ÉTALE COHOMOLOGY

03N1

Contents

1.	Introduction	3
2.	Which sections to skip on a first reading?	3
3.	Prologue	4
4.	The étale topology	4
5.	Feats of the étale topology	5
6.	A computation	5
7.	Nontorsion coefficients	7
8.	Sheaf theory	7
9.	Presheaves	7
10.	Sites	8
11.	Sheaves	9
12.	The example of G-sets	9
13.	Sheafification	10
14.	Cohomology	12
15.	The fpqc topology	12
16.	Faithfully flat descent	14
17.	Quasi-coherent sheaves	16
18.	Čech cohomology	18
19.	The Čech-to-cohomology spectral sequence	20
20.	Big and small sites of schemes	21
21.	The étale topos	23
22.	Cohomology of quasi-coherent sheaves	24
23.	Examples of sheaves	26
24.	Picard groups	27
25.	The étale site	27
26.	Étale morphisms	28
27.	Étale coverings	29
28.	Kummer theory	30
29.	Neighborhoods, stalks and points	34
30.	Points in other topologies	40
31.	Supports of abelian sheaves	41
32.	Henselian rings	42
33.	Stalks of the structure sheaf	44
34.	Functoriality of small étale topos	45
35.	Direct images	46
36.	Inverse image	47
37.	Functoriality of big topoi	49
38.	Functoriality and sheaves of modules	49

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39. Comparing topologies 50
40. Recovering morphisms 52
41. Push and pull 58
42. Property (A) 58
43. Property (B) 60
44. Property (C) 62
45. Topological invariance of the small étale site 64
46. Closed immersions and pushforward 67
47. Integral universally injective morphisms 68
48. Big sites and pushforward 69
49. Exactness of big lower shriek 70
50. Étale cohomology 72
51. Colimits 73
52. Stalks of higher direct images 78
53. The Leray spectral sequence 78
54. Vanishing of finite higher direct images 79
55. Galois action on stalks 82
56. Group cohomology 84
57. Tate's continuous cohomology 87
58. Cohomology of a point 88
59. Cohomology of curves 89
60. Brauer groups 89
61. The Brauer group of a scheme 91
62. The Artin-Schreier sequence 93
63. Locally constant sheaves 96
64. Locally constant sheaves and the fundamental group 98
65. Méthode de la trace 98
66. Galois cohomology 100
67. Higher vanishing for the multiplicative group 104
68. Picard groups of curves 105
69. Extension by zero 108
70. Constructible sheaves 110
71. Auxiliary lemmas on morphisms 114
72. More on constructible sheaves 115
73. Constructible sheaves on Noetherian schemes 121
74. Complexes with constructible cohomology 124
75. Tor finite with constructible cohomology 126
76. Torsion sheaves 127
77. Cohomology with support in a closed subscheme 128
78. Schemes with strictly henselian local rings 130
79. Affine analog of proper base change 133
80. Cohomology of torsion sheaves on curves 137
81. First cohomology of proper schemes 142
82. Preliminaries on base change 143
83. Base change for pushforward 146
84. Base change for higher direct images 149
85. Smooth base change 154
86. Applications of smooth base change 161

87. The proper base change theorem 162
88. Applications of proper base change 168
89. Cohomological dimension 171
90. Finite cohomological dimension 176
91. Künneth in étale cohomology 177
92. Comparing chaotic and Zariski topologies 185
93. Comparing big and small topoi 185
94. Comparing fppf and étale topologies 189
95. Comparing fppf and étale topologies: modules 194
96. Comparing ph and étale topologies 195
97. Comparing h and étale topologies 199
98. Blow up squares and étale cohomology 202
99. Almost blow up squares and the h topology 204
100. Cohomology of the structure sheaf in the h topology 205
101. Other chapters 206
References 207

1. Introduction

03N2 This chapter is the first in a series of chapter on the étale cohomology of schemes. In this chapter we discuss the very basics of the étale topology and cohomology of abelian sheaves in this topology. Many of the topics discussed may be safely skipped on a first reading; please see the advice in the next section as to how to decide what to skip.

The initial version of this chapter was formed by the notes of the first part of a course on étale cohomology taught by Johan de Jong at Columbia University in the Fall of 2009. The original note takers were Thibaut Pugin, Zachary Maddock and Min Lee. The second part of the course can be found in the chapter on the trace formula, see The Trace Formula, Section 1.

2. Which sections to skip on a first reading?

04JG We want to use the material in this chapter for the development of theory related to algebraic spaces, Deligne-Mumford stacks, algebraic stacks, etc. Thus we have added some pretty technical material to the original exposition of étale cohomology for schemes. The reader can recognize this material by the frequency of the word "topos", or by discussions related to set theory, or by proofs dealing with very general properties of morphisms of schemes. Some of these discussions can be skipped on a first reading.

In particular, we suggest that the reader skip the following sections:

(1) Comparing big and small topoi, Section 93.
(2) Recovering morphisms, Section 40.
(3) Push and pull, Section 41.
(4) Property (A), Section 42.
(5) Property (B), Section 43.
(6) Property (C), Section 44.
(7) Topological invariance of the small étale site, Section 45.

(8) Integral universally injective morphisms, Section 47.
(9) Big sites and pushforward, Section 48.
(10) Exactness of big lower shriek, Section 49.

Besides these sections there are some sporadic results that may be skipped that the reader can recognize by the keywords given above.

