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1. Introduction

These are the notes of a course on étale cohomology taught by Johan de Jong at Columbia University in the Fall of 2009. The original note takers were Thibaut Pugin, Zachary Maddock and Min Lee. Over time we will add references to background material in the rest of the Stacks project and provide rigorous proofs of all the statements.

2. Which sections to skip on a first reading?

We want to use the material in this chapter for the development of theory related to algebraic spaces, Deligne-Mumford stacks, algebraic stacks, etc. Thus we have added some pretty technical material to the original exposition of étale cohomology for schemes. The reader can recognize this material by the frequency of the word “topos”, or by discussions related to set theory, or by proofs dealing with very general properties of morphisms of schemes. Some of these discussions can be skipped on a first reading.

In particular, we suggest that the reader skip the following sections:

(1) Comparing big and small topoi, Section 91
(2) Recovering morphisms, Section 40
(3) Push and pull, Section 41
(4) Property (A), Section 42
(5) Property (B), Section 43
(6) Property (C), Section 44
(7) Topological invariance of the small étale site, Section 45
(8) Integral universally injective morphisms, Section 47
(9) Big sites and pushforward, Section 48
(10) Exactness of big lower shriek, Section 49

Besides these sections there are some sporadic results that may be skipped that the reader can recognize by the keywords given above.

3. Prologue

These lectures are about another cohomology theory. The first thing to remark is that the Zariski topology is not entirely satisfactory. One of the main reasons that it fails to give the results that we would want is that if $X$ is a complex variety and $F$ is a constant sheaf then

$$H^i(X, F) = 0, \quad \text{for all } i > 0.$$ 

The reason for that is the following. In an irreducible scheme (a variety in particular), any two nonempty open subsets meet, and so the restriction mappings of a constant sheaf are surjective. We say that the sheaf is flasque. In this case, all higher Čech cohomology groups vanish, and so do all higher Zariski cohomology groups. In other words, there are “not enough” open sets in the Zariski topology to detect this higher cohomology.
On the other hand, if $X$ is a smooth projective complex variety, then
\[ H_{\text{Betti}}^{2 \dim X}(X(\mathbb{C}), \Lambda) = \Lambda \quad \text{for } \Lambda = \mathbb{Z}, \mathbb{Z}/n\mathbb{Z}, \]
where $X(\mathbb{C})$ means the set of complex points of $X$. This is a feature that would be nice to replicate in algebraic geometry. In positive characteristic in particular.

4. The étale topology

03N4 It is very hard to simply “add” extra open sets to refine the Zariski topology. One efficient way to define a topology is to consider not only open sets, but also some schemes that lie over them. To define the étale topology, one considers all morphisms $\varphi : U \to X$ which are étale. If $X$ is a smooth projective variety over $\mathbb{C}$, then this means

1. $U$ is a disjoint union of smooth varieties, and
2. $\varphi$ is (analytically) locally an isomorphism.

The word “analytically” refers to the usual (transcendental) topology over $\mathbb{C}$. So the second condition means that the derivative of $\varphi$ has full rank everywhere (and in particular all the components of $U$ have the same dimension as $X$).

A double cover – loosely defined as a finite degree 2 map between varieties – for example

\[ \text{Spec}(\mathbb{C}[t]) \to \text{Spec}(\mathbb{C}[t]), \quad t \mapsto t^2 \]
will not be an étale morphism if it has a fibre consisting of a single point. In the example this happens when $t = 0$. For a finite map between varieties over $\mathbb{C}$ to be étale all the fibers should have the same number of points. Removing the point $t = 0$ from the source of the map in the example will make the morphism étale. But we can remove other points from the source of the morphism also, and the morphism will still be étale. To consider the étale topology, we have to look at all such morphisms. Unlike the Zariski topology, these need not be merely open subsets of $X$, even though their images always are.

03N5 **Definition 4.1.** A family of morphisms $\{\varphi_i : U_i \to X\}_{i \in I}$ is called an étale covering if each $\varphi_i$ is an étale morphism and their images cover $X$, i.e., $X = \bigcup_{i \in I} \varphi_i(U_i)$.

This “defines” the étale topology. In other words, we can now say what the sheaves are. An étale sheaf $\mathcal{F}$ of sets (resp. abelian groups, vector spaces, etc) on $X$ is the data:

1. for each étale morphism $\varphi : U \to X$ a set (resp. abelian group, vector space, etc) $\mathcal{F}(U)$,
2. for each pair $U, U'$ of étale schemes over $X$, and each morphism $U \to U'$ over $X$ (which is automatically étale) a restriction map $\rho_{U'}^U : \mathcal{F}(U') \to \mathcal{F}(U)$

These data have to satisfy the condition that $\rho_U^U = \text{id}$ in case of the identity morphism $U \to U$ and that $\rho_U^U \circ \rho_{U''}^U = \rho_U^{U''}$ when we have morphisms $U \to U' \to U''$ of schemes étale over $X$ as well as the following sheaf axiom:

(*) for every étale covering $\{\varphi_i : U_i \to U\}_{i \in I}$, the diagram

\[ \emptyset \to \mathcal{F}(U) \to \prod_{i \in I} \mathcal{F}(U_i) \to \prod_{i,j \in I} \mathcal{F}(U_i \times_U U_j) \]

is exact in the category of sets (resp. abelian groups, vector spaces, etc).
Remark 4.2. In the last statement, it is essential not to forget the case where \( i = j \) which is in general a highly nontrivial condition (unlike in the Zariski topology). In fact, frequently important coverings have only one element.

Since the identity is an étale morphism, we can compute the global sections of an étale sheaf, and cohomology will simply be the corresponding right-derived functors. In other words, once more theory has been developed and statements have been made precise, there will be no obstacle to defining cohomology.

5. Feats of the étale topology

For a natural number \( n \in \mathbb{N} = \{1, 2, 3, 4, \ldots\} \) it is true that
\[
H^2_{\text{étale}}(\mathbb{P}^1_C, \mathbb{Z}/n\mathbb{Z}) = \mathbb{Z}/n\mathbb{Z}.
\]
More generally, if \( X \) is a complex variety, then its étale Betti numbers with coefficients in a finite field agree with the usual Betti numbers of \( X(\mathbb{C}) \), i.e.,
\[
\dim_{\mathbb{F}} H^2_{\text{étale}}(X, \mathbb{F}_q) = \dim_{\mathbb{F}} H^2_{\text{Betti}}(X(\mathbb{C}), \mathbb{F}_q).
\]
This is extremely satisfactory. However, these equalities only hold for torsion coefficients, not in general. For integer coefficients, one has
\[
H^2_{\text{étale}}(\mathbb{P}^1_C, \mathbb{Z}) = 0.
\]
By contrast, \( H^2_{\text{Betti}}(\mathbb{P}^1(\mathbb{C}), \mathbb{Z}) = \mathbb{Z} \) as the topological space \( \mathbb{P}^1(\mathbb{C}) \) is homeomorphic to a 2-sphere. There are ways to get back to nontorsion coefficients from torsion ones by a limit procedure which we will come to shortly.

6. A computation

How do we compute the cohomology of \( \mathbb{P}^1_C \) with coefficients \( \Lambda = \mathbb{Z}/n\mathbb{Z} \)? We use Čech cohomology. A covering of \( \mathbb{P}^1_C \) is given by the two standard opens \( U_0, U_1 \), which are both isomorphic to \( \mathbb{A}^1_C \), and whose intersection is isomorphic to \( \mathbb{A}^1_C \setminus \{0\} = \mathbb{G}_m, C \). It turns out that the Mayer-Vietoris sequence holds in étale cohomology. This gives an exact sequence
\[
H^{i-1}_{\text{étale}}(U_0 \cap U_1, \Lambda) \to H^i_{\text{étale}}(\mathbb{P}^1_C, \Lambda) \to H^i_{\text{étale}}(U_0, \Lambda) \oplus H^i_{\text{étale}}(U_1, \Lambda) \to H^i_{\text{étale}}(U_0 \cap U_1, \Lambda).
\]
To get the answer we expect, we would need to show that the direct sum in the third term vanishes. In fact, it is true that, as for the usual topology,
\[
H^3_{\text{étale}}(\mathbb{A}^1_C, \Lambda) = 0 \quad \text{for } q \geq 1,
\]
and
\[
H^q_{\text{étale}}(\mathbb{A}^1_C \setminus \{0\}, \Lambda) = \begin{cases} \Lambda & \text{if } q = 1, \\ 0 & \text{for } q \geq 2. \end{cases}
\]
These results are already quite hard (what is an elementary proof?). Let us explain how we would compute this once the machinery of étale cohomology is at our disposal.

Higher cohomology. This is taken care of by the following general fact: if \( X \) is an affine curve over \( C \), then
\[
H^q_{\text{étale}}(X, \mathbb{Z}/n\mathbb{Z}) = 0 \quad \text{for } q \geq 2.
\]
This is proved by considering the generic point of the curve and doing some Galois cohomology. So we only have to worry about the cohomology in degree 1.
Cohomology in degree 1. We use the following identifications:

\[ H^1_{\text{étale}}(X, \mathbb{Z}/n\mathbb{Z}) = \left\{ \text{sheaves of sets } \mathcal{F} \text{ on the étale site } \mathcal{X}_{\text{étale}} \text{ endowed with an action } \mathbb{Z}/n\mathbb{Z} \times \mathcal{F} \to \mathcal{F} \text{ such that } \mathcal{F} \text{ is a } \mathbb{Z}/n\mathbb{Z}\text{-torsor.} \right\} / \cong \]

\[ = \left\{ \text{morphisms } Y \to X \text{ which are finite étale together with a free } \mathbb{Z}/n\mathbb{Z} \text{ action such that } X = Y/(\mathbb{Z}/n\mathbb{Z}). \right\} / \cong. \]

The first identification is canonical, the second isn't, see Remark 68.5. Since the proof of Riemann-Hurwitz does not use the computation of cohomology, the above actually constitutes a proof (provided we fill in the details on vanishing, etc).
7. Nontorsion coefficients

To study nontorsion coefficients, one makes the following definition:

\[ H^i_{\text{étale}}(X, \mathbb{Q}_\ell) := \left( \lim_n H^i_{\text{étale}}(X, \mathbb{Z}/\ell^n\mathbb{Z}) \right) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell. \]

The symbol \( \lim_n \) denote the limit of the system of cohomology groups \( H^i_{\text{étale}}(X, \mathbb{Z}/\ell^n\mathbb{Z}) \) indexed by \( n \), see Categories, Section 21. Thus we will need to study systems of sheaves satisfying some compatibility conditions.

8. Sheaf theory

At this point we start talking about sites and sheaves in earnest. There is an amazing amount of useful abstract material that could fit in the next few sections. Some of this material is worked out in earlier chapters, such as the chapter on sites, modules on sites, and cohomology on sites. We try to refrain from adding too much material here, just enough so the material later in this chapter makes sense.

9. Presheaves

A reference for this section is Sites, Section 2.

Definition 9.1. Let \( C \) be a category. A **presheaf of sets** (respectively, an **abelian presheaf**) on \( C \) is a functor \( C^{\text{opp}} \to \text{Sets} \) (resp. \( \text{Ab} \)).

Terminology. If \( U \in \text{Ob}(C) \), then elements of \( F(U) \) are called sections of \( F \) over \( U \). For \( \varphi : V \to U \) in \( C \), the map \( F(\varphi) : F(U) \to F(V) \) is called the restriction map and is often denoted \( s \mapsto s|_V \) or sometimes \( s \mapsto \varphi^*s \). The notation \( s|_V \) is ambiguous since the restriction map depends on \( \varphi \), but it is a standard abuse of notation. We also use the notation \( \Gamma(U, F) = F(U) \).

Saying that \( F \) is a functor means that if \( W \to V \to U \) are morphisms in \( C \) and \( s \in \Gamma(U, F) \) then \( (s|_V)|_W = s|_W \), with the abuse of notation just seen. Moreover, the restriction mappings corresponding to the identity morphisms \( \text{id}_U : U \to U \) are the identity.

The category of presheaves of sets (respectively of abelian presheaves) on \( C \) is denoted \( PSh(C) \) (resp. \( PAb(C) \)). It is the category of functors from \( C^{\text{opp}} \to \text{Sets} \) (resp. \( \text{Ab} \)), which is to say that the morphisms of presheaves are natural transformations of functors. We only consider the categories \( PSh(C) \) and \( PAb(C) \) when the category \( C \) is small. (Our convention is that a category is small unless otherwise mentioned, and if it isn’t small it should be listed in Categories, Remark 2.2.)

Example 9.2. Given an object \( X \in \text{Ob}(C) \), we consider the functor

\[
\begin{align*}
h_X : & \quad C^{\text{opp}} \to \text{Sets} \\
& U \quad \mapsto \quad h_X(U) = \text{Mor}_C(U, X) \\
& V \xrightarrow{\varphi} U \quad \mapsto \quad \varphi \circ - : h_X(U) \to h_X(V).
\end{align*}
\]

It is a presheaf, called the **representable presheaf associated to** \( X \). It is not true that representable presheaves are sheaves in every topology on every site.

Lemma 9.3 (Yoneda). Let \( C \) be a category, and \( X, Y \in \text{Ob}(C) \). There is a natural bijection

\[
\begin{align*}
\text{Mor}_C(X, Y) & \quad \to \quad \text{Mor}_{PSh(C)}(h_X, h_Y) \\
\psi & \quad \mapsto \quad h_\psi = \psi \circ - : h_X \to h_Y.
\end{align*}
\]

Proof. See Categories, Lemma 3.5. \( \square \)
10. Sites

**Definition 10.1.** Let $C$ be a category. A *family of morphisms with fixed target* $U = \{ \varphi_i : U_i \to U \}_{i \in I}$ is the data of

1. an object $U \in C$,
2. a set $I$ (possibly empty), and
3. for all $i \in I$, a morphism $\varphi_i : U_i \to U$ of $C$ with target $U$.

There is a notion of a *morphism of families of morphisms with fixed target*. A special case of that is the notion of a refinement. A reference for this material is Sites, Section 8.

**Definition 10.2.** A *site* consists of a category $C$ and a set $\text{Cov}(C)$ consisting of families of morphisms with fixed target called *coverings*, such that

1. (isomorphism) if $\varphi : V \to U$ is an isomorphism in $C$, then $\{ \varphi : V \to U \}$ is a covering,
2. (locality) if $\{ \varphi_i : U_i \to U \}_{i \in I}$ is a covering and for all $i \in I$ we are given a covering $\{ \psi_{ij} : U_{ij} \to U_i \}_{j \in I_i}$, then
   $$\{ \varphi_i \circ \psi_{ij} : U_{ij} \to U \}_{(i,j) \in I_i \times I}$$
   is also a covering, and
3. (base change) if $\{ U_i \to U \}_{i \in I}$ is a covering and $V \to U$ is a morphism in $C$, then
   a. for all $i \in I$ the fibre product $U_i \times_U V$ exists in $C$, and
   b. $\{ U_i \times_U V \to V \}_{i \in I}$ is a covering.

For us the category underlying a site is always “small”, i.e., its collection of objects form a set, and the collection of coverings of a site is a set as well (as in the definition above). We will mostly, in this chapter, leave out the arguments that cut down the collection of objects and coverings to a set. For further discussion, see Sites, Remark 6.3.

**Example 10.3.** If $X$ is a topological space, then it has an associated site $X_{\text{Zar}}$ defined as follows: the objects of $X_{\text{Zar}}$ are the open subsets of $X$, the morphisms between these are the inclusion mappings, and the coverings are the usual topological (surjective) coverings. Observe that if $U, V \subset W \subset X$ are open subsets then $U \times_W V = U \cap V$ exists: this category has fiber products. All the verifications are trivial and everything works as expected.

11. Sheaves

**Definition 11.1.** A presheaf $F$ of sets (resp. abelian presheaf) on a site $C$ is said to be a *separated presheaf* if for all coverings $\{ \varphi_i : U_i \to U \}_{i \in I} \in \text{Cov}(C)$ the map

$$F(U) \to \prod_{i \in I} F(U_i)$$

1What we call a site is a called a category endowed with a pretopology in [AGV71, Exposé II, Définition 1.3]. In [Art62] it is called a category with a Grothendieck topology.
is injective. Here the map is \( s \mapsto (s|_{U_i})_{i \in I} \). The presheaf \( \mathcal{F} \) is a sheaf if for all coverings \( \{ \varphi_i : U_i \to U \}_{i \in I} \in \text{Cov}(\mathcal{C}) \), the diagram

\[
\begin{array}{ccc}
\mathcal{F}(U) & \longrightarrow & \prod_{i \in I} \mathcal{F}(U_i) \\
& & \longrightarrow \\
& & \prod_{i,j \in I} \mathcal{F}(U_i \times_U U_j),
\end{array}
\]

where the first map is \( s \mapsto (s|_{U_i})_{i \in I} \) and the two maps on the right are \((s_i)_{i \in I} \mapsto (s_i|_{U_i \times_U U_j})\) and \((s_i)_{i \in I} \mapsto (s_j|_{U_i \times_U U_j})\), is an equalizer diagram in the category of sets (resp. abelian groups).

**Remark 11.2.** For the empty covering (where \( I = \emptyset \)), this implies that \( \mathcal{F}(\emptyset) \) is an empty product, which is a final object in the corresponding category (a singleton, for both \( \text{Sets} \) and \( \text{Ab} \)).

**Example 11.3.** Working this out for the site \( X_{\text{Zar}} \) associated to a topological space, see Example 10.3 gives the usual notion of sheaves.

**Definition 11.4.** We denote \( \text{Sh}(\mathcal{C}) \) (resp. \( \text{Ab}(\mathcal{C}) \)) the full subcategory of \( \text{PSh}(\mathcal{C}) \) (resp. \( \text{PAb}(\mathcal{C}) \)) whose objects are sheaves. This is the category of sheaves of sets (resp. abelian sheaves) on \( \mathcal{C} \).

12. The example of G-sets

Let \( G \) be a group and define a site \( \mathcal{T}_G \) as follows: the underlying category is the category of \( G \)-sets, i.e., its objects are sets endowed with a left \( G \)-action and the morphisms are equivariant maps; and the coverings of \( \mathcal{T}_G \) are the families \( \{ \varphi_i : U_i \to U \}_{i \in I} \) satisfying \( U = \bigcup_{i \in I} \varphi_i(U_i) \).

There is a special object in the site \( \mathcal{T}_G \), namely the \( G \)-set \( G \) endowed with its natural action by left translations. We denote it \( \cdot_G G \). Observe that there is a natural group isomorphism

\[
\rho : \ G^{\text{opp}} \longrightarrow \text{Aut}_{\text{G-Sets}}(G \cdot_G G) \\
g \longmapsto (h \mapsto hg).
\]

In particular, for any presheaf \( \mathcal{F} \), the set \( \mathcal{F}(G \cdot_G G) \) inherits a \( G \)-action via \( \rho \). (Note that by contravariance of \( \mathcal{F} \), the set \( \mathcal{F}(G \cdot_G G) \) is again a left \( G \)-set.) In fact, the functor

\[
\begin{array}{ccc}
\text{Sh}(\mathcal{T}_G) & \longrightarrow & \text{G-Sets} \\
\mathcal{F} & \longmapsto & \mathcal{F}(G \cdot_G G)
\end{array}
\]

is an equivalence of categories. Its quasi-inverse is the functor \( X \mapsto h_X \). Without giving the complete proof (which can be found in Sites, Section 9) let us try to explain why this is true.

1. If \( S \) is a \( G \)-set, we can decompose it into orbits \( S = \prod_{i \in I} O_i \). The sheaf axiom for the covering \( \{ O_i \to S \}_{i \in I} \) says that

\[
\begin{array}{ccc}
\mathcal{F}(S) & \longrightarrow & \prod_{i \in I} \mathcal{F}(O_i) \\
& & \longrightarrow \\
& & \prod_{i,j \in I} \mathcal{F}(O_i \times_S O_j)
\end{array}
\]

is an equalizer. Observing that fibered products in \( \text{G-Sets} \) are induced from fibered products in \( \text{Sets} \), and using the fact that \( \mathcal{F}(\emptyset) \) is a \( G \)-singleton, we get that

\[
\prod_{i,j \in I} \mathcal{F}(O_i \times_S O_j) = \prod_{i \in I} \mathcal{F}(O_i)
\]

and the two maps above are in fact the same. Therefore the sheaf axiom merely says that \( \mathcal{F}(S) = \prod_{i \in I} \mathcal{F}(O_i) \).
(2) If \( S \) is the \( G \)-set \( S = G/H \) and \( \mathcal{F} \) is a sheaf on \( \mathcal{T}_G \), then we claim that

\[
\mathcal{F}(G/H) = \mathcal{F}(G G)^H
\]

and in particular \( \mathcal{F}(*_G) = \mathcal{F}(G G)^G \). To see this, let’s use the sheaf axiom for the covering \( \{G G \to G/H\} \) of \( S \). We have

\[
G G \times_{G/H} G G \cong G \times H
\]

\[
(g_1, g_2) \mapsto (g_1, g_1 g_2^{-1})
\]

is a disjoint union of copies of \( G G \) (as a \( G \)-set). Hence the sheaf axiom reads

\[
\mathcal{F}(G/H) \to \mathcal{F}(G G) \to \prod_{h \in H} \mathcal{F}(G G)
\]

where the two maps on the right are \( s \mapsto (s)_{h \in H} \) and \( s \mapsto (hs)_{h \in H} \). Therefore \( \mathcal{F}(G/H) = \mathcal{F}(G G)^H \) as claimed.

This doesn’t quite prove the claimed equivalence of categories, but it shows at least that a sheaf \( \mathcal{F} \) is entirely determined by its sections over \( G G \). Details (and set theoretical remarks) can be found in Sites, Section 9.

13. Sheafification

**Definition 13.1.** Let \( \mathcal{F} \) be a presheaf on the site \( \mathcal{C} \) and \( \mathcal{U} = \{U_i \to U\} \in \operatorname{Cov}(\mathcal{C}) \). We define the zeroth Čech cohomology group of \( \mathcal{F} \) with respect to \( \mathcal{U} \) by

\[
\check{H}^0(\mathcal{U}, \mathcal{F}) = \left\{ (s_i)_{i \in I} \in \prod_{i \in I} \mathcal{F}(U_i) \text{ such that } s_i|_{U_i \times_U U_j} = s_j|_{U_i \times_U U_j} \right\}.
\]

There is a canonical map \( \mathcal{F}(U) \to \check{H}^0(\mathcal{U}, \mathcal{F}) \), \( s \mapsto (s|_{U_i})_{i \in I} \). We say that a morphism of coverings from a covering \( \mathcal{V} = \{V_j \to V\}_{j \in J} \) to \( \mathcal{U} \) is a triple \( (\chi, \alpha, \chi_j) \), where \( \chi : \mathcal{V} \to \mathcal{U} \) is a morphism, \( \alpha : J \to I \) is a map of sets, and for all \( j \in J \) the morphism \( \chi_j \) fits into a commutative diagram

\[
\begin{array}{ccc}
V_j & \xrightarrow{\chi_j} & U_{\alpha(j)} \\
\downarrow & & \downarrow \\
V & \xrightarrow{\chi} & U.
\end{array}
\]

Given the data \( \chi, \alpha, \{\chi_j\}_{j \in J} \) we define

\[
\check{H}^0(\mathcal{U}, \mathcal{F}) \to \check{H}^0(\mathcal{V}, \mathcal{F}) \quad (s_i)_{i \in I} \mapsto (\chi_j^*(s_{\alpha(j)}))_{j \in J}.
\]

We then claim that

1. the map is well-defined, and
2. depends only on \( \chi \) and is independent of the choice of \( \alpha, \{\chi_j\}_{j \in J} \).
Étale CoHomoLoGY 12

We omit the proof of the first fact. To see part (2), consider another triple \((\psi, \beta, \psi_j)\) with \(\chi = \psi\). Then we have the commutative diagram

\[
\begin{array}{ccc}
V_j & \to & U_{\alpha(j)} \times_U U_{\beta(j)} \\
\downarrow & & \downarrow \\
V & \to & U.
\end{array}
\]

Given a section \(s \in \mathcal{F}(U)\), its image in \(\mathcal{F}(V_j)\) under the map given by \((\chi, \alpha, \{\chi_j\}_{j \in J})\) is \(\chi_j^* s_{\alpha(j)}\), and its image under the map given by \((\psi, \beta, \{\psi_j\}_{j \in J})\) is \(\psi_j^* s_{\beta(j)}\). These two are equal since by assumption \(s \in \check{H}(U, \mathcal{F})\) and hence both are equal to the pullback of the common value

\[
s_{\alpha(j)}|_{U \cap U_{\beta(j)}} = s_{\beta(j)}|_{U \cap U_{\alpha(j)}}
\]
pulled back by the map \((\chi_j, \psi_j)\) in the diagram.

**Theorem 13.2.** Let \(\mathcal{C}\) be a site and \(\mathcal{F}\) a presheaf on \(\mathcal{C}\).

1. The rule
   \[
   U \mapsto \mathcal{F}^+(U) := \operatorname{colim}_U \text{ covering of } U \check{H}^0(U, \mathcal{F})
   \]
   is a presheaf. And the colimit is a directed one.
2. There is a canonical map of presheaves \(\mathcal{F} \to \mathcal{F}^+\).
3. If \(\mathcal{F}\) is a separated presheaf then \(\mathcal{F}^+\) is a sheaf and the map in (2) is injective.
4. \(\mathcal{F}^+\) is a separated presheaf.
5. \(\mathcal{F}^# = (\mathcal{F}^+)^+\) is a sheaf, and the canonical map induces a functorial isomorphism
   \[
   \operatorname{Hom}_{\text{PSh}(\mathcal{C})}(\mathcal{F}, \mathcal{G}) = \operatorname{Hom}_{\text{Sh}(\mathcal{C})}(\mathcal{F}^#, \mathcal{G})
   \]
   for any \(\mathcal{G} \in \text{Sh}(\mathcal{C})\).

**Proof.** See Sites, Theorem [10.10].

In other words, this means that the natural map \(\mathcal{F} \to \mathcal{F}^#\) is a left adjoint to the forgetful functor \(\text{Sh}(\mathcal{C}) \to \text{PSh}(\mathcal{C})\).

14. Cohomology

The following is the basic result that makes it possible to define cohomology for abelian sheaves on sites.

**Theorem 14.1.** The category of abelian sheaves on a site is an abelian category which has enough injectives.

**Proof.** See Modules on Sites, Lemma [3.4] and Injectives, Theorem [7.4].

So we can define cohomology as the right-derived functors of the sections functor: if \(U \in \text{Ob}(\mathcal{C})\) and \(\mathcal{F} \in \text{Ab}(\mathcal{C})\),

\[
H^p(U, \mathcal{F}) := R^p \Gamma(U, \mathcal{F}) = H^p(\Gamma(U, \mathcal{I}^*))
\]
where \( \mathcal{F} \to \mathcal{I} \) is an injective resolution. To do this, we should check that the functor \( \Gamma(U, -) \) is left exact. This is true and is part of why the category \( \text{Ab} (\mathcal{C}) \) is abelian, see Modules on Sites, Lemma 3.1. For more general discussion of cohomology on sites (including the global sections functor and its right derived functors), see Cohomology on Sites, Section 3.

15. The fpqc topology

03NV Before doing étale cohomology we study a bit the fpqc topology, since it works well for quasi-coherent sheaves.

03NW **Definition 15.1.** Let \( T \) be a scheme. An fpqc covering of \( T \) is a family \( \{ \varphi_i : T_i \to T \}_{i \in I} \) such that

1. each \( \varphi_i \) is a flat morphism and \( \bigcup_{i \in I} \varphi_i(T_i) = T \), and
2. for each affine open \( U \subset T \) there exists a finite set \( K \), a map \( i : K \to I \) and affine opens \( U_{i(k)} \subset T_{i(k)} \) such that \( U = \bigcup_{k \in K} \varphi_{i(k)}(U_{i(k)}) \).

03NX **Remark 15.2.** The first condition corresponds to fp, which stands for fidèlement plat, faithfully flat in french, and the second to qc, quasi-compact. The second part of the first condition is unnecessary when the second condition holds.

03NY **Example 15.3.** Examples of fpqc coverings.

1. Any Zariski open covering of \( T \) is an fpqc covering.
2. A family \( \{ \text{Spec}(B) \to \text{Spec}(A) \} \) is an fpqc covering if and only if \( A \to B \) is a faithfully flat ring map.
3. If \( f : X \to Y \) is flat, surjective and quasi-compact, then \( \{ f : X \to Y \} \) is an fpqc covering.
4. The morphism \( \varphi : \coprod_{x \in \mathbb{A}_k^1} \text{Spec}(\mathcal{O}_{\mathbb{A}_k^1, x}) \to \mathbb{A}_k^1 \), where \( k \) is a field, is flat and surjective. It is not quasi-compact, and in fact the family \( \{ \varphi \} \) is not an fpqc covering.
5. Write \( \mathbb{A}_k^2 = \text{Spec}(k[x, y]) \). Denote \( i_x : D(x) \to \mathbb{A}_k^2 \) and \( i_y : D(y) \hookrightarrow \mathbb{A}_k^2 \) the standard opens. Then the families \( \{ i_x, i_y, \text{Spec}(k[[x, y]]) \to \mathbb{A}_k^2 \} \) and \( \{ i_x, i_y, \text{Spec}(\mathcal{O}_{\mathbb{A}_k^2, 0}) \to \mathbb{A}_k^2 \} \) are fpqc coverings.

03NZ **Lemma 15.4.** The collection of fpqc coverings on the category of schemes satisfies the axioms of site.

**Proof.** See Topologies, Lemma 9.7.

It seems that this lemma allows us to define the fpqc site of the category of schemes. However, there is a set theoretical problem that comes up when considering the fpqc topology, see Topologies, Section 9. It comes from our requirement that sites are “small”, but that no small category of schemes can contain a cofinal system of fpqc coverings of a given nonempty scheme. Although this does not strictly speaking prevent us from defining “partial” fpqc sites, it does not seem prudent to do so. The work-around is to allow the notion of a sheaf for the fpqc topology (see below) but to prohibit considering the category of all fpqc sheaves.

03X6 **Definition 15.5.** Let \( S \) be a scheme. The category of schemes over \( S \) is denoted \( \text{Sch}/S \). Consider a functor \( \mathcal{F} : (\text{Sch}/S)^{opp} \to \text{Sets} \), in other words a presheaf of sets. We say \( \mathcal{F} \) satisfies the sheaf property for the fpqc topology if for every fpqc covering \( \{ U_i \to U \}_{i \in I} \) of schemes over \( S \) the diagram (11.1.1) is an equalizer diagram.
We similarly say that \( \mathcal{F} \) satisfies the sheaf property for the Zariski topology if for every open covering \( U = \bigcup_{i \in I} U_i \) the diagram (11.1.1) is an equalizer diagram. See Schemes, Definition 15.3. Clearly, this is equivalent to saying that for every scheme \( T \) over \( S \) the restriction of \( \mathcal{F} \) to the opens of \( T \) is a (usual) sheaf.

**Lemma 15.6.** Let \( \mathcal{F} \) be a presheaf on \( \text{Sch}/S \). Then \( \mathcal{F} \) satisfies the sheaf property for the fpqc topology if and only if

1. \( \mathcal{F} \) satisfies the sheaf property with respect to the Zariski topology, and
2. for every faithfully flat morphism \( \text{Spec}(B) \to \text{Spec}(A) \) of affine schemes over \( S \), the sheaf axiom holds for the covering \( \{ \text{Spec}(B) \to \text{Spec}(A) \} \).

Namely, this means that

\[
\mathcal{F}(\text{Spec}(A)) \longrightarrow \mathcal{F}(\text{Spec}(B)) \longrightarrow \mathcal{F}(\text{Spec}(B \otimes_A B))
\]

is an equalizer diagram.

**Proof.** See Topologies, Lemma 9.13.

An alternative way to think of a presheaf \( \mathcal{F} \) on \( \text{Sch}/S \) which satisfies the sheaf condition for the fpqc topology is as the following data:

1. for each \( T/S \), a usual (i.e., Zariski) sheaf \( \mathcal{F}_T \) on \( T \) Zar,
2. for every map \( f : T' \to T \) over \( S \), a restriction mapping \( f^{-1} \mathcal{F}_T \to \mathcal{F}_{T'} \),

such that

(a) the restriction mappings are functorial,
(b) if \( f : T' \to T \) is an open immersion then the restriction mapping \( f^{-1} \mathcal{F}_T \to \mathcal{F}_{T'} \) is an isomorphism, and
(c) for every faithfully flat morphism \( \text{Spec}(B) \to \text{Spec}(A) \) over \( S \), the diagram

\[
\mathcal{F}_{\text{Spec}(A)}(\text{Spec}(A)) \longrightarrow \mathcal{F}_{\text{Spec}(B)}(\text{Spec}(B)) \longrightarrow \mathcal{F}_{\text{Spec}(B \otimes_A B)}(\text{Spec}(B \otimes_A B))
\]

is an equalizer.

Data (1) and (2) and conditions (a), (b) give the data of a presheaf on \( \text{Sch}/S \) satisfying the sheaf condition for the Zariski topology. By Lemma 15.6 condition (c) then suffices to get the sheaf condition for the fpqc topology.

**Example 15.7.** Consider the presheaf

\[
\mathcal{F} : (\text{Sch}/S)^{\text{opp}} \longrightarrow \text{Ab}
\]

\[
T/S \longmapsto \Gamma(T, \Omega_{T/S}).
\]

The compatibility of differentials with localization implies that \( \mathcal{F} \) is a sheaf on the Zariski site. However, it does not satisfy the sheaf condition for the fpqc topology. Namely, consider the case \( S = \text{Spec}(\mathbb{F}_p) \) and the morphism

\[
\varphi : V = \text{Spec}(\mathbb{F}_p[u]) \to U = \text{Spec}(\mathbb{F}_p[v])
\]

given by mapping \( u \) to \( v^p \). The family \( \{ \varphi \} \) is an fpqc covering, yet the restriction mapping \( \mathcal{F}(U) \to \mathcal{F}(V) \) sends the generator \( du \) to \( d(v^p) = 0 \), so it is the zero map, and the diagram

\[
\mathcal{F}(U) \longrightarrow \mathcal{F}(V) \longrightarrow \mathcal{F}(V \times_U V)
\]

is not an equalizer. We will see later that \( \mathcal{F} \) does in fact give rise to a sheaf on the étale and smooth sites.
Lemma 15.8. Any representable presheaf on \(\mathcal{Sch}/S\) satisfies the sheaf condition for the fpqc topology.

Proof: See Descent, Lemma 10.7

We will return to this later, since the proof of this fact uses descent for quasi-coherent sheaves, which we will discuss in the next section. A fancy way of expressing the lemma is to say that the fpqc topology is weaker than the canonical topology, or that the fpqc topology is subcanonical. In the setting of sites this is discussed in Sites, Section 12.

Remark 15.9. The fpqc is the finest topology that we will see. Hence any presheaf satisfying the sheaf condition for the fpqc topology will be a sheaf in the subsequent sites (étale, smooth, etc). In particular representable presheaves will be sheaves on the étale site of a scheme for example.

Example 15.10. Let \(S\) be a scheme. Consider the additive group scheme \(G_{a,S} = \mathbb{A}^1_S\) over \(S\), see Groupoids, Example 5.3. The associated representable presheaf is given by

\[
h_{G_{a,S}}(T) = \text{Mor}_S(T, G_{a,S}) = \Gamma(T, \mathcal{O}_T).\]

By the above we now know that this is a presheaf of sets which satisfies the sheaf condition for the fpqc topology. On the other hand, it is clearly a presheaf of rings as well. Hence we can think of this as a functor

\[
\mathcal{O} : (\mathcal{Sch}/S)^{opp} \to \text{Rings}
\]

\(T/S \mapsto \Gamma(T, \mathcal{O}_T)\)

which satisfies the sheaf condition for the fpqc topology. Correspondingly there is a notion of \(\mathcal{O}\)-module, and so on and so forth.

16. Faithfully flat descent

Definition 16.1. Let \(\mathcal{U} = \{t_i : T_i \to T\}_{i \in I}\) be a family of morphisms of schemes with fixed target. A descent datum for quasi-coherent sheaves with respect to \(\mathcal{U}\) is a family \((F_i, \varphi_{ij})_{i,j \in I}\) where

1. for all \(i, F_i\) is a quasi-coherent sheaf on \(T_i\), and
2. for all \(i, j \in I\) the map \(\varphi_{ij} : \text{pr}_0^*F_i \cong \text{pr}_1^*F_j\) is an isomorphism on \(T_i \times_T T_j\) such that the diagrams

\[
\begin{array}{ccc}
\text{pr}_0^*F_i & \overset{\varphi_{ij}}{\longrightarrow} & \text{pr}_1^*F_j \\
\downarrow & & \downarrow \\
\text{pr}_2^*F_j & \rightarrow & \text{pr}_1^*F_j \\
\end{array}
\]

commute on \(T_i \times_T T_j \times_T T_k\).

This descent datum is called effective if there exist a quasi-coherent sheaf \(\mathcal{F}\) over \(T\) and \(\mathcal{O}_{T_i}\)-module isomorphisms \(\varphi_i : t_i^*\mathcal{F} \cong \mathcal{F}\) satisfying the cocycle condition, namely

\[
\varphi_{ij} = \text{pr}_0^*\varphi_j \circ \text{pr}_0^*\varphi_i^{-1}.
\]

In this and the next section we discuss some ingredients of the proof of the following theorem, as well as some related material.
Theorem 16.2. If $V = \{ T_i \to T \}_{i \in I}$ is an fpqc covering, then all descent data for quasi-coherent sheaves with respect to $V$ are effective.

Proof. See Descent, Proposition 5.2.

In other words, the fibered category of quasi-coherent sheaves is a stack on the fpqc site. The proof of the theorem is in two steps. The first one is to realize that for Zariski coverings this is easy (or well-known) using standard glueing of sheaves (see Sheaves, Section 33) and the locality of quasi-coherence. The second step is the case of an fpqc covering of the form $\{ \text{Spec}(B) \to \text{Spec}(A) \}$ where $A \to B$ is a faithfully flat ring map. This is a lemma in algebra, which we now present.

Descent of modules. If $A \to B$ is a ring map, we consider the complex

$$(B/A)_* : B \to B \otimes_A B \to B \otimes_A B \otimes_A B \to \ldots$$

where $B$ is in degree $0$, $B \otimes_A B$ in degree $1$, etc, and the maps are given by

- $b \mapsto 1 \otimes b - b \otimes 1$,
- $b_0 \otimes b_1 \mapsto 1 \otimes b_0 \otimes b_1 - b_0 \otimes 1 \otimes b_1 + b_0 \otimes b_1 \otimes 1$,
- etc.

Lemma 16.3. If $A \to B$ is faithfully flat, then the complex $(B/A)_*$ is exact in positive degrees, and $H^0((B/A)_*) = A$.

Proof. See Descent, Lemma 3.6.

Grothendieck proves this in three steps. Firstly, he assumes that the map $A \to B$ has a section, and constructs an explicit homotopy to the complex where $A$ is the only nonzero term, in degree $0$. Secondly, he observes that to prove the result, it suffices to do so after a faithfully flat base change $A \to A'$, replacing $B$ with $B' = B \otimes_A A'$. Thirdly, he applies the faithfully flat base change $A \to A'$ and remark that the map $A' = B \to B' = B \otimes_A B$ has a natural section.

The same strategy proves the following lemma.

Lemma 16.4. If $A \to B$ is faithfully flat and $M$ is an $A$-module, then the complex $(B/A)_* \otimes_A M$ is exact in positive degrees, and $H^0((B/A)_* \otimes_A M) = M$.

Proof. See Descent, Lemma 3.6.

Definition 16.5. Let $A \to B$ be a ring map and $N$ a $B$-module. A descent datum for $N$ with respect to $A \to B$ is an isomorphism $\varphi : N \otimes_A B \cong B \otimes_A N$ of $B \otimes_A B$-modules such that the diagram of $B \otimes_A B \otimes_A B$-modules

$$
\begin{array}{ccc}
N \otimes_A B & \xrightarrow{\varphi_{01}} & B \otimes_A N \otimes_A B \\
\downarrow{\varphi_{02}} & & \downarrow{\varphi_{12}} \\
B \otimes_A B & \otimes_A N
\end{array}
$$

commutes where $\varphi_{01} = \varphi \otimes \text{id}_B$ and similarly for $\varphi_{12}$ and $\varphi_{02}$.

If $N' = B \otimes_A M$ for some $A$-module $M$, then it has a canonical descent datum given by the map

$$
\varphi_{\text{can}} : N' \otimes_A B \to B \otimes_A N' \\
b_0 \otimes m \otimes b_1 \mapsto b_0 \otimes b_1 \otimes m.
$$
**Definition 16.6.** A descent datum \((N, \varphi)\) is called effective if there exists an \(A\)-module \(M\) such that \((N, \varphi) \cong (B \otimes_A M, \varphi_{\text{can}})\), with the obvious notion of isomorphism of descent data.

Theorem 16.2 is a consequence the following result.

**Theorem 16.7.** If \(A \rightarrow B\) is faithfully flat then descent data with respect to \(A \rightarrow B\) are effective.

**Proof.** See Descent, Proposition 3.9. See also Descent, Remark 3.11 for an alternative view of the proof. \(\square\)

**Remarks 16.8.** The results on descent of modules have several applications:

1. The exactness of the Čech complex in positive degrees for the covering \(\{\text{Spec}(B) \rightarrow \text{Spec}(A)\}\) where \(A \rightarrow B\) is faithfully flat. This will give some vanishing of cohomology.
2. If \((N, \varphi)\) is a descent datum with respect to a faithfully flat map \(A \rightarrow B\), then the corresponding \(A\)-module is given by
   \[
   M = \ker \left( \frac{N}{\varphi(n)} \rightarrow B \otimes_A N \right). 
   \]
   See Descent, Proposition 3.9.

17. Quasi-coherent sheaves

We can apply the descent of modules to study quasi-coherent sheaves.

**Proposition 17.1.** For any quasi-coherent sheaf \(F\) on \(S\) the presheaf
\[
F^a : \text{Sch}/S \rightarrow Ab \\
(f : T \rightarrow S) \mapsto \Gamma(T, f^*F)
\]
is an \(O\)-module which satisfies the sheaf condition for the fpqc topology.

**Proof.** This is proved in Descent, Lemma 8.1. We indicate the proof here. As established in Lemma 15.6, it is enough to check the sheaf property on Zariski coverings and faithfully flat morphisms of affine schemes. The sheaf property for Zariski coverings is standard scheme theory, since \(\Gamma(U, i^*F) = F(U)\) when \(i : U \hookrightarrow S\) is an open immersion.

For \(\{\text{Spec}(B) \rightarrow \text{Spec}(A)\}\) with \(A \rightarrow B\) faithfully flat and \(F|_{\text{Spec}(A)} = \tilde{M}\) this corresponds to the fact that \(M = H^0((B/A)_\bullet \otimes_A M)\), i.e., that
\[
0 \rightarrow M \rightarrow B \otimes_A M \rightarrow B \otimes_A B \otimes_A M
\]
is exact by Lemma 16.4. \(\square\)

There is an abstract notion of a quasi-coherent sheaf on a ringed site. We briefly introduce this here. For more information please consult Modules on Sites, Section 23. Let \(\mathcal{C}\) be a category, and let \(U\) be an object of \(\mathcal{C}\). Then \(\mathcal{C}/U\) indicates the category of objects over \(U\), see Categories, Example 2.13. If \(\mathcal{C}\) is a site, then \(\mathcal{C}/U\) is a site as well, namely the coverings of \(V/U\) are families \(\{V_i/U \rightarrow V/U\}\) of morphisms of \(\mathcal{C}/U\) with fixed target such that \(\{V_i \rightarrow V\}\) is a covering of \(\mathcal{C}\). Moreover, given any sheaf \(F\) on \(\mathcal{C}\) the restriction \(F|_{\mathcal{C}/U}\) (defined in the obvious manner) is a sheaf as well. See Sites, Section 25 for details.
**Definition 17.2.** Let $\mathcal{C}$ be a *ringed site*, i.e., a site endowed with a sheaf of rings $\mathcal{O}$. A sheaf of $\mathcal{O}$-modules $\mathcal{F}$ on $\mathcal{C}$ is called *quasi-coherent* if for all $U \in \text{Ob}(\mathcal{C})$ there exists a covering $\{U_i \to U\}_{i \in I}$ of $\mathcal{C}$ such that the restriction $\mathcal{F}|_{\mathcal{C}/U_i}$ is isomorphic to the cokernel of an $\mathcal{O}$-linear map of free $\mathcal{O}$-modules

$$\bigoplus_{k \in K} \mathcal{O}|_{\mathcal{C}/U_i} \to \bigoplus_{l \in L} \mathcal{O}|_{\mathcal{C}/U_i}.$$ 

The direct sum over $K$ is the sheaf associated to the presheaf $V \mapsto \bigoplus_{k \in K} \mathcal{O}(V)$ and similarly for the other.

Although it is useful to be able to give a general definition as above this notion is not well behaved in general.

**Remark 17.3.** In the case where $\mathcal{C}$ has a final object, e.g. $S$, it suffices to check the condition of the definition for $U = S$ in the above statement. See Modules on Sites, Lemma 23.3.

**Theorem 17.4** (Meta theorem on quasi-coherent sheaves). Let $S$ be a scheme. Let $\mathcal{C}$ be a site. Assume that

1. the underlying category $\mathcal{C}$ is a full subcategory of $\text{Sch}/S$,
2. any Zariski covering of $T \in \text{Ob}(\mathcal{C})$ can be refined by a covering of $\mathcal{C}$,
3. $S/S$ is an object of $\mathcal{C}$,
4. every covering of $\mathcal{C}$ is an fpqc covering of schemes.

Then the presheaf $\mathcal{O}$ is a sheaf on $\mathcal{C}$ and any quasi-coherent $\mathcal{O}$-module on $(\mathcal{C}, \mathcal{O})$ is of the form $\mathcal{F}^a$ for some quasi-coherent sheaf $\mathcal{F}$ on $S$.

**Proof.** After some formal arguments this is exactly Theorem [16.2]. Details omitted. In Descent, Proposition 8.11 we prove a more precise version of the theorem for the big Zariski, fppf, étale, smooth, and syntomic sites of $S$, as well as the small Zariski and étale sites of $S$.

In other words, there is no difference between quasi-coherent modules on the scheme $S$ and quasi-coherent $\mathcal{O}$-modules on sites $\mathcal{C}$ as in the theorem. More precise statements for the big and small sites $(\text{Sch}/S)_{\text{fppf}}$, $S_{\text{étale}}$, etc can be found in Descent, Section 8. In this chapter we will sometimes refer to a ‘site as in Theorem 17.4’ in order to conveniently state results which hold in any of those situations.

18. Čech cohomology

**Definition 18.1.** Let $\mathcal{C}$ be a category, $\mathcal{U} = \{U_i \to U\}_{i \in I}$ a family of morphisms of $\mathcal{C}$ with fixed target, and $\mathcal{F} \in \text{PAb}(\mathcal{C})$ an abelian presheaf. We define the Čech complex $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})$ by

$$\prod_{i_0 \in I} \mathcal{F}(U_{i_0}) \to \prod_{i_0, i_1 \in I} \mathcal{F}(U_{i_0} \times_U U_{i_1}) \to \prod_{i_0, i_1, i_2 \in I} \mathcal{F}(U_{i_0} \times_U U_{i_1} \times_U U_{i_2}) \to \ldots$$

where the first term is in degree 0, and the maps are the usual ones. Again, it is essential to allow the case $i_0 = i_1$ etc. The Čech cohomology groups are defined by

$$\check{H}^p(\mathcal{U}, \mathcal{F}) = H^p(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})).$$
Lemma 18.2. The functor $\check{C}^\bullet(U, \cdot)$ is exact on the category $PAb(C)$.

In other words, if $0 \to F_1 \to F_2 \to F_3 \to 0$ is a short exact sequence of presheaves of abelian groups, then

$$0 \to \check{C}^\bullet(U, F_1) \to \check{C}^\bullet(U, F_2) \to \check{C}^\bullet(U, F_3) \to 0$$

is a short exact sequence of complexes.

Proof. This follows at once from the definition of a short exact sequence of presheaves. Namely, as the category of abelian presheaves is the category of functors on some category with values in $Ab$, it is automatically an abelian category: a sequence $F_1 \to F_2 \to F_3$ is exact in $PAb$ if and only if for all $U \in \text{Ob}(C)$, the sequence $F_1(U) \to F_2(U) \to F_3(U)$ is exact in $Ab$. So the complex above is merely a product of short exact sequences in each degree. See also Cohomology on Sites, Lemma 10.1.

This shows that $\check{H}^\bullet(U, \cdot)$ is a $\delta$-functor. We now proceed to show that it is a universal $\delta$-functor. We thus need to show that it is an effaceable functor. We start by recalling the Yoneda lemma.

Lemma 18.3 (Yoneda Lemma). For any presheaf $F$ on a category $C$ there is a functorial isomorphism

$$\text{Hom}_{PSh(C)}(h_U, F) = F(U).$$

Proof. See Categories, Lemma 3.5.

This follows from the formula above. See Cohomology on Sites, Lemma 10.3.

Definition 18.4. Let $C$ be a category. Given a presheaf of sets $G$, we define the free abelian presheaf on $G$, denoted $Z_G$, by the rule

$$Z_G(U) = Z[G(U)],$$

for $U \in \text{Ob}(C)$ with restriction maps induced by the restriction maps of $G$. In the special case $G = h_U$ we write simply $Z_U = Z[h_U]$.

The functor $G \mapsto Z_G$ is left adjoint to the forgetful functor $PAb(C) \to PSh(C)$. Thus, for any presheaf $F$, there is a canonical isomorphism

$$\text{Hom}_{PAb(C)}(Z_U, F) = \text{Hom}_{PSh(C)}(h_U, F) = F(U)$$

the last equality by the Yoneda lemma. In particular, we have the following result.

Lemma 18.5. The Čech complex $\check{C}^\bullet(U, F)$ can be described explicitly as follows

$$\check{C}^\bullet(U, F) = \left( \prod_{i_0 \in I} \text{Hom}_{PAb(C)}(Z_{U_{i_0}}, F) \xrightarrow{} \prod_{i_0, i_1 \in I} \text{Hom}_{PAb(C)}(Z_{U_{i_0} \times_U U_{i_1}}, F) \xrightarrow{} \cdots \right)$$

Proof. This follows from the formula above. See Cohomology on Sites, Lemma 10.3.
The complex of abelian presheaves

Let Observe that all of the preceding statements are about presheaves.

Lemma 18.6. The complex of abelian presheaves

\[ Z^*_d \] is exact in all degrees except 0 in \( \text{PAb}(\mathcal{C}) \).

Proof. For any \( V \in \text{Ob}(\mathcal{C}) \) the complex of abelian groups \( Z^*_d(V) \) is

\[ Z \left( \prod_{i_0 \in I} \text{Mor}_\mathcal{C}(V, U_{i_0}) \right) \leftarrow Z \left( \prod_{i_0, i_1 \in I} \text{Mor}_\mathcal{C}(V, U_{i_0} \times U_{i_1}) \right) \leftarrow \cdots = \\
\bigoplus_{\varphi : V \to U} \left( Z \left( \prod_{i_0 \in I} \text{Mor}_\varphi(V, U_{i_0}) \right) \leftarrow Z \left( \prod_{i_0, i_1 \in I} \text{Mor}_\varphi(V, U_{i_0} \times U_{i_1}) \right) \leftarrow \cdots \right) \]

where \( \text{Mor}_\varphi(V, U_i) = \{ V \to U_i \text{ such that } V \to U_i \to U \text{ equals } \varphi \} \).

Set \( S_\varphi = \prod_{i \in I} \text{Mor}_\varphi(V, U_i) \), so that

\[ Z^*_d(V) = \bigoplus_{\varphi : V \to U} \left( Z[S_\varphi] \leftarrow Z[S_\varphi \times S_\varphi] \leftarrow Z[S_\varphi \times S_\varphi \times S_\varphi] \leftarrow \cdots \right) . \]

Thus it suffices to show that for each \( S = S_\varphi \), the complex

\[ Z[S] \leftarrow Z[S \times S] \leftarrow Z[S \times S \times S] \leftarrow \cdots \]

is exact in negative degrees. To see this, we can give an explicit homotopy. Fix \( s \in S \) and define \( K : n(s_0, \ldots, s_p) \mapsto n(s, s_0, \ldots, s_p) \). One easily checks that \( K \) is a nullhomotopy for the operator

\[ \delta : \eta(s_0, \ldots, s_p) \mapsto \sum_{i=0}^p (-1)^p \eta(s_0, \ldots, s_i, \ldots, s_p) . \]

See Cohomology on Sites, Lemma 10.4 for more details.

Lemma 18.7. Let \( \mathcal{C} \) be a category. If \( \mathcal{I} \) is an injective object of \( \text{PAb}(\mathcal{C}) \) and \( \mathcal{U} \) is a family of morphisms with fixed target in \( \mathcal{C} \), then \( \check{H}^p(\mathcal{U}, \mathcal{I}) = 0 \) for all \( p > 0 \).

Proof. The Čech complex is the result of applying the functor \( \text{Hom}_{\text{PAb}(\mathcal{C})}(-, \mathcal{I}) \) to the complex \( Z^*_d \), i.e.,

\[ \check{H}^p(\mathcal{U}, \mathcal{I}) = H^p(\text{Hom}_{\text{PAb}(\mathcal{C})}(Z^*_d, \mathcal{I})) . \]

But we have just seen that \( Z^*_d \) is exact in negative degrees, and the functor \( \text{Hom}_{\text{PAb}(\mathcal{C})}(-, \mathcal{I}) \) is exact, hence \( \text{Hom}_{\text{PAb}(\mathcal{C})}(Z^*_d, \mathcal{I}) \) is exact in positive degrees.

Theorem 18.8. On \( \text{PAb}(\mathcal{C}) \) the functors \( \check{H}^p(\mathcal{U}, -) \) are the right derived functors of \( \check{H}^0(\mathcal{U}, -) \).

Proof. By the Lemma 18.7 the functors \( \check{H}^p(\mathcal{U}, -) \) are universal \( \delta \)-functors since they are effaceable. So are the right derived functors of \( \check{H}^0(\mathcal{U}, -) \). Since they agree in degree 0, they agree by the universal property of universal \( \delta \)-functors. For more details see Cohomology on Sites, Lemma 10.6.

Remark 18.9. Observe that all of the preceding statements are about presheaves so we haven’t made use of the topology yet.
19. The Čech-to-cohomology spectral sequence

This spectral sequence is fundamental in proving foundational results on cohomology of sheaves.

Lemma 19.1. The forgetful functor $\text{Ab}(C) \to \text{PAb}(C)$ transforms injectives into injectives.

Proof. This is formal using the fact that the forgetful functor has a left adjoint, namely sheafification, which is an exact functor. For more details see Cohomology on Sites, Lemma 11.1. □

Theorem 19.2. Let $C$ be a site. For any covering $U = \{U_i \to U\}_{i \in I}$ of $U \in \text{Ob}(C)$ and any abelian sheaf $F$ on $C$ there is a spectral sequence

$$E_2^{p,q} = \tilde{H}^p(U, H^q(F)) \Rightarrow H^{p+q}(U, F),$$

where $H^q(F)$ is the abelian presheaf $V \mapsto H^q(V, F)$.

Proof. Choose an injective resolution $F \to I^\bullet$ in $\text{Ab}(C)$, and consider the double complex $\tilde{C}^\bullet(U, I^\bullet)$ and the maps

$$\Gamma(U, I^\bullet) \longrightarrow \tilde{C}^\bullet(U, I^\bullet) \longrightarrow C^\bullet(U, F)$$

Here the horizontal map is the natural map $\Gamma(U, I^\bullet) \to \tilde{C}^0(U, I^\bullet)$ to the left column, and the vertical map is induced by $F \to I^0$ and lands in the bottom row. By assumption, $I^\bullet$ is a complex of injectives in $\text{Ab}(C)$, hence by Lemma 19.1 it is a complex of injectives in $\text{PAb}(C)$. Thus, the rows of the double complex are exact in positive degrees (Lemma 18.7), and the kernel of $\tilde{C}^0(U, I^\bullet) \to \tilde{C}^1(U, I^\bullet)$ is equal to $\Gamma(U, I^\bullet)$, since $I^\bullet$ is a complex of sheaves. In particular, the cohomology of the total complex is the standard cohomology of the global sections functor $H^0(U, F)$.

For the vertical direction, the $q$th cohomology group of the $p$th column is

$$\prod_{i_0, \ldots, i_p} H^q(U_{i_0} \times_U \ldots \times_U U_{i_p}, F) = \prod_{i_0, \ldots, i_p} H^q(F)(U_{i_0} \times_U \ldots \times_U U_{i_p})$$

in the entry $E_1^{p,q}$. So this is a standard double complex spectral sequence, and the $E_2$-page is as prescribed. For more details see Cohomology on Sites, Lemma 11.6. □

Remark 19.3. This is a Grothendieck spectral sequence for the composition of functors

$$\text{Ab}(C) \longrightarrow \text{PAb}(C) \longrightarrow \text{Ab}.$$
of quasi-coherent sheaves it is convenient to introduce the big \( \tau \)-site and in case \( \tau \in \{ \text{étale}, \text{Zariski} \} \), the small \( \tau \)-site of \( S \). In order to do this we first introduce the notion of a \( \tau \)-covering.

**Definition 20.1.** (See Topologies, Definitions [7.1, 6.1, 5.1, 4.1, and 3.1]) Let \( \tau \in \{ \text{fppf}, \text{syntomic}, \text{smooth, étale, Zariski} \} \). A family of morphisms of schemes \( \{ f_i : T_i \to T \}_{i \in I} \) with fixed target is called a \( \tau \)-covering if and only if each \( f_i \) is flat of finite presentation, syntomic, smooth, étale, resp. an open immersion, and we have \( \bigcup f_i(T_i) = T \).

It turns out that the class of all \( \tau \)-coverings satisfies the axioms (1), (2) and (3) of Definition [10.2](our definition of a site), see Topologies, Lemmas [7.3, 6.3, 5.3, 4.3, and 3.2]. In order to be able to compare any of these new topologies to the fpqc topology and to use the preceding results on descent on modules we single out a special class of \( \tau \)-coverings of affine schemes called standard coverings.

**Definition 20.2.** (See Topologies, Definitions [7.5, 6.5, 5.5, 4.5, and 3.4]) Let \( \tau \in \{ \text{fppf}, \text{syntomic, smooth, étale, Zariski} \} \). Let \( T \) be an affine scheme. A **standard** \( \tau \)-covering of \( T \) is a family \( \{ f_j : U_j \to T \}_{j=1,\ldots,m} \) with each \( U_j \) affine, and each \( f_j \) flat and of finite presentation, standard syntomic, standard smooth, étale, resp. the immersion of a standard principal open in \( T \) and \( T = \bigcup f_j(U_j) \).

**Lemma 20.3.** Let \( \tau \in \{ \text{fppf}, \text{syntomic, smooth, étale, Zariski} \} \). Any \( \tau \)-covering of an affine scheme can be refined by a standard \( \tau \)-covering.

**Proof.** See Topologies, Lemmas [7.4, 6.4, 5.4, 4.4, and 3.3]. \( \square \)

Finally, we come to our definition of the sites we will be working with. This is actually somewhat involved since, contrary to what happens in [AGV71], we do not want to choose a universe. Instead we pick a “partial universe” (which is a suitably large set as in Sets, Section 5), and consider all schemes contained in this set. Of course we make sure that our favorite base scheme \( S \) is contained in the partial universe. Having picked the underlying category we pick a suitably large set of \( \tau \)-coverings which turns this into a site. The details are in the chapter on topologies on schemes; there is a lot of freedom in the choices made, but in the end the actual choices made will not affect the étale (or other) cohomology of \( S \) (just as in [AGV71] the actual choice of universe doesn’t matter at the end). Moreover, the way the material is written the reader who is happy using strongly inaccessible cardinals (i.e., universes) can do so as a substitute.

**Definition 20.4.** Let \( S \) be a scheme. Let \( \tau \in \{ \text{fppf, syntomic, smooth, étale, Zariski} \} \).

1. A **big \( \tau \)-site of \( S \)** is any of the sites \( (\text{Sch}/S)_\tau \) constructed as explained above and in more detail in Topologies, Definitions [7.8, 6.8, 5.8, 4.8, and 3.7].
2. If \( \tau \in \{ \text{étale, Zariski} \} \), then the **small \( \tau \)-site of \( S \)** is the full subcategory \( S_\tau \) of \( (\text{Sch}/S)_\tau \) whose objects are schemes \( T \) over \( S \) whose structure morphism \( T \to S \) is étale, resp. an open immersion. A covering in \( S_\tau \) is a covering \( \{ U_i \to U \} \) in \( (\text{Sch}/S)_\tau \) such that \( U \) is an object of \( S_\tau \).

The underlying category of the site \( (\text{Sch}/S)_\tau \) has reasonable “closure” properties, i.e., given a scheme \( T \) in it any locally closed subscheme of \( T \) is isomorphic to an object of \( (\text{Sch}/S)_\tau \). Other such closure properties are: closed under fibre products of schemes, taking countable disjoint unions, taking finite type schemes over a given
scheme, given an affine scheme Spec(R) one can complete, localize, or take the quotient of R by an ideal while staying inside the category, etc. On the other hand, for example arbitrary disjoint unions of schemes in (Sch/S)τ will take you outside of it. Also note that, given an object T of (Sch/S)τ there will exist τ-coverings \( \{T_i \to T\}_{i \in I} \) (as in Definition 20.1) which are not coverings in (Sch/S)τ for example because the schemes \( T_i \) are not objects of the category (Sch/S)τ. But our choice of the sites (Sch/S)τ is such that there always does exist a covering \( \{U_j \to T\}_{j \in J} \) of (Sch/S)τ which refines the covering \( \{T_i \to T\}_{i \in I} \), see Topologies, Lemmas [7.7 6.7 5.7 4.7] and [3.6]. We will mostly ignore these issues in this chapter.

If \( \mathcal{F} \) is a sheaf on (Sch/S)τ or Sτ, then we denote

\[ H^p(U, \mathcal{F}) \]

the cohomology groups of \( \mathcal{F} \) over the object U of the site, see Section [14]. Thus we have \( H^p_{fppf}(S, \mathcal{F}), H^p_{syntomic}(S, \mathcal{F}), H^p_{smooth}(S, \mathcal{F}), H^p_{étale}(S, \mathcal{F}), H^p_{Zar}(S, \mathcal{F}) \). The last two are potentially ambiguous since they might refer to either the big or small étale or Zariski site. However, this ambiguity is harmless by the following lemma.

03YX **Lemma 20.5.** Let \( \tau \in \{ \text{étale, Zariski} \} \). If \( \mathcal{F} \) is an abelian sheaf defined on (Sch/S)τ, then the cohomology groups of \( \mathcal{F} \) over S agree with the cohomology groups of \( \mathcal{F}|_S \) over S.

**Proof.** By Topologies, Lemmas [3.13 and 4.13] the functors \( S_\tau \to (Sch/S)_\tau \) satisfy the hypotheses of Sites, Lemma [21.8]. Hence our lemma follows from Cohomology on Sites, Lemma [8.2]. □

For completeness we state and prove the invariance under choice of partial universe of the cohomology groups we are considering. We will prove invariance of the small étale topos in Lemma [21.3] below. For notation and terminology used in this lemma we refer to Topologies, Section [12].

03YY **Lemma 20.6.** Let \( \tau \in \{ fppf, syntomic, smooth, étale, Zariski \} \). Let S be a scheme. Let (Sch/S)τ and (Sch′/S)τ be two big \( \tau \)-sites of S, and assume that the first is contained in the second. In this case

1. for any abelian sheaf \( \mathcal{F}' \) defined on (Sch′/S)τ and any object U of (Sch/S)τ we have

\[ H^p(U, \mathcal{F}'|_{(Sch/S)_\tau}) = H^p(U, \mathcal{F}') \]

In words: the cohomology of \( \mathcal{F}' \) over U computed in the bigger site agrees with the cohomology of \( \mathcal{F}' \) restricted to the smaller site over U.

(2) for any abelian sheaf \( \mathcal{F} \) on (Sch/S)τ there is an abelian sheaf \( \mathcal{F}' \) on (Sch/S)τ whose restriction to (Sch/S)τ is isomorphic to \( \mathcal{F} \).

**Proof.** By Topologies, Lemma [12.2] the inclusion functor (Sch/S)τ \( \to (Sch'/S)_\tau \) satisfies the assumptions of Sites, Lemma [21.8]. This implies (2) and (1) follows from Cohomology on Sites, Lemma [8.2]. □

21. The étale topos

04HP A *topos* is the category of sheaves of sets on a site, see Sites, Definition [15.1]. Hence it is customary to refer to the use the phrase “étale topos of a scheme” to refer to the category of sheaves on the small étale site of a scheme. Here is the formal definition.
Definition 21.1. Let $S$ be a scheme.

1. The étale topos, or the small étale topos of $S$ is the category $\text{Sh}(\mathcal{S}_{\text{étale}})$ of sheaves of sets on the small étale site of $S$.
2. The Zariski topos, or the small Zariski topos of $S$ is the category $\text{Sh}(\mathcal{S}_{\text{Zar}})$ of sheaves of sets on the small Zariski site of $S$.
3. For $\tau \in \{fppf, \text{syntomic, smooth, étale, Zariski}\}$ a big $\tau$-topos is the category of sheaves of sets on a big $\tau$-topos of $S$. 

Note that the small Zariski topos of $S$ is simply the category of sheaves of sets on the underlying topological space of $S$, see Topologies, Lemma 3.11. Whereas the small étale topos does not depend on the choices made in the construction of the small étale site, in general the big topoi do depend on those choices.

Here is a lemma, which is one of many possible lemmas expressing the fact that it doesn’t matter too much which site we choose to define the small étale topos of a scheme.

Lemma 21.2. Let $S$ be a scheme. Let $\mathcal{S}_{\text{affine, étale}}$ denote the full subcategory of $\mathcal{S}_{\text{étale}}$ whose objects are those $U/S \in \text{Ob}(\mathcal{S}_{\text{étale}})$ with $U$ affine. A covering of $\mathcal{S}_{\text{affine, étale}}$ will be a standard étale covering, see Topologies, Definition 4.5. Then restriction

$$\mathcal{F} \mapsto \mathcal{F}|_{\mathcal{S}_{\text{affine, étale}}}$$

defines an equivalence of topoi $\text{Sh}(\mathcal{S}_{\text{étale}}) \cong \text{Sh}(\mathcal{S}_{\text{affine, étale}})$.

Proof. This you can show directly from the definitions, and is a good exercise. But it also follows immediately from Sites, Lemma 29.1 by checking that the inclusion functor $\mathcal{S}_{\text{affine, étale}} \to \mathcal{S}_{\text{étale}}$ is a special cocontinuous functor (see Sites, Definition 29.2). □

Lemma 21.3. Let $S$ be a scheme. The étale topos of $S$ is independent (up to canonical equivalence) of the construction of the small étale site in Definition 20.4.

Proof. We have to show, given two big étale sites $\mathcal{S}_{\text{étale}}$ and $\mathcal{S}'_{\text{étale}}$ containing $S$, then $\text{Sh}(\mathcal{S}_{\text{étale}}) \cong \text{Sh}(\mathcal{S}'_{\text{étale}})$ with obvious notation. By Topologies, Lemma 12.1 we may assume $\mathcal{S}_{\text{étale}} \subset \mathcal{S}'_{\text{étale}}$. By Sets, Lemma 9.9 any affine scheme étale over $S$ is isomorphic to an object of both $\mathcal{S}_{\text{étale}}$ and $\mathcal{S}'_{\text{étale}}$. Thus the induced functor $\mathcal{S}_{\text{affine, étale}} \to \mathcal{S}'_{\text{affine, étale}}$ is an equivalence. Moreover, it is clear that both this functor and a quasi-inverse map transform standard étale coverings into standard étale coverings. Hence the result follows from Lemma 21.2. □

Lemma 21.4. Let $S$ be a scheme. Let $\mathcal{S}_{\text{affine, Zar}}$ denote the full subcategory of $\mathcal{S}_{\text{Zar}}$ consisting of affine objects. A covering of $\mathcal{S}_{\text{affine, Zar}}$ will be a standard Zariski covering, see Topologies, Definition 3.4. Then restriction

$$\mathcal{F} \mapsto \mathcal{F}|_{\mathcal{S}_{\text{affine, Zar}}}$$

defines an equivalence of topoi $\text{Sh}(\mathcal{S}_{\text{Zar}}) \cong \text{Sh}(\mathcal{S}_{\text{affine, Zar}})$.

Proof. Please skip the proof of this lemma. It follows immediately from Sites, Lemma 29.1 by checking that the inclusion functor $\mathcal{S}_{\text{affine, Zar}} \to \mathcal{S}_{\text{Zar}}$ is a special cocontinuous functor (see Sites, Definition 29.2). □
ÉTALE COHOMOLOGY

22. Cohomology of quasi-coherent sheaves

We start with a simple lemma (which holds in greater generality than stated). It says that the Čech complex of a standard covering is equal to the Čech complex of an fpqc covering of the form \{Spec(B) \to Spec(A)\} with A \to B faithfully flat.

Lemma 22.1. Let \( \tau \in \{fpf, syntomic, smooth, étale, Zariski\} \). Let \( S \) be a scheme. Let \( F \) be an abelian sheaf on \((Sch/S)_\tau\), or on \( S_\tau \) in case \( \tau = \text{étale} \), and let \( U = \{U_i \to U\}_{i \in I} \) be a standard \( \tau \)-covering of this site. Let \( V = \coprod_{i \in I} U_i \). Then

1. \( V \) is an affine scheme,
2. \( V = \{V \to U\} \) is a \( \tau \)-covering and an fpqc covering,
3. the Čech complexes \( \check{C}^*(U, F) \) and \( C^*(V, F) \) agree.

Proof. As the covering is a standard \( \tau \)-covering each of the schemes \( U_i \) is affine and \( I \) is a finite set. Hence \( V \) is an affine scheme. It is clear that \( V \to U \) is flat and surjective, hence \( V \) is an fpqc covering, see Example 15.3. Note that \( U \) is a refinement of \( V \) and hence there is a map of Čech complexes \( \check{C}^*(V, F) \to C^*(U, F) \), see Cohomology on Sites, Equation (9.2.1). Next, we observe that if \( T = \coprod_{j \in J} T_j \) is a disjoint union of schemes in the site on which \( F \) is defined then the family of morphisms with fixed target \( \{T_j \to T\}_{j \in J} \) is a Zariski covering, and so

\[
\mathcal{F}(T) = \mathcal{F}(\coprod_{j \in J} T_j) = \prod_{j \in J} \mathcal{F}(T_j)
\]

by the sheaf condition of \( \mathcal{F} \). This implies the map of Čech complexes above is an isomorphism in each degree because

\[
V \times_U \ldots \times_U V = \coprod_{i_0, \ldots, i_p} U_{i_0} \times_U \ldots \times_U U_{i_p}
\]
as schemes.

Note that Equality \((22.1.1)\) is false for a general presheaf. Even for sheaves it does not hold on any site, since coproducts may not lead to coverings, and may not be disjoint. But it does for all the usual ones (at least all the ones we will study).

Remark 22.2. In the statement of Lemma 22.1 the covering \( U \) is a refinement of \( V \) but not the other way around. Coverings of the form \( \{V \to U\} \) do not form an initial subcategory of the category of all coverings of \( U \). Yet it is still true that we can compute Čech cohomology \( \check{H}^n(U, F) \) (which is defined as the colimit over the opposite of the category of coverings \( U \) of \( U \) of the Čech cohomology groups of \( F \) with respect to \( U \)) in terms of the coverings \( \{V \to U\} \). We will formulate a precise lemma (it only works for sheaves) and add it here if we ever need it.

Lemma 22.3 (Locality of cohomology). Let \( C \) be a site, \( F \) an abelian sheaf on \( C \), \( U \) an object of \( C \), \( p > 0 \) an integer and \( \xi \in H^p(U, F) \). Then there exists a covering \( U = \{U_i \to U\}_{i \in I} \) of \( U \) in \( C \) such that \( \xi|_{U_i} = 0 \) for all \( i \in I \).

Proof. Choose an injective resolution \( \mathcal{F} \to \mathcal{I}^* \). Then \( \xi \) is represented by a cocycle \( \xi \in \mathcal{I}^p(U) \) with \( d^p(\xi) = 0 \). By assumption, the sequence \( \mathcal{I}^{p-1} \to \mathcal{I}^p \to \mathcal{I}^{p+1} \) is exact in \( Ab(C) \), which means that there exists a covering \( U = \{U_i \to U\}_{i \in I} \) such that \( \xi|_{U_i} = d^{p-1}(\xi_i) \) for some \( \xi_i \in \mathcal{I}^{p-1}(U_i) \). Since the cohomology class \( \xi|_{U_i} \) is represented by the cocycle \( \xi|_{U_i} \) which is a coboundary, it vanishes. For more details see Cohomology on Sites, Lemma 8.3.
Theorem 22.4. Let $S$ be a scheme and $\mathcal{F}$ a quasi-coherent $\mathcal{O}_S$-module. Let $\mathcal{C}$ be either $(\text{Sch}/S)_\tau$ for $\tau \in \{\text{fpqc}, \text{syntomic}, \text{smooth}, \text{étale}, \text{Zariski}\}$ or $S_{\text{étale}}$. Then

$$H^p(S, \mathcal{F}) = H^p(S, \mathcal{F}^a)$$

for all $p \geq 0$ where

1. the left hand side indicates the usual cohomology of the sheaf $\mathcal{F}$ on the underlying topological space of the scheme $S$, and
2. the right hand side indicates cohomology of the abelian sheaf $\mathcal{F}^a$ (see Proposition [17.1]) on the site $\mathcal{C}$.

Proof. We are going to show that $H^p(U, f^*\mathcal{F}) = H^p(U, \mathcal{F}^a)$ for any object $f : U \to S$ of the site $\mathcal{C}$. The result is true for $p = 0$ by the sheaf property.

Assume that $U$ is affine. Then we want to prove that $H^p(U, \mathcal{F}^a) = 0$ for all $p > 0$. We use induction on $p$.

$p = 1$ Pick $\xi \in H^1(U, \mathcal{F}^a)$. By Lemma 22.3 there exists an fpqc covering $U = \{U_i \to U\}_{i \in I}$ such that $\xi|_{U_i} = 0$ for all $i \in I$. Up to refining $\mathcal{U}$, we may assume that $\mathcal{U}$ is a standard $\tau$-covering. Applying the spectral sequence of Theorem 19.2, we see that $\xi$ comes from a cohomology class $\xi \in H^1(\mathcal{U}, \mathcal{F}^a)$. Consider the covering $\mathcal{V} = \coprod_{i \in I} U_i \to U$. By Lemma 22.1 $H^p(\mathcal{U}, \mathcal{F}^a) = H^p(\mathcal{V}, \mathcal{F}^a)$. On the other hand, since $\mathcal{V}$ is a covering of the form $\{\text{Spec}(B) \to \text{Spec}(A)\}$ and $f^*\mathcal{F} = \mathcal{M}$ for some $A$-module $M$, we see the Čech complex $\check{\mathcal{C}}^\bullet(\mathcal{V}, \mathcal{F})$ is none other than the complex $(B/A, \otimes_A M)$. Now by Lemma 16.4 $H^p((B/A, \otimes_A M) = 0$ for $p > 0$, hence $\xi = 0$ and so $\xi = 0$.

$p > 1$ Pick $\xi \in H^p(U, \mathcal{F}^a)$. By Lemma 22.3 there exists an fpqc covering $U = \{U_i \to U\}_{i \in I}$ such that $\xi|_{U_i} = 0$ for all $i \in I$. Up to refining $\mathcal{U}$, we may assume that $\mathcal{U}$ is a standard $\tau$-covering. We apply the spectral sequence of Theorem 19.2. Observe that the intersections $U_{i_0} \times_U \ldots \times_U U_{i_p}$ are affine, so that by induction hypothesis the cohomology groups

$$E^{p,q}_2 = H^p(\mathcal{U}, \check{\mathcal{H}}^q(\mathcal{F}^a))$$

vanish for all $0 < q < p$. We see that $\xi$ must come from a $\xi' \in \check{\mathcal{H}}^p(\mathcal{U}, \mathcal{F}^a)$. Replacing $\mathcal{U}$ with the covering $\mathcal{V}$ containing only one morphism and using Lemma 16.4 again, we see that the Čech cohomology class $\xi'$ must be zero, hence $\xi = 0$.

Next, assume that $U$ is separated. Choose an affine open covering $U = \bigcup_{i \in I} U_i$ of $U$. The family $\mathcal{U} = \{U_i \to U\}_{i \in I}$ is then an fpqc covering, and all the intersections $U_{i_0} \times_U \ldots \times_U U_{i_p}$ are affine since $U$ is separated. So all rows of the spectral sequence of Theorem 19.2 are zero, except the zeroth row. Therefore

$$H^p(U, \mathcal{F}^a) = \check{H}^p(\mathcal{U}, \mathcal{F}^a) = \check{H}^p(\mathcal{U}, \mathcal{F}) = H^p(U, \mathcal{F})$$

where the last equality results from standard scheme theory, see Cohomology of Schemes, Lemma [2.6].

The general case is technical and (to extend the proof as given here) requires a discussion about maps of spectral sequences, so we won’t treat it. It follows from Descent, Proposition 8.10 (whose proof takes a slightly different approach) combined with Cohomology on Sites, Lemma 8.1.
Remark 22.5. Comment on Theorem 22.4. Since $S$ is a final object in the category $\mathcal{C}$, the cohomology groups on the right-hand side are merely the right derived functors of the global sections functor. In fact the proof shows that $H^p(U, f^* \mathcal{F}) = H^p_U(U, \mathcal{F}^a)$ for any object $f : U \to S$ of the site $\mathcal{C}$.

23. Examples of sheaves

Let $S$ and $\tau$ be as in Section 20. We have already seen that any representable presheaf is a sheaf on $(\text{Sch}/S)_\tau$ or $S_\tau$, see Lemma 15.8 and Remark 15.9. Here are some special cases.

Definition 23.1. On any of the sites $(\text{Sch}/S)_\tau$ or $S_\tau$ of Section 20.

1. The sheaf $T \mapsto \Gamma(T, \mathcal{O}_T)$ is denoted $\mathcal{O}_S$, or $\mathbf{G}_a$, or $\mathbf{G}_{a,S}$ if we want to indicate the base scheme.

2. Similarly, the sheaf $T \mapsto \Gamma(T, \mathcal{O}_T^*)$ is denoted $\mathcal{O}_S^*$, or $\mathbf{G}_m$, or $\mathbf{G}_{m,S}$ if we want to indicate the base scheme.

3. The constant sheaf $\mathbb{Z}/n\mathbb{Z}$ on any site is the sheafification of the constant presheaf $U \mapsto \mathbb{Z}/n\mathbb{Z}$. The first is a sheaf by Theorem 17.4 for example. The second is a sub presheaf of the first, which is easily seen to be a sheaf itself. The third is a sheaf by definition. Note that each of these sheaves is representable. The first and second by the schemes $\mathbf{G}_{a,S}$ and $\mathbf{G}_{m,S}$, see Groupoids, Section 4. The third by the finite étale group scheme $\mathbb{Z}/n\mathbb{Z}_S$ sometimes denoted $(\mathbb{Z}/n\mathbb{Z})_S$ which is just $n$ copies of $S$ endowed with the obvious group scheme structure over $S$, see Groupoids, Example 5.6 and the following remark.

Remark 23.2. Let $G$ be an abstract group. On any of the sites $(\text{Sch}/S)_\tau$ or $S_\tau$ of Section 20 the sheafification $\mathcal{G}$ of the constant presheaf associated to $G$ in the Zariski topology of the site already gives

$$\Gamma(U, \mathcal{G}) = \{\text{Zariski locally constant maps } U \to G\}$$

This Zariski sheaf is representable by the group scheme $G_S$ according to Groupoids, Example 5.6. By Lemma 15.8 any representable presheaf satisfies the sheaf condition for the $\tau$-topology as well, and hence we conclude that the Zariski sheafification $\mathcal{G}$ above is also the $\tau$-sheafification.

Definition 23.3. Let $S$ be a scheme. The structure sheaf of $S$ is the sheaf of rings $\mathcal{O}_S$ on any of the sites $S_{\text{Zar}}, S_{\text{étale}},$ or $(\text{Sch}/S)_\tau$ discussed above.

If there is some possible confusion as to which site we are working on then we will indicate this by using indices. For example we may use $\mathcal{O}_{S_{\text{étale}}}$ to stress the fact that we are working on the small étale site of $S$.

Remark 23.4. In the terminology introduced above a special case of Theorem 22.4 is

$$H^p_{fppf}(X, \mathbf{G}_a) = H^p_{\text{étale}}(X, \mathbf{G}_a) = H^p_{2\text{ar}}(X, \mathbf{G}_a) = H^p(X, \mathcal{O}_X)$$

for all $p \geq 0$. Moreover, we could use the notation $H^p_{fppf}(X, \mathcal{O}_X)$ to indicate the cohomology of the structure sheaf on the big fppf site of $X$. 


24. Picard groups

The following theorem is sometimes called “Hilbert 90”.

**Theorem 24.1.** For any scheme $X$ we have canonical identifications

$$H^1_{fppf}(X, \mathbb{G}_m) = H^1_{\text{syntomic}}(X, \mathbb{G}_m) = H^1_{\text{smooth}}(X, \mathbb{G}_m) = H^1_{\text{étale}}(X, \mathbb{G}_m) = H^1_{\text{Zar}}(X, \mathbb{G}_m) = \text{Pic}(X) = H^1(X, \mathcal{O}_X^*)$$

**Proof.** Let $\tau$ be one of the topologies considered in Section 20. By Cohomology on Sites, Lemma 7.1 we see that $H^1_{\tau}(X, \mathbb{G}_m) = H^1_{\tau}(X, \mathcal{O}_\tau^*) = \text{Pic}(\mathcal{O}_\tau)$ where $\mathcal{O}_\tau$ is the structure sheaf of the site $(\text{Sch}/X)_\tau$. Now an invertible $\mathcal{O}_\tau$-module is a quasi-coherent $\mathcal{O}_\tau$-module. By Theorem 17.4 or the more precise Descent, Proposition 8.11 we see that $\text{Pic}(\mathcal{O}_\tau) = \text{Pic}(X)$. The last equality is proved in the same way. □

25. The étale site

At this point we start exploring the étale site of a scheme in more detail. As a first step we discuss a little the notion of an étale morphism.

26. Étale morphisms

For more details, see Morphisms, Section 34 for the formal definition and Étale Morphisms, Sections 11, 12, 13, 14, 16, and 19 for a survey of interesting properties of étale morphisms.

Recall that an algebra $A$ over an algebraically closed field $k$ is smooth if it is of finite type and the module of differentials $\Omega_{A/k}$ is finite locally free of rank equal to the dimension. A scheme $X$ over $k$ is smooth over $k$ if it is locally of finite type and each affine open is the spectrum of a smooth $k$-algebra. If $k$ is not algebraically closed then an $A$-algebra is said to be a smooth $k$-algebra if $A \otimes_k \overline{k}$ is a smooth $\overline{k}$-algebra. A ring map $A \rightarrow B$ is smooth if it is flat, finitely presented, and for all primes $\mathfrak{p} \subset A$ the fibre ring $\kappa(\mathfrak{p}) \otimes_A B$ is smooth over the residue field $\kappa(\mathfrak{p})$. More generally, a morphism of schemes is smooth if it is flat, locally of finite presentation, and the geometric fibers are smooth.

For these facts please see Morphisms, Section 32. Using this we may define an étale morphism as follows.

**Definition 26.1.** A morphism of schemes is étale if it is smooth of relative dimension 0.

In particular, a morphism of schemes $X \rightarrow S$ is étale if it is smooth and $\Omega_{X/S} = 0$.

**Proposition 26.2.** Facts on étale morphisms.

1. Let $k$ be a field. A morphism of schemes $U \rightarrow \text{Spec}(k)$ is étale if and only if $U \cong \coprod_{i \in I} \text{Spec}(k_i)$ such that for each $i \in I$ the ring $k_i$ is a field which is a finite separable extension of $k$.
(2) Let $\varphi : U \rightarrow S$ be a morphism of schemes. The following conditions are equivalent:
   (a) $\varphi$ is étale,
   (b) $\varphi$ is locally finitely presented, flat, and all its fibres are étale,
   (c) $\varphi$ is flat, unramified and locally of finite presentation.
(3) A ring map $A \rightarrow B$ is étale if and only if $B \cong A[x_1, \ldots, x_n]/(f_1, \ldots, f_n)$ such that $\Delta = \det \left( \frac{\partial f_i}{\partial x_j} \right)$ is invertible in $B$.
(4) The base change of an étale morphism is étale.
(5) Compositions of étale morphisms are étale.
(6) Fibre products and products of étale morphisms are étale.
(7) An étale morphism has relative dimension 0.
(8) Let $Y \rightarrow X$ be an étale morphism. If $X$ is reduced (respectively regular) then so is $Y$.
(9) Étale morphisms are open.
(10) If $X \rightarrow S$ and $Y \rightarrow S$ are étale, then any $S$-morphism $X \rightarrow Y$ is also étale.

Proof. We have proved these facts (and more) in the preceding chapters. Here is a list of references: (1) Morphisms, Lemma 34.7. (2) Morphisms, Lemmas 34.8 and 34.16. (3) Algebra, Lemma 141.2. (4) Morphisms, Lemma 34.4. (5) Morphisms, Lemma 34.3. (6) Follows formally from (4) and (5). (7) Morphisms, Lemmas 34.6 and 28.5. (8) See Algebra, Lemmas 157.7 and 157.5, see also more results of this kind in Étale Morphisms, Section 19. (9) See Morphisms, Lemma 24.9 and 34.12. (10) See Morphisms, Lemma 34.18.

03PD Definition 26.3. A ring map $A \rightarrow B$ is called standard étale if $B \cong (A[t]/(f))_g$ with $f, g \in A[t]$, with $f$ monic, and $df/dt$ invertible in $B$.

It is true that a standard étale ring map is étale. Namely, suppose that $B = (A[t]/(f))_g$ with $f, g \in A[t]$, with $f$ monic, and $df/dt$ invertible in $B$. Then $A[t]/(f)$ is a finite free $A$-module of rank equal to the degree of the monic polynomial $f$. Hence $B$, as a localization of this free algebra is finitely presented and flat over $A$. To finish the proof that $B$ is étale it suffices to show that the fibre rings

$$\kappa(p) \otimes_A B \cong \kappa(p) \otimes_A (A[t]/(f))_g \cong \kappa(p)[t, 1/\overline{g}]/(\overline{f})$$

are finite products of finite separable field extensions. Here $\overline{f}, \overline{g} \in \kappa(p)[t]$ are the images of $f$ and $g$. Let

$$\overline{f} = \overline{f}_1 \cdots \overline{f}_a \overline{f}_{a+1} \cdots \overline{f}_{a+b}$$

be the factorization of $\overline{f}$ into powers of pairwise distinct irreducible monic factors $\overline{f}_i$ with $\epsilon_1, \ldots, \epsilon_b > 0$. By assumption $d\overline{f}/dt$ is invertible in $\kappa(p)[t, 1/\overline{g}]$. Hence we see that at least all the $\overline{f}_i$, $i > a$ are invertible. We conclude that

$$\kappa(p)[t, 1/\overline{g}]/(\overline{f}) \cong \prod_{i \in I} \kappa(p)[t]/(\overline{f}_i)$$

where $I \subset \{1, \ldots, a\}$ is the subset of indices $i$ such that $\overline{f}_i$ does not divide $\overline{g}$. Moreover, the image of $d\overline{f}/dt$ in the factor $\kappa(p)[t]/(\overline{f}_i)$ is clearly equal to a unit times $d\overline{f}_i/dt$. Hence we conclude that $\kappa_i = \kappa(p)[t]/(\overline{f}_i)$ is a finite field extension of $\kappa(p)$ generated by one element whose minimal polynomial is separable, i.e., the field extension $\kappa(p) \subset \kappa_i$ is finite separable as desired.
It turns out that any étale ring map is locally standard étale. To formulate this we introduce the following notation. A ring map $A \to B$ is étale at a prime $q$ of $B$ if there exists $h \in B, h \notin q$ such that $A \to B_h$ is étale. Here is the result.

**Theorem 26.4.** A ring map $A \to B$ is étale at a prime $q$ if and only if there exists $g \in B, g \notin q$ such that $B \to B_g$ is standard étale over $A$.

**Proof.** See Algebra, Proposition 141.16.

27. Étale coverings

We recall the definition.

**Definition 27.1.** An étale covering of a scheme $U$ is a family of morphisms of schemes \(\{\varphi_i : U_i \to U\}_{i \in I}\) such that

1. each $\varphi_i$ is an étale morphism,
2. the $U_i$ cover $U$, i.e., $U = \bigcup_{i \in I} \varphi_i(U_i)$.

**Lemma 27.2.** Any étale covering is an fpqc covering.

**Proof.** (See also Topologies, Lemma 9.6.) Let \(\{\varphi_i : U_i \to U\}_{i \in I}\) be an étale covering. Since an étale morphism is flat, and the elements of the covering should cover its target, the property fp (faithfully flat) is satisfied. To check the property qc (quasi-compact), let $V \subset U$ be an affine open, and write $\varphi_i^{-1}(V) = \bigcup_{j \in J_i} V_{ij}$ for some affine opens $V_{ij} \subset U_i$. Since $\varphi_i$ is open (as étale morphisms are open), we see that $V = \bigcup_{i \in I} \bigcup_{j \in J_i} \varphi_i(V_{ij})$ is an open covering of $V$. Further, since $V$ is quasi-compact, this covering has a finite refinement. □

So any statement which is true for fpqc coverings remains true a fortiori for étale coverings. For instance, the étale site is subcanonical.

Let $S$ be a scheme. The big étale site over $S$ is the site $(\text{Sch}/S)_{\text{étale}}$, see Definition 20.4. The small étale site over $S$ is the site $S_{\text{étale}}$, see Definition 20.4. We define similarly the big and small Zariski sites on $S$, denoted $(\text{Sch}/S)_{\text{Zar}}$ and $S_{\text{Zar}}$.

Loosely speaking the big étale site of $S$ is made up out of schemes over $S$ and coverings the étale coverings. The small étale site of $S$ is made up out of schemes étale over $S$ with coverings the étale coverings. Actually any morphism between objects of $S_{\text{étale}}$ is étale, in virtue of Proposition 26.2, hence to check that $\{U_i \to U\}_{i \in I}$ in $S_{\text{étale}}$ is a covering it suffices to check that $\coprod U_i \to U$ is surjective.

The small étale site has fewer objects than the big étale site, it contains only the “opens” of the étale topology on $S$. It is a full subcategory of the big site, and its topology is induced from the topology on the big site. Hence it is true that the restriction functor from the big étale site to the small one is exact and maps injectives to injectives. This has the following consequence.

**Proposition 27.4.** Let $S$ be a scheme and $F$ an abelian sheaf on $(\text{Sch}/S)_{\text{étale}}$. Then $F|_{S_{\text{étale}}}$ is a sheaf on $S_{\text{étale}}$ and

$$H^p_{\text{étale}}(S, F|_{S_{\text{étale}}}) = H^p_{\text{étale}}(S, F)$$

for all $p \geq 0$.

**Proof.** This is a special case of Lemma 20.5.

□
In accordance with the general notation introduced in Section 20 we write $H^p_{\text{étale}}(S, \mathcal{F})$ for the above cohomology group.

## 28. Kummer theory

03PK Let $n \in \mathbb{N}$ and consider the functor $\mu_n$ defined by

$$
\begin{align*}
\text{Sch}^{\text{opp}} & \to \text{Ab} \\
S & \mapsto \mu_n(S) = \{ t \in \Gamma(S, \mathcal{O}_S^*) \mid t^n = 1 \}.
\end{align*}
$$

By Groupoids, Example 5.2 this is a representable functor, and the scheme representing it is denoted $\mu_n$ also. By Lemma 15.8 this functor satisfies the sheaf condition for the fpqc topology (in particular, it also satisfies the sheaf condition for the étale, Zariski, etc. topology).

**Lemma 28.1.** If $n \in \mathcal{O}_S^*$ then

$$
0 \to \mu_n \to G_{m,S} \xrightarrow{(\cdot)^n} G_{m,S} \to 0
$$

is a short exact sequence of sheaves on both the small and big étale site of $S$.

**Proof.** By definition the sheaf $\mu_{n,S}$ is the kernel of the map $(\cdot)^n$. Hence it suffices to show that the last map is surjective. Let $U$ be a scheme over $S$. Let $f \in G_{m}(U) = \Gamma(U, \mathcal{O}_U^*)$. We need to show that we can find an étale cover of $U$ over the members of which the restriction of $f$ is an $n$th power. Set $U' = \text{Spec}_U (\mathcal{O}_U[T]/(T^n - f)) \to U$.

(See Constructions, Section 3 or 4 for a discussion of the relative spectrum.) Let $\text{Spec}(A) \subset U$ be an affine open, and say $f|_{\text{Spec}(A)}$ corresponds to the unit $a \in A^*$. Then $\pi^{-1}(\text{Spec}(A)) = \text{Spec}(B)$ with $B = A[T]/(T^n - a)$. The ring map $A \to B$ is finite free of rank $n$, hence it is faithfully flat, and hence we conclude that $\text{Spec}(B) \to \text{Spec}(A)$ is surjective. Since this holds for every affine open in $U$ we conclude that $\pi$ is surjective. In addition, $n$ and $T^{n-1}$ are invertible in $B$, so $nT^{n-1} \in B^*$ and the ring map $A \to B$ is standard étale, in particular étale. Since this holds for every affine open of $U$ we conclude that $\pi$ is étale. Hence $U = \{ \pi : U' \to U \}$ is an étale covering. Moreover, $f|_{U'} = (f')^n$ where $f'$ is the class of $T$ in $\Gamma(U', \mathcal{O}_{U'}^*)$, so $U$ has the desired property. \[\square\]

03PM **Remark 28.2.** Lemma 28.1 is false when “étale” is replaced with “Zariski”. Since the étale topology is coarser than the smooth topology, see Topologies, Lemma 5.2 it follows that the sequence is also exact in the smooth topology.

By Theorem 24.1 and Lemma 28.1 and general properties of cohomology we obtain the long exact cohomology sequence

$$
\begin{align*}
0 & \to H^0_{\text{étale}}(S, \mu_n) \to \Gamma(S, \mathcal{O}_S^*) \xrightarrow{(\cdot)^n} \Gamma(S, \mathcal{O}_S^*) \\
& \xrightarrow{\; \downarrow \;} H^1_{\text{étale}}(S, \mu_n) \to \text{Pic}(S) \xrightarrow{(\cdot)^n} \text{Pic}(S) \\
& \xrightarrow{\; \downarrow \;} H^2_{\text{étale}}(S, \mu_n) \to \ldots
\end{align*}
$$
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at least if $n$ is invertible on $S$. When $n$ is not invertible on $S$ we can apply the following lemma.

**Lemma 28.3.** For any $n \in \mathbb{N}$ the sequence

$$0 \to \mu_{n,S} \to G_{m,S} \xrightarrow{(\cdot)^n} G_{m,S} \to 0$$

is a short exact sequence of sheaves on the site $(\text{Sch}/S)_{fppf}$ and $(\text{Sch}/S)_{\text{syntomic}}$.

**Proof.** By definition the sheaf $\mu_{n,S}$ is the kernel of the map $(\cdot)^n$. Hence it suffices to show that the last map is surjective. Since the syntomic topology is weaker than the fppf topology, see Topologies, Lemma 7.2, it suffices to prove this for the syntomic topology. Let $U$ be a scheme over $S$. Let $f \in G_{m}(U) = \Gamma(U, \mathcal{O}_U)$. We need to show that we can find a syntomic cover of $U$ over the members of which the restriction of $f$ is an $n$th power. Set

$$U' = \text{Spec}_{U'}(\mathcal{O}_U[T]/(T^n - f)) \xrightarrow{\pi} U.$$ (See Constructions, Section 3 or 4 for a discussion of the relative spectrum.) Let $\text{Spec}(A) \subset U$ be an affine open, and say $f|_{\text{Spec}(A)}$ corresponds to the unit $a \in A^*$. Then $\pi^{-1}(\text{Spec}(A)) = \text{Spec}(B)$ with $B = A[T]/(T^n - a)$. The ring map $A \to B$ is finite free of rank $n$, hence it is faithfully flat, and hence we conclude that $\text{Spec}(B) \to \text{Spec}(A)$ is surjective. Since this holds for every affine open in $U$ we conclude that $\pi$ is surjective. In addition, $B$ is a global relative complete intersection over $A$, so the ring map $A \to B$ is standard syntomic, in particular syntomic. Since this holds for every affine open of $U$ we conclude that $\pi$ is syntomic. Hence $U = \{ \pi : U' \to U \}$ is a syntomic covering. Moreover, $f|_{U'} = (f')^n$ where $f'$ is the class of $T$ in $\Gamma(U', \mathcal{O}_{U'})$, so $U$ has the desired property. \qed

**Remark 28.4.** Lemma 28.3 is false for the smooth, étale, or Zariski topology.

By Theorem 24.1 and Lemma 28.3 and general properties of cohomology we obtain the long exact cohomology sequence

$$0 \to H^0_{fppf}(S, \mu_{n,S}) \to \Gamma(S, \mathcal{O}_S) \xrightarrow{(\cdot)^n} \Gamma(S, \mathcal{O}_S^n)$$

$$H^1_{fppf}(S, \mu_{n,S}) \to \text{Pic}(S) \xrightarrow{(\cdot)^n} \text{Pic}(S)$$

$$H^2_{fppf}(S, \mu_{n,S}) \to \ldots$$

for any scheme $S$ and any integer $n$. Of course there is a similar sequence with syntomic cohomology.

Let $n \in \mathbb{N}$ and let $S$ be any scheme. There is another more direct way to describe the first cohomology group with values in $\mu_n$. Consider pairs $(\mathcal{L}, \alpha)$ where $\mathcal{L}$ is an invertible sheaf on $S$ and $\alpha : \mathcal{L}^{\otimes n} \to \mathcal{O}_S$ is a trivialization of the $n$th tensor power of $\mathcal{L}$. Let $(\mathcal{L}', \alpha')$ be a second such pair. An isomorphism $\varphi : (\mathcal{L}, \alpha) \to (\mathcal{L}', \alpha')$ is
an isomorphism \( \varphi : \mathcal{L} \to \mathcal{L}' \) of invertible sheaves such that the diagram

\[
\begin{array}{ccc}
\mathcal{L} \otimes n & \xrightarrow{\alpha} & \mathcal{O}_S \\
\varphi \otimes n & \downarrow & \\
(\mathcal{L}') \otimes n & \xrightarrow{\alpha'} & \mathcal{O}_S \\
\end{array}
\]

commutes. Thus we have

040P (28.4.1) \( \text{Isoms}((\mathcal{L}, \alpha), (\mathcal{L}', \alpha')) = \begin{cases} \\
\emptyset & \text{if they are not isomorphic} \\
H^0(S, \mu_{n, S} \cdot \varphi) & \text{if } \varphi \text{ isomorphism of pairs}
\end{cases} \)

Moreover, given two pairs \((\mathcal{L}, \alpha), (\mathcal{L}', \alpha')\) the tensor product

\((\mathcal{L}, \alpha) \otimes (\mathcal{L}', \alpha') = (\mathcal{L} \otimes \mathcal{L}', \alpha \otimes \alpha')\)

is another pair. The pair \((\mathcal{O}_S, 1)\) is an identity for this tensor product operation, and an inverse is given by

\((\mathcal{L}, \alpha)^{-1} = (\mathcal{L}^{-1}, \alpha^{-1})\).

Hence the collection of isomorphism classes of pairs forms an abelian group. Note that

\((\mathcal{L}, \alpha)^{\otimes n} = (\mathcal{L}^{\otimes n}, \alpha^{\otimes n}) \xrightarrow{\alpha} (\mathcal{O}_S, 1)\)

is an isomorphism hence every element of this group has order dividing \(n\). We warn the reader that this group is in general \textbf{not} the \(n\)-torsion in \(\text{Pic}(S)\).

040Q \textbf{Lemma 28.5.} Let \(S\) be a scheme. There is a canonical identification

\[H^1_{\text{étale}}(S, \mu_n) = \text{group of pairs } (\mathcal{L}, \alpha) \text{ up to isomorphism as above if } n \text{ is invertible on } S.\]

In general we have

\[H^1_{\text{fppf}}(S, \mu_n) = \text{group of pairs } (\mathcal{L}, \alpha) \text{ up to isomorphism as above.}\]

The same result holds with fppf replaced by syntomic.

\textbf{Proof.} We first prove the second isomorphism. Let \((\mathcal{L}, \alpha)\) be a pair as above. Choose an affine open covering \(S = \bigcup U_i\) such that \(\mathcal{L}|_{U_i} \cong \mathcal{O}_{U_i}\). Say \(s_i \in \mathcal{L}(U_i)\) is a generator. Then \(\alpha(s_i^{\otimes n}) = f_i \in \mathcal{O}^*_S(U_i)\). Writing \(U_i = \text{Spec}(A_i)\) we see there exists a global relative complete intersection \(A_i \to B_i = A_i[T]/(T^n - f_i)\) such that \(f_i\) maps to an \(n\)th power in \(B_i\). In other words, setting \(V_i = \text{Spec}(B_i)\) we obtain a syntomic covering \(V = \{V_i \to S\}_{i \in I}\) and trivializations \(\varphi_i : (\mathcal{L}, \alpha)|_{V_i} \to (\mathcal{O}_V, 1)\).

We will use this result (the existence of the covering \(V\)) to associate to this pair a cohomology class in \(H^1_{\text{syntomic}}(S, \mu_n, S)\). We give two (equivalent) constructions.

First construction: using Čech cohomology. Over the double overlaps \(V_i \times_S V_j\) we have the isomorphism

\[
(\mathcal{O}_{V_i \times_S V_j}, 1) \xrightarrow{\text{pr}_i^* \varphi_i^{-1}} (\mathcal{O}|_{V_i \times_S V_j}, \alpha|_{V_i \times_S V_j}) \xrightarrow{\text{pr}_j^* \varphi_j} (\mathcal{O}_{V_i \times_S V_j}, 1)
\]

of pairs. By (28.4.1) this is given by an element \(\zeta_{ij} \in \mu_n(V_i \times_S V_j)\). We omit the verification that these \(\zeta_{ij}\)’s give a \(1\)-cocycle, i.e., give an element \((\zeta_{ijk}) \in \check{C}(\mathcal{V}, \mu_n)\) with \(d(\zeta_{ijk}) = 0\). Thus its class is an element in \(H^1(V, \mu_n)\) and by Theorem 19.2 it maps to a cohomology class in \(H^1_{\text{syntomic}}(S, \mu_n, S)\).

Second construction: Using torsors. Consider the presheaf

\[
\mu_n(\mathcal{L}, \alpha) : U \mapsto \text{Isom}_U((\mathcal{O}_U, 1), (\mathcal{L}, \alpha)|_U)
\]
on \((\text{Sch}/S)_{\text{syntomic}}\). We may view this as a subpresheaf of \(\mathcal{H}om_{\mathcal{O}}(\mathcal{O}, \mathcal{L})\) (internal hom sheaf, see Modules on Sites, Section 27). Since the conditions defining this subpresheaf are local, we see that it is a sheaf. By (28.4.1) this sheaf has a free action of the sheaf \(\mu_{n,S}\). Hence the only thing we have to check is that it locally has sections. This is true because of the existence of the trivializing cover \(V\). Hence \(\mu_n(\mathcal{L}, \alpha)\) is a \(\mu_{n,S}\)-torsor and by Cohomology on Sites, Lemma 5.3 we obtain a corresponding element of \(H^1_{\text{syntomic}}(S, \mu_{n,S})\).

Ok, now we have to still show the following

1. The two constructions give the same cohomology class.
2. Isomorphic pairs give rise to the same cohomology class.
3. The cohomology class of \((\mathcal{L}, \alpha) \otimes (\mathcal{L}', \alpha')\) is the sum of the cohomology classes of \((\mathcal{L}, \alpha)\) and \((\mathcal{L}', \alpha')\).
4. If the cohomology class is trivial, then the pair is trivial.
5. Any element of \(H^1_{\text{syntomic}}(S, \mu_{n,S})\) is the cohomology class of a pair.

We omit the proof of (1). Part (2) is clear from the second construction, since isomorphic torsors give the same cohomology classes. Part (3) is clear from the first construction, since the resulting Čech classes add up. Part (4) is clear from the second construction since a torsor is trivial if and only if it has a global section, see Cohomology on Sites, Lemma 5.2.

Part (5) can be seen as follows (although a direct proof would be preferable). Suppose \(\xi \in H^1_{\text{syntomic}}(S, \mu_{n,S})\). Then \(\xi\) maps to an element \(\bar{\xi} \in H^1_{\text{syntomic}}(S, \mathbb{G}_m,S)\) with \(n\bar{\xi} = 0\). By Theorem 24.1 we see that \(\bar{\xi}\) corresponds to an invertible sheaf \(\mathcal{L}\) whose \(n\)th tensor power is isomorphic to \(\mathcal{O}_S\). Hence there exists a pair \((\mathcal{L}, \alpha')\) whose cohomology class \(\xi'\) has the same image \(\bar{\xi}'\) in \(H^1_{\text{syntomic}}(S, \mathbb{G}_m,S)\). Thus it suffices to show that \(\xi - \xi'\) is the class of a pair. By construction, and the long exact cohomology sequence above, we see that \(\xi - \xi' = \partial(f)\) for some \(f \in H^0(S, \mathcal{O}_S^*)\). Consider the pair \((\mathcal{O}_S, f)\). We omit the verification that the cohomology class of this pair is \(\partial(f)\), which finishes the proof of the first identification (with fppf replaced with syntomic).

To see the first, note that if \(n\) is invertible on \(S\), then the covering \(\mathcal{V}\) constructed in the first part of the proof is actually an étale covering (compare with the proof of Lemma 28.1). The rest of the proof is independent of the topology, apart from the very last argument which uses that the Kummer sequence is exact, i.e., uses Lemma 28.1.

29. Neighborhoods, stalks and points

We can associate to any geometric point of \(S\) a stalk functor which is exact. A map of sheaves on \(S_{\text{étale}}\) is an isomorphism if and only if it is an isomorphism on all these stalks. A complex of abelian sheaves is exact if and only if the complex of stalks is exact at all geometric points. Altogether this means that the small étale site of a scheme \(S\) has enough points. It also turns out that any point of the small étale topos of \(S\) (an abstract notion) is given by a geometric point. Thus in some sense the small étale topos of \(S\) can be understood in terms of geometric points and neighbourhoods.

Definition 29.1. Let \(S\) be a scheme.
(1) A geometric point of $S$ is a morphism $\text{Spec}(k) \to S$ where $k$ is algebraically closed. Such a point is usually denoted $\pi$, i.e., by an overlined small case letter. We often use $\bar{\pi}$ to denote the scheme $\text{Spec}(k)$ as well as the morphism, and we use $\kappa(\pi)$ to denote $k$.

(2) We say $\pi$ lies over $s$ to indicate that $s \in S$ is the image of $\pi$.

(3) An étale neighborhood of a geometric point $\pi$ of $S$ is a commutative diagram

$$
\begin{array}{ccc}
U & \xrightarrow{\varphi} & S \\
\downarrow{\bar{\pi}} & & \\
\pi & \xrightarrow{} & S
\end{array}
$$

where $\varphi$ is an étale morphism of schemes. We write $(U, \pi) \to (S, \pi)$.

(4) A morphism of étale neighborhoods $(U, \pi) \to (U', \pi')$ is an $S$-morphism $h : U \to U'$ such that $\pi' = h \circ \pi$.

**Remark 29.2.** Since $U$ and $U'$ are étale over $S$, any $S$-morphism between them is also étale, see Proposition 26.2. In particular all morphisms of étale neighborhoods are étale.

**Remark 29.3.** Let $S$ be a scheme and $s \in S$ a point. In More on Morphisms, Definition 31.1, we defined the notion of an étale neighbourhood $(U, u) \to (S, s)$ of $(S, s)$. If $\pi$ is a geometric point of $S$ lying over $s$, then any étale neighbourhood $(U, \pi) \to (S, \pi)$ gives rise to an étale neighbourhood $(U, u)$ of $(S, s)$ by taking $u \in U$ to be the unique point of $U$ such that $\pi$ lies over $u$. Conversely, given an étale neighbourhood $(U, u)$ of $(S, s)$ the residue field extension $\kappa(s) \subset \kappa(u)$ is finite separable (see Proposition 26.2) and hence we can find an embedding $\kappa(u) \subset \kappa(\pi)$ over $\kappa(s)$. In other words, we can find a geometric point $\pi$ of $U$ lying over $u$ such that $(U, \pi)$ is an étale neighbourhood of $(S, \pi)$. We will use these observations to go between the two types of étale neighbourhoods.

**Lemma 29.4.** Let $S$ be a scheme, and let $\pi$ be a geometric point of $S$. The category of étale neighborhoods is cofiltered. More precisely:

1. Let $(U_i, \pi_i)_{i=1,2}$ be two étale neighborhoods of $\pi$ in $S$. Then there exists a third étale neighborhood $(U, \pi)$ and morphisms $(U, \pi) \to (U_i, \pi_i)$, $i = 1, 2$.

2. Let $h_1, h_2 : (U, \pi) \to (U', \pi')$ be two morphisms between étale neighborhoods of $\pi$. Then there exist an étale neighborhood $(U'', \pi'')$ and a morphism $h : (U'', \pi'') \to (U, \pi)$ which equalizes $h_1$ and $h_2$, i.e., such that $h_1 \circ h = h_2 \circ h$.

**Proof.** For part (1), consider the fibre product $U = U_1 \times_S U_2$. It is étale over both $U_1$ and $U_2$ because étale morphisms are preserved under base change, see Proposition 26.2. The map $\pi \to U$ defined by $(\bar{\pi}_1, \bar{\pi}_2)$ gives it the structure of an étale neighborhood mapping to both $U_1$ and $U_2$. For part (2), define $U''$ as the fibre product

$$
\begin{array}{ccc}
U'' & \xrightarrow{} & U \\
\downarrow & & \\
U' & \xrightarrow{(h_1, h_2)} & U' \times_S U'.
\end{array}
$$

Since $\pi$ and $\pi'$ agree over $S$ with $\pi$, we see that $\pi'' = (\pi, \pi')$ is a geometric point of $U''$. In particular $U'' \neq \emptyset$. Moreover, since $U'$ is étale over $S$, so is the fibre product $U' \times_S U'$ (see Proposition 26.2). Hence the vertical arrow $(h_1, h_2)$ is étale.
by Remark 29.2 above. Therefore $U''$ is étale over $U'$ by base change, and hence also étale over $S$ (because compositions of étale morphisms are étale). Thus $(U'', \overline{U}'')$ is a solution to the problem. 

\textbf{Lemma 29.5.} Let $S$ be a scheme. Let $\overline{s}$ be a geometric point of $S$. Let $(U, \overline{u})$ be an étale neighborhood of $\overline{s}$. Let $U = \{ \varphi_i : U \to U \}_{i \in I}$ be an étale covering. Then there exist $i \in I$ and $\overline{u}_i : \overline{s} \to U_i$ such that $\varphi_i : (U_i, \overline{u}_i) \to (U, \overline{u})$ is a morphism of étale neighborhoods.

\textbf{Proof.} As $U = \bigcup_{i \in I} \varphi_i(U_i)$, the fibre product $\overline{s} \times_{U, \overline{u}} U_i$ is not empty for some $i$. Then look at the cartesian diagram

$$
\begin{array}{ccc}
\overline{s} \times_{U, \overline{u}} U_i & \to & U_i \\
\downarrow pr_1 & & \downarrow \varphi_i \\
Spec(k) = \overline{s} & \to & U
\end{array}
$$

The projection $pr_1$ is the base change of an étale morphisms so it is étale, see Proposition 26.2. Therefore, $\overline{s} \times_{U, \overline{u}} U_i$ is a disjoint union of finite separable extensions of $k$, by Proposition 26.2. Here $\overline{s} = Spec(k)$. But $k$ is algebraically closed, so all these extensions are trivial, and there exists a section $\sigma$ of $pr_1$. The composition $pr_2 \circ \sigma$ gives a map compatible with $\overline{s}$.

\textbf{Definition 29.6.} Let $S$ be a scheme. Let $\mathcal{F}$ be a presheaf on $S_{\text{étale}}$. Let $\overline{s}$ be a geometric point of $S$. The stalk of $\mathcal{F}$ at $\overline{s}$ is

$$
\mathcal{F}_{\overline{s}} = \colim_{(U, \overline{u})} \mathcal{F}(U)
$$

where $(U, \overline{u})$ runs over all étale neighborhoods of $\overline{s}$ in $S$.

By Lemma 29.4 this colimit is over a filtered index category, namely the opposite of the category of étale neighbourhoods. In other words, an element of $\mathcal{F}_{\overline{s}}$ can be thought of as a triple $(U, \overline{u}, \sigma)$ where $\sigma \in \mathcal{F}(U)$. Two triples $(U, \overline{u}, \sigma)$, $(U', \overline{u}', \sigma')$ define the same element of the stalk if there exists a third étale neighbourhood $(U'', \overline{u}'')$ and morphisms of étale neighbourhoods $h : (U'', \overline{u}'') \to (U, \overline{u})$, $h' : (U'', \overline{u}'') \to (U', \overline{u}')$ such that $h^* \sigma = (h')^* \sigma'$ in $\mathcal{F}(U'')$. See Categories, Section 19.

\textbf{Lemma 29.7.} Let $S$ be a scheme. Let $\overline{s}$ be a geometric point of $S$. Consider the functor

$$
u : S_{\text{étale}} \to \text{Sets},
U \mapsto \{ \overline{u} \text{ such that } (U, \overline{u}) \text{ is an étale neighbourhood of } \overline{s} \}.
$$

Here $|U_{\overline{s}}|$ denotes the underlying set of the geometric fibre. Then $\nu$ defines a point $p$ of the site $S_{\text{étale}}$ (Sites, Definition 32.4) and its associated stalk functor $\mathcal{F} \mapsto \mathcal{F}_p$ (Sites, Equation 32.1.1) is the functor $\mathcal{F} \mapsto \mathcal{F}_{\overline{s}}$ defined above.

\textbf{Proof.} In the proof of Lemma 29.5 we have seen that the scheme $U_{\overline{s}}$ is a disjoint union of schemes isomorphic to $\overline{s}$. Thus we can also think of $|U_{\overline{s}}|$ as the set of geometric points of $U$ lying over $\overline{s}$, i.e., as the collection of morphisms $\overline{u} : \overline{s} \to U$ fitting into the diagram of Definition 29.1. From this it follows that $\nu(S)$ is a singleton, and that $\nu(U \times_V W) = \nu(U) \times_{\nu(V)} \nu(W)$ whenever $U \to V$ and $W \to V$ are morphisms in $S_{\text{étale}}$. And, given a covering $(U_i \to U)_{i \in I}$ in $S_{\text{étale}}$ we see that $\prod \nu(U_i) \to \nu(U)$ is surjective by Lemma 29.5. Hence Sites, Proposition 33.2.
applies, so \( p \) is a point of the site \( S_{\text{étale}} \). Finally, our functor \( F \mapsto F_p \) is given by exactly the same colimit as the functor \( F \mapsto F_p \) associated to \( p \) in Sites, Equation 32.1.1 which proves the final assertion. \( \square \)

**Remark 29.8.** Let \( S \) be a scheme and let \( \overline{\sigma} : \text{Spec}(k) \rightarrow S \) and \( \overline{\sigma}' : \text{Spec}(k') \rightarrow S \) be two geometric points of \( S \). A morphism \( a : \overline{\sigma} \rightarrow \overline{\sigma}' \) of geometric points is simply a morphism \( a : \text{Spec}(k) \rightarrow \text{Spec}(k') \) such that \( a \circ \overline{\sigma} = \overline{\sigma}' \). Given such a morphism we obtain a functor from the category of étale neighbourhoods of \( \overline{\sigma} \) to the category of étale neighbourhoods of \( \overline{\sigma}' \) by the rule \( (U, \overline{\sigma}') \mapsto (U, \overline{\sigma} \circ a) \). Hence we obtain a canonical map

\[
F_{\overline{\sigma}} = \text{colim}_{(U, \overline{\sigma})} F(U) \longrightarrow \text{colim}_{(U, \overline{\sigma})} F(U) = \mathcal{F}_{\overline{\sigma}}
\]

from Categories, Lemma 14.7. Using the description of elements of stalks as triples this maps the element of \( \mathcal{F}_{\overline{\sigma}} \) represented by the triple \( (U, \overline{\sigma}, \sigma) \) to the element of \( \mathcal{F}_{\overline{\sigma}'} \) represented by the triple \( (U, \overline{\sigma}' \circ a, \sigma) \). Since the functor above is clearly an equivalence we conclude that this canonical map is an isomorphism of stalk functors.

Let us make sure we have the map of stalks corresponding to a pointing in the correct direction. Note that the above means, according to Sites, Definition 37.2 that a defines a morphism \( a : p \rightarrow p' \) between the points \( p, p' \) of the site \( S_{\text{étale}} \) associated to \( \overline{\sigma}, \overline{\sigma}' \) by Lemma 29.7. There are more general morphisms of points (corresponding to specializations of points of \( S \)) which we will describe later, and which will not be isomorphisms (insert future reference here).

