

# DERIVED CATEGORIES OF VARIETIES

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## 1. Introduction

0FY1 In this chapter we continue the discussion started in *Derived Categories of Schemes*, Section 1. We will discuss Fourier-Mukai transforms, first studied by Mukai in [Muk81]. We will prove Orlov's theorem on derived equivalences ([Or197]). We also discuss the countability of derived equivalence classes proved by Anel and Toën in [AT09].

A good introduction to this material is the book [Huy06] by Daniel Huybrechts. Some other papers which helped popularize this topic are

- (1) the paper by Bondal and Kapranov, see [BK89]
- (2) the paper by Bondal and Orlov, see [BO01]
- (3) the paper by Bondal and Van den Bergh, see [BV03]

- (4) the papers by Beilinson, see [Bei78] and [Bei84]
- (5) the paper by Orlov, see [Orl02]
- (6) the paper by Orlov, see [Orl05]
- (7) the paper by Rouquier, see [Rou08]
- (8) there are many more we could mention here.

## 2. Conventions and notation

0FY2 Let  $k$  be a field. A  $k$ -linear triangulated category  $\mathcal{T}$  is a triangulated category (Derived Categories, Section 3) which is endowed with a  $k$ -linear structure (Differential Graded Algebra, Section 24) such that the translation functors  $[n] : \mathcal{T} \rightarrow \mathcal{T}$  are  $k$ -linear for all  $n \in \mathbf{Z}$ .

Let  $k$  be a field. We denote  $\text{Vect}_k$  the category of  $k$ -vector spaces. For a  $k$ -vector space  $V$  we denote  $V^\vee$  the  $k$ -linear dual of  $V$ , i.e.,  $V^\vee = \text{Hom}_k(V, k)$ .

Let  $X$  be a scheme. We denote  $D_{\text{perf}}(\mathcal{O}_X)$  the full subcategory of  $D(\mathcal{O}_X)$  consisting of perfect complexes (Cohomology, Section 45). If  $X$  is Noetherian then  $D_{\text{perf}}(\mathcal{O}_X) \subset D_{\text{Coh}}^b(\mathcal{O}_X)$ , see Derived Categories of Schemes, Lemma 10.6. If  $X$  is Noetherian and regular, then  $D_{\text{perf}}(\mathcal{O}_X) = D_{\text{Coh}}^b(\mathcal{O}_X)$ , see Derived Categories of Schemes, Lemma 10.8.

Let  $k$  be a field. Let  $X$  and  $Y$  be schemes over  $k$ . In this situation we will write  $X \times Y$  instead of  $X \times_{\text{Spec}(k)} Y$ .

Let  $S$  be a scheme. Let  $X, Y$  be schemes over  $S$ . Let  $\mathcal{F}$  be a  $\mathcal{O}_X$ -module and let  $\mathcal{G}$  be a  $\mathcal{O}_Y$ -module. We set

$$\mathcal{F} \boxtimes \mathcal{G} = \text{pr}_1^* \mathcal{F} \otimes_{\mathcal{O}_{X \times_S Y}} \text{pr}_2^* \mathcal{G}$$

as  $\mathcal{O}_{X \times_S Y}$ -modules. If  $K \in D(\mathcal{O}_X)$  and  $M \in D(\mathcal{O}_Y)$  then we set

$$K \boxtimes M = L\text{pr}_1^* K \otimes_{\mathcal{O}_{X \times_S Y}}^{\mathbf{L}} L\text{pr}_2^* M$$

as an object of  $D(\mathcal{O}_{X \times_S Y})$ . Thus our notation is potentially ambiguous, but context should make it clear which of the two is meant.

## 3. Serre functors

0FY3 The material in this section is taken from [BK89].

0FY4 **Lemma 3.1.** *Let  $k$  be a field. Let  $\mathcal{T}$  be a  $k$ -linear triangulated category such that  $\dim_k \text{Hom}_{\mathcal{T}}(X, Y) < \infty$  for all  $X, Y \in \text{Ob}(\mathcal{T})$ . The following are equivalent*

- (1) *there exists a  $k$ -linear equivalence  $S : \mathcal{T} \rightarrow \mathcal{T}$  and  $k$ -linear isomorphisms  $c_{X, Y} : \text{Hom}_{\mathcal{T}}(X, Y) \rightarrow \text{Hom}_{\mathcal{T}}(Y, S(X))^\vee$  functorial in  $X, Y \in \text{Ob}(\mathcal{T})$ ,*
- (2) *for every  $X \in \text{Ob}(\mathcal{T})$  the functor  $Y \mapsto \text{Hom}_{\mathcal{T}}(X, Y)^\vee$  is representable and the functor  $Y \mapsto \text{Hom}_{\mathcal{T}}(Y, X)^\vee$  is corepresentable.*

**Proof.** Condition (1) implies (2) since given  $(S, c)$  and  $X \in \text{Ob}(\mathcal{T})$  the object  $S(X)$  represents the functor  $Y \mapsto \text{Hom}_{\mathcal{T}}(X, Y)^\vee$  and the object  $S^{-1}(X)$  corepresents the functor  $Y \mapsto \text{Hom}_{\mathcal{T}}(Y, X)^\vee$ .

Assume (2). We will repeatedly use the Yoneda lemma, see Categories, Lemma 3.5. For every  $X$  denote  $S(X)$  the object representing the functor  $Y \mapsto \text{Hom}_{\mathcal{T}}(X, Y)^\vee$ . Given  $\varphi : X \rightarrow X'$ , we obtain a unique arrow  $S(\varphi) : S(X) \rightarrow S(X')$  determined by the corresponding transformation of functors  $\text{Hom}_{\mathcal{T}}(X, -)^\vee \rightarrow \text{Hom}_{\mathcal{T}}(X', -)^\vee$ .

Thus  $S$  is a functor and we obtain the isomorphisms  $c_{X,Y}$  by construction. It remains to show that  $S$  is an equivalence. For every  $X$  denote  $S'(X)$  the object corepresenting the functor  $Y \mapsto \text{Hom}_{\mathcal{T}}(Y, X)^\vee$ . Arguing as above we find that  $S'$  is a functor. We claim that  $S'$  is quasi-inverse to  $S$ . To see this observe that

$$\text{Hom}_{\mathcal{T}}(X, Y) = \text{Hom}_{\mathcal{T}}(Y, S(X))^\vee = \text{Hom}_{\mathcal{T}}(S'(S(X)), Y)$$

bifunctorially, i.e., we find  $S' \circ S \cong \text{id}_{\mathcal{T}}$ . Similarly, we have

$$\text{Hom}_{\mathcal{T}}(Y, X) = \text{Hom}_{\mathcal{T}}(S'(X), Y)^\vee = \text{Hom}_{\mathcal{T}}(Y, S(S'(X)))$$

and we find  $S \circ S' \cong \text{id}_{\mathcal{T}}$ .  $\square$

0FY5 **Definition 3.2.** Let  $k$  be a field. Let  $\mathcal{T}$  be a  $k$ -linear triangulated category such that  $\dim_k \text{Hom}_{\mathcal{T}}(X, Y) < \infty$  for all  $X, Y \in \text{Ob}(\mathcal{T})$ . We say a *Serre functor exists* if the equivalent conditions of Lemma 3.1 are satisfied. In this case a *Serre functor* is a  $k$ -linear equivalence  $S : \mathcal{T} \rightarrow \mathcal{T}$  endowed with  $k$ -linear isomorphisms  $c_{X,Y} : \text{Hom}_{\mathcal{T}}(X, Y) \rightarrow \text{Hom}_{\mathcal{T}}(Y, S(X))^\vee$  functorial in  $X, Y \in \text{Ob}(\mathcal{T})$ .

0FY6 **Lemma 3.3.** *In the situation of Definition 3.2. If a Serre functor exists, then it is unique up to unique isomorphism and it is an exact functor of triangulated categories.*

**Proof.** Given a Serre functor  $S$  the object  $S(X)$  represents the functor  $Y \mapsto \text{Hom}_{\mathcal{T}}(X, Y)^\vee$ . Thus the object  $S(X)$  together with the functorial identification  $\text{Hom}_{\mathcal{T}}(X, Y)^\vee = \text{Hom}_{\mathcal{T}}(Y, S(X))$  is determined up to unique isomorphism by the Yoneda lemma (Categories, Lemma 3.5). Moreover, for  $\varphi : X \rightarrow X'$ , the arrow  $S(\varphi) : S(X) \rightarrow S(X')$  is uniquely determined by the corresponding transformation of functors  $\text{Hom}_{\mathcal{T}}(X, -)^\vee \rightarrow \text{Hom}_{\mathcal{T}}(X', -)^\vee$ .

For objects  $X, Y$  of  $\mathcal{T}$  we have

$$\begin{aligned} \text{Hom}(Y, S(X)[1])^\vee &= \text{Hom}(Y[-1], S(X))^\vee \\ &= \text{Hom}(X, Y[-1]) \\ &= \text{Hom}(X[1], Y) \\ &= \text{Hom}(Y, S(X[1]))^\vee \end{aligned}$$

By the Yoneda lemma we conclude that there is a unique isomorphism  $S(X[1]) \rightarrow S(X)[1]$  inducing the isomorphism from top left to bottom right. Since each of the isomorphisms above is functorial in both  $X$  and  $Y$  we find that this defines an isomorphism of functors  $S \circ [1] \rightarrow [1] \circ S$ .

Let  $(A, B, C, f, g, h)$  be a distinguished triangle in  $\mathcal{T}$ . We have to show that the triangle  $(S(A), S(B), S(C), S(f), S(g), S(h))$  is distinguished. Here we use the canonical isomorphism  $S(A[1]) \rightarrow S(A)[1]$  constructed above to identify the target  $S(A[1])$  of  $S(h)$  with  $S(A)[1]$ . We first observe that for any  $X$  in  $\mathcal{T}$  the triangle  $(S(A), S(B), S(C), S(f), S(g), S(h))$  induces a long exact sequence

$$\dots \rightarrow \text{Hom}(X, S(A)) \rightarrow \text{Hom}(X, S(B)) \rightarrow \text{Hom}(X, S(C)) \rightarrow \text{Hom}(X, S(A)[1]) \rightarrow \dots$$

of finite dimensional  $k$ -vector spaces. Namely, this sequence is  $k$ -linear dual of the sequence

$$\dots \leftarrow \text{Hom}(A, X) \leftarrow \text{Hom}(B, X) \leftarrow \text{Hom}(C, X) \leftarrow \text{Hom}(A[1], X) \leftarrow \dots$$

which is exact by Derived Categories, Lemma 4.2. Next, we choose a distinguished triangle  $(S(A), E, S(C), i, p, S(h))$  which is possible by axioms TR1 and TR2. We want to construct the dotted arrow making following diagram commute

$$\begin{array}{ccccccccc} S(C)[-1] & \xrightarrow{S(h[-1])} & S(A) & \xrightarrow{S(f)} & S(B) & \xrightarrow{S(g)} & S(C) & \xrightarrow{S(h)} & S(A)[1] \\ \parallel & & \parallel & & \uparrow \varphi & & \parallel & & \parallel \\ S(C)[-1] & \xrightarrow{S(h[-1])} & S(A) & \xrightarrow{i} & E & \xrightarrow{p} & S(C) & \xrightarrow{S(h)} & S(A)[1] \end{array}$$

Namely, if we have  $\varphi$ , then we claim for any  $X$  the resulting map  $\text{Hom}(X, E) \rightarrow \text{Hom}(X, S(B))$  will be an isomorphism of  $k$ -vector spaces. Namely, we will obtain a commutative diagram

$$\begin{array}{ccccccccc} \text{Hom}(X, S(C)[-1]) & \longrightarrow & \text{Hom}(X, S(A)) & \longrightarrow & \text{Hom}(X, S(B)) & \longrightarrow & \text{Hom}(X, S(C)) & \longrightarrow & \text{Hom}(X, S(A)[1]) \\ \parallel & & \parallel & & \uparrow \varphi & & \parallel & & \parallel \\ \text{Hom}(X, S(C)[-1]) & \longrightarrow & \text{Hom}(X, S(A)) & \longrightarrow & \text{Hom}(X, E) & \longrightarrow & \text{Hom}(X, S(C)) & \longrightarrow & \text{Hom}(X, S(A)[1]) \end{array}$$

with exact rows (see above) and we can apply the 5 lemma (Homology, Lemma 5.20) to see that the middle arrow is an isomorphism. By the Yoneda lemma we conclude that  $\varphi$  is an isomorphism. To find  $\varphi$  consider the following diagram

$$\begin{array}{ccc} \text{Hom}(E, S(C)) & \longrightarrow & \text{Hom}(S(A), S(C)) \\ \uparrow & & \uparrow \\ \text{Hom}(E, S(B)) & \longrightarrow & \text{Hom}(S(A), S(B)) \end{array}$$

The elements  $p$  and  $S(f)$  in positions  $(0, 1)$  and  $(1, 0)$  define a cohomology class  $\xi$  in the total complex of this double complex. The existence of  $\varphi$  is equivalent to whether  $\xi$  is zero. If we take  $k$ -linear duals of this and we use the defining property of  $S$  we obtain

$$\begin{array}{ccc} \text{Hom}(C, E) & \longleftarrow & \text{Hom}(C, S(A)) \\ \downarrow & & \downarrow \\ \text{Hom}(B, E) & \longleftarrow & \text{Hom}(B, S(A)) \end{array}$$

Since both  $A \rightarrow B \rightarrow C$  and  $S(A) \rightarrow E \rightarrow S(C)$  are distinguished triangles, we know by TR3 that given elements  $\alpha \in \text{Hom}(C, E)$  and  $\beta \in \text{Hom}(B, S(A))$  mapping to the same element in  $\text{Hom}(B, E)$ , there exists an element in  $\text{Hom}(C, S(A))$  mapping to both  $\alpha$  and  $\beta$ . In other words, the cohomology of the total complex associated to this double complex is zero in degree 1, i.e., the degree corresponding to  $\text{Hom}(C, E) \oplus \text{Hom}(B, S(A))$ . Taking duals the same must be true for the previous one which concludes the proof.  $\square$

#### 4. Examples of Serre functors

0FY7 The lemma below is the standard example.

0FY8 **Lemma 4.1.** *Let  $k$  be a field. Let  $X$  be a proper scheme over  $k$  which is Gorenstein. Consider the complex  $\omega_X^\bullet$  of Duality for Schemes, Lemmas 27.1. Then the functor*

$$S : D_{\text{perf}}(\mathcal{O}_X) \longrightarrow D_{\text{perf}}(\mathcal{O}_X), \quad K \longmapsto S(K) = \omega_X^\bullet \otimes_{\mathcal{O}_X}^{\mathbf{L}} K$$

is a Serre functor.

**Proof.** The statement make sense because  $\dim \operatorname{Hom}_X(K, L) < \infty$  for  $K, L \in D_{\text{perf}}(\mathcal{O}_X)$  by Derived Categories of Schemes, Lemma 10.7. Since  $X$  is Gorenstein the dualizing complex  $\omega_X^\bullet$  is an invertible object of  $D(\mathcal{O}_X)$ , see Duality for Schemes, Lemma 24.4. In particular, locally on  $X$  the complex  $\omega_X^\bullet$  has one nonzero cohomology sheaf which is an invertible module, see Cohomology, Lemma 47.2. Thus  $S(K)$  lies in  $D_{\text{perf}}(\mathcal{O}_X)$ . On the other hand, the invertibility of  $\omega_X^\bullet$  clearly implies that  $S$  is a self-equivalence of  $D_{\text{perf}}(\mathcal{O}_X)$ . Finally, we have to find an isomorphism

$$c_{K,L} : \operatorname{Hom}_X(K, L) \longrightarrow \operatorname{Hom}_X(L, \omega_X^\bullet \otimes_{\mathcal{O}_X}^{\mathbf{L}} K)^\vee$$

bifunctorially in  $K, L$ . To do this we use the canonical isomorphisms

$$\operatorname{Hom}_X(K, L) = H^0(X, L \otimes_{\mathcal{O}_X}^{\mathbf{L}} K^\vee)$$

and

$$\operatorname{Hom}_X(L, \omega_X^\bullet \otimes_{\mathcal{O}_X}^{\mathbf{L}} K) = H^0(X, \omega_X^\bullet \otimes_{\mathcal{O}_X}^{\mathbf{L}} K \otimes_{\mathcal{O}_X}^{\mathbf{L}} L^\vee)$$

given in Cohomology, Lemma 46.4. Since  $(L \otimes_{\mathcal{O}_X}^{\mathbf{L}} K^\vee)^\vee = (K^\vee)^\vee \otimes_{\mathcal{O}_X}^{\mathbf{L}} L^\vee$  and since there is a canonical isomorphism  $K \rightarrow (K^\vee)^\vee$  we find these  $k$ -vector spaces are canonically dual by Duality for Schemes, Lemma 27.4. This produces the isomorphisms  $c_{K,L}$ . We omit the proof that these isomorphisms are functorial.  $\square$

## 5. Characterizing coherent modules

0FY9 This section is in some sense a continuation of the discussion in Derived Categories of Schemes, Section 32 and More on Morphisms, Section 61.

Before we can state the result we need some notation. Let  $k$  be a field. Let  $n \geq 0$  be an integer. Let  $S = k[X_0, \dots, X_n]$ . For an integer  $e$  denote  $S_e \subset S$  the homogeneous polynomials of degree  $e$ . Consider the (noncommutative)  $k$ -algebra

$$R = \begin{pmatrix} S_0 & S_1 & S_2 & \dots & \dots \\ 0 & S_0 & S_1 & \dots & \dots \\ 0 & 0 & S_0 & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & \dots & S_0 \end{pmatrix}$$

(with  $n + 1$  rows and columns) with obvious multiplication and addition.

0FYA **Lemma 5.1.** *With  $k, n,$  and  $R$  as above, for an object  $K$  of  $D(R)$  the following are equivalent*

- (1)  $\sum_{i \in \mathbf{Z}} \dim_k H^i(K) < \infty$ , and
- (2)  $K$  is a compact object.

**Proof.** If  $K$  is a compact object, then  $K$  can be represented by a complex  $M^\bullet$  which is finite projective as a graded  $R$ -module, see Differential Graded Algebra, Lemma 36.6. Since  $\dim_k R < \infty$  we conclude  $\sum \dim_k M^i < \infty$  and a fortiori  $\sum \dim_k H^i(M^\bullet) < \infty$ . (One can also easily deduce this implication from the easier Differential Graded Algebra, Proposition 36.4.)

Assume  $K$  satisfies (1). Consider the distinguished triangle of truncations  $\tau_{\leq m} K \rightarrow K \rightarrow \tau_{\geq m+1} K$ , see Derived Categories, Remark 12.4. It is clear that both  $\tau_{\leq m} K$  and  $\tau_{\geq m+1} K$  satisfy (1). If we can show both are compact, then so is  $K$ , see Derived Categories, Lemma 37.2. Hence, arguing on the number of nonzero cohomology

modules of  $K$  we may assume  $H^i(K)$  is nonzero only for one  $i$ . Shifting, we may assume  $K$  is given by the complex consisting of a single finite dimensional  $R$ -module  $M$  sitting in degree 0.

Since  $\dim_k(M) < \infty$  we see that  $M$  is Artinian as an  $R$ -module. Thus it suffices to show that every simple  $R$ -module represents a compact object of  $D(R)$ . Observe that

$$I = \begin{pmatrix} 0 & S_1 & S_2 & \dots & \dots \\ 0 & 0 & S_1 & \dots & \dots \\ 0 & 0 & 0 & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & \dots & 0 \end{pmatrix}$$

is a nilpotent two sided ideal of  $R$  and that  $R/I$  is a commutative  $k$ -algebra isomorphic to a product of  $n+1$  copies of  $k$  (placed along the diagonal in the matrix, i.e.,  $R/I$  can be lifted to a  $k$ -subalgebra of  $R$ ). It follows that  $R$  has exactly  $n+1$  isomorphism classes of simple modules  $M_0, \dots, M_n$  (sitting along the diagonal). Consider the right  $R$ -module  $P_i$  of row vectors

$$P_i = (0 \quad \dots \quad 0 \quad S_0 \quad \dots \quad S_{i-1} \quad S_i)$$

with obvious multiplication  $P_i \times R \rightarrow P_i$ . Then we see that  $R \cong P_0 \oplus \dots \oplus P_n$  as a right  $R$ -module. Since clearly  $R$  is a compact object of  $D(R)$ , we conclude each  $P_i$  is a compact object of  $D(R)$ . (We of course also conclude each  $P_i$  is projective as an  $R$ -module, but this isn't what we have to show in this proof.) Clearly,  $P_0 = M_0$  is the first of our simple  $R$ -modules. For  $P_1$  we have a short exact sequence

$$0 \rightarrow P_0^{\oplus n+1} \rightarrow P_1 \rightarrow M_1 \rightarrow 0$$

which proves that  $M_1$  fits into a distinguished triangle whose other members are compact objects and hence  $M_1$  is a compact object of  $D(R)$ . More generally, there exists a short exact sequence

$$0 \rightarrow C_i \rightarrow P_i \rightarrow M_i \rightarrow 0$$

where  $C_i$  is a finite dimensional  $R$ -module whose simple constituents are isomorphic to  $S_j$  for  $j < i$ . By induction, we first conclude that  $C_i$  determines a compact object of  $D(R)$  whereupon we conclude that  $M_i$  does too as desired.  $\square$

0FYB **Lemma 5.2.** *Let  $k$  be a field. Let  $n \geq 0$ . Let  $K \in D_{QCoh}(\mathcal{O}_{\mathbf{P}_k^n})$ . The following are equivalent*

- (1)  $K$  is in  $D_{Coh}^b(\mathcal{O}_{\mathbf{P}_k^n})$ ,
- (2)  $\sum_{i \in \mathbf{Z}} \dim_k H^i(\mathbf{P}_k^n, E \otimes^{\mathbf{L}} K) < \infty$  for each perfect object  $E$  of  $D(\mathcal{O}_{\mathbf{P}_k^n})$ ,
- (3)  $\sum_{i \in \mathbf{Z}} \dim_k \text{Ext}_{\mathbf{P}_k^n}^i(E, K) < \infty$  for each perfect object  $E$  of  $D(\mathcal{O}_{\mathbf{P}_k^n})$ ,
- (4)  $\sum_{i \in \mathbf{Z}} \dim_k H^i(\mathbf{P}_k^n, K \otimes^{\mathbf{L}} \mathcal{O}_{\mathbf{P}_k^n}(d)) < \infty$  for  $d = 0, 1, \dots, n$ .

**Proof.** Parts (2) and (3) are equivalent by Cohomology, Lemma 46.4. If (1) is true, then for  $E$  perfect the derived tensor product  $E \otimes^{\mathbf{L}} K$  is in  $D_{Coh}^b(\mathcal{O}_{\mathbf{P}_k^n})$  and we see that (2) holds by Derived Categories of Schemes, Lemma 10.3. It is clear that (2) implies (4) as  $\mathcal{O}_{\mathbf{P}_k^n}(d)$  can be viewed as a perfect object of the derived category of  $\mathbf{P}_k^n$ . Thus it suffices to prove that (4) implies (1).

Assume (4). Let  $R$  be as in Lemma 5.1. Let  $P = \bigoplus_{d=0, \dots, n} \mathcal{O}_{\mathbf{P}_k^n}(-d)$ . Recall that  $R = \text{End}_{\mathbf{P}_k^n}(P)$  whereas all other self-Exts of  $P$  are zero and that  $P$  determines

an equivalence  $-\otimes^{\mathbf{L}} P : D(R) \rightarrow D_{QCoh}(\mathcal{O}_{\mathbf{P}_k^n})$  by Derived Categories of Schemes, Lemma 19.1. Say  $K$  corresponds to  $L$  in  $D(R)$ . Then

$$\begin{aligned} H^i(L) &= \text{Ext}_{D(R)}^i(R, L) \\ &= \text{Ext}_{\mathbf{P}_k^n}^i(P, K) \\ &= H^i(\mathbf{P}_k^n, K \otimes P^\vee) \\ &= \bigoplus_{d=0, \dots, n} H^i(\mathbf{P}_k^n, K \otimes \mathcal{O}(d)) \end{aligned}$$

by Differential Graded Algebra, Lemma 35.4 (and the fact that  $-\otimes^{\mathbf{L}} P$  is an equivalence) and Cohomology, Lemma 46.4. Thus our assumption (4) implies that  $L$  satisfies condition (2) of Lemma 5.1 and hence is a compact object of  $D(R)$ . Therefore  $K$  is a compact object of  $D_{QCoh}(\mathcal{O}_{\mathbf{P}_k^n})$ . Thus  $K$  is perfect by Derived Categories of Schemes, Proposition 16.1. Since  $D_{perf}(\mathcal{O}_{\mathbf{P}_k^n}) = D_{Coh}^b(\mathcal{O}_{\mathbf{P}_k^n})$  by Derived Categories of Schemes, Lemma 10.8 we conclude (1) holds.  $\square$

0FYC **Lemma 5.3.** *Let  $X$  be a scheme proper over a field  $k$ . Let  $K \in D_{Coh}^b(\mathcal{O}_X)$  and let  $E$  in  $D(\mathcal{O}_X)$  be perfect. Then  $\sum_{i \in \mathbf{Z}} \dim_k \text{Ext}_X^i(E, K) < \infty$ .*

**Proof.** This follows for example by combining Derived Categories of Schemes, Lemmas 10.7 and 17.2. Alternative proof: combine Derived Categories of Schemes, Lemmas 10.6 and 10.3.  $\square$

0FYD **Lemma 5.4.** *Let  $X$  be a projective scheme over a field  $k$ . Let  $K \in \text{Ob}(D_{QCoh}(\mathcal{O}_X))$ . The following are equivalent* [Rou08, Lemma 7.46] and implicit in [BV03, Theorem A.1]

- (1)  $K \in D_{Coh}^b(\mathcal{O}_X)$ , and
- (2)  $\sum_{i \in \mathbf{Z}} \dim_k \text{Ext}_X^i(E, K) < \infty$  for all perfect  $E$  in  $D(\mathcal{O}_X)$ .

**Proof.** The implication (1)  $\Rightarrow$  (2) follows from Lemma 5.3.

Assume (2). Choose a closed immersion  $i : X \rightarrow \mathbf{P}_k^n$ . It suffices to show that  $Ri_*K$  is in  $D_{Coh}^b(\mathbf{P}_k^n)$  since a quasi-coherent module  $\mathcal{F}$  on  $X$  is coherent, resp. zero if and only if  $i_*\mathcal{F}$  is coherent, resp. zero. For a perfect object  $E$  of  $D(\mathcal{O}_{\mathbf{P}_k^n})$  we have

$$\text{Ext}_{\mathbf{P}_k^n}^q(E, Ri_*K) = \text{Ext}_X^q(Li^*E, K) = F(Li^*E[q])$$

Hence by our assumption we see that  $\sum_{q \in \mathbf{Z}} \dim_k \text{Ext}_{\mathbf{P}_k^n}^q(E, Ri_*K) < \infty$ . We conclude by Lemma 5.2.  $\square$

## 6. A representability theorem

0FYE The material in this section is taken from [BV03].

Let  $\mathcal{T}$  be a  $k$ -linear triangulated category. In this section we consider  $k$ -linear cohomological functors  $H$  from  $\mathcal{T}$  to the category of  $k$ -vector spaces. This will mean  $H$  is a functor

$$H : \mathcal{T}^{opp} \longrightarrow \text{Vect}_k$$

which is  $k$ -linear such that for any distinguished triangle  $X \rightarrow Y \rightarrow Z$  in  $\mathcal{T}$  the sequence  $H(Z) \rightarrow H(Y) \rightarrow H(X)$  is an exact sequence of  $k$ -vector spaces. See Derived Categories, Definition 3.5 and Differential Graded Algebra, Section 24.

0FYF **Lemma 6.1.** *Let  $\mathcal{D}$  be a triangulated category. Let  $\mathcal{D}' \subset \mathcal{D}$  be a full triangulated subcategory. Let  $X \in \text{Ob}(\mathcal{D})$ . The category of arrows  $E \rightarrow X$  with  $E \in \text{Ob}(\mathcal{D}')$  is filtered.*

**Proof.** We check the conditions of Categories, Definition 19.1. The category is nonempty because it contains  $0 \rightarrow X$ . If  $E_i \rightarrow X$ ,  $i = 1, 2$  are objects, then  $E_1 \oplus E_2 \rightarrow X$  is an object and there are morphisms  $(E_i \rightarrow X) \rightarrow (E_1 \oplus E_2 \rightarrow X)$ . Finally, suppose that  $a, b : (E \rightarrow X) \rightarrow (E' \rightarrow X)$  are morphisms. Choose a distinguished triangle  $E \xrightarrow{a-b} E' \rightarrow E''$  in  $\mathcal{D}'$ . By Axiom TR3 we obtain a morphism of triangles

$$\begin{array}{ccccc} E & \xrightarrow{a-b} & E' & \longrightarrow & E'' \\ \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & X & \longrightarrow & X \end{array}$$

and we find that the resulting arrow  $(E' \rightarrow X) \rightarrow (E'' \rightarrow X)$  equalizes  $a$  and  $b$ .  $\square$

0FYG **Lemma 6.2.** *Let  $k$  be a field. Let  $\mathcal{D}$  be a  $k$ -linear triangulated category which has direct sums and is compactly generated. Denote  $\mathcal{D}_c$  the full subcategory of compact objects. Let  $H : \mathcal{D}_c^{ppp} \rightarrow \text{Vect}_k$  be a  $k$ -linear cohomological functor such that  $\dim_k H(X) < \infty$  for all  $X \in \text{Ob}(\mathcal{D}_c)$ . Then  $H$  is isomorphic to the functor  $X \mapsto \text{Hom}(X, Y)$  for some  $Y \in \text{Ob}(\mathcal{D})$ .* [CKN01, Lemma 2.14]

**Proof.** We will use Derived Categories, Lemma 37.2 without further mention. Denote  $G : \mathcal{D}_c \rightarrow \text{Vect}_k$  the  $k$ -linear homological functor which sends  $X$  to  $H(X)^\vee$ . For any object  $Y$  of  $\mathcal{D}$  we set

$$G'(Y) = \text{colim}_{X \rightarrow Y, X \in \text{Ob}(\mathcal{D}_c)} G(X)$$

The colimit is filtered by Lemma 6.1. We claim that  $G'$  is a  $k$ -linear homological functor, the restriction of  $G'$  to  $\mathcal{D}_c$  is  $G$ , and  $G'$  sends direct sums to direct sums.

Namely, suppose that  $Y_1 \rightarrow Y_2 \rightarrow Y_3$  is a distinguished triangle. Let  $\xi \in G'(Y_2)$  map to zero in  $G'(Y_3)$ . Since the colimit is filtered  $\xi$  is represented by some  $X \rightarrow Y_2$  with  $X \in \text{Ob}(\mathcal{D}_c)$  and  $g \in G(X)$ . The fact that  $\xi$  maps to zero in  $G'(Y_3)$  means the composition  $X \rightarrow Y_2 \rightarrow Y_3$  factors as  $X \rightarrow X' \rightarrow Y_3$  with  $X' \in \mathcal{D}_c$  and  $g$  mapping to zero in  $G(X')$ . Choose a distinguished triangle  $X'' \rightarrow X \rightarrow X'$ . Then  $X'' \in \text{Ob}(\mathcal{D}_c)$ . Since  $G$  is homological we find that  $g$  is the image of some  $g'' \in G(X'')$ . By Axiom TR3 the maps  $X \rightarrow Y_2$  and  $X' \rightarrow Y_3$  fit into a morphism of distinguished triangles  $(X'' \rightarrow X \rightarrow X') \rightarrow (Y_1 \rightarrow Y_2 \rightarrow Y_3)$  and we find that indeed  $\xi$  is the image of the element of  $G'(Y_1)$  represented by  $X'' \rightarrow Y_1$  and  $g'' \in G(X'')$ .

If  $Y \in \text{Ob}(\mathcal{D}_c)$ , then  $\text{id} : Y \rightarrow Y$  is the final object in the category of arrows  $X \rightarrow Y$  with  $X \in \text{Ob}(\mathcal{D}_c)$ . Hence we see that  $G'(Y) = G(Y)$  in this case and the statement on restriction holds. Let  $Y = \bigoplus_{i \in I} Y_i$  be a direct sum. Let  $a : X \rightarrow Y$  with  $X \in \text{Ob}(\mathcal{D}_c)$  and  $g \in G(X)$  represent an element  $\xi$  of  $G'(Y)$ . The morphism  $a : X \rightarrow Y$  can be uniquely written as a sum of morphisms  $a_i : X \rightarrow Y_i$  almost all zero as  $X$  is a compact object of  $\mathcal{D}$ . Let  $I' = \{i \in I \mid a_i \neq 0\}$ . Then we can factor  $a$  as the composition

$$X \xrightarrow{(1, \dots, 1)} \bigoplus_{i \in I'} X \xrightarrow{\bigoplus_{i \in I'} a_i} \bigoplus_{i \in I} Y_i = Y$$

We conclude that  $\xi = \sum_{i \in I'} \xi_i$  is the sum of the images of the elements  $\xi_i \in G'(Y_i)$  corresponding to  $a_i : X \rightarrow Y_i$  and  $g \in G(X)$ . Hence  $\bigoplus G'(Y_i) \rightarrow G'(Y)$  is surjective. We omit the (trivial) verification that it is injective.



