaning of the \( \gamma \)'s, see below). To do this we choose a compactification \( \overline{Z} \) of \( Z \) over \( T \), then a compactification \( \overline{Y} \) of \( Y \) over \( Z \), and then a compactification \( \overline{X} \) of \( X \) over \( \overline{Y} \). This uses More on Flatness, Theorem 33.8 and Lemma 32.2. Let \( W \subset \overline{Y} \) be the inverse image of \( Z \) under \( \overline{Y} \to \overline{Z} \) and let \( U \subset V \subset \overline{X} \) be the inverse images of \( Y \subset W \) under \( \overline{X} \to \overline{Y} \). This produces the following diagram

\[
\begin{array}{ccc}
X & \longrightarrow & U \\
\ \\ \downarrow f \\
Y & \longrightarrow & V \\
\ \\ \downarrow A \\
Y & \longrightarrow & W \\
\ \\ \downarrow B \\
Z & \longrightarrow & Z \\
\ \\ \downarrow C \\
Z & \longrightarrow & \overline{Z}
\end{array}
\]

Without introducing tons of notation but arguing exactly as in the proof of Lemma 16.3 we see that the maps in the first displayed diagram use the maps (4.1.1) for the rectangles \( A + B, \ B + C, \ A, \) and \( C \) as indicated. Since by Lemmas 5.1 and 5.2
In Situation 16.1 let \( f : X \to Y \) be a morphism of FTS\(_S\). There are canonical maps
\[
\mu_{f,K} : Lf^*K \otimes \mathcal{O}_X \to f^!K
\]
functorial in \( K \) in \( D^+_{Q\text{Coh}}(\mathcal{O}_Y) \). If \( g : Y \to Z \) is another morphism of FTS\(_S\), then the diagram
\[
Lf^*(Lg^*K \otimes \mathcal{O}_Y) \otimes \mathcal{O}_X \xrightarrow{\mu_f} f^!(Lg^*K \otimes \mathcal{O}_Y) \xrightarrow{f^!g^!} f^!g^!K
\]
commutes for all \( K \in D^+_{Q\text{Coh}}(\mathcal{O}_Z) \).

**Proof.** If \( f \) is proper, then \( f^! = a \) and we can use \( 8.0.1 \) and if \( g \) is also proper, then Lemma \( \ref{lem:8.4} \) proves the commutativity of the diagram (in greater generality).

In general, choose a compactification \( j : X \to \overline{X} \) of \( X \) over \( Y \). Since \( f^! \) is defined as \( j^* \circ \pi \) we obtain \( \mu_f \) as the restriction of the map \( 8.0.1 \)
\[
Lf^+ K \otimes \mathcal{O}_\overline{X} \to \pi(K)
\]
to \( X \). To see this is independent of the choice of the compactification, we may assume given a morphism \( g : X_1 \to X_2 \) between compactifications \( j_i : X \to X_i \) over \( Y \). But now we know that the maps
\[
Lf_1^+ K \otimes \mathcal{O}_{X_1} \to \pi_1(K) \quad \text{and} \quad Lf_2^+ K \otimes \mathcal{O}_{X_2} \to \pi_2(K)
\]
fit into a commutative diagram by Lemma \( \ref{lem:8.3} \) with two other maps given by \( \mu_g \) which restrict to an isomorphism on \( X \) by Lemma \( \ref{lem:8.3} \). This implies the two displayed maps above restrict to the same map on the open, via the identification \( \pi_1(K)|_X = \pi_2(K)|_X \) used in the definition of \( f^! \). Having said this, the commutativity of the diagram follows from the construction of the isomorphism \( (g \circ f)^! \to f^! \circ g^! \) (first part of the proof of Lemma \( \ref{lem:16.3} \) using \( \overline{X} \to \overline{Y} \to Z \)) and the result of Lemma \( \ref{lem:8.4} \) for \( X \to Y \to Z \). \( \square \)

### 17. Properties of upper shriek functors

**Lemma 17.1.** In Situation 16.1 let \( Y \) be an object of FTS\(_S\) and let \( j : X \to Y \) be an open immersion. Then there is a canonical isomorphism \( j^! = j^* \) of functors.

**Proof.** In this case we may choose \( X = Y \) as our compactification. Then the right adjoint of Lemma \( \ref{lem:3.1} \) for \( \text{id} : Y \to Y \) is the identity functor and hence \( j^! = j^* \) by definition. \( \square \)

**Lemma 17.2.** In Situation 16.1 let \( Y \) be an object of FTS\(_S\) and let \( f : X = \mathbb{A}^1_Y \to Y \) be the projection. Then there is a (noncanonical) isomorphism \( f^!(\_)[1] \equiv Lf^*(-)[1] \) of functors.
Proof. Since $X = \mathbb{A}^1_Y \subset \mathbb{P}^1_Y$ and since $\mathcal{O}_{\mathbb{P}^1_Y}(-2)|_X \cong \mathcal{O}_X$ this follows from Lemmas 15.1 and 13.3. □

Lemma 17.3. In Situation 16.1 let $Y$ be an object of $\text{FTS}_S$ and let $i : X \to Y$ be a closed immersion. Then there is a canonical isomorphism $i^!(-) = R\text{Hom}(\mathcal{O}_X, -)$ of functors.

Proof. This is a restatement of Lemma 9.7. □

Remark 17.4 (Local description upper shriek). In Situation 16.1 let $f : X \to Y$ be a morphism of $\text{FTS}_S$. Using the lemmas above we can compute $f^!$ locally as follows. Suppose that we are given affine opens

$$
\begin{array}{ccc}
U & \xrightarrow{g} & X \\
\downarrow f & & \downarrow f \\
V & \xrightarrow{i} & Y
\end{array}
$$

Since $j^! \circ f^! = g^! \circ i^!$ (Lemma 16.3) and since $j^!$ and $i^!$ are given by restriction (Lemma 17.1) we see that $(f^! E)|_U = g^! (E|_V)$ for any $E \in D^+_{\text{QCoh}}(\mathcal{O}_X)$. Write $U = \text{Spec} (A)$ and $V = \text{Spec} (R)$ and let $\varphi : R \to A$ be the finite type ring map corresponding to $g$. Choose a presentation $A = P/I$ where $P = R[x_1, \ldots, x_n]$ is a polynomial algebra in $n$ variables over $R$. Choose an object $K \in D^+(R)$ corresponding to $E|_V$ (Derived Categories of Schemes, Lemma 3.5). Then we claim that $f^! E|_U$ corresponds to

$$
\varphi^!(K) = R\text{Hom}(A, K \otimes_R P[n])
$$

where $R\text{Hom}(A, -) : D(P) \to D(A)$ is the functor of Dualizing Complexes, Section 13 and where $\varphi^! : D(R) \to D(A)$ is the functor of Dualizing Complexes, Section 24. Namely, the choice of presentation gives a factorization

$$
\begin{array}{ccc}
U & \xrightarrow{A^1_V} & A^n_V \\
\downarrow & & \downarrow \varphi^!
\end{array}
$$

where $A^n_V \to V$ and $A^1_V \to V$. Applying Lemma 17.2 exactly $n$ times we see that $(A^n_V \to V)^!(E|_V)$ corresponds to $K \otimes_R P[n]$. By Lemmas 9.5 and 17.3 the last step corresponds to applying $R\text{Hom}(A, -)$.

Lemma 17.5. In Situation 16.1 let $f : X \to Y$ be a morphism of $\text{FTS}_S$. Then $f^!$ maps $D^+_{\text{Coh}}(\mathcal{O}_Y)$ into $D^+_{\text{Coh}}(\mathcal{O}_X)$.

Proof. The question is local on $X$ hence we may assume that $X$ and $Y$ are affine schemes. In this case we can factor $f : X \to Y$ as

$$
\begin{array}{ccc}
X & \xrightarrow{i} & A^n_Y \\
\downarrow & & \downarrow \varphi \\
A^1_Y & \to & Y
\end{array}
$$

where $i$ is a closed immersion. The lemma follows from By Lemmas 17.2 and 9.6 and Dualizing Complexes, Lemma 15.10 and induction. □

Lemma 17.6. In Situation 16.1 let $f : X \to Y$ be a morphism of $\text{FTS}_S$. If $K$ is a dualizing complex for $Y$, then $f^! K$ is a dualizing complex for $X$.
In Situation 16.1 let $\mathcal{O}_X$ for $\rem 33.8$ and Lemma 32.2. Let $\mathcal{O}_Y$ for $\rem 33.8$ and Lemma 32.2. Let

$$X \xrightarrow{i} \mathbb{A}^n \rightarrow \mathbb{A}^{n-1} \rightarrow \cdots \rightarrow \mathbb{A}^1 \rightarrow Y$$

where $i$ is a closed immersion. By Lemma 17.2 and Dualizing Complexes, Lemma 15.10 and induction we see that we obtain a dualizing complex on $\mathbb{A}^n$ where $p : \mathbb{A}^n \rightarrow Y$ is the projection. Similarly, by Dualizing Complexes, Lemma 15.9 and Lemmas 9.5 and 17.3 we see that $i^!$ transforms dualizing complexes into dualizing complexes.

0AU2 **Lemma 17.7.** In Situation 16.1 let $f : X \rightarrow Y$ be a morphism of FTS. Let $K$ be a dualizing complex on $Y$. Set $D_Y(M) = R\mathcal{H}om_{\mathcal{O}_Y}(M, K)$ for $M \in D_{\text{Coh}}(\mathcal{O}_Y)$ and $D_X(E) = R\mathcal{H}om_{\mathcal{O}_X}(E, f^! K)$ for $E \in D_{\text{Coh}}(\mathcal{O}_X)$. Then there is a canonical isomorphism

$$f^! M \longrightarrow D_X(Lf^* D_Y(M))$$

for $M \in D_{\text{Coh}}^+(\mathcal{O}_Y)$.

**Proof.** Choose compactification $j : X \subset \overline{X}$ of $X$ over $Y$ (More on Flatness, Theorem 33.8 and Lemma 32.2). Let $a$ be the right adjoint of Lemma 3.1 for $X \rightarrow Y$. Set $D_{\overline{X}}(E) = R\mathcal{H}om_{\mathcal{O}_{\overline{X}}}(E, a(K))$ for $E \in D_{\text{Coh}}(\mathcal{O}_{\overline{X}})$. Since formation of $R\mathcal{H}om$ commutes with restriction to opens and since $f^! = j^* \circ a$ we see that it suffices to prove that there is a canonical isomorphism

$$a(M) \longrightarrow D_{\overline{X}}(Lf^* f_Y(M))$$

for $M \in D_{\text{Coh}}(\mathcal{O}_X)$. For $F \in D_{\text{QCoh}}(\mathcal{O}_X)$ we have

$$\text{Hom}_{\mathcal{O}_X}(F, D_{\overline{X}}(Lf^* f_Y(M))) = \text{Hom}_{\mathcal{O}_X}(F \otimes_{\mathcal{O}_X} Lf^* f_Y(M), a(K))$$

$$= \text{Hom}_{\mathcal{O}_Y}(Rf_* (F \otimes_{\mathcal{O}_X} Lf^* f_Y(M)), K)$$

$$= \text{Hom}_{\mathcal{O}_Y}(Rf_* (F \otimes_{\mathcal{O}_X} f_Y(M)), K)$$

$$= \text{Hom}_{\mathcal{O}_Y}(Rf_* (F), f_Y(M))$$

$$= \text{Hom}_{\mathcal{O}_Y}(F, a(M))$$

The first equality by Cohomology, Lemma 36.2 The second by definition of $a$. The third by Derived Categories of Schemes, Lemma 21.1 The fourth equality by Cohomology, Lemma 36.2 and the definition of $f_Y$. The fifth equality by Lemma 24. The final equality by definition of $a$. Hence we see that $a(M) = D_{\overline{X}}(Lf^* f_Y(M))$ by Yoneda’s lemma.

0B6U **Lemma 17.8.** In Situation 16.1 let $f : X \rightarrow Y$ be a morphism of FTS. Assume $f$ is perfect (e.g., flat). Then

(a) $f^!$ maps $D_{\text{Coh}}^b(\mathcal{O}_Y)$ into $D_{\text{Coh}}^b(\mathcal{O}_X)$,

(b) the map $\mu_{f,K} : Lf^* K \otimes_{\mathcal{O}_X} f^! \mathcal{O}_Y \rightarrow f^! K$ of Lemma 16.3 is an isomorphism for all $K \in D_{\text{QCoh}}^+(\mathcal{O}_Y)$.

**Proof.** (A flat morphism of finite presentation is perfect, see More on Morphisms, Lemma 53.5) We begin with a series of preliminary remarks.

1. We already know that $f^!$ sends $D_{\text{Coh}}^+(\mathcal{O}_Y)$ into $D_{\text{Coh}}^+(\mathcal{O}_X)$, see Lemma 17.5

DUALITY FOR SCHEMES 45
(2) If \( f \) is an open immersion, then (a) and (b) are true because we can take \( X = Y \) in the construction of \( f^! \) and \( \mu_f \). See also Lemma 17.1

(3) If \( f \) is a perfect proper morphism, then (b) is true by Lemma 13.3

(4) If there exists an open covering \( X = \bigcup U_i \) and (a) is true for \( U_i \to Y \), then (a) is true for \( X \to Y \). Same for (b). This holds because the construction of \( f^! \) and \( \mu_f \) commutes with passing to open subschemes.

(5) If \( g : Y \to Z \) is a second perfect morphism in \( \text{FTS}_S \) and (b) holds for \( f \) and \( g \), then \( f^! g^! \mathcal{O}_Z = L f^* g^! \mathcal{O}_Z \otimes^\mathbb{L}_{\mathcal{O}_X} f^! \mathcal{O}_Y \) and (b) holds for \( g \circ f \) by the commutative diagram of Lemma 16.5

(6) If (a) and (b) hold for both \( f \) and \( g \), then (a) and (b) hold for \( g \circ f \). Namely, then \( f^! g^! \mathcal{O}_Z \) is bounded above (by the previous point) and \( L(g \circ f)^* \) has finite cohomological dimension and (a) follows from (b) which we saw above.

From these points we see it suffices to prove the result in case \( X \) is affine. Choose an immersion \( X \to \mathbb{A}_Y^n \) (Morphisms, Lemma 37.2) which we factor as \( X \to U \to \mathbb{A}_Y^n \to Y \) where \( X \to U \) is a closed immersion and \( U \subset \mathbb{A}_Y^n \) is open. Note that \( X \to U \) is a perfect closed immersion by More on Morphisms, Lemma 53.8. Thus it suffices to prove the lemma for a perfect closed immersion and for the projection \( \mathbb{A}_Y^n \to Y \).