**Lemma 29.9.** Let \( S \) be a scheme. Let \( \overline{\sigma} \) be a geometric point of \( S \).

1. The stalk functor \( \text{PAb}(S_{\text{étale}}) \rightarrow \text{Ab}, F \mapsto \mathcal{F}_{\overline{\sigma}} \) is exact.
2. We have \( (\mathcal{F}^\#)_{\overline{\sigma}} = \mathcal{F}_{\overline{\sigma}} \) for any presheaf of sets \( \mathcal{F} \) on \( S_{\text{étale}} \).
3. The functor \( \text{Ab}(S_{\text{étale}}) \rightarrow \text{Ab}, F \mapsto \mathcal{F}_{\overline{\sigma}} \) is exact.
4. Similarly the functors \( \text{PSh}(S_{\text{étale}}) \rightarrow \text{Sets} \) and \( \text{Sh}(S_{\text{étale}}) \rightarrow \text{Sets} \) given by the stalk functor \( F \mapsto \mathcal{F}_{\overline{\sigma}} \) are exact (see Categories, Definition 23.1) and commute with arbitrary colimits.

**Proof.** Before we indicate how to prove this by direct arguments we note that the result follows from the general material in Modules on Sites, Section 35. This is true because \( \mathcal{F} \mapsto \mathcal{F}_{\overline{\sigma}} \) comes from a point of the small étale site of \( S \), see Lemma 29.7. We will only give a direct proof of (1), (2) and (3), and omit a direct proof of (4).

Exactness as a functor on \( \text{PAb}(S_{\text{étale}}) \) is formal from the fact that directed colimits commute with all colimits and with finite limits. The identification of the stalks in (2) is via the map

\[
\kappa : \mathcal{F}_{\overline{\sigma}} \longrightarrow (\mathcal{F}^\#)_{\overline{\sigma}}
\]

induced by the natural morphism \( \mathcal{F} \rightarrow \mathcal{F}^\# \), see Theorem 13.2. We claim that this map is an isomorphism of abelian groups. We will show injectivity and omit the proof of surjectivity.

Let \( \sigma \in \mathcal{F}_{\overline{\sigma}} \). There exists an étale neighborhood \( (U, \overline{\sigma}) \rightarrow (S, \overline{\sigma}) \) such that \( \sigma \) is the image of some section \( s \in \mathcal{F}(U) \). If \( \kappa(\sigma) = 0 \) in \( (\mathcal{F}^\#)_{\overline{\sigma}} \) then there exists a morphism of étale neighborhoods \( (U', \overline{\sigma}') \rightarrow (U, \overline{\sigma}) \) such that \( s|_{U'} = 0 \in \mathcal{F}^\#(U') \). It follows there exists an étale covering \( \{U_i' \rightarrow U'_i \}_{i \in I} \) such that \( s|_{U_i'} = 0 \in \mathcal{F}(U_i') \) for all \( i \). By Lemma 29.5 there exist \( i \in I \) and a morphism \( \overline{\sigma}'_i : \overline{\sigma} \rightarrow U'_i \) such that
Let \((U'_i, \overline{\sigma}_i) \to (U'_j, \overline{\sigma}_j) \to (U, \overline{\sigma})\) are morphisms of étale neighborhoods. Hence \(\sigma = 0\) since \((U'_i, \overline{\sigma}_i) \to (U, \overline{\sigma})\) is a morphism of étale neighbourhoods such that we have \(s|_{U'_i} = 0\). This proves \(\kappa\) is injective.

To show that the functor \(\text{Ab}(S_{\text{étale}}) \to \text{Ab}\) is exact, consider any short exact sequence in \(\text{Ab}(S_{\text{étale}})\): \(0 \to F \to G \to H \to 0\). This gives us the exact sequence of presheaves

\[0 \to F \to G \to H \to \text{H}/\text{P}G \to 0,\]

where \(\text{P}\) denotes the quotient in \(\text{PAb}(S_{\text{étale}})\). Taking stalks at \(\overline{s}\), we see that \((H/\text{P}G)_\overline{s} = (\text{H}/\text{P}G)_\overline{s} = 0\), since the sheafification of \(\text{H}/\text{P}G\) is 0. Therefore,

\[0 \to F_\overline{s} \to G_\overline{s} \to H_\overline{s} \to 0 = (\text{H}/\text{P}G)_\overline{s}\]

is exact, since taking stalks is exact as a functor from presheaves. \(\square\)

\textbf{Theorem 29.10.} Let \(S\) be a scheme. A map \(\alpha : F \to G\) of sheaves of sets is injective (resp. surjective) if and only if the map on stalks \(\alpha_\overline{s} : F_\overline{s} \to G_\overline{s}\) is injective (resp. surjective) for all geometric points of \(S\). A sequence of abelian sheaves on \(S_{\text{étale}}\) is exact if and only if it is exact on all stalks at geometric points of \(S\).

\textbf{Proof.} The necessity of exactness on stalks follows from Lemma 29.9. For the converse, it suffices to show that a map of sheaves is surjective (respectively injective) if and only if it is surjective (respectively injective) on all stalks. We prove this in the case of surjectivity, and omit the proof in the case of injectivity.

Let \(\alpha : F \to G\) be a map of sheaves such that \(F_\overline{s} \to G_\overline{s}\) is surjective for all geometric points. Fix \(U \in \text{Ob}(S_{\text{étale}})\) and \(s \in G(U)\). For every \(u \in U\) choose some \(\overline{\sigma} \to U\) lying over \(u\) and an étale neighborhood \((V_u, \overline{\tau}_u) \to (U, \overline{\sigma})\) such that \(s|_{V_u} = \alpha(s|_{V_u})\) for some \(s|_{V_u} \in F(V_u)\). This is possible since \(\alpha\) is surjective on stalks. Then \(\{V_u \to U\}_{u \in U}\) is an étale covering on which the restrictions of \(s\) are in the image of the map \(\alpha\). Thus, \(\alpha\) is surjective, see Sites, Section 11. \(\square\)

\textbf{Remarks 29.11.} On points of the geometric sites.

(1) Theorem 29.10 says that the family of points of \(S_{\text{étale}}\) given by the geometric points of \(S\) (Lemma 29.7) is conservative, see Sites, Definition 38.1. In particular \(S_{\text{étale}}\) has enough points.

(2) Suppose \(F\) is a sheaf on the big étale site of \(S\). Let \(T \to S\) be an object of the big étale site of \(S\), and let \(\overline{t}\) be a geometric point of \(T\). Then we define \(F_\overline{t}\) as the stalk of the restriction \(F|_{T_{\text{étale}}}\) of \(F\) to the small étale site of \(T\). In other words, we can define the stalk of \(F\) at any geometric point of any scheme \(T/S \in \text{Ob}((\text{Sch}/S)_{\text{étale}})\).

(3) The big étale site of \(S\) also has enough points, by considering all geometric points of all objects of this site, see (2).

The following lemma should be skipped on a first reading.

\textbf{Lemma 29.12.} Let \(S\) be a scheme.

(1) Let \(p\) be a point of the small étale site \(S_{\text{étale}}\) of \(S\) given by a functor \(u : S_{\text{étale}} \to \text{Sets}\). Then there exists a geometric point \(\overline{s}\) of \(S\) such that \(p\) is isomorphic to the point of \(S_{\text{étale}}\) associated to \(\overline{s}\) in Lemma 29.7.

(2) Let \(p: \text{Sh}(pt) \to \text{Sh}(S_{\text{étale}})\) be a point of the small étale topos of \(S\). Then \(p\) comes from a geometric point of \(S\), i.e., the stalk functor \(F \to F_p\) is isomorphic to a stalk functor as defined in Definition 29.6.
Proof. By Sites, Lemma [32.7] there is a one to one correspondence between points of the site and points of the associated topos, hence it suffices to prove (1). By Sites, Proposition [33.2] the functor $u$ has the following properties: (a) $u(S) = \{*, \emptyset\}$, (b) $u(U \times S V) = u(U) \times_{u(S)} u(V)$, and (c) if $\{U \rightarrow U\}$ is an étale covering, then $\coprod u(U_i) \rightarrow u(U)$ is surjective. In particular, if $U' \subset U$ is an open subscheme, then $u(U') \subset u(U)$. Moreover, by Sites, Lemma [32.7] we can write $u(U) = p^{-1}(h^U_\emptyset)$, in other words $u(U)$ is the stalk of the representable sheaf $h_U$. If $U = V \amalg W$, then we see that $h_U = (h_V \amalg h_W)^\emptyset$ and we get $u(U) = u(V) \amalg u(W)$ since $p^{-1}$ is exact.

Consider the restriction of $u$ to $S_{\text{zar}}$. By Sites, Examples [33.4] and [33.5] there exists a unique point $s \in S$ such that for $S' \subset S$ open we have $u(S') = \{s\}$ if $s \in S'$ and $u(S') = \emptyset$ if $s \not\in S'$. Note that if $\varphi : U \rightarrow S$ is an object of $S_{\text{etale}}$ then $\varphi(U) \subset S$ is open (see Proposition [26.2] and $(U \rightarrow \varphi(U))$ is an étale covering. Hence we conclude that $u(U) = \emptyset \leftrightarrow s \in \varphi(U)$.

Pick a geometric point $\overline{s} : \overline{s} \rightarrow S$ lying over $s$, see Definition [29.1] for customary abuse of notation. Suppose that $\varphi : U \rightarrow S$ is an object of $S_{\text{etale}}$ with $U$ affine. Note that $\varphi$ is separated, and that the fibre $U_s$ of $\varphi$ over $s$ is an affine scheme over $\text{Spec}(\kappa(s))$ which is the spectrum of a finite product of finite separable extensions $k_i$ of $\kappa(s)$. Hence we may apply Étale Morphisms, Lemma [18.2] to get an étale neighbourhood $(V, \overline{\nu})$ of $(S, \overline{s})$ such that

$$U \times_S V = U_1 \amalg \ldots \amalg U_n \amalg W$$

with $U_i \rightarrow V$ an isomorphism and $W$ having no point lying over $\overline{s}$. Thus we conclude that

$$u(U) \times u(V) = u(U \times_S V) = u(U_1) \amalg \ldots \amalg u(U_n) \amalg u(W)$$

and of course also $u(U_i) = u(V)$. After shrinking $V$ a bit we can assume that $V$ has exactly one point lying over $s$, and hence $W$ has no point lying over $s$. By the above this then gives $u(W) = \emptyset$. Hence we obtain

$$u(U) \times u(V) = u(U_1) \amalg \ldots \amalg u(U_n) = \coprod_{i=1, \ldots, n} u(V)$$

Note that $u(V) \neq \emptyset$ as $s$ is in the image of $V \rightarrow S$. In particular, we see that in this situation $u(U)$ is a finite set with $n$ elements.

Consider the limit

$$\lim_{(V, \overline{\nu})} u(V)$$

over the category of étale neighbourhoods $(V, \overline{\nu})$ of $\overline{s}$. It is clear that we get the same value when taking the limit over the subcategory of $(V, \overline{\nu})$ with $V$ affine. By the previous paragraph (applied with the roles of $V$ and $U$ switched) we see that in this case $u(V)$ is always a finite nonempty set. Moreover, the limit is cofiltered, see Lemma [29.4] Hence by Categories, Section [20] the limit is nonempty. Pick an element $x$ from this limit. This means we obtain a $x_{V, \overline{\nu}} \in u(V)$ for every étale neighbourhood $(V, \overline{\nu})$ of $(S, \overline{s})$ such that for every morphism of étale neighbourhoods $\varphi : (V', \overline{\nu'}) \rightarrow (V, \overline{\nu})$ we have $u(\varphi)(x_{V', \overline{\nu'}}) = x_{V, \overline{\nu}}$.

We will use the choice of $x$ to construct a functorial bijective map

$$e : [U_{\overline{s}}] \rightarrow u(U)$$

for $U \in \text{Ob}(S_{\text{etale}})$ which will conclude the proof. See Lemma [29.7] and its proof for a description of $[U_{\overline{s}}]$. First we claim that it suffices to construct the map for $U$.
In this section we briefly discuss the existence of points for some sites other than the étale site of a scheme. We refer to Sites, Section 38 and Topologies, Section 2 for the terminology used in this section. All of the geometric sites have enough points.

Consider the diagram

\[\begin{array}{ccc}
V' & \xrightarrow{\varphi} & V \\
\sigma' \downarrow & & \sigma \downarrow \\
U \times_S V' & \xrightarrow{1 \times \varphi} & U \times_S V
\end{array}\]

Now, it may not be the case that this diagram commutes. The reason is that the schemes \(V'\) and \(V\) may not be connected, and hence the decompositions used to construct \(\sigma'\) and \(\sigma\) above may not be unique. But we do know that \(\sigma \circ \varphi \circ \overline{\tau} = (1 \times \varphi) \circ \sigma' \circ \overline{\tau}\) by construction. Hence, since \(U \times_S V\) is étale over \(S\), there exists an open neighbourhood \(V'' \subset V'\) of \(\overline{\tau}\) such that the diagram does commute when restricted to \(V''\), see Morphisms, Lemma 33.17. This means we may extend the diagram above to

\[\begin{array}{ccc}
V'' & \xrightarrow{\varphi} & V \\
\sigma'_{|V''} \downarrow & & \sigma' \downarrow \\
U \times_S V'' & \xrightarrow{1 \times \varphi} & U \times_S V
\end{array}\]

such that the left square and the outer rectangle commute. Since \(u\) is a functor this implies that \(x_{V'', \overline{\tau}}\) maps to the same element in \(u(U \times_S V)\) no matter which route we take through the diagram. On the other hand, it maps to the elements \(x_{V', \overline{\tau}}\) and \(x_{V, \overline{\tau}}\) in \(u(V')\) and \(u(V)\). This implies the desired equality \(u(\sigma')(x_{V', \overline{\tau}}) = u(\sigma)(x_{V, \overline{\tau}})\).

In a similar manner one proves that the construction \(c : [U_{\overline{\tau}}] \to u(U)\) is functorial in \(U\); details omitted. And finally, by the results of the third paragraph it is clear that the map \(c\) is bijective which ends the proof of the lemma.

\[\square\]

30. Points in other topologies

06VW In this section we briefly discuss the existence of points for some sites other than the étale site of a scheme. We refer to Sites, Section 38 and Topologies, Section 2 for the terminology used in this section. All of the geometric sites have enough points.

06VX Lemma 30.1. Let \(S\) be a scheme. All of the following sites have enough points (Sch/S)zar, (Sch/S)etale, (Aff/S)zar, (Aff/S)etale, (Sch/S)smooth, (Aff/S)smooth, (Sch/S)syntomic, (Aff/S)syntomic, (Sch/S)fppf, and (Aff/S)fppf.

Proof. For each of the big sites the associated topos is equivalent to the topos defined by the site (Aff/S)zar, see Topologies, Lemmas 3.10, 4.11, 5.9, 6.9, and 7.11.
The result for the sites \((\text{Aff}/S)_\text{et}\) follows immediately from Deligne’s result Sites, Lemma \([39.4]\).

The result for \(S_{\text{Zar}}\) is clear. The result for \(S_{\text{ét}}\) either follows from (the proof of) Theorem \([29.10]\) or from Lemma \([21.2]\) and Deligne’s result applied to \(S_{\text{affine, ét}}\). □

The lemma above guarantees the existence of points, but it doesn’t tell us what these points look like. We can explicitly construct some points as follows. Suppose \(\tilde{\pi} : \text{Spec}(k) \to S\) is a geometric point with \(k\) algebraically closed. Consider the functor

\[
u : (\text{Sch}/S)_{\text{fppf}} \to \text{Sets}, \quad \nu(U) = U(k) = \text{Mor}_S(\text{Spec}(k), U).
\]

Note that \(U \mapsto U(k)\) commutes with finite limits as \(S(k) = \{\overline{s}\}\) and \((U_1 \times_U U_2)(k) = U_1(k) \times_{U(k)} U_2(k)\). Moreover, if \(\{U_i \to U\}\) is an fppf covering, then \(\coprod U_i(k) \to U(k)\) is surjective. By Sites, Proposition \([33.2]\) we see that \(\nu\) defines a point \(p\) of \((\text{Sch}/S)_{\text{fppf}}\) with stalks

\[\mathcal{F}_p = \text{colim}_{(U,x)} \mathcal{F}(U)\]

where the colimit is over pairs \(U \to S, x \in U(k)\) as usual. But... this category has an initial object, namely \((\text{Spec}(k), \text{id})\), hence we see that

\[\mathcal{F}_p = \mathcal{F}(\text{Spec}(k))\]

which isn’t terribly interesting! In fact, in general these points won’t form a conservative family of points. A more interesting type of point is described in the following remark.

**Remark 30.2.** Let \(S = \text{Spec}(A)\) be an affine scheme. Let \((p, u)\) be a point of the site \((\text{Aff}/S)_{\text{fppf}}\), see Sites, Sections \([32]\) and \([33]\). Let \(B = \mathcal{O}_p\) be the stalk of the structure sheaf at the point \(p\). Recall that

\[B = \text{colim}_{(U,x)} \mathcal{O}(U) = \text{colim}_{(\text{Spec}(C), x_C)} C\]

where \(x_C \in u(\text{Spec}(C))\). It can happen that \(\text{Spec}(B)\) is an object of \((\text{Aff}/S)_{\text{fppf}}\) and that there is an element \(x_B \in u(\text{Spec}(B))\) mapping to the compatible system \(x_C\). In this case the system of neighbourhoods has an initial object and it follows that \(\mathcal{F}_p = \mathcal{F}(\text{Spec}(B))\) for any sheaf \(\mathcal{F}\) on \((\text{Aff}/S)_{\text{fppf}}\). It is straightforward to see that if \(\mathcal{F} \to \mathcal{F}(\text{Spec}(B))\) defines a point of \(\text{Sh}((\text{Aff}/S)_{\text{fppf}})\), then \(B\) has to be a local \(A\)-algebra such that for every faithfully flat, finitely presented ring map \(B \to B'\) there is a section \(B' \to B\). Conversely, for any such \(A\)-algebra \(B\) the functor \(\mathcal{F} \mapsto \mathcal{F}(\text{Spec}(B))\) is the stalk functor of a point. Details omitted. It is not clear what a general point of the site \((\text{Aff}/S)_{\text{fppf}}\) looks like.

### 31. Supports of abelian sheaves

**Lemma 31.1.** Let \(S\) be a scheme. Let \(\mathcal{F}\) be a subsheaf of the final object of the étale topos of \(S\) (see Sites, Example \([10.2]\)). Then there exists a unique open \(W \subset S\) such that \(\mathcal{F} = h_W\).

**Proof.** The condition means that \(\mathcal{F}(U)\) is a singleton or empty for all \(\varphi : U \to S\) in \(\text{Ob}(S_{\text{ét}})\). In particular local sections always glue. If \(\mathcal{F}(U) \neq \emptyset\), then \(\mathcal{F}(\varphi(U)) \neq \emptyset\) because \(\{\varphi : U \to \varphi(U)\}\) is a covering. Hence we can take \(W = \bigcup_{\varphi : U \to S, \mathcal{F}(U) \neq \emptyset} \varphi(U)\). □
Let $S$ be a scheme. Let $\mathcal{F}$ be an abelian sheaf on $\mathcal{S}_{\text{étale}}$. Let $\sigma \in \mathcal{F}(U)$ be a local section. There exists an open subset $W \subseteq U$ such that

1. $W \subseteq U$ is the largest Zariski open subset of $U$ such that $\sigma|_W = 0$,
2. for every $\phi : V \to U$ in $\mathcal{S}_{\text{étale}}$ we have $\sigma|_V = 0$ if $\phi(V) \subseteq W$,
3. for every geometric point $\overline{s}$ of $U$ we have $(U, \overline{s}, \sigma) = 0$ in $\mathcal{F}_{\overline{s}}$ if $\overline{s} \in W$.

where $\overline{s} = (U \to S) \circ \overline{\pi}$.

Proof. Since $\mathcal{F}$ is a sheaf in the étale topology the restriction of $\mathcal{F}$ to $U_{\text{Zar}}$ is a sheaf on $U$ in the Zariski topology. Hence there exists a Zariski open $W$ having property (1), see Modules, Lemma 31.2. Let $\phi : V \to U$ be an arrow of $\mathcal{S}_{\text{étale}}$. Note that $\phi(V) \subseteq U$ is an open subset and that $\{V \to \phi(V)\}$ is an étale covering. Hence if $\sigma|_V = 0$, then by the sheaf condition for $\mathcal{F}$ we see that $\sigma|_{\phi(V)} = 0$. This proves (2). To prove (3) we have to show that if $(U, \pi, \sigma)$ defines the zero element of $\mathcal{F}_{\pi}$, then $\pi \in W$. This is true because the assumption means there exists a morphism of étale neighbourhoods $(V, \pi) \to (U, \overline{\pi})$ such that $\sigma|_V = 0$. Hence by (2) we see that $V \to U$ maps into $W$, and hence $\pi \in W$. □

Let $S$ be a scheme. Let $s \in S$. Let $\mathcal{F}$ be a sheaf on $\mathcal{S}_{\text{étale}}$. By Remark 29.8 the isomorphism class of the stalk of the sheaf $\mathcal{F}$ at a geometric points lying over $s$ is well defined.

Definition 31.3. Let $S$ be a scheme. Let $\mathcal{F}$ be an abelian sheaf on $\mathcal{S}_{\text{étale}}$.

1. The support of $\mathcal{F}$ is the set of points $s \in S$ such that $\mathcal{F}_{\overline{s}} \neq 0$ for any (some) geometric point $\overline{s}$ lying over $s$.
2. Let $\sigma \in \mathcal{F}(U)$ be a section. The support of $\sigma$ is the closed subset $U \setminus W$, where $W \subseteq U$ is the largest open subset of $U$ on which $\sigma$ restricts to zero (see Lemma 31.2).

In general the support of an abelian sheaf is not closed. For example, suppose that $S = \text{Spec}(\mathbb{A}_k^1)$. Let $i_t : \text{Spec}(\mathbb{C}) \to S$ be the inclusion of the point $t \in \mathbb{C}$. We will see later that $\mathcal{F}_t = i_{t,*}(\mathbb{Z}/2\mathbb{Z})$ is an abelian sheaf whose support is exactly $\{t\}$, see Section 46. Then

$$\bigoplus_{n \in \mathbb{N}} \mathcal{F}_n$$

is an abelian sheaf with support $\{1, 2, 3, \ldots\} \subseteq S$. This is true because taking stalks commutes with colimits, see Lemma 29.9. Thus an example of an abelian sheaf whose support is not closed. Here are some basic facts on supports of sheaves and sections.

Lemma 31.4. Let $S$ be a scheme. Let $\mathcal{F}$ be an abelian sheaf on $\mathcal{S}_{\text{étale}}$. Let $U \in \text{Ob} (\mathcal{S}_{\text{étale}})$ and $\sigma \in \mathcal{F}(U)$.

1. The support of $\sigma$ is closed in $U$.
2. The support of $\sigma + \sigma'$ is contained in the union of the supports of $\sigma, \sigma' \in \mathcal{F}(U)$.
3. If $\phi : \mathcal{F} \to \mathcal{G}$ is a map of abelian sheaves on $\mathcal{S}_{\text{étale}}$, then the support of $\phi(\sigma)$ is contained in the support of $\sigma \in \mathcal{F}(U)$.
4. The support of $\mathcal{F}$ is the union of the images of the supports of all local sections of $\mathcal{F}$. 

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(5) If \( F \to G \) is surjective then the support of \( G \) is a subset of the support of \( F \).
(6) If \( F \to G \) is injective then the support of \( F \) is a subset of the support of \( G \).

Proof. Part (1) holds by definition. Parts (2) and (3) hold because they hold for the restriction of \( F \) and \( G \) to \( U_{	ext{Zar}} \), see Modules, Lemma 5.2. Part (4) is a direct consequence of Lemma 31.2 part (3). Parts (5) and (6) follow from the other parts. □

Lemma 31.5. The support of a sheaf of rings on \( S_{	ext{étale}} \) is closed.

Proof. This is true because (according to our conventions) a ring is 0 if and only if 1 = 0, and hence the support of a sheaf of rings is the support of the unit section. □

32. Henselian rings

We begin by stating a theorem which has already been used many times in the Stacks project. There are many versions of this result; here we just state the algebraic version.

Theorem 32.1. Let \( A \to B \) be finite type ring map and \( p \subset A \) a prime ideal. Then there exist an \( \text{étale} \) ring map \( A \to A' \) and a prime \( p' \subset A' \) lying over \( p \) such that

\[
\begin{align*}
(1) & \quad \kappa(p) = \kappa(p'), \\
(2) & \quad B \otimes_A A' = B_1 \times \ldots \times B_r \times C, \\
(3) & \quad A' \to B_i \text{ is finite and there exists a unique prime } q_i \subset B_i \text{ lying over } p', \text{ and} \\
(4) & \quad \text{all irreducible components of the fibre } \text{Spec}(C \otimes_{A'} \kappa(p')) \text{ of } C \text{ over } p' \text{ have dimension at least 1.}
\end{align*}
\]

Proof. See Algebra, Lemma 141.22 or see [GD67, Théorème 18.12.1]. For a slew of versions in terms of morphisms of schemes, see More on Morphisms, Section 36. □

Recall Hensel’s lemma. There are many versions of this lemma. Here are two:

(f) if \( f \in \mathbb{Z}_p[T] \) monic and \( f \mod p = g_0h_0 \) with \( \gcd(g_0, h_0) = 1 \) then \( f \) factors as \( f = gh \) with \( \bar{g} = g_0 \) and \( \bar{h} = h_0 \),

(r) if \( f \in \mathbb{Z}_p[T] \), monic \( a_0 \in \mathbb{F}_p \), \( \bar{f}(a_0) = 0 \) but \( \bar{f}'(a_0) \neq 0 \) then there exists \( a \in \mathbb{Z}_p \) with \( f(a) = 0 \) and \( \bar{a} = a_0 \).

Both versions are true (we will see this later). The first version asks for lifts of factorizations into coprime parts, and the second version asks for lifts of simple roots modulo the maximal ideal. It turns out that requiring these conditions for a general local ring are equivalent, and are equivalent to many other conditions. We use the root lifting property as the definition of a henselian local ring as it is often the easiest one to check.

Definition 32.2. (See Algebra, Definition 148.1) A local ring \( (R, m, \kappa) \) is called henselian if for all \( f \in R[T] \) monic, for all \( a_0 \in \kappa \) such that \( \bar{f}(a_0) = 0 \) and \( \bar{f}'(a_0) \neq 0 \), there exists an \( a \in R \) such that \( f(a) = 0 \) and \( a \mod m = a_0 \).

A good example of henselian local rings to keep in mind is complete local rings. Recall (Algebra, Definition 154.1) that a complete local ring is a local ring \( (R, m) \) such that \( R \cong \lim_n R/m^n \), i.e., it is complete and separated for the \( m \)-adic topology.
Theorem 32.3. Complete local rings are henselian.


Theorem 32.4. Let \((R, m, \kappa)\) be a local ring. The following are equivalent:

1. \(R\) is henselian,
2. for any \(f \in R[T]\) and any factorization \(f = g_0h_0\) in \(\kappa[T]\) with \(\gcd(g_0, h_0) = 1\), there exists a factorization \(f = gh\) in \(R[T]\) with \(\bar{g} = g_0\) and \(\bar{h} = h_0\),
3. any finite \(R\)-algebra \(S\) is isomorphic to a finite product of local rings finite over \(R\),
4. any finite type \(R\)-algebra \(A\) is isomorphic to a product \(A \cong A' \times C\) where \(A' \cong A_1 \times \ldots \times A_r\) is a product of finite local \(R\)-algebras and all the irreducible components of \(C \otimes_R \kappa\) have dimension at least 1,
5. if \(A\) is an étale \(R\)-algebra and \(n\) is a maximal ideal of \(A\) lying over \(m\) such that \(\kappa \cong A/n\), then there exists an isomorphism \(\varphi : A \cong R \times A'\) such that \(\varphi(n) = m \times A' \subset R \times A'\).

Proof. This is just a subset of the results from Algebra, Lemma 148.3. Note that part (5) above corresponds to part (8) of Algebra, Lemma 148.3 but is formulated slightly differently.

Lemma 32.5. If \(R\) is henselian and \(A\) is a finite \(R\)-algebra, then \(A\) is a finite product of henselian local rings.

Proof. See Algebra, Lemma 148.4.

Definition 32.6. A local ring \(R\) is called strictly henselian if it is henselian and its residue field is separably closed.

Example 32.7. In the case \(R = \mathbb{C}[[t]]\), the étale \(R\)-algebras are finite products of the trivial extension \(R \to R\) and the extensions \(R \to R[X, X^{-1}]/(X^n - t)\). The latter ones factor through the open \(D(t) \subset \text{Spec}(R)\), so any étale covering can be refined by the covering \(\{\text{id} : \text{Spec}(R) \to \text{Spec}(R)\}\). We will see below that this is a somewhat general fact on étale coverings of spectra of henselian rings. This will show that higher étale cohomology of the spectrum of a strictly henselian ring is zero.

Theorem 32.8. Let \((R, m, \kappa)\) be a local ring and \(\kappa \subset \kappa^{sep}\) a separable algebraic closure. There exist canonical flat local ring maps \(R \to R^h \to R^{sh}\) where

1. \(R^h, R^{sh}\) are filtered colimits of étale \(R\)-algebras,
2. \(R^h\) is henselian, \(R^{sh}\) is strictly henselian,
3. \(mR^h\) (resp. \(mR^{sh}\)) is the maximal ideal of \(R^h\) (resp. \(R^{sh}\)), and
4. \(\kappa = R^h/mR^h\), and \(\kappa^{sep} = R^{sh}/mR^{sh}\) as extensions of \(\kappa\).

Proof. The structure of \(R^h\) and \(R^{sh}\) is described in Algebra, Lemmas 150.1 and 150.2.

The rings constructed in Theorem 32.8 are called respectively the henselization and the strict henselization of the local ring \(R\), see Algebra, Definition 150.3. Many of the properties of \(R\) are reflected in its (strict) henselization, see More on Algebra, Section 44.
33. Stalks of the structure sheaf

In this section we identify the stalk of the structure sheaf at a geometric point with the strict henselization of the local ring at the corresponding “usual” point.

**Lemma 33.1.** Let $S$ be a scheme. Let $\overline{s}$ be a geometric point of $S$ lying over $s \in S$. Let $\kappa = \kappa(s)$ and let $\kappa \subset \kappa^{sep} \subset \kappa(\overline{s})$ denote the separable algebraic closure of $\kappa$ in $\kappa(\overline{s})$. Then there is a canonical identification

$$(O_{S,s})^{sh} \cong O_{S,\overline{s}}$$

where the left hand side is the strict henselization of the local ring $O_{S,s}$ as described in Theorem 32.8 and right hand side is the stalk of the structure sheaf $O_S$ on $S_{\text{étale}}$ at the geometric point $\overline{s}$.

**Proof.** Let $\text{Spec}(A) \subset S$ be an affine neighbourhood of $s$. Let $p \subset A$ be the prime ideal corresponding to $s$. With these choices we have canonical isomorphisms $O_{S,s} = A_p$ and $\kappa(s) = \kappa(p)$. Thus we have $\kappa(p) \subset \kappa^{sep} \subset \kappa(\overline{s})$. Recall that

$$O_{S,\overline{s}} = \colim_{(U,\overline{u})} O(U)$$

where the limit is over the étale neighbourhoods of $(S, \overline{s})$. A cofinal system is given by those étale neighbourhoods $(U, \overline{u})$ such that $U$ is affine and $U \to S$ factors through $\text{Spec}(A)$. In other words, we see that

$$O_{S,\overline{s}} = \colim_{(B,q,\phi)} B$$

where the colimit is over étale $A$-algebras $B$ endowed with a prime $q$ lying over $p$ and a $\kappa(p)$-algebra map $\phi : \kappa(q) \to \kappa(\overline{s})$. Note that since $\kappa(q)$ is finite separable over $\kappa(p)$ the image of $\phi$ is contained in $\kappa^{sep}$. Via these translations the result of the lemma is equivalent to the result of Algebra, Lemma 150.13. □

**Definition 33.2.** Let $S$ be a scheme. Let $\overline{s}$ be a geometric point of $S$ lying over the point $s \in S$.

1. The étale local ring of $S$ at $\overline{s}$ is the stalk of the structure sheaf $O_S$ on $S_{\text{étale}}$ at $\overline{s}$. We sometimes call this the strict henselization of $O_{S,s}$ relative to the geometric point $\overline{s}$. Notation used: $O_{S,\overline{s}} = O_{S,s}^{sh}$.
2. The henselization of $O_{S,s}$ is the henselization of the local ring of $S$ at $s$. See Algebra, Definition 150.3 and Theorem 32.8. Notation: $O_{S,s}^h$.
3. The strict henselization of $S$ at $\overline{s}$ is the scheme $\text{Spec}(O_{S,s}^{sh})$.
4. The henselization of $S$ at $s$ is the scheme $\text{Spec}(O_{S,s}^h)$.

**Lemma 33.3.** Let $S$ be a scheme. Let $s \in S$. Then we have

$$O_{S,s}^h = \colim_{(U,u)} O(U)$$

where the colimit is over the filtered category of étale neighbourhoods $(U,u)$ of $(S,s)$ such that $\kappa(s) = \kappa(u)$.

**Proof.** This lemma is a copy of More on Morphisms, Lemma 31.5. □

**Remark 33.4.** Let $S$ be a scheme. Let $s \in S$. If $S$ is locally noetherian then $O_{S,s}^h$ is also noetherian and it has the same completion:

$$\widehat{O_{S,s}} \cong \widehat{O_{S,s}^h}.$$
In particular, $\mathcal{O}_{S,s} \subset \mathcal{O}^h_{S,s} \subset \widehat{\mathcal{O}}_{S,s}$. The henselization of $\mathcal{O}_{S,s}$ is in general much smaller than its completion and inherits many of its properties. For example, if $\mathcal{O}_{S,s}$ is reduced, then so is $\mathcal{O}^h_{S,s}$, but this is not true for the completion in general. Insert future references here.

**Lemma 33.5.** Let $S$ be a scheme. The small étale site $S_{\text{étale}}$ endowed with its structure sheaf $\mathcal{O}_S$ is a locally ringed site, see Modules on Sites, Definition 39.4.

**Proof.** This follows because the stalks $\mathcal{O}^h_{S,s} = \mathcal{O}_{S,s}$ are local, and because $S_{\text{étale}}$ has enough points, see Lemma 33.1, Theorem 29.10, and Remarks 29.11. See Modules on Sites, Lemmas 39.2 and 39.3 for the fact that this implies the small étale site is locally ringed. □

### 34. Functoriality of small étale topos

So far we haven’t yet discussed the functoriality of the étale site, in other words what happens when given a morphism of schemes. A precise formal discussion can be found in Topologies, Section 4. In this and the next sections we discuss this material briefly specifically in the setting of small étale sites.

Let $f : X \to Y$ be a morphism of schemes. We obtain a functor

$$u : Y_{\text{étale}} \to X_{\text{étale}}, \quad V/Y \mapsto X \times_Y V/X.$$

This functor has the following important properties

1. $u$(final object) = final object,
2. $u$ preserves fibre products,
3. if $\{V_j \to V\}$ is a covering in $Y_{\text{étale}}$, then $\{u(V_j) \to u(V)\}$ is a covering in $X_{\text{étale}}$.

Each of these is easy to check (omitted). As a consequence we obtain what is called a morphism of sites

$$f_{\text{small}} : X_{\text{étale}} \to Y_{\text{étale}},$$

see Sites, Definition 14.1 and Sites, Proposition 14.7. It is not necessary to know about the abstract notion in detail in order to work with étale sheaves and étale cohomology. It usually suffices to know that there are functors $f_{\text{small,*}}$ (pushforward) and $f^{-1}_{\text{small}}$ (pullback) on étale sheaves, and to know some of their simple properties. We will discuss these properties in the next sections, but we will sometimes refer to the more abstract material for proofs since that is often the natural setting to prove them.

### 35. Direct images

Let us define the pushforward of a presheaf.

**Definition 35.1.** Let $f : X \to Y$ be a morphism of schemes. Let $\mathcal{F}$ a presheaf of sets on $X_{\text{étale}}$. The direct image, or pushforward of $\mathcal{F}$ (under $f$) is

$$f_* \mathcal{F} : Y_{\text{étale}}^{\text{opp}} \to \text{Sets}, \quad (V/Y) \mapsto \mathcal{F}(X \times_Y V/X).$$

We sometimes write $f_* = f_{\text{small,*}}$ to distinguish from other direct image functors (such as usual Zariski pushforward or $f_{\text{big,*}}$).
This is a well-defined étale presheaf since the base change of an étale morphism is again étale. A more categorical way of saying this is that $f_* F$ is the composition of functors $F \circ u$ where $u$ is as in Equation (34.0.1). This makes it clear that the construction is functorial in the presheaf $F$ and hence we obtain a functor

$$f_* = f_{small,*} : PSh(X_{étale}) \longrightarrow PSh(Y_{étale})$$

Note that if $F$ is a presheaf of abelian groups, then $f_* F$ is also a presheaf of abelian groups and we obtain

$$f_* = f_{small,*} : PAb(X_{étale}) \longrightarrow PAb(Y_{étale})$$

as before (i.e., defined by exactly the same rule).

**Remark 35.2.** We claim that the direct image of a sheaf is a sheaf. Namely, if $\{V_j \to V\}$ is an étale covering in $Y_{étale}$ then $\{X \times_Y V_j \to X \times_Y V\}$ is an étale covering in $X_{étale}$. Hence the sheaf condition for $F$ with respect to $\{X \times_Y V_i \to X \times_Y V\}$ is equivalent to the sheaf condition for $f_* F$ with respect to $\{V_i \to V\}$. Thus if $F$ is a sheaf, so is $f_* F$.

**Definition 35.3.** Let $f : X \to Y$ be a morphism of schemes. Let $F$ a sheaf of sets on $X_{étale}$. The direct image, or pushforward of $F$ (under $f$) is

$$f_* : Y_{étale}^{opp} \longrightarrow Sets, \quad (V/Y) \mapsto F(X \times_Y V/X)$$

which is a sheaf by Remark 35.2. We sometimes write $f_* = f_{small,*}$ to distinguish from other direct image functors (such as usual Zariski pushforward or $f_{big,*}$).

The exact same discussion as above applies and we obtain functors

$$f_* = f_{small,*} : Sh(X_{étale}) \longrightarrow Sh(Y_{étale})$$

and

$$f_* = f_{small,*} : Ab(X_{étale}) \longrightarrow Ab(Y_{étale})$$

called direct image again.

The functor $f_*$ on abelian sheaves is left exact. (See Homology, Section 7 for what it means for a functor between abelian categories to be left exact.) Namely, if $0 \to F_1 \to F_2 \to F_3$ is exact on $X_{étale}$, then for every $U/X \in \text{Ob}(X_{étale})$ the sequence of abelian groups $0 \to f_* F_1(U) \to f_* F_2(U) \to f_* F_3(U)$ is exact. Hence for every $V/Y \in \text{Ob}(Y_{étale})$ the sequence of abelian groups $0 \to f_* F_1(V) \to f_* F_2(V) \to f_* F_3(V)$ is exact, because this is the previous sequence with $U = X \times_Y V$.

**Definition 35.4.** Let $f : X \to Y$ be a morphism of schemes. The right derived functors $\{R^p f_*\}_{p \geq 1}$ of $f_* : Ab(X_{étale}) \to Ab(Y_{étale})$ are called higher direct images.

The higher direct images and their derived category variants are discussed in more detail in (insert future reference here).

### 36. Inverse image

In this section we briefly discuss pullback of sheaves on the small étale sites. The precise construction of this is in Topologies, Section 4.
Definition 36.1. Let $f : X \to Y$ be a morphism of schemes. The inverse image, or pullback functors are the functors

$$f^{-1} = f^{-1}_{small} : Sh(Y_{étale}) \to Sh(X_{étale})$$

and

$$f^{-1} = f^{-1}_{small} : Ab(Y_{étale}) \to Ab(X_{étale})$$

which are left adjoint to $f_* = f_{small,*}$. Thus $f^{-1}$ thus characterized by the fact

$$\text{Hom}_{Sh(X_{étale})}(f^{-1}_* G, F) = \text{Hom}_{Sh(Y_{étale})}(G, f_* F)$$

for $F \in Ab(X_{étale})$ and $G \in Ab(Y_{étale})$. We similarly have

$$\text{Hom}_{Ab(Y_{étale})}(f^{-1}_* G, F) = \text{Hom}_{Ab(Y_{étale})}(G, f_* F)$$

for $F \in Ab(X_{étale})$ and $G \in Ab(Y_{étale})$.

It is not trivial that such an adjoint exists. On the other hand, it exists in a fairly general setting, see Remark 36.3 below. The general machinery shows that $f^{-1} G$ is the sheaf associated to the presheaf

$$(36.1.1) \quad U/X \mapsto \text{colim}_{U \to X \times_Y V} G(V/Y)$$

where the colimit is over the category of pairs $(V/Y, \varphi : U/X \to X \times_Y V/X)$. To see this apply Sites, Proposition 14.7 to the functor $u$ of Equation (34.0.1) and use the description of $u_* = (u_y)_\#$ in Sites, Sections 13 and 5. We will occasionally use this formula for the pullback in order to prove some of its basic properties.

Lemma 36.2. Let $f : X \to Y$ be a morphism of schemes.

1. The functor $f^{-1} : Ab(Y_{étale}) \to Ab(X_{étale})$ is exact.
2. The functor $f^{-1} : Sh(Y_{étale}) \to Sh(X_{étale})$ is exact, i.e., it commutes with finite limits and colimits, see Categories, Definition 23.7.
3. Let $\overline{p} \to X$ be a geometric point. Let $G$ be a sheaf on $Y_{étale}$. Then there is a canonical identification

$$\left(f^{-1} G\right)_\overline{p} = G_{\overline{p}}.$$  

where $\overline{p} = f \circ \overline{p}$.
4. For any $V \to Y$ étale we have $f^{-1} h_V = h_{X \times_Y V}$.

Proof. The exactness of $f^{-1}$ on sheaves of sets is a consequence of Sites, Proposition 14.7 applied to our functor $u$ of Equation (34.0.1). In fact the exactness of pullback is part of the definition of a morphism of topoi (or sites if you like). Thus we see (2) holds. It implies part (1) since given an abelian sheaf $G$ on $Y_{étale}$ the underlying sheaf of sets of $f^{-1} F$ is the same as $f^{-1}$ of the underlying sheaf of sets of $F$, see Sites, Section 14. See also Modules on Sites, Lemma 30.2. In the literature (1) and (2) are sometimes deduced from (3) via Theorem 29.10.

Part (3) is a general fact about stalks of pullbacks, see Sites, Lemma 34.1. We will also prove (3) directly as follows. Note that by Lemma 29.9 taking stalks commutes with sheafification. Now recall that $f^{-1} G$ is the sheaf associated to the presheaf

$$U \mapsto \text{colim}_{U \to X \times_Y V} G(V),$$

We use the notation $f^{-1}$ for pullbacks of sheaves of sets or sheaves of abelian groups, and we reserve $f^*$ for pullbacks of sheaves of modules via a morphism of ringed sites/topoi.
see Equation \([\text{36.1.1}]\). Thus we have
\[
(f^{-1}\mathcal{G})_{\pi} = \text{colim}_{(U,\pi)} f^{-1}\mathcal{G}(U)
= \text{colim}_{(U,\pi)} \text{colim}_{a:U \to X \times_Y V} \mathcal{G}(V)
= \text{colim}_{(V,\tau)} \mathcal{G}(V)
= \mathcal{G}_{\tau}
\]
in the third equality the pair \((U,\pi)\) and the map \(a:U \to X \times_Y V\) corresponds to the pair \((V,a \circ \pi)\).

Part (4) can be proved in a similar manner by identifying the colimits which define \(f^{-1}h_V\). Or you can use Yoneda’s lemma (Categories, Lemma \([3.5]\)) and the functorial equalities
\[
\text{Mor}_{\text{Sh}(X_{\text{etale}})}(f^{-1}h_V,F) = \text{Mor}_{\text{Sh}(Y_{\text{etale}})}(h_V,f_*F) = f_*F(V) = F(X \times_Y V)
\]
combined with the fact that representable presheaves are sheaves. See also Sites, Lemma \([13.5]\) for a completely general result. □

The pair of functors \((f_*,f^{-1})\) define a morphism of small étale topoi
\[
f_{\text{small}} : \text{Sh}(X_{\text{etale}}) \to \text{Sh}(Y_{\text{etale}})
\]
Many generalities on cohomology of sheaves hold for topoi and morphisms of topoi. We will try to point out when results are general and when they are specific to the étale topos.

03Q2 Remark 36.3. More generally, let \(C_1, C_2\) be sites, and assume they have final objects and fibre products. Let \(u : C_2 \to C_1\) be a functor satisfying:

1. if \(\{V_i \to V\}\) is a covering of \(C_2\), then \(\{u(V_i) \to u(V)\}\) is a covering of \(C_1\) (we say that \(u\) is continuous), and
2. \(u\) commutes with finite limits (i.e., \(u\) is left exact, i.e., \(u\) preserves fibre products and final objects).

Then one can define \(f_* : \text{Sh}(C_1) \to \text{Sh}(C_2)\) by \(f_*F(V) = F(u(V))\). Moreover, there exists an exact functor \(f^{-1}\) which is left adjoint to \(f_*\), see Sites, Definition \([14.1]\) and Proposition \([14.7]\). Warning: It is not enough to require simply that \(u\) is continuous and commutes with fibre products in order to get a morphism of topoi.

37. Functoriality of big topoi

04DI Given a morphism of schemes \(f : X \to Y\) there are a whole host of morphisms of topoi associated to \(f\), see Topologies, Section \([11]\) for a list. Perhaps the most used ones are the morphisms of topoi
\[
f_{\text{big}} = f_{\text{big},\tau} : \text{Sh}((\text{Sch}/X)_\tau) \to \text{Sh}((\text{Sch}/Y)_\tau)
\]
where \(\tau \in \{\text{Zariski, étale, smooth, syntomic, fppf}\}\). These each correspond to a continuous functor
\[
(\text{Sch}/Y)_\tau \to (\text{Sch}/X)_\tau, \quad V/Y \mapsto X \times_Y V/X
\]
which preserves final objects, fibre products and covering, and hence defines a morphism of sites
\[
f_{\text{big}} : (\text{Sch}/X)_\tau \to (\text{Sch}/Y)_\tau.
\]
See Topologies, Sections 3, 4, 6, and 7. In particular, pushforward along $f_{big}$ is given by the rule

$$(f_{big,*}F)(V/Y) = F(X \times Y V/X)$$

It turns out that these morphisms of topoi have an inverse image functor $f_{big}^{-1}$ which is very easy to describe. Namely, we have

$$(f_{big}^{-1}G)(U/X) = G(U/Y)$$

where the structure morphism of $U/Y$ is the composition of the structure morphism $U \to X$ with $f$, see Topologies, Lemmas 3.15, 4.15, 5.10, 6.10, and 7.12.

### 38. Functoriality and sheaves of modules

04I4 In this section we are going to reformulate some of the material explained in Descent, Section 8 in the setting of étale topologies. Let $f : X \to Y$ be a morphism of schemes. We have seen above, see Sections 34, 35, and 36 that this induces a morphism $f_{small}$ of small étale sites. In Descent, Remark 8.4 we have seen that $f$ also induces a natural map

$$f^\sharp_{small} : \mathcal{O}_Y^{\text{étale}} \to f_{small,*} \mathcal{O}_X^{\text{étale}}$$

of sheaves of rings on $Y_{\text{étale}}$ such that $(f_{small}, f^\sharp_{small})$ is a morphism of ringed sites. Let us just recall here that $f^\sharp_{small}$ is defined by the compatible system of maps

$$\text{pr}^V_X : \mathcal{O}(V) \to \mathcal{O}(X \times Y V)$$

for $V$ varying over the objects of $Y_{\text{étale}}$.

It is clear that this construction is compatible with compositions of morphisms of schemes. More precisely, if $f : X \to Y$ and $g : Y \to Z$ are morphisms of schemes, then we have

$$(g_{small}, g^\sharp_{small}) \circ (f_{small}, f^\sharp_{small}) = ((g \circ f)_{small}, (g \circ f)^\sharp_{small})$$

as morphisms of ringed topoi. Moreover, by Modules on Sites, Definition 6.1 we see that given a morphism $f : X \to Y$ of schemes we get well defined pullback and direct image functors

$$f^*_{small} : \text{Mod}(\mathcal{O}_Y^{\text{étale}}) \to \text{Mod}(\mathcal{O}_X^{\text{étale}}),$$
$$f_{small,*} : \text{Mod}(\mathcal{O}_X^{\text{étale}}) \to \text{Mod}(\mathcal{O}_Y^{\text{étale}})$$

which are adjoint in the usual way. If $g : Y \to Z$ is another morphism of schemes, then we have $(g \circ f)^*_{small} = f^*_{small} \circ g^*_{small}$ and $(g \circ f)_{small,*} = g_{small,*} \circ f_{small,*}$ because of what we said about compositions.

There is quite a bit of difference between the category of all $\mathcal{O}_X$ modules on $X$ and the category between all $\mathcal{O}_{X_{\text{étale}}}$-modules on $X_{\text{étale}}$. But the results of Descent, Section 8 tell us that there is not much difference between considering quasi-coherent modules on $S$ and quasi-coherent modules on $S_{\text{étale}}$. (We have already seen this in Theorem 17.4 for example.) In particular, if $f : X \to Y$ is any morphism of schemes, then the pullback functors $f^*_{small}$ and $f^*$ match for quasi-coherent sheaves, see Descent, Proposition 8.14 Moreover, the same is true for pushforward provided $f$ is quasi-compact and quasi-separated, see Descent, Lemma 8.15.

A few words about functoriality of the structure sheaf on big sites. Let $f : X \to Y$ be a morphism of schemes. Choose any of the topologies $\tau \in \{\text{Zariski, étale},$
smooth, syntomic, fppf). Then the morphism \( f_{big} : (\text{Sch}/\mathcal{X})_r \to (\text{Sch}/\mathcal{Y})_r \) becomes a morphism of ringed sites by a map
\[
f_{big}^* : \mathcal{O}_Y \to f_{big,*}\mathcal{O}_X
\]
see Descent, Remark \(8.4\). In fact it is given by the same construction as in the case of small sites explained above.

39. Comparing topologies

In this section we start studying what happens when you compare sheaves with respect to different topologies.

**Lemma 39.1.** Let \( S \) be a scheme. Let \( \mathcal{F} \) be a sheaf of sets on \( S_{\text{étale}} \). Let \( s, t \in \mathcal{F}(S) \). Then there exists an open \( W \subset S \) characterized by the following property: A morphism \( f : T \to S \) factors through \( W \) if and only if \( s|_T = t|_T \) (restriction is pullback by \( f_{\text{small}} \)).

**Proof.** Consider the presheaf which assigns to \( U \in \text{Ob}(S_{\text{étale}}) \) the empty set if \( s|_U \neq t|_U \) and a singleton else. It is clear that this is a subsheaf of the final object \( Sh(S_{\text{étale}}) \). By Lemma \(3.11\) we find an open \( W \subset S \) representing this presheaf. For a geometric point \( \pi \) of \( S \) we see that \( \pi \in W \) if and only if the stalks of \( s \) and \( t \) at \( \pi \) agree. By the description of stalks of pullbacks in Lemma \(3.12\) we see that \( W \) has the desired property. \( \square \)

**Lemma 39.2.** Let \( S \) be a scheme. Let \( \pi \in \{\text{Zariski, étale}\} \). Consider the morphism
\[
\pi_S : (\text{Sch}/S)_r \to S_r
\]
of Topologies, Lemma \(3.13\) or \(4.13\). Let \( \mathcal{F} \) be a sheaf on \( S_r \). Then \( \pi_S^{-1}\mathcal{F} \) is given by the rule
\[
(\pi_S^{-1}\mathcal{F})(T) = \Gamma(T, f_{\text{small}}^{-1}\mathcal{F})
\]
where \( f : T \to S \). Moreover, \( \pi_S^{-1}\mathcal{F} \) satisfies the sheaf condition with respect to fpqc coverings.

**Proof.** Observe that we have a morphism \( i_f : Sh(T_r) \to Sh(Sh(S)/S)_r \) such that \( \pi_S \circ i_f = f_{\text{small}} \) as morphisms \( T_r \to S_r \), see Topologies, Lemmas \(3.12\) and \(4.12\). Since pullback is transitive we see that \( i_f^{-1}\pi_S^{-1}\mathcal{F} = f_{\text{small}}^{-1}\mathcal{F} \) as desired.

Let \( \{g_i : T_i \to T\}_{i \in I} \) be an fpqc covering. The final statement means the following: Given a sheaf \( \mathcal{G} \) on \( T_r \) and given sections \( s_i \in \Gamma(T_i, g_{i,\text{small}}^{-1}\mathcal{G}) \) whose pullbacks to \( T_i \times_T T_j \) agree, there is a unique section \( s \) of \( \mathcal{G} \) over \( T \) whose pullback to \( T_i \) agrees with \( s_i \).

Let \( V \to T \) be an object of \( T_r \) and let \( t \in \mathcal{G}(V) \). For every \( i \) there is a largest open \( W_i \subset T_i \times_T V \) such that the pullbacks of \( s_i \) and \( t \) agree as sections of the pullback of \( \mathcal{G} \) to \( W_i \subset T_i \times_T V \), see Lemma \(39.1\). Because \( s_i \) and \( s_j \) agree over \( T_i \times_T T_j \) we find that \( W_i \) and \( W_j \) pullback to the same open over \( T_i \times_T T_j \). By Descent, Lemma \(10.6\) we find an open \( W \subset V \) whose inverse image to \( T_i \times_T V \) recovers \( W_i \).

By construction of \( g_{i,\text{small}}^{-1}\mathcal{G} \) there exists a \( \tau \)-covering \( \{T_{ij} \to T_i\}_{j \in J_i} \), for each \( j \) an open immersion or étale morphism \( V_{ij} \to T \), a section \( t_{ij} \in \mathcal{G}(V_{ij}) \), and
commutative diagrams

\[
\begin{array}{ccc}
T_{ij} & \longrightarrow & V_{ij} \\
\downarrow & & \downarrow \\
T_i & \longrightarrow & T
\end{array}
\]

such that \( s_i|_{T_{ij}} \) is the pullback of \( t_{ij} \). In other words, after replacing the covering \( \{ T_i \to T \} \) by \( \{ T_{ij} \to T \} \) with \( V_i \in \text{Ob}(T_\tau) \) and sections \( t_i \in G(V_i) \) pulling back to \( s_i \) over \( T_i \). By the result of the previous paragraph we find opens \( W_i \subset V_i \) such that \( t_i|_{W_i} \) “agrees with” every \( s_j \) over \( T_j \times_T W_i \). Note that \( T_i \to V_i \) factors through \( W_i \). Hence \( \{ W_i \to T \} \) is a \( \tau \)-covering and the lemma is proven. \( \square \)

**Lemma 39.3.** Let \( S \) be a scheme. Let \( f : T \to S \) be a morphism such that

1. \( f \) is flat and quasi-compact, and
2. the geometric fibres of \( f \) are connected.

Let \( F \) be a sheaf on \( S_{\text{etale}} \). Then \( \Gamma(S, F) = \Gamma(T, f^{-1}_{\text{small}} F) \).

**Proof.** There is a canonical map \( \Gamma(S, F) \to \Gamma(T, f^{-1}_{\text{small}} F) \). Since \( f \) is surjective (because its fibres are connected) we see that this map is injective.

To show that the map is surjective, let \( \alpha \in \Gamma(T, f^{-1}_{\text{small}} F) \). Since \( \{ T \to S \} \) is an fpqc covering we can use Lemma 39.2 to see that suffices to prove that \( \alpha \) pulls back to the same section over \( T \times_S T \) by the two projections. Let \( \tau \to S \) be a geometric point. It suffices to show the agreement holds over \( (T \times_S T)_{\tau} \) as every geometric point of \( T \times_S T \) is contained in one of these geometric fibres. In other words, we are trying to show that \( \alpha|_{T_{\tau}} \) pulls back to the same section over

\[
(T \times_S T)_{\tau} = T_{\tau} \times_{\tau} T_{\tau}
\]

by the two projections to \( T_{\tau} \). However, since \( F|_{T_{\tau}} \) is the pullback of \( F|_{\tau} \) it is a constant sheaf with value \( F_{\tau} \). Since \( T_{\tau} \) is connected by assumption, any section of a constant sheaf is constant. Hence \( \alpha|_{T_{\tau}} \) corresponds to an element of \( F_{\tau} \). Thus the two pullbacks to \( (T \times_S T)_{\tau} \) both correspond to this same element and we conclude. \( \square \)

Here is a version of Lemma 39.3 where we do not assume that the morphism is flat.

**Lemma 39.4.** Let \( S \) be a scheme. Let \( f : X \to S \) be a morphism such that

1. \( f \) is submersive, and
2. the geometric fibres of \( f \) are connected.

Let \( F \) be a sheaf on \( S_{\text{etale}} \). Then \( \Gamma(S, F) = \Gamma(X, f^{-1}_{\text{small}} F) \).

**Proof.** There is a canonical map \( \Gamma(S, F) \to \Gamma(X, f^{-1}_{\text{small}} F) \). Since \( f \) is surjective (because its fibres are connected) we see that this map is injective.

To show that the map is surjective, let \( \tau \in \Gamma(X, f^{-1}_{\text{small}} F) \). It suffices to find an étale covering \( \{ U_i \to S \} \) and sections \( \sigma_i \in F(U_i) \) such that \( \sigma_i \) pulls back to \( \tau|_{X \times_S U_i} \). Namely, the injectivity shown above guarantees that \( \sigma_i \) and \( \sigma_j \) restrict to the same section of \( F \) over \( U_i \times_S U_j \). Thus we obtain a unique section \( \sigma \in F(S) \) which restricts to \( \sigma_i \) over \( U_i \). Then the pullback of \( \sigma \) to \( X \) is \( \tau \) because this is true locally.
Let $\pi$ be a geometric point of $X$ with image $\pi$ in $S$. Consider the image of $\tau$ in the stalk

$$(f_{\text{small}}^{-1}\mathcal{F})_\pi = \mathcal{F}_\pi$$

See Lemma 36.2. We can find an étale neighbourhood $U \to S$ of $\pi$ and a section $\sigma \in \mathcal{F}(U)$ mapping to this image in the stalk. Thus after replacing $S$ by $U$ and $X$ by $X \times_S U$ we may assume there exits a section $\sigma$ of $\mathcal{F}$ over $S$ whose image in $(f_{\text{small}}^{-1}\mathcal{F})_\pi$ is the same as $\tau$.

By Lemma 39.1 there exists a maximal open $W \subset X$ such that $f_{\text{small}}^{-1}\sigma$ and $\tau$ agree over $W$ and the formation of $W$ commutes with further pullback. Observe that the pullback of $\mathcal{F}$ to the geometric fibre $X_\pi$ is the pullback of $\mathcal{F}_\pi$ viewed as a sheaf on $\pi$ by $X_\pi \to \pi$. Hence we see that $\tau$ and $\sigma$ give sections of the constant sheaf with value $\mathcal{F}_\pi$ on $X_\pi$ which agree in one point. Since $X_\pi$ is connected by assumption, we conclude that $W$ contains $X_\pi$. The same argument for different geometric fibres shows that $W$ contains every fibre it meets. Since $f$ is submersive, we conclude that $W$ is the inverse image of an open neighbourhood of $s$ in $S$. This finishes the proof. □

0A3I Lemma 39.5. Let $k \subset K$ be an extension of fields with $k$ separably algebraically closed. Let $S$ be a scheme over $k$. Denote $p : S_K = S \times_{\text{Spec}(k)} \text{Spec}(K) \to S$ the projection. Let $\mathcal{F}$ be a sheaf on $S_{\text{étale}}$. Then $\Gamma(S, \mathcal{F}) = \Gamma(S_K, p_{\text{small}}^{-1}\mathcal{F})$.

Proof. Follows from Lemma 39.3. Namely, it is clear that $p$ is flat and quasi-compact as the base change of $\text{Spec}(K) \to \text{Spec}(k)$. On the other hand, if $\pi : \text{Spec}(L) \to S$ is a geometric point, then the fibre of $p$ over $\pi$ is the spectrum of $K \otimes_k L$ which is irreducible hence connected by Algebra, Lemma 46.2 □

40. Recovering morphisms

04JH In this section we prove that the rule which associates to a scheme its locally ringed small étale topos is fully faithful in a suitable sense, see Theorem 40.5.

04I5 Lemma 40.1. Let $f : X \to Y$ be a morphism of schemes. The morphism of ringed sites $(f_{\text{small}}^{-1}f, f_{\text{étale}}^{-1}f)$ associated to $f$ is a morphism of locally ringed sites, see Modules on Sites, Definition 39.9.

Proof. Note that the assertion makes sense since we have seen that $(X_{\text{étale}}, \mathcal{O}_{X_{\text{étale}}})$ and $(Y_{\text{étale}}, \mathcal{O}_{Y_{\text{étale}}})$ are locally ringed sites, see Lemma 33.5. Moreover, we know that $X_{\text{étale}}$ has enough points, see Theorem 29.10 and Remarks 29.11. Hence it suffices to prove that $(f_{\text{small}}^{-1}f, f_{\text{étale}}^{-1}f)$ satisfies condition (3) of Modules on Sites, Lemma 39.8. To see this take a point $p$ of $X_{\text{étale}}$. By Lemma 29.12 $p$ corresponds to a geometric point $\pi$ of $X$. By Lemma 36.2 the point $q = f_{\text{small}} \circ p$ corresponds to the geometric point $\pi = f \circ \pi$ of $Y$. Hence the assertion we have to prove is that the induced map of stalks

$$\mathcal{O}_{Y, \pi} \longrightarrow \mathcal{O}_{X, \pi}$$

is a local ring map. Suppose that $a \in \mathcal{O}_{Y, \pi}$ is an element of the left hand side which maps to an element of the maximal ideal of the right hand side. Suppose that $a$ is the equivalence class of a triple $(V, \pi, a)$ with $V \to Y$ étale, $\pi : \pi \to V$ over $Y$, and $a \in \mathcal{O}(V)$. It maps to the equivalence class of $(X \times_Y V, \pi \times \pi, \text{pr}_V^*(a))$ in the local ring $\mathcal{O}_{X, \pi}$. But it is clear that being in the maximal ideal means that pulling back
Let $X, Y$ be schemes. Let $f: X \to Y$ be a morphism of schemes. Let $t$ be a 2-morphism from $(f_{small}, f'_{small})$ to itself, see Modules on Sites, Definition 04.01. Then $t = id$.

**Proof.** This means that $t : f_{small}^{-1} \to f_{small}^{-1}$ is a transformation of functors such that the diagram

$$
\begin{array}{ccc}
\pi_{small}^{-1}O_Y & \cong & \pi_{small}^{-1}O_Y \\
\downarrow t & & \downarrow \pi_{small}^{-1}O_Y \\
\pi_{small}^{-1}O_X & \cong & \pi_{small}^{-1}O_X
\end{array}
$$

is commutative. Suppose $V \to Y$ is étale with $V$ affine. By Morphisms, Lemma 04.02 we may choose an immersion $i : V \to \mathcal{A}_X^V$ over $Y$. In terms of sheaves this means that $i$ induces an injection $h_i : h_V \to \prod_{i=1}^n O_Y$ of sheaves. The base change $i'$ of $i$ to $X$ is an immersion (Schemes, Lemma 04.03). Hence $i' : X \times_Y V \to \mathcal{A}_X^V$ is an immersion, which in turn means that $h_{i'} : h_X \times Y \to \prod_{j=1}^n O_X$ is an injection of sheaves. Via the identification $f_{small}^{-1}h_V \cong h_X \times Y$ of Lemma 04.01, the map $h_{i'}$ is equal to

$$f_{small}^{-1}h_V \xrightarrow{f_{small}^{-1}h_i} \prod_{j=1}^n f_{small}^{-1}O_Y \xrightarrow{\prod f_{small}'} \prod_{j=1}^n O_X$$

(verification omitted). This means that the map $t : f_{small}^{-1}h_V \to f_{small}^{-1}h_V$ fits into the commutative diagram

$$
\begin{array}{ccc}
f_{small}^{-1}h_V & \xrightarrow{f_{small}^{-1}h_i} & \prod_{j=1}^n f_{small}^{-1}O_Y \\
\downarrow t & & \downarrow \prod f_{small}' \\
\prod_{j=1}^n f_{small}^{-1}O_Y & \xrightarrow{\prod f_{small}'} & \prod_{j=1}^n O_X
\end{array}
$$

The commutativity of the right square holds by our assumption on $t$ explained above. Since the composition of the horizontal arrows is injective by the discussion above we conclude that the left vertical arrow is the identity map as well. Any sheaf of sets on $Y_{étale}$ admits a surjection from a (huge) coproduct of sheaves of the form $h_V$ with $V$ affine (combine Lemma 04.02 with Sites, Lemma 04.03). Thus we conclude that $t : f_{small}^{-1} \to f_{small}^{-1}$ is the identity transformation as desired. 

**Lemma 04.03.** Let $X, Y$ be schemes. Any two morphisms $a, b : X \to Y$ of schemes for which there exists a 2-isomorphism $(a_{small}, a'_{small}) \cong (b_{small}, b'_{small})$ in the 2-category of ringed topoi are equal.

**Proof.** Let us argue this carefully since it is a bit confusing. Let $t : a_{small}^{-1} \to b_{small}^{-1}$ be the 2-isomorphism. Consider any open $V \subset Y$. Note that $h_V$ is a subsheaf of the final sheaf $\mathcal{O}$. Thus both $a_{small}^{-1}h_V = h_{a^{-1}(V)}$ and $b_{small}^{-1}h_V = h_{b^{-1}(V)}$ are subsheaves of the final sheaf. Thus the isomorphism

$$t : a_{small}^{-1}h_V = h_{a^{-1}(V)} \to b_{small}^{-1}h_V = h_{b^{-1}(V)}$$

Consider any open $V \subset Y$. Note that $h_V$ is a subsheaf of the final sheaf $\mathcal{O}$. Thus both $a_{small}^{-1}h_V = h_{a^{-1}(V)}$ and $b_{small}^{-1}h_V = h_{b^{-1}(V)}$ are subsheaves of the final sheaf. Thus the isomorphism

$$t : a_{small}^{-1}h_V = h_{a^{-1}(V)} \to b_{small}^{-1}h_V = h_{b^{-1}(V)}$$
has to be the identity, and $a^{-1}(V) = b^{-1}(V)$. It follows that $a$ and $b$ are equal on underlying topological spaces. Next, take a section $f \in \mathcal{O}_Y(V)$. This determines and is determined by a map of sheaves of sets $f : h_V \rightarrow \mathcal{O}_Y$. Pull this back and apply $t$ to get a commutative diagram

\[
\begin{array}{ccc}
  b^{-1}_{\text{small}}h_V & \xrightarrow{t} & a^{-1}_{\text{small}}h_V \\
  \downarrow a^{-1}_{\text{small}}(f) & & \downarrow a^{-1}_{\text{small}}(f) \\
  b^{-1}_{\text{small}}\mathcal{O}_Y & \xrightarrow{t} & a^{-1}_{\text{small}}\mathcal{O}_Y \\
  \downarrow b^t & & \downarrow a^t \\
  \mathcal{O}_X & & \mathcal{O}_X
\end{array}
\]

where the triangle is commutative by definition of a 2-isomorphism in Modules on Sites, Section 8. Above we have seen that the composition of the top horizontal arrows comes from the identity $a^{-1}(V) = b^{-1}(V)$. Thus the commutativity of the diagram tells us that $a^{-1}_{\text{small}}(f) = b^{-1}_{\text{small}}(f)$ in $\mathcal{O}_X(a^{-1}(V)) = \mathcal{O}_X(b^{-1}(V))$. Since this holds for every open $V$ and every $f \in \mathcal{O}_Y(V)$ we conclude that $a = b$ as morphisms of schemes.

\begin{lemma}
Let $X$, $Y$ be affine schemes. Let

\[(g, g^\#) : (\text{Sh}(X_{\text{etale}}), \mathcal{O}_X) \longrightarrow (\text{Sh}(Y_{\text{etale}}), \mathcal{O}_Y)\]

be a morphism of locally ringed topoi. Then there exists a unique morphism of schemes $f : X \rightarrow Y$ such that $(g, g^\#)$ is 2-isomorphic to $(f_{\text{small}}, f_{\text{small}}^\#)$, see Modules on Sites, Definition 8.1.

\end{lemma}

\begin{proof}
In this proof we write $\mathcal{O}_X$ for the structure sheaf of the small étale site $X_{\text{étale}}$, and similarly for $\mathcal{O}_Y$. Say $Y = \text{Spec}(B)$ and $X = \text{Spec}(A)$. Since $B = \Gamma(Y_{\text{étale}}, \mathcal{O}_Y)$, $A = \Gamma(X_{\text{étale}}, \mathcal{O}_X)$ we see that $g^\#$ induces a ring map $\varphi : B \rightarrow A$. Let $f = \text{Spec}(\varphi) : X \rightarrow Y$ be the corresponding morphism of affine schemes. We will show this $f$ does the job.

Let $V \rightarrow Y$ be an affine scheme étale over $Y$. Thus we may write $V = \text{Spec}(C)$ with $C$ an étale $B$-algebra. We can write

\[C = B[x_1, \ldots, x_n]/(P_1, \ldots, P_n)\]

with $P_i$ polynomials such that $\Delta = \det(\partial P_i/\partial x_j)$ is invertible in $C$, see for example Algebra, Lemma 14.12. If $T$ is a scheme over $Y$, then a $T$-valued point of $V$ is given by $n$ sections of $\Gamma(T, \mathcal{O}_T)$ which satisfy the polynomial equations $P_1 = 0, \ldots, P_n = 0$. In other words, the sheaf $h_V$ on $Y_{\text{étale}}$ is the equalizer of the two maps

\[\prod_{i=1,\ldots,n} \mathcal{O}_Y \xrightarrow{a} \prod_{j=1,\ldots,n} \mathcal{O}_Y \]

where $b(h_1, \ldots, h_n) = 0$ and $a(h_1, \ldots, h_n) = (P_1(h_1, \ldots, h_n), \ldots, P_n(h_1, \ldots, h_n))$.

Since $g^{-1}$ is exact we conclude that the top row of the following solid commutative
diagram is an equalizer diagram as well:

\[
\begin{array}{ccc}
\prod_{i=1}^{g-1} h_{V} & \longrightarrow & \prod_{i=1}^{g-1} g_{V}^{-1} \mathcal{O}_{Y} \\
\downarrow & & \downarrow \\
\prod_{i=1}^{g-1} \mathcal{O}_{X} & \longrightarrow & \prod_{i=1}^{g-1} \mathcal{O}_{X}
\end{array}
\]

Here \( b' \) is the zero map and \( a' \) is the map defined by the images \( P_{i}^{\ell} = \varphi(P_{i}) \in A[x_{1}, \ldots, x_{n}] \) via the same rule \( a'(h_{1}, \ldots, h_{n}) = (P_{i}^{\ell}(h_{1}, \ldots, h_{n}), \ldots, P_{n}^{\ell}(h_{1}, \ldots, h_{n})) \). that \( a \) was defined by. The commutativity of the diagram follows from the fact that \( \varphi = g^{\ell} \) on global sections. The lower row is an equalizer diagram also, by exactly the same arguments as before since \( X \times_{V} Y \) is the affine scheme \( \text{Spec}(A \otimes B C) \) and \( A \otimes B C = A[x_{1}, \ldots, x_{n}]/(P_{1}, \ldots, P_{n}) \). Thus we obtain a unique dotted arrow \( g^{-1} h_{V} \rightarrow h_{X \times_{V} Y} \) fitting into the diagram

We claim that the map of sheaves \( g^{-1} h_{V} \rightarrow h_{X \times_{V} Y} \) is an isomorphism. Since the small étale site of \( X \) has enough points (Theorem 29.10) it suffices to prove this on stalks. Hence let \( \pi \) be a geometric point of \( X \), and denote \( p \) the associate point of the small étale topos of \( X \). Set \( q = g \circ p \). This is a point of the small étale topos of \( Y \). By Lemma 29.12 we see that \( q \) corresponds to a geometric point \( \pi \) of \( Y \). Consider the map of stalks

\[
(g^{\ell})_{p} : \mathcal{O}_{Y, \pi} = \mathcal{O}_{Y, q} = (g^{-1} \mathcal{O}_{Y})_{p} \longrightarrow \mathcal{O}_{X, p} = \mathcal{O}_{X, \pi}
\]

Since \( (g, g^{\ell}) \) is a morphism of locally ringed topos \( (g^{\ell})_{p} \) is a local ring homomorphism of strictly henselian local rings. Applying localization to the big commutative diagram above and Algebra, Lemma 29.12 we conclude that \( (g^{-1} h_{V})_{p} \rightarrow (h_{X \times_{V} Y})_{p} \) is an isomorphism as desired.

We claim that the isomorphisms \( g^{-1} h_{V} \rightarrow h_{X \times_{V} Y} \) are functorial. Namely, suppose that \( V_{1} \rightarrow V_{2} \) is a morphism of affine schemes étale over \( Y \). Write \( \text{Spec}(C_{i}) \) with

\[
C_{i} = B[x_{i,1}, \ldots, x_{i,n_{i}}]/(P_{i,1}, \ldots, P_{i,n_{i}})
\]

The morphism \( V_{1} \rightarrow V_{2} \) is given by a \( B \)-algebra map \( C_{2} \rightarrow C_{1} \) which in turn is given by some polynomials \( Q_{j} \in B[x_{i,1}, \ldots, x_{i,n_{i}}] \) for \( j = 1, \ldots, n_{2} \). Then it is an easy matter to show that the diagram of sheaves

\[
\begin{array}{ccc}
h_{V_{1}} & \longrightarrow & \prod_{i=1}^{Q_{1},\ldots,Q_{n_{2}}} \mathcal{O}_{Y} \\
\downarrow & & \downarrow \\
h_{V_{2}} & \longrightarrow & \prod_{i=1}^{Q_{1},\ldots,Q_{n_{2}}} \mathcal{O}_{Y}
\end{array}
\]
is commutative, and pulling back to $X_{\text{étale}}$ we obtain the solid commutative diagram

\[
\begin{array}{c}
\includegraphics[scale=0.5]{diagram.png}
\end{array}
\]

where $Q'_j \in A[x_{1,1}, \ldots, x_{1,n_1}]$ is the image of $Q_j$ via $\varphi$. Since the dotted arrows exist, make the two squares commute, and the horizontal arrows are injective we see that the whole diagram commutes. This proves functoriality (and also that the construction of $g^{-1}h_V \to h_{X \times Y}V$ is independent of the choice of the presentation, although we strictly speaking do not need to show this).

At this point we are able to show that $f_{\text{small},*} \cong g_*$. Namely, let $\mathcal{F}$ be a sheaf on $X_{\text{étale}}$. For every $V \in \text{Ob}(X_{\text{étale}})$ affine we have

\[
(g_*\mathcal{F})(V) = \text{Mor}_{\text{Sh}(X_{\text{étale}})}(h_V, g_*\mathcal{F}) = \text{Mor}_{\text{Sh}(X_{\text{étale}})}(g^{-1}h_V, \mathcal{F}) = \text{Mor}_{\text{Sh}(X_{\text{étale}})}(h_{X \times Y}V, \mathcal{F}) = \mathcal{F}(X \times Y V) = f_{\text{small},*}\mathcal{F}(V)
\]

where in the third equality we use the isomorphism $g^{-1}h_V \cong h_{X \times Y}V$ constructed above. These isomorphisms are clearly functorial in $\mathcal{F}$ and functorial in $V$ as the isomorphisms $g^{-1}h_V \cong h_{X \times Y}V$ are functorial. Now any sheaf on $Y_{\text{étale}}$ is determined by the restriction to the subcategory of affine schemes (Lemma 21.2), and hence we obtain an isomorphism of functors $f_{\text{small},*} \cong g_*$ as desired.

Finally, we have to check that, via the isomorphism $f_{\text{small},*} \cong g_*$ above, the maps $f_{\text{small}}^*$ and $g^*$ agree. By construction this is already the case for the global sections of $\mathcal{O}_Y$, i.e., for the elements of $B$. We only need to check the result on sections over an affine $V$ étale over $Y$ (by Lemma 21.2 again). Writing $V = \text{Spec}(C)$, $C = B[x_i]/(P_j)$ as before it suffices to check that the coordinate functions $x_i$ are mapped to the same sections of $\mathcal{O}_X$ over $X \times_Y V$. And this is exactly what it means that the diagram

\[
\begin{array}{c}
\includegraphics[scale=0.5]{diagram.png}
\end{array}
\]

commutes. Thus the lemma is proved.

\[\square\]

Here is a version for general schemes.
**Theorem 40.5.** Let $X, Y$ be schemes. Let

$$(g, g^\#) : (\mathcal{S}(X_{\text{etale}}), \mathcal{O}_X) \to (\mathcal{S}(Y_{\text{etale}}), \mathcal{O}_Y)$$

be a morphism of locally ringed topoi. Then there exists a unique morphism of schemes $f : X \to Y$ such that $(g, g^\#)$ is isomorphic to $(f_{\text{small}}, f^\sharp_{\text{small}})$. In other words, the construction

$$\text{Sch} \to \text{Locally ringed topoi}, \quad X \mapsto (X_{\text{etale}}, \mathcal{O}_X)$$

is fully faithful (morphisms up to 2-isomorphisms on the right hand side).

**Proof.** You can prove this theorem by carefully adjusting the arguments of the proof of Lemma 40.4 to the global setting. However, we want to indicate how we can glue the result of that lemma to get a global morphism due to the rigidity provided by the result of Lemma 40.2. Unfortunately, this is a bit messy.

Let us prove existence when $Y$ is affine. In this case choose an affine open covering $X = \bigcup U_i$. For each $i$ the inclusion morphism $j_i : U_i \to X$ induces a morphism of locally ringed topoi $(j_i, \text{small}, j^\sharp_{i, \text{small}}) : (\mathcal{S}(U_i_{\text{etale}}), \mathcal{O}_{U_i}) \to (\mathcal{S}(X_{\text{etale}}), \mathcal{O}_X)$ by Lemma 40.1. We can compose this with $(g, g^\sharp)$ to obtain a morphism of locally ringed topoi

$$(g, g^\sharp) \circ (j_i, \text{small}, j^\sharp_{i, \text{small}}) : (\mathcal{S}(U_i_{\text{etale}}), \mathcal{O}_{U_i}) \to (\mathcal{S}(X_{\text{etale}}), \mathcal{O}_X)$$

see Modules on Sites, Lemma 39.10. By Lemma 40.4 there exists a unique morphism of schemes $f_i : U_i \to Y$ and a 2-isomorphism

$$t_i : (f_i, \text{small}, f^\sharp_{i, \text{small}}) \to (g, g^\sharp) \circ (j_i, \text{small}, j^\sharp_{i, \text{small}}).$$

Set $U_{i, i'} = U_i \cap U_{i'}$, and denote $j_{i, i'} : U_{i, i'} \to U_i$ the inclusion morphism. Since we have $j_i \circ j_{i, i'} = j_{i'} \circ j_{i', i}$ we see that

$$(g, g^\sharp) \circ (j_i, \text{small}, j^\sharp_{i, \text{small}}) \circ (j_{i, i'}, \text{small}, j^\sharp_{i, i', \text{small}}) = (g, g^\sharp) \circ (j_{i', \text{small}}, j^\sharp_{i', \text{small}}) \circ (j_{i', i}, \text{small}, j^\sharp_{i', i, \text{small}}).$$

Hence by uniqueness (see Lemma 40.3) we conclude that $f_i \circ j_{i, i'} = f_{i'} \circ j_{i', i}$, in other words the morphisms of schemes $f_i = f \circ j_i$ are the restrictions of a global morphism of schemes $f : X \to Y$. Consider the diagram of 2-isomorphisms (where we drop the components $t$ to ease the notation)

$$\begin{array}{ccc}
g \circ j_{i, \text{small}} \circ j_{i', \text{small}} & \xrightarrow{t \circ \text{id}_{j_{i, i', \text{small}}}} & f_{\text{small}} \circ j_{i, \text{small}} \circ j_{i', \text{small}} \\
\phantom{g \circ j_{i, \text{small}} \circ j_{i', \text{small}}} & | & | \\
g \circ j_{i', \text{small}} \circ j_{i', \text{small}} & \xrightarrow{t \circ \text{id}_{j_{i', i, \text{small}}}} & f_{\text{small}} \circ j_{i', \text{small}} \circ j_{i, \text{small}}
\end{array}$$

The notation $\ast$ indicates horizontal composition, see Categories, Definition 28.1 in general and Sites, Section 36 for our particular case. By the result of Lemma 40.2 this diagram commutes. Hence for any sheaf $\mathcal{G}$ on $Y_{\text{etale}}$ the isomorphisms $t_i : f_{\text{small}}^{-1} \mathcal{G}|_{U_i} \to g^{-1} \mathcal{G}|_{U_i}$ agree over $U_{i, i'}$ and we obtain a global isomorphism $t : f_{\text{small}}^{-1} \mathcal{G} \to g^{-1} \mathcal{G}$. It is clear that this isomorphism is functorial in $\mathcal{G}$ and is compatible with the maps $f^\sharp_{\text{small}}$ and $g^\sharp$ (because it is compatible with these maps locally). This proves the theorem in case $Y$ is affine.
In the general case, let $V \subset Y$ be an affine open. Then $h_V$ is a subsheaf of the final sheaf $*$ on $Y_{etale}$. As $g$ is exact we see that $g^{-1}h_V$ is a subsheaf of the final sheaf on $X_{etale}$. Hence by Lemma 31.1 there exists an open subscheme $W \subset X$ such that $g^{-1}h_V = h_W$. By Modules on Sites, Lemma 39.12 there exists a commutative diagram of morphisms of locally ringed topoi

$$
\begin{array}{ccc}
(\text{Sh}(W_{etale}), \mathcal{O}_W) & \longrightarrow & (\text{Sh}(X_{etale}), \mathcal{O}_X) \\
g' \downarrow & & \downarrow g \\
(\text{Sh}(V_{etale}), \mathcal{O}_V) & \longrightarrow & (\text{Sh}(Y_{etale}), \mathcal{O}_Y)
\end{array}
$$

where the horizontal arrows are the localization morphisms (induced by the inclusion morphisms $V \rightarrow Y$ and $W \rightarrow X$) and where $g'$ is induced from $g$. By the result of the preceding paragraph we obtain a morphism of schemes $f' : W \rightarrow V$ and a 2-isomorphism $t : (f'_{smallest}, (f'_{smallest})^2) \rightarrow (g', (g')^2)$. Exactly as before these morphisms $f'$ (for varying affine opens $V \subset Y$) agree on overlaps by uniqueness, so we get a morphism $f : X \rightarrow Y$. Moreover, the 2-isomorphisms $t$ are compatible on overlaps by Lemma 40.2 again and we obtain a global 2-isomorphism $(f_{smallest}, (f_{smallest})^2) \rightarrow (g, (g)^2)$. as desired. Some details omitted. \qed

41. Push and pull

04C6 Let $f : X \rightarrow Y$ be a morphism of schemes. Here is a list of conditions we will consider in the following:

(A) For every étale morphism $U \rightarrow X$ and $u \in U$ there exist an étale morphism $V \rightarrow Y$ and a disjoint union decomposition $X \times_Y V = W \amalg W'$ and a morphism $h : W \rightarrow U$ over $X$ with $u$ in the image of $h$.

(B) For every $V \rightarrow Y$ étale, and every étale covering $\{U_i \rightarrow X \times_Y V\}$ there exists an étale covering $\{V_j \rightarrow V\}$ such that for each $j$ we have $X \times_Y V_j = \amalg_i W_{ij}$ where $W_{ij} \rightarrow X \times_Y V$ factors through $U_i \rightarrow X \times_Y V$ for some $i$.

(C) For every $U \rightarrow X$ étale, there exists a $V \rightarrow Y$ étale and a surjective morphism $X \times_Y V \rightarrow U$ over $X$.

It turns out that each of these properties has meaning in terms of the behaviour of the functor $f_{smallest,*}$. We will work this out in the next few sections.

42. Property (A)

04DJ Please see Section 41 for the definition of property (A).

04DK Lemma 42.1. Let $f : X \rightarrow Y$ be a morphism of schemes. Assume (A).

1. $f_{smallest,*} : \text{Ab}(X_{etale}) \rightarrow \text{Ab}(Y_{etale})$ reflects injections and surjections,

2. $f_{smallest,*}^\ast \mathcal{F} \rightarrow \mathcal{F}$ is surjective for any abelian sheaf $\mathcal{F}$ on $X_{etale}$.

3. $f_{smallest,*} : \text{Ab}(X_{etale}) \rightarrow \text{Ab}(Y_{etale})$ is faithful.

Proof. Let $\mathcal{F}$ be an abelian sheaf on $X_{etale}$. Let $U$ be an object of $X_{etale}$. By assumption we can find a covering $\{W_i \rightarrow U\}$ in $X_{etale}$ such that each $W_i$ is an open and closed subscheme of $X \times_Y V_i$ for some object $V_i$ of $Y_{etale}$. The sheaf condition shows that

$$
\mathcal{F}(U) \subset \prod \mathcal{F}(W_i)
$$

and that $\mathcal{F}(W_i)$ is a direct summand of $\mathcal{F}(X \times_Y V_i) = f_{smallest,*}\mathcal{F}(V_i)$. Hence it is clear that $f_{smallest,*}$ reflects injections.
Next, suppose that \( a : G \rightarrow \mathcal{F} \) is a map of abelian sheaves such that \( f_{small,*}a \) is surjective. Let \( s \in \mathcal{F}(U) \) with \( U \) as above. With \( W_i, V_i \) as above we see that it suffices to show that \( s_{|W_i} \) is étale locally the image of a section of \( G \) under \( a \). Since \( \mathcal{F}(W_i) \) is a direct summand of \( \mathcal{F}(X \times_Y V_i) \) it suffices to show that for any \( V \in \text{Ob}(\mathcal{Y}_{\text{étale}}) \) any element \( s \in \mathcal{F}(X \times_Y V) \) is étale locally on \( X \times_Y V \) the image of a section of \( G \) under \( a \). Since \( \mathcal{F}(X \times_Y V) = f_{small,*}\mathcal{F}(V) \) we see by assumption that there exists a covering \( \{ V_j \rightarrow V \} \) such that \( s \) is the image of \( \iota_j \in f_{small,*}G(V_j) = G(X \times_Y V_j) \). This proves \( f_{small,*} \) reflects surjections.

Parts (2), (3) follow formally from part (1), see Modules on Sites, Lemma \([15.1]\). □

**Lemma 42.2.** Let \( f : X \rightarrow Y \) be a separated locally quasi-finite morphism of schemes. Then property (A) above holds.

**Proof.** Let \( U \rightarrow X \) be an étale morphism and \( u \in U \). The geometric statement (A) reduces directly to the case where \( U \) and \( Y \) are affine schemes. Denote \( x \in X \) and \( y \in Y \) the images of \( u \). Since \( X \rightarrow Y \) is locally quasi-finite, and \( U \rightarrow X \) is locally quasi-finite (see Morphisms, Lemma \([34.6]\)) we see that \( U \rightarrow Y \) is locally quasi-finite (see Morphisms, Lemma \([19.12]\)). Moreover both \( X \rightarrow Y \) and \( U \rightarrow Y \) are separated. Thus More on Morphisms, Lemma \([36.5]\) applies to both morphisms. This means we may pick an étale neighbourhood \( (V, v) \rightarrow (Y, y) \) such that

\[
X \times_Y V = W \amalg R, \quad U \times_Y V = W' \amalg R'
\]

and points \( w \in W, w' \in W' \) such that

1. \( W, R \) are open and closed in \( X \times_Y V \),
2. \( W', R' \) are open and closed in \( U \times_Y V \),
3. \( W \rightarrow V \) and \( W' \rightarrow V \) are finite,
4. \( w, w' \) map to \( v \),
5. \( \kappa(v) \subset \kappa(w) \) and \( \kappa(v) \subset \kappa(w') \) are purely inseparable, and
6. no other point of \( W \) or \( W' \) maps to \( v \).

Here is a commutative diagram

\[
\begin{array}{ccc}
U & \leftarrow & U \times_Y V & \leftarrow & W' \amalg R' \\
\downarrow & & \downarrow & & \downarrow \\
X & \leftarrow & X \times_Y V & \leftarrow & W \amalg R \\
\downarrow & & \downarrow & & \downarrow \\
Y & \leftarrow & \quad & \quad & \quad \\
\end{array}
\]

After shrinking \( V \) we may assume that \( W' \) maps into \( W \): just remove the image the inverse image of \( R \) in \( W' \); this is a closed set (as \( W' \rightarrow V \) is finite) not containing \( v \). Then \( W' \rightarrow W \) is finite because both \( W \rightarrow V \) and \( W' \rightarrow V \) are finite. Hence \( W' \rightarrow W \) is finite étale, and there is exactly one point in the fibre over \( w \) with \( \kappa(w) = \kappa(w') \). Hence \( W' \rightarrow W \) is an isomorphism in an open neighbourhood \( W^o \) of \( w \), see Étale Morphisms, Lemma \([14.2]\). Since \( W \rightarrow V \) is finite the image of \( W \setminus W^o \) is a closed subset \( T \) of \( V \) not containing \( v \). Thus after replacing \( V \) by \( V \setminus T \) we may assume that \( W' \rightarrow W \) is an isomorphism. Now the decomposition \( X \times_Y V = W \amalg R \) and the morphism \( W \rightarrow U \) are as desired and we win. □
Lemma 42.3. Let \( f : X \rightarrow Y \) be an integral morphism of schemes. Then property (A) holds.

Proof. Let \( U \rightarrow X \) be étale, and let \( u \in U \) be a point. We have to find \( V \rightarrow Y \) étale, a disjoint union decomposition \( X \times_Y V = W \amalg W' \) and an \( X \)-morphism \( W \rightarrow U \) with \( u \) in the image. We may shrink \( U \) and \( Y \) and assume \( U \) and \( Y \) are affine. In this case also \( X \) is affine, since an integral morphism is affine by definition.

Write \( Y = \text{Spec}(A) \), \( X = \text{Spec}(B) \) and \( U = \text{Spec}(C) \). Then \( A \rightarrow B \) is an integral ring map, and \( B \rightarrow C \) is an étale ring map. By Algebra, Lemma 141.3 we can find a finite \( A \)-subalgebra \( B' \subset B \) and an étale ring map \( B' \rightarrow C' \) such that \( C = B \otimes_B C' \). Thus the question reduces to the étale morphism \( U' = \text{Spec}(C') \rightarrow X' = \text{Spec}(B') \) over the finite morphism \( X' \rightarrow Y \).

In this case the result follows from Lemma 42.2.

\( \square \)

Lemma 42.4. Let \( f : X \rightarrow Y \) be a morphism of schemes. Denote \( f_{\text{small}} : \text{Sh}(X_{\text{étale}}) \rightarrow \text{Sh}(Y_{\text{étale}}) \) the associated morphism of small étale topoi. Assume at least one of the following

(1) \( f \) is integral, or
(2) \( f \) is separated and locally quasi-finite.

Then the functor \( f_{\text{small}},* : \text{Ab}(X_{\text{étale}}) \rightarrow \text{Ab}(Y_{\text{étale}}) \) has the following properties

(1) the map \( f_{\text{small}}^{-1}f_{\text{small}},*F \rightarrow F \) is always surjective,
(2) \( f_{\text{small}},* \) is faithful, and
(3) \( f_{\text{small}},* \) reflects injections and surjections.

Proof. Combine Lemmas 42.2, 42.3, and 42.1

\( \square \)

43. Property (B)

Please see Section 41 for the definition of property (B).

Lemma 43.1. Let \( f : X \rightarrow Y \) be a morphism of schemes. Assume (B) holds. Then the functor \( f_{\text{small}},* : \text{Sh}(X_{\text{étale}}) \rightarrow \text{Sh}(Y_{\text{étale}}) \) transforms surjections into surjections.

Proof. This follows from Sites, Lemma 41.2

\( \square \)

Lemma 43.2. Let \( f : X \rightarrow Y \) be a morphism of schemes. Suppose

(1) \( V \rightarrow Y \) is an étale morphism of schemes,
(2) \( \{U_i \rightarrow X \times_Y V\} \) is an étale covering, and
(3) \( v \in V \) is a point.

Assume that for any such data there exists an étale neighbourhood \( (V',v') \rightarrow (V,v) \), a disjoint union decomposition \( X \times_Y V' = \bigsqcup W'_i \), and morphisms \( W'_i \rightarrow U_i \) over \( X \times_Y V \). Then property (B) holds.

Proof. Omitted.

\( \square \)

Lemma 43.3. Let \( f : X \rightarrow Y \) be a finite morphism of schemes. Then property (B) holds.

Proof. Consider \( V \rightarrow Y \) étale, \( \{U_i \rightarrow X \times_Y V\} \) an étale covering, and \( v \in V \). We have to find a \( V' \rightarrow V \) and decomposition and maps as in Lemma 43.2. We may shrink \( V \) and \( Y \), hence we may assume that \( V \) and \( Y \) are affine. Since \( X \) is finite over \( Y \), this also implies that \( X \) is affine. During the proof we may (finitely often)
replace \((V, v)\) by an étale neighbourhood \((V', v')\) and correspondingly the covering \(\{U_i \to X \times_Y V\}\) by \(\{V' \times_Y U_i \to X \times_Y V'\}\).

Since \(X \times_Y V \to V\) is finite there exist finitely many (pairwise distinct) points \(x_1, \ldots, x_n \in X \times_Y V\) mapping to \(v\). We may apply More on Morphisms, Lemma \[36.5\] to \(X \times_Y V \to V\) and the points \(x_1, \ldots, x_n\) lying over \(v\) and find an étale neighbourhood \((V', v') \to (V, v)\) such that

\[
X \times_Y V' = R \amalg \bigsqcup T_a
\]

with \(T_a \to V'\) finite with exactly one point \(p_a\) lying over \(v'\) and moreover \(\kappa(v') \subset \kappa(p_a)\) purely inseparable, and such that \(R \to V'\) has empty fibre over \(v'\). Because \(X \to Y\) is finite, also \(R \to V'\) is finite. Hence after shrinking \(V'\) we may assume that \(R = \emptyset\). Thus we may assume that \(X \times_Y V = X_1 \amalg \ldots \amalg X_n\) with exactly one point \(x_i \in X_i\) lying over \(v\) with moreover \(\kappa(v) \subset \kappa(x_i)\) purely inseparable. Note that this property is preserved under refinement of the étale neighbourhood \((V, v)\).

For each \(l\) choose an \(i_l\) and a point \(u_l \in U_{i_l}\) mapping to \(x_l\). Now we apply property (A) for the finite morphism \(X \times_Y V \to V\) and the étale morphisms \(U_{i_l} \to X \times_Y V\) and the points \(u_l\). This is permissible by Lemma \[42.3\]. This gives produces an étale neighbourhood \((V', v') \to (V, v)\) and decompositions

\[
X \times_Y V' = W_{i_l} \amalg R_l
\]

and \(X\)-morphisms \(a_l : W_i \to U_{i_l}\) whose image contains \(u_{i_l}\). Here is a picture:

\[
\begin{array}{cccccccc}
W_i & \to & W_{i_l} \amalg R_l & \to & X \times_Y V' & \to & X \times_Y V & \to & X \\
& & & & V' & \downarrow & V & \downarrow & Y \\
& & & & & & & \\
U_{i_l} & \to & \end{array}
\]

After replacing \((V, v)\) by \((V', v')\) we conclude that each \(x_l\) is contained in an open and closed neighbourhood \(W_l\) such that the inclusion morphism \(W_l \to X \times_Y V\) factors through \(U_{i_l} \to X \times_Y V\) for some \(i_l\). Replacing \(W_l\) by \(W_l \cap X\) we see that these open and closed sets are disjoint and moreover that \(\{x_1, \ldots, x_n\} \subset W_1 \cup \ldots \cup W_n\). Since \(X \times_Y V \to V\) is finite we may shrink \(V\) and assume that \(X \times_Y V = W_1 \amalg \ldots \amalg W_n\) as desired. \(\square\)

04DR **Lemma 43.4.** Let \(f : X \to Y\) be an integral morphism of schemes. Then property (B) holds.

**Proof.** Consider \(V \to Y\) étale, \(\{U_i \to X \times_Y V\}\) an étale covering, and \(v \in V\). We have to find a \(V' \to V\) and decomposition and maps as in Lemma \[43.2\]. We may shrink \(V\) and \(Y\), hence we may assume that \(V\) and \(Y\) are affine. Since \(X\) is integral over \(Y\), this also implies that \(X\) and \(X \times_Y V\) are affine. We may refine the covering \(\{U_i \to X \times_Y V\}\), and hence we may assume that \(\{U_i \to X \times_Y V\}_{i=1,\ldots,n}\) is a standard étale covering. Write \(Y = \text{Spec}(A), X = \text{Spec}(B), V = \text{Spec}(C)\), and \(U_i = \text{Spec}(B_i)\). Then \(A \to B\) is an integral ring map, and \(B \otimes A C \to B_i\) are étale ring maps. By Algebra, Lemma \[141.3\] we can find a finite \(A\)-subalgebra \(B' \subset B\) and an étale ring map \(B' \otimes A C \to B'_i\) for \(i = 1,\ldots,n\) such that \(B_i = B \otimes B' B'_i\).
Thus the question reduces to the étale covering \{\text{Spec}(B'_i) \to X' \times_Y V\}_{i=1,\ldots,n} with \(X' = \text{Spec}(B')\) finite over \(Y\). In this case the result follows from Lemma \ref{lem:43.3}.

\begin{lemma}
Let \(f : X \to Y\) be a morphism of schemes. Assume \(f\) is integral (for example finite). Then
\begin{enumerate}
\item \(f_{\text{small}*}\) transforms surjections into surjections (on sheaves of sets and on abelian sheaves),
\item \(f^{-1}_{\text{small}}f_{\text{small}*}\mathcal{F} \to \mathcal{F}\) is surjective for any abelian sheaf \(\mathcal{F}\) on \(X_{\text{étale}}\),
\item \(f_{\text{small}*} : \text{Ab}(X_{\text{étale}}) \to \text{Ab}(Y_{\text{étale}})\) is faithful and reflects injections and surjections, and
\item \(f_{\text{small}*} : \text{Ab}(X_{\text{étale}}) \to \text{Ab}(Y_{\text{étale}})\) is exact.
\end{enumerate}
\end{lemma}

\begin{proof}
Parts (2), (3) we have seen in Lemma \ref{lem:42}. Part (1) follows from Lemmas \ref{lem:43} and \ref{lem:43.1}. Part (4) is a consequence of part (1), see Modules on Sites, Lemma \ref{lem:44.3}.
\end{proof}

\section*{44. Property (C)}

\begin{lemma}
Let \(f : X \to Y\) be a morphism of schemes. Assume (C) holds. Then the functor \(Y_{\text{étale}} \to X_{\text{étale}}, V \mapsto X \times_Y V\) satisfies the assumption of Sites, Lemma \ref{lem:41.4}.
\end{lemma}

\begin{remark}
Property (C) holds if \(f : X \to Y\) is an open immersion. Namely, if \(U \in \text{Ob}(X_{\text{étale}})\), then we can view \(U\) also as an object of \(Y_{\text{étale}}\) and \(U \times_Y X = U\). Hence property (C) does not imply that \(f_{\text{small}*}\) is exact as this is not the case for open immersions (in general).
\end{remark}

\begin{lemma}
Let \(f : X \to Y\) be a morphism of schemes. Assume that for any \(V \to Y\) étale we have that
\begin{enumerate}
\item \(X \times_Y V \to V\) has property (C), and
\item \(X \times_Y V \to V\) is closed.
\end{enumerate}
Then the functor \(Y_{\text{étale}} \to X_{\text{étale}}, V \mapsto X \times_Y V\) is almost cocontinuous, see Sites, Definition \ref{def:42.3}.
\end{lemma}

\begin{proof}
Let \(V \to Y\) be an object of \(Y_{\text{étale}}\) and let \(\{U_i \to X \times_Y V\}_{i \in I}\) be a covering of \(X_{\text{étale}}\). By assumption (1) for each \(i\) we can find an étale morphism \(h_i : V_i \to V\) and a surjective morphism \(X \times_Y V_i \to U_i\) over \(X \times_Y V\). Note that \(\bigcup h_i(V_i) \subset V\) is an open set containing the closed set \(Z = \text{Im}(X \times_Y V \to V)\). Let \(h_0 : V_0 = V \setminus Z \to V\) be the open immersion. It is clear that \(\{V_i \to V\}_{i \in I \cup \{0\}}\) is an étale covering such that for each \(i \in I \cup \{0\}\) we have either \(V_i \times_Y X = \emptyset\) (namely if \(i = 0\)), or \(V_i \times_Y X \to V \times_Y X\) factors through \(U_i \to X \times_Y V\) (if \(i \neq 0\)). Hence the functor \(Y_{\text{étale}} \to X_{\text{étale}}\) is almost cocontinuous.
\end{proof}

\begin{lemma}
Let \(f : X \to Y\) be an integral morphism of schemes which defines a homeomorphism of \(X\) with a closed subset of \(Y\). Then property (C) holds.
\end{lemma}
Proof. Let \( g : U \to X \) be an étale morphism. We need to find an object \( V \to Y \) of \( Y_{\text{ét}} \) and a surjective morphism \( X \times_Y V \to U \) over \( X \). Suppose that for every \( u \in U \) we can find an object \( V_u \to Y \) of \( Y_{\text{ét}} \) and a morphism \( h_u : X \times Y V_u \to U \) over \( X \) with \( u \in \text{Im}(h_u) \). Then we can take \( V = \coprod V_u \) and \( h = \coprod h_u \) and we win. Hence given a point \( u \in U \) we find a pair \( (V_u, h_u) \) as above. To do this we may shrink \( U \) and assume that \( U \) is affine. In this case \( g : U \to X \) is locally quasi-finite. Let \( g^{-1}(g(\{u\})) = \{u, u_2, \ldots, u_n\} \). Since there are no specializations \( u_i \leadsto u \) we may replace \( U \) by an affine neighbourhood so that \( g^{-1}(g(\{u\})) = \{u\} \).

The image \( g(U) \subset X \) is open, hence \( f(g(U)) \) is locally closed in \( Y \). Choose an open \( V \subset Y \) such that \( f(g(U)) = f(X) \cap V \). It follows that \( g \) factors through \( X \times_Y V \) and that the resulting \( \{U \to X \times_Y V\} \) is an étale covering. Since \( f \) has property (B), see Lemma 43.4, we see that there exists an étale covering \( \{V_j \to V\} \) such that \( X \times_Y V_j \to X \times_Y V \) factor through \( U \). This implies that \( V' = \coprod V_j \) is étale over \( Y \) and that there is a morphism \( h : X \times_Y V' \to U \) whose image surjects onto \( g(U) \). Since \( u \) is the only point in its fibre it must be in the image of \( h \) and we win.

We urge the reader to think of the following lemma as a way station on the journey towards the ultimate truth regarding \( f_{\text{small}*} \) for integral universally injective morphisms.

**Lemma 44.5.** Let \( f : X \to Y \) be a morphism of schemes. Assume that \( f \) is universally injective and integral (for example a closed immersion). Then

\[
\begin{align*}
(1) \quad f_{\text{small}*} & : \text{Sh}(X_{\text{ét}}) \to \text{Sh}(Y_{\text{ét}}) \text{ reflects injections and surjections,} \\
(2) \quad f_{\text{small}*} & : \text{Sh}(X_{\text{ét}}) \to \text{Sh}(Y_{\text{ét}}) \text{ commutes with pushouts and coequalizers (and more generally finite connected colimits),} \\
(3) \quad f_{\text{small}*} & \text{ transforms surjections into surjections (on sheaves of sets and on abelian sheaves),} \\
(4) \quad \text{the map } f_{\text{small}*}^{-1} \mathcal{F} & \to \mathcal{F} \text{ is surjective for any sheaf (of sets or of abelian groups) } \mathcal{F} \text{ on } X_{\text{ét}}, \\
(5) \quad f_{\text{small}*} & \text{ is faithful (on sheaves of sets and on abelian sheaves),} \\
(6) \quad f_{\text{small}*} & : \text{Ab}(X_{\text{ét}}) \to \text{Ab}(Y_{\text{ét}}) \text{ is exact, and} \\
(7) \quad \text{the functor } Y_{\text{ét}} & \to X_{\text{ét}} , V \mapsto X \times_Y V \text{ is almost cocontinuous.}
\end{align*}
\]

**Proof.** By Lemmas 42.3, 43.4, and 44.4 we know that the morphism \( f \) has properties (A), (B), and (C). Moreover, by Lemma 44.3 we know that the functor \( Y_{\text{ét}} \to X_{\text{ét}} \) is almost cocontinuous. Now we have

\[
\begin{align*}
(1) \quad \text{property (C) implies (1) by Lemma 44.1} \\
(2) \quad \text{almost continuous implies (2) by Sites, Lemma 42.6} \\
(3) \quad \text{property (B) implies (3) by Lemma 43.1}
\end{align*}
\]

Properties (4), (5), and (6) follow formally from the first three, see Sites, Lemma 41.1 and Modules on Sites, Lemma 15.2 Property (7) we saw above.

45. **Topological invariance of the small étale site**

In the following theorem we show that the small étale site is a topological invariant in the following sense: If \( f : X \to Y \) is a morphism of schemes which is a universal homeomorphism, then \( X_{\text{ét}} \cong Y_{\text{ét}} \) as sites. This improves the result of Étale...
Morphisms, Theorem 15.2. We first prove the result for morphisms and then we state the result for categories.

**Theorem 45.1.** Let $X$ and $Y$ be two schemes over a base scheme $S$. Let $S' \to S$ be a universal homeomorphism. Denote $X'$ (resp. $Y'$) the base change to $S'$. If $X$ is étale over $S$, then the map

$$\text{Mor}_S(Y, X) \to \text{Mor}_{S'}(Y', X')$$

is bijective.

**Proof.** After base changing via $Y \to S$, we may assume that $Y = S$. Thus we may and do assume both $X$ and $Y$ are étale over $S$. In other words, the theorem states that the base change functor is a fully faithful functor from the category of schemes étale over $S$ to the category of schemes étale over $S'$.

Consider the forgetful functor

$$0\text{BTZ}(X', \varphi') \to \text{descent data} \text{ relative to } S'/S$$

with $X'$ étale over $S'$. We claim this functor is an equivalence. On the other hand, the functor

$$0\text{BU}(X \text{ étale over } S) \to \text{descent data} \text{ relative to } S'/S$$

with $X'$ étale over $S'$ is fully faithful by Étale Morphisms, Lemma 20.3. Thus the claim implies the theorem.

**Proof of the claim.** Recall that a universal homeomorphism is the same thing as an integral, universally injective, surjective morphism, see Morphisms, Lemma 43.5. In particular, the diagonal $\Delta : S' \to S' \times_S S'$ is a thickening by Morphisms, Lemma 10.2. Thus by Étale Morphisms, Theorem 15.1 we see that given $X' \to S'$ étale there is a unique isomorphism

$$\varphi' : X' \times_S S' \to S' \times_S X'$$

of schemes étale over $S' \times_S S'$ which pulls back under $\Delta$ to $\text{id} : X' \to X'$ over $S'$. Since $S' \to S' \times_S S' \times_S S'$ is a thickening as well (it is bijective and a closed immersion) we conclude that $(X', \varphi')$ is a descent datum relative to $S'/S$. The canonical nature of the construction of $\varphi'$ shows that it is compatible with morphisms between schemes étale over $S'$. In other words, we obtain a quasi-inverse $X' \mapsto (X', \varphi')$ of the functor (45.1.1). This proves the claim and finishes the proof of the theorem. 

**Theorem 45.2.** Let $f : X \to Y$ be a morphism of schemes. Assume $f$ is integral, universally injective and surjective (i.e., $f$ is a universal homeomorphism, see Morphisms, Lemma 43.5). The functor

$$V \mapsto V_X = X \times_Y V$$

defines an equivalence of categories

$$\{\text{ schemes } V \text{ étale over } Y\} \leftrightarrow \{\text{ schemes } U \text{ étale over } X\}$$

We give two proofs. The first uses effectivity of descent for quasi-compact, separated, étale morphisms relative to surjective integral morphisms. The second uses the material on properties (A), (B), and (C) discussed earlier in the chapter.
First proof. By Theorem 45.1 we see that the functor is fully faithful. It remains to show that the functor is essentially surjective. Let $U \to X$ be an étale morphism of schemes.

Suppose that the result holds if $U$ and $Y$ are affine. In that case, we choose an affine open covering $U = \bigcup U_i$ such that each $U_i$ maps into an affine open of $Y$. By assumption (affine case) we can find étale morphisms $V_i \to Y$ such that $X \times_Y V_i \cong U_i$ as schemes over $X$. Let $V_{i,i'} \subset V_i$ be the open subscheme whose underlying topological space corresponds to $U_i \cap U_{i'}$. Because we have isomorphisms

$$X \times_Y V_{i,i'} \cong U_i \cap U_{i'} \cong X \times_Y V_{i,i'},$$

as schemes over $X$ we see by fully faithfulness that we obtain isomorphisms $\theta_{i,i'} : V_{i,i'} \to V_{i,i'}$ of schemes over $Y$. We omit the verification that these isomorphisms satisfy the cocycle condition of Schemes, Section 14. Applying Schemes, Lemma 14.2 we obtain a scheme $V \to Y$ by glueing the schemes $V_i$ along the identifications $\theta_{i,i'}$. It is clear that $V \to Y$ is étale and $X \times_Y V \cong U$ by construction.

Thus it suffices to show the lemma in case $U$ and $Y$ are affine. Recall that in the proof of Theorem 45.1 we showed that $U$ comes with a unique descent datum $(U, \varphi)$ relative to $X/Y$. By Étale Morphisms, Proposition 20.6 (which applies because $U \to X$ is quasi-compact and separated as well as étale by our reduction to the affine case) there exists an étale morphism $V \to Y$ such that $X \times_Y V \cong U$ and the proof is complete. □

Second proof. By Theorem 45.1 we see that the functor is fully faithful. It remains to show that the functor is essentially surjective. Let $U \to X$ be an étale morphism of schemes.

Suppose that the result holds if $U$ and $Y$ are affine. In that case, we choose an affine open covering $U = \bigcup U_i$ such that each $U_i$ maps into an affine open of $Y$. By assumption (affine case) we can find étale morphisms $V_i \to Y$ such that $X \times_Y V_i \cong U_i$ as schemes over $X$. Let $V_{i,i'} \subset V_i$ be the open subscheme whose underlying topological space corresponds to $U_i \cap U_{i'}$. Because we have isomorphisms

$$X \times_Y V_{i,i'} \cong U_i \cap U_{i'} \cong X \times_Y V_{i,i'},$$

as schemes over $X$ we see by fully faithfulness that we obtain isomorphisms $\theta_{i,i'} : V_{i,i'} \to V_{i,i'}$ of schemes over $Y$. We omit the verification that these isomorphisms satisfy the cocycle condition of Schemes, Section 14. Applying Schemes, Lemma 14.2 we obtain a scheme $V \to Y$ by glueing the schemes $V_i$ along the identifications $\theta_{i,i'}$. It is clear that $V \to Y$ is étale and $X \times_Y V \cong U$ by construction.

Thus it suffices to prove that the functor

04E0 (45.2.1) \{affine schemes $V$ étale over $Y$\} $\leftrightarrow$ \{affine schemes $U$ étale over $X$\}

is essentially surjective when $X$ and $Y$ are affine.

Let $U \to X$ be an affine scheme étale over $X$. We have to find $V \to Y$ étale (and affine) such that $X \times_Y V$ is isomorphic to $U$ over $X$. Note that an étale morphism of affines has universally bounded fibres, see Morphisms, Lemmas 34.6 and 54.10. Hence we can do induction on the integer $n$ bounding the degree of the fibres of $U \to X$. See Morphisms, Lemma 54.9 for a description of this integer in the case of an étale morphism. If $n = 1$, then $U \to X$ is an open immersion (see Étale Morphisms, Theorem 14.1), and the result is clear. Assume $n > 1$. 
By Lemma 44.4 there exists an étale morphism of schemes $W \to Y$ and a surjective morphism $W_X \to U$ over $X$. As $U$ is quasi-compact we may replace $W$ by a disjoint union of finitely many affine opens of $W$, hence we may assume that $W$ is affine as well. Here is a diagram

$$
\begin{align*}
U & \to U \times_Y W \\ & \to W_X \\
X & \to W_X \\
Y & \to W
\end{align*}
$$

The disjoint union decomposition arises because by construction the étale morphism of affine schemes $U \times_Y W \to W_X$ has a section. OK, and now we see that the morphism $R \to X \times_Y W$ is an étale morphism of affine schemes whose fibres have degree universally bounded by $n - 1$. Hence by induction assumption there exists a scheme $V' \to W$ étale such that $R \cong W_X \times_W V'$. Taking $V'' = W \times V'$ we find a scheme $V''$ étale over $W$ whose base change to $W_X$ is isomorphic to $U \times_Y W$ over $X \times_Y W$.

At this point we can use descent to find $V$ over $Y$ whose base change to $X$ is isomorphic to $U$ over $X$. Namely, by the fully faithfulness of the functor (45.2.1) corresponding to the universal homeomorphism $X \times_Y (W \times_Y W) \to (W \times_Y W)$ there exists a unique isomorphism $\varphi : V'' \times_Y W \to W \times_Y V''$ whose base change to $X \times_Y (W \times_Y W)$ is the canonical descent datum for $U \times_Y W$ over $X \times_Y W$. In particular $\varphi$ satisfies the cocycle condition. Hence by Descent, Lemma 34.1 we see that $\varphi$ is effective (recall that all schemes above are affine). Thus we obtain $V \to Y$ and an isomorphism $V'' \cong W \times_Y V$ such that the canonical descent datum on $W \times_Y V/W/Y$ agrees with $\varphi$. Note that $V \to Y$ is étale, by Descent, Lemma 20.29. Moreover, there is an isomorphism $V_X \cong U$ which comes from descending the isomorphism

$$V_X \times_X W_X = X \times_Y (W \times_Y W) \cong W \times_Y V'' \cong U \times_Y W$$

which we have by construction. Some details omitted.  

\[\text{05YX Remark 45.3.} \quad \text{In the situation of Theorem 45.2 it is also true that } V \to V_X \text{ induces an equivalence between those étale morphisms } V \to Y \text{ with } V \text{ affine and those étale morphisms } U \to X \text{ with } U \text{ affine. This follows for example from Limits, Proposition 11.2.}\]

\[\text{03SI Proposition 45.4 (Topological invariance of étale cohomology).} \quad \text{Let } X_0 \to X \text{ be a universal homeomorphism of schemes (for example the closed immersion defined by a nilpotent sheaf of ideals). Then}\]

\[\begin{align*}
(1) \ & \text{the étale sites } X_{\text{étale}} \text{ and } (X_0)_{\text{étale}} \text{ are isomorphic,} \\
(2) \ & \text{the étale topoi } \text{Sh}(X_{\text{étale}}) \text{ and } \text{Sh}((X_0)_{\text{étale}}) \text{ are equivalent, and} \\
(3) \ & H^n_{\text{étale}}(X, F) = H^n_{\text{étale}}(X_0, F|_{X_0}) \text{ for all } q \text{ and for any abelian sheaf } F \text{ on } X_{\text{étale}}.
\end{align*}\]

\[\text{Proof.} \quad \text{The equivalence of categories } X_{\text{étale}} \to (X_0)_{\text{étale}} \text{ is given by Theorem 45.2.} \]

\[\text{We omit the proof that under this equivalence the étale coverings correspond. Hence}\]

\[\text{(1) holds. Parts (2) and (3) follow formally from (1).}\]
46. Closed immersions and pushforward

Before stating and proving Proposition 46.4 in its correct generality we briefly state and prove it for closed immersions. Namely, some of the preceding arguments are quite a bit easier to follow in the case of a closed immersion and so we repeat them here in their simplified form.

In the rest of this section \(i : Z \to X\) is a closed immersion. The functor

\[
\text{Sch}/X \to \text{Sch}/Z, \quad U \mapsto U_Z = Z \times_X U
\]

will be denoted \(U \mapsto U_Z\) as indicated. Since being a closed immersion is preserved under arbitrary base change the scheme \(U_Z\) is a closed subscheme of \(U\).

**Lemma 46.1.** Let \(i : Z \to X\) be a closed immersion of schemes. Let \(U, U'\) be schemes étale over \(X\). Let \(h : U_Z \to U'_Z\) be a morphism over \(Z\). Then there exists a diagram

\[
U \xleftarrow{a} W \xrightarrow{b} U'
\]

such that \(a_Z : W_Z \to U_Z\) is an isomorphism and \(h = b_Z \circ (a_Z)^{-1}\).