It follows that the functor  $Y \mapsto G'(Y)^\vee$  is cohomological and sends direct sums to direct products. Hence by Brown representability, see Derived Categories, Proposition 38.2 we conclude that there exists a  $Y \in \text{Ob}(\mathcal{D})$  and an isomorphism  $G'(Z)^\vee = \text{Hom}(Z, Y)$  functorially in  $Z$ . For  $X \in \text{Ob}(\mathcal{D}_c)$  we have  $G'(X)^\vee = G(X)^\vee = (H(X)^\vee)^\vee = H(X)$  because  $\dim_k H(X) < \infty$  and the proof is complete.  $\square$

0FYH **Theorem 6.3.** *Let  $X$  be a projective scheme over a field  $k$ . Let  $F : D_{\text{perf}}(\mathcal{O}_X)^{\text{opp}} \rightarrow \text{Vect}_k$  be a  $k$ -linear cohomological functor such that* [BV03, Theorem A.1]

$$\sum_{n \in \mathbf{Z}} \dim_k F(E[n]) < \infty$$

for all  $E \in D_{\text{perf}}(\mathcal{O}_X)$ . Then  $F$  is isomorphic to a functor of the form  $E \mapsto \text{Hom}_X(E, K)$  for some  $K \in D_{\text{Coh}}^b(\mathcal{O}_X)$ .

**Proof.** The derived category  $D_{\text{QCoh}}(\mathcal{O}_X)$  has direct sums, is compactly generated, and  $D_{\text{perf}}(\mathcal{O}_X)$  is the full subcategory of compact objects, see Derived Categories of Schemes, Lemma 3.1, Theorem 14.3, and Proposition 16.1. By Lemma 6.2 we may assume  $F(E) = \text{Hom}_X(E, K)$  for some  $K \in \text{Ob}(D_{\text{QCoh}}(\mathcal{O}_X))$ . Then it follows that  $K$  is in  $D_{\text{Coh}}^b(\mathcal{O}_X)$  by Lemma 5.4.  $\square$

## 7. Representability in the regular proper case

0FYI Theorem 6.3 also holds for regular (for example smooth) proper varieties. This is proven in [BV03] using their general characterization of “right saturated”  $k$ -linear triangulated categories. In this section we give a quick and dirty proof of this result using a little bit of duality.

0FYJ **Lemma 7.1.** *Let  $f : X' \rightarrow X$  be a proper birational morphism of integral Noetherian schemes with  $X$  regular. The map  $\mathcal{O}_X \rightarrow \text{Rf}_* \mathcal{O}_{X'}$  canonically splits in  $D(\mathcal{O}_X)$ .*

The proof given here follows the argument given in [MS20, Remark 3.4]

**Proof.** Set  $E = \text{Rf}_* \mathcal{O}_{X'}$  in  $D(\mathcal{O}_X)$ . Observe that  $E$  is in  $D_{\text{Coh}}^b(\mathcal{O}_X)$  by Derived Categories of Schemes, Lemma 10.3. By Derived Categories of Schemes, Lemma 10.8 we find that  $E$  is a perfect object of  $D(\mathcal{O}_X)$ . Since  $\mathcal{O}_{X'}$  is a sheaf of algebras, we have the relative cup product  $\mu : E \otimes_{\mathcal{O}_X}^{\mathbf{L}} E \rightarrow E$  by Cohomology, Remark 28.7. Let  $\sigma : E \otimes E^\vee \rightarrow E^\vee \otimes E$  be the commutativity constraint on the symmetric monoidal category  $D(\mathcal{O}_X)$  (Cohomology, Lemma 46.5). Denote  $\eta : \mathcal{O}_X \rightarrow E \otimes E^\vee$  and  $\epsilon : E^\vee \otimes E \rightarrow \mathcal{O}_X$  the maps constructed in Cohomology, Example 46.6. Then we can consider the map

$$E \xrightarrow{\eta \otimes 1} E \otimes E^\vee \otimes E \xrightarrow{\sigma \otimes 1} E^\vee \otimes E \otimes E \xrightarrow{1 \otimes \mu} E^\vee \otimes E \xrightarrow{\epsilon} \mathcal{O}_X$$

We claim that this map is a one sided inverse to the map in the statement of the lemma. To see this it suffices to show that the composition  $\mathcal{O}_X \rightarrow \mathcal{O}_X$  is the identity map. This we may do in the generic point of  $X$  (or on an open subscheme of  $X$  over which  $f$  is an isomorphism). In this case  $E = \mathcal{O}_X$  and  $\mu$  is the usual multiplication map and the result is clear.  $\square$

0FYK **Lemma 7.2.** *Let  $X$  be a proper scheme over a field  $k$  which is regular. Let  $K \in \text{Ob}(D_{\text{QCoh}}(\mathcal{O}_X))$ . The following are equivalent*

- (1)  $K \in D_{\text{Coh}}^b(\mathcal{O}_X) = D_{\text{perf}}(\mathcal{O}_X)$ , and
- (2)  $\sum_{i \in \mathbf{Z}} \dim_k \text{Ext}_X^i(E, K) < \infty$  for all perfect  $E$  in  $D(\mathcal{O}_X)$ .

**Proof.** The equality in (1) holds by Derived Categories of Schemes, Lemma 10.8. The implication (1)  $\Rightarrow$  (2) follows from Lemma 5.3.

Assume (2). We may replace  $X$  by one of its connected components and assume  $X$  is a proper regular variety over  $k$ . By Chow's lemma we can choose a birational proper morphism  $f : X' \rightarrow X$  with  $X'$  a projective variety over  $k$ , see Cohomology of Schemes, Lemma 18.1 and Remark 18.2. Denote  $a : D_{QCoh}(\mathcal{O}_X) \rightarrow D_{QCoh}(\mathcal{O}_{X'})$  the right adjoint to  $Rf_*$ , see Duality for Schemes, Lemma 3.1. If  $E$  in  $D(\mathcal{O}_{X'})$  is perfect, then so is  $Rf_*E$  in  $D(\mathcal{O}_X)$  by Derived Categories of Schemes, Lemmas 10.3 and 10.8. Hence we see that

$$\mathrm{Ext}_{X'}^i(E, a(K)) = \mathrm{Ext}_X^i(Rf_*E, K)$$

is finite dimensional and nonzero for only a finite number of  $i$  by our assumption on  $K$ . By Lemma 5.4 we find that  $a(K)$  is in  $D_{Coh}^b(\mathcal{O}_{X'})$ . By Duality for Schemes, Lemma 3.6 we have

$$Rf_*a(K) = R\mathcal{H}om_{\mathcal{O}_X}(Rf_*\mathcal{O}_{X'}, K)$$

The left hand side is in  $D_{Coh}^b(\mathcal{O}_X)$  by the already used Derived Categories of Schemes, Lemmas 10.3. By Lemma 7.1 we have that  $Rf_*\mathcal{O}_{X'}$  contains  $\mathcal{O}_X$  as a summand. Hence we conclude that  $K$  is in  $D_{Coh}^b(\mathcal{O}_X)$  as desired.  $\square$

0FYL **Lemma 7.3.** *Let  $X$  be a proper scheme over a field  $k$  which is regular.*

- (1) *Let  $F : D_{perf}(\mathcal{O}_X)^{opp} \rightarrow \mathrm{Vect}_k$  be a  $k$ -linear cohomological functor such that*

$$\sum_{n \in \mathbf{Z}} \dim_k F(E[n]) < \infty$$

*for all  $E \in D_{perf}(\mathcal{O}_X)$ . Then  $F$  is isomorphic to a functor of the form  $E \mapsto \mathrm{Hom}_X(E, K)$  for some  $K \in D_{perf}(\mathcal{O}_X)$ .*

- (2) *Let  $G : D_{perf}(\mathcal{O}_X) \rightarrow \mathrm{Vect}_k$  be a  $k$ -linear homological functor such that*

$$\sum_{n \in \mathbf{Z}} \dim_k G(E[n]) < \infty$$

*for all  $E \in D_{perf}(\mathcal{O}_X)$ . Then  $G$  is isomorphic to a functor of the form  $E \mapsto \mathrm{Hom}_X(K, E)$  for some  $K \in D_{perf}(\mathcal{O}_X)$ .*

**Proof.** Proof of (1). The derived category  $D_{QCoh}(\mathcal{O}_X)$  has direct sums, is compactly generated, and  $D_{perf}(\mathcal{O}_X)$  is the full subcategory of compact objects, see Derived Categories of Schemes, Lemma 3.1, Theorem 14.3, and Proposition 16.1. By Lemma 6.2 we may assume  $F(E) = \mathrm{Hom}_X(E, K)$  for some  $K \in \mathrm{Ob}(D_{QCoh}(\mathcal{O}_X))$ . Then it follows that  $K$  is in  $D_{Coh}^b(\mathcal{O}_X)$  by Lemma 7.2.

Proof of (2). Consider the contravariant functor  $E \mapsto E^\vee$  on  $D_{perf}(\mathcal{O}_X)$ , see Cohomology, Lemma 46.4. This functor is an exact anti-self-equivalence of  $D_{perf}(\mathcal{O}_X)$ . Hence we may apply part (1) to the functor  $F(E) = G(E^\vee)$  to find  $K \in D_{perf}(\mathcal{O}_X)$  such that  $G(E^\vee) = \mathrm{Hom}_X(E, K)$ . It follows that  $G(E) = \mathrm{Hom}_X(E^\vee, K) = \mathrm{Hom}_X(K^\vee, E)$  and we conclude that taking  $K^\vee$  works.  $\square$

## 8. Existence of adjoints

0FYM As a consequence of the results in the paper of Bondal and van den Bergh we get the following automatic existence of adjoints.

0FYN **Lemma 8.1.** *Let  $k$  be a field. Let  $X$  and  $Y$  be proper schemes over  $k$ . If  $X$  is regular, then  $k$ -linear any exact functor  $F : D_{\text{perf}}(\mathcal{O}_X) \rightarrow D_{\text{perf}}(\mathcal{O}_Y)$  has an exact right adjoint and an exact left adjoint.*

**Proof.** If an adjoint exists it is an exact functor by the very general Derived Categories, Lemma 7.1.

Let us prove the existence of a right adjoint. To see existence, it suffices to show that for  $M \in D_{\text{perf}}(\mathcal{O}_Y)$  the contravariant functor  $K \mapsto \text{Hom}_Y(F(K), M)$  is representable. This functor is contravariant,  $k$ -linear, and cohomological. Hence by Lemma 7.3 part (1) it suffices to show that

$$\sum_{i \in \mathbf{Z}} \dim_k \text{Ext}_Y^i(F(K), M) < \infty$$

This follows from Lemma 5.3.

For the existence of the left adjoint we argue in the same manner using part (2) of Lemma 7.3.  $\square$

### 9. Fourier-Mukai functors

0FYF These functors were first introduced in [Muk81].

0FYQ **Definition 9.1.** Let  $S$  be a scheme. Let  $X$  and  $Y$  be schemes over  $S$ . Let  $K \in D(\mathcal{O}_{X \times_S Y})$ . The exact functor

$$\Phi_K : D(\mathcal{O}_X) \longrightarrow D(\mathcal{O}_Y), \quad M \longmapsto R\text{pr}_{2,*}(L\text{pr}_1^* M \otimes_{\mathcal{O}_{X \times_S Y}}^{\mathbf{L}} K)$$

of triangulated categories is called a *Fourier-Mukai functor* and  $K$  is called a *Fourier-Mukai kernel* for this functor. Moreover,

- (1) if  $\Phi_K$  sends  $D_{\text{QCoh}}(\mathcal{O}_X)$  into  $D_{\text{QCoh}}(\mathcal{O}_Y)$  then the resulting exact functor  $\Phi_K : D_{\text{QCoh}}(\mathcal{O}_X) \rightarrow D_{\text{QCoh}}(\mathcal{O}_Y)$  is called a Fourier-Mukai functor,
- (2) if  $\Phi_K$  sends  $D_{\text{perf}}(\mathcal{O}_X)$  into  $D_{\text{perf}}(\mathcal{O}_Y)$  then the resulting exact functor  $\Phi_K : D_{\text{perf}}(\mathcal{O}_X) \rightarrow D_{\text{perf}}(\mathcal{O}_Y)$  is called a Fourier-Mukai functor, and
- (3) if  $X$  and  $Y$  are Noetherian and  $\Phi_K$  sends  $D_{\text{Coh}}^b(\mathcal{O}_X)$  into  $D_{\text{Coh}}^b(\mathcal{O}_Y)$  then the resulting exact functor  $\Phi_K : D_{\text{Coh}}^b(\mathcal{O}_X) \rightarrow D_{\text{Coh}}^b(\mathcal{O}_Y)$  is called a Fourier-Mukai functor. Similarly for  $D_{\text{Coh}}$ ,  $D_{\text{Coh}}^+$ ,  $D_{\text{Coh}}^-$ .

0FYR **Lemma 9.2.** *Let  $S$  be a scheme. Let  $X$  and  $Y$  be schemes over  $S$ . Let  $K \in D(\mathcal{O}_{X \times_S Y})$ . The corresponding Fourier-Mukai functor  $\Phi_K$  sends  $D_{\text{QCoh}}(\mathcal{O}_X)$  into  $D_{\text{QCoh}}(\mathcal{O}_Y)$  if  $K$  is in  $D_{\text{QCoh}}(\mathcal{O}_{X \times_S Y})$  and  $X \rightarrow S$  is quasi-compact and quasi-separated.*

**Proof.** This follows from the fact that derived pullback preserves  $D_{\text{QCoh}}$  (Derived Categories of Schemes, Lemma 3.8), derived tensor products preserve  $D_{\text{QCoh}}$  (Derived Categories of Schemes, Lemma 3.9), the projection  $\text{pr}_2 : X \times_S Y \rightarrow Y$  is quasi-compact and quasi-separated (Schemes, Lemmas 19.3 and 21.12), and total direct image along a quasi-separated and quasi-compact morphism preserves  $D_{\text{QCoh}}$  (Derived Categories of Schemes, Lemma 4.1).  $\square$

0FYF **Lemma 9.3.** *Let  $S$  be a scheme. Let  $X, Y, Z$  be schemes over  $S$ . Assume  $X \rightarrow S$ ,  $Y \rightarrow S$ , and  $Z \rightarrow S$  are quasi-compact and quasi-separated. Let  $K \in D_{\text{QCoh}}(\mathcal{O}_{X \times_S Y})$ . Let  $K' \in D_{\text{QCoh}}(\mathcal{O}_{Y \times_S Z})$ . Consider the Fourier-Mukai functors*

$\Phi_K : D_{QCoh}(\mathcal{O}_X) \rightarrow D_{QCoh}(\mathcal{O}_Y)$  and  $\Phi_{K'} : D_{QCoh}(\mathcal{O}_Y) \rightarrow D_{QCoh}(\mathcal{O}_Z)$ . If  $X$  and  $Z$  are tor independent over  $S$  and  $Y \rightarrow S$  is flat, then

$$\Phi_{K'} \circ \Phi_K = \Phi_{K''} : D_{QCoh}(\mathcal{O}_X) \longrightarrow D_{QCoh}(\mathcal{O}_Z)$$

where

$$K'' = Rpr_{13,*}(Lpr_{12}^*K \otimes_{\mathcal{O}_{X \times_S Y \times_S Z}}^{\mathbf{L}} Lpr_{23}^*K')$$

in  $D_{QCoh}(\mathcal{O}_{X \times_S Z})$ .

**Proof.** The statement makes sense by Lemma 9.2. We are going to use Derived Categories of Schemes, Lemmas 3.8, 3.9, and 4.1 and Schemes, Lemmas 19.3 and 21.12 without further mention. By Derived Categories of Schemes, Lemma 21.4 we see that  $X \times_S Y$  and  $Y \times_S Z$  are tor independent over  $Y$ . This means that we have base change for the cartesian diagram

$$\begin{array}{ccc} X \times_S Y \times_S Z & \longrightarrow & Y \times_S Z \\ \downarrow & & \downarrow p_Y^{YZ} \\ X \times_S Y & \xrightarrow{p_Y^{XY}} & Y \end{array}$$

for complexes with quasi-coherent cohomology sheaves, see Derived Categories of Schemes, Lemma 21.5. Abbreviating  $p^* = Lp^*$ ,  $p_* = Rp_*$  and  $\otimes = \otimes^{\mathbf{L}}$  we have for  $M \in D_{QCoh}(\mathcal{O}_X)$  the sequence of equalities

$$\begin{aligned} \Phi_{K'}(\Phi_K(M)) &= p_{Z,*}^{YZ}(p_Y^{YZ,*}p_Y^{XY,*}(p_X^{XY,*}M \otimes K) \otimes K') \\ &= p_{Z,*}^{YZ}(\mathrm{pr}_{23,*}\mathrm{pr}_{12}^*(p_X^{XY,*}M \otimes K) \otimes K') \\ &= p_{Z,*}^{YZ}(\mathrm{pr}_{23,*}(\mathrm{pr}_1^*M \otimes \mathrm{pr}_{12}^*K) \otimes K') \\ &= p_{Z,*}^{YZ}(\mathrm{pr}_{23,*}(\mathrm{pr}_1^*M \otimes \mathrm{pr}_{12}^*K \otimes \mathrm{pr}_{23}^*K')) \\ &= \mathrm{pr}_{3,*}(\mathrm{pr}_1^*M \otimes \mathrm{pr}_{12}^*K \otimes \mathrm{pr}_{23}^*K') \\ &= p_{Z,*}^{XZ}\mathrm{pr}_{13,*}(\mathrm{pr}_1^*M \otimes \mathrm{pr}_{12}^*K \otimes \mathrm{pr}_{23}^*K') \\ &= p_{Z,*}^{XZ}(p_X^{XZ,*}M \otimes \mathrm{pr}_{13,*}(\mathrm{pr}_{12}^*K \otimes \mathrm{pr}_{23}^*K')) \end{aligned}$$

as desired. Here we have used the remark on base change in the second equality and we have use Derived Categories of Schemes, Lemma 21.1 in the 4th and last equality.  $\square$

0FYT **Lemma 9.4.** *Let  $S$  be a scheme. Let  $X$  and  $Y$  be schemes over  $S$ . Let  $K \in D(\mathcal{O}_{X \times_S Y})$ . The corresponding Fourier-Mukai functor  $\Phi_K$  sends  $D_{\mathrm{perf}}(\mathcal{O}_X)$  into  $D_{\mathrm{perf}}(\mathcal{O}_Y)$  if at least one of the following conditions is satisfied:*

- (1)  $S$  is Noetherian,  $X \rightarrow S$  and  $Y \rightarrow S$  are of finite type,  $K \in D_{\mathrm{Coh}}^b(\mathcal{O}_{X \times_S Y})$ , the support of  $H^i(K)$  is proper over  $Y$  for all  $i$ , and  $K$  has finite tor dimension as an object of  $D(\mathrm{pr}_2^{-1}\mathcal{O}_Y)$ ,
- (2)  $X \rightarrow S$  is of finite presentation and  $K$  can be represented by a bounded complex  $\mathcal{K}^\bullet$  of finitely presented  $\mathcal{O}_{X \times_S Y}$ -modules, flat over  $Y$ , with support proper over  $Y$ ,
- (3)  $X \rightarrow S$  is a proper flat morphism of finite presentation and  $K$  is perfect,
- (4)  $S$  is Noetherian,  $X \rightarrow S$  is flat and proper, and  $K$  is perfect
- (5)  $X \rightarrow S$  is a proper flat morphism of finite presentation and  $K$  is  $Y$ -perfect,
- (6)  $S$  is Noetherian,  $X \rightarrow S$  is flat and proper, and  $K$  is  $Y$ -perfect.

**Proof.** If  $M$  is perfect on  $X$ , then  $Lpr_1^*M$  is perfect on  $X \times_S Y$ , see Cohomology, Lemma 45.6. We will use this without further mention below. We will also use that if  $X \rightarrow S$  is of finite type, or proper, or flat, or of finite presentation, then the same thing is true for the base change  $pr_2 : X \times_S Y \rightarrow Y$ , see Morphisms, Lemmas 15.4, 40.5, 25.8, and 21.4.

Part (1) follows from Derived Categories of Schemes, Lemma 25.1 combined with Derived Categories of Schemes, Lemma 10.6.

Part (2) follows from Derived Categories of Schemes, Lemma 28.1.

Part (3) follows from Derived Categories of Schemes, Lemma 28.4.

Part (4) follows from part (3) and the fact that a finite type morphism of Noetherian schemes is of finite presentation by Morphisms, Lemma 21.9.

Part (5) follows from Derived Categories of Schemes, Lemma 33.10 combined with Derived Categories of Schemes, Lemma 33.5.

Part (6) follows from part (5) in the same way that part (4) follows from part (3).  $\square$

0FYU **Lemma 9.5.** *Let  $S$  be a Noetherian scheme. Let  $X$  and  $Y$  be schemes of finite type over  $S$ . Let  $K \in D_{Coh}^b(\mathcal{O}_{X \times_S Y})$ . The corresponding Fourier-Mukai functor  $\Phi_K$  sends  $D_{Coh}^b(\mathcal{O}_X)$  into  $D_{Coh}^b(\mathcal{O}_Y)$  if at least one of the following conditions is satisfied:*

- (1) *the support of  $H^i(K)$  is proper over  $Y$  for all  $i$ , and  $K$  has finite tor dimension as an object of  $D(pr_1^{-1}\mathcal{O}_X)$ ,*
- (2)  *$K$  can be represented by a bounded complex  $\mathcal{K}^\bullet$  of coherent  $\mathcal{O}_{X \times_S Y}$ -modules, flat over  $X$ , with support proper over  $Y$ ,*
- (3) *the support of  $H^i(K)$  is proper over  $Y$  for all  $i$  and  $X$  is a regular scheme,*
- (4)  *$K$  is perfect, the support of  $H^i(K)$  is proper over  $Y$  for all  $i$ , and  $Y \rightarrow S$  is flat.*

Furthermore in each case the support condition is automatic if  $X \rightarrow S$  is proper.

**Proof.** Let  $M$  be an object of  $D_{Coh}^b(\mathcal{O}_X)$ . In each case we will use Derived Categories of Schemes, Lemma 10.3 to show that

$$\Phi_K(M) = Rpr_{2,*}(Lpr_1^*M \otimes_{\mathcal{O}_{X \times_S Y}}^{\mathbf{L}} K)$$

is in  $D_{Coh}^b(\mathcal{O}_Y)$ . The derived tensor product  $Lpr_1^*M \otimes_{\mathcal{O}_{X \times_S Y}}^{\mathbf{L}} K$  is a pseudo-coherent object of  $D(\mathcal{O}_{X \times_S Y})$  (by Cohomology, Lemma 43.3, Derived Categories of Schemes, Lemma 9.3, and Cohomology, Lemma 43.5) whence has coherent cohomology sheaves (by Derived Categories of Schemes, Lemma 9.3 again). In each case the supports of the cohomology sheaves  $H^i(Lpr_1^*M \otimes_{\mathcal{O}_{X \times_S Y}}^{\mathbf{L}} K)$  is proper over  $Y$  as these supports are contained in the union of the supports of the  $H^i(K)$ . Hence in each case it suffices to prove that this tensor product is bounded below.

Case (1). By Cohomology, Lemma 27.4 we have

$$Lpr_1^*M \otimes_{\mathcal{O}_{X \times_S Y}}^{\mathbf{L}} K \cong pr_1^{-1}M \otimes_{pr_1^{-1}\mathcal{O}_X}^{\mathbf{L}} K$$

with obvious notation. Hence the assumption on tor dimension and the fact that  $M$  has only a finite number of nonzero cohomology sheaves, implies the bound we want.

Case (2) follows because here the assumption implies that  $K$  has finite tor dimension as an object of  $D(\mathrm{pr}_1^{-1}\mathcal{O}_X)$  hence the argument in the previous paragraph applies.

In Case (3) it is also the case that  $K$  has finite tor dimension as an object of  $D(\mathrm{pr}_1^{-1}\mathcal{O}_X)$ . Namely, choose affine opens  $U = \mathrm{Spec}(A)$  and  $V = \mathrm{Spec}(B)$  of  $X$  and  $Y$  mapping into the affine open  $W = \mathrm{Spec}(R)$  of  $S$ . Then  $K|_{U \times V}$  is given by a bounded complex of finite  $A \otimes_R B$ -modules  $M^\bullet$ . Since  $A$  is a regular ring of finite dimension we see that each  $M^i$  has finite projective dimension as an  $A$ -module (Algebra, Lemma 109.8) and hence finite tor dimension as an  $A$ -module. Thus  $M^\bullet$  has finite tor dimension as a complex of  $A$ -modules (More on Algebra, Lemma 63.8). Since  $X \times Y$  is quasi-compact we conclude there exist  $[a, b]$  such that for every point  $z \in X \times Y$  the stalk  $K_z$  has tor amplitude in  $[a, b]$  over  $\mathcal{O}_{X, \mathrm{pr}_1(z)}$ . This implies  $K$  has bounded tor dimension as an object of  $D(\mathrm{pr}_1^{-1}\mathcal{O}_X)$ , see Cohomology, Lemma 44.5. We conclude as in the previous paragraphs.

Case (4). With notation as above, the ring map  $R \rightarrow B$  is flat. Hence the ring map  $A \rightarrow A \otimes_R B$  is flat. Hence any projective  $A \otimes_R B$ -module is  $A$ -flat. Thus any perfect complex of  $A \otimes_R B$ -modules has finite tor dimension as a complex of  $A$ -modules and we conclude as before.  $\square$

0FYV **Example 9.6.** Let  $X \rightarrow S$  be a separated morphism of schemes. Then the diagonal  $\Delta : X \rightarrow X \times_S X$  is a closed immersion and hence  $\mathcal{O}_\Delta = \Delta_*\mathcal{O}_X = R\Delta_*\mathcal{O}_X$  is a quasi-coherent  $\mathcal{O}_{X \times_S X}$ -module of finite type which is flat over  $X$  (under either projection). The Fourier-Mukai functor  $\Phi_{\mathcal{O}_\Delta}$  is equal to the identity in this case. Namely, if we represent any  $M \in D(\mathcal{O}_X)$  by a K-flat complex  $\mathcal{M}^\bullet$ , then  $\mathrm{pr}_1^{-1}\mathcal{M}^\bullet$  is a K-flat complex of  $\mathrm{pr}_1^{-1}\mathcal{O}_X$ -modules (Cohomology, Lemma 26.7) and hence we have

$$L\mathrm{pr}_1^*M \otimes_{\mathcal{O}_{X \times_S X}}^{\mathbf{L}} \mathcal{O}_\Delta = \mathrm{pr}_1^{-1}\mathcal{M}^\bullet \otimes_{\mathrm{pr}_1^{-1}\mathcal{O}_X}^{\mathbf{L}} \mathcal{O}_\Delta = R\Delta_*(\mathcal{M}^\bullet)$$

The first equality by Cohomology, Lemma 27.4 and the second equality by Cohomology, Lemma 49.4 for the morphism of ringed spaces  $\Delta : (X, \mathcal{O}_X) \rightarrow (X \times_S X, \mathrm{pr}_1^{-1}\mathcal{O}_X)$ . If we push this to  $X$  using  $R\mathrm{pr}_{2,*}$  we obtain  $\mathcal{M}^\bullet$  as desired. Some details omitted.

0FYW **Lemma 9.7.** Let  $X \rightarrow S$  and  $Y \rightarrow S$  be morphisms of quasi-compact and quasi-separated schemes. Let  $\Phi : D_{QCoh}(\mathcal{O}_X) \rightarrow D_{QCoh}(\mathcal{O}_Y)$  be a Fourier-Mukai functor with pseudo-coherent kernel  $K \in D_{QCoh}(\mathcal{O}_{X \times_S Y})$ . Let  $a : D_{QCoh}(\mathcal{O}_Y) \rightarrow D_{QCoh}(\mathcal{O}_{X \times_S Y})$  be the right adjoint to  $R\mathrm{pr}_{2,*}$ , see Duality for Schemes, Lemma 3.1. Denote

$$K' = (Y \times_S X \rightarrow X \times_S Y)^* R\mathrm{Hom}_{\mathcal{O}_{X \times_S Y}}(K, a(\mathcal{O}_Y)) \in D_{QCoh}(\mathcal{O}_{Y \times_S X})$$

and denote  $\Phi' : D_{QCoh}(\mathcal{O}_Y) \rightarrow D_{QCoh}(\mathcal{O}_X)$  the corresponding Fourier-Mukai transform. There is a canonical map

$$\mathrm{Hom}_X(M, \Phi'(N)) \longrightarrow \mathrm{Hom}_Y(\Phi(M), N)$$

functorial in  $M$  in  $D_{QCoh}(\mathcal{O}_X)$  and  $N$  in  $D_{QCoh}(\mathcal{O}_Y)$  which is an isomorphism if

- (1)  $N$  is perfect, or
- (2)  $K$  is perfect and  $X \rightarrow S$  is proper flat and of finite presentation.

**Proof.** By Lemma 9.2 we obtain a functor  $\Phi$  as in the statement. Observe that  $a(\mathcal{O}_Y)$  is in  $D_{QCoh}^+(\mathcal{O}_{X \times_S Y})$  by Duality for Schemes, Lemma 3.5. Hence for  $K$

Compare with discussion in [Riz17].

pseudo-coherent we have  $K' \in D_{QCoh}(\mathcal{O}_{Y \times_S X})$  by Derived Categories of Schemes, Lemma 9.8 we obtain  $\Phi'$  as indicated.

We abbreviate  $\otimes^{\mathbf{L}} = \otimes_{\mathcal{O}_{X \times_S Y}}^{\mathbf{L}}$  and  $\mathcal{H}om = R\mathcal{H}om_{\mathcal{O}_{X \times_S Y}}$ . Let  $M$  be in  $D_{QCoh}(\mathcal{O}_X)$  and let  $N$  be in  $D_{QCoh}(\mathcal{O}_Y)$ . We have

$$\begin{aligned} \mathrm{Hom}_Y(\Phi(M), N) &= \mathrm{Hom}_Y(R\mathrm{pr}_{2,*}(L\mathrm{pr}_1^*M \otimes^{\mathbf{L}} K), N) \\ &= \mathrm{Hom}_{X \times_S Y}(L\mathrm{pr}_1^*M \otimes^{\mathbf{L}} K, a(N)) \\ &= \mathrm{Hom}_{X \times_S Y}(L\mathrm{pr}_1^*M, R\mathcal{H}om(K, a(N))) \\ &= \mathrm{Hom}_X(M, R\mathrm{pr}_{1,*}R\mathcal{H}om(K, a(N))) \end{aligned}$$

where we have used Cohomology, Lemmas 38.2 and 28.1. There are canonical maps

$$L\mathrm{pr}_2^*N \otimes^{\mathbf{L}} R\mathcal{H}om(K, a(\mathcal{O}_Y)) \xrightarrow{\alpha} R\mathcal{H}om(K, L\mathrm{pr}_2^*N \otimes^{\mathbf{L}} a(\mathcal{O}_Y)) \xrightarrow{\beta} R\mathcal{H}om(K, a(N))$$

Here  $\alpha$  is Cohomology, Lemma 38.6 and  $\beta$  is Duality for Schemes, Equation (8.0.1). Combining all of these arrows we obtain the functorial displayed arrow in the statement of the lemma.

The arrow  $\alpha$  is an isomorphism by Derived Categories of Schemes, Lemma 9.9 as soon as either  $K$  or  $N$  is perfect. The arrow  $\beta$  is an isomorphism if  $N$  is perfect by Duality for Schemes, Lemma 8.1 or in general if  $X \rightarrow S$  is flat proper of finite presentation by Duality for Schemes, Lemma 12.3.  $\square$

**OFYX Lemma 9.8.** *Let  $S$  be a Noetherian scheme. Let  $Y \rightarrow S$  be a flat proper Gorenstein morphism and let  $X \rightarrow S$  be a finite type morphism. Denote  $\omega_{Y/S}^\bullet$  the relative dualizing complex of  $Y$  over  $S$ . Let  $\Phi : D_{QCoh}(\mathcal{O}_X) \rightarrow D_{QCoh}(\mathcal{O}_Y)$  be a Fourier-Mukai functor with perfect kernel  $K \in D_{QCoh}(\mathcal{O}_{X \times_S Y})$ . Denote* Compare with discussion in [Riz17].

$$K' = (Y \times_S X \rightarrow X \times_S Y)^*(K^\vee \otimes_{\mathcal{O}_{X \times_S Y}}^{\mathbf{L}} L\mathrm{pr}_2^*\omega_{Y/S}^\bullet) \in D_{QCoh}(\mathcal{O}_{Y \times_S X})$$

and denote  $\Phi' : D_{QCoh}(\mathcal{O}_Y) \rightarrow D_{QCoh}(\mathcal{O}_X)$  the corresponding Fourier-Mukai transform. There is a canonical isomorphism

$$\mathrm{Hom}_Y(N, \Phi(M)) \longrightarrow \mathrm{Hom}_X(\Phi'(N), M)$$

functorial in  $M$  in  $D_{QCoh}(\mathcal{O}_X)$  and  $N$  in  $D_{QCoh}(\mathcal{O}_Y)$ .