Let \( f : X \to Y \) be a perfect closed immersion. We already know (b) holds. Let \( K \in \mathcal{D}^{\text{coh}}_{\text{Coh}}(\mathcal{O}_Y) \). Then \( f^! K = R \mathcal{H}om(\mathcal{O}_X, K) \) (Lemma 17.3) and \( f_* f^! K = R \mathcal{H}om_{\mathcal{O}_Y}(f_* \mathcal{O}_X, K) \). Since \( f \) is perfect, the complex \( f_* \mathcal{O}_X \) is perfect and hence \( R \mathcal{H}om_{\mathcal{O}_Y}(f_* \mathcal{O}_X, K) \) is bounded above. This proves that (a) holds. Some details omitted.

Let \( f : \mathbb{A}_Y^n \to Y \) be the projection. Then (a) holds by repeated application of Lemma 17.2. Finally, (b) is true because it holds for \( \mathbb{P}_Y^n \to Y \) (flat and proper) and because \( \mathbb{A}_Y^n \subset \mathbb{P}_Y^n \) is an open.

**Lemma 17.9.** In Situation 16.1 let \( f : X \to Y \) be a morphism of \( \text{FTS}_S \). If \( f \) is flat, then \( f^! \mathcal{O}_Y \) is a \( Y \)-perfect object of \( D(\mathcal{O}_X) \) and \( \mathcal{O}_X \to R \mathcal{H}om_{\mathcal{O}_X}(f^! \mathcal{O}_Y, f^! \mathcal{O}_Y) \) is an isomorphism.

**Proof.** Both assertions are local on \( X \). Thus we may assume \( X \) and \( Y \) are affine. Then Remark 17.4 turns the lemma into an algebra lemma, namely Dualizing Complexes, Lemma 25.2. (Use Derived Categories of Schemes, Lemma 31.3 to match the languages.)

**Lemma 17.10.** In Situation 16.1 let \( f : X \to Y \) be a morphism of \( \text{FTS}_S \). Assume \( f : X \to Y \) is a local complete intersection morphism. Then

1. \( f^! \mathcal{O}_Y \) is an invertible object of \( D(\mathcal{O}_X) \), and
2. \( f^! \) maps perfect complexes to perfect complexes.

**Proof.** Recall that a local complete intersection morphism is perfect, see More on Morphisms, Lemma 54.4. By Lemma 17.8 it suffices to show that \( f^! \mathcal{O}_Y \) is an invertible object in \( D(\mathcal{O}_X) \). This question is local on \( X \) and \( Y \). Hence we may assume that \( X \to Y \) factors as \( X \to \mathbb{A}_Y^n \to Y \) where the first arrow is a Koszul regular immersion. See More on Morphisms, Section 54. The result holds for \( \mathbb{A}_Y^n \to Y \) by Lemma 17.2. Thus it suffices to prove the lemma when \( f \) is a Koszul regular immersion. Working locally once again we reduce to the case \( X = \text{Spec}(A) \) and \( Y = \text{Spec}(B) \), where \( A = B/(f_1, \ldots, f_r) \) for some regular
sequence \( f_1, \ldots, f_r \in B \) (use that for Noetherian local rings the notion of Koszul regular and regular are the same, see More on Algebra, Lemma 29.7). Thus \( X \to Y \) is a composition
\[
X = X_r \to X_{r-1} \to \cdots \to X_1 \to X_0 = Y
\]
where each arrow is the inclusion of an effective Cartier divisor. In this way we reduce to the case of an inclusion of an effective Cartier divisor \( i : D \to X \). In this case \( i^* \mathcal{O}_X = \mathcal{N}[1] \) by Lemma 14.1 and the proof is complete. 

\[ \square \]

18. Base change for upper shriek

In Situation 16.1 let

\[
\begin{align*}
X' & \xrightarrow{g'} X \\
f' & \xrightarrow{g} f \\
Y' & \xrightarrow{g} Y
\end{align*}
\]

be a cartesian diagram in \( FTS_S \) such that \( X \) and \( Y' \) are Tor independent over \( Y \). Our setup is currently not sufficient to construct a base change map \( L(g')^* \circ f' \to (f')^! \circ Lg^* \) in this generality. The reason is that in general it will not be possible to choose a compactification \( j : X \to \overline{X} \) over \( Y \) such that \( X \) and \( Y' \) are tor independent over \( Y \) and hence our construction of the base change map in Section 5 does not apply.

**Lemma 18.1.** In Situation 16.1 let

\[
\begin{align*}
X' & \xrightarrow{g'} X \\
f' & \xrightarrow{g} f \\
Y' & \xrightarrow{g} Y
\end{align*}
\]

be a cartesian diagram of \( FTS_S \) with \( g \) flat. Then there is an isomorphism \( L(g')^* \circ f' \to (f')^! \circ Lg^* \) on \( D^+_{QCoh}(\mathcal{O}_Y) \).

**Proof.** Namely, because \( g \) is flat, for every choice of compactification \( j : X \to \overline{X} \) of \( X \) over \( Y \) the scheme \( \overline{X} \) is Tor independent of \( Y' \). Denote \( j' : X' \to \overline{X} \) the base change of \( j \) and \( \overline{g'} : \overline{X} \to \overline{X} \) the projection. We define the base change map as the composition
\[
L(g')^* \circ f' = L(g')^* \circ j^* \circ a = (j')^* \circ L(\overline{g'})^* \circ a \to (j')^* \circ a' \circ Lg^* = (f')^! \circ Lg^*
\]
where the middle arrow is the base change map (5.0.1) and \( a \) and \( a' \) are the right adjoints to pushforward of Lemma 3.1 for \( \overline{X} \to Y \) and \( X' \to Y' \). This construction is independent of the choice of compactification (we will formulate a precise lemma and prove it, if we ever need this result).

To finish the proof it suffices to show that the base change map \( L(g')^* \circ a \to a' \circ Lg^* \) is an isomorphism on \( D^+_{QCoh}(\mathcal{O}_Y) \). By Lemma 4.4 formation of \( a \) and \( a' \) commutes with restriction to affine opens of \( Y \) and \( Y' \). Thus by Remark 6.1 we may assume that \( Y \) and \( Y' \) are affine. Thus the result by Lemma 6.2.

---

4 The reader who is well versed with derived algebraic geometry will realize this is not a “real” problem. Namely, taking \( \overline{X} \) to be the derived fibre product of \( \overline{X} \) and \( Y' \) over \( Y \), one can argue exactly as in the proof of Lemma 18.1 to define this map. After all, the Tor independence of \( X \) and \( Y' \) guarantees that \( X' \) will be an open subscheme of the derived scheme \( \overline{X} \).
In the rest of this section, we formulate some easy to prove results which would be consequences of a good theory of the base change map.

**Lemma 18.2** (Poor man’s base change). In Situation 16.1 let

\[
\begin{array}{ccc}
X' & \xrightarrow{g'} & X \\
\downarrow{f'} & & \downarrow{f} \\
Y' & \xrightarrow{g} & Y
\end{array}
\]

be a cartesian diagram of \(\mathcal{S}\). Let \(E \in D^+_{\mathcal{Q}Coh} (\mathcal{O}_Y)\). If \(f\) is flat, then \(L(g')^* f^! E\) and \((f')^! Lg^* E\) restrict to isomorphic objects of \(D(\mathcal{O}_{U'})\) for \(U' \subset X'\) affine open mapping into affine opens of \(Y', Y'\), and \(X\).

**Proof.** By our assumptions we immediately reduce to the case where \(X, Y, Y', \) and \(X'\) are affine. Say \(Y = \text{Spec}(R), Y' = \text{Spec}(R'), X = \text{Spec}(A), \) and \(X' = \text{Spec}(A')\). Then \(A' = A \otimes_R R'\). Let \(E\) correspond to \(K \in D^+(R)\). Denoting \(\varphi : R \rightarrow A\) and \(\varphi' : R' \rightarrow A'\) the given maps we see from Remark 17.4 that \(L(g')^* f^! E\) and \((f')^!* Lg^* E\) correspond to \(\varphi(K) \otimes_A A'\) and \((\varphi')^! (K \otimes_R R')\) where \(\varphi'\) and \((\varphi')^!\) are the functors from Dualizing Complexes, Section 24. The result follows from Dualizing Complexes, Lemma 24.6.

**Lemma 18.3.** In Situation 16.1 let \(f : X \rightarrow Y\) be a morphism of \(\mathcal{S}\). Assume \(f\) is flat. Set \(\omega^\bullet_{X/Y} = f^! \mathcal{O}_Y\) in \(D^b_{\mathcal{Q}Coh}(X)\). Let \(y \in Y\) and \(h : X_y \rightarrow X\) the projection. Then \(Lh^* \omega^\bullet_{X/Y}\) is a dualizing complex on \(X_y\).

**Proof.** The complex \(\omega^\bullet_{X/Y}\) is in \(D^b_{\mathcal{Q}Coh}\) by Lemma 17.8. Being a dualizing complex is a local property. Hence by Lemma 18.2, it suffices to show that \((X_y \rightarrow y)^! \mathcal{O}_y\) is a dualizing complex on \(X_y\). This follows from Lemma 17.6.

19. A duality theory

In this section we spell out what kind of a duality theory our very general results above give for finite type separated schemes over a fixed Noetherian base scheme. Recall that a dualizing complex on a Noetherian scheme \(X\), is an object of \(D(\mathcal{O}_X)\) which affine locally gives a dualizing complex for the corresponding rings, see Definition 2.2.

Given a Noetherian scheme \(S\) denote \(\mathcal{S}\) the category of schemes which are of finite type and separated over \(S\). Then:

1. the functors \(f^!\) turn \(D^+_{\mathcal{Q}Coh}\) into a pseudo functor on \(\mathcal{S}\),
2. if \(f : X \rightarrow Y\) is a proper morphism in \(\mathcal{S}\), then \(f^!\) is the restriction of the right adjoint of \(Rf_* : D_{\mathcal{Q}Coh} (\mathcal{O}_X) \rightarrow D_{\mathcal{Q}Coh}(\mathcal{O}_Y)\) to \(D^+_{\mathcal{Q}Coh}(\mathcal{O}_Y)\) and there is a canonical isomorphism
   \[ Rf_* R\text{Hom}_{\mathcal{O}_X} (K, f^! M) \rightarrow R\text{Hom}_{\mathcal{O}_Y} (Rf_* K, M) \]
   for all \(K \in D_{\mathcal{Q}Coh}(\mathcal{O}_X)\) and \(M \in D^+_{\mathcal{Q}Coh}(\mathcal{O}_Y),\)
3. if an object \(X\) of \(\mathcal{S}\) has a dualizing complex \(\omega^\bullet_X\), then the functor \(D_X = R\text{Hom}_{\mathcal{O}_X} (-, \omega^\bullet_X)\) defines an involution of \(D_{\mathcal{Coh}} (\mathcal{O}_X)\) switching \(D^+_{\mathcal{Coh}}(\mathcal{O}_X)\) and \(D^-_{\mathcal{Coh}}(\mathcal{O}_X)\) and fixing \(D^0_{\mathcal{Coh}}(\mathcal{O}_X),\)
4. if \(f : X \rightarrow Y\) is a morphism of \(\mathcal{S}\) and \(\omega^\bullet_Y\) is a dualizing complex on \(Y\), then
We will now use glueing of dualizing complexes to get a theory which works for all finite type schemes over $S$, see Lemmas 2.4, 3.6, 9.7, 11.4, 14.2, 15.6, 15.7, 16.3, 16.4, 17.3, 17.6, 17.7, and 17.8.

We have obtained our functors by a very abstract procedure which finally rests on invoking an existence theorem (Derived Categories, Proposition 35.2). This means we have, in general, no explicit description of the functors $f^!$. This can sometimes be a problem. But in fact, it is often enough to know the existence of a dualizing complex and the duality isomorphism to pin down $f^!$.

20. Glueing dualizing complexes

0AU5 We will now use glueing of dualizing complexes to get a theory which works for all finite type schemes over $S$ given a pair $(S, \omega^\bullet_S)$ as in Situation 20.1. This is similar to [Har06, Remark on page 310].

0AU4 **Situation 20.1.** Here $S$ is a Noetherian scheme and $\omega^\bullet_S$ is a dualizing complex.

In Situation 20.1 let $X$ be a scheme of finite type over $S$. Let $\mathcal{U} : X = \bigcup_{i=1}^n U_i$ be a finite open covering of $X$ by objects of $FTS_S$, see Situation 16.1. All this means is that the morphisms $U_i \to S$ are separated (as they are already of finite type).

Every affine scheme of finite type over $S$ is an object of $FTS_S$ by Schemes, Lemma 21.13, hence such open coverings certainly exist. Then for each $i, j, k \in \{1, \ldots, n\}$ the morphisms $p_i : U_i \to S$, $p_{ij} : U_i \cap U_j \to S$, and $p_{ijk} : U_i \cap U_j \cap U_k \to S$ are separated and each of these schemes is an object of $FTS_S$. From such an open covering we obtain

(a) $\omega^\bullet_{X/U} = f^!\omega^\bullet_S$ is a dualizing complex for $X$.

(b) $f^!M = D_X(Lf^*D_Y(M))$ canonically for $M \in D^+_{Coh}(O_Y)$, and

(c) if in addition $f$ is proper then

$$Rf_*R\mathcal{H}om_{O_X}(K, \omega^\bullet_X) = R\mathcal{H}om_{O_Y}(Rf_*K, \omega^\bullet_Y)$$

(5) if $f : X \to Y$ is a closed immersion in $FTS_S$, then $f^!(-) = R\mathcal{H}om_{O_X}(-, -)$,

(6) if $f : Y \to X$ is a finite morphism in $FTS_S$, then $f_*f^!(-) = R\mathcal{H}om_{O_X}(f_*O_Y, -)$,

(7) if $f : X \to Y$ is the inclusion of an effective Cartier divisor into an object of $FTS_S$, then $f^!(-) \cong Lf^*(-) \otimes_{O_X} O_Y(-X)[-1]$,

(8) if $f : X \to Y$ is a Koszul regular immersion of codimension $c$ into an object of $FTS_S$, then $f^!(-) \cong Lf^*(-) \otimes_{O_X} \mathcal{K}^c_{N}[-c]$, and

(9) if $f : X \to Y$ is a smooth proper morphism of relative dimension $d$ in $FTS_S$, then $f^!(-) \cong Lf^* \otimes_{O_X} \mathcal{K}^d_{X/Y}[d]$.

We have obtained our functors by a very abstract procedure which finally rests on invoking an existence theorem (Derived Categories, Proposition 35.2). This means we have, in general, no explicit description of the functors $f^!$. This can sometimes be a problem. But in fact, it is often enough to know the existence of a dualizing complex and the duality isomorphism to pin down $f^!$.

In the situation just described a *dualizing complex normalized relative to $\omega^\bullet_S$ and $\mathcal{U}$* is a pair $(K, \alpha_i)$ where $K \in D(O_X)$ and $\alpha_i : K|_{U_i} \to \omega^\bullet_U$ are isomorphisms such
In Situation 20.1 let \( X \) be a scheme of finite type over \( S \) and let \( \mathcal{U} \) be a finite open covering of \( X \) by schemes separated over \( S \). If there exists a dualizing complex normalized relative to \( \omega_S^* \) and \( \mathcal{U} \), then it is unique up to unique isomorphism.