**Proof.** Consider the scheme \(M = U \times_Y U'\). The graph \(\Gamma_h \subset M\) of \(h\) is open. This is true for example as \(\Gamma_h\) is the image of a section of the étale morphism \(\text{pr}_{1,Z} : M_Z \to U_Z\), see Étale Morphisms, Proposition 6.1. Hence there exists an open subscheme \(W \subset M\) whose intersection with the closed subset \(M_Z\) is \(\Gamma_h\). Set \(a = \text{pr}_1|_W\) and \(b = \text{pr}_2|_W\). \(\square\)

**Lemma 46.2.** Let \(i : Z \to X\) be a closed immersion of schemes. Let \(V \to Z\) be an étale morphism of schemes. There exist étale morphisms \(U_i \to X\) and morphisms \(U_i,Z \to V\) such that \(\{U_i,Z \to V\}\) is a Zariski covering of \(V\).

**Proof.** Since we only have to find a Zariski covering of \(V\) consisting of schemes of the form \(U_Z\) with \(U\) étale over \(X\), we may Zariski localize on \(X\) and \(V\). Hence we may assume \(X\) and \(V\) affine. In the affine case this is Algebra, Lemma \(\text{[141.10]}\). \(\square\)

If \(\pi : \text{Spec}(k) \to X\) is a geometric point of \(X\), then either \(\pi\) factors (uniquely) through the closed subscheme \(Z\), or \(Z \pi = \emptyset\). If \(\pi\) factors through \(Z\) we say that \(\pi\) is a geometric point of \(Z\) (because it is) and we use the notation “\(\pi \in Z\)” to indicate this.

**Lemma 46.3.** Let \(i : Z \to X\) be a closed immersion of schemes. Let \(\mathcal{G}\) be a sheaf of sets on \(Z\) étale. Let \(\pi\) be a geometric point of \(X\). Then

\[
(i_{\text{small}}, \mathcal{G})_\pi = \begin{cases} 
* & \text{if } \pi \notin Z \\
\mathcal{G}_\pi & \text{if } \pi \in Z
\end{cases}
\]

where \(*\) denotes a singleton set.

**Proof.** Note that \((i_{\text{small}}, \mathcal{G})|_{\text{étale}} = *\) is the final object in the category of étale sheaves on \(U\), i.e., the sheaf which associates a singleton set to each scheme étale over \(U\). This explains the value of \((i_{\text{small}}, \mathcal{G})_\pi\) if \(\pi \notin Z\).

Next, suppose that \(\pi \in Z\). Note that

\[
(i_{\text{small}}, \mathcal{G})_\pi = \text{colim}_{(U, \pi)} \mathcal{G}(U_Z)
\]

and on the other hand

\[
\mathcal{G}_\pi = \text{colim}_{(V, \pi)} \mathcal{G}(V).
\]
Let $\mathcal{C}_1 = \{(U, \pi)\}^{opp}$ be the opposite of the category of étale neighbourhoods of $x$ in $X$, and let $\mathcal{C}_2 = \{(V, \pi)\}^{opp}$ be the opposite of the category of étale neighbourhoods of $x$ in $Z$. The canonical map

$$G \to (i_{\text{small}})_*G$$

corresponds to the functor $F: \mathcal{C}_1 \to \mathcal{C}_2$, $F(U, \pi) = (U_Z, \pi)$. Now Lemmas 46.2 and 46.1 imply that $\mathcal{C}_1$ is cofinal in $\mathcal{C}_2$, see Categories, Definition 17.1. Hence it follows that the displayed arrow is an isomorphism, see Categories, Lemma 17.2. □

**Proposition 46.4.** Let $i: Z \to X$ be a closed immersion of schemes.

1. The functor

$$i_{\text{small}}_* : \text{Sh}(Z_{\text{étale}}) \to \text{Sh}(X_{\text{étale}})$$

is fully faithful and its essential image is those sheaves of sets $F$ on $X_{\text{étale}}$ whose restriction to $X \setminus Z$ is isomorphic to $*$, and

2. the functor

$$i_{\text{small}}_* : \text{Ab}(Z_{\text{étale}}) \to \text{Ab}(X_{\text{étale}})$$

is fully faithful and its essential image is those abelian sheaves on $X_{\text{étale}}$ whose support is contained in $Z$.

In both cases $i_{\text{small}}^{-1}$ is a left inverse to the functor $i_{\text{small}}_*$. 

**Proof.** Let’s discuss the case of sheaves of sets. For any sheaf $G$ on $Z$ the morphism $i_{\text{small}}^{-1}i_{\text{small}}_*G \to G$ is an isomorphism by Lemma 46.3 (and Theorem 29.10). This implies formally that $i_{\text{small}}_*G|_{U_{\text{étale}}} \cong *$ where $U = X \setminus Z$. Conversely, suppose that $F$ is a sheaf of sets on $X$ such that $F|_{U_{\text{étale}}} \cong *$. Consider the adjunction mapping

$$F \to i_{\text{small}}_*i_{\text{small}}^{-1}F$$

Combining Lemmas 46.3 and 36.2 we see that it is an isomorphism. This finishes the proof of (1). The proof of (2) is identical. □

### 47. Integral universally injective morphisms

**Proposition 47.1.** Let $f: X \to Y$ be a morphism of schemes which is integral and universally injective.

1. The functor

$$f_{\text{small}}_* : \text{Sh}(X_{\text{étale}}) \to \text{Sh}(Y_{\text{étale}})$$

is fully faithful and its essential image is those sheaves of sets $F$ on $Y_{\text{étale}}$ whose restriction to $Y \setminus f(X)$ is isomorphic to $*$, and

2. the functor

$$f_{\text{small}}_* : \text{Ab}(X_{\text{étale}}) \to \text{Ab}(Y_{\text{étale}})$$

is fully faithful and its essential image is those abelian sheaves on $Y_{\text{étale}}$ whose support is contained in $f(X)$.

In both cases $f_{\text{small}}^{-1}$ is a left inverse to the functor $f_{\text{small}}_*$. 


Proof. We may factor $f$ as

$$X \xrightarrow{h} Z \xrightarrow{i} Y$$

where $h$ is integral, universally injective and surjective and $i : Z \to Y$ is a closed immersion. Apply Proposition 46.4 to $i$ and apply Theorem 45.2 to $h$. □

48. Big sites and pushforward

Lemma 48.1. Let $\tau \in \{\text{Zariski, étale, smooth, syntomic, fppf}\}$. Let $f : X \to Y$ be a monomorphism of schemes. Then the canonical map $f_{\text{big}}^{-1} f_{\text{big}*} F \to F$ is an isomorphism for any sheaf $F$ on $(\text{Sch}/X)_\tau$.

Proof. In this case the functor $(\text{Sch}/X)_\tau \to (\text{Sch}/Y)_\tau$ is continuous, cocontinuous and fully faithful. Hence the result follows from Sites, Lemma 21.7. □

Remark 48.2. In the situation of Lemma 48.1 it is true that the canonical map $F \to f_{\text{big}}^{-1} f_{\text{big}*} F$ is an isomorphism for any sheaf of sets $F$ on $(\text{Sch}/X)_\tau$. The proof is the same. This also holds for sheaves of abelian groups. However, note that the functor $f_{\text{big}}$ for sheaves of abelian groups is defined in Modules on Sites, Section 16 and is in general different from $f_{\text{big}}$ on sheaves of sets. The result for sheaves of abelian groups follows from Modules on Sites, Lemma 16.4.

Lemma 48.3. Let $f : X \to Y$ be a closed immersion of schemes. Let $U \to X$ be a syntomic (resp. smooth, resp. étale) morphism. Then there exist syntomic (resp. smooth, resp. étale) morphisms $V_i \to Y$ and morphisms $V_i \times_Y X \to U$ such that $\{V_i \times_Y X \to U\}$ is a Zariski covering of $U$.

Proof. Let us prove the lemma when $\tau = \text{syntomic}$. The question is local on $U$. Thus we may assume that $U$ is an affine scheme mapping into an affine of $Y$. Hence we reduce to proving the following case: $Y = \text{Spec}(A)$, $X = \text{Spec}(A/I)$, and $U = \text{Spec}(B)$, where $A/I \to B$ be a syntomic ring map. By Algebra, Lemma 134.18 we can find elements $\overline{g}_i \in B$ such that $B\overline{g}_i = A_i/IA_i$ for certain syntomic ring maps $A \to A_i$. This proves the lemma in the syntomic case. The proof of the smooth case is the same except it uses Algebra, Lemma 135.19. In the étale case use Algebra, Lemma 141.10. □

Lemma 48.4. Let $f : X \to Y$ be a closed immersion of schemes. Let $\{U_i \to X\}$ be a syntomic (resp. smooth, resp. étale) covering. There exists a syntomic (resp. smooth, resp. étale) covering $\{V_j \to Y\}$ such that for each $j$, either $V_j \times_Y X = \emptyset$, or the morphism $V_j \times_Y X \to X$ factors through $U_i$ for some $i$.

Proof. For each $i$ we can choose syntomic (resp. smooth, resp. étale) morphisms $g_{ij} : V_{ij} \to Y$ and morphisms $V_{ij} \times_Y X \to U_i$ over $X$, such that $\{V_{ij} \times_Y X \to U_i\}$ are Zariski coverings, see Lemma 48.3. This in particular implies that $\bigcup_{ij} g_{ij}(V_{ij})$ contains the closed subset $f(X)$. Hence the family of syntomic (resp. smooth, resp. étale) maps $g_{ij}$ together with the open immersion $Y \setminus f(X) \to Y$ forms the desired syntomic (resp. smooth, resp. étale) covering of $Y$. □
Lemma 48.5. Let \( f : X \to Y \) be a closed immersion of schemes. Let \( \tau \in \{ \text{syntomic, smooth, étale} \} \). The functor \( V \mapsto X \times_Y V \) defines an almost cocontinuous functor (see Sites, Definition 42.3) \( (\text{Sch}/Y)_{\tau} \to (\text{Sch}/X)_{\tau} \) between big \( \tau \) sites.

Proof. We have to show the following: given a morphism \( V \to Y \) and any syntomic (resp. smooth, resp. étale) covering \( \{ U_i \to X \times_Y V \} \), there exists a smooth (resp. smooth, resp. étale) covering \( \{ V_j \to V \} \) such that for each \( j \), either \( X \times_Y V_j \) is empty, or \( X \times_Y V_j \to Z \times_Y V \) factors through one of the \( U_i \). This follows on applying Lemma 48.4 above to the closed immersion \( X \times_Y V \to V \). \( \square \)

Lemma 48.6. Let \( f : X \to Y \) be a closed immersion of schemes. Let \( \tau \in \{ \text{syntomic, smooth, étale} \} \).

1. The pushforward \( f_\text{big,*} : \text{Sh}((\text{Sch}/X)_{\tau}) \to \text{Sh}((\text{Sch}/Y)_{\tau}) \) commutes with coequalizers and pushouts.
2. The pushforward \( f_\text{big,*} : \text{Ab}((\text{Sch}/X)_{\tau}) \to \text{Ab}((\text{Sch}/Y)_{\tau}) \) is exact.

Proof. This follows from Sites, Lemma 42.6, Modules on Sites, Lemma 15.3, and Lemma 48.5 above. \( \square \)

Remark 48.7. In Lemma 48.6 the case \( \tau = \text{fppf} \) is missing. The reason is that given a ring \( A \), an ideal \( I \) and a faithfully flat, finitely presented ring map \( A/I \to B \), there is no reason to think that one can find any flat finitely presented ring map \( A \to B \) with \( B/IB \neq 0 \) such that \( A/I \to B/IB \) factors through \( B \). Hence the proof of Lemma 48.5 does not work for the fppf topology. In fact it is likely false that \( f_\text{big,*} : \text{Ab}((\text{Sch}/X)_{\text{fppf}}) \to \text{Ab}((\text{Sch}/Y)_{\text{fppf}}) \) is exact when \( f \) is a closed immersion. If you know an example, please email stacks.project@gmail.com.

49. Exactness of big lower shriek

This is just the following technical result. Note that the functor \( f_\text{big!} \) has nothing whatsoever to do with cohomology with compact support in general.

Lemma 49.1. Let \( \tau \in \{ \text{Zariski, étale, smooth, syntomic, fppf} \} \). Let \( f : X \to Y \) be a morphism of schemes. Let

\[
  f_\text{big} : \text{Sh}((\text{Sch}/X)_{\tau}) \to \text{Sh}((\text{Sch}/Y)_{\tau})
\]

be the corresponding morphism of topoi as in Topologies, Lemma 3.13, 4.13, 5.10, 6.10, or 7.12.

1. The functor \( f_\text{big}^{-1} : \text{Ab}((\text{Sch}/Y)_{\tau}) \to \text{Ab}((\text{Sch}/X)_{\tau}) \) has a left adjoint

\[
  f_\text{big!} : \text{Ab}((\text{Sch}/X)_{\tau}) \to \text{Ab}((\text{Sch}/Y)_{\tau})
\]

which is exact.
2. The functor \( f_\text{big}^* : \text{Mod}((\text{Sch}/Y)_{\tau}, \mathcal{O}) \to \text{Mod}((\text{Sch}/X)_{\tau}, \mathcal{O}) \) has a left adjoint

\[
  f_\text{big!} : \text{Mod}((\text{Sch}/X)_{\tau}, \mathcal{O}) \to \text{Mod}((\text{Sch}/Y)_{\tau}, \mathcal{O})
\]

which is exact.

Moreover, the two functors \( f_\text{big!} \) agree on underlying sheaves of abelian groups.
Proof. Recall that $f_{big}$ is the morphism of topoi associated to the continuous and cocontinuous functor $u : (\text{Sch}/X)_{\tau} \to (\text{Sch}/Y)_{\tau}$, $U/X \mapsto U/Y$. Moreover, we have $f_{big}^{-1} \mathcal{O} = \mathcal{O}$. Hence the existence of $f_{big}$ follows from Modules on Sites, Lemma 16.2 respectively Modules on Sites, Lemma 40.1. Note that if $U$ is an object of $(\text{Sch}/X)_{\tau}$ then the functor $u$ induces an equivalence of categories

$$u' : (\text{Sch}/X)_{\tau}/U \to (\text{Sch}/Y)_{\tau}/U$$

because both sides of the arrow are equal to $(\text{Sch}/U)_{\tau}$. Hence the agreement of $f_{big}$ on underlying abelian sheaves follows from the discussion in Modules on Sites, Remark 40.2. The exactness of $f_{big}$ follows from Modules on Sites, Lemma 16.3 as the functor $u$ above which commutes with fibre products and equalizers. □

Next, we prove a technical lemma that will be useful later when comparing sheaves of modules on different sites associated to algebraic stacks.

Lemma 49.2. Let $X$ be a scheme. Let $\tau \in \{\text{Zariski, étale, smooth, syntomic, fppf}\}$. Let $\mathcal{C}_1 \subset \mathcal{C}_2 \subset (\text{Sch}/X)_{\tau}$ be full subcategories with the following properties:

1. For an object $U/X$ of $\mathcal{C}_1$,
   a) if $\{U_i \to U\}$ is a covering of $(\text{Sch}/X)_{\tau}$, then $U_i/X$ is an object of $\mathcal{C}_1$,
   b) $U \times \mathbf{A}^1/X$ is an object of $\mathcal{C}_1$.
2. $X/X$ is an object of $\mathcal{C}_1$.

We endow $\mathcal{C}_1$ with the structure of a site whose coverings are exactly those coverings $\{U_i \to U\}$ of $(\text{Sch}/X)_{\tau}$ with $U \in \text{Ob}(\mathcal{C}_1)$. Then

1. The functor $\mathcal{C}_1 \to \mathcal{C}_2$ is fully faithful, continuous, and cocontinuous.

Denote $g : \text{Sh}(\mathcal{C}_1) \to \text{Sh}(\mathcal{C}_2)$ the corresponding morphism of topoi. Denote $\mathcal{O}_1$ the restriction of $\mathcal{O}$ to $\mathcal{C}_1$. Denote $g_!$ the functor of Modules on Sites, Definition 16.1.

2. The canonical map $g_!\mathcal{O}_1 \to \mathcal{O}_2$ is an isomorphism.

Proof. Assertion (a) is immediate from the definitions. In this proof all schemes are schemes over $X$ and all morphisms of schemes are morphisms of schemes over $X$. Note that $g^{-1}$ is given by restriction, so that for an object $U$ of $\mathcal{C}_1$ we have $\mathcal{O}_1(U) = \mathcal{O}_2(U) = \mathcal{O}(U)$. Recall that $g_!\mathcal{O}_1$ is the sheaf associated to the presheaf $g_!\mathcal{O}_1$ which associates to $V$ in $\mathcal{C}_2$ the group

$$\text{colim}_{V \to U} \mathcal{O}(U)$$

where $U$ runs over the objects of $\mathcal{C}_1$ and the colimit is taken in the category of abelian groups. Below we will use frequently that if

$$V \to U \to U'$$

are morphisms with $U, U' \in \text{Ob}(\mathcal{C}_1)$ and if $f' \in \mathcal{O}(U')$ restricts to $f \in \mathcal{O}(U)$, then $(V \to U, f)$ and $(V \to U', f')$ define the same element of the colimit. Also, $g_!\mathcal{O}_1 \to \mathcal{O}_2$ maps the element $(V \to U, f)$ simply to the pullback of $f$ to $V$.

Surjectivity. Let $V$ be a scheme and let $h \in \mathcal{O}(V)$. Then we obtain a morphism $V \to X \times \mathbf{A}^1$ induced by $h$ and the structure morphism $V \to X$. Writing $\mathbf{A}^1 = \text{Spec}(\mathbb{Z}[x])$ we see the element $x \in \mathcal{O}(X \times \mathbf{A}^1)$ pulls back to $h$. Since $X \times \mathbf{A}^1$ is an object of $\mathcal{C}_1$ by assumptions (1)(b) and (2) we obtain the desired surjectivity.

Injectivity. Let $V$ be a scheme. Let $s = \sum_{i=1}^{n}(V \to U_i, f_i)$ be an element of the colimit displayed above. For any $i$ we can use the morphism $f_i : U_i \to X \times \mathbf{A}^1$ to see
that \((V \rightarrow U_i, f_i)\) defines the same element of the colimit as \((f_i : V \rightarrow X \times A^1, x)\). Then we can consider
\[
f_1 \times \ldots \times f_n : V \rightarrow X \times A^n
\]
and we see that \(s\) is equivalent in the colimit to
\[
\sum_{i=1}^{n} (f_1 \times \ldots \times f_n : V \rightarrow X \times A^n, x_i) = (f_1 \times \ldots \times f_n : V \rightarrow X \times A^n, x_1 + \ldots + x_n)
\]
Now, if \(x_1 + \ldots + x_n\) restricts to zero on \(V\), then we see that \(f_1 \times \ldots \times f_n\) factors through \(X \times A^{n-1} = V(x_1 + \ldots + x_n)\). Hence we see that \(s\) is equivalent to zero in the colimit. \(\square\)

50. Étale cohomology

03Q3 In the following sections we prove some basic results on étale cohomology. Here is an example of something we know for cohomology of topological spaces which also holds for étale cohomology.

0A50 **Lemma 50.1** (Mayer-Vietoris for étale cohomology). Let \(X\) be a scheme. Suppose that \(X = U \cup V\) is a union of two opens. For any abelian sheaf \(F\) on \(X_{\text{ét}}\) there exists a long exact cohomology sequence
\[
0 \rightarrow H^0_{\text{ét}}(X, F) \rightarrow H^0_{\text{ét}}(U, F) \oplus H^0_{\text{ét}}(V, F) \rightarrow H^0_{\text{ét}}(U \cap V, F) \\
\rightarrow H^1_{\text{ét}}(X, F) \rightarrow H^1_{\text{ét}}(U, F) \oplus H^1_{\text{ét}}(V, F) \rightarrow H^1_{\text{ét}}(U \cap V, F) \rightarrow \ldots
\]
This long exact sequence is functorial in \(F\).

**Proof.** Observe that if \(\mathcal{I}\) is an injective abelian sheaf, then
\[
0 \rightarrow \mathcal{I}(X) \rightarrow \mathcal{I}(U) \oplus \mathcal{I}(V) \rightarrow \mathcal{I}(U \cap V) \rightarrow 0
\]
is exact. This is true in the first and middle spots as \(\mathcal{I}\) is a sheaf. It is true on the right, because \(\mathcal{I}(U) \rightarrow \mathcal{I}(U \cap V)\) is surjective by Cohomology on Sites, Lemma 13.6. Another way to prove it would be to show that the cokernel of the map \(\mathcal{I}(U) \oplus \mathcal{I}(V) \rightarrow \mathcal{I}(U \cap V)\) is the first Čech cohomology group of \(\mathcal{I}\) with respect to the covering \(X = U \cup V\) which vanishes by Lemmas 18.7 and 19.1. Thus, if \(F \rightarrow \mathcal{I}^*\) is an injective resolution, then
\[
0 \rightarrow \mathcal{I}^*(X) \rightarrow \mathcal{I}^*(U) \oplus \mathcal{I}^*(V) \rightarrow \mathcal{I}^*(U \cap V) \rightarrow 0
\]
is a short exact sequence of complexes and the associated long exact cohomology sequence is the sequence of the statement of the lemma. \(\square\)

0EYK **Lemma 50.2** (Relative Mayer-Vietoris). Let \(f : X \rightarrow Y\) be a morphism of schemes. Suppose that \(X = U \cup V\) is a union of two open subschemes. Denote \(a = f|_U : U \rightarrow Y\), \(b = f|_V : V \rightarrow Y\), and \(c = f|_{U \cap V} : U \cap V \rightarrow Y\). For every abelian sheaf \(F\) on \(X_{\text{ét}}\) there exists a long exact sequence
\[
0 \rightarrow f_*F \rightarrow a_*F|_U \oplus b_*F|_V \rightarrow c_*F|_{U \cap V} \rightarrow R^1f_*F \rightarrow \ldots
\]
on \(Y_{\text{ét}}\). This long exact sequence is functorial in \(F\).

**Proof.** Let \(F \rightarrow \mathcal{I}^*\) be an injective resolution of \(F\) on \(X_{\text{ét}}\). We claim that we get a short exact sequence of complexes
\[
0 \rightarrow f_*\mathcal{I}^* \rightarrow a_*\mathcal{I}^*|_U \oplus b_*\mathcal{I}^*|_V \rightarrow c_*\mathcal{I}^*|_{U \cap V} \rightarrow 0.
\]
Namely, for any $W$ in $Y_{\text{étale}}$, and for any $n \geq 0$ the corresponding sequence of groups of sections over $W$

$$0 \to \mathcal{I}^n(W \times_Y X) \to \mathcal{I}^n(W \times_Y U) \oplus \mathcal{I}^n(W \times_Y V) \to \mathcal{I}^n(W \times_Y (U \cap V)) \to 0$$

was shown to be short exact in the proof of Lemma 50.1. The lemma follows by taking cohomology sheaves and using the fact that $\mathcal{I}^n_U$ is an injective resolution of $\mathcal{F}|_U$ and similarly for $\mathcal{I}^n|_V$, $\mathcal{I}^n|_{U \cap V}$.

51. Colimits

0EZL **Definition 51.1.** Let $I$ be a preordered set. Let $(X_i, f_{\nu i})$ be an inverse system of schemes over $I$. A system $(\mathcal{F}_i, \varphi_{\nu i})$ of sheaves on $(X_i, f_{\nu i})$ is given by

1. a sheaf $\mathcal{F}_i$ on $(X_i)_{\text{étale}}$ for all $i \in I$,
2. for $i' \geq i$ a map $\varphi_{\nu i} : f^{-1}_{\nu i} \mathcal{F}_i \to \mathcal{F}_{i'}$ of sheaves on $(X_{i'})_{\text{étale}}$ such that $\varphi_{\nu i} = \varphi_{\nu i'} \circ f^{-1}_{\nu i'} \varphi_{\nu i}$ whenever $i'' \geq i' \geq i$.

In the situation of Definition 51.1 assume $I$ is a directed set and the transition morphisms $f_{\nu i}$ affine. Let $X = \lim_i X_i$ be the limit in the category of schemes, see Limits, Section 2. Denote $f_i : X \to X_i$ the projection morphisms and consider the maps

$$f_i^{-1} \mathcal{F}_i = f_{\nu i}^{-1} f^{-1}_{\nu i} \mathcal{F}_i \xrightarrow{f_{\nu i}^{-1} \varphi_{\nu i}} f_{\nu i}^{-1} \mathcal{F}_{i'}$$

This turns $f_i^{-1} \mathcal{F}_i$ into a system of sheaves on $X_{\text{étale}}$ over $I$ (it is a good exercise to check this). We often want to know whether there is an isomorphism

$$H^q_{\text{étale}}(X, \text{colim} f_i^{-1} \mathcal{F}_i) = \text{colim} H^q_{\text{étale}}(X_i, \mathcal{F}_i)$$

It will turn out this is true if $X_i$ is quasi-compact and quasi-separated for all $i$, see Theorem 51.3.

0EYL **Lemma 51.2.** Let $I$ be a directed set. Let $(X_i, f_{\nu i})$ be an inverse system of schemes over $I$ with affine transition morphisms. Let $X = \lim_{i \in I} X_i$. With notation as in Lemma 21.2 we have

$$X_{\text{affine, étale}} = \text{colim}(X_i)_{\text{affine, étale}}$$

as sites in the sense of Sites, Lemma 18.2.

**Proof.** Let us first prove this when $X$ and $X_i$ are quasi-compact and quasi-separated for all $i$ (as this is true in all cases of interest). In this case any object of $X_{\text{affine, étale}}$, resp. $(X_i)_{\text{affine, étale}}$ is of finite presentation over $X$. Moreover, the category of schemes of finite presentation over $X$ is the colimit of the categories of schemes of finite presentation over $X_i$, see Limits, Lemma 10.1. The same holds for the subcategories of affine objects étale over $X$ by Limits, Lemmas 4.13 and 8.10. Finally,
if \( \{U^j \to U\} \) is a covering of \( X_{\text{affine, étale}} \) and if \( U^j_i \to U_i \) is a morphism of affine schemes étale over \( X_i \) whose base change to \( X \) is \( U^j \to U \), then we see that the base change of \( \{U^j_i \to U_i\} \) to some \( X_{U'} \) is a covering for \( i' \) large enough, see Limits, Lemma 8.13.\[09YQ\]

In the general case, let \( U \) be an object of \( X_{\text{affine, étale}} \). Then \( U \to X \) is étale and separated (as \( U \) is separated) but in general not quasi-compact. Still, \( U \to X \) is locally of finite presentation and hence by Limits, Lemma 10.5 there exists an i, a quasi-compact and quasi-separated scheme \( U_i \), and a morphism \( U_i \to X_i \) which is locally of finite presentation whose base change to \( X \) is \( U \to X \). Then \( U = \lim_{i \geq i} U_i \) where \( U_i = U_i \times X, X_i \). After increasing \( i \) we may assume \( U_i \) is affine, see Limits, Lemma 14.13. To check that \( U_i \to X_i \) is étale for \( i \) sufficiently large, choose a finite affine open covering \( U_i = U_{i,1} \cup \ldots \cup U_{i,m} \) such that \( U_{i,j} \to U_i \to X_i \) maps into an affine open \( W_{i,j} \subset X_i \). Then we can apply Limits, Lemma 8.10 to see that \( U_{i,j} \to W_{i,j} \) is étale after possibly increasing \( i \). In this way we see that the functor \( \lim \) \( (X_i)_{\text{affine, étale}} \to X_{\text{affine, étale}} \) is essentially surjective. Fully faithfulness follows directly from the already used Limits, Lemma 10.5. The statement on coverings is proved in exactly the same manner as done in the first paragraph of the proof. \[\square\]

Using the above we get the following general result on colimits and cohomology.

**Theorem 5.3.** Let \( X = \lim_{i \in I} X_i \) be a limit of a directed system of schemes with affine transition morphisms \( f_{i,i} : X_i \to X_i \). We assume that \( X_i \) is quasi-compact and quasi-separated for all \( i \in I \). Let \( (F_i, \varphi_{i,j}) \) be a system of abelian sheaves on \( (X_i, f_{i,i}) \). Denote \( f_i : X \to X_i \) the projection and set \( F = \lim \) \( f_i^{-1} F_i \). Then

\[
\lim_{i \in I} H^p_{\text{étale}}(X_i, F_i) = H^p_{\text{étale}}(X, F).
\]

for all \( p \geq 0 \).

**Proof.** By Lemma 21.2 we can compute the cohomology of \( F \) on \( X_{\text{affine, étale}} \). Thus the result by a combination of Lemma 51.2 and Cohomology on Sites, Lemma 17.3. \[\square\]

The following two results are special cases of the theorem above.

**Lemma 5.4.** Let \( X \) be a quasi-compact and quasi-separated scheme. Let \( I \) be a directed set. Let \( (F_i, \varphi_{i,j}) \) be a system of abelian sheaves on \( X_{\text{étale}} \) over \( I \). Then

\[
\lim_{i \in I} H^p_{\text{étale}}(X, F_i) = H^p_{\text{étale}}(X, \lim_{i \in I} F_i).
\]

**Proof.** This is a special case of Theorem 5.3. We also sketch a direct proof. We prove it for all \( X \) at the same time, by induction on \( p \).

1. (1) For any quasi-compact and quasi-separated scheme \( X \) and any étale covering \( \mathcal{U} \) of \( X \), show that there exists a refinement \( \mathcal{V} = \{V_j \to X\}_{j \in J} \) with \( J \) finite and each \( V_j \) quasi-compact and quasi-separated such that all \( V_{j_0} \times_X \ldots \times_X V_{j_p} \) are also quasi-compact and quasi-separated.

2. (2) Using the previous step and the definition of colimits in the category of sheaves, show that the theorem holds for \( p = 0 \) and all \( X \).

3. (3) Using the locality of cohomology (Lemma 22.3), the Čech-to-cohomology spectral sequence (Theorem 19.2) and the fact that the induction hypothesis applies to all \( V_{j_0} \times_X \ldots \times_X V_{j_p} \) in the above situation, prove the induction step \( p \to p + 1 \).
**Lemma 51.5.** Let $A$ be a ring, $(I, \leq)$ a directed set and $(B_i, \varphi_{ij})$ a system of $A$-algebras. Set $B = \colim_{i \in I} B_i$. Let $X \to \Spec(A)$ be a quasi-compact and quasi-separated morphism of schemes. Let $\mathcal{F}$ an abelian sheaf on $X_{\text{étale}}$. Denote $Y_i = X \times_{\Spec(A)} \Spec(B_i)$, $Y = X \times_{\Spec(A)} \Spec(B)$, $\mathcal{G}_i = (Y_i \to X)^{-1}\mathcal{F}$ and $\mathcal{G} = (Y \to X)^{-1}\mathcal{F}$. Then

$$H^p_{\text{étale}}(Y, \mathcal{G}) = \colim_{i \in I} H^p_{\text{étale}}(Y_i, \mathcal{G}_i).$$

**Proof.** This is a special case of Theorem 51.3. We also outline a direct proof as follows.

1. Given $V \to Y$ étale with $V$ quasi-compact and quasi-separated, there exist $i \in I$ and $V_i \to Y_i$ such that $V = V_i \times_{Y_i} Y$. If all the schemes considered were affine, this would correspond to the following algebra statement: if $B = \colim B_i$ and $B \to C$ is étale, then there exist $i \in I$ and $B_i \to C_i$ étale such that $C \cong B \otimes_{B_i} C_i$. This is proved in Algebra, Lemma 141.3.

2. In the situation of (1) show that $\mathcal{G}(V) = \colim_{i \geq 1} \mathcal{G}_i(V_i)$ where $V_i$ is the base change of $V_i$ to $Y_i$.

3. By (1), we see that for every étale covering $V = \{V_j \to Y\}_{j \in J}$ with $J$ finite and the $V_j$s quasi-compact and quasi-separated, there exists $i \in I$ and an étale covering $V_i = \{V_{ij} \to Y_i\}_{j \in J}$ such that $V \cong V_i \times_{Y_i} Y$.

4. Show that (2) and (3) imply

$$H^i(V, \mathcal{G}) = \colim_{i \in I} H^i(V_i, \mathcal{G}_i).$$

5. Cleverly use the Čech-to-cohomology spectral sequence (Theorem 49.2).

**Lemma 51.6.** Let $f : X \to Y$ be a morphism of schemes and $\mathcal{F} \in \text{Ab}(X_{\text{étale}})$. Then $R^p f_* \mathcal{F}$ is the sheaf associated to the presheaf

$$(V \to Y) \mapsto H^p_{\text{étale}}(X \times_Y V, \mathcal{F}|_{X \times_Y V}).$$

**Proof.** This lemma is valid for topological spaces, and the proof in this case is the same. See Cohomology on Sites, Lemma 8.4 for details.

**Lemma 51.7.** Let $S$ be a scheme. Let $X = \lim_{i \in I} X_i$ be a limit of a directed system of schemes over $S$ with affine transition morphisms $f_{ij} : X_i \to X_j$. We assume the structure morphisms $g_i : X_i \to S$ and $g : X \to S$ are quasi-compact and quasi-separated. Let $(\mathcal{F}_i, \varphi_{ij})$ be a system of abelian sheaves on $(X_i, f_{ij})$. Denote $f_i : X \to X_i$ the projection and set $\mathcal{F} = \colim f_i^{-1}\mathcal{F}_i$. Then

$$\colim_{i \in I} R^p g_{i*}\mathcal{F}_i = R^p g_*\mathcal{F}$$

for all $p \geq 0$.

**Proof.** Recall (Lemma 51.6) that $R^p g_{i*}\mathcal{F}_i$ is the sheaf associated to the presheaf $U \mapsto H^p_{\text{étale}}(U \times_S X_i, \mathcal{F}_i)$ and similarly for $R^p g_*\mathcal{F}$. Moreover, the colimit of a system of sheaves is the sheafification of the colimit on the level of presheaves. Note that every object of $S_{\text{étale}}$ has a covering by quasi-compact and quasi-separated objects (e.g., affine schemes). Moreover, if $U$ is a quasi-compact and quasi-separated object, then we have

$$\colim H^p_{\text{étale}}(U \times_S X_i, \mathcal{F}_i) = H^p_{\text{étale}}(U \times_S X, \mathcal{F})$$

by Theorem 51.3. Thus the lemma follows.
Let $I$ be a directed set. Let $g_i : X_i \to S_i$ be an inverse system of morphisms of schemes over $I$. Assume $g_i$ is quasi-compact and quasi-separated and for $i' \geq i$ the transition morphisms $f_{i'i} : X_{i'} \to X_i$ and $h_{i'i} : S_{i'} \to S_i$ are affine. Let $g : X \to S$ be the limit of the morphisms $g_i$, see Limits, Section 3. Denote $f_i : X \to X_i$ and $h_i : S \to S_i$ the projections. Let $(\mathcal{F}_i, \varphi_{i'i})$ be a system of sheaves on $(X_i, f_{i'i})$. Set $\mathcal{F} = \colim f_i^{-1}\mathcal{F}_i$. Then

$$R^p g_* \mathcal{F} = \colim_{i \in I} h_i^{-1} R^p g_i_* \mathcal{F}_i$$

for all $p \geq 0$.

**Proof.** How is the map of the lemma constructed? For $i' \geq i$ we have a commutative diagram

$$
\begin{array}{ccc}
X & \xrightarrow{f_{i'}} & X_{i'} \\
g \downarrow & & \downarrow g_{i'} \\
S & \xrightarrow{h_{i'}} & S_{i'}
\end{array}
\quad 
\begin{array}{ccc}
X & \xrightarrow{f_{i'i}} & X_i \\
g \downarrow & & \downarrow g_{i'i} \\
S & \xrightarrow{h_{i'i}} & S_i
\end{array}
$$

If we combine the base change map $h_{i'i}^{-1} Rg_{i'i} \mathcal{F}_{i'} \to Rg_{i'i} f_{i'i}^{-1} \mathcal{F}_i$ (Cohomology on Sites, Lemma 16.1 or Remark 20.3) with the map $Rg_{i'i} \varphi_{i'i}$, then we obtain $\psi_{i'i} : h_{i'i}^{-1} Rg_{i'i} \mathcal{F}_{i'} \to Rg_{i'i} \mathcal{F}_i$. Similarly, using the left square in the diagram we obtain maps $\psi_i : h_i^{-1} Rg_i \mathcal{F}_i \to Rg_i \mathcal{F}$. The maps $h_{i'i}^{-1}$ $\psi_{i'i}$ and $\psi_i$ are the maps used in the statement of the lemma. For this to make sense, we have to check that $\psi_{i'i} = \psi_{i'i'} \circ h_{i'i'}^{-1} \psi_{i'}$ and $\psi_i \circ Rg_{i'i} \psi_{i'i} = \psi_i$; this follows from Cohomology on Sites, Remark 20.5.

Proof of the equality. First proof using dimension shifting. For any $U$ affine and étale over $X$ by Theorem 51.3 we have

$$g_* \mathcal{F}(U) = H^0(U \times_S X, \mathcal{F}) = \colim H^0(U_i \times_{S_i} X_i, \mathcal{F}_i) = \colim g_{i'}_* \mathcal{F}_i(U_i)$$

where the colimit is over $i$ large enough such that there exists an $i$ and $U_i$ affine étale over $S_i$ whose base change is equal to $U$ over $S$ (see Lemma 51.2). The right hand side is equal to $(\colim h_i^{-1} g_{i'} \mathcal{F}_i)(U)$ by Sites, Lemma 18.4. This proves the lemma for $p = 0$. If $(G_i, \varphi_{i'i})$ is a system with $G = \colim f_i^{-1} G_{i'}$ such that $G_i$ is an injective abelian sheaf on $X_i$ for all $i$, then for any $U$ affine and étale over $X$ by Theorem 51.3 we have

$$H^p(U \times_S X, G) = \colim H^p(U_i \times_{S_i} X_i, G_i) = 0$$

for $p > 0$ (same colimit as before). Hence $R^p g_* G = 0$ and we get the result for $p > 0$ for such a system. In general we may choose a short exact sequence of systems

$$
0 \to (\mathcal{F}_i, \varphi_{i'i}) \to (G_i, \varphi_{i'i}) \to (Q_i, \varphi_{i'i}) \to 0
$$

where $(G_i, \varphi_{i'i})$ is as above, see Cohomology on Sites, Lemma 17.2. By induction the lemma holds for $p - 1$ and by the above we have vanishing for $p$ and $(G_i, \varphi_{i'i})$. Hence the result for $p$ and $(\mathcal{F}_i, \varphi_{i'i})$ by the long exact sequence of cohomology.

Second proof. Recall that $S_{affine, étale} = \colim(S_i)_{affine, étale}$, see Lemma 51.2. Thus if $U$ is an object of $S_{affine, étale}$, then we can write $U = U_i \times_{S_i} S$ for some $i$ and some $U_i$ in $(S_i)_{affine, étale}$ and

$$(\colim_{i \in I} h_i^{-1} Rg_i_* \mathcal{F}_i)(U) = \colim_{i' \geq i} (R^{i'} g_{i'}^{-1} \mathcal{F}_{i'})(U_i \times_{S_i} S_{i'})$$

You can also use this method to produce the maps in the lemma.

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by Sites, Lemma [18.4] and the construction of the transition maps in the system described above. Since $R^pg_{i^*}, F_{i'}$ is the sheaf associated to the presheaf $U_i \mapsto H^p(U_i \times_{S_i} X_{i'}, F_{i'})$ and since $R^pg_s, F$ is the sheaf associated to the presheaf $U \mapsto H^p(U \times_{S} X, F)$ (Lemma [51.6]) we obtain a canonical commutative diagram

$$\colim_{i' \geq i} H^p(U_i \times_{S_i} X_{i'}, F_{i'}) \xrightarrow{\xi} \colim_{i' \geq i}(R^pg_{i^*}, F_{i'}))(U_i \times_{S_i} S_{i'})$$

$$H^p(U \times_{S} X, F) \xrightarrow{\xi} R^pg_s, F(U)$$

Observe that the left hand vertical arrow is an isomorphism by Theorem [51.3]. We’re trying to show that the right hand vertical arrow is an isomorphism. However, we already know that the source and target of this arrow are sheaves on $S_{\text{affine, étale}}$. Hence it suffices to show: (1) an element in the target, locally comes from an element in the source and (2) an element in the source which maps to zero in the target locally vanishes. Part (1) follows immediately from the above and the fact that the lower horizontal arrow comes from a map of presheaves which becomes an isomorphism after sheafification. For part (2), say $\xi \in \colim_{i' \geq i}(R^pg_{i^*}, F_{i'}))(U_i \times_{S_i} S_{i'})$ is in the kernel. Choose an $i' \geq i$ and $\xi_{i'} \in (R^pg_{i^*}, F_{i'}))(U_i \times_{S_i} S_{i'})$ representing $\xi$. Choose a standard étale covering $\{U_{i', k} \rightarrow U_i \times_{S_i} S_{i'}\}_{k=1, \ldots, m}$ such that $\xi_{i'}|U_{i', k}$ comes from $\xi_{i', k} \in H^p(U_{i', k} \times_{S_{i'}} X_{i'}, F_{i'})$. Since it is enough to prove that $\xi$ dies locally, we may replace $U$ by the members of the étale covering $\{U_{i', k} \times_{S_{i'}} X \rightarrow U = U_i \times_{S_i} S\}$. After this replacement we see that $\xi$ is the image of an element $\xi'$ of the group $\colim_{i' \geq i} H^p(U_i \times_{S_i} X_{i'}, F_{i'})$ in the diagram above. Since $\xi'$ maps to zero in $R^pg_s, F(U)$ we can do another replacement and assume that $\xi'$ maps to zero in $H^p(U \times_{S} X, F)$. However, since the left vertical arrow is an isomorphism we then conclude $\xi' = 0$ hence $\xi = 0$ as desired.

0EYN Lemma 51.9. Let $X = \lim_{i \in I} X_i$ be a directed limit of schemes with affine transition morphisms $f_{i'}^i$ and projection morphisms $f_i : X \rightarrow X_i$. Let $F$ be a sheaf on $X_{\text{étale}}$. Then

1. there are canonical maps $\varphi_{i'} : f_{i'}^{-1} f_i, s F \rightarrow f_{i'}, s F$ such that $(f_{i'}, s F, \varphi_{i'})$ is a system of sheaves on $(X_i, f_{i'}^i)$ as in Definition 51.1 and
2. $F = \colim f_{i}^{-1} f_i, s F$.

Proof. Via Lemmas 21.2 and 51.2 this is a special case of Sites, Lemma 18.5.

0DV2 Lemma 51.10. Let $I$ be a directed set. Let $g_i : X_i \rightarrow S_i$ be an inverse system of morphisms of schemes over $I$. Assume $g_i$ is quasi-compact and quasi-separated and for $i' \geq i$ the transition morphisms $X_{i'} \rightarrow X_i$ and $S_{i'} \rightarrow S_i$ are affine. Let $g : X \rightarrow S$ be the limit of the morphisms $g_i$, see Limits, Section 3. Denote $f_i : X_i \rightarrow X$ and $h_i : S \rightarrow S_i$ the projections. Let $F$ be an abelian sheaf on $X$. Then we have

$$R^pg_s, F = \colim_{i \in I} h_i^{-1} R^pg_i, (f_i, s F)$$

Proof. Formal combination of Lemmas 51.8 and 51.9.

0EZZ Remark 51.11. Many of the results above have variants for bounded below complexes, but one has to be careful that the bounds have to be uniform. We explain this in the simplest case. Let $X$ be a quasi-compact and quasi-separated scheme. Let $I$ be a directed set. Let $F^*_{\text{•}}$ be a system over $I$ of complexes of sheaves on
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Assume there is an integer $a$ such that $F^n = 0$ for $n < a$ and all $i \in I$. Then we have

$$H^p_{etale}(X, \text{colim } F^i) = \text{colim} H^p_{etale}(X, F^i)$$

If we ever need this we will state a precise lemma with full proof here.

52. Stalks of higher direct images

**Theorem 52.1.** Let $f : X \to S$ be a quasi-compact and quasi-separated morphism of schemes, $F$ an abelian sheaf on $X_{etale}$, and $s$ a geometric point of $S$ lying over $s \in S$. Then

$$(R^n f_\ast F)_s = H^n_{etale}(X \times_S \text{Spec}(O^{sh}_{S,s}), p^{-1} F)$$

where $p : X \times_S \text{Spec}(O^{sh}_{S,s}) \to X$ is the projection.

**Proof.** Let $\mathcal{I}$ be the category of étale neighborhoods of $s$ on $S$. By Lemma 51.6 we have

$$(R^n f_\ast F)_s = \text{colim}_{(V, \pi) \in \mathcal{I}^{opp}} H^n_{etale}(X \times_S V, F|_{X \times_S V})$$

We may replace $\mathcal{I}$ by the initial subcategory consisting of affine étale neighborhoods of $s$. Observe that

$$\text{Spec}(O^{sh}_{S,s}) = \lim_{(V, \pi) \in \mathcal{I}} V$$

by Lemma 33.1 and Limits, Lemma 2.1. Since fibre products commute with limits we also obtain

$$X \times_S \text{Spec}(O^{sh}_{S,s}) = \lim_{(V, \pi) \in \mathcal{I}} X \times_S V$$

We conclude by Lemma 51.5.

53. The Leray spectral sequence

**Lemma 53.1.** Let $f : X \to Y$ be a morphism and $\mathcal{I}$ an injective object of $Ab(X_{etale})$. Let $V \in \text{Ob}(Y_{etale})$. Then

1. for any covering $\mathcal{V} = \{V_j \to V\}_{j \in J}$ we have $\check{H}^p(\mathcal{V}, f_\ast \mathcal{I}) = 0$ for all $p > 0$,
2. $f_\ast \mathcal{I}$ is acyclic for the functor $\Gamma(V, -)$, and
3. if $g : Y \to Z$, then $f_\ast \mathcal{I}$ is acyclic for $g_\ast$.

**Proof.** Observe that $\check{C}^\bullet(\mathcal{V}, f_\ast \mathcal{I}) = \check{C}^\bullet(\mathcal{V} \times_Y X, \mathcal{I})$ which has vanishing higher cohomology groups by Lemma 18.7. This proves (1). The second statement follows as a sheaf which has vanishing higher Čech cohomology groups for any covering has vanishing higher cohomology groups. This a wonderful exercise in using the Čech-to-cohomology spectral sequence, but see Cohomology on Sites, Lemma 11.9 for details and a more precise and general statement. Part (3) is a consequence of (2) and the description of $R^q g_\ast$ in Lemma 51.6.

Using the formalism of Grothendieck spectral sequences, this gives the following.

**Proposition 53.2 (Leray spectral sequence).** Let $f : X \to Y$ be a morphism of schemes and $F$ an étale sheaf on $X$. Then there is a spectral sequence

$$E_2^{p,q} = H^p_{etale}(Y, R^q f_\ast F) \Rightarrow H^{p+q}_{etale}(X, F).$$

**Proof.** See Lemma 53.1 and see Derived Categories, Section 22.
54. Vanishing of finite higher direct images

The next goal is to prove that the higher direct images of a finite morphism of schemes vanish.

Lemma 54.1. Let \( R \) be a strictly henselian local ring. Set \( S = \text{Spec}(R) \) and let \( \overline{s} \) be its closed point. Then the global sections functor \( \Gamma(S, -) : \text{Ab}(S_{\text{etale}}) \to \text{Ab} \) is exact. In fact we have \( \Gamma(S, F) = F_{\overline{s}} \) for any sheaf of sets \( F \). In particular

\[
\forall p \geq 1, \quad H^p_{\text{etale}}(S, F) = 0
\]

for all \( F \in \text{Ab}(S_{\text{etale}}) \).

Proof. If we show that \( \Gamma(S, F) = F_{\overline{s}} \) then \( \Gamma(S, -) \) is exact as the stalk functor is exact. Let \( (U, u) \) be an étale neighbourhood of \( \overline{s} \). Pick an affine open neighborhood \( \text{Spec}(A) \) of \( u \) in \( U \). Then \( R \to A \) is étale and \( \kappa(\overline{s}) = \kappa(\overline{u}) \). By Theorem 32.4 we see that \( A \cong R \times A' \) as an \( R \)-algebra compatible with maps to \( \kappa(\overline{s}) = \kappa(\overline{u}) \). Hence we get a section

\[
\text{Spec}(A) \to U \quad \downarrow \quad \downarrow \quad S
\]

It follows that in the system of étale neighbourhoods of \( \overline{s} \) the identity map \( (S, \overline{s}) \to (S, \overline{s}) \) is cofinal. Hence \( \Gamma(S, F) = F_{\overline{s}} \). The final statement of the lemma follows as the higher derived functors of an exact functor are zero, see Derived Categories, Lemma 17.9. \( \square \)

Proposition 54.2. Let \( f : X \to Y \) be a finite morphism of schemes.

1. For any geometric point \( \overline{y} : \text{Spec}(k) \to Y \) we have

\[
(f_*F)_{\overline{y}} = \prod_{\overline{x} : \text{Spec}(k) \to X, \ f(\overline{x}) = \overline{y}} F_{\overline{x}}.
\]

for \( F \) in \( \text{Sh}(X_{\text{etale}}) \) and

\[
(f_*F)_{\overline{y}} = \bigoplus_{\overline{x} : \text{Spec}(k) \to X, \ f(\overline{x}) = \overline{y}} F_{\overline{x}}.
\]

for \( F \) in \( \text{Ab}(X_{\text{etale}}) \).

2. For any \( q \geq 1 \) we have \( R^q f_* F = 0 \).

Proof. Let \( X^h_y \) denote the fiber product \( X \times_Y \text{Spec}(O^h_{Y, \overline{y}}) \). By Theorem 52.1 the stalk of \( R^q f_* F \) at \( \overline{y} \) is computed by \( H^q_{\text{etale}}(X^h_{\overline{y}}, F) \). Since \( f \) is finite, \( X^h_{\overline{y}} \) is finite over \( \text{Spec}(O^h_{Y, \overline{y}}) \), thus \( X^h_{\overline{y}} = \text{Spec}(A) \) for some ring \( A \) finite over \( O^h_{Y, \overline{y}} \). Since the latter is strictly henselian, Lemma 32.5 implies that \( A \) is a finite product of henselian local rings \( A = A_1 \times \ldots \times A_r \). Since the residue field of \( O^h_{Y, \overline{y}} \) is separably closed the same is true for each \( A_i \). Hence \( A_i \) is strictly henselian. This implies that \( X^h_{\overline{y}} = \prod_{i=1}^r \text{Spec}(A_i) \). The vanishing of Lemma 54.1 implies that \( (R^q f_* F)_{\overline{y}} = 0 \) for \( q > 0 \) which implies (2) by Theorem 29.10. Part (1) follows from the corresponding statement of Lemma 54.1. \( \square \)
Lemma 54.3. Consider a cartesian square

\[
\begin{array}{ccc}
X' & \xrightarrow{g'} & X \\
\downarrow f' & & \downarrow f \\
Y' & \xrightarrow{g} & Y
\end{array}
\]

of schemes with \( f \) a finite morphism. For any sheaf of sets \( F \) on \( X_{\text{étale}} \) we have \( f'_*(g')^{-1} F = g^{-1} f_* F \).

Proof. In great generality there is a pullback map \( g^{-1} f_* F \rightarrow f'_*(g')^{-1} F \), see Sites, Section 45. It suffices to check on stalks (Theorem 29.10). Let \( \overline{y}' : \text{Spec}(k) \rightarrow Y' \) be a geometric point. We have

\[
(f'_*(g')^{-1} F)_{\overline{y}'} = \prod_{\overline{x}' : \text{Spec}(k) \rightarrow X', \ f' \circ \overline{x}' = \overline{y}'} ((g')^{-1} F)_{\overline{x}'} = \prod_{\overline{x} : \text{Spec}(k) \rightarrow X, \ f \circ \overline{x} = g \circ \overline{y}'} \mathcal{F}_{\overline{x}} = (f_* F)_{g \circ \overline{y}'} = (g^{-1} f_* F)_{\overline{y}'}
\]

The first equality by Proposition 54.2. The second equality by Lemma 36.2. The third equality holds because the diagram is a cartesian square and hence the map

\[
\{ \overline{x}' : \text{Spec}(k) \rightarrow X', \ f' \circ \overline{x}' = \overline{y}' \} \rightarrow \{ \overline{x} : \text{Spec}(k) \rightarrow X, \ f \circ \overline{x} = g \circ \overline{y} \}
\]

sending \( \overline{x}' \) to \( g' \circ \overline{x}' \) is a bijection. The fourth equality by Proposition 54.2. The fifth equality by Lemma 36.2. □

Lemma 54.4. Consider a cartesian square

\[
\begin{array}{ccc}
X' & \xrightarrow{g'} & X \\
\downarrow f' & & \downarrow f \\
Y' & \xrightarrow{g} & Y
\end{array}
\]

of schemes with \( f \) an integral morphism. For any sheaf of sets \( F \) on \( X_{\text{étale}} \) we have \( f'_*(g')^{-1} F = g^{-1} f_* F \).

Proof. The question is local on \( Y \) and hence we may assume \( Y \) is affine. Then we can write \( X = \lim X_i \) with \( f_i : X_i \rightarrow Y \) finite (this is easy in the affine case, but see Limits, Lemma 37.2 for a reference). Denote \( p_{ij} : X_{ij} \rightarrow X_i \) the transition morphisms and \( p_i : X \rightarrow X_i \) the projections. Setting \( F_i = p_{i,*} F \) we obtain from Lemma 51.9 a system \((F_i, \varphi_{ij})\) with \( F = \colim F_i \). We get \( f_* F = \colim f_{i,*} F_i \) from Lemma 51.7. Set \( X'_i = Y' \times_Y X_i \) with projections \( f'_i \) and \( g'_i \). Then \( X' = \lim X'_i \)
as limits commute with limits. Denote $p'_i : X' \to X'_i$ the projections. We have
\[
g^{-1} f_* \mathcal{F} = g^{-1} \text{colim } f_{i*} \mathcal{F}_i \\
= \text{colim } g^{-1} f_{i*} \mathcal{F}_i \\
= \text{colim } f'_{i*} (g'_i)^{-1} \mathcal{F}_i \\
= f'_{i*} (\text{colim } (g'_i)^{-1} (g'_i)^{-1} \mathcal{F}_i) \\
= f'_{i*} (\text{colim } (g'_i)^{-1} g_i^{-1} \mathcal{F}_i) \\
= f'_{i*} (g_i^{-1} \text{colim } p_i^{-1} \mathcal{F}_i) \\
= f'_{i*} (g_i^{-1} \mathcal{F})
\]
as desired. For the first equality see above. For the second use that pullback commutes with colimits. For the third use the finite case, see Lemma 54.3. For the fourth use Lemma 54.5. For the fifth use that $g'_i \circ p'_i = p_i \circ g_i'. For the sixth use that pullback commutes with colimits. For the seventh use $\mathcal{F} = \text{colim } p_i^{-1} \mathcal{F}_i$. □

The following lemma is a case of cohomological descent dealing with étale sheaves and finite surjective morphisms. We will significantly generalize this result once we prove the proper base change theorem.

**Lemma 54.5.** Let $f : X \to Y$ be a surjective finite morphism of schemes. Set $f_n : X_n \to Y$ equal to the $(n+1)$-fold fibre product of $X$ over $Y$. For $\mathcal{F} \in \text{Ab}(Y_{\text{étale}})$ set $\mathcal{F}_n = f_{n*} f_{n-1}^{-1} \mathcal{F}$. There is an exact sequence
\[
0 \to \mathcal{F} \to \mathcal{F}_0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \ldots
\]
on $X_{\text{étale}}$. Moreover, there is a spectral sequence
\[
E_1^{p,q} = H_{\text{étale}}^q(X, f_p^{-1} \mathcal{F})
\]
converging to $H^{p+q}(Y_{\text{étale}}, \mathcal{F})$. This spectral sequence is functorial in $\mathcal{F}$.

**Proof.** If we prove the first statement of the lemma, then we obtain a spectral sequence with $E_1^{p,q} = H_{\text{étale}}^q(Y, \mathcal{F})$ converging to $H^{p+q}(Y_{\text{étale}}, \mathcal{F})$, see Derived Categories, Lemma 21.3. On the other hand, since $R^i f_p* f_p^{-1} \mathcal{F} = 0$ for $i > 0$ (Proposition 54.2), we get
\[
H_{\text{étale}}^p(X, f_p^{-1} \mathcal{F}) = H_{\text{étale}}^p(Y, f_p* f_p^{-1} \mathcal{F}) = H_{\text{étale}}^p(Y, \mathcal{F}_p)
\]
by Proposition 53.2 and we get the spectral sequence of the lemma.

To prove the first statement of the lemma, observe that $X_n$ forms a simplicial scheme over $Y$, see Simplicial, Example 3.5. Observe moreover, that for each of the projections $d_j : X_{n+1} \to X_n$ there is a map $d_{j*}^{-1} f_n^{-1} \mathcal{F} \to f_{n+1}^{-1} \mathcal{F}$. These maps induce maps
\[
\delta_j : \mathcal{F}_n \to \mathcal{F}_{n+1}
\]
for $j = 0, \ldots, n + 1$. We use the alternating sum of these maps to define the differentials $\mathcal{F}_n \to \mathcal{F}_{n+1}$. Similarly, there is a canonical augmentation $\mathcal{F} \to \mathcal{F}_0$, namely this is just the canonical map $\mathcal{F} \to f_* f^{-1} \mathcal{F}$. To check that this sequence of sheaves is an exact complex it suffices to check on stalks at geometric points (Theorem 29.10). Thus we let $\overline{\eta} : \text{Spec}(k) \to Y$ be a geometric point. Let $E = \{ \overline{x} : \text{Spec}(k) \to X | \overline{f(x)} = \overline{\eta} \}$. Then $E$ is a finite nonempty set and we see that
\[
(\mathcal{F}_n)_\overline{\eta} = \bigoplus_{\overline{x} \in E_{n+1}} \mathcal{F}_\overline{x}
\]
by Proposition 54.2 and Lemma 36.2. Thus we have to see that given an abelian group $M$ the sequence

$$0 \to M \to \bigoplus_{e \in E} M \to \bigoplus_{e \in E} M \to \ldots$$

is exact. Here the first map is the diagonal map and the map $\bigoplus_{e \in E^{n+1}} M \to \bigoplus_{e \in E^{n+2}} M$ is the alternating sum of the maps induced by the $(n + 2)$ projections $E^{n+2} \to E^{n+1}$. This can be shown directly or deduced by applying Simplicial, Lemma 26.9 to the map $E \to \{\ast\}$.

\[ \square \]

**Remark 54.6.** In the situation of Lemma 54.5 if $G$ is a sheaf of sets on $Y_{\text{etale}}$, then we have

$$\Gamma(Y, G) = \text{Equalizer}( \Gamma(X_0, f_0^{-1}G) \longrightarrow \Gamma(X_1, f_1^{-1}G) )$$

This is proved in exactly the same way, by showing that the sheaf $G$ is the equalizer of the two maps $f_0 \ast f_0^{-1}G \to f_1 \ast f_1^{-1}G$.

### 55. Galois action on stalks

Galois action on stalks. Let $S$ be a scheme. Let $\pi$ be a geometric point of $S$. Let $\sigma \in \text{Aut}(\kappa(\pi)/\kappa(s))$. Define an action of $\sigma$ on the stalk $F_\pi$ of a sheaf $F$ as follows

$$(55.0.1) \quad F_\pi \longrightarrow F_\pi \quad (U, \pi, t) \longmapsto (U, \pi \circ \text{Spec}(\sigma), t).$$

where we use the description of elements of the stalk in terms of triples as in the discussion following Definition 29.6. This is a left action, since if $\sigma_i \in \text{Aut}(\kappa(\pi)/\kappa(s))$ then

$$\sigma_1 \cdot (\sigma_2 \cdot (U, \pi, t)) = \sigma_1 \cdot (U, \pi \circ \text{Spec}(\sigma_2), t) = (U, \pi \circ \text{Spec}(\sigma_2) \circ \text{Spec}(\sigma_1), t) = (U, \pi \circ \text{Spec}(\sigma_1 \circ \sigma_2), t) = (\sigma_1 \circ \sigma_2) \cdot (U, \pi, t)$$

It is clear that this action is functorial in the sheaf $F$. We note that we could have defined this action by referring directly to Remark 29.8.

**Definition 55.1.** Let $S$ be a scheme. Let $\pi$ be a geometric point lying over the point $s$ of $S$. Let $\kappa(s) \subset \kappa(s)^{\text{sep}} \subset \kappa(\pi)$ denote the separable algebraic closure of $\kappa(s)$ in the algebraically closed field $\kappa(\pi)$.

1. In this situation the **absolute Galois group** of $\kappa(s)$ is $\text{Gal}(\kappa(s)^{\text{sep}}/\kappa(s))$. It is sometimes denoted $\text{Gal}_{\kappa(s)}$.
2. The geometric point $\pi$ is called **algebraic** if $\kappa(s) \subset \kappa(\pi)$ is an algebraic closure of $\kappa(s)$.

**Example 55.2.** The geometric point $\text{Spec}(\mathbb{C}) \to \text{Spec}(\mathbb{Q})$ is not algebraic. Let $\kappa(s) \subset \kappa(s)^{\text{sep}} \subset \kappa(\pi)$ be as in the definition. Note that as $\kappa(\pi)$ is algebraically closed the map

$$\text{Aut}(\kappa(\pi)/\kappa(s)) \longrightarrow \text{Gal}(\kappa(s)^{\text{sep}}/\kappa(s)) = \text{Gal}_{\kappa(s)}$$

is surjective. Suppose \((U, \pi)\) is an \(\text{étale}\) neighbourhood of \(\pi\), and say \(\pi\) lies over the point \(u\) of \(U\). Since \(U \to S\) is \(\text{étale}\), the residue field extension \(\kappa(s) \subset \kappa(u)\) is finite separable. This implies the following

1. If \(\sigma \in \text{Aut}(\kappa(\pi)/\kappa(s)_{\text{sep}})\) then \(\sigma\) acts trivially on \(\mathcal{F}_\pi\).
2. More precisely, the action of \(\text{Aut}(\kappa(\pi)/\kappa(s))\) determines and is determined by an action of the absolute Galois group \(\text{Gal}_K\) on \(\mathcal{F}_\pi\).
3. Given \((U, \pi, t)\) representing an element \(\xi\) of \(\mathcal{F}_\pi\) any element of \(\text{Gal}(\kappa(s)_{\text{sep}}/K)\) acts trivially, where \(\kappa(s) \subset K \subset \kappa(s)_{\text{sep}}\) is the image of \(\pi^t : \kappa(u) \to \kappa(\pi)\).

 Altogether we see that \(\mathcal{F}_\pi\) becomes a \(\text{Gal}_K\)-set (see Fundamental Groups, Definition 2.1). Hence we may think of the stalk functor as a functor

\[
\text{Sh}(\text{étale}) \to \text{Gal}_{\kappa(s)}\text{-Sets}, \quad \mathcal{F} \mapsto \mathcal{F}_\pi
\]

and from now on we usually do think about the stalk functor in this way.

**Theorem 55.3.** Let \(S = \text{Spec}(K)\) with \(K\) a field. Let \(\pi\) be a geometric point of \(S\). Let \(G = \text{Gal}_K\) denote the absolute Galois group. Taking stalks induces an equivalence of categories

\[
\text{Sh}(\text{étale}) \to G\text{-Sets}, \quad \mathcal{F} \mapsto \mathcal{F}_\pi.
\]

**Proof.** Let us construct the inverse to this functor. In Fundamental Groups, Lemma 2.2 we have seen that given a \(G\)-set \(M\) there exists an \(\text{étale}\) morphism \(X \to \text{Spec}(K)\) such that \(\text{Mor}_K(\text{Spec}(K)_{\text{sep}}, X)\) is isomorphic to \(M\) as a \(G\)-set. Consider the sheaf \(\mathcal{F}\) on \(\text{Spec}(K)_{\text{étale}}\) defined by the rule \(U \mapsto \text{Mor}_K(U, X)\). This is a sheaf as the \(\text{étale}\) topology is subcanonical. Then we see that \(\mathcal{F}_\pi = \text{Mor}_K(\text{Spec}(K)_{\text{sep}}, X) = M\) as \(G\)-sets (details omitted). This gives the inverse of the functor and we win. \(\square\)

**Remark 55.4.** Another way to state the conclusion of Theorem 55.3 and Fundamental Groups, Lemma 2.2 is to say that every sheaf on \(\text{Spec}(K)_{\text{étale}}\) is representable by a scheme \(X\) \(\text{étale}\) over \(\text{Spec}(K)\). This does not mean that every sheaf is representable in the sense of Sites, Definition 12.3. The reason is that in our construction of \(\text{Spec}(K)_{\text{étale}}\) we chose a sufficiently large set of schemes \(\text{étale}\) over \(\text{Spec}(K)\), whereas sheaves on \(\text{Spec}(K)_{\text{étale}}\) form a proper class.

**Lemma 55.5.** Assumptions and notations as in Theorem 55.3. There is a functorial bijection

\[
\Gamma(S, \mathcal{F}) = (\mathcal{F}_\pi)^G
\]

**Proof.** We can prove this using formal arguments and the result of Theorem 55.3 as follows. Given a sheaf \(\mathcal{F}\) corresponding to the \(G\)-set \(M = \mathcal{F}_\pi\) we have

\[
\Gamma(S, \mathcal{F}) = \text{Mor}_{\text{Sh}(\text{étale})}(h_{\text{Spec}(K)}, \mathcal{F}) = \text{Mor}_{\text{G-Sets}}(\{\ast\}, M) = M^G
\]

Here the first identification is explained in Sites, Sections 2 and 12, the second results from Theorem 55.3 and the third is clear. We will also give a direct proof. Suppose that \(t \in \Gamma(S, \mathcal{F})\) is a global section. Then the triple \((S, \pi, t)\) defines an element of \(\mathcal{F}_\pi\) which is clearly invariant under the action of \(G\). Conversely, suppose \(5\) For the doubting Thomases out there.
that \((U, \pi, t)\) defines an element of \(\mathcal{F}_\pi\) which is invariant. Then we may shrink \(U\) and assume \(U = \text{Spec}(L)\) for some finite separable field extension of \(K\), see Proposition 26.2. In this case the map \(\mathcal{F}(U) \to \mathcal{F}_\pi\) is injective, because for any morphism of étale neighbourhoods \((U', \pi') \to (U, \pi)\) the restriction map \(\mathcal{F}(U) \to \mathcal{F}(U')\) is injective since \(U' \to U\) is a covering of \(S_{\text{étale}}\). After enlarging \(L\) a bit we may assume \(K \subset L\) is a finite Galois extension. At this point we use that \(\text{Spec}(L) \times_{\text{Spec}(K)} \text{Spec}(L)\) is the disjoint union of \(\sigma \in \text{Gal}(L/K)\) \(\text{Spec}(L)\) with the maps \(\text{Spec}(L) \to \text{Spec}(L \otimes_K L)\) coming from the ring maps \(a \otimes b \mapsto a\sigma(b)\). Hence we see that the condition that \((U, \pi, t)\) is invariant under all of \(G\) implies that \(t \in \mathcal{F}(\text{Spec}(L))\) maps to the same element of \(\mathcal{F}(\text{Spec}(L) \times_{\text{Spec}(K)} \text{Spec}(L))\) via restriction by either projection (this uses the injectivity mentioned above; details omitted). Hence the sheaf condition of \(\mathcal{F}\) for the étale covering \(\{\text{Spec}(L) \to \text{Spec}(K)\}\) kicks in and we conclude that \(t\) comes from a unique section of \(\mathcal{F}\) over \(\text{Spec}(K)\). □

04JN **Remark 55.6.** Let \(S\) be a scheme and let \(\pi : \text{Spec}(k) \to S\) be a geometric point of \(S\). By definition this means that \(k\) is algebraically closed. In particular the absolute Galois group of \(k\) is trivial. Hence by Theorem 55.3 the category of sheaves on \(\text{Spec}(k)_{\text{étale}}\) is equivalent to the category of sets. The equivalence is given by taking sections over \(\text{Spec}(k)\). This finally provides us with an alternative definition of the stalk functor. Namely, the functor

\[
\text{Sh}(\text{Spec}(k)_{\text{étale}}) \to \text{Sets}, \ F \mapsto F^*\]

is isomorphic to the functor

\[
\text{Sh}(\text{Spec}(k)_{\text{étale}}) \to \text{Sh}(\text{Spec}(k)_{\text{étale}}) = \text{Sets}, \ F \mapsto \pi^*F
\]

To prove this rigorously one can use Lemma 36.2 part (3) with \(f = \pi\). Moreover, having said this the general case of Lemma 36.2 part (3) follows from functoriality of pullbacks.

### 56. Group cohomology

0A2H If we write \(H^i(G, M)\) we will mean that \(G\) is a topological group and \(M\) a discrete \(G\)-module with continuous \(G\)-action. This includes the case of an abstract group \(G\), which simply means that \(G\) is viewed as a topological group with the discrete topology.

When the module has a nondiscrete topology, we will use the notation \(H^i_{\text{cont}}(G, M)\) to indicate the continuous group cohomology groups introduced in [Tat76], see Section 57.

04JP **Definition 56.1.** Let \(G\) be a topological group.

1. A \(G\)-module, sometimes called a **discrete \(G\)-module**, is an abelian group \(M\) endowed with a left action \(a : G \times M \to M\) by group homomorphisms such that \(a\) is continuous when \(M\) is given the discrete topology.

2. A morphism of \(G\)-modules \(f : M \to N\) is a \(G\)-equivariant homomorphism from \(M\) to \(N\).

3. The category of \(G\)-modules is denoted \(\text{Mod}_G\).

Let \(R\) be a ring.
(1) An $R$-module $M$ is an $R$-module $M$ endowed with a left action $a: G \times R \to M$ by $R$-linear maps such that $a$ is continuous when $M$ is given the discrete topology.

(2) A morphism of $R$-modules $f: M \to N$ is a $G$-equivariant $R$-module map from $M$ to $N$.

(3) The category of $R$-modules is denoted $\text{Mod}_{R,G}$.

The condition that $a: G \times M \to M$ is continuous is equivalent with the condition that the stabilizer of any $x \in M$ is open in $G$. If $G$ is an abstract group then this corresponds to the notion of an abelian group endowed with a $G$-action provided we endow $G$ with the discrete topology. Observe that $\text{Mod}_{Z,G} = \text{Mod}_G$.

The category $\text{Mod}_G$ has enough injectives, see Injectives, Lemma 3.1. Consider the left exact functor

$$\text{Mod}_G \to \text{Ab}, \quad M \mapsto M^G = \{x \in M \mid g \cdot x = x \ \forall g \in G\}$$

We sometimes denote $M^G = H^0(G, M)$ and sometimes we write $M^G = \Gamma_G(M)$. This functor has a total right derived functor $R\Gamma_G(M)$ and $i$th right derived functor $R^i\Gamma_G(M) = H^i(G, M)$ for any $i \geq 0$.

The same construction works for $H^0(G, -) : \text{Mod}_{R,G} \to \text{Mod}_R$. We will see in Lemma 56.3 that this agrees with the cohomology of the underlying $G$-module.

**Definition 56.2.** Let $G$ be a topological group. Let $M$ be a $G$-module as in Definition 56.1.

1. The right derived functors $H^i(G, M)$ are called the continuous group cohomology groups of $M$.
2. If $G$ is an abstract group endowed with the discrete topology then the $H^i(G, M)$ are called the group cohomology groups of $M$.
3. If $G$ is a Galois group, then the groups $H^i(G, M)$ are called the Galois cohomology groups of $M$.
4. If $G$ is the absolute Galois group of a field $K$, then the groups $H^i(G, M)$ are sometimes called the Galois cohomology groups of $K$ with coefficients in $M$. In this case we sometimes write $H^i(K, M)$ instead of $H^i(G, M)$.

**Lemma 56.3.** Let $G$ be a topological group. Let $R$ be a ring. For every $i \geq 0$ the diagram

$$\begin{array}{ccc}
\text{Mod}_{R,G} & \xrightarrow{H^i(G, -)} & \text{Mod}_R \\
\downarrow & & \downarrow \\
\text{Mod}_G & \xrightarrow{H^i(G, -)} & \text{Ab}
\end{array}$$

whose vertical arrows are the forgetful functors is commutative.

**Proof.** Let us denote the forgetful functor $F : \text{Mod}_{R,G} \to \text{Mod}_G$. Then $F$ has a left adjoint $H : \text{Mod}_G \to \text{Mod}_{R,G}$ given by $H(M) = M \otimes_Z R$. Observe that every object of $\text{Mod}_G$ is a quotient of a direct sum of modules of the form $Z[G/U]$ where $U \subset G$ is an open subgroup. Here $Z[G/U]$ denotes the $G$-modules of finite $Z$-linear combinations of right $U$ congruence classes in $G$ endowed with left $G$-action. Thus every bounded above complex in $\text{Mod}_G$ is quasi-isomorphic to a bounded above complex in $\text{Mod}_G$ whose underlying terms are flat $Z$-modules (Derived Categories, Lemma 16.5). Thus it is clear that $LH$ exists on $D^- (\text{Mod}_G)$ and is computed by
evaluating $H$ on any complex whose terms are flat $\mathbb{Z}$-modules; this follows from Derived Categories, Lemma \[16.7\] and Proposition \[17.8\] We conclude from Derived Categories, Lemma \[28.4\] that

$$\text{Ext}^i(\mathbb{Z}, F(M)) = \text{Ext}^i(R, M)$$

for $M$ in $\text{Mod}_{R,G}$. Observe that $H^0(G, -) = \text{Hom}(\mathbb{Z}, -)$ on $\text{Mod}_G$ where $\mathbb{Z}$ denotes the $G$-module with trivial action. Hence $H^i(G, -) = \text{Ext}^i(\mathbb{Z}, -)$ on $\text{Mod}_G$. Similarly we have $H^i(G, -) = \text{Ext}^i(R, -)$ on $\text{Mod}_{R,G}$. Combining all we see that the lemma is true. \hfill $\square$

\textbf{Lemma 56.4.} Let $G$ be a topological group. Let $R$ be a ring. Let $M$, $N$ be $R$-$G$-modules. If $M$ is finite projective as an $R$-module, then $\text{Ext}^i(M, N) = H^i(G, M^\vee \otimes_R N)$ (for notation see proof).

\textbf{Proof.} The module $M^\vee = \text{Hom}_R(M, R)$ endowed with the contragredient action of $G$. Namely $(g \cdot \lambda)(m) = \lambda(g^{-1} \cdot m)$ for $g \in G$, $\lambda \in M^\vee$, $m \in M$. The action of $G$ on $M^\vee \otimes_R N$ is the diagonal one, i.e., given by $g \cdot (\lambda \otimes n) = g \cdot \lambda \otimes g \cdot n$. Note that for a third $R$-$G$-module $E$ we have $\text{Hom}(E, M^\vee \otimes_R N) = \text{Hom}(E \otimes_R E, N)$. Namely, this is true on the level of $R$-modules by Algebra, Lemmas \[11.8\] and \[17.8\] and the definitions of $G$-actions are chosen such that it remains true for $R$-$G$-modules. It follows that $M^\vee \otimes_R N$ is an injective $R$-$G$-module if $N$ is an injective $R$-$G$-module. Hence if $N \to N^\bullet$ is an injective resolution, then $M^\vee \otimes_R N \to M^\vee \otimes_R N^\bullet$ is an injective resolution. Then

$$\text{Hom}(M, N^\bullet) = \text{Hom}(R, M^\vee \otimes_R N^\bullet) = (M^\vee \otimes_R N^\bullet)^G$$

Since the left hand side computes $\text{Ext}^i(M, N)$ and the right hand side computes $H^i(G, M^\vee \otimes_R N)$ the proof is complete. \hfill $\square$

\textbf{Lemma 56.5.} Let $G$ be a topological group. Let $k$ be a field. Let $V$ be a $k$-$G$-module. If $G$ is topologically finitely generated and $\dim_k(V) < \infty$, then $\dim_k H^1(G, V) < \infty$.

\textbf{Proof.} Let $g_1, \ldots, g_r \in G$ be elements which topologically generate $G$, i.e., this means that the subgroup generated by $g_1, \ldots, g_r$ is dense. By Lemma \[56.4\] we see that $H^1(G, V)$ is the $k$-vector space of extensions

$$0 \to V \to E \to k \to 0$$

of $k$-$G$-modules. Choose $e \in E$ mapping to $1 \in k$. Write

$$g_i \cdot e = v_i + e$$

for some $v_i \in V$. This is possible because $g_i \cdot 1 = 1$. We claim that the list of elements $v_1, \ldots, v_r \in V$ determine the isomorphism class of the extension $E$. Once we prove this the lemma follows as this means that our Ext vector space is isomorphic to a subquotient of the $k$-vector space $V^{\oplus r}$; some details omitted. Since $E$ is an object of the category defined in Definition \[56.1\] we know there is an open subgroup $U$ such that $u \cdot e = e$ for all $u \in U$. Now pick any $g \in G$. Then $gU$ contains a word $w$ in the elements $g_1, \ldots, g_r$. Say $gu = w$. Since the element $w \cdot e$ is determined by $v_1, \ldots, v_r$, we see that $g \cdot e = (gu) \cdot e = w \cdot e$ is too. \hfill $\square$

\textbf{Lemma 56.6.} Let $G$ be a profinite topological group. Then

1. $H^i(G, M)$ is torsion for $i > 0$ and any $G$-module $M$, and
2. $H^1(G, M) = 0$ if $M$ is a $\mathbb{Q}$-vector space.
Proof. Proof of (1). By dimension shifting we see that it suffices to show that $H^1(G, M)$ is torsion for every $G$-module $M$. Choose an exact sequence $0 \to M \to I \to N \to 0$ with $I$ an injective object of the category of $G$-modules. Then any element of $H^1(G, M)$ is the image of an element $y \in N^G$. Choose $x \in I$ mapping to $y$. The stabilizer $U \subset G$ of $x$ is open, hence has finite index $r$. Let $g_1, \ldots, g_r \in G$ be a system of representatives for $G/U$. Then $\sum g_i(x)$ is an invariant element of $I$ which maps to $ry$. Thus $r$ kills the element of $H^1(G, M)$ we started with. Part (2) follows as then $H^i(G, M)$ is both a $\mathbb{Q}$-vector space and torsion. \[\square\]

57. Continuous group cohomology

The continuous group cohomology is defined by the complex of inhomogeneous cochains. We can define this when $M$ is an arbitrary topological abelian group endowed with a continuous $G$-action. Namely, we consider the complex

$$C^\bullet_{\text{cont}}(G, M) : M \to \text{Maps}_{\text{cont}}(G, M) \to \text{Maps}_{\text{cont}}(G \times G, M) \to \cdots$$

where the boundary map is defined for $n \geq 1$ by the rule

$$d(f)(g_1, \ldots, g_{n+1}) = g_1(f(g_2, \ldots, g_{n+1})) + \sum_{j=1, \ldots, n}^{(-1)^j} f(g_1, \ldots, g_{j}g_{j+1}, \ldots, g_{n+1}) + (-1)^{n+1} f(g_1, \ldots, g_n)$$

and for $n = 0$ sends $m \in M$ to the map $g \mapsto g(m) - m$. We define

$$H^i_{\text{cont}}(G, M) = H^i(C^\bullet_{\text{cont}}(G, M))$$

Since the terms of the complex involve continuous maps from $G$ and self products of $G$ into the topological module $M$, it is not clear that this turns a short exact sequence of topological modules into a long exact cohomology sequence. Another difficulty is that the category of topological abelian groups isn’t an abelian category!

However, a short exact sequence of discrete $G$-modules does give rise to a short exact sequence of complexes of continuous cochains and hence a long exact cohomology sequence of continuous cohomology groups $H^i_{\text{cont}}(G, -)$. Therefore, on the category $\text{Mod}_G$ the functors $H^i_{\text{cont}}(G, M)$ form a cohomological $\delta$-functor (Homology, Section \[11\]). Since $H^i(G, M)$ is a universal $\delta$-functor (Derived Categories, Lemma \[17.6\]) we obtain canonical maps

$$H^i(G, M) \longrightarrow H^i_{\text{cont}}(G, M)$$

for $M \in \text{Mod}_G$. It is known that these maps are isomorphisms when $G$ is an abstract group (i.e., $G$ has the discrete topology) or when $G$ is a profinite group (insert future reference here). If you know an example showing this map is not an isomorphism for a topological group $G$ and $M \in \text{Ob} \text{Mod}_G$ please email stacks.project@gmail.com.

58. Cohomology of a point

As a consequence of the discussion in the preceding sections we obtain the equivalence of étale cohomology of the spectrum of a field with Galois cohomology.

03QQ

\[\text{Lemma 58.1.} \] Let $S = \text{Spec}(K)$ with $K$ a field. Let $\pi$ be a geometric point of $S$. Let $G = \text{Gal}_k$ denote the absolute Galois group. The stalk functor induces an equivalence of categories

$$\text{Ab}(S_{\text{étale}}) \longrightarrow \text{Mod}_G, \quad \mathcal{F} \mapsto \mathcal{F}_\pi.$$
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Proof. In Theorem 55.3 we have seen the equivalence between sheaves of sets and $G$-sets. The current lemma follows formally from this as an abelian sheaf is just a sheaf of sets endowed with a commutative group law, and a $G$-module is just a $G$-set endowed with a commutative group law.

\textbf{Lemma 58.2.} Notation and assumptions as in Lemma 58.1. Let $F$ be an abelian sheaf on $\text{Spec}(K)_{\text{étale}}$ which corresponds to the $G$-module $M$. Then

1. in $D(\text{Ab})$ we have a canonical isomorphism $R\Gamma(S, F) = R\Gamma_G(M),$
2. $H^0_{\text{étale}}(S, F) = M^G,$ and
3. $H^i_{\text{étale}}(S, F) = H^i(G, M).$

\textbf{Proof.} Combine Lemma 58.1 with Lemma 55.5.

\textbf{Example 58.3.} Sheaves on $\text{Spec}(K)_{\text{étale}}$. Let $G = \text{Gal}(K^{\text{sep}}/K)$ be the absolute Galois group of $K$.

1. The constant sheaf $\mathbb{Z}/n\mathbb{Z}$ corresponds to the module $\mathbb{Z}/n\mathbb{Z}$ with trivial $G$-action,
2. the sheaf $G_m|_{\text{Spec}(K)_{\text{étale}}}$ corresponds to $(K^{\text{sep}})^*$ with its $G$-action,
3. the sheaf $G_a|_{\text{Spec}(K^{\text{sep}})}$ corresponds to $(K^{\text{sep}}, +)$ with its $G$-action, and
4. the sheaf $\mu_n|_{\text{Spec}(K^{\text{sep}})}$ corresponds to $\mu_n(K^{\text{sep}})$ with its $G$-action.

By Remark 23.4 and Theorem 24.1 we have the following identifications for cohomology groups:

$$H^0_{\text{étale}}(S_{\text{étale}}, G_m) = \Gamma(S, O_S^*)$$
$$H^1_{\text{étale}}(S_{\text{étale}}, G_m) = H^1_{\text{Zar}}(S, O_S^*) = \text{Pic}(S)$$
$$H^1_{\text{étale}}(S_{\text{étale}}, G_a) = H^1_{\text{Zar}}(S, O_S)$$

Also, for any quasi-coherent sheaf $F$ on $S_{\text{étale}}$ we have

$$H^i(S_{\text{étale}}, F) = H^i_{\text{Zar}}(S, F),$$

see Theorem 22.4. In particular, this gives the following sequence of equalities

$$0 = \text{Pic}(\text{Spec}(K)) = H^1_{\text{étale}}(\text{Spec}(K)_{\text{étale}}, G_m) = H^1(G, (K^{\text{sep}})^*)$$

which is none other than Hilbert’s 90 theorem. Similarly, for $i \geq 1,$

$$0 = H^i(\text{Spec}(K), O) = H^i_{\text{étale}}(\text{Spec}(K)_{\text{étale}}, G_a) = H^i(G, K^{\text{sep}})$$

where the $K^{\text{sep}}$ indicates $K^{\text{sep}}$ as a Galois module with addition as group law. In this way we may consider the work we have done so far as a complicated way of computing Galois cohomology groups.

The following result is a curiosity and should be skipped on a first reading.

\textbf{Lemma 58.4.} Let $R$ be a local ring of dimension 0. Let $S = \text{Spec}(R)$. Then every $O_S$-module on $S_{\text{étale}}$ is quasi-coherent.

\textbf{Proof.} Let $F$ be an $O_S$-module on $S_{\text{étale}}$. We have to show that $F$ is determined by the $R$-module $M = \Gamma(S, F)$. More precisely, if $\pi : X \to S$ is étale we have to show that $\Gamma(X, F) = \Gamma(X, \pi^*M)$.

Let $m \subset R$ be the maximal ideal and let $\kappa$ be the residue field. By Algebra, Lemma 148.10 the local ring $R$ is henselian. If $X \to S$ is étale, then the underlying topological space of $X$ is discrete by Morphisms, Lemma 31.7 and hence $X$ is a disjoint union of affine schemes each having one point. Moreover, if $X = \text{Spec}(A)$
is affine and has one point, then $R \to A$ is finite étale by Algebra, Lemma \ref{lemma-affine-covering-finite-etale}. We have to show that $\Gamma(X, F) = M \otimes_R A$ in this case.

The functor $A \mapsto A/mA$ defines an equivalence of the category of finite étale $R$-algebras with the category of finite separable $\kappa$-algebras by Algebra, Lemma \ref{lemma-equivalence-finite-etale}. Let us first consider the case where $A/mA$ is a Galois extension of $\kappa$ with Galois group $G$. For each $\sigma \in G$ let $\sigma : A \to A$ denote the corresponding automorphism of $A$ over $R$. Let $N = \Gamma(X, F)$. Then $\text{Spec}(\sigma) : X \to X$ is an automorphism over $S$ and hence pullback by this defines a map $\sigma : N \to N$ which is a $\sigma$-linear map: $\sigma(an) = \sigma(a)\sigma(n)$ for $a \in A$ and $n \in N$. We will apply Galois descent to the quasi-coherent module $\tilde{N}$ on $X$ endowed with the isomorphisms coming from the action on $\sigma$ on $N$. See Descent, Lemma \ref{lemma-descent-module}. This lemma tells us there is an isomorphism $N = N^G \otimes_R A$. On the other hand, it is clear that $N^G = M$ by the sheaf property for $F$. Thus the required isomorphism holds.

The general case (with $A$ local and finite étale over $R$) is deduced from the Galois case as follows. Choose $A \to B$ finite étale such that $B$ is local with residue field Galois over $\kappa$. Let $G = \text{Aut}(B/R) = \text{Gal}(\kappa_B/\kappa)$. Let $H \subset G$ be the Galois group corresponding to the Galois extension $\kappa_B/\kappa_A$. Then as above one shows that $\Gamma(X, F) = \Gamma(\text{Spec}(B), F)^H$. By the result for Galois extensions (used twice) we get

$$\Gamma(X, F) = (M \otimes_R B)^H = M \otimes_R A$$

as desired. \hfill\Box

59. Cohomology of curves

03R0 The next task at hand is to compute the étale cohomology of a smooth curve over an algebraically closed field with torsion coefficients, and in particular show that it vanishes in degree at least 3. To prove this, we will compute cohomology at the generic point, which amounts to some Galois cohomology.

60. Brauer groups

03R1 Brauer groups of fields are defined using finite central simple algebras. In this section we review the relevant facts about Brauer groups, most of which are discussed in the chapter Brauer Groups, Section \ref{brauer-groups}. For other references, see [Ser62], [Ser97] or [Wei48].

03R2 \textbf{Theorem 60.1.} Let $K$ be a field. For a unital, associative (not necessarily commutative) $K$-algebra $A$ the following are equivalent

1. $A$ is finite central simple $K$-algebra,
2. $A$ is a finite dimensional $K$-vector space, $K$ is the center of $A$, and $A$ has no nontrivial two-sided ideal,
3. there exists $d \geq 1$ such that $A \otimes_K \bar{K} \cong \text{Mat}(d \times d, \bar{K})$,
4. there exists $d \geq 1$ such that $A \otimes_K K^\text{sep} \cong \text{Mat}(d \times d, K^\text{sep})$,
5. there exist $d \geq 1$ and a finite Galois extension $K \subset K'$ such that $A \otimes_K K' \cong \text{Mat}(d \times d, K')$,
6. there exist $n \geq 1$ and a finite central skew field $D$ over $K$ such that $A \cong \text{Mat}(n \times n, D)$.

The integer $d$ is called the degree of $A$.

\textbf{Proof.} This is a copy of Brauer Groups, Lemma \ref{brauer-groups}. \hfill\Box
Let $A$ be a finite central simple algebra over $K$. Then
\[ A \otimes_K A^{\text{opp}} \to \text{End}_K(A) \]
\[ a \otimes a' \mapsto (x \mapsto axa') \]
is an isomorphism of algebras over $K$.

**Proof.** See Brauer Groups, Lemma [4.10]

Two finite central simple algebras $A_1$ and $A_2$ over $K$ are called similar, or equivalent if there exist $m, n \geq 1$ such that $\text{Mat}(n \times n, A_1) \cong \text{Mat}(m \times m, A_2)$. We write $A_1 \sim A_2$.

By Brauer Groups, Lemma [5.1] this is an equivalence relation.

Let $K$ be a field. The Brauer group of $K$ is the set $\text{Br}(K)$ of similarity classes of finite central simple algebras over $K$, endowed with the group law induced by tensor product (over $K$). The class of $A$ in $\text{Br}(K)$ is denoted by $[A]$. The neutral element is $[K] = [\text{Mat}(d \times d, K)]$ for any $d \geq 1$.

The previous lemma implies that inverses exist and that $-[A] = [A^{\text{opp}}]$. The Brauer group of a field is always torsion. In fact, we will see that $[A]$ has order dividing $\deg(A)$ for any finite central simple algebra $A$ (see Lemma [61.2]). In general the Brauer group is not finitely generated, for example the Brauer group of a non-Archimedean local field is $\mathbb{Q}/\mathbb{Z}$. The Brauer group of $C(x,y)$ is uncountable.

Let $K$ be a field and let $K^{\text{sep}}$ be a separable algebraic closure. Then the set of isomorphism classes of central simple algebras of degree $d$ over $K$ is in bijection with the non-abelian cohomology $H^1(\text{Gal}(K^{\text{sep}}/K), \text{PGL}_d(K^{\text{sep}}))$.

**Sketch of proof.** The Skolem-Noether theorem (see Brauer Groups, Theorem [6.1]) implies that for any field $L$ the group $\text{Aut}_{L-\text{Algebras}}(\text{Mat}_d(L))$ equals $\text{PGL}_d(L)$. By Theorem [60.1] we see that central simple algebras of degree $d$ correspond to forms of the $K$-algebra $\text{Mat}_d(K)$. Combined we see that isomorphism classes of degree $d$ central simple algebras correspond to elements of $H^1(\text{Gal}(K^{\text{sep}}/K), \text{PGL}_d(K^{\text{sep}}))$.

For more details on twisting, see example [5186].

If $A$ is a finite central simple algebra of degree $d$ over a field $K$, we denote $\xi_A$ the corresponding cohomology class in $H^1(\text{Gal}(K^{\text{sep}}/K), \text{PGL}_d(K^{\text{sep}}))$. Consider the short exact sequence
\[ 1 \to (K^{\text{sep}})^* \to \text{GL}_d(K^{\text{sep}}) \to \text{PGL}_d(K^{\text{sep}}) \to 1, \]
which gives rise to a long exact cohomology sequence (up to degree 2) with coboundary map
\[ \delta_d : H^1(\text{Gal}(K^{\text{sep}}/K), \text{PGL}_d(K^{\text{sep}})) \to H^2(\text{Gal}(K^{\text{sep}}/K), (K^{\text{sep}})^*). \]

Explicitly, this is given as follows: if $\xi$ is a cohomology class represented by the 1-cocycle $(g_\sigma)$, then $\delta_d(\xi)$ is the class of the 2-cocycle
\[ (\sigma, \tau) \mapsto \tilde{g}_\sigma^{-1} \tilde{g}_{\sigma \tau} \xi(\tilde{g}_\tau)^{-1} \in (K^{\text{sep}})^* \]
where $\tilde{g}_\sigma \in \text{GL}_d(K^{\text{sep}})$ is a lift of $g_\sigma$. Using this we can make explicit the map
\[ \delta : \text{Br}(K) \to H^2(\text{Gal}(K^{\text{sep}}/K), (K^{\text{sep}})^*), \quad [A] \mapsto \delta_{\text{deg}, A}(\xi_A) \]
as follows. Assume $A$ has degree $d$ over $K$. Choose an isomorphism $\phi : \text{Mat}_d(K^{\text{sep}}) \to A \otimes_K K^{\text{sep}}$. For $\sigma \in \text{Gal}(K^{\text{sep}}/K)$ choose an element $\tilde{g}_\sigma \in \text{GL}_d(K^{\text{sep}})$ such that
Let $\varphi^{-1} \circ \sigma(\varphi)$ is equal to the map $x \mapsto \tilde{g}_x x \tilde{g}_x^{-1}$. The class in $H^2$ is defined by the two cocycle (60.5.1).

\begin{tcolorbox}[colback=blue!20]
**Theorem 60.6.** Let $K$ be a field with separable algebraic closure $K^{sep}$. The map 
$$
\delta : \mathrm{Br}(K) \to H^2(\mathrm{Gal}(K^{sep}/K), (K^{sep})^*)
$$
defined above is a group isomorphism.
\end{tcolorbox}

**Sketch of proof.** In the abelian case ($d = 1$), one has the identification
$$H^1(\mathrm{Gal}(K^{sep}/K), \mathrm{GL}_d(K^{sep})) = H^1_{\text{étale}}(\text{Spec}(K), \mathrm{GL}_d(\mathcal{O}))$$
the latter of which is trivial by fpqc descent. If this were true in the non-abelian case, this would readily imply injectivity of $\delta$. (See [Del77].) Rather, to prove this, one can reinterpret $\delta([A])$ as the obstruction to the existence of a $K$-vector space $V$ with a left $A$-module structure and such that $\dim_K V = \deg A$. In the case where $V$ exists, one has $A \cong \mathrm{End}_K(V)$. For surjectivity, pick a cohomology class $\xi \in H^2(\mathrm{Gal}(K^{sep}/K), (K^{sep})^*)$, then there exists a finite Galois extension $K \subset K' \subset K^{sep}$ such that $\xi$ is the image of some $\xi' \in H^2(\mathrm{Gal}(K'/K), (K')^*)$. Then write down an explicit central simple algebra over $K$ using the data $K', \xi'$.

\section{61. The Brauer group of a scheme}

0A2J Let $S$ be a scheme. An $\mathcal{O}_S$-algebra $A$ is called Azumaya if it is étale locally a matrix algebra, i.e., if there exists an étale covering $U = \{ \varphi_i : U_i \to S \}_{i \in I}$ such that $\varphi_i^* A \cong \mathrm{Mat}_{d_i}(\mathcal{O}_{U_i})$ for some $d_i \geq 1$. Two such $A$ and $B$ are called equivalent if there exist finite locally free $\mathcal{O}_S$-modules $\mathcal{F}$ and $\mathcal{G}$ which have positive rank at every $s \in S$ such that
$$A \otimes_{\mathcal{O}_S} \mathrm{Hom}_{\mathcal{O}_S}(\mathcal{F}, \mathcal{F}) \cong B \otimes_{\mathcal{O}_S} \mathrm{Hom}_{\mathcal{O}_S}(\mathcal{G}, \mathcal{G})$$
as $\mathcal{O}_S$-algebras. The Brauer group of $S$ is the set $\mathrm{Br}(S)$ of equivalence classes of Azumaya $\mathcal{O}_S$-algebras with the operation induced by tensor product (over $\mathcal{O}_S$).

0A2K **Lemma 61.1.** Let $S$ be a scheme. Let $\mathcal{F}$ and $\mathcal{G}$ be finite locally free sheaves of $\mathcal{O}_S$-modules of positive rank. If there exists an isomorphism $\mathrm{Hom}_{\mathcal{O}_S}(\mathcal{F}, \mathcal{F}) \cong \mathrm{Hom}_{\mathcal{O}_S}(\mathcal{G}, \mathcal{G})$ of $\mathcal{O}_S$-algebras, then there exists an invertible sheaf $\mathcal{L}$ on $S$ such that $\mathcal{F} \otimes_{\mathcal{O}_S} \mathcal{L} \cong \mathcal{G}$ and such that this isomorphism induces the given isomorphism of endomorphism algebras.

**Proof.** Fix an isomorphism $\mathrm{Hom}_{\mathcal{O}_S}(\mathcal{F}, \mathcal{F}) \to \mathrm{Hom}_{\mathcal{O}_S}(\mathcal{G}, \mathcal{G})$. Consider the sheaf $\mathcal{L} \subset \mathrm{Hom}(\mathcal{F}, \mathcal{G})$ generated as an $\mathcal{O}_S$-module by the local isomorphisms $\varphi : \mathcal{F} \to \mathcal{G}$ such that conjugation by $\varphi$ is the given isomorphism of endomorphism algebras. A local calculation (reducing to the case that $\mathcal{F}$ and $\mathcal{G}$ are finite free and $S$ is affine) shows that $\mathcal{L}$ is invertible. Another local calculation shows that the evaluation map
$$\mathcal{F} \otimes_{\mathcal{O}_S} \mathcal{L} \longrightarrow \mathcal{G}$$
is an isomorphism.

The argument given in the proof of the following lemma can be found in [Sal81].

0A2L **Lemma 61.2.** Let $S$ be a scheme. Let $A$ be an Azumaya algebra which is locally free of rank $d^2$ over $S$. Then the class of $A$ in the Brauer group of $S$ is annihilated by $d$. Argument taken from [Sal81].
Proof. Choose an étale covering \( \{ U_i \to S \} \) and choose isomorphisms \( A|_{U_i} \to \mathcal{H}om(F_i, F_i) \) for some locally free \( \mathcal{O}_{U_i} \)-modules \( F_i \) of rank \( d \). (We may assume \( F_i \) is free.) Consider the composition
\[
p_i : F_i^{\otimes d} \to \wedge^d F_i \to F_i^{\otimes d}
\]
The first arrow is the usual projection and the second arrow is the isomorphism of the top exterior power of \( F_i \) with the submodule of sections of \( F_i^{\otimes d} \) which transform according to the sign character under the action of the symmetric group on \( d \) letters. Then \( p_i^2 = d!p_i \) and the rank of \( p_i \) is 1. Using the given isomorphism \( A|_{U_i} \to \mathcal{H}om(F_i, F_i) \) and the canonical isomorphism
\[
\mathcal{H}om(F_i, F_i)^{\otimes d} = \mathcal{H}om(F_i^{\otimes d}, F_i^{\otimes d})
\]
we may think of \( p_i \) as sections of \( \mathcal{A}^{\otimes d} \) over \( U_i \). We claim that \( p_i|_{U_i \times s U_j} = p_j|_{U_i \times s U_j} \) as sections of \( \mathcal{A}^{\otimes d} \). Namely, applying Lemma 61.1 we obtain an invertible sheaf \( L_{ij} \) and a canonical isomorphism
\[
F_i|_{U_i \times s U_j} \otimes L_{ij} \to F_j|_{U_i \times s U_j}.
\]
Using this isomorphism we see that \( p_i \) maps to \( p_j \). Since \( \mathcal{A}^{\otimes d} \) is a sheaf on \( S_{\text{étale}} \) (Proposition 17.1) we find a canonical global section \( p \in \Gamma(S, \mathcal{A}^{\otimes d}) \). A local calculation shows that
\[
\mathcal{H} = \text{Im}(\mathcal{A}^{\otimes d} \to \mathcal{A}^{\otimes d}, f \mapsto fp)
\]
is a locally free module of rank \( d^d \) and that (left) multiplication by \( \mathcal{A}^{\otimes d} \) induces an isomorphism \( \mathcal{A}^{\otimes d} \to \mathcal{H}om(\mathcal{H}, \mathcal{H}) \). In other words, \( \mathcal{A}^{\otimes d} \) is the trivial element of the Brauer group of \( S \) as desired. \( \square \)

In this setting, the analogue of the isomorphism \( \delta \) of Theorem 60.6 is a map
\[
\delta_S : \text{Br}(S) \to H^2_{\text{étale}}(S, \mathbb{G}_m).
\]
It is true that \( \delta_S \) is injective. If \( S \) is quasi-compact or connected, then \( \text{Br}(S) \) is a torsion group, so in this case the image of \( \delta_S \) is contained in the cohomological Brauer group of \( S \)
\[
\text{Br}'(S) := H^2_{\text{étale}}(S, \mathbb{G}_m)_{\text{torsion}}.
\]
So if \( S \) is quasi-compact or connected, there is an inclusion \( \text{Br}(S) \subset \text{Br}'(S) \). This is not always an equality: there exists a nonseparated singular surface \( S \) for which \( \text{Br}(S) \subset \text{Br}'(S) \) is a strict inclusion. If \( S \) is quasi-projective, then \( \text{Br}(S) = \text{Br}'(S) \).

However, it is not known whether this holds for a smooth proper variety over \( \mathbb{C} \), say.

### 62. The Artin-Schreier sequence

0A3J Let \( p \) be a prime number. Let \( S \) be a scheme in characteristic \( p \). The Artin-Schreier sequence is the short exact sequence
\[
0 \to \mathbf{Z}/p\mathbf{Z}_S \to G_{a, S} \xrightarrow{F-1} G_{a, S} \to 0
\]
where \( F-1 \) is the map \( x \to x^p-x \).

0A3K \underline{Lemma 62.1.} Let \( p \) be a prime. Let \( S \) be a scheme of characteristic \( p \).

1. If \( S \) is affine, then \( H^q_{\text{étale}}(S, \mathbf{Z}/p\mathbf{Z}) = 0 \) for all \( q \geq 2 \).
2. If \( S \) is a quasi-compact and quasi-separated scheme of dimension \( d \), then \( H^q_{\text{étale}}(S, \mathbf{Z}/p\mathbf{Z}) = 0 \) for all \( q \geq 2 + d \).
Let \( \mathcal{O}_X \) be a separated scheme of finite type over a field \( k \). Let \( \mathcal{F} \) be a coherent sheaf of \( \mathcal{O}_X \)-modules. Then \( \dim_k H^d(X, \mathcal{F}) < \infty \) where \( d = \dim(X) \).

Proof. We will prove this by induction on \( d \). The case \( d = 0 \) holds because in that case \( X \) is the spectrum of a finite dimensional \( k \)-algebra \( A \) (Varieties, Lemma \([20.2]\)) and every coherent sheaf \( \mathcal{F} \) corresponds to a finite \( A \)-module \( M = H^0(X, \mathcal{F}) \) which has \( \dim_k M < \infty \).

Assume \( d > 0 \) and the result has been shown for separated schemes of finite type of dimension \( < d \). The scheme \( X \) is Noetherian. Consider the property \( \mathcal{P} \) of coherent sheaves on \( X \) defined by the rule

\[
\mathcal{P}(\mathcal{F}) \iff \dim_k H^d(X, \mathcal{F}) < \infty
\]
We are going to use the result of Cohomology of Schemes, Lemma \[12.4\] to prove that \( P \) holds for every coherent sheaf on \( X \).

Let 

\[ 0 \to F_1 \to F \to F_2 \to 0 \]

be a short exact sequence of coherent sheaves on \( X \). Consider the long exact sequence of cohomology

\[ H^d(X, F_1) \to H^d(X, F) \to H^d(X, F_2) \]

Thus if \( P \) holds for \( F_1 \) and \( F_2 \), then it holds for \( F \).

Let \( Z \subset X \) be an integral closed subscheme. Let \( I \) be a coherent sheaf of ideals on \( Z \). To finish the proof have to show that \( H^d(X, i^*O_Z) = H^d(Z, O_Z) \) is finite dimensional.

If \( \dim(Z) < d \), then the result holds because the cohomology group will be zero (Cohomology, Proposition \[21.7\]). In this way we reduce to the situation discussed in the following paragraph.

Assume \( X \) is a variety of dimension \( d \) and \( F = I \) is a coherent ideal sheaf. In this case we have a short exact sequence

\[ 0 \to I \to O_X \to i^*O_Z \to 0 \]

where \( i : Z \to X \) is the closed subscheme defined by \( I \). By induction hypothesis we see that \( H^{d-1}(Z, O_Z) = H^{d-1}(X, i^*O_Z) \) is finite dimensional. Thus we see that it suffices to prove the result for the structure sheaf.

We can apply Chow’s lemma (Cohomology of Schemes, Lemma \[18.1\]) to the morphism \( X \to \text{Spec}(k) \). Thus we get a diagram

\[
\begin{array}{ccc}
X & \xrightarrow{\pi} & X' \\
\downarrow{g} & & \downarrow{g'} \\
\text{Spec}(k) & & \\
\end{array}
\]

as in the statement of Chow’s lemma. Also, let \( U \subset X \) be the dense open subscheme such that \( \pi^{-1}(U) \to U \) is an isomorphism. We may assume \( X' \) is a variety as well, see Cohomology of Schemes, Remark \[18.2\]. The morphism \( i' = (i, \pi) : X' \to \mathbb{P}^n_k \) is a closed immersion (loc. cit.). Hence

\[ L = i^*O_{\mathbb{P}^n_k}(1) \cong (i')^*O_{\mathbb{P}^n_X}(1) \]

is \( \pi \)-relatively ample (for example by Morphisms, Lemma \[37.1\]). Hence by Cohomology of Schemes, Lemma \[16.2\] there exists an \( n \geq 0 \) such that \( R^p\pi_*O_{\mathbb{P}^n} = 0 \) for all \( p > 0 \). Set \( \mathcal{G} = \pi_*O_{\mathbb{P}^n} \). Choose any nonzero global section \( s \) of \( \mathcal{L}^\otimes n \). Since \( \mathcal{G} = \pi_*O_{\mathbb{P}^n} \), the section \( s \) corresponds to section of \( \mathcal{G} \), i.e., a map \( O_X \to \mathcal{G} \). Since \( s|_U \neq 0 \) as \( X' \) is a variety and \( \mathcal{L} \) invertible, we see that \( O_X|_U \to \mathcal{G}|_U \) is nonzero. As \( \mathcal{G}|_U = \mathcal{K} \mathcal{L}^\otimes n|_{\pi^{-1}(U)} \) is invertible we conclude that we have a short exact sequence

\[ 0 \to O_X \to \mathcal{G} \to \mathcal{Q} \to 0 \]

where \( \mathcal{Q} \) is coherent and supported on a proper closed subscheme of \( X \). Arguing as before using our induction hypothesis, we see that it suffices to prove \( \dim H^d(X, \mathcal{G}) < \infty \).

By the Leray spectral sequence (Cohomology, Lemma \[14.6\]) we see that \( H^d(X, \mathcal{G}) = H^d(X', \mathcal{L}^\otimes n) \). Let \( \overline{X}' \subset \mathbb{P}^n_k \) be the closure of \( X' \). Then \( \overline{X}' \) is a projective variety.
of dimension $d$ over $k$ and $X' \subset X$ is a dense open. The invertible sheaf $\mathcal{L}$ is the restriction of $\mathcal{O}_X(n)$ to $X$. By Cohomology, Proposition \ref{coh:23.4} the map

$$H^d(X', \mathcal{O}_X(n)) \rightarrow H^d(X', \mathcal{L}^\otimes n)$$

is surjective. Since the cohomology group on the left has finite dimension by Cohomology of Schemes, Lemma \ref{coh:14.1} the proof is complete. \hfill \Box

\textbf{0A3N} \textbf{Lemma 62.4.} Let $X$ be separated of finite type over an algebraically closed field $k$ of characteristic $p > 0$. Then $H^q_{\text{étale}}(X, \mathbb{Z}/p\mathbb{Z}) = 0$ for $q \geq \dim(X) + 1$.

\textbf{Proof.} Let $d = \dim(X)$. By the vanishing established in Lemma \ref{coh:62.1} it suffices to show that $H^{d+1}_{\text{étale}}(X, \mathbb{Z}/p\mathbb{Z}) = 0$. By Lemma \ref{coh:62.3} we see that $H^d(X, \mathcal{O}_X)$ is a finite dimensional $k$-vector space. Hence the long exact cohomology sequence associated to the Artin-Schreier sequence ends with

$$H^d(X, \mathcal{O}_X) \xrightarrow{F - 1} H^d(X, \mathcal{O}_X) \rightarrow H^{d+1}_{\text{étale}}(X, \mathbb{Z}/p\mathbb{Z}) \rightarrow 0$$

By Lemma \ref{coh:62.2} the map $F - 1$ in this sequence is surjective. This proves the lemma. \hfill \Box

\textbf{0A3P} \textbf{Lemma 62.5.} Let $X$ be a proper scheme over an algebraically closed field $k$ of characteristic $p > 0$. Then

1. $H^q_{\text{étale}}(X, \mathbb{Z}/p\mathbb{Z})$ is a finite $\mathbb{Z}/p\mathbb{Z}$-module for all $q$, and
2. $H^q_{\text{étale}}(X, \mathcal{O}_X) \rightarrow H^q_{\text{étale}}(X_{k'}, \mathcal{O}_X)$ is an isomorphism if $k \subset k'$ is an extension of algebraically closed fields.

\textbf{Proof.} By Cohomology of Schemes, Lemma \ref{coh:19.2} and the comparison of cohomology of Theorem \ref{coh:22.4} the cohomology groups $H^q_{\text{étale}}(X, \mathbb{G}_a) = H^q(X, \mathcal{O}_X)$ are finite dimensional $k$-vector spaces. Hence by Lemma \ref{coh:62.2} the long exact cohomology sequence associated to the Artin-Schreier sequence, splits into short exact sequences

$$0 \rightarrow H^q_{\text{étale}}(X, \mathbb{Z}/p\mathbb{Z}) \rightarrow H^q(X, \mathcal{O}_X) \xrightarrow{F - 1} H^q(X, \mathcal{O}_X) \rightarrow 0$$

and moreover the $\mathbb{F}_p$-dimension of the cohomology groups $H^q_{\text{étale}}(X, \mathbb{Z}/p\mathbb{Z})$ is equal to the $k$-dimension of the vector space $H^q(X, \mathcal{O}_X)$. This proves the first statement. The second statement follows as $H^q(X, \mathcal{O}_X) \otimes_k k' \rightarrow H^q(X_{k'}, \mathcal{O}_{X_{k'}})$ is an isomorphism by flat base change (Cohomology of Schemes, Lemma \ref{coh:5.2}). \hfill \Box

\section{Locally constant sheaves}

\textbf{09Y8} This section is the analogue of Modules on Sites, Section \ref{modules:42} for the étale site.

\textbf{Definition 63.1.} Let $X$ be a scheme. Let $\mathcal{F}$ be a sheaf of sets on $X_{\text{étale}}$.

1. Let $E$ be a set. We say $\mathcal{F}$ is the constant sheaf with value $E$ if $\mathcal{F}$ is the sheafification of the presheaf $U \mapsto E$. Notation: $\underline{E}_X$ or $\underline{E}$.
2. We say $\mathcal{F}$ is a constant sheaf if it is isomorphic to a sheaf as in (1).
3. We say $\mathcal{F}$ is locally constant if there exists a covering $\{U_i \to X\}$ such that $\mathcal{F}|_{U_i}$ is a constant sheaf.
4. We say that $\mathcal{F}$ is finite locally constant if it is locally constant and the values are finite sets.

Let $\mathcal{F}$ be a sheaf of abelian groups on $X_{\text{étale}}$. 

(1) Let $A$ be an abelian group. We say $\mathcal{F}$ is the \textit{constant sheaf with value $A$} if $\mathcal{F}$ is the sheafification of the presheaf $U \mapsto A$. Notation: $\mathcal{A}_X$ or $A$.

(2) We say $\mathcal{F}$ is a \textit{constant sheaf} if it is isomorphic as an abelian sheaf to a sheaf as in (1).

(3) We say $\mathcal{F}$ is \textit{locally constant} if there exists a covering $\{U_i \to X\}$ such that $\mathcal{F}|_{U_i}$ is a constant sheaf.

(4) We say that $\mathcal{F}$ is \textit{finite locally constant} if it is locally constant and the values are finite abelian groups.

Let $\Lambda$ be a ring. Let $\mathcal{F}$ be a sheaf of $\Lambda$-modules on $X$.\textit{étale}.

(1) Let $M$ be a $\Lambda$-module. We say $\mathcal{F}$ is the \textit{constant sheaf with value $M$} if $\mathcal{F}$ is the sheafification of the presheaf $U \mapsto M$. Notation: $\mathcal{M}_X$ or $M$.

(2) We say $\mathcal{F}$ is a \textit{constant sheaf} if it is isomorphic as a sheaf of $\Lambda$-modules to a sheaf as in (1).

(3) We say $\mathcal{F}$ is \textit{locally constant} if there exists a covering $\{U_i \to X\}$ such that $\mathcal{F}|_{U_i}$ is a constant sheaf.

**Lemma 63.2.** Let $f : X \to Y$ be a morphism of schemes. If $\mathcal{G}$ is a locally constant sheaf of sets, abelian groups, or $\Lambda$-modules on $Y$\textit{étale}, the same is true for $f^{-1}\mathcal{G}$ on $X$\textit{étale}.

**Proof.** Holds for any morphism of topoi, see Modules on Sites, Lemma \ref{modules-on-sites-lemma}.

**Lemma 63.3.** Let $f : X \to Y$ be a finite étale morphism of schemes. If $\mathcal{F}$ is a (finite) locally constant sheaf of sets, (finite) locally constant sheaf of abelian groups, or (finite type) locally constant sheaf of $\Lambda$-modules on $X$\textit{étale}, the same is true for $f_*\mathcal{F}$ on $Y$\textit{étale}.

**Proof.** The construction of $f_*$ commutes with étale localization. A finite étale morphism is locally isomorphic to a disjoint union of isomorphisms, see Étale Morphisms, Lemma \ref{étale-morphisms-lemma}. Thus the lemma says that if $\mathcal{F}_i$, $i = 1, \ldots, n$ are (finite) locally constant sheaves of sets, then $\prod_{i=1,\ldots,n} \mathcal{F}_i$ is too. This is clear. Similarly for sheaves of abelian groups and modules.

**Lemma 63.4.** Let $X$ be a scheme and $\mathcal{F}$ a sheaf of sets on $X$\textit{étale}. Then the following are equivalent

(1) $\mathcal{F}$ is finite locally constant, and

(2) $\mathcal{F} = h_U$ for some finite étale morphism $U \to X$.

**Proof.** A finite étale morphism is locally isomorphic to a disjoint union of isomorphisms, see Étale Morphisms, Lemma \ref{étale-morphisms-lemma}. Thus (2) implies (1). Conversely, if $\mathcal{F}$ is finite locally constant, then there exists an étale covering $\{X_i \to X\}$ such that $\mathcal{F}|_{X_i}$ is representable by $U_i \to X_i$ finite étale. Arguing exactly as in the proof of Descent, Lemma \ref{descent-lemma} we obtain a descent datum for schemes $(U_i, \varphi_{ij})$ relative to $\{X_i \to X\}$ (details omitted). This descent datum is effective for example by Descent, Lemma \ref{descent-lemma-2} and the resulting morphism of schemes $U \to X$ is finite étale by Descent, Lemmas \ref{descent-lemma-2} and \ref{descent-lemma-29}.

**Lemma 63.5.** Let $X$ be a scheme.

(1) Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a map of locally constant sheaves of sets on $X$\textit{étale}. If $\mathcal{F}$ is finite locally constant, there exists an étale covering $\{U_i \to X\}$ such that $\varphi|_{U_i}$ is the map of constant sheaves associated to a map of sets.
(2) Let \( \varphi : F \rightarrow G \) be a map of locally constant sheaves of abelian groups on \( X_{\text{étale}} \). If \( F \) is finite locally constant, there exists an étale covering \( \{U_i \rightarrow X\} \) such that \( \varphi|_{U_i} \) is the map of constant abelian sheaves associated to a map of abelian groups.

(3) Let \( \Lambda \) be a ring. Let \( \varphi : F \rightarrow G \) be a map of locally constant sheaves of \( \Lambda \)-modules on \( X_{\text{étale}} \). If \( F \) is of finite type, then there exists an étale covering \( \{U_i \rightarrow X\} \) such that \( \varphi|_{U_i} \) is the map of constant sheaves of \( \Lambda \)-modules associated to a map of \( \Lambda \)-modules.

Proof. This holds on any site, see Modules on Sites, Lemma 42.3.

03RX Lemma 63.6. Let \( X \) be a scheme.

(1) The category of finite locally constant sheaves of sets is closed under finite limits and colimits inside \( \text{Sh}(X_{\text{étale}}) \).

(2) The category of finite locally constant abelian sheaves is a weak Serre subcategory of \( \text{Ab}(X_{\text{étale}}) \).

(3) Let \( \Lambda \) be a Noetherian ring. The category of finite type, locally constant sheaves of \( \Lambda \)-modules on \( X_{\text{étale}} \) is a weak Serre subcategory of \( \text{Mod}(X_{\text{étale}}, \Lambda) \).

Proof. This holds on any site, see Modules on Sites, Lemma 42.5.

095D Lemma 63.7. Let \( X \) be a scheme. Let \( \Lambda \) be a ring. The tensor product of two locally constant sheaves of \( \Lambda \)-modules on \( X_{\text{étale}} \) is a locally constant sheaf of \( \Lambda \)-modules.

Proof. This holds on any site, see Modules on Sites, Lemma 42.6.