**Proof.** By Lemma 9.2 we obtain a functor  $\Phi$  as in the statement.

Observe that formation of the relative dualizing complex commutes with base change in our setting, see Duality for Schemes, Remark 12.5. Thus  $L\mathrm{pr}_2^*\omega_{Y/S}^\bullet = \omega_{X \times_S Y/X}^\bullet$ . Moreover, we observe that  $\omega_{Y/S}^\bullet$  is an invertible object of the derived category, see Duality for Schemes, Lemma 25.10, and a fortiori perfect.

To actually prove the lemma we're going to cheat. Namely, we will show that if we replace the roles of  $X$  and  $Y$  and  $K$  and  $K'$  then these are as in Lemma 9.7 and we get the result. It is clear that  $K'$  is perfect as a tensor product of perfect objects so that the discussion in Lemma 9.7 applies to it. To show that the procedure of Lemma 9.7 applied to  $K'$  on  $Y \times_S X$  produces a complex isomorphic to  $K$  it suffices (details omitted) to show that

$$R\mathcal{H}om(R\mathcal{H}om(K, \omega_{X \times_S Y/X}^\bullet), \omega_{X \times_S Y/X}^\bullet) = K$$

This is clear because  $K$  is perfect and  $\omega_{X \times_S Y/X}^\bullet$  is invertible; details omitted. Thus Lemma 9.7 produces a map

$$\mathrm{Hom}_Y(N, \Phi(M)) \longrightarrow \mathrm{Hom}_X(\Phi'(N), M)$$

functorial in  $M$  in  $D_{QCoh}(\mathcal{O}_X)$  and  $N$  in  $D_{QCoh}(\mathcal{O}_Y)$  which is an isomorphism because  $K'$  is perfect. This finishes the proof.  $\square$

0FYF **Lemma 9.9.** *Let  $S$  be a Noetherian scheme.*

- (1) *For  $X, Y$  proper and flat over  $S$  and  $K$  in  $D_{perf}(\mathcal{O}_{X \times_S Y})$  we obtain a Fourier-Mukai functor  $\Phi_K : D_{perf}(\mathcal{O}_X) \rightarrow D_{perf}(\mathcal{O}_Y)$ .*
- (2) *For  $X, Y, Z$  proper and flat over  $S$ ,  $K \in D_{perf}(\mathcal{O}_{X \times_S Y})$ ,  $K' \in D_{perf}(\mathcal{O}_{Y \times_S Z})$  the composition  $\Phi_{K'} \circ \Phi_K : D_{perf}(\mathcal{O}_X) \rightarrow D_{perf}(\mathcal{O}_Z)$  is equal to  $\Phi_{K''}$  with  $K'' \in D_{perf}(\mathcal{O}_{X \times_S Z})$  computed as in Lemma 9.3,*
- (3) *For  $X, Y, K, \Phi_K$  as in (1) if  $X \rightarrow S$  is Gorenstein, then  $\Phi_{K'} : D_{perf}(\mathcal{O}_Y) \rightarrow D_{perf}(\mathcal{O}_X)$  is a right adjoint to  $\Phi_K$  where  $K' \in D_{perf}(\mathcal{O}_{Y \times_S X})$  is the pull-back of  $Lpr_1^* \omega_{X/S}^\bullet \otimes_{\mathcal{O}_{X \times_S Y}}^\mathbf{L} K^\vee$  by  $Y \times_S X \rightarrow X \times_S Y$ .*
- (4) *For  $X, Y, K, \Phi_K$  as in (1) if  $Y \rightarrow S$  is Gorenstein, then  $\Phi_{K''} : D_{perf}(\mathcal{O}_Y) \rightarrow D_{perf}(\mathcal{O}_X)$  is a left adjoint to  $\Phi_K$  where  $K'' \in D_{perf}(\mathcal{O}_{Y \times_S X})$  is the pull-back of  $Lpr_2^* \omega_{Y/S}^\bullet \otimes_{\mathcal{O}_{X \times_S Y}}^\mathbf{L} K^\vee$  by  $Y \times_S X \rightarrow X \times_S Y$ .*

**Proof.** Part (1) is immediate from Lemma 9.4 part (4).

Part (2) follows from Lemma 9.3 and the fact that  $K'' = Rpr_{13,*} (Lpr_{12}^* K \otimes_{\mathcal{O}_{X \times_S Y \times_S Z}}^\mathbf{L} Lpr_{23}^* K')$  is perfect for example by Derived Categories of Schemes, Lemma 25.4.

The adjointness in part (3) on all complexes with quasi-coherent cohomology sheaves follows from Lemma 9.7 with  $K'$  equal to the pullback of  $R\mathcal{H}om_{\mathcal{O}_{X \times_S Y}}(K, a(\mathcal{O}_Y))$  by  $Y \times_S X \rightarrow X \times_S Y$  where  $a$  is the right adjoint to  $Rpr_{2,*} : D_{QCoh}(\mathcal{O}_{X \times_S Y}) \rightarrow D_{QCoh}(\mathcal{O}_Y)$ . Denote  $f : X \rightarrow S$  the structure morphism of  $X$ . Since  $f$  is proper the functor  $f^! : D_{QCoh}^+(\mathcal{O}_S) \rightarrow D_{QCoh}^+(\mathcal{O}_X)$  is the restriction to  $D_{QCoh}^+(\mathcal{O}_S)$  of the right adjoint to  $Rf_* : D_{QCoh}(\mathcal{O}_X) \rightarrow D_{QCoh}(\mathcal{O}_S)$ , see Duality for Schemes, Section 16. Hence the relative dualizing complex  $\omega_{X/S}^\bullet$  as defined in Duality for Schemes, Remark 12.5 is equal to  $\omega_{X/S}^\bullet = f^! \mathcal{O}_S$ . Since formation of the relative dualizing complex commutes with base change (see Duality for Schemes, Remark 12.5) we see that  $a(\mathcal{O}_Y) = Lpr_1^* \omega_{X/S}^\bullet$ . Thus

$$R\mathcal{H}om_{\mathcal{O}_{X \times_S Y}}(K, a(\mathcal{O}_Y)) \cong Lpr_1^* \omega_{X/S}^\bullet \otimes_{\mathcal{O}_{X \times_S Y}}^\mathbf{L} K^\vee$$

by Cohomology, Lemma 46.4. Finally, since  $X \rightarrow S$  is assumed Gorenstein the relative dualizing complex is invertible: this follows from Duality for Schemes, Lemma 25.10. We conclude that  $\omega_{X/S}^\bullet$  is perfect (Cohomology, Lemma 47.2) and hence  $K'$  is perfect. Therefore  $\Phi_{K'}$  does indeed map  $D_{perf}(\mathcal{O}_Y)$  into  $D_{perf}(\mathcal{O}_X)$  which finishes the proof of (3).

The proof of (4) is the same as the proof of (3) except one uses Lemma 9.8 instead of Lemma 9.7.  $\square$

## 10. Resolutions and bounds

0FYZ The diagonal of a smooth proper scheme has a nice resolution.



0FZ0 **Lemma 10.1.** *Let  $R$  be a Noetherian ring. Let  $X, Y$  be finite type schemes over  $R$  having the resolution property. For any coherent  $\mathcal{O}_{X \times_R Y}$ -module  $\mathcal{F}$  there exist a surjection  $\mathcal{E} \boxtimes \mathcal{G} \rightarrow \mathcal{F}$  where  $\mathcal{E}$  is a finite locally free  $\mathcal{O}_X$ -module and  $\mathcal{G}$  is a finite locally free  $\mathcal{O}_Y$ -module.*

**Proof.** Let  $U \subset X$  and  $V \subset Y$  be affine open subschemes. Let  $\mathcal{I} \subset \mathcal{O}_X$  be the ideal sheaf of the reduced induced closed subscheme structure on  $X \setminus U$ . Similarly, let  $\mathcal{I}' \subset \mathcal{O}_Y$  be the ideal sheaf of the reduced induced closed subscheme structure on  $Y \setminus V$ . Then the ideal sheaf

$$\mathcal{J} = \text{Im}(\text{pr}_1^* \mathcal{I} \otimes_{\mathcal{O}_{X \times_R Y}} \text{pr}_2^* \mathcal{I}' \rightarrow \mathcal{O}_{X \times_R Y})$$

satisfies  $V(\mathcal{J}) = X \times_R Y \setminus U \times_R V$ . For any section  $s \in \mathcal{F}(U \times_R V)$  we can find an integer  $n > 0$  and a map  $\mathcal{J}^n \rightarrow \mathcal{F}$  whose restriction to  $U \times_R V$  gives  $s$ , see Cohomology of Schemes, Lemma 10.4. By assumption we can choose surjections  $\mathcal{E} \rightarrow \mathcal{I}$  and  $\mathcal{G} \rightarrow \mathcal{I}'$ . These produce corresponding surjections

$$\mathcal{E} \boxtimes \mathcal{G} \rightarrow \mathcal{J} \quad \text{and} \quad \mathcal{E}^{\otimes n} \boxtimes \mathcal{G}^{\otimes n} \rightarrow \mathcal{J}^n$$

and hence a map  $\mathcal{E}^{\otimes n} \boxtimes \mathcal{G}^{\otimes n} \rightarrow \mathcal{F}$  whose image contains the section  $s$  over  $U \times_R V$ . Since we can cover  $X \times_R Y$  by a finite number of affine opens of the form  $U \times_R V$  and since  $\mathcal{F}|_{U \times_R V}$  is generated by finitely many sections (Properties, Lemma 16.1) we conclude that there exists a surjection

$$\bigoplus_{j=1, \dots, N} \mathcal{E}_j^{\otimes n_j} \boxtimes \mathcal{G}_j^{\otimes n_j} \rightarrow \mathcal{F}$$

where  $\mathcal{E}_j$  is finite locally free on  $X$  and  $\mathcal{G}_j$  is finite locally free on  $Y$ . Setting  $\mathcal{E} = \bigoplus \mathcal{E}_j^{\otimes n_j}$  and  $\mathcal{G} = \bigoplus \mathcal{G}_j^{\otimes n_j}$  we conclude that the lemma is true.  $\square$

0FZ1 **Lemma 10.2.** *Let  $R$  be a ring. Let  $X, Y$  be quasi-compact and quasi-separated schemes over  $R$  having the resolution property. For any finite type quasi-coherent  $\mathcal{O}_{X \times_R Y}$ -module  $\mathcal{F}$  there exist a surjection  $\mathcal{E} \boxtimes \mathcal{G} \rightarrow \mathcal{F}$  where  $\mathcal{E}$  is a finite locally free  $\mathcal{O}_X$ -module and  $\mathcal{G}$  is a finite locally free  $\mathcal{O}_Y$ -module.*

**Proof.** Follows from Lemma 10.1 by a limit argument. We urge the reader to skip the proof. Since  $X \times_R Y$  is a closed subscheme of  $X \times_{\mathbf{Z}} Y$  it is harmless if we replace  $R$  by  $\mathbf{Z}$ . We can write  $\mathcal{F}$  as the quotient of a finitely presented  $\mathcal{O}_{X \times_R Y}$ -module by Properties, Lemma 22.7. Hence we may assume  $\mathcal{F}$  is of finite presentation. Next we can write  $X = \lim X_i$  with  $X_i$  of finite presentation over  $\mathbf{Z}$  and similarly  $Y = \lim Y_j$ , see Limits, Proposition 5.4. Then  $\mathcal{F}$  will descend to  $\mathcal{F}_{ij}$  on some  $X_i \times_R Y_j$  (Limits, Lemma 10.2) and so does the property of having the resolution property (Derived Categories of Schemes, Lemma 34.7). Then we apply Lemma 10.1 to  $\mathcal{F}_{ij}$  and we pullback.  $\square$

0FZ2 **Lemma 10.3.** *Let  $R$  be a Noetherian ring. Let  $X$  be a separated finite type scheme over  $R$  which has the resolution property. Set  $\mathcal{O}_\Delta = \Delta_*(\mathcal{O}_X)$  where  $\Delta : X \rightarrow X \times_R X$  is the diagonal of  $X/k$ . There exists a resolution*

$$\dots \rightarrow \mathcal{E}_2 \boxtimes \mathcal{G}_2 \rightarrow \mathcal{E}_1 \boxtimes \mathcal{G}_1 \rightarrow \mathcal{E}_0 \boxtimes \mathcal{G}_0 \rightarrow \mathcal{O}_\Delta \rightarrow 0$$

where each  $\mathcal{E}_i$  and  $\mathcal{G}_i$  is a finite locally free  $\mathcal{O}_X$ -module.

**Proof.** Since  $X$  is separated, the diagonal morphism  $\Delta$  is a closed immersion and hence  $\mathcal{O}_\Delta$  is a coherent  $\mathcal{O}_{X \times_R X}$ -module (Cohomology of Schemes, Lemma 9.8). Thus the lemma follows immediately from Lemma 10.1.  $\square$

0FZ3 **Lemma 10.4.** *Let  $X$  be a regular Noetherian scheme of dimension  $d < \infty$ . Then*

- (1) *for  $\mathcal{F}, \mathcal{G}$  coherent  $\mathcal{O}_X$ -modules we have  $\mathrm{Ext}_X^n(\mathcal{F}, \mathcal{G}) = 0$  for  $n > d$ , and*
- (2) *for  $K, L \in D_{\mathrm{Coh}}^b(\mathcal{O}_X)$  and  $a \in \mathbf{Z}$  if  $H^i(K) = 0$  for  $i < a + d$  and  $H^i(L) = 0$  for  $i \geq a$  then  $\mathrm{Hom}_X(K, L) = 0$ .*

**Proof.** To prove (1) we use the spectral sequence

$$H^p(X, \mathcal{E}xt^q(\mathcal{F}, \mathcal{G})) \Rightarrow \mathrm{Ext}_X^{p+q}(\mathcal{F}, \mathcal{G})$$

of Cohomology, Section 39. Recall that taking  $\mathcal{E}xt$  of coherent modules agrees with the algebra construction affine locally, see Derived Categories of Schemes, Lemma 9.8. Let  $x \in X$  be a point. A regular local ring is Gorenstein, see Dualizing Complexes, Lemma 21.3. Hence  $\mathcal{O}_{X,x}[d_x]$  where  $d_x = \dim(\mathcal{O}_{X,x})$  is a normalized dualizing complex for  $\mathcal{O}_{X,x}$ . It follows that the modules  $\mathcal{E}xt^q(\mathcal{F}, \mathcal{G})$  have support of dimension at most  $d - q$ , see Dualizing Complexes, Lemma 16.5. Hence we have  $H^p(X, \mathcal{E}xt^q(\mathcal{F}, \mathcal{G})) = 0$  for  $p > d - q$  by Cohomology, Proposition 20.7. This proves (1).

Proof of (2). We may use induction on the number of nonzero cohomology sheaves of  $K$  and  $L$ . The case where these numbers are 0, 1 follows from (1). If the number of nonzero cohomology sheaves of  $K$  is  $> 1$ , then we let  $i \in \mathbf{Z}$  be minimal such that  $H^i(K)$  is nonzero. We obtain a distinguished triangle

$$H^i(K)[-i] \rightarrow K \rightarrow \tau_{\geq i+1}K$$

(Derived Categories, Remark 12.4) and we get the vanishing of  $\mathrm{Hom}(K, L)$  from the vanishing of  $\mathrm{Hom}(H^i(K)[-i], L)$  and  $\mathrm{Hom}(\tau_{\geq i+1}K, L)$  by Derived Categories, Lemma 4.2. Similarly if  $L$  has more than one nonzero cohomology sheaf.  $\square$

0FZ4 **Lemma 10.5.** *Let  $X$  be a regular Noetherian scheme of dimension  $d < \infty$ . Let  $K \in D_{\mathrm{Coh}}^b(\mathcal{O}_X)$  and  $a \in \mathbf{Z}$ . If  $H^i(K) = 0$  for  $a < i < a + d$ , then  $K = \tau_{\leq a}K \oplus \tau_{\geq a+d}K$ .*

**Proof.** We have  $\tau_{\leq a}K = \tau_{\leq a+d-1}K$  by the assumed vanishing of cohomology sheaves. By Derived Categories, Remark 12.4 we have a distinguished triangle

$$\tau_{\leq a}K \rightarrow K \rightarrow \tau_{\geq a+d}K \xrightarrow{\delta} (\tau_{\leq a}K)[1]$$

By Derived Categories, Lemma 4.11 it suffices to show that the morphism  $\delta$  is zero. This follows from Lemma 10.4.  $\square$

0FZ5 **Lemma 10.6.** *Let  $k$  be a field. Let  $X$  be a quasi-compact separated smooth scheme over  $k$ . There exist finite locally free  $\mathcal{O}_X$ -modules  $\mathcal{E}$  and  $\mathcal{G}$  such that*

$$\mathcal{O}_\Delta \in \langle \mathcal{E} \boxtimes \mathcal{G} \rangle$$

*in  $D(\mathcal{O}_{X \times X})$  where the notation is as in Derived Categories, Section 36.*

**Proof.** Recall that  $X$  is regular by Varieties, Lemma 25.3. Hence  $X$  has the resolution property by Derived Categories of Schemes, Lemma 34.6. Hence we may choose a resolution as in Lemma 10.3. Say  $\dim(X) = d$ . Since  $X \times X$  is smooth over  $k$  it is regular. Hence  $X \times X$  is a regular Noetherian scheme with  $\dim(X \times X) = 2d$ . The object

$$K = (\mathcal{E}_{2d} \boxtimes \mathcal{G}_{2d} \rightarrow \dots \rightarrow \mathcal{E}_0 \boxtimes \mathcal{G}_0)$$

of  $D_{\text{perf}}(\mathcal{O}_{X \times X})$  has cohomology sheaves  $\mathcal{O}_\Delta$  in degree 0 and  $\text{Ker}(\mathcal{E}_{2d} \boxtimes \mathcal{G}_{2d} \rightarrow \mathcal{E}_{2d} \boxtimes \mathcal{G}_{2d})$  in degree  $-2d$  and zero in all other degrees. Hence by Lemma 10.5 we see that  $\mathcal{O}_\Delta$  is a summand of  $K$  in  $D_{\text{perf}}(\mathcal{O}_{X \times X})$ . Clearly, the object  $K$  is in

$$\left\langle \bigoplus_{i,j=0,\dots,-2d} \mathcal{E}_i \boxtimes \mathcal{G}_i \right\rangle \subset \left\langle \left( \bigoplus_{i,j=0,\dots,-2d} \mathcal{E}_i \right) \boxtimes \left( \bigoplus_{i,j=0,\dots,-2d} \mathcal{G}_i \right) \right\rangle$$

which finishes the proof. (The reader may consult Derived Categories, Lemmas 36.1 and 35.7 to see that our object is contained in this category.)  $\square$

0FZ6 **Lemma 10.7.** *Let  $k$  be a field. Let  $X$  be a scheme proper and smooth over  $k$ . Then  $D_{\text{perf}}(\mathcal{O}_X)$  has a strong generator.*

**Proof.** Using Lemma 10.6 choose finite locally free  $\mathcal{O}_X$ -modules  $\mathcal{E}$  and  $\mathcal{G}$  such that  $\mathcal{O}_\Delta \in \langle \mathcal{E} \boxtimes \mathcal{G} \rangle$  in  $D(\mathcal{O}_{X \times X})$ . We claim that  $\mathcal{G}$  is a strong generator for  $D_{\text{perf}}(\mathcal{O}_X)$ . With notation as in Derived Categories, Section 35 choose  $m, n \geq 1$  such that

$$\mathcal{O}_\Delta \in \text{smd}(\text{add}(\mathcal{E} \boxtimes \mathcal{G}[-m, m])^{*n})$$

This is possible by Derived Categories, Lemma 36.2. Let  $K$  be an object of  $D_{\text{perf}}(\mathcal{O}_X)$ . Since  $L\text{pr}_1^* K \otimes_{\mathcal{O}_{X \times Y}}^{\mathbf{L}} -$  is an exact functor and since

$$L\text{pr}_1^* K \otimes_{\mathcal{O}_{X \times Y}}^{\mathbf{L}} (\mathcal{E} \boxtimes \mathcal{G}) = (K \otimes_{\mathcal{O}_X}^{\mathbf{L}} \mathcal{E}) \boxtimes \mathcal{G}$$

we conclude from Derived Categories, Remark 35.5 that

$$L\text{pr}_1^* K \otimes_{\mathcal{O}_{X \times Y}}^{\mathbf{L}} \mathcal{O}_\Delta \in \text{smd}(\text{add}((K \otimes_{\mathcal{O}_X}^{\mathbf{L}} \mathcal{E}) \boxtimes \mathcal{G}[-m, m])^{*n})$$

Applying the exact functor  $R\text{pr}_{2,*}$  and observing that

$$R\text{pr}_{2,*}((K \otimes_{\mathcal{O}_X}^{\mathbf{L}} \mathcal{E}) \boxtimes \mathcal{G}) = R\Gamma(X, K \otimes_{\mathcal{O}_X}^{\mathbf{L}} \mathcal{E}) \otimes_k \mathcal{G}$$

by Derived Categories of Schemes, Lemma 21.1 we conclude that

$$K = R\text{pr}_{2,*}(L\text{pr}_1^* K \otimes_{\mathcal{O}_{X \times Y}}^{\mathbf{L}} \mathcal{O}_\Delta) \in \text{smd}(\text{add}(R\Gamma(X, K \otimes_{\mathcal{O}_X}^{\mathbf{L}} \mathcal{E}) \otimes_k \mathcal{G}[-m, m])^{*n})$$

The equality follows from the discussion in Example 9.6. Since  $K$  is perfect, there exist  $a \leq b$  such that  $H^i(X, K)$  is nonzero only for  $i \in [a, b]$ . We conclude that the right hand side is contained in  $\text{smd}(\text{add}(\mathcal{G}[-m+a, m+b])^{*n})$  which is itself contained in  $\langle \mathcal{G} \rangle_n$  by one of the references given above. This finishes the proof.  $\square$

0FZ7 **Lemma 10.8.** *Let  $k$  be a field. Let  $X$  be a proper smooth scheme over  $k$ . There exists integers  $m, n \geq 1$  and a finite locally free  $\mathcal{O}_X$ -module  $\mathcal{G}$  such that every coherent  $\mathcal{O}_X$ -module is contained in  $\text{smd}(\text{add}(\mathcal{G}[-m, m])^{*n})$  with notation as in Derived Categories, Section 35.*

**Proof.** In the proof of Lemma 10.7 we have shown that there exist  $m', n \geq 1$  such that

$$\mathcal{F} \in \text{smd}(\text{add}(\mathcal{G}[-m' + a, m' + b])^{*n})$$

for any  $a \leq b$  such that  $H^i(X, \mathcal{F})$  is zero only for  $i \in [a, b]$ . Thus we can take  $a = 0$  and  $b = \dim(X)$ . Taking  $m = \max(m', m' + b)$  finishes the proof.  $\square$

The following lemma is the boundedness result referred to in the title of this section.

0FZ8 **Lemma 10.9.** *Let  $k$  be a field. Let  $X$  be a smooth proper scheme over  $k$ . Let  $\mathcal{A}$  be an abelian category. Let  $H : D_{\text{perf}}(\mathcal{O}_X) \rightarrow \mathcal{A}$  be a homological functor (Derived Categories, Definition 3.5) such that for all  $K$  in  $D_{\text{perf}}(\mathcal{O}_X)$  the object  $H^i(K)$  is nonzero for only a finite number of  $i \in \mathbf{Z}$ . Then there exists an integer  $m \geq 1$  such that  $H^i(F(\mathcal{F})) = 0$  for  $i \notin [-m, m]$ . Similarly for cohomological functors.*

**Proof.** Combine Lemma 10.8 with Derived Categories, Lemma 35.8.  $\square$

0FZ9 **Lemma 10.10.** *Let  $k$  be a field. Let  $X, Y$  be finite type schemes over  $k$ . Let  $K_0 \rightarrow K_1 \rightarrow K_2 \rightarrow \dots$  be a system of objects of  $D_{\text{perf}}(\mathcal{O}_{X \times Y})$  and  $m \geq 0$  an integer such that*

- (1)  $H^q(K_i)$  is nonzero only for  $q \leq m$ ,
- (2) for every coherent  $\mathcal{O}_X$ -module  $\mathcal{F}$  with  $\dim(\text{Supp}(\mathcal{F})) = 0$  the object

$$R\text{pr}_{2,*}(\text{pr}_1^* \mathcal{F} \otimes_{\mathcal{O}_{X \times Y}}^{\mathbf{L}} K_n)$$

*has vanishing cohomology sheaves in degrees outside  $[-m, m] \cup [-m-n, m-n]$  and for  $n > 2m$  the transition maps induce isomorphisms on cohomology sheaves in degrees in  $[-m, m]$ .*

*Then  $K_n$  has vanishing cohomology sheaves in degrees outside  $[-m, m] \cup [-m-n, m-n]$  and for  $n > 2m$  the transition maps induce isomorphisms on cohomology sheaves in degrees in  $[-m, m]$ . Moreover, if  $X$  and  $Y$  are smooth over  $k$ , then for  $n$  large enough we find  $K_n = K \oplus C_n$  in  $D_{\text{perf}}(\mathcal{O}_{X \times Y})$  where  $K$  has cohomology only in degrees  $[-m, m]$  and  $C_n$  only in degrees  $[-m-n, m-n]$  and the transition maps define isomorphisms between various copies of  $K$ .*

**Proof.** Let  $Z$  be the scheme theoretic support of an  $\mathcal{F}$  as in (2). Then  $Z \rightarrow \text{Spec}(k)$  is finite, hence  $Z \times Y \rightarrow Y$  is finite. It follows that for an object  $M$  of  $D_{\text{QCoh}}(\mathcal{O}_{X \times Y})$  with cohomology sheaves supported on  $Z \times Y$  we have  $H^i(R\text{pr}_{2,*}(M)) = \text{pr}_{2,*}H^i(M)$  and the functor  $\text{pr}_{2,*}$  is faithful on quasi-coherent modules supported on  $Z \times Y$ ; details omitted. Hence we see that the objects

$$\text{pr}_1^* \mathcal{F} \otimes_{\mathcal{O}_{X \times Y}}^{\mathbf{L}} K_n$$

in  $D_{\text{perf}}(\mathcal{O}_{X \times Y})$  have vanishing cohomology sheaves outside  $[-m, m] \cup [-m-n, m-n]$  and for  $n > 2m$  the transition maps induce isomorphisms on cohomology sheaves in  $[-m, m]$ . Let  $z \in X \times Y$  be a closed point mapping to the closed point  $x \in X$ . Then we know that

$$K_{n,z} \otimes_{\mathcal{O}_{X \times Y,z}}^{\mathbf{L}} \mathcal{O}_{X \times Y,z} / \mathfrak{m}_x^t \mathcal{O}_{X \times Y,z}$$

has nonzero cohomology only in the intervals  $[-m, m] \cup [-m-n, m-n]$ . We conclude by More on Algebra, Lemma 93.2 that  $K_{n,z}$  only has nonzero cohomology in degrees  $[-m, m] \cup [-m-n, m-n]$ . Since this holds for all closed points of  $X \times Y$ , we conclude  $K_n$  only has nonzero cohomology sheaves in degrees  $[-m, m] \cup [-m-n, m-n]$ . In exactly the same way we see that the maps  $K_n \rightarrow K_{n+1}$  are isomorphisms on cohomology sheaves in degrees  $[-m, m]$  for  $n > 2m$ .

If  $X$  and  $Y$  are smooth over  $k$ , then  $X \times Y$  is smooth over  $k$  and hence regular by Varieties, Lemma 25.3. Thus we will obtain the direct sum decomposition of  $K_n$  as soon as  $n > 2m + \dim(X \times Y)$  from Lemma 10.5. The final statement is clear from this.  $\square$

## 11. Functors between categories of quasi-coherent modules

0FZA In this section we briefly study functors between categories of quasi-coherent modules.

0FZB **Example 11.1.** Let  $R$  be a ring. Let  $X$  and  $Y$  be schemes over  $R$  with  $X$  quasi-compact and quasi-separated. Let  $\mathcal{K}$  be a quasi-coherent  $\mathcal{O}_{X \times_R Y}$ -module. Then we can consider the functor

$$0FZC \quad (11.1.1) \quad F : QCoh(\mathcal{O}_X) \longrightarrow QCoh(\mathcal{O}_Y), \quad \mathcal{F} \longmapsto \mathrm{pr}_{2,*}(\mathrm{pr}_1^* \mathcal{F} \otimes_{\mathcal{O}_{X \times_R Y}} \mathcal{K})$$

The morphism  $\mathrm{pr}_2$  is quasi-compact and quasi-separated (Schemes, Lemmas 19.3 and 21.12). Hence pushforward along this morphism preserves quasi-coherent modules, see Schemes, Lemma 24.1. Moreover, our functor is  $R$ -linear and commutes with arbitrary direct sums, see Cohomology of Schemes, Lemma 6.1.

0FZD **Lemma 11.2.** *Let  $R$  be a ring. Let  $X$  and  $Y$  be schemes over  $R$  with  $X$  affine. There is an equivalence of categories between*

- (1) *the category of  $R$ -linear functors  $F : QCoh(\mathcal{O}_X) \rightarrow QCoh(\mathcal{O}_Y)$  which are right exact and commute with arbitrary direct sums, and*
- (2) *the category  $QCoh(\mathcal{O}_{X \times_R Y})$*

*given by sending  $\mathcal{K}$  to the functor  $F$  in (11.1.1).*

**Proof.** First we observe that since  $\mathrm{pr}_2 : X \times_R Y \rightarrow Y$  is affine (Morphisms, Lemma 11.8) the functor  $\mathrm{pr}_{2,*}$  is exact (see for example Cohomology of Schemes, Lemma 2.3). Hence the functor (11.1.1) is right exact in this case.

Let us construct the quasi-inverse to the construction. Let  $F$  be as in (1). Say  $X = \mathrm{Spec}(A)$ . Consider the quasi-coherent  $\mathcal{O}_Y$ -module  $\mathcal{G} = F(\mathcal{O}_X)$ . Every element  $a \in A$  induces an endomorphism of  $\mathcal{G}$  and this defines an  $R$ -linear map  $A \rightarrow \mathrm{End}_{\mathcal{O}_Y}(\mathcal{G})$ . Hence we see that  $\mathcal{G}$  is a sheaf of modules over

$$A \otimes_R \mathcal{O}_Y = \mathrm{pr}_{2,*} \mathcal{O}_{X \times_R Y}$$

By Morphisms, Lemma 11.6 we find that there is a unique quasi-coherent module  $\mathcal{K}$  on  $X \times_R Y$  such that  $\mathcal{G} = \mathrm{pr}_{2,*} \mathcal{K}$  compatible with action of  $A$  and  $\mathcal{O}_Y$ . Commutation with direct sums shows that  $F(\bigoplus_{i \in I} \mathcal{O}_X) = \bigoplus_{i \in I} \mathcal{G}$ . Finally, since  $X = \mathrm{Spec}(A)$  for every quasi-coherent  $\mathcal{O}_X$ -module  $\mathcal{F}$  we can choose an exact sequence

$$\bigoplus_{j \in J} \mathcal{O}_X \rightarrow \bigoplus_{i \in I} \mathcal{O}_X \rightarrow \mathcal{F} \rightarrow 0$$

This leads to an exact sequence

$$\bigoplus_{j \in J} \mathcal{K} \rightarrow \bigoplus_{i \in I} \mathcal{K} \rightarrow \mathrm{pr}_1^* \mathcal{F} \otimes_{\mathcal{O}_{X \times_R Y}} \mathcal{K} \rightarrow 0$$

which using the exact functor  $\mathrm{pr}_{2,*}$  gives the exact sequence

$$\bigoplus_{j \in J} \mathcal{G} \rightarrow \bigoplus_{i \in I} \mathcal{G} \rightarrow \mathrm{pr}_{2,*}(\mathrm{pr}_1^* \mathcal{F} \otimes_{\mathcal{O}_{X \times_R Y}} \mathcal{K}) \rightarrow 0$$

which as  $F$  commutes with direct sums we may rewrite as

$$F(\bigoplus_{j \in J} \mathcal{O}_X) \rightarrow F(\bigoplus_{i \in I} \mathcal{O}_X) \rightarrow \mathrm{pr}_{2,*}(\mathrm{pr}_1^* \mathcal{F} \otimes_{\mathcal{O}_{X \times_R Y}} \mathcal{K}) \rightarrow 0$$

By right exactness of  $F$  we conclude  $F$  is isomorphic to the functor (11.1.1).  $\square$

0FZE **Remark 11.3.** Below we will use that for an affine morphism  $h : T \rightarrow S$  we have  $h_* \mathcal{G} \otimes \mathcal{H} = h_*(\mathcal{G} \otimes h^* \mathcal{H})$  for  $\mathcal{G} \in QCoh(\mathcal{O}_T)$  and  $\mathcal{H} \in QCoh(\mathcal{O}_S)$ . This follows immediately on translating into algebra.