3. Prologue

03N3 These lectures are about another cohomology theory. The first thing to remark is that the Zariski topology is not entirely satisfactory. One of the main reasons that it fails to give the results that we would want is that if X is a complex variety and $\mathcal{F}$ is a constant sheaf then

$$H^i(X, \mathcal{F}) = 0, \quad \text{for all } i > 0.$$

The reason for that is the following. In an irreducible scheme (a variety in particular), any two nonempty open subsets meet, and so the restriction mappings of a constant sheaf are surjective. We say that the sheaf is *flasque*. In this case, all higher Čech cohomology groups vanish, and so do all higher Zariski cohomology groups. In other words, there are "not enough" open sets in the Zariski topology to detect this higher cohomology.

On the other hand, if X is a smooth projective complex variety, then

$$H^{2\dim X}_{Betti}(X(\mathbf{C}), \Lambda) = \Lambda \quad \text{for } \Lambda = \mathbf{Z},\ \mathbf{Z}/n\mathbf{Z},$$

where $X(\mathbf{C})$ means the set of complex points of X. This is a feature that would be nice to replicate in algebraic geometry. In positive characteristic in particular.

4. The étale topology

03N4 It is very hard to simply "add" extra open sets to refine the Zariski topology. One efficient way to define a topology is to consider not only open sets, but also some schemes that lie over them. To define the étale topology, one considers all morphisms $\varphi : U \to X$ which are étale. If X is a smooth projective variety over $\mathbf{C}$, then this means

(1) U is a disjoint union of smooth varieties, and
(2) φ is (analytically) locally an isomorphism.

The word "analytically" refers to the usual (transcendental) topology over $\mathbf{C}$. So the second condition means that the derivative of φ has full rank everywhere (and in particular all the components of U have the same dimension as X).

A double cover – loosely defined as a finite degree 2 map between varieties – for example

$$\mathrm{Spec}(\mathbf{C}[t]) \longrightarrow \mathrm{Spec}(\mathbf{C}[t]), \quad t \longmapsto t^2$$

will not be an étale morphism if it has a fibre consisting of a single point. In the example this happens when $t = 0$. For a finite map between varieties over $\mathbf{C}$ to be étale all the fibers should have the same number of points. Removing the point $t = 0$ from the source of the map in the example will make the morphism étale. But we can remove other points from the source of the morphism also, and the morphism will still be étale. To consider the étale topology, we have to look at all such morphisms. Unlike the Zariski topology, these need not be merely open subsets of X, even though their images always are.

03N5 **Definition 4.1.** A family of morphisms $\{\varphi_i : U_i \to X\}_{i \in I}$ is called an *étale covering* if each φ_i is an étale morphism and their images cover X, i.e., $X = \bigcup_{i \in I} \varphi_i(U_i)$.

This "defines" the étale topology. In other words, we can now say what the sheaves are. An *étale sheaf* $\mathcal{F}$ of sets (resp. abelian groups, vector spaces, etc) on X is the data:

(1) for each étale morphism $\varphi : U \to X$ a set (resp. abelian group, vector space, etc) $\mathcal{F}(U)$,
(2) for each pair U, U' of étale schemes over X, and each morphism $U \to U'$ over X (which is automatically étale) a restriction map $\rho^{U'}_U : \mathcal{F}(U') \to \mathcal{F}(U)$

These data have to satisfy the condition that $\rho^U_U = \mathrm{id}$ in case of the identity morphism $U \to U$ and that $\rho^{U'}_U \circ \rho^{U''}_{U'} = \rho^{U''}_U$ when we have morphisms $U \to U' \to U''$ of schemes étale over X as well as the following *sheaf axiom*:

(*) for every étale covering $\{\varphi_i : U_i \to U\}_{i \in I}$, the diagram

$$\emptyset \longrightarrow \mathcal{F}(U) \longrightarrow \Pi_{i \in I} \mathcal{F}(U_i) \rightrightarrows \Pi_{i,j \in I} \mathcal{F}(U_i \times_U U_j)$$

is exact in the category of sets (resp. abelian groups, vector spaces, etc).

03N6 **Remark 4.2.** In the last statement, it is essential not to forget the case where $i = j$ which is in general a highly nontrivial condition (unlike in the Zariski topology). In fact, frequently important coverings have only one element.

Since the identity is an étale morphism, we can compute the global sections of an étale sheaf, and cohomology will simply be the corresponding right-derived functors. In other words, once more theory has been developed and statements have been made precise, there will be no obstacle to defining cohomology.

5. Feats of the étale topology

03N7 For a natural number $n \in \mathbf{N} = \{1, 2, 3, 4, \ldots\}$ it is true that

$$H^2_{étale}(\mathbf{P}^1_{\mathbf{C}}, \mathbf{Z}/n\mathbf{Z}) = \mathbf{Z}/n\mathbf{Z}.$$

More generally, if X is a complex variety, then its étale Betti numbers with coefficients in a finite field agree with the usual Betti numbers of $X(\mathbf{C})$, i.e.,

$$\dim_{\mathbf{F}_q} H^{2i}_{étale}(X, \mathbf{F}_q) = \dim_{\mathbf{F}_q} H^{2i}_{Betti}(X(\mathbf{C}), \mathbf{F}_q).$$

This is extremely satisfactory. However, these equalities only hold for torsion coefficients, not in general. For integer coefficients, one has

$$H^2_{étale}(\mathbf{P}^1_{\mathbf{C}}, \mathbf{Z}) = 0.$$

By contrast $H^2_{Betti}(\mathbf{P}^1(\mathbf{C}), \mathbf{Z}) = \mathbf{Z}$ as the topological space $\mathbf{P}^1(\mathbf{C})$ is homeomorphic to a 2-sphere. There are ways to get back to nontorsion coefficients from torsion ones by a limit procedure which we will come to shortly.