**Proof.** If \((K, \alpha_i)\) and \((K', \alpha'_i)\) are two, then we consider \( L = R\hom_{\mathcal{O}_X}(K, K') \). By Lemma 20.2 and its proof, this is an invertible object of \( D(\mathcal{O}_X) \). Using \( \alpha_i \) and \( \alpha'_i \) we obtain an isomorphism

\[
\alpha_i \otimes \alpha'_i : L|_{U_i} \longrightarrow R\hom_{\mathcal{O}_X}(\omega_S^*, \omega_S^*) = \mathcal{O}_{U_i}[0]
\]

This already implies that \( L = H^0(L)[0] \) in \( D(\mathcal{O}_X) \). Moreover, \( H^0(L) \) is an invertible sheaf with given trivializations on the opens \( U_i \) of \( X \). Finally, the condition that \( \alpha_j|_{U_i \cap U_j} \circ \alpha_i^{-1}|_{U_i \cap U_j} \) and \( \alpha'_j|_{U_i \cap U_j} \circ (\alpha'_i)^{-1}|_{U_i \cap U_j} \) both give \( \varphi_{ij} \) implies that the transition maps are 1 and we get an isomorphism \( H^0(L) = \mathcal{O}_X \). \( \square \)

**Lemma 20.3.** In Situation 20.1 let \( X \) be a scheme of finite type over \( S \) and let \( \mathcal{U}, \mathcal{V} \) be two finite open coverings of \( X \) by schemes separated over \( S \). If there exists a dualizing complex normalized relative to \( \omega_S^* \) and \( \mathcal{U} \), then there exists a dualizing complex normalized relative to \( \omega_S^* \) and \( \mathcal{V} \) and these complexes are canonically isomorphic.

**Proof.** It suffices to prove this when \( \mathcal{U} \) is given by the opens \( U_1, \ldots, U_n \) and \( \mathcal{V} \) by the opens \( U_1, \ldots, U_{n+m} \). In fact, we may and do even assume \( m = 1 \). To go from a dualizing complex \((K, \alpha_i)\) normalized relative to \( \omega_S^* \) and \( \mathcal{U} \) to a dualizing complex normalized relative to \( \omega_S^* \) and \( \mathcal{U} \), we need to construct a map \( \alpha_{n+1} : K|_{U_{n+1}} \rightarrow \omega_{n+1}^* \) satisfying the required condition. To do this we observe that \( U_{n+1} = \bigcup U_i \cap U_{n+1} \) is an open covering. It is clear that \( (K|_{U_{n+1}}, \alpha_i|_{U_i \cap U_{n+1}}) \) is another dualizing complex normalized relative to \( \omega_S^* \) and the covering \( U_{n+1} = \bigcup U_i \cap U_{n+1} \). On the other hand, by condition (3) the pair \( (\omega_{n+1}^*|_{U_{n+1}}, \varphi_{n+1}) \) is another dualizing complex normalized relative to \( \omega_S^* \) and the covering \( U_{n+1} = \bigcup U_i \cap U_{n+1} \). By Lemma 20.2 we obtain a unique isomorphism

\[
\alpha_{n+1} : K|_{U_{n+1}} \longrightarrow \omega_{n+1}^*
\]

compatible with the given local isomorphisms. It is a pleasant exercise to show that this means it satisfies the required property. \( \square \)

**Lemma 20.4.** In Situation 20.1 let \( X \) be a scheme of finite type over \( S \) and let \( \mathcal{U} \) be a finite open covering of \( X \) by schemes separated over \( S \). Then there exists a dualizing complex normalized relative to \( \omega_S^* \) and \( \mathcal{U} \).

**Proof.** Say \( \mathcal{U} : X = \bigcup_{i=1}^{n} U_i \). We prove the lemma by induction on \( n \). The base case \( n = 1 \) is immediate. Assume \( n > 1 \). Set \( X' = U_1 \cup \ldots \cup U_{n-1} \) and let \((K', \alpha'_i)_{i=1,\ldots,n-1}\) be a dualizing complex normalized relative to \( \omega_{S'}^* \) and \( \mathcal{U}' \) : \( X' = \bigcup_{i=1}^{n-1} U_i \). It is clear that \((K'|_{X' \cap U_n}, \alpha_i'|_{U_i \cap U_n})\) is a dualizing complex normalized relative to \( \omega_{S'}^* \), and the covering \( X' \cap U_n = \bigcup_{i=1}^{n-1} U_i \cap U_n \). On the other hand, by condition (3) the pair \((\omega_{n+1}^*|_{X' \cap U_n}, \varphi_{n+1})\) is another dualizing complex.
normalized relative to $\omega_S^\bullet$ and the covering $X' \cap U_n = \bigcup_{i=1,\ldots,n-1} U_i \cap U_n$. By Lemma 20.2 we obtain a unique isomorphism

$$\epsilon : K'|_{X' \cap U_n} \longrightarrow \omega_i^\bullet|_{X' \cap U_n}$$

compactible with the given local isomorphisms. By Cohomology, Lemma 39.1 we obtain $K \in D(O_X)$ together with isomorphisms $\beta : K|_{X'} \rightarrow K'$ and $\gamma : K|_{U_n} \rightarrow \omega_n^\bullet$ such that $\epsilon = \gamma|_{X' \cap U_n} \circ \beta|_{X' \cap U_n}^{-1}$. Then we define

$$\alpha_i = \alpha'_i \circ \beta|_{U_i}, \ i = 1, \ldots, n-1, \text{ and } \alpha_n = \gamma$$

We still need to verify that $\varphi_{ij}$ is given by $\alpha_j|_{U_i \cap U_j} \circ \alpha_i^{-1}|_{U_i \cap U_j}$. For $i,j \leq n-1$ this follows from the corresponding condition for $\alpha'_i$. For $i = j = n$ it is clear as well. If $i < j = n$, then we get

$$\alpha_n|_{U_i \cap U_n} \circ \alpha_i^{-1}|_{U_i \cap U_n} = \gamma|_{U_i \cap U_n} \circ \beta^{-1}|_{U_i \cap U_n} \circ (\alpha'_i)^{-1}|_{U_i \cap U_n} = \epsilon|_{U_i \cap U_n} \circ (\alpha'_i)^{-1}|_{U_i \cap U_n}$$

This is equal to $\alpha_{in}$ exactly because $\epsilon$ is the unique map compatible with the maps $\alpha'_i$ and $\alpha_{ni}$.

Let $(S, \omega_S^\bullet)$ be as in Situation 20.1. The upshot of the lemmas above is that given any scheme $X$ of finite type over $S$, there is a pair $(K,\alpha_U)$ given up to unique isomorphism, consisting of an object $K \in D(O_X)$ and isomorphisms $\alpha_U : K|_U \rightarrow \omega_U^\bullet$ for every open subscheme $U \subset X$ which is separated over $S$. Here $\omega_U^\bullet = (U \rightarrow S)^! \omega_S^\bullet$ is a dualizing complex on $U$, see Section 19. Moreover, if $U : X = \bigcup U_i$ is an open covering by opens which are separated over $S$, then $(K,\alpha_U)$ is a dualizing complex normalized relative to $\omega_S^\bullet$ and $U$. Namely, uniqueness up to unique isomorphism by Lemma 20.2, existence for one open covering by Lemma 20.3, and the fact that $K$ then works for all open coverings is Lemma 20.4.

\[ \text{Definition 20.5. Let } S \text{ be a Noetherian scheme and let } \omega_S^\bullet \text{ be a dualizing complex on } S. \text{ Let } X \text{ be a scheme of finite type over } S. \text{ The complex } K \text{ constructed above is called the dualizing complex normalized relative to } \omega_S^\bullet \text{ and is denoted } \omega_X^\bullet. \]

As the terminology suggest, a dualizing complex normalized relative to $\omega_S^\bullet$ is not just an object of the derived category of $X$ but comes equipped with the local isomorphisms described above. This does not conflict with setting $\omega_X^\bullet = p^! \omega_S^\bullet$ where $p : X \rightarrow S$ is the structure morphism if $X$ is separated over $S$. More generally we have the following sanity check.

\[ \text{Lemma 20.6. Let } (S, \omega_S^\bullet) \text{ be as in Situation 20.1. Let } f : X \rightarrow Y \text{ be a morphism of finite type schemes over } S. \text{ Let } \omega_X^\bullet \text{ and } \omega_Y^\bullet \text{ be dualizing complexes normalized relative to } \omega_S^\bullet. \text{ Then } \omega_X^\bullet \text{ is a dualizing complex normalized relative to } \omega_Y^\bullet. \]

\[ \text{Proof. This is just a matter of bookkeeping. Choose a finite affine open covering } \mathcal{V} : Y = \bigcup V_j. \text{ For each } j \text{ choose a finite affine open covering } f^{-1}(V_j) = U_{ji}. \text{ Set } \mathcal{U} : X = \bigcup U_{ji}. \text{ The schemes } V_j \text{ and } U_{ji} \text{ are separated over } S, \text{ hence we have the upper shriek functors for } q_j : V_j \rightarrow S, p_{ji} : U_{ji} \rightarrow S \text{ and } f_{ji} : U_{ji} \rightarrow V_j \text{ and } f_{ji}^{-1} : V_j \rightarrow U_{ji}. \text{ Let } (L,\beta_j) \text{ be a dualizing complex normalized relative to } \omega_S^\bullet \text{ and } \mathcal{V}. \text{ Let } (K,\gamma_{ji}) \text{ be a dualizing complex normalized relative to } \omega_S^\bullet \text{ and } \mathcal{U}. \text{ (In other words, } L = \omega_X^\bullet \text{ and } K = \omega_X^\bullet). \text{ We can define } \]

$$\alpha_{ji} : K|_{U_{ji}} \stackrel{\gamma_{ji}}{\longrightarrow} p_{ji}^! \omega_S^\bullet = f_{ji}^! q_j^! \omega_S^\bullet \longrightarrow f_{ji}^! (L|_{V_j}) = (f_{ji}^!)^!(L)$$
To finish the proof we have to show that \( \alpha_{ij}|_{U_{ij} \cap U_{i'j'}} \circ \alpha_{i'j'}^{-1}|_{U_{ij} \cap U_{i'j'}} \) is the canonical isomorphism \((f'_{ij})^!(L)|_{U_{ij} \cap U_{i'j'}} = (f'_{i'j'})^!(L)|_{U_{ij} \cap U_{i'j'}}\). This is formal and we omit the details. \(\square\)

**Lemma 20.7.** Let \((S, \omega^*_S)\) be as in Situation \(20.1\). Let \(j : X \to Y\) be an open immersion of schemes of finite type over \(S\). Let \(\omega_X^*\) and \(\omega_Y^*\) be dualizing complexes normalized relative to \(\omega_S^*\). Then there is a canonical isomorphism \(\omega_X^* = \omega_Y^*|_X\).

**Proof.** Immediate from the construction of normalized dualizing complexes given just above Definition \(20.3\). \(\square\)

**Lemma 20.8.** Let \((S, \omega_S^*)\) be as in Situation \(20.1\). Let \(f : X \to Y\) be a proper morphism of schemes of finite type over \(S\). Let \(\omega_X^*\) and \(\omega_Y^*\) be dualizing complexes normalized relative to \(\omega_S^*\). Let a be the right adjoint of Lemma \(3.1\) for \(f\). Then there is a canonical isomorphism \(a(\omega_Y^*) = \omega_X^*\).

**Proof.** Let \(p : X \to S\) and \(q : Y \to S\) be the structure morphisms. If \(X\) and \(Y\) are separated over \(S\), then this follows from the fact that \(\omega_X^* = p^!\omega_S^*\), \(\omega_Y^* = q^!\omega_S^*\), \(f^! = a\), and \(f^! \circ q^! = p^!\) (Lemma \(16.3\)). In the general case we first use Lemma \(20.6\) to reduce to the case \(Y = S\). In this case \(X\) and \(Y\) are separated over \(S\) and we've just seen the result. \(\square\)

Let \((S, \omega^*_S)\) be as in Situation \(20.1\). For a scheme \(X\) of finite type over \(S\) denote \(\omega^*_X\) the dualizing complex for \(X\) normalized relative to \(\omega^*_S\). Define \(D_X(-) = R\text{Hom}_{\mathcal{O}_X}(-, \omega^*_X)\) as in Lemma \(2.4\). Let \(f : X \to Y\) be a morphism of finite type schemes over \(S\). Define

\[
f^!_{\text{new}} = D_X \circ Lf^* \circ D_Y : D^+_Coh(\mathcal{O}_Y) \to D^+_Coh(\mathcal{O}_X)
\]

If \(f : X \to Y\) and \(g : Y \to Z\) are composable morphisms between schemes of finite type over \(S\), define

\[
(g \circ f)^!_{\text{new}} = D_X \circ L(g \circ f)^* \circ D_Z
= D_X \circ Lf^* \circ Lg^* \circ D_Z
\to D_X \circ Lf^* \circ D_Y \circ Lg^* \circ D_Z
= f^!_{\text{new}} \circ g^!_{\text{new}}
\]

where the arrow is defined in Lemma \(2.3\). We collect the results together in the following lemma.

**Lemma 20.9.** Let \((S, \omega^*_S)\) be as in Situation \(20.1\). With \(f^!_{\text{new}}\) and \(\omega^*_X\) defined for all (morphisms of) schemes of finite type over \(S\) as above:

1. The functors \(f^!_{\text{new}}\) and the arrows \((g \circ f)^!_{\text{new}} = f^!_{\text{new}} \circ g^!_{\text{new}}\) turn \(D^+_Coh\) into a pseudo functor from the category of schemes of finite type over \(S\) into the 2-category of categories,
2. \(\omega^*_X = (X \to S)^!_{\text{new}} \omega^*_S\),
3. The functor \(D_X\) defines an involution of \(D^+_Coh(\mathcal{O}_X)\) switching \(D^+_Coh(\mathcal{O}_X)\) and \(D^+_Coh(\mathcal{O}_X)\) and fixing \(D^+_Coh(\mathcal{O}_X)\),
4. \(\omega^*_X = f^!_{\text{new}} \omega^*_Y\) for \(f : X \to Y\) a morphism of finite type schemes over \(S\),
5. \(f^!_{\text{new}} M = D_X(Lf^*D_Y(M))\) for \(M \in D^+_Coh(\mathcal{O}_Y)\), and
Lemma 8.1. Hence, if \( f \) is a morphism between schemes separated over \( S \), then there is a canonical isomorphism of schemes of finite type over \( S \).