09BF Lemma 63.8. Let \( X \) be a connected scheme. Let \( \Lambda \) be a ring and let \( F \) be a locally constant sheaf of \( \Lambda \)-modules. Then there exists a \( \Lambda \)-module \( M \) and an étale covering \( \{U_i \rightarrow X\} \) such that \( F|_{U_i} \cong M|_{U_i} \).

Proof. Choose an étale covering \( \{U_i \rightarrow X\} \) such that \( F|_{U_i} \) is constant, say \( F|_{U_i} \cong M_i|_{U_i} \). Observe that \( U_i \times_X U_j \) is empty if \( M_i \) is not isomorphic to \( M_j \). For each \( \Lambda \)-module \( M \) let \( I_M = \{i \in I \mid M_i \cong M\} \). As étale morphisms are open we see that \( U_M = \bigcup_{i \in I_M} \text{Im}(U_i \rightarrow X) \) is an open subset of \( X \). Then \( X = \bigsqcup U_M \) is a disjoint open covering of \( X \). As \( X \) is connected only one \( U_M \) is nonempty and the lemma follows.

64. Locally constant sheaves and the fundamental group

0DV4 We can relate locally constant sheaves to the fundamental group of a scheme in some cases.

0DV5 Lemma 64.1. Let \( X \) be a connected scheme. Let \( \pi \) be a geometric point of \( X \).

(1) There is an equivalence of categories

\[
\begin{array}{c}
\text{finite locally constant sheaves of sets on } X_{\text{étale}} \\
\leftrightarrow \{ \text{finite } \pi_1(X, \pi)-\text{sets} \}
\end{array}
\]

(2) There is an equivalence of categories

\[
\begin{array}{c}
\text{finite locally constant sheaves of abelian groups on } X_{\text{étale}} \\
\leftrightarrow \{ \text{finite } \pi_1(X, \pi)\text{-modules} \}
\end{array}
\]
Let $\Lambda$ be a finite ring. There is an equivalence of categories
\[
\text{finite type, locally constant sheaves of $\Lambda$-modules on } X_{\text{étale}} \leftrightarrow \text{finite } \pi_1(X, \pi)\text{-modules endowed with commuting $\Lambda$-module structure}
\]

**Proof.** We observe that $\pi_1(X, \pi)$ is a profinite topological group, see Fundamental Groups, Definition 6.1. The left hand categories are defined in Section 63. The notation used in the right hand categories is taken from Fundamental Groups, Definition 2.1 for sets and Definition 56.1 for abelian groups. This explains the notation.

Assertion (1) follows from Lemma 63.4 and Fundamental Groups, Theorem 6.2. Parts (2) and (3) follow immediately from this by endowing the underlying (sheaves of) sets with additional structure. For example, a finite locally constant sheaf of abelian groups on $X_{\text{étale}}$ is the same thing as a finite locally constant sheaf of sets $F$ together with a map $+: F \times F \to F$ satisfying the usual axioms. The equivalence in (1) sends products to products and hence sends $+$ to an addition on the corresponding finite $\pi_1(X, \pi)$-set. Since $\pi_1(X, \pi)$-modules are the same thing as $\pi_1(X, \pi)$-sets with a compatible abelian group structure we obtain (2). Part (3) is proved in exactly the same way. \[ \Box \]

**Remark 64.2.** The equivalences of Lemma 64.1 are compatible with pullbacks. More precisely, suppose $f : Y \to X$ is a morphism of connected schemes. Let $\overline{y}$ be geometric point of $Y$ and set $x = f(\overline{y})$. Then the diagram
\[
\begin{array}{ccc}
\text{finite locally constant sheaves of sets on } Y_{\text{étale}} & \longrightarrow & \text{finite } \pi_1(Y, \overline{y})\text{-sets} \\
& \downarrow f^{-1} & \\
\text{finite locally constant sheaves of sets on } X_{\text{étale}} & \longrightarrow & \text{finite } \pi_1(X, x)\text{-sets}
\end{array}
\]
is commutative, where the vertical arrow on the right comes from the continuous homomorphism $\pi_1(Y, \overline{y}) \to \pi_1(X, \pi)$ induced by $f$. This follows immediately from the commutative diagram in Fundamental Groups, Theorem 6.2.

65. Méthode de la trace

A reference for this section is [AGV71, Exposé IX, §5]. The material here will be used in the proof of Lemma 78.8 below.

Let $f : Y \to X$ be an étale morphism of schemes. There is a sequence
\[
f_!, f^{-1}, f_*
\]
of adjoint functors between $\text{Ab}(X_{\text{étale}})$ and $\text{Ab}(Y_{\text{étale}})$. The functor $f_!$ is discussed in Section 69. The adjunction map $\text{id} \to f_*f^{-1}$ is called restriction. The adjunction map $f_!f^{-1} \to \text{id}$ is often called the trace map. If $f$ is finite étale, then $f_* = f_!$ (Lemma 69.5) and we can view this as a map $f_*f^{-1} \to \text{id}$.

**Definition 65.1.** Let $f : Y \to X$ be a finite étale morphism of schemes. The map $f_*f^{-1} \to \text{id}$ described above and below is called the trace.

Let $f : Y \to X$ be a finite étale morphism of schemes. The trace map is characterized by the following two properties:

1. It commutes with étale localization on $X$ and
2. If $Y = \coprod_{i=1}^d X$ then the trace map is the sum map $f_*f^{-1}F = F^{\oplus d} \to F$.  

By Étale Morphisms, Lemma 18.3 every finite étale morphism \( f : Y \to X \) is étale locally on \( X \) of the form given in (2) for some integer \( d \geq 0 \). Hence we can define the trace map using the characterization given; in particular we do not need to know about the existence of \( f_! \) and the agreement of \( f_! \) with \( f_* \) in order to construct the trace map. This description shows that if \( f \) has constant degree \( d \), then the composition

\[
\mathcal{F} \xrightarrow{res} f_* f^{-1} \mathcal{F} \xrightarrow{\text{trace}} \mathcal{F}
\]

is multiplication by \( d \). The “méthode de la trace” is the following observation: if \( \mathcal{F} \) is an abelian sheaf on \( X_{\text{étale}} \) such that multiplication by \( d \) on \( \mathcal{F} \) is an isomorphism, then the map

\[
H^n_{\text{étale}}(X, \mathcal{F}) \to H^n_{\text{étale}}(Y, f^{-1} \mathcal{F})
\]

is injective. Namely, we have

\[
H^n_{\text{étale}}(Y, f^{-1} \mathcal{F}) = H^n_{\text{étale}}(X, f_* f^{-1} \mathcal{F})
\]

by the vanishing of the higher direct images (Proposition 64.2 and the Leray spectral sequence (Proposition 53.2). Thus we can consider the maps

\[
H^n_{\text{étale}}(X, \mathcal{F}) \to H^n_{\text{étale}}(Y, f^{-1} \mathcal{F}) = H^n_{\text{étale}}(X, f_* f^{-1} \mathcal{F}) \xrightarrow{\text{trace}} H^n_{\text{étale}}(X, \mathcal{F})
\]

and the composition is an isomorphism (under our assumption on \( \mathcal{F} \) and \( f \)). In particular, if \( H^n_{\text{étale}}(Y, f^{-1} \mathcal{F}) = 0 \) then \( H^n_{\text{étale}}(X, \mathcal{F}) = 0 \) as well. Indeed, multiplication by \( d \) induces an isomorphism on \( H^n_{\text{étale}}(X, \mathcal{F}) \) which factors through \( H^n_{\text{étale}}(Y, f^{-1} \mathcal{F}) = 0 \).

This is often combined with the following.

**Lemma 65.2.** Let \( S \) be a connected scheme. Let \( \ell \) be a prime number. Let \( \mathcal{F} \) a finite type, locally constant sheaf of \( \mathbf{F}_\ell \)-vector spaces on \( S_{\text{étale}} \). Then there exists a finite étale morphism \( f : T \to S \) of degree prime to \( \ell \) such that \( f^{-1} \mathcal{F} \) has a finite filtration whose successive quotients are \( \mathbf{Z}/\ell \mathbf{Z}_\ell \).

**Proof.** Choose a geometric point \( \overline{s} \) of \( S \). Via the equivalence of Lemma 64.1 the sheaf \( \mathcal{F} \) corresponds to a finite dimensional \( \mathbf{F}_\ell \)-vector space \( V \) with a continuous \( \pi_1(S, \overline{s}) \)-action. Let \( G \subset \text{Aut}(V) \) be the image of the homomorphism \( \rho : \pi_1(S, \overline{s}) \to \text{Aut}(V) \) giving the action. Observe that \( G \) is finite. The surjective continuous homomorphism \( \overline{\rho} : \pi_1(S, \overline{s}) \to G \) corresponds to a Galois object \( Y \to S \) of \( F\text{Ét}_S \) with automorphism group \( G = \text{Aut}(Y/S) \), see Fundamental Groups, Section 7. Let \( H \subset G \) be an \( \ell \)-Sylow subgroup. We claim that \( T = Y/H \to S \) works. Namely, let \( \overline{t} \in T \) be a geometric point over \( \overline{s} \). The image of \( \pi_1(T, \overline{t}) \to \pi_1(S, \overline{s}) \) is \( (\overline{\rho})^{-1}(H) \) as follows from the functorial nature of fundamental groups. Hence the action of \( \pi_1(T, \overline{t}) \) on \( V \) corresponding to \( f^{-1} \mathcal{F} \) is through the map \( \pi_1(T, \overline{t}) \to H \), see Remark 64.2. As \( H \) is a finite \( \ell \)-group, the irreducible constituents of the representation \( \rho|_{\pi_1(T, \overline{t})} \) are each trivial of rank 1 (this is a simple lemma on representation theory of finite groups; insert future reference here). Via the equivalence of Lemma 64.1 this means \( f^{-1} \mathcal{F} \) is a successive extension of constant sheaves with value \( \mathbf{Z}/\ell \mathbf{Z}_\ell \).

Moreover the degree of \( T = Y/H \to S \) is prime to \( \ell \) as it is equal to the index of \( H \) in \( G \). \( \square \)
66. Galois cohomology

In this section we prove a result on Galois cohomology (Proposition 66.4) using étale cohomology and the trick from Section 65. This will allow us to prove vanishing of higher étale cohomology groups over the spectrum of a field.

Lemma 66.1. Let ℓ be a prime number and n an integer > 0. Let S be a quasi-compact and quasi-separated scheme. Let \( X = \lim_{i \in I} X_i \) be the limit of a directed system of S-schemes each \( X_i \to S \) being finite étale of constant degree relatively prime to ℓ. For any abelian ℓ-power torsion sheaf \( G \) on \( S \) such that \( \text{H}^n_{\text{étale}}(S, G) \neq 0 \) there exists an ℓ-power torsion sheaf \( \mathcal{F} \) on \( X \) such that \( \text{H}^n_{\text{étale}}(X, \mathcal{F}) \neq 0 \).

Proof. Let \( g : X \to S \) and \( g_i : X_i \to S \) denote the structure morphisms. Fix an ℓ-power torsion sheaf \( G \) on \( S \) with \( \text{H}^n_{\text{étale}}(S, G) \neq 0 \). The system given by \( G_i = g_i^{-1}G \) satisfies the conditions of Theorem 51.3 with colimit sheaf given by \( g^{-1}G \). This tells us that:

\[
\text{colim}_{i \in I} \text{H}^p_{\text{étale}}(X_i, g_i^{-1}G) = \text{H}^p_{\text{étale}}(X, G)
\]

By virtue of the \( g_i \) being finite étale morphism of degree prime to ℓ we can apply “la méthode de la trace” and we find the maps

\[
\text{H}^p_{\text{étale}}(S, G) \to \text{H}^p_{\text{étale}}(X, g_i^{-1}G)
\]

are all injective (and compatible with the transition maps). See Section 65. Thus, the colimit is non-zero, i.e., \( \text{H}^n(X, g^{-1}G) \neq 0 \), giving us the desired result with \( \mathcal{F} = g^{-1}G \).

Conversely, suppose given an ℓ-power torsion sheaf \( \mathcal{F} \) on \( X \) with \( \text{H}^n_{\text{étale}}(X, \mathcal{F}) \neq 0 \). We note that since the \( g_i \) are finite morphisms the higher direct images vanish (Proposition 54.2). Then, by applying Lemma 51.7 we may also conclude the same for \( g \). The vanishing of the higher direct images tells us that \( \text{H}^n_{\text{étale}}(X, \mathcal{F}) = \text{H}^n(S, g_*\mathcal{F}) \neq 0 \) by Leray (Proposition 53.2) giving us what we want with \( \mathcal{G} = g_\mathcal{F} \).

Lemma 66.2. Let ℓ be a prime number and n an integer > 0. Let \( K \) be a field with \( G = \text{Gal}(K^{\text{sep}}/K) \) and let \( H \subset G \) be a maximal pro-ℓ subgroup with \( L/K \) being the corresponding field extension. Then \( \text{H}^n_{\text{étale}}(\text{Spec}(K), \mathcal{F}) = 0 \) for all ℓ-power torsion \( \mathcal{F} \) if and only if \( \text{H}^n_{\text{étale}}(\text{Spec}(L), \mathbb{Z}/\ell\mathbb{Z}) = 0 \).

Proof. Write \( L = \bigcup L_i \) as the union of its finite subextensions over \( K \). Our choice of \( H \) implies that \( [L_i : K] \) is prime to ℓ. Thus \( \text{Spec}(L) = \lim_{i \in I} \text{Spec}(L_i) \) as in Lemma 66.1. Thus we may replace \( K \) by \( L \) and assume that the absolute Galois group \( G \) of \( K \) is a profinite pro-ℓ group.

Assume \( \text{H}^n(\text{Spec}(K), \mathbb{Z}/\ell\mathbb{Z}) = 0 \). Let \( \mathcal{F} \) be an ℓ-power torsion sheaf on \( \text{Spec}(K)_{\text{étale}} \). We will show that \( \text{H}^n_{\text{étale}}(\text{Spec}(K), \mathcal{F}) = 0 \). By the correspondence specified in Lemma 58.1 our sheaf \( \mathcal{F} \) corresponds to an ℓ-power torsion \( G \)-module \( M \). Any finite set of elements \( x_1, \ldots, x_m \in M \) must be fixed by an open subgroup \( U \) by continuity. Let \( M' \) be the module spanned by the orbits of \( x_1, \ldots, x_m \). This is a finite abelian ℓ-group as each \( x_i \) is killed by a power of ℓ and the orbits are finite. Since \( M \) is the filtered colimit of these submodules \( M' \), we see that \( \mathcal{F} \) is the filtered colimit of the corresponding subsheaves \( \mathcal{F}' \subset \mathcal{F} \). Applying Theorem 51.3 to this colimit, we reduce to the case where \( \mathcal{F} \) is a finite locally constant sheaf.
Let $M$ be a finite abelian $\ell$-group with a continuous action of the profinite pro-$\ell$ group $G$. Then there is a $G$-invariant filtration

$$0 = M_0 \subset M_1 \subset \ldots \subset M_r = M$$

such that $M_{i+1}/M_i \cong \mathbb{Z}/\ell\mathbb{Z}$ with trivial $G$-action (this is a simple lemma on representation theory of finite groups; insert future reference here). Thus the corresponding sheaf $\mathcal{F}$ has a filtration

$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \ldots \subset \mathcal{F}_r = \mathcal{F}$$

with successive quotients isomorphic to $\mathbb{Z}/\ell\mathbb{Z}$. Thus by induction and the long exact cohomology sequence we conclude. □

0DV9 Lemma 66.3. Let $\ell$ be a prime number and $n$ an integer $> 0$. Let $K$ be a field with $G = \text{Gal}(K^{\text{sep}}/K)$ and let $H \subset G$ be a maximal pro-$\ell$ subgroup with $L/K$ being the corresponding field extension. Then $H^q_{\text{étale}}(\text{Spec}(K), \mathcal{F}) = 0$ for $q \geq n$ and all $\ell$-torsion sheaves $\mathcal{F}$ if and only if $H^q_{\text{étale}}(\text{Spec}(L), \mathbb{Z}/\ell\mathbb{Z}) = 0$.

Proof. The forward direction is trivial, so we need only prove the reverse direction. We proceed by induction on $q$. The case of $q = n$ is Lemma 66.2. Now let $\mathcal{F}$ be an $\ell$-power torsion sheaf on $\text{Spec}(K)$. Let $f : \text{Spec}(K^{\text{sep}}) \to \text{Spec}(K)$ be the inclusion of a geometric point. Then consider the exact sequence:

$$0 \to \mathcal{F} \xrightarrow{f_*} f_*f^{-1}\mathcal{F} \to f_*f^{-1}\mathcal{F}/\mathcal{F} \to 0$$

Note that $K^{\text{sep}}$ may be written as the filtered colimit of finite separable extensions. Thus $f$ is the limit of a directed system of finite étale morphisms. We may, as was seen in the proof of Lemma 66.1, conclude that $f$ has vanishing higher direct images. Thus, we may express the higher cohomology of $f_*f^{-1}\mathcal{F}$ as the higher cohomology on the geometric point which clearly vanishes. Hence, as everything here is still $\ell$-torsion, we may use the inductive hypothesis in conjunction with the long-exact cohomology sequence to conclude the result for $q + 1$. □

03R8 Proposition 66.4. Let $K$ be a field with separable algebraic closure $K^{\text{sep}}$. Assume that for any finite extension $K'$ of $K$ we have $\text{Br}(K') = 0$. Then

1. $H^q(\text{Gal}(K^{\text{sep}}/K), (K^{\text{sep}})^*) = 0$ for all $q \geq 1$, and
2. $H^q(\text{Gal}(K^{\text{sep}}/K), M) = 0$ for any torsion $\text{Gal}(K^{\text{sep}}/K)$-module $M$ and any $q \geq 2$.

Proof. Set $p = \text{char}(K)$. By Lemma 58.2, Theorem 60.6, and Example 58.3 the proposition is equivalent to showing that if $H^2(\text{Spec}(K'), \mathbb{G}_m|_{\text{Spec}(K')_{\text{étale}}}) = 0$ for all finite extensions $K'/K$ then:

- $H^q(\text{Spec}(K), \mathbb{G}_m|_{\text{Spec}(K)_{\text{étale}}}) = 0$ for all $q \geq 1$, and
- $H^q(\text{Spec}(K), \mathcal{F}) = 0$ for any torsion sheaf $\mathcal{F}$ and any $q \geq 2$.

We prove the second part first. Since $\mathcal{F}$ is a torsion sheaf, we may use the $\ell$-primary decomposition as well as the compatibility of cohomology with colimits (i.e., direct sums, see Theorem 51.3) to reduce to showing $H^q(\text{Spec}(K), \mathcal{F}) = 0$, $q \geq 2$ for all $\ell$-power torsion sheaves for every prime $\ell$. This allows us to analyze each prime individually.

Suppose that $\ell \neq p$. For any extension $K'/K$ consider the Kummer sequence (Lemma 28.1)

$$0 \to \mu_{\ell, \text{Spec } K'} \to \mathbb{G}_m|_{\text{Spec } K'} \xrightarrow{\ell \cdot f} \mathbb{G}_m|_{\text{Spec } K'} \to 0$$

Ser97 Chapter II, Section 3, Proposition 5]
Since $H^q(\Spec K', G_m|_{\Spec(K')_{etale}}) = 0$ for $q = 2$ by assumption and for $q = 1$ by Theorem 21.41 combined with Pic($K$) = {0}. Thus, by the long-exact cohomology sequence we may conclude that $H^2(\Spec K', \mu_p) = 0$ for any separable $K'/K$. Now let $H$ be a maximal pro-$\ell$ subgroup of the absolute Galois group of $K$ and let $L$ be the corresponding extension. We can write $L$ as the colimit of finite extensions, applying Theorem 51.3 to this colimit we see that $H^2(\Spec(L), \mu_\ell) = 0$. Now $\mu_\ell$ must be the constant sheaf. If it weren’t, that would imply there exists a Galois extension of degree relatively prime to $\ell$ of $L$ which is not true by definition of $L$ (namely, the extension one gets by adjoining the $\ell$th roots of unity to $L$). Hence, via Lemma 66.3 we conclude the result for $\ell \neq p$.

Now suppose that $\ell = p$. We consider the Artin-Schrier exact sequence (Section 62)

$$0 \to \mathbb{Z}/p\mathbb{Z}_{\Spec K} \to G_{a,\Spec K} \xrightarrow{F-1} G_{a,\Spec K} \to 0$$

where $F-1$ is the map $x \mapsto x^p - x$. Then note that the higher Cohomology of $G_{a,\Spec K}$ vanishes, by Remark 23.4 and the vanishing of the higher cohomology of the structure sheaf of an affine scheme (Cohomology of Schemes, Lemma 2.2). Note this can be applied to any field of characteristic $p$. In particular, we can apply it to the field extension $L$ defined by a maximal pro-$p$ subgroup $H$. This allows us to conclude $H^n(\Spec L, \mathbb{Z}/p\mathbb{Z}_{\Spec L}) = 0$ for $n \geq 2$, from which the result follows for $\ell = p$, by Lemma 66.3.

To finish the proof we still have to show that $H^q(Gal(K^{sep}/K), (K^{sep})^*) = 0$ for all $q \geq 1$. Set $G = Gal(K^{sep}/K)$ and set $M = (K^{sep})^*$ viewed as a $G$-module. We have already shown (above) that $H^1(G, M) = 0$ and $H^2(G, M) = 0$. Consider the exact sequence

$$0 \to A \to M \to M \otimes Q \to B \to 0$$

of $G$-modules. By the above we have $H^i(G, A) = 0$ and $H^i(G, B) = 0$ for $i > 1$ since $A$ and $B$ are torsion $G$-modules. By Lemma 56.6 we have $H^i(G, M \otimes Q) = 0$ for $i > 0$. It is a pleasant exercise to see that this implies that $H^i(G, M) = 0$ also for $i \geq 3$. \hfill $\square$

03R9 **Definition 66.5.** A field $K$ is called $C_r$ if for every $0 < \delta < n$ and every $f \in K[T_1, \ldots, T_n]$ homogeneous of degree $d$, there exist $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\alpha_i \in K$ not all zero, such that $f(\alpha) = 0$. Such an $\alpha$ is called a nontrivial solution of $f$.

03RA **Example 66.6.** An algebraically closed field is $C_r$.

In fact, we have the following simple lemma.

03RB **Lemma 66.7.** Let $k$ be an algebraically closed field. Let $f_1, \ldots, f_s \in k[T_1, \ldots, T_n]$ be homogeneous polynomials of degree $d_1, \ldots, d_s$ with $d_i > 0$. If $s < n$, then $f_1 = \ldots = f_s = 0$ have a common nontrivial solution.

**Proof.** This follows from dimension theory, for example in the form of Varieties, Lemma 33.2 applied $s - 1$ times. \hfill $\square$

The following result computes the Brauer group of $C_1$ fields.

03RC **Theorem 66.8.** Let $K$ be a $C_1$ field. Then $Br(K) = 0$. 

Proof. Let $D$ be a finite dimensional division algebra over $K$ with center $K$. We have seen that
\[ D \otimes_K K^{sep} \cong \text{Mat}_d(K^{sep}) \]
uniquely up to inner isomorphism. Hence the determinant $\det : \text{Mat}_d(K^{sep}) \to K^{sep}$ is Galois invariant and descends to a homogeneous degree $d$ map
\[ \det = N_{\text{red}} : D \to K \]
called the reduced norm. Since $K$ is $C_1$, if $d > 1$, then there exists a nonzero $x \in D$ with $N_{\text{red}}(x) = 0$. This clearly implies that $x$ is not invertible, which is a contradiction. Hence $\text{Br}(K) = 0$. \qed

**Definition 66.9.** Let $k$ be a field. A *variety* is a separated, integral scheme of finite type over $k$. A *curve* is a variety of dimension 1.

**Theorem 66.10** (Tsen’s theorem). The function field of a variety of dimension $r$ over an algebraically closed field $k$ is $C_r$.

Proof. For projective space one can show directly that the field $k(x_1, \ldots, x_r)$ is $C_r$ (exercise).

General case. Without loss of generality, we may assume $X$ to be projective. Let $f \in K[T_1, \ldots, T_n]$ with $0 < d^r < n$. Say the coefficients of $f$ are in $\Gamma(X, \mathcal{O}_X(H))$ for some ample $H \subset X$. Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ with $\alpha_i \in \Gamma(X, \mathcal{O}_X(eH))$. Then $f(\alpha) \in \Gamma(X, \mathcal{O}_X((de+1)H))$. Consider the system of equations $f(\alpha) = 0$. Then by asymptotic Riemann-Roch (Varieties, Proposition 44.13) there exists a $c > 0$ such that

- the number of variables is $n \dim_K \Gamma(X, \mathcal{O}_X(eH)) \sim ne^rc$ and
- the number of equations is $\dim_K \Gamma(X, \mathcal{O}_X((de+1)H)) \sim (de+1)^rc$.

Since $n > d^r$, there are more variables than equations. The equations are homogeneous hence there is a solution by Lemma 66.7. \qed

**Lemma 66.11.** Let $C$ be a curve over an algebraically closed field $k$. Then the Brauer group of the function field of $C$ is zero: $\text{Br}(k(C)) = 0$.

Proof. This is clear from Tsen’s theorem, Theorem 66.10 and Theorem 66.8. \qed

**Lemma 66.12.** Let $k$ be an algebraically closed field and $k \subset K$ a field extension of transcendence degree 1. Then for all $q \geq 1$, $H^q_{\text{étale}}(\text{Spec}(K), \mathbb{G}_m) = 0$.

**Proof.** Recall that $H^q_{\text{étale}}(\text{Spec}(K), \mathbb{G}_m) = H^q(\text{Gal}(K^{sep}/K), (K^{sep})^*)$ by Lemma 58.2. Thus by Proposition 66.4 it suffices to show that if $K \subset K'$ is a finite field extension, then $\text{Br}(K') = 0$. Now observe that $K' = \text{colim} K''$, where $K''$ runs over the finitely generated subextensions of $k$ contained in $K'$ of transcendence degree 1. Note that $\text{Br}(K') = \text{colim} \text{Br}(K'')$ which reduces us to a finitely generated field extension $K''/k$ of transcendence degree 1. Such a field is the function field of a curve over $k$, hence has trivial Brauer group by Lemma 66.11. \qed

67. Higher vanishing for the multiplicative group

In this section, we fix an algebraically closed field $k$ and a smooth curve $X$ over $k$. We denote $i_x : x \to X$ the inclusion of a closed point of $X$ and $j : \eta \to X$ the inclusion of the generic point. We also denote $X_0$ the set of closed points of $X$. 

Theorem 67.1 (The Fundamental Exact Sequence). There is a short exact sequence of étale sheaves on $X$

$$0 \to \mathbb{G}_m, X \to j_* \mathbb{G}_m, \eta \to \bigoplus_{x \in X_0} i_x, \mathbb{Z} \to 0.$$ 

Proof. Let $\varphi : U \to X$ be an étale morphism. Then by properties of étale morphisms (Proposition 26.2), $U = \coprod_i U_i$ where $U_i$ is a smooth curve mapping to $X$. The above sequence for $U$ is a product of the corresponding sequences for each $U_i$, so it suffices to treat the case where $U$ is connected, hence irreducible. In this case, there is a well known exact sequence

$$1 \to \Gamma(U, \mathcal{O}_U^*) \to k(U)^* \to \bigoplus_{y \in U^0} \mathbb{Z}_y.$$ 

This amounts to a sequence

$$0 \to \Gamma(U, \mathcal{O}_U^*) \to \Gamma(\eta \times_X U, \mathcal{O}_{\eta \times_X U}^*) \to \bigoplus_{x \in X_0} \Gamma(x \times_X U, \mathbb{Z})$$

which, unfolding definitions, is nothing but a sequence

$$0 \to \mathbb{G}_m(U) \to j_* \mathbb{G}_m, \eta(U) \to \left( \bigoplus_{x \in X_0} i_x, \mathbb{Z}\right)(U).$$

This defines the maps in the Fundamental Exact Sequence and shows it is exact except possibly at the last step. To see surjectivity, let us recall that if $U$ is a nonsingular curve and $D$ is a divisor on $U$, then there exists a Zariski open covering $\{U_j \to U\}$ of $U$ such that $D|_{U_j} = \text{div}(f_j)$ for some $f_j \in k(U)^*$. \hfill \Box

Lemma 67.2. For any $q \geq 1$, $R^q j_* \mathbb{G}_m, \eta = 0$.

Proof. We need to show that $(R^q j_* \mathbb{G}_m, \eta)_x = 0$ for every geometric point $x$ of $X$. Assume that $x$ lies over a closed point $y$ of $X$. Let $\text{Spec}(A)$ be an affine open neighborhood of $y$ in $X$, and $K$ the fraction field of $A$. Then

$$\text{Spec}(\mathcal{O}_{X,x}^{\text{sh}}) \times_X \eta = \text{Spec}(\mathcal{O}_{X,x}^{\text{sh}} \otimes_A K).$$

The ring $\mathcal{O}_{X,x}^{\text{sh}} \otimes_A K$ is a localization of the discrete valuation ring $\mathcal{O}_{X,x}^{\text{sh}}$, so it is either $\mathcal{O}_{X,x}^{\text{sh}}$ again, or its fraction field $K_{x}^{\text{sh}}$. But since some local uniformizer gets inverted, it must be the latter. Hence

$$(R^q j_* \mathbb{G}_m, \eta)(\text{Spec}(K_{x}^{\text{sh}}), \mathbb{G}_m) = H_{\text{étale}}^q(\text{Spec}(K_{x}^{\text{sh}}), \mathbb{G}_m).$$

Now recall that $\mathcal{O}_{X,x}^{\text{sh}} = \text{colim}_{(U, \bar{\eta}) \to x} \mathcal{O}(U) = \text{colim}_{A \to B} B$ where $A \to B$ is étale, hence $K_{x}^{\text{sh}}$ is an algebraic extension of $K = k(X)$, and we may apply Lemma 66.12 to get the vanishing.

Assume that $x = \bar{\eta}$ lies over the generic point $\eta$ of $X$ (in fact, this case is superfluous). Then $\mathcal{O}_{X, \bar{\eta}} = \kappa(\eta)^{\text{sep}}$ and thus

$$\begin{align*}
(R^q j_* \mathbb{G}_m, \eta)_{\bar{\eta}} &= H_{\text{étale}}^q(\text{Spec}(\kappa(\eta)^{\text{sep}}) \times_X \eta, \mathbb{G}_m) \\
&= H_{\text{étale}}^q(\text{Spec}(\kappa(\eta)^{\text{sep}}), \mathbb{G}_m) \\
&= 0 \text{ for } q \geq 1
\end{align*}$$

since the corresponding Galois group is trivial. \hfill \Box

Lemma 67.3. For all $p \geq 1$, $H_{\text{étale}}^p(X, j_* \mathbb{G}_m, \eta) = 0$. 

Proof. The Leray spectral sequence reads

\[ E_{2}^{p,q} = H_{\text{étale}}^{p}(X, R^{q}j_{*}G_{m,\eta}) \Rightarrow H_{\text{étale}}^{p+q}(\eta, G_{m,\eta}), \]

which vanishes for \( p + q \geq 1 \) by Lemma 66.12. Taking \( q = 0 \), we get the desired vanishing. \( \square \)

Lemma 67.4. For all \( q \geq 1 \), \( H_{\text{étale}}^{q}(X, \bigoplus_{x \in X_{0}} i_{x*}\mathbb{Z}) = 0 \).

Proof. For \( X \) quasi-compact and quasi-separated, cohomology commutes with colimits, so it suffices to show the vanishing of \( H_{\text{étale}}^{q}(X, i_{x*}\mathbb{Z}) \). But then the inclusion \( i_{x} \) of a closed point is finite so \( R^{p}i_{x*}\mathbb{Z} = 0 \) for all \( p \geq 1 \) by Proposition 54.2. Applying the Leray spectral sequence, we see that \( H_{\text{étale}}^{q}(X, i_{x*}\mathbb{Z}) = H_{\text{étale}}^{q}(x, \mathbb{Z}) \). Finally, since \( x \) is the spectrum of an algebraically closed field, all higher cohomology on \( x \) vanishes. \( \square \)

Concluding this series of lemmata, we get the following result.

Theorem 67.5. Let \( X \) be a smooth curve over an algebraically closed field. Then \( H_{\text{étale}}^{q}(X, G_{m}) = 0 \) for all \( q \geq 2 \).

Proof. See discussion above. \( \square \)

We also get the cohomology long exact sequence

\[ 0 \rightarrow H_{\text{étale}}^{0}(X, G_{m}) \rightarrow H_{\text{étale}}^{0}(X, j_{*}G_{m,\eta}) \rightarrow H_{\text{étale}}^{0}(X, \bigoplus_{x \in X_{0}} i_{x*}\mathbb{Z}) \rightarrow H_{\text{étale}}^{1}(X, G_{m}) \rightarrow 0 \]

although this is the familiar

\[ 0 \rightarrow H_{\text{zar}}^{0}(X, \mathcal{O}_{X}^{*}) \rightarrow k(X)^{*} \rightarrow \text{Div}(X) \rightarrow \text{Pic}(X) \rightarrow 0. \]

68. Picard groups of curves

Our next step is to use the Kummer sequence to deduce some information about the cohomology group of a curve with finite coefficients. In order to get vanishing in the long exact sequence, we review some facts about Picard groups.

Let \( X \) be a smooth projective curve over an algebraically closed field \( k \). Let \( g = \dim_{k} H^{1}(X, \mathcal{O}_{X}) \) be the genus of \( X \). There exists a short exact sequence

\[ 0 \rightarrow \text{Pic}^{0}(X) \rightarrow \text{Pic}(X) \xrightarrow{\text{deg}} \mathbb{Z} \rightarrow 0. \]

The abelian group \( \text{Pic}^{0}(X) \) can be identified with \( \text{Pic}^{0}(X) = \text{Pic}_{X/k}(k) \), i.e., the \( k \)-valued points of an abelian variety \( \text{Pic}_{X/k}^{0} \) over \( k \) of dimension \( g \). Consequently, if \( n \in k^{*} \) then \( \text{Pic}^{0}(X)[n] \cong (\mathbb{Z}/n\mathbb{Z})^{2g} \) as abelian groups. See Picard Schemes of Curves, Section 6 and Groupoids, Section 9. This key fact, namely the description of the torsion in the Picard group of a smooth projective curve over an algebraically closed field does not appear to have an elementary proof.

Lemma 68.1. Let \( X \) be a smooth projective curve of genus \( g \) over an algebraically closed field \( k \) and let \( n \geq 1 \) be invertible in \( k \). Then there are canonical identifications

\[ H_{\text{étale}}^{0}(X, \mu_{n}) = \begin{cases} \mu_{n}(k) & \text{if } q = 0, \\ \text{Pic}^{0}(X)[n] & \text{if } q = 1, \\ \mathbb{Z}/n\mathbb{Z} & \text{if } q = 2, \\ 0 & \text{if } q \geq 3. \end{cases} \]
Since $\mu_n \cong \mathbb{Z}/n\mathbb{Z}$, this gives (noncanonical) identifications

$$H^q_{\text{étale}}(X, \mathbb{Z}/n\mathbb{Z}) \cong \begin{cases} \mathbb{Z}/n\mathbb{Z} & \text{if } q = 0, \\ (\mathbb{Z}/n\mathbb{Z})^{2g} & \text{if } q = 1, \\ \mathbb{Z}/n\mathbb{Z} & \text{if } q = 2, \\ 0 & \text{if } q \geq 3. \end{cases}$$

**Proof.** Theorems 24.1 and 67.5 determine the étale cohomology of $\mathbb{G}_m$ on $X$ in terms of the Picard group of $X$. The Kummer sequence $0 \to \mu_n, X \to \mathbb{G}_m, X \to 0$ (Lemma 28.1) then gives us the long exact cohomology sequence

$$\cdots \to H^1_{\text{étale}}(X, \mu_n) \to \text{Pic}(X) \xrightarrow{(\cdot)^n} \text{Pic}(X) \to H^2_{\text{étale}}(X, \mu_n) \to 0 \to \cdots$$

The $n$th power map $k^* \to k^*$ is surjective since $k$ is algebraically closed. So we need to compute the kernel and cokernel of the map $n : \text{Pic}(X) \to \text{Pic}(X)$. Consider the commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \to & \text{Pic}^0(X) & \to & \text{Pic}(X) & \xrightarrow{\deg} & \mathbb{Z} & \to & 0 \\
& & \downarrow{(\cdot)^n} & & \downarrow{\deg} & & \downarrow{n} & & \\
0 & \to & \text{Pic}^0(X) & \to & \text{Pic}(X) & \xrightarrow{\deg} & \mathbb{Z} & \to & 0 \end{array}$$

The group $\text{Pic}^0(X)$ is the $k$-points of the group scheme $\text{Pic}^0_{X/k}$; see Picard Schemes of Curves, Lemma 6.7. The same lemma tells us that $\text{Pic}^0_{X/k}$ is a $g$-dimensional abelian variety over $k$ as defined in Groupoids, Definition 9.1. Hence the left vertical map is surjective by Groupoids, Proposition 9.11. Applying the snake lemma gives canonical identifications as stated in the lemma.

To get the noncanonical identifications of the lemma we need to show the kernel of $n : \text{Pic}^0(X) \to \text{Pic}^0(X)$ is isomorphic to $(\mathbb{Z}/n\mathbb{Z})^{\oplus 2g}$. This is also part of Groupoids, Proposition 9.11. □

**Lemma 68.2.** Let $\pi : X \to Y$ be a nonconstant morphism of smooth projective curves over an algebraically closed field $k$ and let $n \geq 1$ be invertible in $k$. The map

$$\pi^* : H^2_{\text{étale}}(Y, \mu_n) \to H^2_{\text{étale}}(X, \mu_n)$$

is given by multiplication by the degree of $\pi$.

**Proof.** Observe that the statement makes sense as we have identified both cohomology groups $H^2_{\text{étale}}(Y, \mu_n)$ and $H^2_{\text{étale}}(X, \mu_n)$ with $\mathbb{Z}/n\mathbb{Z}$ in Lemma 68.1. In fact, if $\mathcal{L}$ is a line bundle of degree 1 on $Y$ with class $[\mathcal{L}] \in H^1_{\text{étale}}(Y, \mathbb{G}_m)$, then the coboundary of $[\mathcal{L}]$ is the generator of $H^2_{\text{étale}}(Y, \mu_n)$. Here the coboundary is the coboundary of the long exact sequence of cohomology associated to the Kummer sequence. Thus the result of the lemma follows from the fact that the degree of the line bundle $\pi^* \mathcal{L}$ on $X$ is $\deg(\pi)$. Some details omitted. □
03RR \textbf{Lemma 68.3.} Let $X$ be an affine smooth curve over an algebraically closed field $k$ and $n \in k^*$. Then

\begin{enumerate}
\item $H^0_{\text{étale}}(X, \mu_n) = \mu_n(k)$;
\item $H^1_{\text{étale}}(X, \mu_n) \cong (\mathbb{Z}/n\mathbb{Z})^{2g+r-1}$, where $r$ is the number of points in $\overline{X} - X$ for some smooth projective compactification $\overline{X}$ of $X$, and
\item for all $q \geq 2$, $H^q_{\text{étale}}(X, \mu_n) = 0$.
\end{enumerate}

\textbf{Proof.} Write $X = \overline{X} - \{x_1, \ldots, x_r\}$. Then $\text{Pic}(X) = \text{Pic}(\overline{X})/R$, where $R$ is the subgroup generated by $\mathcal{O}_X(x_i)$, $1 \leq i \leq r$. Since $r \geq 1$, we see that $\text{Pic}^0(X) \to \text{Pic}(X)$ is surjective, hence $\text{Pic}(X)$ is divisible. Applying the Kummer sequence, we get (1) and (3). For (2), recall that

\[ H^1_{\text{étale}}(X, \mu_n) = \{(\mathcal{L}, \alpha) | \mathcal{L} \in \text{Pic}(X), \alpha : \mathcal{L}^\otimes_n \to \mathcal{O}_X\}/\cong \]

where $\mathcal{L} \in \text{Pic}^0(\overline{X})$, $D$ is a divisor on $\overline{X}$ supported on $\{x_1, \ldots, x_r\}$ and $\overline{\alpha} : \mathcal{L}^\otimes_n \cong \mathcal{O}_X(D)$ is an isomorphism. Note that $D$ must have degree 0. Further $\overline{R}$ is the subgroup of triples of the form $(\mathcal{O}_X(D'), nD', 1^\otimes)$ where $D'$ is supported on $\{x_1, \ldots, x_r\}$ and has degree 0. Thus, we get an exact sequence

\[ 0 \to H^1_{\text{étale}}(\overline{X}, \mu_n) \to H^1_{\text{étale}}(X, \mu_n) \to \bigoplus_{i=1}^r \mathbb{Z}/n\mathbb{Z} \xrightarrow{\sum} \mathbb{Z}/n\mathbb{Z} \to 0 \]

where the middle map sends the class of a triple $(\mathcal{L}, D, \overline{\alpha})$ with $D = \sum_{i=1}^r a_i(x_i)$ to the $r$-tuple $(a_i)_{i=1}^r$. It now suffices to use Lemma 68.1 to count ranks. \hfill \Box

03RS \textbf{Remark 68.4.} The “natural” way to prove the previous corollary is to excise $X$ from $\overline{X}$. This is possible, we just haven’t developed that theory.

0A44 \textbf{Remark 68.5.} Let $k$ be an algebraically closed field. Let $n$ be an integer prime to the characteristic of $k$. Recall that

\[ G_{m,k} = A^1_k \setminus \{0\} = P^1_k \setminus \{0, \infty\} \]

We claim there is a canonical isomorphism

\[ H^1_{\text{étale}}(G_{m,k}, \mu_n) = \mathbb{Z}/n\mathbb{Z} \]

What does this mean? This means there is an element $1_k$ in $H^1_{\text{étale}}(G_{m,k}, \mu_n)$ such that for every morphism $\text{Spec}(k') \to \text{Spec}(k)$ the pullback map on étale cohomology for the map $G_{m,k'} \to G_{m,k}$ maps $1_k$ to $1_{k'}$. (In particular this element is fixed under all automorphisms of $k$.) To see this, consider the $\mu_n, z$-torsor $G_{m,z} \to G_{m,z}$, $x \mapsto x^n$. By the identification of torsors with first cohomology, this pulls back to give our canonical elements $1_k$. Twisting back we see that there are canonical identifications

\[ H^1_{\text{étale}}(G_{m,k}, \mathbb{Z}/n\mathbb{Z}) = \text{Hom}(\mu_n(k), \mathbb{Z}/n\mathbb{Z}), \]

i.e., these isomorphisms are compatible with respect to maps of algebraically closed fields, in particular with respect to automorphisms of $k$. 

69. Extension by zero

The general material in Modules on Sites, Section 19 allows us to make the following definition.

**Definition 69.1.** Let \( j : U \rightarrow X \) be an étale morphism of schemes.

1. The restriction functor \( j^{-1} : \text{Sh}(X_{\text{étale}}) \rightarrow \text{Sh}(U_{\text{étale}}) \) has a left adjoint \( j^! : \text{Sh}(U_{\text{étale}}) \rightarrow \text{Sh}(X_{\text{étale}}) \).
2. The restriction functor \( j^{-1} : \text{Ab}(X_{\text{étale}}) \rightarrow \text{Ab}(U_{\text{étale}}) \) has a left adjoint which is denoted \( j_! : \text{Ab}(U_{\text{étale}}) \rightarrow \text{Ab}(X_{\text{étale}}) \) and called *extension by zero*.
3. Let \( \Lambda \) be a ring. The restriction functor \( j^{-1} : \text{Mod}(X_{\text{étale}}, \Lambda) \rightarrow \text{Mod}(U_{\text{étale}}, \Lambda) \) has a left adjoint which is denoted \( j_! : \text{Mod}(U_{\text{étale}}, \Lambda) \rightarrow \text{Mod}(X_{\text{étale}}, \Lambda) \) and called *extension by zero*.

If \( F \) is an abelian sheaf on \( X_{\text{étale}} \), then \( j_! F \neq j^! \text{Sh} F \) in general. On the other hand \( j_! \) for sheaves of \( \Lambda \)-modules agrees with \( j_! \) on underlying abelian sheaves (Modules on Sites, Remark 19.6). The functor \( j_! \) is characterized by the functorial isomorphism

\[
\text{Hom}_X(j_! F, G) = \text{Hom}_U(F, j^{-1} G)
\]

for all \( F \in \text{Ab}(U_{\text{étale}}) \) and \( G \in \text{Ab}(X_{\text{étale}}) \). Similarly for sheaves of \( \Lambda \)-modules.

To describe it more explicitly, recall that \( j^{-1} \) is just the restriction via the functor \( U_{\text{étale}} \rightarrow X_{\text{étale}} \). In other words, \( j^{-1} G(U') = G(U') \) for \( U' \) étale over \( U \). For \( F \in \text{Ab}(U_{\text{étale}}) \) we consider the presheaf

\[
j_!^P F : X_{\text{étale}} \rightarrow \text{Ab}, \quad V \mapsto \bigoplus_{V \rightarrow U} F(V)
\]

Then \( j_! F \) is the sheafification of \( j_!^P F \).

**Exercise 69.2.** Prove directly that \( j_! \) is left adjoint to \( j^{-1} \) and that \( j_* \) is right adjoint to \( j^{-1} \).

**Proposition 69.3.** Let \( j : U \rightarrow X \) be an étale morphism of schemes. Let \( F \) in \( \text{Ab}(U_{\text{étale}}) \). If \( \bar{\pi} : \text{Spec}(k) \rightarrow X \) is a geometric point of \( X \), then

\[
(j_! F)_{\bar{\pi}} = \bigoplus_{\bar{\pi} : \text{Spec}(k) \rightarrow U, \ j(\bar{\pi}) = \bar{u}} F_{\bar{u}}.
\]

In particular, \( j_! \) is an exact functor.

**Proof.** Exactness of \( j_* \) is very general, see Modules on Sites, Lemma 19.3. Of course it does also follow from the description of stalks. The formula for the stalk of \( j_! F \) can be deduced directly from the explicit description of \( j_! \) given above. On the other hand, we can deduce it from the very general Modules on Sites, Lemma 37.1 and the description of points of the small étale site in terms of geometric points, see Lemma 29.12. \( \square \)

**Lemma 69.4** (Extension by zero commutes with base change). Let \( f : Y \rightarrow X \) be a morphism of schemes. Let \( j : V \rightarrow X \) be an étale morphism. Consider the fibre product

\[
V' = Y \times_X V \xrightarrow{j'} Y
\]

Then we have \( j_! j'^{-1} = f^{-1} j_! \) on abelian sheaves and on sheaves of modules.
Proof. This is true because \( j'_! f'^{-1} \) is left adjoint to \( f'_! (j')^{-1} \) and \( f^{-1} j_! \) is left adjoint to \( j^{-1} f_! \). Further \( f'_! (j')^{-1} = j^{-1} f_! \) because \( f_* \) commutes with étale localization (by construction). In fact, the lemma holds very generally in the setting of a morphism of sites, see Modules on Sites, Lemma 20.1. \( \square \)

Lemma 69.5. Let \( j : U \to X \) be finite and étale. Then \( j_! = j_* \) on abelian sheaves and sheaves of \( \Lambda \)-modules.

Proof. We prove this in the case of abelian sheaves. We claim there is a natural construction. In fact, the lemma holds very generally in the setting of a morphism \( \Lambda \rightarrow A \) of functors \( X \subset \mathcal{Z} \) and sheaves of \( \Lambda \)-modules. See Modules on Sites, Lemma 20.1.

For a geometric point \( x \in X \) we have either \( s_x = 0 \) or \( s_x = 1 \) for \( j_! F = \bigoplus_{x \in X} F(V_x \to U) \) and \( j_* F = F(V \times_X U) \). We leave it to the reader to see that this construction is compatible with restriction mappings.

It suffices to check \( j_! F \to j_* F \) is an isomorphism étale locally on \( X \). Thus we may assume \( U \to X \) is a finite disjoint union of isomorphisms, see Étale Morphisms, Lemma 18.3. We omit the proof in this case. \( \square \)

Lemma 69.6. Let \( X \) be a scheme. Let \( Z \subset X \) be a closed subscheme and let \( U \subset X \) be the complement. Denote \( i : Z \to X \) and \( j : U \to X \) the inclusion morphisms. For every abelian sheaf \( F \) on \( X_{\text{ét}} \) there is a canonical short exact sequence

\[
0 \to j_! j^{-1} F \to F \to i_* i^{-1} F \to 0
\]
on \( X_{\text{ét}} \).

Proof. We obtain the maps by the adjointness properties of the functors involved. For a geometric point \( \tau \in X \) we have either \( \tau \in U \) in which case the map on the left hand side is an isomorphism on stalks and the stalk of \( i_* i^{-1} F \) is zero or \( \tau \in Z \) in which case the map on the right hand side is an isomorphism on stalks and the stalk of \( j_! j^{-1} F \) is zero. Here we have used the description of stalks of Lemma 16.3 and Proposition 69.3. \( \square \)

70. Constructible sheaves

Let \( X \) be a scheme. A constructible locally closed subscheme of \( X \) is a locally closed subscheme \( T \subset X \) such that the underlying topological space of \( T \) is a constructible subset of \( X \). If \( T, T' \subset X \) are locally closed subschemes with the same underlying topological space, then \( T_{\text{ét}} \cong T'_{\text{ét}} \) by the topological invariance of the étale site (Theorem 45.2). Thus in the following definition we may assume our locally closed subschemes are reduced.
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03RW **Definition 70.1.** Let $X$ be a scheme.

(1) A sheaf of sets on $X_{\text{étale}}$ is **constructible** if for every affine open $U \subset X$ there exists a finite decomposition of $U$ into constructible locally closed subschemes $U = \bigsqcup_i U_i$ such that $\mathcal{F}|_{U_i}$ is finite locally constant for all $i$.

(2) A sheaf of abelian groups on $X_{\text{étale}}$ is **constructible** if for every affine open $U \subset X$ there exists a finite decomposition of $U$ into constructible locally closed subschemes $U = \bigsqcup_i U_i$ such that $\mathcal{F}|_{U_i}$ is finite locally constant for all $i$.

(3) Let $\Lambda$ be a Noetherian ring. A sheaf of $\Lambda$-modules on $X_{\text{étale}}$ is **constructible** if for every affine open $U \subset X$ there exists a finite decomposition of $U$ into constructible locally closed subschemes $U = \bigsqcup_i U_i$ such that $\mathcal{F}|_{U_i}$ is of finite type and locally constant for all $i$.

It seems that this is the accepted definition. An alternative, which lends itself more readily to generalizations beyond the étale site of a scheme, would have been to define constructible sheaves by starting with $h_U$, $j_U^! \mathbb{Z}/n \mathbb{Z}$, and $j_U^! \Lambda$ where $U$ runs over all quasi-compact and quasi-separated objects of $X_{\text{étale}}$, and then take the smallest full subcategory of $\text{Sh}(X_{\text{étale}})$, $\text{Ab}(X_{\text{étale}})$, and $\text{Mod}(X_{\text{étale}}, \Lambda)$ containing these and closed under finite limits and colimits. It follows from Lemma 70.6 and Lemmas 72.5, 72.7, and 72.6 that this produces the same category if $X$ is quasi-compact and quasi-separated. In general this does not produce the same category however.

A disjoint union decomposition $U = \bigsqcup_i U_i$ of a scheme by locally closed subschemes will be called a *partition* of $U$ (compare with Topology, Section 28).

095E **Lemma 70.2.** Let $X$ be a quasi-compact and quasi-separated scheme. Let $\mathcal{F}$ be a sheaf of sets on $X_{\text{étale}}$. The following are equivalent

(1) $\mathcal{F}$ is constructible,

(2) there exists an open covering $X = \bigcup U_i$ such that $\mathcal{F}|_{U_i}$ is constructible, and

(3) there exists a partition $X = \bigcup X_i$ by constructible locally closed subschemes such that $\mathcal{F}|_{X_i}$ is finite locally constant.

A similar statement holds for abelian sheaves and sheaves of $\Lambda$-modules if $\Lambda$ is Noetherian.

**Proof.** It is clear that (1) implies (2).

Assume (2). For every $x \in X$ we can find an $i$ and an affine open neighbourhood $V_x \subset U_i$ of $x$. Hence we can find a finite affine open covering $X = \bigcup V_j$ such that for each $j$ there exists a finite decomposition $V_j = \bigsqcup V_{j,k}$ by locally closed constructible subsets such that $\mathcal{F}|_{V_{j,k}}$ is finite locally constant. By Topology, Lemma 15.5 each $V_{j,k}$ is constructible as a subset of $X$. By Topology, Lemma 28.7 we can find a finite stratification $X = \bigsqcup X_i$ with constructible locally closed strata such that each $V_{j,k}$ is a union of $X_i$. Thus (3) holds.

Assume (3) holds. Let $U \subset X$ be an affine open. Then $U \cap X_i$ is a constructible locally closed subset of $U$ (for example by Properties, Lemma 2.1) and $U = \bigsqcup U \cap X_i$ is a partition of $U$ as in Definition 70.1. Thus (1) holds.

09YR **Lemma 70.3.** Let $X$ be a quasi-compact and quasi-separated scheme. Let $\mathcal{F}$ be a sheaf of sets, abelian groups, $\Lambda$-modules (with $\Lambda$ Noetherian) on $X_{\text{étale}}$. If there
exist constructible locally closed subschemes $T_i \subset X$ such that (a) $X = \bigcup T_j$ and (b) $F|_{T_j}$ is constructible, then $F$ is constructible.

**Proof.** First, we can assume the covering is finite as $X$ is quasi-compact in the spectral topology (Topology, Lemma 23.2 and Properties, Lemma 2.4). Observe that each $T_i$ is a quasi-compact and quasi-separated scheme in its own right (because it is constructible in $X$; details omitted). Thus we can find a finite partition $T_i = \bigsqcup T_{i,j}$ into locally closed constructible parts of $T_i$ such that $F|_{T_{i,j}}$ is finite locally constant (Lemma 70.2). By Topology, Lemma 15.12 we see that $T_{i,j}$ is a constructible locally closed subscheme of $X$. Then we can apply Topology, Lemma 28.7 to $X = \bigcup T_{i,j}$ to find the desired partition of $X$. □

**Lemma 70.4.** Let $X$ be a scheme. Checking constructibility of a sheaf of sets, abelian groups, $\Lambda$-modules (with $\Lambda$ Noetherian) can be done Zariski locally on $X$.

**Proof.** The statement means if $X = \bigcup U_i$ is an open covering such that $F|_{U_i}$ is constructible, then $F$ is constructible. If $U \subset X$ is affine open, then $U = \bigcup U \cap U_i$ and $F|_{U \cap U_i}$ is constructible (it is trivial that the restriction of a constructible sheaf to an open is constructible). It follows from Lemma 70.2 that $F|_{U}$ is constructible, i.e., a suitable partition of $U$ exists. □

**Lemma 70.5.** Let $f : X \rightarrow Y$ be a morphism of schemes. If $F$ is a constructible sheaf of sets, abelian groups, or $\Lambda$-modules (with $\Lambda$ Noetherian) on $Y_{\text{étale}}$, the same is true for $f^{-1}F$ on $X_{\text{étale}}$.

**Proof.** By Lemma 70.4 this reduces to the case where $X$ and $Y$ are affine. By Lemma 70.2 it suffices to find a finite partition of $X$ by constructible locally closed subschemes such that $f^{-1}F$ is finite locally constant on each of them. To find it we just pull back the partition of $Y$ adapted to $F$ and use Lemma 63.2. □

**Lemma 70.6.** Let $X$ be a scheme.

1. The category of constructible sheaves of sets is closed under finite limits and colimits inside $\text{Sh}(X_{\text{étale}})$.
2. The category of constructible abelian sheaves is a weak Serre subcategory of $\text{Ab}(X_{\text{étale}})$.
3. Let $\Lambda$ be a Noetherian ring. The category of constructible sheaves of $\Lambda$-modules on $X_{\text{étale}}$ is a weak Serre subcategory of $\text{Mod}(X_{\text{étale}}, \Lambda)$.

**Proof.** We prove (3). We will use the criterion of Homology, Lemma 9.3. Suppose that $\varphi : F \rightarrow G$ is a map of constructible sheaves of $\Lambda$-modules. We have to show that $\mathcal{K} = \text{Ker}(\varphi)$ and $\mathcal{Q} = \text{Coker}(\varphi)$ are constructible. Similarly, suppose that $0 \rightarrow F \rightarrow E \rightarrow G \rightarrow 0$ is a short exact sequence of sheaves of $\Lambda$-modules with $F$, $G$ constructible. We have to show that $E$ is constructible. In both cases we can replace $X$ with the members of an affine open covering. Hence we may assume $X$ is affine. The we may further replace $X$ by the members of a finite partition of $X$ by constructible locally closed subschemes on which $F$ and $G$ are of finite type and locally constant. Thus we may apply Lemma 63.6 to conclude.

The proofs of (1) and (2) are very similar and are omitted. □

**Lemma 70.7.** Let $X$ be a scheme. Let $\Lambda$ be a Noetherian ring. The tensor product of two constructible sheaves of $\Lambda$-modules on $X_{\text{étale}}$ is a constructible sheaf of $\Lambda$-modules.
Proof. The question immediately reduces to the case where $X$ is affine. Since any two partitions of $X$ with constructible locally closed strata have a common refinement of the same type and since pullbacks commute with tensor product we reduce to Lemma 63.7.

**Lemma 70.8.** Let $X$ be a quasi-compact and quasi-separated scheme.

1. Let $\mathcal{F} \to \mathcal{G}$ be a map of constructible sheaves of sets on $X_{\text{étale}}$. Then the set of points $x \in X$ where $\mathcal{F}_x \to \mathcal{G}_x$ is surjective, resp. injective, resp. is isomorphic to a given map of sets, is constructible in $X$.

2. Let $\mathcal{F}$ be a constructible abelian sheaf on $X_{\text{étale}}$. The support of $\mathcal{F}$ is constructible.

3. Let $\Lambda$ be a Noetherian ring. Let $\mathcal{F}$ be a constructible sheaf of $\Lambda$-modules on $X_{\text{étale}}$. The support of $\mathcal{F}$ is constructible.

Proof. Proof of (1). Let $X = \coprod X_i$ be a partition of $X$ by locally closed constructible subschemes such that both $\mathcal{F}$ and $\mathcal{G}$ are finite locally constant over the parts (use Lemma 70.2 for both $\mathcal{F}$ and $\mathcal{G}$ and choose a common refinement). Then apply Lemma 63.5 to the restriction of the map to each part.

The proof of (2) and (3) is omitted. □

The following lemma will turn out to be very useful later on. It roughly says that the category of constructible sheaves has a kind of weak “Noetherian” property.

**Lemma 70.9.** Let $X$ be a quasi-compact and quasi-separated scheme. Let $\mathcal{F} = \colim_{i \in I} \mathcal{F}_i$ be a filtered colimit of sheaves of sets, abelian sheaves, or sheaves of modules.

1. If $\mathcal{F}$ and $\mathcal{F}_i$ are constructible sheaves of sets, then the ind-object $\mathcal{F}_i$ is essentially constant with value $\mathcal{F}$.

2. If $\mathcal{F}$ and $\mathcal{F}_i$ are constructible sheaves of abelian groups, then the ind-object $\mathcal{F}_i$ is essentially constant with value $\mathcal{F}$.

3. Let $\Lambda$ be a Noetherian ring. If $\mathcal{F}$ and $\mathcal{F}_i$ are constructible sheaves of $\Lambda$-modules, then the ind-object $\mathcal{F}_i$ is essentially constant with value $\mathcal{F}$.

Proof. Proof of (1). We will use without further mention that finite limits and colimits of constructible sheaves are constructible (Lemma 63.6). For each $i$ let $T_i \subset X$ be the set of points $x \in X$ where $\mathcal{F}_i \to \mathcal{F}_x$ is not surjective. Because $\mathcal{F}_i$ and $\mathcal{F}$ are constructible $T_i$ is a constructible subset of $X$ (Lemma 70.8). Since the stalks of $\mathcal{F}$ are finite and since $\mathcal{F} = \colim_{i \in I} \mathcal{F}_i$ we see that for all $x \in X$ we have $x \not\in T_i$ for $i$ large enough. Thus $\mathcal{F}_i \to \mathcal{F}$ is surjective for $i$ large enough. Since $X$ is a spectral space by Properties, Lemma 2.4 the constructible topology on $X$ is quasi-compact by Topology, Lemma 23.2. Thus $T_i = \emptyset$ for $i$ large enough. Assume now that $\mathcal{F}_i \to \mathcal{F}$ is surjective for all $i$. Choose $i \in I$. For $i' \geq i$ denote $S_{i'} \subset X$ the set of points $x$ such that the number of elements in $\text{Im}(\mathcal{F}_i \to \mathcal{F}_x)$ is equal to the number of elements in $\text{Im}(\mathcal{F}_i \to \mathcal{F}_{i'} \to \mathcal{F}_x)$. Because $\mathcal{F}_i$, $\mathcal{F}_{i'}$ and $\mathcal{F}$ are constructible $S_{i'}$ is a constructible subset of $X$ (details omitted; hint: use Lemma 70.8). Since the stalks of $\mathcal{F}_i$ and $\mathcal{F}$ are finite and since $\mathcal{F} = \colim_{i' \geq i} \mathcal{F}_{i'}$ we see that for all $x \in X$ we have $x \not\in S_{i'}$ for $i'$ large enough. By the same argument as above we can find a large $i'$ such that $S_{i'} = \emptyset$. Thus $\mathcal{F}_i \to \mathcal{F}_{i'}$ factors through $\mathcal{F}$ as desired.

Proof of (2). Observe that a constructible abelian sheaf is a constructible sheaf of sets. Thus case (2) follows from (1).
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Proof of (3). We will use without further mention that the category of constructible sheaves of $\Lambda$-modules is abelian (Lemma [63.6]). For each $i$ let $Q_i$ be the cokernel of the map $F_i \to F$. The support $T_i$ of $Q_i$ is a constructible subset of $X$ as $Q_i$ is constructible (Lemma [70.8]). Since the stalks of $F$ are finite $\Lambda$-modules and since $F = \text{colim}_{i \in I} F_i$ we see that for all $x \in X$ we have $x \notin T_i$ for $i$ large enough. Since $X$ is a spectral space by Properties, Lemma [2.4] the constructible topology on $X$ is quasi-compact by Topology, Lemma [23.2] Thus $T_i = \emptyset$ for $i$ large enough. This proves the first assertion. For the second, assume now that $F_i \to F$ is surjective for all $i$. Choose $i \in I$. For $i' \geq i$ denote $K_{i'}$ the image of $\text{Ker}(F_i \to F)$ in $F_{i'}$. The support $S_{i'}$ of $K_{i'}$ is a constructible subset of $X$ as $K_{i'}$ is constructible. Since the stalks of $\text{Ker}(F_i \to F)$ are finite $\Lambda$-modules and since $F = \text{colim}_{i' \geq i} F_{i'}$ we see that for all $x \in X$ we have $x \notin S_{i'}$ for $i'$ large enough. By the same argument as above we can find a large $i'$ such that $S_{i'} = \emptyset$. Thus $F_i \to F_{i'}$ factors through $F$ as desired. □

71. Auxiliary lemmas on morphisms

095J Some lemmas that are useful for proving functoriality properties of constructible sheaves.

03S0 [Lemma 71.1.] Let $U \to X$ be an étale morphism of quasi-compact and quasi-separated schemes (for example an étale morphism of Noetherian schemes). Then there exists a partition $X = \bigsqcup_i X_i$ by constructible locally closed subschemes such that $X_i \times_X U \to X_i$ is finite étale for all $i$.

**Proof.** If $U \to X$ is separated, then this is More on Morphisms, Lemma [38.9] In general, we may assume $X$ is affine. Choose a finite affine open covering $U = \bigsqcup U_j$. Apply the previous case to all the morphisms $U_j \to X$ and $U_j \cap U_j' \to X$ and choose a common refinement $X = \bigsqcup X_i$ of the resulting partitions. After refining the partition further we may assume $X_i$ affine as well. Fix $i$ and set $V = U \times_X X_i$. The morphisms $V_j = U_j \times_X X_i \to X_i$ and $V_j' = (U_j \cap U_j') \times_X X_i \to X_i$ are finite étale. Hence $V_j$ and $V_{j'}$ are affine schemes and $V_j \cap V_j' \subseteq V_j$ is closed as well as open (since $V_{j'} \to X_i$ is proper, so Morphisms, Lemma [39.7] applies). Then $V = \bigcup V_j$ is separated because $\mathcal{O}(V_j) \to \mathcal{O}(V_{j'})$ is surjective, see Schemes, Lemma [21.7]. Thus the previous case applies to $V \to X_i$ and we can further refine the partition if needed (it actually isn’t but we don’t need this). □

In the Noetherian case one can prove the preceding lemma by Noetherian induction and the following amusing lemma.

03S1 [Lemma 71.2.] Let $f : X \to Y$ be a morphism of schemes which is quasi-compact, quasi-separated, and locally of finite type. If $\eta$ is a generic point of an irreducible component of $Y$ such that $f^{-1}(\eta)$ is finite, then there exists an open $V \subseteq Y$ containing $\eta$ such that $f^{-1}(V) \subseteq V$ is finite.

**Proof.** This is Morphisms, Lemma [49.1]. □

The statement of the following lemma can be strengthened a bit.

095K [Lemma 71.3.] Let $f : Y \to X$ be a quasi-finite and finitely presented morphism of affine schemes.

1. There exists a surjective morphism of affine schemes $X' \to X$ and a closed subscheme $Z' \subseteq Y' = X' \times_X Y$ such that
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(a) $Z' \subset Y'$ is a thickening, and
(b) $Z' \to X'$ is a finite étale morphism.

(2) There exists a finite partition $X = \coprod X_i$ by locally closed, constructible, affine strata, and surjective finite locally free morphisms $X'_i \to X_i$ such that the reduction of $Y'_i = X'_i \times_X Y$ is isomorphic to $\coprod_{i=1}^{n_i} (X'_i)_{\text{red}} \to (X_i)_{\text{red}}$ for some $n_i$.

Proof. Setting $X' = \coprod X'_i$ we see that (2) implies (1). Write $X = \text{Spec}(A)$ and $Y = \text{Spec}(B)$. Write $A$ as a filtered colimit of finite type $\mathbb{Z}$-algebras $A_i$. Since $B$ is an $A$-algebra of finite presentation, we see that there exists $0 \in I$ and a finite type ring map $A_0 \to B_0$ such that $B = \text{colim} B_i$ with $B_i = A_i \otimes_{A_0} B_0$, see Algebra, Lemma[12.6.8]. For $i$ sufficiently large we see that $A_i \to B_i$ is quasi-finite, see Limits, Lemma[16.2]. Thus we reduce to the case of finite type algebras over $\mathbb{Z}$, in particular we reduce to the Noetherian case. (Details omitted.)

Assume $X$ and $Y$ Noetherian. In this case any locally closed subset of $X$ is constructible. By Lemma[17.2] and Noetherian induction we see that there is a finite partition $X = \coprod X_i$ of $X$ by locally closed strata such that $Y \times_X X_i \to X_i$ is finite. We can refine this partition to get affine strata. Thus after replacing $X$ by $X' = \coprod X_i$ we may assume $Y \to X$ is finite.

Assume $X$ and $Y$ Noetherian and $Y \to X$ finite. Suppose that we can prove (2) after base change by a surjective, flat, quasi-finite morphism $U \to X$. Thus we have a partition $U = \coprod U_i$ and finite locally free morphisms $U'_i \to U_i$ such that $U'_i \times_X Y \to U'_i$ is isomorphic to $\coprod_{i=1}^{n_i} (U'_i)_{\text{red}} \to (U_i)_{\text{red}}$ for some $n_i$. Then, by the argument in the previous paragraph, we can find a partition $X = \coprod X_j$ with locally closed affine strata such that $X_j \times_X U_i \to X_j$ is finite for all $i, j$. By Morphisms, Lemma[46.2] each $X_j \times_X U_i \to X_j$ is locally free. Hence $X_j \times_X U'_i \to X_j$ is locally free (Morphisms, Lemma[46.3]). It follows that $X = \coprod X_j$ and $X'_j = \coprod X_j \times_X U'_i$ is a solution for $Y \to X$. Thus it suffices to prove the result (in the Noetherian case) after a surjective flat quasi-finite base change.

Applying Morphisms, Lemma[46.6] we see we may assume that $Y$ is a closed subscheme of an affine scheme $Z$ which is (set theoretically) a union of closed subschemes mapping isomorphically to $X$. In this case we will find a finite partition of $X = \coprod X_j$ with affine locally closed strata that works (in other words $X'_j = X_j$). Set $T_i = Y \cap Z_i$. This is a closed subscheme of $X$. As $X$ is Noetherian we can find a finite partition of $X = \coprod X_j$ by affine locally closed subschemes, such that each $X_j \times_X T_i$ is (set theoretically) a union of strata $X_j \times_X Z_i$. Replacing $X$ by $X_j$ we see that we may assume $I = I_1 \sqcup I_2$ with $Z_i \subset Y$ for $i \in I_1$ and $Z_i \cap Y = \emptyset$ for $i \in I_2$. Replacing $Z$ by $\bigcup_{i \in I_1} Z_i$ we see that we may assume $Y = Z$. Finally, we can replace $X$ again by the members of a partition as above such that for every $i, i' \subset I$ the intersection $Z_i \cap Z_{i'}$ is either empty or (set theoretically) equal to $Z_i$ and $Z_{i'}$. This clearly means that $Y$ is (set theoretically) equal to a disjoint union of the $Z_i$ which is what we wanted to show.

72. More on constructible sheaves

Let $\Lambda$ be a Noetherian ring. Let $X$ be a scheme. We often consider $X_{\text{etale}}$ as a ringed site with sheaf of rings $\Lambda$. In case of abelian sheaves we often take $\Lambda = \mathbb{Z}/n\mathbb{Z}$ for a suitable integer $n$. 

095M
Lemma 72.1. Let $j : U \to X$ be an étale morphism of quasi-compact and quasi-separated schemes.

1. The sheaf $h_U$ is a constructible sheaf of sets.
2. The sheaf $j_! M$ is a constructible abelian sheaf for a finite abelian group $M$.
3. If $\Lambda$ is a Noetherian ring and $M$ is a finite $\Lambda$-module, then $j_! M$ is a constructible sheaf of $\Lambda$-modules on $X_{\text{étale}}$.

Proof. By Lemma 71.1 there is a partition $\coprod_i X_i$ such that $\pi_i : j^{-1}(X_i) \to X_i$ is finite étale. The restriction of $h_U$ to $X_i$ is $h_{j^{-1}(X_i)}$ which is finite locally constant by Lemma 63.4. For cases (2) and (3) we note that $j_!(M)|_{X_i} = \pi_i_!(M) = \pi_i_*(M)$ by Lemmas 69.4 and 69.5. Thus it suffices to show the lemma for $\pi : Y \to X$ finite étale. This is Lemma 63.3. □

Lemma 72.2. Let $X$ be a quasi-compact and quasi-separated scheme.

1. Let $F$ be a sheaf of sets on $X_{\text{étale}}$. Then $F$ is a filtered colimit of constructible sheaves of sets.
2. Let $F$ be an abelian sheaf on $X_{\text{étale}}$. Then $F$ is a filtered colimit of constructible abelian sheaves.
3. Let $\Lambda$ be a Noetherian ring and $F$ a sheaf of $\Lambda$-modules on $X_{\text{étale}}$. Then $F$ is a filtered colimit of constructible sheaves of $\Lambda$-modules.

Proof. Let $\mathcal{B}$ be the collection of quasi-compact and quasi-separated objects of $X_{\text{étale}}$. By Modules on Sites, Lemma 29.6 any sheaf of sets is a filtered colimit of sheaves of the form

$$\text{Coequalizer} \left( \coprod_{j=1,\ldots,m} h_{V_j} \rightrightarrows \coprod_{i=1,\ldots,n} j_{U_i} \right)$$

with $V_j$ and $U_i$ quasi-compact and quasi-separated objects of $X_{\text{étale}}$. By Lemmas 72.1 and 70.6 these coequalizers are constructible. This proves (1).

Let $\Lambda$ be a Noetherian ring. By Modules on Sites, Lemma 29.6 $\Lambda$-modules $F$ is a filtered colimit of modules of the form

$$\text{Coker} \left( \bigoplus_{j=1,\ldots,m} j_{V_j}! \Delta_{V_j} \to \bigoplus_{i=1,\ldots,n} j_{U_i}! \Delta_{U_i} \right)$$

with $V_j$ and $U_i$ quasi-compact and quasi-separated objects of $X_{\text{étale}}$. By Lemmas 72.1 and 70.6 these cokernels are constructible. This proves (3).

Proof of (2). First write $F = \bigcup F[n]$ where $F[n]$ is the $n$-torsion subsheaf. Then we can view $F[n]$ as a sheaf of $\mathbb{Z}/n\mathbb{Z}$-modules and apply (3). □

Lemma 72.3. Let $f : X \to Y$ be a surjective morphism of quasi-compact and quasi-separated schemes.

1. Let $F$ be a sheaf of sets on $Y_{\text{étale}}$. Then $F$ is constructible if and only if $f^{-1}F$ is constructible.
2. Let $F$ be an abelian sheaf on $Y_{\text{étale}}$. Then $F$ is constructible if and only if $f^{-1}F$ is constructible.
3. Let $\Lambda$ be a Noetherian ring. Let $F$ be sheaf of $\Lambda$-modules on $Y_{\text{étale}}$. Then $F$ is constructible if and only if $f^{-1}F$ is constructible.
Proof. One implication follows from Lemma \[70.5\] for the converse, assume \( f^{-1}\mathcal{F} \) is constructible. Write \( \mathcal{F} = \text{colim} \mathcal{F}_{i} \) as a filtered colimit of constructible sheaves (of sets, abelian groups, or modules) using Lemma \[72.2\]. Since \( f^{-1} \) is a left adjoint it commutes with colimits (Categories, Lemma \[24.5\]) and we see that \( f^{-1}\mathcal{F} = \text{colim} f^{-1}\mathcal{F}_{i} \). By Lemma \[70.9\] we see that \( f^{-1}\mathcal{F}_{i} \to f^{-1}\mathcal{F} \) is surjective for all \( i \) large enough. Since \( f \) is surjective we conclude (by looking at stalks using Lemma \[36.2\] and Theorem \[29.10\]) that \( \mathcal{F}_{i} \to \mathcal{F} \) is surjective for all \( i \) large enough. Thus \( \mathcal{F} \) is the quotient of a constructible sheaf \( \mathcal{G} \). Applying the argument once more to \( \mathcal{G} \to \mathcal{F} \) we conclude using that \( f^{-1} \) is exact and that the category of constructible sheaves of sets (of sets, abelian groups, or modules) is preserved under finite (co)limits or (co)kernels inside \( \text{Sh}(\mathcal{Y}_{\text{etale}}) \), \( \text{Ab}(\mathcal{Y}_{\text{etale}}) \), \( \text{Ab}(X_{\text{etale}}) \), \( \text{Mod}(\mathcal{Y}_{\text{etale}}, \Lambda) \), and \( \text{Mod}(X_{\text{etale}}, \Lambda) \), see Lemma \[70.6\].

\[095H\] Lemma 72.4. Let \( f : X \to Y \) be a finite étale morphism of schemes. Let \( \Lambda \) be a Noetherian ring. If \( \mathcal{F} \) is a constructible sheaf of sets, constructible sheaf of abelian groups, or constructible sheaf of \( \Lambda \)-modules on \( X_{\text{etale}} \), the same is true for \( f_{*}\mathcal{F} \) on \( Y_{\text{etale}} \).

Proof. By Lemma \[70.4\] it suffices to check this Zariski locally on \( Y \) and by Lemma \[72.3\] we may replace \( Y \) by an étale cover (the construction of \( f_{*} \) commutes with étale localization). A finite étale morphism is étale locally isomorphic to a disjoint union of isomorphisms, see Étale Morphisms, Lemma \[18.3\]. Thus, in the case of sheaves of sets, the lemma says that if \( \mathcal{F}_{i}, i = 1, \ldots, n \) are constructible sheaves of sets, then \( \prod_{i=1}^{n} \mathcal{F}_{i} \) is too. This is clear. Similarly for sheaves of abelian groups and modules.

\[09Y9\] Lemma 72.5. Let \( X \) be a quasi-compact and quasi-separated scheme. The category of constructible sheaves of sets is the full subcategory of \( \text{Sh}(X_{\text{etale}}) \) consisting of sheaves \( \mathcal{F} \) which are coequalizers

\[
\begin{array}{ccc}
\mathcal{F}_{1} & \longrightarrow & \mathcal{F}_{0} \\
\downarrow & & \downarrow \\
\mathcal{F} & \longrightarrow & \mathcal{F}
\end{array}
\]

such that \( \mathcal{F}_{i}, i = 0, 1 \) is a finite coproduct of sheaves of the form \( h_{U} \) with \( U \) a quasi-compact and quasi-separated object of \( X_{\text{etale}} \).

Proof. In the proof of Lemma \[72.2\] we have seen that sheaves of this form are constructible. For the converse, suppose that for every constructible sheaf of sets \( \mathcal{F} \) we can find a surjection \( \mathcal{F}_{0} \to \mathcal{F} \) with \( \mathcal{F}_{0} \) as in the lemma. Then we find our surjection \( \mathcal{F}_{1} \to \mathcal{F}_{0} \times_{\mathcal{F}} \mathcal{F}_{0} \) because the latter is constructible by Lemma \[70.6\].

By Topology, Lemma \[28.7\] we may choose a finite stratification \( X = \coprod_{i \in I} X_{i} \) such that \( \mathcal{F} \) is finite locally constant on each stratum. We will prove the result by induction on the cardinality of \( I \). Let \( i \in I \) be a minimal element in the partial ordering of \( I \). Then \( X_{i} \subset X \) is closed. By induction, there exist finitely many quasi-compact and quasi-separated objects \( U_{\alpha} \) of \( (X \setminus X_{i})_{\text{etale}} \) and a surjective map \( \prod h_{U_{\alpha}} \to \mathcal{F}|_{X \setminus X_{i}} \). These determine a map

\[
\prod h_{U_{\alpha}} \to \mathcal{F}
\]

which is surjective after restricting to \( X \setminus X_{i} \). By Lemma \[63.4\] we see that \( \mathcal{F}|_{X_{i}} = h_{V} \) for some scheme \( V \) finite étale over \( X_{i} \). Let \( \pi \) be a geometric point of \( V \) lying over \( \pi \in X_{i} \). We may think of \( \pi \) as an element of the stalk \( \mathcal{F}_{\pi} = V_{\pi} \). Thus we can find an étale neighbourhood \( (U, \pi) \) of \( \pi \) and a section \( s \in \mathcal{F}(U) \) whose stalk at \( \pi \) gives
Thinking of $s$ as a map $s : h_U \to \mathcal{F}$, restricting to $X_i$ we obtain a morphism $s|_{X_i} : U \times_X X_i \to V$ over $X_i$ which maps $\overline{s}$ to $\overline{v}$. Since $V$ is quasi-compact (finite over the closed subscheme $X_i$ of the quasi-compact scheme $X$) a finite number $s^{(1)}, \ldots, s^{(m)}$ of these sections of $\mathcal{F}$ over $U^{(1)}, \ldots, U^{(m)}$ will determine a jointly surjective map

$$\prod s^{(j)}|_{X_i} : \prod U^{(j)} \times_X X_i \to V$$

Then we obtain the surjection

$$\prod h_{U_i} \to \mathcal{F}$$

as desired.

**Lemma 72.6.** Let $X$ be a quasi-compact and quasi-separated scheme. Let $\Lambda$ be a Noetherian ring. The category of constructible sheaves of $\Lambda$-modules is exactly the category of modules of the form

$$\text{Coker} \left( \bigoplus_{j=1,\ldots,m} j_{V_j!} \Delta_{V_j} \to \bigoplus_{i=1,\ldots,n} j_{U_i!} \Delta_{U_i} \right)$$

with $V_j$ and $U_i$ quasi-compact and quasi-separated objects of $X_{\text{étale}}$. In fact, we can even assume $U_i$ and $V_j$ affine.

**Proof.** In the proof of Lemma 72.2 we have seen modules of this form are constructible. Since the category of constructible modules is abelian (Lemma 70.6) it suffices to prove that given a constructible module $\mathcal{F}$ there is a surjection

$$\bigoplus_{i=1,\ldots,n} j_{U_i!} \Delta_{U_i} \to \mathcal{F}$$

for some affine objects $U_i$ in $X_{\text{étale}}$. By Modules on Sites, Lemma 29.6 there is a surjection

$$\Psi : \bigoplus_{i \in I} j_{U_i!} \Delta_{U_i} \to \mathcal{F}$$

with $U_i$ affine and the direct sum over a possibly infinite index set $I$. For every finite subset $I' \subset I$ set

$$T_{I'} = \text{Supp}(\text{Coker}(\bigoplus_{i \in I'} j_{U_i!} \Delta_{U_i} \to \mathcal{F}))$$

By the very definition of constructible sheaves, the set $T_{I'}$ is a constructible subset of $X$. We want to show that $T_{I'} = \emptyset$ for some $I'$. Since every stalk $\mathcal{F}_x$ is a finite type $\Lambda$-module and since $\Psi$ is surjective, for every $x \in X$ there is an $I'$ such that $x \notin T_{I'}$. In other words we have $\emptyset = \bigcap_{I' \subset I \text{ finite}} T_{I'}$. Since $X$ is a spectral space by Properties, Lemma 2.4 the constructible topology on $X$ is quasi-compact by Topology, Lemma 23.2. Thus $T_{I'} = \emptyset$ for some $I' \subset I$ finite as desired. \hfill $\square$

**Lemma 72.7.** Let $X$ be a quasi-compact and quasi-separated scheme. The category of constructible abelian sheaves is exactly the category of abelian sheaves of the form

$$\text{Coker} \left( \bigoplus_{j=1,\ldots,m} j_{V_j!} \mathbb{Z}/m_j \mathbb{Z}_{V_j} \to \bigoplus_{i=1,\ldots,n} j_{U_i!} \mathbb{Z}/n_i \mathbb{Z}_{U_i} \right)$$

with $V_j$ and $U_i$ quasi-compact and quasi-separated objects of $X_{\text{étale}}$ and $m_j$, $n_i$ positive integers. In fact, we can even assume $U_i$ and $V_j$ affine.

**Proof.** This follows from Lemma 72.6 applied with $\Lambda = \mathbb{Z}/n \mathbb{Z}$ and the fact that, since $X$ is quasi-compact, every constructible abelian sheaf is annihilated by some positive integer $n$ (details omitted). \hfill $\square$
09Z4 **Lemma 72.8.** Let $X$ be a quasi-compact and quasi-separated scheme. Let $\Lambda$ be a Noetherian ring. Let $\mathcal{F}$ be a constructible sheaf of sets, abelian groups, or $\Lambda$-modules on $X_{\text{etale}}$. Let $\mathcal{G} = \text{colim} \mathcal{G}_i$ be a filtered colimit of sheaves of sets, abelian groups, or $\Lambda$-modules. Then

$$\text{Mor}(\mathcal{F}, \mathcal{G}) = \text{colim} \text{Mor}(\mathcal{F}, \mathcal{G}_i)$$

in the category of sheaves of sets, abelian groups, or $\Lambda$-modules on $X_{\text{etale}}$.

**Proof.** The case of sheaves of sets. By Lemma 72.5, it suffices to prove the lemma for $h_U$ where $U$ is a quasi-compact and quasi-separated object of $X_{\text{etale}}$. Recall that $\text{Mor}(h_U, \mathcal{G}) = \mathcal{G}(U)$. Hence the result follows from Sites, Lemma 17.5. In the case of abelian sheaves or sheaves of modules, the result follows in the same way using Lemmas 72.7 and 72.6. For the case of abelian sheaves, we add that $\text{Mor}(j_! \mathbb{Z}/n \mathbb{Z}, \mathcal{G})$ is equal to the $n$-torsion elements of $\mathcal{G}(U)$.

095R **Lemma 72.9.** Let $f : X \to Y$ be a finite and finitely presented morphism of schemes. Let $\Lambda$ be a Noetherian ring. If $\mathcal{F}$ is a constructible sheaf of sets, abelian groups, or $\Lambda$-modules on $X_{\text{etale}}$, then $f_* \mathcal{F}$ is too.

**Proof.** It suffices to prove this when $X$ and $Y$ are affine by Lemma 70.4. By Lemmas 51.3 and 72.3, we may base change to any affine scheme surjective over $X$. By Lemma 71.3, this reduces us to the case of a finite étale morphism (because a thickening leads to an equivalence of étale topoi and even small étale sites, see Theorem 45.2). The finite étale case is Lemma 72.4.

09YU **Lemma 72.10.** Let $X = \lim_{i \in I} X_i$ be a limit of a directed system of schemes with affine transition morphisms. We assume that $X_i$ is quasi-compact and quasi-separated for all $i \in I$.

1. The category of constructible sheaves of sets on $X_{\text{etale}}$ is the colimit of the categories of constructible sheaves of sets on $(X_i)_{\text{etale}}$.
2. The category of constructible abelian sheaves on $X_{\text{etale}}$ is the colimit of the categories of constructible abelian sheaves on $(X_i)_{\text{etale}}$.
3. Let $\Lambda$ be a Noetherian ring. The category of constructible sheaves of $\Lambda$-modules on $X_{\text{etale}}$ is the colimit of the categories of constructible sheaves of $\Lambda$-modules on $(X_i)_{\text{etale}}$.

**Proof.** Proof of (1). Denote $f_i : X \to X_i$ the projection maps. There are 3 parts to the proof corresponding to “faithful”, “fully faithful”, and “essentially surjective”. Faithful. Choose $0 \in I$ and let $\mathcal{F}_0, \mathcal{G}_0$ be constructible sheaves on $X_0$. Suppose that $a, b : \mathcal{F}_0 \to \mathcal{G}_0$ are maps such that $f_0^{-1}a = f_0^{-1}b$. Let $E \subset X_0$ be the set of points $x \in X_0$ such that $a_x = b_x$. By Lemma 70.8, the subset $E \subset X_0$ is constructible. By assumption $X \to X_0$ maps into $E$. By Limits, Lemma 4.10, we find an $i \geq 0$ such that $X_i \to X_0$ maps into $E$. Hence $f_0^{-1}a = f_0^{-1}b$.

Fully faithful. Choose $0 \in I$ and let $\mathcal{F}_0, \mathcal{G}_0$ be constructible sheaves on $X_0$. Suppose that $a : f_0^{-1} \mathcal{F}_0 \to f_0^{-1} \mathcal{G}_0$ is a map. We claim there is an $i$ and a map $a_i : f_0^{-1} \mathcal{F}_0 \to f_0^{-1} \mathcal{G}_0$ which pulls back to $a$ on $X$. By Lemma 72.5, we can replace $\mathcal{F}_0$ by a finite coproduct of sheaves represented by quasi-compact and quasi-separated objects of $(X_0)_{\text{etale}}$. Thus we have to show: If $U_0 \to X_0$ is such an object of $(X_0)_{\text{etale}}$, then

$$f_0^{-1} \mathcal{G}(U) = \text{colim}_{i \geq 0} f_0^{-1} \mathcal{G}(U_i)$$
where $U = X \times X, U_0$ and $U_i = X_i \times X_i U_0$. This is a special case of Theorem 51.3
Essentially surjective. We have to show every constructible $\mathcal{F}$ on $X$ is isomorphic to $f^{-1}_i \mathcal{F}$ for some constructible $\mathcal{F}_i$ on $X_i$. Applying Lemma 72.2 and using the results of the previous two paragraphs, we see that it suffices to prove this for $h_U$ for some quasi-compact and quasi-separated object $U$ of $X_{\text{étale}}$. In this case we have to show that $U$ is the base change of a quasi-compact and quasi-separated scheme étale over $X_i$ for some $i$. This follows from Limits, Lemmas 10.1 and 8.10

Proof of (3). The argument is very similar to the argument for sheaves of sets, but using Lemma 72.6 instead of Lemma 72.5. Details omitted. Part (2) follows from part (3) because every constructible abelian sheaf over a quasi-compact scheme is a constructible sheaf of $\mathbb{Z}/n\mathbb{Z}$-modules for some $n$. □

**Lemma 72.11.** Let $X$ be an irreducible scheme with generic point $\eta$.

1. Let $S' \subset S$ be an inclusion of sets. If we have $S' \subset S \subset \mathcal{S}$ in $\text{Sh}(X_{\text{étale}})$ and $S' = \mathcal{S}'$, then $\mathcal{G} = \mathcal{S}'$.
2. Let $A' \subset A$ be an inclusion of abelian groups. If we have $A' \subset \mathcal{G} \subset \mathcal{A}$ in $\text{Ab}(X_{\text{étale}})$ and $A' = \mathcal{G}_{\mathcal{P}}$, then $\mathcal{G} = A'$.  
3. Let $M' \subset M$ be an inclusion of modules over a ring $\Lambda$. If we have $M' \subset \mathcal{G} \subset \mathcal{M}$ in $\text{Mod}(X_{\text{étale}}, \Lambda)$ and $M' = \mathcal{G}_{\mathcal{P}}$, then $\mathcal{G} = M'$.  

**Proof.** This is true because for every étale morphism $U \to X$ with $U \neq \emptyset$ the point $\eta$ is in the image. □

**Lemma 72.12.** Let $X$ be an integral normal scheme with function field $K$. Let $E$ be a set.

1. Let $g : \text{Spec}(K) \to X$ be the inclusion of the generic point. Then $g_* E = E$.
2. Let $j : U \to X$ be the inclusion of a nonempty open. Then $j_* E = E$.

**Proof.** Proof of (1). Let $x \in X$ be a point. Let $\mathcal{O}_{X,x}$ be a strict henselization of $\mathcal{O}_{X,x}$. By More on Algebra, Lemma 14.6 we see that $\mathcal{O}_{X,x}$ is a normal domain. Hence $\text{Spec}(K) \times_X \text{Spec}(\mathcal{O}_{X,x})$ is irreducible. It follows that the stalk $(g_* E)_x$ is equal to $E$, see Theorem 52.1.

Proof of (2). Since $g$ factors through $j$ there is a map $j_* E \to g_* E$. This map is injective because for every scheme $V$ étale over $X$ the set $\text{Spec}(K) \times_X V$ is dense in $U \times_X V$. On the other hand, we have a map $E \to j_* E$ and we conclude. □

**Lemma 72.13.** Let $X$ be a quasi-compact and quasi-separated scheme. Let $\eta \in X$ be a generic point of an irreducible component of $X$.

1. Let $\mathcal{F}$ be a torsion abelian sheaf on $X_{\text{étale}}$ whose stalk $\mathcal{F}_\eta$ is zero. Then $\mathcal{F} = \text{colim} \mathcal{F}_i$ is a filtered colimit of constructible abelian sheaves $\mathcal{F}_i$ such that for each $i$ the support of $\mathcal{F}_i$ is contained in a closed subscheme not containing $\eta$.
2. Let $\Lambda$ be a Noetherian ring and $\mathcal{F}$ a sheaf of $\Lambda$-modules on $X_{\text{étale}}$ whose stalk $\mathcal{F}_\eta$ is zero. Then $\mathcal{F} = \text{colim} \mathcal{F}_i$ is a filtered colimit of constructible sheaves of $\Lambda$-modules $\mathcal{F}_i$ such that for each $i$ the support of $\mathcal{F}_i$ is contained in a closed subscheme not containing $\eta$.

**Proof.** Proof of (1). We can write $\mathcal{F} = \text{colim}_{i \in I} \mathcal{F}_i$ with $\mathcal{F}_i$ constructible abelian by Lemma 72.2. Choose $i \in I$. Since $\mathcal{F}_i|_\eta$ is zero by assumption, we see that there exists an $i'(i) \geq i$ such that $\mathcal{F}_i|_\eta \to \mathcal{F}_{i'(i)}|_\eta$ is zero, see Lemma 70.9. Then
Étale cohomology

Proposition 73.1. Let \( X \) be a Noetherian scheme. Let \( \Lambda \) be a Noetherian ring.

1. Any sub or quotient sheaf of a constructible sheaf of sets is constructible.
2. The category of constructible abelian sheaves on \( X_{\text{étale}} \) is a (strong) Serre subcategory of \( \text{Ab}(X_{\text{étale}}) \). In particular, every sub and quotient sheaf of a constructible abelian sheaf on \( X_{\text{étale}} \) is constructible.
3. The category of constructible sheaves of \( \Lambda \)-modules on \( X_{\text{étale}} \) is a (strong) Serre subcategory of \( \text{Mod}(X_{\text{étale}}, \Lambda) \). In particular, every submodule and quotient module of a constructible sheaf of \( \Lambda \)-modules on \( X_{\text{étale}} \) is constructible.

Proof. Proof of (1). Let \( \mathcal{G} \subset \mathcal{F} \) with \( \mathcal{F} \) a constructible sheaf of sets on \( X_{\text{étale}} \). Let \( \eta \in X \) be a generic point of an irreducible component of \( X \). By Noetherian induction it suffices to find an open neighbourhood \( U \) of \( \eta \) such that \( \mathcal{G}|_U \) is locally constant. To do this we may replace \( X \) by an étale neighbourhood of \( \eta \). Hence we may assume \( \mathcal{F} \) is constant and \( X \) is irreducible.

Say \( \mathcal{F} = \mathcal{S} \) for some finite set \( S \). Then \( S' = \mathcal{G}\eta \subset S \) say \( S' = \{ s_1, \ldots, s_t \} \). Pick an étale neighbourhood \( (U, \varpi) \) of \( \eta \) and sections \( \sigma_1, \ldots, \sigma_t \in \mathcal{G}(U) \) which map to \( s_i \) in \( \mathcal{G}\eta \subset S \). Since \( \sigma_i \) maps to an element \( s_i \in S' \subset S = \Gamma(X, \mathcal{F}) \) we see that the two pullbacks of \( \sigma_i \) to \( U \times_X U \) are the same as sections of \( \mathcal{G} \). By the sheaf condition for \( \mathcal{G} \) we find that \( \sigma_i \) comes from a section of \( \mathcal{G} \) over the open \( \text{Im}(U \to X) \) of \( X \). Shrinking \( X \) we may assume \( S' \subset \mathcal{G} \subset S \). Then we see that \( S' = \mathcal{G} \) by Lemma 72.11.

Let \( \mathcal{F} \to \mathcal{Q} \) be a surjection with \( \mathcal{F} \) a constructible sheaf of sets on \( X_{\text{étale}} \). Then set \( \mathcal{G} = \mathcal{F} \times_{\mathcal{Q}} \mathcal{F} \). By the first part of the proof we see that \( \mathcal{G} \) is constructible as a subsheaf of \( \mathcal{F} \times \mathcal{F} \). This in turn implies that \( \mathcal{Q} \) is constructible, see Lemma 70.6.

Proof of (3). we already know that constructible sheaves of modules form a weak Serre subcategory, see Lemma 70.6. Thus it suffices to show the statement on submodules.
Let $G \subset F$ be a submodule of a constructible sheaf of $\Lambda$-modules on $X_{\text{étale}}$. Let $\eta \in X$ be a generic point of an irreducible component of $X$. By Noetherian induction it suffices to find an open neighbourhood $U$ of $\eta$ such that $G|_U$ is locally constant. To do this we may replace $X$ by an étale neighbourhood of $\eta$. Hence we may assume $F$ is constant and $X$ is irreducible.

Say $F = M$ for some finite $\Lambda$-module $M$. Then $M' = G_{\eta} \subset M$. Pick finitely many elements $s_1, \ldots, s_t$ generating $M'$ as a $\Lambda$-module. (This is possible as $\Lambda$ is Noetherian and $M$ is finite.) Pick an étale neighbourhood $(U, \pi)$ of $\eta$ and sections $\sigma_1, \ldots, \sigma_t \in G(U)$ which map to $s_i$ in $G_{\eta} \subset M$. Since $\sigma_i$ maps to an element $s_i \in M' \subset M = \Gamma(X, F)$ we see that the two pullbacks of $\sigma_i$ to $U \times_X U$ are the same as sections of $G$. By the sheaf condition for $G$ we find that $\sigma_i$ comes from a section of $G$ over the open $\Im(U \to X)$ of $X$. Shrinking $X$ we may assume $M' \subset G \subset M$. Then we see that $M' = G$ by Lemma 72.11.

Proof of (2). This follows in the usual manner from (3). Details omitted. □

The following lemma tells us that every object of the abelian category of constructible sheaves on $X$ is “Noetherian”, i.e., satisfies a.c.c. for subobjects.

**Lemma 73.2.** Let $X$ be a Noetherian scheme. Let $\Lambda$ be a Noetherian ring. Consider inclusions

$$F_1 \subset F_2 \subset F_3 \subset \ldots \subset F$$

in the category of sheaves of sets, abelian groups, or $\Lambda$-modules. If $F$ is constructible, then for some $n$ we have $F_n = F_{n+1} = F_{n+2} = \ldots$.

**Proof.** By Proposition 73.1 we see that $F_i$ and colim $F_i$ are constructible. Then the lemma follows from Lemma 70.9. □

**Lemma 73.3.** Let $X$ be a Noetherian scheme.

1. Let $F$ be a constructible sheaf of sets on $X_{\text{étale}}$. There exist an injective map of sheaves

$$F \longrightarrow \coprod_{i=1, \ldots, n} f_i^* E_i$$

where $f_i : Y_i \to X$ is a finite morphism and $E_i$ is a finite set.

2. Let $F$ be a constructible abelian sheaf on $X_{\text{étale}}$. There exist an injective map of abelian sheaves

$$F \longrightarrow \bigoplus_{i=1, \ldots, n} f_i^* M_i$$

where $f_i : Y_i \to X$ is a finite morphism and $M_i$ is a finite abelian group.

3. Let $\Lambda$ be a Noetherian ring. Let $F$ be a constructible sheaf of $\Lambda$-modules on $X_{\text{étale}}$. There exist an injective map of sheaves of modules

$$F \longrightarrow \bigoplus_{i=1, \ldots, n} f_i^* M_i$$

where $f_i : Y_i \to X$ is a finite morphism and $M_i$ is a finite $\Lambda$-module.

Moreover, we may assume each $Y_i$ is irreducible, reduced, maps onto an irreducible and reduced closed subscheme $Z_i \subset X$ such that $Y_i \to Z_i$ is finite étale over a nonempty open of $Z_i$. 

Proof. Proof of (1). Because we have the ascending chain condition for subsheaves of \( \mathcal{F} \) (Lemma 73.2), it suffices to show that for every point \( x \in X \) we can find a map \( \varphi : \mathcal{F} \to f_\ast \mathcal{E} \) where \( f : Y \to X \) is finite and \( E \) is a finite set such that \( \varphi_\pi : \mathcal{F}_\pi \to (f_\ast \mathcal{E})_\pi \) is injective. (This argument can be avoided by picking a partition of \( X \) as in Lemma 70.2 and constructing a \( Y_i \to X \) for each irreducible component of each part.) Let \( Z \subset X \) be the induced reduced scheme structure (Schemes, Definition 12.5) on \( \{x\} \). Since \( \mathcal{F} \) is constructible, there is a finite separable extension \( \kappa(x) \subset \text{Spec}(K) \) such that \( \mathcal{F}|_{\text{Spec}(K)} \) is the constant sheaf with value \( E \) for some finite set \( E \). Let \( Y \to Z \) be the normalization of \( Z \) in \( \text{Spec}(K) \). By Morphisms, Lemma 51.13 we see that \( Y \) is a normal integral scheme. As \( \kappa(x) \subset K \) is finite, it is clear that \( K \) is the function field of \( Y \). Denote \( g : \text{Spec}(K) \to Y \) the inclusion. The map \( \mathcal{F}|_{\text{Spec}(K)} \to E \) is adjoint to a map \( \mathcal{F}|_Y \to g_\ast E = E \) (Lemma 72.12). This in turn is adjoint to a map \( \varphi : \mathcal{F} \to f_\ast E \). Observe that the stalk of \( \varphi \) at a geometric point \( \pi \) is injective: we may take a lift \( y \in Y \) of \( x \) and the commutative diagram

\[
\begin{array}{ccc}
\mathcal{F}_\pi & \to & (\mathcal{F}|_Y)_\pi \\
\downarrow & & \downarrow \\
(f_\ast E)_\pi & \to & E_{\pi}
\end{array}
\]

proves the injectivity. We are not yet done, however, as the morphism \( f : Y \to Z \) is integral but in general not finite.

To fix the problem stated in the last sentence of the previous paragraph, we write \( Y = \text{lim}_{i \in I} Y_i \) with \( Y_i \) irreducible, integral, and finite over \( Z \). Namely, apply Properties, Lemma 22.13 to \( f_\ast \mathcal{O}_Y \) viewed as a sheaf of \( \mathcal{O}_Z \)-algebras and apply the functor \( \text{Spec}_\pi \). Then \( f_\ast E = \text{colim} \ f_i_\ast E \) by Lemma 51.7. By Lemma 72.8 the map \( \mathcal{F} \to f_\ast E \) factors through \( f_i_\ast E \) for some \( i \). Since \( Y_i \to Z \) is a finite morphism of integral schemes and since the function field extension induced by this morphism is finite separable, we see that the morphism is finite étale over a nonempty open of \( Z \) (use Algebra, Lemma 138.9; details omitted). This finishes the proof of (1).

The proofs of (2) and (3) are identical to the proof of (1).

In the following lemma we use a standard trick to reduce a very general statement to the Noetherian case.

\textbf{Lemma 73.4.} Let \( X \) be a quasi-compact and quasi-separated scheme.

1. Let \( \mathcal{F} \) be a constructible sheaf of sets on \( X_{\text{étale}} \). There exist an injective map of sheaves

\[ \mathcal{F} \to \prod_{i=1,...,n} f_i_\ast E_i \]

where \( f_i : Y_i \to X \) is a finite and finitely presented morphism and \( E_i \) is a finite set.

2. Let \( \mathcal{F} \) be a constructible abelian sheaf on \( X_{\text{étale}} \). There exist an injective map of abelian sheaves

\[ \mathcal{F} \to \bigoplus_{i=1,...,n} f_i_\ast M_i \]

where \( f_i : Y_i \to X \) is a finite and finitely presented morphism and \( M_i \) is a finite abelian group.

\textsuperscript{6}If \( X \) is a Nagata scheme, for example of finite type over a field, then \( Y \to Z \) is finite.
(3) Let $\Lambda$ be a Noetherian ring. Let $F$ be a constructible sheaf of $\Lambda$-modules on $X_{\text{étale}}$. There exist an injective map of sheaves of modules

$$F \rightarrow \bigoplus_{i=1}^{n} f_{i,*} M_i$$

where $f_i : Y_i \rightarrow X$ is a finite and finitely presented morphism and $M_i$ is a finite $\Lambda$-module.

**Proof.** We will reduce this lemma to the Noetherian case by absolute Noetherian approximation. Namely, by Limits, Proposition 5.4 we can write $X = \lim_{t \in T} X_t$ with each $X_t$ of finite type over $\text{Spec}(\mathbb{Z})$ and with affine transition morphisms. By Lemma 72.10 the category of constructible sheaves (of sets, abelian groups, or $\Lambda$-modules) on $X_{\text{étale}}$ is the colimit of the corresponding categories for $X_t$. Thus our constructible sheaf $F$ is the pullback of a similar constructible sheaf $F_t$ over $X_t$ for some $t$. Then we apply the Noetherian case (Lemma 73.3) to find an injection $F_t \rightarrow \bigoplus_{i=1}^{n} f_{i,*} M_i$ over $X_t$ for some finite morphisms $f_i : Y_i \rightarrow X_t$. Since $X_t$ is Noetherian the morphisms $f_i$ are of finite presentation. Since pullback is exact and since formation of $f_{i,*}$ commutes with base change (Lemma 54.3), we conclude. \[\square\]

**Lemma 73.5.** Let $X$ be a Noetherian scheme. Let $E \subset X$ be a subset closed under specialization.

(1) Let $F$ be a torsion abelian sheaf on $X_{\text{étale}}$ whose support is contained in $E$. Then $F = \text{colim} F_i$ is a filtered colimit of constructible abelian sheaves $F_i$ such that for each $i$ the support of $F_i$ is contained in a closed subset contained in $E$.

(2) Let $\Lambda$ be a Noetherian ring and $F$ a sheaf of $\Lambda$-modules on $X_{\text{étale}}$ whose support is contained in $E$. Then $F = \text{colim} F_i$ is a filtered colimit of constructible sheaves of $\Lambda$-modules $F_i$ such that for each $i$ the support of $F_i$ is contained in a closed subset contained in $E$.

**Proof.** Proof of (1). We can write $F = \text{colim}_{i \in I} F_i$ with $F_i$ constructible abelian by Lemma 72.2. By Proposition 73.1 the image $F_i' \subset F$ of the map $F_i \rightarrow F$ is constructible. Thus $F = \text{colim} F_i'$ and the support of $F_i'$ is contained in $E$. Since the support of $F_i'$ is constructible (by our definition of constructible sheaves), we see that its closure is also contained in $E$, see for example Topology, Lemma 23.5. The proof in case (2) is exactly the same and we omit it. \[\square\]

74. Torsion sheaves

A brief section on torsion abelian sheaves and their étale cohomology. Let $C$ be a site. We have shown in Cohomology on Sites, Lemma 20.7 that any object in $D(C)$ whose cohomology sheaves are torsion sheaves, can be represented by a complex all of whose terms are torsion. In particular, if $X$ is a scheme, then $\text{Coh}^+_t(X)$ is the full subcategory of $\text{Coh}(X)$ consisting of torsion sheaves. For $n \geq 0$ let $\text{Coh}^+_t(X)^n$ be the set of torsion sheaves of $\Lambda$-modules on $X_{\text{étale}}$ whose cohomology groups $H^i(X,F)$ are torsion for all $i$. Consider the category $\text{Coh}^+_t(X)$ with tensor products. Then it is a category, see Additivity, Section 44.1

**Lemma 74.1.** Let $X$ be a quasi-compact and quasi-separated scheme.

(1) If $\mathcal{F}$ is a torsion abelian sheaf on $X_{\text{étale}}$, then $H^n_{\text{étale}}(X, \mathcal{F})$ is a torsion abelian group for all $n$.

(2) If $K$ in $D^+(X_{\text{étale}})$ has torsion cohomology sheaves, then $H^n_{\text{étale}}(X, K)$ is a torsion abelian group for all $n$. 

Proof. To prove (1) we write $\mathcal{F} = \bigcup \mathcal{F}[n]$ where $\mathcal{F}[d]$ is the $d$-torsion subsheaf.

By Lemma 51.4 we have $H^n_{\text{étale}}(X, \mathcal{F}) = \text{colim} H^n_{\text{étale}}(X, \mathcal{F}[d])$. This proves (1) as $H^n_{\text{étale}}(X, \mathcal{F}[d])$ is annihilated by $d$.

To prove (2) we can use the spectral sequence $E^{pq}_2 = H^p_{\text{étale}}(X, H^q(K))$ converging to $H^n_{\text{étale}}(X, K)$ (Derived Categories, Lemma 21.3) and the result for sheaves. □

Lemma 74.2. Let $f : X \to Y$ be a quasi-compact and quasi-separated morphism of schemes.

1. If $\mathcal{F}$ is a torsion abelian sheaf on $X_{\text{étale}}$, then $R^n f_* \mathcal{F}$ is a torsion abelian sheaf on $Y_{\text{étale}}$ for all $n$.
2. If $K \in D^+(X_{\text{étale}})$ has torsion cohomology sheaves, then $Rf_* K$ is an object of $D^+(Y_{\text{étale}})$ whose cohomology sheaves are torsion abelian sheaves.

Proof. Proof of (1). Recall that $R^n f_* \mathcal{F}$ is the sheaf associated to the presheaf $V \mapsto H^n_{\text{étale}}(X \times_Y V, \mathcal{F})$ on $Y_{\text{étale}}$. See Cohomology on Sites, Lemma 8.4. If we choose $V$ affine, then $X \times_Y V$ is quasi-compact and quasi-separated because $f$ is, hence we can apply Lemma 74.1 to see that $H^n_{\text{étale}}(X \times_Y V, \mathcal{F})$ is torsion.

Proof of (2). Recall that $R^n f_* K$ is the sheaf associated to the presheaf $V \mapsto H^n_{\text{étale}}(X \times_Y V, K)$ on $Y_{\text{étale}}$. See Cohomology on Sites, Lemma 21.6. If we choose $V$ affine, then $X \times_Y V$ is quasi-compact and quasi-separated because $f$ is, hence we can apply Lemma 74.1 to see that $H^n_{\text{étale}}(X \times_Y V, K)$ is torsion. □

75. Cohomology with support in a closed subscheme

Let $X$ be a scheme and let $Z \subset X$ be a closed subscheme. Let $\mathcal{F}$ be an abelian sheaf on $X_{\text{étale}}$. We let

$$\Gamma_Z(X, \mathcal{F}) = \{ s \in \mathcal{F}(X) \mid \text{Supp}(s) \subset Z \}$$

be the sections with support in $Z$ (Definition 31.3). This is a left exact functor which is not exact in general. Hence we obtain a derived functor

$$R\Gamma_Z(X, -) : D(X_{\text{étale}}) \to D(\text{Ab})$$

and cohomology groups with support in $Z$ defined by $H^i_Z(X, \mathcal{F}) = R^i \Gamma_Z(X, \mathcal{F})$.

Let $\mathcal{I}$ be an injective abelian sheaf on $X_{\text{étale}}$. Let $U = X \setminus Z$. Then the restriction map $\mathcal{I}(X) \to \mathcal{I}(U)$ is surjective (Cohomology on Sites, Lemma 13.6) with kernel $\Gamma_Z(X, \mathcal{I})$. It immediately follows that for $K \in D(X_{\text{étale}})$ there is a distinguished triangle

$$R\Gamma_Z(X, K) \to R\Gamma(X, K) \to R\Gamma(U, K) \to R\Gamma_Z(X, K)[1]$$

in $D(\text{Ab})$. As a consequence we obtain a long exact cohomology sequence

$$\cdots \to H^i_Z(X, K) \to H^i(X, K) \to H^i(U, K) \to H^{i+1}_Z(X, K) \to \cdots$$

for any $K$ in $D(X_{\text{étale}})$.

For an abelian sheaf $\mathcal{F}$ on $X_{\text{étale}}$ we can consider the subsheaf of sections with support in $Z$, denoted $\mathcal{H}_Z(\mathcal{F})$, defined by the rule

$$\mathcal{H}_Z(\mathcal{F})(U) = \{ s \in \mathcal{F}(U) \mid \text{Supp}(s) \subset U \times_X Z \}$$
Let $i : Z \to X$ be a closed immersion of schemes. Let $\mathcal{I}$ be an injective abelian sheaf on $X_{\text{étale}}$. Then $\mathcal{H}_{Z}(\mathcal{I})$ is an injective abelian sheaf on $Z_{\text{étale}}$.

**Proof.** Observe that for any abelian sheaf $\mathcal{G}$ on $Z_{\text{étale}}$ we have

$$\text{Hom}_{Z}(\mathcal{G}, \mathcal{H}_{Z}(\mathcal{F})) = \text{Hom}_{X}(i_{*}\mathcal{G}, \mathcal{F})$$

because after all any section of $i_{*}\mathcal{G}$ has support in $Z$. Since $i_{*}$ is exact (Section 46) and as $\mathcal{I}$ is injective on $X_{\text{étale}}$ we conclude that $\mathcal{H}_{Z}(\mathcal{I})$ is injective on $Z_{\text{étale}}$. 

Denote

$$R\mathcal{H}_{Z} : D(X_{\text{étale}}) \to D(Z_{\text{étale}})$$

the derived functor. We set $\mathcal{H}_{Z}^{p}(\mathcal{F}) = R^{p}\mathcal{H}_{Z}(\mathcal{F})$ so that $\mathcal{H}_{Z}^{0}(\mathcal{F}) = \mathcal{H}_{Z}(\mathcal{F})$. By the lemma above we have a Grothendieck spectral sequence

$$E_{2}^{p,q} = H^{p}(Z, \mathcal{H}_{Z}^{q}(\mathcal{F})) \Rightarrow H^{p+q}_{Z}(X, \mathcal{F})$$

**Lemma 75.2.** Let $i : Z \to X$ be a closed immersion of schemes. Let $\mathcal{G}$ be an injective abelian sheaf on $Z_{\text{étale}}$. Then $\mathcal{H}_{Z}^{p}(i_{*}\mathcal{G}) = 0$ for $p > 0$.

**Proof.** This is true because the functor $i_{*}$ is exact and transforms injective abelian sheaves into injective abelian sheaves (Cohomology on Sites, Lemma 15.2).

Let $X$ be a scheme and let $Z \subset X$ be a closed subscheme. We denote $D_{Z}(X_{\text{étale}})$ the strictly full saturated triangulated subcategory of $D(X_{\text{étale}})$ consisting of complexes whose cohomology sheaves are supported on $Z$. Note that $D_{Z}(X_{\text{étale}})$ only depends on the underlying closed subset of $X$.

**Lemma 75.4.** Let $i : Z \to X$ be a closed immersion of schemes. The map $Ri_{small, *} : D(Z_{\text{étale}}) \to D(X_{\text{étale}})$ induces an equivalence $D(Z_{\text{étale}}) \to D_{Z}(X_{\text{étale}})$ with quasi-inverse

$$i_{small}^{-1}|_{D_{Z}(X_{\text{étale}})} = R\mathcal{H}_{Z}|_{D_{Z}(X_{\text{étale}})}$$
Proof. Recall that \( i_{\text{small}}^{-1} \) and \( i_{\text{small},*} \) is an adjoint pair of exact functors such that \( i_{\text{small}}^{-1} i_{\text{small},*} \) is isomorphic to the identity functor on abelian sheaves. See Proposition \([46.4]\) and Lemma \([36.2]\). Thus \( i_{\text{small},*} : D(Z_{\text{étale}}) \to D_Z(X_{\text{étale}}) \) is fully faithful and \( i_{\text{small}}^{-1} \) determines a left inverse. On the other hand, suppose that \( K \) is an object of \( D_Z(X_{\text{étale}}) \) and consider the adjunction map \( K \to i_{\text{small},*} i_{\text{small}}^{-1} K \).

Using exactness of \( i_{\text{small},*} \) and \( i_{\text{small}}^{-1} \) this induces the adjunction maps \( H^n(K) \to i_{\text{small},*} i_{\text{small}}^{-1} H^n(K) \) on cohomology sheaves. Since these cohomology sheaves are supported on \( Z \) we see these adjunction maps are isomorphisms and we conclude that \( D(Z_{\text{étale}}) \to D_Z(X_{\text{étale}}) \) is an equivalence.

To finish the proof we have to show that \( RH_Z(K) = i_{\text{small}}^{-1} K \) if \( K \) is an object of \( D_Z(X_{\text{étale}}) \). To do this we can use that \( K = i_{\text{small},*} i_{\text{small}}^{-1} K \) as we’ve just proved this is the case. Then we can choose a \( K \)-injective representative \( I^\bullet \) for \( i_{\text{small}}^{-1} K \).

Since \( i_{\text{small},*} \) is the right adjoint to the exact functor \( i_{\text{small}}^{-1} \), the complex \( i_{\text{small},*} I^\bullet \) is \( K \)-injective (Derived Categories, Lemma \([29.9]\)). We see that \( RH_Z(K) \) is computed by \( H_Z(i_{\text{small},*} I^\bullet) = I^\bullet \) as desired. □

**Lemma 75.5.** Let \( X \) be a scheme. Let \( Z \subset X \) be a closed subscheme. Let \( F \) be a quasi-coherent \( O_X \)-module and denote \( F^a \) the associated quasi-coherent sheaf on the small étale site of \( X \) (Proposition \([17.1]\)). Then

1. \( H^q_Z(X,F) \) agrees with \( H^q_Z(X_{\text{étale}},F^a) \),
2. if the complement of \( Z \) is retrocompact in \( X \), then \( i_* H^q_Z(F^a) \) is a quasi-coherent sheaf of \( O_X \)-modules equal to \( (i_* H^q_Z(F))^a \).

**Proof.** Let \( j : U \to X \) be the inclusion of the complement of \( Z \). The statement (1) on cohomology groups follows from the long exact sequences for cohomology with supports and the agreements \( H^q(X_{\text{étale}},F^a) = H^q(X,F) \) and \( H^q(U_{\text{étale}},F^a) = H^q(U,F) \), see Theorem \([22.4]\). If \( j : U \to X \) is a quasi-compact morphism, i.e., if \( U \subset X \) is retrocompact, then \( R^q j_* \) transforms quasi-coherent sheaves into quasi-coherent sheaves (Cohomology of Schemes, Lemma \([4.5]\)) and commutes with taking associated sheaf on étale sites (Descent, Lemma \([8.15]\)). We conclude by applying Lemma \([75.3]\). □

### 76. Schemes with strictly henselian local rings

**Lemma 76.1.** Let \( S \) be a scheme all of whose local rings are strictly henselian. Then for any abelian sheaf \( F \) on \( S_{\text{étale}} \) we have \( H^i(S_{\text{étale}},F) = H^i(S_{\text{Zar}},F) \).