6. A computation

03N8 How do we compute the cohomology of $\mathbf{P}^1_{\mathbf{C}}$ with coefficients $\Lambda = \mathbf{Z}/n\mathbf{Z}$? We use Čech cohomology. A covering of $\mathbf{P}^1_{\mathbf{C}}$ is given by the two standard opens U_0, U_1, which are both isomorphic to $\mathbf{A}^1_{\mathbf{C}}$, and whose intersection is isomorphic

to $\mathbf{A}^1_{\mathbf{C}} \setminus \{0\} = \mathbf{G}_{m,\mathbf{C}}$. It turns out that the Mayer-Vietoris sequence holds in étale cohomology. This gives an exact sequence

$$H^{i-1}_{\acute{e}tale}(U_0 \cap U_1, \Lambda) \to H^i_{\acute{e}tale}(\mathbf{P}^1_C, \Lambda) \to H^i_{\acute{e}tale}(U_0, \Lambda) \oplus H^i_{\acute{e}tale}(U_1, \Lambda) \to H^i_{\acute{e}tale}(U_0 \cap U_1, \Lambda).$$

To get the answer we expect, we would need to show that the direct sum in the third term vanishes. In fact, it is true that, as for the usual topology,

$$H^q_{\acute{e}tale}(\mathbf{A}^1_{\mathbf{C}}, \Lambda) = 0 \quad \text{for } q \geq 1,$$

and

$$H^q_{\acute{e}tale}(\mathbf{A}^1_{\mathbf{C}} \setminus \{0\}, \Lambda) = \begin{cases} \Lambda & \text{if } q = 1, \text{ and} \\ 0 & \text{for } q \geq 2. \end{cases}$$

These results are already quite hard (what is an elementary proof?). Let us explain how we would compute this once the machinery of étale cohomology is at our disposal.

Higher cohomology. This is taken care of by the following general fact: if X is an affine curve over $\mathbf{C}$, then

$$H^q_{\acute{e}tale}(X, \mathbf{Z}/n\mathbf{Z}) = 0 \quad \text{for } q \geq 2.$$

This is proved by considering the generic point of the curve and doing some Galois cohomology. So we only have to worry about the cohomology in degree 1.

Cohomology in degree 1. We use the following identifications:

$$\begin{aligned} H^1_{\acute{e}tale}(X, \mathbf{Z}/n\mathbf{Z}) &= \left\{ \begin{array}{c} \text{sheaves of sets } \mathcal{F} \text{ on the étale site } X_{\acute{e}tale} \text{ endowed with an} \\ \text{action } \mathbf{Z}/n\mathbf{Z} \times \mathcal{F} \to \mathcal{F} \text{ such that } \mathcal{F} \text{ is a } \mathbf{Z}/n\mathbf{Z}\text{-torsor.} \end{array} \right\} \Big/ \cong \\ &= \left\{ \begin{array}{c} \text{morphisms } Y \to X \text{ which are finite étale together} \\ \text{with a free } \mathbf{Z}/n\mathbf{Z} \text{ action such that } X = Y/(\mathbf{Z}/n\mathbf{Z}). \end{array} \right\} \Big/ \cong . \end{aligned}$$

The first identification is very general (it is true for any cohomology theory on a site) and has nothing to do with the étale topology. The second identification is a consequence of descent theory. The last set describes a collection of geometric objects on which we can get our hands.

The curve $\mathbf{A}^1_{\mathbf{C}}$ has no nontrivial finite étale covering and hence $H^1_{\acute{e}tale}(\mathbf{A}^1_{\mathbf{C}}, \mathbf{Z}/n\mathbf{Z}) = 0$. This can be seen either topologically or by using the argument in the next paragraph.

Let us describe the finite étale coverings $\varphi : Y \to \mathbf{A}^1_{\mathbf{C}} \setminus \{0\}$. It suffices to consider the case where Y is connected, which we assume. We are going to find out what Y can be by applying the Riemann-Hurwitz formula (of course this is a bit silly, and you can go ahead and skip the next section if you like). Say that this morphism is n to 1, and consider a projective compactification

$$\begin{tikzcd} Y \arrow[r, hook] \arrow[d, "\varphi"] & \bar{Y} \arrow[d, "\bar{\varphi}"] \\ \mathbf{A}^1_{\mathbf{C}} \setminus \{0\} \arrow[r, hook] & \mathbf{P}^1_{\mathbf{C}} \end{tikzcd}$$

Even though φ is étale and does not ramify, $\bar{\varphi}$ may ramify at 0 and ∞. Say that the preimages of 0 are the points $y_1, \ldots, y_r$ with indices of ramification $e_1, \ldots e_r$, and that the preimages of ∞ are the points $y'_1, \ldots, y'_s$ with indices of ramification

$d_1, \dots d_s$. In particular, $\sum e_i = n = \sum d_j$. Applying the Riemann-Hurwitz formula, we get

$$2g_Y - 2 = -2n + \sum (e_i - 1) + \sum (d_j - 1)$$

and therefore $g_Y = 0$, $r = s = 1$ and $e_1 = d_1 = n$. Hence $Y \cong \mathbf{A}^1_{\mathbf{C}} \setminus \{0\}$, and it is easy to see that $\varphi(z) = \lambda z^n$ for some $\lambda \in \mathbf{C}^*$. After reparametrizing Y we may assume $\lambda = 1$. Thus our covering is given by taking the nth root of the coordinate on $\mathbf{A}^1_{\mathbf{C}} \setminus \{0\}$.