Proof. Let \( f : X \to Y \), \( g : Y \to Z \), \( h : Z \to T \) be morphisms of schemes of finite type over \( S \). We have to show that

\[
\begin{array}{ccc}
(h \circ g \circ f)_\text{new}^! & \to & f^! \circ (h \circ g)_\text{new}^! \\
\downarrow & & \downarrow \\
(g \circ f)_\text{new}^! \circ h^! & \to & f^! \circ g^! \circ h^!
\end{array}
\]

is commutative. Let \( \eta_Y : \text{id} \to D^b \) and \( \eta_Z : \text{id} \to D^b \) be the canonical isomorphisms of Lemma 3.1. Then, using Categories, Lemma 27.2, a computation (omitted) shows that both arrows \( (h \circ g \circ f)_\text{new}^! \to f^! \circ g^! \circ h^! \) are given by

\[
1 \ast \eta_Y \ast 1 \ast \eta_Z \ast 1 : D_X \otimes Lf^* \otimes Lg^* \otimes Dh^* \otimes DT \to D_X \otimes Lf^* \otimes D_Y^* \otimes Lg^* \otimes D_Z^* \otimes Dh^* \otimes DT
\]

This proves (1). Part (2) is immediate from the definition of \( (X \to S)_\text{new}^! \) and the fact that \( D_S(\omega^!_S) = \mathcal{O}_S \). Part (3) is Lemma 2.4. Part (4) follows by the same argument as part (2). Part (5) is the definition of \( f^! \).

Proof of (6). Let \( a \) be the right adjoint of Lemma 3.1 for the proper morphism \( f : X \to Y \) of schemes of finite type over \( S \). The issue is that we do not know \( X \) or \( Y \) is separated over \( S \); (and in general this won’t be true) hence we cannot immediately apply Lemma 17.7 to \( f \) over \( S \). To get around this we use the canonical identification \( \omega^!_X = a(\omega^!_S) \) of Lemma 20.8. Hence \( f^! \) is the restriction of \( a \) to \( D^+(\mathcal{O}_Y) \) by Lemma 17.7 applied to \( f : X \to Y \) over the base scheme \( Y \). Thus the result is true by Lemma 3.6.

The final assertions follow from the construction of normalized dualizing complexes and the already used Lemma 17.7.

Remark 20.10. Let \( S \) be a Noetherian scheme which has a dualizing complex. Let \( f : X \to Y \) be a morphism of schemes of finite type over \( S \). Then the functor

\[
f^!_\text{new} : D^+(\mathcal{O}_Y) \to D^+(\mathcal{O}_X)
\]

is independent of the choice of the dualizing complex \( \omega^!_S \) up to canonical isomorphism. We sketch the proof. Any second dualizing complex is of the form \( \omega^!_S \otimes_{\mathcal{O}_S} \mathcal{L} \) with \( \mathcal{L} \) an invertible object of \( D(\mathcal{O}_S) \), see Lemma 2.4. For any separated morphism \( p : U \to S \) of finite type we have \( p^!(\omega^!_S \otimes_{\mathcal{O}_S} \mathcal{L}) = p^!(\omega^!_S) \otimes_{\mathcal{O}_U} Lp^* \mathcal{L} \) by Lemma 8.1. Hence, if \( \omega^!_X \) and \( \omega^!_Y \) are the dualizing complexes normalized relative to \( \omega^!_S \), we see that \( \omega^!_X \otimes_{\mathcal{O}_X} Lp^* \mathcal{L} \) and \( \omega^!_Y \otimes_{\mathcal{O}_Y} Lp^* \mathcal{L} \) are the dualizing complexes.

We haven’t checked that these are compatible with the isomorphisms \( (g \circ f)^! \to f^! \circ g^! \) and \( (g \circ f)_\text{new}^! \rightarrow f^! \circ g^! \circ h^! \). We will do this here if we need this later.
normalized relative to $\omega_S^* \otimes_{\mathcal{O}_S} \mathcal{L}$ (where $a : X \to S$ and $b : Y \to S$ are the structure morphisms). Then the result follows as

$$R \mathcal{H}om_{\mathcal{O}_X}(Lf^* R \mathcal{H}om_{\mathcal{O}_Y}(K, \omega_Y^* \otimes_{\mathcal{O}_Y} \mathcal{L}), \omega_X^* \otimes_{\mathcal{O}_X} \mathcal{L})$$

for $K \in D^+_\mathcal{Qcoh}(\mathcal{O}_Y)$. The last equality because $La^* \mathcal{L}$ is invertible in $D(\mathcal{O}_X)$.

**Example 20.11.** Let $\mathcal{S}$ be a Noetherian scheme and let $\omega_S^*$ be a dualizing complex. Let $f : X \to Y$ be a proper morphism of finite type schemes over $S$. Let $\omega_X^*$ and $\omega_Y^*$ be dualizing complexes normalized relative to $\omega_S^*$. In this situation we have $a(\omega_Y^*) = \omega_X^*$ (Lemma 20.8) and hence the trace map (Section 7) is a canonical arrow

$$\text{Tr}_f : Rf_* \omega_X^* \to \omega_Y^*$$

which produces the isomorphisms (Lemma 20.9)

$$\text{Hom}_X(L, \omega_X^*) = \text{Hom}_Y(Rf_* L, \omega_Y^*)$$

and

$$Rf_* R \mathcal{H}om_{\mathcal{O}_X}(L, \omega_X^*) = R \mathcal{H}om_{\mathcal{O}_Y}(Rf_* L, \omega_Y^*)$$

for $L$ in $D^+_\mathcal{Qcoh}(\mathcal{O}_X)$.

**Remark 20.12.** Let $\mathcal{S}$ be a Noetherian scheme and let $\omega_S^*$ be a dualizing complex. Let $f : X \to Y$ be a finite morphism between schemes of finite type over $S$. Let $\omega_X^*$ and $\omega_Y^*$ be dualizing complexes normalized relative to $\omega_S^*$. Then we have $f_* \omega_X^* = R \mathcal{H}om(f_* \mathcal{O}_X, \omega_Y^*)$ in $D^+_\mathcal{Qcoh}(f_* \mathcal{O}_X)$ by Lemmas 11.4 and 20.8 and the trace map of Example 20.11 is the map

$$\text{Tr}_f : Rf_* \omega_X^* = f_* \omega_X^* = R \mathcal{H}om(f_* \mathcal{O}_X, \omega_Y^*) \to \omega_Y^*$$

which often goes under the name "evaluation at 1".

**Remark 20.13.** Let $f : X \to Y$ be a flat proper morphism of finite type schemes over a pair $(\mathcal{S}, \omega_S^*)$ as in Situation 20.1. The relative dualizing complex (Remark 12.5) is $\omega_{X/Y}^* = a(\mathcal{O}_Y)$. By Lemma 20.8 we have the first canonical isomorphism in $D(\mathcal{O}_X)$.

$$\omega_X^* = a(\omega_Y^*) = Lf^* \omega_Y^* \otimes_{\mathcal{O}_X} \omega_{X/Y}^*$$

in $D(\mathcal{O}_X)$. The second canonical isomorphism follows from the discussion in Remark 12.5.

21. Dimension functions

We need a bit more information about how the dimension functions change when passing to a scheme of finite type over another.
Proof. We may replace $X$ by an affine neighbourhood of $x$, hence we may and
do assume that $f : X \to S$ is separated. Then $\omega^*_X = f^*\omega^*_S$. We have to show that $R\text{Hom}_{O_X}(\kappa(x),\omega^*_X)$ is sitting in degree 0. Let $i_x : x \to X$ denote the inclusion morphism which is a closed immersion as $x$ is a closed point. Hence $R\text{Hom}_{O_X}(\kappa(x),\omega^*_X)$ represents $i_x^!\omega^*_X$ by Lemma 17.3. Consider the commutative diagram

$$
\begin{array}{ccc}
x & \xrightarrow{i_x} & X \\
\downarrow{\pi} & & \downarrow{f} \\
S & \xrightarrow{i_x} & S
\end{array}
$$

By Morphisms, Lemma 19.3 the extension $\kappa(s) \subset \kappa(x)$ is finite and hence $\pi$ is a finite morphism. We conclude that

$$i_x^!\omega^*_X = i_x^!f^!\omega^*_S = \pi^!i_x^!\omega^*_S$$

Thus if $\omega^*_S$ is a normalized dualizing complex over $O_{S,s}$, then $i_x^!\omega^*_S = \kappa(s)[0]$ by the same reasoning as above. We have

$$R\pi_*(\pi^!(\kappa(s)[0])) = R\text{Hom}_{O_S}(R\pi_*(\kappa(s)[0]),\kappa(s)[0]) = \text{Hom}_{\kappa(s)}(\kappa(x),\kappa(s))$$

The first equality by Lemma 3.6, applied with $L = \kappa(x)[0]$. The second equality holds because $\pi_*$ is exact. Thus $\pi_*(\kappa(s)[0])$ is supported in degree 0 and we win. □

0AWM Lemma 21.2. Let $S$ be a Noetherian scheme and let $\omega^*_S$ be a dualizing complex. Let $f : X \to S$ be of finite type and let $\omega^*_X$ be the dualizing complex normalized relative to $\omega^*_S$. For all $x \in X$ we have

$$\delta_X(x) - \delta_S(f(x)) = \text{trdeg}_{\kappa(f(x))}(\kappa(x))$$

where $\delta_S$, resp. $\delta_X$ is the dimension function of $\omega^*_S$, resp. $\omega^*_X$, see Lemma 2.6

Proof. We may replace $X$ by an affine neighbourhood of $x$. Hence we may and
do assume there is a compactification $X \subset \overline{X}$ over $S$. Then we may replace $X$ by $\overline{X}$ and assume that $X$ is proper over $S$. We may also assume $X$ is connected by replacing $X$ by the connected component of $X$ containing $x$. Next, recall that both $\delta_X$ and the function $x \mapsto \delta_S(f(x)) + \text{trdeg}_{\kappa(f(x))}(\kappa(x))$ are dimension functions on $X$, see Morphisms, Lemma 50.3 and the fact that $S$ is universally catenary by Lemma 20.3. By Topology, Lemma 20.3 we see that the difference is locally constant, hence constant as $X$ is connected. Thus it suffices to prove equality in any point of $X$. By Properties, Lemma 5.9 the scheme $X$ has a closed point $x$. Since $X \to S$ is proper the image $s$ of $x$ is closed in $S$. Thus we may apply Lemma 21.1 to conclude. □

0BV5 Lemma 21.3. In Situation 16.1 let $f : X \to Y$ be a morphism of FTS$_S$. Let $x \in X$ with image $y \in Y$. Then

$$H^i(f^!O_Y)_x \neq 0 \Rightarrow -\dim_x(X_y) \leq i.$$

Proof. Since the statement is local on $X$ we may assume $X$ and $Y$ are affine schemes. Write $X = \text{Spec}(A)$ and $Y = \text{Spec}(R)$. Then $f^!O_Y$ corresponds to the relative dualizing complex $\omega^*_{A/R}$ of Dualizing Complexes, Section 25 by Remark 17.4. Thus the lemma follows from Dualizing Complexes, Lemma 25.4. □
Lemma 21.4. In Situation 16.1 let \( f : X \to Y \) be a morphism of FTS\(_S\). Let \( x \in X \) with image \( y \in Y \). If \( f \) is flat, then
\[
H^i(f^!\mathcal{O}_Y)_{x} \neq 0 \Rightarrow -\dim_x(X_y) \leq i \leq 0.
\]

In fact, if all fibres of \( f \) have dimension \( \leq d \), then \( f^!\mathcal{O}_Y \) has tor-amplitude in \([-d,0]\) as an object of \( D(X,f^{-1}\mathcal{O}_Y) \).

Proof. Arguing exactly as in the proof of Lemma 21.3 this follows from Dualizing Complexes, Lemma 25.8.

Lemma 21.5. In Situation 16.1 let \( f : X \to Y \) be a morphism of FTS\(_S\). Let \( x \in X \) with image \( y \in Y \). Assume
1. \( \mathcal{O}_{Y,y} \) is Cohen-Macaulay, and
2. \( \text{trdeg}_{\kappa(\xi)}(\kappa(\xi)) \leq r \) for any generic point \( \xi \) of an irreducible component of \( X \) containing \( x \).

Then
\[
H^i(f^!\mathcal{O}_Y)_{x} \neq 0 \Rightarrow -r \leq i
\]
and the stalk \( H^{-r}(f^!\mathcal{O}_Y)_{x} \) is \((S_2)\) as an \( \mathcal{O}_{X,x} \)-module.

Proof. After replacing \( X \) by an open neighbourhood of \( x \), we may assume every irreducible component of \( X \) passes through \( x \). Then arguing exactly as in the proof of Lemma 21.3 this follows from Dualizing Complexes, Lemma 25.9.

Lemma 21.6. In Situation 16.1 let \( f : X \to Y \) be a morphism of FTS\(_S\). If \( f \) is flat and quasi-finite, then
\[
f^!\mathcal{O}_Y = \omega_{X/Y}[0]
\]
for some coherent \( \mathcal{O}_X \)-module \( \omega_{X/Y} \) flat over \( Y \).

Proof. Consequence of Lemma 21.4 and the fact that the cohomology sheaves of \( f^!\mathcal{O}_Y \) are coherent by Lemma 17.5.

Lemma 21.7. In Situation 16.1 let \( f : X \to Y \) be a morphism of FTS\(_S\). If \( f \) is Cohen-Macaulay (More on Morphisms, Definition 20.1), then
\[
f^!\mathcal{O}_Y = \omega_{X/Y}[d]
\]
for some coherent \( \mathcal{O}_X \)-module \( \omega_{X/Y} \) flat over \( Y \) where \( d \) is the locally constant function on \( X \) which gives the relative dimension of \( X \) over \( Y \).

Proof. The relative dimension \( d \) is well defined and locally constant by Morphisms, Lemma 28.4. The cohomology sheaves of \( f^!\mathcal{O}_Y \) are coherent by Lemma 17.5. We will get flatness of \( \omega_{X/Y} \) from Lemma 21.4 if we can show the other cohomology sheaves of \( f^!\mathcal{O}_Y \) are zero.

The question is local on \( X \), hence we may assume \( X \) and \( Y \) are affine and the morphism has relative dimension \( d \). If \( d = 0 \), then the result follows directly from Lemma 21.6. If \( d > 0 \), then we may assume there is a factorization
\[
X \xrightarrow{g} \mathbb{A}^d_Y \xrightarrow{p} Y
\]
with \( g \) quasi-finite and flat, see More on Morphisms, Lemma 20.8. Then \( f^! = g^! \circ p^! \).

By Lemma 17.2 we see that \( p^!\mathcal{O}_Y \cong \mathcal{O}_{\mathbb{A}^d_Y}[-d] \). We conclude by the case \( d = 0 \).
Remark 21.8. Let $S$ be a Noetherian scheme endowed with a dualizing complex $\omega_S^\bullet$. In this case Lemmas 21.3, 21.4, 21.6, and 21.7 are true for any morphism $f : X \to Y$ of finite type schemes over $S$ but with $f^!$ replaced by $f^!_{\text{new}}$. This is clear because in each case the proof reduces immediately to the affine case and then $f^! = f^!_{\text{new}}$ by Lemma 20.9.

22. Dualizing modules

0AWH This section is a continuation of Dualizing Complexes, Section 19.