0FZF **Lemma 11.4.** *In Lemma 11.2 let  $F$  correspond to  $\mathcal{K}$  in  $QCoh(\mathcal{O}_{X \times_R Y})$ . We have*

- (1) *If  $f : X' \rightarrow X$  is an affine morphism, then  $F \circ f_*$  corresponds to  $(f \times \mathrm{id}_Y)^* \mathcal{K}$ .*

- (2) If  $g : Y' \rightarrow Y$  is a quasi-compact and quasi-separated flat morphism, then  $g^* \circ F$  corresponds to  $(\text{id}_X \times g)^* \mathcal{K}$ .
- (3) If  $j : V \rightarrow Y$  is an open immersion, then  $j^* \circ F$  corresponds to  $\mathcal{K}|_{X \times_R V}$ .

**Proof.** For part (1) let  $\mathcal{F}'$  be a quasi-coherent module on  $X'$ . With obvious notation we have

$$\begin{aligned} \text{pr}_{2,*}(\text{pr}_1^* f_* \mathcal{F}' \otimes_{\mathcal{O}_{X \times_R Y}} \mathcal{K}) &= \text{pr}_{2,*}((f \times \text{id}_Y)^*(\text{pr}'_1)^* \mathcal{F}' \otimes_{\mathcal{O}_{X \times_R Y}} \mathcal{K}) \\ &= \text{pr}_{2,*}(f \times \text{id}_Y)_* \left( (\text{pr}'_1)^* \mathcal{F}' \otimes_{\mathcal{O}_{X' \times_R Y}} (f \times \text{id}_Y)^* \mathcal{K} \right) \\ &= \text{pr}'_{2,*}((\text{pr}'_1)^* \mathcal{F}' \otimes_{\mathcal{O}_{X' \times_R Y}} (f \times \text{id}_Y)^* \mathcal{K}) \end{aligned}$$

Here the first equality is affine base change, see Cohomology of Schemes, Lemma 5.1. The second equality hold by Remark 11.3. The third equality is functoriality of pushforwards for modules. For part (2) we have

$$g^* \text{pr}_{2,*}(\text{pr}_1^* \mathcal{F} \otimes_{\mathcal{O}_{X \times_R Y}} \mathcal{K}) = \text{pr}'_{2,*}((\text{pr}'_1)^* \mathcal{F} \otimes_{\mathcal{O}_{X \times_R Y'}} (\text{id}_X \times g)^* \mathcal{K})$$

by flat base change, see Cohomology of Schemes, Lemma 5.2. For part (3) we only have to remark that formation of  $\text{pr}_2$  commutes with localization on the target.  $\square$

0FZG **Lemma 11.5.** *In Lemma 11.2 if  $F$  is an exact functor, then the corresponding object  $\mathcal{K}$  of  $QCoh(\mathcal{O}_{X \times_R Y})$  is flat over  $X$ .*

**Proof.** By Lemma 11.4 we may assume  $Y$  is affine. In this case we can translate the statement into algebra as follows: Given a ring  $R$  and  $R$ -algebras  $A, B$  for an  $A \otimes_R B$ -module  $K$  the functor  $\text{Mod}_A \rightarrow \text{Mod}_B, M \mapsto M \otimes_A K$  is exact if and only if  $K$  is flat as an  $A$ -module. This is obvious.  $\square$

0FZH **Lemma 11.6.** *Let  $R$  be a ring. Let  $X$  and  $Y$  be schemes over  $R$ . Assume  $X$  is quasi-compact and that the diagonal morphism of  $X$  is affine. There is an equivalence of categories between*

- (1) *the category of  $R$ -linear exact functors  $F : QCoh(\mathcal{O}_X) \rightarrow QCoh(\mathcal{O}_Y)$  which commute with arbitrary direct sums, and*
- (2) *the full subcategory of  $QCoh(\mathcal{O}_{X \times_R Y})$  consisting of  $\mathcal{K}$  such that*
  - (a)  *$\mathcal{K}$  is flat over  $X$ ,*
  - (b) *for  $\mathcal{F} \in QCoh(\mathcal{O}_X)$  we have  $R^q \text{pr}_{2,*}(\text{pr}_1^* \mathcal{F} \otimes_{\mathcal{O}_{X \times_R Y}} \mathcal{K}) = 0$  for  $q > 0$ .*

*given by sending  $\mathcal{K}$  to the functor  $F$  in (11.1.1).*

**Proof.** Let  $\mathcal{K}$  be as in (2). The functor  $F$  in (11.1.1) commutes with direct sums. Since by (1) (a) the modules  $\mathcal{K}$  is  $X$ -flat, we see that given a short exact sequence  $0 \rightarrow \mathcal{F}_1 \rightarrow \mathcal{F}_2 \rightarrow \mathcal{F}_3 \rightarrow 0$  we obtain a short exact sequence

$$0 \rightarrow \text{pr}_1^* \mathcal{F}_1 \otimes_{\mathcal{O}_{X \times_R Y}} \mathcal{K} \rightarrow \text{pr}_1^* \mathcal{F}_2 \otimes_{\mathcal{O}_{X \times_R Y}} \mathcal{K} \rightarrow \text{pr}_1^* \mathcal{F}_3 \otimes_{\mathcal{O}_{X \times_R Y}} \mathcal{K} \rightarrow 0$$

Since by (2)(b) the higher direct image  $R^1 \text{pr}_{2,*}$  on the first term is zero, we conclude that  $0 \rightarrow F(\mathcal{F}_1) \rightarrow F(\mathcal{F}_2) \rightarrow F(\mathcal{F}_3) \rightarrow 0$  and we see that  $F$  is as in (1).

Let us construct the quasi-inverse to the construction. Let  $F$  be as in (1). Choose an affine open covering  $X = \bigcup_{i=1, \dots, n} U_i$ . Since the diagonal of  $X$  is affine, we see that the intersections  $U_{i_0 \dots i_p} = U_{i_0} \cap \dots \cap U_{i_p}$  are affine and that the inclusion morphisms  $j_{i_0 \dots i_p} : U_{i_0 \dots i_p} \rightarrow X$  are affine. See Morphisms, Lemma 11.11. In particular, the composition

$$QCoh(\mathcal{O}_{U_{i_0 \dots i_p}}) \xrightarrow{j_{i_0 \dots i_p}^*} QCoh(\mathcal{O}_X) \xrightarrow{F} QCoh(\mathcal{O}_Y)$$

is an exact functor commuting with direct sums as a composition of such functors. By Lemmas 11.2 and 11.5 this functor is given by a quasi-coherent module  $\mathcal{K}_{i_0 \dots i_p}$  on  $U_{i_0 \dots i_p} \times_R Y$  flat over  $U_{i_0 \dots i_p}$ . Since

$$QCoh(\mathcal{O}_{U_{i_0 \dots i_p i_{p+1}}}) \xrightarrow{(U_{i_0 \dots i_p i_{p+1}} \rightarrow U_{i_0 \dots i_p})^*} QCoh(\mathcal{O}_{U_{i_0 \dots i_p}}) \xrightarrow{j_{i_0 \dots i_p}^*} QCoh(\mathcal{O}_X)$$

is equal to  $j_{i_0 \dots i_p i_{p+1}}^*$  we conclude from Lemma 11.4 and the equivalence of categories of the already used Lemma 11.2 that we obtain identifications

$$\mathcal{K}_{i_0 \dots i_p i_{p+1}} = \mathcal{K}_{i_0 \dots i_p} |_{U_{i_0 \dots i_p i_{p+1}} \times_R Y}$$

which satisfy the usual compatibilities for glueing. In other words, there exists a unique  $\mathcal{K} \in QCoh(\mathcal{O}_{X \times_R Y})$  flat over  $X$  which restricts to each  $\mathcal{K}_{i_0 \dots i_p}$  on  $U_{i_0 \dots i_p} \times_R Y$  compatible with these identifications. For every quasi-coherent  $\mathcal{O}_X$ -module we have the sheafified Čech complex

$$0 \rightarrow \mathcal{F} \rightarrow \bigoplus_{i_0} j_{i_0}^* \mathcal{F} |_{U_{i_0}} \rightarrow \bigoplus_{i_0 i_1} j_{i_0 i_1}^* \mathcal{F} |_{U_{i_0 i_1}} \rightarrow \dots$$

which is exact. See Cohomology, Lemma 24.1. Applying the exact functor  $F$  we find that  $F(\mathcal{F})$  maps quasi-isomorphically to the relative Čech complex with terms

$$\bigoplus_{i_0 \dots i_p} (U_{i_0 \dots i_p} \times_R Y \rightarrow Y)_*(\mathrm{pr}_1^* \mathcal{F} \otimes_{\mathcal{O}_{X \times_R Y}} \mathcal{K}) |_{U_{i_0 \dots i_p} \times_R Y}$$

Since this Čech complex computes the pushforward and higher direct images of  $\mathrm{pr}_1^* \mathcal{F} \otimes_{\mathcal{O}_{X \times_R Y}} \mathcal{K}$  by  $\mathrm{pr}_2$  (by Cohomology of Schemes, Lemma 7.1) we conclude  $F$  and  $\mathcal{K}$  correspond and that we have property (2)(b).  $\square$

0FZI **Lemma 11.7.** *Let  $R, X, Y$ , and  $\mathcal{K}$  be as in Lemma 11.6 part (2). Then for any scheme  $T$  over  $R$  we have*

$$R^q \mathrm{pr}_{13,*} (\mathrm{pr}_{12}^* \mathcal{F} \otimes_{\mathcal{O}_{T \times_R X \times_R Y}} \mathrm{pr}_{23}^* \mathcal{K}) = 0$$

for  $\mathcal{F}$  quasi-coherent on  $T \times_R X$  and  $q > 0$ .

**Proof.** The question is local on  $T$  hence we may assume  $T$  is affine. In this case we can consider the diagram

$$\begin{array}{ccccc} T \times_R X & \longleftarrow & T \times_R X \times_R Y & \longrightarrow & T \times_R Y \\ \downarrow & & \downarrow & & \downarrow \\ X & \longleftarrow & X \times_R Y & \longrightarrow & Y \end{array}$$

whose vertical arrows are affine. In particular the pushforward along  $T \times_R Y \rightarrow Y$  is faithful and exact. Chasing around in the diagram using that higher direct images along affine morphisms vanish we see that it suffices to prove

$$R^q \mathrm{pr}_{2,*} (\mathrm{pr}_{23,*} (\mathrm{pr}_{12}^* \mathcal{F} \otimes_{\mathcal{O}_{T \times_R X \times_R Y}} \mathrm{pr}_{23}^* \mathcal{K})) = R^q \mathrm{pr}_{2,*} (\mathrm{pr}_{23,*} (\mathrm{pr}_{12}^* \mathcal{F}) \otimes_{\mathcal{O}_{X \times_R Y}} \mathcal{K})$$

is zero which is true by assumption on  $\mathcal{K}$ . The equality holds by Remark 11.3.  $\square$

0FZJ **Lemma 11.8.** *In Lemma 11.6 let  $F$  and  $\mathcal{K}$  correspond. If  $X$  is separated and flat over  $R$ , then there is a surjection  $\mathcal{O}_X \boxtimes F(\mathcal{O}_X) \rightarrow \mathcal{K}$ .*

**Proof.** Let  $\Delta : X \rightarrow X \times_R X$  be the diagonal morphism and set  $\mathcal{O}_\Delta = \Delta_* \mathcal{O}_X$ . Since  $\Delta$  is a closed immersion have a short exact sequence

$$0 \rightarrow \mathcal{I} \rightarrow \mathcal{O}_{X \times_R X} \rightarrow \mathcal{O}_\Delta \rightarrow 0$$

Since  $\mathcal{K}$  is flat over  $X$ , the pullback  $\mathrm{pr}_{23}^*\mathcal{K}$  to  $X \times_R X \times_R Y$  is flat over  $X \times_R X$  and we obtain a short exact sequence

$$0 \rightarrow \mathrm{pr}_{12}^*\mathcal{I} \otimes \mathrm{pr}_{23}^*\mathcal{K} \rightarrow \mathrm{pr}_{12}^*\mathcal{O}_{X \times_R X} \otimes \mathrm{pr}_{23}^*\mathcal{K} \rightarrow \mathrm{pr}_{12}^*\mathcal{O}_\Delta \otimes \mathrm{pr}_{23}^*\mathcal{K} \rightarrow 0$$

on  $X \times_R X \times_R Y$ . Thus, by Lemma 11.7 we obtain a surjection

$$\mathrm{pr}_{13,*}(\mathrm{pr}_{12}^*\mathcal{O}_{X \times_R X} \otimes \mathrm{pr}_{23}^*\mathcal{K}) \rightarrow \mathrm{pr}_{13,*}(\mathrm{pr}_{12}^*\mathcal{O}_\Delta \otimes \mathrm{pr}_{23}^*\mathcal{K})$$

By flat base change (Cohomology of Schemes, Lemma 5.2) the source of this arrow is equal to  $\mathcal{O}_X \boxtimes F(\mathcal{O}_X)$ . On the other hand the target is equal to

$$\mathrm{pr}_{13,*}(\mathrm{pr}_{12}^*\mathcal{O}_\Delta \otimes \mathrm{pr}_{23}^*\mathcal{K}) = \mathrm{pr}_{13,*}(\Delta \times \mathrm{id}_Y)_*\mathcal{K} = \mathcal{K}$$

which finishes the proof. The first equality holds for example by Cohomology, Lemma 49.4 and the fact that  $\mathrm{pr}_{12}^*\mathcal{O}_\Delta = (\Delta \times \mathrm{id}_Y)_*\mathcal{O}_{X \times_R Y}$ .  $\square$

## 12. Functors between categories of coherent modules

0FZK We need a supply of lemmas telling us certain exact functors have a certain shape.

0FZL **Lemma 12.1.** *Let  $X$  and  $Y$  be Noetherian schemes. Let  $F : \mathrm{Coh}(\mathcal{O}_X) \rightarrow \mathrm{Coh}(\mathcal{O}_Y)$  be a functor. Then  $F$  extends uniquely to a functor  $QCoh(\mathcal{O}_X) \rightarrow QCoh(\mathcal{O}_Y)$  which commutes with filtered colimits. If  $F$  is additive, then its extension commutes with arbitrary direct sums. If  $F$  is exact, left exact, or right exact, so is its extension.*

**Proof.** The existence and uniqueness of the extension is a general fact, see Categories, Lemma 26.2. To see that the lemma applies observe that coherent modules are of finite presentation (Modules, Lemma 12.2) and hence categorically compact objects of  $\mathrm{Mod}(\mathcal{O}_X)$  by Modules, Lemma 11.6. Finally, every quasi-coherent module is a filtered colimit of coherent ones for example by Properties, Lemma 22.3.

Assume  $F$  is additive. If  $\mathcal{F} = \bigoplus_{j \in J} \mathcal{H}_j$  with  $\mathcal{H}_j$  quasi-coherent, then  $\mathcal{F} = \mathrm{colim}_{J' \subset J \text{ finite}} \bigoplus_{j \in J'} \mathcal{H}_j$ . Denoting the extension of  $F$  also by  $F$  we obtain

$$\begin{aligned} F(\mathcal{F}) &= \mathrm{colim}_{J' \subset J \text{ finite}} F\left(\bigoplus_{j \in J'} \mathcal{H}_j\right) \\ &= \mathrm{colim}_{J' \subset J \text{ finite}} \bigoplus_{j \in J'} F(\mathcal{H}_j) \\ &= \bigoplus_{j \in J} F(\mathcal{H}_j) \end{aligned}$$

Thus  $F$  commutes with arbitrary direct sums.

Suppose  $0 \rightarrow \mathcal{F} \rightarrow \mathcal{F}' \rightarrow \mathcal{F}'' \rightarrow 0$  is a short exact sequence of quasi-coherent  $\mathcal{O}_X$ -modules. Then we write  $\mathcal{F}' = \bigcup \mathcal{F}'_i$  as the union of its coherent submodules, see Properties, Lemma 22.3. Denote  $\mathcal{F}''_i \subset \mathcal{F}''$  the image of  $\mathcal{F}'_i$  and denote  $\mathcal{F}_i = \mathcal{F} \cap \mathcal{F}'_i = \mathrm{Ker}(\mathcal{F}'_i \rightarrow \mathcal{F}''_i)$ . Then it is clear that  $\mathcal{F} = \bigcup \mathcal{F}_i$  and  $\mathcal{F}'' = \bigcup \mathcal{F}''_i$  and that we have short exact sequences

$$0 \rightarrow \mathcal{F}_i \rightarrow \mathcal{F}'_i \rightarrow \mathcal{F}''_i \rightarrow 0$$

Since the extension commutes with filtered colimits we have  $F(\mathcal{F}) = \mathrm{colim}_{i \in I} F(\mathcal{F}_i)$ ,  $F(\mathcal{F}') = \mathrm{colim}_{i \in I} F(\mathcal{F}'_i)$ , and  $F(\mathcal{F}'') = \mathrm{colim}_{i \in I} F(\mathcal{F}''_i)$ . Since filtered colimits are exact (Modules, Lemma 3.2) we conclude that exactness properties of  $F$  are inherited by its extension.  $\square$



0FZM **Lemma 12.2.** *Let  $f : V \rightarrow X$  be a quasi-finite separated morphism of Noetherian schemes. If there exists a coherent  $\mathcal{O}_V$ -module  $\mathcal{K}$  whose support is  $V$  such that  $f_*\mathcal{K}$  is coherent and  $R^q f_*\mathcal{K} = 0$ , then  $f$  is finite.*

**Proof.** By Zariski's main theorem we can find an open immersion  $j : V \rightarrow Y$  over  $X$  with  $\pi : Y \rightarrow X$  finite, see More on Morphisms, Lemma 38.3. Since  $\pi$  is affine the functor  $\pi_*$  is exact and faithful on the category of coherent  $\mathcal{O}_X$ -modules. Hence we see that  $j_*\mathcal{K}$  is coherent and that  $R^q j_*\mathcal{K}$  is zero for  $q > 0$ . In other words, we reduce to the case discussed in the next paragraph.

Assume  $f$  is an open immersion. We may replace  $X$  by the scheme theoretic closure of  $V$ . Assume  $X \setminus V$  is nonempty to get a contradiction. Choose a generic point  $\xi \in X \setminus V$  of an irreducible component of  $X \setminus V$ . Looking at the situation after base change by  $\text{Spec}(\mathcal{O}_{X,\xi}) \rightarrow X$  using flat base change and using Local Cohomology, Lemma 8.2 we reduce to the algebra problem discussed in the next paragraph.

Let  $(A, \mathfrak{m})$  be a Noetherian local ring. Let  $M$  be a finite  $A$ -module whose support is  $\text{Spec}(A)$ . Then  $H_{\mathfrak{m}}^i(A) \neq 0$  for some  $i$ . This is true by Dualizing Complexes, Lemma 11.1 and the fact that  $M$  is not zero hence has finite depth.  $\square$

0FZN **Lemma 12.3.** *Let  $k$  be a field. Let  $X, Y$  be finite type schemes over  $k$  with  $X$  separated. There is an equivalence of categories between*

- (1) *the category of  $k$ -linear exact functors  $F : \text{Coh}(\mathcal{O}_X) \rightarrow \text{Coh}(\mathcal{O}_Y)$ , and*
- (2) *the category of coherent  $\mathcal{O}_{X \times Y}$ -modules  $\mathcal{K}$  which are flat over  $X$  and have support finite over  $Y$*

*given by sending  $\mathcal{K}$  to the restriction of the functor (11.1.1) to  $\text{Coh}(\mathcal{O}_X)$ .*

**Proof.** Let  $\mathcal{K}$  be as in (2). By Lemma 11.6 the functor  $F$  given by (11.1.1) is exact and  $k$ -linear. Moreover,  $F$  sends  $\text{Coh}(\mathcal{O}_X)$  into  $\text{Coh}(\mathcal{O}_Y)$  for example by Cohomology of Schemes, Lemma 26.10.

Let us construct the quasi-inverse to the construction. Let  $F$  be as in (1). By Lemma 12.1 we can extend  $F$  to a  $k$ -linear exact functor on the categories of quasi-coherent modules which commutes with arbitrary direct sums. By Lemma 11.6 the extension corresponds to a unique quasi-coherent module  $\mathcal{K}$ , flat over  $X$ , such that  $R^q \text{pr}_{2,*}(\text{pr}_1^* \mathcal{F} \otimes_{\mathcal{O}_{X \times Y}} \mathcal{K}) = 0$  for  $q > 0$  for all quasi-coherent  $\mathcal{O}_X$ -modules  $\mathcal{F}$ . Since  $F(\mathcal{O}_X)$  is a coherent  $\mathcal{O}_Y$ -module, we conclude from Lemma 11.8 that  $\mathcal{K}$  is coherent.

For a closed point  $x \in X$  denote  $\mathcal{O}_x$  the skyscraper sheaf at  $x$  with value the residue field of  $x$ . We have

$$F(\mathcal{O}_x) = \text{pr}_{2,*}(\text{pr}_1^* \mathcal{O}_x \otimes \mathcal{K}) = (x \times Y \rightarrow Y)_*(\mathcal{K}|_{x \times Y})$$

Since  $x \times Y \rightarrow Y$  is finite, we see that the pushforward along this morphism is faithful. Hence if  $y \in Y$  is in the image of the support of  $\mathcal{K}|_{x \times Y}$ , then  $y$  is in the support of  $F(\mathcal{O}_x)$ .

Let  $Z \subset X \times Y$  be the scheme theoretic support  $Z$  of  $\mathcal{K}$ , see Morphisms, Definition 5.5. We first prove that  $Z \rightarrow Y$  is quasi-finite, by proving that its fibres over closed points are finite. Namely, if the fibre of  $Z \rightarrow Y$  over a closed point  $y \in Y$  has dimension  $> 0$ , then we can find infinitely many pairwise distinct closed points  $x_1, x_2, \dots$  in the image of  $Z_y \rightarrow X$ . Since we have a surjection  $\mathcal{O}_X \rightarrow \bigoplus_{i=1, \dots, n} \mathcal{O}_{x_i}$  we obtain a surjection

$$F(\mathcal{O}_X) \rightarrow \bigoplus_{i=1, \dots, n} F(\mathcal{O}_{x_i})$$

By what we said above, the point  $y$  is in the support of each of the coherent modules  $F(\mathcal{O}_{x_i})$ . Since  $F(\mathcal{O}_X)$  is a coherent module, this will lead to a contradiction because the stalk of  $F(\mathcal{O}_X)$  at  $y$  will be generated by  $< n$  elements if  $n$  is large enough. Hence  $Z \rightarrow Y$  is quasi-finite. Since  $\mathrm{pr}_{2,*}\mathcal{K}$  is coherent and  $R^q\mathrm{pr}_{2,*}\mathcal{K} = 0$  for  $q > 0$  we conclude that  $Z \rightarrow Y$  is finite by Lemma 12.2.  $\square$

0FZP **Lemma 12.4.** *Let  $f : X \rightarrow Y$  be a finite type separated morphism of schemes. Let  $\mathcal{F}$  be a finite type quasi-coherent module on  $X$  with support finite over  $Y$  and with  $\mathcal{L} = f_*\mathcal{F}$  an invertible  $\mathcal{O}_X$ -module. Then there exists a section  $s : Y \rightarrow X$  such that  $\mathcal{F} \cong s_*\mathcal{L}$ .*

**Proof.** Looking affine locally this translates into the following algebra problem. Let  $A \rightarrow B$  be a ring map and let  $N$  be a  $B$ -module which is invertible as an  $A$ -module. Then the annihilator  $J$  of  $N$  in  $B$  has the property that  $A \rightarrow B/J$  is an isomorphism. We omit the details.  $\square$

0FZQ **Lemma 12.5.** *Let  $f : X \rightarrow Y$  be a finite type separated morphism of schemes with a section  $s : Y \rightarrow X$ . Let  $\mathcal{F}$  be a finite type quasi-coherent module on  $X$ , set theoretically supported on  $s(Y)$  with  $\mathcal{L} = f_*\mathcal{F}$  an invertible  $\mathcal{O}_X$ -module. If  $Y$  is reduced, then  $\mathcal{F} \cong s_*\mathcal{L}$ .*

**Proof.** By Lemma 12.4 there exists a section  $s' : Y \rightarrow X$  such that  $\mathcal{F} = s'_*\mathcal{L}$ . Since  $s'(Y)$  and  $s(Y)$  have the same underlying closed subset and since both are reduced closed subschemes of  $X$ , they have to be equal. Hence  $s = s'$  and the lemma holds.  $\square$

0FZR **Lemma 12.6.** *Let  $k$  be a field. Let  $X, Y$  be finite type schemes over  $k$  with  $X$  separated and  $Y$  reduced. If there is a  $k$ -linear equivalence  $F : \mathrm{Coh}(\mathcal{O}_X) \rightarrow \mathrm{Coh}(\mathcal{O}_Y)$  of categories, then there is an isomorphism  $f : Y \rightarrow X$  over  $k$  and an invertible  $\mathcal{O}_Y$ -module  $\mathcal{L}$  such that  $F(\mathcal{F}) = f_*\mathcal{F} \otimes \mathcal{L}$ .*

**Proof.** By Lemma 12.3 we obtain a coherent  $\mathcal{O}_{X \times Y}$ -module  $\mathcal{K}$  which is flat over  $X$  with support finite over  $Y$  such that  $F$  is given by the restriction of the functor (11.1.1) to  $\mathrm{Coh}(\mathcal{O}_X)$ . If we can show that  $F(\mathcal{O}_X)$  is an invertible  $\mathcal{O}_Y$ -module, then by Lemma 12.4 we see that  $\mathcal{K} = s_*\mathcal{L}$  for some section  $s : Y \rightarrow X \times Y$  of  $\mathrm{pr}_2$  and some invertible  $\mathcal{O}_Y$ -module  $\mathcal{L}$ . This will show that  $F$  has the form indicated with  $f = \mathrm{pr}_1 \circ s$ . Some details omitted.

It remains to show that  $F(\mathcal{O}_X)$  is invertible. We only sketch the proof and we omit some of the details. For a closed point  $x \in X$  we denote  $\mathcal{O}_x$  in  $\mathrm{Coh}(\mathcal{O}_X)$  the skyscraper sheaf at  $x$  with value  $\kappa(x)$ . First we observe that the only simple objects of the category  $\mathrm{Coh}(\mathcal{O}_X)$  are these skyscraper sheaves  $\mathcal{O}_x$ . The same is true for  $Y$ . Hence for every closed point  $y \in Y$  there exists a closed point  $x \in X$  such that  $\mathcal{O}_y \cong F(\mathcal{O}_x)$ . Moreover, looking at endomorphisms we find that  $\kappa(x) \cong \kappa(y)$  as finite extensions of  $k$ . Then

$$\mathrm{Hom}_Y(F(\mathcal{O}_X), \mathcal{O}_y) \cong \mathrm{Hom}_Y(F(\mathcal{O}_X), F(\mathcal{O}_x)) \cong \mathrm{Hom}_X(\mathcal{O}_X, \mathcal{O}_x) \cong \kappa(x) \cong \kappa(y)$$

This implies that the stalk of the coherent  $\mathcal{O}_Y$ -module  $F(\mathcal{O}_X)$  at  $y \in Y$  can be generated by 1 generator (and no less) for each closed point  $y \in Y$ . It follows immediately that  $F(\mathcal{O}_X)$  is locally generated by 1 element (and no less) and since  $Y$  is reduced this indeed tells us it is an invertible module.  $\square$

Weak version of the result in [Gab62] stating that the category of quasi-coherent modules determines the isomorphism class of a scheme.

### 13. Sibling functors

0FZS In this section we prove some categorical result on the following notion.

0FZT **Definition 13.1.** Let  $\mathcal{A}$  be an abelian category. Let  $\mathcal{D}$  be a triangulated category. We say two exact functors of triangulated categories

$$F, F' : D^b(\mathcal{A}) \longrightarrow \mathcal{D}$$

are *siblings*, or we say  $F'$  is a *sibling* of  $F$ , if the following two conditions are satisfied

- (1) the functors  $F \circ i$  and  $F' \circ i$  are isomorphic where  $i : \mathcal{A} \rightarrow D^b(\mathcal{A})$  is the inclusion functor, and
- (2)  $F(K) \cong F'(K)$  for any  $K$  in  $D^b(\mathcal{A})$ .

Sometimes the second condition is a consequence of the first.

0FZU **Lemma 13.2.** *Let  $\mathcal{A}$  be an abelian category. Let  $\mathcal{D}$  be a triangulated category. Let  $F, F' : D^b(\mathcal{A}) \longrightarrow \mathcal{D}$  be exact functors of triangulated categories. Assume*

- (1) *the functors  $F \circ i$  and  $F' \circ i$  are isomorphic where  $i : \mathcal{A} \rightarrow D^b(\mathcal{A})$  is the inclusion functor, and*
- (2) *for all  $X, Y \in \text{Ob}(\mathcal{A})$  we have  $\text{Ext}_{\mathcal{D}}^q(F(X), F(Y)) = 0$  for  $q < 0$  (for example if  $F$  is fully faithful).*

*Then  $F$  and  $F'$  are siblings.*

**Proof.** Let  $K \in D^b(\mathcal{A})$ . We will show  $F(K)$  is isomorphic to  $F'(K)$ . We can represent  $K$  by a bounded complex  $A^\bullet$  of objects of  $\mathcal{A}$ . After replacing  $K$  by a translation we may assume  $A^i = 0$  for  $i > 0$ . Choose  $n \geq 0$  such that  $A^{-i} = 0$  for  $i > n$ . The objects

$$M_i = (A^{-i} \rightarrow \dots \rightarrow A^0)[-i], \quad i = 0, \dots, n$$

form a Postnikov system in  $D^b(\mathcal{A})$  for the complex  $A^\bullet = A^{-n} \rightarrow \dots \rightarrow A^0$  in  $D^b(\mathcal{A})$ . See Derived Categories, Example 40.2. Since both  $F$  and  $F'$  are exact functors of triangulated categories both

$$F(M_i) \quad \text{and} \quad F'(M_i)$$

form a Postnikov system in  $\mathcal{D}$  for the complex

$$F(A^{-n}) \rightarrow \dots \rightarrow F(A^0) = F'(A^{-n}) \rightarrow \dots \rightarrow F'(A^0)$$

Since all negative Exts between these objects vanish by assumption we conclude by uniqueness of Postnikov systems (Derived Categories, Lemma 40.6) that  $F(K) = F(M_n[n]) \cong F'(M_n[n]) = F'(K)$ .  $\square$

0FZV **Lemma 13.3.** *Let  $F$  and  $F'$  be siblings as in Definition 13.1. Then*

- (1) *if  $F$  is essentially surjective, then  $F'$  is essentially surjective,*
- (2) *if  $F$  is fully faithful, then  $F'$  is fully faithful.*

**Proof.** Part (1) is immediate from property (2) for siblings.

Assume  $F$  is fully faithful. Denote  $\mathcal{D}' \subset \mathcal{D}$  the essential image of  $F$  so that  $F : D^b(\mathcal{A}) \rightarrow \mathcal{D}'$  is an equivalence. Since the functor  $F'$  factors through  $\mathcal{D}'$  by property (2) for siblings, we can consider the functor  $H = F^{-1} \circ F' : D^b(\mathcal{A}) \rightarrow D^b(\mathcal{A})$ . Observe that  $H$  is a sibling of the identity functor. Since it suffices to prove that  $H$  is fully faithful, we reduce to the problem discussed in the next paragraph.

Set  $\mathcal{D} = D^b(\mathcal{A})$ . We have to show a sibling  $F : \mathcal{D} \rightarrow \mathcal{D}$  of the identity functor is fully faithful. Denote  $a_X : X \rightarrow F(X)$  the functorial isomorphism for  $X \in \text{Ob}(\mathcal{A})$  given to us by Definition 13.1. For any  $K$  in  $\mathcal{D}$  and distinguished triangle  $K_1 \rightarrow K_2 \rightarrow K_3$  of  $\mathcal{D}$  if the maps

$$F : \text{Hom}(K, K_i[n]) \rightarrow \text{Hom}(F(K), F(K_i[n]))$$

are isomorphisms for all  $n \in \mathbf{Z}$  and  $i = 1, 3$ , then the same is true for  $i = 2$  and all  $n \in \mathbf{Z}$ . This uses the 5-lemma Homology, Lemma 5.20 and Derived Categories, Lemma 4.2; details omitted. Similarly, if the maps

$$F : \text{Hom}(K_i[n], K) \rightarrow \text{Hom}(F(K_i[n]), F(K))$$

are isomorphisms for all  $n \in \mathbf{Z}$  and  $i = 1, 3$ , then the same is true for  $i = 2$  and all  $n \in \mathbf{Z}$ . Using the canonical truncations and induction on the number of nonzero cohomology objects, we see that it is enough to show

$$F : \text{Ext}^q(X, Y) \rightarrow \text{Ext}^q(F(X), F(Y))$$

is bijective for all  $X, Y \in \text{Ob}(\mathcal{A})$  and all  $q \in \mathbf{Z}$ . Since  $F$  is a sibling of id we have  $F(X) \cong X$  and  $F(Y) \cong Y$  hence the right hand side is zero for  $q < 0$ . The case  $q = 0$  is OK by our assumption that  $F$  is a sibling of the identity functor. It remains to prove the cases  $q > 0$ .