Remember that we need to classify the coverings of $\mathbf{A}^1_{\mathbf{C}} \setminus \{0\}$ together with free $\mathbf{Z}/n\mathbf{Z}$-actions on them. In our case any such action corresponds to an automorphism of Y sending z to $\zeta_n z$, where ζ_n is a primitive nth root of unity. There are $\phi(n)$ such actions (here $\phi(n)$ means the Euler function). Thus there are exactly $\phi(n)$ connected finite étale coverings with a given free $\mathbf{Z}/n\mathbf{Z}$-action, each corresponding to a primitive nth root of unity. We leave it to the reader to see that the disconnected finite étale degree n coverings of $\mathbf{A}^1_{\mathbf{C}} \setminus \{0\}$ with a given free $\mathbf{Z}/n\mathbf{Z}$-action correspond one-to-one with nth roots of 1 which are not primitive. In other words, this computation shows that

$$H^1_{\acute{e}tale}(\mathbf{A}^1_{\mathbf{C}} \setminus \{0\}, \mathbf{Z}/n\mathbf{Z}) = \mathrm{Hom}(\mu_n(\mathbf{C}), \mathbf{Z}/n\mathbf{Z}) \cong \mathbf{Z}/n\mathbf{Z}.$$

The first identification is canonical, the second isn't, see Remark 68.5. Since the proof of Riemann-Hurwitz does not use the computation of cohomology, the above actually constitutes a proof (provided we fill in the details on vanishing, etc).

7. Nontorsion coefficients

03N9 To study nontorsion coefficients, one makes the following definition:

$$H^i_{\acute{e}tale}(X, \mathbf{Q}_\ell) := \left(\lim\nolimits_n H^i_{\acute{e}tale}(X, \mathbf{Z}/\ell^n\mathbf{Z})\right) \otimes_{\mathbf{Z}_\ell} \mathbf{Q}_\ell.$$

The symbol $\lim_n$ denote the *limit* of the system of cohomology groups $H^i_{\acute{e}tale}(X, \mathbf{Z}/\ell^n\mathbf{Z})$ indexed by n, see Categories, Section 21. Thus we will need to study systems of sheaves satisfying some compatibility conditions.

8. Sheaf theory

03NA At this point we start talking about sites and sheaves in earnest. There is an amazing amount of useful abstract material that could fit in the next few sections. Some of this material is worked out in earlier chapters, such as the chapter on sites, modules on sites, and cohomology on sites. We try to refrain from adding too much material here, just enough so the material later in this chapter makes sense.

9. Presheaves

03NB A reference for this section is Sites, Section 2.

03NC **Definition 9.1.** Let $\mathcal{C}$ be a category. A *presheaf of sets* (respectively, an *abelian presheaf*) on $\mathcal{C}$ is a functor $\mathcal{C}^{opp} \to$ *Sets* (resp. *Ab*).

Terminology. If $U \in \mathrm{Ob}(\mathcal{C})$, then elements of $\mathcal{F}(U)$ are called *sections* of $\mathcal{F}$ over U. For $\varphi : V \to U$ in $\mathcal{C}$, the map $\mathcal{F}(\varphi) : \mathcal{F}(U) \to \mathcal{F}(V)$ is called the *restriction map* and is often denoted $s \mapsto s|_V$ or sometimes $s \mapsto \varphi^* s$. The notation $s|_V$ is ambiguous since the restriction map depends on φ, but it is a standard abuse of notation. We also use the notation $\Gamma(U, \mathcal{F}) = \mathcal{F}(U)$.

Saying that $\mathcal{F}$ is a functor means that if $W \to V \to U$ are morphisms in $\mathcal{C}$ and $s \in \Gamma(U, \mathcal{F})$ then $(s|_V)|_W = s|_W$, with the abuse of notation just seen. Moreover, the restriction mappings corresponding to the identity morphisms $\mathrm{id}_U : U \to U$ are the identity.

The category of presheaves of sets (respectively of abelian presheaves) on $\mathcal{C}$ is denoted $PSh(\mathcal{C})$ (resp. $PAb(\mathcal{C})$). It is the category of functors from $\mathcal{C}^{opp}$ to *Sets* (resp. *Ab*), which is to say that the morphisms of presheaves are natural transformations of functors. We only consider the categories $PSh(\mathcal{C})$ and $PAb(\mathcal{C})$ when the category $\mathcal{C}$ is small. (Our convention is that a category is small unless otherwise mentioned, and if it isn't small it should be listed in Categories, Remark 2.2.)

03ND **Example 9.2.** Given an object $X \in \mathrm{Ob}(\mathcal{C})$, we consider the functor

$$\begin{array}{cccc} h_X : & \mathcal{C}^{opp} & \longrightarrow & Sets \\ & U & \longmapsto & h_X(U) = Mor_{\mathcal{C}}(U, X) \\ & V \xrightarrow{\varphi} U & \longmapsto & \varphi \circ - : h_X(U) \to h_X(V). \end{array}$$

It is a presheaf, called the *representable presheaf associated to X*. It is not true that representable presheaves are sheaves in every topology on every site.

03NE **Lemma 9.3** (Yoneda)**.** *Let $\mathcal{C}$ be a category, and $X, Y \in \mathrm{Ob}(\mathcal{C})$. There is a natural bijection*

$$\begin{array}{ccc} Mor_{\mathcal{C}}(X, Y) & \longrightarrow & Mor_{PSh(\mathcal{C})}(h_X, h_Y) \\ \psi & \longmapsto & h_\psi = \psi \circ - : h_X \to h_Y. \end{array}$$

Proof. See Categories, Lemma 3.5. $\square$

10. Sites

03NF

03NG **Definition 10.1.** Let $\mathcal{C}$ be a category. A *family of morphisms with fixed target* $\mathcal{U} = \{\varphi_i : U_i \to U\}_{i \in I}$ is the data of

1. an object $U \in \mathcal{C}$,
2. a set I (possibly empty), and
3. for all $i \in I$, a morphism $\varphi_i : U_i \to U$ of $\mathcal{C}$ with target U.