Let $X$ be a Noetherian scheme and let $\omega_X^\bullet$ be a dualizing complex. Let $n \in \mathbb{Z}$ be the smallest integer such that $H^n(\omega_X^\bullet)$ is nonzero. In other words, $-n$ is the maximal value of the dimension function associated to $\omega_X^\bullet$ (Lemma 2.6). Sometimes $H^n(\omega_X^\bullet)$ is called a dualizing module or dualizing sheaf for $X$ and then it is often denoted by $\omega_X$. We will say “let $\omega_X$ be a dualizing module” to indicate the above.

Care has to be taken when using dualizing modules $\omega_X$ on Noetherian schemes $X$:

(1) the integer $n$ may change when passing from $X$ to an open $U$ of $X$ and then it won’t be true that $\omega_X|_U = \omega_U$,

(2) the dualizing complex isn’t unique; the dualizing module is only unique up to tensoring by an invertible module.

The second problem will often be irrelevant because we will work with $X$ of finite type over a base change $S$ which is endowed with a fixed dualizing complex $\omega_S^\bullet$ and $\omega_X^\bullet$ will be the dualizing complex normalized relative to $\omega_S^\bullet$. The first problem will not occur if $X$ is equidimensional, more precisely, if the dimension function associated to $\omega_X^\bullet$ (Lemma 2.6) maps every generic point of $X$ to the same integer.

Example 22.1. Say $S = \text{Spec}(A)$ with $(A, m, \kappa)$ a local Noetherian ring, and $\omega_S^\bullet$ corresponds to a normalized dualizing complex $\omega_A^\bullet$. Then if $f : X \to S$ is proper over $S$ and $\omega_X^\bullet = f^!\omega_S^\bullet$ the coherent sheaf

$$\omega_X = H^{-\dim(X)}(\omega_X^\bullet)$$

is a dualizing module and is often called the dualizing module of $X$ (with $S$ and $\omega_S^\bullet$ being understood). We will see that this has good properties.

Example 22.2. Say $X$ is an equidimensional scheme of finite type over a field $k$. Then it is customary to take $\omega_X^\bullet$ the dualizing complex normalized relative to $k[0]$ and to refer to

$$\omega_X = H^{-\dim(X)}(\omega_X^\bullet)$$

as the dualizing module of $X$. If $X$ is separated over $k$, then $\omega_X^\bullet = f^!O_{\text{Spec}(k)}$ where $f : X \to \text{Spec}(k)$ is the structure morphism by Lemma 20.9. If $X$ is proper over $k$, then this is a special case of Example 22.1.

Lemma 22.3. Let $X$ be a connected Noetherian scheme and let $\omega_X$ be a dualizing module on $X$. The support of $\omega_X$ is the union of the irreducible components of maximal dimension with respect to any dimension function and $\omega_X$ is a coherent $O_X$-module having property $(S_2)$.

Proof. By our conventions discussed above there exists a dualizing complex $\omega_X^\bullet$ such that $\omega_X$ is the leftmost nonvanishing cohomology sheaf. Since $X$ is connected, any two dimension functions differ by a constant (Topology, Lemma 20.3). Hence we may use the dimension function associated to $\omega_X^\bullet$ (Lemma 2.6). With these
and the definitions (in particular Cohomology of Schemes, Definition 11.1). □

\textbf{Lemma 22.4.} Let $X/A$ with $\omega^* \cdot X$ and $\omega_X$ be as in Example 22.1. Then
\begin{enumerate}
  \item $H^i(\omega^\bullet_X) \neq 0 \Rightarrow i \in \{-\dim(X), \ldots, 0\}$,
  \item the dimension of the support of $H^i(\omega^\bullet_X)$ is at most $-i$,
  \item $\operatorname{Supp}(\omega_X)$ is the union of the components of dimension $\dim(X)$, and
  \item $\omega_X$ has property $(S_2)$.
\end{enumerate}

\textbf{Proof.} Let $\delta_X$ and $\delta_S$ be the dimension functions associated to $\omega^\bullet_X$ and $\omega^\bullet_S$ as in Lemma 21.2. As $X$ is proper over $A$, every closed subscheme of $X$ contains a closed point $x$ which maps to the closed point $s \in S$ and $\delta_X(x) = \delta_S(s) = 0$. Hence $\delta_X(\xi) = \dim(\{\xi\})$ for any point $\xi \in X$. Hence we can check each of the statements of the lemma by looking at what happens over $\operatorname{Spec}(\mathcal{O}_{X,x})$ in which case the result follows from Dualizing Complexes, Lemmas 16.5 and 17.5. Some details omitted.

The last two statements can also be deduced from Lemma 22.3. □

\textbf{Lemma 22.5.} Let $X/A$ with dualizing module $\omega_X$ be as in Example 22.1. Let $d = \dim(X_s)$ be the dimension of the closed fibre. If $\dim(X) = d + \dim(A)$, then the dualizing module $\omega_X$ represents the functor

$$F \mapsto \operatorname{Hom}_A(H^d(X, F), \omega_A)$$

on the category of coherent $\mathcal{O}_X$-modules.

\textbf{Proof.} We have

$$\operatorname{Hom}_X(\mathcal{F}, \omega_X) = \operatorname{Ext}^{-\dim(X)}_X(\mathcal{F}, \omega_X^\bullet) = \operatorname{Hom}_X(\mathcal{F}[\dim(X)], \omega_X^\bullet) = \operatorname{Hom}_X(\mathcal{F}[\dim(X)], f^! (\omega_A^\bullet)) = \operatorname{Hom}_S(Rf_* \mathcal{F}[\dim(X)], \omega_A^\bullet) = \operatorname{Hom}_A(H^d(X, \mathcal{F}), \omega_A)$$

The first equality because $H^i(\omega^\bullet_X) = 0$ for $i < -\dim(X)$, see Lemma 22.4 and Derived Categories, Lemma 27.3. The second equality is follows from the definition of Ext groups. The third equality is our choice of $\omega^\bullet_X$. The fourth equality holds because $f^!$ is the right adjoint of Lemma 33.1 for $f$, see Section 19. The final equality holds because $R^i f_* \mathcal{F}$ is zero for $i > d$ (Cohomology of Schemes, Lemma 20.9) and $H^j(\omega_A^\bullet)$ is zero for $j < -\dim(A)$. □

23. Cohen-Macaulay schemes

This section is the continuation of Dualizing Complexes, Section 20. Duality takes a particularly simple form for Cohen-Macaulay schemes.

\textbf{Lemma 23.1.} Let $X$ be a locally Noetherian scheme with dualizing complex $\omega_X^\bullet$.
\begin{enumerate}
  \item $X$ is Cohen-Macaulay $\Leftrightarrow \omega_X^\bullet$ locally has a unique nonzero cohomology sheaf,
  \item $\mathcal{O}_{X,x}$ is Cohen-Macaulay $\Leftrightarrow \omega_{X,x}^\bullet$ has a unique nonzero cohomology,
  \item $U = \{x \in X \mid \mathcal{O}_{X,x}$ is Cohen-Macaulay$\}$ is open and Cohen-Macaulay.
\end{enumerate}
If $X$ is connected and Cohen-Macaulay, then there is an integer $n$ and a coherent Cohen-Macaulay $\mathcal{O}_X$-module $\omega_X$ such that $\omega_X^\bullet = \omega_X[-n]$.
Proof. By definition and Dualizing Complexes, Lemma 15.6 for every \( x \in X \) the complex \( \omega^*_X \) is a dualizing complex over \( \mathcal{O}_{X,x} \). By Dualizing Complexes, Lemma 20.2 we see that (2) holds.

To see (3) assume that \( \mathcal{O}_{X,x} \) is Cohen-Macaulay. Let \( n_x \) be the unique integer such that \( H^{n_x}(\omega^*_X)|_V \) is nonzero. For an affine neighbourhood \( V \subset X \) of \( x \) we have \( \omega^*_X|_V \) is in \( D^b_{\text{Coh}}(\mathcal{O}_V) \) hence there are finitely many nonzero coherent modules \( H^i(\omega^*_X)|_V \). Thus after shrinking \( V \) we may assume only \( H^{n_x} \) is nonzero, see Modules, Lemma 9.5. In this way we see that \( \mathcal{O}_{X,v} \) is Cohen-Macaulay for every \( v \in V \). This proves that \( U \) is open as well as a Cohen-Macaulay scheme.

Proof of (1). The implication \( \Leftarrow \) follows from (2). The implication \( \Rightarrow \) follows from the discussion in the previous paragraph, where we showed that if \( \mathcal{O}_{X,x} \) is Cohen-Macaulay, then in a neighbourhood of \( x \) the complex \( \omega^*_X \) has only one nonzero cohomology sheaf.

Assume \( X \) is connected and Cohen-Macaulay. The above shows that the map \( x \mapsto n_x \) is locally constant. Since \( X \) is connected it is constant, say equal to \( n \). Setting \( \omega_X = H^n(\omega^*_X) \) we see that the lemma holds because \( \omega_X \) is Cohen-Macaulay by Dualizing Complexes, Lemma 20.2 (and Cohomology of Schemes, Definition 11.4).

\( 0AWU \) Lemma 23.2. Let \( X \) be a locally Noetherian scheme. If there exists a coherent sheaf \( \omega_X \) such that \( \omega_X[0] \) is a dualizing complex on \( X \), then \( X \) is a Cohen-Macaulay scheme.

**Proof.** This follows immediately from Dualizing Complexes, Lemma 20.3 and our definitions.

\( 0C0Z \) Lemma 23.3. In Situation 16.1 let \( f : X \to Y \) be a morphism of \( \text{FTS}_S \). Let \( x \in X \). If \( f \) is flat, then the following are equivalent

1. \( f \) is Cohen-Macaulay at \( x \),
2. \( f^*\mathcal{O}_Y \) has a unique nonzero cohomology sheaf in a neighbourhood of \( x \).

**Proof.** One direction of the lemma follows from Lemma 21.7. To prove the converse, we may assume \( f^*\mathcal{O}_Y \) has a unique nonzero cohomology sheaf. Let \( y = f(x) \). Let \( \xi_1, \ldots, \xi_n \in X_y \) be the generic points of the fibre \( X_y \) specializing to \( x \). Let \( d_1, \ldots, d_n \) be the dimensions of the corresponding irreducible components of \( X_y \)

The morphism \( f : X \to Y \) is Cohen-Macaulay at \( \eta_i \) by More on Morphisms, Lemma 20.7. Hence by Lemma 21.7 we see that \( d_1 = \ldots = d_n \). If \( d \) denotes the common value, then \( d = \dim_x(X_y) \). After shrinking \( X \) we may assume all fibres have dimension at most \( d \) (Morphisms, Lemma 27.4). Then the only nonzero cohomology sheaf \( \omega = H^{-d}(f^*\mathcal{O}_Y) \) is flat over \( Y \) by Lemma 21.4. Hence, if \( h : X_y \to X \) denotes the canonical morphism, then \( Lh^*(f^*\mathcal{O}_Y) = Lh^*(\omega[d]) = (h^*\omega)[d] \) by Derived Categories of Schemes, Lemma 21.6. Thus \( h^*\omega[d] \) is the dualizing complex of \( X_y \) by Lemma 18.3. Hence \( X_y \) is Cohen-Macaulay by Lemma 23.1. This proves \( f \) is Cohen-Macaulay at \( x \) as desired.

\( 0C10 \) Remark 23.4. In Situation 16.1 let \( f : X \to Y \) be a morphism of \( \text{FTS}_S \). Assume \( f \) is a Cohen-Macaulay morphism of relative dimension \( d \). Let \( \omega_{X/Y} = H^{-d}(f^*\mathcal{O}_Y) \) be the unique nonzero cohomology sheaf of \( f^*\mathcal{O}_Y \), see Lemma 21.7. Then there is
a canonical isomorphism

\[ f^! K = Lf^* K \otimes_{\mathcal{O}_X} \omega_{X/Y}[d] \]

for \( K \in D_{\text{QCoh}}^+(\mathcal{O}_Y) \), see Lemma 17.8. In particular, if \( S \) has a dualizing complex \( \omega_S, \omega_Y = (Y \to S)^! \omega_S \), and \( \omega_X = (X \to S)^! \omega_S \) then we have

\[ \omega_X = Lf^* \omega_Y \otimes_{\mathcal{O}_X} \omega_{X/Y}[d] \]

Thus if further \( X \) and \( Y \) are connected and Cohen-Macaulay and if \( \omega_Y \) and \( \omega_X \) denote the unique nonzero cohomology sheaves of \( \omega_Y^* \) and \( \omega_X^* \), then we have

\[ \omega_X = f^* \omega_Y \otimes_{\mathcal{O}_X} \omega_{X/Y}. \]

Similar results hold for \( X \) and \( Y \) arbitrary finite type schemes over \( S \) (i.e., not necessarily separated over \( S \)) with dualizing complexes normalized with respect to \( \omega_S \) as in Section 20.

24. Gorenstein schemes

Definition 24.1. Let \( X \) be a scheme. We say \( X \) is Gorenstein if \( X \) is locally Noetherian and \( \mathcal{O}_{X,x} \) is Gorenstein for all \( x \in X \).

This definition makes sense because a Noetherian ring is said to be Gorenstein if and only if all of its local rings are Gorenstein, see Dualizing Complexes, Definition 21.1.


Proof. Looking affine locally this follows from the corresponding result in algebra, namely Dualizing Complexes, Lemma 21.2.

Lemma 24.3. A regular scheme is Gorenstein.

Proof. Looking affine locally this follows from the corresponding result in algebra, namely Dualizing Complexes, Lemma 21.3.

Lemma 24.4. Let \( X \) be a locally Noetherian scheme.

(1) If \( X \) has a dualizing complex \( \omega_X^* \), then
   (a) \( X \) is Gorenstein \iff \( \omega_X^* \) is an invertible object of \( D(\mathcal{O}_X) \),
   (b) \( \mathcal{O}_{X,x} \) is Gorenstein \iff \( \omega_{X,x}^* \) is an invertible object of \( D(\mathcal{O}_{X,x}) \),
   (c) \( U = \{ x \in X \mid \mathcal{O}_{X,x} \text{ is Gorenstein} \} \) is an open Gorenstein subscheme.

(2) If \( X \) is Gorenstein, then \( X \) has a dualizing complex if and only if \( \mathcal{O}_X[0] \) is a dualizing complex.

Proof. Looking affine locally this follows from the corresponding result in algebra, namely Dualizing Complexes, Lemma 21.4.

Lemma 24.5. If \( f : Y \to X \) is a local complete intersection morphism with \( X \) a Gorenstein scheme, then \( Y \) is Gorenstein.

Proof. By More on Morphisms, Lemma 54.5 it suffices to prove the corresponding statement about ring maps. This is Dualizing Complexes, Lemma 21.7.