The case  $q = 1$ : Injectivity. An element  $\xi$  of  $\text{Ext}^1(X, Y)$  gives rise to a distinguished triangle

$$Y \rightarrow E \rightarrow X \xrightarrow{\xi} Y[1]$$

Observe that  $E \in \text{Ob}(\mathcal{A})$ . Since  $F$  is a sibling of the identity functor we obtain a commutative diagram

$$\begin{array}{ccc} E & \longrightarrow & X \\ \downarrow & & \downarrow \\ F(E) & \longrightarrow & F(X) \end{array}$$

whose vertical arrows are the isomorphisms  $a_E$  and  $a_X$ . By TR3 the distinguished triangle associated to  $\xi$  we started with is isomorphic to the distinguished triangle

$$F(Y) \rightarrow F(E) \rightarrow F(X) \xrightarrow{F(\xi)} F(Y[1]) = F(Y)[1]$$

Thus  $\xi = 0$  if and only if  $F(\xi)$  is zero, i.e., we see that  $F : \text{Ext}^1(X, Y) \rightarrow \text{Ext}^1(F(X), F(Y))$  is injective.

The case  $q = 1$ : Surjectivity. Let  $\theta$  be an element of  $\text{Ext}^1(F(X), F(Y))$ . This defines an extension of  $F(X)$  by  $F(Y)$  in  $\mathcal{A}$  which we may write as  $F(E)$  as  $F$  is a sibling of the identity functor. We thus get a distinguished triangle

$$F(Y) \xrightarrow{F(\alpha)} F(E) \xrightarrow{F(\beta)} F(X) \xrightarrow{\theta} F(Y[1]) = F(Y)[1]$$

for some morphisms  $\alpha : Y \rightarrow E$  and  $\beta : E \rightarrow X$ . Since  $F$  is a sibling of the identity functor, the sequence  $0 \rightarrow Y \rightarrow E \rightarrow X \rightarrow 0$  is a short exact sequence in  $\mathcal{A}$ ! Hence we obtain a distinguished triangle

$$Y \xrightarrow{\alpha} E \xrightarrow{\beta} X \xrightarrow{\delta} Y[1]$$

for some morphism  $\delta : X \rightarrow Y[1]$ . Applying the exact functor  $F$  we obtain the distinguished triangle

$$F(Y) \xrightarrow{F(\alpha)} F(E) \xrightarrow{F(\beta)} F(X) \xrightarrow{F(\delta)} F(Y)[1]$$

Arguing as above, we see that these triangles are isomorphic. Hence there exists a commutative diagram

$$\begin{array}{ccc} F(X) & \xrightarrow{F(\delta)} & F(Y[1]) \\ \downarrow \gamma & & \downarrow \epsilon \\ F(X) & \xrightarrow{\theta} & F(Y[1]) \end{array}$$

for some isomorphisms  $\gamma, \epsilon$  (we can say more but we won't need more information). We may write  $\gamma = F(\gamma')$  and  $\epsilon = F(\epsilon')$ . Then we have  $\theta = F(\epsilon' \circ \delta \circ (\gamma')^{-1})$  and we see the surjectivity holds.

The case  $q > 1$ : surjectivity. Using Yoneda extensions, see Derived Categories, Section 27, we find that for any element  $\xi$  in  $\text{Ext}^q(F(X), F(Y))$  we can find  $F(X) = B_0, B_1, \dots, B_{q-1}, B_q = F(Y) \in \text{Ob}(\mathcal{A})$  and elements

$$\xi_i \in \text{Ext}^1(B_{i-1}, B_i)$$

such that  $\xi$  is the composition  $\xi_q \circ \dots \circ \xi_1$ . Write  $B_i = F(A_i)$  (of course we have  $A_i = B_i$  but we don't need to use this) so that

$$\xi_i = F(\eta_i) \in \text{Ext}^1(F(A_{i-1}), F(A_i)) \quad \text{with} \quad \eta_i \in \text{Ext}^1(A_{i-1}, A_i)$$

by surjectivity for  $q = 1$ . Then  $\eta = \eta_q \circ \dots \circ \eta_1$  is an element of  $\text{Ext}^q(X, Y)$  with  $F(\eta) = \xi$ .

The case  $q > 1$ : injectivity. An element  $\xi$  of  $\text{Ext}^q(X, Y)$  gives rise to a distinguished triangle

$$Y[q-1] \rightarrow E \rightarrow X \xrightarrow{\xi} Y[q]$$

Applying  $F$  we obtain a distinguished triangle

$$F(Y)[q-1] \rightarrow F(E) \rightarrow F(X) \xrightarrow{F(\xi)} F(Y)[q]$$

If  $F(\xi) = 0$ , then  $F(E) \cong F(Y)[q-1] \oplus F(X)$  in  $\mathcal{D}$ , see Derived Categories, Lemma 4.11. Since  $F$  is a sibling of the identity functor we have  $E \cong F(E)$  and hence

$$E \cong F(E) \cong F(Y)[q-1] \oplus F(X) \cong Y[q-1] \oplus X$$

In other words,  $E$  is isomorphic to the direct sum of its cohomology objects. This implies that the initial distinguished triangle is split, i.e.,  $\xi = 0$ .  $\square$

Let us make a nonstandard definition. Let  $\mathcal{A}$  be an abelian category. Let us say  $\mathcal{A}$  *has enough negative objects* if given any  $X \in \text{Ob}(\mathcal{A})$  there exists an object  $N$  such that

- (1) there is a surjection  $N \rightarrow X$ ,
- (2)  $\text{Ext}^q(N, X) = 0$  for  $q > 0$ ,
- (3)  $\text{Hom}(X, N) = 0$ .

We encourage the reader to read the original argument of the follows proposition, see [Orl97, Proposition 2.16].

0FZW **Proposition 13.4.** *Let  $F$  and  $F'$  be siblings as in Definition 13.1. Assume that  $F$  is fully faithful and that  $\mathcal{A}$  has enough negative objects (see above). Then  $F$  and  $F'$  are isomorphic functors.* [Orl97, Proposition 2.16]

**Proof.** By part (2) of Definition 13.1 the image of the functor  $F'$  is contained in the essential image of the functor  $F$ . Hence the functor  $H = F^{-1} \circ F'$  is a sibling of the identity functor. This reduces us to the case described in the next paragraph.

Let  $\mathcal{D} = D^b(\mathcal{A})$ . We have to show a sibling  $F : \mathcal{D} \rightarrow \mathcal{D}$  of the identity functor is isomorphic to the identity functor. Given an object  $X$  of  $\mathcal{D}$  let us say  $X$  has width  $w = w(X)$  if  $w \geq 0$  is minimal such that there exists an integer  $a \in \mathbf{Z}$  with  $H^i(X) = 0$  for  $i \notin [a, a + w - 1]$ . Since  $F$  is a sibling of the identity and since  $F \circ [n] = [n] \circ F$  we are already given isomorphisms

$$c_X : X \rightarrow F(X)$$

for  $w(X) \leq 1$  compatible with shifts. Moreover, if  $X = A[-a]$  and  $X' = A'[-a]$  for some  $A, A' \in \text{Ob}(\mathcal{A})$  then for any morphism  $f : X \rightarrow X'$  the diagram

$$0FZX \quad (13.4.1) \quad \begin{array}{ccc} X & \xrightarrow{\quad f \quad} & X' \\ c_X \downarrow & & \downarrow c_{X'} \\ F(X) & \xrightarrow{F(f)} & F(X') \end{array}$$

is commutative.

Next, let us show that for any morphism  $f : X \rightarrow X'$  with  $w(X), w(X') \leq 1$  the diagram (13.4.1) commutes. If  $X$  or  $X'$  is zero, this is clear. If not then we can write  $X = A[-a]$  and  $X' = A'[-a']$  for unique  $A, A'$  in  $\mathcal{A}$  and  $a, a' \in \mathbf{Z}$ . The case  $a = a'$  was discussed above/ If  $a' > a$ , then  $f = 0$  (Derived Categories, Lemma 27.3 and the result is clear. If  $a' < a$  then  $f$  corresponds to an element  $\xi \in \text{Ext}^q(A, A')$  with  $q = a - a'$ . Using Yoneda extensions, see Derived Categories, Section 27, we can find  $A = A_0, A_1, \dots, A_{q-1}, A_q = A' \in \text{Ob}(\mathcal{A})$  and elements

$$\xi_i \in \text{Ext}^1(A_{i-1}, A_i)$$

such that  $\xi$  is the composition  $\xi_q \circ \dots \circ \xi_1$ . In other words, setting  $X_i = A_i[-a + i]$  we obtain morphisms

$$X = X_0 \xrightarrow{f_1} X_1 \rightarrow \dots \rightarrow X_{q-1} \xrightarrow{f_q} X_q = X'$$

whose composition is  $f$ . Since the commutativity of (13.4.1) for  $f_1, \dots, f_q$  implies it for  $f$ , this reduces us to the case  $q = 1$ . In this case after shifting we may assume we have a distinguished triangle

$$A' \rightarrow E \rightarrow A \xrightarrow{f} A'[1]$$

Observe that  $E$  is an object of  $\mathcal{A}$ . Consider the following diagram

$$\begin{array}{ccccccc} E & \longrightarrow & A & \xrightarrow{\quad f \quad} & A'[1] & \longrightarrow & E[1] \\ c_E \downarrow & & c_A \downarrow & \swarrow \epsilon & \downarrow c_{A'[1]} & & \downarrow c_{E[1]} \\ F(E) & \longrightarrow & F(A) & \xrightarrow{F(f)} & F(A')[1] & \longrightarrow & F(E)[1] \end{array}$$

whose rows are distinguished triangles. The square on the right commutes already but we don't yet know that the middle square does. By the axioms of a triangulated

category we can find a morphism  $\delta$  which does make the diagram commute. Then  $\gamma - c_{A'}[1]$  composed with  $F(A')[1] \rightarrow F(E)[1]$  is zero hence we can find  $\epsilon : A'[1] \rightarrow F(A)$  such that  $\gamma - c_{A'}[1] = F(f) \circ \epsilon$ . However, any arrow  $A'[1] \rightarrow F(A)$  is zero as it is a negative ext class between objects of  $\mathcal{A}$ . Hence  $\gamma = c_{A'}[1]$  and we conclude the middle square commutes too which is what we wanted to show.

To finish the proof we are going to argue by induction on  $w$  that there exist isomorphisms  $c_X : X \rightarrow F(X)$  for all  $X$  with  $w(X) \leq w$  compatible with all morphisms between such objects. The base case  $w = 1$  was shown above. Assume we know the result for some  $w \geq 1$ .

Let  $X$  be an object with  $w(X) = w + 1$ . Pick  $a \in \mathbf{Z}$  with  $H^i(X) = 0$  for  $i \notin [a, a + w]$ . Set  $b = a + w$  so that  $H^b(X)$  is nonzero. Pick  $N$  in  $\mathcal{A}$  such that there exists a surjection  $N \rightarrow H^b(X)$ , such that  $\text{Hom}(H^b(X), N) = 0$  and such that  $\text{Ext}^q(N, H^i(X)) = 0$  for  $i \in \mathbf{Z}$  and  $q > 0$ . This is possible because  $\mathcal{A}$  has enough negative objects by applying the definition to  $\bigoplus H^i(X)$ . By the vanishing of Exts we can lift the surjection  $N \rightarrow H^b(X)$  to a morphism  $N[-b] \rightarrow X$ ; details omitted. Let us call a morphism  $N[-b] \rightarrow X$  constructed in this manner a *good morphism*. Given a good morphism  $N[-b] \rightarrow X$  choose a distinguished diagram

$$N[-b] \rightarrow X \rightarrow Y \rightarrow N[-b + 1]$$

Computing the long exact cohomology sequence we find  $w(Y) \leq w$ . Hence by induction we find the solid arrows in the following diagram

$$\begin{array}{ccccccc} N[-b] & \longrightarrow & X & \longrightarrow & Y & \longrightarrow & N[-b + 1] \\ \downarrow c_{N[-b]} & & \downarrow c_{N[-b] \rightarrow X} & & \downarrow c_Y & & \downarrow c_{N[-b+1]} \\ F(N)[-b] & \longrightarrow & F(X) & \longrightarrow & F(Y) & \longrightarrow & F(N)[-b + 1] \end{array}$$

We obtain the dotted arrow  $c_{N[-b] \rightarrow X}$ . By Derived Categories, Lemma 4.8 the dotted arrow is unique because  $\text{Hom}(X, F(N)[-b]) \cong \text{Hom}(X, N[-b]) = 0$  by our choice of  $N$ . In fact,  $c_{N[-b] \rightarrow X}$  is the unique dotted arrow making the square with vertices  $X, Y, F(X), F(Y)$  commute. Our goal is to show that  $c_{N[-b] \rightarrow X}$  is independent of the choice of good morphism  $N[-b] \rightarrow X$  and that the diagrams (13.4.1) commute.

Independence of the choice of good morphism. Given two good morphisms  $N[-b] \rightarrow X$  and  $N'[-b] \rightarrow X$  we get another good morphism, namely  $(N \oplus N')[-b] \rightarrow X$ . Thus we may assume  $N'[-b] \rightarrow X$  factors as  $N'[-b] \rightarrow N[-b] \rightarrow X$  for some morphism  $N' \rightarrow N$ . Choose distinguished triangles  $N[-b] \rightarrow X \rightarrow Y \rightarrow N[-b + 1]$  and  $N'[-b] \rightarrow X \rightarrow Y' \rightarrow N'[-b + 1]$ . By axiom TR3 we can find a morphism  $g : Y' \rightarrow Y$  which joint with  $\text{id}_X$  and  $N' \rightarrow N$  forms a morphism of triangles. Since we have (13.4.1) for  $g$  we conclude that

$$(F(X) \rightarrow F(Y)) \circ c_{N'[-b] \rightarrow X} = (F(X) \rightarrow F(Y)) \circ c_{N[-b] \rightarrow X}$$

The uniqueness of  $c_{N[-b] \rightarrow X}$  pointed out in the construction above now shows that  $c_{N'[-b] \rightarrow X} = c_{N[-b] \rightarrow X}$ .

Let  $f : X \rightarrow X'$  be a morphism of objects with  $w(X) \leq w + 1$  and  $w(X') \leq w + 1$ . Choose  $a \leq b \leq a + w$  such that  $H^i(X) = 0$  for  $i \notin [a, b]$  and  $a' \leq b' \leq a' + w$  such that  $H^i(X') = 0$  for  $i \notin [a', b']$ . We will use induction on  $(b' - a') + (b - a)$  to show

this. (The base case is when this number is zero which is OK because  $w \geq 1$ .) We distinguish two cases.

Case I:  $b' < b$ . In this case we choose a good morphism  $N[-b] \rightarrow X$  such that in addition  $\text{Ext}^q(N, H^i(X')) = 0$  for  $q > 0$  and all  $i$ . Choose a distinguished triangle  $N[-b] \rightarrow X \rightarrow Y \rightarrow N[-b+1]$ . Since  $\text{Hom}(N[-b], X') = 0$  by our choice of  $N$  and we find that  $f$  factors as  $X \rightarrow Y \rightarrow X'$ . Since  $H^i(Y)$  is nonzero only for  $i \in [a, b-1]$  we see by induction that (13.4.1) commutes for  $Y \rightarrow X'$ . The diagram (13.4.1) commutes for  $X \rightarrow Y$  by construction if  $w(X) = w+1$  and by our first induction hypothesis if  $w(X) \leq w$ . Hence (13.4.1) commutes for  $f$ .

Case II:  $b' \geq b$ . In this case we choose a good morphism  $N'[-b'] \rightarrow X'$  such that  $\text{Hom}(H^{b'}(X), N') = 0$  (this is relevant only if  $b' = b$ ). We choose a distinguished triangle  $N'[-b'] \rightarrow X' \rightarrow Y' \rightarrow N'[-b'+1]$ . Since  $\text{Hom}(X, X') \rightarrow \text{Hom}(X, Y')$  is injective by our choice of  $N'$  (details omitted) the same is true for  $\text{Hom}(X, F(X')) \rightarrow \text{Hom}(X, F(Y'))$ . Hence it suffices in this case to check that (13.4.1) commutes for the composition  $X \rightarrow Y'$  of the morphisms  $X \rightarrow X' \rightarrow Y'$ . Since  $H^i(Y')$  is nonzero only for  $i \in [a', b'-1]$  we conclude by induction hypothesis.  $\square$

#### 14. Deducing fully faithfulness

0G23 It will be useful for us to know when a functor is fully faithful we offer the following variant of [Orl97, Lemma 2.15].

0G24 **Lemma 14.1.** *Let  $F : \mathcal{D} \rightarrow \mathcal{D}'$  be an exact functor of triangulated categories. Let  $S \subset \text{Ob}(\mathcal{D})$  be a set of objects. Assume* Variant of [Orl97, Lemma 2.15]

- (1)  $F$  has both right and left adjoints,
- (2) for  $K \in \mathcal{D}$  if  $\text{Hom}(E, K[i]) = 0$  for all  $E \in S$  and  $i \in \mathbf{Z}$  then  $K = 0$ ,
- (3) for  $K \in \mathcal{D}$  if  $\text{Hom}(K, E[i]) = 0$  for all  $E \in S$  and  $i \in \mathbf{Z}$  then  $K = 0$ ,
- (4) the map  $\text{Hom}(E, E'[i]) \rightarrow \text{Hom}(F(E), F(E')[i])$  induced by  $F$  is bijective for all  $E, E' \in S$  and  $i \in \mathbf{Z}$ .

Then  $F$  is fully faithful.

**Proof.** Denote  $F_r$  and  $F_l$  the right and left adjoints of  $F$ . For  $E \in S$  choose a distinguished triangle

$$E \rightarrow F_r(F(E)) \rightarrow C \rightarrow E[1]$$

where the first arrow is the unit of the adjunction. For  $E' \in S$  we have

$$\text{Hom}(E', F_r(F(E))[i]) = \text{Hom}(F(E'), F(E)[i]) = \text{Hom}(E', E[i])$$

The last equality holds by assumption (4). Hence applying the homological functor  $\text{Hom}(E', -)$  (Derived Categories, Lemma 4.2) to the distinguished triangle above we conclude that  $\text{Hom}(E', C[i]) = 0$  for all  $i \in \mathbf{Z}$  and  $E' \in S$ . By assumption (2) we conclude that  $C = 0$  and  $E = F_r(F(E))$ .

For  $K \in \text{Ob}(\mathcal{D})$  choose a distinguished triangle

$$F_l(F(K)) \rightarrow K \rightarrow C \rightarrow F_l(F(K))[1]$$

where the first arrow is the counit of the adjunction. For  $E \in S$  we have

$$\text{Hom}(F_l(F(K)), E[i]) = \text{Hom}(F(K), F(E)[i]) = \text{Hom}(K, F_r(F(E))[i]) = \text{Hom}(K, E[i])$$

where the last equality holds by the result of the first paragraph. Thus we conclude as before that  $\text{Hom}(C, E[i]) = 0$  for all  $E \in S$  and  $i \in \mathbf{Z}$ . Hence  $C = 0$  by assumption (3). Thus  $F$  is fully faithful by Categories, Lemma 24.3.  $\square$



0G02 **Lemma 14.2.** *Let  $k$  be a field. Let  $X$  be a scheme of finite type over  $k$  which is regular. Let  $x \in X$  be a closed point. For a coherent  $\mathcal{O}_X$ -module  $\mathcal{F}$  supported at  $x$  denote  $\mathcal{F}'$  coherent  $\mathcal{O}_X$ -module supported at  $x$  such that  $\mathcal{F}_x$  and  $\mathcal{F}'_x$  are Matlis dual. Then there is an isomorphism*

$$\mathrm{Hom}_X(\mathcal{F}, M) = H^0(X, M \otimes_{\mathcal{O}_X}^{\mathbf{L}} \mathcal{F}'[-d_x])$$

where  $d_x = \dim(\mathcal{O}_{X,x})$  functorial in  $M$  in  $D_{\mathrm{perf}}(\mathcal{O}_X)$ .

**Proof.** Since  $\mathcal{F}$  is supported at  $x$  we have

$$\mathrm{Hom}_X(\mathcal{F}, M) = \mathrm{Hom}_{\mathcal{O}_{X,x}}(\mathcal{F}_x, M_x)$$

and similarly we have

$$H^0(X, M \otimes_{\mathcal{O}_X}^{\mathbf{L}} \mathcal{F}'[-d_x]) = \mathrm{Tor}_{d_x}^{\mathcal{O}_{X,x}}(M_x, \mathcal{F}'_x)$$

Thus it suffices to show that given a Noetherian regular local ring  $A$  of dimension  $d$  and a finite length  $A$ -module  $N$ , if  $N'$  is the Matlis dual to  $N$ , then there exists a functorial isomorphism

$$\mathrm{Hom}_A(N, K) = \mathrm{Tor}_d^A(K, N')$$

for  $K$  in  $D_{\mathrm{perf}}(A)$ . We can write the left hand side as  $H^0(R\mathrm{Hom}_A(N, A) \otimes_A^{\mathbf{L}} K)$  by More on Algebra, Lemma 70.14 and the fact that  $N$  determines a perfect object of  $D(A)$ . Hence the formula holds because

$$R\mathrm{Hom}_A(N, A) = R\mathrm{Hom}_A(N, A[d])[-d] = N'[-d]$$

by Dualizing Complexes, Lemma 16.4 and the fact that  $A[d]$  is a normalized dualizing complex over  $A$  ( $A$  is Gorenstein by Dualizing Complexes, Lemma 21.3).  $\square$

0G03 **Lemma 14.3.** *Let  $k$  be a field. Let  $X$  be a scheme of finite type over  $k$  which is regular. Let  $x \in X$  be a closed point and denote  $\mathcal{O}_x$  the skyscraper sheaf at  $x$  with value  $\kappa(x)$ . Let  $K$  in  $D_{\mathrm{perf}}(\mathcal{O}_X)$ .*

- (1) *If  $\mathrm{Ext}_X^i(\mathcal{O}_x, K) = 0$  then there exists an open neighbourhood  $U$  of  $x$  such that  $H^{i-d_x}(K)|_U = 0$  where  $d_x = \dim(\mathcal{O}_{X,x})$ .*
- (2) *If  $\mathrm{Hom}_X(\mathcal{O}_x, K[i]) = 0$  for all  $i \in \mathbf{Z}$ , then  $K$  is zero in an open neighbourhood of  $x$ .*
- (3) *If  $\mathrm{Ext}_X^i(K, \mathcal{O}_x) = 0$  then there exists an open neighbourhood  $U$  of  $x$  such that  $H^i(K^\vee)|_U = 0$ .*
- (4) *If  $\mathrm{Hom}_X(K, \mathcal{O}_x[i]) = 0$  for all  $i \in \mathbf{Z}$ , then  $K$  is zero in an open neighbourhood of  $x$ .*
- (5) *If  $H^i(X, K \otimes_{\mathcal{O}_X}^{\mathbf{L}} \mathcal{O}_x) = 0$  then there exists an open neighbourhood  $U$  of  $x$  such that  $H^i(K)|_U = 0$ .*
- (6) *If  $H^i(X, K \otimes_{\mathcal{O}_X}^{\mathbf{L}} \mathcal{O}_x) = 0$  for  $i \in \mathbf{Z}$  then  $K$  is zero in an open neighbourhood of  $x$ .*

**Proof.** Observe that  $H^i(X, K \otimes_{\mathcal{O}_X}^{\mathbf{L}} \mathcal{O}_x)$  is equal to  $K_x \otimes_{\mathcal{O}_{X,x}}^{\mathbf{L}} \kappa(x)$ . Hence part (5) follows from More on Algebra, Lemma 72.4. Part (6) follows from part (5). Part (1) follows from part (5), Lemma 14.2, and the fact that the Matlis dual of  $\kappa(x)$  is  $\kappa(x)$ . Part (2) follows from part (1). Part (3) follows from part (5) and the fact that  $\mathrm{Ext}_X^i(K, \mathcal{O}_x) = H^i(X, K^\vee \otimes_{\mathcal{O}_X}^{\mathbf{L}} \mathcal{O}_x)$  by Cohomology, Lemma 46.4. Part (4) follows from part (3) and the fact that  $K \cong (K^\vee)^\vee$  by the lemma just cited.  $\square$

0G25 **Lemma 14.4.** *Let  $k$  be a field. Let  $X$  and  $Y$  be proper schemes over  $k$ . Assume  $X$  is regular. Then a  $k$ -linear exact functor  $F : D_{\text{perf}}(\mathcal{O}_X) \rightarrow D_{\text{perf}}(\mathcal{O}_Y)$  is fully faithful if and only if for any closed points  $x, x' \in X$  the maps*

$$F : \text{Ext}_X^i(\mathcal{O}_x, \mathcal{O}_{x'}) \longrightarrow \text{Ext}_Y^i(F(\mathcal{O}_x), F(\mathcal{O}_{x'}))$$

are isomorphisms for all  $i \in \mathbf{Z}$ . Here  $\mathcal{O}_x$  is the skyscraper sheaf at  $x$  with value  $\kappa(x)$ .

**Proof.** By Lemma 8.1 the functor  $F$  has both a left and a right adjoint. Thus we may apply the criterion of Lemma 14.1 because assumptions (2) and (3) of that lemma follow from Lemma 14.3.  $\square$

0G26 **Lemma 14.5.** *Let  $k$  be a field. Let  $X$  be a smooth proper scheme over  $k$ . Let  $F : D_{\text{perf}}(\mathcal{O}_X) \rightarrow D_{\text{perf}}(\mathcal{O}_X)$  be a  $k$ -linear exact functor. Assume for every coherent  $\mathcal{O}_X$ -module  $\mathcal{F}$  with  $\dim(\text{Supp}(\mathcal{F})) = 0$  there is an isomorphism  $\mathcal{F} \cong F(\mathcal{F})$ . Then  $F$  is fully faithful.*

Email from Noah Olander of Jun 9, 2020

**Proof.** By Lemma 14.4 it suffices to show that the maps

$$F : \text{Ext}_X^i(\mathcal{O}_x, \mathcal{O}_{x'}) \longrightarrow \text{Ext}_Y^i(F(\mathcal{O}_x), F(\mathcal{O}_{x'}))$$

are isomorphisms for all  $i \in \mathbf{Z}$  and all closed points  $x, x' \in X$ . By assumption, the source and the target are isomorphic. If  $x \neq x'$ , then both sides are zero and the result is true. If  $x = x'$ , then it suffices to prove that the map is either injective or surjective. For  $i < 0$  both sides are zero and the result is true. For  $i = 0$  any nonzero map  $\alpha : \mathcal{O}_x \rightarrow \mathcal{O}_x$  of  $\mathcal{O}_X$ -modules is an isomorphism. Hence  $F(\alpha)$  is an isomorphism too and so  $F(\alpha)$  is nonzero. Thus the result for  $i = 0$ . For  $i = 1$  a nonzero element  $\xi$  in  $\text{Ext}^1(\mathcal{O}_x, \mathcal{O}_x)$  corresponds to a nonsplit short exact sequence

$$0 \rightarrow \mathcal{O}_x \rightarrow \mathcal{F} \rightarrow \mathcal{O}_x \rightarrow 0$$

Since  $F(\mathcal{F}) \cong \mathcal{F}$  we see that  $F(\mathcal{F})$  is a nonsplit extension of  $\mathcal{O}_x$  by  $\mathcal{O}_x$  as well. Since  $\mathcal{O}_x \cong F(\mathcal{O}_x)$  is a simple  $\mathcal{O}_X$ -module and  $\mathcal{F} \cong F(\mathcal{F})$  has length 2, we see that in the distinguished triangle

$$F(\mathcal{O}_x) \rightarrow F(\mathcal{F}) \rightarrow F(\mathcal{O}_x) \xrightarrow{F(\xi)} F(\mathcal{O}_x)[1]$$

the first two arrows must form a short exact sequence which must be isomorphic to the above short exact sequence and hence is nonsplit. It follows that  $F(\xi)$  is nonzero and we conclude for  $i = 1$ . For  $i > 1$  composition of ext classes defines a surjection

$$\text{Ext}^1(F(\mathcal{O}_x), F(\mathcal{O}_x)) \otimes \dots \otimes \text{Ext}^1(F(\mathcal{O}_x), F(\mathcal{O}_x)) \longrightarrow \text{Ext}^i(F(\mathcal{O}_x), F(\mathcal{O}_x))$$

See Duality for Schemes, Lemma 15.4. Hence surjectivity in degree 1 implies surjectivity for  $i > 0$ . This finishes the proof.  $\square$

## 15. Special functors

0FZY In this section we prove some results on functors of a special type that we will use later in this chapter.

0FZZ **Definition 15.1.** Let  $k$  be a field. Let  $X, Y$  be finite type schemes over  $k$ . Recall that  $D_{\text{Coh}}^b(\mathcal{O}_X) = D^b(\text{Coh}(\mathcal{O}_X))$  by Derived Categories of Schemes, Proposition 10.2. We say two  $k$ -linear exact functors

$$F, F' : D_{\text{Coh}}^b(\mathcal{O}_X) = D^b(\text{Coh}(\mathcal{O}_X)) \longrightarrow D_{\text{Coh}}^b(\mathcal{O}_Y)$$

are *siblings*, or we say  $F'$  is a *sibling* of  $F$  if  $F$  and  $F'$  are siblings in the sense of Definition 13.1 with abelian category being  $\text{Coh}(\mathcal{O}_X)$ . If  $X$  is regular then  $D_{\text{perf}}(\mathcal{O}_X) = D_{\text{Coh}}^b(\mathcal{O}_X)$  by Derived Categories of Schemes, Lemma 10.6 and we use the same terminology for  $k$ -linear exact functors  $F, F' : D_{\text{perf}}(\mathcal{O}_X) \rightarrow D_{\text{perf}}(\mathcal{O}_Y)$ .

0G00 **Lemma 15.2.** *Let  $k$  be a field. Let  $X, Y$  be finite type schemes over  $k$  with  $X$  separated. Let  $F : D_{\text{Coh}}^b(\mathcal{O}_X) \rightarrow D_{\text{Coh}}^b(\mathcal{O}_Y)$  be a  $k$ -linear exact functor sending  $\text{Coh}(\mathcal{O}_X) \subset D_{\text{Coh}}^b(\mathcal{O}_X)$  into  $\text{Coh}(\mathcal{O}_Y) \subset D_{\text{Coh}}^b(\mathcal{O}_Y)$ . Then there exists a Fourier-Mukai functor  $F' : D_{\text{Coh}}^b(\mathcal{O}_X) \rightarrow D_{\text{Coh}}^b(\mathcal{O}_Y)$  whose kernel is a coherent  $\mathcal{O}_{X \times Y}$ -module  $\mathcal{K}$  flat over  $X$  and with support finite over  $Y$  which is a sibling of  $F$ .*

**Proof.** Denote  $H : \text{Coh}(\mathcal{O}_X) \rightarrow \text{Coh}(\mathcal{O}_Y)$  the restriction of  $F$ . Since  $F$  is an exact functor of triangulated categories, we see that  $H$  is an exact functor of abelian categories. Of course  $H$  is  $k$ -linear as  $F$  is. By Lemma 12.3 we obtain a coherent  $\mathcal{O}_{X \times Y}$ -module  $\mathcal{K}$  which is flat over  $X$  and has support finite over  $Y$ . Let  $F'$  be the Fourier-Mukai functor defined using  $\mathcal{K}$  so that  $F'$  restricts to  $H$  on  $\text{Coh}(\mathcal{O}_X)$ . The functor  $F'$  sends  $D_{\text{Coh}}^b(\mathcal{O}_X)$  into  $D_{\text{Coh}}^b(\mathcal{O}_Y)$  by Lemma 9.5. Observe that  $F$  and  $F'$  satisfy the first and second condition of Lemma 13.2 and hence are siblings.  $\square$

0G01 **Remark 15.3.** If  $F, F' : D_{\text{Coh}}^b(\mathcal{O}_X) \rightarrow \mathcal{D}$  are siblings,  $F$  is fully faithful, and  $X$  is reduced and projective over  $k$  then  $F \cong F'$ ; this follows from Proposition 13.4 via the argument given in the proof of Theorem 16.3. However, in general we do not know whether siblings are isomorphic. Even in the situation of Lemma 15.2 it seems difficult to prove that the siblings  $F$  and  $F'$  are isomorphic functors. In fact, we don't even know in general if a sibling of the identity functor on  $D_{\text{Coh}}^b(\mathcal{O}_X)$  is isomorphic to the identity functor, even if  $X$  is smooth and proper over  $k$ . If you have a proof or a counter example, please email [stacks.project@gmail.com](mailto:stacks.project@gmail.com).