There is a notion of a *morphism of families of morphisms with fixed target*. A special case of that is the notion of a *refinement*. A reference for this material is Sites, Section 8.

03NH **Definition 10.2.** A *site*[1] consists of a category $\mathcal{C}$ and a set $\mathrm{Cov}(\mathcal{C})$ consisting of families of morphisms with fixed target called *coverings*, such that

1. (isomorphism) if $\varphi : V \to U$ is an isomorphism in $\mathcal{C}$, then $\{\varphi : V \to U\}$ is a covering,
2. (locality) if $\{\varphi_i : U_i \to U\}_{i \in I}$ is a covering and for all $i \in I$ we are given a covering $\{\psi_{ij} : U_{ij} \to U_i\}_{j \in I_i}$, then
$$\{\varphi_i \circ \psi_{ij} : U_{ij} \to U\}_{(i,j) \in \prod_{i \in I} \{i\} \times I_i}$$
is also a covering, and

[1] What we call a site is a called a category endowed with a pretopology in [AGV71, Exposé II, Définition 1.3]. In [Art62] it is called a category with a Grothendieck topology.

(3) (base change) if $\{U_i \to U\}_{i\in I}$ is a covering and $V \to U$ is a morphism in $\mathcal{C}$, then
   (a) for all $i \in I$ the fibre product $U_i \times_U V$ exists in $\mathcal{C}$, and
   (b) $\{U_i \times_U V \to V\}_{i\in I}$ is a covering.

For us the category underlying a site is always "small", i.e., its collection of objects form a set, and the collection of coverings of a site is a set as well (as in the definition above). We will mostly, in this chapter, leave out the arguments that cut down the collection of objects and coverings to a set. For further discussion, see Sites, Remark 6.3.

03NI **Example 10.3.** If X is a topological space, then it has an associated site X_{Zar} defined as follows: the objects of X_{Zar} are the open subsets of X, the morphisms between these are the inclusion mappings, and the coverings are the usual topological (surjective) coverings. Observe that if $U, V \subset W \subset X$ are open subsets then $U \times_W V = U \cap V$ exists: this category has fiber products. All the verifications are trivial and everything works as expected.

11. Sheaves

03NJ

03NK **Definition 11.1.** A presheaf $\mathcal{F}$ of sets (resp. abelian presheaf) on a site $\mathcal{C}$ is said to be a *separated presheaf* if for all coverings $\{\varphi_i : U_i \to U\}_{i\in I} \in \text{Cov}(\mathcal{C})$ the map

$$\mathcal{F}(U) \longrightarrow \prod\nolimits_{i\in I} \mathcal{F}(U_i)$$

is injective. Here the map is $s \mapsto (s|_{U_i})_{i\in I}$. The presheaf $\mathcal{F}$ is a *sheaf* if for all coverings $\{\varphi_i : U_i \to U\}_{i\in I} \in \text{Cov}(\mathcal{C})$, the diagram

03NL (11.1.1)
$$\mathcal{F}(U) \longrightarrow \prod\nolimits_{i\in I} \mathcal{F}(U_i) \rightrightarrows \prod\nolimits_{i,j\in I} \mathcal{F}(U_i \times_U U_j),$$

where the first map is $s \mapsto (s|_{U_i})_{i\in I}$ and the two maps on the right are $(s_i)_{i\in I} \mapsto (s_i|_{U_i \times_U U_j})$ and $(s_i)_{i\in I} \mapsto (s_j|_{U_i \times_U U_j})$, is an equalizer diagram in the category of sets (resp. abelian groups).

03NM **Remark 11.2.** For the empty covering (where $I = \emptyset$), this implies that $\mathcal{F}(\emptyset)$ is an empty product, which is a final object in the corresponding category (a singleton, for both *Sets* and *Ab*).

03NN **Example 11.3.** Working this out for the site X_{Zar} associated to a topological space, see Example 10.3, gives the usual notion of sheaves.

03NO **Definition 11.4.** We denote $Sh(\mathcal{C})$ (resp. $Ab(\mathcal{C})$) the full subcategory of $PSh(\mathcal{C})$ (resp. $PAb(\mathcal{C})$) whose objects are sheaves. This is the *category of sheaves of sets* (resp. *abelian sheaves*) on $\mathcal{C}$.

12. The example of G-sets

03NP Let G be a group and define a site $\mathcal{T}_G$ as follows: the underlying category is the category of G-sets, i.e., its objects are sets endowed with a left G-action and the morphisms are equivariant maps; and the coverings of $\mathcal{T}_G$ are the families $\{\varphi_i : U_i \to U\}_{i\in I}$ satisfying $U = \bigcup_{i\in I} \varphi_i(U_i)$.

There is a special object in the site $\mathcal{T}_G$, namely the G-set G endowed with its natural action by left translations. We denote it ${}_GG$. Observe that there is a natural group isomorphism

$$\begin{array}{rccc} \rho : & G^{opp} & \longrightarrow & \mathrm{Aut}_{G\text{-}Sets}({}_GG) \\ & g & \longmapsto & (h \mapsto hg). \end{array}$$

In particular, for any presheaf $\mathcal{F}$, the set $\mathcal{F}({}_GG)$ inherits a G-action via ρ. (Note that by contravariance of $\mathcal{F}$, the set $\mathcal{F}({}_GG)$ is again a left G-set.) In fact, the functor

$$\begin{array}{ccc} Sh(\mathcal{T}_G) & \longrightarrow & G\text{-}Sets \\ \mathcal{F} & \longmapsto & \mathcal{F}({}_GG) \end{array}$$

is an equivalence of categories. Its quasi-inverse is the functor $X \mapsto h_X$. Without giving the complete proof (which can be found in Sites, Section 9) let us try to explain why this is true.