Lemma 24.6. The property \( \mathcal{P}(S) = \text{“} S \text{ is Gorenstein” } \) is local in the syntomic topology.
Proof. Let \( \{ S_i \to S \} \) be a syntomic covering. The scheme \( S \) is locally Noetherian if and only if each \( S_i \) is Noetherian, see Descent, Lemma 13.1. Thus we may now assume \( S \) and \( S_i \) are locally Noetherian. If \( S \) is Gorenstein, then each \( S_i \) is Gorenstein by Lemma 24.5. Conversely, if each \( S_i \) is Gorenstein, then for each point \( s \in S \) we can pick \( i \) and \( t \in S_i \) mapping to \( s \). Then \( \mathcal{O}_{S,s} \to \mathcal{O}_{S_i,t} \) is a flat local ring homomorphism with \( \mathcal{O}_{S_i,t} \) Gorenstein. Hence \( \mathcal{O}_{S,s} \) is Gorenstein by Dualizing Complexes, Lemma 21.8. □

25. Gorenstein morphisms

This section is one in a series. The corresponding sections for normal morphisms, regular morphisms, and Cohen-Macaulay morphisms can be found in More on Morphisms, Sections 18, 19, and 20.

The following lemma says that it does not make sense to define geometrically Gorenstein schemes, since these would be the same as Gorenstein schemes.

Lemma 25.1. Let \( X \) be a locally Noetherian scheme over the field \( k \). Let \( k \subset k' \) be a finitely generated field extension. Let \( x \in X \) be a point, and let \( x' \in X_{k'} \) be a point lying over \( x \). Then we have

\[
\mathcal{O}_{X,x} \text{ is Gorenstein } \iff \mathcal{O}_{X_{k'},x'} \text{ is Gorenstein}
\]

If \( X \) is locally of finite type over \( k \), the same holds for any field extension \( k \subset k' \).

Proof. In both cases the ring map \( \mathcal{O}_{X,x} \to \mathcal{O}_{X_{k'},x'} \) is a faithfully flat local morphism of Noetherian local rings. Thus if \( \mathcal{O}_{X_{k'},x'} \) is Gorenstein, then so is \( \mathcal{O}_{X,x} \) by Dualizing Complexes, Lemma 21.8. To go up, we use Dualizing Complexes, Lemma 21.8 as well. Thus we have to show that

\[
\mathcal{O}_{X_{k'},x'}/m_x \mathcal{O}_{X_{k'},x'} = \kappa(x) \otimes_k k'
\]

is Gorenstein. Note that in the first case \( k \to k' \) is finitely generated and in the second case \( k \to \kappa(x) \) is finitely generated. Hence this follows as property (A) holds for Gorenstein, see Dualizing Complexes, Lemma 23.1. □

The lemma above guarantees that the following is the correct definition of Gorenstein morphisms.

Definition 25.2. Let \( f : X \to Y \) be a morphism of schemes. Assume that all the fibres \( X_y \) are locally Noetherian schemes.

1. Let \( x \in X \), and \( y = f(x) \). We say that \( f \) is Gorenstein at \( x \) if \( f \) is flat at \( x \), and the local ring of the scheme \( X_y \) at \( x \) is Gorenstein.

2. We say \( f \) is a Gorenstein morphism if \( f \) is Gorenstein at every point of \( X \).

Here is a translation.

Lemma 25.3. Let \( f : X \to Y \) be a morphism of schemes. Assume all fibres of \( f \) are locally Noetherian. The following are equivalent

1. \( f \) is Gorenstein, and
2. \( f \) is flat and its fibres are Gorenstein schemes.

Proof. This follows directly from the definitions. □


Proof. Follows from Lemma 24.2 and the definitions. □
Lemma 25.5. A syntomic morphism is Gorenstein. Equivalently a flat local complete intersection morphism is Gorenstein.

Proof. Recall that a syntomic morphism is flat and its fibres are local complete intersections over fields, see Morphisms, Lemma 29.11. Since a local complete intersection over a field is a Gorenstein scheme by Lemma 24.5 we conclude. The properties “syntomic” and “flat and local complete intersection morphism” are equivalent by More on Morphisms, Lemma 54.8.

Lemma 25.6. Let \( f : X \to Y \) and \( g : Y \to Z \) be morphisms. Assume that the fibres \( X_y, Y_z \) and \( X_z \) of \( f, g \), and \( g \circ f \) are locally Noetherian.

1. If \( f \) is Gorenstein at \( x \) and \( g \) is Gorenstein at \( f(x) \), then \( g \circ f \) is Gorenstein at \( x \).
2. If \( f \) and \( g \) are Gorenstein, then \( g \circ f \) is Gorenstein.
3. If \( g \circ f \) is Gorenstein at \( x \) and \( f \) is flat at \( x \) and \( g \) is Gorenstein at \( f(x) \).
4. If \( f \circ g \) is Gorenstein and \( f \) is flat, then \( f \) is Gorenstein at every point in the image of \( f \).

Proof. After translating into algebra this follows from Dualizing Complexes, Lemma 21.8.

Lemma 25.7. Let \( f : X \to Y \) be a flat morphism of locally Noetherian schemes. If \( X \) is Gorenstein, then \( f \) is Gorenstein and \( \mathcal{O}_{Y,f(x)} \) is Gorenstein for all \( x \in X \).

Proof. After translating into algebra this follows from Dualizing Complexes, Lemma 21.8.

Lemma 25.8. Let \( f : X \to Y \) be a morphism of schemes. Assume that all the fibres \( X_y \) are locally Noetherian schemes. Let \( Y' \to Y \) be locally of finite type. Let \( f' : X' \to Y' \) be the base change of \( f \). Let \( x' \in X' \) be a point with image \( x \in X \).

1. If \( f \) is Gorenstein at \( x \), then \( f' : X' \to Y' \) is Gorenstein at \( x' \).
2. If \( f \) is flat and \( x \) and \( f' \) is Gorenstein at \( x' \), then \( f \) is Gorenstein at \( x \).
3. If \( Y' \to Y \) is flat at \( f'(x') \) and \( f' \) is Gorenstein at \( x' \), then \( f \) is Gorenstein at \( x \).

Proof. Note that the assumption on \( Y' \to Y \) implies that for \( y' \in Y' \) mapping to \( y \in Y \) the field extension \( \kappa(y) \subset \kappa(y') \) is finitely generated. Hence also all the fibres \( X'_{y'} = (X_y)_{\kappa(y')} \) are locally Noetherian, see Varieties, Lemma 11.1. Thus the lemma makes sense. Set \( y' = f'(x') \) and \( y = f(x) \). Hence we get the following commutative diagram of local rings

\[
\begin{array}{ccc}
\mathcal{O}_{X',x'} & \leftarrow & \mathcal{O}_{X,x} \\
\uparrow & & \uparrow \\
\mathcal{O}_{Y',y'} & \leftarrow & \mathcal{O}_{Y,y}
\end{array}
\]

where the upper left corner is a localization of the tensor product of the upper right and lower left corners over the lower right corner.

Assume \( f \) is Gorenstein at \( x \). The flatness of \( \mathcal{O}_{Y,Y} \to \mathcal{O}_{X,x} \) implies the flatness of \( \mathcal{O}_{Y',y'} \to \mathcal{O}_{X',x'} \), see Algebra, Lemma 99.1. The fact that \( \mathcal{O}_{X,x}/\mathfrak{m}_y\mathcal{O}_{X,x} \) is
Gorenstein implies that $\mathcal{O}_{X',x'}/m_y\mathcal{O}_{X',x'}$ is Gorenstein, see Lemma 25.1. Hence we see that $f'$ is Gorenstein at $x'$.

Assume $f$ is flat at $x$ and $f'$ is Gorenstein at $x'$. The fact that $\mathcal{O}_{X',x'}/m_y\mathcal{O}_{X',x'}$ is Gorenstein implies that $\mathcal{O}_{X,x}/m_y\mathcal{O}_{X,x}$ is Gorenstein, see Lemma 25.1. Hence we see that $f$ is Gorenstein at $x$.

Assume $Y' \to Y$ is flat at $y'$ and $f'$ is Gorenstein at $x'$. The flatness of $\mathcal{O}_{Y',y'} \to \mathcal{O}_{X',x'}$ and $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ implies the flatness of $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$, see Algebra, Lemma 99.1. The fact that $\mathcal{O}_{X',x'}/m_y\mathcal{O}_{X',x'}$ is Gorenstein implies that $\mathcal{O}_{X,x}/m_y\mathcal{O}_{X,x}$ is Gorenstein, see Lemma 25.1. Hence we see that $f$ is Gorenstein at $x$.

**Lemma 25.9.** Let $f : X \to Y$ be a morphism of schemes which is flat and locally of finite type. Then formation of the set $\{x \in X \mid f \text{ is Gorenstein at } x\}$ commutes with arbitrary base change.

**Proof.** The assumption implies any fibre of $f$ is locally of finite type over a field and hence locally Noetherian and the same is true for any base change. Thus the statement makes sense. Looking at fibres we reduce to the following problem: let $X$ be a scheme locally of finite type over a field $k$, let $K/k$ be a field extension, and let $x_K \in X_K$ be a point with image $x \in X$. Problem: show that $\mathcal{O}_{X_K,x_K}$ is Gorenstein if and only if $\mathcal{O}_{X,x}$ is Gorenstein.

The problem can be solved using a bit of algebra as follows. Choose an affine open $\text{Spec}(A) \subset X$ containing $x$. Say $x$ corresponds to $p \subset A$. With $A_K = A \otimes_k K$ we see that $\text{Spec}(A_K) \subset X_K$ contains $x_K$. Say $x_K$ corresponds to $p_K \subset A_K$. Let $\omega_A$ be a dualizing complex for $A$. By Dualizing Complexes, Lemma 25.3 $\omega_A \otimes_A A_K$ is a dualizing complex for $A_K$. Now we are done because $A_p \to (A_K)_{p_K}$ is a flat local homomorphism of Noetherian rings and hence $(\omega_A)_p$ is an invertible object of $D(A_p)$ if and only if $(\omega_A)_p \otimes_{A_p} (A_K)_{p_K}$ is an invertible object of $D((A_K)_{p_K})$.

Some details omitted; hint: look at cohomology modules.

**Lemma 25.10.** In Situation 16.1 let $f : X \to Y$ be a morphism of FTS. Let $x \in X$. If $f$ is flat, then the following are equivalent:

1. $f$ is Gorenstein at $x$,
2. $f^!\mathcal{O}_X$ is isomorphic to an invertible object in a neighbourhood of $x$.

In particular, the set of points where $f$ is Gorenstein is open in $X$.

**Proof.** Set $\omega^* = f^!\mathcal{O}_Y$. By Lemma 18.3 this is a bounded complex with coherent cohomology sheaves whose derived restriction $Lh^*\omega^*$ to the fibre $X_y$ is a dualizing complex on $X_y$. Denote $i : x \to X_y$ the inclusion of a point. Then the following are equivalent:

1. $f$ is Gorenstein at $x$,
2. $\mathcal{O}_{X_y,x}$ is Gorenstein,
3. $Lh^*\omega^*$ is invertible in a neighbourhood of $x$,
4. $L(h \circ i)^*\omega^*$ has exactly one nonzero cohomology of dimension 1 over $\kappa(x)$,
5. $L(h \circ i)^*\omega^*$ has exactly one nonzero cohomology of dimension 1 over $\kappa(x)$,
6. $\omega^*$ is invertible in a neighbourhood of $x$.

The equivalence of (1) and (2) is by definition (as $f$ is flat). The equivalence of (2) and (3) follows from Lemma 24.4. The equivalence of (3) and (4) follows from More on Algebra, Lemma 71.5. The equivalence of (4) and (5) holds because
The property \( P(f) = \text{"the fibres of } f \text{ are locally Noetherian and } f \text{ is Gorenstein"} \) is local in the fppf topology on the target and local in the syntomic topology on the source.  

\footnote{If }
Proof. We have \( P(f) = P_1(f) \land P_2(f) \) where \( P_1(f) = \text{“} f \text{ is flat} \text{”} \), and \( P_2(f) = \text{“} the fibres of } f \text{ are locally Noetherian and Gorenstein} \text{”} \). We know that \( P_1 \) is local in the fppf topology on the source and the target, see Descent, Lemmas 20.15 and 24.1. Thus we have to deal with \( P_2 \).

Let \( f : X \to Y \) be a morphism of schemes. Let \( \{ \varphi_i : Y_i \to Y \}_{i \in I} \) be an fppf covering of \( Y \). Denote \( f_i : X_i \to Y_i \) the base change of \( f \) by \( \varphi_i \). Let \( i \in I \) and let \( y_i \in Y_i \) be a point. Set \( y = \varphi_i(y_i) \). Note that \( X_{i,y_i} = \text{Spec}(\kappa(y_i)) \times_{\text{Spec}(\kappa(y))} X_y \).

and that \( \kappa(y) \subset \kappa(y_i) \) is a finitely generated field extension. Hence if \( X_y \) is locally Noetherian, then \( X_{i,y_i} \) is locally Noetherian, see Varieties, Lemma 11.1. And if in addition \( X_y \) is Gorenstein, then \( X_{i,y_i} \) is Gorenstein, see Lemma 25.1. Thus \( P_2 \) is fppf local on the target.

Let \( \{ X_i \to X \} \) be a syntomic covering of \( X \). Let \( y \in Y \). In this case \( \{ X_{i,y} \to X_y \} \) is a syntomic covering of the fibre. Hence the locality of \( P_2 \) for the syntomic topology on the source follows from Lemma 24.6. \( \square \)

26. More on dualizing complexes

0E4M Some lemmas which don’t fit anywhere else very well.

0E4N Lemma 26.1. Let \( f : X \to Y \) be a morphism of locally Noetherian schemes. Assume

(1) \( f \) is syntomic and surjective, or
(2) \( f \) is a surjective flat local complete intersection morphism, or
(3) \( f \) is a surjective Gorenstein morphism of finite type.

Then \( K \in D_{\text{QCoh}}(\mathcal{O}_Y) \) is a dualizing complex on \( Y \) if and only if \( Lf^*K \) is a dualizing complex on \( X \).

Proof. Taking affine opens and using Derived Categories of Schemes, Lemma 3.5 this translates into Dualizing Complexes, Lemma 26.2. \( \square \)

27. Relative dualizing complexes

0E2S For a proper, flat morphism of finite presentation we have a rigid relative dualizing complex, see Remark 12.5 and Lemma 12.7. For a separated and finite type morphism \( f : X \to Y \) of Noetherian schemes, we can consider \( f^!\mathcal{O}_Y \). In this section we define relative dualizing complexes for morphisms which are flat and locally of finite presentation (but not necessarily quasi-separated or quasi-compact) between schemes (not necessarily locally Noetherian). We show such complexes exist, are unique up to unique isomorphism, and agree with the cases mentioned above. Before reading this section, please read Dualizing Complexes, Section 27.