0G04 **Lemma 15.4.** *Let  $k$  be a field. Let  $X$  be a separated scheme of finite type over  $k$  which is regular. Let  $F : D_{\text{perf}}(\mathcal{O}_X) \rightarrow D_{\text{perf}}(\mathcal{O}_X)$  be a  $k$ -linear exact functor. Assume for every coherent  $\mathcal{O}_X$ -module  $\mathcal{F}$  with  $\dim(\text{Supp}(\mathcal{F})) = 0$  there is an isomorphism of  $k$ -vector spaces*

$$\text{Hom}_X(\mathcal{F}, M) = \text{Hom}_X(\mathcal{F}, F(M))$$

*functorial in  $M$  in  $D_{\text{perf}}(\mathcal{O}_X)$ . Then there exists an automorphism  $f : X \rightarrow X$  over  $k$  which induces the identity on the underlying topological space<sup>1</sup> and an invertible  $\mathcal{O}_X$ -module  $\mathcal{L}$  such that  $F$  and  $F'(M) = f^*M \otimes_{\mathcal{O}_X}^{\mathbf{L}} \mathcal{L}$  are siblings.*

**Proof.** By Lemma 14.2 we conclude that for every coherent  $\mathcal{O}_X$ -module  $\mathcal{F}$  whose support is a closed point there are isomorphisms

$$H^0(X, M \otimes_{\mathcal{O}_X}^{\mathbf{L}} \mathcal{F}) = H^0(X, F(M) \otimes_{\mathcal{O}_X}^{\mathbf{L}} \mathcal{F})$$

functorial in  $M$ .

Let  $x \in X$  be a closed point and apply the above with  $\mathcal{F} = \mathcal{O}_x$  the skyscraper sheaf with value  $\kappa(x)$  at  $x$ . We find

$$\dim_{\kappa(x)} \text{Tor}_p^{\mathcal{O}_x, x}(M_x, \kappa(x)) = \dim_{\kappa(x)} \text{Tor}_p^{\mathcal{O}_x, x}(F(M)_x, \kappa(x))$$

for all  $p \in \mathbf{Z}$ . In particular, if  $H^i(M) = 0$  for  $i > 0$ , then  $H^i(F(M)) = 0$  for  $i > 0$  by Lemma 14.3.

<sup>1</sup>This often forces  $f$  to be the identity, see Lemma 15.5.

If  $\mathcal{E}$  is locally free of rank  $r$ , then  $F(\mathcal{E})$  is locally free of rank  $r$ . This is true because a perfect complex  $K$  over  $\mathcal{O}_{X,x}$  with

$$\dim_{\kappa(x)} \mathrm{Tor}_i^{\mathcal{O}_{X,x}}(K, \kappa(x)) = \begin{cases} r & \text{if } i = 0 \\ 0 & \text{if } i \neq 0 \end{cases}$$

is equal to a free module of rank  $r$  placed in degree 0. See for example More on Algebra, Lemma 71.6.

If  $M$  is supported on a closed subscheme  $Z \subset X$ , then  $F(M)$  is also supported on  $Z$ . This is clear because we will have  $M \otimes_{\mathcal{O}_X}^{\mathbf{L}} \mathcal{O}_x = 0$  for  $x \notin Z$  and hence the same will be true for  $F(M)$  and hence we get the conclusion from Lemma 14.3.

In particular  $F(\mathcal{O}_x)$  is supported at  $\{x\}$ . Let  $i \in \mathbf{Z}$  be the minimal integer such that  $H^i(\mathcal{O}_x) \neq 0$ . We know that  $i \leq 0$ . If  $i < 0$ , then there is a morphism  $\mathcal{O}_x[-i] \rightarrow F(\mathcal{O}_x)$  which contradicts the fact that all morphisms  $\mathcal{O}_x[-i] \rightarrow \mathcal{O}_x$  are zero. Thus  $F(\mathcal{O}_x) = \mathcal{H}[0]$  where  $\mathcal{H}$  is a skyscraper sheaf at  $x$ .

Let  $\mathcal{G}$  be a coherent  $\mathcal{O}_X$ -module with  $\dim(\mathrm{Supp}(\mathcal{G})) = 0$ . Then there exists a filtration

$$0 = \mathcal{G}_0 \subset \mathcal{G}_1 \subset \dots \subset \mathcal{G}_n = \mathcal{G}$$

such that for  $n \geq i \geq 1$  the quotient  $\mathcal{G}_i/\mathcal{G}_{i-1}$  is isomorphic to  $\mathcal{O}_{x_i}$  for some closed point  $x_i \in X$ . Then we get distinguished triangles

$$F(\mathcal{G}_{i-1}) \rightarrow F(\mathcal{G}_i) \rightarrow F(\mathcal{O}_{x_i})$$

and using induction we find that  $F(\mathcal{G}_i)$  is a coherent sheaf placed in degree 0.

Let  $\mathcal{G}$  be a coherent  $\mathcal{O}_X$ -module. We know that  $H^i(F(\mathcal{G})) = 0$  for  $i > 0$ . To get a contradiction assume that  $H^i(F(\mathcal{G}))$  is nonzero for some  $i < 0$ . We choose  $i$  minimal with this property so that we have a morphism  $H^i(F(\mathcal{G}))[-i] \rightarrow F(\mathcal{G})$  in  $D_{\mathrm{perf}}(\mathcal{O}_X)$ . Choose a closed point  $x \in X$  in the support of  $H^i(F(\mathcal{G}))$ . By More on Algebra, Lemma 93.2 there exists an  $n > 0$  such that

$$H^i(F(\mathcal{G}))_x \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{X,x}/\mathfrak{m}_x^n \longrightarrow \mathrm{Tor}_{-i}^{\mathcal{O}_{X,x}}(F(\mathcal{G})_x, \mathcal{O}_{X,x}/\mathfrak{m}_x^n)$$

is nonzero. Next, we take  $m \geq 1$  and we consider the short exact sequence

$$0 \rightarrow \mathfrak{m}_x^m \mathcal{G} \rightarrow \mathcal{G} \rightarrow \mathcal{G}/\mathfrak{m}_x^m \mathcal{G} \rightarrow 0$$

By the above we know that  $F(\mathcal{G}/\mathfrak{m}_x^m \mathcal{G})$  is a sheaf placed in degree 0. Hence  $H^i(F(\mathfrak{m}_x^m \mathcal{G})) \rightarrow H^i(F(\mathcal{G}))$  is an isomorphism. Consider the commutative diagram

$$\begin{array}{ccc} H^i(F(\mathfrak{m}_x^m \mathcal{G}))_x \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{X,x}/\mathfrak{m}_x^n & \longrightarrow & \mathrm{Tor}_{-i}^{\mathcal{O}_{X,x}}(F(\mathfrak{m}_x^m \mathcal{G})_x, \mathcal{O}_{X,x}/\mathfrak{m}_x^n) \\ \downarrow & & \downarrow \\ H^i(F(\mathcal{G}))_x \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{X,x}/\mathfrak{m}_x^n & \longrightarrow & \mathrm{Tor}_{-i}^{\mathcal{O}_{X,x}}(F(\mathcal{G})_x, \mathcal{O}_{X,x}/\mathfrak{m}_x^n) \end{array}$$

Since the left vertical arrow is an isomorphism and the bottom arrow is nonzero, we conclude that the right vertical arrow is nonzero for all  $m \geq 1$ . On the other hand, by the first paragraph of the proof, we know this arrow is isomorphic to the arrow

$$\mathrm{Tor}_{-i}^{\mathcal{O}_{X,x}}(\mathfrak{m}_x^m \mathcal{G}_x, \mathcal{O}_{X,x}/\mathfrak{m}_x^n) \longrightarrow \mathrm{Tor}_{-i}^{\mathcal{O}_{X,x}}(\mathcal{G}_x, \mathcal{O}_{X,x}/\mathfrak{m}_x^n)$$

However, this arrow is zero for  $m \gg n$  by More on Algebra, Lemma 94.5 which is the contradiction we're looking for.

Thus we know that  $F$  preserves coherent modules. By Lemma 15.2 we find  $F$  is a sibling to the Fourier-Mukai functor  $F'$  given by a coherent  $\mathcal{O}_{X \times X}$ -module  $\mathcal{K}$  flat over  $X$  via  $\text{pr}_1$  and finite over  $X$  via  $\text{pr}_2$ . Since  $F(\mathcal{O}_X)$  is an invertible  $\mathcal{O}_X$ -module  $\mathcal{L}$  placed in degree 0 we see that

$$\mathcal{L} \cong F(\mathcal{O}_X) \cong F'(\mathcal{O}_X) \cong \text{pr}_{2,*}\mathcal{K}$$

Thus by Lemma 12.4 there is a morphism  $s : X \rightarrow X \times X$  with  $\text{pr}_2 \circ s = \text{id}_X$  such that  $\mathcal{K} = s_*\mathcal{L}$ . Set  $f = \text{pr}_1 \circ s$ . Then we have

$$\begin{aligned} F'(M) &= R\text{pr}_{2,*}(L\text{pr}_1^*K \otimes \mathcal{K}) \\ &= R\text{pr}_{2,*}(L\text{pr}_1^*M \otimes s_*\mathcal{L}) \\ &= R\text{pr}_{2,*}(Rs_*(Lf^*M \otimes \mathcal{L})) \\ &= Lf^*M \otimes \mathcal{L} \end{aligned}$$

where we have used Derived Categories of Schemes, Lemma 21.1 in the third step. Since for all closed points  $x \in X$  the module  $F(\mathcal{O}_x)$  is supported at  $x$ , we see that  $f$  induces the identity on the underlying topological space of  $X$ . We still have to show that  $f$  is an isomorphism which we will do in the next paragraph.

Let  $x \in X$  be a closed point. For  $n \geq 1$  denote  $\mathcal{O}_{x,n}$  the skyscraper sheaf at  $x$  with value  $\mathcal{O}_{X,x}/\mathfrak{m}_x^n$ . We have

$$\text{Hom}_X(\mathcal{O}_{x,m}, \mathcal{O}_{x,n}) \cong \text{Hom}_X(\mathcal{O}_{x,m}, F(\mathcal{O}_{x,n})) \cong \text{Hom}_X(\mathcal{O}_{x,m}, f^*\mathcal{O}_{x,n} \otimes \mathcal{L})$$

functorially with respect to  $\mathcal{O}_X$ -module homomorphisms between the  $\mathcal{O}_{x,n}$ . (The first isomorphism exists by assumption and the second isomorphism because  $F$  and  $F'$  are siblings.) For  $m \geq n$  we have  $\mathcal{O}_{X,x}/\mathfrak{m}_x^n = \text{Hom}_X(\mathcal{O}_{x,m}, \mathcal{O}_{x,n})$  via the action on  $\mathcal{O}_{x,n}$  we conclude that  $f^\sharp : \mathcal{O}_{X,x}/\mathfrak{m}_x^n \rightarrow \mathcal{O}_{X,x}/\mathfrak{m}_x^n$  is bijective for all  $n$ . Thus  $f$  induces isomorphisms on complete local rings at closed points and hence is étale (Étale Morphisms, Lemma 11.3). Looking at closed points we see that  $\Delta_f : X \rightarrow X \times_{f,X,f} X$  (which is an open immersion as  $f$  is étale) is bijective hence an isomorphism. Hence  $f$  is a monomorphism. Finally, we conclude  $f$  is an isomorphism as Descent, Lemma 22.1 tells us it is an open immersion.  $\square$

0G05 **Lemma 15.5.** *Let  $X$  be a reduced scheme of finite type over a field  $k$ . Let  $f : X \rightarrow X$  be an automorphism over  $k$  which induces the identity map on the underlying topological space of  $X$ . Then*

- (1)  $f^*\mathcal{F} \cong \mathcal{F}$  for every coherent  $\mathcal{O}_X$ -module, and
- (2) if  $\dim(Z) > 0$  for every irreducible component  $Z \subset X$ , then  $f$  is the identity.

**Proof.** Part (1) follows from part (2) and the fact that the connected components of  $X$  of dimension 0 are spectra of fields.

Let  $Z \subset X$  be an irreducible component viewed as an integral closed subscheme. Clearly  $f(Z) \subset Z$  and  $f|_Z : Z \rightarrow Z$  is an automorphism over  $k$  which induces the identity map on the underlying topological space of  $Z$ . Since  $X$  is reduced, it suffices to show that the arrows  $f|_Z : Z \rightarrow Z$  are the identity. This reduces us to the case discussed in the next paragraph.

Assume  $X$  is irreducible of dimension  $> 0$ . Choose a nonempty affine open  $U \subset X$ . Since  $f(U) \subset U$  and since  $U \subset X$  is scheme theoretically dense it suffices to prove that  $f|_U : U \rightarrow U$  is the identity.

Assume  $X = \text{Spec}(A)$  is affine, irreducible, of dimension  $> 0$  and  $k$  is an infinite field. Let  $g \in A$  be nonconstant. The set

$$S = \bigcup_{\lambda \in k} V(g - \lambda)$$

is dense in  $X$  because it is the inverse image of the dense subset  $\mathbf{A}_k^1(k)$  by the nonconstant morphism  $g : X \rightarrow \mathbf{A}_k^1$ . If  $x \in S$ , then the image  $g(x)$  of  $g$  in  $\kappa(x)$  is in the image of  $k \rightarrow \kappa(x)$ . Hence  $f^\sharp : \kappa(x) \rightarrow \kappa(x)$  fixes  $g(x)$ . Thus the image of  $f^\sharp(g)$  in  $\kappa(x)$  is equal to  $g(x)$ . We conclude that

$$S \subset V(g - f^\sharp(g))$$

and since  $X$  is reduced and  $S$  is dense we conclude  $g = f^\sharp(g)$ . This proves  $f^\sharp = \text{id}_A$  as  $A$  is generated as a  $k$ -algebra by elements  $g$  as above (details omitted; hint: the set of constant functions is a finite dimensional  $k$ -subvector space of  $A$ ). We conclude that  $f = \text{id}_X$ .

Assume  $X = \text{Spec}(A)$  is affine, irreducible, of dimension  $> 0$  and  $k$  is a finite field. If for every 1-dimensional integral closed subscheme  $C \subset X$  the restriction  $f|_C : C \rightarrow C$  is the identity, then  $f$  is the identity. This reduces us to the case where  $X$  is a curve. A curve over a finite field has a finite automorphism group (details omitted). Hence  $f$  has finite order, say  $n$ . Then we pick  $g : X \rightarrow \mathbf{A}_k^1$  nonconstant as above and we consider

$$S = \{x \in X \text{ closed such that } [\kappa(g(x)) : k] \text{ is prime to } n\}$$

Arguing as before we find that  $S$  is dense in  $X$ . Since for  $x \in X$  closed the map  $f^\sharp : \kappa(x) \rightarrow \kappa(x)$  is an automorphism of order dividing  $n$  we see that for  $x \in S$  this automorphism acts trivially on the subfield generated by the image of  $g$  in  $\kappa(x)$ . Thus we conclude that  $S \subset V(g - f^\sharp(g))$  and we win as before.  $\square$

0G27 **Lemma 15.6.** *Let  $k$  be a field. Let  $X$  be a smooth proper scheme over  $k$ . Let  $F : D_{\text{perf}}(\mathcal{O}_X) \rightarrow D_{\text{perf}}(\mathcal{O}_X)$  be a  $k$ -linear exact functor. Assume for every coherent  $\mathcal{O}_X$ -module  $\mathcal{F}$  with  $\dim(\text{Supp}(\mathcal{F})) = 0$  there is an isomorphism  $\mathcal{F} \cong F(\mathcal{F})$ . Then there exists an automorphism  $f : X \rightarrow X$  over  $k$  which induces the identity on the underlying topological space<sup>2</sup> and an invertible  $\mathcal{O}_X$ -module  $\mathcal{L}$  such that  $F$  and  $F'(M) = f^*M \otimes_{\mathcal{O}_X}^{\mathbf{L}} \mathcal{L}$  are siblings.*

Email from Noah Olander of Jun 8, 2020

**First proof.** By Lemma 14.5 the functor  $F$  is fully faithful. We claim that Lemma 15.4 applies to  $F$ . Namely, for every coherent  $\mathcal{O}_X$ -module  $\mathcal{F}$  with  $\dim(\text{Supp}(\mathcal{F})) = 0$  there is an isomorphism of  $k$ -vector spaces

$$\text{Hom}_X(\mathcal{F}, M) = \text{Hom}_X(F(\mathcal{F}), F(M)) \cong \text{Hom}_X(\mathcal{F}, F(M))$$

functorial in  $M$  in  $D_{\text{perf}}(\mathcal{O}_X)$ . The first equality because  $F$  is fully faithful.  $\square$

**Second proof.** The proof is similar to the proof of Lemma 15.4 and we only sketch the arguments.

Step 1. By Lemma 14.5 the functor  $F$  is fully faithful.

Step 2. For a closed point  $x \in X$  we denote  $\mathcal{O}_x$  the skyscraper sheaf with value  $\kappa(x)$  at  $x$ . An object  $K$  of  $D_{\text{perf}}(\mathcal{O}_X)$  is isomorphic to a locally free  $\mathcal{O}_X$ -module of rank  $r$  placed in degree 0 if and only if

$$\dim \text{Ext}_X^i(K, \mathcal{O}_x) = 0 \text{ unless } i = 0 \text{ and } \dim \text{Ext}_X^i(K, \mathcal{O}_x) = r$$

<sup>2</sup>This often forces  $f$  to be the identity, see Lemma 15.5.

Since  $F(\mathcal{O}_x) \cong \mathcal{O}_x$  and  $F$  is fully faithful we find

$$\dim \operatorname{Ext}_X^i(K, \mathcal{O}_x) = \dim \operatorname{Ext}^i(F(K), \mathcal{O}_x)$$

for all  $i \in \mathbf{Z}$ . Thus, if  $\mathcal{E}$  is a locally free  $\mathcal{O}_X$ -module of rank  $r$ , then  $F(\mathcal{E})$  is a locally free  $\mathcal{O}_X$ -module of rank  $r$  placed in degree 0. In particular, we have  $F(\mathcal{O}_X) = \mathcal{L}[0]$  for some invertible  $\mathcal{O}_X$ -module  $\mathcal{L}$ .

Step 3. Let  $U \subset X$  be a dense affine open with complement  $D \subset X$ . Then  $D = \sum_{i=1, \dots, r} D_i$  is a sum of prime divisors each of which is an effective Cartier divisor, see Divisors, Lemma 16.7 and its proof. By Step 2 we have invertible modules  $\mathcal{L}_n$  such that  $F(\mathcal{O}_X(-nD)) = \mathcal{L}_n[0]$ . The functor  $F$  transforms the maps

$$\mathcal{O}_X(-(n+1)D) \rightarrow \mathcal{O}_X(-nD) \rightarrow \dots \rightarrow \mathcal{O}_X(-D) \rightarrow \mathcal{O}_X$$

into maps  $\mathcal{L}_{n+1} \rightarrow \mathcal{L}_n$  such that the induced map

$$\operatorname{Hom}_X(\mathcal{L}_{n+1}, \mathcal{O}_x) \rightarrow \operatorname{Hom}_X(\mathcal{L}_n, \mathcal{O}_x)$$

is nonzero if  $x \in U$  and zero if  $x \in D$ . We conclude that we may write

$$\mathcal{L}_n = \mathcal{L}(\sum e_i(n)D_i)$$

with  $e_i(n+1) < e_i(n) < \dots < e_i(0) = 0$ . It follows that for  $K \in D_{\text{perf}}(\mathcal{O}_X)$  we have

$$\begin{aligned} \Gamma(U, K) &\cong \operatorname{colim}_n \operatorname{Hom}(\mathcal{O}_X(-nD), K) \\ &\cong \operatorname{colim}_n \operatorname{Hom}(\mathcal{L}_n, F(K)) \\ &\cong \operatorname{colim}_n \operatorname{Hom}(\mathcal{O}_X(-nD), F(K) \otimes \mathcal{L}^{\otimes -1}) \\ &\cong \Gamma(U, F(K) \otimes \mathcal{L}^{\otimes -1}) \end{aligned}$$

The first and last isomorphisms follow from Cohomology of Schemes, Lemma 10.4 (details omitted). The second isomorphism holds because  $F$  is fully faithful. The third isomorphism holds by comparing the system of divisors  $-nD$  for varying  $n$  with the system of divisors  $\sum e_i(n)D_i$  for varying  $n$ .

Step 4. The functor  $F$  sends  $\operatorname{Coh}(\mathcal{O}_X)$  to  $\operatorname{Coh}(\mathcal{O}_X)$ . Namely, this is clear from the formula in Step 3.

Step 5. By Lemma 15.2 we find  $F$  is a sibling to the Fourier-Mukai functor  $F'$  given by a coherent  $\mathcal{O}_{X \times X}$ -module  $\mathcal{K}$  flat over  $X$  via  $\operatorname{pr}_1$  and finite over  $X$  via  $\operatorname{pr}_2$ . Since  $F(\mathcal{O}_X) = \mathcal{L}[0]$  we see that

$$\mathcal{L} \cong F(\mathcal{O}_X) \cong F'(\mathcal{O}_X) \cong \operatorname{pr}_{2,*} \mathcal{K}$$

Thus by Lemma 12.4 there is a morphism  $s : X \rightarrow X \times X$  with  $\operatorname{pr}_2 \circ s = \operatorname{id}_X$  such that  $\mathcal{K} = s_* \mathcal{L}$ . Set  $f = \operatorname{pr}_1 \circ s$ . Then we have

$$\begin{aligned} F'(M) &= R\operatorname{pr}_{2,*}(L\operatorname{pr}_1^* K \otimes \mathcal{K}) \\ &= R\operatorname{pr}_{2,*}(L\operatorname{pr}_1^* M \otimes s_* \mathcal{L}) \\ &= R\operatorname{pr}_{2,*}(Rs_*(Lf^* M \otimes \mathcal{L})) \\ &= Lf^* M \otimes \mathcal{L} \end{aligned}$$

where we have used Derived Categories of Schemes, Lemma 21.1 in the third step. Since for all closed points  $x \in X$  the module  $F(\mathcal{O}_x)$  is supported at  $x$ , we see that  $f$  induces the identity on the underlying topological space of  $X$ .

Step 6. Let  $x \in X$  be a closed point. For  $n \geq 1$  denote  $\mathcal{O}_{x,n}$  the skyscraper sheaf at  $x$  with value  $\mathcal{O}_{X,x}/\mathfrak{m}_x^n$ . Then  $\text{End}_X(\mathcal{O}_{x,n}) = \mathcal{O}_{X,x}/\mathfrak{m}_x^n$ . Since  $F$  and  $F'$  are siblings and since  $F$  is fully faithful we see that

$$\text{End}_X(\mathcal{O}_{x,n}) \rightarrow \text{End}_X(F(\mathcal{O}_{x,n})) \cong \text{End}_X(f^*\mathcal{O}_{x,n} \otimes \mathcal{L}) \cong \text{End}_X(f^*\mathcal{O}_{x,n})$$

are bijections. We conclude that  $f^\sharp : \mathcal{O}_{X,x}/\mathfrak{m}_x^n \rightarrow \mathcal{O}_{X,x}/\mathfrak{m}_x^n$  is bijective for all  $n$ . Thus  $f$  induces isomorphisms on complete local rings at closed points and hence is étale (Étale Morphisms, Lemma 11.3). Looking at closed points we see that  $\Delta_f : X \rightarrow X \times_{f,X,f} X$  (which is an open immersion as  $f$  is étale) is bijective hence an isomorphism. Hence  $f$  is a monomorphism. Finally, we conclude  $f$  is an isomorphism as Descent, Lemma 22.1 tells us it is an open immersion.  $\square$

0G06 **Lemma 15.7.** *Let  $k$  be a field. Let  $X, Y$  be smooth proper schemes over  $k$ . Let  $F, G : D_{\text{perf}}(\mathcal{O}_X) \rightarrow D_{\text{perf}}(\mathcal{O}_Y)$  be  $k$ -linear exact functors such that*

- (1)  $F(\mathcal{F}) \cong G(\mathcal{F})$  for any coherent  $\mathcal{O}_X$ -module  $\mathcal{F}$  with  $\dim(\text{Supp}(\mathcal{F})) = 0$ ,
- (2)  $F$  is fully faithful, and
- (3)  $G$  is a Fourier-Mukai functor whose kernel is in  $D_{\text{perf}}(\mathcal{O}_{X \times Y})$ .

*Then there exists a Fourier-Mukai functor  $F' : D_{\text{perf}}(\mathcal{O}_X) \rightarrow D_{\text{perf}}(\mathcal{O}_Y)$  whose kernel is in  $D_{\text{perf}}(\mathcal{O}_{X \times Y})$  such that  $F$  and  $F'$  are siblings.*

**First proof.** Recall that  $G$  and  $F$  have both adjoints, see Lemma 8.1. In particular the essential image  $\mathcal{A} \subset D_{\text{perf}}(\mathcal{O}_Y)$  of  $F$  satisfies the equivalent conditions of Derived Categories, Lemma 39.5. We claim that  $G$  factors through  $\mathcal{A}$ . Since  $\mathcal{A} = {}^\perp(\mathcal{A}^\perp)$  by Derived Categories, Lemma 39.5 it suffices to show that  $\text{Hom}_Y(G(M), N) = 0$  for all  $M$  in  $D_{\text{perf}}(\mathcal{O}_X)$  and  $N \in \mathcal{A}^\perp$ . We have

$$\text{Hom}_Y(G(M), N) = \text{Hom}_X(M, G_r(N))$$

where  $G_r$  is the right adjoint to  $G$ . Since  $G(\mathcal{F}) \cong F(\mathcal{F})$  for  $\mathcal{F}$  as in (1) we see that  $\text{Hom}_X(\mathcal{F}, G_r(N)) = 0$  by the same formula and the fact that  $N$  is in the right orthogonal to the essential image  $\mathcal{A}$  of  $F$ . Of course, the same vanishing holds for  $\text{Hom}_X(\mathcal{F}, G_r(N)[i])$  for any  $i \in \mathbf{Z}$ . Thus  $G_r(N) = 0$  by Lemma 14.3 and the claim holds.

For  $\mathcal{F}$  as in (1) we have

$$\begin{aligned} \text{Hom}_X(\mathcal{F}, M) &= \text{Hom}_Y(F(\mathcal{F}), F(M)) \\ &\cong \text{Hom}_Y(G(\mathcal{F}), F(M)) \\ &= \text{Hom}_X(\mathcal{F}, G_r(F(M))) \end{aligned}$$

functorially in  $M$  in  $D_{\text{perf}}(\mathcal{O}_X)$ . Thus the functor  $G_r \circ F$  satisfies the assumptions of Lemma 15.4. Let  $f : X \rightarrow X$  and  $\mathcal{L}$  be the automorphism and invertible module found in Lemma 15.4. We will use below that  $f^*\mathcal{F} \cong \mathcal{F}$  for every coherent  $\mathcal{O}_X$ -module  $\mathcal{F}$ , see Lemma 15.5.



Let  $G_l$  be the left adjoint to  $G$ . Denote  $S_X : D_{perf}(\mathcal{O}_X) \rightarrow D_{perf}(\mathcal{O}_X)$  a Serre functor, see Lemma 4.1. We have

$$\begin{aligned} \mathrm{Hom}_X(\mathcal{F}, S_X(G_l(G(M))))^\vee &= \mathrm{Hom}_X(G_l(G(M)), \mathcal{F}) \\ &= \mathrm{Hom}_Y(G(M), G(\mathcal{F})) \\ &\cong \mathrm{Hom}_Y(G(M), F(\mathcal{F})) \\ &= \mathrm{Hom}_X(M, G_r(F(\mathcal{F}))) \\ &\cong \mathrm{Hom}_X(M, \mathcal{F}) \\ &= \mathrm{Hom}_X(\mathcal{F}, S_X(M))^\vee \end{aligned}$$

functorially in  $M$ . The first isomorphism uses  $F(\mathcal{F}) \cong G(\mathcal{F})$ . The second isomorphism uses  $(G_r \circ F)(\mathcal{F}) \cong f^* \mathcal{F} \otimes_{\mathcal{O}_X}^{\mathbf{L}} \mathcal{L} \cong \mathcal{F} \otimes_{\mathcal{O}_X}^{\mathbf{L}} \mathcal{L} \cong \mathcal{F}$  by the result of the previous paragraph. We conclude Lemma 15.4 applies to the functor  $S_X \circ G_l \circ G \circ S_X^{-1}$ . In particular, we see that  $S_X \circ G_l \circ G \circ S_X^{-1}$  is an equivalence as a sibling of an equivalence (Lemma 13.3). Hence  $G_l \circ G$  is an equivalence. By Categories, Lemma 24.5 we see that  $G$  is fully faithful.

Since  $G$  is fully faithful, we see that  $H = F_r \circ G$  is a sibling of a functor  $H' : M \mapsto (f')^* M \otimes_{\mathcal{O}_X}^{\mathbf{L}} \mathcal{L}'$  for some automorphism  $f' : X \rightarrow X$  and invertible  $\mathcal{O}_X$ -module  $\mathcal{L}'$  by the same argument as in the second paragraph of the proof with roles of  $F$  and  $G$  switched. It follows that  $H$  is an auto-equivalence by Lemma 13.3 and that  $H$  induces an auto-equivalence of  $\mathrm{Coh}(\mathcal{O}_X)$  (as this is true for its sibling functor  $H'$ ). Thus the quasi-inverses  $H^{-1}$  and  $(H')^{-1}$  exist, are siblings (small detail omitted), and  $(H')^{-1}$  sends  $M$  to  $(f')^{-1}{}^*(M \otimes_{\mathcal{O}_X}^{\mathbf{L}} (\mathcal{L}')^{-1})$  which is a Fourier-Mukai functor (details omitted).

Since  $F$  is fully faithful, the counit  $F \circ F_r \rightarrow \mathrm{id}$  is an isomorphism when evaluated on objects of the essential image of  $F$ . Since we've shown in the first paragraph that  $G(M)$  is in the essential image of  $F$  we conclude that the counit maps

$$F(H(M)) = F(F_r(G(M))) \longrightarrow G(M)$$

are isomorphisms functorial in  $M$ ! Thus

$$F \cong G \circ H^{-1}$$

is a sibling of the functor  $G \circ (H')^{-1}$ . This finishes the proof because  $G \circ (H')^{-1}$  is a Fourier-Mukai functor as compositions of Fourier-Mukai functors are Fourier-Mukai by Lemma 9.3.  $\square$

**Second proof.** Recall that  $F$  has both adjoints, see Lemma 8.1. In particular the essential image  $\mathcal{A} \subset D_{perf}(\mathcal{O}_Y)$  of  $F$  satisfies the equivalent conditions of Derived Categories, Lemma 39.5. We claim that  $G$  factors through  $\mathcal{A}$ . Since  $\mathcal{A} = {}^\perp(\mathcal{A}^\perp)$  by Derived Categories, Lemma 39.5 it suffices to show that  $\mathrm{Hom}_Y(G(M), N) = 0$  for all  $M$  in  $D_{perf}(\mathcal{O}_X)$  and  $N \in \mathcal{A}^\perp$ . We have

$$\mathrm{Hom}_Y(G(M), N) = \mathrm{Hom}_X(M, G_r(N))$$

where  $G_r$  is the right adjoint to  $G$ . Since  $G(\mathcal{F}) \cong F(\mathcal{F})$  for  $\mathcal{F}$  as in (1) we see that  $\mathrm{Hom}_X(\mathcal{F}, G_r(N)) = 0$  by the same formula and the fact that  $N$  is in the right orthogonal to the essential image  $\mathcal{A}$  of  $F$ . Of course, the same vanishing holds for  $\mathrm{Hom}_X(\mathcal{F}, G_r(N)[i])$  for any  $i \in \mathbf{Z}$ . Thus  $G_r(N) = 0$  by Lemma 14.3 and the claim holds.

Apply Lemma 15.6 to the functor  $H = F^{-1} \circ G$  which makes sense because the essential image of  $G$  is contained in the essential image of  $F$  by the previous paragraph and because  $F$  is fully faithful. We obtain an automorphism  $f : X \rightarrow X$  and an invertible  $\mathcal{O}_X$ -module  $\mathcal{L}$  such that the functor  $H' : K \mapsto f^*K \otimes \mathcal{L}$  is a sibling of  $H$ . In particular  $H$  is an auto-equivalence by Lemma 13.3 and  $H$  induces an auto-equivalence of  $\text{Coh}(\mathcal{O}_X)$  (as this is true for its sibling functor  $H'$ ). Thus the quasi-inverses  $H^{-1}$  and  $(H')^{-1}$  exist, are siblings (small detail omitted), and  $(H')^{-1}$  sends  $M$  to  $(f^{-1})^*(M \otimes_{\mathcal{O}_X}^L \mathcal{L}^{\otimes -1})$  which is a Fourier-Mukai functor (details omitted). Then of course  $F = G \circ H^{-1}$  is a sibling of  $G \circ (H')^{-1}$ . Since compositions of Fourier-Mukai functors are Fourier-Mukai by Lemma 9.3 we conclude.  $\square$

## 16. Fully faithful functors

0G07 Our goal is to prove fully faithful functors between derived categories are siblings of Fourier-Mukai functors, following [Orl97] and [Bal08].

0G08 **Situation 16.1.** Here  $k$  is a field. We have proper smooth schemes  $X$  and  $Y$  over  $k$ . We have a  $k$ -linear, exact, fully faithful functor  $F : D_{\text{perf}}(\mathcal{O}_X) \rightarrow D_{\text{perf}}(\mathcal{O}_Y)$ .

Before reading on, it makes sense to read at least some of Derived Categories, Section 40.