(1) If S is a G-set, we can decompose it into orbits $S = \coprod_{i \in I} O_i$. The sheaf axiom for the covering $\{O_i \to S\}_{i \in I}$ says that

$$\mathcal{F}(S) \longrightarrow \prod\nolimits_{i \in I} \mathcal{F}(O_i) \rightrightarrows \prod\nolimits_{i,j \in I} \mathcal{F}(O_i \times_S O_j)$$

is an equalizer. Observing that fibered products in G-*Sets* are induced from fibered products in *Sets*, and using the fact that $\mathcal{F}(\emptyset)$ is a G-singleton, we get that

$$\prod_{i,j \in I} \mathcal{F}(O_i \times_S O_j) = \prod_{i \in I} \mathcal{F}(O_i)$$

and the two maps above are in fact the same. Therefore the sheaf axiom merely says that $\mathcal{F}(S) = \prod_{i \in I} \mathcal{F}(O_i)$.

(2) If S is the G-set $S = G/H$ and $\mathcal{F}$ is a sheaf on $\mathcal{T}_G$, then we claim that

$$\mathcal{F}(G/H) = \mathcal{F}({}_GG)^H$$

and in particular $\mathcal{F}(\{*\}) = \mathcal{F}({}_GG)^G$. To see this, let's use the sheaf axiom for the covering $\{{}_GG \to G/H\}$ of S. We have

$$\begin{array}{ccc} {}_GG \times_{G/H} {}_GG & \cong & G \times H \\ (g_1, g_2) & \longmapsto & (g_1, g_1 g_2^{-1}) \end{array}$$

is a disjoint union of copies of ${}_GG$ (as a G-set). Hence the sheaf axiom reads

$$\mathcal{F}(G/H) \longrightarrow \mathcal{F}({}_GG) \rightrightarrows \prod\nolimits_{h \in H} \mathcal{F}({}_GG)$$

where the two maps on the right are $s \mapsto (s)_{h \in H}$ and $s \mapsto (hs)_{h \in H}$. Therefore $\mathcal{F}(G/H) = \mathcal{F}({}_GG)^H$ as claimed.

This doesn't quite prove the claimed equivalence of categories, but it shows at least that a sheaf $\mathcal{F}$ is entirely determined by its sections over ${}_GG$. Details (and set theoretical remarks) can be found in Sites, Section 9.

13. Sheafification

03NQ

03NR **Definition 13.1.** Let $\mathcal{F}$ be a presheaf on the site $\mathcal{C}$ and $\mathcal{U} = \{U_i \to U\} \in \mathrm{Cov}(\mathcal{C})$. We define the *zeroth Čech cohomology group* of $\mathcal{F}$ with respect to $\mathcal{U}$ by

$$\check{H}^0(\mathcal{U}, \mathcal{F}) = \left\{ (s_i)_{i \in I} \in \prod\nolimits_{i \in I} \mathcal{F}(U_i) \text{ such that } s_i|_{U_i \times_U U_j} = s_j|_{U_i \times_U U_j} \right\}.$$

There is a canonical map $\mathcal{F}(U) \to \check{H}^0(\mathcal{U}, \mathcal{F})$, $s \mapsto (s|_{U_i})_{i \in I}$. We say that a *morphism of coverings* from a covering $\mathcal{V} = \{V_j \to V\}_{j \in J}$ to $\mathcal{U}$ is a triple (χ, α, χ_j), where $\chi : V \to U$ is a morphism, $\alpha : J \to I$ is a map of sets, and for all $j \in J$ the morphism χ_j fits into a commutative diagram

$$\begin{tikzcd} V_j \arrow[r, "\chi_j"'] \arrow[d] & U_{\alpha(j)} \arrow[d] \\ V \arrow[r, "\chi"] & U. \end{tikzcd}$$

Given the data $\chi, \alpha, \{\chi_j\}_{j \in J}$ we define

$$\begin{array}{rcl} \check{H}^0(\mathcal{U}, \mathcal{F}) & \longrightarrow & \check{H}^0(\mathcal{V}, \mathcal{F}) \\ (s_i)_{i \in I} & \longmapsto & \left(\chi_j^* \left(s_{\alpha(j)}\right)\right)_{j \in J}. \end{array}$$

We then claim that

(1) the map is well-defined, and
(2) depends only on χ and is independent of the choice of $\alpha, \{\chi_j\}_{j \in J}$.

We omit the proof of the first fact. To see part (2), consider another triple (ψ, β, ψ_j) with $\chi = \psi$. Then we have the commutative diagram

$$\begin{tikzcd} V_j \arrow[rr, "{(\chi_j, \psi_j)}"'] \arrow[dd] & & U_{\alpha(j)} \times_U U_{\beta(j)} \arrow[dl] \arrow[dr] & \\ & U_{\alpha(j)} \arrow[dr] & & U_{\beta(j)} \arrow[dl] \\ V \arrow[rr, "\chi = \psi"] & & U. & \end{tikzcd}$$

Given a section $s \in \mathcal{F}(\mathcal{U})$, its image in $\mathcal{F}(V_j)$ under the map given by $(\chi, \alpha, \{\chi_j\}_{j \in J})$ is $\chi_j^* s_{\alpha(j)}$, and its image under the map given by $(\psi, \beta, \{\psi_j\}_{j \in J})$ is $\psi_j^* s_{\beta(j)}$. These two are equal since by assumption $s \in \check{H}(\mathcal{U}, \mathcal{F})$ and hence both are equal to the pullback of the common value

$$s_{\alpha(j)}|_{U_{\alpha(j)} \times_U U_{\beta(j)}} = s_{\beta(j)}|_{U_{\alpha(j)} \times_U U_{\beta(j)}}$$

pulled back by the map (χ_j, ψ_j) in the diagram.