**Proof.** Let \( \epsilon : S_{\text{étale}} \to S_{\text{Zar}} \) be the morphism of sites given by the inclusion functor. The Zariski sheaf \( R^p \epsilon_* F \) is the sheaf associated to the presheaf \( U \mapsto H^p_{\text{étale}}(U,F) \). Thus the stalk at \( x \in X \) is colim \( H^p_{\text{étale}}(U,F) = H^p_{\text{étale}}(\Spec(O_{X,x},G_x)) \) where \( G_x \) denotes the pullback of \( F \) to \( \Spec(O_{X,x}) \), see Lemma \([51.5]\). Thus the higher direct images of \( R^p \epsilon_* F \) are zero by Lemma \([54.1]\) and we conclude by the Leray spectral sequence. □

**Lemma 76.2.** Let \( S \) be an affine scheme such that (1) all points are closed, and (2) all residue fields are separably algebraically closed. Then for any abelian sheaf \( F \) on \( S_{\text{étale}} \) we have \( H^i(S_{\text{étale}},F) = 0 \) for \( i > 0 \).
Proof. Condition (1) implies that the underlying topological space of $S$ is profinite, see Algebra, Lemma \[5.5\] Thus the higher cohomology groups of an abelian sheaf on the topological space $S$ (i.e., Zariski cohomology) is trivial, see Cohomology, Lemma \[23.3\]. The local rings are strictly henselian by Algebra, Lemma \[148.10\] Thus étale cohomology of $S$ is computed by Zariski cohomology by Lemma \[76.1\] and the proof is done.

The spectrum of an absolutely integrally closed ring is an example of a scheme all of whose local rings are strictly henselian, see More on Algebra, Lemma \[14.7\]. It turns out that normal domains with separably closed fraction fields have an even stronger property as explained in the following lemma.

**Lemma 76.3.** Let $X$ be an integral normal scheme with separably closed function field.

1. A separated étale morphism $U \to X$ is a disjoint union of open immersions.
2. All local rings of $X$ are strictly henselian.

Proof. Let $R$ be a normal domain whose fraction field is separably algebraically closed. Let $R \to A$ be an étale ring map. Then $A \otimes_R K$ is as a $K$-algebra a finite product $\prod_{i=1,\ldots,n} K$ of copies of $K$. Let $e_i$, $i = 1, \ldots, n$ be the corresponding idempotents of $A \otimes_R K$. Since $A$ is normal (Algebra, Lemma \[157.9\]) the idempotents $e_i$ are in $A$ (Algebra, Lemma \[36.12\]). Hence $A = \prod A e_i$ and we may assume $A \otimes_R K = K$. Since $A \subset A \otimes_R K = K$ (by flatness of $R \to A$ and since $R \subset K$) we conclude that $A$ is a domain. By the same argument we conclude that $A \otimes_R A \subset (A \otimes_R A) \otimes_R K = K$. It follows that the map $A \otimes_R A \to A$ is injective as well as surjective. Thus $R \to A$ defines an open immersion by Morphisms, Lemma \[10.2\] and Étale Morphisms, Theorem \[14.1\].

Let $f : U \to X$ be a separated étale morphism. Let $\eta \in X$ be the generic point and let $f^{-1}(\{\eta\}) = \{\xi_i\}_{i \in I}$. The result of the previous paragraph shows the following: For any affine open $U' \subset U$ whose image in $X$ is contained in an affine we have $U' = \coprod_{i \in I} U'_i$ where $U'_i$ is the set of point of $U'$ which are specializations of $\xi_i$. Moreover, the morphism $U'_i \to X$ is an open immersion. It follows that $U_i = \{\xi_i\}$ is an open and closed subscheme of $U$ and that $U_i \to X$ is locally on the source an isomorphism. By Morphisms, Lemma \[47.7\] the fact that $U_i \to X$ is separated, implies that $U_i \to X$ is injective and we conclude that $U_i \to X$ is an open immersion, i.e., (1) holds.

Part (2) follows from part (1) and the description of the strict henselization of $O_{X,x}$ as the local ring at $\mathfrak{p}$ on the étale site of $X$ (Lemma \[33.1\]). It can also be proved directly, see Fundamental Groups, Lemma \[12.2\].

**Lemma 76.4.** Let $f : X \to Y$ be a morphism of schemes where $X$ is an integral normal scheme with separably closed function field. Then $R^q f_* M = 0$ for $q > 0$ and any abelian group $M$.

Proof. Recall that $R^q f_* M$ is the sheaf associated to the presheaf $V \mapsto H^q_{\text{ét}}(V \times_Y X, M)$ on $Y_{\text{ét}},$ see Lemma \[61.6\]. If $V$ is affine, then $V \times_Y X \to X$ is separated and étale. Hence $V \times_Y X = \coprod U_i$ is a disjoint union of open subschemes $U_i$ of $X$, see Lemma \[76.3\]. By Lemma \[76.1\] we see that $H^q_{\text{ét}}(U_i, M)$ is equal to $H^q_{\text{Zar}}(U_i, M)$. This vanishes by Cohomology, Lemma \[21.2\].
09ZA **Lemma 76.5.** Let $X$ be an affine integral normal scheme with separably closed function field. Let $Z \subset X$ be a closed subscheme. Let $V \to Z$ be an étale morphism with $V$ affine. Then $V$ is a finite disjoint union of open subschemes of $Z$. If $V \to Z$ is surjective and finite étale, then $V \to Z$ has a section.

**Proof.** By Algebra, Lemma 141.10 we can lift $V$ to an affine scheme $U$ étale over $X$. Apply Lemma 76.3 to $U \to X$ to get the first statement.

The final statement is a consequence of the first. Let $V = \coprod_{i=1}^{n} V_i$ be a finite decomposition into open and closed subschemes with $V_i \to Z$ an open immersion. As $V \to Z$ is finite we see that $V_i \to Z$ is also closed. Let $U_i \subset Z$ be the image. Then we have a decomposition into open and closed subschemes

$$Z = \coprod_{(A,B)} \bigcap_{i \in A} U_i \cap \bigcap_{i \in B} U_i^c,$$

where the disjoint union is over $\{1, \ldots, n\} = A \amalg B$ where $A$ has at least one element. Each of the strata is contained in a single $U_i$ and we find our section. □

09ZB **Lemma 76.6.** Let $X$ be a normal integral affine scheme with separably closed function field. Let $Z \subset X$ be a closed subscheme. For any finite abelian group $M$ we have $\check{H}^1_{\text{étale}}(Z, M) = 0$.

**Proof.** By Cohomology on Sites, Lemma 5.3 an element of $\check{H}^1_{\text{étale}}(Z, M)$ corresponds to a $M$-torsor $F$ on $Z_{\text{étale}}$. Such a torsor is clearly a finite locally constant sheaf. Hence $F$ is representable by a scheme $V$ finite étale over $Z$, Lemma 63.4. Of course $V \to Z$ is surjective as a torsor is locally trivial. Since $V \to Z$ has a section by Lemma 76.5 we are done. □

09ZC **Lemma 76.7.** Let $X$ be a normal integral affine scheme with separably closed function field. Let $Z \subset X$ be a closed subscheme. For any finite abelian group $M$ we have $\check{H}^q_{\text{étale}}(Z, M) = 0$ for $q \geq 1$.

**Proof.** We have seen that the result is true for $H^1$ in Lemma 76.6. We will prove the result for $q \geq 2$ by induction on $q$. Let $\xi \in \check{H}^q_{\text{étale}}(Z, M)$.

Let $X = \text{Spec}(R)$. Let $I \subset R$ be the set of elements $f \in R$ such that $\xi|_{Z \cap D(f)} = 0$. All local rings of $Z$ are strictly henselian by Lemma 76.3 and Algebra, Lemma 150.10. Hence étale cohomology on $Z$ or open subschemes of $Z$ is equal to Zariski cohomology, see Lemma 76.1. In particular $\xi$ is Zariski locally trivial. It follows that for every prime $p$ of $R$ there exists an $f \in I$ with $f \not\in p$. Thus if we can show that $I$ is an ideal, then $1 \in I$ and we’re done. It is clear that $f \in I$, $r \in R$ implies $rf \in I$. Thus we now assume that $f, g \in I$ and we show that $f + g \in I$. Note that

$$D(f + g) \cap Z = D(f(f + g)) \cap Z \cup D(g(f + g)) \cap Z.$$
By Mayer-Vietoris (Cohomology, Lemma 9.2 which applies as étale cohomology on open subschemes of $Z$ equals Zariski cohomology) we have an exact sequence

$$H^q_{\text{étale}}(D(fg(f + g)) \cap Z, M) \to H^q_{\text{étale}}(D(f + g) \cap Z, M) \to H^q_{\text{étale}}(D(f(g + g)) \cap Z, M) \oplus H^q_{\text{étale}}(D(g(f + g)) \cap Z, M)$$

and the result follows as the first group is zero by induction. □

**Lemma 76.8.** Let $X$ be an affine scheme.

1. There exists an integral surjective morphism $X' \to X$ such that for every closed subscheme $Z' \subset X'$ every finite abelian group $M$, and every $q \geq 1$ we have $H^q_{\text{étale}}(Z', M) = 0$.

2. For any closed subscheme $Z \subset X$, finite abelian group $M$, $q \geq 1$, and $\xi \in H^q_{\text{étale}}(Z, M)$ there exists a finite surjective morphism $X' \to X$ of finite presentation such that $\xi$ pulls back to zero in $H^q_{\text{étale}}(X' \times_X Z, M)$.

**Proof.** Write $X = \text{Spec}(A)$. Write $A = \mathbb{Z}[x_i]/J$ for some ideal $J$. Let $R$ be the integral closure of $\mathbb{Z}[x_i]$ in an algebraic closure of the fraction field of $\mathbb{Z}[x_i]$. Let $A' = R/JR$ and set $X' = \text{Spec}(A')$. This gives an example as in (1) by Lemma 76.7.

Proof of (2). Let $X' \to X$ be the integral surjective morphism we found above. Certainly, $\xi$ maps to zero in $H^q_{\text{étale}}(X' \times_X Z, M)$. We may write $X'$ as a limit $X' = \lim_i X'_i$ of schemes finite and of finite presentation over $X$; this is easy to do in our current affine case, but it is a special case of the more general Limits, Lemma 7.2. By Lemma 51.5 we see that $\xi$ maps to zero in $H^q_{\text{étale}}(X'_i \times_X Z, M)$ for some $i$ large enough. □

**77. Affine analog of proper base change**

In this section we discuss a result by Ofer Gabber, see [Gab94]. This was also proved by Roland Huber, see [Hub93]. We have already done some of the work needed for Gabber’s proof in Section 76.

**Lemma 77.1.** Let $X$ be an affine scheme. Let $F$ be a torsion abelian sheaf on $X_{\text{étale}}$. Let $Z \subset X$ be a closed subscheme. Let $\xi \in H^q_{\text{étale}}(Z, F|_Z)$ for some $q \geq 0$. Then there exists an injective map $\mathcal{F} \to \mathcal{F}'$ of torsion abelian sheaves on $X_{\text{étale}}$ such that the image of $\xi$ in $H^q_{\text{étale}}(Z, \mathcal{F}'|_Z)$ is zero.

**Proof.** By Lemmas 72.2 and 51.4 we can find a map $\mathcal{G} \to \mathcal{F}$ with $\mathcal{G}$ a constructible abelian sheaf and $\xi$ coming from an element $\zeta$ of $H^q_{\text{étale}}(Z, \mathcal{G}|_Z)$. Suppose we can find an injective map $\mathcal{G} \to \mathcal{G}'$ of torsion abelian sheaves on $X_{\text{étale}}$ such that the image of $\zeta$ in $H^q_{\text{étale}}(Z, \mathcal{G}'|_Z)$ is zero. Then we can take $\mathcal{F}'$ to be the pushout

$$\mathcal{F}' = \mathcal{G}' \amalg_{\mathcal{G}} \mathcal{F}$$

and we conclude the result of the lemma holds. (Observe that restriction to $Z$ is exact, so commutes with finite limits and colimits and moreover it commutes with
arbitrary colimits as a left adjoint to pushforward.) Thus we may assume \( \mathcal{F} \) is constructible.

Assume \( \mathcal{F} \) is constructible. By Lemma \ref{73.4} it suffices to prove the result when \( \mathcal{F} \) is of the form \( f_* \mathcal{M} \) where \( \mathcal{M} \) is a finite abelian group and \( f : Y \to X \) is a finite morphism of finite presentation (such sheaves are still constructible by Lemma \ref{72.9} but we won’t need this). Since formation of \( f_* \) commutes with any base change (Lemma \ref{54.3}) we see that the restriction of \( f_* \mathcal{M} \) to \( Z \) is equal to the pushforward of \( \mathcal{M} \) via \( Y \times_X Z \to Z \). By the Leray spectral sequence (Proposition \ref{53.2}) and vanishing of higher direct images (Proposition \ref{54.2}), we find

\[
H^n_{\text{étale}}(Z, f_* \mathcal{M}|_z) = H^n_{\text{étale}}(Y \times_X Z, \mathcal{M}).
\]

By Lemma \ref{76.8} we can find a finite surjective morphism \( Y' \to Y \) of finite presentation such that \( \xi \) maps to zero in \( H^3(Y' \times_X Z, \mathcal{M}) \). Denoting \( f' : Y' \to X \) the composition \( Y' \to Y \to X \) we claim the map

\[
f_* \mathcal{M} \to f'_* \mathcal{M}
\]

is injective which finishes the proof by what was said above. To see the desired injectivity we can look at stalks. Namely, if \( \overline{x} : \text{Spec}(k) \to X \) is a geometric point, then

\[
(f_* \mathcal{M})_{\overline{x}} = \bigoplus_{f(\overline{y}) = \overline{x}} M
\]

by Proposition \ref{54.2} and similarly for the other sheaf. Since \( Y' \to Y \) is surjective and finite we see that the induced map on geometric points lifting \( \overline{x} \) is surjective too and we conclude. \( \square \)

The lemma above will take care of higher cohomology groups in Gabber’s result. The following lemma will be used to deal with global sections.

\begin{lem}
Let \( X \) be a quasi-compact and quasi-separated scheme. Let \( i : Z \to X \) be a closed immersion. Assume that

1. for any sheaf \( \mathcal{F} \) on \( X_{\text{Zar}} \) the map \( \Gamma(X, \mathcal{F}) \to \Gamma(Z, i^{-1} \mathcal{F}) \) is bijective, and
2. for any finite morphism \( X' \to X \) assumption (1) holds for \( Z \times_X X' \to X' \).

Then for any sheaf \( \mathcal{F} \) on \( X_{\text{étale}} \) we have \( \Gamma(X, \mathcal{F}) = \Gamma(Z, i^{-1}_{\text{small}} \mathcal{F}) \).
\end{lem}

\textbf{Proof.} Let \( \mathcal{F} \) be a sheaf on \( X_{\text{étale}} \). There is a canonical (base change) map

\[
i^{-1}(\mathcal{F}|_{X_{\text{Zar}}}) \to (i^{-1}_{\text{small}} \mathcal{F})|_{Z_{\text{Zar}}}
\]

of sheaves on \( Z_{\text{Zar}} \). This map is injective as can be seen by looking on stalks. The stalk on the left hand side at \( z \in Z \) is the stalk of \( \mathcal{F}|_{X_{\text{Zar}}} \) at \( z \). The stalk on the right hand side is the colimit over all elementary étale neighbourhoods \( (U, u) \to (X, z) \) such that \( U \times_X Z \to Z \) has a section over a neighbourhood of \( z \). As étale morphisms are open, the image of \( U \to X \) is an open neighbourhood of \( z \) in \( X \) and injectivity follows.

It follows from this and assumption (1) that the map \( \Gamma(X, \mathcal{F}) \to \Gamma(Z, i^{-1}_{\text{small}} \mathcal{F}) \) is injective. By (2) the same thing is true on all \( X' \) finite over \( X \).

Let \( s \in \Gamma(Z, i^{-1}_{\text{small}} \mathcal{F}) \). By construction of \( i^{-1}_{\text{small}} \mathcal{F} \) there exists an étale covering \( \{V_j \to Z\} \), étale morphisms \( U_j \to X \), sections \( s_j \in \mathcal{F}(U_j) \) and morphisms \( V_j \to U_j \) over \( X \) such that \( s|_{V_j} \) is the pullback of \( s_j \). Observe that every nonempty closed subscheme \( T \subset X \) meets \( Z \) by assumption (1) applied to the sheaf \( (T \to X)_* \mathcal{F} \) for example. Thus we see that \( \coprod U_j \to X \) is surjective. By More on Morphisms,
Lemma [8.13] we can find a finite surjective morphism $X' \to X$ such that $X' \to X$ Zariski locally factors through $\coprod U_j \to X$. It follows that $s|_{Z'}$ Zariski locally comes from a section of $\mathcal{F}|_{X'}$. In other words, $s|_{Z'}$ comes from $t' \in \Gamma(X', \mathcal{F}|_{X'})$ by assumption (2). By injectivity we conclude that the two pullbacks of $t'$ to $X' \times_X X'$ are the same (after all this is true for the pullbacks of $s$ to $Z' \times_Z Z'$). Hence we conclude $t'$ comes from a section of $\mathcal{F}$ over $X$ by Remark [54.6].

**Lemma 77.3.** Let $Z \subseteq X$ be a closed subset of a topological space $X$. Assume

1. $X$ is a spectral space (Topology, Definition [23.1]), and

2. for $x \in X$ the intersection $Z \cap \{x\}$ is connected (in particular nonempty).

If $Z = Z_1 \amalg Z_2$ with $Z_i$ closed in $Z$, then there exists a decomposition $X = X_1 \amalg X_2$ with $X_i$ closed in $X$ and $Z_i = Z \cap X_i$.

**Proof.** Observe that $Z_i$ is quasi-compact. Hence the set of points $W_i$ specializing to $Z_i$ is closed in the constructible topology by Topology, Lemma [23.7]. Assumption (2) implies that $X = W_1 \amalg W_2$. Let $x \in W_1$. By Topology, Lemma [23.5] part (1) there exists a specialization $x_1 \leadsto x$ with $x_1 \in W_1$. Thus $\{x\} \subseteq \{x_1\}$ and we see that $x \in W_1$. In other words, setting $X_i = W_i$ does the job.

**Lemma 77.4.** Let $Z \subseteq X$ be a closed subset of a topological space $X$. Assume

1. $X$ is a spectral space (Topology, Definition [23.1]), and

2. for $x \in X$ the intersection $Z \cap \{x\}$ is connected (in particular nonempty).

Then for any sheaf $\mathcal{F}$ on $X$ we have $\Gamma(X, \mathcal{F}) = \Gamma(Z, \mathcal{F}|_Z)$.

**Proof.** If $x \leadsto x'$ is a specialization of points, then there is a canonical map $\mathcal{F}_{x'} \to \mathcal{F}_x$ compatible with sections over opens and functorial in $\mathcal{F}$. Since every point of $X$ specializes to a point of $Z$ it follows that $\Gamma(X, \mathcal{F}) \to \Gamma(Z, \mathcal{F}|_Z)$ is injective. The difficult part is to show that it is surjective.

Denote $\mathcal{B}$ be the set of all quasi-compact opens of $X$. Write $\mathcal{F}$ as a filtered colimit $\mathcal{F} = \colim \mathcal{F}_i$ where each $\mathcal{F}_i$ is as in Modules, Equation [17.2.1]. See Modules, Lemma [17.2]. Then $\mathcal{F}|_Z = \colim \mathcal{F}_i|_Z$ as restriction to $Z$ is a left adjoint (Categories, Lemma [24.5] and Sheaves, Lemma [21.8]). By Sheaves, Lemma [29.1] the functors $\Gamma(X, -)$ and $\Gamma(Z, -)$ commute with filtered colimits. Hence we may assume our sheaf $\mathcal{F}$ is as in Modules, Equation [17.2.1].

Suppose that we have an embedding $\mathcal{F} \subseteq \mathcal{G}$. Then we have

$$\Gamma(X, \mathcal{F}) = \Gamma(Z, \mathcal{F}|_Z) \cap \Gamma(X, \mathcal{G})$$

where the intersection takes place in $\Gamma(Z, \mathcal{G}|_Z)$. This follows from the first remark of the proof because we can check whether a global section of $\mathcal{G}$ is in $\mathcal{F}$ by looking at the stalks and because every point of $X$ specializes to a point of $Z$.

By Modules, Lemma [17.4] there is an injection $\mathcal{F} \to \coprod (Z_i \to X)_{*} S_i$ where the product is finite, $Z_i \subseteq X$ is closed, and $S_i$ is finite. Thus it suffices to prove surjectivity for the sheaves $(Z_i \to X)_{*} S_i$. Observe that

$$\Gamma(X, (Z_i \to X)_{*} S_i) = \Gamma(Z_i, S_i) \quad \text{and} \quad \Gamma(X, (Z_i \to X)_{*} S_i|_Z) = \Gamma(Z \cap Z_i, S_i)$$

Moreover, conditions (1) and (2) are inherited by $Z_i$; this is clear for (2) and follows from Topology, Lemma [23.4] for (1). Thus it suffices to prove the lemma in the case of a (finite) constant sheaf. This case is a restatement of Lemma 77.3 which finishes the proof.
0CAF **Example 77.5.** Lemma 77.4 is false if $X$ is not spectral. Here is an example: Let $Y$ be a $T_1$ topological space, and $y \in Y$ a non-open point. Let $X = Y \amalg \{x\}$, endowed with the topology whose closed sets are $\emptyset, \{y\}$, and all $F \amalg \{x\}$, where $F$ is a closed subset of $Y$. Then $Z = \{x,y\}$ is a closed subset of $X$, which satisfies assumption (2) of Lemma 77.4. But $X$ is connected, while $Z$ is not. The conclusion of the lemma thus fails for the constant sheaf with value $\{0,1\}$ on $X$.

09ZH **Lemma 77.6.** Let $(A,I)$ be a henselian pair. Set $X = \text{Spec}(A)$ and $Z = \text{Spec}(A/I)$. For any sheaf $\mathcal{F}$ on $X_{\text{étale}}$ we have $\Gamma(X,F) = \Gamma(Z,F_{|Z})$.

**Proof.** Recall that the spectrum of any ring is a spectral space, see Algebra, Lemma 25.2. By More on Algebra, Lemma 11.12 we see that $\{x\} \cap Z$ is connected for every $x \in X$. By Lemma 77.4 we see that the statement is true for sheaves on $X_{\text{Zar}}$. For any finite morphism $X' \to X$ we have $X' = \text{Spec}(A')$ and $Z \times_X X' = \text{Spec}(A'/IA')$ with $(A',IA')$ a henselian pair, see More on Algebra, Lemma 11.8 and we get the same statement for sheaves on $(X')_{\text{Zar}}$. Thus we can apply Lemma 77.2 to conclude.

Finally, we can state and prove Gabber’s theorem.

09ZI **Theorem 77.7 (Gabber).** Let $(A,I)$ be a henselian pair. Set $X = \text{Spec}(A)$ and $Z = \text{Spec}(A/I)$. For any torsion abelian sheaf $\mathcal{F}$ on $X_{\text{étale}}$ we have $H^q_{\text{étale}}(X, \mathcal{F}) = H^q_{\text{étale}}(Z, \mathcal{F}_{|Z})$.

**Proof.** The result holds for $q = 0$ by Lemma 77.6. Let $q \geq 1$. Suppose the result has been shown in all degrees $< q$. Let $\mathcal{F}$ be a torsion abelian sheaf. Let $\mathcal{F} \to \mathcal{F}'$ be an injective map of torsion abelian sheaves (to be chosen later) with cokernel $\mathcal{Q}$ so that we have the short exact sequence

$$0 \to \mathcal{F} \to \mathcal{F}' \to \mathcal{Q} \to 0$$

of torsion abelian sheaves on $X_{\text{étale}}$. This gives a map of long exact cohomology sequences over $X$ and $Z$ part of which looks like

$$
\begin{array}{cccccccc}
H^{q-1}_{\text{étale}}(X, \mathcal{F}') & \longrightarrow & H^{q-1}_{\text{étale}}(X, \mathcal{Q}) & \longrightarrow & H^q_{\text{étale}}(X, \mathcal{F}) & \longrightarrow & H^q_{\text{étale}}(X, \mathcal{F}') \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
H^{q-1}_{\text{étale}}(Z, \mathcal{F}'_{|Z}) & \longrightarrow & H^{q-1}_{\text{étale}}(Z, \mathcal{Q}_{|Z}) & \longrightarrow & H^q_{\text{étale}}(Z, \mathcal{F}_{|Z}) & \longrightarrow & H^q_{\text{étale}}(Z, \mathcal{F}'_{|Z}) 
\end{array}
$$

Using this commutative diagram of abelian groups with exact rows we will finish the proof.

Injectivity for $\mathcal{F}$. Let $\xi$ be a nonzero element of $H^q_{\text{étale}}(X, \mathcal{F})$. By Lemma 77.1 applied with $Z = X$ (!) we can find $\mathcal{F} \subseteq \mathcal{F}'$ such that $\xi$ maps to zero in $H^q_{\text{étale}}(Z, \mathcal{F}_{|Z})$. Then $\xi$ is the image of an element of $H^{q-1}_{\text{étale}}(X, \mathcal{Q})$ and bijectivity for $q - 1$ implies $\xi$ does not map to zero in $H^q_{\text{étale}}(Z, \mathcal{F}_{|Z})$.

Surjectivity for $\mathcal{F}$. Let $\xi$ be an element of $H^q_{\text{étale}}(Z, \mathcal{F}_{|Z})$. By Lemma 77.1 applied with $Z = Z$ we can find $\mathcal{F} \subseteq \mathcal{F}'$ such that $\xi$ maps to zero in $H^q_{\text{étale}}(Z, \mathcal{F}_{|Z})$. Then $\xi$ is the image of an element of $H^{q-1}_{\text{étale}}(Z, \mathcal{Q}_{|Z})$ and bijectivity for $q - 1$ implies $\xi$ is in the image of the vertical map. □

0A5I **Lemma 77.8.** Let $X$ be a scheme with affine diagonal which can be covered by $n + 1$ affine opens. Let $Z \subseteq X$ be a closed subscheme. Let $\mathcal{A}$ be a torsion sheaf
of rings on $X_{\text{étale}}$ and let $\mathcal{I}$ be an injective sheaf of $A$-modules on $X_{\text{étale}}$. Then $H^q_{\text{étale}}(Z, \mathcal{I}|_Z) = 0$ for $q > n$.

**Proof.** We will prove this by induction on $n$. If $n = 0$, then $X$ is affine. Say $X = \text{Spec}(A)$ and $Z = \text{Spec}(A/I)$. Let $A^h$ be the filtered colimit of étale $A$-algebras $B$ such that $A/I \to B/IB$ is an isomorphism. Then $(A^h, IA^h)$ is a henselian pair and $A/I = A^h/IA^h$, see More on Algebra, Lemma 12.1 and its proof. Set $X^h = \text{Spec}(A^h)$. By Theorem 77.7 we see that

$$H^q_{\text{étale}}(Z, \mathcal{I}|_Z) = H^q_{\text{étale}}(X^h, \mathcal{I}|_{X^h})$$

By Theorem 51.3 we have

$$H^q_{\text{étale}}(X^h, \mathcal{I}|_{X^h}) = \colim_{A \to B} H^q_{\text{étale}}(\text{Spec}(B), \mathcal{I}|_{\text{Spec}(B)})$$

where the colimit is over the $A$-algebras $B$ as above. Since the morphisms $\text{Spec}(B) \to \text{Spec}(A)$ are étale, the restriction $\mathcal{I}|_{\text{Spec}(B)}$ is an injective sheaf of $A|_{\text{Spec}(B)}$-modules (Cohomology on Sites, Lemma 8.1). Thus the cohomology groups on the right are zero and we get the result in this case.

Induction step. We can use Mayer-Vietoris to do the induction step. Namely, suppose that $X = U \cup V$ where $U$ is a union of $n$ affine opens and $V$ is affine. Then, using that the diagonal of $X$ is affine, we see that $U \cap V$ is the union of $n$ affine opens. Mayer-Vietoris gives an exact sequence

$$H^{q-1}_{\text{étale}}(U \cap V \cap Z, \mathcal{I}|_Z) \to H^q_{\text{étale}}(Z, \mathcal{I}|_Z) \to H^q_{\text{étale}}(U \cap Z, \mathcal{I}|_Z) \oplus H^q_{\text{étale}}(V \cap Z, \mathcal{I}|_Z)$$

and by our induction hypothesis we obtain vanishing for $q > n$ as desired. $\square$

### 78. Cohomology of torsion sheaves on curves

03SB The goal of this section is to prove Theorem 78.9.

0A52 **Situation 78.1.** Here $k$ is an algebraically closed field, $X$ is a separated, finite type scheme of dimension $\leq 1$ over $k$, and $\mathcal{F}$ is a torsion abelian sheaf on $X_{\text{étale}}$.

In Situation 78.1 we want to prove the following statements

0A53 (1) $H^q_{\text{étale}}(X, \mathcal{F}) = 0$ for $q > 2$,

0A54 (2) $H^q_{\text{étale}}(X, \mathcal{F}) = 0$ for $q > 1$ if $X$ is affine,

0A55 (3) $H^q_{\text{étale}}(X, \mathcal{F}) = 0$ for $q > 1$ if $p = \text{char}(k) > 0$ and $\mathcal{F}$ is $p$-power torsion,

0A56 (4) $H^0_{\text{étale}}(X, \mathcal{F})$ is finite if $\mathcal{F}$ is constructible and torsion prime to $\text{char}(k)$,

0A57 (5) $H^1_{\text{étale}}(X, \mathcal{F})$ is finite if $X$ is proper and $\mathcal{F}$ constructible,

0A58 (6) $H^0_{\text{étale}}(X, \mathcal{F}) \to H^0_{\text{étale}}(X_{k'},\mathcal{F}|_{X_{k'}})$ is an isomorphism for any extension $k \subset k'$ of algebraically closed fields if $\mathcal{F}$ is torsion prime to $\text{char}(k)$,

0A59 (7) $H^1_{\text{étale}}(X, \mathcal{F}) \to H^1_{\text{étale}}(X_{k'},\mathcal{F}|_{X_{k'}})$ is an isomorphism for any extension $k \subset k'$ of algebraically closed fields if $X$ is proper,

0A5A (8) $H^q_{\text{étale}}(X, \mathcal{F}) \to H^q_{\text{étale}}(U, \mathcal{F})$ is surjective for all $U \subset X$ open.

Given any Situation 78.1 we will say that “statements [1] – [8] hold” if those statements that apply to the given situation are true. We start the proof with the following consequence of our computation of cohomology with constant coefficients.

0A5B **Lemma 78.2.** In Situation 78.1 assume $X$ is smooth and $\mathcal{F} = \mathbb{Z}/\ell\mathbb{Z}$ for some prime number $\ell$. Then statements [1] – [8] hold for $\mathcal{F}$. 

Proof. Since $X$ is smooth, we see that $X$ is a finite disjoint union of smooth curves. Hence we may assume $X$ is a smooth curve.

Case I: $\ell$ different from the characteristic of $k$. This case follows from Lemma 68.1 (projective case) and Lemma 68.3 (affine case). Statement (6) on cohomology and extension of algebraically closed ground field follows from the fact that the genus $g$ and the number of “punctures” $r$ do not change when passing from $k$ to $k'$. Statement (8) follows as $H^2_{\text{étale}}(U, F)$ is zero as soon as $U \neq X$, because then $U$ is affine (Varieties, Lemmas 42.2 and 42.7).

Case II: $\ell$ is equal to the characteristic of $k$. Vanishing by Lemma 62.4. Statements (5) and (7) follow from Lemma 62.5. □

Remark 78.3 (Invariance under extension of algebraically closed ground field).

Let $k$ be an algebraically closed field of characteristic $p > 0$. In Section 62 we have seen that there is an exact sequence

$$k[x] \to k[x] \to H^1_{\text{étale}}(A^1_k, \mathbb{Z}/p\mathbb{Z}) \to 0$$

where the first arrow maps $f(x)$ to $f^p - f$. A set of representatives for the cokernel is formed by the polynomials

$$\sum_{\text{p|n}} \lambda_n x^n$$

with $\lambda_n \in k$. (If $k$ is not algebraically closed you have to add some constants to this as well.) In particular when $k' \supset k$ is an algebraically closed overfield, then the map

$$H^1_{\text{étale}}(A^1_k, \mathbb{Z}/p\mathbb{Z}) \to H^1_{\text{étale}}(A^1_{k'}, \mathbb{Z}/p\mathbb{Z})$$

is not an isomorphism in general. In particular, the map $\pi_1(A^1_k) \to \pi_1(A^1_{k'})$ between étale fundamental groups (insert future reference here) is not an isomorphism either. Thus the étale homotopy type of the affine line depends on the algebraically closed ground field. From Lemma 78.2 above we see that this is a phenomenon which only happens in characteristic $p$ with $p$-power torsion coefficients.

Lemma 78.4. Let $k$ be an algebraically closed field. Let $X$ be a separated finite type scheme over $k$ of dimension $\leq 1$. Let $0 \to F_1 \to F \to F_2 \to 0$ be a short exact sequence of torsion abelian sheaves on $X$. If statements (1) – (8) hold for $F_1$ and $F_2$, then they hold for $F$.

Proof. This is mostly immediate from the definitions and the long exact sequence of cohomology. Also observe that $F$ is constructible (resp. of torsion prime to the characteristic of $k$) if and only if both $F_1$ and $F_2$ are constructible (resp. of torsion prime to the characteristic of $k$). See Proposition 73.1. Some details omitted. □

Lemma 78.5. Let $k$ be an algebraically closed field. Let $f : X \to Y$ be a finite morphism of separated finite type schemes over $k$ of dimension $\leq 1$. Let $F$ be a torsion abelian sheaf on $X$. If statements (1) – (8) hold for $F$, then they hold for $f_*, F$.

Proof. Follows from the vanishing of the higher direct images $R^q f_*$ (Proposition 54.2), the Leray spectral sequence (Proposition 53.2), and the fact that formation of $f_*$ commutes with arbitrary base change (Lemma 54.3). □
Lemma 78.6. In Situation 78.1 assume $X$ is smooth. Let $j : U \to X$ an open immersion. Let $\ell$ be a prime number. Let $\mathcal{F} = j_! \mathbb{Z}/\ell \mathbb{Z}$. Then statements (1) – (8) hold for $\mathcal{F}$.

**Proof.** Consider the short exact sequence

$$0 \to j_! \mathbb{Z}/\ell \mathbb{Z}_{|U} \to \mathbb{Z}/\ell \mathbb{Z}_X \to \bigoplus_{x \in X \setminus U} i_x^* (\mathbb{Z}/\ell \mathbb{Z}) \to 0.$$ 

Statements (1) – (5) hold for $\mathbb{Z}/\ell \mathbb{Z}$ by Lemma 78.2. Since the inclusion morphisms $i_x : x \to X$ are finite and since $x$ is the spectrum of an algebraically closed field, we see that $H^q_{\text{étale}}(X, i_x^* \mathbb{Z}/\ell \mathbb{Z})$ is zero for $q > 0$ and equal to $\mathbb{Z}/\ell \mathbb{Z}$ for $q = 0$. Thus we get from the long exact cohomology sequence

$$0 \to H^0_{\text{étale}}(X, \mathcal{F}) \to H^0(X, \mathbb{Z}/\ell \mathbb{Z}) \to \bigoplus_{x \in X \setminus U} \mathbb{Z}/\ell \mathbb{Z}$$

and $H^q_{\text{étale}}(X, \mathcal{F}) \to H^q_{\text{étale}}(X, \mathbb{Z}/\ell \mathbb{Z})$ for $q \geq 2$. Each of the statements (1) – (8) follows by inspection. \(\square\)

Lemma 78.7. In Situation 78.1 assume $X$ reduced. Let $j : U \to X$ an open immersion. Let $\ell$ be a prime number and $\mathcal{F} = j_! \mathbb{Z}/\ell \mathbb{Z}$. Then statements (1) – (8) hold for $\mathcal{F}$.

**Proof.** The difference with Lemma 78.6 is that here we do not assume $X$ is smooth. Let $\nu : X' \to X$ be the normalization morphism which is finite as varieties are Nagata schemes. Let $j' : U' \to X'$ be the inverse image of $U$. By Lemma 78.6 the result holds for $j'_! \mathbb{Z}/\ell \mathbb{Z}$. By Lemma 78.5 the result holds for $\nu_* j'_! \mathbb{Z}/\ell \mathbb{Z}$. In general it won’t be true that $\nu_* j'_! \mathbb{Z}/\ell \mathbb{Z}$ is equal to $j_! \mathbb{Z}/\ell \mathbb{Z}$, but there will be a canonical injective map

$$j_! \mathbb{Z}/\ell \mathbb{Z} \to \nu_* j'_! \mathbb{Z}/\ell \mathbb{Z}$$

whose cokernel is of the form $\bigoplus_{x \in Z} i_x^* M_x$ where $Z \subset X$ is a finite set of closed points and $M_x$ is a finite dimensional $\mathbb{F}_\ell$-vector space for each $x \in Z$. We obtain a short exact sequence

$$0 \to j_! \mathbb{Z}/\ell \mathbb{Z} \to \nu_* j'_! \mathbb{Z}/\ell \mathbb{Z} \to \bigoplus_{x \in Z} i_x^* M_x \to 0$$

and we can argue exactly as in the proof of Lemma 78.6 to finish the argument. Some details omitted. \(\square\)

Lemma 78.8. In Situation 78.1 assume $X$ reduced. Let $j : U \to X$ an open immersion with $U$ connected. Let $\ell$ be a prime number. Let $\mathcal{G}$ a finite locally constant sheaf of $\mathbb{F}_\ell$-vector spaces on $U$. Let $\mathcal{F} = j_! \mathcal{G}$. Then statements (1) – (8) hold for $\mathcal{F}$.

**Proof.** Let $f : V \to U$ be a finite étale morphism of degree prime to $\ell$ as in Lemma 65.2 The discussion in Section 65 gives maps

$$\mathcal{G} \to f_* f^{-1} \mathcal{G} \to \mathcal{G}$$
whose composition is an isomorphism. Hence it suffices to prove the lemma with $\mathcal{F} = jj'f^*f^{-1}\mathcal{G}$. By Zariski’s Main theorem (More on Morphisms, Lemma 54.3) we can choose a diagram

$$
\begin{array}{ccc}
V & \xrightarrow{f} & Y \\
\downarrow{j'} & & \downarrow{j} \\
U & \xrightarrow{j} & X
\end{array}
$$

with $j : Y \to X$ finite and $j'$ an open immersion with dense image. We may replace $Y$ by its reduction (this does not change $V$ as $V$ is reduced being étale over $U$). By Lemma 78.5 it suffices to prove the lemma for $j$ by its reduction (this does not change $Y$). Hence it suffices to prove the lemma with $\mathcal{F}$ whose composition is an isomorphism. Thus we see that it suffices to assume that $\mathcal{F}$ is $\ell$-primary for some prime $\ell$. Now consider the exact sequence

$$
0 \to \mathcal{F} \to \mathcal{F} \to \mathcal{F}/\mathcal{F}[\ell] \to 0.
$$

Thus we see that it suffices to assume that $\mathcal{F}$ is $\ell$-torsion. This means that $\mathcal{F}$ is a constructible sheaf of $\mathbf{F}_\ell$-vector spaces for some prime number $\ell$.

By definition this means there is a dense open $U \subset X$ such that $\mathcal{F}|_U$ is finite locally constant sheaf of $\mathbf{F}_\ell$-vector spaces. Since $\dim(X) \leq 1$ we may assume, after shrinking $U$, that $U = U_1 \amalg \ldots \amalg U_n$ is a disjoint union of irreducible schemes (just remove the closed points which lie in the intersections of $\geq 2$ components of $U$). Consider the short exact sequence

$$
0 \to jj'^{-1}\mathcal{F} \to \mathcal{F} \to \bigoplus_{x \in \mathbb{Z}} i_{x*}M_x \to 0
$$

03SC Theorem 78.9. If $k$ is an algebraically closed field, $X$ is a separated, finite type scheme of dimension $\leq 1$ over $k$, and $\mathcal{F}$ is a torsion abelian sheaf on $X_{\text{étale}}$, then

1. $H^2_{\text{étale}}(X, \mathcal{F}) = 0$ for $q > 2$,
2. $H^2_{\text{étale}}(X, \mathcal{F}) = 0$ for $q > 1$ if $X$ is affine,
3. $H^q_{\text{étale}}(X, \mathcal{F}) = 0$ for $q > 1$ if $p = \text{char}(k) > 0$ and $\mathcal{F}$ is $p$-power torsion,
4. $H^q_{\text{étale}}(X, \mathcal{F})$ is finite if $\mathcal{F}$ is constructible and torsion prime to $\text{char}(k)$,
5. $H^q_{\text{étale}}(X, \mathcal{F})$ is finite if $X$ is proper and $\mathcal{F}$ constructible,
6. $H^q_{\text{étale}}(X, \mathcal{F}) \to H^q_{\text{étale}}(X_{k'}, \mathcal{F}|_{X_{k'}})$ is an isomorphism for any extension $k \subset k'$ of algebraically closed fields if $\mathcal{F}$ is torsion prime to $\text{char}(k)$,
7. $H^q_{\text{étale}}(X, \mathcal{F}) \to H^q_{\text{étale}}(X_{k'}, \mathcal{F}|_{X_{k'}})$ is an isomorphism for any extension $k \subset k'$ of algebraically closed fields if $X$ is proper,
8. $H^2_{\text{étale}}(X, \mathcal{F}) \to H^2_{\text{étale}}(U, \mathcal{F})$ is surjective for all $U \subset X$ open.

Proof. The theorem says that in Situation 78.1 statements (1) – (8) hold. Our first step is to replace $X$ by its reduction, which is permissible by Proposition 54.4. By Lemma 72.2 we can write $\mathcal{F}$ as a filtered colimit of constructible abelian sheaves. Taking cohomology commutes with colimits, see Lemma 51.4. Moreover, pullback via $X_{k'} \to X$ commutes with colimits as a left adjoint. Thus it suffices to prove the statements for a constructible sheaf.

In this paragraph we use Lemma 78.4 without further mention. Writing $\mathcal{F} = \mathcal{F}_1 \oplus \ldots \oplus \mathcal{F}_r$ where $\mathcal{F}_i$ is $\ell_i$-primary for some prime $\ell_i$, we may assume that $\ell^n$ kills $\mathcal{F}$ for some prime $\ell$. Now consider the exact sequence

$$
0 \to \mathcal{F}[\ell] \to \mathcal{F} \to \mathcal{F}/\mathcal{F}[\ell] \to 0.
$$

Thus we see that it suffices to assume that $\mathcal{F}$ is $\ell$-torsion. This means that $\mathcal{F}$ is a constructible sheaf of $\mathbf{F}_\ell$-vector spaces for some prime number $\ell$.

By definition this means there is a dense open $U \subset X$ such that $\mathcal{F}|_U$ is finite locally constant sheaf of $\mathbf{F}_\ell$-vector spaces. Since $\dim(X) \leq 1$ we may assume, after shrinking $U$, that $U = U_1 \amalg \ldots \amalg U_n$ is a disjoint union of irreducible schemes (just remove the closed points which lie in the intersections of $\geq 2$ components of $U$). Consider the short exact sequence

$$
0 \to jj'^{-1}\mathcal{F} \to \mathcal{F} \to \bigoplus_{x \in \mathbb{Z}} i_{x*}M_x \to 0
$$
where $Z = X \setminus U$ and $M_\ell$ is a finite dimensional $\mathbf{F}_\ell$ vector space, see Lemma 69.6. Since the étale cohomology of $i_* M_\ell$ vanishes in degrees $\geq 1$ and is equal to $M_\ell$ in degree 0 it suffices to prove the theorem for $j_! j^{-1} \mathcal{F}$ (argue exactly as in the proof of Lemma 78.6). Thus we reduce to the case $\mathcal{F} = j_! \mathcal{G}$ where $\mathcal{G}$ is a finite locally constant sheaf of $\mathbf{F}_\ell$-vector spaces on $U$.

Since we chose $U = U_1 \amalg \ldots \amalg U_n$ with $U_i$ irreducible we have

$$j_! \mathcal{G} = j_{1!}(\mathcal{G}|_{U_1}) \oplus \ldots \oplus j_{n!}(\mathcal{G}|_{U_n})$$

where $j_i : U_i \to X$ is the inclusion morphism. The case of $j_{n!}(\mathcal{G}|_{U_n})$ is handled in Lemma 78.8. 

**Theorem 78.10.** Let $X$ be a finite type, dimension 1 scheme over an algebraically closed field $k$. Let $\mathcal{F}$ be a torsion sheaf on $X_{\text{étale}}$. Then

$$H^q_{\text{étale}}(X, \mathcal{F}) = 0, \quad \forall q \geq 3.$$ 

If $X$ affine then also $H^3_{\text{étale}}(X, \mathcal{F}) = 0$.

**Proof.** If $X$ is separated, this follows immediately from the more precise Theorem 78.9. If $X$ is nonseparated, choose an affine open covering $X = X_1 \cup \ldots \cup X_n$. By induction on $n$ we may assume the vanishing holds over $U = X_1 \cup \ldots \cup X_{n-1}$. Then Mayer-Vietoris (Lemma 50.1) gives

$$H^3_{\text{étale}}(U, \mathcal{F}) \oplus H^3_{\text{étale}}(X_n, \mathcal{F}) \to H^3_{\text{étale}}(U \cap X_n, \mathcal{F}) \to H^3_{\text{étale}}(X, \mathcal{F}) \to 0$$

However, since $U \cap X_n$ is an open of an affine scheme and hence affine by our dimension assumption, the group $H^3_{\text{étale}}(U \cap X_n, \mathcal{F})$ vanishes by Theorem 78.9. 

**Lemma 78.11.** Let $k \subset k'$ be an extension of separably closed fields. Let $X$ be a proper scheme over $k$ of dimension $\leq 1$. Let $\mathcal{F}$ be a torsion abelian sheaf on $X$. Then the map $H^3_{\text{étale}}(X, \mathcal{F}) \to H^3_{\text{étale}}(X_{k'}, \mathcal{F}|_{X_{k'}})$ is an isomorphism for $q \geq 0$.

**Proof.** We have seen this for algebraically closed fields in Theorem 78.9. Given $k \subset k'$ as in the statement of the lemma we can choose a diagram

$$
\begin{array}{ccc}
  k' & \longrightarrow & \overline{k'} \\
  \uparrow & & \uparrow \\
  k & \longrightarrow & \overline{k}
\end{array}
$$

where $k \subset \overline{k}$ and $k' \subset \overline{k'}$ are the algebraic closures. Since $k$ and $k'$ are separably closed the field extensions $k \subset \overline{k}$ and $k' \subset \overline{k'}$ are algebraic and purely inseparable. In this case the morphisms $X_{\overline{k}} \to X$ and $X_{\overline{k'}} \to X_{k'}$ are universal homeomorphisms. Thus the cohomology of $\mathcal{F}$ may be computed on $X_{\overline{k}}$ and the cohomology of $\mathcal{F}|_{X_{k'}}$ may be computed on $X_{\overline{k'}}$, see Proposition 15.4. Hence we deduce the general case from the case of algebraically closed fields. 

**79. First cohomology of proper schemes**

In Fundamental Groups, Section 9 we have seen, in some sense, that taking $R^1 f_* G$ commutes with base change if $f : X \to Y$ is a proper morphism and $G$ is a finite group (not necessarily commutative). In this section we deduce a useful consequence of these results.
0A5G  **Lemma 79.1.** Let $A$ be a henselian local ring. Let $X$ be a proper scheme over $A$ with closed fibre $X_0$. Let $M$ be a finite abelian group. Then $H^1_{\text{étale}}(X, M) = H^1_{\text{étale}}(X_0, M)$.

**Proof.** By Cohomology on Sites, Lemma 5.3 an element of $H^1_{\text{étale}}(X, M)$ corresponds to a $M$-torsor $F$ on $X_{\text{étale}}$. Such a torsor is clearly a finite locally constant sheaf. Hence $F$ is representable by a scheme $V$ finite étale over $X$, Lemma 63.4. Conversely, a scheme $V$ finite étale over $X$ with an $M$-action which turns it into an $M$-torsor over $X$ gives rise to a cohomology class. The same translation between cohomology classes over $X_0$ and torsors finite étale over $X_0$ holds. Thus the lemma is a consequence of the equivalence of categories of Fundamental Groups, Lemma 9.1.

The following technical lemma is a key ingredient in the proof of the proper base change theorem. The argument works word for word for any proper scheme over $A$ whose special fibre has dimension $\leq 1$, but in fact the conclusion will be a consequence of the proper base change theorem and we only need this particular version in its proof.

0A5H  **Lemma 79.2.** Let $A$ be a henselian local ring. Let $X = P^1_A$. Let $X_0 \subset X$ be the closed fibre. Let $\ell$ be a prime number. Let $I$ be an injective sheaf of $\mathcal{O}/\ell\mathcal{O}$-modules on $X_{\text{étale}}$. Then $H^q_{\text{étale}}(X_0, I|_{X_0}) = 0$ for $q > 0$.

**Proof.** Observe that $X$ is a separated scheme which can be covered by 2 affine opens. Hence for $q > 1$ this follows from Gabber’s affine variant of the proper base change theorem, see Lemma 77.8. Thus we may assume $q = 1$. Let $\xi \in H^1_{\text{étale}}(X_0, I|_{X_0})$. Goal: show that $\xi$ is 0. By Lemmas 72.2 and 51.4 we can find a map $F \to I$ with $F$ a constructible sheaf of $\mathcal{O}/\ell\mathcal{O}$-modules and $\xi$ coming from an element $\eta$ of $H^1_{\text{étale}}(X_0, F|_{X_0})$. Suppose we have an injective map $F \to F'$ of sheaves of $\mathcal{O}/\ell\mathcal{O}$-modules on $X_{\text{étale}}$. Since $I$ is injective we can extend the given map $F \to I$ to a map $F' \to I$. In this situation we may replace $F$ by $F'$ and $\xi$ by the image of $\eta$ in $H^1_{\text{étale}}(X_0, F'|_{X_0})$. Also, if $F = F_1 \oplus F_2$ is a direct sum, then we may replace $F$ by $F_i$ and $\eta$ by the image of $\eta$ in $H^1_{\text{étale}}(X_0, F_i|_{X_0})$.

By Lemma 73.4 and the remarks above we may assume $F$ is of the form $f_*M$ where $M$ is a finite $\mathcal{O}/\ell\mathcal{O}$-module and $f : Y \to X$ is a finite morphism of finite presentation (such sheaves are still constructible by Lemma 72.9 but we won’t need this). Since formation of $f_*$ commutes with any base change (Lemma 54.3) we see that the restriction of $f_*M$ to $X_0$ is equal to the pushforward of $M$ via the induced morphism $Y_0 \to X_0$ of special fibres. By the Leray spectral sequence (Proposition 53.2) and vanishing of higher direct images (Proposition 54.2), we find

$$H^1_{\text{étale}}(X_0, f_*M|_{X_0}) = H^1_{\text{étale}}(Y_0, M).$$

Since $Y \to \text{Spec}(A)$ is proper we can use Lemma 79.1 to see that the $H^1_{\text{étale}}(Y_0, M)$ is equal to $H^1_{\text{étale}}(Y, M)$. Thus we see that our cohomology class $\xi$ lifts to a cohomology class

$$\tilde{\xi} \in H^1_{\text{étale}}(Y, M) = H^1_{\text{étale}}(X, f_*M).$$
However, $\tilde{\zeta}$ maps to zero in $H^1_{\text{étale}}(X, I)$ as $I$ is injective and by commutativity of

$$
\begin{array}{ccc}
H^1_{\text{étale}}(X, f_* M) & \rightarrow & H^1_{\text{étale}}(X, I) \\
\downarrow & & \downarrow \\
H^1_{\text{étale}}(X_0, (f_* M)|_{X_0}) & \rightarrow & H^1_{\text{étale}}(X_0, I|_{X_0})
\end{array}
$$

we conclude that the image $\xi$ of $\zeta$ is zero as well. \hfill \square

### 80. Preliminaries on base change

0EZQ If you are interested in either the smooth base change theorem or the proper base change theorem, you should skip directly to the corresponding sections. In this section and the next few sections we consider commutative diagrams

$$
\begin{array}{ccc}
X & \xleftarrow{h} & Y \\
\downarrow{f} & & \downarrow{e} \\
S & \xleftarrow{g} & T
\end{array}
$$

of schemes; we usually assume this diagram is cartesian, i.e., $Y = X \times_S T$. A commutative diagram as above gives rise to a commutative diagram

$$
\begin{array}{ccc}
X_{\text{étale}} & \xleftarrow{h_{\text{small}}} & Y_{\text{étale}} \\
\downarrow{f_{\text{small}}} & & \downarrow{e_{\text{small}}} \\
S_{\text{étale}} & \xleftarrow{g_{\text{small}}} & T_{\text{étale}}
\end{array}
$$

of small étale sites. Let us use the notation

$$
f^{-1} = f_{\text{small}}^{-1}, \quad g_* = g_{\text{small},*}, \quad e^{-1} = e_{\text{small}}^{-1}, \quad \text{and} \quad h_* = h_{\text{small},*}.
$$

By Sites, Section [15] we get a base change or pullback map

$$
f^{-1} g_* \mathcal{F} \rightarrow h_* e^{-1} \mathcal{F}
$$

for a sheaf $\mathcal{F}$ on $T_{\text{étale}}$. If $\mathcal{F}$ is an abelian sheaf on $T_{\text{étale}}$, then we get a derived base change map

$$
f^{-1} Rg_* \mathcal{F} \rightarrow R h_* e^{-1} \mathcal{F}
$$

see Cohomology on Sites, Lemma [16.1]. Finally, if $K$ is an arbitrary object of $D(T_{\text{étale}})$ there is a base change map

$$
f^{-1} Rg_* K \rightarrow R h_* e^{-1} K
$$

see Cohomology on Sites, Remark [20.3].

0EZR **Lemma 80.1.** Consider a cartesian diagram of schemes

$$
\begin{array}{ccc}
X & \xleftarrow{h} & Y \\
\downarrow{f} & & \downarrow{e} \\
S & \xleftarrow{g} & T
\end{array}
$$
and a sheaf $\mathcal{F}$ on $T_{\text{étale}}$. Let $\{U_i \to X\}$ be an étale covering such that $U_i \to S$ factors as $U_i \to V_i \to S$ with $V_i \to S$ étale and consider the cartesian diagrams

$$
\begin{align*}
U_i & \xleftarrow{h_i} U_i \times_X Y \\
f_i & \downarrow e_i \\
V_i & \xleftarrow{g_i} V_i \times_S T
\end{align*}
$$

Set $\mathcal{F}_i = \mathcal{F}|_{V_i \times_S T}$.

1. If $f^{-1}_i g_i_* \mathcal{F}_i = h_i_* e_i^{-1} \mathcal{F}_i$ for all $i$, then $f^{-1}_i g_* \mathcal{F} = h_* e^{-1} \mathcal{F}$.
2. If $\mathcal{F}$ is an abelian sheaf and $f^{-1}_i R^q g_i_* \mathcal{F}_i = R^q h_i_* e_i^{-1} \mathcal{F}_i$ for all $i$, then $f^{-1}_i R^q g_* \mathcal{F} = R^q h_* e^{-1} \mathcal{F}$.

**Proof.** We have $f^{-1}_i R^q g_* \mathcal{F}|_{U_i} = f^{-1}_i R^q g_i_* \mathcal{F}_i$ and $R^q h_* e^{-1} \mathcal{F}|_{U_i} = R^q h_i_* e_i^{-1} \mathcal{F}_i$ as follows from the compatibility of localization with morphisms of sites, see Sites, Lemma 28.2 and Cohomology on Sites, Lemma 21.4.

**Lemma 80.2.** Consider a tower of cartesian diagrams of schemes

$$
\begin{align*}
W & \xleftarrow{j} Z \\
i & \downarrow k \\
X & \xleftarrow{h} Y \\
f & \downarrow e \\
S & \xleftarrow{g} T
\end{align*}
$$

Let $K$ in $D(T_{\text{étale}})$. If

$$
\begin{align*}
f^{-1}_i R^q g_* K & \to R^q h_* e^{-1} K \\
i^{-1}_i R^q h_* e^{-1} K & \to R^q j_* k^{-1} e^{-1} K
\end{align*}
$$

are isomorphisms, then $(f \circ i)^{-1} R^q g_* K \to R^q j_* (e \circ k)^{-1} K$ is an isomorphism. Similarly, if $\mathcal{F}$ is an abelian sheaf on $T_{\text{étale}}$ and if

$$
\begin{align*}
f^{-1}_i R^q g_* \mathcal{F} & \to R^q h_* e^{-1} \mathcal{F} \\
i^{-1}_i R^q h_* e^{-1} \mathcal{F} & \to R^q j_* k^{-1} e^{-1} \mathcal{F}
\end{align*}
$$

are isomorphisms, then $(f \circ i)^{-1} R^q g_* \mathcal{F} \to R^q j_* (e \circ k)^{-1} \mathcal{F}$ is an isomorphism.

**Proof.** This is formal, provided one checks that the composition of these base change maps is the base change map for the outer rectangle, see Cohomology on Sites, Remark 20.5.

**Lemma 80.3.** Let $I$ be a directed set. Consider an inverse system of cartesian diagrams of schemes

$$
\begin{align*}
X_i & \xleftarrow{h_i} Y_i \\
f_i & \downarrow e_i \\
S_i & \xleftarrow{g_i} T_i
\end{align*}
$$

with affine transition morphisms and with $g_i$ quasi-compact and quasi-separated. Set $X = \lim X_i$, $S = \lim S_i$, $T = \lim T_i$ and $Y = \lim Y_i$ to obtain the cartesian diagram

$$
\begin{align*}
X & \xleftarrow{h} Y \\
f & \downarrow e \\
S & \xleftarrow{g} T
\end{align*}
$$
Let \((\mathcal{F}_i, \varphi_{i,j})\) be a system of sheaves on \((T_i)\) as in Definition \[51.1\]. Set \(\mathcal{F} = \colim p_i^{-1}\mathcal{F}_i\) on \(T\) where \(p_i : T \to T_i\) is the projection. Then we have the following

1. If \(f_i^{-1}g_{i,*}\mathcal{F}_i = h_{i,*}e_i^{-1}\mathcal{F}_i\) for all \(i\), then \(f_{-1}g_{*}\mathcal{F} = h_{*}e^{-1}\mathcal{F}\).
2. If \(\mathcal{F}_i\) is an abelian sheaf for all \(i\) and \(f_i^{-1}R^qg_{i,*}\mathcal{F}_i = R^qh_{i,*}e_i^{-1}\mathcal{F}_i\) for all \(i\), then \(f_{-1}R^qg_{*}\mathcal{F} = R^qh_{*}e^{-1}\mathcal{F}\).

**Proof.** We prove (2) and we omit the proof of (1). We will use without further mention that pullback of sheaves commutes with colimits as it is a left adjoint. Observe that \(h_i\) is quasi-compact and quasi-separated as a base change of \(g_i\). Denoting \(g : Y \to Y_i\) the projections, observe that \(e^{-1}\mathcal{F} = \colim e^{-1}p_i^{-1}\mathcal{F}_i = \colim q_i^{-1}e_i^{-1}\mathcal{F}_i\).

By Lemma \[51.8\] this gives
\[
R^qh_{*}e^{-1}\mathcal{F} = \colim r_i^{-1}R^qh_{i,*}e_i^{-1}\mathcal{F}_i
\]
where \(r_i : X \to X_i\) is the projection. Similarly, we have
\[
f_{-1}R^qg_{*}\mathcal{F} = f_{-1}\colim s_i^{-1}R^qg_{i,*}\mathcal{F}_i = \colim r_i^{-1}f_i^{-1}R^qg_{i,*}\mathcal{F}_i
\]
where \(s_i : S \to S_i\) is the projection. The lemma follows. \(\square\)

**Lemma 80.4.** Consider a cartesian diagram of schemes

\[
\begin{array}{ccc}
X & \leftarrow & Y \\
\downarrow f & & \downarrow e \\
S & \leftarrow & T
\end{array}
\]

where \(g : T \to S\) is quasi-compact and quasi-separated. Let \(\mathcal{F}\) be an abelian sheaf on \(T_{etale}\). Let \(q \geq 0\). The following are equivalent:

1. For every geometric point \(\overline{x}\) of \(X\) with image \(\overline{y} = f(\overline{x})\) we have
\[
H^q(\Spec(\mathcal{O}_{X,\overline{x}}^{h}) \times_S T, \mathcal{F}) = H^q(\Spec(\mathcal{O}_{S,\overline{y}}^{h}) \times_S T, \mathcal{F})
\]
2. \(f_{-1}R^qg_{*}\mathcal{F} \to R^qh_{*}e^{-1}\mathcal{F}\) is an isomorphism.

**Proof.** Since \(Y = X \times_T Y\) we have \(\Spec(\mathcal{O}_{X,\overline{x}}^{h}) \times_X Y = \Spec(\mathcal{O}_{S,\overline{y}}^{h}) \times_S T\). Thus the map in (1) is the map of stalks at \(\overline{x}\) for the map in (2) by Theorem \[52.1\] (and Lemma \[36.2\]). Thus the result by Theorem \[29.10\]. \(\square\)

**Lemma 80.5.** Let \(f : X \to S\) be a morphism of schemes. Let \(\overline{x}\) be a geometric point of \(X\) with image \(\overline{y} = f(\overline{x})\) in \(S\). Let \(\Spec(K) \to \Spec(\mathcal{O}_{S,\overline{y}}^{h})\) be a morphism with \(K\) a separably closed field. Let \(\mathcal{F}\) be an abelian sheaf on \(\Spec(K)_{etale}\). Let \(q \geq 0\). The following are equivalent:

1. \(H^q(\Spec(\mathcal{O}_{X,\overline{x}}^{h}) \times_S \Spec(K), \mathcal{F}) = H^q(\Spec(\mathcal{O}_{S,\overline{y}}^{h}) \times_S \Spec(K), \mathcal{F})\)
2. \(H^q(\Spec(\mathcal{O}_{X,\overline{x}}^{h}) \times_{\Spec(\mathcal{O}_{S,\overline{y}}^{h})} \Spec(K), \mathcal{F}) = H^q(\Spec(K), \mathcal{F})\)

**Proof.** Observe that \(\Spec(K) \times_S \Spec(\mathcal{O}_{S,\overline{y}}^{h})\) is the spectrum of a filtered colimit of étale algebras over \(K\). Since \(K\) is separably closed, each étale \(K\)-algebra is a finite product of copies of \(K\). Thus we can write
\[
\Spec(K) \times_S \Spec(\mathcal{O}_{S,\overline{y}}^{h}) = \lim_{i \in I} \prod_{a \in A_i} \Spec(K)
\]
as a cofiltered limit where each term is a disjoint union of copies of \( \text{Spec}(K) \) over a finite set \( A_i \). Note that \( A_i \) is nonempty as we are given \( \text{Spec}(K) \to \text{Spec}(\mathcal{O}_{S,T}^{\text{sh}}) \).

It follows that

\[
\text{Spec}(\mathcal{O}_{X,T}^{\text{sh}}) \times_S \text{Spec}(K) = \text{colim}_{i \in I} \coprod_{a \in A_i} \text{Spec}(\mathcal{O}_{X,a}^{\text{sh}}) \times_S \text{Spec}(K)
\]

Since taking cohomology in our setting commutes with limits of schemes (Theorem 51.3) we conclude. \( \square \)

81. Base change for pushforward

This section is preliminary and should be skipped on a first reading. In this section we discuss for what morphisms \( f : X \to S \) we have \( f^{-1}g_* = h_*e^{-1} \) on all sheaves (of sets) for every cartesian diagram

\[
\begin{array}{ccc}
X & \leftarrow & Y \\
\downarrow f & & \downarrow e \\
S & \leftarrow & T \\
\end{array}
\]

with \( g \) quasi-compact and quasi-separated.

**Lemma 81.1.** Consider the cartesian diagram of schemes

\[
\begin{array}{ccc}
X & \leftarrow & Y \\
\downarrow f & & \downarrow e \\
S & \leftarrow & T \\
\end{array}
\]

Assume that \( f \) is flat and every object \( U \) of \( X_{\text{étale}} \) has a covering \( \{ U_i \to U \} \) such that \( U_i \to S \) factors as \( U_i \to V_i \to S \) with \( V_i \to S \) étale and \( U_i \to V_i \) quasi-compact with geometrically connected fibres. Then for any sheaf \( F \) of sets on \( T_{\text{étale}} \) we have \( f^{-1}g_*F = h_*e^{-1}F \).

**Proof.** Let \( U \to X \) be an étale morphism such that \( U \to S \) factors as \( U \to V \to S \) with \( V \to S \) étale and \( U \to V \) quasi-compact with geometrically connected fibres. Observe that \( U \to V \) is flat (More on Flatness, Lemma 2.3). We claim that

\[
f^{-1}g_*F(U) = g_*F(V) = F(V \times_S T) = e^{-1}F(U \times_X Y) = h_*e^{-1}F(U)
\]

Namely, thinking of \( U \) as an object of \( X_{\text{étale}} \) and \( V \) as an object of \( S_{\text{étale}} \) we see that the first equality follows from Lemma [39.3]. Thinking of \( V \times_S T \) as an object of \( T_{\text{étale}} \) the second equality follows from the definition of \( g_* \). Observe that \( U \times_Y Y = U \times_S T \) (because \( Y = X \times_S T \)) and hence \( U \times_X Y \to V \times_S T \) has geometrically connected fibres as a base change of \( U \to V \). Thinking of \( U \times_Y Y \) as an object of \( Y_{\text{étale}} \), we see that the third equality follows from Lemma [39.3] as before. Finally, the fourth equality follows from the definition of \( h_* \).

7Strictly speaking, we are also using that the restriction of \( f^{-1}g_*F \) to \( U_{\text{étale}} \) is the pullback via \( U \to V \) of the restriction of \( g_*F \) to \( V_{\text{étale}} \). See Sites, Lemma 28.2.
Since by assumption every object of $X_{\text{étale}}$ has an étale covering to which the argument of the previous paragraph applies we see that the lemma is true. □

Lemma 81.2. Consider a cartesian diagram of schemes

$$
\begin{array}{ccc}
X & \xleftarrow{h} & Y \\
\downarrow{f} & & \downarrow{e} \\
S & \xleftarrow{g} & T
\end{array}
$$

where $f$ is flat and locally of finite presentation with geometrically reduced fibres. Then $f^{-1}g_*\mathcal{F} = h_*e^{-1}\mathcal{F}$ for any sheaf $\mathcal{F}$ on $T_{\text{étale}}$.

Proof. Combine Lemma 81.1 with More on Morphisms, Lemma 39.3 □

Lemma 81.3. Consider the cartesian diagrams of schemes

$$
\begin{array}{ccc}
X & \xleftarrow{h} & Y \\
\downarrow{f} & & \downarrow{e} \\
S & \xleftarrow{g} & T
\end{array}
$$

Assume that $S$ is the spectrum of a separably closed field. Then $f^{-1}g_*\mathcal{F} = h_*e^{-1}\mathcal{F}$ for any sheaf $\mathcal{F}$ on $T_{\text{étale}}$.

Proof. We may work locally on $X$. Hence we may assume $X$ is affine. Then we can write $X$ as a cofiltered limit of affine schemes of finite type over $S$. By Lemma 80.3 we may assume that $X$ is of finite type over $S$. Then Lemma 81.1 applies because any scheme of finite type over a separably closed field is a finite disjoint union of connected and geometrically connected schemes (see Varieties, Lemma 7.6). □

Lemma 81.4. Consider a cartesian diagram of schemes

$$
\begin{array}{ccc}
X & \xleftarrow{h} & Y \\
\downarrow{f} & & \downarrow{e} \\
S & \xleftarrow{g} & T
\end{array}
$$

Assume that

1. $f$ is flat and open,
2. the residue fields of $S$ are separably algebraically closed,
3. given an étale morphism $U \to X$ with $U$ affine we can write $U$ as a finite disjoint union of open subschemes of $X$ (for example if $X$ is a normal integral scheme with separably closed function field),
4. any nonempty open of a fibre $X_s$ of $f$ is connected (for example if $X_s$ is irreducible or empty).

Then for any sheaf $\mathcal{F}$ of sets on $T_{\text{étale}}$ we have $f^{-1}g_*\mathcal{F} = h_*e^{-1}\mathcal{F}$.

Proof. Omitted. Hint: the assumptions almost trivially imply the condition of Lemma 81.1. The for example in part (3) follows from Lemma 76.3 □

The following lemma doesn’t really belong here but there does not seem to be a good place for it anywhere.
Lemma 81.5. Let \( f : X \to S \) be a morphism of schemes which is flat and locally of finite presentation with geometrically reduced fibres. Then \( f^{-1} : \text{Sh}(\text{S_{\text{etale}}}) \to \text{Sh}(X_{\text{etale}}) \) commutes with products.

Proof. Let \( I \) be a set and let \( G_i \) be a sheaf on \( S_{\text{etale}} \) for \( i \in I \). Let \( U \to X \) be an \( \text{etale} \) morphism such that \( U \to S \) factors as \( U \to V \to S \) with \( V \to S \) \( \text{etale} \) and \( U \to V \) flat of finite presentation with geometrically connected fibres. Then we have

\[
\begin{align*}
 f^{-1}(\prod G_i)(U) &= (\prod G_i)(V) \\
 &= \prod G_i(V) \\
 &= \prod f^{-1}G_i(U) \\
 &= (\prod f^{-1}G_i)(U)
\end{align*}
\]

where we have used Lemma 39.3 in the first and third equality (we are also using that the restriction of \( f^{-1}G \) to \( U_{\text{etale}} \) is the pullback via \( U \to V \) of the restriction of \( G \) to \( V_{\text{etale}} \), see Sites, Lemma 28.2). By More on Morphisms, Lemma 39.3 every object \( U \) of \( X_{\text{etale}} \) has an \( \text{etale} \) covering \( \{ U_i \to U \} \) such that the discussion in the previous paragraph applies to \( U_i \). The lemma follows.

Lemma 81.6. Let \( f : X \to S \) be a flat morphism of schemes such that for every geometric point \( \overline{x} \) of \( X \) the map

\[
\mathcal{O}^{sh}_{S,f(\overline{x})} \to \mathcal{O}^{sh}_{X,\overline{x}}
\]

has geometrically connected fibres. Then for every cartesian diagram of schemes

\[
\begin{array}{ccc}
 X & \leftarrow & Y \\
 f \downarrow & & \downarrow e \\
 S & \leftarrow & T
\end{array}
\]

with \( g \) quasi-compact and quasi-separated we have \( f^{-1}g_*\mathcal{F} = h_*e^{-1}\mathcal{F} \) for any sheaf \( \mathcal{F} \) of sets on \( T_{\text{etale}} \).

Proof. It suffices to check equality on stalks, see Theorem 29.10. By Theorem 52.1 we have

\[
(h_*e^{-1}\mathcal{F})_\overline{x} = \Gamma(\text{Spec}(\mathcal{O}^{sh}_{X,\overline{x}}) \times_X Y, e^{-1}\mathcal{F})
\]

and we have similarly

\[
(f^{-1}g_*^{-1}\mathcal{F})_\overline{x} = (g_*^{-1}\mathcal{F})_{f(\overline{x})} = \Gamma(\text{Spec}(\mathcal{O}^{sh}_{S,f(\overline{x})}) \times_S T, \mathcal{F})
\]

These sets are equal by an application of Lemma 39.3 to the morphism

\[
\text{Spec}(\mathcal{O}^{sh}_{X,\overline{x}}) \times_X Y \to \text{Spec}(\mathcal{O}^{sh}_{S,f(\overline{x})}) \times_S T
\]

which is a base change of \( \text{Spec}(\mathcal{O}^{sh}_{X,\overline{x}}) \to \text{Spec}(\mathcal{O}^{sh}_{S,f(\overline{x})}) \) because \( Y = X \times_S T \). □
82. Base change for higher direct images

This section is the analogue of Section 81 for higher direct images. This section is preliminary and should be skipped on a first reading.

**Remark 82.1.** Let \( f : X \to S \) be a morphism of schemes. Let \( n \) be an integer. We will say \( BC(f, n, q_0) \) is true if for every commutative diagram

\[
\begin{array}{ccc}
X & \leftarrow & X' \\
\downarrow f & & \downarrow f' \\
S & \leftarrow & S'
\end{array}
\]

with \( X' = X \times_S S' \) and \( Y = X' \times_S T \) and \( g \) quasi-compact and quasi-separated, and every abelian sheaf \( F \) on \( T_{\text{étale}} \) annihilated by \( n \) the base change map

\[
(f')^{-1} R^n g_* F \to R^n h_* e^{-1} F
\]

is an isomorphism for \( q \leq q_0 \).

**Lemma 82.2.** With \( f : X \to S \) and \( n \) as in Remark 82.1 assume for some \( q \geq 1 \) we have \( BC(f, n, q - 1) \). Then for every commutative diagram

\[
\begin{array}{ccc}
X & \leftarrow & X' \\
\downarrow f & & \downarrow f' \\
S & \leftarrow & S'
\end{array}
\]

with \( X' = X \times_S S' \) and \( Y = X' \times_S T \) and \( g \) quasi-compact and quasi-separated, and every abelian sheaf \( F \) on \( T_{\text{étale}} \) annihilated by \( n \)

1. the base change map \((f')^{-1} R^n g_* F \to R^n h_* e^{-1} F\) is injective,
2. if \( F \subset G \) where \( G \) on \( T_{\text{étale}} \) is annihilated by \( n \), then

\[
\text{Coker} \left( (f')^{-1} R^n g_* F \to R^n h_* e^{-1} F \right) \subset \text{Coker} \left( (f')^{-1} R^n g_* G \to R^n h_* e^{-1} G \right)
\]

3. if in (2) the sheaf \( G \) is an injective sheaf of \( \mathbb{Z}/n\mathbb{Z} \)-modules, then

\[
\text{Coker} \left( (f')^{-1} R^n g_* F \to R^n h_* e^{-1} F \right) \subset R^n h_* e^{-1} G
\]

**Proof.** Choose a short exact sequence \( 0 \to F \to I \to Q \to 0 \) where \( I \) is an injective sheaf of \( \mathbb{Z}/n\mathbb{Z} \)-modules. Consider the induced diagram

\[
\begin{array}{ccc}
(f')^{-1} R^{e-1} g_* I & \rightarrow & (f')^{-1} R^{e-1} g_* Q \\
\uparrow \cong & & \uparrow \cong \\
R^{e-1} h_* e^{-1} I & \rightarrow & R^{e-1} h_* e^{-1} Q \rightarrow R^n h_* e^{-1} F \rightarrow R^n h_* e^{-1} I
\end{array}
\]

with exact rows. We have the zero in the right upper corner as \( I \) is injective. The left two vertical arrows are isomorphisms by \( BC(f, n, q - 1) \). We conclude that part (1) holds. The above also shows that

\[
\text{Coker} \left( (f')^{-1} R^n g_* F \to R^n h_* e^{-1} F \right) \subset R^n h_* e^{-1} I
\]

hence part (3) holds. To prove (2) choose \( F \subset G \subset I \). \( \square \)
Lemma 82.3. With $f : X \to S$ and $n$ as in Remark 82.1 assume for some $q \geq 1$ we have $BC(f,n,q-1)$. Consider commutative diagrams

$$
\begin{array}{ccc}
X & \xleftarrow{X'} & Y \\
\xleftarrow{f} & & \xleftarrow{e} \\
S & \xleftarrow{g} & T
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
X' & \xleftarrow{h} & Y' \\
\xleftarrow{f'} & & \xleftarrow{e'} \\
S' & \xleftarrow{g'} & T'
\end{array}
$$

where all squares are cartesian, $g$ quasi-compact and quasi-separated, and $\pi$ is integral surjective. Let $F$ be an abelian sheaf on $\mathcal{T}_{\text{etale}}$ annihilated by $n$ and set $F' = \pi^{-1}F$. If the base change map

$$(f)^{-1}R^qg_*F' \to R^qh'_*(e')^{-1}F'$$

is an isomorphism, then the base change map $(f)^{-1}R^qg_*F \to R^qh_*e^{-1}F$ is an isomorphism.

Proof. Observe that $F \to \pi_*\pi^{-1}F'$ is injective as $\pi$ is surjective (check on stalks). Thus by Lemma 82.2 we see that it suffices to show that the base change map

$$(f)^{-1}R^qg_*\pi_*F' \to R^qh_*=e^{-1}\pi_*F'$$

is an isomorphism. This follows from the assumption because we have $R^qg_*\pi_*F' = R^qg'_*F'$, we have $e^{-1}\pi_*F' = \pi'_*(e')^{-1}F'$, and we have $R^qh_*\pi'_*(e')^{-1}F' = R^qh'_*(e')^{-1}F'$. This follows from Lemmas 54.4 and 43.5 and the relative Leray spectral sequence (Cohomology on Sites, Lemma 15.7).

Lemma 82.4. With $f : X \to S$ and $n$ as in Remark 82.1 assume for some $q \geq 1$ we have $BC(f,n,q-1)$. Consider commutative diagrams

$$
\begin{array}{ccc}
X & \xleftarrow{X'} & X'' \\
\xleftarrow{f} & & \xleftarrow{h'} \\
S & \xleftarrow{\pi} & S''
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
X' & \xleftarrow{h=\pi'} & Y' \\
\xleftarrow{f'} & & \xleftarrow{e} \\
S' & \xleftarrow{g'} & T'
\end{array}
$$

where all squares are cartesian, $g'$ quasi-compact and quasi-separated, and $\pi$ is integral. Let $F$ be an abelian sheaf on $\mathcal{T}_{\text{etale}}$ annihilated by $n$. If the base change map

$$(f')^{-1}R^qg_*F \to R^qh_*=e^{-1}F$$

is an isomorphism, then the base change map $(f'')^{-1}R^qg'_*F \to R^qh'_*=e^{-1}F$ is an isomorphism.

Proof. Since $\pi$ and $\pi'$ are integral we have $R\pi_* = \pi_*$ and $R\pi'_* = \pi'_*$, see Lemma 43.5. We also have $(f')^{-1}\pi_* = \pi'_*(f'')^{-1}$. Thus we see that $\pi'_*(f'')^{-1}R^qg'_*F = (f')^{-1}R^qg_*F$ and $\pi'_*R^qh'_*=e^{-1}F = R^qh_*e^{-1}F$. Thus the assumption means that our map becomes an isomorphism after applying the functor $\pi'_*$. Hence we see that it is an isomorphism by Lemma 43.5.

Lemma 82.5. Let $T$ be a quasi-compact and quasi-separated scheme. Let $P$ be a property for quasi-compact and quasi-separated schemes over $T$. Assume

1. If $T'' \to T'$ is a thickening of quasi-compact and quasi-separated schemes over $T$, then $P(T'')$ if and only if $P(T')$.
2. If $T' = \lim_i T_i$ is a limit of an inverse system of quasi-compact and quasi-separated schemes over $T$ with affine transition morphisms and $P(T_i)$ holds for all $i$, then $P(T')$ holds.
(3) If \( Z \subset T' \) is a closed subscheme with quasi-compact complement \( V \subset T' \) and \( P(T') \) holds, then either \( P(V) \) or \( P(Z) \) holds.

Then \( P(T) \) implies \( P(\text{Spec}(K)) \) for some morphism \( \text{Spec}(K) \to T \) where \( K \) is a field.

**Proof.** Consider the set \( \mathcal{T} \) of closed subschemes \( T' \subset T \) such that \( P(T') \). By assumption (2) this set has a minimal element, say \( T' \). By assumption (1) we see that \( T' \) is reduced. Let \( \eta \in T' \) be the generic point of an irreducible component of \( T' \). Then \( \eta = \text{Spec}(K) \) for some field \( K \) and \( \eta = \lim V \) where the limit is over the affine open subschemes \( V \subset T' \) containing \( \eta \). By assumption (3) and the minimality of \( T' \) we see that \( P(V) \) holds for all these \( V \). Hence \( P(\eta) \) by (2) and the proof is complete. \( \square \)

**Lemma 82.6.** With \( f : X \to S \) and \( n \) as in Remark 82.1 assume for some \( q \geq 1 \) we have that \( BC(f,n,q-1) \) is true, but \( BC(f,n,q) \) is not. Then there exist a commutative diagram

\[
\begin{array}{ccc}
X & \xleftarrow{h} & Y \\
\downarrow{f} & & \downarrow{e} \\
S & \xleftarrow{g} & \text{Spec}(K)
\end{array}
\]

where \( X' = X \times_S S' \), \( Y = X' \times_{S'} \text{Spec}(K) \), \( K \) is a field, and \( F \) is an abelian sheaf on \( \text{Spec}(K) \) annihilated by \( n \) such that \((f')^{-1}R^q\pi_*F \to R^q\eta_*e^{-1}F \) is not an isomorphism.

**Proof.** Choose a commutative diagram

\[
\begin{array}{ccc}
X & \xleftarrow{h} & Y \\
\downarrow{f} & & \downarrow{e} \\
S & \xleftarrow{g} & T
\end{array}
\]

with \( X' = X \times_S S' \) and \( Y = X' \times_{S'} T \) and \( g \) quasi-compact and quasi-separated, and an abelian sheaf \( F \) on \( T_{\text{étale}} \) annihilated by \( n \) such that the base change map \((f')^{-1}R^q\pi_*F \to R^q\eta_*e^{-1}F \) is not an isomorphism. Of course we may and do replace \( S' \) by an affine open of \( S' \); this implies that \( T \) is quasi-compact and quasi-separated. By Lemma 82.2 we see \((f')^{-1}R^q\pi_*F \to R^q\eta_*e^{-1}F \) is injective. Pick a geometric point \( \pi \) of \( X' \) and an element \( \xi \) of \((R^q\eta_*q^{-1}F)_{\pi} \) which is not in the image of the map \((f')^{-1}R^q\pi_*F \to (R^q\eta_*e^{-1}F)_{\pi} \).

Consider a morphism \( \pi : T' \to T \) with \( T' \) quasi-compact and quasi-separated and denote \( F' = \pi^{-1}F \). Denote \( \pi' : Y' = Y \times_T T' \to Y \) the base change of \( \pi \) and \( e' : Y' \to T' \) the base change of \( e \). Picture

\[
\begin{array}{ccc}
X' & \xleftarrow{h} & Y' \\
\downarrow{f'} & & \downarrow{e'} \\
S' & \xleftarrow{g'} & T'
\end{array}
\]

and

\[
\begin{array}{ccc}
X' & \xleftarrow{h'=h\circ\pi} & Y' \\
\downarrow{f'} & & \downarrow{e'} \\
S' & \xleftarrow{g'=g\circ\pi} & T'
\end{array}
\]
Using pullback maps we obtain a canonical commutative diagram
\[
\begin{array}{c}
(f')^{-1}R^qg_*\mathcal{F} \\
\downarrow \\
R^qh_{i*e^{-1}}\mathcal{F}
\end{array}
\rightarrow
\begin{array}{c}
(f')^{-1}R^qg'_*\mathcal{F}' \\
\downarrow \\
R^qh'_i(e')^{-1}\mathcal{F}'
\end{array}
\]
of abelian sheaves on \(X'\). Let \(P(T')\) be the property

- The image \(\xi'\) of \(\xi\) in \((R^qh'_i(e')^{-1}\mathcal{F}')_{\pi}\) is not in the image of the map \((f^{-1}R^qg'_i\mathcal{F})_{\pi} \to (R^qh'_i(e')^{-1}\mathcal{F}')_{\pi}\).

We claim that hypotheses (1), (2), and (3) of Lemma \(82.5\) hold for \(P\) which proves our lemma.

Condition (1) of Lemma \(82.5\) holds for \(P\) because the étale topology of a scheme and a thickening of the scheme is the same. See Proposition \(45.4\).

Suppose that \(I\) is a directed set and that \(T_i\) is an inverse system over \(I\) of quasi-compact and quasi-separated schemes over \(T\) with affine transition morphisms. Set \(T' = \lim T_i\). Denote \(\mathcal{F}'\) and \(\mathcal{F}_i\) the pullback of \(\mathcal{F}\) to \(T'\), resp. \(T_i\). Consider the diagrams

\[
\begin{array}{c}
\begin{array}{c}
X \leftarrow h \rightarrow Y \xrightarrow{\pi_i} Y_i \\
S \leftarrow g \rightarrow T \xrightarrow{\pi_i} T_i
\end{array}
\end{array}
\quad \text{and} \quad
\begin{array}{c}
\begin{array}{c}
X \leftarrow h_1 \rightarrow Y \xrightarrow{\pi_i} Y_i \\
S \leftarrow g_1 \rightarrow T \xrightarrow{\pi_i} T_i
\end{array}
\end{array}
\]
as in the previous paragraph. It is clear that \(\mathcal{F}'\) on \(T'\) is the colimit of the pullbacks of \(\mathcal{F}_i\) to \(T'\) and that \((e')^{-1}\mathcal{F}'\) is the colimit of the pullbacks of \(e_i^{-1}\mathcal{F}_i\) to \(Y'\). By Lemma \(51.8\) we have
\[
R^qh'_i(e')^{-1}\mathcal{F}' = \text{colim} R^qh_{i*e_i}^{-1}\mathcal{F}_i \quad \text{and} \quad (f')^{-1}R^qg'_*\mathcal{F}' = \text{colim}(f')^{-1}R^qg_{i*}\mathcal{F}_i
\]
It follows that if \(P(T_i)\) is true for all \(i\), then \(P(T')\) holds. Thus condition (2) of Lemma \(82.5\) holds for \(P\).

The most interesting is condition (3) of Lemma \(82.5\). Assume \(T'\) is a quasi-compact and quasi-separated scheme over \(T\) such that \(P(T')\) is true. Let \(Z \subset T'\) be a closed subscheme with complement \(V \subset T'\) quasi-compact. Consider the diagram

\[
\begin{array}{c}
Y' \times T' \xrightarrow{j} Y' \xrightarrow{\xi' e'} Y' \times T' \xrightarrow{ev} Y' \\
\downarrow i \downarrow \xi' \downarrow \downarrow j \downarrow \downarrow j' \\
Z \xrightarrow{i} V \xrightarrow{ev} V
\end{array}
\]
Choose an injective map \(j^{-1}\mathcal{F}' \to \mathcal{J}\) where \(\mathcal{J}\) is an injective sheaf of \(\mathbf{Z}/n\mathbf{Z}\)-modules on \(V\). Looking at stalks we see that the map
\[
\mathcal{F}' \to \mathcal{G} = j_*\mathcal{J} \oplus i_*j^{-1}\mathcal{F}'
\]
is injective. Thus \(\xi'\) maps to a nonzero element of
\[
\begin{align*}
\text{Coker} \left( (f')^{-1}R^qg'_*\mathcal{G} \right)_{\pi} & \to (R^qh'_i(e')^{-1}\mathcal{G})_{\pi} \\
\text{Coker} \left( (f')^{-1}R^qg'_*\mathcal{J} \right)_{\pi} & \to (R^qh'_i(e')^{-1}\mathcal{J})_{\pi} \oplus \\
\text{Coker} \left( (f')^{-1}R^qg'_*i_*j^{-1}\mathcal{F}' \right)_{\pi} & \to (R^qh'_i(e')^{-1}i_*j^{-1}\mathcal{F}')_{\pi}
\end{align*}
\]
by part (2) of Lemma 82.2. If \( \xi' \) does not map to zero in the second summand, then we use

\[
(f')^{-1}R^qg'_\ast i'_{\ast i^{-1}}\mathcal{F}' = (f')^{-1}R^q(g' \circ i)_\ast i^{-1}\mathcal{F}'
\]

(because \( Ri_\ast = i_\ast \) by Proposition 54.2) and

\[
R^qh'_\ast (e')^{-1}i'_{\ast i^{-1}}\mathcal{F} = R^qh'_\ast i'_{\ast i^{-1}}\mathcal{F} = R^q(h' \circ \xi')_\ast e_Z^{-1}i^{-1}\mathcal{F}'
\]

(first equality by Lemma 54.3 and the second because \( Ri'_\ast = i'_\ast \) by Proposition 54.2) to see that we have \( P(Z) \). Finally, suppose \( \xi' \) does not map to zero in the first summand. We have

\[
(e')^{-1}j'_{\ast \mathcal{J}} = j'_V e^{-1}_V \mathcal{J} \quad \text{and} \quad R^qj'_V e^{-1}_V \mathcal{J} = 0, \quad a = 1, \ldots, q - 1
\]

by \( BC(f, n, q - 1) \) applied to the diagram

\[
\begin{array}{ccc}
X & & Y' \\
\downarrow f & & \downarrow e' \\
S & & T' \\
\end{array}
\begin{array}{c}
Y \\
\downarrow e_V \\
V
\end{array}
\]

and the fact that \( \mathcal{J} \) is injective. By the relative Leray spectral sequence for \( h' \circ j' \)
(Cohomology on Sites, Lemma 15.7) we deduce that

\[
R^qh'_\ast (e')^{-1}j_{\ast \mathcal{J}} = R^qh'_\ast j'_V e^{-1}_V \mathcal{J} \rightarrow R^q(h' \circ j')_\ast e^{-1}_V \mathcal{J}
\]

is injective. Thus \( \xi \) maps to a nonzero element of \( (R^q(h' \circ j')_\ast e^{-1}_V \mathcal{J})_\ast \). Applying part (3) of Lemma 82.2 to the injection \( j^{-1}\mathcal{F} \rightarrow \mathcal{J} \) we conclude that \( P(V) \) holds. \( \Box \)

\textbf{Lemma 82.7.} With \( f : X \to S \) and \( n \) as in Remark 82.1 assume for some \( q \geq 1 \) we have that \( BC(f, n, q - 1) \) is true, but \( BC(f, n, q) \) is not. Then there exist a commutative diagram

\[
\begin{array}{ccc}
X & & X' \\
\downarrow f & & \downarrow h \\
S & & Spec(K)
\end{array}
\begin{array}{c}
Y \\
\downarrow \\
Spec(K)
\end{array}
\]

with both squares cartesian, where

- (1) \( S' \) is affine, integral, and normal with algebraically closed function field,
- (2) \( K \) is algebraically closed and \( Spec(K) \to S' \) is dominant (in other words \( K \) is an extension of the function field of \( S' \))

and there exists an integer \( d \mid n \) such that \( R^qh_\ast(Z/dZ) \) is nonzero.

Conversely, nonvanishing of \( R^qh_\ast(Z/dZ) \) in the lemma implies \( BC(f, n, q) \) isn’t true as Lemma 82.4 shows that \( R^q(Spec(K) \to S'), Z/dZ = 0 \).

\textbf{Proof.} First choose a diagram and \( \mathcal{F} \) as in Lemma 82.6. We may and do assume \( S' \) is affine (this is obvious, but see proof of the lemma in case of doubt). By Lemma 82.3 we may assume \( K \) is algebraically closed. Then \( \mathcal{F} \) corresponds to a \( Z/nZ \)-module. Such a modules is a direct sum of copies of \( Z/dZ \) for varying \( d/n \) hence we may assume \( \mathcal{F} \) is constant with value \( Z/dZ \). By Lemma 82.4 we may replace \( S' \) by the normalization of \( S' \) in \( Spec(K) \) which finishes the proof. \( \Box \)
0EYT Lemma 83.1. Let $K/k$ be an extension of fields. Let $X$ be a smooth affine curve over $k$ with a rational point $x \in X(k)$. Let $\mathcal{F}$ be an abelian sheaf on $\text{Spec}(K)$ annihilated by an integer $n$ invertible in $k$. Let $q > 0$ and

$$\xi \in H^q(X_K, (X_K \to \text{Spec}(K))^{-1}\mathcal{F})$$

There exist

1. finite extensions $K'/K$ and $k'/k$ with $k' \subset K'$,
2. a finite étale Galois cover $Z \to X_{k'}$ with group $G$

such that the order of $G$ divides a power of $n$, such that $Z \to X_{k'}$ is split over $x_{k'}$, and such that $\xi$ dies in $H^q(Z_{K'}, (Z_{K'} \to \text{Spec}(K))^{-1}\mathcal{F})$.

Proof. For $q > 1$ we know that $\xi$ dies in $H^q(X_{K'}, (X_{K'} \to \text{Spec}(K))^{-1}\mathcal{F})$ (Theorem 78.9). By Lemma 51.5 we see that this means there is a finite extension $K'/K$ such that $\xi$ dies in $H^q(Z_{K'}, (Z_{K'} \to \text{Spec}(K))^{-1}\mathcal{F})$. Thus we can take $k' = k$ and $Z = X$ in this case.

Assume $q = 1$. Recall that $\mathcal{F}$ corresponds to a discrete module $M$ with continuous $\text{Gal}_K$-action, see Lemma 83.1. Since $M$ is $n$-torsion, it is the union of finite $\text{Gal}_K$-stable subgroups. Thus we reduce to the case where $M$ is a finite abelian group annihilated by $n$, see Lemma 51.4. After replacing $K$ by a finite extension we may assume that the action of $\text{Gal}_K$ on $M$ is trivial. Thus we may assume $\mathcal{F} = M$ is the constant sheaf with value a finite abelian group $M$ annihilated by $n$.

We can write $M$ as a direct sum of cyclic groups. Any two finite étale Galois coverings whose Galois groups have order invertible in $k$, can be dominated by a third one whose Galois group has order invertible in $k$ (Fundamental Groups, Section 7). Thus it suffices to prove the lemma when $M = \mathbb{Z}/d\mathbb{Z}$ where $d|n$.

Assume $M = \mathbb{Z}/d\mathbb{Z}$ where $d|n$. In this case $\xi = \xi|_{X_T}$ is an element of

$$H^1(X_T, \mathbb{Z}/d\mathbb{Z}) = H^1(X_T, \mathbb{Z}/d\mathbb{Z})$$

See Theorem 78.9. This group classifies $\mathbb{Z}/d\mathbb{Z}$-torsors, see Cohomology on Sites, Lemma 5.3. The torsor corresponding to $\xi$ (viewed as a sheaf on $X_{T, \text{étale}}$) in turn gives rise to a finite étale morphism $T \to X_T$ endowed an action of $\mathbb{Z}/d\mathbb{Z}$ transitive on the fibre of $T$ over $x_T$, see Lemma 63.4. Choose a connected component $T' \subset T$ (if $\xi$ has order $d$, then $T$ is already connected). Then $T' \to X_T$ is a finite étale Galois cover whose Galois group is a subgroup $G \subset \mathbb{Z}/d\mathbb{Z}$ (small detail omitted). Moreover the element $\xi$ maps to zero under the map $H^1(X_{T'}, \mathbb{Z}/d\mathbb{Z}) \to H^1(T', \mathbb{Z}/d\mathbb{Z})$ as this is one of the defining properties of $T$.

Next, we use a limit argument to choose a finite extension $k'/k$ contained in $K$ such that $T' \to X_T$ descends to a finite étale Galois cover $Z \to X_{k'}$ with group $G$. See Limits, Lemmas 10.1.8.3 and 10.10. After increasing $k'$ we may assume that $Z$ splits over $x_{k'}$. The image of $\xi$ in $H^1(Z_{K'}, \mathbb{Z}/d\mathbb{Z})$ is zero by construction. Thus by Lemma 51.5 we can find a finite subextension $\overline{K}/K'$ containing $k'$ such that $\xi$ dies in $H^1(Z_{K'}, \mathbb{Z}/d\mathbb{Z})$ and this finishes the proof. $\square$
**Theorem 83.2** (Smooth base change). Consider a cartesian diagram of schemes

\[
\begin{array}{ccc}
X & \xleftarrow{h} & Y \\
f \downarrow & & \downarrow e \\
S & \xleftarrow{g} & T
\end{array}
\]

where \( f \) is smooth and \( g \) quasi-compact and quasi-separated. Then

\[
f^{-1} R^q g_* F = R^q h_* e^{-1} F
\]

for any \( q \) and any abelian sheaf \( F \) on \( T \) étale all of whose stalks at geometric points are torsion of orders invertible on \( S \).

**First proof of smooth base change.** This proof is very long but more direct (using less general theory) than the second proof given below.

The theorem is local on \( X \) étale. More precisely, suppose we have \( U \to X \) étale such that \( U \to S \) factors as \( U \to V \to S \) with \( V \to S \) étale. Then we can consider the cartesian square

\[
\begin{array}{ccc}
U & \xleftarrow{h'} & U \times_X Y \\
f' \downarrow & & \downarrow e' \\
V & \xleftarrow{g'} & V \times_S T
\end{array}
\]

and setting \( F' = F|_{V \times_S T} \) we have \( f^{-1} R^q g_* F|_U = (f')^{-1} R^q g'_* F' \) and \( R^q h_* e^{-1} F|_U = R^q h'_* (e')^{-1} F' \) (as follows from the compatibility of localization with morphisms of sites, see Sites, Lemma \( \text{28.2} \) and and Cohomology on Sites, Lemma \( \text{21.4} \)). Thus it suffices to produce an étale covering of \( X \) by \( U \to X \) and factorizations \( U \to V \to S \) as above such that the theorem holds for the diagram with \( f' \), \( h' \), \( g' \), \( e' \).

By the local structure of smooth morphisms, see Morphisms, Lemma \( \text{34.20} \) we may assume \( X \) and \( S \) are affine and \( X \to S \) factors through an étale morphism \( X \to A^d_S \). If we have a tower of cartesian diagrams

\[
\begin{array}{ccc}
W & \xleftarrow{j} & Z \\
& \downarrow k & \\
X & \xleftarrow{h} & Y \\
& \downarrow e & \\
S & \xleftarrow{g} & T
\end{array}
\]

and the theorem holds for the bottom and top squares, then the theorem holds for the outer rectangle; this is formal. Writing \( X \to S \) as the composition

\[
X \to A^{d-1}_S \to A^{d-2}_S \to \ldots \to A^1_S \to S
\]

we conclude that it suffices to prove the theorem when \( X \) and \( S \) are affine and \( X \to S \) has relative dimension 1.

For every \( n \geq 1 \) invertible on \( S \), let \( F[n] \) be the subsheaf of sections of \( F \) annihilated by \( n \). Then \( F = \text{colim} F[n] \) by our assumption on the stalks of \( F \). The functors \( e^{-1} \) and \( f^{-1} \) commute with colimits as they are left adjoints. The functors \( R^q h_* \) and \( R^q g_* \) commute with filtered colimits by Lemma \( \text{51.7} \). Thus it suffices to prove
the theorem for \( F[n] \). From now on we fix an integer \( n \), we work with sheaves of \( \mathbb{Z}/n\mathbb{Z} \)-modules and we assume \( S \) is a scheme over \( \text{Spec}(\mathbb{Z}[1/n]) \).

Next, we reduce to the case where \( T \) is affine. Since \( g \) is quasi-compact and quasi-separated and \( S \) is affine, the scheme \( T \) is quasi-compact and quasi-separated. Thus we can use the induction principle of Cohomology of Schemes, Lemma 4.1. Hence it suffices to that if \( T = W \cup W' \) is an open covering and the theorem holds for the squares

\[
\begin{array}{ccc}
X & \xleftarrow{i} & e^{-1}(W) \\
\downarrow & & \downarrow \\
S & \xleftarrow{a} & W
\end{array} \quad \begin{array}{ccc}
X & \xleftarrow{j} & e^{-1}(W') \\
\downarrow & & \downarrow \\
S & \xleftarrow{b} & W'
\end{array} \quad \begin{array}{ccc}
X & \xleftarrow{k} & e^{-1}(W \cap W') \\
\downarrow & & \downarrow \\
S & \xleftarrow{c} & W \cap W'
\end{array}
\]

then the theorem holds for the original diagram. To see this we consider the diagram

\[
\begin{array}{c}
\cdots \rightarrow f^{-1}R^{q-1}c_*\mathcal{F}|_{W \cap W'} \\
\downarrow \sim \quad \downarrow \sim \\
R^qk_*e^{-1}\mathcal{F}|_{e^{-1}(W \cap W')} \rightarrow R^qh_*e^{-1}\mathcal{F}
\end{array}
\]

whose rows are exact sequences by Lemma 50.2. Thus the 5-lemma gives the desired conclusion.

Summarizing, we may assume \( S, X, T, \) and \( Y \) affine, \( \mathcal{F} \) is \( n \) torsion, \( X \rightarrow S \) is smooth of relative dimension 1, and \( S \) is a scheme over \( \mathbb{Z}[1/n] \). We will prove the theorem by induction on \( q \). The base case \( q = 0 \) is handled by Lemma 81.2. Assume \( q > 0 \) and the theorem holds for all smaller degrees. Choose a short exact sequence \( 0 \rightarrow \mathcal{F} \rightarrow \mathcal{I} \rightarrow \mathcal{Q} \rightarrow 0 \) where \( \mathcal{I} \) is an injective sheaf of \( \mathbb{Z}/n\mathbb{Z} \)-modules. Consider the induced diagram

\[
\begin{array}{ccc}
f^{-1}R^{q-1}g_*\mathcal{I} & \rightarrow & f^{-1}R^{q-1}g_*\mathcal{Q} \\
\downarrow \sim & & \downarrow \sim \\
R^qh_*e^{-1}\mathcal{I} & \rightarrow & R^qh_*e^{-1}\mathcal{Q}
\end{array}
\]

with exact rows. We have the zero in the right upper corner as \( \mathcal{I} \) is injective. The left two vertical arrows are isomorphisms by induction hypothesis. Thus it suffices to prove that \( R^qh_*e^{-1}\mathcal{I} = 0 \).

Write \( S = \text{Spec}(A) \) and \( T = \text{Spec}(B) \) and say the morphism \( T \rightarrow S \) is given by the ring map \( A \rightarrow B \). We can write \( A \rightarrow B = \text{colim}_{i \in I}(A_i \rightarrow B_i) \) as a filtered colimit of maps of rings of finite type over \( \mathbb{Z}[1/n] \) (see Algebra, Lemma 126.14). For \( i \in I \) we set \( S_i = \text{Spec}(A_i) \) and \( T_i = \text{Spec}(B_i) \). For \( i \) large enough we can find a smooth morphism \( X_i \rightarrow S_i \) of relative dimension 1 such that \( X = X_i \times_{S_i} S \), see Limits, Lemmas 10.1, 8.9, and 16.3. Set \( Y_i = X_i \times_{S_i} T_i \) to get squares

\[
\begin{array}{ccc}
X_i & \xleftarrow{h_i} & Y_i \\
\downarrow f_i & & \downarrow e_i \\
S_i & \xleftarrow{g_i} & T_i
\end{array}
\]
Observe that $\mathcal{I}_i = (T \to T_i)_{\mathcal{I}}$ is an injective sheaf of $\mathbb{Z}/n\mathbb{Z}$-modules on $T_i$, see Cohomology on Sites, Lemma [15.2]. We have $\mathcal{I} = \text{colim}(T \to T_i)^{-1}\mathcal{I}_i$ by Lemma 51.9. Pulling back by $e$ we get $e^{-1}\mathcal{I} = \text{colim}(Y \to Y_i)^{-1}e_i^{-1}\mathcal{I}_i$. By Lemma 51.8 applied to the system of morphisms $Y_i \to X_i$ with limit $Y \to X$ we have

$$R^qh_*e^{-1}\mathcal{I} = \text{colim}(X \to X_i)^{-1}R^qh_i_*e_i^{-1}\mathcal{I}_i.$$ 

This reduces us to the case where $T$ and $S$ are affine of finite type over $\mathbb{Z}[1/n]$. Summarizing, we have an integer $q \geq 1$ such that the theorem holds in degrees $< q$, the schemes $S$ and $T$ affine of finite type over $\mathbb{Z}[1/n]$, we have $X \to S$ smooth of relative dimension 1 with $X$ affine, and $\mathcal{I}$ is an injective sheaf of $\mathbb{Z}/n\mathbb{Z}$-modules and we have to show that $R^qh_*e^{-1}\mathcal{I} = 0$. We will do this by induction on $\dim(T)$.

The base case is $T = \emptyset$, i.e., $\dim(T) < 0$. If you don't like this, you can take as your base case the case $\dim(T) = 0$. In this case $T \to S$ is finite (in fact even $T \to \text{Spec}(\mathbb{Z}[1/n])$ is finite as the target is Jacobson; details omitted), so $h$ is finite too and hence has vanishing higher direct images (see references below).

Assume $\dim(T) = d \geq 0$ and we know the result for all situations where $T$ has lower dimension. Pick $U$ affine and étale over $X$ and a section $\xi$ of $R^qh_*e^{-1}\mathcal{I}$ over $U$. We have to show that $\xi$ is zero. Of course, we may replace $X$ by $U$ (and correspondingly $Y$ by $U \times_X Y$) and assume $\xi \in H^0(X, R^qh_*e^{-1}\mathcal{I})$. Moreover, since $R^qh_*e^{-1}\mathcal{I}$ is a sheaf, it suffices to prove that $\xi$ is zero locally on $X$. Hence we may replace $X$ by the members of an étale covering. In particular, using Lemma 51.6 we may assume that $\xi$ is the image of an element $\xi' \in H^q(Y, e^{-1}\mathcal{I})$. In terms of $\xi'$ our task is to show that $\xi'$ dies in $H^q(U, Y, e^{-1}\mathcal{I})$ for some étale covering $\{U, X \to Y\}$.

By More on Morphisms, Lemma 34.8 we may assume that $X \to S$ factors as $X \to V \to S$ where $V \to S$ is étale and $X \to V$ is a smooth morphism of affine schemes of relative dimension 1, has a section, and has geometrically connected fibres. Observe that $\dim(V \times_S T) \leq \dim(T) = d$ for example by More on Algebra, Lemma 33.2. Hence we may then replace $S$ by $V$ and $T$ by $V \times_S T$ (exactly as in the discussion in the first paragraph of the proof). Thus we may assume $X \to S$ is smooth of relative dimension 1, geometrically connected fibres, and has a section $\sigma : S \to X$.

Let $\pi : T' \to T$ be a finite surjective morphism. We will use below that $\dim(T') \leq \dim(T) = d$, see Algebra, Lemma 111.3. Choose an injective map $\pi^{-1}\mathcal{I} \to T'$ into an injective sheaf of $\mathbb{Z}/n\mathbb{Z}$-modules. Then $\mathcal{I} \to \pi_*\mathcal{I}'$ is injective and hence has a splitting (as $\mathcal{I}$ is an injective sheaf of $\mathbb{Z}/n\mathbb{Z}$-modules). Denote $\pi' : Y' = Y \times_T T' \to Y$ the base change of $\pi$ and $e' : Y' \to T'$ the base change of $e$. Picture

$$
\begin{array}{ccc}
X & \leftarrow & Y \\
\downarrow f & & \downarrow \pi' \\
S & \leftarrow & T \\
\downarrow e & & \downarrow e' \\
T' & \rightarrow & Y'
\end{array}
$$

By Proposition 54.2 and Lemma 54.3 we have $R^{q'}(e')^{-1}\mathcal{I}' = e^{-1}\pi_*\mathcal{I}'$. Thus by the Leray spectral sequence (Cohomology on Sites, Lemma 15.5) we have

$$H^q(Y', (e')^{-1}\mathcal{I}') = H^q(Y, e^{-1}\pi_*\mathcal{I}') \supset H^q(Y, e^{-1}\mathcal{I})$$

and this remains true after base change by any $U \to X$ étale. Thus we may replace $T$ by $T'$, $\mathcal{I}$ by $\mathcal{I}'$ and $\hat{\xi}$ by its image in $H^q(Y', (e')^{-1}\mathcal{I}')$. 

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Suppose we have a factorization $T \rightarrow S' \rightarrow S$ where $\pi : S' \rightarrow S$ is finite. Setting $X' = S' \times_S X$ we can consider the induced diagram

$$
\begin{array}{cccc}
X & \xleftarrow{\pi} & X' & \xleftarrow{\kappa} & Y \\
\downarrow f & & \downarrow f' & & \downarrow e \\
S & \xleftarrow{\pi} & S' & \xleftarrow{\sigma} & T
\end{array}
$$

Since $\pi'$ has vanishing higher direct images we see that $R^qh_*e^{-1}\mathcal{I} = \pi'_*R^qh'_*e^{-1}\mathcal{I}$ by the Leray spectral sequence. Hence $H^0(X, R^qh_*e^{-1}\mathcal{I}) = H^0(X', R^qh'_*e^{-1}\mathcal{I})$. Thus $\xi$ is zero if and only if the corresponding section of $R^qh'_*e^{-1}\mathcal{I}$ is zero. Thus we may replace $S$ by $S'$ and $X$ by $X'$. Observe that $\sigma : S \rightarrow X$ base changes to $\sigma' : S' \rightarrow X'$ and hence after this replacement it is still true that $X \rightarrow S$ has a section $\sigma$ and geometrically connected fibres.

We will use that $S$ and $T$ are Nagata schemes, see Algebra, Proposition 156.16 which will guarantee that various normalizations are finite, see Morphisms, Lemmas 51.15 and 52.10. In particular, we may first replace $T$ by its normalization and then replace $S$ by the normalization of $S$ in $T$. Then $T \rightarrow S$ is a disjoint union of dominant morphisms of integral normal schemes, see Morphisms, Lemma 51.13. Clearly we may argue one connected component at a time, hence we may assume $T \rightarrow S$ is a dominant morphism of integral normal schemes.

Let $s \in S$ and $t \in T$ be the generic points. By Lemma 83.1 there exist finite field extensions $K/\kappa(t)$ and $k/\kappa(s)$ such that $k$ is contained in $K$ and a finite étale Galois covering $Z \rightarrow X_1$ with Galois group $G$ of order dividing a power of $n$ split over $\sigma(\text{Spec}(k))$ such that $\xi$ maps to zero in $H^q(Z_K, e^{-1}\mathcal{I}|_{Z_K})$. Let $T' \rightarrow T$ be the normalization of $T$ in $\text{Spec}(K)$ and let $S' \rightarrow S$ be the normalization of $S$ in $\text{Spec}(k)$. Then we obtain a commutative diagram

$$
\begin{array}{cccc}
S' & \xleftarrow{T'} & T' \\
\downarrow & & \downarrow \\
S & \xleftarrow{T} & T
\end{array}
$$

whose vertical arrows are finite. By the arguments given above we may and do replace $S$ and $T$ by $S'$ and $T'$ (and correspondingly $X$ by $X \times_S S'$ and $Y$ by $Y \times_T T'$). After this replacement we conclude we have a finite étale Galois covering $Z \rightarrow X_1$ of the generic fibre of $X \rightarrow S$ with Galois group $G$ of order dividing a power of $n$ split over $\sigma(s)$ such that $\xi$ maps to zero in $H^q(Z_t, (Z_t \rightarrow Y)^{-1}\mathcal{I})$. Here $Z_t = Z \times_S t = Z \times t = Z \times X_1 Y_1$. Since $n$ is invertible on $S$, by Fundamental Groups, Lemma 31.7 we can find a finite étale morphism $U \rightarrow X$ whose restriction to $X_1$ is $Z$.

At this point we replace $X$ by $U$ and $Y$ by $U \times_X Y$. After this replacement it may no longer be the case that the fibres of $X \rightarrow S$ are geometrically connected.

---

8This step can also be seen another way. Namely, we have to show that there is an étale covering $(U_i \rightarrow X)$ such that $\xi$ dies in $H^q(U_i \times_X Y, e^{-1}\mathcal{I})$. However, if we prove there is an étale covering $(U'_j \rightarrow X')$ such that $\xi$ dies in $H^q(U'_j \times_Y Y, e^{-1}\mathcal{I})$, then by property (B) for $X' \rightarrow X$ (Lemma 43.4) there exists an étale covering $(U_i \rightarrow X)$ such that $U_i \times_X X'$ is a disjoint union of schemes over $X'$ each of which factors through $U'_j$ for some $j$. Thus we see that $\xi$ dies in $H^q(U_i \times_X Y, e^{-1}\mathcal{I})$ as desired.
(there still is a section but we won’t use this), but what we gain is that after this replacement \( \tilde{\xi} \) maps to zero in \( H^q(Y, e^{-1}\mathcal{I}) \), i.e., \( \tilde{\xi} \) restricts to zero on the generic fibre of \( Y \to T \).

Recall that \( t \) is the spectrum of the function field of \( T \), i.e., as a scheme \( t \) is the limit of the nonempty affine open subschemes of \( T \). By Lemma 51.5 we conclude there exists a nonempty open subscheme \( V \subset T \) such that \( \tilde{\xi} \) maps to zero in \( H^q(Y \times_T V, e^{-1}\mathcal{I}|_{Y \times_T V}) \).

Denote \( Z = T \setminus V \). Consider the diagram

\[
\begin{array}{ccc}
Y \times_T Z & \xrightarrow{i'} & Y \\
\downarrow{e_Z} & & \downarrow{e} \\
Z & \xrightarrow{i} & T \\
\downarrow{j} & & \downarrow{e_V} \\
& & V
\end{array}
\]

Choose an injection \( i^{-1}\mathcal{I} \to \mathcal{I}' \) into an injective sheaf of \( \mathbb{Z}/n\mathbb{Z} \)-modules on \( Z \).

Looking at stalks we see that the map

\[
\tilde{\xi} \text{ is injective. Thus the vanishing of the image of } \tilde{\xi} \text{ maps to zero in } H^q(Y, e^{-1}\mathcal{I}).
\]

at least after replacing \( X \) by the members of an étale covering. Observe that

\[
e^{-1}j_*\mathcal{I}|_V = j'_*e^{-1}_V\mathcal{I}|_V, \quad e^{-1}i_*\mathcal{I}' = i'_*e^{-1}_Z\mathcal{I}'
\]

By induction hypothesis on \( q \) we see that

\[
R^aq'_*e^{-1}_V\mathcal{I}|_V = 0, \quad a = 1, \ldots, q - 1
\]

By the Leray spectral sequence for \( j' \) and the vanishing above it follows that

\[
H^q(Y, j'_*(e^{-1}_V\mathcal{I}|_V)) \to H^q(Y \times_T V, e^{-1}_V\mathcal{I}_V) = H^q(Y \times_T V, e^{-1}\mathcal{I}|_{Y \times_T V})
\]

is injective. Thus the vanishing of the image of \( \tilde{\xi} \) in the first summand above because we know \( \tilde{\xi} \) vanishes in \( H^q(Y \times_T V, e^{-1}\mathcal{I}|_{Y \times_T V}) \). Since \( \dim(Z) < \dim(T) = d \) by induction the image of \( \tilde{\xi} \) in the second summand

\[
H^q(Y, e^{-1}i_*\mathcal{I}') = H^q(Y, i'_*e^{-1}_Z\mathcal{I}') = H^q(Y \times_T Z, e^{-1}_Z\mathcal{I}')
\]

dies after replacing \( X \) by the members of a suitable étale covering. This finishes the proof of the smooth base change theorem. \( \square \)

**Second proof of smooth base change.** This proof is the same as the longer first proof; it is shorter only in that we have split out the arguments used in a number of lemmas.

The case of \( q = 0 \) is Lemma 81.2. Thus we may assume \( q > 0 \) and the result is true for all smaller degrees.

For every \( n \geq 1 \) invertible on \( S \), let \( \mathcal{F}[n] \) be the subsheaf of sections of \( \mathcal{F} \) annihilated by \( n \). Then \( \mathcal{F} = \text{colim} \mathcal{F}[n] \) by our assumption on the stalks of \( \mathcal{F} \). The functors \( e^{-1} \) and \( f^{-1} \) commute with colimits as they are left adjoints. The functors \( R^qh_* \) and \( R^qg_* \) commute with filtered colimits by Lemma 51.7. Thus it suffices to prove the theorem for \( \mathcal{F}[n] \). From now on we fix an integer \( n \) invertible on \( S \) and we work with sheaves of \( \mathbb{Z}/n\mathbb{Z} \)-modules.
By Lemma \[80.1\] the question is étale local on \(X\) and \(S\). By the local structure of smooth morphisms, see Morphisms, Lemma \[34.20\] we may assume \(X\) and \(S\) are affine and \(X \to S\) factors through an étale morphism \(X \to \mathbf{A}_S^d\). Writing \(X \to S\) as the composition

\[
X \to \mathbf{A}_S^{d-1} \to \mathbf{A}_S^{d-2} \to \ldots \to \mathbf{A}_S^1 \to S
\]

we conclude from Lemma \[80.2\] that it suffices to prove the theorem when \(X\) and \(S\) are affine and \(X \to S\) has relative dimension 1.

By Lemma \[82.7\] it suffices to show that \(R^q h_* \mathbb{Z}/d\mathbb{Z} = 0\) for \(d|n\) whenever we have a cartesian diagram

\[
\begin{array}{ccc}
X & \xrightarrow{h} & Y \\
\downarrow & & \downarrow \\
S & \xleftarrow{} & \text{Spec}(K)
\end{array}
\]

where \(X \to S\) is affine and smooth of relative dimension 1, \(S\) is the spectrum of a normal domain \(A\) with algebraically closed fraction field \(L\), and \(K/L\) is an extension of algebraically closed fields.

Recall that \(R^q h_* \mathbb{Z}/d\mathbb{Z}\) is the sheaf associated to the presheaf

\[
U \mapsto H^q(U \times_X Y, \mathbb{Z}/d\mathbb{Z}) = H^q(U \times_S \text{Spec}(K), \mathbb{Z}/d\mathbb{Z})
\]

on \(X_{\text{étale}}\) (Lemma \[51.6\]). Thus it suffices to show: given \(U\) and \(\xi \in H^q(U \times_S \text{Spec}(K), \mathbb{Z}/d\mathbb{Z})\) there exists an étale covering \(\{U_i \to U\}\) such that \(\xi\) dies in \(H^q(U_i \times_S \text{Spec}(K), \mathbb{Z}/d\mathbb{Z})\).

Of course we may take \(U\) affine. Then \(U \times_S \text{Spec}(K)\) is a (smooth) affine curve over \(K\) and hence we have vanishing for \(q > 1\) by Theorem \[78.9\].