Recall that  $X$  is regular and hence has the resolution property (Varieties, Lemma 25.3 and Derived Categories of Schemes, Lemma 34.6). Thus on  $X \times X$  we may choose a resolution

$$\dots \rightarrow \mathcal{E}_2 \boxtimes \mathcal{G}_2 \rightarrow \mathcal{E}_1 \boxtimes \mathcal{G}_1 \rightarrow \mathcal{E}_0 \boxtimes \mathcal{G}_0 \rightarrow \mathcal{O}_\Delta \rightarrow 0$$

where each  $\mathcal{E}_i$  and  $\mathcal{G}_i$  is a finite locally free  $\mathcal{O}_X$ -module, see Lemma 10.3. Using the complex

0G09 (16.1.1) 
$$\dots \rightarrow \mathcal{E}_2 \boxtimes \mathcal{G}_2 \rightarrow \mathcal{E}_1 \boxtimes \mathcal{G}_1 \rightarrow \mathcal{E}_0 \boxtimes \mathcal{G}_0$$

in  $D_{\text{perf}}(\mathcal{O}_{X \times X})$  as in Derived Categories, Example 40.2 if for each  $n$  we denote

$$M_n = (\mathcal{E}_n \boxtimes \mathcal{G}_n \rightarrow \dots \rightarrow \mathcal{E}_0 \boxtimes \mathcal{G}_0)[-n]$$

we obtain an infinite Postnikov system for the complex (16.1.1). This means the morphisms  $M_0 \rightarrow M_1[1] \rightarrow M_2[2] \rightarrow \dots$  and  $M_n \rightarrow \mathcal{E}_n \boxtimes \mathcal{G}_n$  and  $\mathcal{E}_n \boxtimes \mathcal{G}_n \rightarrow M_{n-1}$  satisfy certain conditions documented in Derived Categories, Definition 40.1. Set

$$\mathcal{F}_n = \text{Ker}(\mathcal{E}_n \boxtimes \mathcal{G}_n \rightarrow \mathcal{E}_{n-1} \boxtimes \mathcal{G}_{n-1})$$

Observe that since  $\mathcal{O}_\Delta$  is flat over  $X$  via  $\text{pr}_1$  the same is true for  $\mathcal{F}_n$  for all  $n$  (this is a convenient though not essential observation). We have

$$H^q(M_n[n]) = \begin{cases} \mathcal{O}_\Delta & \text{if } q = 0 \\ \mathcal{F}_n & \text{if } q = -n \\ 0 & \text{if } q \neq 0, -n \end{cases}$$

Thus for  $n \geq \dim(X \times X)$  we have

$$M_n[n] \cong \mathcal{O}_\Delta \oplus \mathcal{F}_n[n]$$

in  $D_{\text{perf}}(\mathcal{O}_{X \times X})$  by Lemma 10.5.

We are interested in the complex

0G0A (16.1.2) 
$$\dots \rightarrow \mathcal{E}_2 \boxtimes F(\mathcal{G}_2) \rightarrow \mathcal{E}_1 \boxtimes F(\mathcal{G}_1) \rightarrow \mathcal{E}_0 \boxtimes F(\mathcal{G}_0)$$

in  $D_{\text{perf}}(\mathcal{O}_{X \times Y})$  as the “totalization” of this complex should give us the kernel of the Fourier-Mukai functor we are trying to construct. For all  $i, j \geq 0$  we have

$$\begin{aligned} \text{Ext}_{X \times Y}^q(\mathcal{E}_i \boxtimes F(\mathcal{G}_i), \mathcal{E}_j \boxtimes F(\mathcal{G}_j)) &= \bigoplus_p \text{Ext}_X^{q+p}(\mathcal{E}_i, \mathcal{E}_j) \otimes_k \text{Ext}_Y^{-p}(F(\mathcal{G}_i), F(\mathcal{G}_j)) \\ &= \bigoplus_p \text{Ext}_X^{q+p}(\mathcal{E}_i, \mathcal{E}_j) \otimes_k \text{Ext}_X^{-p}(\mathcal{G}_i, \mathcal{G}_j) \end{aligned}$$

The second equality holds because  $F$  is fully faithful and the first by Derived Categories of Schemes, Lemma 23.1. We find these  $\text{Ext}^q$  are zero for  $q < 0$ . Hence by Derived Categories, Lemma 40.6 we can build an infinite Postnikov system  $K_0, K_1, K_2, \dots$  in  $D_{\text{perf}}(\mathcal{O}_{X \times Y})$  for the complex (16.1.2). Parallel to what happens with  $M_0, M_1, M_2, \dots$  this means we obtain morphisms  $K_0 \rightarrow K_1[1] \rightarrow K_2[2] \rightarrow \dots$  and  $K_n \rightarrow \mathcal{E}_n \boxtimes F(\mathcal{G}_n)$  and  $\mathcal{E}_n \boxtimes F(\mathcal{G}_n) \rightarrow K_{n-1}$  in  $D_{\text{perf}}(\mathcal{O}_{X \times Y})$  satisfying certain conditions documented in Derived Categories, Definition 40.1.

Let  $\mathcal{F}$  be a coherent  $\mathcal{O}_X$ -module whose support has a finite number of points, i.e., with  $\dim(\text{Supp}(\mathcal{F})) = 0$ . Consider the exact functor of triangulated categories

$$D_{\text{perf}}(\mathcal{O}_{X \times Y}) \longrightarrow D_{\text{perf}}(\mathcal{O}_Y), \quad N \longmapsto R\text{pr}_{2,*}(\text{pr}_1^* \mathcal{F} \otimes_{\mathcal{O}_{X \times Y}}^{\mathbf{L}} N)$$

It follows that the objects  $R\text{pr}_{2,*}(\text{pr}_1^* \mathcal{F} \otimes_{\mathcal{O}_{X \times Y}}^{\mathbf{L}} K_i)$  form a Postnikov system for the complex in  $D_{\text{perf}}(\mathcal{O}_Y)$  with terms

$$R\text{pr}_{2,*}((\mathcal{F} \otimes \mathcal{E}_i) \boxtimes F(\mathcal{G}_i)) = \Gamma(X, \mathcal{F} \otimes \mathcal{E}_i) \otimes_k F(\mathcal{G}_i) = F(\Gamma(X, \mathcal{F} \otimes \mathcal{E}_i) \otimes_k \mathcal{G}_i)$$

Here we have used that  $\mathcal{F} \otimes \mathcal{E}_i$  has vanishing higher cohomology as its support has dimension 0. On the other hand, applying the exact functor

$$D_{\text{perf}}(\mathcal{O}_{X \times X}) \longrightarrow D_{\text{perf}}(\mathcal{O}_Y), \quad N \longmapsto F(R\text{pr}_{2,*}(\text{pr}_1^* \mathcal{F} \otimes_{\mathcal{O}_{X \times X}}^{\mathbf{L}} N))$$

we find that the objects  $F(R\text{pr}_{2,*}(\text{pr}_1^* \mathcal{F} \otimes_{\mathcal{O}_{X \times X}}^{\mathbf{L}} M_n))$  form a second infinite Postnikov system for the complex in  $D_{\text{perf}}(\mathcal{O}_Y)$  with terms

$$F(R\text{pr}_{2,*}((\mathcal{F} \otimes \mathcal{E}_i) \boxtimes \mathcal{G}_i)) = F(\Gamma(X, \mathcal{F} \otimes \mathcal{E}_i) \otimes_k \mathcal{G}_i)$$

This is the same as before! By uniqueness of Postnikov systems (Derived Categories, Lemma 40.6) which applies because

$$\text{Ext}_Y^q(F(\Gamma(X, \mathcal{F} \otimes \mathcal{E}_i) \otimes_k \mathcal{G}_i), F(\Gamma(X, \mathcal{F} \otimes \mathcal{E}_j) \otimes_k \mathcal{G}_j)) = 0, \quad q < 0$$

as  $F$  is fully faithful, we find a system of isomorphisms

$$F(R\text{pr}_{2,*}(\text{pr}_1^* \mathcal{F} \otimes_{\mathcal{O}_{X \times X}}^{\mathbf{L}} M_n[n])) \cong R\text{pr}_{2,*}(\text{pr}_1^* \mathcal{F} \otimes_{\mathcal{O}_{X \times Y}}^{\mathbf{L}} K_n[n])$$

in  $D_{\text{perf}}(\mathcal{O}_Y)$  compatible with the morphisms in  $D_{\text{perf}}(\mathcal{O}_Y)$  induced by the morphisms

$$\begin{aligned} M_{n-1}[n-1] &\rightarrow M_n[n] & \text{and} & & K_{n-1}[n-1] &\rightarrow K_n[n] \\ M_n &\rightarrow \mathcal{E}_n \boxtimes \mathcal{G}_n & \text{and} & & K_n &\rightarrow \mathcal{E}_n \boxtimes F(\mathcal{G}_n) \\ \mathcal{E}_n \boxtimes \mathcal{G}_n &\rightarrow M_{n-1} & \text{and} & & \mathcal{E}_n \boxtimes F(\mathcal{G}_n) &\rightarrow K_{n-1} \end{aligned}$$

which are part of the structure of Postnikov systems. For  $n$  sufficiently large we obtain a direct sum decomposition

$$F(R\text{pr}_{2,*}(\text{pr}_1^* \mathcal{F} \otimes_{\mathcal{O}_{X \times X}}^{\mathbf{L}} M_n[n])) = F(\mathcal{F}) \oplus F(R\text{pr}_{2,*}(\text{pr}_1^* \mathcal{F} \otimes_{\mathcal{O}_{X \times Y}}^{\mathbf{L}} \mathcal{F}_n))[n]$$

corresponding to the direct sum decomposition of  $M_n$  constructed above (we are using the flatness of  $\mathcal{F}_n$  over  $X$  via  $\text{pr}_1$  to write a usual tensor product in the formula above, but this isn't essential for the argument). By Lemma 10.9 we find there exists an integer  $m \geq 0$  such that the first summand in this direct sum

decomposition has nonzero cohomology sheaves only in the interval  $[-m, m]$  and the second summand in this direct sum decomposition has nonzero cohomology sheaves only in the interval  $[-m - n, m + \dim(X) - n]$ . We conclude the system  $K_0 \rightarrow K_1[1] \rightarrow K_2[2] \rightarrow \dots$  in  $D_{\text{perf}}(\mathcal{O}_{X \times Y})$  satisfies the assumptions of Lemma 10.10 after possibly replacing  $m$  by a larger integer. We conclude we can write

$$K_n[n] = K \oplus C_n$$

for  $n \gg 0$  compatible with transition maps and with  $C_n$  having nonzero cohomology sheaves only in the range  $[-m - n, m - n]$ . Denote  $G$  the Fourier-Mukai functor corresponding to  $K$ . Putting everything together we find

$$\begin{aligned} G(\mathcal{F}) \oplus R\text{pr}_{2,*}(\text{pr}_1^* \mathcal{F} \otimes_{\mathcal{O}_{X \times Y}}^{\mathbf{L}} C_n) &\cong \\ R\text{pr}_{2,*}(\text{pr}_1^* \mathcal{F} \otimes_{\mathcal{O}_{X \times Y}}^{\mathbf{L}} K_n[n]) &\cong \\ F(R\text{pr}_{2,*}(\text{pr}_1^* \mathcal{F} \otimes_{\mathcal{O}_{X \times X}}^{\mathbf{L}} M_n[n])) &\cong \\ F(\mathcal{F}) \oplus F(R\text{pr}_{2,*}(\text{pr}_1^* \mathcal{F} \otimes_{\mathcal{O}_{X \times Y}} \mathcal{F}_n)) &[n] \end{aligned}$$

Looking at the degrees that objects live in we conclude that for  $n \gg m$  we obtain an isomorphism

$$F(\mathcal{F}) \cong G(\mathcal{F})$$

Moreover, recall that this holds for every coherent  $\mathcal{F}$  on  $X$  whose support has dimension 0.

0G0B **Lemma 16.2.** *Let  $k$  be a field. Let  $X$  and  $Y$  be smooth proper schemes over  $k$ . Given a  $k$ -linear, exact, fully faithful functor  $F : D_{\text{perf}}(\mathcal{O}_X) \rightarrow D_{\text{perf}}(\mathcal{O}_Y)$  there exists a Fourier-Mukai functor  $F' : D_{\text{perf}}(\mathcal{O}_X) \rightarrow D_{\text{perf}}(\mathcal{O}_Y)$  whose kernel is in  $D_{\text{perf}}(\mathcal{O}_{X \times Y})$  which is a sibling to  $F$ .*

**Proof.** Apply Lemma 15.7 to  $F$  and the functor  $G$  constructed above.  $\square$

0G0C **Theorem 16.3** (Orlov). *Let  $k$  be a field. Let  $X$  and  $Y$  be smooth proper schemes over  $k$  with  $X$  projective over  $k$ . Any  $k$ -linear fully faithful exact functor  $F : D_{\text{perf}}(\mathcal{O}_X) \rightarrow D_{\text{perf}}(\mathcal{O}_Y)$  is a Fourier-Mukai functor for some kernel in  $D_{\text{perf}}(\mathcal{O}_{X \times Y})$ .* [Orl97, Theorem 2.2]

**Proof.** Let  $F'$  be the Fourier-Mukai functor which is a sibling of  $F$  as in Lemma 16.2. By Proposition 13.4 we have  $F \cong F'$  provided we can show that  $\text{Coh}(\mathcal{O}_X)$  has enough negative objects. However, if  $X = \text{Spec}(k)$  for example, then this isn't true. Thus we first decompose  $X = \coprod X_i$  into its connected (and irreducible) components and we argue that it suffices to prove the result for each of the (fully faithful) composition functors

$$F_i : D_{\text{perf}}(\mathcal{O}_{X_i}) \rightarrow D_{\text{perf}}(\mathcal{O}_X) \rightarrow D_{\text{perf}}(\mathcal{O}_Y)$$

Details omitted. Thus we may assume  $X$  is irreducible.

The case  $\dim(X) = 0$ . Here  $X$  is the spectrum of a finite (separable) extension  $k'/k$  and hence  $D_{\text{perf}}(\mathcal{O}_X)$  is equivalent to the category of graded  $k'$ -vector spaces such that  $\mathcal{O}_X$  corresponds to the trivial 1-dimensional vector space in degree 0. It is straightforward to see that any two siblings  $F, F' : D_{\text{perf}}(\mathcal{O}_X) \rightarrow D_{\text{perf}}(\mathcal{O}_Y)$  are isomorphic. Namely, we are given an isomorphism  $F(\mathcal{O}_X) \cong F'(\mathcal{O}_X)$  compatible the action of the  $k$ -algebra  $k' = \text{End}_{D_{\text{perf}}(\mathcal{O}_X)}(\mathcal{O}_X)$  which extends canonically to an isomorphism on any graded  $k'$ -vector space.

The case  $\dim(X) > 0$ . Here  $X$  is a projective smooth variety of dimension  $> 1$ . Let  $\mathcal{F}$  be a coherent  $\mathcal{O}_X$ -module. We have to show there exists a coherent module  $\mathcal{N}$  such that

- (1) there is a surjection  $\mathcal{N} \rightarrow \mathcal{F}$ ,
- (2)  $\text{Ext}^q(\mathcal{N}, \mathcal{F}) = 0$  for  $q > 0$ ,
- (3)  $\text{Hom}(\mathcal{F}, \mathcal{N}) = 0$ .

Choose an ample invertible  $\mathcal{O}_X$ -module  $\mathcal{L}$ . We claim that  $\mathcal{N} = (\mathcal{L}^{\otimes n})^{\oplus r}$  will work for  $n \ll 0$  and  $r$  large enough. Condition (1) follows from Properties, Proposition 26.13. Condition (2) follows from  $\text{Ext}^q(\mathcal{L}^{\otimes n}, \mathcal{F}) = H^q(X, \mathcal{F} \otimes \mathcal{L}^{\otimes -n})$  and Cohomology of Schemes, Lemma 17.1. Finally, we have

$$\text{Hom}(\mathcal{F}, \mathcal{L}^{\otimes n}) = H^0(X, \text{Hom}(\mathcal{F}, \mathcal{L}^{\otimes n})) = H^0(X, \text{Hom}(\mathcal{F}, \mathcal{O}_X) \otimes \mathcal{L}^{\otimes n})$$

Since the dual  $\text{Hom}(\mathcal{F}, \mathcal{O}_X)$  is torsion free, this vanishes for  $n \ll 0$  by Varieties, Lemma 47.1. This finishes the proof.  $\square$

0G0D **Proposition 16.4.** *Let  $k$  be a field. Let  $X$  and  $Y$  be smooth proper schemes over  $k$ . If  $F : D_{\text{perf}}(\mathcal{O}_X) \rightarrow D_{\text{perf}}(\mathcal{O}_Y)$  is a  $k$ -linear exact equivalence of triangulated categories then there exists a Fourier-Mukai functor  $F' : D_{\text{perf}}(\mathcal{O}_X) \rightarrow D_{\text{perf}}(\mathcal{O}_Y)$  whose kernel is in  $D_{\text{perf}}(\mathcal{O}_{X \times Y})$  which is an equivalence and a sibling of  $F$ .*

**Proof.** The functor  $F'$  of Lemma 16.2 is an equivalence by Lemma 13.3.  $\square$

0G0E **Lemma 16.5.** *Let  $k$  be a field. Let  $X$  be a smooth proper scheme over  $k$ . Let  $K \in D_{\text{perf}}(\mathcal{O}_{X \times X})$ . If the Fourier-Mukai functor  $\Phi_K : D_{\text{perf}}(\mathcal{O}_X) \rightarrow D_{\text{perf}}(\mathcal{O}_X)$  is isomorphic to the identity functor, then  $K \cong \Delta_* \mathcal{O}_X$  in  $D_{\text{perf}}(\mathcal{O}_{X \times X})$ .*

**Proof.** Let  $i$  be the minimal integer such that the cohomology sheaf  $H^i(K)$  is nonzero. Let  $\mathcal{E}$  and  $\mathcal{G}$  be finite locally free  $\mathcal{O}_X$ -modules. Then

$$\begin{aligned} H^i(X \times X, K \otimes_{\mathcal{O}_{X \times X}}^{\mathbf{L}} (\mathcal{E} \boxtimes \mathcal{G})) &= H^i(X, R\text{pr}_{2,*}(K \otimes_{\mathcal{O}_{X \times X}}^{\mathbf{L}} (\mathcal{E} \boxtimes \mathcal{G}))) \\ &= H^i(X, \Phi_K(\mathcal{E}) \otimes_{\mathcal{O}_X}^{\mathbf{L}} \mathcal{G}) \\ &\cong H^i(X, \mathcal{E} \otimes \mathcal{G}) \end{aligned}$$

which is zero if  $i < 0$ . On the other hand, we can choose  $\mathcal{E}$  and  $\mathcal{G}$  such that there is a surjection  $\mathcal{E}^\vee \boxtimes \mathcal{G}^\vee \rightarrow H^i(K)$  by Lemma 10.1. In this case the left hand side of the equalities is nonzero. Hence we conclude that  $H^i(K) = 0$  for  $i < 0$ .

Let  $i$  be the maximal integer such that  $H^i(K)$  is nonzero. The same argument with  $\mathcal{E}$  and  $\mathcal{G}$  support of dimension 0 shows that  $i \leq 0$ . Hence we conclude that  $K$  is given by a single coherent  $\mathcal{O}_{X \times X}$ -module  $\mathcal{K}$  sitting in degree 0.

Since  $R\text{pr}_{2,*}(\text{pr}_1^* \mathcal{F} \otimes \mathcal{K})$  is  $\mathcal{F}$ , by taking  $\mathcal{F}$  supported at closed points we see that the support of  $\mathcal{K}$  is finite over  $X$  via  $\text{pr}_2$ . Since  $R\text{pr}_{2,*}(\mathcal{K}) \cong \mathcal{O}_X$  we conclude by Lemma 12.4 that  $\mathcal{K} = s_* \mathcal{O}_X$  for some section  $s : X \rightarrow X \times X$  of the second projection. Then  $\Phi_K(M) = f^* M$  where  $f = \text{pr}_1 \circ s$  and this can happen only if  $s$  is the diagonal morphism as desired.  $\square$

## 17. A category of Fourier-Mukai kernels

0G0F Let  $S$  be a scheme. We claim there is a category with

- (1) Objects are proper smooth schemes over  $S$ .
- (2) Morphisms from  $X$  to  $Y$  are isomorphism classes of objects of  $D_{\text{perf}}(\mathcal{O}_{X \times_S Y})$ .
- (3) Composition of the isomorphism class of  $K \in D_{\text{perf}}(\mathcal{O}_{X \times_S Y})$  and the isomorphism class of  $K' \in D_{\text{perf}}(\mathcal{O}_{Y \times_S Z})$  is the isomorphism class of

$$R\text{pr}_{13,*}(\text{Lpr}_{12}^* K \otimes_{\mathcal{O}_{X \times_S Y \times_S Z}}^{\mathbf{L}} \text{Lpr}_{23}^* K')$$

- which is in  $D_{perf}(\mathcal{O}_{X \times_S Z})$  by Derived Categories of Schemes, Lemma 28.4.
- (4) The identity morphism from  $X$  to  $X$  is the isomorphism class of  $\Delta_{X/S,*}\mathcal{O}_X$  which is in  $D_{perf}(\mathcal{O}_{X \times_S X})$  by More on Morphisms, Lemma 53.12 and the fact that  $\Delta_{X/S}$  is a perfect morphism by Divisors, Lemma 22.11 and More on Morphisms, Lemma 53.7.

Let us check that associativity of composition of morphisms holds; we omit verifying that the identity morphisms are indeed identities. To see this suppose we have  $X, Y, Z, W$  and  $c \in D_{perf}(\mathcal{O}_{X \times_S Y})$ ,  $c' \in D_{perf}(\mathcal{O}_{Y \times_S Z})$ , and  $c'' \in D_{perf}(\mathcal{O}_{Z \times_S W})$ . Then we have

$$\begin{aligned}
c'' \circ (c' \circ c) &\cong \mathrm{pr}_{14,*}^{134} (\mathrm{pr}_{13}^{134,*} \mathrm{pr}_{13,*}^{123} (\mathrm{pr}_{12}^{123,*} c \otimes \mathrm{pr}_{23}^{123,*} c') \otimes \mathrm{pr}_{34}^{134,*} c'') \\
&\cong \mathrm{pr}_{14,*}^{134} (\mathrm{pr}_{134,*}^{1234} \mathrm{pr}_{123}^{1234,*} (\mathrm{pr}_{12}^{123,*} c \otimes \mathrm{pr}_{23}^{123,*} c') \otimes \mathrm{pr}_{34}^{134,*} c'') \\
&\cong \mathrm{pr}_{14,*}^{134} (\mathrm{pr}_{134,*}^{1234} (\mathrm{pr}_{12}^{1234,*} c \otimes \mathrm{pr}_{23}^{1234,*} c') \otimes \mathrm{pr}_{34}^{134,*} c'') \\
&\cong \mathrm{pr}_{14,*}^{134} \mathrm{pr}_{134,*}^{1234} ((\mathrm{pr}_{12}^{1234,*} c \otimes \mathrm{pr}_{23}^{1234,*} c') \otimes \mathrm{pr}_{34}^{1234,*} c'') \\
&\cong \mathrm{pr}_{14,*}^{1234} ((\mathrm{pr}_{12}^{1234,*} c \otimes \mathrm{pr}_{23}^{1234,*} c') \otimes \mathrm{pr}_{34}^{1234,*} c'')
\end{aligned}$$

Here we use the notation

$$p_{134}^{1234} : X \times_S Y \times_S Z \times_S W \rightarrow X \times_S Z \times_S W \quad \text{and} \quad p_{14}^{134} : X \times_S Z \times_S W \rightarrow X \times_S W$$

the projections and similarly for other indices. We also write  $\mathrm{pr}_*$  instead of  $R\mathrm{pr}_*$  and  $\mathrm{pr}^*$  instead of  $L\mathrm{pr}^*$  and we drop all super and sub scripts on  $\otimes$ . The first equality is the definition of the composition. The second equality holds because  $\mathrm{pr}_{13}^{134,*} \mathrm{pr}_{13,*}^{123} = \mathrm{pr}_{134,*}^{1234} \mathrm{pr}_{123}^{1234,*}$  by base change (Derived Categories of Schemes, Lemma 21.5). The third equality holds because pullbacks compose correctly and pass through tensor products, see Cohomology, Lemmas 27.2 and 27.3. The fourth equality follows from the ‘‘projection formula’’ for  $p_{134}^{1234}$ , see Derived Categories of Schemes, Lemma 21.1. The fifth equality is that proper pushforward is compatible with composition, see Cohomology, Lemma 28.2. Since tensor product is associative this concludes the proof of associativity of composition.

0G0G **Lemma 17.1.** *Let  $S' \rightarrow S$  be a morphism of schemes. The rule which sends*

- (1) *a smooth proper scheme  $X$  over  $S$  to  $X' = S' \times_S X$ , and*
- (2) *the isomorphism class of an object  $K$  of  $D_{perf}(\mathcal{O}_{X \times_S Y})$  to the isomorphism class of  $L(X' \times_{S'} Y' \rightarrow X \times_S Y)^* K$  in  $D_{perf}(\mathcal{O}_{X' \times_{S'} Y'})$*

*is a functor from the category defined for  $S$  to the category defined for  $S'$ .*

**Proof.** To see this suppose we have  $X, Y, Z$  and  $K \in D_{perf}(\mathcal{O}_{X \times_S Y})$  and  $M \in D_{perf}(\mathcal{O}_{Y \times_S Z})$ . Denote  $K' \in D_{perf}(\mathcal{O}_{X' \times_{S'} Y'})$  and  $M' \in D_{perf}(\mathcal{O}_{Y' \times_{S'} Z'})$  their pullbacks as in the statement of the lemma. The diagram

$$\begin{array}{ccc}
X' \times_{S'} Y' \times_{S'} Z' & \longrightarrow & X \times_S Y \times_S Z \\
\mathrm{pr}'_{13} \downarrow & & \downarrow \mathrm{pr}_{13} \\
X' \times_{S'} Z' & \longrightarrow & X \times_S Z
\end{array}$$

is cartesian and  $\mathrm{pr}_{13}$  is proper and smooth. By Derived Categories of Schemes, Lemma 28.4 we see that the derived pullback by the lower horizontal arrow of the composition

$$R\mathrm{pr}_{13,*} (L\mathrm{pr}'_{12,*} K \otimes_{\mathcal{O}_{X \times_S Y \times_S Z}}^{\mathbf{L}} L\mathrm{pr}'_{23,*} M)$$

indeed is (canonically) isomorphic to

$$R\mathrm{pr}'_{13,*}(L(\mathrm{pr}'_{12})^*K' \otimes_{\mathcal{O}_{X' \times_S Y' \times_S Z'}}^{\mathbf{L}} L(\mathrm{pr}'_{23})^*M')$$

as desired. Some details omitted.  $\square$

### 18. Relative equivalences

0G0H In this section we prove some lemmas about the following concept.

0G0I **Definition 18.1.** Let  $S$  be a scheme. Let  $X \rightarrow S$  and  $Y \rightarrow S$  be smooth proper morphisms. An object  $K \in D_{\mathrm{perf}}(\mathcal{O}_{X \times_S Y})$  is said to be *the Fourier-Mukai kernel of a relative equivalence from  $X$  to  $Y$  over  $S$*  if there exist an object  $K' \in D_{\mathrm{perf}}(\mathcal{O}_{X \times_S Y})$  such that

$$\Delta_{X/S,*}\mathcal{O}_X \cong R\mathrm{pr}_{13,*}(L\mathrm{pr}_{12}^*K \otimes_{\mathcal{O}_{X \times_S Y \times_S X}}^{\mathbf{L}} L\mathrm{pr}_{23}^*K')$$

in  $D(\mathcal{O}_{X \times_S X})$  and

$$\Delta_{Y/S,*}\mathcal{O}_Y \cong R\mathrm{pr}_{13,*}(L\mathrm{pr}_{12}^*K' \otimes_{\mathcal{O}_{Y \times_S X \times_S Y}}^{\mathbf{L}} L\mathrm{pr}_{23}^*K)$$

in  $D(\mathcal{O}_{Y \times_S Y})$ . In other words, the isomorphism class of  $K$  defines an invertible arrow in the category defined in Section 17.

The language is intentionally cumbersome.

0G0J **Lemma 18.2.** *With notation as in Definition 18.1 let  $K$  be the Fourier-Mukai kernel of a relative equivalence from  $X$  to  $Y$  over  $S$ . Then the corresponding Fourier-Mukai functors  $\Phi_K : D_{\mathrm{QCoh}}(\mathcal{O}_X) \rightarrow D_{\mathrm{QCoh}}(\mathcal{O}_Y)$  (Lemma 9.2) and  $\Phi_K : D_{\mathrm{perf}}(\mathcal{O}_X) \rightarrow D_{\mathrm{perf}}(\mathcal{O}_Y)$  (Lemma 9.4) are equivalences.*

**Proof.** Immediate from Lemma 9.3 and Example 9.6.  $\square$

0G0K **Lemma 18.3.** *With notation as in Definition 18.1 let  $K$  be the Fourier-Mukai kernel of a relative equivalence from  $X$  to  $Y$  over  $S$ . Let  $S_1 \rightarrow S$  be a morphism of schemes. Let  $X_1 = S_1 \times_S X$  and  $Y_1 = S_1 \times_S Y$ . Then the pullback  $K_1 = L(X_1 \times_{S_1} Y_1 \rightarrow X \times_S Y)^*K$  is the Fourier-Mukai kernel of a relative equivalence from  $X_1$  to  $Y_1$  over  $S_1$ .*

**Proof.** Let  $K' \in D_{\mathrm{perf}}(\mathcal{O}_{Y \times_S X})$  be the object assumed to exist in Definition 18.1. Denote  $K'_1$  the pullback of  $K'$  by  $Y_1 \times_{S_1} X_1 \rightarrow Y \times_S X$ . Then it suffices to prove that we have

$$\Delta_{X_1/S_1,*}\mathcal{O}_{X_1} \cong R\mathrm{pr}_{13,*}(L\mathrm{pr}_{12}^*K_1 \otimes_{\mathcal{O}_{X_1 \times_{S_1} Y_1 \times_{S_1} X_1}}^{\mathbf{L}} L\mathrm{pr}_{23}^*K'_1)$$

in  $D(\mathcal{O}_{X_1 \times_{S_1} X_1})$  and similarly for the other condition. Since

$$\begin{array}{ccc} X_1 \times_{S_1} Y_1 \times_{S_1} X_1 & \longrightarrow & X \times_S Y \times_S X \\ \mathrm{pr}_{13} \downarrow & & \downarrow \mathrm{pr}_{13} \\ X_1 \times_{S_1} X_1 & \longrightarrow & X \times_S X \end{array}$$

is cartesian it suffices by Derived Categories of Schemes, Lemma 28.4 to prove that

$$\Delta_{X_1/S_1,*}\mathcal{O}_{X_1} \cong L(X_1 \times_{S_1} X_1 \rightarrow X \times_S X)^*\Delta_{X/S,*}\mathcal{O}_X$$

This in turn will be true if  $X$  and  $X_1 \times_{S_1} X_1$  are tor independent over  $X \times_S X$ , see Derived Categories of Schemes, Lemma 21.5. This tor independence can be seen

directly but also follows from the more general More on Morphisms, Lemma 61.1 applied to the square with corners  $X, X, X, S$  and its base change by  $S_1 \rightarrow S$ .  $\square$

0G0L **Lemma 18.4.** *Let  $S = \lim_{i \in I} S_i$  be a limit of a directed system of schemes with affine transition morphisms  $g_{i'} : S_{i'} \rightarrow S_i$ . We assume that  $S_i$  is quasi-compact and quasi-separated for all  $i \in I$ . Let  $0 \in I$ . Let  $X_0 \rightarrow S_0$  and  $Y_0 \rightarrow S_0$  be smooth proper morphisms. We set  $X_i = S_i \times_{S_0} X_0$  for  $i \geq 0$  and  $X = S \times_{S_0} X_0$  and similarly for  $Y_0$ . If  $K$  is the Fourier-Mukai kernel of a relative equivalence from  $X$  to  $Y$  over  $S$  then for some  $i \geq 0$  there exists a Fourier-Mukai kernel of a relative equivalence from  $X_i$  to  $Y_i$  over  $S_i$ .*

**Proof.** Let  $K' \in D_{perf}(\mathcal{O}_{Y \times_S X})$  be the object assumed to exist in Definition 18.1. Since  $X \times_S Y = \lim X_i \times_{S_i} Y_i$  there exists an  $i$  and objects  $K_i$  and  $K'_i$  in  $D_{perf}(\mathcal{O}_{Y_i \times_{S_i} X_i})$  whose pullbacks to  $Y \times_S X$  give  $K$  and  $K'$ . See Derived Categories of Schemes, Lemma 27.3. By Derived Categories of Schemes, Lemma 28.4 the object

$$R\mathrm{pr}_{13,*}(L\mathrm{pr}_{12}^* K_i \otimes_{\mathcal{O}_{X_i \times_{S_i} Y_i \times_{S_i} X_i}}^{\mathbf{L}} L\mathrm{pr}_{23}^* K'_i)$$

is perfect and its pullback to  $X \times_S X$  is equal to

$$R\mathrm{pr}_{13,*}(L\mathrm{pr}_{12}^* K \otimes_{\mathcal{O}_{X \times_S Y \times_S X}}^{\mathbf{L}} L\mathrm{pr}_{23}^* K') \cong \Delta_{X/S,*} \mathcal{O}_X$$

See proof of Lemma 18.3. On the other hand, since  $X_i \rightarrow S$  is smooth and separated the object

$$\Delta_{i,*} \mathcal{O}_{X_i}$$

of  $D(\mathcal{O}_{X_i \times_{S_i} X_i})$  is also perfect (by More on Morphisms, Lemmas 54.18 and 53.13) and its pullback to  $X \times_S X$  is equal to

$$\Delta_{X/S,*} \mathcal{O}_X$$

See proof of Lemma 18.3. Thus by Derived Categories of Schemes, Lemma 27.3 after increasing  $i$  we may assume that

$$\Delta_{i,*} \mathcal{O}_{X_i} \cong R\mathrm{pr}_{13,*}(L\mathrm{pr}_{12}^* K_i \otimes_{\mathcal{O}_{X_i \times_{S_i} Y_i \times_{S_i} X_i}}^{\mathbf{L}} L\mathrm{pr}_{23}^* K'_i)$$

as desired. The same works for the roles of  $K$  and  $K'$  reversed.  $\square$

## 19. No deformations

0G0M The title of this section refers to Lemma 19.4

0G0N **Lemma 19.1.** *Let  $(R, \mathfrak{m}, \kappa) \rightarrow (A, \mathfrak{n}, \lambda)$  be a flat local ring homomorphism of local rings which is essentially of finite presentation. Let  $\bar{f}_1, \dots, \bar{f}_r \in \mathfrak{n}/\mathfrak{m}A \subset A/\mathfrak{m}A$  be a regular sequence. Let  $K \in D(A)$ . Assume*

(1)  $K$  is perfect,

(2)  $K \otimes_A^{\mathbf{L}} A/\mathfrak{m}A$  is isomorphic in  $D(A/\mathfrak{m}A)$  to the Koszul complex on  $\bar{f}_1, \dots, \bar{f}_r$ .