03NS **Theorem 13.2.** *Let $\mathcal{C}$ be a site and $\mathcal{F}$ a presheaf on $\mathcal{C}$.*

(1) *The rule*
$$U \mapsto \mathcal{F}^+(U) := \mathrm{colim}_{\mathcal{U} \text{ covering of } U} \check{H}^0(\mathcal{U}, \mathcal{F})$$
is a presheaf. And the colimit is a directed one.
(2) *There is a canonical map of presheaves $\mathcal{F} \to \mathcal{F}^+$.*
(3) *If $\mathcal{F}$ is a separated presheaf then $\mathcal{F}^+$ is a sheaf and the map in (2) is injective.*
(4) *$\mathcal{F}^+$ is a separated presheaf.*
(5) *$\mathcal{F}^\# = (\mathcal{F}^+)^+$ is a sheaf, and the canonical map induces a functorial isomorphism*
$$\mathrm{Hom}_{PSh(\mathcal{C})}(\mathcal{F}, \mathcal{G}) = \mathrm{Hom}_{Sh(\mathcal{C})}(\mathcal{F}^\#, \mathcal{G})$$
for any $\mathcal{G} \in Sh(\mathcal{C})$.

Proof. See Sites, Theorem 10.10. □

In other words, this means that the natural map $\mathcal{F} \to \mathcal{F}^\#$ is a left adjoint to the forgetful functor $Sh(\mathcal{C}) \to PSh(\mathcal{C})$.

14. Cohomology

03NT The following is the basic result that makes it possible to define cohomology for abelian sheaves on sites.

03NU **Theorem 14.1.** *The category of abelian sheaves on a site is an abelian category which has enough injectives.*

Proof. See Modules on Sites, Lemma 3.1 and Injectives, Theorem 7.4. □

So we can define cohomology as the right-derived functors of the sections functor: if $U \in \text{Ob}(\mathcal{C})$ and $\mathcal{F} \in Ab(\mathcal{C})$,

$$H^p(U, \mathcal{F}) := R^p\Gamma(U, \mathcal{F}) = H^p(\Gamma(U, \mathcal{I}^\bullet))$$

where $\mathcal{F} \to \mathcal{I}^\bullet$ is an injective resolution. To do this, we should check that the functor $\Gamma(U, -)$ is left exact. This is true and is part of why the category $Ab(\mathcal{C})$ is abelian, see Modules on Sites, Lemma 3.1. For more general discussion of cohomology on sites (including the global sections functor and its right derived functors), see Cohomology on Sites, Section 2.

15. The fpqc topology

03NV Before doing étale cohomology we study a bit the fpqc topology, since it works well for quasi-coherent sheaves.

03NW **Definition 15.1.** Let T be a scheme. An *fpqc covering* of T is a family $\{\varphi_i : T_i \to T\}_{i \in I}$ such that

1. each φ_i is a flat morphism and $\bigcup_{i \in I} \varphi_i(T_i) = T$, and
2. for each affine open $U \subset T$ there exists a finite set K, a map $\mathbf{i} : K \to I$ and affine opens $U_{\mathbf{i}(k)} \subset T_{\mathbf{i}(k)}$ such that $U = \bigcup_{k \in K} \varphi_{\mathbf{i}(k)}(U_{\mathbf{i}(k)})$.

03NX **Remark 15.2.** The first condition corresponds to fp, which stands for *fidèlement plat*, faithfully flat in french, and the second to qc, *quasi-compact*. The second part of the first condition is unnecessary when the second condition holds.

03NY **Example 15.3.** Examples of fpqc coverings.

1. Any Zariski open covering of T is an fpqc covering.
2. A family $\{\text{Spec}(B) \to \text{Spec}(A)\}$ is an fpqc covering if and only if $A \to B$ is a faithfully flat ring map.
3. If $f : X \to Y$ is flat, surjective and quasi-compact, then $\{f : X \to Y\}$ is an fpqc covering.
4. The morphism $\varphi : \coprod_{x \in \mathbf{A}^1_k} \text{Spec}(\mathcal{O}_{\mathbf{A}^1_k, x}) \to \mathbf{A}^1_k$, where k is a field, is flat and surjective. It is not quasi-compact, and in fact the family $\{\varphi\}$ is not an fpqc covering.
5. Write $\mathbf{A}^2_k = \text{Spec}(k[x, y])$. Denote $i_x : D(x) \to \mathbf{A}^2_k$ and $i_y : D(y) \hookrightarrow \mathbf{A}^2_k$ the standard opens. Then the families $\{i_x, i_y, \text{Spec}(k[[x, y]]) \to \mathbf{A}^2_k\}$ and $\{i_x, i_y, \text{Spec}(\mathcal{O}_{\mathbf{A}^2_k, 0}) \to \mathbf{A}^2_k\}$ are fpqc coverings.

03NZ **Lemma 15.4.** *The collection of fpqc coverings on the category of schemes satisfies the axioms of site.*

Proof. See Topologies, Lemma 9.7. □

It seems that this lemma allows us to define the fpqc site of the category of schemes. However, there is a set theoretical problem that comes up when considering the fpqc topology, see Topologies, Section 9. It comes from our requirement that sites are "small", but that no small category of schemes can contain a cofinal system of fpqc coverings of a given nonempty scheme. Although this does not strictly speaking prevent us from defining "partial" fpqc sites, it does not seem prudent to do so. The work-around is to allow the notion of a sheaf for the fpqc topology (see below) but to prohibit considering the category of all fpqc sheaves.