Final case: \(q = 1\). We may replace \(U\) by the members of an étale covering as in More on Morphisms, Lemma \[34.8\] Then \(U \to S\) factors as \(U \to V \to S\) where \(U \to V\) has geometrically connected fibres, \(U, V\) are affine, \(V \to S\) is étale, and there is a section \(\sigma : V \to U\). By Lemma \[76.3\] we see that \(V\) is isomorphic to a (finite) disjoint union of (affine) open subschemes of \(S\). Clearly we may replace \(S\) by one of these and \(X\) by the corresponding component of \(U\). Thus we may assume \(X \to S\) has geometrically connected fibres, has a section \(\sigma\), and \(\xi \in H^1(Y, \mathbb{Z}/d\mathbb{Z})\).

Since \(K\) and \(L\) are algebraically closed we have

\[
H^1(X_L, \mathbb{Z}/d\mathbb{Z}) = H^1(Y, \mathbb{Z}/d\mathbb{Z})
\]

See Lemma \[78.11\] Thus there is a finite étale Galois covering \(Z \to X_L\) with Galois group \(G \subset \mathbb{Z}/d\mathbb{Z}\) which annihilates \(\xi\). You can either see this by looking at the statement or proof of Lemma \[83.1\] or by using directly that \(\xi\) corresponds to a \(\mathbb{Z}/d\mathbb{Z}\)-torsor over \(X_L\). Finally, by Fundamental Groups, Lemma \[31.8\] we find a (necessarily surjective) finite étale morphism \(X' \to X\) whose restriction to \(X_L\) is \(Z \to X_L\). Since \(\xi\) dies in \(X'_K\) this finishes the proof. \(\square\)

The following immediate consequence of the smooth base change theorem is what is often used in practice.
Lemma 83.3. Let $S$ be a scheme. Let $S' = \lim S_i$ be a directed inverse limit of schemes $S_i$ smooth over $S$ with affine transition morphisms. Let $f : X \to S$ be quas-compact and quasi-separated and form the fibre square

$$\begin{array}{ccc}
X' & \longrightarrow & X \\
\downarrow f' & & \downarrow f \\
S' & \longrightarrow & S
\end{array}$$

Then

$$g^{-1}Rf_*E = R(f'_*) (g')^{-1} E$$

for any $E \in D^+(X_{\text{étale}})$ whose cohomology sheaves $H^q(E)$ have stalks which are torsion of orders invertible on $S$.

Proof. Consider the spectral sequences

$$E_2^{p,q} = R^p f_* H^q (E) \quad \text{and} \quad E_2^{p,q} = R^p f'_* H^q ((g')^{-1} E) = R^p f'_* (g')^{-1} H^q (E)$$

converging to $R^p f_* E$ and $R^p f'_* (g')^{-1} E$. These spectral sequences are constructed in Derived Categories, Lemma [21.3]. Combining the smooth base change theorem (Theorem 83.2) with Lemma 80.3 we see that

$$g^{-1} R^p f_* H^q (E) = R^p (f'_*) (g')^{-1} H^q (E)$$

Combining all of the above we get the lemma. □

84. Applications of smooth base change

In this section we discuss some more or less immediate consequences of the smooth base change theorem.

Lemma 84.1. Let $L/K$ be an extension of fields. Let $g : T \to S$ be a quas-compact and quasi-separated morphism of schemes over $K$. Denote $g_* : T_* \to S_*$ the base change of $g$ to $\text{Spec}(L)$. Let $E \in D^+(T_{\text{étale}})$ have cohomology sheaves whose stalks are torsion of orders invertible in $K$. Let $E_L$ be the pullback of $E$ to $(T_L)_{\text{étale}}$. Then $Rg_* E_L$ is the pullback of $Rg_* E$ to $S_L$.

Proof. If $L/K$ is separable, then $L$ is a filtered colimit of smooth $K$-algebras, see Algebra, Lemma 152.11. Thus the lemma in this case follows immediately from Lemma 83.3. In the general case, let $K'$ and $L'$ be the perfect closures (Algebra, Definition 44.5) of $K$ and $L$. Then $\text{Spec}(K') \to \text{Spec}(K)$ and $\text{Spec}(L') \to \text{Spec}(L)$ are universal homeomorphisms as $K'/K$ and $L'/L$ are purely inseparable (see Algebra, Lemma 45.7). Thus we have $(T_{K'})_{\text{étale}} = T_{\text{étale}}, (S_{K'})_{\text{étale}} = S_{\text{étale}}, (T_{L'})_{\text{étale}} = (T_L)_{\text{étale}},$ and $(S_{L'})_{\text{étale}} = (S_L)_{\text{étale}}$ by the topological invariance of étale cohomology, see Proposition 45.4. This reduces the lemma to the case of the field extension $L'/K'$ which is separable (by definition of perfect fields, see Algebra, Definition 44.1). □

Lemma 84.2. Let $K/k$ be an extension of separably closed fields. Let $X$ be a quas-compact and quasi-separated scheme over $k$. Let $E \in D^+(X_{\text{étale}})$ have cohomology sheaves whose stalks are torsion of orders invertible in $k$. Then

1. the maps $H^3_{\text{étale}}(X, E) \to H^3_{\text{étale}}(X_K, E|_{X_K})$ are isomorphisms, and
2. $E \to R(X_K \to X)_* E|_{X_K}$ is an isomorphism.
Proof. Proof of (1). First let \( \overline{k} \) and \( \overline{K} \) be the algebraic closures of \( k \) and \( K \). The morphisms \( \text{Spec}(k) \to \text{Spec}(k) \) and \( \text{Spec}(\overline{K}) \to \text{Spec}(K) \) are universal homeomorphisms as \( k/k \) and \( \overline{K}/K \) are purely inseparable (see Algebra, Lemma 45.7). Thus \( H^q_{\text{ét}}(X, \mathcal{F}) = H^q_{\text{ét}}(X_{\overline{k}}, \mathcal{F}_{X_{\overline{k}}}) \) by the topological invariance of étale cohomology, see Proposition 45.4. Similarly for \( X_K \) and \( X_{\overline{K}} \). Thus we may assume \( k \) and \( K \) are algebraically closed. In this case \( K \) is a limit of smooth \( k \)-algebras, see Algebra, Lemma 152.11. We conclude our lemma is a special case of Theorem 83.2 as reformulated in Lemma 83.3.

Proof of (2). For any quasi-compact and quasi-separated \( U \) in \( X_{\text{ét}} \) the above shows that the restriction of the map \( E \to R(X_K \to X)_* E|_{X_K} \) determines an isomorphism on cohomology. Since every object of \( X_{\text{ét}} \) has an étale covering by such \( U \) this proves the desired statement. □

**Lemma 84.3.** With \( f : X \to S \) and \( n \) as in Remark 82.1 assume \( n \) is invertible on \( S \) and that for some \( q \geq 1 \) we have that \( BC(f, n, q - 1) \) is true, but \( BC(f, n, q) \) is not. Then there exist a commutative diagram

\[
\begin{array}{ccc}
X & \leftarrow & X' \\
\downarrow f & & \downarrow h \\
S & \leftarrow & S' \\
& \downarrow & \\
& \text{Spec}(K) & \\
\end{array}
\]

with both squares cartesian, where \( S' \) is affine, integral, and normal with algebraically closed function field \( K \) and there exists an integer \( d|n \) such that \( Rh'_* (\mathbb{Z}/d\mathbb{Z}) \) is nonzero.

**Proof.** First choose a diagram and \( F \) as in Lemma 82.7. We may and do assume \( S' \) is affine (this is obvious, but see proof of the lemma in case of doubt). Let \( K' \) be the function field of \( S' \) and let \( Y' = X' \times_S \text{Spec}(K') \) to get the diagram

\[
\begin{array}{ccc}
X & \leftarrow & X' \\
\downarrow f & & \downarrow h' \\
S & \leftarrow & S' \\
& \downarrow & \\
& \text{Spec}(K') & \\
& \downarrow & \\
& \text{Spec}(K) & \\
\end{array}
\]

By Lemma 84.2 the total direct image \( R(Y \to Y')_* \mathbb{Z}/d\mathbb{Z} \) is isomorphic to \( \mathbb{Z}/d\mathbb{Z} \) in \( D(Y'_{\text{ét}}) \); here we use that \( n \) is invertible on \( S \). Thus \( Rh'_* \mathbb{Z}/d\mathbb{Z} = Rh_* \mathbb{Z}/d\mathbb{Z} \) by the relative Leray spectral sequence. This finishes the proof. □

85. The proper base change theorem

The proper base change theorem is stated and proved in this section. Our approach follows roughly the proof in [AGV71, XII, Theorem 5.1] using Gabber’s ideas (from the affine case) to slightly simplify the arguments.

**Lemma 85.1.** Let \( (A, I) \) be a henselian pair. Let \( f : X \to \text{Spec}(A) \) be a proper morphism of schemes. Let \( Z = X \times_{\text{Spec}(A)} \text{Spec}(A/I) \). For any sheaf \( \mathcal{F} \) on the topological space associated to \( X \) we have \( \Gamma(X, \mathcal{F}) = \Gamma(Z, \mathcal{F}|_Z) \).

**Proof.** We will use Lemma 7.4 to prove this. First observe that the underlying topological space of \( X \) is spectral by Properties, Lemma 2.4. Let \( Y \subset X \) be an irreducible closed subscheme. To finish the proof we show that \( Y \cap Z = Y \times_{\text{Spec}(A)} \)
Spec(A/I) is connected. Replacing X by Y we may assume that X is irreducible and we have to show that Z is connected. Let X \to \text{Spec}(B) \to \text{Spec}(A) be the Stein factorization of f (More on Morphisms, Theorem [16.5]). Then A \to B is integral and (B, \mathcal{IB}) is a henselian pair (More on Algebra, Lemma [11.8]). Thus we may assume the fibres of X \to \text{Spec}(A) are geometrically connected. On the other hand, the image T \subset \text{Spec}(A) of f is irreducible and closed as X is proper over A. Hence T \cap V(I) is connected by More on Algebra, Lemma [11.12]. Now \(Y \times_{\text{Spec}(A)} \text{Spec}(A/I) \to T \cap V(I)\) is a surjective closed map with connected fibres. The result now follows from Topology, Lemma [7.4]. □

**Lemma 85.2.** Let \((A, I)\) be a henselian pair. Let \(f : X \to \text{Spec}(A)\) be a proper morphism of schemes. Let \(i : Z \to X\) be the closed immersion of \(X \times_{\text{Spec}(A)} \text{Spec}(A/I)\) into X. For any sheaf \(\mathcal{F}\) on \(X_{\text{étale}}\) we have \(\Gamma(X, \mathcal{F}) = \Gamma(Z, i^{-1}_{\text{small}}\mathcal{F})\).

**Proof.** This follows from Lemma [77.2] and [85.1] and the fact that any scheme finite over Spec(A). □

**Lemma 85.3.** Let \(A\) be a henselian local ring. Let \(f : X \to \text{Spec}(A)\) be a proper morphism of schemes. Let \(X_0 \subset X\) be the fibre of \(f\) over the closed point. For any sheaf \(\mathcal{F}\) on \(X_{\text{étale}}\) we have \(\Gamma(X, \mathcal{F}) = \Gamma(X_0, \mathcal{F}|_{X_0})\).

**Proof.** This is a special case of Lemma [85.2]. □

Let \(f : X \to S\) be a morphism of schemes. Let \(\bar{s} : \text{Spec}(k) \to S\) be a geometric point. The fibre of \(f\) at \(\bar{s}\) is the scheme \(X_{\bar{s}} = \text{Spec}(k) \times_S X\) viewed as a scheme over \(\text{Spec}(k)\). If \(\mathcal{F}\) is a sheaf on \(X_{\text{étale}}\), then denote \(\mathcal{F}_{\bar{s}} = p^{-1}_{\text{small}}\mathcal{F}\) the pullback of \(\mathcal{F}\) to \((X_{\bar{s}})_{\text{étale}}\). In the following we will consider the set

\[\Gamma(X_{\bar{s}}, \mathcal{F}_{\bar{s}})\]

Let \(s \in S\) be the image point of \(\bar{s}\). Let \(k(s)^{\text{sep}}\) be the separable algebraic closure of \(k(s)\) in \(k\) as in Definition [55.1]. By Lemma [39.5] pullback defines a bijection

\[\Gamma(X_{k(s)^{\text{sep}}}, p^{-1}_{\text{sep}}\mathcal{F}) \to \Gamma(X_{\bar{s}}, \mathcal{F}_{\bar{s}})\]

where \(p_{\text{sep}} : X_{k(s)^{\text{sep}}} = \text{Spec}(k(s)^{\text{sep}}) \times_S X \to X\) is the projection.

**Lemma 85.4.** Let \(f : X \to S\) be a proper morphism of schemes. Let \(\bar{s} : S \to \text{Spec}(A)\) be a geometric point. For any sheaf \(\mathcal{F}\) on \(X_{\text{étale}}\) the canonical map

\[(f_* \mathcal{F})_{\bar{s}} \to \Gamma(X_{\bar{s}}, \mathcal{F}_{\bar{s}})\]

is bijective.

**Proof.** By Theorem [52.1] (for sheaves of sets) we have

\[(f_* \mathcal{F})_{\bar{s}} = \Gamma(X \times_S \text{Spec}(O_{S, \bar{s}}^{sh}), p_{\text{small}}^{-1}\mathcal{F})\]

where \(p : X \times_S \text{Spec}(O_{S, \bar{s}}^{sh}) \to X\) is the projection. Since the residue field of the strictly henselian local ring \(O_{S, \bar{s}}^{sh}\) is \(k(s)^{\text{sep}}\) we conclude from the discussion above the lemma and Lemma [85.3]. □

**Lemma 85.5.** Let \(f : X \to Y\) be a proper morphism of schemes. Let \(g : Y' \to Y\) be a morphism of schemes. Set \(X' = Y' \times_Y X\) with projections \(f' : X' \to Y'\) and \(g' : X' \to X\). Let \(\mathcal{F}\) be any sheaf on \(X_{\text{étale}}\). Then \(g^{-1} f_* \mathcal{F} = f'_*(g')^{-1} \mathcal{F}\).
Proof. There is a canonical map \( g^{-1}f_*F \rightarrow f'_*(g')^{-1}F \). Namely, it is adjoint to the map
\[
f_*F \rightarrow g_*f'_*(g')^{-1}F = f_*g'_*(g')^{-1}F
\]
which is \( f_* \) applied to the canonical map \( F \rightarrow g'_*(g')^{-1}F \). To check this map is an isomorphism we can compute what happens on stalks. Let \( y' : \text{Spec}(k) \rightarrow Y' \) be a geometric point with image \( y \) in \( Y \). By Lemma 85.4 the stalks are \( \Gamma(X'_y, F_y') \) and \( \Gamma(X_y, F_y) \) respectively. Here the sheaves \( F_y \) and \( F_y' \) are the pullbacks of \( F \) by the projections \( X_y \rightarrow X \) and \( X'_y \rightarrow X \). Thus we see that the groups agree by Lemma 39.5. We omit the verification that this isomorphism is compatible with our map. \( \square \)

At this point we start discussing the proper base change theorem. To do so we introduce some notation. Consider a commutative diagram
\[
\begin{array}{ccc}
X' & \to & X \\
\downarrow f' & & \downarrow f \\
Y' & \to & Y
\end{array}
\] (85.5.1)
of morphisms of schemes. Then we obtain a commutative diagram of sites
\[
\begin{array}{ccc}
X'_{\text{etale}} & \to & X_{\text{etale}} \\
\downarrow f'_{\text{small}} & & \downarrow f_{\text{small}} \\
Y'_{\text{etale}} & \to & Y_{\text{etale}}
\end{array}
\]

For any object \( E \) of \( D(X_{\text{etale}}) \) we obtain a canonical base change map
\[
g_{\text{smal}l}^{-1}Rf_{\text{smal}ll_*}E \rightarrow Rf'_{\text{smal}l_*}(g'_{\text{smal}l})^{-1}E
\]
in \( D(Y'_{\text{etale}}) \). See Cohomology on Sites, Remark 20.3 where we use the constant sheaf \( \mathbb{Z} \) as our sheaf of rings. We will usually omit the subscripts \( \text{smal}l \) in this formula. For example, if \( E = F[0] \) where \( F \) is an abelian sheaf on \( X_{\text{etale}} \), the base change map is a map
\[
g^{-1}Rf_*F \rightarrow Rf'_*(g')^{-1}F
\]
in \( D(Y'_{\text{etale}}) \).

The map (85.5.2) has no chance of being an isomorphism in the generality given above. The goal is to show it is an isomorphism if the diagram (85.5.1) is cartesian, \( f : X \rightarrow Y \) proper, and the cohomology sheaves of \( E \) are torsion. To study this question we introduce the following terminology. Let us say that cohomology commutes with base change for \( f : X \rightarrow Y \) if (85.5.3) is an isomorphism for every diagram (85.5.1) where \( X' = Y' \times_Y X \) and every torsion abelian sheaf \( F \).

**Lemma 85.6.** Let \( f : X \rightarrow Y \) be a proper morphism of schemes. The following are equivalent

1. cohomology commutes with base change for \( f \) (see above),
2. for every prime number \( \ell \) and every injective sheaf of \( \mathbb{Z}/\ell\mathbb{Z} \)-modules \( I \) on \( X_{\text{etale}} \) and every diagram (85.5.1) where \( X' = Y' \times_Y X \) the sheaves \( R^qf'_*(g')^{-1}I \) are zero for \( q > 0 \).
It is clear that (1) implies (2). Conversely, assume (2) and let $\mathcal{F}$ be a torsion abelian sheaf on $X_{\text{ét}}$. Let $Y' \to Y$ be a morphism of schemes and let $X' = Y' \times_Y X$ with projections $g' : X' \to X$ and $f' : X' \to Y'$ as in diagram $[\text{85.5.1}]$. We want to show the maps of sheaves

$$g^{-1} R^q f_* \mathcal{F} \to R^q f'_* (g')^{-1} \mathcal{F}$$

are isomorphisms for all $q \geq 0$.

For every $n \geq 1$, let $\mathcal{F}[n]$ be the subsheaf of sections of $\mathcal{F}$ annihilated by $n$. Then $\mathcal{F} = \text{colim} \mathcal{F}[n]$. The functors $g^{-1}$ and $(g')^{-1}$ commute with arbitrary colimits (as left adjoints). Taking higher direct images along $f$ or $f'$ commutes with filtered colimits by Lemma $[\text{51.7}]$. Hence we see that

$$g^{-1} R^q f_* \mathcal{F} = \text{colim} g^{-1} R^q f_* \mathcal{F}[n] \quad \text{and} \quad R^q f'_* (g')^{-1} \mathcal{F} = \text{colim} R^q f'_* (g')^{-1} \mathcal{F}[n]$$

Thus it suffices to prove the result in case $\mathcal{F}$ is annihilated by a positive integer $n$.

If $n = \ell n'$ for some prime number $\ell$, then we obtain a short exact sequence

$$0 \to \mathcal{F}[\ell] \to \mathcal{F} \to \mathcal{F}/\mathcal{F}[\ell] \to 0$$

Observe that $\mathcal{F}/\mathcal{F}[\ell]$ is annihilated by $n'$. Moreover, if the result holds for both $\mathcal{F}[\ell]$ and $\mathcal{F}/\mathcal{F}[\ell]$, then the result holds by the long exact sequence of higher direct images (and the 5 lemma). In this way we reduce to the case that $\mathcal{F}$ is annihilated by a prime number $\ell$.

Assume $\mathcal{F}$ is annihilated by a prime number $\ell$. Choose an injective resolution $\mathcal{F} \to \mathcal{I}^\bullet$ in $D(X_{\text{ét}}, \mathbb{Z}/\ell \mathbb{Z})$. Applying assumption (2) and Leray’s acyclicity lemma (Derived Categories, Lemma $[17.7]$) we see that

$$f'_* (g')^{-1} \mathcal{I}^\bullet$$

computes $Rf'_* (g')^{-1} \mathcal{F}$. We conclude by applying Lemma $[\text{85.5}]$.}

**Theorem 85.7.** Let $f : X \to Y$ and $g : Y \to Z$ be proper morphisms of schemes.

**Assume**

1. cohomology commutes with base change for $f$,
2. cohomology commutes with base change for $g \circ f$, and
3. $f$ is surjective.

Then cohomology commutes with base change for $g$.

**Proof.** We will use the equivalence of Lemma $[\text{85.6}]$ without further mention. Let $\ell$ be a prime number. Let $\mathcal{I}$ be an injective sheaf of $\mathbb{Z}/\ell \mathbb{Z}$-modules on $Y_{\text{ét}}$. Choose an injective map of sheaves $f^{-1} \mathcal{I} \to \mathcal{J}$ where $\mathcal{J}$ is an injective sheaf of $\mathbb{Z}/\ell \mathbb{Z}$-modules on $Z_{\text{ét}}$. Since $f$ is surjective the map $\mathcal{I} \to f_* \mathcal{J}$ is injective (look at stalks in geometric points). Since $\mathcal{I}$ is injective we see that $\mathcal{I}$ is a direct summand of $f_* \mathcal{J}$. Thus it suffices to prove the desired vanishing for $f_* \mathcal{J}$.

Let $Z' \to Z$ be a morphism of schemes and set $Y' = Z' \times_Z Y$ and $X' = Z' \times_Z X = Y' \times_Y X$. Denote $a : X' \to X$, $b : Y' \to Y$, and $c : Z' \to Z$ the projections. Similarly for $f' : X' \to Y'$ and $g' : Y' \to Z'$. By Lemma $[\text{85.5}]$ we have $b^{-1} f_* \mathcal{J} = f'_* a^{-1} \mathcal{J}$. On the other hand, we know that $R^p f'_* a^{-1} \mathcal{J}$ and $R^q (g' \circ f')_* a^{-1} \mathcal{J}$ are zero for $q > 0$. Using the spectral sequence (Cohomology on Sites, Lemma $[15.7]$)

$$R^p g'_* R^q f'_* a^{-1} \mathcal{J} \Rightarrow R^{p+q} (g' \circ f')_* a^{-1} \mathcal{J}$$

we conclude that $R^p g'_* (b^{-1} f_* \mathcal{J}) = R^p g'_* (f'_* a^{-1} \mathcal{J}) = 0$ for $p > 0$ as desired. □
Let $f : X \to Y$ and $g : Y \to Z$ be proper morphisms of schemes.

Assume

1. cohomology commutes with base change for $f$,
2. cohomology commutes with base change for $g$.

Then cohomology commutes with base change for $g \circ f$.

**Proof.** We will use the equivalence of Lemma [85.6](#) without further mention. Let $\ell$ be a prime number. Let $I$ be an injective sheaf of $\mathbb{Z}/\ell\mathbb{Z}$-modules on $X_{\text{étale}}$. Then $f_*I$ is an injective sheaf of $\mathbb{Z}/\ell\mathbb{Z}$-modules on $Y_{\text{étale}}$ (Cohomology on Sites, Lemma [15.2](#)). The result follows formally from this, but we will also spell it out.

Let $Z' \to Z$ be a morphism of schemes and set $Y' = Z' \times_Z Y$ and $X' = Z' \times_Z X = Y' \times_Y X$. Denote $a : X' \to X$, $b : Y' \to Y$, and $c : Z' \to Z$ the projections. Similarly for $f' : X' \to Y'$ and $g' : Y' \to Z'$. By Lemma [85.5](#) we have $b^{-1}f_*I = f'_*a^{-1}I$. On the other hand, we know that $R^pf'_*a^{-1}I$ and $R^q(b^{-1}f_*I)$ are zero for $q > 0$.

Using the spectral sequence (Cohomology on Sites, Lemma [15.7](#))

$$R^p g'_* R^q f'_* a^{-1} I \Rightarrow R^{p+q} (g' \circ f'_*) a^{-1} I$$

we conclude that $R^p (g' \circ f'_*) a^{-1} I = 0$ for $p > 0$ as desired. □

**Lemma 85.9.** Let $f : X \to Y$ be a finite morphism of schemes. Then cohomology commutes with base change for $f$.

**Proof.** Observe that a finite morphism is proper, see Morphisms, Lemma [42.11](#) Moreover, the base change of a finite morphism is finite, see Morphisms, Lemma [42.6](#) Thus the result follows from Lemma [85.6](#) combined with Proposition [54.2](#) □

**Lemma 85.10.** To prove that cohomology commutes with base change for every proper morphism of schemes it suffices to prove it holds for the morphism $\mathbb{P}^1_S \to S$ for every scheme $S$.

**Proof.** Let $f : X \to Y$ be a proper morphism of schemes. Let $Y = \bigcup Y_i$ be an affine open covering and set $X_i = f^{-1}(Y_i)$. If we can prove cohomology commutes with base change for $X_i \to Y_i$, then cohomology commutes with base change for $f$. Namely, the formation of the higher direct images commutes with Zariski (and even étale) localization on the base, see Lemma [51.6](#) Thus we may assume $Y$ is affine.

Let $Y$ be an affine scheme and let $X \to Y$ be a proper morphism. By Chow’s lemma there exists a commutative diagram

$$
\begin{array}{ccc}
X & \xrightarrow{\pi} & X' \\
\downarrow & & \downarrow \\
Y & & \mathbb{P}^n_Y
\end{array}
$$

where $X' \to \mathbb{P}^n_Y$ is an immersion, and $\pi : X' \to X$ is proper and surjective, see Limits, Lemma [12.1](#) Since $X \to Y$ is proper, we find that $X' \to Y$ is proper (Morphisms, Lemma [39.4](#)). Hence $X' \to \mathbb{P}^n_Y$ is a closed immersion (Morphisms, Lemma [39.7](#)). It follows that $X' \to X \times_Y \mathbb{P}^n_Y = \mathbb{P}^n_X$ is a closed immersion (as an immersion with closed image).

By Lemma [85.7](#) it suffices to prove cohomology commutes with base change for $\pi$ and $X' \to Y$. These morphisms both factor as a closed immersion followed by
a projection $\mathbb{P}_S^1 \to S$ for some $S$). By Lemma 85.9 the result holds for closed immersions (as closed immersions are finite). By Lemma 85.8 it suffices to prove the result for projections $\mathbb{P}_S^n \to S$.

For every $n \geq 1$ there is a finite surjective morphism

$$
\mathbb{P}_S^1 \times_S \ldots \times_S \mathbb{P}_S^1 \longrightarrow \mathbb{P}_S^n
$$

given on coordinates by

$$(x_1 : y_1), (x_2 : y_2), \ldots, (x_n : y_n) \mapsto (F_0 : \ldots : F_n)
$$

where $F_0, \ldots, F_n$ in $x_1, \ldots, y_n$ are the polynomials with integer coefficients such that

$$
\prod (x_i + y_i) = F_0 t^n + F_1 t^{n-1} + \ldots + F_n
$$

Applying Lemmas 85.7, 85.9 and 85.8 one more time we conclude that the lemma is true. □

**Theorem 85.11.** Let $f : X \to Y$ be a proper morphism of schemes. Let $g : Y' \to Y$ be a morphism of schemes. Set $X' = Y' \times_Y X$ and consider the cartesian diagram

$$
\begin{array}{ccc}
X' & \xrightarrow{g'} & X \\
\downarrow f' & & \downarrow f \\
Y' & \xrightarrow{g} & Y
\end{array}
$$

Let $\mathcal{F}$ be an abelian torsion sheaf on $X_{\text{étale}}$. Then the base change map

$$
g'^{-1}Rf_*\mathcal{F} \to Rf'_*(g')^{-1}\mathcal{F}
$$

is an isomorphism.

**Proof.** In the terminology introduced above, this means that cohomology commutes with base change for every proper morphism of schemes. By Lemma 85.10 it suffices to prove that cohomology commutes with base change for the morphism $\mathbb{P}_S^1 \to S$ for every scheme $S$.

Let $S$ be the spectrum of a strictly henselian local ring with closed point $s$. Set $X = \mathbb{P}_S^1$ and $X_0 = X_s = \mathbb{P}_s^1$. Let $\mathcal{F}$ be a sheaf of $\mathbb{Z}/\ell\mathbb{Z}$-modules on $X_{\text{étale}}$. The key to our proof is that

$$
H^q_{\text{étale}}(X, \mathcal{F}) = H^q_{\text{étale}}(X_0, \mathcal{F}|_{X_0}).
$$

Namely, choose a resolution $\mathcal{F} \to \mathcal{T}^\bullet$ by injective sheaves of $\mathbb{Z}/\ell\mathbb{Z}$-modules. Then $\mathcal{T}^\bullet|_{X_0}$ is a resolution of $\mathcal{F}|_{X_0}$ by right $H^q_{\text{étale}}(X_0, -)$-acyclic objects, see Lemma 79.2. Leray’s acyclicity lemma tells us the right hand side is computed by the complex $H^0_{\text{étale}}(X_0, \mathcal{T}^\bullet|_{X_0})$ which is equal to $H^0_{\text{étale}}(X, \mathcal{T}^\bullet)$ by Lemma 85.3. This complex computes the left hand side.

Assume $S$ is general and $\mathcal{F}$ is a sheaf of $\mathbb{Z}/\ell\mathbb{Z}$-modules on $X_{\text{étale}}$. Let $\pi : \text{Spec}(k) \to S$ be a geometric point of $S$ lying over $s \in S$. We have

$$
(R^q f_\pi)_\pi = H^q_{\text{étale}}(\mathbb{P}_S^1, \mathcal{F}|_{\mathbb{P}_S^1}) = H^q_{\text{étale}}(\mathbb{P}_s^1, \mathcal{F}|_{\mathbb{P}_s^1})
$$

where $\kappa(s)^{\text{sep}}$ is the residue field of $\mathcal{O}_S^h$, i.e., the separable algebraic closure of $\kappa(s)$ in $k$. The first equality by Theorem 52.1 and the second equality by the displayed formula in the previous paragraph.
Finally, consider any morphism of schemes \( g : T \to S \) where \( S \) and \( F \) are as above. Set \( f' : \mathbf{P}^1_T \to T \) the projection and let \( g' : \mathbf{P}^1_T \to \mathbf{P}^1_S \) the morphism induced by \( g \). Consider the base change map

\[
g^{-1}R^qf_*F \to R^qf'_*(g')^{-1}F
\]

Let \( \overline{t} \) be a geometric point of \( T \) with image \( \overline{s} = g(\overline{t}) \). By our discussion above the map on stalks at \( \overline{t} \) is the map

\[
H^q_{\text{etale}}(\mathbf{P}^1_{\kappa(s)^{\text{sep}}}, F|_{\mathbf{P}^1_{\kappa(s)^{\text{sep}}}}) \to H^q_{\text{etale}}(\mathbf{P}^1_{\kappa(t)^{\text{sep}}}, F|_{\mathbf{P}^1_{\kappa(t)^{\text{sep}}}})
\]

Since \( \kappa(s)^{\text{sep}} \subset \kappa(t)^{\text{sep}} \) this map is an isomorphism by Lemma \[78.11\]. This proves cohomology commutes with base change for \( \mathbf{P}^1_S \to S \) and sheaves of \( \mathbb{Z}/\ell\mathbb{Z} \)-modules. In particular, for an injective sheaf of \( \mathbb{Z}/\ell\mathbb{Z} \)-modules the higher direct images of any base change are zero. In other words, condition (2) of Lemma \[85.6\] holds and the proof is complete.

\[86.1\] \textbf{Lemma 86.1.} Let \( f : X \to Y \) be a proper morphism of schemes. Let \( g : Y' \to Y \) be a morphism of schemes. Set \( X' = Y' \times_Y X \) and denote \( f' : X' \to Y' \) and \( g' : X' \to X \) the projections. Let \( E \in D^+(X_{\text{etale}}) \) have torsion cohomology sheaves. Then the base change map \( (85.5.2) \)

\[
g^{-1}Rf_*E \to Rf'_*(g')^{-1}E
\]

is an isomorphism.

\[86.2\] \textbf{Lemma 86.2.} Let \( f : X \to Y \) be a proper morphism of schemes all of whose fibres have dimension \( \leq n \). Then for any abelian torsion sheaf \( F \) on \( X_{\text{etale}} \) we have \( R^qf_*F = 0 \) for \( q > 2n \).
Proof. We will prove this by induction on \( n \) for all proper morphisms.

If \( n = 0 \), then \( f \) is a finite morphism (More on Morphisms, Lemma \textcolor{blue}{[38.4]} and the result is true by Proposition \textcolor{blue}{[54.2]}).

If \( n > 0 \), then using Lemma \textcolor{blue}{[85.13]} we see that it suffices to prove \( H^i_{\text{étale}}(X, \mathcal{F}) = 0 \) for \( i > 2n \) and \( X \) a proper scheme, \( \dim(X) \leq n \) over an algebraically closed field \( k \) and \( \mathcal{F} \) is a torsion abelian sheaf on \( X \).

If \( n = 1 \) this follows from Theorem \textcolor{blue}{[78.10]} Assume \( n > 1 \). By Proposition \textcolor{blue}{[45.4]} we may replace \( X \) by its reduction. Let \( \nu : X' \to X \) be the normalization. This is a surjective birational finite morphism (see Varieties, Lemma \textcolor{blue}{[27.1]} and hence an isomorphism over a dense open \( U \subset X \) (Morphisms, Lemma \textcolor{blue}{[48.5]}). Then we see that \( c : \mathcal{F} \to \nu_*\nu^!\mathcal{F} \) is injective (as \( \nu \) is surjective) and an isomorphism over \( U \). Denote \( i : Z \to X \) the inclusion of the complement of \( U \). Since \( U \) is dense in \( X \) we have \( \dim(Z) < \dim(X) = n \). By Proposition \textcolor{blue}{[46.4]} have \( \text{Coker}(c) = i_*\mathcal{G} \) for some abelian torsion sheaf \( \mathcal{G} \) on \( Z_{\text{étale}} \). Then \( H^q_{\text{étale}}(X, \text{Coker}(c)) = H^q_{\text{étale}}(Z, \mathcal{F}) \) (by Proposition \textcolor{blue}{[54.2]} and the Leray spectral sequence) and by induction hypothesis we conclude that the cokernel of \( c \) has cohomology in degrees \( \leq 2(n-1) \). Thus it suffices to prove the result for \( \nu_*\nu^!\mathcal{F} \). As \( \nu \) is finite this reduces us to showing that \( H^i_{\text{étale}}(X', \nu^!\mathcal{F}) \) is zero for \( i > 2n \). This case is treated in the next paragraph.

Assume \( X \) is integral normal proper scheme over \( k \) of dimension \( n \). Choose a nonconstant rational function \( f \) on \( X \). The graph \( X' \subset X \times \mathbb{P}^1_k \) of \( f \) sits into a diagram

\[
X \xrightarrow{b} X' \to \mathbb{P}^1_k
\]

Observe that \( b \) is an isomorphism over an open subscheme \( U \subset X \) whose complement is a closed subscheme \( Z \subset X \) of codimension \( \geq 2 \). Namely, \( U \) is the domain of definition of \( f \) which contains all codimension 1 points of \( X \), see Morphisms, Lemmas \textcolor{blue}{[47.9]} and \textcolor{blue}{[40.5]} (combined with Serre’s criterion for normality, see Properties, Lemma \textcolor{blue}{[12.3]}). Moreover the fibres of \( b \) have dimension \( \leq 1 \) (as closed subschemes of \( \mathbb{P}^1 \)). Hence \( R^ib_*b^{-1}\mathcal{F} \) is nonzero only if \( i \in \{0, 1, 2\} \) by induction. Choose a distinguished triangle

\[
\mathcal{F} \to Rb_*b^{-1}\mathcal{F} \to Q \to \mathcal{F}[1]
\]

Using that \( \mathcal{F} \to b_*b^{-1}\mathcal{F} \) is injective as before and using what we just said, we see that \( Q \) has nonzero cohomology sheaves only in degrees \( 0, 1, 2 \) sitting on \( Z \). Moreover, these cohomology sheaves are torsion by Lemma \textcolor{blue}{[74.2]} By induction we see that \( H^i(X, Q) \) is zero for \( i > 2 + 2\dim(Z) \leq 2 + 2(n-2) = 2n - 2 \). Thus it suffices to prove that \( H^i(X', b^{-1}\mathcal{F}) = 0 \) for \( i > 2n \). At this point we use the morphism

\[
f : X' \to \mathbb{P}^1_k
\]

whose fibres have dimension \( < n \). Hence by induction we see that \( R^if_*b^{-1}\mathcal{F} = 0 \) for \( i > 2(n-1) \). We conclude by the Leray spectral sequence

\[
H^i(\mathbb{P}^1_k, R^jf_*b^{-1}\mathcal{F}) \Rightarrow H^{i+j}(X', b^{-1}\mathcal{F})
\]

and the fact that \( \dim(\mathbb{P}^1_k) = 1 \).

When working with mod \( n \) coefficients we can do proper base change for unbounded complexes.

\textbf{Lemma 86.3.} Let \( f : X \to Y \) be a morphism of finite type with \( Y \) quasi-compact. Then the dimension of the fibres of \( f \) is bounded.
Proof. By Morphisms, Lemma 27.4 the set $U_n \subset X$ of points where the dimension of the fibre is $\leq n$ is open. Since $f$ is of finite type, every point is contained in some $U_n$. Since $Y$ is quasi-compact and $f$ is of finite type, we see that $X$ is quasi-compact. Hence $X = U_n$ for some $n$. □

**Lemma 86.4.** Let $f : X \to Y$ be a proper morphism of schemes. Let $g : Y' \to Y$ be a morphism of schemes. Set $X' = Y' \times_Y X$ and denote $f' : X' \to Y'$ and $g' : X' \to X$ the projections. Let $n \geq 1$ be an integer. Let $E \in D(X_{\text{étale}}, \mathbb{Z}/n\mathbb{Z})$. Then the base change map (85.5.2) $g^{-1}Rf_*E \to Rf'_*(g')^{-1}E$ is an isomorphism.

Proof. It is enough to prove this when $Y$ and $Y'$ are quasi-compact. By Lemma 86.3 we see that the dimension of the fibres of $f : X \to Y$ and $f' : X' \to Y'$ are bounded. Thus Lemma 86.2 implies that

$$f_* : \text{Mod}(X_{\text{étale}}, \mathbb{Z}/n\mathbb{Z}) \to \text{Mod}(Y_{\text{étale}}, \mathbb{Z}/n\mathbb{Z})$$

and

$$f'_* : \text{Mod}(X'_{\text{étale}}, \mathbb{Z}/n\mathbb{Z}) \to \text{Mod}(Y'_{\text{étale}}, \mathbb{Z}/n\mathbb{Z})$$

have finite cohomological dimension in the sense of Derived Categories, Lemma 30.2. Choose a K-injective complex $\mathcal{I}^n$ of $\mathbb{Z}/n\mathbb{Z}$-modules each of whose terms $\mathcal{I}_i$ is an injective sheaf of $\mathbb{Z}/n\mathbb{Z}$-modules representing $E$. See Injectives, Theorem 12.6. By the usual proper base change theorem we find that $R^qf'_*(g')^{-1}\mathcal{I}_i = 0$ for $q > 0$, see Theorem 85.11. Hence we conclude by Derived Categories, Lemma 30.2 that we may compute $Rf'_*(g')^{-1}E$ by the complex $f'_*(g')^{-1}\mathcal{I}^n$. Another application of the usual proper base change theorem shows that this is equal to $g^{-1}f_*\mathcal{I}^n$ as desired. □

**Lemma 86.5.** Let $X$ be a quasi-compact and quasi-separated scheme. Let $E \in D^+(X_{\text{étale}})$ and $K \in D^+(\mathbb{Z})$. Then

$$R\Gamma(X, E \otimes_{\mathbb{Z}} K) = R\Gamma(X, E) \otimes_{\mathbb{Z}} K$$

Proof. Say $H^i(E) = 0$ for $i \geq a$ and $H^j(K) = 0$ for $j \geq b$. We may represent $K$ by a bounded below complex $K^\bullet$ of torsion free $\mathbb{Z}$-modules. (Choose a K-flat complex $L^\bullet$ representing $K$ and then take $K^\bullet = \tau_{\geq -b+1}L^\bullet$. This works because $\mathbb{Z}$ has global dimension 1. See More on Algebra, Lemma 63.2.) We may represent $E$ by a bounded below complex $\mathcal{E}^\bullet$. Then $E \otimes_{\mathbb{Z}} K$ is represented by

$$\text{Tot}(\mathcal{E}^\bullet \otimes_{\mathbb{Z}} K)$$

Using distinguished triangles

$$\sigma_{\geq -b+n+1}K^\bullet \to K^\bullet \to \sigma_{\leq -b+n}K^\bullet$$

and the trivial vanishing

$$H^a(X, \text{Tot}(\mathcal{E}^\bullet \otimes_{\mathbb{Z}} \sigma_{\geq -a+n+1}K^\bullet)) = 0$$

and

$$H^a(R\Gamma(X, E) \otimes_{\mathbb{Z}} \sigma_{\geq -a+n+1}K^\bullet) = 0$$

we reduce to the case where $K^\bullet$ is a bounded complex of flat $\mathbb{Z}$-modules. Repeating the argument we reduce to the case where $K^\bullet$ is equal to a single flat $\mathbb{Z}$-module sitting in some degree. Next, using the stupid truncations for $\mathcal{E}^\bullet$ we reduce in exactly the same manner to the case where $\mathcal{E}^\bullet$ is a single abelian sheaf sitting in some degree. Thus it suffices to show that

$$H^a(X, \mathcal{E} \otimes_{\mathbb{Z}} M) = H^a(X, \mathcal{E}) \otimes_{\mathbb{Z}} M$$
when $M$ is a flat $\mathbb{Z}$-module and $E$ is an abelian sheaf on $X$. In this case we write $M$ is a filtered colimit of finite free $\mathbb{Z}$-modules (Lazard’s theorem, see Algebra, Theorem 80.4). By Theorem 51.3 this reduces us to the case of finite free $\mathbb{Z}$-module $M$ in which case the result is trivially true. □

Lemma 86.6. Let $f : X \to Y$ be a proper morphism of schemes. Let $E \in D^+(\mathcal{X}_{\text{etale}})$ have torsion cohomology sheaves. Let $K \in D^+(\mathcal{Y}_{\text{etale}})$. Then

$$Rf_*E \otimes^L_{\mathbb{Z}} K = Rf_*(E \otimes^L_{\mathbb{Z}} f^{-1}K)$$

in $D^+(\mathcal{Y}_{\text{etale}})$.

Proof. There is a canonical map from left to right by Cohomology on Sites, Section 46. We will check the equality on stalks. Recall that computing derived tensor products commutes with pullbacks. See Cohomology on Sites, Lemma 19.4. Thus we have

$$(E \otimes^L_{\mathbb{Z}} f^{-1}K)_{\bar{y}} = E_{\bar{x}} \otimes^L_{\mathbb{Z}} K_{\bar{y}}$$

where $\bar{y}$ is the image of $\bar{x}$ in $Y$. Since $\mathbb{Z}$ has global dimension 1 we see that this complex has vanishing cohomology in degree $< -1 + a + b$ if $H^i(E) = 0$ for $i \geq a$ and $H^j(K) = 0$ for $j \geq b$. Moreover, since $H^i(E)$ is a torsion abelian sheaf for each $i$, the same is true for the cohomology sheaves of the complex $E \otimes^L_{\mathbb{Z}} K$. Namely, we have

$$(E \otimes^L_{\mathbb{Z}} f^{-1}K) \otimes^L_{\mathbb{Z}} Q = (E \otimes^L_{\mathbb{Z}} Q) \otimes^L_{\mathbb{Q}} (f^{-1}K \otimes^L_{\mathbb{Z}} Q)$$

which is zero in the derived category. In this way we see that Lemma 85.13 applies to both sides to see that it suffices to show

$$R\Gamma(X_{\mathbb{Z}},E|_{X_{\mathbb{Z}}} \otimes^L_{\mathbb{Z}} (X_{\mathbb{Z}} \to Y)^{-1}K_{\mathbb{Z}}) = R\Gamma(X_{\mathbb{Z}},E|_{X_{\mathbb{Z}}} \otimes^L_{\mathbb{Z}} K_{\mathbb{Z}})$$

This is shown in Lemma 86.5. □

87. Cohomological dimension

We can deduce some bounds on the cohomological dimension of schemes and on the cohomological dimension of fields using the results in Section 78 and one, seemingly innocuous, application of the proper base change theorem (in the proof of Proposition 87.6).

Definition 87.1. Let $X$ be a quasi-compact and quasi-separated scheme. The **cohomological dimension** of $X$ is the smallest element

$$\text{cd}(X) \in \{0, 1, 2, \ldots\} \cup \{\infty\}$$

such that for any abelian torsion sheaf $F$ on $X_{\text{etale}}$ we have $H^i_{\text{etale}}(X,F) = 0$ for $i > \text{cd}(X)$. If $X = \text{Spec}(A)$ we sometimes call this the cohomological dimension of $A$.

If the scheme is in characteristic $p$, then we often can obtain sharper bounds for the vanishing of cohomology of $p$-power torsion sheaves. We will address this elsewhere (insert future reference here).

Lemma 87.2. Let $X = \lim X_i$ be a directed limit of a system of quasi-compact and quasi-separated schemes with affine transition morphisms. Then $\text{cd}(X) \leq \max \text{cd}(X_i)$.
Proof. Denote $f_i : X \to X_i$ the projections. Let $\mathcal{F}$ be an abelian torsion sheaf on $X_{\text{étale}}$. Then we have $\mathcal{F} = \lim_{\rightarrow} f_i^{-1} \mathcal{F}_i$ by Lemma 87.2. Thus $H^q_{\text{étale}}(X, \mathcal{F}) = \text{colim} H^q_{\text{étale}}(X_i, f_i^* \mathcal{F})$ by Theorem 87.3. The lemma follows.

\begin{lemma} \label{lemma-87.3}
Let $K$ be a field. Let $X$ be a 1-dimensional affine scheme of finite type over $K$. Then $\text{cd}(X) \leq 1 + \text{cd}(K)$.
\end{lemma}

\begin{proof}
Let $\mathcal{F}$ be an abelian torsion sheaf on $X_{\text{étale}}$. Consider the Leray spectral sequence for the morphism $f : X \to \text{Spec}(K)$. We obtain
$$E_2^{p,q} = H^p(\text{Spec}(K), R^q f_* \mathcal{F})$$
converging to $H^{p+q}_{\text{étale}}(X, \mathcal{F})$. The stalk of $R^q f_* \mathcal{F}$ at a geometric point $\text{Spec}(K) \to \text{Spec}(K)$ is the cohomology of the pullback of $\mathcal{F}$ to $X_{\text{K}}$. Hence it vanishes in degrees $\geq 2$ by Theorem 78.9.
\end{proof}

\begin{lemma} \label{lemma-87.4}
Let $L/K$ be a field extension. Then we have $\text{cd}(L) \leq \text{cd}(K) + \text{trdeg}_K(L)$.
\end{lemma}

\begin{proof}
If $\text{trdeg}_K(L) = \infty$, then this is clear. If not then we can find a sequence of extensions $L = L_r/L_{r-1}/\ldots/L_1/L_0 = K$ such that $\text{trdeg}_L(L_{i+1}) = 1$ and $r = \text{trdeg}_K(L)$. Hence it suffices to prove the lemma in the case that $r = 1$. In this case we can write $L = \text{colim} \, A_i$ as a filtered colimit of its finite type $K$-subalgebras. By Lemma 87.3 it suffices to prove that $\text{cd}(A_i) \leq 1 + \text{cd}(K)$. This follows from Lemma 87.3.
\end{proof}

\begin{lemma} \label{lemma-87.5}
Let $K$ be a field. Let $X$ be a scheme of finite type over $K$. Let $x \in X$. Set $a = \text{trdeg}_K(\kappa(x))$ and $d = \dim_x(X)$. Then there is a map
$$K(t_1,\ldots,t_a)^{\text{sep}} \to O_{X,x}^{\text{sh}}$$
such that
\begin{enumerate}
\item the residue field of $O_{X,x}^{\text{sh}}$ is a purely inseparable extension of $K(t_1,\ldots,t_a)^{\text{sep}}$,
\item $O_{X,x}^{\text{sh}}$ is a filtered colimit of finite type $K(t_1,\ldots,t_a)^{\text{sep}}$-algebras of dimension $\leq d - a$.
\end{enumerate}
\end{lemma}

\begin{proof}
We may assume $X$ is affine. By Noether normalization, after possibly shrinking $X$ again, we can choose a finite morphism $\pi : X \to \mathbf{A}_K^d$, see Algebra, Lemma 114.5. Since $\kappa(x)$ is a finite extension of the residue field of $\pi(x)$, this residue field has transcendence degree $a$ over $K$ as well. Thus we can find a finite morphism $\pi' : \mathbf{A}_K^d \to \mathbf{A}_K^d$ such that $\pi'(\pi(x))$ corresponds to the generic point of the linear subspace $\mathbf{A}_K^a \subseteq \mathbf{A}_K^d$ given by setting the last $d - a$ coordinates equal to zero. Hence the composition
$$X \xrightarrow{\pi' \circ \pi} \mathbf{A}_K^d \xrightarrow{p} \mathbf{A}_K^a$$
of $\pi' \circ \pi$ and the projection $p$ onto the first $a$ coordinates maps $x$ to the generic point $\eta \in \mathbf{A}_K^a$. The induced map
$$K(t_1,\ldots,t_a)^{\text{sep}} = O_{\mathbf{A}_K^d,\eta} \to O_{X,x}^{\text{sh}}$$
on étale local rings satisfies (1) since it is clear that the residue field of $O_{X,x}^{\text{sh}}$ is an algebraic extension of the separably closed field $K(t_1,\ldots,t_a)^{\text{sep}}$. On the other hand, if $X = \text{Spec}(B)$, then $O_{X,x}^{\text{sh}} = \text{colim} \, B_j$ is a filtered colimit of étale $B$-algebras $B_j$. Observe that $B_j$ is quasi-finite over $K[t_1,\ldots,t_d]$ as $B$ is finite over $K[t_1,\ldots,t_d]$. We may similarly write $K(t_1,\ldots,t_a)^{\text{sep}} = \text{colim} \, A_i$ as a filtered
colimit of étale $K[t_1, \ldots, t_a]$-algebras. For every $i$ we can find an $j$ such that $A_i \to K(t_1, \ldots, t_a)^{\mathrm{sep}} \to \mathcal{O}_{X,x}^{\mathrm{sh}}$ factors through a map $\psi_{i,j} : A_i \to B_j$. Then $B_j$ is quasi-finite over $A_i[t_{a+1}, \ldots, t_d]$. Hence

$$B_{i,j} = B_j \otimes_{\psi_{i,j}, A_i} K(t_1, \ldots, t_a)^{\mathrm{sep}}$$

has dimension $\leq d - a$ as it is quasi-finite over $K(t_1, \ldots, t_a)^{\mathrm{sep}}[t_{a+1}, \ldots, t_d]$. The proof of (2) is now finished as $\mathcal{O}_{X,x}^{\mathrm{sh}}$ is a filtered colimit of the algebras $B_{i,j}$. Some details omitted. □

0F0V Proposition 87.6. Let $K$ be a field. Let $X$ be an affine scheme of finite type over $K$. Then we have $\operatorname{cd}(X) \leq \dim(X) + \operatorname{cd}(K)$.

Proof. We will prove this by induction on $\dim(X)$. Let $\mathcal{F}$ be an abelian torsion sheaf on $X_{\text{étale}}$.

The case $\dim(X) = 0$. In this case the structure morphism $f : X \to \text{Spec}(K)$ is finite. Hence we see that $R^nf_*\mathcal{F} = 0$ for $i > 0$, see Proposition 54.2. Thus $H^i_{\text{étale}}(X, \mathcal{F}) = H^i_{\text{étale}}(\text{Spec}(K), f_*\mathcal{F})$ by the Leray spectral sequence for $f$ (Cohomology on Sites, Lemma 15.5) and the result is clear.

The case $\dim(X) = 1$. This is Lemma 87.3.

Assume $d = \dim(X) > 1$ and the proposition holds for finite type affine schemes of dimension $< d$ over fields. By Noether normalization, see for example Varieties, Lemma 18.2 there exists a finite morphism $f : X \to \mathbb{A}^d_K$. Recall that $R^nf_*\mathcal{F} = 0$ for $i > 0$ by Proposition 54.2. By the Leray spectral sequence for $f$ (Cohomology on Sites, Lemma 15.5) we conclude that it suffices to prove the result for $\pi_*\mathcal{F}$ on $\mathbb{A}^d_K$.

Interlude I. Let $j : X \to Y$ be an open immersion of smooth $d$-dimensional varieties over $K$ (not necessarily affine) whose complement is the support of an effective Cartier divisor $D$. The sheaves $R^iq_*\mathcal{F}$ for $q > 0$ are supported on $D$. We claim that $(R^iq_*\mathcal{F})_\pi = 0$ for $a = \text{trdeg}_K(\kappa(y)) > d - q$. Namely, by Theorem 52.1 we have

$$(R^iq_*\mathcal{F})_\pi = H^q(\text{Spec}(\mathcal{O}_{Y,y}^{\mathrm{sh}}) \times_Y X, \mathcal{F})$$

Choose a local equation $f \in m_y = \mathcal{O}_{Y,y}$ for $D$. Then we have

$$\text{Spec}(\mathcal{O}_{Y,y}^{\mathrm{sh}}) \times_Y X = \text{Spec}(\mathcal{O}_{Y,y}^{\mathrm{sh}}[1/f])$$

Using Lemma 87.5 we get an embedding

$$K(t_1, \ldots, t_a)^{\mathrm{sep}}(x) = K(t_1, \ldots, t_a)^{\mathrm{sep}}[x][1/x] \to \mathcal{O}_{Y,y}^{\mathrm{sh}}[1/f]$$

Since the transcendence degree over $K$ of the fraction field of $\mathcal{O}_{Y,y}^{\mathrm{sh}}$ is $d$, we see that $\mathcal{O}_{Y,y}^{\mathrm{sh}}[1/f]$ is a filtered colimit of $(d - a - 1)$-dimensional finite type algebras over the field $K(t_1, \ldots, t_a)^{\mathrm{sep}}(x)$ which itself has cohomological dimension 1 by Lemma 87.4. Thus by induction hypothesis and Lemma 87.2 we obtain the desired vanishing.

Interlude II. Let $Z$ be a smooth variety over $K$ of dimension $d - 1$. Let $E_a \subset Z$ be the set of points $z \in Z$ with $\text{trdeg}_K(\kappa(z)) \leq a$. Observe that $E_a$ is closed under

---

Let $R$ be a ring. Let $A = \operatorname{colim}_{i \in I} A_i$ be a filtered colimit of finitely presented $R$-algebras. Let $B = \operatorname{colim}_{j \in J} B_j$ be a filtered colimit of $R$-algebras. Let $A \to B$ be an $R$-algebra map. Assume that for all $i \in I$ there is a $j \in J$ and an $R$-algebra map $\psi_{i,j} : A_i \to B_j$. Say $(i', j, \psi_{i', j}) \geq (i, j, \psi_{i, j})$ if $i' \geq i$, $j \geq j$, and $\psi_{i', j}$ and $\psi_{i, j}$ are compatible. Then the collection of triples forms a directed set and $B = \operatorname{colim} B_j \otimes_{\psi_{i, j}, A_i} A$. 

specialization, see Varieties, Lemma 20.3. Suppose that $\mathcal{G}$ is a torsion abelian sheaf on $Z$ whose support is contained in $E_a$. Then we claim that $H^b_{\text{étale}}(Z, \mathcal{G}) = 0$ for $b > a + \text{cd}(K)$. Namely, we can write $\mathcal{G} = \colim \mathcal{G}_i$ with $\mathcal{G}_i$ a torsion abelian sheaf supported on a closed subscheme $Z_i$ contained in $E_a$, see Lemma 73.5. Then the induction hypothesis kicks in to imply the desired vanishing for $\mathcal{G}_i^{10}$. Finally, we conclude by Theorem 51.3.

Consider the commutative diagram

$$
\begin{array}{ccc}
\mathbb{A}^d_K & \xrightarrow{j} & \mathbb{P}^1_K \times_K \mathbb{A}^{d-1}_K \\
\downarrow f & & \downarrow g \\
\mathbb{A}^{d-1}_K & & \mathbb{A}^{d-1}_K
\end{array}
$$

Observe that $j$ is an open immersion of smooth $d$-dimensional varieties whose complement is an effective Cartier divisor $D$. Thus we may use the results obtained in interlude I. We are going to study the relative Leray spectral sequence

$$E_2^{p,q} = R^pg_*R^qj_*\mathcal{F} \Rightarrow R^{p+q}f_*\mathcal{F}$$

Since $R^qj_*\mathcal{F}$ for $q > 0$ is supported on $D$ and since $g|_D : D \to \mathbb{A}^d_K$ is an isomorphism, we find $R^pg_*R^qj_*\mathcal{F} = 0$ for $p > 0$ and $q > 0$. Moreover, we have $R^qj_*\mathcal{F} = 0$ for $q > d$. On the other hand, $g$ is a proper morphism of relative dimension 1. Hence by Lemma 86.2 we see that $R^qj_*\mathcal{F} = 0$ for $q > 2$. Thus the $E_2$-page of the spectral sequence looks like this

$$
\begin{array}{cccc}
g_*R^d j_* \mathcal{F} & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots \\
g_*R^2 j_* \mathcal{F} & 0 & 0 \\
g_*R^1 j_* \mathcal{F} & 0 & 0 \\
g_*j_* \mathcal{F} & R^1 g_*j_* \mathcal{F} & R^2 g_*j_* \mathcal{F}
\end{array}
$$

We conclude that $R^q f_* \mathcal{F} = g_*R^q j_* \mathcal{F}$ for $q > 2$. By interlude I we see that the support of $R^q f_* \mathcal{F}$ for $q > 2$ is contained in the set of points of $\mathbb{A}^{d-1}_K$ whose residue field has transcendence degree $\leq d - q$. By interlude II

$$H^p(\mathbb{A}^{d-1}_K, R^q f_* \mathcal{F}) = 0 \text{ for } p > d - q + \text{cd}(K) \text{ and } q > 2$$

On the other hand, by Theorem 52.1 we have $R^2 f_* \mathcal{F}_\pi = H^2(\mathbb{A}^{d-1}_K, \mathcal{F}) = 0$ (vanishing by the case of dimension 1). Hence by interlude II again we see

$$H^p(\mathbb{A}^{d-1}_K, R^2 f_* \mathcal{F}) = 0 \text{ for } p > d - 2 + \text{cd}(K)$$

Finally, we have

$$H^p(\mathbb{A}^{d-1}_K, R^q f_* \mathcal{F}) = 0 \text{ for } p > d - 1 + \text{cd}(K) \text{ and } q = 0, 1$$

by induction hypothesis. Combining everything we just said with the Leray spectral sequence $H^p(\mathbb{A}^{d-1}_K, R^q f_* \mathcal{F}) \Rightarrow H^{p+q}(\mathbb{A}^{d-1}_K, \mathcal{F})$ we conclude. 

\[10\] Here we first use Proposition 46.4 to write $\mathcal{G}_i$ as the pushforward of a sheaf on $Z_i$, the induction hypothesis gives the vanishing for this sheaf on $Z_i$, and the Leray spectral sequence for $Z_i \to Z$ gives the vanishing for $\mathcal{G}_i$. 

Lemma 87.7. Let $K$ be a field. Let $X$ be an affine scheme of finite type over $K$. Let $E_a \subset X$ be the set of points $x \in X$ with $\operatorname{trdeg}_K(\kappa(x)) \leq a$. Let $\mathcal{F}$ be an abelian torsion sheaf on $X_{\text{étale}}$ whose support is contained in $E_a$. Then $H^b_{\text{étale}}(X, \mathcal{F}) = 0$ for $b > a + cd(K)$.

Proof. We can write $\mathcal{F} = \colim \mathcal{F}_i$ with $\mathcal{F}_i$ a torsion abelian sheaf supported on a closed subscheme $Z_i$ contained in $E_a$, see Lemma 73.5. Then Proposition 87.6 gives the desired vanishing for $\mathcal{F}_i$. Details omitted; hints: first use Proposition 46.4 to write $\mathcal{F}_i$ as the pushforward of a sheaf on $Z_i$, use the vanishing for this sheaf on $Z_i$, and use the Leray spectral sequence for $Z_i \to Z$ to get the vanishing for $\mathcal{F}_i$.

Finally, we conclude by Theorem 51.3.

Lemma 87.8. Let $f : X \to Y$ be an affine morphism of schemes of finite type over a field $K$. Let $E_a(X)$ be the set of points $x \in X$ with $\operatorname{trdeg}_K(\kappa(x)) \leq a$. Let $\mathcal{F}$ be an abelian torsion sheaf on $X_{\text{étale}}$ whose support is contained in $E_a$. Then $R^q f_* \mathcal{F}$ has support contained in $E_{a-q}(Y)$.

Proof. The question is local on $Y$ hence we can assume $Y$ is affine. Then $X$ is affine too and we can choose a diagram

$$
\begin{array}{ccc}
X & \xrightarrow{i} & A^{n+m}_K \\
\downarrow{f} & & \downarrow{\text{pr}} \\
Y & \xrightarrow{j} & A^n_K \\
\end{array}
$$

where the horizontal arrows are closed immersions and the vertical arrow on the right is the projection (details omitted). Then $j_* R^q f_* \mathcal{F} = R^q \text{pr}_* i_* \mathcal{F}$ by the vanishing of the higher direct images of $i$ and $j$, see Proposition 31.2. Moreover, the description of the stalks of $j_*$ in the proposition shows that it suffices to prove the vanishing for $j_* R^q f_* \mathcal{F}$. Thus we may assume $f$ is the projection morphism $\text{pr} : A^{n+m}_K \to A^n_K$ and an abelian torsion sheaf $\mathcal{F}$ on $A^{n+m}_K$ satisfying the assumption in the statement of the lemma.

Let $y$ be a point in $A^n_K$. By Theorem 52.1 we have

$$(R^q \text{pr}_* \mathcal{F})_Y = H^q(A^{n+m}_K \times_{A^n_K} \text{Spec}(O^{\text{sh}}_{Y,y}), \mathcal{F}) = H^q(A^m_{O^{\text{sh}}_{Y,y}}, \mathcal{F})$$

Say $b = \operatorname{trdeg}_K(\kappa(y))$. From Lemma 87.5 we get an embedding

$$L = K(t_1, \ldots, t_b)_{\text{sep}} \to O^{\text{sh}}_{Y,y}$$

Write $O^{\text{sh}}_{Y,y} = \colim B_i$ as the filtered colimit of finite type $L$-subalgebras $B_i \subset O^{\text{sh}}_{Y,y}$ containing the ring $K[T_1, \ldots, T_n]$ of regular functions on $A^n_K$. Then we get

$$A^m_{O^{\text{sh}}_{Y,y}} = \lim A^m_{B_i}$$

If $z \in A^m_{B_i}$ is a point in the support of $\mathcal{F}$, then the image $x$ of $z$ in $A^{m+n}_K$ satisfies $\operatorname{trdeg}_K(\kappa(x)) \leq a$ by our assumption on $\mathcal{F}$ in the lemma. Since $O^{\text{sh}}_{Y,y}$ is a filtered colimit of étale algebras over $K[T_1, \ldots, T_n]$ and since $B_i \subset O^{\text{sh}}_{Y,y}$ we see that $\kappa(z)/\kappa(x)$ is algebraic (some details omitted). Then $\operatorname{trdeg}_K(\kappa(z)) \leq a$ and hence $\operatorname{trdeg}_L(\kappa(z)) \leq a - b$. By Lemma 87.7 we see that

$$H^q(A^m_{B_i}, \mathcal{F}) = 0$$

for $q > a - b$.

Thus by Theorem 51.3 we get $(Rf_* \mathcal{F})_Y = 0$ for $q > a - b$ as desired.
88. Finite cohomological dimension

0F0Y We continue the discussion started in Section 87.

0F0Z Definition 88.1. Let $f : X \to Y$ be a quasi-compact and quasi-separated morphism of schemes. The cohomological dimension of $f$ is the smallest element

$$\text{cd}(f) \in \{0, 1, 2, \ldots\} \cup \{\infty\}$$

such that for any abelian torsion sheaf $\mathcal{F}$ on $X_{\text{étale}}$ we have $R^i f_* \mathcal{F} = 0$ for $i > \text{cd}(f)$.

0F10 Lemma 88.2. Let $K$ be a field.

1. If $f : X \to Y$ is a morphism of finite type schemes over $K$, then $\text{cd}(f) < \infty$.
2. If $\text{cd}(K) < \infty$, then $\text{cd}(X) < \infty$ for any finite type scheme $X$ over $K$.

Proof. Proof of (1). We may assume $Y$ is affine. We will use the induction principle of Cohomology of Schemes, Lemma 4.1 to prove this. If $X$ is affine too, then the result holds by Lemma 87.8. Thus it suffices to show that if $X = U \cup V$ and the result is true for $U \to Y$, $V \to Y$, and $U \cap V \to Y$, then it is true for $f$. This follows from the relative Mayer-Vietoris sequence, see Lemma 50.2.

Proof of (2). We will use the induction principle of Cohomology of Schemes, Lemma 4.1 to prove this. If $X$ is affine, then the result holds by Proposition 87.6. Thus it suffices to show that if $X = U \cup V$ and the result is true for $U$, $V$, and $U \cap V$, then it is true for $X$. This follows from the Mayer-Vietoris sequence, see Lemma 50.1.

0F11 Lemma 88.3. Cohomology and direct sums. Let $n \geq 1$ be an integer.

1. Let $f : X \to Y$ be a quasi-compact and quasi-separated morphism of schemes with $\text{cd}(f) < \infty$. Then the functor

$$Rf_* : D(X_{\text{étale}}, \mathbb{Z}/n\mathbb{Z}) \to D(Y_{\text{étale}}, \mathbb{Z}/n\mathbb{Z})$$

commutes with direct sums.
2. Let $X$ be a quasi-compact and quasi-separated scheme with $\text{cd}(X) < \infty$. Then the functor

$$R\Gamma(X, -) : D(X_{\text{étale}}, \mathbb{Z}/n\mathbb{Z}) \to D(\mathbb{Z}/n\mathbb{Z})$$

commutes with direct sums.

Proof. Proof of (1). Since $\text{cd}(f) < \infty$ we see that

$$f_* : \text{Mod}(X_{\text{étale}}, \mathbb{Z}/n\mathbb{Z}) \to \text{Mod}(Y_{\text{étale}}, \mathbb{Z}/n\mathbb{Z})$$

has finite cohomological dimension in the sense of Derived Categories, Lemma 30.2.

Let $I$ be a set and for $i \in I$ let $E_i$ be an object of $D(X_{\text{étale}}, \mathbb{Z}/n\mathbb{Z})$. Choose a $K$-injective complex $T^*_i$ of $\mathbb{Z}/n\mathbb{Z}$-modules each of whose terms $T^*_i$ is an injective sheaf of $\mathbb{Z}/n\mathbb{Z}$-modules representing $E_i$. See Injectives, Theorem 12.6. Then $\bigoplus E_i$ is represented by the complex $\bigoplus T^*_i$ (termwise direct sum), see Injectives, Lemma 13.4. By Lemma 51.7 we have

$$R^q f_* (\bigoplus T^*_i) = \bigoplus R^q f_* (T^*_i) = 0$$

for $q > 0$ and any $n$. Hence we conclude by Derived Categories, Lemma 30.2 that we may compute $Rf_* (\bigoplus E_i)$ by the complex

$$f_* (\bigoplus T^*_i) = \bigoplus f_* (T^*_i)$$
(equality again by Lemma 51.7) which represents $\bigoplus Rf_*E_i$ by the already used Injectives, Lemma 13.4.

Proof of (2). This is identical to the proof of (1) and we omit it. \qed

**Lemma 88.4.** Let $f : X \to Y$ be a proper morphism of schemes. Let $n \geq 1$ be an integer. Then the functor

$$Rf_* : D(X_{\text{etale}}, \mathbb{Z}/n\mathbb{Z}) \to D(Y_{\text{etale}}, \mathbb{Z}/n\mathbb{Z})$$

commutes with direct sums.

**Proof.** It is enough to prove this when $Y$ is quasi-compact. By Lemma 88.3 we see that the dimension of the fibres of $f : X \to Y$ is bounded. Thus Lemma 86.2 implies that $\text{cd}(f) < \infty$. Hence the result by Lemma 88.3. \qed

**Lemma 88.5.** Let $X$ be a quasi-compact and quasi-separated scheme such that $\text{cd}(X) < \infty$. Let $n \geq 1$ be an integer. Let $E \in D(X_{\text{etale}}, \mathbb{Z}/n\mathbb{Z})$ and $K \in D(\mathbb{Z}/n\mathbb{Z})$. Then

$$R\Gamma(X, E \otimes^L_{\mathbb{Z}/n\mathbb{Z}} K) = R\Gamma(X, E) \otimes^L_{\mathbb{Z}/n\mathbb{Z}} K$$

in $D(Y_{\text{etale}}, \mathbb{Z}/n\mathbb{Z})$.

**Proof.** There is a canonical map from left to right by Cohomology on Sites, Section 46. We will check the equality on stalks at $y$. By the proper base change (in the form of Lemma 86.4 where $Y' = \overline{y}$) this reduces to the case where $Y$ is the spectrum of an algebraically closed field. This is shown in Lemma 88.5 where we use that $\text{cd}(X) < \infty$ by Lemma 86.2. \qed

**Lemma 88.6.** Let $f : X \to Y$ be a proper morphism of schemes. Let $n \geq 1$ be an integer. Let $E \in D(X_{\text{etale}}, \mathbb{Z}/n\mathbb{Z})$ and $K \in D(\mathbb{Z}/n\mathbb{Z})$. Then

$$Rf_* E \otimes^L_{\mathbb{Z}/n\mathbb{Z}} K = Rf_*(E \otimes^L_{\mathbb{Z}/n\mathbb{Z}} f^{-1}K)$$

in $D(Y_{\text{etale}}, \mathbb{Z}/n\mathbb{Z})$.

**Proof.** There is a canonical map from left to right by Cohomology on Sites, Section 46. We will check the equality on stalks at $y$. By the proper base change (in the form of Lemma 86.4 where $Y' = \overline{y}$) this reduces to the case where $Y$ is the spectrum of an algebraically closed field. This is shown in Lemma 88.5 where we use that $\text{cd}(X) < \infty$ by Lemma 86.2. \qed

**89. Küneth in étale cohomology**

We first prove a Küneth formula in case one of the factors is proper. Then we use this formula to prove a base change property for open immersions. This then gives a “base change by morphisms towards spectra of fields” (akin to smooth base change). Finally we use this to get a more general Küneth formula.

**Remark 89.1.** Consider a cartesian diagram in the category of schemes:

$$\begin{array}{ccc}
X \times_S Y & \to & Y \\
p \downarrow & & \downarrow g \\
X & \to & S
\end{array}$$

$$p \quad f$$

or consider the following diagram:

$$\begin{array}{ccc}
X \times Y & \to & Y \\
p \downarrow & & \downarrow g \\
X & \to & S
\end{array}$$

or consider the following diagram:

$$\begin{array}{ccc}
X \times_S Y & \to & Y \\
p \downarrow & & \downarrow g \\
X & \to & S
\end{array}$$

where $f$ is a morphism of schemes, $g$ is a morphism of schemes, and $p$ is a morphism of schemes.
Let $\Lambda$ be a ring and let $E \in D(X_{\text{étale}}, \Lambda)$ and $K \in D(Y_{\text{étale}}, \Lambda)$. Then there is a canonical map

$$Rf_* E \otimes^L \Lambda K \to R\varepsilon_* (p^{-1} E \otimes^L \Lambda q^{-1} K)$$

For example we can define this using the canonical maps $Rf_* E \to R\varepsilon_* p^{-1} E$ and $Rg_* K \to R\varepsilon_* q^{-1} K$ and the relative cup product defined in Cohomology on Sites, Remark [20.6] or you can use the adjoint to the map

$$c^{-1}(Rf_* E \otimes^L \Lambda K) = p^{-1} f^{-1} Rf_* E \otimes^L \Lambda q^{-1} Rg_* K \to p^{-1} E \otimes^L \Lambda q^{-1} K$$

which uses the adjunction maps $f^{-1} Rf_* E \to E$ and $g^{-1} Rg_* K \to K$.

0F14 Lemma 89.2. Let $k$ be a separably closed field. Let $X$ be a proper scheme over $k$.

Let $Y$ be a quasi-compact and quasi-separated scheme over $k$.

(1) If $E \in D^+(X_{\text{étale}})$ has torsion cohomology sheaves and $K \in D^+(Y_{\text{étale}})$, then

$$R\Gamma(X \times_{\text{Spec}(k)} Y, pr_1^* E \otimes^L \Lambda pr_2^* K) = R\Gamma(X, E) \otimes^L \Lambda R\Gamma(Y, K)$$

(2) If $n \geq 1$ is an integer, $Y$ is of finite type over $k$, $E \in D(X_{\text{étale}}, \mathcal{Z}/n\mathcal{Z})$, and $K \in D(Y_{\text{étale}}, \mathcal{Z}/n\mathcal{Z})$, then

$$R\Gamma(X \times_{\text{Spec}(k)} Y, pr_1^* E \otimes^L \Lambda pr_2^* K) = R\Gamma(X, E) \otimes^L \Lambda R\Gamma(Y, K)$$

Proof. Proof of (1). By Lemma [86.6] we have

$$Rpr_{2,*}(pr_1^* E \otimes^L \Lambda pr_2^* K) = Rpr_{2,*}(pr_1^* E) \otimes^L \Lambda K$$

By proper base change (in the form of Lemma [85.12]) this is equal to the object

$$R\Gamma(X, E) \otimes^L \Lambda K$$

of $D(Y_{\text{étale}})$. Taking $R\Gamma(Y, –)$ on this object reproduces the left hand side of the equality in (1) by the Leray spectral sequence for $pr_2$. Thus we conclude by Lemma [86.5].

Proof of (2). This is exactly the same as the proof of (1) except that we use Lemmas [88.6] and [86.4] as well as cd$(Y) < \infty$ by Lemma [88.2].

0F1F Lemma 89.3. Let $K$ be a separably closed field. Let $X$ be a scheme of finite type over $K$. Let $\mathcal{F}$ be an abelian sheaf on $X_{\text{étale}}$ whose support is contained in the set of closed points of $X$. Then $H^q(X, \mathcal{F}) = 0$ for $q > 0$ and $\mathcal{F}$ is globally generated.

Proof. (If $\mathcal{F}$ is torsion, then the vanishing follows immediately from Lemma [87.7])

By Lemma [73.5] we can write $\mathcal{F}$ as a filtered colimit of constructible sheaves $\mathcal{F}_i$ of $\mathcal{Z}$-modules whose supports $Z_i \subset X$ are finite sets of closed points. By Proposition [46.4] such a sheaf is of the form $(Z_i \to X)_* \mathcal{G}_i$, where $\mathcal{G}_i$ is a sheaf on $Z_i$. As $K$ is separably closed, the scheme $Z_i$ is a finite disjoint union of spectra of separably closed fields. Recall that $H^q(Z_i, \mathcal{G}_i) = H^q(X, \mathcal{F}_i)$ by the Leray spectral sequence for $Z_i \to X$ and vanishing of higher direct images for this morphism (Proposition [54.2]). By Lemmas [58.1] and [58.2] we see that $H^q(Z_i, \mathcal{G}_i)$ is zero for $q > 0$ and that $H^0(Z_i, \mathcal{G}_i)$ generates $\mathcal{G}_i$. We conclude the vanishing of $H^q(X, \mathcal{F}_i)$ for $q > 0$ and that $\mathcal{F}_i$ is generated by global sections. By Theorem [51.3] we see that $H^q(X, \mathcal{F}) = 0$ for $q > 0$. The proof is now done because a filtered colimit of globally generated sheaves of abelian groups is globally generated (details omitted).
0F1G **Lemma 89.4.** Let $K$ be a separably closed field. Let $X$ be a scheme of finite type over $K$. Let $Q \in D(X_{\text{étale}})$. Assume that $Q_{\mathfrak{p}}$ is nonzero only if $x$ is a closed point of $X$. Then

$$Q = 0 \iff H^i(X, Q) = 0 \text{ for all } i$$

**Proof.** The implication from left to right is trivial. Thus we need to prove the reverse implication.

Assume $Q$ is bounded below; this cases suffices for almost all applications. If $Q$ is not zero, then we can look at the smallest $i$ such that the cohomology sheaf $H^i(Q)$ is nonzero. By Lemma 89.3 we have $H^i(X, Q) = H^0(X, H^i(Q)) \neq 0$ and we conclude.

**General case.** Let $\mathcal{B} \subset \text{Ob}(X_{\text{étale}})$ be the quasi-compact objects. By Lemma 89.3 the assumptions of Cohomology on Sites, Lemma 23.11 are satisfied. We conclude that $H^i(U, Q) = H^0(U, H^i(Q))$ for all $U \in \mathcal{B}$. In particular, this holds for $U = X$. Thus the conclusion by Lemma 89.3 as $Q$ is zero in $D(X_{\text{étale}})$ if and only if $H^i(Q)$ is zero for all $q$. □

0F1H **Lemma 89.5.** Let $K$ be a field. Let $j : U \to X$ be an open immersion of schemes of finite type over $K$. Let $Y$ be a scheme of finite type over $K$. Consider the diagram

$$
\begin{array}{ccc}
Y \times_{\text{Spec}(K)} X & \xleftarrow{h} & Y \times_{\text{Spec}(K)} U \\
\downarrow q & & \downarrow p \\
X & \xleftarrow{j} & U
\end{array}
$$

Then the base change map $q^{-1}Rj_*\mathcal{F} \to Rh_*p^{-1}\mathcal{F}$ is an isomorphism for $\mathcal{F}$ an abelian sheaf on $U_{\text{étale}}$ whose stalks are torsion of orders invertible in $K$.

**Proof.** Write $\mathcal{F} = \text{colim} \mathcal{F}[n]$ where the colimit is over the multiplicative system of integers invertible in $K$. Since cohomology commutes with filtered colimits in our situation (for a precise reference see Lemma 89.3), it suffices to prove the lemma for $\mathcal{F}[n]$. Thus we may assume $\mathcal{F}$ is a sheaf of $\mathbb{Z}/n\mathbb{Z}$-modules for some $n$ invertible in $K$ (we will use this at the very end of the proof). In the proof we use the short hand $X \times_K Y$ for the fibre product over $\text{Spec}(K)$. We will prove the lemma by induction on $\dim(X) + \dim(Y)$. The lemma is trivial if $\dim(X) \leq 0$, since in this case $U$ is an open and closed subscheme of $X$. Choose a point $z \in X \times_K Y$. We will show the stalk at $z$ is an isomorphism.

Suppose that $z \mapsto x \in X$ and assume $\text{trdeg}_K(\kappa(x)) > 0$. Set $X' = \text{Spec}(O_{X,z}^\text{sh})$ and denote $U' \subset X'$ the inverse image of $U$. Consider the base change

$$
\begin{array}{ccc}
Y \times_K X' & \xleftarrow{h'} & Y \times_K U' \\
\downarrow q' & & \downarrow p' \\
X' & \xleftarrow{j'} & U'
\end{array}
$$

of our diagram by $X' \to X$. Observe that $X' \to X$ is a filtered colimit of étale morphisms. By smooth base change in the form of Lemma 83.3 the pullback of $q^{-1}Rj_*\mathcal{F} \to Rh_*p^{-1}\mathcal{F}$ to $X'$ to $Y \times_K X'$ is the map $(q')^{-1}Rj'_*\mathcal{F}' \to Rj'_*(p')^{-1}\mathcal{F}'$ where $\mathcal{F}'$ is the pullback of $\mathcal{F}$ to $U'$. (In this step it would suffice to use étale base change which is an essentially trivial result.) So it suffices to show that
$$(q')^{-1}Rj'_sF' \to Rj'_s(p')^{-1}F'$$ is an isomorphism in order to prove that our original map is an isomorphism on stalks at $\mathfrak{z}$. By Lemma \ref{87.3} there is a separably closed field $L/K$ such that $X' = \varinjlim X_i$ with $X_i$ affine of finite type over $L$ and $\dim(X_i) < \dim(X)$. For $i$ large enough there exists an open $U_i \subset X_i$ restricting to $U'$ in $X'$. We may apply the induction hypothesis to the diagram

$$
Y \times_K X_i \leftarrow Y \times_K U_i \quad \begin{array}{c}
q_i \\
\downarrow h_i \end{array}
X_i \leftarrow j_i \quad \begin{array}{c}
p_i \\
\downarrow \end{array}
U_i \quad \begin{array}{c}
\text{equal to} \\
\end{array}
Y_L \times_L X_i \leftarrow Y_L \times_L U_i \quad \begin{array}{c}
q_i \\
\downarrow h_i \end{array}
X_i \leftarrow j_i \quad \begin{array}{c}
p_i \\
\downarrow \end{array}
U_i
$$

over the field $L$ and the pullback of $F$ to these diagrams. By Lemma \ref{80.3} we conclude that the map $(q')^{-1}Rj'_sF' \to Rj'_s(p')^{-1}F$ is an isomorphism.

Suppose that $z \mapsto y \in Y$ and assume $\text{trdeg}_K(\kappa(y)) > 0$. Let $Y' = \text{Spec}(\mathcal{O}^{sh}_{X,z})$. By Lemma \ref{87.3} there is a separably closed field $L/K$ such that $Y' = \varinjlim Y_i$ with $Y_i$ affine of finite type over $L$ and $\dim(Y_i) < \dim(Y)$. In particular $Y'$ is a scheme over $L$. Denote with a subscript $L$ the base change from schemes over $K$ to schemes over $L$. Consider the commutative diagrams

$$
Y' \times_K X \leftarrow Y' \times_K U \quad \begin{array}{c}
f \\
\downarrow h' \end{array}
Y \times_K X \leftarrow Y \times_K U \quad \begin{array}{c}
f \\
\downarrow h \end{array}
X \leftarrow j \quad \begin{array}{c}
p \\
\downarrow \end{array}
U \quad \begin{array}{c}
\text{and} \\
\end{array}
Y' \times_L X_L \leftarrow Y' \times_L U_L \quad \begin{array}{c}
q' \\
\downarrow h' \end{array}
X_L \leftarrow j_L \quad \begin{array}{c}
p_L \\
\downarrow \end{array}
U_L
$$

and observe the top and bottom rows are the same on the left and the right. By smooth base change we see that $f^{-1}Rh_*p^{-1}F = Rh'_*(f')^{-1}p^{-1}F$ (similarly to the previous paragraph). By smooth base change for $\text{Spec}(L) \to \text{Spec}(K)$ (Lemma \ref{84.1}) we see that $Rj_{L,*}F_L$ is the pullback of $Rj_*F$ to $X_L$. Combining these two observations, we conclude that it suffices to prove the base change map for the upper square in the diagram on the right is an isomorphism in order to prove that our original map is an isomorphism on stalks at $\mathfrak{z}$. Then using that $Y' = \varinjlim Y_i$ and arguing exactly as in the previous paragraph we see that the induction hypothesis forces our map over $Y' \times_K X$ to be an isomorphism.

Thus any counter example with $\dim(X) + \dim(Y)$ minimal would only have non-isomorphisms $q^{-1}Rj_*F \to Rh_*p^{-1}F$ on stalks at closed points of $X \times_K Y$ (because a point $z$ of $X \times_K Y$ is a closed point if and only if both the image of $z$ in $X$ and in $Y$ are closed). Since it is enough to prove the isomorphism locally, we may assume $X$ and $Y$ are affine. However, then we can choose an open dense immersion $Y \to Y'$ with $Y'$ projective. (Choose a closed immersion $Y \to \mathbb{A}^n_K$ and let $Y'$ be the scheme theoretic closure of $Y$ in $\mathbb{P}^n_K$.) Then $\dim(Y') = \dim(Y)$ and hence we get a “minimal” counter example with $Y$ projective over $K$. In the next paragraph we show that this can’t happen.

Consider a diagram as in the statement of the lemma such that $q^{-1}Rj_*F \to Rh_*p^{-1}F$ is an isomorphism at all non-closed points of $X \times_K Y$ and such that

\footnote{Here we use that a “vertical composition” of base change maps is a base change map as explained in Cohomology on Sites, Remark \ref{20.4}.}
Y is projective. The restriction of the map to $(X \times_{K} Y)_{\text{sep}}$ is the corresponding map for the diagram of the lemma base changed to $K_{\text{sep}}$. Thus we may and do assume $K$ is separably algebraically closed. Choose a distinguished triangle

$$q^{-1}Rj_{*}\mathcal{F} \to Rh_{*}p^{-1}\mathcal{F} \to Q \to (q^{-1}Rj_{*}\mathcal{F})[1]$$

in $D((X \times_{K} Y)_{\text{etale}})$. Since $Q$ is supported in closed points we see that it suffices to prove $H^{i}(X \times_{K} Y, Q) = 0$ for all $i$, see Lemma 89.4. Thus it suffices to prove that $q^{-1}Rj_{*}\mathcal{F} \to Rh_{*}p^{-1}\mathcal{F}$ induces an isomorphism on cohomology. Recall that $\mathcal{F}$ is annihilated by $n$ invertible in $K$. By the Künneth formula of Lemma 89.2 we have

$$R\Gamma(X \times_{K} Y, q^{-1}Rj_{*}\mathcal{F}) = R\Gamma(X, Rj_{*}\mathcal{F}) \otimes_{\mathbb{Z}/n\mathbb{Z}} R\Gamma(Y, \mathbb{Z}/n\mathbb{Z})$$

and

$$R\Gamma(X \times_{K} Y, Rh_{*}p^{-1}\mathcal{F}) = R\Gamma(U \times_{K} Y, p^{-1}\mathcal{F}) = R\Gamma(U, \mathcal{F}) \otimes_{\mathbb{Z}/n\mathbb{Z}} R\Gamma(Y, \mathbb{Z}/n\mathbb{Z})$$

This finishes the proof. □

**Lemma 89.6.** Let $K$ be a field. For any commutative diagram

$$
\begin{array}{ccc}
X & \xleftarrow{h} & X' \\
\downarrow & & \downarrow f' \\
\text{Spec}(K) & \xleftarrow{g} & T \\
\end{array}
$$

of schemes over $K$ with $X' = X \times_{\text{Spec}(K)} S'$ and $Y = X' \times_{S} T$ and $g$ quasi-compact and quasi-separated, and every abelian sheaf $\mathcal{F}$ on $T_{\text{etale}}$ whose stalks are torsion of orders invertible in $K$ the base change map

$$(f')^{-1}Rg_{*}\mathcal{F} \longrightarrow Rh_{*}e^{-1}\mathcal{F}$$

is an isomorphism.

**Proof.** The question is local on $X$, hence we may assume $X$ is affine. By Limits, Lemma 7.1 we can write $X = \text{lim} X_{i}$ as a cofiltered limit with affine transition morphisms of schemes $X_{i}$ of finite type over $K$. Denote $X'_{i} = X_{i} \times_{\text{Spec}(K)} S'$ and $Y_{i} = X'_{i} \times_{S} T$. By Lemma 80.3 it suffices to prove the statement for the squares with corners $X_{i}, Y_{i}, S_{i}, T_{i}$. Thus we may assume $X$ is of finite type over $K$. Similarly, we may write $\mathcal{F} = \text{colim} \mathcal{F}[n]$ where the colimit is over the multiplicative system of integers invertible in $K$. The same lemma used above reduces us to the case where $\mathcal{F}$ is a sheaf of $\mathbb{Z}/n\mathbb{Z}$-modules for some $n$ invertible in $K$.

We may replace $K$ by its algebraic closure $\overline{K}$. Namely, formation of direct image commutes with base change to $\overline{K}$ according to Lemma 84.1 (works for both $g$ and $h$). And it suffices to prove the agreement after restriction to $X'_{\overline{K}}$. Next, we may replace $X$ by its reduction as we have the topological invariance of étale cohomology, see Proposition 45.4. After this replacement the morphism $X \to \text{Spec}(K)$ is flat, finite presentation, with geometrically reduced fibres and the same is true for any base change, in particular for $X' \to S'$. Hence $(f')^{-1}g_{*}\mathcal{F} \to Rh_{*}e^{-1}\mathcal{F}$ is an isomorphism by Lemma 81.5.
At this point we may apply Lemma \ref{lem:base-change} to see that it suffices to prove: given a commutative diagram

\[
\begin{array}{ccc}
X & \xleftarrow{p} & X' \\
\downarrow f & & \downarrow h \\
\text{Spec}(K) & \xleftarrow{i} & S' \\
\end{array}
\]

with both squares cartesian, where \( S' \) is affine, integral, and normal with algebraically closed function field \( K \), then \( R^i h_* (\mathbb{Z}/d\mathbb{Z}) \) is zero for \( q > 0 \) and \( d|n \).

Observe that this vanishing is equivalent to the statement that

\[(f')^{-1} R^i (\text{Spec}(L) \to S')_* \mathbb{Z}/d\mathbb{Z} \to R^i h_* \mathbb{Z}/d\mathbb{Z}\]

is an isomorphism, because the left hand side is zero for example by Lemma \ref{lem:lemma6.4}. Write \( S' = \text{Spec}(B) \) so that \( L \) is the fraction field of \( B \). Write \( B = \bigcup_{i \in I} B_i \) as the union of its finite type \( K \)-subalgebras \( B_i \). Let \( J \) be the set of pairs \((i, g)\) where \( i \in I \) and \( g \in B_i \) nonzero with ordering \((i', g') \geq (i, g)\) if and only if \( i' \geq i \) and \( g \) maps to an invertible element of \((B_i')_g\). Then \( L = \text{colim}_{(i, g) \in J(B_i)_g}. \) For \( j = (i, g) \in J \) set \( S_j = \text{Spec}(B_i) \) and \( U_j = \text{Spec}((B_i)_g). \) Then

\[
\begin{array}{ccc}
X' & \xleftarrow{h} & Y \\
\downarrow & & \downarrow & & \downarrow \\
\text{Spec}(L) & & \text{Spec}(L) & & \text{Spec}(L) \\
\end{array}
\]

Thus we may apply Lemma \ref{lem:lemma80.3} to see that it suffices to prove base change holds in the diagrams on the right which is what we proved in Lemma \ref{lem:lemma89.5}. \hfill \Box

**Lemma 89.7.** Let \( K \) be a field. Let \( n \geq 1 \) be invertible in \( K \). Consider a commutative diagram

\[
\begin{array}{ccc}
X & \xleftarrow{p} & X' \\
\downarrow f' & & \downarrow h \\
\text{Spec}(K) & \xleftarrow{i} & S' \\
\end{array}
\]

of schemes with \( X' = X \times_{\text{Spec}(K)} S' \) and \( Y = X' \times_S T \) and \( g \) quasi-compact and quasi-separated. The canonical map

\[p^{-1} E \otimes_{\mathbb{Z}/n\mathbb{Z}} (f')^{-1} Rg_* F \to Rh_*(h^{-1} p^{-1} E \otimes_{\mathbb{Z}/n\mathbb{Z}} e^{-1} F)\]

is an isomorphism if \( E \in D^+(X_{\text{étale}}, \mathbb{Z}/n\mathbb{Z}) \) has tor amplitude in \([a, \infty]\) for some \( a \in \mathbb{Z} \) and \( F \in D^+(T_{\text{étale}}, \mathbb{Z}/n\mathbb{Z}). \)

**Proof.** This lemma is a generalization of Lemma \ref{lem:lemma89.6} to objects of the derived category; the assertion of our lemma is true because in Lemma \ref{lem:lemma89.6} the scheme \( X \) over \( K \) is arbitrary. We strongly urge the reader to skip the laborious proof (alternative: read only the last paragraph).

We may represent \( E \) by a bounded below \( K \)-flat complex \( E^\bullet \) consisting of flat \( \mathbb{Z}/n\mathbb{Z} \)-modules. See Cohomology on Sites, Lemma \ref{lem:cohomology-on-sites}. Choose an integer \( b \) such that \( H^i(F) = 0 \) for \( i < b \). Choose a large integer \( N \) and consider the short exact sequence

\[0 \to \sigma_{\geq N+1} E^\bullet \to E^\bullet \to \sigma_{\leq N} E^\bullet \to 0\]
of stupid truncations. This produces a distinguished triangle $E'' \to E \to E' \to E''[1]$ in $D(X_{\text{étale}}, \mathbb{Z}/n\mathbb{Z})$. For fixed $F$ both sides of the arrow in the statement of the lemma are exact functors in $E$. Observe that

$$p^{-1}E'' \otimes_{\mathbb{Z}/n\mathbb{Z}} (f')^{-1}Rg_*F \quad \text{and} \quad Rh_*(h^{-1}p^{-1}E'' \otimes_{\mathbb{Z}/n\mathbb{Z}} e^{-1}F)$$

are sitting in degrees $\geq N + b$. Hence, if we can prove the lemma for the object $E'$, then we see that the lemma holds in degrees $\leq N + b$ and we will conclude. Some details omitted. Thus we may assume $E$ is represented by a bounded complex of flat $\mathbb{Z}/n\mathbb{Z}$-modules. Doing another argument of the same nature, we may assume $E$ is given by a single flat $\mathbb{Z}/n\mathbb{Z}$-module $\mathcal{E}$.

Next, we use the same arguments for the variable $F$ to reduce to the case where $F$ is given by a single sheaf of $\mathbb{Z}/n\mathbb{Z}$-modules $\mathcal{F}$. Say $\mathcal{F}$ is annihilated by an integer $m | n$. If $\ell$ is a prime number dividing $m$ and $m > \ell$, then we can look at the short exact sequence $0 \to \mathcal{F}[\ell] \to \mathcal{F} \to \mathcal{F}/\mathcal{F}[\ell] \to 0$ and reduce to smaller $m$. This finally reduces us to the case where $\mathcal{F}$ is annihilated by a prime number $\ell$ dividing $n$. In this case observe that

$$p^{-1}\mathcal{E} \otimes_{\mathbb{Z}/n\mathbb{Z}} (f')^{-1}Rg_*\mathcal{F} = p^{-1}(\mathcal{E}/\ell\mathcal{E}) \otimes_{\mathbb{F}_\ell} (f')^{-1}Rg_*\mathcal{F}$$

by the flatness of $\mathcal{E}$. Similarly for the other term. This reduces us to the case where we are working with sheaves of $\mathbb{F}_\ell$-vector spaces which is discussed.

Assume $\ell$ is a prime number invertible in $K$. Assume $\mathcal{E}, \mathcal{F}$ are sheaves of $\mathbb{F}_\ell$-vector spaces on $X_{\text{étale}}$ and $T_{\text{étale}}$. We want to show that

$$p^{-1}\mathcal{E} \otimes_{\mathbb{F}_\ell} (f')^{-1}Rg_*\mathcal{F} \longrightarrow R^qh_*(h^{-1}p^{-1}\mathcal{E} \otimes_{\mathbb{F}_\ell} e^{-1}\mathcal{F})$$

is an isomorphism for every $q \geq 0$. This question is local on $X$ hence we may assume $X$ is affine. We can write $\mathcal{E}$ as a filtered colimit of constructible sheaves of $\mathbb{F}_\ell$-vector spaces on $X_{\text{étale}}$, see Lemma [72.2]. Since tensor products commute with filtered colimits and since higher direct images do too (Lemma [51.7]) we may assume $\mathcal{E}$ is a constructible sheaf of $\mathbb{F}_\ell$-vector spaces on $X_{\text{étale}}$. Then we can choose an integer $m$ and finite and finitely presented morphisms $\pi_i : X_i \to X$, $i = 1, \ldots, m$ such that there is an injective map

$$\mathcal{E} \to \bigoplus_{i=1,\ldots,m} \pi_{i,*}\mathbb{F}_\ell$$

See Lemma [73.4]. Observe that the direct sum is a constructible sheaf as well (Lemma [72.9]). Thus the cokernel is constructible too (Lemma [70.6]). By dimension shifting, i.e., induction on $q$, on the category of constructible sheaves of $\mathbb{F}_\ell$-vector spaces on $X_{\text{étale}}$, it suffices to prove the result for the sheaves $\pi_{i,*}\mathbb{F}_\ell$ (details omitted; hint: start with proving injectivity for $q = 0$ for all constructible $\mathcal{E}$). To prove this case we extend the diagram of the lemma to

$$\begin{array}{ccccccccc}
X_i & \xleftarrow{p_i} & X_i' & \xleftarrow{h_i} & Y_i \\
\downarrow{\pi_i} & & \downarrow{\pi_i'} & & \downarrow{p_i} \\
X & \xrightarrow{p} & X' & \xrightarrow{h} & Y \\
\downarrow{f'} & & \downarrow{e} & & \\
\text{Spec}(K) & \xleftarrow{g} & S' & \xrightarrow{g} & T
\end{array}$$
with all squares cartesian. In the equations below we are going to use that \( R\pi_{i,*} = \pi_{i,*} \) and similarly for \( \pi'_i, \rho_i \), we are going to use the Leray spectral sequence, we are going to use Lemma \[54.3\] and we are going to use Lemma \[88.6\] (although this lemma is almost trivial for finite morphisms) for \( \pi_i, \pi'_i, \rho_i \). Doing so we see that

\[
p^{-1}\pi_{i,*}F_\ell \otimes_{F_\ell} (f')^{-1}R^qg_*F = \pi'_{i,*}F_\ell \otimes_{F_\ell} (f')^{-1}R^qg_*F = (\pi'_{i,*})^{-1}(f')^{-1}R^qg_*F
\]

Similarly, we have

\[
R^qh_*(h^{-1}p^{-1}\pi_{i,*}F_\ell \otimes_{F_\ell} e^{-1}F) = R^qh_*(\rho_{i,*}F_\ell \otimes_{F_\ell} e^{-1}F) = R^qh_*(\rho_i^{-1}e^{-1}F) = \pi'_{i,*}R^qh_{i,*}\rho_i^{-1}e^{-1}F
\]

Since \( R^qh_{i,*}\rho_i^{-1}e^{-1}F = (\pi'_i)^{-1}(f')^{-1}R^qg_*F \) by Lemma \[89.6\] we conclude.

**Lemma 89.8.** Let \( K \) be a field. Let \( n \geq 1 \) be invertible in \( K \). Consider a commutative diagram

\[
\begin{array}{ccc}
X & \xleftarrow{p} & X' \xleftarrow{f'} \xrightarrow{e} Y \\
& \downarrow \scriptstyle h & \downarrow \scriptstyle e \\
\text{Spec}(K) & \xleftarrow{g} & T
\end{array}
\]

of schemes of finite type over \( K \) with \( X' = X \times_{\text{Spec}(K)} S' = Y \times_\pi T \). The canonical map

\[
p^{-1}E \otimes_{\mathbf{Z}/n\mathbf{Z}} (f')^{-1}Rg_*F \rightarrow R\pi_*(h^{-1}p^{-1}E \otimes_{\mathbf{Z}/n\mathbf{Z}} e^{-1}F)
\]

is an isomorphism for \( E \) in \( D(X_{\text{etale}}, \mathbf{Z}/n\mathbf{Z}) \) and \( F \) in \( D(Y_{\text{etale}}, \mathbf{Z}/n\mathbf{Z}) \).

**Proof.** We will reduce this to Lemma \[89.7\] using that our functors commute with direct sums. We suggest the reader skip the proof. Recall that derived tensor product commutes with direct sums. Recall that (derived) pullback commutes with direct sums. Recall that \( R\pi_* \) and \( Rg_* \) commute with direct sums, see Lemmas \[88.2\] and \[88.3\] (this is where we use our schemes are of finite type over \( K \)).

To finish the proof we can argue as follows. First we write \( E = \hocolim \tau_{\leq N} E \). Since our functors commute with direct sums, they commute with homotopy colimits. Hence it suffices to prove the lemma for \( E \) bounded above. Similarly for \( F \) we may assume \( F \) is bounded above. Then we can represent \( E \) by a bounded above complex \( \mathcal{E}^\bullet \) of sheaves of \( \mathbf{Z}/n\mathbf{Z} \)-modules. Then

\[
\mathcal{E}^\bullet = \colim \sigma_{\geq -N} \mathcal{E}^\bullet
\]

(stupid truncations). Thus we may assume \( \mathcal{E}^\bullet \) is a bounded complex of sheaves of \( \mathbf{Z}/n\mathbf{Z} \)-modules. For \( F \) we choose a bounded above complex of flat(!) sheaves of \( \mathbf{Z}/n\mathbf{Z} \)-modules. Then we reduce to the case where \( F \) is represented by a bounded complex of flat sheaves of \( \mathbf{Z}/n\mathbf{Z} \)-modules. At this point Lemma \[89.7\] kicks in and we conclude.

**Lemma 89.9.** Let \( k \) be a separably closed field. Let \( X \) and \( Y \) be finite type schemes over \( k \). Let \( n \geq 1 \) be an integer invertible in \( k \). Then for \( E \in D(X_{\text{etale}}, \mathbf{Z}/n\mathbf{Z}) \) and \( K \in D(Y_{\text{etale}}, \mathbf{Z}/n\mathbf{Z}) \) we have

\[
R\Gamma(X \times_{\text{Spec}(k)} Y, pr_1^{-1}E \otimes_{\mathbf{Z}/n\mathbf{Z}} pr_2^{-1}K) = R\Gamma(X, E) \otimes_{\mathbf{Z}/n\mathbf{Z}} R\Gamma(Y, K)
\]
Proof. By Lemma 89.8 we have
\[ R\Gamma_{\text{pr}_1 \ast}(\text{pr}_1^{-1}E \otimes_{\mathbb{Z}/n\mathbb{Z}} \text{pr}_2^{-1}K) = E \otimes_{\mathbb{Z}/n\mathbb{Z}} R\Gamma(Y,K) \]
We conclude by Lemma 88.5 which we may use because \( \text{cd}(X) < \infty \) by Lemma 88.2. \( \square \)

90. Comparing chaotic and Zariski topologies

When constructing the structure sheaf of an affine scheme, we first construct the values on affine opens, and then we extend to all opens. A similar construction is often useful for constructing complexes of abelian groups on a scheme \( X \). Recall that \( X_{\text{affine},\text{Zar}} \) denotes the category of affine opens of \( X \) with topology given by standard Zariski coverings, see Lemma 21.4. Let’s denote \( X_{\text{affine},\text{chaotic}} \) the same underlying category with the chaotic topology, i.e., such that sheaves agree with presheaves. We obtain a morphisms of sites
\[ \epsilon : X_{\text{affine},\text{Zar}} \to X_{\text{affine},\text{chaotic}} \]
as in Cohomology on Sites, Section 27.

**Lemma 90.1.** In the situation above let \( K \) be an object of \( D^+(X_{\text{affine},\text{chaotic}}) \). Then \( K \) is in the essential image of the (fully faithful) functor \( R\epsilon_* : D(X_{\text{affine},\text{Zar}}) \to D(X_{\text{affine},\text{chaotic}}) \) if and only if the following two conditions hold

1. \( R\Gamma(\emptyset, K) \) is zero in \( D(Ab) \), and
2. if \( U = V \cup W \) with \( U,V,W \subset X \) affine open and \( V,W \subset U \) standard open (Algebra, Definition 16.3), then the map \( c^K_{U;V,W;V\cap W} \) of Cohomology on Sites, Lemma 26.1 is a quasi-isomorphism.

**Proof.** Except for a snafu having to do with the empty set, this follows from the very general Cohomology on Sites, Lemma 29.2 whose hypotheses hold by Schemes, Lemma 11.7 and Cohomology on Sites, Lemma 29.3.

To get around the snafu, denote \( X_{\text{affine},\text{almost-chaotic}} \) the site where the empty object \( \emptyset \) has two coverings, namely, \( \{ \emptyset \to \emptyset \} \) and the empty covering (see Sites, Example 6.4 for a discussion). Then we have morphisms of sites
\[ X_{\text{affine},\text{Zar}} \to X_{\text{affine},\text{almost-chaotic}} \to X_{\text{affine},\text{chaotic}} \]
The argument above works for the first arrow. Then we leave it to the reader to see that an object \( K \) of \( D^+(X_{\text{affine},\text{chaotic}}) \) is in the essential image of the (fully faithful) functor \( D(X_{\text{affine},\text{almost-chaotic}}) \to D(X_{\text{affine},\text{chaotic}}) \) if and only if \( R\Gamma(\emptyset, K) \) is zero in \( D(Ab) \). \( \square \)

91. Comparing big and small topoi

Let \( S \) be a scheme. In Topologies, Lemma 4.13 we have introduced comparison morphisms \( \pi_S : (\text{Sch}/S)_{\text{étale}} \to S_{\text{étale}} \) and \( i_S : \text{Sh}(S_{\text{étale}}) \to \text{Sh}((\text{Sch}/S)_{\text{étale}}) \) with \( \pi_S \circ i_S = \text{id} \) and \( \pi_{S*} = i_{S*}^{-1} \). More generally, if \( f : T \to S \) is an object of \( (\text{Sch}/S)_{\text{étale}} \), then there is a morphism \( i_f : \text{Sh}(T_{\text{étale}}) \to \text{Sh}((\text{Sch}/S)_{\text{étale}}) \) such that \( f_{\text{small}} = \pi_S \circ i_f \), see Topologies, Lemmas 4.12 and 4.16. In Descent, Remark 8.4 we have extended these to a morphism of ringed sites
\[ \pi_S : ((\text{Sch}/S)_{\text{étale}}, \mathcal{O}) \to (S_{\text{étale}}, \mathcal{O}_S) \]
and morphisms of ringed topoi

\[ i_S : (\text{Sh}(S_{\text{étale}}), \mathcal{O}_S) \to (\text{Sh}((\text{Sch}/S)_{\text{étale}}), \mathcal{O}) \]

and

\[ i_f : (\text{Sh}(T_{\text{étale}}), \mathcal{O}_T) \to (\text{Sh}((\text{Sch}/S)_{\text{étale}}), \mathcal{O}_T) \]

Note that the restriction \( i^{-1}_S \pi_{S*} \) (see Topologies, Definition 4.14) transforms \( \mathcal{O} \) into \( \mathcal{O}_S \). Similarly, \( i^{-1}_f \) transforms \( \mathcal{O} \) into \( \mathcal{O}_T \). See Descent, Remark 8.4. Hence \( i^*_S \mathcal{F} = i^{-1}_S \mathcal{F} \) and \( i^*_f \mathcal{F} = i^{-1}_f \mathcal{F} \) for any \( \mathcal{O} \)-module \( \mathcal{F} \) on \((\text{Sch}/S)_{\text{étale}}\). In particular \( i^*_S \) and \( i^*_f \) are exact functors. The functor \( i^*_S \) is often denoted \( \mathcal{F} \mapsto \mathcal{F}|_{S_{\text{étale}}} \) (and this does not conflict with the notation in Topologies, Definition 4.14).

**Lemma 91.1.** Let \( S \) be a scheme. Let \( T \) be an object of \((\text{Sch}/S)_{\text{étale}}\).

1. If \( \mathcal{I} \) is injective in \( \text{Ab}((\text{Sch}/S)_{\text{étale}}) \), then
   (a) \( i^{-1}_f \mathcal{I} \) is injective in \( \text{Ab}(T_{\text{étale}}) \),
   (b) \( \mathcal{I}|_{S_{\text{étale}}} \) is injective in \( \text{Ab}(S_{\text{étale}}) \),
2. If \( \mathcal{I}^* \) is a \( K \)-injective complex in \( \text{Ab}((\text{Sch}/S)_{\text{étale}}) \), then
   (a) \( i^{-1}_f \mathcal{I}^* \) is a \( K \)-injective complex in \( \text{Ab}(T_{\text{étale}}) \),
   (b) \( \mathcal{I}^*|_{S_{\text{étale}}} \) is a \( K \)-injective complex in \( \text{Ab}(S_{\text{étale}}) \),

The corresponding statements for modules do not hold.

**Proof.** Parts (1)(b) and (2)(b) follow formally from the fact that the restriction functor \( \pi_{S*} = i^{-1}_S \) is a right adjoint of the exact functor \( \pi_{S*} \), see Homology, Lemma 26.1 and Derived Categories, Lemma 29.9.

Parts (1)(a) and (2)(a) can be seen in two ways. First proof: We can use that \( i^{-1}_f \) is a right adjoint of the exact functor \( i_f \). This functor is constructed in Topologies, Lemma 4.12 for sheaves of sets and for abelian sheaves in Modules on Sites, Lemma 16.2. It is shown in Modules on Sites, Lemma 16.3 that it is exact. Second proof. We can use that \( i_f = i_T \circ f_{\text{big}} \) as is shown in Topologies, Lemma 4.16. Since \( f_{\text{big}} \) is a localization, we see that pullback by it preserves injectives and \( K \)-injectives, see Cohomology on Sites, Lemmas 8.1 and 21.1. Then we apply the already proved parts (1)(b) and (2)(b) to the functor \( i^{-1}_T \) to conclude.

Let \( S = \text{Spec}(\mathbb{Z}) \) and consider the map \( 2 : \mathcal{O}_S \to \mathcal{O}_S \). This is an injective map of \( \mathcal{O}_S \)-modules on \( S_{\text{étale}} \). However, the pullback \( \pi_{S*}(2) : \mathcal{O} \to \mathcal{O} \) is not injective as we see by evaluating on \( \text{Spec}(\mathbb{F}_2) \). Now choose an injection \( \alpha : \mathcal{O} \to \mathcal{I} \) into an injective \( \mathcal{O} \)-module \( \mathcal{I} \) on \((\text{Sch}/S)_{\text{étale}}\). Then consider the diagram

\[
\begin{array}{ccc}
\mathcal{O}_S & \xrightarrow{\alpha|_{S_{\text{étale}}}} & \mathcal{I}|_{S_{\text{étale}}} \\
2 \downarrow & & \downarrow \\
\mathcal{O}_S & \xrightarrow{\alpha|_{S_{\text{étale}}}} & \mathcal{I}|_{S_{\text{étale}}}
\end{array}
\]

Then the dotted arrow cannot exist in the category of \( \mathcal{O}_S \)-modules because it would mean (by adjunction) that the injective map \( \alpha \) factors through the noninjective map \( \pi_{S*}(2) \) which cannot be the case. Thus \( \mathcal{I}|_{S_{\text{étale}}} \) is not an injective \( \mathcal{O}_S \)-module. \( \square \)
Let $f : T \to S$ be a morphism of schemes. The commutative diagram of Topologies, Lemma \[4.16\](3) leads to a commutative diagram of ringed sites

$$
\begin{array}{ccc}
(T_{\text{étale}}, \mathcal{O}_T) & \xleftarrow{\pi_T} & ((\text{Sch}/T)_{\text{étale}}, \mathcal{O}) \\
\downarrow f_{\text{small}} & & \downarrow f_{\text{big}} \\
(S_{\text{étale}}, \mathcal{O}_S) & \xleftarrow{\pi_S} & ((\text{Sch}/S)_{\text{étale}}, \mathcal{O})
\end{array}
$$

as one easily sees by writing out the definitions of $f_{\text{small}}^*, f_{\text{big}}^*, \pi_S^*$, and $\pi_T^*$. In particular this means that

$$(91.1.1) \quad (f_{\text{big}})_* \mathcal{F}|_{\text{étale}} = f_{\text{small}}_* (f_* \mathcal{F})|_{\text{étale}}$$

for any sheaf $\mathcal{F}$ on $((\text{Sch}/T)_{\text{étale}}$ and if $\mathcal{F}$ is a sheaf of $\mathcal{O}$-modules, then \[91.1.1\] is an isomorphism of $\mathcal{O}_S$-modules on $S_{\text{étale}}$.

075A **Lemma 91.2.** Let $f : T \to S$ be a morphism of schemes.

1. For $K$ in $D((\text{Sch}/T)_{\text{étale}})$ we have $(Rf_{\text{big}})_*(K|_{\text{étale}}) = Rf_{\text{small}}_* (K|_{\text{étale}})$ in $D(S_{\text{étale}})$.
2. For $K$ in $D((\text{Sch}/T)_{\text{étale}}, \mathcal{O})$ we have $(Rf_{\text{big}})_*(K|_{\text{étale}}) = Rf_{\text{small}}_* (K|_{\text{étale}})$ in $D(D(S_{\text{étale}}, \mathcal{O}_S))$.

More generally, let $g : S' \to S$ be an object of $((\text{Sch}/S)_{\text{étale}}$. Consider the fibre product

$$
\begin{array}{ccc}
T' & \xrightarrow{g'} & T \\
\downarrow f' & & \downarrow f \\
S' & \xrightarrow{g} & S
\end{array}
$$

Then

3. For $K$ in $D((\text{Sch}/T)_{\text{étale}})$ we have $i_g^*(Rf_{\text{big}}^* K) = Rf'_{\text{small}}_* (i_g^{-1} K)$ in $D(S'_{\text{étale}})$.
4. For $K$ in $D((\text{Sch}/T)_{\text{étale}}, \mathcal{O})$ we have $i_g^*(Rf_{\text{big}}^* K) = Rf'_* (i_g^* K)$ in $D(D(S'_{\text{étale}}, \mathcal{O}_{S'}))$.
5. For $K$ in $D((\text{Sch}/T)_{\text{étale}})$ we have $g_{\text{big}}^{-1}(Rf_{\text{big}}^* K) = Rf'_{\text{big}}_* ((g_{\text{big}})^{-1} K)$ in $D(S'_{\text{étale}})$.
6. For $K$ in $D((\text{Sch}/T)_{\text{étale}}, \mathcal{O})$ we have $g_{\text{big}}^* (Rf_{\text{big}}^* K) = Rf'_{\text{big}}_* ((g_{\text{big}})^* K)$ in $D(D(S'_{\text{étale}}, \mathcal{O}_{S'}))$.

**Proof.** Part (1) follows from Lemma \[91.1\] and \[91.1.1\] on choosing a K-injective complex of abelian sheaves representing $K$.

Part (3) follows from Lemma \[91.1\] and Topologies, Lemma \[4.18\] on choosing a K-injective complex of abelian sheaves representing $K$.

Part (5) follows from Cohomology on Sites, Lemmas \[8.1\] and \[21.1\] and Topologies, Lemma \[4.18\] on choosing a K-injective complex of abelian sheaves representing $K$.

Part (6): Observe that $g_{\text{big}}$ and $g'_{\text{big}}$ are localizations and hence $g_{\text{big}}^{-1} = g_{\text{big}}$ and $(g_{\text{big}})^{-1} = (g_{\text{big}})^*$ are the restriction functors. Hence (6) follows from Cohomology on Sites, Lemmas \[8.1\] and \[21.1\] and Topologies, Lemma \[4.18\] on choosing a K-injective complex of modules representing $K$.

Part (2) can be proved as follows. Above we have seen that $\pi_S \circ f_{\text{big}} = f_{\text{small}} \circ \pi_T$ as morphisms of ringed sites. Hence we obtain $R\pi_S^* \circ Rf_{\text{big}} = Rf_{\text{small}} \circ R\pi_T^*$.
Let \(0DDH\). \(\square\) Since the restriction functors \(\pi_{S,*}\) and \(\pi_{T,*}\) are exact, we conclude.

Part (4) follows from part (6) and part (2) applied to \(f' : T' \to S'\).

Let \(S\) be a scheme and let \(\mathcal{H}\) be an abelian sheaf on \((\text{Sch}/S)_{\text{ét}}\). Recall that \(H^0_{\text{ét}}(U, \mathcal{H})\) denotes the cohomology of \(\mathcal{H}\) over an object \(U\) of \((\text{Sch}/S)_{\text{ét}}\).

**Lemma 91.3.** Let \(f : T \to S\) be a morphism of schemes. Then

1. For \(K \in D(S_{\text{ét}})\) we have \(H^n_{\text{ét}}(S, \pi_{S}^{-1}K) = H^n(S_{\text{ét}}, K)\).
2. For \(K \in D(S_{\text{ét}}, \mathcal{O}\_S)\) we have \(H^{n}_{\text{ét}}(S, L\pi_{S}^{*}K) = H^{n}(S_{\text{ét}}, K)\).
3. For \(K \in D(S_{\text{ét}})\) we have \(H^{n}_{\text{ét}}(T, \pi_{S}^{-1}K) = H^{n}(T_{\text{ét}}, f^{-1}_{\text{small}}K)\).
4. For \(K \in D(S_{\text{ét}}, \mathcal{O}\_S)\) we have \(H^{n}_{\text{ét}}(T, L\pi_{S}^{*}K) = H^{n}(T_{\text{ét}}, Lf^{*}_{\text{small}}K)\).
5. For \(M \in D((\text{Sch}/S)_{\text{ét}})\) we have \(H^{n}_{\text{ét}}(T, M) = H^{n}(T_{\text{ét}}, i^{1}_{f}M)\).
6. For \(M \in D((\text{Sch}/S)_{\text{ét}}, \mathcal{O})\) we have \(H^{n}_{\text{ét}}(T, M) = H^{n}(T_{\text{ét}}, i^{*}_{f}M)\).

**Proof.** To prove (5) represent \(M\) by a \(K\)-injective complex of abelian sheaves and apply Lemma 91.1 and work out the definitions. Part (3) follows from this as \(i^{1}_{f} \pi_{S}^{-1} = f^{*}_{\text{small}}\). Part (1) is a special case of (3).

Part (6) follows from the very general Cohomology on Sites, Lemma 36.5. Then part (4) follows because \(Lf^{*}_{\text{small}} = i^{*}_{f} \circ L\pi_{S}^{*}\). Part (2) is a special case of (4). \(\square\)

**Lemma 91.4.** Let \(S\) be a scheme. For \(K \in D(S_{\text{ét}})\) the map

\[K \to R\pi_{S,*}\pi_{S}^{-1}K\]

is an isomorphism.