0E2T Definition 27.1. Let \( X \to S \) be a morphism of schemes which is flat and locally of finite presentation. Let \( W \subset X \times_S X \) be any open such that the diagonal \( \Delta_{X/S} : X \times_S X \) factors through a closed immersion \( \Delta : X \to W \). A relative dualizing complex is a pair \((K, \xi)\) consisting of an object \( K \in D(\mathcal{O}_X) \) and a map

\[ \xi : \Delta_*\mathcal{O}_X \to \text{Lpr}_1^*K|_W \]

in \( D(\mathcal{O}_W) \) such that

(1) \( K \) is \( S \)-perfect (Derived Categories of Schemes, Definition 31.1), and
(2) \( \xi \) defines an isomorphism of \( \Delta_* \mathcal{O}_X \) with \( R \mathcal{H}om_{\mathcal{O}_W}(\Delta_* \mathcal{O}_X, \text{Lpr}_1 K|_W) \).

By Lemma 9.3 condition (2) is equivalent to the existence of an isomorphism

\[
\mathcal{O}_X \to R \mathcal{H}om(\mathcal{O}_X, \text{Lpr}_1 K|_W)
\]

in \( D(\mathcal{O}_X) \) whose pushforward via \( \Delta \) is equal to \( \xi \). Since \( R \mathcal{H}om(\mathcal{O}_X, \text{Lpr}_1 K|_W) \) is independent of the choice of the open \( W \), so is the category of pairs \((K, \xi)\). If \( X \to S \) is separated, then we can choose \( W = X \times_S X \). We will reduce many of the arguments to the case of rings using the following lemma.

**Lemma 27.2.** Let \( X \to S \) be a morphism of schemes which is flat and locally of finite presentation. Let \((K, \xi)\) be a relative dualizing complex. Then for any commutative diagram

\[
\begin{array}{ccc}
\text{Spec}(A) & \longrightarrow & X \\
\downarrow & & \downarrow \\
\text{Spec}(R) & \longrightarrow & S
\end{array}
\]

whose horizontal arrows are open immersions, the restriction of \( K \) to \( \text{Spec}(A) \) corresponds via Derived Categories of Schemes, Lemma 3.5 to a relative dualizing complex for \( R \to A \) in the sense of Dualizing Complexes, Definition 27.1.

**Proof.** Since formation of \( R \mathcal{H}om \) commutes with restrictions to opens we may as well assume \( X = \text{Spec}(A) \) and \( S = \text{Spec}(R) \). Observe that relatively perfect objects of \( D(\mathcal{O}_X) \) are pseudo-coherent and hence are in \( D_{Qcoh}(\mathcal{O}_X) \) (Derived Categories of Schemes, Lemma 9.1). Thus the statement makes sense. Observe that taking \( \Delta_* \), \( \text{Lpr}_1 \), and \( R \mathcal{H}om \) is compatible with what happens on the algebraic side by Derived Categories of Schemes, Lemmas 3.7, 3.8, 9.8. For the last one we observe that \( \text{Lpr}_1 K \) is \( S \)-perfect (hence bounded below) and that \( \Delta_* \mathcal{O}_X \) is a pseudo-coherent object of \( D(\mathcal{O}_W) \); translated into algebra this means that \( A \) is pseudo-coherent as an \( A \otimes_R A \)-module which follows from More on Algebra, Lemma 75.8 applied to \( R \to A \otimes_R A \to A \). Thus we recover exactly the conditions in Dualizing Complexes, Definition 27.1.

**Lemma 27.3.** Let \( X \to S \) be a morphism of schemes which is flat and locally of finite presentation. Let \((K, \xi)\) be a relative dualizing complex. Then \( \mathcal{O}_X \to R \mathcal{H}om_{\mathcal{O}_X}(K, K) \) is an isomorphism.

**Proof.** Looking affine locally this reduces using Lemma 27.2 to the algebraic case which is Dualizing Complexes, Lemma 27.9.

**Lemma 27.4.** Let \( X \to S \) be a morphism of schemes which is flat and locally of finite presentation. If \((K, \xi)\) and \((L, \eta)\) are two relative dualizing complexes on \( X/S \), then there is a unique isomorphism \( K \to L \) sending \( \xi \) to \( \eta \).

**Proof.** Let \( U \subset X \) be an affine open mapping into an affine open of \( S \). Then there is an isomorphism \( K|_U \to L|_U \) by Lemma 27.2 and Dualizing Complexes, Lemma 27.2. The reader can reuse the argument of that lemma in the schemes case to obtain a proof in this case. We will instead use a gluing argument.

Suppose we have an isomorphism \( \alpha : K \to L \). Then \( \alpha(\xi) = \eta \) for some invertible section \( \eta \in H^0(W, \Delta_* \mathcal{O}_X) = H^0(X, \mathcal{O}_X) \). (Because both \( \eta \) and \( \alpha(\xi) \) are generators of an invertible \( \Delta_* \mathcal{O}_X \)-module by assumption.) Hence after replacing \( \alpha \) by \( u^{-1} \alpha \)
we see that $\alpha(\xi) = \eta$. Since the automorphism group of $K$ is $H^0(X, O_X^*)$ by Lemma \ref{lem:automorphisms}, there is at most one such $\alpha$.

Let $\mathcal{B}$ be the collection of affine opens of $X$ which map into an affine open of $S$. For each $U \in \mathcal{B}$ we have a unique isomorphism $\alpha_U : K|_U \to L|_U$ mapping $\xi$ to $\eta$ by the discussion in the previous two paragraphs. Observe that $\text{Ext}^i(K|_U, K|_U) = 0$ for $i < 0$ and any open $U$ of $X$ by Lemma \ref{lem:isomorphisms} By Cohomology, Lemma \ref{lem:cohomology} applied to $id : X \to X$ we get a unique morphism $\alpha : K \to L$ agreeing with $\alpha_U$ for all $U \in \mathcal{B}$. Then $\alpha$ sends $\xi$ to $\eta$ as this is true locally. \hfill $\square$

**Lemma 27.5.** Let $X \to S$ be a morphism of schemes which is flat and locally of finite presentation. There exists a relative dualizing complex $(K, \xi)$.

**Proof.** Let $\mathcal{B}$ be the collection of affine opens of $X$ which map into an affine open of $S$. For each $U$ we have a relative dualizing complex $(K_U, \xi_U)$ for $U$ over $S$. Namely, choose an affine open $V \subset S$ such that $U \to X \to S$ factors through $V$. Write $U = \text{Spec}(A)$ and $V = \text{Spec}(R)$. By Dualizing Complexes, Lemma \ref{lem:dualizing complexes} there exists a relative dualizing complex $K_A \in D(A)$ for $R \to A$. Arguing backwards through the proof of Lemma \ref{lem:existence}, this determines an $V$-perfect object $K_U \in D(O_U)$ and a map

$$\xi : \Delta_* O_U \to \text{Lpr}_U^* K_U$$

in $D(O_{U \times V})$. Since being $V$-perfect is the same as being $S$-perfect and since $U \times_V U = U \times_S U$ we find that $(K_U, \xi_U)$ is as desired.

If $U' \subset U \subset X$ with $U', U \in \mathcal{B}$, then we have a unique isomorphism $\rho_U^U : K_U|_{U'} \to K_{U'}$ in $D(O_{U'})$ sending $\xi_{U'}|_{U' \times_S U'}$ to $\xi_{U'}$ by Lemma \ref{lem:dualizing complexes} (note that trivially the restriction of a relative dualizing complex to an open is a relative dualizing complex). The uniqueness guarantees that $\rho_{U''}^U = \rho_{U''}^U \circ \rho_{U'}^U|_{U''}$ for $U'' \subset U' \subset U$ in $\mathcal{B}$. Observe that $\text{Ext}^i(K_U, K_U) = 0$ for $i < 0$ for $U \in \mathcal{B}$ by Lemma \ref{lem:isomorphisms} applied to $U/S$ and $K_U$. Thus the BBD glueing lemma (Cohomology, Theorem \ref{thm:BD-glue}) tells us there is a unique solution, namely, an object $K \in D(O_X)$ and isomorphisms $\rho_U : K|_U \to K_U$ such that we have $\rho_{U''}^U \circ \rho_{U'}^U|_{U''} = \rho_{U'}^U$ for all $U' \subset U, U, U' \in \mathcal{B}$.

To finish the proof we have to construct the map

$$\xi : \Delta_* O_X \longrightarrow \text{Lpr}_U^* K|_W$$

in $D(O_W)$ inducing an isomorphism from $\Delta_* O_X$ to $R \text{Hom}_{O_W}(\Delta_* O_X, \text{Lpr}_U^* K|_W)$. Since we may change $W$, we choose $W = \bigcup_{U \in \mathcal{B}} U \times_S U$. We can use $\rho_U$ to get isomorphisms

$$R \text{Hom}_{O_W}(\Delta_* O_X, \text{Lpr}_U^* K|_W)|_{U \times_S U} \overset{\rho_U}{\longrightarrow} R \text{Hom}_{O_{U \times_S U}}(\Delta_* O_U, \text{Lpr}_U^* K_U)$$

As $W$ is covered by the opens $U \times_S U$ we conclude that the cohomology sheaves of $R \text{Hom}_{O_W}(\Delta_* O_X, \text{Lpr}_U^* K|_W)$ are zero except in degree 0. Moreover, we obtain isomorphisms

$$H^0(U \times_S U, R \text{Hom}_{O_W}(\Delta_* O_X, \text{Lpr}_U^* K|_W)) \overset{\rho_U}{\longrightarrow} H^0(U \times_S U, R \text{Hom}_{O_{U \times_S U}}(\Delta_* O_U, \text{Lpr}_U^* K_U))$$

Let $\tau_U$ in the LHS be an element mapping to $\xi_U$ under this map. The compatibilities between $\rho_{U''}^U, \xi_{U''}, \rho_{U'}^U$, and $\rho_{U'}^U$ for $U' \subset U \subset X$ open $U', U \in \mathcal{B}$ imply that $\tau_{U'}|_{U' \times_S U'} = \tau_{U''}$. Thus we get a global section $\tau$ of the 0th cohomology sheaf $H^0(R \text{Hom}_{O_W}(\Delta_* O_X, \text{Lpr}_U^* K|_W))$. Since the other cohomology sheaves
Consider a cartesian square

\[
\begin{array}{ccc}
X' & \xrightarrow{g'} & X \\
\downarrow f' & & \downarrow f \\
S' & \xrightarrow{g} & S
\end{array}
\]

of schemes. Assume \( X \to S \) is flat and locally of finite presentation. Let \((K, \xi)\) be a relative dualizing complex for \( f \). Set \( K' = L(g')^*K \). Let \( \xi' \) be the derived base change of \( \xi \) (see proof). Then \((K', \xi')\) is a relative dualizing complex for \( f' \).

**Proof.** Consider the cartesian square

\[
\begin{array}{ccc}
X' & \xrightarrow{g' \times g'} & X \\
\downarrow \Delta_{X/S'} & & \downarrow \Delta_{X/S} \\
X' \times S' & \xrightarrow{\Delta_{X/S} \times \Delta_{X/S'}} & X \times_S X
\end{array}
\]

Choose \( W \subset X \times_S X \) open such that \( \Delta_{X/S} \) factors through a closed immersion \( \Delta : X \to W \). Choose \( W' \subset X' \times_S X' \) open such that \( \Delta_{X'/S'} \) factors through a closed immersion \( \Delta' : X \to W' \) and such that \((g' \times g')(W') \subset W\). Let us still denote \( g' \times g' : W' \to W \) the induced morphism. We have

\[
L((g' \times g')^*\Delta_{X'} \Omega_{X'}) = \Delta_{X/S}' \Omega_{X'} \quad \text{and} \quad L((g' \times g')^*\text{Lpr}_{1}^*K|_{W'}) = \text{Lpr}_{1}^*K'|_{W'}
\]

The first equality holds because \( X \) and \( X' \times_{S'} X' \) are tor independent over \( X \times_S X \) (see for example More on Morphisms, Lemma \[61.1\]). The second holds by transitivity of derived pullback (Cohomology, Lemma \[28.2\]). Thus \( \xi' = L((g' \times g')^*\xi) \) can be viewed as a map

\[
\xi' : \Delta_{X'}/\Omega_{X'} \longrightarrow \text{Lpr}_{1}^*K'|_{W'}
\]

Having said this the proof of the lemma is straightforward. First, \( K' \) is \( S' \)-perfect by Derived Categories of Schemes, Lemma \[31.6\]. To check that \( \xi' \) induces an isomorphism of \( \Delta_{X'/S}' \Omega_{X'/S} \) to \( R\text{Hom}_{\mathcal{O}_{W'}}(\Delta_{X'/S}'\Omega_{X'/S}, \text{Lpr}_{1}^*K'|_{W'}) \) we may work affine locally. By Lemma \[27.7\] we reduce to the corresponding statement in algebra which is proven in Dualizing Complexes, Lemma \[27.4\].

**Lemma 27.7.** Let \( S \) be a quasi-compact and quasi-separated scheme. Let \( f : X \to S \) be a proper, flat morphism of finite presentation. The relative dualizing complex \( \omega_{X/S'}^{\bullet} \) of Remark \[12.5\] together with \([12.7.1]\) is a relative dualizing complex in the sense of Definition \[27.1\].

**Proof.** In Lemma \[12.6\] we proved that \( \omega_{X/S}^{\bullet} \) is \( S \)-perfect. Let \( c \) be the right adjoint of Lemma \[3.1\] for the diagonal \( \Delta : X \to X \times_{S} X \). Then we can apply \( \Delta_{*} \) to \([12.7.1]\) to get an isomorphism

\[
\Delta_{*}\Omega_{X} \to \Delta_{*}(c(\text{Lpr}_{1}^*\omega_{X/S}^{\bullet})) = R\text{Hom}_{\mathcal{O}_{X \times_{S} X}}(\Delta_{*}\Omega_{X}, \text{Lpr}_{1}^*\omega_{X/S}^{\bullet})
\]

The equality holds by Lemmas \[9.7\] and \[9.3\]. This finishes the proof. \( \square \)
0E4P **Remark 27.8.** Let $X \to S$ be a morphism of schemes which is flat, proper, and of finite presentation. By Lemma 27.5 there exists a relative dualizing complex $(\omega_{X/S}^\bullet, \xi)$ in the sense of Definition 27.1. Consider any morphism $g : S' \to S$ where $S'$ is quasi-compact and quasi-separated (for example an affine open of $S$). By Lemma 27.6 we see that $(L(g)'^* \omega_{X/S}^\bullet, L(g)'^* \xi)$ is a relative dualizing complex for the base change $f' : X' \to S'$ in the sense of Definition 27.1. Let $\omega_{X'/S'}^\bullet$ be the relative dualizing complex for $X' \to S'$ in the sense of Remark 12.5. Combining Lemmas 27.7 and 27.4 we see that there is a unique isomorphism

$$\omega_{X'/S'}^\bullet \to L(g)'^* \omega_{X/S}^\bullet$$

compatible with (12.7.1) and $L(g)'^* \xi$. These isomorphisms are compatible with morphisms between quasi-compact and quasi-separated schemes over $S$ and the base change isomorphisms of Lemma 12.4 (if we ever need this compatibility we will carefully state and prove it here).