*Then  $K$  is isomorphic in  $D(A)$  to a Koszul complex on a regular sequence  $f_1, \dots, f_r \in A$  lifting the given elements  $\bar{f}_1, \dots, \bar{f}_r$ . Moreover,  $A/(f_1, \dots, f_r)$  is flat over  $R$ .*

**Proof.** Let us use chain complexes in the proof of this lemma. The Koszul complex  $K_\bullet(\bar{f}_1, \dots, \bar{f}_r)$  is defined in More on Algebra, Definition 28.2. By More on Algebra, Lemma 71.4 we can represent  $K$  by a complex

$$K_\bullet : R \rightarrow R^{\oplus r} \rightarrow \dots \rightarrow R^{\oplus r} \rightarrow R$$



whose tensor product with  $R/I$  is equal (!) to  $K_{\bullet}(\bar{f}_1, \dots, \bar{f}_r)$ . Denote  $f_1, \dots, f_r \in R$  the components of the arrow  $R^{\oplus r} \rightarrow R$ . These  $f_i$  are lifts of the  $\bar{f}_i$ . By Algebra, Lemma 127.6  $f_1, \dots, f_r$  form a regular sequence in  $A$  and  $A/(f_1, \dots, f_r)$  is flat over  $R$ . Let  $J = (f_1, \dots, f_r) \subset A$ . Consider the diagram

$$\begin{array}{ccc} K_{\bullet} & \xrightarrow{\varphi_{\bullet}} & K_{\bullet}(f_1, \dots, f_r) \\ & \searrow & \swarrow \\ & A/J & \end{array}$$

Since  $f_1, \dots, f_r$  is a regular sequence the south-east arrow is a quasi-isomorphism (see More on Algebra, Lemma 29.2). Hence we can find the dotted arrow making the diagram commute for example by Algebra, Lemma 70.4. Reducing modulo  $\mathfrak{m}$  we obtain a commutative diagram

$$\begin{array}{ccc} K_{\bullet}(\bar{f}_1, \dots, \bar{f}_r) & \xrightarrow{\bar{\varphi}_{\bullet}} & K_{\bullet}(\bar{f}_1, \dots, \bar{f}_r) \\ & \searrow & \swarrow \\ & (A/\mathfrak{m}A)/(\bar{f}_1, \dots, \bar{f}_r) & \end{array}$$

by our choice of  $K_{\bullet}$ . Thus  $\bar{\varphi}$  is an isomorphism in the derived category  $D(A/\mathfrak{m}A)$ . It follows that  $\bar{\varphi} \otimes_{A/\mathfrak{m}A}^{\mathbf{L}} \lambda$  is an isomorphism. Since  $\bar{f}_i \in \mathfrak{n}/\mathfrak{m}A$  we see that

$$\mathrm{Tor}_i^{A/\mathfrak{m}A}(K_{\bullet}(\bar{f}_1, \dots, \bar{f}_r), \lambda) = K_i(\bar{f}_1, \dots, \bar{f}_r) \otimes_{A/\mathfrak{m}A} \lambda$$

Hence  $\varphi_i \bmod \mathfrak{n}$  is invertible. Since  $A$  is local this means that  $\varphi_i$  is an isomorphism and the proof is complete.  $\square$

0G0P **Lemma 19.2.** *Let  $R \rightarrow S$  be a finite type flat ring map of Noetherian rings. Let  $\mathfrak{q} \subset S$  be a prime ideal lying over  $\mathfrak{p} \subset R$ . Let  $K \in D(S)$  be perfect. Let  $f_1, \dots, f_r \in \mathfrak{q}S_{\mathfrak{q}}$  be a regular sequence such that  $S_{\mathfrak{q}}/(f_1, \dots, f_r)$  is flat over  $R$  and such that  $K \otimes_S^{\mathbf{L}} S_{\mathfrak{q}}$  is isomorphic to the Koszul complex on  $f_1, \dots, f_r$ . Then there exists a  $g \in S$ ,  $g \notin \mathfrak{q}$  such that*

- (1)  $f_1, \dots, f_r$  are the images of  $f'_1, \dots, f'_r \in S_g$ ,
- (2)  $f'_1, \dots, f'_r$  form a regular sequence in  $S_g$ ,
- (3)  $S_g/(f'_1, \dots, f'_r)$  is flat over  $R$ ,
- (4)  $K \otimes_S^{\mathbf{L}} S_g$  is isomorphic to the Koszul complex on  $f_1, \dots, f_r$ .

**Proof.** We can find  $g \in S$ ,  $g \notin \mathfrak{q}$  with property (1) by the definition of localizations. After replacing  $g$  by  $gg'$  for some  $g' \in S$ ,  $g' \notin \mathfrak{q}$  we may assume (2) holds, see Algebra, Lemma 67.6. By Algebra, Theorem 128.4 we find that  $S_g/(f'_1, \dots, f'_r)$  is flat over  $R$  in an open neighbourhood of  $\mathfrak{q}$ . Hence after once more replacing  $g$  by  $gg'$  for some  $g' \in S$ ,  $g' \notin \mathfrak{q}$  we may assume (3) holds as well. Finally, we get (4) for a further replacement by More on Algebra, Lemma 70.16.  $\square$

For a generalization of the following lemma, please see More on Morphisms of Spaces, Lemma 49.6.

0G0Q **Lemma 19.3.** *Let  $S$  be a Noetherian scheme. Let  $s \in S$ . Let  $p : X \rightarrow Y$  be a morphism of schemes over  $S$ . Assume*

- (1)  $Y \rightarrow S$  and  $X \rightarrow S$  proper,
- (2)  $X$  is flat over  $S$ ,

(3)  $X_s \rightarrow Y_s$  an isomorphism.

Then there exists an open neighbourhood  $U \subset S$  of  $s$  such that the base change  $X_U \rightarrow Y_U$  is a closed immersion.

**Proof.** The morphism  $p$  is proper by Morphisms, Lemma 40.6. By Cohomology of Schemes, Lemma 21.2 there is an open  $Y_s \subset V \subset Y$  such that  $p|_{p^{-1}(V)} : p^{-1}(V) \rightarrow V$  is finite. By More on Morphisms, Theorem 16.1 there is an open  $X_s \subset U \subset X$  such that  $p|_U : U \rightarrow Y$  is flat. After removing the images of  $X \setminus U$  and  $Y \setminus V$  (which are closed subsets not containing  $s$ ) we may assume  $p$  is flat and finite. Then  $p$  is open (Morphisms, Lemma 25.10) and  $Y_s \subset p(X) \subset Y$  hence after shrinking  $S$  we may assume  $p$  is surjective. As  $p_s : X_s \rightarrow Y_s$  is an isomorphism, the map

$$p^\sharp : \mathcal{O}_Y \longrightarrow p_* \mathcal{O}_X$$

of coherent  $\mathcal{O}_Y$ -modules ( $p$  is finite) becomes an isomorphism after pullback by  $i : X_s \rightarrow X$  (by Cohomology of Schemes, Lemma 5.1 for example). By Nakayama's lemma, this implies that  $\mathcal{O}_{Y,y} \rightarrow (p_* \mathcal{O}_X)_y$  is surjective for all  $y \in X_s$ . Hence there is an open  $Y_s \subset V \subset Y$  such that  $p^\sharp|_V$  is surjective (Modules, Lemma 9.4). Hence after shrinking  $S$  once more we may assume  $p^\sharp$  is surjective which means that  $p$  is a closed immersion (as  $p$  is already finite). Thus now  $p$  is a surjective flat closed immersion of Noetherian schemes and hence an isomorphism, see Morphisms, Section 26.  $\square$

0G0R **Lemma 19.4.** *Let  $k$  be a field. Let  $S$  be a finite type scheme over  $k$  with  $k$ -rational point  $s$ . Let  $Y \rightarrow S$  be a smooth proper morphism. Let  $X = Y_s \times S \rightarrow S$  be the constant family with fibre  $Y_s$ . Let  $K$  be the Fourier-Mukai kernel of a relative equivalence from  $X$  to  $Y$  over  $S$ . Assume the restriction*

$$L(Y_s \times_S Y_s \rightarrow X \times_S Y)^* K \cong \Delta_{Y_s/k,*} \mathcal{O}_{Y_s}$$

in  $D(\mathcal{O}_{Y_s \times Y_s})$ . Then there is an open neighbourhood  $s \in U \subset S$  such that  $Y|_U$  is isomorphic to  $Y_s \times U$  over  $U$ .

**Proof.** Denote  $i : Y_s \times Y_s = X_s \times Y_s \rightarrow X \times_S Y$  the natural closed immersion. (We will write  $Y_s$  and not  $X_s$  for the fibre of  $X$  over  $s$  from now on.) Let  $z \in Y_s \times Y_s = (X \times_S Y)_s \subset X \times_S Y$  be a closed point. As indicated we think of  $z$  both as a closed point of  $Y_s \times Y_s$  as well as a closed point of  $X \times_S Y$ .

Case I:  $z \notin \Delta_{Y_s/k}(Y_s)$ . Denote  $\mathcal{O}_z$  the coherent  $\mathcal{O}_{Y_s \times Y_s}$ -module supported at  $z$  whose value is  $\kappa(z)$ . Then  $i_* \mathcal{O}_z$  is the coherent  $\mathcal{O}_{X \times_S Y}$ -module supported at  $z$  whose value is  $\kappa(z)$ . Our assumption means that

$$K \otimes_{\mathcal{O}_{X \times_S Y}}^{\mathbf{L}} i_* \mathcal{O}_z = Li^* K \otimes_{\mathcal{O}_{Y_s \times Y_s}}^{\mathbf{L}} \mathcal{O}_z = 0$$

Hence by Lemma 14.3 we find an open neighbourhood  $U(z) \subset X \times_S Y$  of  $z$  such that  $K|_{U(z)} = 0$ . In this case we set  $Z(z) = \emptyset$  as closed subscheme of  $U(z)$ .

Case II:  $z \in \Delta_{Y_s/k}(Y_s)$ . Since  $Y_s$  is smooth over  $k$  we know that  $\Delta_{Y_s/k} : Y_s \rightarrow Y_s \times Y_s$  is a regular immersion, see More on Morphisms, Lemma 54.18. Choose a regular sequence  $\bar{f}_1, \dots, \bar{f}_r \in \mathcal{O}_{Y_s \times Y_s, z}$  cutting out the ideal sheaf of  $\Delta_{Y_s/k}(Y_s)$ . Since a regular sequence is Koszul-regular (More on Algebra, Lemma 29.2) our assumption means that

$$K_z \otimes_{\mathcal{O}_{X \times_S Y, z}}^{\mathbf{L}} \mathcal{O}_{Y_s \times Y_s, z} \in D(\mathcal{O}_{Y_s \times Y_s, z})$$

is represented by the Koszul complex on  $\bar{f}_1, \dots, \bar{f}_r$  over  $\mathcal{O}_{Y_s \times Y_s, z}$ . By Lemma 19.1 applied to  $\mathcal{O}_{S, s} \rightarrow \mathcal{O}_{X \times_S Y, z}$  we conclude that  $K_z \in D(\mathcal{O}_{X \times_S Y, z})$  is represented by the Koszul complex on a regular sequence  $f_1, \dots, f_r \in \mathcal{O}_{X \times_S Y, z}$  lifting the regular sequence  $\bar{f}_1, \dots, \bar{f}_r$  such that moreover  $\mathcal{O}_{X \times_S Y} / (f_1, \dots, f_r)$  is flat over  $\mathcal{O}_{S, s}$ . By some limit arguments (Lemma 19.2) we conclude that there exists an affine open neighbourhood  $U(z) \subset X \times_S Y$  of  $z$  and a closed subscheme  $Z(z) \subset U(z)$  such that

- (1)  $Z(z) \rightarrow U(z)$  is a regular closed immersion,
- (2)  $K|_{U(z)}$  is quasi-isomorphic to  $\mathcal{O}_{Z(z)}$ ,
- (3)  $Z(z) \rightarrow S$  is flat,
- (4)  $Z(z)_s = \Delta_{Y_s/k}(Y_s) \cap U(z)_s$  as closed subschemes of  $U(z)_s$ .

By property (2), for  $z, z' \in Y_s \times Y_s$ , we find that  $Z(z) \cap U(z') = Z(z') \cap U(z)$  as closed subschemes. Hence we obtain an open neighbourhood

$$U = \bigcup_{z \in Y_s \times Y_s \text{ closed}} U(z)$$

of  $Y_s \times Y_s$  in  $X \times_S Y$  and a closed subscheme  $Z \subset U$  such that (1)  $Z \rightarrow U$  is a regular closed immersion, (2)  $Z \rightarrow S$  is flat, and (3)  $Z_s = \Delta_{Y_s/k}(Y_s)$ . Since  $X \times_S Y \rightarrow S$  is proper, after replacing  $S$  by an open neighbourhood of  $s$  we may assume  $U = X \times_S Y$ . Since the projections  $Z_s \rightarrow Y_s$  and  $Z_s \rightarrow X_s$  are isomorphisms, we conclude that after shrinking  $S$  we may assume  $Z \rightarrow Y$  and  $Z \rightarrow X$  are isomorphisms, see Lemma 19.3. This finishes the proof.  $\square$

0G0S **Lemma 19.5.** *Let  $k$  be an algebraically closed field. Let  $X$  be a smooth proper scheme over  $k$ . Let  $f : Y \rightarrow S$  be a smooth proper morphism with  $S$  of finite type over  $k$ . Let  $K$  be the Fourier-Mukai kernel of a relative equivalence from  $X \times S$  to  $Y$  over  $S$ . Then  $S$  can be covered by open subschemes  $U$  such that there is a  $U$ -isomorphism  $f^{-1}(U) \cong Y_0 \times U$  for some  $Y_0$  proper and smooth over  $k$ .*

**Proof.** Choose a closed point  $s \in S$ . Since  $k$  is algebraically closed this is a  $k$ -rational point. Set  $Y_0 = Y_s$ . The restriction  $K_0$  of  $K$  to  $X \times Y_0$  is the Fourier-Mukai kernel of a relative equivalence from  $X$  to  $Y_0$  over  $\text{Spec}(k)$  by Lemma 18.3. Let  $K'_0$  in  $D_{\text{perf}}(\mathcal{O}_{Y_0 \times X})$  be the object assumed to exist in Definition 18.1. Then  $K'_0$  is the Fourier-Mukai kernel of a relative equivalence from  $Y_0$  to  $X$  over  $\text{Spec}(k)$  by the symmetry inherent in Definition 18.1. Hence by Lemma 18.3 we see that the pullback

$$M = (Y_0 \times X \times S \rightarrow Y_0 \times X)^* K'_0$$

on  $(Y_0 \times S) \times_S (X \times S) = Y_0 \times X \times S$  is the Fourier-Mukai kernel of a relative equivalence from  $Y_0 \times S$  to  $X \times S$  over  $S$ . Now consider the kernel

$$K_{\text{new}} = R\text{pr}_{13,*} (L\text{pr}_{12}^* M \otimes_{\mathcal{O}_{(Y_0 \times S) \times_S (X \times S) \times_S Y}}^{\mathbf{L}}} L\text{pr}_{23}^* K)$$

on  $(Y_0 \times S) \times_S Y$ . This is the Fourier-Mukai kernel of a relative equivalence from  $Y_0 \times S$  to  $Y$  over  $S$  since it is the composition of two invertible arrows in the category constructed in Section 17. Moreover, this composition passes through base change (Lemma 17.1). Hence we see that the pullback of  $K_{\text{new}}$  to  $((Y_0 \times S) \times_S Y)_s = Y_0 \times Y_0$  is equal to the composition of  $K_0$  and  $K'_0$  and hence equal to the identity in this category. In other words, we have

$$L(Y_0 \times Y_0 \rightarrow (Y_0 \times S) \times_S Y)^* K_{\text{new}} \cong \Delta_{Y_0/k,*} \mathcal{O}_{Y_0}$$

Thus by Lemma 19.4 we conclude that  $Y \rightarrow S$  is isomorphic to  $Y_0 \times S$  in an open neighbourhood of  $s$ . This finishes the proof.  $\square$

## 20. Countability

0G0T In this section we prove some elementary lemmas about countability of certain sets. Let  $\mathcal{C}$  be a category. In this section we will say that  $\mathcal{C}$  is *countable* if

- (1) for any  $X, Y \in \text{Ob}(\mathcal{C})$  the set  $\text{Mor}_{\mathcal{C}}(X, Y)$  is countable, and
- (2) the set of isomorphism classes of objects of  $\mathcal{C}$  is countable.

0G0U **Lemma 20.1.** *Let  $R$  be a countable Noetherian ring. Then the category of schemes of finite type over  $R$  is countable.*

**Proof.** Omitted. □

0G0V **Lemma 20.2.** *Let  $\mathcal{A}$  be a countable abelian category. Then  $D^b(\mathcal{A})$  is countable.*

**Proof.** It suffices to prove the statement for  $D(\mathcal{A})$  as the others are full subcategories of this one. Since every object in  $D(\mathcal{A})$  is a complex of objects of  $\mathcal{A}$  it is immediate that the set of isomorphism classes of objects of  $D^b(\mathcal{A})$  is countable. Moreover, for bounded complexes  $A^\bullet$  and  $B^\bullet$  of  $\mathcal{A}$  it is clear that  $\text{Hom}_{K^b(\mathcal{A})}(A^\bullet, B^\bullet)$  is countable. We have

$$\text{Hom}_{D^b(\mathcal{A})}(A^\bullet, B^\bullet) = \text{colim}_{s: (A')^\bullet \rightarrow A^\bullet \text{ qis and } (A')^\bullet \text{ bounded}} \text{Hom}_{K^b(\mathcal{A})}((A')^\bullet, B^\bullet)$$

by Derived Categories, Lemma 11.6. Thus this is a countable set as a countable colimit of □

0G0W **Lemma 20.3.** *Let  $X$  be a scheme of finite type over a countable Noetherian ring. Then the categories  $D_{\text{perf}}(\mathcal{O}_X)$  and  $D_{\text{Coh}}^b(\mathcal{O}_X)$  are countable.*

**Proof.** Observe that  $X$  is Noetherian by Morphisms, Lemma 15.6. Hence  $D_{\text{perf}}(\mathcal{O}_X)$  is a full subcategory of  $D_{\text{Coh}}^b(\mathcal{O}_X)$  by Derived Categories of Schemes, Lemma 10.6. Thus it suffices to prove the result for  $D_{\text{Coh}}^b(\mathcal{O}_X)$ . Recall that  $D_{\text{Coh}}^b(\mathcal{O}_X) = D^b(\text{Coh}(\mathcal{O}_X))$  by Derived Categories of Schemes, Proposition 10.2. Hence by Lemma 20.2 it suffices to prove that  $\text{Coh}(\mathcal{O}_X)$  is countable. This we omit. □

0G0X **Lemma 20.4.** *Let  $K$  be an algebraically closed field. Let  $S$  be a finite type scheme over  $K$ . Let  $X \rightarrow S$  and  $Y \rightarrow S$  be finite type morphisms. There exists a countable set  $I$  and for  $i \in I$  a pair  $(S_i \rightarrow S, h_i)$  with the following properties*

- (1)  $S_i \rightarrow S$  is a morphism of finite type, set  $X_i = X \times_S S_i$  and  $Y_i = Y \times_S S_i$ ,
- (2)  $h_i : X_i \rightarrow Y_i$  is an isomorphism over  $S_i$ , and
- (3) for any closed point  $s \in S(K)$  if  $X_s \cong Y_s$  over  $K = \kappa(s)$  then  $s$  is in the image of  $S_i \rightarrow S$  for some  $i$ .

**Proof.** The field  $K$  is the filtered union of its countable subfields. Dually,  $\text{Spec}(K)$  is the cofiltered limit of the spectra of the countable subfields of  $K$ . Hence Limits, Lemma 10.1 guarantees that we can find a countable subfield  $k$  and morphisms  $X_0 \rightarrow S_0$  and  $Y_0 \rightarrow S_0$  of schemes of finite type over  $k$  such that  $X \rightarrow S$  and  $Y \rightarrow S$  are the base changes of these.

By Lemma 20.1 there is a countable set  $I$  and pairs  $(S_{0,i} \rightarrow S_0, h_{0,i})$  such that

- (1)  $S_{0,i} \rightarrow S_0$  is a morphism of finite type, set  $X_{0,i} = X_0 \times_{S_0} S_{0,i}$  and  $Y_{0,i} = Y_0 \times_{S_0} S_{0,i}$ ,
- (2)  $h_{0,i} : X_{0,i} \rightarrow Y_{0,i}$  is an isomorphism over  $S_{0,i}$ .

such that every pair  $(T \rightarrow S_0, h_T)$  with  $T \rightarrow S_0$  of finite type and  $h_T : X_0 \times_{S_0} T \rightarrow Y_0 \times_{S_0} T$  an isomorphism is isomorphic to one of these. Denote  $(S_i \rightarrow S, h_i)$  the base change of  $(S_{0,i} \rightarrow S_0, h_{0,i})$  by  $\text{Spec}(K) \rightarrow \text{Spec}(k)$ . We claim this works.

Let  $s \in S(K)$  and let  $h_s : X_s \rightarrow Y_s$  be an isomorphism over  $K = \kappa(s)$ . We can write  $K$  as the filtered union of its finitely generated  $k$ -subalgebras. Hence by Limits, Proposition 6.1 and Lemma 10.1 we can find such a finitely generated  $k$ -subalgebra  $K \supset A \supset k$  such that

- (1) there is a commutative diagram

$$\begin{array}{ccc} \text{Spec}(K) & \longrightarrow & \text{Spec}(A) \\ s \downarrow & & \downarrow s' \\ S & \longrightarrow & S_0 \end{array}$$

for some morphism  $s' : \text{Spec}(A) \rightarrow S_0$  over  $k$ ,

- (2)  $h_s$  is the base change of an isomorphism  $h_{s'} : X_0 \times_{S_0, s'} \text{Spec}(A) \rightarrow X_0 \times_{S_0, s'} \text{Spec}(A)$  over  $A$ .

Of course, then  $(s' : \text{Spec}(A) \rightarrow S_0, h_{s'})$  is isomorphic to the pair  $(S_{0,i} \rightarrow S_0, h_{0,i})$  for some  $i \in I$ . This concludes the proof because the commutative diagram in (1) shows that  $s$  is in the image of the base change of  $s'$  to  $\text{Spec}(K)$ .  $\square$

0G0Y **Lemma 20.5.** *Let  $K$  be an algebraically closed field. There exists a countable set  $I$  and for  $i \in I$  a pair  $(S_i/K, X_i \rightarrow S_i, Y_i \rightarrow S_i, M_i)$  with the following properties*

- (1)  $S_i$  is a scheme of finite type over  $K$ ,
- (2)  $X_i \rightarrow S_i$  and  $Y_i \rightarrow S_i$  are proper smooth morphisms of schemes,
- (3)  $M_i \in D_{\text{perf}}(\mathcal{O}_{X_i \times_{S_i} Y_i})$  is the Fourier-Mukai kernel of a relative equivalence from  $X_i$  to  $Y_i$  over  $S_i$ , and
- (4) for any smooth proper schemes  $X$  and  $Y$  over  $K$  such that there is a  $K$ -linear exact equivalence  $D_{\text{perf}}(\mathcal{O}_X) \rightarrow D_{\text{perf}}(\mathcal{O}_Y)$  there exists an  $i \in I$  and a  $s \in S_i(K)$  such that  $X \cong (X_i)_s$  and  $Y \cong (Y_i)_s$ .

**Proof.** Choose a countable subfield  $k \subset K$  for example the prime field. By Lemmas 20.1 and 20.3 there exists a countable set of isomorphism classes of systems over  $k$  satisfying parts (1), (2), (3) of the lemma. Thus we can choose a countable set  $I$  and for each  $i \in I$  such a system

$$(S_{0,i}/k, X_{0,i} \rightarrow S_{0,i}, Y_{0,i} \rightarrow S_{0,i}, M_{0,i})$$

over  $k$  such that each isomorphism class occurs at least once. Denote  $(S_i/K, X_i \rightarrow S_i, Y_i \rightarrow S_i, M_i)$  the base change of the displayed system to  $K$ . This system has properties (1), (2), (3), see Lemma 18.3. Let us prove property (4).

Consider smooth proper schemes  $X$  and  $Y$  over  $K$  such that there is a  $K$ -linear exact equivalence  $F : D_{\text{perf}}(\mathcal{O}_X) \rightarrow D_{\text{perf}}(\mathcal{O}_Y)$ . By Proposition 16.4 we may assume that there exists an object  $M \in D_{\text{perf}}(\mathcal{O}_{X \times Y})$  such that  $F = \Phi_M$  is the corresponding Fourier-Mukai functor. By Lemma 9.9 there is an  $M'$  in  $D_{\text{perf}}(\mathcal{O}_{Y \times X})$  such that  $\Phi_{M'}$  is the right adjoint to  $\Phi_M$ . Since  $\Phi_M$  is an equivalence, this means that  $\Phi_{M'}$  is the quasi-inverse to  $\Phi_M$ . By Lemma 9.9 we see that the Fourier-Mukai functors defined by the objects

$$A = R\text{pr}_{13,*}(L\text{pr}_{12}^* M \otimes_{\mathcal{O}_{X \times Y \times X}}^{\mathbf{L}} L\text{pr}_{23}^* M')$$

in  $D_{perf}(\mathcal{O}_{X \times X})$  and

$$B = R\mathrm{pr}_{13,*}(L\mathrm{pr}_{12}^*M' \otimes_{\mathcal{O}_{Y \times X \times Y}}^{\mathbf{L}} L\mathrm{pr}_{23}^*M)$$

in  $D_{perf}(\mathcal{O}_{Y \times Y})$  are isomorphic to  $\mathrm{id} : D_{perf}(\mathcal{O}_X) \rightarrow D_{perf}(\mathcal{O}_X)$  and  $\mathrm{id} : D_{perf}(\mathcal{O}_Y) \rightarrow D_{perf}(\mathcal{O}_Y)$ . Hence  $A \cong \Delta_{X/K,*}\mathcal{O}_X$  and  $B \cong \Delta_{Y/K,*}\mathcal{O}_Y$  by Lemma 16.5. Hence we see that  $M$  is the Fourier-Mukai kernel of a relative equivalence from  $X$  to  $Y$  over  $K$  by definition.

We can write  $K$  as the filtered colimit of its finite type  $k$ -subalgebras  $A \subset K$ . By Limits, Lemma 10.1 we can find  $X_0, Y_0$  of finite type over  $A$  whose base changes to  $K$  produces  $X$  and  $Y$ . By Limits, Lemmas 13.1 and 8.9 after enlarging  $A$  we may assume  $X_0$  and  $Y_0$  are smooth and proper over  $A$ . By Lemma 18.4 after enlarging  $A$  we may assume  $M$  is the pullback of some  $M_0 \in D_{perf}(\mathcal{O}_{X_0 \times_{\mathrm{Spec}(A)} Y_0})$  which is the Fourier-Mukai kernel of a relative equivalence from  $X_0$  to  $Y_0$  over  $\mathrm{Spec}(A)$ . Thus we see that  $(S_0/k, X_0 \rightarrow S_0, Y_0 \rightarrow S_0, M_0)$  is isomorphic to  $(S_{0,i}/k, X_{0,i} \rightarrow S_{0,i}, Y_{0,i} \rightarrow S_{0,i}, M_{0,i})$  for some  $i \in I$ . Since  $S_i = S_{0,i} \times_{\mathrm{Spec}(k)} \mathrm{Spec}(K)$  we conclude that (4) is true with  $s : \mathrm{Spec}(K) \rightarrow S_i$  induced by the morphism  $\mathrm{Spec}(K) \rightarrow \mathrm{Spec}(A) \cong S_{0,i}$  we get from  $A \subset K$ .  $\square$

## 21. Countability of derived equivalent varieties

0G0Z In this section we prove a result of Anel and Toën, see [AT09].

0G10 **Definition 21.1.** Let  $k$  be a field. Let  $X$  and  $Y$  be smooth projective schemes over  $k$ . We say  $X$  and  $Y$  are *derived equivalent* if there exists a  $k$ -linear exact equivalence  $D_{perf}(\mathcal{O}_X) \rightarrow D_{perf}(\mathcal{O}_Y)$ .

Here is the result

0G11 **Theorem 21.2.** *Let  $K$  be an algebraically closed field. Let  $\mathbf{X}$  be a smooth proper scheme over  $K$ . There are at most countably many isomorphism classes of smooth proper schemes  $\mathbf{Y}$  over  $K$  which are derived equivalent to  $\mathbf{X}$ .* Slight improvement of [AT09]

**Proof.** Choose a countable set  $I$  and for  $i \in I$  systems  $(S_i/K, X_i \rightarrow S_i, Y_i \rightarrow S_i, M_i)$  satisfying properties (1), (2), (3), and (4) of Lemma 20.5. Pick  $i \in I$  and set  $S = S_i$ ,  $X = X_i$ ,  $Y = Y_i$ , and  $M = M_i$ . Clearly it suffices to show that the set of isomorphism classes of fibres  $Y_s$  for  $s \in S(K)$  such that  $X_s \cong \mathbf{X}$  is countable. This we prove in the next paragraph.

Let  $S$  be a finite type scheme over  $K$ , let  $X \rightarrow S$  and  $Y \rightarrow S$  be proper smooth morphisms, and let  $M \in D_{perf}(\mathcal{O}_{X \times_S Y})$  be the Fourier-Mukai kernel of a relative equivalence from  $X$  to  $Y$  over  $S$ . We will show the set of isomorphism classes of fibres  $Y_s$  for  $s \in S(K)$  such that  $X_s \cong \mathbf{X}$  is countable. By Lemma 20.4 applied to the families  $\mathbf{X} \times S \rightarrow S$  and  $X \rightarrow S$  there exists a countable set  $I$  and for  $i \in I$  a pair  $(S_i \rightarrow S, h_i)$  with the following properties

- (1)  $S_i \rightarrow S$  is a morphism of finite type, set  $X_i = X \times_S S_i$ ,
- (2)  $h_i : \mathbf{X} \times S_i \rightarrow X_i$  is an isomorphism over  $S_i$ , and
- (3) for any closed point  $s \in S(K)$  if  $\mathbf{X} \cong X_s$  over  $K = \kappa(s)$  then  $s$  is in the image of  $S_i \rightarrow S$  for some  $i$ .

Set  $Y_i = Y \times_S S_i$ . Denote  $M_i \in D_{perf}(\mathcal{O}_{X_i \times_{S_i} Y_i})$  the pullback of  $M$ . By Lemma 18.3  $M_i$  is the Fourier-Mukai kernel of a relative equivalence from  $X_i$  to  $Y_i$  over  $S_i$ . Since  $I$  is countable, by property (3) it suffices to prove that the set of isomorphism

classes of fibres  $Y_{i,s}$  for  $s \in S_i(K)$  is countable. In fact, this number is finite by Lemma 19.5 and the proof is complete.  $\square$

## 22. Other chapters

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