**Proof.** This is true because both \(\pi_{S}^{-1}\) and \(\pi_{S,*} = i^{1}_{S}\) are exact functors and the composition \(\pi_{S,*} \circ \pi_{S}^{-1}\) is the identity functor. \(\square\)

**Lemma 91.5.** Let \(f : T \to S\) be a proper morphism of schemes. Then we have

1. \(\pi_{S}^{-1} \circ f^{*}_{\text{small,*}} = f^{*}_{\text{big,*}} \circ \pi_{T}^{-1}\) as functors \(\text{Sh}(T_{\text{ét}}) \to \text{Sh}(S_{\text{ét}})\).
2. \(\pi_{S}^{-1} Rf^{*}_{\text{small,*}} K = Rf^{*}_{\text{big,*}} \pi_{T}^{-1} K\) for \(K \in D^{+}(T_{\text{ét}})\) whose cohomology sheaves are torsion, and
3. \(\pi_{S}^{-1} Rf^{*}_{\text{small,*}} K = Rf^{*}_{\text{big,*}} \pi_{T}^{-1} K\) for all \(K \in D(T_{\text{ét}})\) if \(f\) is finite.

**Proof.** Proof of (1). Let \(\mathcal{F}\) be a sheaf on \(T_{\text{ét}}\). Let \(g : S' \to S\) be an object of \((\text{Sch}/S)_{\text{ét}}\). Consider the fibre product

\[
\begin{array}{ccc}
T' & \xrightarrow{f'} & S' \\
\downarrow{g'} & & \downarrow{g} \\
T & \xrightarrow{f} & S
\end{array}
\]

Then we have

\((f_{\text{big,*}} \pi_{T}^{-1} \mathcal{F})(S') = (\pi_{S}^{-1} \mathcal{F})(T') = ((g')_{\text{small}})^{-1} \mathcal{F})(T') = (f_{\text{small,*}} \pi_{T}^{-1} \mathcal{F})(S')\)

the second equality by Lemma 39.2. On the other hand

\((\pi_{S}^{-1} f_{\text{small,*}} \mathcal{F})(S') = (g_{\text{small}})^{-1} \mathcal{F})(S')\)

again by Lemma 39.2. Hence by proper base change for sheaves of sets (Lemma 85.5) we conclude the two sets are canonically isomorphic. The isomorphism is
compatible with restriction mappings and defines an isomorphism \( \pi_S^{-1} f_{\text{small},*} \mathcal{F} = f_{\text{big},*} \pi_T^{-1} \mathcal{F} \). Thus an isomorphism of functors \( \pi_S^{-1} \circ f_{\text{small},*} = f_{\text{big},*} \circ \pi_T^{-1} \).

Proof of (2). There is a canonical base change map \( \pi_S^{-1} Rf_{\text{small},*} K \to Rf_{\text{big},*} \pi_T^{-1} K \) for any \( K \) in \( D(T_{\text{étale}}) \), see Cohomology on Sites, Remark [20.3]. To prove it is an isomorphism, it suffices to prove the pull back of the base change map by \( i_g : Sh(S'_{\text{étale}}) \to Sh((Sch/S)_{\text{étale}}) \) is an isomorphism for any object \( g : S' \to S \) of \( (Sch/S)_{\text{étale}} \). Let \( T', g', f' \) be as in the previous paragraph. The pullback of the base change map is

\[
g_{\text{small}}^{-1} Rf_{\text{small},*} K = i_g^{-1} \pi_S^{-1} Rf_{\text{small},*} K
\]

\[
\to i_g^{-1} Rf_{\text{big},*} \pi_T^{-1} K
\]

\[
= Rf'_{\text{small},*}(i_g^{-1} \pi_T^{-1} K)
\]

\[
= Rf'_{\text{small},*}((g')_{\text{small}})^{-1} K
\]

where we have used \( \pi_S \circ i_g = g_{\text{small}} \), \( \pi_T \circ i_{g'} = (g')_{\text{small}} \), and Lemma [91.2]. This map is an isomorphism by the proper base change theorem (Lemma [85.12]) provided \( K \) is bounded below and the cohomology sheaves of \( K \) are torsion.

Proof of (3). If \( f \) is finite, then the functors \( f_{\text{small},*} \) and \( f_{\text{big},*} \) are exact. This follows from Proposition [54.2] for \( f_{\text{small}} \). Since any base change \( f' \) of \( f \) is finite too, we conclude from Lemma [91.2] part (3) that \( f_{\text{big},*} \) is exact too (as the higher derived functors are zero). Thus this case follows from part (1).

### 92. Comparing fppf and étale topologies

0DDK A model for this section is the section on the comparison of the usual topology and the qc topology on locally compact topological spaces as discussed in Cohomology on Sites, Section [31]. We first review some material from Topologies, Sections [11] and [4].

Let \( S \) be a scheme and let \( (Sch/S)_{\text{fppf}} \) be an fppf site. On the same underlying category we have a second topology, namely the étale topology, and hence a second site \( (Sch/S)_{\text{étale}} \). The identity functor \( (Sch/S)_{\text{étale}} \to (Sch/S)_{\text{fppf}} \) is continuous and defines a morphism of sites

\[ \epsilon_S : (Sch/S)_{\text{fppf}} \to (Sch/S)_{\text{étale}} \]

See Cohomology on Sites, Section [27]. Please note that \( \epsilon_S \) is the identity functor on underlying presheaves and that \( \epsilon_S^{-1} \) associates to an étale sheaf the fppf sheafification. Let \( S_{\text{étale}} \) be the small étale site. There is a morphism of sites

\[ \pi_S : (Sch/S)_{\text{étale}} \to S_{\text{étale}} \]

given by the continuous functor \( S_{\text{étale}} \to (Sch/S)_{\text{étale}}, U \mapsto U \). Namely, \( S_{\text{étale}} \) has fibre products and a final object and the functor above commutes with these and Sites, Proposition [14.4] applies.

0DDL Lemma 92.1. With notation as above. Let \( \mathcal{F} \) be a sheaf on \( S_{\text{étale}} \). The rule

\[ (Sch/S)_{\text{fppf}} \to \text{Sets, } (f : X \to S) \mapsto \Gamma(X, f^{-1}_{\text{small}} \mathcal{F}) \]

is a sheaf and a fortiori a sheaf on \( (Sch/S)_{\text{étale}} \). In fact this sheaf is equal to \( \pi_S^{-1} \mathcal{F} \) on \( (Sch/S)_{\text{étale}} \) and \( \epsilon_S^{-1} \pi_S^{-1} \mathcal{F} \) on \( (Sch/S)_{\text{fppf}} \).
Proof. The statement about the étale topology is the content of Lemma 39.2. To finish the proof it suffices to show that $\pi^{-1}_S F$ is a sheaf for the fppf topology. This is shown in Lemma 39.2 as well.

In the situation of Lemma 92.1 the composition of $\epsilon_S$ and $\pi_S$ and the equality determine a morphism of sites

$$a_S : (\text{Sch}/S)_{\text{fppf}} \rightarrow S_{\text{étale}}$$

Lemma 92.2. With notation as above. Let $f : X \rightarrow Y$ be a morphism of $(\text{Sch}/S)_{\text{fppf}}$. Then there are commutative diagrams of topoi

$$\begin{array}{ccc}
\text{Sh}((\text{Sch}/X)_{\text{fppf}}) & \xrightarrow{\epsilon_X} & \text{Sh}((\text{Sch}/Y)_{\text{fppf}}) \\
\downarrow f_{\text{big},\text{fppf}} & & \downarrow \epsilon_Y \\
\text{Sh}((\text{Sch}/X)_{\text{étale}}) & \xrightarrow{f_{\text{big},\text{étale}}} & \text{Sh}((\text{Sch}/Y)_{\text{étale}})
\end{array}$$

and

$$\begin{array}{ccc}
\text{Sh}((\text{Sch}/X)_{\text{fppf}}) & \xrightarrow{a_X} & \text{Sh}(\text{X}_{\text{étale}}) \\
\downarrow f_{\text{big},\text{fppf}} & & \downarrow \alpha_Y \\
\text{Sh}((\text{Sch}/Y)_{\text{fppf}}) & \xrightarrow{f_{\text{big},\text{étale}}} & \text{Sh}(\text{Y}_{\text{étale}})
\end{array}$$

with $a_X = \pi_X \circ \epsilon_X$ and $a_Y = \pi_Y \circ \epsilon_Y$.

Proof. The commutativity of the diagrams follows from the discussion in Topologies, Section 11.

Lemma 92.3. In Lemma 92.2 if $f$ is proper, then we have $a_Y^{-1} \circ f_{\text{small},*} = f_{\text{big},\text{fppf},*} \circ a_X^{-1}$.

Proof. You can prove this by repeating the proof of Lemma 91.5 part (1); we will instead deduce the result from this. As $\epsilon_Y,*$ is the identity functor on underlying presheaves, it reflects isomorphisms. The description in Lemma 92.1 shows that $\epsilon_Y,* \circ a_Y^{-1} = \pi_Y^{-1}$ and similarly for $X$. To show that the canonical map $a_Y^{-1} f_{\text{small},*} F \rightarrow f_{\text{big},\text{fppf},*} a_X^{-1} F$ is an isomorphism, it suffices to show that

$$\pi_Y^{-1} f_{\text{small},*} F = \epsilon_Y,* a_Y^{-1} f_{\text{small},*} F \rightarrow \epsilon_Y,* f_{\text{big},\text{fppf},*} a_X^{-1} F = f_{\text{big},\text{étale},*} \epsilon_X, * a_X^{-1} F = f_{\text{big},\text{étale},*} \pi_X^{-1} F$$

is an isomorphism. This is part (1) of Lemma 91.5.

Lemma 92.4. In Lemma 92.2 assume $f$ is flat, locally of finite presentation, and surjective. Then the functor

$$\text{Sh}(\text{Y}_{\text{étale}}) \rightarrow \left\{ (G, H, \alpha) \mid G \in \text{Sh}(\text{X}_{\text{étale}}), H \in \text{Sh}((\text{Sch}/Y)_{\text{fppf}}), \alpha : a_X^{-1} G \rightarrow f_{\text{big},\text{fppf}}^{-1} H \text{ an isomorphism} \right\}$$

sending $F$ to $(f_{\text{small}}^{-1} F, a_Y^{-1} F, \text{can})$ is an equivalence.
Proof. The functor $a_X^{-1}$ is fully faithful (as $a_X, a_X^{-1} = \text{id}$ by Lemma 92.1). Hence the forgetful functor $(\mathcal{G}, \mathcal{H}, \alpha) \mapsto \mathcal{H}$ identifies the category of triples with a full subcategory of $Sh((Sch/Y)_{fppf})$. Moreover, the functor $a_Y^{-1}$ is fully faithful, hence the functor in the lemma is fully faithful as well.

Suppose that we have an étale covering $\{Y_i \to Y\}$. Let $f_i : X_i \to Y_i$ be the base change of $f$. Denote $f_{ij} = f_i \times f_j : X_i \times_X X_j \to Y_i \times_Y Y_j$. Claim: if the lemma is true for $f_i$ and $f_{ij}$ for all $i, j$, then the lemma is true for $f$. To see this, note that the given étale covering determines an étale covering of the final object in each of the four sites $Y_{\text{étale}}, X_{\text{étale}}, (Sch/Y)_{fppf}, (Sch/X)_{fppf}$. Thus the category of sheaves is equivalent to the category of glueing data for this covering (Sites, Lemma 26.5) in each of the four cases. A huge commutative diagram of categories then finishes the proof of the claim. We omit the details. The claim shows that we may work étale locally on $Y$.

Note that $\{X \to Y\}$ is an fpqc covering. Working étale locally on $Y$, we may assume there exists a morphism $s : X' \to X$ such that the composition $f' = f \circ s : X' \to Y$ is surjective finite locally free, see More on Morphisms, Lemma 41.1. Claim: if the lemma is true for $f'$, then it is true for $f$. Namely, given a triple $(\mathcal{G}, \mathcal{H}, \alpha)$ for $f$, we can pullback by $s$ to get a triple $(s_{\text{small}}^{-1}\mathcal{G}, \mathcal{H}, s_{\text{big}, fppf}^{-1}\alpha)$ for $f'$. A solution for this triple gives a sheaf $\mathcal{F}$ on $Y_{\text{étale}}$ with $a_Y^{-1}\mathcal{F} = \mathcal{H}$. By the first paragraph of the proof this means the triple is in the essential image. This reduces us to the case described in the next paragraph.

Assume $f$ is surjective finite locally free. Let $(\mathcal{G}, \mathcal{H}, \alpha)$ be a triple. In this case consider the triple

$$(\mathcal{G}_1, \mathcal{H}_1, \alpha_1) = (f^{-1}_{\text{small}} f_{\text{small}}, \mathcal{G}, f_{\text{big}, fppf}, a_X^{-1} f_{\text{big}, fppf} \mathcal{H}, \alpha_1)$$

where $\alpha_1$ comes from the identifications

$$a_X^{-1} f^{-1}_{\text{small}} f_{\text{small}}, \mathcal{G} = f^{-1}_{\text{big}, fppf} a_Y^{-1} f_{\text{small}}, \mathcal{G}$$
$$= f^{-1}_{\text{big}, fppf} f_{\text{big}, fppf}, a_X^{-1} \mathcal{G}$$
$$\to f^{-1}_{\text{big}, fppf} f_{\text{big}, fppf}, \mathcal{H}$$

where the third equality is Lemma 92.3 and the arrow is given by $\alpha$. This triple is in the image of our functor because $\mathcal{F}_1 = f_{\text{small}}, \mathcal{F}$ is a solution (to see this use Lemma 92.3 again; details omitted). There is a canonical map of triples

$$(\mathcal{G}, \mathcal{H}, \alpha) \to (\mathcal{G}_1, \mathcal{H}_1, \alpha_1)$$

which uses the unit id $\to f_{\text{big}, fppf}, f^{-1}_{\text{big}, fppf}$ on the second entry (it is enough to prescribe morphisms on the second entry by the first paragraph of the proof). Since $\{f : X \to Y\}$ is an fpqc covering the map $\mathcal{H} \to \mathcal{H}_1$ is injective (details omitted). Set

$$\mathcal{G}_2 = \mathcal{G}_1 \amalg \mathcal{G}_1 \quad \mathcal{H}_2 = \mathcal{H}_1 \amalg \mathcal{H} \mathcal{H}_1$$

and let $\alpha_2$ be the induced isomorphism (pullback functors are exact, so this makes sense). Then $\mathcal{H}$ is the equalizer of the two maps $\mathcal{H}_1 \to \mathcal{H}_2$. Repeating the arguments above for the triple $(\mathcal{G}_2, \mathcal{H}_2, \alpha_2)$ we find an injective morphism of triples

$$(\mathcal{G}_2, \mathcal{H}_2, \alpha_2) \to (\mathcal{G}_3, \mathcal{H}_3, \alpha_3)$$

such that this last triple is in the image of our functor. Say it corresponds to $\mathcal{F}_3$ in $Sh(Y_{\text{étale}})$. By fully faithfulness we obtain two maps $\mathcal{F}_1 \to \mathcal{F}_3$ and we can let $\mathcal{F}$
be the equalizer of these two maps. By exactness of the pullback functors involved we find that \( a_X^{-1}F = H \) as desired. \( \square \)

**Lemma 92.5.** Consider the comparison morphism \( \epsilon : (\mathcal{S}/S)_{fppf} \to (\mathcal{S}/S)_{\text{étale}} \). Let \( \mathcal{P} \) denote the class of finite morphisms of schemes. For \( X \) in \((\mathcal{S}/S)_{\text{étale}}\) denote \( \mathcal{A}_X' \subset \text{Ab}(\mathcal{S}/X)_{\text{étale}} \) the full subcategory consisting of sheaves of the form \( \pi_X^{-1}F \) with \( F \) in \( \text{Ab}(X_{\text{étale}}) \). Then Cohomology on Sites, Properties (1), (2), (3), (4), and (5) of Cohomology on Sites, Situation 30.1 hold.

**Proof.** We first show that \( \mathcal{A}_X' \subset \text{Ab}(\mathcal{S}/X)_{\text{étale}} \) is a weak Serre subcategory by checking conditions (1), (2), (3), and (4) of Homology, Lemma 9.3. Parts (1), (2), (3) are immediate as \( \pi_X^{-1} \) is exact and fully faithful for example by Lemma 91.4. If \( 0 \to \pi_X^{-1}F \to G \to \pi_X^{-1}F' \to 0 \) is a short exact sequence in \( \text{Ab}(\mathcal{S}/X)_{\text{étale}} \) then \( 0 \to F \to \pi_X_*G \to \pi_X_*F' \to 0 \) is exact by Lemma 91.4. Hence \( \mathcal{G} = \pi_X^{-1}\pi_X_*G \) is in \( \mathcal{A}_X' \) which checks the final condition.

Cohomology on Sites, Property (1) holds by the existence of fibre products of schemes and the fact that the base change of a finite morphism of schemes is a finite morphism of schemes, see Morphisms, Lemma 42.6.

Cohomology on Sites, Property (2) follows from the commutative diagram (3) in finite morphism of schemes, see Morphisms, Lemma 42.6.

Cohomology on Sites, Property (3) is Lemma 92.1.

Cohomology on Sites, Property (4) holds by Lemma 91.5 part (3).

Cohomology on Sites, Property (5) is implied by More on Morphisms, Lemma 41.1. \( \square \)

**Lemma 92.6.** With notation as above.

1. For \( X \in \text{Ob}((\mathcal{S}/S)_{fppf}) \) and an abelian sheaf \( F \) on \( X_{\text{étale}} \) we have \( \epsilon_{X,*} a_X^{-1}F = \pi_X^{-1}F \) and \( R^i \epsilon_{X,*}(a_X^{-1}F) = 0 \) for \( i > 0 \).
2. For a finite morphism \( f : X \to Y \) in \((\mathcal{S}/S)_{fppf}\) and abelian sheaf \( F \) on \( X \) we have \( a_Y^{-1}(Rf_{small,*}F) = Rf_{big,fppf,*}(a_X^{-1}F) \) for all \( i \).
3. For a scheme \( X \) and \( K \) in \( D^+(X_{\text{étale}}) \) the map \( \pi_X^{-1}K \to R\epsilon_{X,*}(a_X^{-1}K) \) is an isomorphism.
4. For a finite morphism \( f : X \to Y \) of schemes and \( K \) in \( D^+(X_{\text{étale}}) \) we have \( a_Y^{-1}(Rf_{small,*}K) = Rf_{big,fppf,*}(a_X^{-1}K) \).
5. For a proper morphism \( f : X \to Y \) of schemes and \( K \) in \( D^+(X_{\text{étale}}) \) with torsion cohomology sheaves we have \( a_Y^{-1}(Rf_{small,*}K) = Rf_{big,fppf,*}(a_X^{-1}K) \).

**Proof.** By Lemma 92.5 the lemmas in Cohomology on Sites, Section 30 all apply to our current setting. To translate the results observe that the category \( \mathcal{A}_X \) of Cohomology on Sites, Lemma 30.2 is the essential image of \( a_X^{-1} : \text{Ab}(X_{\text{étale}}) \to \text{Ab}((\mathcal{S}/X)_{fppf}) \).

Part (1) is equivalent to \((V_n)\) for all \( n \) which holds by Cohomology on Sites, Lemma 30.8.

Part (2) follows by applying \( \epsilon_Y^{-1} \) to the conclusion of Cohomology on Sites, Lemma 30.5.

Part (3) follows from Cohomology on Sites, Lemma 30.8 part (1) because \( \pi_X^{-1}K \) is in \( D^+_{\mathcal{A}_X}((\mathcal{S}/X)_{\text{étale}}) \) and \( a_X^{-1} = \epsilon_X^{-1} \circ a_X^{-1} \). 

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Part (4) follows from Cohomology on Sites, Lemma 30.8 part (2) for the same reason.

Part (5). We use that

$$R\epsilon_{Y,*}Rf_{big, fppf,*}a_X^{-1}K = Rf_{big, \acute{e}tale,*}R\epsilon_{X,*}a_X^{-1}K$$

$$= Rf_{big, \acute{e}tale,*}\pi_X^{-1}K$$

$$= \pi_Y^{-1}Rf_{small,*}K$$

$$= R\epsilon_{Y,*}a_Y^{-1}Rf_{small,*}K$$

The first equality by the commutative diagram in Lemma 92.2 and Cohomology on Sites,Lemma 92.8. The second equality is (3). The third is Lemma 92.7 part (2). The fourth is (3) again. Thus the base change map

$$a_Y^{-1}(Rf_{small,*}K) \rightarrow Rf_{big, fppf,*}(a_X^{-1}K)$$

induces an isomorphism

$$R\epsilon_{Y,*}a_Y^{-1}Rf_{small,*}K \rightarrow R\epsilon_{Y,*}Rf_{big, fppf,*}a_X^{-1}K$$

The proof is finished by the following remark: a map $\alpha : a_Y^{-1}L \rightarrow M$ with $L$ in $D^+(Y_{\acute{e}tale})$ and $M$ in $D^+(\text{Sch}/Y)_{fppf}$ such that $R\epsilon_{Y,*}\alpha$ is an isomorphism, is an isomorphism. Namely, we show by induction on $i$ that $H^i(\alpha)$ is an isomorphism. This is true for all sufficiently small $i$. If it holds for $i \leq i_0$, then we see that $R^i\epsilon_{Y,*}H^j(M) = 0$ for $j > 0$ and $i \leq i_0$ by (1) because $H^i(M) = a_Y^{-1}H^i(L)$ in this range. Hence $\epsilon_{Y,*}H^{i+1}(M) = H^i(M) = a_Y^{-1}H^i(L)$ by a spectral sequence argument. Thus $\epsilon_{Y,*}H^{i+1}(M) = \pi_Y^{-1}H^{i+1}(L) = \epsilon_{Y,*}a_Y^{-1}H^{i+1}(L)$. This implies $H^{i+1}(\alpha)$ is an isomorphism (because $\epsilon_{Y,*}$ reflects isomorphisms as it is the identity on underlying presheaves) as desired.

**Lemma 92.7.** Let $X$ be a scheme. For $K \in D^+(X_{\acute{e}tale})$ the map

$$K \rightarrow R\pi_{X,*}a_X^{-1}K$$

is an isomorphism with $a_X : \text{Sh}(\text{Sch}/X)_{fppf} \rightarrow \text{Sh}(X_{\acute{e}tale})$ as above.

**Proof.** We first reduce the statement to the case where $K$ is given by a single abelian sheaf. Namely, represent $K$ by a bounded below complex $F^*$. By the case of a sheaf we see that $F^q = a_X^{-1}F^q$ and that the sheaves $R^ia_X^{-1}F^q$ are zero for $q > 0$. By Leray’s acyclicity lemma (Derived Categories, Lemma 17.7) applied to $a_X^{-1}F^*$ and the functor $a_X^{-1}$ we conclude. From now on assume $K = F$.

By Lemma 92.1 we have $a_X^{-1}a_X^{-1}F = F$. Thus it suffices to show that $R^q a_X^{-1}F = 0$ for $q > 0$. For this we can use $a_X = \epsilon_X \circ \pi_X$ and the Leray spectral sequence (Cohomology on Sites, Lemma 15.7). By Lemma 92.6 we have $R^q \epsilon_{X,*}(a_X^{-1}F) = 0$ for $i > 0$ and $\epsilon_{X,*}a_X^{-1}F = \pi_X^{-1}F$. By Lemma 91.4 we have $R^q \pi_X^{-1}(\pi_X^{-1}F) = 0$ for $j > 0$. This concludes the proof.

**Lemma 92.8.** For a scheme $X$ and $a_X : \text{Sh}(\text{Sch}/X)_{fppf} \rightarrow \text{Sh}(X_{\acute{e}tale})$ as above:

1. $H^q(X_{\acute{e}tale}, F) = H^q_{fppf}(X, a_X^{-1}F)$ for an abelian sheaf $F$ on $X_{\acute{e}tale}$,
2. $H^q(X_{\acute{e}tale}, K) = H^q_{fppf}(X, a_X^{-1}K)$ for $K \in D^+(X_{\acute{e}tale})$.

**Example:** if $A$ is an abelian group, then $H^q_{\acute{e}tale}(X, A) = H^q_{fppf}(X, A)$.

**Proof.** This follows from Lemma 92.7 by Cohomology on Sites, Remark 15.4.

\[ \square \]
93. Comparing fppf and étale topologies: modules

0DEV We continue the discussion in Section 92 but in this section we briefly discuss what happens for sheaves of modules.

Let $S$ be a scheme. The morphisms of sites $\epsilon_S$, $\pi_S$, and their composition $a_S$ introduced in Section 92 have natural enhancements to morphisms of ringed sites. The first is written as

$$\epsilon_S : ((\text{Sch}/S)_{\text{fppf}}, O) \longrightarrow ((\text{Sch}/S)_{\text{étale}}, O)$$

Note that we can use the same symbol for the structure sheaf as indeed the sheaves have the same underlying presheaf. The second is

$$\pi_S : ((\text{Sch}/S)_{\text{étale}}, O) \longrightarrow (S_{\text{étale}}, O_S)$$

The third is the morphism

$$a_S : ((\text{Sch}/S)_{\text{fppf}}, O) \longrightarrow (S_{\text{étale}}, O_S)$$

We already know that the category of quasi-coherent modules on the scheme $S$ is the same as the category of quasi-coherent modules on $(S_{\text{étale}}, O_S)$, see Descent, Proposition 8.11. Since we are interested in stating a comparison between étale and fppf cohomology, we will in the rest of this section think of quasi-coherent sheaves in terms of the small étale site. Let us review what we already know about quasi-coherent modules on these sites.

**Lemma 93.1.** Let $S$ be a scheme. Let $F$ be a quasi-coherent $O_S$-module on $S_{\text{étale}}$.

1. The rule

$$F^a : (\text{Sch}/S)_{\text{étale}} \longrightarrow \text{Ab}, \quad (f : T \rightarrow S) \longmapsto \Gamma(T, f^* F)$$

satisfies the sheaf condition for fppf and a fortiori étale coverings,

2. $F^a = \pi_S^* F$ on $(\text{Sch}/S)_{\text{étale}},$

3. $F^a = a_S^* F$ on $(\text{Sch}/S)_{\text{fppf}},$

4. the rule $F \mapsto F^a$ defines an equivalence between quasi-coherent $O_S$-modules and quasi-coherent modules on $((\text{Sch}/S)_{\text{étale}}, O),$

5. the rule $F \mapsto F^a$ defines an equivalence between quasi-coherent $O_S$-modules and quasi-coherent modules on $((\text{Sch}/S)_{\text{fppf}}, O),$

6. we have $\epsilon_S_* a^*_S F = \pi^*_S F$ and $a^*_S a^*_S F = F,$

7. we have $R^i \epsilon_S_* (a^*_S F) = 0$ and $R^i a^*_S a^*_S (a^*_S F) = 0$ for $i > 0.$

**Proof.** We urge the reader to find their own proof of these results based on the material in Descent, Section 8.

We first explain why the notation in this lemma is consistent with our earlier use of the notation $F^a$ in Sections 17 and 22 and in Descent, Section 8. Namely, we know by Descent, Proposition 8.11 that there exists a quasi-coherent module $F_0$ on the scheme $S$ (in other words on the small Zariski site) such that $F$ is the restriction of the rule

$$F^a_0 : (\text{Sch}/S)_{\text{étale}} \longrightarrow \text{Ab}, \quad (f : T \rightarrow S) \longmapsto \Gamma(T, f^* F)$$

to the subcategory $S_{\text{étale}} \subset (\text{Sch}/S)_{\text{étale}}$ where here $f^*$ denotes usual pullback of sheaves of modules on schemes. Since $F^a_0$ is pullback by the morphism of ringed sites

$$((\text{Sch}/S)_{\text{étale}}, O) \longrightarrow (S_{\text{Zar}}, O_{S_{\text{Zar}}})$$
by Descent, Remark 8.6 it follows immediately (from composition of pullbacks) that $F^a = F^a_0$. This proves the sheaf property even for fqc coverings by Descent, Lemma 8.1 (see also Proposition 17.1). Then (2) and (3) follow again by Descent, Remark 8.6 and (4) and (5) follow from Descent, Proposition 8.11 (see also the meta result Theorem 17.4).

Part (6) is immediate from the description of the sheaf $F^a = \pi_\ast^S F = a_\ast^S F$.

For any abelian $H$ on $(\text{Sch}/S)_{fppf}$ the higher direct image $R^p \epsilon_\ast S, \ast H$ is the sheaf associated to the presheaf $U \mapsto H^p_{fppf}(U, H)$ on $(\text{Sch}/S)_{\text{étale}}$. See Cohomology on Sites, Lemma 8.4. Hence to prove $R^p a_\ast^S S, \ast F^a = 0$ for $p > 0$ it suffices to show that any scheme $U$ over $S$ has an étale covering $\{U_i \to U\}_{i \in I}$ such that $H^p_{fppf}(U_i, F^a) = 0$ for $p > 0$. If we take an open covering by affines, then the required vanishing follows from comparison with usual cohomology (Descent, Proposition 8.10 or Theorem 22.4) and the vanishing of cohomology of quasi-coherent sheaves on affine schemes afforded by Cohomology of Schemes, Lemma 2.2.

To show that $R^p a_\ast^S S, \ast 1 F^a = R^p a_\ast^S F^a = 0$ for $p > 0$ we argue in exactly the same manner. This finishes the proof. □

**Lemma 93.2.** Let $S$ be a scheme. For $F$ a quasi-coherent $O_S$-module on $S_{\text{étale}}$ the maps

$$\pi_\ast^S F \to R\epsilon_\ast S, \ast a_\ast^S F \quad \text{and} \quad F \to R\epsilon_\ast S, \ast 1 a_\ast^S F$$

are isomorphisms with $a_\ast : \text{Sh}((\text{Sch}/S)_{fppf}) \to \text{Sh}(S_{\text{étale}})$ as above.

**Proof.** This is an immediate consequence of parts (6) and (7) of Lemma 93.1 □

94. Comparing ph and étale topologies

A model for this section is the section on the comparison of the usual topology and the qc topology on locally compact topological spaces as discussed in Cohomology on Sites, Section 31. We first review some material from Topologies, Sections 11 and 4.

Let $S$ be a scheme and let $(\text{Sch}/S)_{ph}$ be a ph site. On the same underlying category with have a second topology, namely the étale topology, and hence a second site $(\text{Sch}/S)_{\text{étale}}$. The identity functor $(\text{Sch}/S)_{\text{étale}} \to (\text{Sch}/S)_{ph}$ is continuous (by More on Morphisms, Lemma 41.7 and Topologies, Lemma 7.2) and defines a morphism of sites

$$\epsilon_S : (\text{Sch}/S)_{ph} \to (\text{Sch}/S)_{\text{étale}}$$

See Cohomology on Sites, Section 27. Please note that $\epsilon_\ast S, \ast$ is the identity functor on underlying presheaves and that $\epsilon_\ast S, \ast$ associates to an étale sheaf the ph sheafification. Let $S_{\text{étale}}$ be the small étale site. There is a morphism of sites

$$\pi_\ast : (\text{Sch}/S)_{\text{étale}} \to S_{\text{étale}}$$

given by the continuous functor $S_{\text{étale}} \to (\text{Sch}/S)_{\text{étale}}, U \mapsto U$. Namely, $S_{\text{étale}}$ has fibre products and a final object and the functor above commutes with these and Sites, Proposition 14.7 applies.

**Lemma 94.1.** With notation as above. Let $F$ be a sheaf on $S_{\text{étale}}$. The rule

$$(\text{Sch}/S)_{ph} \to \text{Sets}, \quad (f : X \to S) \mapsto \Gamma(X, f_\ast^{-1} F)$$

is an isomorphism of sites.
is a sheaf and a fortiori a sheaf on \((\text{Sch}/S)_{\text{étale}}\). In fact this sheaf is equal to \(\pi^!_SF\) on \((\text{Sch}/S)_{\text{étale}}\) and \(\epsilon^!_S\pi^!_SF\) on \((\text{Sch}/S)_{\text{ph}}\).

**Proof.** The statement about the étale topology is the content of Lemma 39.2. To finish the proof it suffices to show that \(\pi^!_SF\) is a sheaf for the ph topology. By Topologies, Lemma 8.15 it suffices to show that given a proper surjective morphism \(V \to U\) of schemes over \(S\) we have an equalizer diagram

\[
(\pi^!_SF)(U) \longrightarrow (\pi^!_SF)(V) \longrightarrow (\pi^!_SF)(V \times_U V)
\]

Set \(G = \pi^!_SF|_{U_{\text{étale}}}\). Consider the commutative diagram

\[
\begin{array}{ccc}
V \times_U V & \longrightarrow & V \\
\downarrow & & \downarrow \\
V & \longrightarrow & U
\end{array}
\]

We have

\[
(\pi^!_SF)(V) = \Gamma(V, f^{-1}G) = \Gamma(U, f_* f^{-1}G)
\]

where we use \(f_*\) and \(f^{-1}\) to denote functorialities between small étale sites. Second, we have

\[
(\pi^!_SF)(V \times_U V) = \Gamma(V \times_U V, g^{-1}G) = \Gamma(U, g_* g^{-1}G)
\]

The two maps in the equalizer diagram come from the two maps

\[
f_* f^{-1}G \longrightarrow g_* g^{-1}G
\]

Thus it suffices to prove \(G\) is the equalizer of these two maps of sheaves. Let \(\pi\) be a geometric point of \(U\). Set \(\Omega = G_\pi\). Taking stalks at \(\pi\) by Lemma 85.3 we obtain the two maps

\[
H^0(V_\pi, \Omega) \longrightarrow H^0((V \times_U V)_\pi, \Omega) = H^0(V_\pi \times_\pi V_\pi, \Omega)
\]

where \(\Omega\) indicates the constant sheaf with value \(\Omega\). Of course these maps are the pullback by the projection maps. Then it is clear that the sections coming from pullback by projection onto the first factor are constant on the fibres of the first projection, and sections coming from pullback by projection onto the first factor are constant on the fibres of the first projection. The sections in the intersection of the images of these pullback maps are constant on all of \(V_\pi \times_\pi V_\pi\), i.e., these come from elements of \(\Omega\) as desired. \(\square\)

In the situation of Lemma 94.1, the composition of \(\epsilon_S\) and \(\pi_S\) and the equality determine a morphism of sites

\[
a_S : (\text{Sch}/S)_{\text{ph}} \longrightarrow S_{\text{étale}}
\]

0DDX **Lemma 94.2.** With notation as above. Let \(f : X \to Y\) be a morphism of \((\text{Sch}/S)_{\text{ph}}\). Then there are commutative diagrams of topoi

\[
\begin{array}{ccc}
\text{Sh}((\text{Sch}/X)_{\text{ph}}) & \longrightarrow & \text{Sh}((\text{Sch}/Y)_{\text{ph}}) \\
\downarrow \epsilon_X & & \downarrow \epsilon_Y \\
\text{Sh}((\text{Sch}/X)_{\text{étale}}) & \longrightarrow & \text{Sh}((\text{Sch}/Y)_{\text{étale}})
\end{array}
\]
and

\[
\begin{array}{ccc}
\text{Sh}((\text{Sch}/X)_{ph}) & \xrightarrow{f_{\text{big,ph}}} & \text{Sh}((\text{Sch}/Y)_{ph}) \\
an_X & & an_Y \\
\text{Sh}(X_{\text{étale}}) & \xrightarrow{f_{\text{small}}} & \text{Sh}(Y_{\text{étale}})
\end{array}
\]

with \(a_X = \pi_X \circ \epsilon_X\) and \(a_Y = \pi_Y \circ \epsilon_Y\).

**Proof.** The commutativity of the diagrams follows from the discussion in Topologies, Section 11. \(\square\)

**Lemma 94.3.** In Lemma 94.2 if \(f\) is proper, then we have \(a_Y^{-1} \circ f_{\text{small,}\ast} = f_{\text{big,ph,}\ast} \circ a_X^{-1}\).

**Proof.** You can prove this by repeating the proof of Lemma 91.5 part (1); we will instead deduce the result from this. As \(\epsilon_{Y,\ast}\) is the identity functor on underlying presheaves, it reflects isomorphisms. The description in Lemma 94.1 shows that \(\epsilon_{Y,\ast} \circ a_Y^{-1} = \pi_Y^{-1}\) and similarly for \(X\). To show that the canonical map \(a_Y^{-1} f_{\text{small,}\ast} F \rightarrow f_{\text{big,ph,}\ast} a_X^{-1} F\) is an isomorphism, it suffices to show that \(\pi_Y^{-1} f_{\text{small,}\ast} F = \epsilon_Y, \ast a_Y^{-1} f_{\text{small,}\ast} F\) \(\rightarrow \epsilon_Y, \ast f_{\text{big,ph,}\ast} a_X^{-1} F\) \(= f_{\text{big,étale,}\ast} \epsilon_{X,\ast} a_X^{-1} F\) \(= f_{\text{big,étale,}\ast} \pi_X^{-1} F\) is an isomorphism. This is part (1) of Lemma 91.5. \(\square\)

**Lemma 94.4.** Consider the comparison morphism \(\epsilon : (\text{Sch}/S)_{ph} \rightarrow (\text{Sch}/S)_{\text{étale}}\).

Let \(P\) denote the class of proper morphisms of schemes. For \(X\) in \((\text{Sch}/S)_{\text{étale}}\) denote \(A'_X \subset \text{Ab}((\text{Sch}/X)_{\text{étale}})\) the full subcategory consisting of sheaves of the form \(\pi_X^{-1} F\) where \(F\) is a torsion abelian sheaf on \(X_{\text{étale}}\). Then Cohomology on Sites, Properties (1), (2), (3), (4), and (5) of Cohomology on Sites, Situation 30.1 hold.

**Proof.** We first show that \(A'_X \subset \text{Ab}((\text{Sch}/X)_{\text{étale}})\) is a weak Serre subcategory by checking conditions (1), (2), (3), and (4) of Homology, Lemma 9.3. Parts (1), (2), (3) are immediate as \(\pi_X^{-1}\) is exact and fully faithful for example by Lemma 91.1. If \(0 \rightarrow \pi_X^{-1} F \rightarrow G \rightarrow \pi_X^{-1} F' \rightarrow 0\) is a short exact sequence in \(\text{Ab}((\text{Sch}/X)_{\text{étale}})\) then \(0 \rightarrow F \rightarrow \pi_X, \ast G \rightarrow F' \rightarrow 0\) is exact by Lemma 91.1. In particular we see that \(\pi_X, \ast G\) is an abelian torsion sheaf on \(X_{\text{étale}}\). Hence \(G = \pi_X^{-1} \pi_X, \ast G\) is in \(A'_X\) which checks the final condition.

Cohomology on Sites, Property (1) holds by the existence of fibre products of schemes and the fact that the base change of a proper morphism of schemes is a proper morphism of schemes, see Morphisms, Lemma 39.5.

Cohomology on Sites, Property (2) follows from the commutative diagram (3) in Topologies, Lemma 4.16.

Cohomology on Sites, Property (3) is Lemma 94.1.

Cohomology on Sites, Property (4) holds by Lemma 91.5 part (2) and the fact that \(R^i f_{\text{small}} \ast F\) is torsion if \(F\) is an abelian torsion sheaf on \(X_{\text{étale}}\), see Lemma 74.2.
Cohomology on Sites, Property [3] follows from More on Morphisms, Lemma 41.1 combined with the fact that a finite morphism is proper and a surjective proper morphism defines a ph covering, see Topologies, Lemma 8.6.

**Lemma 94.5.** With notation as above.

1. For $X \in \text{Ob}((\text{Sch}/S)_{ph})$ and an abelian torsion sheaf $\mathcal{F}$ on $X_{\text{etale}}$ we have $\epsilon_{X,*}a_X^{-1}\mathcal{F} = \pi_X^{-1}\mathcal{F}$ and $R^i\epsilon_{X,*}(a_X^{-1}\mathcal{F}) = 0$ for $i > 0$.

2. For a proper morphism $f : X \to Y$ in $(\text{Sch}/S)_{ph}$ and abelian torsion sheaf $\mathcal{F}$ on $X$ we have $a_Y^{-1}(R^if_{small,*}\mathcal{F}) = R^if_{big,ph,*}(a_X^{-1}\mathcal{F})$ for all $i$.

3. For a scheme $X$ and $K$ in $D^+(X_{\text{etale}})$ with torsion cohomology sheaves the map $\pi_X^{-1}K \to R\epsilon_{X,*}(a_X^{-1}K)$ is an isomorphism.

4. For a proper morphism $f : X \to Y$ of schemes and $K$ in $D^+(X_{\text{etale}})$ with torsion cohomology sheaves we have $a_Y^{-1}(Rf_{small,*}K) = Rf_{big,ph,*}(a_X^{-1}K)$.

**Proof.** By Lemma 94.4 the lemmas in Cohomology on Sites, Section 30 all apply to our current setting. To translate the results observe that the category $A_X$ of Cohomology on Sites, Lemma 30.2 is the full subcategory of $\text{Ab}((\text{Sch}/X)_{ph})$ consisting of sheaves of the form $a_X^{-1}\mathcal{F}$ where $\mathcal{F}$ is an abelian torsion sheaf on $X_{\text{etale}}$.

Part (1) is equivalent to $(V_n)$ for all $n$ which holds by Cohomology on Sites, Lemma 30.8.

Part (2) follows by applying $\epsilon_Y^{-1}$ to the conclusion of Cohomology on Sites, Lemma 30.3.

Part (3) follows from Cohomology on Sites, Lemma 30.8 part (1) because $\pi_X^{-1}K$ is in $D^+_{A_X}((\text{Sch}/X)_{\text{etale}})$ and $a_X^{-1} = \epsilon_X^{-1} \circ a_X^{-1}$.

Part (4) follows from Cohomology on Sites, Lemma 30.8 part (2) for the same reason.

**Lemma 94.6.** Let $X$ be a scheme. For $K \in D^+(X_{\text{etale}})$ with torsion cohomology sheaves the map $K \to Ra_{X,*}a_X^{-1}K$ is an isomorphism with $a_X : \text{Sh}((\text{Sch}/X)_{ph}) \to \text{Sh}(X_{\text{etale}})$ as above.

**Proof.** We first reduce the statement to the case where $K$ is given by a single abelian sheaf. Namely, represent $K$ by a bounded below complex $\mathcal{F}^\bullet$ of torsion abelian sheaves. This is possible by Cohomology on Sites, Lemma 20.7. By the case of a sheaf we see that $\mathcal{F}^n = a_X^{-1}\mathcal{F}^n$ and that the sheaves $R^i\epsilon_{X,*}a_X^{-1}\mathcal{F}^n$ are zero for $q > 0$. By Leray’s acyclicity lemma (Derived Categories, Lemma 17.1) applied to $a_X^{-1}\mathcal{F}^\bullet$ and the functor $a_X,*$ we conclude. From now on assume $K = \mathcal{F}$ where $\mathcal{F}$ is a torsion abelian sheaf.

By Lemma 94.1 we have $a_X,*a_X^{-1}\mathcal{F} = \mathcal{F}$. Thus it suffices to show that $R^i\epsilon_{X,*}a_X^{-1}\mathcal{F} = 0$ for $q > 0$. For this we can use $a_X = \epsilon_X \circ \pi_X$ and the Leray spectral sequence (Cohomology on Sites, Lemma 15.7). By Lemma 94.5 we have $R^i\epsilon_{X,*}(a_X^{-1}\mathcal{F}) = 0$ for $i > 0$ and $\epsilon_{X,*}a_X^{-1}\mathcal{F} = \pi_X^{-1}\mathcal{F}$. By Lemma 94.1 we have $R^j\pi_{X,*}(\pi_X^{-1}\mathcal{F}) = 0$ for $j > 0$. This concludes the proof.

**Lemma 94.7.** For a scheme $X$ and $a_X : \text{Sh}((\text{Sch}/X)_{ph}) \to \text{Sh}(X_{\text{etale}})$ as above:

1. $H^q(X_{\text{etale}}, \mathcal{F}) = H^q_{ph}(X, a_X^{-1}\mathcal{F})$ for a torsion abelian sheaf $\mathcal{F}$ on $X_{\text{etale}},$
\( H^q(X_{\text{étale}}, K) = H^q_{\text{ph}}(X, a_X^{-1}K) \) for \( K \in D^+(X_{\text{étale}}) \) with torsion cohomology sheaves.

**Example:** if \( A \) is a torsion abelian group, then \( H^q_{\text{étale}}(X, A) = H^q_{\text{ph}}(X, A) \).

**Proof.** This follows from Lemma 94.6 by Cohomology on Sites, Remark 15.4. \( \square \)

### 95. Comparing \( h \) and étale topologies

A model for this section is the section on the comparison of the usual topology and the qc topology on locally compact topological spaces as discussed in Cohomology on Sites, Section 31. Moreover, this section is almost word for word the same as the section comparing the \( \text{ph} \) and étale topologies. We first review some material from Topologies, Sections 11 and 4 and More on Flatness, Section 32.

Let \( S \) be a scheme and let \( (\text{Sch}/S)_h \) be an \( h \) site. On the same underlying category \( (\text{Sch}/S)_{\text{étale}} \), there is a second site \( (\text{Sch}/S)_{\text{étale}} \). The identity functor \( (\text{Sch}/S)_h \to (\text{Sch}/S)_{\text{étale}} \) is continuous (by More on Flatness, Lemma 32.5 and Topologies, Lemma 7.2) and defines a morphism of sites

\[ \epsilon_S : (\text{Sch}/S)_h \to (\text{Sch}/S)_{\text{étale}} \]

See Cohomology on Sites, Section 27. Please note that \( \epsilon_S \), the identity functor on underlying presheaves and that \( \epsilon_S^{-1} \) associates to an étale sheaf the \( h \) sheafification.

Let \( S_{\text{étale}} \) be the small étale site. There is a morphism of sites

\[ \pi_S : (\text{Sch}/S)_{\text{étale}} \to S_{\text{étale}} \]

given by the continuous functor \( S_{\text{étale}} \to (\text{Sch}/S)_{\text{étale}}, U \mapsto U \). Namely, \( S_{\text{étale}} \) has fibre products and a final object and the functor above commutes with these and Sites, Proposition 14.7 applies.

**Lemma 95.1.** With notation as above. Let \( F \) be a sheaf on \( S_{\text{étale}} \). The rule

\[ (\text{Sch}/S)_h \to \text{Sets}, (f : X \to S) \mapsto \Gamma(X, f^{-1}_{\text{small}}F) \]

is a sheaf and a fortiori a sheaf on \( (\text{Sch}/S)_{\text{étale}} \). In fact this sheaf is equal to \( \pi_S^{-1}F \) on \( (\text{Sch}/S)_{\text{étale}} \) and \( \epsilon_S^{-1} \pi_S^{-1}F \) on \( (\text{Sch}/S)_h \).

**Proof.** The statement about the étale topology is the content of Lemma 39.2. To finish the proof it suffices to show that \( \pi_S^{-1}F \) is a sheaf for the \( h \) topology. However, in Lemma 94.1 we have shown that \( \pi_S^{-1}F \) is a sheaf even in the stronger \( \text{ph} \) topology. \( \square \)

In the situation of Lemma 95.1 the composition of \( \epsilon_S \) and \( \pi_S \) and the equality determine a morphism of sites

\[ a_S : (\text{Sch}/S)_h \to S_{\text{étale}} \]

**Lemma 95.2.** With notation as above. Let \( f : X \to Y \) be a morphism of \( (\text{Sch}/S)_h \). Then there are commutative diagrams of topoi

\[
\begin{array}{ccc}
Sh((\text{Sch}/X)_h) & \xrightarrow{f_{\text{rig}, h}} & Sh((\text{Sch}/Y)_h) \\
\epsilon_X & & \epsilon_Y \\
Sh((\text{Sch}/X)_{\text{étale}}) & \xrightarrow{f_{\text{rig, étale}}} & Sh((\text{Sch}/Y)_{\text{étale}})
\end{array}
\]
and

\[ Sh((\text{Sch}/X)_h) \xrightarrow{a_X} Sh((\text{Sch}/Y)_h) \]

\[ Sh(X_{\text{étale}}) \xrightarrow{a_Y} Sh(Y_{\text{étale}}) \]

with \( a_X = \pi_X \circ \epsilon_X \) and \( a_Y = \pi_X \circ \epsilon_X \).

**Proof.** The commutativity of the diagrams follows similarly to what was said in Topologies, Section 11. □

**Lemma 95.3.** In Lemma 95.2 if \( f \) is proper, then we have \( a_Y^{-1} \circ f_{\text{small},*} = f_{\text{big},*} \circ a_X^{-1} \).

**Proof.** You can prove this by repeating the proof of Lemma 91.5 part (1); we will instead deduce the result from this. As \( \epsilon_{Y,*} \) is the identity functor on underlying presheaves, it reflects isomorphisms. The description in Lemma 95.1 shows that \( \epsilon_{Y,*} \circ a_Y^{-1} = \pi_Y^{-1} \) and similarly for \( X \). To show that the canonical map \( a_Y^{-1} f_{\text{small},*} F \to f_{\text{big},*} a_X^{-1} F \) is an isomorphism, it suffices to show that

\[ \pi_Y^{-1} f_{\text{small},*} F = \epsilon_{Y,*} a_Y^{-1} f_{\text{small},*} F \]

\[ = f_{\text{big},*} a_X^{-1} F \]

is an isomorphism. This is part (1) of Lemma 91.5 □

**Lemma 95.4.** Consider the comparison morphism \( \epsilon : (\text{Sch}/S)_h \to (\text{Sch}/S)_{\text{étale}} \). Let \( \mathcal{P} \) denote the class of proper morphisms. For \( X \) in \( (\text{Sch}/S)_{\text{étale}} \) denote \( A'_X \subset Ab((\text{Sch}/X)_{\text{étale}}) \) the full subcategory consisting of sheaves of the form \( \pi_X^{-1} F \) where \( F \) is a torsion abelian sheaf on \( X_{\text{étale}} \) Then Cohomology on Sites, Properties (1), (2), (3), (4), and (5) of Cohomology on Sites, Situation 30.1 hold.

**Proof.** We first show that \( A'_X \subset Ab((\text{Sch}/X)_{\text{étale}}) \) is a weak Serre subcategory by checking conditions (1), (2), (3), and (4) of Homology, Lemma 93.3 Parts (1), (2), (3) are immediate as \( \pi_X^{-1} \) is exact and fully faithful for example by Lemma 91.4. If \( 0 \to F \to G \to \pi_X^{-1} F' \to 0 \) is a short exact sequence in \( Ab((\text{Sch}/X)_{\text{étale}}) \) then \( 0 \to F \to G \to \pi_X^{-1} F' \to 0 \) is exact by Lemma 91.4. In particular we see that \( \pi_X^{-1} G \) is an abelian torsion sheaf on \( X_{\text{étale}} \). Hence \( \mathcal{G} = \pi_X^{-1} \pi_X^{-1} \mathcal{G} \) is in \( A'_X \) which checks the final condition.

Cohomology on Sites, Property (1) holds by the existence of fibre products of schemes, the fact that the base change of a proper morphism of schemes is a proper morphism of schemes, see Morphisms, Lemma 39.5, and the fact that the base change of a morphism of finite presentation is a morphism of finite presentation, see Morphisms, Lemma 20.4.

Cohomology on Sites, Property (2) follows from the commutative diagram (3) in Topologies, Lemma 4.16.

Cohomology on Sites, Property (3) is Lemma 95.1.

Cohomology on Sites, Property (4) holds by Lemma 91.5 part (2) and the fact that \( R^if_{\text{small}} F \) is torsion if \( F \) is an abelian torsion sheaf on \( X_{\text{étale}} \), see Lemma 74.2.
Cohomology on Sites, Property \( \ref{H_{etale}} \) is implied by More on Morphisms, Lemma \( \ref{More on Flat-combined} \) combined with the fact that a surjective finite locally free morphism is surjective, proper, and of finite presentation and hence defines a h-covering by More on Flatness, Lemma \( \ref{More on Flatness} \).

\begin{lemma}
With notation as above.
\begin{enumerate}
\item For \( X \in \text{Ob}((\text{Sch}/S)_h) \) and an abelian torsion sheaf \( \mathcal{F} \) on \( X_{\text{etale}} \) we have \( \epsilon_{X,*} a_{X}^{-1} \mathcal{F} = \pi_X^{-1} \mathcal{F} \) and \( R^i \epsilon_{X,*} (a_{X}^{-1} \mathcal{F}) = 0 \) for \( i > 0 \).
\item For a proper morphism \( f : X \to Y \) in \((\text{Sch}/S)_h\) and abelian torsion sheaf \( \mathcal{F} \) on \( X \) we have \( a_Y^{-1}(R^i f_{\text{small},*} \mathcal{F}) = R^i f_{\text{big},h,*} (a_X^{-1} \mathcal{F}) \) for all \( i \).
\item For a scheme \( X \) and \( K \) in \( D^+(X_{\text{etale}}) \) with torsion cohomology sheaves the map \( \pi_X^{-1} K \to R\epsilon_{X,*}(a_X^{-1}K) \) is an isomorphism.
\item For a proper morphism \( f : X \to Y \) of schemes and \( K \) in \( D^+(X_{\text{etale}}) \) with torsion cohomology sheaves we have \( a_Y^{-1}(Rf_{\text{small},*}K) = Rf_{\text{big},h,*}(a_X^{-1}K) \).
\end{enumerate}
\end{lemma}

\textbf{Proof.} By Lemma \( \ref{More on Flat-combined} \) the lemmas in Cohomology on Sites, Section \( \ref{Cohomology on Sites} \) all apply to our current setting. To translate the results observe that the category \( \mathcal{A}_X \) of Cohomology on Sites, Lemma \( \ref{Cohomology on Sites} \) is the full subcategory of \( \text{Ab}((\text{Sch}/X)_h) \) consisting of sheaves of the form \( a_X^{-1} \mathcal{F} \) where \( \mathcal{F} \) is an abelian torsion sheaf on \( X_{\text{etale}} \).

Part (1) is equivalent to \((V_n)\) for all \( n \) which holds by Cohomology on Sites, Lemma \( \ref{Cohomology on Sites} \).

Part (2) follows by applying \( \epsilon_Y^{-1} \) to the conclusion of Cohomology on Sites, Lemma \( \ref{Cohomology on Sites} \).

Part (3) follows from Cohomology on Sites, Lemma \( \ref{Cohomology on Sites} \) part (1) because \( \pi_X^{-1} K \) is in \( D^+_{\mathcal{A}_X}((\text{Sch}/X)_{\text{etale}}) \) and \( a_X^{-1} = \epsilon_X^{-1} \circ a_X^{-1} \).

Part (4) follows from Cohomology on Sites, Lemma \( \ref{Cohomology on Sites} \) part (2) for the same reason.

\begin{lemma}
Let \( X \) be a scheme. For \( K \in D^+(X_{\text{etale}}) \) with torsion cohomology sheaves the map
\[ K \to R\epsilon_{X,*}a_X^{-1}K \]
is an isomorphism with \( a_X : \text{Sh}((\text{Sch}/X)_h) \to \text{Sh}(X_{\text{etale}}) \) as above.
\end{lemma}

\textbf{Proof.} We first reduce the statement to the case where \( K \) is given by a single abelian sheaf. Namely, represent \( K \) by a bounded below complex \( \mathcal{F}^\bullet \) of torsion abelian sheaves. This is possible by Cohomology on Sites, Lemma \( \ref{Cohomology on Sites} \). By the case of a sheaf we see that \( F^q = a_X^{-1} F^q \) and that the sheaves \( R^i a_X^{-1} F^q \) are zero for \( q > 0 \). By Leray’s acyclicity lemma (Derived Categories, Lemma \( \ref{Derived Categories} \)) applied to \( a_X^{-1} \mathcal{F}^\bullet \) and the functor \( a_X,* \) we conclude. From now on assume \( K = \mathcal{F} \) where \( \mathcal{F} \) is a torsion abelian sheaf.

By Lemma \( \ref{Cohomology on Sites} \) we have \( a_X^{-1} \mathcal{F} = \mathcal{F} \). Thus it suffices to show that \( R^q a_X^{-1} \mathcal{F} = 0 \) for \( q > 0 \). For this we can use the \( \epsilon_X : \pi_X \circ \epsilon_X \) and the Leray spectral sequence (Cohomology on Sites, Lemma \( \ref{Cohomology on Sites} \)). By Lemma \( \ref{Cohomology on Sites} \) we have \( R^i \epsilon_{X,*}(a_X^{-1} \mathcal{F}) = 0 \) for \( i > 0 \) and \( \epsilon_X,a_X^{-1} \mathcal{F} = \pi_X^{-1} \mathcal{F} \). By Lemma \( \ref{Cohomology on Sites} \) we have \( R^j \pi_X,*(\pi_X^{-1} \mathcal{F}) = 0 \) for \( j > 0 \). This concludes the proof.

\begin{lemma}
For a scheme \( X \) and \( a_X : \text{Sh}((\text{Sch}/X)_h) \to \text{Sh}(X_{\text{etale}}) \) as above:
\begin{enumerate}
\item \( H^q(X_{\text{etale}}, \mathcal{F}) = H^q_h(X, a_X^{-1} \mathcal{F}) \) for a torsion abelian sheaf \( \mathcal{F} \) on \( X_{\text{etale}} \).
\end{enumerate}
\end{lemma}

\textbf{Proof.}
(2) \( H^q(X_{\text{étale}}, K) = H^q_h(X, a_X^{-1}K) \) for \( K \in D^+(X_{\text{étale}}) \) with torsion cohomology sheaves.

Example: if \( A \) is a torsion abelian group, then \( H^q_{\text{étale}}(X, A) = H^q_h(X, A) \).

Proof. This follows from Lemma \ref{95.6} by Cohomology on Sites, Remark \ref{15.4} \( \square \)

### 96. Blow up squares and étale cohomology

0EW4 Blow up squares are introduced in More on Flatness, Section \ref{34}. Using the proper base change theorem we can see that we have a Mayer-Vietoris type result for blow up squares.

0EW5 \textbf{Lemma 96.1.} Let \( X \) be a scheme and let \( Z \subset X \) be a closed subscheme cut out by a quasi-coherent ideal of finite type. Consider the corresponding blow up square

\[
\begin{array}{ccc}
E & \rightarrow & X' \\
\downarrow \pi & & \downarrow b \\
Z & \rightarrow & X
\end{array}
\]

For \( K \in D^+(X_{\text{étale}}) \) with torsion cohomology sheaves we have a distinguished triangle

\[
K \rightarrow Ri_*(K|_Z) \oplus Rb_*(K|_{X'}) \rightarrow Rc_*(K|_E) \rightarrow K[1]
\]

in \( D(X_{\text{étale}}) \) where \( c = i \circ \pi = b \circ j \).

Proof. The notation \( K|_{X'} \) stands for \( b^{-1}_{\text{small}}K \). Choose a bounded below complex \( \mathcal{F}^\bullet \) of abelian sheaves representing \( K \). Observe that \( i_*(- \mathcal{F}^\bullet|_Z) \) represents \( Ri_*(K|_Z) \) because \( i_* \) is exact (Proposition \ref{54.2}). Choose a quasi-isomorphism \( b^{-1}_{\text{small}} \mathcal{F}^\bullet \rightarrow \mathcal{I}^\bullet \) where \( \mathcal{I}^\bullet \) is a bounded below complex of injective abelian sheaves on \( X'_{\text{étale}} \). This map is adjoint to a map \( \mathcal{F}^\bullet \rightarrow b_*(\mathcal{I}^\bullet) \) and \( b_*(\mathcal{I}^\bullet) \) represents \( Rb_*(K|_{X'}) \). We have \( \pi_*(\mathcal{I}^\bullet|_E) = (b_*(\mathcal{I}^\bullet)|_Z \) by Lemma \ref{85.5} and by Lemma \ref{85.12} this complex represents \( R\pi_*(K|_E) \). Hence the map

\[
Ri_*(K|_Z) \oplus Rb_*(K|_{X'}) \rightarrow Rc_*(K|_E)
\]

is represented by the surjective map of bounded below complexes

\[
i_*(\mathcal{F}^\bullet|_Z) \oplus b_*(\mathcal{I}^\bullet) \rightarrow i_*(b_*(\mathcal{I}^\bullet)|_Z)
\]

To get our distinguished triangle it suffices to show that the canonical map \( \mathcal{F}^\bullet \rightarrow i_*(\mathcal{F}^\bullet|_Z) \oplus b_*(\mathcal{I}^\bullet) \) maps quasi-isomorphically onto the kernel of the map of complexes displayed above (namely a short exact sequence of complexes determines a distinguished triangle in the derived category, see Derived Categories, Section \ref{12}). We may check this on stalks at a geometric point \( \overline{x} \) of \( X \). If \( \overline{x} \) is not in \( Z \), then \( X' \rightarrow X \) is an isomorphism over an open neighbourhood of \( \overline{x} \). Thus, if \( \overline{x}' \) denotes the corresponding geometric point of \( X' \) in this case, then we have to show that

\[
\mathcal{F}^\bullet_{\overline{x}} \rightarrow \mathcal{I}^\bullet_{\overline{x}}
\]

is a quasi-isomorphism. This is true by our choice of \( \mathcal{I}^\bullet \). If \( \overline{x} \) is in \( Z \), then \( b_*(\mathcal{I}^\bullet)|_Z \rightarrow i_*(b_*(\mathcal{I}^\bullet)|_Z)_{\overline{x}} \) is an isomorphism of complexes of abelian groups. Hence the kernel is equal to \( i_*(\mathcal{F}^\bullet|_Z)_{\overline{x}} = \mathcal{F}^\bullet_{\overline{x}} \) as desired. \( \square \)
Lemma 96.2. Let $X$ be a scheme and let $K \in D^+(X_{\text{étale}})$ have torsion cohomology sheaves. Let $Z \subset X$ be a closed subscheme cut out by a quasi-coherent ideal of finite type. Consider the corresponding blow up square

$$
\begin{array}{ccc}
E & \rightarrow & X' \\
\downarrow & & \downarrow b \\
Z & \rightarrow & X
\end{array}
$$

Then there is a canonical long exact sequence

$$
H^p_{\text{étale}}(X, K) \rightarrow H^p_{\text{étale}}(X', K|_{X'}) \oplus H^p_{\text{étale}}(Z, K|_Z) \rightarrow H^p_{\text{étale}}(E, K|_E) \rightarrow H^{p+1}_{\text{étale}}(X, K)
$$

First proof. This follows immediately from Lemma 96.1 and the fact that $R\Gamma(X, Rb_*(K|_{X'})) = R\Gamma(X', K|_{X'})$ (see Cohomology on Sites, Section 15) and similarly for the others. □

Second proof. By Lemma 94.7 these cohomology groups are the cohomology of $X, X', E, Z$ with values in some complex of abelian sheaves on the site $(\text{Sch}/X)_{\text{ph}}$. (Namely, the object $a^{X'}_X K$ of the derived category, see Lemma 94.1 above and recall that $K|_{X'} = b^{-1}_{\text{smooth}} K$.) By More on Flatness, Lemma 34.1 the ph sheafification of the diagram of representable presheaves is cocartesian. Thus the lemma follows from the very general Cohomology on Sites, Lemma 26.3 applied to the site $(\text{Sch}/X)_{\text{ph}}$ and the commutative diagram of the lemma. □

Lemma 96.3. Let $X$ be a scheme and let $Z \subset X$ be a closed subscheme cut out by a quasi-coherent ideal of finite type. Consider the corresponding blow up square

$$
\begin{array}{ccc}
E & \rightarrow & X' \\
\downarrow \pi & & \downarrow b \\
Z & \rightarrow & X
\end{array}
$$

Suppose given

1. an object $K'$ of $D^+(X'_{\text{étale}})$ with torsion cohomology sheaves,
2. an object $L$ of $D^+(Z_{\text{étale}})$ with torsion cohomology sheaves, and
3. an isomorphism $\gamma : K'|_E \rightarrow L|_E$.

Then there exists an object $K$ of $D^+(X_{\text{étale}})$ and isomorphisms $f : K|_{X'} \rightarrow K'$, $g : K|_Z \rightarrow L$ such that $\gamma = g|_E \circ f^{-1}|_E$. Moreover, given

1. an object $M$ of $D^+(X_{\text{étale}})$ with torsion cohomology sheaves,
2. a morphism $\alpha : K' \rightarrow M|_{X'}$ of $D(X'_{\text{étale}})$,
3. a morphism $\beta : L \rightarrow M|_Z$ of $D(Z_{\text{étale}})$,

such that

$$
\alpha|_E = \beta|_E \circ \gamma.
$$

Then there exists a morphism $M \rightarrow K$ in $D(X_{\text{étale}})$ whose restriction to $X'$ is $\alpha \circ f$ and whose restriction to $Z$ is $b \circ g$.

Proof. If $K$ exists, then Lemma 96.1 tells us a distinguished triangle that it fits in. Thus we simply choose a distinguished triangle

$$
K \rightarrow R\pi_*(L) \oplus Rb_*(K') \rightarrow Rc_*(L|_E) \rightarrow K[1]
$$
where \( c = i \circ \pi = b \circ j \). Here the map \( R\iota_*(L) \to R\imath_*(L|_E) \) is \( R\iota_* \) applied to the adjunction mapping \( E \to R\pi_*(L|_E) \). The map \( Rb_*(K') \to R\iota_*(L|_E) \) is the composition of the canonical map \( Rb_*(K') \to R\iota_*(K'|_E) \) = \( R \) and \( R\iota_*(\gamma) \). The maps \( g \) and \( f \) of the statement of the lemma are the adjoints of these maps. If we restrict this distinguished triangle to \( Z \) then the map \( Rb_*(K) \to R\imath_*(L|_E) \) becomes an isomorphism by the base change theorem (Lemma 85.12) and hence the map \( g : K|_Z \to L \) is an isomorphism. Looking at the distinguished triangle we see that \( f : K'|_{X'} \to K' \) is an isomorphism over \( X' \setminus E = X \setminus Z \). Moreover, we have \( \gamma \circ f|_E = g|_E \) by construction. Then since \( \gamma \) and \( g \) are isomorphisms we conclude that \( f \) induces isomorphisms on stalks at geometric points of \( E \) as well. Thus \( f \) is an isomorphism.

For the final statement, we may replace \( K' \) by \( K|_{X'} \), \( L \) by \( K|_Z \), and \( \gamma \) by the canonical identification. Observe that \( \alpha \) and \( \beta \) induce a commutative square

Thus by the axioms of a derived category we get a dotted arrow producing a morphism of distinguished triangles. □

97. Almost blow up squares and the h topology

0EWL In this section we continue the discussion in More on Flatness, Section 35. For the convenience of the reader we recall that an almost blow up square is a commutative diagram

\[
E \longrightarrow X' \\
n \downarrow \quad \quad \downarrow b \\
Z \longrightarrow X
\]

of schemes satisfying the following conditions:

1. \( Z \to X \) is a closed immersion of finite presentation,
2. \( E = b^{-1}(Z) \) is a locally principal closed subscheme of \( X' \),
3. \( b \) is proper and of finite presentation,
4. the closed subscheme \( X'' \subset X' \) cut out by the quasi-coherent ideal of sections of \( \mathcal{O}_{X'} \) supported on \( E \) (Properties, Lemma 24.5) is the blow up of \( X \) in \( Z \).

It follows that the morphism \( b \) induces an isomorphism \( X' \setminus E \to X \setminus Z \).

We are going to give a criterion for “h sheafiness” for objects in the derived category of the big fppf site \((\text{Sch}/S)_{\text{fppf}}\) of a scheme \( S \). On the same underlying category we have a second topology, namely the h topology (More on Flatness, Section 32). Recall that fppf coverings are h coverings (More on Flatness, Lemma 32.5). Hence we may consider the morphism

\[ \epsilon : (\text{Sch}/S)_h \longrightarrow (\text{Sch}/S)_{\text{fppf}} \]

See Cohomology on Sites, Section 27. In particular, we have a fully faithful functor

\[ R\epsilon_* : D((\text{Sch}/S)_h) \longrightarrow D((\text{Sch}/S)_{\text{fppf}}) \]
and we can ask: what is the essential image of this functor?

**Lemma 97.1.** With notation as above, if \( K \) is in the essential image of \( R_{\epsilon_*} \), then the maps \( c^K_{X,Z,X',E} \) of Cohomology on Sites, Lemma \([26.1]\) are quasi-isomorphisms.

**Proof.** Denote \( \# \) sheafification in the \( h \) topology. We have seen in More on Flatness, Lemma \([35.7]\) that \( h^\#_X = h^\#_Z \amalg h^\#_E \). On the other hand, the map \( h^\#_E \to h^\#_X \) is injective as \( E \to X' \) is a monomorphism. Thus this lemma is a special case of Cohomology on Sites, Lemma \([26.3]\) (which itself is a formal consequence of Cohomology on Sites, Lemma \([29.3]\)). \( \square \)

**Proposition 97.2.** Let \( K \) be an object of \( D^+((\text{Sch}/S)_{fppf}) \). Then \( K \) is in the essential image of \( R_{\epsilon_*} : D((\text{Sch}/S)_h) \to D((\text{Sch}/S)_{fppf}) \) if and only if \( c^K_{X,X',Z,E} \) is a quasi-isomorphism for every almost blow up square \([97.0.1]\) in \((\text{Sch}/S)_h\) with \( X \) affine.

**Proof.** We prove this by applying Cohomology on Sites, Lemma \([29.2]\) whose hypotheses hold by Lemma \([97.1]\) and More on Flatness, Proposition \([35.9]\). \( \square \)

**Lemma 97.3.** Let \( K \) be an object of \( D^+((\text{Sch}/S)_{fppf}) \). Then \( K \) is in the essential image of \( R_{\epsilon_*} : D((\text{Sch}/S)_h) \to D((\text{Sch}/S)_{fppf}) \) if and only if \( c^K_{X,X',Z,E} \) is a quasi-isomorphism for every almost blow up square as in More on Flatness, Examples \([35.10]\) and \([35.11]\).

**Proof.** We prove this by applying Cohomology on Sites, Lemma \([29.2]\) whose hypotheses hold by Lemma \([97.1]\) and More on Flatness, Lemma \([35.12]\). \( \square \)

### 98. Cohomology of the structure sheaf in the \( h \) topology

**Lemma 98.1.** Let \( p \) be a prime number. Let \( (\mathcal{C}, \mathcal{O}) \) be a ringed site with \( p\mathcal{O} = 0 \). Then we set \( \colim F \mathcal{O} \) equal to the colimit in the category of sheaves of rings of the system

\[
\mathcal{O} \xrightarrow{F} \mathcal{O} \xrightarrow{F} \mathcal{O} \xrightarrow{F} \ldots
\]

where \( F : \mathcal{O} \to \mathcal{O}, f \mapsto f^p \) is the Frobenius endomorphism.

**Lemma 98.2.** Let \( p \) be a prime number. Let \( S \) be a scheme over \( \mathbf{F}_p \). Consider the sheaf \( \mathcal{O}^{perf} = \colim F \mathcal{O} \) on \((\text{Sch}/S)_{fppf}\). Then \( \mathcal{O}^{perf} \) is in the essential image of \( R_{\epsilon_*} : D((\text{Sch}/S)_h) \to D((\text{Sch}/S)_{fppf}) \).

**Proof.** We prove this using the criterion of Lemma \([97.3]\). Before check the conditions, we note that for a quasi-compact and quasi-separated object \( X \) of \((\text{Sch}/S)_{fppf}\) we have

\[
H^i_{fppf}(X, \mathcal{O}^{perf}) = \colim F H^i_{fppf}(X, \mathcal{O})
\]

See Cohomology on Sites, Lemma \([17.1]\). We will also use that \( H^i_{fppf}(X, \mathcal{O}) = H^i(X, \mathcal{O}) \), see Descent, Proposition \([8.10]\).

Let \( A, f, J \) be as in More on Flatness, Example \([35.10]\) and consider the associated almost blow up square. Since \( X, X', Z, E \) are affine, we have no higher cohomology of \( \mathcal{O} \). Hence we only have to check that

\[
0 \to \mathcal{O}^{perf}(X) \to \mathcal{O}^{perf}(X') \oplus \mathcal{O}^{perf}(Z) \to \mathcal{O}^{perf}(E) \to 0
\]

is a short exact sequence. This was shown in (the proof of) More on Flatness, Lemma \([36.2]\).
Let $X, X', Z, E$ be as in More on Flatness, Example 35.11. Since $X$ and $Z$ are affine we have $H^p(X, \mathcal{O}_X) = H^p(Z, \mathcal{O}_X) = 0$ for $p > 0$. By More on Flatness, Lemma 36.1 we have $H^p(X', \mathcal{O}_{X'}) = 0$ for $p > 0$. Since $E = \mathbb{P}^1_Z$ and $Z$ is affine we also have $H^p(E, \mathcal{O}_E) = 0$ for $p > 0$. As in the previous paragraph we reduce to checking that

$$0 \to \mathcal{O}^{\text{perf}}(X) \to \mathcal{O}^{\text{perf}}(X') \oplus \mathcal{O}^{\text{perf}}(Z) \to \mathcal{O}^{\text{perf}}(E) \to 0$$

is a short exact sequence. This was shown in (the proof of) More on Flatness, Lemma 36.2. □

**Proposition 98.2.** Let $p$ be a prime number. Let $S$ be a quasi-compact and quasi-separated scheme over $\mathbb{F}_p$. Then

$$H^i((\text{Sch}/S)_h, \mathcal{O}_h) = \operatorname{colim}_F H^i(S, \mathcal{O})$$

Here on the left hand side by $\mathcal{O}_h$ we mean the $h$ sheafification of the structure sheaf.

**Proof.** This is just a reformulation of Lemma 98.1. Recall that $\mathcal{O}_h = \mathcal{O}^{\text{perf}} = \operatorname{colim}_F \mathcal{O}$, see More on Flatness, Lemma 36.7. By Lemma 98.1 we see that $\mathcal{O}^{\text{perf}}$ viewed as an object of $D((\text{Sch}/S)_{\text{fppf}})$ is of the form $R\epsilon_* K$ for some $K \in D((\text{Sch}/S)_h)$. Then $K = \epsilon^{-1} \mathcal{O}^{\text{perf}}$ which is actually equal to $\mathcal{O}^{\text{perf}}$ because $\mathcal{O}^{\text{perf}}$ is an $h$ sheaf. See Cohomology on Sites, Section 27. Hence $R\epsilon_* \mathcal{O}^{\text{perf}} = \mathcal{O}^{\text{perf}}$ (with apologies for the confusing notation). Thus the lemma now follows from Leray

$$R\Gamma_h(S, \mathcal{O}^{\text{perf}}) = R\Gamma_{\text{fppf}}(S, R\epsilon_* \mathcal{O}^{\text{perf}}) = R\Gamma_{\text{fppf}}(S, \mathcal{O}^{\text{perf}})$$

and the fact that

$$H^i_{\text{fppf}}(S, \mathcal{O}^{\text{perf}}) = H^i_{\text{fppf}}(S, \operatorname{colim}_F \mathcal{O}) = \operatorname{colim}_F H^i_{\text{fppf}}(S, \mathcal{O})$$

as $S$ is quasi-compact and quasi-separated (see proof of Lemma 98.1). □

**99. The trace formula**

A typical course in étale cohomology would normally state and prove the proper and smooth base change theorems, purity and Poincaré duality. All of these can be found in [Del77, Arcata]. Instead, we are going to study the trace formula for the Frobenius, following the account of Deligne in [Del77, Rapport]. We will only look at dimension 1, but using proper base change this is enough for the general case. Since all the cohomology groups considered will be étale, we drop the subscript 'étale'. Let us now describe the formula we are after. Let $X$ be a finite type scheme of dimension 1 over a finite field $k$, $\ell$ a prime number and $F$ a constructible, flat $\mathbb{Z}/\ell^n\mathbb{Z}$ sheaf. Then

$$\sum_{x \in X(k)} \text{Tr}(\text{Frob}|_{\mathcal{F}_x}) = \sum_{i=0}^2 (-1)^i \text{Tr}(\pi_X|_{H^i(X \otimes_k \bar{k}, \mathcal{F}))}$$

as elements of $\mathbb{Z}/\ell^n\mathbb{Z}$. As we will see, this formulation is slightly wrong as stated. Let us nevertheless describe the symbols that occur therein.

**100. Frobenii**

In this section we will prove a “baffling” theorem. A topological analogue of the baffling theorem is the following.

**Exercise 100.1.** Let $X$ be a topological space and $g : X \to X$ a continuous map such that $g^{-1}(U) = U$ for all opens $U$ of $X$. Then $g$ induces the identity on cohomology on $X$ (for any coefficients).
We now turn to the statement for the étale site.

**Lemma 100.2.** Let $X$ be a scheme and $g : X \to X$ a morphism. Assume that for all $\varphi : U \to X$ étale, there is an isomorphism

$$
\begin{array}{ccc}
U & \overset{\sim}{\longrightarrow} & U \times_{\varphi, X, g} X \\
\downarrow \varphi & & \downarrow \text{pr}_2 \\
X & & \end{array}
$$

functorial in $U$. Then $g$ induces the identity on cohomology (for any sheaf).

**Proof.** The proof is formal and without difficulty. \qed

Please see Varieties, Section 35 for a discussion of different variants of the Frobenius morphism.

**Theorem 100.3** (The Baffling Theorem). Let $X$ be a scheme in characteristic $p > 0$. Then the absolute Frobenius induces (by pullback) the trivial map on cohomology, i.e., for all integers $j \geq 0$,

$$
F_X^* : H^j(X, \mathbb{Z}/n\mathbb{Z}) \to H^j(X, \mathbb{Z}/n\mathbb{Z})
$$

is the identity.

This theorem is purely formal. It is a good idea, however, to review how to compute the pullback of a cohomology class. Let us simply say that in the case where cohomology agrees with Čech cohomology, it suffices to pull back (using the fiber products on a site) the Čech cocycles. The general case is quite technical, see Hypercoverings, Theorem 10.1. To prove the theorem, we merely verify that the assumption of Lemma 100.2 holds for the Frobenius.

**Proof of Theorem 100.3.** We need to verify the existence of a functorial isomorphism as above. For an étale morphism $\varphi : U \to X$, consider the diagram

$$
\begin{array}{ccc}
U & \overset{\sim}{\longrightarrow} & U \times_{\varphi, X, F_X} X \\
\downarrow \varphi & & \downarrow \text{pr}_2 \\
X & & \end{array}
$$

The dotted arrow is an étale morphism and a universal homeomorphism, so it is an isomorphism. See Étale Morphisms, Lemma 14.3. \qed

**Definition 100.4.** Let $k$ be a finite field with $q = p^f$ elements. Let $X$ be a scheme over $k$. The *geometric Frobenius* of $X$ is the morphism $\pi_X : X \to X$ over $\text{Spec}(k)$ which equals $F_X^f$.

Since $\pi_X$ is a morphism over $k$, we can base change it to any scheme over $k$. In particular we can base change it to the algebraic closure $\bar{k}$ and get a morphism $\pi_X : X_{\bar{k}} \to X_{\bar{k}}$. Using $\pi_X$ also for this base change should not be confusing as $X_{\bar{k}}$ does not have a geometric Frobenius of its own.
Lemma 100.5. Let $\mathcal{F}$ be a sheaf on $X_{\text{étale}}$. Then there are canonical isomorphisms
\[ \pi_X^{-1}\mathcal{F} \cong \mathcal{F} \] and $\mathcal{F} \cong \pi_X^*\mathcal{F}$.

This is false for the fppf site.

Proof. Let $\varphi : U \to X$ be étale. Recall that $\pi_X^*\mathcal{F}(U) = \mathcal{F}(U \times_{\varphi, X, \pi_X} X)$. Since $\pi_X = F_X$, it follows from the proof of Theorem 100.3 that there is a functorial isomorphism
\[ \gamma_U \downarrow \downarrow \gamma_U \rightarrow \mathcal{F}(U) \]
where $\gamma_U = (\varphi, F_U^f)$. Now we define an isomorphism
\[ \mathcal{F}(U) \to \pi_X^*\mathcal{F}(U) = \mathcal{F}(U \times_{\varphi, X, \pi_X} X) \]
by taking the restriction map of $\mathcal{F}$ along $\gamma_U^{-1}$. The other isomorphism is analogous. □

Remark 100.6. It may or may not be the case that $F_U^f$ equals $\pi_U^f$.

We continue discussion cohomology of sheaves on our scheme $X$ over the finite field $k$ with $q = p^f$ elements. Fix an algebraic closure $\bar{k}$ of $k$ and write $G_k = \text{Gal}(\bar{k}/k)$ for the absolute Galois group of $k$. Let $\mathcal{F}$ be an abelian sheaf on $X_{\text{étale}}$. We will define a left $G_k$-module structure cohomology group $H^j(X_{\bar{k}}, \mathcal{F}|_{X_{\bar{k}}})$ as follows: if $\sigma \in G_k$, the diagram
\[ \begin{array}{ccc}
X_{\bar{k}} & \xrightarrow{\Spec(\sigma) \times \text{id}_X} & X_{\bar{k}} \\
\downarrow & & \downarrow \\
X & \xrightarrow{\Spec(\sigma) \times \text{id}_X} & X
\end{array} \]
commutes. Thus we can set, for $\xi \in H^j(X_{\bar{k}}, \mathcal{F}|_{X_{\bar{k}}})$
\[ \sigma \cdot \xi := (\Spec(\sigma) \times \text{id}_X)^*\xi \in H^j(X_{\bar{k}}, (\Spec(\sigma) \times \text{id}_X)^{-1}\mathcal{F}|_{X_{\bar{k}}}) = H^j(X_{\bar{k}}, \mathcal{F}|_{X_{\bar{k}}}), \]
where the last equality follows from the commutativity of the previous diagram. This endows the latter group with the structure of a $G_k$-module.

Lemma 100.7. In the situation above denote $\alpha : X \to \Spec(k)$ the structure morphism. Consider the stalk $(R^j\alpha_*, \mathcal{F})_{\Spec(\bar{k})}$ endowed with its natural Galois action as in Section 55. Then the identification
\[ (R^j\alpha_*, \mathcal{F})_{\Spec(\bar{k})} \cong H^j(X_{\bar{k}}, \mathcal{F}|_{X_{\bar{k}}}) \]
from Theorem 52.1 is an isomorphism of $G_k$-modules.

A similar result holds comparing $(R^j\alpha_*, \mathcal{F})_{\Spec(\bar{k})}$ with $H^j_{\text{c}}(X_{\bar{k}}, \mathcal{F}|_{X_{\bar{k}}})$.

Proof. Omitted. □

Definition 100.8. The arithmetic frobenius is the map $\text{frob}_k : \bar{k} \to \bar{k}, x \mapsto x^q$ of $G_k$.

Theorem 100.9. Let $\mathcal{F}$ be an abelian sheaf on $X_{\text{étale}}$. Then for all $j \geq 0$, $\text{frob}_k$ acts on the cohomology group $H^j(X_{\bar{k}}, \mathcal{F}|_{X_{\bar{k}}})$ as the inverse of the map $\pi_X^*$.
The map \( \pi_X^* \) is defined by the composition

\[
H^j(X_k, \mathcal{F}|_{X_k}) \xrightarrow{\pi_X^*} H^j(X_k, (\pi_X^{-1}\mathcal{F})|_{X_k}) \cong H^j(X_k, \mathcal{F}|_{X_k}).
\]

where the last isomorphism comes from the canonical isomorphism \( \pi_X^{-1}\mathcal{F} \cong \mathcal{F} \) of Lemma [100.5]

**Proof.** The composition \( X_k \xrightarrow{\text{Spec(frob}_k)} X_k \xrightarrow{\pi} X_k \) is equal to \( F_{X_k} \), hence the result follows from the baffling theorem suitably generalized to nontrivial coefficients. Note that the previous composition commutes in the sense that \( F_{X_k} = \pi_X \circ \text{Spec(frob}_k) = \text{Spec(frob}_k) \circ \pi_X \).

---

**Definition 100.10.** If \( x \in X(k) \) is a rational point and \( \bar{x} : \text{Spec}(\bar{k}) \to X \) the geometric point lying over \( x \), we let \( \pi_x : \mathcal{F}_x \to \mathcal{F}_{\bar{x}} \) denote the action by \( \text{frob}_k \) and call it the *geometric frobenius*\(^{12} \)

We can now make a more precise statement (albeit a false one) of the trace formula \([99.0.1]\). Let \( X \) be a finite type scheme of dimension 1 over a finite field \( k, \ell \) a prime number and \( \mathcal{F} \) a constructible, flat \( \mathbb{Z}/\ell^n\mathbb{Z} \) sheaf. Then

\[
\sum_{x \in X(k)} \text{Tr}(\pi_x|_{\mathcal{F}_x}) = \sum_{i=0}^{2} (-1)^i \text{Tr}(\pi_x^*|H^i(X_{\bar{k}}, \mathcal{F}))
\]
as elements of \( \mathbb{Z}/\ell^n\mathbb{Z} \). The reason this equation is wrong is that the trace in the right-hand side does not make sense for the kind of sheaves considered. Before addressing this issue, we try to motivate the appearance of the geometric frobenius (apart from the fact that it is a natural morphism!).

Let us consider the case where \( X = \mathbb{P}^1_k \) and \( \mathcal{F} = \mathbb{Z}/\ell^n\mathbb{Z} \). For any point, the Galois module \( \mathcal{F}_x \) is trivial, hence for any morphism \( \varphi \) acting on \( \mathcal{F}_x \), the left-hand side is

\[
\sum_{x \in X(k)} \text{Tr}(\varphi|_{\mathcal{F}_x}) = \# \mathbb{P}^1_k(k) = q + 1.
\]

Now \( \mathbb{P}^1_k \) is proper, so compactly supported cohomology equals standard cohomology, and so for a morphism \( \pi : \mathbb{P}^1_k \to \mathbb{P}^1_k \), the right-hand side equals

\[
\text{Tr}(\pi^*|H^0(\mathbb{P}^1_k, \mathbb{Z}/\ell^n\mathbb{Z})) + \text{Tr}(\pi^*|H^2(\mathbb{P}^1_k, \mathbb{Z}/\ell^n\mathbb{Z})).
\]

The Galois module \( H^0(\mathbb{P}^1_k, \mathbb{Z}/\ell^n\mathbb{Z}) = \mathbb{Z}/\ell^n\mathbb{Z} \) is trivial, since the pullback of the identity is the identity. Hence the first trace is 1, regardless of \( \pi \). For the second trace, we need to compute the pullback \( \pi^* : H^2(\mathbb{P}^1_k, \mathbb{Z}/\ell^n\mathbb{Z}) \) for a map \( \pi : \mathbb{P}^1_k \to \mathbb{P}^1_k \). This is a good exercise and the answer is multiplication by the degree of \( \pi \) (for a proof see Lemma [68.2]). In other words, this works as in the familiar situation of complex cohomology. In particular, if \( \pi \) is the geometric frobenius we get

\[
\text{Tr}(\pi_X^*|H^2(\mathbb{P}^1_k, \mathbb{Z}/\ell^n\mathbb{Z})) = q
\]

and if \( \pi \) is the arithmetic frobenius then we get

\[
\text{Tr}(\text{frob}_k^*|H^2(\mathbb{P}^1_k, \mathbb{Z}/\ell^n\mathbb{Z})) = q^{-1}.
\]

The latter option is clearly wrong.

---

\(^{12}\)This notation is not standard. This operator is denoted \( F_\pi \) in [Del77]. We will likely change this notation in the future.
Remark 100.11. The computation of the degrees can be done by lifting (in some obvious sense) to characteristic 0 and considering the situation with complex coefficients. This method almost never works, since lifting is in general impossible for schemes which are not projective space.

The question remains as to why we have to consider compactly supported cohomology. In fact, in view of Poincaré duality, it is not strictly necessary for smooth varieties, but it involves adding in certain powers of \(q\). For example, let us consider the case where \(X = \mathbb{A}^1_k\) and \(F = \mathbb{Z}/\ell\mathbb{Z}\). The action on stalks is again trivial, so we only need look at the action on cohomology. But then \(\pi_X^*\) acts as the identity on \(H^0(\mathbb{A}^1_k, \mathbb{Z}/\ell\mathbb{Z})\) and as multiplication by \(q\) on \(H^2_{\text{c}}(\mathbb{A}^1_k, \mathbb{Z}/\ell\mathbb{Z})\).

101. Traces

We now explain how to take the trace of an endomorphism of a module over a noncommutative ring. Fix a finite ring \(\Lambda\) with cardinality prime to \(p\). Typically, \(\Lambda\) is the group ring \((\mathbb{Z}/\ell^n\mathbb{Z})[G]\) for some finite group \(G\). By convention, all the \(\Lambda\)-modules considered will be left \(\Lambda\)-modules.

We introduce the following notation: We set \(\Lambda^\natural\) to be the quotient of \(\Lambda\) by its additive subgroup generated by the commutators (i.e., the elements of the form \(ab - ba\), \(a, b \in \Lambda\)). Note that \(\Lambda^\natural\) is not a ring.

For instance, the module \((\mathbb{Z}/\ell^n\mathbb{Z})[G]^\natural\) is the dual of the class functions, so

\[
(\mathbb{Z}/\ell^n\mathbb{Z})[G]^\natural = \bigoplus_{\text{conjugacy classes of } G} \mathbb{Z}/\ell^n\mathbb{Z}.
\]

For a free \(\Lambda\)-module, we have \(\text{End}_\Lambda(\Lambda^\oplus m) = \text{Mat}_n(\Lambda)\). Note that since the modules are left modules, representation of endomorphism by matrices is a right action: if \(a \in \text{End}(\Lambda^\oplus m)\) has matrix \(A\) and \(v \in \Lambda\), then \(a(v) = vA\).

Definition 101.1. The trace of the endomorphism \(a\) is the sum of the diagonal entries of a matrix representing it. This defines an additive map \(\text{Tr} : \text{End}_\Lambda(\Lambda^\oplus m) \to \Lambda^\natural\).

Exercise 101.2. Given maps

\[
\Lambda^\oplus n \xrightarrow{a} \Lambda^\oplus n \xrightarrow{b} \Lambda^\oplus m
\]

show that \(\text{Tr}(ab) = \text{Tr}(ba)\).

We extend the definition of the trace to a finite projective \(\Lambda\)-module \(P\) and an endomorphism \(\varphi\) of \(P\) as follows. Write \(P\) as the summand of a free \(\Lambda\)-module, i.e., consider maps \(P \xrightarrow{a} \Lambda^\oplus n \xrightarrow{b} P\) with

1. \(\Lambda^\oplus n = \text{Im}(a) \oplus \text{Ker}(b)\);
2. \(b \circ a = \text{id}_P\).

Then we set \(\text{Tr}(\varphi) = \text{Tr}(a \varphi b)\). It is easy to check that this is well-defined, using the previous exercise.
102. Why derived categories?

With this definition of the trace, let us now discuss another issue with the formula as stated. Let $C$ be a smooth projective curve over $k$. Then there is a correspondence between finite locally constant sheaves $\mathcal{F}$ on $\text{C}_\text{étale}$ whose stalks are isomorphic to $(\mathbb{Z}/\ell^n\mathbb{Z})^{\oplus m}$ on the one hand, and continuous representations $\rho : \pi_1(C, \bar{c}) \to \text{GL}_m(\mathbb{Z}/\ell^n\mathbb{Z}))$ (for some fixed choice of $\bar{c}$) on the other hand. We denote $\mathcal{F}_\rho$ the sheaf corresponding to $\rho$. Then $H^2(C_k, \mathcal{F}_\rho)$ is the group of coinvariants for the action of $\rho(\pi_1(C, \bar{c}))$ on $(\mathbb{Z}/\ell^n\mathbb{Z})^{\oplus m}$, and there is a short exact sequence

$$0 \longrightarrow \pi_1(C_k, \bar{c}) \longrightarrow \pi_1(C, \bar{c}) \longrightarrow G_k \longrightarrow 0.$$ 

For instance, let $\mathbb{Z} = \mathbb{Z}\sigma$ act on $\mathbb{Z}/\ell^2\mathbb{Z}$ via $\sigma(x) = (1 + \ell)x$. The coinvariants are $(\mathbb{Z}/\ell^2\mathbb{Z})_{\rho} = \mathbb{Z}/\ell\mathbb{Z}$, which is not a flat $\mathbb{Z}/\ell^2\mathbb{Z}$-module. Hence we cannot take the trace of some action on $H^2(C_k, \mathcal{F}_\rho)$, at least not in the sense of the previous section.

In fact, our goal is to consider a trace formula for $\ell$-adic coefficients. But $\mathbb{Q}_\ell = \mathbb{Z}_\ell[1/\ell]$ and $\mathbb{Z}_\ell = \lim \mathbb{Z}/\ell^n\mathbb{Z}$, and even for a flat $\mathbb{Z}/\ell^n\mathbb{Z}$ sheaf, the individual cohomology groups may not be flat, so we cannot compute traces. One possible remedy is consider the total derived complex $\mathbb{R}\Gamma(C_k, \mathcal{F}_\rho)$ in the derived category $\text{D}(\mathbb{Z}/\ell^n\mathbb{Z})$ and show that it is a perfect object, which means that it is quasi-isomorphic to a finite complex of finite free module. For such complexes, we can define the trace, but this will require an account of derived categories.

103. Derived categories

To set up notation, let $\mathcal{A}$ be an abelian category. Let $\text{Comp}(\mathcal{A})$ be the abelian category of complexes in $\mathcal{A}$. Let $K(\mathcal{A})$ be the category of complexes up to homotopy, with objects equal to complexes in $\mathcal{A}$ and morphisms equal to homotopy classes of morphisms of complexes. This is not an abelian category. Loosely speaking, $D(\mathcal{A})$ is defined to be the category obtained by inverting all quasi-isomorphisms in $\text{Comp}(\mathcal{A})$ or, equivalently, in $K(\mathcal{A})$. Moreover, we can define $\text{Comp}^+(\mathcal{A}), K^+(\mathcal{A}), D^+(\mathcal{A})$ analogously using only bounded below complexes. Similarly, we can define $\text{Comp}^-(\mathcal{A}), K^-(\mathcal{A}), D^-(\mathcal{A})$ using bounded above complexes, and we can define $\text{Comp}^b(\mathcal{A}), K^b(\mathcal{A}), D^b(\mathcal{A})$ using bounded complexes.

**Remark 103.1.** Notes on derived categories.

1. There are some set-theoretical problems when $\mathcal{A}$ is somewhat arbitrary, which we will happily disregard.
2. The categories $K(\mathcal{A})$ and $D(\mathcal{A})$ are endowed with the structure of a triangulated category.
3. The categories $\text{Comp}(\mathcal{A})$ and $K(\mathcal{A})$ can also be defined when $\mathcal{A}$ is an additive category.

The homology functor $H^i : \text{Comp}(\mathcal{A}) \to \mathcal{A}$ taking a complex $K^\bullet \mapsto H^i(K^\bullet)$ extends to functors $H^i : K(\mathcal{A}) \to \mathcal{A}$ and $H^i : D(\mathcal{A}) \to \mathcal{A}$.

**Lemma 103.2.** An object $E$ of $D(\mathcal{A})$ is contained in $D^+(\mathcal{A})$ if and only if $H^i(E) = 0$ for all $i < 0$. Similar statements hold for $D^-$ and $D^+$.

**Proof.** Hint: use truncation functors. See Derived Categories, Lemma 11.5.

**Lemma 103.3.** Morphisms between objects in the derived category.
(1) Let $I^\bullet \in \operatorname{Comp}^+(A)$ with $I^n$ injective for all $n \in \mathbb{Z}$. Then
\[ \operatorname{Hom}_{D(A)}(K^\bullet, I^\bullet) = \operatorname{Hom}_{K(A)}(K^\bullet, I^\bullet). \]

(2) Let $P^\bullet \in \operatorname{Comp}^-(A)$ with $P^n$ projective for all $n \in \mathbb{Z}$. Then
\[ \operatorname{Hom}_{D(A)}(A, P^\bullet, K^\bullet) = \operatorname{Hom}_{K(A)}(A, P^\bullet, K^\bullet). \]

(3) If $A$ has enough injectives and $I \subset A$ is the additive subcategory of injectives, then $D^+(A) \cong K^+(I)$ (as triangulated categories).

(4) If $A$ has enough projectives and $P \subset A$ is the additive subcategory of projectives, then $D^-(A) \cong K^-(P)$.

Proof. Omitted. \[ \square \]

**Definition 103.4.** Let $F : A \to B$ be a left exact functor and assume that $A$ has enough injectives. We define the total right derived functor of $F$ as the functor $RF : D^+(A) \to D^+(B)$ fitting into the diagram
\[
\begin{array}{ccc}
D^+(A) & \xrightarrow{RF} & D^+(B) \\
\uparrow & & \uparrow \\
K^+(I) & \xrightarrow{F} & K^+(B).
\end{array}
\]

This is possible since the left vertical arrow is invertible by the previous lemma. Similarly, let $G : A \to B$ be a right exact functor and assume that $A$ has enough projectives. We define the total left derived functor of $G$ as the functor $LG : D^-(A) \to D^-(B)$ fitting into the diagram
\[
\begin{array}{ccc}
D^-(A) & \xrightarrow{LG} & D^-(B) \\
\uparrow & & \uparrow \\
K^-(P) & \xrightarrow{G} & K^-(B).
\end{array}
\]

This is possible since the left vertical arrow is invertible by the previous lemma.

**Remark 103.5.** In these cases, it is true that $R^iF(K^\bullet) = H^i(RF(K^\bullet))$, where the left hand side is defined to be $i$th homology of the complex $F(K^\bullet)$.

### 04. Filtered derived category

**Definition 104.1.** Let $\mathcal{A}$ be an abelian category.

1. Let $\operatorname{Fil}(\mathcal{A})$ be the category of filtered objects $(A, F)$ of $\mathcal{A}$, where $F$ is a filtration of the form
\[ A \supset \ldots \supset F^nA \supset F^{n+1}A \supset \ldots \supset 0. \]

   This is an additive category.

2. We denote $\operatorname{Fil}^f(\mathcal{A})$ the full subcategory of $\operatorname{Fil}(\mathcal{A})$ whose objects $(A, F)$ have finite filtration. This is also an additive category.

3. An object $I \in \operatorname{Fil}^f(\mathcal{A})$ is called filtered injective (respectively projective) provided that $\operatorname{gr}^p(I) = \operatorname{gr}^p_F(I) = F^pI/F^{p+1}I$ is injective (resp. projective) in $\mathcal{A}$ for all $p$.
(4) The category of complexes $\text{Comp}(\text{Fil}^f(A)) \supset \text{Comp}^+(\text{Fil}^f(A))$ and its homotopy category $\text{K}(\text{Fil}^f(A)) \supset \text{K}^+(\text{Fil}^f(A))$ are defined as before.

(5) A morphism $\alpha : K^\bullet \to L^\bullet$ of complexes in $\text{Comp}(\text{Fil}^f(A))$ is called a filtered quasi-isomorphism provided that $\text{gr}^p(\alpha) : \text{gr}^p(K^\bullet) \to \text{gr}^p(L^\bullet)$ is a quasi-isomorphism for all $p \in \mathbb{Z}$.

(6) We define $DF(A)$ (resp. $DF^+(A)$) by inverting the filtered quasi-isomorphisms in $\text{K}(\text{Fil}^f(A))$ (resp. $\text{K}^+(\text{Fil}^f(A))$).

03TB **Lemma 104.2.** If $\mathcal{A}$ has enough injectives, then $DF^+(A) \cong K^+(I)$, where $I$ is the full additive subcategory of $\text{Fil}^f(A)$ consisting of filtered injective objects. Similarly, if $\mathcal{A}$ has enough projectives, then $DF^-(A) \cong K^+(P)$, where $P$ is the full additive subcategory of $\text{Fil}^f(A)$ consisting of filtered projective objects.

**Proof.** Omitted. \hfill $\square$

105. Filtered derived functors

03TC And then there are the filtered derived functors.

03TD **Definition 105.1.** Let $T : \mathcal{A} \to \mathcal{B}$ be a left exact functor and assume that $\mathcal{A}$ has enough injectives. Define $RT : DF^+(A) \to DF^+(B)$ to fit in the diagram

\[
\begin{array}{ccc}
\text{DF}^+(A) & \xrightarrow{RT} & \text{DF}^+(B) \\
\uparrow & & \uparrow \\
\text{K}^+(I) & \xrightarrow{T} & \text{K}^+(\text{Fil}^f(B)).
\end{array}
\]

This is well-defined by the previous lemma. Let $G : \mathcal{A} \to \mathcal{B}$ be a right exact functor and assume that $\mathcal{A}$ has enough projectives. Define $LG : DF^-(A) \to DF^+(B)$ to fit in the diagram

\[
\begin{array}{ccc}
\text{DF}^-(A) & \xrightarrow{LG} & \text{DF}^-(B) \\
\uparrow & & \uparrow \\
\text{K}^-(P) & \xrightarrow{G} & \text{K}^-(\text{Fil}^f(B)).
\end{array}
\]

Again, this is well-defined by the previous lemma. The functors $RT$, resp. $LG$, are called the filtered derived functor of $T$, resp. $G$.

03TE **Proposition 105.2.** In the situation above, we have

\[
\text{gr}^p \circ RT = RT \circ \text{gr}^p
\]

where the $RT$ on the left is the filtered derived functor while the one on the right is the total derived functor. That is, there is a commuting diagram

\[
\begin{array}{ccc}
\text{DF}^+(A) & \xrightarrow{RT} & \text{DF}^+(B) \\
\text{gr}^p \downarrow & & \text{gr}^p \downarrow \\
\text{D}^+(A) & \xrightarrow{RT} & \text{D}^+(B).
\end{array}
\]

**Proof.** Omitted. \hfill $\square$
Given $K^\bullet \in DF^+(B)$, we get a spectral sequence

\[ E_1^{p,q} = H^{p+q}(\text{gr}^p K^\bullet) \Rightarrow H^{p+q}(\text{forget filt}(K^\bullet)). \]

### 106. Application of filtered complexes

Let $A$ be an abelian category with enough injectives, and $0 \to L \to M \to N \to 0$ a short exact sequence in $A$. Consider $\tilde{M} \in \text{Fil}^1(A)$ to be $M$ along with the filtration defined by

- $F_1 M = L$,
- $F_n M = M$ for $n \leq 0$,
- $F_n M = 0$ for $n \geq 2$.

By definition, we have

- $\text{forget filt}(\tilde{M}) = M$,
- $\text{gr}^0(\tilde{M}) = N$,
- $\text{gr}^1(\tilde{M}) = L$.

Assume that $A$ has enough injectives. Then $RT(\tilde{M}) \in DF^+(B)$ is a filtered complex with

\[
\text{gr}^p(\text{RT}(\tilde{M})) \overset{\text{qis}}{=} \begin{cases} 0 & \text{if } p \neq 0, 1, \\ RT(N) & \text{if } p = 0, \\ RT(L) & \text{if } p = 1. \end{cases}
\]

and forget filt($RT(\tilde{M})$) $\overset{\text{qis}}{=} RT(M)$. The spectral sequence applied to $RT(\tilde{M})$ gives

\[ E_1^{p,q} = R^{p+q}T(\text{gr}^p(\tilde{M})) \Rightarrow R^{p+q}T(\text{forget filt}(\tilde{M})). \]

Unwinding the spectral sequence gives us the long exact sequence

\[
0 \longrightarrow T(L) \longrightarrow T(M) \longrightarrow T(N) \longrightarrow R^1T(L) \longrightarrow R^1T(M) \longrightarrow \ldots
\]

This will be used as follows. Let $X/k$ be a scheme of finite type. Let $F$ be a flat constructible $\mathbb{Z}/\ell^n\mathbb{Z}$-module. Then we want to show that the trace

\[
\text{Tr}(\pi^*_X | R\Gamma_c(X_{\bar{k}}, F)) \in \mathbb{Z}/\ell^n\mathbb{Z}
\]

is additive on short exact sequences. To see this, it will not be enough to work with $R\Gamma_c(X_{\bar{k}}, -) \in D^+(\mathbb{Z}/\ell^n\mathbb{Z})$, but we will have to use the filtered derived category.

### 107. Perfectness

Let $\Lambda$ be a (possibly noncommutative) ring, $\text{Mod}_\Lambda$ the category of left $\Lambda$-modules, $K(\Lambda) = K(\text{Mod}_\Lambda)$ its homotopy category, and $D(\Lambda) = D(\text{Mod}_\Lambda)$ the derived category.

**Definition 107.1.** We denote by $K_{\text{perf}}(\Lambda)$ the category whose objects are bounded complexes of finite projective $\Lambda$-modules, and whose morphisms are morphisms of complexes up to homotopy. The functor $K_{\text{perf}}(\Lambda) \to D(\Lambda)$ is fully faithful (Derived Categories, Lemma 19.8). Denote $D_{\text{perf}}(\Lambda)$ its essential image. An object of $D(\Lambda)$ is called perfect if it is in $D_{\text{perf}}(\Lambda)$.

**Proposition 107.2.** Let $K \in D_{\text{perf}}(\Lambda)$ and $f \in \text{End}_{D(\Lambda)}(K)$. Then the trace $\text{Tr}(f) \in \Lambda^\times$ is well defined.
Proof. We will use Derived Categories, Lemma 19.8 without further mention in this proof. Let $P^\bullet$ be a bounded complex of finite projective $\Lambda$-modules and let $\alpha : P^\bullet \to K$ be an isomorphism in $D(\Lambda)$. Then $\alpha^{-1} \circ f \circ \alpha$ corresponds to a morphism of complexes $f^\bullet : P^\bullet \to P^\bullet$ well defined up to homotopy. Set

$$\text{Tr}(f) = \sum (-1)^i \text{Tr}(f^i : P^i \to P^i) \in \Lambda^3.$$ 

Given $P^\bullet$ and $\alpha$, this is independent of the choice of $f^\bullet$. Namely, any other choice is of the form $\tilde{f}^\bullet = f^\bullet + dh + hd$ for some $h^i : P^i \to P^{i-1} (i \in \mathbb{Z})$. But

$$\text{Tr}(dh) = \sum (-1)^i \text{Tr}(P^i \xrightarrow{dh} P^i) = \sum (-1)^i \text{Tr}(P^{i-1} \xrightarrow{hd} P^{i-1}) = - \sum (-1)^{i-1} \text{Tr}(P^{i-1} \xrightarrow{hd} P^{i-1}) = - \text{Tr}(hd)$$

and so $\sum (-1)^i \text{Tr}((dh+hd)|_P) = 0$. Furthermore, this is independent of the choice of $(P^\bullet, \alpha)$: suppose $(Q^\bullet, \beta)$ is another choice. The compositions

$$Q^\bullet \xrightarrow{\beta} K \xrightarrow{\alpha^{-1}} P^\bullet \quad \text{and} \quad P^\bullet \xrightarrow{\alpha} K \xrightarrow{\beta^{-1}} Q^\bullet$$

are representable by morphisms of complexes $\gamma_1^\bullet$ and $\gamma_2^\bullet$ respectively, such that $\gamma_2^\bullet \circ \gamma_1^\bullet$ is homotopic to the identity. Thus, the morphism of complexes $\gamma_2^\bullet \circ f^\bullet \circ \gamma_1^\bullet : Q^\bullet \to Q^\bullet$ represents the morphism $\beta^{-1} \circ f \circ \beta$ in $D(\Lambda)$. Now

$$\text{Tr}(\gamma_2^\bullet \circ f^\bullet \circ \gamma_1^\bullet|_{Q^\bullet}) = \text{Tr}(\gamma_1^\bullet \circ \gamma_2^\bullet \circ f^\bullet|_{P^\bullet}) = \text{Tr}(f^\bullet|_{P^\bullet})$$

by the fact that $\gamma_1^\bullet \circ \gamma_2^\bullet$ is homotopic to the identity and the independence of the choice of $f^\bullet$ we saw above. \qed

108. Filtrations and perfect complexes

We now present a filtered version of the category of perfect complexes. An object $(M,F)$ of $\text{Fil}^f(\Lambda)$ is called filtered finite projective if for all $p$, $\text{gr}_p(M)$ is finite and projective. We then consider the homotopy category $K\text{F}_{\text{perf}}(\Lambda)$ of bounded complexes of filtered finite projective objects of $\text{Fil}^f(\Lambda)$. We have a diagram of categories

$$KF(\Lambda) \supset K\text{F}_{\text{perf}}(\Lambda) \quad \text{and} \quad DF(\Lambda) \supset D\text{F}_{\text{perf}}(\Lambda)$$

where the vertical functor on the right is fully faithful and the category $D\text{F}_{\text{perf}}(\Lambda)$ is its essential image, as before.

Lemma 108.1 (Additivity). Let $K \in D\text{F}_{\text{perf}}(\Lambda)$ and $f \in \text{End}_{DF}(K)$. Then

$$\text{Tr}(f|_K) = \sum_{p \in \mathbb{Z}} \text{Tr}(f|_{\text{gr}^pK}).$$

Proof. By Proposition 107.2 we may assume we have a bounded complex $P^\bullet$ of filtered finite projectives of $\text{Fil}^f(\Lambda)$ and a map $f^\bullet : P^\bullet \to P^\bullet$ in $\text{Comp}(\text{Fil}^f(\Lambda))$. So the lemma follows from the following result, which proof is left to the reader. \qed
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Lemma 108.2. Let $P \in \text{Fil}^f(\text{Mod}_\Lambda)$ be filtered finite projective, and $f : P \to P$ an endomorphism in $\text{Fil}^f(\text{Mod}_\Lambda)$. Then

$$\text{Tr}(f|_P) = \sum_p \text{Tr}(f|_{\text{gr}^p(P)}).$$

Proof. Omitted. □

109. Characterizing perfect objects

For the commutative case see More on Algebra, Sections 62, 63, and 69.

Definition 109.1. Let $\Lambda$ be a (possibly noncommutative) ring. An object $K \in D(\Lambda)$ has finite Tor-dimension if there exist $a,b \in \mathbb{Z}$ such that for any right $\Lambda$-module $N$, we have $H^i(N \otimes^L \Lambda K) = 0$ for all $i \not\in [a,b]$.

This in particular means that $K \in D^b(\Lambda)$ as we see by taking $N = \Lambda$.

Lemma 109.2. Let $\Lambda$ be a left noetherian ring and $K \in D(\Lambda)$. Then $K$ is perfect if and only if the two following conditions hold:

1. $K$ has finite Tor-dimension, and
2. for all $i \in \mathbb{Z}$, $H^i(K)$ is a finite $\Lambda$-module.

Proof. See More on Algebra, Lemma 69.2 for the proof in the commutative case. □

The reader is strongly urged to try and prove this. The proof relies on the fact that a finite module on a finitely left-presented ring is flat if and only if it is projective.

Remark 109.3. A variant of this lemma is to consider a Noetherian scheme $X$ and the category $D_{\text{perf}}(\mathcal{O}_X)$ of complexes which are locally quasi-isomorphic to a finite complex of finite locally free $\mathcal{O}_X$-modules. Objects $K$ of $D_{\text{perf}}(\mathcal{O}_X)$ can be characterized by having coherent cohomology sheaves and bounded tor dimension.

110. Complexes with constructible cohomology

Let $\Lambda$ be a ring. Let $X$ be a scheme. Let $K(X, \Lambda)$ be the homotopy category of sheaves of $\Lambda$-modules on $X_{\text{étale}}$. Denote $D(X, \Lambda)$ the corresponding derived category. We denote by $D^b(X, \Lambda)$ (respectively $D^+, D^-$) the full subcategory of bounded (resp. above, below) complexes in $D(X, \Lambda)$.

Definition 110.1. Let $X$ be a scheme. Let $\Lambda$ be a Noetherian ring. We denote $D_c(X, \Lambda)$ the full subcategory of $D(X, \Lambda)$ of complexes whose cohomology sheaves are constructible sheaves of $\Lambda$-modules.

This definition makes sense by Lemma 70.6 and Derived Categories, Section 13. Thus we see that $D_c(X, \Lambda)$ is a strictly full, saturated triangulated subcategory of $D_c(X, \Lambda)$.

Lemma 110.2. Let $\Lambda$ be a Noetherian ring. If $j : U \to X$ is an étale morphism of schemes, then

1. $K|_U \in D_c(U, \Lambda)$ if $K \in D_c(X, \Lambda)$, and
2. $j_!M \in D_c(X, \Lambda)$ if $M \in D_c(U, \Lambda)$ and the morphism $j$ is quasi-compact and quasi-separated.

Proof. The first assertion is clear. The second follows from the fact that $j_!$ is exact and Lemma 72.1. □
Lemma 110.3. Let \( \Lambda \) be a Noetherian ring. Let \( f : X \to Y \) be a morphism of schemes. If \( K \in D_{c}(Y, \Lambda) \) then \( Lf^{*}K \in D_{c}(X, \Lambda) \).

Proof. This follows as \( f^{-1} = f^{*} \) is exact and Lemma 70.5. \( \square \)

Lemma 110.4. Let \( X \) be a quasi-compact and quasi-separated scheme. Let \( \Lambda \) be a Noetherian ring. Let \( K \in D(X, \Lambda) \) and \( b \in \mathbb{Z} \) such that \( H^{b}(K) \) is constructible. Then there exist a sheaf \( \mathcal{F} \) which is a finite direct sum of \( j_U! \Lambda \) with \( U \in \text{Ob}(X_{\text{etale}}) \) affine and a map \( \mathcal{F}[−b] \to K \) in \( D(X, \Lambda) \) inducing a surjection \( \mathcal{F} \to H^{b}(K) \).

Proof. Represent \( K \) by a complex \( \mathcal{K}^{\bullet} \) of sheaves of \( \Lambda \)-modules. Consider the surjection

\[
\text{Ker}(\mathcal{K}^{b} \to \mathcal{K}^{b+1}) \to H^{b}(K)
\]

By Modules on Sites, Lemma 29.5 we may choose a surjection \( \bigoplus_{i \in I} j_U! \Lambda \to \text{Ker}(\mathcal{K}^{b} \to \mathcal{K}^{b+1}) \) with \( U_i \) affine. For \( I' \subset I \) finite, denote \( \mathcal{H}_{I'} \subset H^{b}(K) \) the image of \( \bigoplus_{i \in I'} j_U! \Lambda \). By Lemma 70.9 we see that \( \mathcal{H}_{I'} = H^{b}(K) \) for some \( I' \subset I \) finite. The lemma follows taking \( \mathcal{F} = \bigoplus_{i \in I'} j_U! \Lambda \). \( \square \)

Lemma 110.5. Let \( X \) be a quasi-compact and quasi-separated scheme. Let \( \Lambda \) be a Noetherian ring. Let \( K \in D^{-}(X, \Lambda) \). Then the following are equivalent

1. \( K \) is in \( D_{c}(X, \Lambda) \),
2. \( K \) can be represented by a bounded above complex whose terms are finite direct sums of \( j_U! \Lambda \) with \( U \in \text{Ob}(X_{\text{etale}}) \) affine,
3. \( K \) can be represented by a bounded above complex of flat constructible sheaves of \( \Lambda \)-modules.

Proof. It is clear that (2) implies (3) and that (3) implies (1). Assume \( K \) is in \( D^{-}(X, \Lambda) \). Say \( H^{i}(K) = 0 \) for \( i > b \). By induction on \( a \) we will construct a complex \( \mathcal{F}^{a} \to \ldots \to \mathcal{F}^{b} \) such that each \( \mathcal{F}^{i} \) is a finite direct sum of \( j_U! \Lambda \) with \( U \in \text{Ob}(X_{\text{etale}}) \) affine and a map \( \mathcal{F}^{\bullet} \to K \) which induces an isomorphism \( H^{i}(\mathcal{F}^{\bullet}) \to H^{i}(K) \) for \( i > a \) and a surjection \( H^{a}(\mathcal{F}^{\bullet}) \to H^{a}(K) \). For \( a = b \) this can be done by Lemma 110.4. Given such a datum choose a distinguished triangle

\[
\mathcal{F}^{\bullet} \to K \to L \to \mathcal{F}^{\bullet}[1]
\]

Then we see that \( H^{i}(L) = 0 \) for \( i > a \). Choose \( \mathcal{F}^{a-1}[−a+1] \to L \) as in Lemma 110.4. The composition \( \mathcal{F}^{a−1}[-a+1] \to L \to \mathcal{F}^{\bullet} \) corresponds to a map \( \mathcal{F}^{a−1} \to \mathcal{F}^{a} \) such that the composition with \( \mathcal{F}^{a} \to \mathcal{F}^{a+1} \) is zero. By TR4 we obtain a map

\[
(\mathcal{F}^{a−1} \to \ldots \to \mathcal{F}^{b}) \to K
\]

in \( D(X, \Lambda) \). This finishes the induction step and the proof of the lemma. \( \square \)

Lemma 110.6. Let \( X \) be a scheme. Let \( \Lambda \) be a Noetherian ring. Let \( K, L \in D^{-}_{c}(X, \Lambda) \). Then \( K \otimes_{\Lambda} L \) is in \( D^{-}_{c}(X, \Lambda) \).