0E9W **Lemma 27.9.** In Situation 16.1 let $f : X \to Y$ be a morphism of $\text{FTS}_S$. If $f$ is flat, then $f^! O_Y$ is (the first component of) a relative dualizing complex for $X$ over $Y$ in the sense of Definition 27.1.

**Proof.** By Lemma 17.9 we have that $f^! O_Y$ is $Y$-perfect. As $f$ is separated the diagonal $\Delta : X \to X \times_Y X$ is a closed immersion and $\Delta, \Delta^! : R \mathcal{H}om_{O_{X \times_Y X}}(O_X, -)$, see Lemmas 9.7 and 9.3. Hence to finish the proof it suffices to show $\Delta^!(L \mathcal{P}r_1^! f^! O_Y) \cong O_X$ where $\mathcal{P}r_1 : X \times_Y X \to X$ is the first projection. We have

$$O_X = \Delta^! \mathcal{P}r_1^! O_X = \Delta^! \mathcal{P}r_1^! L \mathcal{P}r_2^* O_Y = \Delta^!(L \mathcal{P}r_1^! f^! O_Y)$$

where $\mathcal{P}r_2 : X \times_Y X \to X$ is the second projection and where we have used the base change isomorphism $\mathcal{P}r_1^! \circ L \mathcal{P}r_2^* = L \mathcal{P}r_1^! \circ f^!$ of Lemma 18.1. \(\square\)

0E30 **Lemma 27.10.** Let $f : Y \to X$ and $X \to S$ be morphisms of schemes which are flat and of finite presentation. Let $(K, \xi)$ and $(M, \eta)$ be a relative dualizing complex for $X \to S$ and $Y \to X$. Set $E = M \otimes_B^L f^* K$. Then $(E, \zeta)$ is a relative dualizing complex for $Y \to S$ for a suitable $\zeta$.

**Proof.** Using Lemma 27.2 and the algebraic version of this lemma (Dualizing Complexes, Lemma 27.6) we see that $E$ is affine locally the first component of a relative dualizing complex. In particular we see that $E$ is $S$-perfect since this may be checked affine locally, see Derived Categories of Schemes, Lemma 31.3.

Let us first prove the existence of $\zeta$ in case the morphisms $X \to S$ and $Y \to X$ are separated so that $\Delta_{X/S}, \Delta_{Y/X},$ and $\Delta_{Y/S}$ are closed immersions. Consider the following diagram

$$\begin{align*}
Y & \to Y \\
\downarrow p & \downarrow \delta \\
X & \to X \\
\downarrow \delta & \downarrow \delta \\
X \times_S Y & \to X \times_S Y \\
\downarrow m & \downarrow f \times f \\
X & \to X \times_S X
\end{align*}$$

Let $\Delta_{Y/X}$ be a morphism for which the diagram commutes and $\Delta_{Y/S}$ is a morphism for which the diagram commutes. By the above diagram the morphism $\Delta_{X/S}$ is a morphism for which the diagram commutes. Therefore $\Delta_{X/S}$ is separated.

Let us now prove the existence of $\zeta$ in general. It suffices to do this in the case where $X, Y, S$ are affine. In this case $(E, \zeta) = (E', \zeta')$ is a relative dualizing complex for $Y \to S$ for a suitable $\zeta'$ by the above case. Then by Lemma 27.2 we find a relative dualizing complex $(E'', \zeta'')$ for $Y \to X$ for a suitable $\zeta''$. Combining the above two relative dualizing complexes via Lemmas 27.4 and 27.7 we find a relative dualizing complex $(E', \zeta')$ for $Y \to S$ for a suitable $\zeta'$.
where \( p, q, r \) are the first projections. By Lemma 9.4 we have
\[
R\text{Hom}_{O_Y \times_S Y}(\Delta_{Y/S,*} O_Y, Lp^* E) = R\delta_* \left( R\text{Hom}_{O_Y \times X Y}(\Delta_{Y/X,*} O_Y, R\text{Hom}(O_Y \times X Y, Lp^* E)) \right)
\]
By Lemma 10.3 we have
\[
R\text{Hom}(O_Y \times X Y, Lp^* E) = R\text{Hom}(O_Y \times X Y, L(f \times f)^* Lr^* K) \otimes_{O_Y \times S Y} Lq^* M
\]
By Lemma 10.2 we have
\[
R\text{Hom}(O_Y \times X Y, L(f \times f)^* Lr^* K) = Lm^* R\text{Hom}(O_X, Lr^* K)
\]
The last expression is isomorphic (via \( \eta \)) to \( Lm^* O_X = O_Y \times X Y \). Hence the expression preceding is isomorphic to \( Lq^* M \). Hence
\[
R\text{Hom}_{O_Y \times S Y}(\Delta_{Y/S,*} O_Y, Lp^* E) = R\delta_* \left( R\text{Hom}_{O_Y \times X Y}(\Delta_{Y/X,*} O_Y, R\text{Hom}(O_Y \times X Y, Lq^* M)) \right)
\]
The material inside the parentheses is isomorphic to \( \Delta_{Y/X,*} O_X \) via \( \eta \). This finishes the proof in the separated case.

In the general case we choose an open \( W \subset X \times S X \) such that \( \Delta_{X/S} \) factors through a closed immersion \( \Delta : X \rightarrow W \) and we choose an open \( V \subset Y \times X Y \) such that \( \Delta_{Y/X} \) factors through a closed immersion \( \Delta' : Y \rightarrow V \). Finally, choose an open \( W' \subset Y \times S Y \) whose intersection with \( Y \times X Y \) gives \( V \) and which maps into \( W \).

Then we consider the diagram
\[
\begin{array}{ccc}
Y & \xrightarrow{q} & Y \\
\downarrow{p} & & \downarrow{f} \\
V & \xrightarrow{\delta} & W' \\
\downarrow{m} & & \downarrow{f \times f} \\
X & \xrightarrow{\Delta} & W
\end{array}
\]
and we use exactly the same argument as before. \( \square \)

28. The fundamental class of an lci morphism

0E9X In this section we will use the computations made in Section 15. Thus our result will suffer from the same kind of non-uniqueness as we have in that section.

0E9Y Lemma 28.1. Let \( X \) be a locally ringed space. Let
\[
\varepsilon_1 \xrightarrow{\alpha} \varepsilon_0 \rightarrow \mathcal{F} \rightarrow 0
\]
be a short exact sequence of \( O_X \)-modules. Assume \( \varepsilon_1 \) and \( \varepsilon_0 \) are locally free of ranks \( r_1, r_0 \). Then there is a canonical map
\[
\wedge^{r_0 - r_1} \mathcal{F} \rightarrow \wedge^{r_1}(\varepsilon_1^\vee) \otimes \wedge^{r_0} \varepsilon_0
\]
which is an isomorphism on the stalk at \( x \in X \) if and only if \( \mathcal{F} \) is locally free of rank \( r_0 - r_1 \) in an open neighbourhood of \( x \).

Proof. If \( r_1 > r_0 \) then \( \wedge^{r_0 - r_1} \mathcal{F} = 0 \) by convention and the unique map cannot be an isomorphism. Thus we may assume \( r = r_0 - r_1 \geq 0 \). Define the map by the formula
\[
s_1 \wedge \ldots \wedge s_r \mapsto t_1^\vee \wedge \ldots \wedge t_{r_1}^\vee \otimes \alpha(t_1) \wedge \ldots \wedge \alpha(t_{r_1}) \wedge s_1 \wedge \ldots \wedge s_r
\]
where \( t_1, \ldots, t_r \) is a local basis for \( \mathcal{E}_1 \), correspondingly \( t'_1, \ldots, t'_r \) is the dual basis for \( \mathcal{E}'_1 \), and \( s_i' \) is a local lift of \( s_i \) to a section of \( \mathcal{E}_0 \). We omit the proof that this is well defined.

If \( \mathcal{F} \) is locally free of rank \( r \), then it is straightforward to verify that the map is an isomorphism. Conversely, assume the map is an isomorphism on stalks at \( x \). This implies that \( \mathcal{F}_x \) is generated by at most \( r \) elements. This can only happen if \( \alpha \) has rank \( r \) modulo \( m_x \), i.e., \( \alpha \) has maximal rank modulo \( m_x \). This implies that \( \alpha \) has maximal rank in a neighbourhood of \( x \) and hence \( \mathcal{F} \) is locally free of rank \( r \) in a neighbourhood as desired.

\begin{lemma}
Let \( Y \) be a Noetherian scheme. Let \( f : X \to Y \) be a local complete intersection morphism. Let \( r \) be the locally constant function on \( X \) such that \( \omega_{Y/X} = H^{-r}(f^*\mathcal{O}_Y) \) is the unique nonzero cohomology sheaf of \( f^*\mathcal{O}_Y \), see Lemma 17.10. Assume \( f \) factors as an immersion \( X \to P \) followed by a proper smooth morphism \( P \to Y \). Then there is a map

\[ \wedge^r \Omega_{X/Y} \to \omega_{Y/X} \]

which is an isomorphism on the stalk at a point \( x \) if and only if \( f \) is smooth at \( x \).

\end{lemma}

\begin{proof}
The assumption implies that \( X \) is compactifyable over \( Y \) hence \( f' \) is defined, see Section 16. Let \( j : W \to P \) be an open subscheme such that \( X \to P \) factors through a closed immersion \( i : X \to W \). Moreover, we have \( f' = i^* \circ j \circ g \) where \( g : P \to Y \) is the given morphism. We have \( g^*\mathcal{O}_Y = \wedge^d \Omega_{P/Y}[d] \) by Lemma 15.7 where \( d \) is the locally constant function giving the relative dimension of \( P \) over \( Y \). We have \( j^* = j^* \). We have \( i^*\mathcal{O}_W = \wedge^c N[-c] \) where \( c \) is the codimension of \( X \) in \( W \) (a locally constant function on \( X \)) and where \( N \) is the normal sheaf of the Koszul-regular immersion \( i \), see Lemma 15.6. Combining the above we find

\[ f'\mathcal{O}_Y = (\wedge^c N \otimes_{\mathcal{O}_X} \wedge^d \Omega_{P/Y}[X])[d - c] \]

where we have also used Lemma 17.8. Thus \( r = d|X - c \) as locally constant functions on \( X \). The conormal sheaf of \( X \to P \) is the module \( \mathcal{I}/\mathcal{I}^2 \) where \( \mathcal{I} \subseteq \mathcal{O}_W \) is the ideal sheaf of \( i \), see Morphisms, Section 30. Consider the canonical exact sequence

\[ \mathcal{I}/\mathcal{I}^2 \to \Omega_{P/Y}|_X \to \Omega_{X/Y} \to 0 \]

of Morphisms, Lemma 31.15. We obtain our map by an application of Lemma 28.1 and the fact that \( \Omega_{X/Y} \) is locally free at \( x \) of rank \( r \). Conversely, assume that our map is an isomorphism on stalks at \( x \). Then the lemma shows that \( \Omega_{X/Y} \) is free of rank \( r \) after replacing \( X \) by an open neighbourhood of \( x \). On the other hand, we may also assume that \( X = \text{Spec}(A) \) and \( Y = \text{Spec}(R) \) where \( A = R[x_1, \ldots, x_n]/(f_1, \ldots, f_m) \) and where \( f_1, \ldots, f_m \) is a Koszul regular sequence (this follows from the definition of local complete intersection morphisms). Clearly this implies \( r = n - m \). We conclude that the rank of the matrix of partials \( \partial f_j/\partial x_i \) in the residue field at \( x \) is \( m \). Thus after reordering the variables we may assume the determinant of \( (\partial f_j/\partial x_i)_{1 \leq i,j \leq m} \) is invertible in an open neighbourhood of \( x \). It follows that \( R \to A \) is smooth at this point, see for example Algebra, Example 13.5.8.

\end{proof}

\begin{lemma}
Let \( f : X \to Y \) be a morphism of schemes. Let \( r \geq 0 \). Assume

1. \( Y \) is Cohen-Macaulay (Properties, Definition 8.1),

\end{lemma}
(2) \( f \) factors as \( X \to P \to Y \) where the first morphism is an immersion and the second is smooth and proper,

(3) if \( x \in X \) and \( \dim(\mathcal{O}_{X,x}) \leq 1 \), then \( f \) is Koszul at \( x \) (More on Morphisms, Definition 54.2), and

(4) if \( \xi \) is a generic point of an irreducible component of \( X \), then we have \( \text{trdeg}_{\kappa(\xi)}(\kappa(\xi)) = r \).

Then with \( \omega_{Y/X} = H^{-r}(f^!\mathcal{O}_Y) \) there is a map

\[ \bigwedge^r \Omega_X^{1/Y} \to \omega_{Y/X} \]

which is an isomorphism on the locus where \( f \) is smooth.

**Proof.** Let \( U \subset X \) be the open subscheme over which \( f \) is a local complete intersection morphism. Since \( f \) has relative dimension \( r \) at all generic points by assumption (4) we see that the locally constant function of Lemma \( 28.2 \) is constant with value \( r \) and we obtain a map

\[ \bigwedge^r \Omega_X^{1/Y}|_U = \bigwedge^r \Omega_U^{1/Y} \to \omega_{U/Y} = \omega_{X/Y}|_U \]

which is an isomorphism in the smooth points of \( f \) (this locus is contained in \( U \) because a smooth morphism is a local complete intersection morphism). By Lemma \( 21.5 \) and the assumption that \( Y \) is Cohen-Macaulay the module \( \omega_{X/Y} \) is \((S_2)\). Since \( U \) contains all the points of codimension 1 by condition (3) and using Divisors, Lemma \( 5.11 \) we see that \( j_*\omega_{U/Y} = \omega_{X/Y} \). Hence the map over \( U \) extends to \( X \) and the proof is complete. \( \square \)

29. Other chapters

Preliminaries

(1) Introduction
(2) Conventions
(3) Set Theory
(4) Categories
(5) Topology
(6) Sheaves on Spaces
(7) Sites and Sheaves
(8) Stacks
(9) Fields
(10) Commutative Algebra
(11) Brauer Groups
(12) Homological Algebra
(13) Derived Categories
(14) Simplicial Methods
(15) More on Algebra
(16) Smoothing Ring Maps
(17) Sheaves of Modules
(18) Modules on Sites
(19) Injectives
(20) Cohomology of Sheaves
(21) Cohomology on Sites
(22) Differential Graded Algebra

(23) Divided Power Algebra
(24) Hypercoverings
(25) Schemes
(26) Constructions of Schemes
(27) Properties of Schemes
(28) Morphisms of Schemes
(29) Cohomology of Schemes
(30) Divisors
(31) Limits of Schemes
(32) Varieties
(33) Topologies on Schemes
(34) Descent
(35) Derived Categories of Schemes
(36) More on Morphisms
(37) More on Flatness
(38) Groupoid Schemes
(39) More on Groupoid Schemes
(40) Etale Morphisms of Schemes
(41) Chow Homology
(42) Intersection Theory
(43) Picard Schemes of Curves
## References