Proof. This follows from Lemmas 110.5 and 70.7 \( \square \)

Definition 110.7. Let \( X \) be a scheme. Let \( \Lambda \) be a Noetherian ring. We denote \( D^{\text{ctf}}_{c}(X, \Lambda) \) the full subcategory of \( D_{c}(X, \Lambda) \) consisting of objects having locally finite tor dimension.
This is a strictly full, saturated triangulated subcategory of $D_c(X, \Lambda)$ and $D(X, \Lambda)$. By our conventions, see Cohomology on Sites, Definition 44.1, we see that

$$D_{ctf}(X, \Lambda) \subset D^b(X, \Lambda)$$

if $X$ is quasi-compact. A good way to think about objects of $D_{ctf}(X, \Lambda)$ is given in Lemma 110.9.

**Remark 110.8.** The situation with objects of $D_{ctf}(X, \Lambda)$ is different from $D_{perf}(\mathcal{O}_X)$ in Remark 109.3. Namely, it can happen that a complex of $\mathcal{O}_X$-modules is locally quasi-isomorphic to a finite complex of locally free $\mathcal{O}_X$-modules, without being globally quasi-isomorphic to a bounded complex of locally free $\mathcal{O}_X$-modules. The following lemma shows this does not happen for $D_{ctf}$ on a Noetherian scheme.

**Lemma 110.9.** Let $\Lambda$ be a Noetherian ring. Let $X$ be a quasi-compact and quasi-separated scheme. Let $K \in D(X, \Lambda)$. The following are equivalent

1. $K \in D_{ctf}(X, \Lambda)$, and
2. $K$ can be represented by a finite complex of constructible flat sheaves of $\Lambda$-modules.

In fact, if $K$ has tor amplitude in $[a, b]$ then we can represent $K$ by a complex $F^a \to \ldots \to F^b$ with $F^b$ a constructible flat sheaf of $\Lambda$-modules.

**Proof.** It is clear that a finite complex of constructible flat sheaves of $\Lambda$-modules has finite tor dimension. It is also clear that it is an object of $D_c(X, \Lambda)$. Thus we see that (2) implies (1).

Assume (1). Choose $a, b \in \mathbb{Z}$ such that $H^i(K \otimes_\Lambda \mathcal{G}) = 0$ if $i \not\in [a, b]$ for all sheaves of $\Lambda$-modules $\mathcal{G}$. We will prove the final assertion holds by induction on $b - a$. If $a = b$, then $K = H^a(K)[-a]$ is a flat constructible sheaf and the result holds. Next, assume $b > a$. Represent $K$ by a complex $K^\bullet$ of sheaves of $\Lambda$-modules. Consider the surjection

$$\text{Ker}(K^b \to K^{b+1}) \to H^b(K)$$

By Lemma 72.6 we can find finitely many affine schemes $U_i$ étale over $X$ and a surjection $\bigoplus j_{U_i!} \Lambda_{U_i} \to H^b(K)$. After replacing $U_i$ by standard étale coverings $\{U_{ij} \to U_i\}$ we may assume this surjection lifts to a map $F = \bigoplus j_{U_i!} \Lambda_{U_i} \to \text{Ker}(K^b \to K^{b+1})$. This map determines a distinguished triangle

$$F[-b] \to K \to L \to F[-b+1]$$

in $D(X, \Lambda)$. Since $D_{ctf}(X, \Lambda)$ is a triangulated subcategory we see that $L$ is in it too. In fact $L$ has tor amplitude in $[a, b - 1]$ as $F$ surjects onto $H^b(K)$ (details omitted). By induction hypothesis we can find a finite complex $F^a \to \ldots \to F^{b-1}$ of flat constructible sheaves of $\Lambda$-modules representing $L$. The map $L \to F[-b+1]$ corresponds to a map $F^b \to F$ annihilating the image of $F^{b-1} \to F^b$. Then it follows from axiom TR3 that $K$ is represented by the complex

$$F^a \to \ldots \to F^{b-1} \to F^b$$

which finishes the proof. □

**Remark 110.10.** Let $\Lambda$ be a Noetherian ring. Let $X$ be a scheme. For a bounded complex $K^\bullet$ of constructible flat $\Lambda$-modules on $X_{\text{étale}}$ each stalk $K^p_{\mathfrak{p}}$ is a finite projective $\Lambda$-module. Hence the stalks of the complex are perfect complexes of $\Lambda$-modules.
03TU Remark 110.11. Lemma 110.9 can be used to prove that if \( f : X \to Y \) is a separated, finite type morphism of schemes and \( Y \) is noetherian, then \( Rf_* \) induces a functor \( D_{ctf}(X, \Lambda) \to D_{ctf}(Y, \Lambda) \). We only need this fact in the case where \( Y \) is the spectrum of a field and \( X \) is a curve.

0962 Lemma 110.12. Let \( \Lambda \) be a Noetherian ring. If \( j : U \to X \) is an étale morphism of schemes, then

1. \( K|_U \in D_{ctf}(U, \Lambda) \) if \( K \in D_{ctf}(X, \Lambda) \), and
2. \( j_! M \in D_{ctf}(X, \Lambda) \) if \( M \in D_{ctf}(U, \Lambda) \) and the morphism \( j \) is quasi-compact and quasi-separated.

Proof. Perhaps the easiest way to prove this lemma is to reduce to the case where \( X \) is affine and then apply Lemma 110.9 to translate it into a statement about finite complexes of flat constructible sheaves of \( \Lambda \)-modules where the result follows from Lemma 72.1.

0963 Lemma 110.13. Let \( \Lambda \) be a Noetherian ring. Let \( f : X \to Y \) be a morphism of schemes. If \( K \in D_{ctf}(Y, \Lambda) \) then \( Lf^* K \in D_{ctf}(X, \Lambda) \).

Proof. Apply Lemma 110.9 to reduce this to a question about finite complexes of flat constructible sheaves of \( \Lambda \)-modules. Then the statement follows as \( f^{-1} = f^* \) is exact and Lemma 70.5.

09BI Lemma 110.14. Let \( X \) be a connected scheme. Let \( \Lambda \) be a Noetherian ring. Let \( K \in D_{ctf}(X, \Lambda) \) have locally constant cohomology sheaves. Then there exists a finite complex of finite projective \( \Lambda \)-modules \( M^\bullet \) and an étale covering \( \{U_i \to X\} \) such that \( K|_{U_i} \cong M^\bullet|_{U_i} \) in \( D(U_i, \Lambda) \).

Proof. Choose an étale covering \( \{U_i \to X\} \) such that \( K|_{U_i} \) is constant, say \( K|_{U_i} \cong M^\bullet|_{U_i} \) for some finite complex of finite \( \Lambda \)-modules \( M^\bullet \). See Cohomology on Sites, Lemma 49.1. Observe that \( U_i \times_X U_j \) is empty if \( M^\bullet \) is not isomorphic to \( M^\bullet \) in \( D(\Lambda) \). For each complex of \( \Lambda \)-modules \( M^\bullet \) let \( \text{Im}(M^\bullet) = \{ i \in I | M^\bullet_i \cong M^\bullet \} \in D(\Lambda) \). As étale morphisms are open we see that \( U_{M^\bullet} = \bigcup_{i \in \text{Im}(M^\bullet)} \text{Im}(U_i \to X) \) is an open subset of \( X \). Then \( X = \coprod U_{M^\bullet} \) is a disjoint open covering of \( X \). As \( X \) is connected only one \( U_{M^\bullet} \) is nonempty. As \( K \) is in \( D_{ctf}(X, \Lambda) \) we see that \( M^\bullet \) is a perfect complex of \( \Lambda \)-modules, see More on Algebra, Lemma 69.2. Hence we may assume \( M^\bullet \) is a finite complex of finite projective \( \Lambda \)-modules.

111. Cohomology of nice complexes

0964 The following is a special case of a more general result about compactly supported cohomology of objects of \( D_{ctf}(X, \Lambda) \).

03TV Proposition 111.1. Let \( X \) be a projective curve over a field \( k \), \( \Lambda \) a finite ring and \( K \in D_{ctf}(X, \Lambda) \). Then \( R\Gamma(X_k, K) \in D_{perf}(\Lambda) \).

Sketch of proof. The first step is to show:

1. The cohomology of \( R\Gamma(X_k, K) \) is bounded.

Consider the spectral sequence
\[
H^i(X_k, H^j(K)) \Rightarrow H^{i+j}(R\Gamma(X_k, K)).
\]
Since \( K \) is bounded and \( \Lambda \) is finite, the sheaves \( H^j(K) \) are torsion. Moreover, \( X_k \) has finite cohomological dimension, so the left-hand side is nonzero for finitely many \( i \) and \( j \) only. Therefore, so is the right-hand side.
(2) The cohomology groups $H^{i+j}(R\Gamma(X, K))$ are finite.

Since the sheaves $H^i(K)$ are constructible, the groups $H^i(X, H^j(K))$ are finite (Section 78) so it follows by the spectral sequence again.

(3) $R\Gamma(X, K)$ has finite Tor-dimension.

Let $N$ be a right $\Lambda$-module (in fact, since $\Lambda$ is finite, it suffices to assume that $N$ is finite). By the projection formula (change of module),

$$N \otimes_{\Lambda} R\Gamma(X, K) = R\Gamma(X, N \otimes_{\Lambda} K).$$

Therefore,

$$H^i(N \otimes_{\Lambda} R\Gamma(X, K)) = H^i(R\Gamma(X, N \otimes_{\Lambda} K)).$$

Now consider the spectral sequence

$$H^i(X, H^j(N \otimes_{\Lambda} K)) \Rightarrow H^{i+j}(R\Gamma(X, N \otimes_{\Lambda} K)).$$

Since $K$ has finite Tor-dimension, $H^j(N \otimes_{\Lambda} K)$ vanishes universally for $j$ small enough, and the left-hand side vanishes whenever $i < 0$. Therefore $R\Gamma(X, K)$ has finite Tor-dimension, as claimed. So it is a perfect complex by Lemma 109.2. □

112. Lefschetz numbers

The fact that the total cohomology of a constructible complex of finite tor dimension is a perfect complex is the key technical reason why cohomology behaves well, and allows us to define rigorously the traces occurring in the trace formula.

Definition 112.1. Let $\Lambda$ be a finite ring, $X$ a projective curve over a finite field $k$ and $K \in D_{ctf}(X, \Lambda)$ (for instance $K = \Lambda$). There is a canonical map $c_K : \pi^{-1}_X K \to K$, and its base change $c_K|_{X_k}$ induces an action denoted $\pi_X^*$ on the perfect complex $R\Gamma(X_k, K)$. The global Lefschetz number of $K$ is the trace $\text{Tr}(\pi_X^*|_{R\Gamma(X_k, K)})$ of that action. It is an element of $\Lambda^\natural$.

Definition 112.2. With $\Lambda, X, k, K$ as in Definition 112.1. Since $K \in D_{ctf}(X, \Lambda)$, for any geometric point $\bar{x}$ of $X$, the complex $K_{\bar{x}}$ is a perfect complex (in $D_{\text{perf}}(\Lambda)$). As we have seen in Section 100, the Frobenius $\pi_X$ acts on $K_{\bar{x}}$. The local Lefschetz number of $K$ is the sum

$$\sum_{x \in X(k)} \text{Tr}(\pi_X |_{K_{\bar{x}}})$$

which is again an element of $\Lambda^\natural$.

At last, we can formulate precisely the trace formula.

Theorem 112.3 (Lefschetz Trace Formula). Let $X$ be a projective curve over a finite field $k$, $\Lambda$ a finite ring and $K \in D_{ctf}(X, \Lambda)$. Then the global and local Lefschetz numbers of $K$ are equal, i.e.,

$$\text{Tr}(\pi_X |_{R\Gamma(X_k, K)}) = \sum_{x \in X(k)} \text{Tr}(\pi_X |_{K_{\bar{x}}})$$

in $\Lambda^\natural$.

Proof. See discussion below. □
We will use, rather than prove, the trace formula. Nevertheless, we will give quite a few details of the proof of the theorem as given in [Del77] (some of the things that are not adequately explained are listed in Section 119).

We only stated the formula for curves, and in some weak sense it is a consequence of the following result.

**Theorem 112.4 (Weil).** Let $C$ be a nonsingular projective curve over an algebraically closed field $k$, and $\varphi : C \to C$ a $k$-endomorphism of $C$ distinct from the identity. Let $V(\varphi) = \Delta_C \cdot \Gamma_{\varphi}$, where $\Delta_C$ is the diagonal, $\Gamma_{\varphi}$ is the graph of $\varphi$, and the intersection number is taken on $C \times C$. Let $J = \text{Pic}^0_C/k$ be the jacobian of $C$ and denote $\varphi^* : J \to J$ the action induced by $\varphi$ by taking pullbacks. Then

$$V(\varphi) = 1 - \text{Tr}_J(\varphi^*) + \deg \varphi.$$

**Proof.** The number $V(\varphi)$ is the number of fixed points of $\varphi$, it is equal to

$$V(\varphi) = \sum_{c \in C : \varphi(c) = c} m_{\text{Fix}}(\varphi)(c)$$

where $m_{\text{Fix}}(\varphi)(c)$ is the multiplicity of $c$ as a fixed point of $\varphi$, namely the order or vanishing of the image of a local uniformizer under $\varphi - \text{id}_C$. Proofs of this theorem can be found in [Lan02] and [Wei48]. □

**Example 112.5.** Let $C = E$ be an elliptic curve and $\varphi = [n]$ be multiplication by $n$. Then $\varphi^* = \varphi^t$ is multiplication by $n$ on the jacobian, so it has trace $2n$ and degree $n^2$. On the other hand, the fixed points of $\varphi$ are the points $p \in E$ such that $np = p$, which is the $(n-1)$-torsion, which has cardinality $(n-1)^2$. So the theorem reads

$$(n-1)^2 = 1 - 2n + n^2.$$

**Jacobians.** We now discuss without proofs the correspondence between a curve and its jacobian which is used in Weil’s proof. Let $C$ be a nonsingular projective curve over an algebraically closed field $k$ and choose a base point $c_0 \in C(k)$. Denote by $A^1(C \times C)$ (or $\text{Pic}(C \times C)$, or $\text{CaCl}(C \times C)$) the abelian group of codimension 1 divisors of $C \times C$. Then

$$A^1(C \times C) = \text{pr}_1^*(A^1(C)) \oplus \text{pr}_2^*(A^1(C)) \oplus R$$

where

$$R = \{Z \in A^1(C \times C) \mid Z|_{C \times \{c_0\}} \sim_{\text{rat}} 0 \text{ and } Z|_{\{c_0\} \times C} \sim_{\text{rat}} 0\}.$$

In other words, $R$ is the subgroup of line bundles which pull back to the trivial one under either projection. Then there is a canonical isomorphism of abelian groups $R \cong \text{End}(J)$ which maps a divisor $Z$ in $R$ to the endomorphism

$$J \mapsto \text{pr}_1(Z) \mapsto (\text{pr}_1|_Z)^*(\text{pr}_2|_Z)^*(D).$$
The aforementioned correspondence is the following. We denote by \(\sigma\) the automorphism of \(C \times C\) that switches the factors.

<table>
<thead>
<tr>
<th>(\text{End}(J))</th>
<th>(R)</th>
</tr>
</thead>
<tbody>
<tr>
<td>composition of (\alpha, \beta)</td>
<td>(\text{pr}<em>{13*}(\text{pr}</em>{12*}(\alpha) \circ \text{pr}_{23*}(\beta)))</td>
</tr>
<tr>
<td>(\text{id}_J)</td>
<td>(\Delta_C - {c_0} \times C - C \times {c_0})</td>
</tr>
<tr>
<td>(\varphi^*)</td>
<td>(\Gamma_{\varphi} - C \times {\varphi(c_0)} - \sum_{\varphi(c) = c_0} {c} \times C)</td>
</tr>
</tbody>
</table>

the trace form \(\alpha, \beta \mapsto \text{Tr}(\alpha \beta)\)

\(\alpha, \beta \mapsto -\int_{C \times C} \alpha \sigma^* \beta\)

the Rosati involution \(\alpha \mapsto \alpha^\dagger\)

\(\alpha \mapsto \sigma^* \alpha\)

positivity of Rosati

\(\text{Tr}(\alpha \alpha^\dagger) > 0\)

R
degree theorem on \(C \times C\)

\(-\int_{C \times C} \alpha \sigma^* \alpha > 0\).

In fact, in light of the Kunneth formula, the subgroup \(R\) corresponds to the 1,1 hodge classes in \(H^1(C) \otimes H^1(C)\).

**Weil’s proof.** Using this correspondence, we can prove the trace formula. We have

\[
V(\varphi) = \int_{C \times C} \Gamma_{\varphi} \cdot \Delta
= \int_{C \times C} \Gamma_{\varphi} \cdot (\Delta_C - \{c_0\} \times C - C \times \{c_0\}) + \int_{C \times C} \Gamma_{\varphi} \cdot (\{c_0\} \times C + C \times \{c_0\}).
\]

Now, on the one hand

\[
\int_{C \times C} \Gamma_{\varphi} \cdot (\{c_0\} \times C + C \times \{c_0\}) = 1 + \deg \varphi
\]

and on the other hand, since \(R\) is the orthogonal of the ample divisor \(\{c_0\} \times C + C \times \{c_0\}\),

\[
\int_{C \times C} \Gamma_{\varphi} \cdot (\Delta_C - \{c_0\} \times C - C \times \{c_0\})
= \int_{C \times C} \left(\Gamma_{\varphi} - C \times \{\varphi(c_0)\} - \sum_{\varphi(c) = c_0} \{c\} \times C\right) \cdot (\Delta_C - \{c_0\} \times C - C \times \{c_0\})
= -\text{Tr}_J(\varphi^* \circ \text{id}_J).
\]

Recapitulating, we have

\[
V(\varphi) = 1 - \text{Tr}_J(\varphi^*) + \deg \varphi
\]

which is the trace formula.
Lemma 112.6. Consider the situation of Theorem 112.4 and let $\ell$ be a prime number invertible in $k$. Then
\[
\sum_{i=0}^{2} (-1)^i \text{Tr}(\varphi^*|_{H^i(C, \mathbb{Z}/\ell^n \mathbb{Z})}) = V(\varphi) \mod \ell^n.
\]

Sketch of proof. Observe first that the assumption makes sense because $H^i(C, \mathbb{Z}/\ell^n \mathbb{Z})$ is a free $\mathbb{Z}/\ell^n \mathbb{Z}$-module for all $i$. The trace of $\varphi^*$ on the 0th degree cohomology is 1. The choice of a primitive $\ell^n$th root of unity in $k$ gives an isomorphism $H^i(C, \mathbb{Z}/\ell^n \mathbb{Z}) \cong H^i(C, \mu_{\ell^n})$ compatibly with the action of the geometric Frobenius. On the other hand, $H^1(C, \mu_{\ell^n}) = \text{Pic}(C)/\ell^n \text{Pic}(C) \cong \mathbb{Z}/\ell^n \mathbb{Z}$ where $\varphi^*$ is multiplication by $\text{deg} \varphi$. Hence
\[
\text{Tr}(\varphi^*|_{H^2(C, \mathbb{Z}/\ell^n \mathbb{Z})}) = \text{deg} \varphi.
\]
Thus we have
\[
\sum_{i=0}^{2} (-1)^i \text{Tr}(\varphi^*|_{H^i(C, \mathbb{Z}/\ell^n \mathbb{Z})}) = 1 - \text{Tr}_J(\varphi^*) + \text{deg} \varphi \mod \ell^n
\]
and the corollary follows from Theorem 112.4.

An alternative way to prove this corollary is to show that
\[
X \mapsto H^*(X, \mathbb{Q}_\ell) = \mathbb{Q}_\ell \otimes \lim_n H^*(X, \mathbb{Z}/\ell^n \mathbb{Z})
\]
defines a Weil cohomology theory on smooth projective varieties over $k$. Then the trace formula
\[
V(\varphi) = \sum_{i=0}^{2} (-1)^i \text{Tr}(\varphi^*|_{H^i(C, \mathbb{Q}_\ell)})
\]
is a formal consequence of the axioms (it’s an exercise in linear algebra, the proof is the same as in the topological case).

113. Preliminaries and sorites

Notation: We fix the notation for this section. We denote by $A$ a commutative ring, $\Lambda$ a (possibly noncommutative) ring with a ring map $A \to \Lambda$ which image lies in the center of $\Lambda$. We let $G$ be a finite group, $\Gamma$ a monoid extension of $G$ by $N$, meaning that there is an exact sequence
\[
1 \to G \to \tilde{\Gamma} \to \mathbb{Z} \to 1
\]
and $\tilde{\Gamma}$ consists of those elements of $\tilde{\Gamma}$ which image is nonnegative. Finally, we let $P$ be an $A[\Gamma]$-module which is finite and projective as an $A[G]$-module, and $M$ a $\Lambda[\Gamma]$-module which is finite and projective as a $\Lambda$-module.

Our goal is to compute the trace of $1 \in N$ acting over $\Lambda$ on the coinvariants of $G$ on $P \otimes_A M$, that is, the number
\[
\text{Tr}_\Lambda (1; (P \otimes_A M)_G) \in \Lambda^2.
\]
The element $1 \in N$ will correspond to the Frobenius.
Lemma 113.1. Let $e \in G$ denote the neutral element. The map

$$\sum \lambda_g \cdot g \mapsto \lambda_e$$

factors through $\Lambda[G]^2$. We denote $\varepsilon : \Lambda[G]^2 \to \Lambda$ the induced map.

Proof. We have to show the map annihilates commutators. One has

$$\left( \sum \lambda_g g \right) \left( \sum \mu_g g \right) - \left( \sum \mu_g g \right) \left( \sum \lambda_g g \right) = \sum_g \left( \sum_{g_1, g_2=g} \lambda_{g_1} \mu_{g_2} - \mu_{g_1} \lambda_{g_2} \right) g$$

The coefficient of $e$ is

$$\sum_g (\lambda_g \mu_{g^{-1}} - \mu_g \lambda_{g^{-1}}) = \sum_g (\lambda_g \mu_{g^{-1}} - \mu_{g^{-1}} \lambda_g)$$

which is a sum of commutators, hence it zero in $\Lambda^2$. \qed

Definition 113.2. Let $f : P \to P$ be an endomorphism of a finite projective $\Lambda[G]$-module $P$. We define

$$\text{Tr}^G_A(f; P) := \varepsilon \left( \text{Tr}_{\Lambda[G]}(f; P) \right)$$

to be the $G$-trace of $f$ on $P$.

Lemma 113.3. Let $f : P \to P$ be an endomorphism of the finite projective $\Lambda[G]$-module $P$. Then

$$\text{Tr}_A(f; P) = \#G \cdot \text{Tr}^G_A(f; P).$$

Proof. By additivity, reduce to the case $P = \Lambda[G]$. In that case, $f$ is given by right multiplication by some element $\sum \lambda_g \cdot g$ of $\Lambda[G]$. In the basis $(g)_{g \in G}$, the matrix of $f$ has coefficient $\lambda_{g^{-1}g_1}$ in the $(g_1, g_2)$ position. In particular, all diagonal coefficients are $\lambda_e$, and there are $\#G$ such coefficients. \qed

Lemma 113.4. The map $A \to \Lambda$ defines an $A$-module structure on $\Lambda^3$.

Proof. This is clear. \qed

Lemma 113.5. Let $P$ be a finite projective $\Lambda[G]$-module and $M$ a $\Lambda[G]$-module, finite projective as a $\Lambda$-module. Then $P \otimes_A M$ is a finite projective $\Lambda[G]$-module, for the structure induced by the diagonal action of $G$.

Note that $P \otimes_A M$ is naturally a $\Lambda$-module since $M$ is. Explicitly, together with the diagonal action this reads

$$\left( \sum \lambda_g g \right) (p \otimes m) = \sum gp \otimes \lambda_g gm.$$

Proof. For any $\Lambda[G]$-module $N$ one has

$$\text{Hom}_{\Lambda[G]}(P \otimes_A M, N) = \text{Hom}_{\Lambda[G]}(P, \text{Hom}_{\Lambda}(M, N))$$

where the $G$-action on $\text{Hom}_{\Lambda}(M, N)$ is given by $(g \cdot \varphi)(m) = g \varphi(g^{-1}m)$. Now it suffices to observe that the right-hand side is a composition of exact functors, because of the projectivity of $P$ and $M$. \qed

Lemma 113.6. With assumptions as in Lemma 113.5, let $u \in \text{End}_{\Lambda[G]}(P)$ and $v \in \text{End}_{\Lambda[G]}(M)$. Then

$$\text{Tr}^G_A(u \otimes v; P \otimes_A M) = \text{Tr}^G_A(u; P) \cdot \text{Tr}_A(v; M).$$
Sketch of proof. Reduce to the case \( P = A[G] \). In that case, \( u \) is right multiplication by some element \( a = \sum a_g g \) of \( A[G] \), which we write \( u = R_a \). There is an isomorphism of \( \Lambda[G] \)-modules

\[
\varphi : \quad (A[G] \otimes_A M) \\ g \otimes m \mapsto g \otimes g^{-1}m
\]

where \( (A[G] \otimes_A M) \) has the module structure given by the left \( G \)-action, together with the \( \Lambda \)-linearity on \( M \). This transport of structure changes \( u \otimes v \) into \( \sum g a_g R_g \otimes g^{-1}v \). In other words,

\[
\varphi \circ (u \otimes v) \circ \varphi^{-1} = \sum_g a_g R_g \otimes g^{-1}v.
\]

Working out explicitly both sides of the equation, we have to show

\[
\text{Tr}_{\Lambda}(\gamma, P) = \#Z_\gamma \cdot \text{Tr}_{\Lambda}(\gamma, P).
\]

This is done by showing that

\[
\text{Tr}_{\Lambda}(a_g R_g \otimes g^{-1}v) = \begin{cases} 0 & \text{if } g \neq e \\ a_e \text{Tr}_{\Lambda}(v; M) & \text{if } g = e \end{cases}
\]

by reducing to \( M = \Lambda \).

\[\square\]

Notation: Consider the monoid extension \( 1 \to G \to \Gamma \to N \to 1 \) and let \( \gamma \in \Gamma \). Then we write \( Z_\gamma = \{ g \in G | g\gamma = \gamma g \} \).

**Lemma 113.7.** Let \( P \) be a \( \Lambda[G] \)-module, finite and projective as a \( \Lambda[G] \)-module, and \( \gamma \in \Gamma \). Then

\[
\text{Tr}_{\Lambda}(\gamma, P) = \#Z_\gamma \cdot \text{Tr}_{\Lambda}(\gamma, P).
\]

**Proof.** This follows readily from Lemma 113.3.

**Lemma 113.8.** Let \( P \) be an \( A[G] \)-module, finite projective as a \( \Lambda[G] \)-module. Let \( M \) be a \( \Lambda[G] \)-module, finite projective as a \( \Lambda \)-module. Then

\[
\text{Tr}_{\Lambda}(Z_\gamma, P \otimes_A M) = \text{Tr}_{\Lambda}(Z_\gamma, P) \cdot \text{Tr}_{\Lambda}(\gamma, M).
\]

**Proof.** This follows directly from Lemma 113.6.

**Lemma 113.9.** Let \( P \) be a \( \Lambda[G] \)-module, finite projective as a \( \Lambda[G] \)-module. Then the coinvariants \( P_G = \Lambda \otimes_{\Lambda[G]} P \) form a finite projective \( \Lambda \)-module, endowed with an action of \( \Gamma/G = N \). Moreover, we have

\[
\text{Tr}_{\Lambda}(1; P_G) = \sum_{\gamma \in \Gamma} \text{Tr}_{\Lambda}(Z_\gamma, P)
\]

where \( \sum'_{\gamma \in \Gamma} \) means taking the sum over the \( G \)-conjugacy classes in \( \Gamma \).

**Sketch of proof.** We first prove this after multiplying by \( \#G \).

\[
\#G \cdot \text{Tr}_{\Lambda}(1; P_G) = \text{Tr}_{\Lambda}(\sum_{\gamma \in \Gamma} \gamma, P_G) = \text{Tr}_{\Lambda}(\sum_{\gamma \in \Gamma} \gamma, P)
\]
where the second equality follows by considering the commutative triangle

\[
\begin{array}{ccc}
P^G & \xleftarrow{c} & P_G \\
\downarrow{a} & & \downarrow{b} \\
P & \xleftarrow{\gamma} & P_Gc
\end{array}
\]

where \(a\) is the canonical inclusion, \(b\) the canonical surjection and \(c = \sum_{\gamma \to 1} \gamma\).

Then we have

\[(\sum_{\gamma \to 1} \gamma)|_P = a \circ c \circ b \quad \text{and} \quad (\sum_{\gamma \to 1} \gamma)|_{P_G} = b \circ a \circ c\]

hence they have the same trace. We then have

\[
\#G \cdot Tr_{\Lambda}(1; P_G) = \sum_{\gamma \to 1} \#G \cdot Tr_{\Lambda}(\gamma, P) = \#G \sum_{\gamma \to 1} Tr_{\Lambda}^{G}(\gamma, P).
\]

To finish the proof, reduce to case \(\Lambda\) torsion-free by some universality argument. See [Del77] for details.

**Remark 113.10.** Let us try to illustrate the content of the formula of Lemma 113.8. Suppose that \(\Lambda\), viewed as a trivial \(\Gamma\)-module, admits a finite resolution

\[0 \to P_r \to \ldots \to P_1 \to P_0 \to \Lambda \to 0\]

by some \(\Lambda[\Gamma]\)-modules \(P_i\) which are finite and projective as \(\Lambda[G]\)-modules. In that case

\[H_*(((P_\bullet)_G) = \text{Tor}^\Lambda[G]_*(\Lambda, \Lambda) = H_*(G, \Lambda)\]

and

\[\text{Tr}_{\Lambda}^{G}(\gamma, P_\bullet) = \frac{1}{\#Z_\gamma} \text{Tr}_{\Lambda}(\gamma, P_\bullet) = \frac{1}{\#Z_\gamma} \text{Tr}(\gamma, \Lambda) = \frac{1}{\#Z_\gamma}.
\]

Therefore, Lemma 113.8 says

\[\text{Tr}_{\Lambda}(1, P_G) = \text{Tr}(1|_{H_*(G, \Lambda)}) = \sum_{\gamma \to 1} \frac{1}{\#Z_\gamma}.
\]

This can be interpreted as a point count on the stack \(BG\). If \(\Lambda = \mathbb{F}_\ell\) with \(\ell\) prime to \#G, then \(H_*(G, \Lambda)\) is \(\mathbb{F}_\ell\) in degree 0 (and 0 in other degrees) and the formula reads

\[1 = \sum_{\sigma \text{-conjugacy classes}} \frac{1}{\#Z_\gamma} \mod \ell.
\]

This is in some sense a “trivial” trace formula for \(G\). Later we will see that (112.3.1) can in some cases be viewed as a highly nontrivial trace formula for a certain type of group, see Section 128.

### 114. Proof of the trace formula

**Theorem 114.1.** Let \(k\) be a finite field and \(X\) a finite type, separated scheme of dimension at most 1 over \(k\). Let \(\Lambda\) be a finite ring whose cardinality is prime to that of \(k\), and \(K \in D_{ctf}(X, \Lambda)\). Then

\[(114.1.1) \quad \text{Tr}(\pi_X|_{RF_c(X, \mathbb{C}, K)}) = \sum_{x \in X(k)} \text{Tr}(\pi_x|_{K_x}) \in \Lambda^5.
\]
Please see Remark 114.2 for some remarks on the statement. Notation: For short, we write

\[ T'(X, K) = \sum_{x \in X(k)} \text{Tr}(\pi_x|_{K_x}) \]

for the right-hand side of 114.1.1 and

\[ T''(X, K) = \text{Tr}(\pi_x^*|_{R\Gamma_c(X,k,K)}) \]

for the left-hand side.

**Proof of Theorem 114.1.** The proof proceeds in a number of steps.

**Step 1.** Let \( j : U \hookrightarrow X \) be an open immersion with complement \( Y = X - U \) and \( i : Y \hookrightarrow X \). Then \( T''(X, K) = T''(U, j^{-1}K) + T''(Y, i^{-1}K) \) and \( T'(X, K) = T'(U, j^{-1}K) + T'(Y, i^{-1}K) \).

This is clear for \( T' \). For \( T'' \) use the exact sequence

\[ 0 \rightarrow j^*j^{-1}K \rightarrow K \rightarrow i_*i^{-1}K \rightarrow 0 \]

to get a filtration on \( K \). This gives rise to an object \( \tilde{K} \in DF(X, \Lambda) \) whose graded pieces are \( j^*j^{-1}K \) and \( i_*i^{-1}K \), both of which lie in \( D_{ctf}(X, \Lambda) \). Then, by filtered derived abstract nonsense (INSERT REFERENCE), \( R\Gamma_c(X, K) \in DF_{\text{perf}}(\Lambda) \), and it comes equipped with \( \pi_*^* \) in \( DF_{\text{perf}}(\Lambda) \). By the discussion of traces on filtered complexes (INSERT REFERENCE) we get

\[
\text{Tr}(\pi_X^*|_{R\Gamma_c(X,K)}) = \text{Tr}(\pi_X^*|_{R\Gamma_c(X,k,j^{-1}K)}) + \text{Tr}(\pi_X^*|_{R\Gamma_c(X,k,i^{-1}K)})
\]

\[
= T''(U, i^{-1}K) + T''(Y, i^{-1}K).
\]

**Step 2.** The theorem holds if \( \dim X \leq 0 \).

Indeed, in that case

\[ R\Gamma_c(X, K) = R\Gamma(X, K) = \Gamma(X, K) = \bigoplus_{\tilde{x} \in X_k} K_{\tilde{x}} \leftarrow \pi_X^* \cdot \]

Since the fixed points of \( \pi_X : X_k \rightarrow X_k \) are exactly the points \( \tilde{x} \in X_k \) which lie over a \( k \)-rational point \( x \in X(k) \) we get

\[ \text{Tr}(\pi_X^*|_{R\Gamma_c(X,K)}) = \sum_{x \in X(k)} \text{Tr}(\pi_{\tilde{x}}|_{K_{\tilde{x}}}). \]

**Step 3.** It suffices to prove the equality \( T''(U, \mathcal{F}) = T''(U, \mathcal{F}) \) in the case where

- \( U \) is a smooth irreducible affine curve over \( k \),
- \( U(k) = \emptyset \),
- \( K = \mathcal{F} \) is a finite locally constant sheaf of \( \Lambda \)-modules on \( U \) whose stalk(s) are finite projective \( \Lambda \)-modules, and
- \( \Lambda \) is killed by a power of a prime \( \ell \) and \( \ell \in k^* \).

Indeed, because of Step 2, we can throw out any finite set of points. But we have only finitely many rational points, so we may assume there are none\(^{13}\). We may assume that \( U \) is smooth irreducible and affine by passing to irreducible components and throwing away the bad points if necessary. The assumptions of \( \mathcal{F} \) come from unwinding the definition of \( D_{ctf}(X, \Lambda) \) and those on \( \Lambda \) from considering its primary decomposition.

\(^{13}\)At this point, there should be an evil laugh in the background.
For the remainder of the proof, we consider the situation

\[
\begin{align*}
\mathcal{V} &\longrightarrow Y \\
\mathcal{U} &\longrightarrow X
\end{align*}
\]

where \( \mathcal{U} \) is as above, \( f \) is a finite étale Galois covering, \( \mathcal{V} \) is connected and the horizontal arrows are projective completions. Denoting \( G = \text{Aut}(\mathcal{V}/\mathcal{U}) \), we also assume (as we may) that \( f^{-1}F = M \) is constant, where the module \( M = \Gamma(\mathcal{V}, f^{-1}F) \) is a \( \Lambda[G] \)-module which is finite and projective over \( \Lambda \). This corresponds to the trivial monoid extension

\[ 1 \to G \to \Gamma = G \times N \to N \to 1. \]

In that context, using the reductions above, we need to show that \( T''(\mathcal{U}, F) = 0 \).

Step 4. There is a natural action of \( G \) on \( f_*f^{-1}F \) and the trace map \( f_*f^{-1}F \to F \) defines an isomorphism

\[ (f_*f^{-1}F) \otimes_{\Lambda[G]} \Lambda = (f_*f^{-1}F)|_G \cong F. \]

To prove this, simply unwind everything at a geometric point.

Step 5. Let \( A = \mathbb{Z}/\ell^n\mathbb{Z} \) with \( n \gg 0 \). Then \( f_*f^{-1}F \cong (f_*A) \otimes_{\Lambda} \bar{M} \) with diagonal \( G \)-action.

Step 6. There is a canonical isomorphism \( (f_*A \otimes_{\Lambda} \bar{M}) \otimes_{\Lambda[G]} \Lambda \cong F \).

In fact, this is a derived tensor product, because of the projectivity assumption on \( F \).

Step 7. There is a canonical isomorphism

\[ R\Gamma_c(\mathcal{U}_k, F) = (R\Gamma_c(\mathcal{U}_k, f_*A) \otimes^{L}_{\Lambda} \bar{M}) \otimes^{L}_{\Lambda[G]} \Lambda, \]

compatible with the action of \( \pi_U^* \).

This comes from the universal coefficient theorem, i.e., the fact that \( R\Gamma_c \) commutes with \( \otimes^{L} \), and the flatness of \( F \) as a \( \Lambda \)-module.

We have

\[ \text{Tr}(\pi_U^*|_{R\Gamma_c(\mathcal{U}_k, F)}) = \sum_{g \in G} ' \text{Tr}_{A}^Z((g, \pi_U^*)|_{R\Gamma_c(\mathcal{U}_k, f_*A)}, \bar{M}) \]

\[ = \sum_{g \in G} ' \text{Tr}_{A}^Z(((g, \pi_U^*)|_{R\Gamma_c(\mathcal{U}_k, f_*A)}) \cdot \text{Tr}_{A}(g|_{M}) \]

where \( \Gamma \) acts on \( R\Gamma_c(\mathcal{U}_k, F) \) by \( G \) and \((e,1)\) acts via \( \pi_U^* \). So the monoidal extension is given by \( \Gamma = G \times N \to N, \gamma \mapsto 1 \). The first equality follows from Lemma \([113.9]\) and the second from Lemma \([113.8]\).

Step 8. It suffices to show that \( \text{Tr}_{A}^Z((g, \pi_U^*)|_{R\Gamma_c(\mathcal{U}_k, f_*A)}) \in A \) maps to zero in \( \Lambda \).

Recall that

\[ \#Z_g \cdot \text{Tr}_{A}^Z((g, \pi_U^*)|_{R\Gamma_c(\mathcal{U}_k, f_*A)}) = \text{Tr}_{A}((g, \pi_U^*)|_{R\Gamma_c(\mathcal{U}_k, f_*A)}) \]

\[ = \text{Tr}_{A}((g^{-1} \pi_V^*)|_{R\Gamma_c(\mathcal{V}_c, A)}). \]

The first equality is Lemma \([113.7]\), the second is the Leray spectral sequence, using the finiteness of \( f \) and the fact that we are only taking traces over \( A \). Now since
A = \mathbb{Z}/\ell^n\mathbb{Z} with \( n \gg 0 \) and \( \#\mathbb{Z}_\ell = \ell^a \) for some (fixed) \( a \), it suffices to show the following result.

Step 9. We have \( \text{Tr}_A((g^{-1}\pi_Y)^*|_{R\Gamma_c(V, A)}) = 0 \) in \( A \).

By additivity again, we have

\[
\text{Tr}_A((g^{-1}\pi_Y)^*|_{R\Gamma_c(V, A)}) + \text{Tr}_A((g^{-1}\pi_Y)^*|_{R\Gamma_c(Y - V, A)}) = \text{Tr}_A((g^{-1}\pi_Y)^*|_{R\Gamma(Y_k, A)})
\]

The latter trace is the number of fixed points of \( g^{-1}\pi_Y \) on \( Y \), by Weil’s trace formula Theorem 112.4. Moreover, by the 0-dimensional case already proven in step 2,

\[
\text{Tr}_A((g^{-1}\pi_Y)^*|_{R\Gamma_c(Y - V, A)})
\]

is the number of fixed points of \( g^{-1}\pi_Y \) on \( Y \). Therefore,

\[
\text{Tr}_A((g^{-1}\pi_Y)^*|_{R\Gamma_c(V, A)})
\]

is the number of fixed points of \( g^{-1}\pi_Y \) on \( V \). But there are no such points: if \( \bar{y} \in Y_k \) is fixed under \( g^{-1}\pi_Y \), then \( \bar{f}(\bar{y}) \in X_k \) is fixed under \( \pi_X \). But \( \mathcal{U} \) has no \( k \)-rational point, so we must have \( \bar{f}(\bar{y}) \notin (X_\overline{k}) \) and so \( \bar{y} \notin V_k \), a contradiction. This finishes the proof.

### Remark 114.2

03UI Remarks on Theorem 114.1

1. This formula holds in any dimension. By a dévissage lemma (which uses proper base change etc.) it reduces to the current statement – in that generality.

2. The complex \( R\Gamma_c(X_{\overline{k}}, K) \) is defined by choosing an open immersion \( j : X \rightarrow \bar{X} \) with \( \bar{X} \) projective over \( k \) of dimension at most 1 and setting

\[
R\Gamma_c(X_{\overline{k}}, K) := R\Gamma(\bar{X}_{\overline{k}}, j!*K).
\]

This is independent of the choice of \( \bar{X} \) follows from (insert reference here).

We define \( H^i_c(X_{\overline{k}}, K) \) to be the \( i \)th cohomology group of \( R\Gamma_c(X_{\overline{k}}, K) \).

### Remark 114.3

03UJ Even though all we did are reductions and mostly algebra, the trace formula Theorem 114.1 is much stronger than Weil’s geometric trace formula (Theorem 112.4) because it applies to coefficient systems (sheaves), not merely constant coefficients.

### 115. Applications

03UK OK, having indicated the proof of the trace formula, let’s try to use it for something.

### 116. On \( \ell \)-adic sheaves

03UL **Definition 116.1.** Let \( X \) be a noetherian scheme. A \( \mathbb{Z}_\ell \)-sheaf on \( X \), or simply an \( \ell \)-adic sheaf \( \mathcal{F} \) is an inverse system \( \{\mathcal{F}_n\}_{n \geq 1} \) where

1. \( \mathcal{F}_n \) is a constructible \( \mathbb{Z}/\ell^n\mathbb{Z} \)-module on \( X_{\text{étale}} \), and
2. the transition maps \( \mathcal{F}_{n+1} \rightarrow \mathcal{F}_n \) induce isomorphisms \( \mathcal{F}_{n+1} \otimes_{\mathbb{Z}/\ell^{n+1}\mathbb{Z}} \mathbb{Z}/\ell^n\mathbb{Z} \cong \mathcal{F}_n \).

We say that \( \mathcal{F} \) is lisse if each \( \mathcal{F}_n \) is locally constant. A morphism of such is merely a morphism of inverse systems.
Lemma 116.2. Let \( \{ G_n \}_{n \geq 1} \) be an inverse system of constructible \( \mathbb{Z}/\ell^n \mathbb{Z} \)-modules. Suppose that for all \( k \geq 1 \), the maps
\[
G_{n+1}/\ell^k G_{n+1} \to G_n/\ell^k G_n
\]
are isomorphisms for all \( n \gg 0 \) (where the bound possibly depends on \( k \)). In other words, assume that the system \( \{ G_n/\ell^k G_n \}_{n \geq 1} \) is eventually constant, and call \( F_k \) the corresponding sheaf. Then the system \( \{ F_k \}_{k \geq 1} \) forms a \( \mathbb{Z}_\ell \)-sheaf on \( X \).

Proof. The proof is obvious. \( \square \)

Lemma 116.3. The category of \( \mathbb{Z}_\ell \)-sheaves on \( X \) is abelian.

Proof. Let \( \Phi = \{ \varphi_n \}_{n \geq 1} : \{ F_n \} \to \{ G_n \} \) be a morphism of \( \mathbb{Z}_\ell \)-sheaves. Set
\[
\text{Coker}(\Phi) = \left\{ \text{Coker} \left( F_n \to G_n \right) \right\}_{n \geq 1}
\]
and \( \text{Ker}(\Phi) \) is the result of Lemma 116.2 applied to the inverse system
\[
\left\{ \bigcap_{m \geq n} \text{Im} \left( \text{Ker}(\varphi_m) \to \text{Ker}(\varphi_n) \right) \right\}_{n \geq 1}
\]
That this defines an abelian category is left to the reader. \( \square \)

Example 116.4. Let \( X = \text{Spec}(\mathbb{C}) \) and \( \Phi : \mathbb{Z}_\ell \to \mathbb{Z}_\ell \) be multiplication by \( \ell \). More precisely,
\[
\Phi = \left\{ \mathbb{Z}/\ell^n \mathbb{Z} \to \mathbb{Z}/\ell^n \mathbb{Z} \right\}_{n \geq 1}.
\]
To compute the kernel, we consider the inverse system
\[
\ldots \to \mathbb{Z}/\ell \mathbb{Z} \to \mathbb{Z}/\ell \mathbb{Z} \to \mathbb{Z}/\ell \mathbb{Z}.
\]
Since the images are always zero, \( \text{Ker}(\Phi) \) is zero as a system.

Remark 116.5. If \( F = \{ F_n \}_{n \geq 1} \) is a \( \mathbb{Z}_\ell \)-sheaf on \( X \) and \( \bar{x} \) is a geometric point then \( M_n = \{ F_{n, \bar{x}} \} \) is an inverse system of finite \( \mathbb{Z}/\ell^n \mathbb{Z} \)-modules such that \( M_{n+1} \to M_n \) is surjective and \( M_n = M_{n+1}/\ell^n M_{n+1} \). It follows that
\[
M = \lim_{n \to \infty} M_n = \lim_{n \to \infty} F_{n, \bar{x}}
\]
is a finite \( \mathbb{Z}_\ell \)-module. Indeed, \( M/\ell M = M_1 \) is finite over \( \mathbb{F}_\ell \), so by Nakayama \( M \) is finite over \( \mathbb{Z}_\ell \). Therefore, \( M \cong \mathbb{Z}_\ell^{\oplus r} \oplus \bigoplus_{i=1}^t \mathbb{Z}/\ell^{e_i} \mathbb{Z} \). The module \( M = F_{\bar{x}} \) is called the stalk of \( F \) at \( \bar{x} \).

Definition 116.6. A \( \mathbb{Z}_\ell \)-sheaf \( F \) is torsion if \( \ell^n : F \to F \) is the zero map for some \( n \). The abelian category of \( \mathbb{Q}_\ell \)-sheaves on \( X \) is the quotient of the abelian category of \( \mathbb{Z}_\ell \)-sheaves by the Serre subcategory of torsion sheaves. In other words, its objects are \( \mathbb{Z}_\ell \)-sheaves on \( X \), and if \( F, G \) are two such, then
\[
\text{Hom}_{\mathbb{Q}_\ell} (F, G) = \text{Hom}_{\mathbb{Z}_\ell} (F, G) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell.
\]
We denote by \( F \mapsto F \otimes \mathbb{Q}_\ell \) the quotient functor (right adjoint to the inclusion). If \( F = F' \otimes \mathbb{Q}_\ell \) where \( F' \) is a \( \mathbb{Z}_\ell \)-sheaf and \( \bar{x} \) is a geometric point, then the stalk of \( F \) at \( \bar{x} \) is \( F_{\bar{x}} = F'_{\bar{x}} \otimes \mathbb{Q}_\ell \).

Remark 116.7. Since a \( \mathbb{Z}_\ell \)-sheaf is only defined on a noetherian scheme, it is torsion if and only if its stalks are torsion.
**Étale Cohomology**

**Definition 116.8.** If $X$ is a separated scheme of finite type over an algebraically closed field $k$ and $F = \{F_n\}_{n \geq 1}$ is a $\mathbb{Z}_\ell$-sheaf on $X$, then we define

$$H^i(X, F) := \lim_{n \to \infty} H^i(X, F_n)$$

and

$$H^i_c(X, F) := \lim_{n \to \infty} H^i_c(X, F_n).$$

If $F = F' \otimes \mathbb{Q}_\ell$ for a $\mathbb{Z}_\ell$-sheaf $F'$ then we set

$$H^i_c(X, F) := H^i_c(X, F') \otimes \mathbb{Z}_\ell \mathbb{Q}_\ell.$$  

We call these the $\ell$-adic cohomology of $X$ with coefficients $F$.

**117. L-functions**

**Definition 117.1.** Let $X$ be a scheme of finite type over a finite field $k$. Let $\Lambda$ be a finite ring of order prime to the characteristic of $k$ and $F$ a constructible flat $\Lambda$-module on $X_{\text{étale}}$. Then we set

$$L(X, F) := \prod_{x \in |X|} \det(1 - \pi_x^* T^{\deg x}|_{F_x})^{-1} \in \Lambda[[T]]$$

where $|X|$ is the set of closed points of $X$, $\deg x = [\kappa(x) : k]$ and $\bar{x}$ is a geometric point lying over $x$. This definition clearly generalizes to the case where $F$ is replaced by a $K \in D\ctf(X, \Lambda)$. We call this the $L$-function of $F$.

**Remark 117.2.** Intuitively, $T$ should be thought of as $T = t^f$ where $p^f = \#k$. The definitions are then independent of the size of the ground field.

**Definition 117.3.** Now assume that $F$ is a $\mathbb{Q}_\ell$-sheaf on $X$. In this case we define

$$L(X, F) := \prod_{x \in |X|} \det(1 - \pi_x^* T^{\deg x}|_{F_x})^{-1} \in \mathbb{Q}_\ell[[T]].$$

Note that this product converges since there are finitely many points of a given degree. We call this the $L$-function of $F$.

118. Cohomological interpretation

**Theorem 118.1 (Finite Coefficients).** Let $X$ be a scheme of finite type over a finite field $k$. Let $\Lambda$ be a finite ring of order prime to the characteristic of $k$ and $F$ a constructible flat $\Lambda$-module on $X_{\text{étale}}$. Then

$$L(X, F) = \det(1 - \pi_X^* T|_{H^i_c(X_{\bar{k}}, F)})^{-1} \in \Lambda[[T]].$$

**Proof.** Omitted. □

Thus far, we don’t even know whether each cohomology group $H^i_c(X_{\bar{k}}, F)$ is free.

**Theorem 118.2 (Adic sheaves).** Let $X$ be a scheme of finite type over a finite field $k$, and $F$ a $\mathbb{Q}_\ell$-sheaf on $X$. Then

$$L(X, F) = \prod \det(1 - \pi_X^* T|_{H^i(X_{\bar{k}}, F)})^{-1} \in \mathbb{Q}_\ell[[T]].$$

**Proof.** This is sketched below. □

**Remark 118.3.** Since we have only developed some theory of traces and not of determinants, Theorem [118.1] is harder to prove than Theorem [118.2]. We will only prove the latter, for the former see [Del77]. Observe also that there is no version of this theorem more general for $\mathbb{Z}_\ell$ coefficients since there is no $\ell$-torsion.
We reduce the proof of Theorem 118.2 to a trace formula. Since $\mathbb{Q}_p$ has characteristic 0, it suffices to prove the equality after taking logarithmic derivatives. More precisely, we apply $T \frac{d}{dT} \log$ to both sides. We have on the one hand

$$T \frac{d}{dT} \log L(X, F) = T \frac{d}{dT} \log \prod_{x \in |X|} \det(1 - \pi_x^* T^{\deg x} |_{F_x})^{-1}$$

$$= \sum_{x \in |X|} T \frac{d}{dT} \log(\det(1 - \pi_x^* T^{\deg x} |_{F_x})^{-1})$$

$$= \sum_{x \in |X|} \deg x \sum_{n \geq 1} \Tr((\pi_x^n)^* |_{F_x}) T^{n \deg x}$$

where the last equality results from the formula

$$T \frac{d}{dT} \log \left( \det(1 - fT |_M)^{-1} \right) = \sum_{n \geq 1} \Tr(f^n |_M) T^n$$

which holds for any commutative ring $\Lambda$ and any endomorphism $f$ of a finite projective $\Lambda$-module $M$. On the other hand, we have

$$T \frac{d}{dT} \log \left( \prod_{i \in I} \det(1 - \pi_X^* T|_{H^i_i(X,k,F)})^{(-1)^{i+1}} \right)$$

$$= \sum_i (-1)^i \sum_{n \geq 1} \Tr((\pi_X^n)^* |_{H^i_i(X,F)}) T^n$$

by the same formula again. Now, comparing powers of $T$ and using the Mobius inversion formula, we see that Theorem 118.2 is a consequence of the following equality

$$\sum_d \frac{d}{|X|} \sum_{x \in d} \Tr((\pi_X^{n/d})^* |_{F_x}) = \sum_i (-1)^i \Tr((\pi_X^n)^* |_{H^i_i(X,F)})$$

Writing $k_n$ for the degree $n$ extension of $k$, $X_n = X \times_{\Spec k} \Spec(k_n)$ and $nF = F |_{X_n}$, this boils down to

$$\sum_{x \in X_n(k_n)} \Tr(\pi_X^* |_{X_n}) = \sum_i (-1)^i \Tr((\pi_X^n)^* |_{H^i_i(X_n,F)})$$

which is a consequence of Theorem 118.5.

**Theorem 118.4.** Let $X/k$ be as above, let $\Lambda$ be a finite ring with $\# \Lambda \in k^*$ and $K \in D_{clf}(X, \Lambda)$. Then $R\Gamma_c(X_k, K) \in D_{perf}(\Lambda)$ and

$$\sum_{x \in X(k)} \Tr(\pi_X^* |_{X_k}) = \Tr(\pi_X^* |_{R\Gamma_c(X_k, K)})$$

**Proof.** Note that we have already proved this (REFERENCE) when $\dim X \leq 1$. The general case follows easily from that case together with the proper base change theorem. □

**Theorem 118.5.** Let $X$ be a separated scheme of finite type over a finite field $k$ and $F$ be a $\mathbb{Q}_p$-sheaf on $X$. Then $\dim_{\mathbb{Q}_p} H^i_c(X_k, F)$ is finite for all $i$, and is nonzero for $0 \leq i \leq 2 \dim X$ only. Furthermore, we have

$$\sum_{x \in X(k)} \Tr(\pi_x^* |_{F_x}) = \sum_i (-1)^i \Tr(\pi_X^* |_{H^i_i(X_k, F)})$$.
Proof. We explain how to deduce this from Theorem \[\text{118.4}\]. We first use some étale cohomology arguments to reduce the proof to an algebraic statement which we subsequently prove.

Let \( \mathcal{F} \) be as in the theorem. We can write \( \mathcal{F} = \mathcal{F}' \otimes \mathbb{Q}_\ell \) where \( \mathcal{F}' = \{ \mathcal{F}'_i \} \) is a \( \mathbb{Z}_\ell \)-sheaf without torsion, i.e., \( \ell : \mathcal{F}' \to \mathcal{F}' \) has trivial kernel in the category of \( \mathbb{Z}_\ell \)-sheaves. Then each \( \mathcal{F}'_i \) is a flat constructible \( \mathbb{Z}/\ell^n \mathbb{Z} \)-module on \( X_{\text{étale}} \), so \( \mathcal{F}'_n \in D_{\text{ctf}}(X, \mathbb{Z}/\ell^n \mathbb{Z}) \) and \( \mathcal{F}'_{n+1} \otimes_{\mathbb{Z}/\ell^{n+1} \mathbb{Z}} \mathcal{F}'_n = \mathcal{F}'_n \). Note that the last equality holds also for standard (non-derived) tensor product, since \( \mathcal{F}'_n \) is flat (it is the same equality). Therefore,

1. the complex \( K_n = R\Gamma'_c(X_\ell, \mathcal{F}'_n) \) is perfect, and it is endowed with an endomorphism \( \pi_n : K_n \to K_n \) in \( D(\mathbb{Z}/\ell^n \mathbb{Z}) \),
2. there are identifications \( K_{n+1} \otimes_{\mathbb{Z}/\ell^{n+1} \mathbb{Z}} \mathbb{Z}/\ell^n \mathbb{Z} = K_n \in D_{\text{perf}}(\mathbb{Z}/\ell^n \mathbb{Z}) \), compatible with the endomorphisms \( \pi_{n+1} \) and \( \pi_n \) (see [Del77, Rapport 4.12]),
3. the equality \( \text{Tr}(\pi_X^n|_{K_n}) = \sum_{x \in X(k)} \text{Tr}(\pi_x|_{\mathcal{F}'_n}) \) holds, and
4. for each \( x \in X(k) \), the elements \( \text{Tr}(\pi_x|_{\mathcal{F}'_n}) \) in \( \mathbb{Z}/\ell^n \mathbb{Z} \) form an element \( \mathcal{Z}_\ell \) which is equal to \( \text{Tr}(\pi_x|_{\mathcal{F}'_n}) \in \mathbb{Q}_\ell \).

It then suffices to prove the following algebra lemma.

\[\boxed{\text{Lemma 118.6.} \quad \text{Suppose we have} \ K_n \in D_{\text{perf}}(\mathbb{Z}/\ell^n \mathbb{Z}), \ \pi_n : K_n \to K_n \ \text{and isomorphisms} \ \varphi_n : K_{n+1} \otimes_{\mathbb{Z}/\ell^{n+1} \mathbb{Z}} \mathbb{Z}/\ell^n \mathbb{Z} \to K_n \ \text{compatible with} \ \pi_{n+1} \ \text{and} \ \pi_n. \ \text{Then}
\]

1. the elements \( t_n = \text{Tr}(\pi_n|_{K_n}) \in \mathbb{Z}/\ell^n \mathbb{Z} \) form an element \( t_\infty = \{ t_n \} \) of \( \mathbb{Z}_\ell \),
2. the \( \mathbb{Z}_\ell \)-module \( H^\infty = \lim_n H^i(K_n) \) is finite and is nonzero for finitely many \( i \) only, and
3. the operators \( H^i(\pi_n) : H^i(K_n) \to H^i(K_n) \) are compatible and define \( \pi^\infty : H^\infty \to H^\infty \) satisfying

\[ 1 = \sum (-1)^i \text{Tr}(\pi^\infty|_{H^i(\mathcal{Z}_\ell \otimes \mathbb{Q}_\ell)}) = t_\infty. \]

Proof. Since \( \mathbb{Z}/\ell^n \mathbb{Z} \) is a local ring and \( K_{n} \) is perfect, each \( K_n \) can be represented by a finite complex \( K_n^\bullet \) of finite free \( \mathbb{Z}_\ell \)-modules such that the map \( K_n^p \to K_n^{p+1} \) has image contained in \( \ell K_n^{p+1} \). It is a fact that such a complex is unique up to isomorphism. Moreover \( \pi_n \) can be represented by a morphism of complexes \( \pi^\bullet_n : K_n^\bullet \to K_n^\bullet \) (which is unique up to homotopy). By the same token the isomorphism \( \varphi_n : K_{n+1} \otimes_{\mathbb{Z}/\ell^{n+1} \mathbb{Z}} \mathbb{Z}/\ell^n \mathbb{Z} \to K_n^\bullet \) is represented by a map of complexes

\[ \varphi^\bullet_n : K_{n+1}^\bullet \otimes_{\mathbb{Z}/\ell^{n+1} \mathbb{Z}} \mathbb{Z}/\ell^n \mathbb{Z} \to K_n^\bullet. \]

In fact, \( \varphi^\bullet_n \) is an isomorphism of complexes, thus we see that

- there exist \( a, b \in \mathbb{Z} \) independent of \( n \) such that \( K_n^i = 0 \) for all \( i \not\in [a, b] \), and
- the rank of \( K_n^i \) is independent of \( n \).

Therefore, the module \( K_n^\bullet = \lim_n \{ K_n^i, \varphi^i_n \} \) is a finite free \( \mathbb{Z}_\ell \)-module and \( K_n^\bullet \) is a finite complex of finite free \( \mathbb{Z}_\ell \)-modules. By induction on the number of nonzero terms, one can prove that \( H^i(K_n^\bullet) = \lim_n H^i(K_n^\bullet) \) (this is not true for unbounded complexes). We conclude that \( H^\infty = H^i(K_n^\bullet) \) is a finite \( \mathbb{Z}_\ell \)-module. This proves
To prove the remainder of the lemma, we need to overcome the possible non-commutativity of the diagrams

\[ K_{n+1}^\bullet \xrightarrow{\varphi_{n+1}^\bullet} K_n^\bullet \]

\[ \pi_{n+1}^\bullet \downarrow \quad \downarrow \pi_n^\bullet \]

\[ K_{n+1}^\bullet \xrightarrow{\varphi_n^\bullet} K_n^\bullet. \]

However, this diagram does commute in the derived category, hence it commutes up to homotopy. We inductively replace \( \pi_n^\bullet \) for \( n \geq 2 \) by homotopic maps of complexes making these diagrams commute. Namely, if \( h_i : K_{n+1}^i \to K_n^{i-1} \) is a homotopy, i.e.,

\[ \pi_n^\bullet \circ \varphi_n^\bullet - \varphi_n^\bullet \circ \pi_{n+1}^\bullet = dh + hd, \]

then we choose \( \tilde{h}_i : K_{n+1}^i \to K_n^{i-1} \) lifting \( h_i \). This is possible because \( K_{n+1}^i \) free and \( K_n^{i-1} \to K_n^{i-1} \) is surjective. Then replace \( \pi_n^\bullet \) by \( \tilde{\pi}_n^\bullet \) defined by

\[ \tilde{\pi}_n^\bullet + 1 = \pi_n^\bullet + d\tilde{h} + \tilde{hd}. \]

With this choice of \( \{ \pi_n^\bullet \} \), the above diagrams commute, and the maps fit together to define an endomorphism \( \pi_\infty^\bullet = \lim_n \pi_n^\bullet \) of \( K_\infty^\bullet \). Then part \( i \) is clear: the elements \( t_n = \sum (-1)^i \text{Tr}(\pi_n^i|_{K_n^i}) \) fit into an element \( t_\infty \) of \( \mathbb{Z}_\ell \). Moreover

\[ t_\infty = \sum (-1)^i \text{Tr}_{\mathbb{Z}_\ell}(\pi_\infty^i|_{K_\infty^i}) \]

\[ = \sum (-1)^i \text{Tr}_{\mathbb{Q}_\ell}(\pi_\infty^i|_{K_\infty^i \otimes \mathbb{Q}_\ell}) \]

\[ = \sum (-1)^i \text{Tr}(\pi_\infty^i|_{H^i(K_\infty^\bullet \otimes \mathbb{Q}_\ell)}) \]

where the last equality follows from the fact that \( \mathbb{Q}_\ell \) is a field, so the complex \( K_\infty^\bullet \otimes \mathbb{Q}_\ell \) is quasi-isomorphic to its cohomology \( H^i(K_\infty^\bullet \otimes \mathbb{Q}_\ell) \). The latter is also equal to \( H^i(K_\infty^\bullet) \otimes \mathbb{Q}_\ell = H^i_\infty \otimes \mathbb{Q}_\ell \), which finishes the proof of the lemma, and also that of Theorem 118.5. \( \square \)

119. List of things which we should add above

03V5 What did we skip the proof of in the lectures so far:

1. curves and their Jacobians,
2. proper base change theorem,
3. inadequate discussion of \( R\Gamma_c \),
4. more generally, given \( f : X \to S \) finite type, separated \( S \) quasi-projective, discussion of \( Rf_! \) on étale sheaves.
5. discussion of \( \otimes^L \)
6. discussion of why \( R\Gamma_c \) commutes with \( \otimes^L \)

120. Examples of \( L \)-functions

03V6 We use Theorem 118.2 for curves to give examples of \( L \)-functions.
121. Constant sheaves

Let $k$ be a finite field, $X$ a smooth, geometrically irreducible curve over $k$ and $\mathcal{F} = \mathbb{Q}_\ell$ the constant sheaf. If $\bar{x}$ is a geometric point of $X$, the Galois module $\mathcal{F}_{\bar{x}} = \mathbb{Q}_\ell$ is trivial, so

$$\det(1 - \pi_\ell^* T^{\deg x}|_{\mathcal{F}_{\bar{x}}})^{-1} = \frac{1}{1 - T^{\deg x}}.$$ 

Applying Theorem [118.2](#) we get

$$L(X, \mathcal{F}) = \prod_{i=0}^2 \det(1 - \pi_X^* T|_{H^i(X_k, \mathbb{Q}_\ell)(-1)^i+1}) = \frac{\det(1 - \pi_X^* T|_{H^2(X_k, \mathbb{Q}_\ell)})}{\det(1 - \pi_X^* T|_{H^0(X_k, \mathbb{Q}_\ell)}) \cdot \det(1 - \pi_X^* T|_{H^2(X_k, \mathbb{Q}_\ell)})}.$$ 

To compute the latter, we distinguish two cases.

**Projective case.** Assume that $X$ is projective, so $H^i(X_k, \mathbb{Q}_\ell) = H^i(X_k, \mathbb{Q}_\ell)$, and we have

$$H^i(X_k, \mathbb{Q}_\ell) = \begin{cases} \mathbb{Q}_\ell & \pi_X^0 \neq 1 \\
\mathbb{Q}_\ell^2 & \pi_X^1 \neq q \\
\mathbb{Q}_\ell & \pi_X^2 = q \end{cases}$$

The identification of the action of $\pi_X^0$ on $H^2$ comes from Lemma [68.2](#) and the fact that the degree of $\pi_X$ is $q = \#(k)$. We do not know much about the action of $\pi_X^1$ on the degree 1 cohomology. Let us call $\alpha_1, \ldots, \alpha_{2g}$ its eigenvalues in $\mathbb{Q}_\ell$. Putting everything together, Theorem [118.2](#) yields the equality

$$\prod_{x \in |X|} \frac{1}{1 - T^{\deg x}} = \text{det}(1 - \pi_X^* T|_{H^1(X_k, \mathbb{Q}_\ell)}) = \frac{(1 - \alpha_1 T) \ldots (1 - \alpha_{2g} T)}{(1 - T)(1 - qT)}$$

from which we deduce the following result.

**Lemma 121.1.** Let $X$ be a smooth, projective, geometrically irreducible curve over a finite field $k$. Then

1. the $L$-function $L(X, \mathbb{Q}_\ell)$ is a rational function,
2. the eigenvalues $\alpha_1, \ldots, \alpha_{2g}$ of $\pi_X^0$ on $H^1(X_k, \mathbb{Q}_\ell)$ are algebraic integers independent of $\ell$,
3. the number of rational points of $X$ on $k_n$, where $[k_n : k] = n$, is

$$\#X(k_n) = 1 - \sum_{i=1}^{2g} \alpha_i^n + q^n,$$

4. for each $i$, $|\alpha_i| < q$.

**Proof.** Part (3) is Theorem [118.5](#) applied to $\mathcal{F} = \mathbb{Q}_\ell$ on $X \otimes k_n$. For part (4), use the following result.

**Exercise 121.2.** Let $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$. Then for any conic sector containing the positive real axis of the form $C_\varepsilon = \{ z \in \mathbb{C} \mid |\arg z| < \varepsilon \}$ with $\varepsilon > 0$, there exists an integer $k \geq 1$ such that $\alpha_1^k, \ldots, \alpha_n^k \in C_\varepsilon$.

Then prove that $|\alpha_i| < q$ for all $i$. Then, use elementary considerations on complex numbers to prove (as in the proof of the prime number theorem) that $|\alpha_i| < q$. In fact, the Riemann hypothesis says that for all $|\alpha_i| = \sqrt{q}$ for all $i$. We will come back to this later.
Affine case. Assume now that $X$ is affine, say $X = \bar{X} = \{x_1, \ldots, x_n\}$ where $j : X \hookrightarrow \bar{X}$ is a projective nonsingular completion. Then $H^0_c(X_\bar{k}, \mathbb{Q}_\ell) = 0$ and $H^2_c(X_\bar{k}, \mathbb{Q}_\ell) = H^2(\bar{X}_k, \mathbb{Q}_\ell)$ so Theorem 118.2 reads

$$L(X, \mathbb{Q}_\ell) = \prod_{x \in [X]} \frac{1}{1 - T^{\deg x}} = \frac{\det(1 - \pi^*_X T | H^1_c(X_\bar{k}, \mathbb{Q}_\ell))}{1 - qT}.$$ 

On the other hand, the previous case gives

$$L(X, \mathbb{Q}_\ell) = L(\bar{X}, \mathbb{Q}_\ell) \prod_{i=1}^n (1 - T^{\deg x_i})$$

$$= \prod_{i=1}^n (1 - T^{\deg x_i}) \prod_{j=1}^{2g} (1 - \alpha_j T).$$

Therefore, we see that $\dim H^1_c(X_\bar{k}, \mathbb{Q}_\ell) = 2g + \sum_{i=1}^n \deg(x_i) - 1$, and the eigenvalues $\alpha_1, \ldots, \alpha_{2g}$ of $\pi^*_X$ acting on the degree 1 cohomology are roots of unity. More precisely, each $x_i$ gives a complete set of $\deg(x_i)$th roots of unity, and one occurrence of 1 is omitted. To see this directly using coherent sheaves, consider the short exact sequence on $\bar{X}$

$$0 \to j_! \mathbb{Q}_\ell \to \mathbb{Q}_\ell \to \bigoplus_{i=1}^n \mathbb{Q}_\ell(x_i) \to 0.$$ 

The long exact cohomology sequence reads

$$0 \to \mathbb{Q}_\ell \to \bigoplus_{i=1}^n \mathbb{Q}_\ell^{\oplus \deg x_i} \to H^1_c(X_\bar{k}, \mathbb{Q}_\ell) \to H^1_c(\bar{X}_k, \mathbb{Q}_\ell) \to 0$$

where the action of Frobenius on $\bigoplus_{i=1}^n \mathbb{Q}_\ell^{\oplus \deg x_i}$ is by cyclic permutation of each term; and $H^2_c(X_\bar{k}, \mathbb{Q}_\ell) = H^2(\bar{X}_k, \mathbb{Q}_\ell)$.

### 122. The Legendre family

Let $k$ be a finite field of odd characteristic, $X = \text{Spec}(k[\lambda, \frac{1}{\lambda(\lambda-1)}])$, and consider the family of elliptic curves $f : E \to X$ on $\mathbb{P}^2_X$, whose affine equation is $y^2 = x(x - 1)(x - \lambda)$. We set $\mathcal{F} = Rf^*_! \mathbb{Q}_\ell = \{R^1f_* \mathbb{Z}/\ell^n \mathbb{Z}\}_{n \geq 1} \otimes \mathbb{Q}_\ell$. In this situation, the following is true:

- for each $n \geq 1$, the sheaf $R^1f_* (\mathbb{Z}/\ell^n \mathbb{Z})$ is finite locally constant - in fact, it is free of rank 2 over $\mathbb{Z}/\ell^n \mathbb{Z}$,
- the system $\{R^1f_* \mathbb{Z}/\ell^n \mathbb{Z}\}_{n \geq 1}$ is a lisse $\ell$-adic sheaf, and
- for all $x \in [X]$, $\det(1 - \pi_x T^{\deg x} | _{F_x}) = (1 - \alpha_T^{\deg x})(1 - \beta_T^{\deg x})$ where $\alpha_x, \beta_x$ are the eigenvalues of the geometric frobenius of $E_x$ acting on $H^1(E_x, \mathbb{Q}_\ell)$.

Note that $E_x$ is only defined over $\kappa(x)$ and not over $k$. The proof of these facts uses the proper base change theorem and the local acyclicity of smooth morphisms. For details, see [Del77]. It follows that

$$L(E/X) := L(X, \mathcal{F}) = \prod_{x \in [X]} \frac{1}{(1 - \alpha_x T^{\deg x})(1 - \beta_x T^{\deg x})}.$$
Applying Theorem 118.2 we get
\[
L(E/X) = \prod_{i=0}^{2} \det (1 - \pi_X^* T|_{H^i_c(X, F)})^{(-1)^{i+1}},
\]
and we see in particular that this is a rational function. Furthermore, it is relatively easy to show that \(H^0_c(X, F) = H^2_c(X, F) = 0\), so we merely have
\[
L(E/X) = \det(1 - \pi_X^* T|_{H^1_c(X, F)}).
\]
To compute this determinant explicitly, consider the Leray spectral sequence for the proper morphism \(f : E \to X\) over \(\mathbb{Q}_\ell\), namely
\[
H^i_c(X_\bar{k}, R^jf_* \mathbb{Q}_\ell) \Rightarrow H^{i+j}_c(E_\bar{k}, \mathbb{Q}_\ell)
\]
which degenerates. We have \(f_! \mathbb{Q}_\ell = \mathbb{Q}_\ell\) and \(R^1f_* \mathbb{Q}_\ell = \mathbb{Q}_\ell\). The sheaf \(R^2f_* \mathbb{Q}_\ell = \mathbb{Q}_\ell(-1)\) is the Tate twist of \(\mathbb{Q}_\ell\), i.e., it is the sheaf \(\mathbb{Q}_\ell\) where the Galois action is given by multiplication by \(\#\kappa(x)\) on the stalk at \(x\). It follows that, for all \(n \geq 1\),
\[
\#E(k_n) = \sum_{x} (-1)^i \text{Tr}(\pi_X^*|_{H^i_c(E_\bar{k}, \mathbb{Q}_\ell)})
\]
\[
= \sum_{i,j} (-1)^{i+j} \text{Tr}(\pi_X^*|_{H^i_c(X_\bar{k}, R^jf_* \mathbb{Q}_\ell)})
\]
\[
= (q^n - 2) + \text{Tr}(\pi_X^*|_{H^1_c(X_\bar{k}, F)}) + q^n(q^n - 2)
\]
\[
= q^{2n} - q^n - 2 + \text{Tr}(\pi_X^*|_{H^1_c(X_\bar{k}, F)})
\]
where the first equality follows from Theorem 118.5, the second one from the Leray spectral sequence and the third one by writing down the higher direct images of \(\mathbb{Q}_\ell\) under \(f\). Alternatively, we could write
\[
\#E(k_n) = \sum_{x \in X(k_n)} \#E_x(k_n)
\]
and use the trace formula for each curve. We can also find the number of \(k_n\)-rational points simply by counting. The zero section contributes \(q^n - 2\) points (we omit the points where \(\lambda = 0,1\)) hence
\[
\#E(k_n) = q^n - 2 + \#\{y^2 = x(x-1)(x-\lambda), \lambda \neq 0,1\}.
\]
Now we have
\[
\#\{y^2 = x(x-1)(x-\lambda), \lambda \neq 0,1\}
\]
\[
= \#\{y^2 = x(x-1)(x-\lambda) \text{ in } \mathbb{A}^3\} - \#\{y^2 = x^2(x-1)\} - \#\{y^2 = x(x-1)^2\}
\]
\[
= \#\{\lambda = \frac{-y^2}{x(x-1)} + x, x \neq 0,1\} + \#\{y^2 = x(x-1)(x-\lambda), x = 0,1\} - 2(q^n - \varepsilon_n)
\]
\[
= q^n(q^n - 2) + 2q^n - 2(q^n - \varepsilon_n)
\]
\[
= q^{2n} - 2q^n + 2\varepsilon_n
\]
where \(\varepsilon_n = 1\) if \(-1\) is a square in \(k_n\), 0 otherwise, i.e.,
\[
\varepsilon_n = \frac{1}{2} \left(1 + \left(\frac{-1}{k_n}\right)\right) = \frac{1}{2} \left(1 + (-1)\frac{q^n-1}{2}\right).
\]
Thus \(\#E(k_n) = q^{2n} - q^n - 2 + 2\varepsilon_n\). Comparing with the previous formula, we find
\[
\text{Tr}(\pi_X^*|_{H^1_c(X_\bar{k}, F)}) = 2\varepsilon_n = 1 + (-1)^{\frac{q^n-1}{2}},
\]
which implies, by elementary algebra of complex numbers, that if $-1$ is a square in $k_n^*$, then $\dim H^1_{et}(X, \mathcal{F}) = 2$ and the eigenvalues are 1 and 1. Therefore, in that case we have

$$L(E/X) = (1 - T)^2.$$ 

123. Exponential sums

A standard problem in number theory is to evaluate sums of the form

$$S_{a,b}(p) = \sum_{x \in \mathbf{F}_p - \{0,1\}} e^{\frac{2\pi i a x^b}{p}}.$$ 

In our context, this can be interpreted as a cohomological sum as follows. Consider the base scheme $S = \text{Spec}(\mathbf{F}_p[x, \frac{1}{x}])$ and the affine curve $f : X \to \mathbf{P}^1 - \{0,1,\infty\}$ over $S$ given by the equation $y^{p-1} = x^a(x-1)^b$. This is a finite étale Galois cover with group $\mathbf{F}_p^*$ and there is a splitting

$$f_* (\mathbf{Q}_\ell^*) = \bigoplus_{\chi} \mathcal{F}_\chi$$

where $\chi$ varies over the characters of $\mathbf{F}_p^*$ and $\mathcal{F}_\chi$ is a rank 1 lisse $\mathbf{Q}_\ell$-sheaf on which $\mathbf{F}_p^*$ acts via $\chi$ on stalks. We get a corresponding decomposition

$$H^1_{et}(X, \mathbf{Q}_\ell) = \bigoplus_{\chi} H^1(\mathbf{P}^1_{\bar{k}} - \{0,1,\infty\}, \mathcal{F}_\chi)$$

and the cohomological interpretation of the exponential sum is given by the trace formula applied to $\mathcal{F}_\chi$ over $\mathbf{P}^1 - \{0,1,\infty\}$ for some suitable $\chi$. It reads

$$S_{a,b}(p) = -\text{Tr}(\pi X^*|_{H^1(\mathbf{P}^1_{\bar{k}} - \{0,1,\infty\}, \mathcal{F}_\chi)}).$$

The general yoga of Weil suggests that there should be some cancellation in the sum. Applying (roughly) the Riemann-Hurwitz formula, we see that

$$2g_X - 2 \approx -2(p-1) + 3(p-2) \approx p$$

so $g_X \approx p/2$, which also suggests that the $\chi$-pieces are small.

124. Trace formula in terms of fundamental groups

In the following sections we reformulate the trace formula completely in terms of the fundamental group of a curve, except if the curve happens to be $\mathbf{P}^1$.

125. Fundamental groups

This material is discussed in more detail in the chapter on fundamental groups. See Fundamental Groups, Section 1. Let $X$ be a connected scheme and let $\pi \to X$ be a geometric point. Consider the functor

$$F_{\pi} : \text{finite \ étale schemes over } X \to \text{finite sets}$$

$$Y/X \mapsto F_\pi(Y) = \left\{ \text{geom points } \overline{\pi} \text{ of } Y \text{ lying over } \pi \right\} = Y_{\overline{\pi}}$$

Set

$$\pi_1(X, \overline{\pi}) = \text{Aut}(F_{\pi}) = \text{set of automorphisms of the functor } F_{\pi}$$

Note that for every finite étale $Y \to X$ there is an action

$$\pi_1(X, \overline{\pi}) \times F_\pi(Y) \to F_\pi(Y)$$
Definition 125.1. A subgroup of the form Stab(\(\overline{y} \in F_\pi(Y)\)) \(\subset \pi_1(X, \overline{x})\) is called open.

Theorem 125.2 (Grothendieck). Let \(X\) be a connected scheme.

1. There is a topology on \(\pi_1(X, \overline{x})\) such that the open subgroups form a fundamental system of open nbhds of \(e \in \pi_1(X, \overline{x})\).
2. With topology of (1) the group \(\pi_1(X, \overline{x})\) is a profinite group.
3. The functor
   \[
   \text{schemes finite étale over } X \quad \rightarrow \quad \text{finite discrete continuous } \pi_1(X, \overline{x})\text{-sets}
   \]
   \[
   Y/X \quad \rightarrow \quad F_\pi(Y) \quad \text{with its natural action}
   \]
   is an equivalence of categories.

Proof. See [Gro71]. \(\square\)

Proposition 125.3. Let \(X\) be an integral normal Noetherian scheme. Let \(\overline{y} \rightarrow X\) be an algebraic geometric point lying over the generic point \(\eta \in X\). Then
\[
\pi_x(X, \eta) = \text{Gal}(M/k(\eta))
\]
(\(\kappa(\eta)\), function field of \(X\)) where
\[
\kappa(\eta) \supset M \supset \kappa(\eta) = k(X)
\]
is the max sub-extension such that for every finite sub extension \(M \supset L \supset \kappa(\eta)\) the normalization of \(X\) in \(L\) is finite étale over \(X\).

Proof. Omitted. \(\square\)

Change of base point. For any \(x_1, x_2\) geom. points of \(X\) there exists an isom. of fibre functions
\[
F_{\overline{x}_1} \cong F_{\overline{x}_2}
\]
(This is a path from \(x_1\) to \(x_2\).) Conjugation by this path gives isom
\[
\pi_1(X, \overline{x}_1) \cong \pi_1(X, \overline{x}_2)
\]
well defined up to inner actions.

Functoriality. For any morphism \(X_1 \rightarrow X_2\) of connected schemes any \(\overline{x} \in X_1\) there is a canonical map
\[
\pi_1(X_1, \overline{x}) \rightarrow \pi_1(X_2, \overline{x})
\]
(Why? because the fibre functor ...)

Base field. Let \(X\) be a variety over a field \(k\). Then we get
\[
\pi_1(X, \overline{x}) \rightarrow \pi_1(\text{Spec}(k), \overline{x}) = \text{prop } \text{Gal}(k^{\text{sep}}/k)
\]
This map is surjective if and only if \(X\) is geometrically connected over \(k\). So in the geometrically connected case we get s.e.s. of profinite groups
\[
1 \rightarrow \pi_1(X_{\overline{\mathbb{F}}}, \overline{x}) \rightarrow \pi_1(X, \overline{x}) \rightarrow \text{Gal}(k^{\text{sep}}/k) \rightarrow 1
\]
(\(\pi_1(X_{\overline{\mathbb{F}}}, \overline{x})\): geometric fundamental group of \(X\), \(\pi_1(X, \overline{x})\): arithmetic fundamental group of \(X\))

Comparison. If \(X\) is a variety over \(\mathbb{C}\) then
\[
\pi_1(X, \overline{x}) = \text{profinite completion of } \pi_1(X(\mathbb{C}))(\text{ usual topology}), x
\]
(have \(x \in X(\mathbb{C})\))
Frobenius. X variety over k, \#k < \infty. For any x ∈ X closed point, let
\[ F_x ∈ π_1(x, \overline{x}) = \text{Gal}(κ(x)^{\text{sep}}/κ(x)) \]
be the geometric frobenius. Let η be an alg. geom. gen. pt. Then
\[ π_1(X, η) \overset{\cong}{\leftarrow} π_1(X, \overline{x}) \leftarrow π_1(x, \overline{x}) \]
Easy fact:
\[ π_1(X, η) \overset{\text{deg}}{\rightarrow} π_1(\text{Spec}(k), η) = \text{Gal}(k^{\text{sep}}/k) \]
Recall: deg(x) = [κ(x) : k]

Fundamental groups and lisse sheaves. Let X be a connected scheme, \overline{x} geom. pt. There are equivalences of categories

\[
\begin{array}{ccc}
\text{(Λ finite ring)} & \text{fin. loc. const. sheaves of} & \text{finite (discrete) Λ-modules with continuous} \\
\text{Λ-modules of} & \text{X}_{\text{etale}} & \text{π}_1(X, \overline{x})-\text{action}
\end{array}
\]

\[
\begin{array}{ccc}
\text{(ℓ a prime)} & \text{lisse ℓ-adic} & \text{finitely generated Zℓ-modules M with continuous} \\
\text{sheaves} & \text{π}_1(X, \overline{x})-\text{action where we use ℓ-adic topology on} & \text{π}_1(X, \overline{x})-\text{action}
\end{array}
\]

In particular lisse Qℓ-sheaves correspond to continuous homomorphisms
\[ π_1(X, \overline{x}) \rightarrow \text{GL}_r(\mathbb{Q}_ℓ), \quad r ≥ 0 \]
Notation: A module with action (M, ρ) corresponds to the sheaf \( F_ρ \).

Trace formulas. X variety over k, \#k < \infty.

1. A finite ring (\#Λ, \#k) = 1
\[ ρ : π_1(X, \overline{x}) \rightarrow \text{GL}_r(\Lambda) \]
continuous. For every n ≥ 1 we have
\[ \sum_{d|n} d \left( \sum_{s ∈ |X|, d \text{deg}(x) = d} \text{Tr}(ρ(F_x^n/d)) \right) = \text{Tr} \left( (π_1^n)^*|_{R^*_r(X, \overline{x}, F_ρ)} \right) \]

2. \( l ≠ \text{char}(k) \) prime, ρ : π_1(X, \overline{x}) → GL_r(Qℓ). For any n ≥ 1
\[ \sum_{d|n} d \left( \sum_{s ∈ |X|, d \text{deg}(x) = d} \text{Tr} \left( ρ(F_x^n/d) \right) \right) = \sum_{i=0}^{2\dim X} (-1)^i \text{Tr} \left( π_X^n|_{H^i_r(X, \overline{x}, F_ρ)} \right) \]

Weil conjectures. (Deligne-Weil I, 1974) X smooth proj. over k, \#k = q, then the eigenvalues of π_X on H^i(X, Qℓ) are algebraic integers α with |α| = q^{1/2}.

Deligne’s conjectures. (almost completely proved by Lafforgue + …) Let X be a normal variety over k finite
\[ ρ : π_1(X, \overline{x}) → \text{GL}_r(Qℓ) \]
continuous. Assume: ρ irreducible det(ρ) of finite order. Then
1. there exists a number field E such that for all x ∈ |X| (closed points) the char. poly of ρ(F_x) has coefficients in E.
2. for any x ∈ |X| the eigenvalues α_{x,i}, i = 1, \ldots, r of ρ(F_x) have complex absolute value 1. (these are algebraic numbers not necessary integers)
(3) for every finite place $\lambda$ (not dividing $p$), of $E$ (maybe after enlarging $E$ a bit) there exists
$$\rho_\lambda : \pi_1(X, x) \to \text{GL}_r(E_\lambda)$$
compatible with $\rho$. (some char. polys of $F_x$’s)

03VH **Theorem 125.4** (Deligne, Weil II). For a sheaf $\mathcal{F}_\rho$ with $\rho$ satisfying the conclusions of the conjecture above then the eigenvalues of $\pi_1^*(X, x)$ on $H^i_c(X_{\overline{\mathbb{F}}}, \mathcal{F}_\rho)$ are algebraic numbers $\alpha$ with absolute values $|\alpha| = q^{w/2}$, for $w \in \mathbb{Z}$, $w \leq i$

Moreover, if $X$ smooth and proj. then $w = i$.

**Proof.** See [Del74]. □

126. Profinite groups, cohomology and homology

03VI Let $G$ be a profinite group.

**Cohomology.** Consider the category of discrete modules with continuous $G$-action. This category has enough injectives and we can define
$$H^i(G, M) = R^iH^0(G, M) = R^i(M \mapsto M^G)$$
Also there is a derived version $RH^0(G, -)$.

**Homology.** Consider the category of compact abelian groups with continuous $G$-action. This category has enough projectives and we can define
$$H_i(G, M) = L_iH_0(G, M) = L_i(M \mapsto M_G)$$
and there is also a derived version.

**Trivial duality.** The functor $M \mapsto M^\wedge = \text{Hom}_{\text{cont}}(M, S^1)$ exchanges the categories above and
$$H^i(G, M)^\wedge = H_i(G, M^\wedge)$$
Moreover, this functor maps torsion discrete $G$-modules to profinite continuous $G$-modules and vice versa, and if $M$ is either a discrete or profinite continuous $G$-module, then $M^\wedge = \text{Hom}(M, \mathbb{Q}/\mathbb{Z})$.

**Notes on Homology.**

1. If we look at $\Lambda$-modules for a finite ring $\Lambda$ then we can identify
$$H_i(G, M) = \text{Tor}_i^\Lambda[[G]](M, \Lambda)$$
where $\Lambda[[G]]$ is the limit of the group algebras of the finite quotients of $G$.

2. If $G$ is a normal subgroup of $\Gamma$, and $\Gamma$ is also profinite then
   - $H^0(G, -)$: discrete $\Gamma$-module $\to$ discrete $\Gamma/G$-modules
   - $H_0(G, -)$: compact $\Gamma$-modules $\to$ compact $\Gamma/G$-modules

and hence the profinite group $\Gamma/G$ acts on the cohomology groups of $G$ with values in a $\Gamma$-module. In other words, there are derived functors
$$RH^0(G, -) : D^+(\text{discrete } \Gamma\text{-modules}) \to D^+(\text{discrete } \Gamma/G\text{-modules})$$
and similarly for $LH_0(G, -)$. 

127. Cohomology of curves, revisited

Let $k$ be a field, $X$ be geometrically connected, smooth curve over $k$. We have the fundamental short exact sequence

$$1 \rightarrow \pi_1(X_k, \eta) \rightarrow \pi_1(X, \eta) \rightarrow \text{Gal}(k^{sep}/k) \rightarrow 1$$

If $\Lambda$ is a finite ring with $\# \Lambda \in k^*$ and $M$ a finite $\Lambda$-module, and we are given $\rho : \pi_1(X, \eta) \rightarrow \text{Aut}_\Lambda(M)$ continuous, then $F_\rho$ denotes the associated sheaf on $X_{\text{etale}}$.

Lemma 127.1. There is a canonical isomorphism

$$H^2_c(X_k, F_\rho) = (M)_{\pi_1(X_k, \eta)}(-1)$$

as $\text{Gal}(k^{sep}/k)$-modules.

Here the subscript $\pi_1(X_k, \eta)$ indicates co-invariants, and $(-1)$ indicates the Tate twist i.e., $\sigma \in \text{Gal}(k^{sep}/k)$ acts via $\chi_{\text{cycl}}(\sigma)^{-1} \cdot \sigma$ on RHS

where $\chi_{\text{cycl}} : \text{Gal}(k^{sep}/k) \rightarrow \prod_{l \neq \text{char}(k)} \mathbb{Z}_l^*$

is the cyclotomic character.

Reformulation (Deligne, Weil II, page 338). For any finite locally constant sheaf $\mathcal{F}$ on $X$ there is a maximal quotient $\mathcal{F} \rightarrow \mathcal{F}''$ with $\mathcal{F}''/X_k$ a constant sheaf, hence

$$\mathcal{F}'' = (X \rightarrow \text{Spec}(k))^{-1} \mathcal{F}''$$

where $\mathcal{F}''$ is a sheaf $\text{Spec}(k)$, i.e., a $\text{Gal}(k^{sep}/k)$-module. Then

$$H^2_c(X_k, \mathcal{F}) \rightarrow H^2_c(X_k, \mathcal{F}'') \rightarrow F''(-1)$$

is an isomorphism.

Proof of Lemma 127.1. Let $Y \rightarrow^\varphi X$ be the finite étale Galois covering corresponding to $\text{Ker}(\rho) \subset \pi_1(X, \eta)$. So

$$\text{Aut}(Y/X) = \text{Ind}(\rho)$$

is Galois group. Then $\varphi^* \mathcal{F}_\rho = M_Y$ and

$$\varphi_* \varphi^* \mathcal{F}_\rho \rightarrow \mathcal{F}_\rho$$

which gives

$$H^2_c(X_k, \varphi_* \varphi^* \mathcal{F}_\rho) \rightarrow H^2_c(X_k, \mathcal{F}_\rho)$$

$$= H^2_c(Y_k, \varphi^* \mathcal{F}_\rho)$$

$$= H^2_c(Y_k, M) = \bigoplus_{\text{irred. comp. of } M}$$

$$\text{Im}(\rho) \rightarrow H^2_c(Y_k, M) = \bigoplus_{\text{irred. comp. of } M} \rightarrow \text{Im}(\rho)\text{equivariant } H^2_c(X_k, \mathcal{F}_\rho) \rightarrow \text{trivial } \text{Im}(\rho)$$

irreducible curve $C/\bar{k}$, $H^2_c(C, M) = M$.

Since

$$\text{set of irreducible components of } Y_k = \frac{\text{Im}(\rho)}{\text{Im}(\rho)|_{\pi_1(X_k, \eta)}}$$
We conclude that $H^2_c(X_k, F_\rho)$ is a quotient of $M_{\pi_1(X, \eta)}$. On the other hand, there is a surjection
\[ F_\rho \to F'' = \text{sheaf on } X \text{ associated to } \pi_1(X, \eta) \]
\[ H^2_c(X_k, F_\rho) \to M_{\pi_1(X, \eta)} \]
The twist in Galois action comes from the fact that $H^2_c(X_k, \mu_n) = \mathbb{Z}/n\mathbb{Z}$. □

Remark 127.2. Thus we conclude that if $X$ is also projective then we have functorially in the representation $\rho$ the identifications
\[ H^0(X_k, F_\rho) = M_{\pi_1(X, \eta)} \]
\[ H^2_c(X_k, F_\rho) = M_{\pi_1(X, \eta)}(-1) \]
Of course if $X$ is not projective, then $H^0_c(X_k, F_\rho) = 0$.

Proposition 127.3. Let $X/k$ as before but $X_k \neq \mathbb{P}^1_k$. The functors $(M, \rho) \mapsto H^{-i}_c(X_k, F_\rho)$ are the left derived functor of $(M, \rho) \mapsto H^2_c(X_k, F_\rho)$ so
\[ H^{2-i}_c(X_k, F_\rho) = H_i(\pi_1(X, \eta), M(-1)) \]
Moreover, there is a derived version, namely
\[ R\Gamma_c(X_k, F_\rho) = LH_0(\pi_1(X, \eta), M(-1)) = M(-1) \otimes_{\Lambda[[\pi_1(X, \eta)]]} \Lambda \]
in $D(\Lambda[[\hat{\mathbb{Z}}]])$. Similarly, the functors $(M, \rho) \mapsto H^i(X_k, F_\rho)$ are the right derived functor of $(M, \rho) \mapsto M_{\pi_1(X, \eta)}$ so
\[ H^i(X_k, F_\rho) = H^i(\pi_1(X, \eta), M) \]
Moreover, in this case there is a derived version too.

Proof. (Idea) Show both sides are universal $\delta$-functors. □

Remark 127.4. By the proposition and Trivial duality then you get
\[ H^{2-i}_c(X_k, F_\rho) \times H^i(X_k, F_\rho^\vee(1)) \to \mathbb{Q}/\mathbb{Z} \]
a perfect pairing. If $X$ is projective then this is Poincare duality.

128. Abstract trace formula

Suppose given an extension of profinite groups,
\[ 1 \to G \to \Gamma \xrightarrow{\text{deg}} \hat{\mathbb{Z}} \to 1 \]
We say $\Gamma$ has an abstract trace formula if and only if there exist
(1) an integer $q \geq 1$, and
(2) for every $d \geq 1$ a finite set $S_d$ and for each $x \in S_d$ a conjugacy class $F_x \in \Gamma$
with $\deg(F_x) = d$
such that the following hold
(1) for all $\ell$ not dividing $q$ have $c_d(G) < \infty$, and
(2) for all finite rings $\Lambda$ with $q \in \Lambda^*$, for all finite projective $\Lambda$-modules $M$ with continuous $\Gamma$-action, for all $n > 0$ we have

$$\sum_{d|n} d \left( \sum_{x \in S_d} \text{Tr}(F^n_x/M) \right) = q^n \text{Tr}(F^n|M \otimes \mathbb{L}^{[\Gamma][G]}\Lambda)$$

in $\Lambda^\#$.

Here $M \otimes \mathbb{L}^{[\Gamma][G]}\Lambda = LH_0(G, M)$ denotes derived homology, and $F = 1$ in $\Gamma/G = \hat{\mathbb{Z}}$.

**Remark 128.1.** Here are some observations concerning this notion.

1. If modeling projective curves then we can use cohomology and we don’t need factor $q^n$.
2. The only examples I know are $\Gamma = \pi_1(X, \eta)$ where $X$ is smooth, geometrically irreducible and $K(\pi, 1)$ over finite field. In this case $q = (\#k)^{\dim X}$.
   Modulo the proposition, we proved this for curves in this course.
3. Given the integer $q$ then the sets $S_d$ are uniquely determined. (You can multiple $q$ by an integer $m$ and then replace $S_d$ by $m^d$ copies of $S_d$ without changing the formula.)

**Example 128.2.** Fix an integer $q \geq 1$

$$1 \rightarrow G = \hat{\mathbb{Z}}(q) \rightarrow \Gamma \rightarrow \hat{\mathbb{Z}} \rightarrow 1$$

$$= \prod_{l|q} \mathbb{Z}_l$$

$F \mapsto 1$

with $Fx/F^{-1} = ux, u \in (\hat{\mathbb{Z}}(q))^*$. Just using the trivial modules $\mathbb{Z}/m\mathbb{Z}$ we see

$$q^n - (qu)^n = \sum_{d|n} d\#S_d$$

in $\mathbb{Z}/m\mathbb{Z}$ for all $(m, q) = 1$ (up to $u \rightarrow u^{-1}$) this implies $qu = a \in \mathbb{Z}$ and $|a| < q$.

The special case $a = 1$ does occur with

$$\Gamma = \pi_1^d(G_{m, F_p, \eta}), \quad \#S_1 = q - 1, \quad \text{and} \quad \#S_2 = \frac{(q^2 - 1) - (q - 1)}{2}$$

**129. Automorphic forms and sheaves**

References: See especially the amazing papers [Dri83, Dri84] and [Dri80] by Drinfeld.

**Unramified cusp forms.** Let $k$ be a finite field of characteristic $p$. Let $X$ geometrically irreducible projective smooth curve over $k$. Set $K = k(X)$ equal to the function field of $X$. Let $v$ be a place of $K$ which is the same thing as a closed point $x \in X$. Let $K_v$ be the completion of $K$ at $v$, which is the same thing as the fraction field of the completion of the local ring of $X$ at $x$. Denote $O_v \subset K_v$ the ring of integers. We further set

$$O = \prod_v O_v \subset \Lambda = \prod_v K_v$$

and we let $\Lambda$ be any ring with $p$ invertible in $\Lambda$.

**Definition 129.1.** An unramified cusp form on $GL_2(\Lambda)$ with values in $\Lambda$ is a function

$$f : GL_2(\Lambda) \rightarrow \Lambda$$

such that

---

This is likely nonstandard notation.
$f(x\gamma) = f(x)$ for all $x \in \text{GL}_2(\mathbb{A})$ and all $\gamma \in \text{GL}_2(K)$

(2) $f(ux) = f(x)$ for all $x \in \text{GL}_2(\mathbb{A})$ and all $u \in \text{GL}_2(O)$

(3) for all $x \in \text{GL}_2(\mathbb{A})$,

$$\int_{\Lambda \mod K} f \left( x \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} \right) dz = 0$$

see [dJ01 Section 4.1] for an explanation of how to make sense out of this for a general ring $\Lambda$ in which $p$ is invertible.

**Hecke Operators.** For $v$ a place of $K$ and $f$ an unramified cusp form we set

$$T_v(f)(x) = \int_{g \in M_v} f(g^{-1}x) dg,$$

and

$$U_v(f)(x) = f \left( \begin{pmatrix} \pi_v^{-1} & 0 \\ 0 & \pi_v^{-1} \end{pmatrix} x \right)$$

Notations used: here $\pi_v \in O_v$ is a uniformizer

$$M_v = \{ h \in Mat(2 \times 2, O_v) | \det h = \pi_v O_v^* \}$$

and $dg$ is the Haar measure on $\text{GL}_2(O_v)$ with $\int_{\text{GL}_2(O_v)} dg = 1$. Explicitly we have

$$T_v(f)(x) = f \left( \begin{pmatrix} \pi_v^{-1} & 0 \\ 0 & 1 \end{pmatrix} x \right) + \sum_{i=1}^{q_v} f \left( \begin{pmatrix} 1 & 0 \\ -\pi_v^{-1} \lambda_i & \pi_v^{-1} \end{pmatrix} x \right)$$

with $\lambda_i \in O_v$ a set of representatives of $O_v/(\pi_v) = \kappa_v$, $q_v = \#\kappa_v$.

**Eigenforms.** An eigenform $f$ is an unramified cusp form such that some value of $f$ is a unit and $T_v f = t_v f$ and $U_v f = u_v f$ for some (uniquely determined) $t_v, u_v \in \Lambda$.

**Theorem 129.2.** Given an eigenform $f$ with values in $\overline{\mathbb{Q}}_l$ and eigenvalues $u_v \in \mathbb{Z}_l^*$ then there exists

$$\rho : \pi_1(X) \to \text{GL}_2(E)$$

continuous, absolutely irreducible where $E$ is a finite extension of $\mathbb{Q}_l$ contained in $\overline{\mathbb{Q}}_l$ such that $t_v = \text{Tr}(\rho(F_v))$, and $u_v = q_v^{-1} \det(\rho(F_v))$ for all places $v$.

**Proof.** See [Dri80].

**Theorem 129.3.** Suppose $\mathbb{Q}_l \subset E$ finite, and

$$\rho : \pi_1(X) \to \text{GL}_2(E)$$

absolutely irreducible, continuous. Then there exists an eigenform $f$ with values in $\overline{\mathbb{Q}}_l$ whose eigenvalues $t_v, u_v$ satisfy the equalities $t_v = \text{Tr}(\rho(F_v))$ and $u_v = q_v^{-1} \det(\rho(F_v))$.

**Proof.** See [Dri83].

**Remark 129.4.** We now have, thanks to Lafforgue and many other mathematicians, complete theorems like this two above for $\text{GL}_n$ and allowing ramification! In other words, the full global Langlands correspondence for $\text{GL}_n$ is known for function fields of curves over finite fields. At the same time this does not mean there aren’t a lot of interesting questions left to answer about the fundamental groups of curves over finite fields, as we shall see below.
**Central character.** If $f$ is an eigenform then

$$
\chi_f : \frac{O^* \backslash A^*}{K^*} \rightarrow \Lambda^* \\
(1, \ldots, \pi_v, 1, \ldots, 1) \mapsto u_v^{-1}
$$

is called the *central character*. If corresponds to the determinant of $\rho$ via normalizations as above. Set

$$
C(\Lambda) = \left\{ \text{unr. cusp forms } f \text{ with coefficients in } \Lambda \right\} \\
such \text{that } U_v f = \varphi_v^{-1} f \forall v
$$

**Proposition 129.5.** If $\Lambda$ is Noetherian then $C(\Lambda)$ is a finitely generated $\Lambda$-module. Moreover, if $\Lambda$ is a field with prime subfield $F \subset \Lambda$ then

$$
C(\Lambda) = (C(F)) \otimes_F \Lambda
$$

compatibly with $T_v$ acting.

**Proof.** See [dJ01, Proposition 4.7].

This proposition trivially implies the following lemma.

**Lemma 129.6.** Algebraicity of eigenvalues. If $\Lambda$ is a field then the eigenvalues $t_v$ for $f \in C(\Lambda)$ are algebraic over the prime subfield $F \subset \Lambda$.

**Proof.** Follows from Proposition 129.5.

Combining all of the above we can do the following very useful trick.

**Lemma 129.7.** Switching $l$. Let $E$ be a number field. Start with

$$
\rho : \pi_1(X) \rightarrow SL_2(E_\Lambda)
$$

absolutely irreducible continuous, where $\lambda$ is a place of $E$ not lying above $p$. Then for any second place $\lambda'$ of $E$ not lying above $p$ there exists a finite extension $E_{\lambda'}$ and a absolutely irreducible continuous representation

$$
\rho' : \pi_1(X) \rightarrow SL_2(E'_{\lambda'})
$$

which is compatible with $\rho$ in the sense that the characteristic polynomials of all Frobenii are the same.

Note how this is an instance of Deligne’s conjecture!

**Proof.** To prove the switching lemma use Theorem 129.3 to obtain $f \in C(Q_\lambda)$ eigenform ass. to $\rho$. Next, use Proposition 129.3 to see that we may choose $f \in C(E')$ with $E \subset E'$ finite. Next we may complete $E'$ to see that we get $f \in C(E'_{\lambda'})$ eigenform with $E'_{\lambda'}$ a finite extension of $E_{\lambda'}$. And finally we use Theorem 129.3 to obtain $\rho' : \pi_1(X) \rightarrow SL_2(E'_{\lambda'})$ abs. irred. and continuous after perhaps enlarging $E'_{\lambda'}$ a bit again.

Speculation: If for a (topological) ring $\Lambda$ we have

$$
\left( \rho : \pi_1(X) \rightarrow SL_2(\Lambda) \text{ abs irred} \right) \leftrightarrow \text{eigen forms in } C(\Lambda)
$$

then all eigenvalues of $\rho(E_v)$ algebraic (won’t work in an easy way if $\Lambda$ is a finite ring. Based on the speculation that the Langlands correspondence works more generally than just over fields one arrives at the following conjecture.
**Conjecture.** (See [dJ01]) For any continuous 
\[ \rho : \pi_1(X) \to \text{GL}_n(\mathbf{F}_l[[t]]) \]
we have \( \#\rho(\pi_1(X)) < \infty. \)

A rephrasing in the language of sheaves: "For any lisse sheaf of \( \mathbf{F}_l((t)) \)-modules the geom monodromy is finite."

**Theorem 129.8.** The Conjecture holds if \( n \leq 2 \).

**Proof.** See [dJ01]. \( \square \)

**Theorem 129.9.** Conjecture holds if \( l > 2n \) modulo some unproven things.

**Proof.** See [Gai07]. \( \square \)

It turns out the conjecture is useful for something. See work of Drinfeld on Katz's conjectures. But there is also the much more down to earth application as follows.

**Theorem 129.10.** (See [dJ01, Theorem 3.5]) Suppose 
\[ \rho_0 : \pi_1(X) \to \text{GL}_n(\mathbf{F}_l) \]
is a continuous, \( l \neq p \). Assume 
1. Conj. holds for \( X \),
2. \( \rho_0|_{\pi_1(X)} \) abs. irred., and
3. \( l \) does not divide \( n \).

Then the universal deformation ring \( R_{\text{univ}} \) of \( \rho_0 \) is finite flat over \( \mathbf{Z}_l \).

Explanation: There is a representation \( \rho_{\text{univ}} : \pi_1(X) \to \text{GL}_n(R_{\text{univ}}) \) (Univ. Defo ring) \( R_{\text{univ}} \) loc. complete, residue field \( \mathbf{F}_l \) and \( (R_{\text{univ}} \to \mathbf{F}_l) \circ \rho_{\text{univ}} \cong \rho_0 \). And given any \( R \to \mathbf{F}_l \), \( R \) local complete and \( \rho : \pi_1(X) \to \text{GL}_n(R) \) then there exists \( \psi : R_{\text{univ}} \to R \) such that \( \psi \circ \rho_{\text{univ}} \cong \rho \). The theorem says that the morphism 
\[ \text{Spec}(R_{\text{univ}}) \to \text{Spec}(\mathbf{Z}_l) \]
is finite and flat. In particular, such a \( \rho_0 \) lifts to a \( \rho : \pi_1(X) \to \text{GL}_n(\mathbf{Q}_l) \).

Notes:
1. The theorem on deformations is easy.
2. Any result towards the conjecture seems hard.
3. It would be interesting to have more conjectures on \( \pi_1(X) \)!

### 130. Counting points

Let \( X \) be a smooth, geometrically irreducible, projective curve over \( k \) and \( q = \#k \).

The trace formula gives: there exists algebraic integers \( w_1, \ldots, w_{2g} \) such that 
\[ \#X(k_n) = q^n - \sum_{i=1}^{2g} w_i^n + 1. \]

If \( \sigma \in \text{Aut}(X) \) then for all \( i \), there exists \( j \) such that \( \sigma(w_i) = w_j \).

**Riemann-Hypothesis.** For all \( i \) we have \( |\omega_i| = \sqrt{q} \).

This was formulated by Emil Artin, in 1924, for hyperelliptic curves. Proved by Weil 1940. Weil gave two proofs

- using intersection theory on \( X \times X \), using the Hodge index theorem, and
• using the Jacobian of $X$.

There is another proof whose initial idea is due to Stephanov, and which was given by Bombieri: it uses the function field $k(X)$ and its Frobenius operator (1969). The starting point is that given $f \in k(X)$ one observes that $f^q - f$ is a rational function which vanishes in all the $\mathbb{F}_q$-rational points of $X$, and that one can try to use this idea to give an upper bound for the number of points.

### 131. Precise form of Chebotarev

As a first application let us prove a precise form of Chebotarev for a finite étale Galois covering of curves. Let $\varphi : Y \to X$ be a finite étale Galois covering with group $G$. This corresponds to a homomorphism

$$\pi_1(X) \to G = \text{Aut}(Y/X)$$

Assume $Y_{\overline{k}}$ is irreducible. If $C \subset G$ is a conjugacy class then for all $n > 0$, we have

$$|\# \{x \in X(k_n) \mid F_x \in C\} - \frac{\#C}{\#G} \cdot \#X(k_n)| \leq (\#C)(2g - 2)\sqrt{q^n}$$

(Warning: Please check the coefficient $\#C$ on the right hand side carefully before using.)

**Sketch.** Write

$$\varphi_*(\mathbb{Q}_l) = \bigoplus_{\pi \in \hat{G}}^F \mathcal{F}_\pi$$

where $\hat{G}$ is the set of isomorphism classes of irreducible representations of $G$ over $\mathbb{Q}_l$. For $\pi \in \hat{G}$ let $\chi_\pi : G \to \mathbb{Q}_l$ be the character of $\pi$. Then

$$H^*(X_{\overline{k}}, \mathcal{F}_\pi) = \bigoplus_{\pi \in \hat{G}} H^*(Y_{\overline{k}}, \mathcal{Q}_l)_{\pi} = (\varphi \text{ finite}) \bigoplus_{\pi \in \hat{G}} H^*(X_{\overline{k}}, \mathcal{F}_\pi)$$

If $\pi \neq 1$ then we have

$$H^0(X_{\overline{k}}, \mathcal{F}_\pi) = H^2(X_{\overline{k}}, \mathcal{F}_\pi) = 0, \quad \dim H^1(X_{\overline{k}}, \mathcal{F}_\pi) = (2g - 2)d_\pi^2$$

(can get this from trace formula for acting on ...) and we see that

$$|\sum_{x \in X(k_n)} \chi_\pi(F_x)| \leq (2g - 2)d_\pi^2 \sqrt{q^n}$$

Write $1_C = \sum_\pi a_\pi \chi_\pi$, then $a_\pi = \langle 1_C, \chi_\pi \rangle$, and $a_1 = \langle 1_C, \chi_1 \rangle = \frac{\#C}{\#G}$ where

$$\langle f, h \rangle = \frac{1}{\#G} \sum_{g \in G} f(g) \overline{h(g)}$$

Thus we have the relation

$$\frac{\#C}{\#G} = ||1_C||^2 = \sum |a_\pi|^2$$
Final step:

\[
\# \{ x \in X(k_n) \mid F_x \in C \} = \sum_{x \in X(k_n)} 1_C(x) = \sum_{x \in X(k_n)} \sum_{\pi} a_\pi \chi_\pi(F_x) = \frac{\# C}{\# G} \# X(k_n) + \sum_{\pi \neq 1} a_\pi \sum_{x \in X(k_n)} \chi_\pi(F_x)
\]

We can bound the error term by

\[
|E| \leq \sum_{\pi \in \hat{G}, \pi \neq 1} |a_\pi|(2g - 2)d_\pi \sqrt{q^n}
\]

By Weil’s conjecture, \#X(k_n) \sim q^n. □

132. How many primes decompose completely?

This section gives a second application of the Riemann Hypothesis for curves over a finite field. For number theorists it may be nice to look at the paper by Ihara, entitled “How many primes decompose completely in an infinite unramified Galois extension of a global field?”, see [Iha83]. Consider the fundamental exact sequence

\[
1 \to \pi_1(X) \to \pi_1(X) \xrightarrow{\deg} \hat{\mathbb{Z}} \to 1
\]

**Proposition 132.1.** There exists a finite set \(x_1, \ldots, x_n\) of closed points of \(X\) such that set of all Frobenius elements corresponding to these points topologically generate \(\pi_1(X)\).

Another way to state this is: There exist \(x_1, \ldots, x_n \in |X|\) such that the smallest normal closed subgroup \(\Gamma\) of \(\pi_1(X)\) containing 1 Frobenius element for each \(x_i\) is all of \(\pi_1(X)\). i.e., \(\Gamma = \pi_1(X)\).

**Proof.** Pick \(N \gg 0\) and let

\[
\{x_1, \ldots, x_n\} = \text{set of all closed points of } X \text{ of degree } \leq N \text{ over } k
\]

Let \(\Gamma \subseteq \pi_1(X)\) be as in the variant statement for these points. Assume \(\Gamma \neq \pi_1(X)\). Then we can pick a normal open subgroup \(U\) of \(\pi_1(X)\) containing \(\Gamma\) with \(U \neq \pi_1(X)\). By R.H. for \(X\) our set of points will have some \(x_i\) of degree \(N\), some \(x_{i_2}\) of degree \(N - 1\). This shows \(\deg : \Gamma \to \hat{\mathbb{Z}}\) is surjective and so the same holds for \(U\). This exactly means if \(Y \to X\) is the finite étale Galois covering corresponding to \(U\), then \(Y\) irreducible. Set \(G = \text{Aut}(Y/X)\). Picture

\[
Y \xrightarrow{G} X, \quad G = \pi_1(X)/U
\]

By construction all points of \(X\) of degree \(\leq N\), split completely in \(Y\). So, in particular

\[
\#Y(k_N) \geq (#G) \#X(k_N)
\]
Use R.H. on both sides. So you get
\[ q^N + 1 + 2g_Y q^{N/2} \geq \#G \#X(k_N) \geq \#G(q^N + 1 - 2g_X q^{N/2}) \]
Since \( 2g_Y - 2 = (\#G)(2g_X - 2) \), this means
\[ q^N + 1 + (\#G)(2g_X - 1)q^{N/2} \geq \#G(q^N + 1 - 2g_X q^{N/2}) \]
Thus we see that \( G \) has to be the trivial group if \( N \) is large enough. \( \Box \)

**Weird Question.** Set \( W_X = \deg^{-1}(\mathbb{Z}) \subset \pi_1(X) \). Is it true that for some finite set of closed points \( x_1, \ldots, x_n \) of \( X \) the set of all Frobenii corresponding to these points algebraically generate \( W_X \)?

By a Baire category argument this translates into the same question for all Frobenii.

**133. How many points are there really?**

If the genus of the curve is large relative to \( q \), then the main term in the formula
\[ \#X(k) = q - \sum \omega_i + 1 \] is not \( q \) but the second term \( \sum \omega_i \) which can (a priori) have size about \( 2g_X \sqrt{q} \). In the paper \([VD83]\) the authors Drinfeld and Vladut show that this maximum is (as predicted by Ihara earlier) actually at most about \( g \sqrt{q} \).

Fix \( q \) and let \( k \) be a field with \( k \) elements. Set
\[ A(q) = \limsup_{gX \to \infty} \frac{\#X(k)}{gX} \]
where \( X \) runs over geometrically irreducible smooth projective curves over \( k \). With this definition we have the following results:

- RH \( \Rightarrow \) \( A(q) \leq 2\sqrt{q} \)
- Ihara \( \Rightarrow \) \( A(q) \leq \sqrt{2q} \)
- DV \( \Rightarrow \) \( A(q) \leq \sqrt{q} - 1 \) (actually this is sharp if \( q \) is a square)

**Proof.** Given \( X \) let \( w_1, \ldots, w_{2g} \) and \( g = g_X \) be as before. Set \( \alpha_i = \frac{w_i}{\sqrt{q}} \), so \( |\alpha_i| = 1 \).

If \( \alpha_i \) occurs then \( \overline{\alpha}_i = \alpha_i^{-1} \) also occurs. Then
\[ N = \#X(k) \leq X(k_r) = q^r + 1 - (\sum_i \alpha_i^r)q^{r/2} \]
Rewriting we see that for every \( r \geq 1 \)
\[ - \sum_i \alpha_i^r \geq Nq^{-r/2} - q^{r/2} - q^{-r/2} \]
Observe that
\[ 0 \leq |\alpha_i^n + \alpha_i^{n-1} + \ldots + \alpha_i + 1|^2 = (n + 1) + \sum_{j=1}^{n} (n + 1 - j)(\alpha_i^j + \alpha_i^{-j}) \]
So
\[ 2g(n + 1) \geq - \sum_i \left( \sum_{j=1}^{n} (n + 1 - j)(\alpha_i^j + \alpha_i^{-j}) \right) \]
\[ = - \sum_{j=1}^{n} (n + 1 - j) \left( \sum_i \alpha_i^j + \sum_i \alpha_i^{-j} \right) \]
Take half of this to get

\[ g(n + 1) \geq -\sum_{j=1}^{n} (n + 1 - j)(\sum_i \alpha_i^j) \]

\[ \geq N\sum_{j=1}^{n} (n + 1 - j)q^{-j/2} - \sum_{j=1}^{n} (n + 1 - j)(q^{j/2} + q^{-j/2}) \]

This gives

\[ \frac{N}{g} \leq \left( \sum_{j=1}^{n} \frac{n + 1 - j}{n + 1} q^{-j/2} \right)^{-1} \cdot \left( 1 + \frac{1}{g} \sum_{j=1}^{n} \frac{n + 1 - j}{n + 1} (q^{j/2} + q^{-j/2}) \right) \]

Fix \( n \) let \( g \to \infty \)

\[ A(q) \leq \left( \sum_{j=1}^{n} \frac{n + 1 - j}{n + 1} q^{-j/2} \right)^{-1} \]

So

\[ A(q) \leq \lim_{n \to \infty} \ldots = \left( \sum_{j=1}^{\infty} q^{-j/2} \right)^{-1} = \sqrt{q} - 1 \]

\[ \square \]

134. Other chapters

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41. Chow Homology
42. Intersection Theory
43. Picard Schemes of Curves
44. Adequate Modules
45. Dualizing Complexes
## References


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[Tat76] John Tate, Relations between $K_2$ and galois cohomology, Inventiones mathematicae 36 (1976), 257–274.