03X6 **Definition 15.5.** Let S be a scheme. The category of schemes over S is denoted Sch/S. Consider a functor $\mathcal{F} : (Sch/S)^{opp} \to Sets$, in other words a presheaf of sets. We say $\mathcal{F}$ *satisfies the sheaf property for the fpqc topology* if for every fpqc covering $\{U_i \to U\}_{i \in I}$ of schemes over S the diagram (11.1.1) is an equalizer diagram.

We similarly say that $\mathcal{F}$ *satisfies the sheaf property for the Zariski topology* if for every open covering $U = \bigcup_{i \in I} U_i$ the diagram (11.1.1) is an equalizer diagram. See Schemes, Definition 15.3. Clearly, this is equivalent to saying that for every scheme T over S the restriction of $\mathcal{F}$ to the opens of T is a (usual) sheaf.

03O1 **Lemma 15.6.** *Let $\mathcal{F}$ be a presheaf on Sch/S. Then $\mathcal{F}$ satisfies the sheaf property for the fpqc topology if and only if*

(1) *$\mathcal{F}$ satisfies the sheaf property with respect to the Zariski topology, and*
(2) *for every faithfully flat morphism $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ of affine schemes over S, the sheaf axiom holds for the covering $\{\mathrm{Spec}(B) \to \mathrm{Spec}(A)\}$. Namely, this means that*

$$\mathcal{F}(\mathrm{Spec}(A)) \longrightarrow \mathcal{F}(\mathrm{Spec}(B)) \rightrightarrows \mathcal{F}(\mathrm{Spec}(B \otimes_A B))$$

is an equalizer diagram.

Proof. See Topologies, Lemma 9.13. □

An alternative way to think of a presheaf $\mathcal{F}$ on Sch/S which satisfies the sheaf condition for the fpqc topology is as the following data:

(1) for each T/S, a usual (i.e., Zariski) sheaf $\mathcal{F}_T$ on T_{Zar},
(2) for every map $f : T' \to T$ over S, a restriction mapping $f^{-1}\mathcal{F}_T \to \mathcal{F}_{T'}$

such that

(a) the restriction mappings are functorial,
(b) if $f : T' \to T$ is an open immersion then the restriction mapping $f^{-1}\mathcal{F}_T \to \mathcal{F}_{T'}$ is an isomorphism, and
(c) for every faithfully flat morphism $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ over S, the diagram

$$\mathcal{F}_{\mathrm{Spec}(A)}(\mathrm{Spec}(A)) \longrightarrow \mathcal{F}_{\mathrm{Spec}(B)}(\mathrm{Spec}(B)) \rightrightarrows \mathcal{F}_{\mathrm{Spec}(B \otimes_A B)}(\mathrm{Spec}(B \otimes_A B))$$

is an equalizer.

Data (1) and (2) and conditions (a), (b) give the data of a presheaf on Sch/S satisfying the sheaf condition for the Zariski topology. By Lemma 15.6 condition (c) then suffices to get the sheaf condition for the fpqc topology.

03O2 **Example 15.7.** Consider the presheaf

$$\begin{array}{llll} \mathcal{F}: & (Sch/S)^{opp} & \longrightarrow & Ab \\ & T/S & \longmapsto & \Gamma(T, \Omega_{T/S}). \end{array}$$

The compatibility of differentials with localization implies that $\mathcal{F}$ is a sheaf on the Zariski site. However, it does not satisfy the sheaf condition for the fpqc topology. Namely, consider the case $S = \mathrm{Spec}(\mathbf{F}_p)$ and the morphism

$$\varphi : V = \mathrm{Spec}(\mathbf{F}_p[v]) \to U = \mathrm{Spec}(\mathbf{F}_p[u])$$

given by mapping u to v^p. The family $\{\varphi\}$ is an fpqc covering, yet the restriction mapping $\mathcal{F}(U) \to \mathcal{F}(V)$ sends the generator $\mathrm{d}u$ to $\mathrm{d}(v^p) = 0$, so it is the zero map, and the diagram

$$\mathcal{F}(U) \xrightarrow{0} \mathcal{F}(V) \rightrightarrows \mathcal{F}(V \times_U V)$$

is not an equalizer. We will see later that $\mathcal{F}$ does in fact give rise to a sheaf on the étale and smooth sites.

03O3 **Lemma 15.8.** *Any representable presheaf on Sch/S satisfies the sheaf condition for the fpqc topology.*

Proof. See Descent, Lemma 10.7. □

We will return to this later, since the proof of this fact uses descent for quasi-coherent sheaves, which we will discuss in the next section. A fancy way of expressing the lemma is to say that *the fpqc topology is weaker than the canonical topology*, or that the fpqc topology is *subcanonical*. In the setting of sites this is discussed in Sites, Section 12.

03O4 **Remark 15.9.** The fpqc is the finest topology that we will see. Hence any presheaf satisfying the sheaf condition for the fpqc topology will be a sheaf in the subsequent sites (étale, smooth, etc). In particular representable presheaves will be sheaves on the étale site of a scheme for example.

03O5 **Example 15.10.** Let S be a scheme. Consider the additive group scheme $\mathbf{G}_{a,S} = \mathbf{A}^1_S$ over S, see Groupoids, Example 5.3. The associated representable presheaf is given by

$$h_{\mathbf{G}_{a,S}}(T) = Mor_S(T, \mathbf{G}_{a,S}) = \Gamma(T, \mathcal{O}_T).$$

By the above we now know that this is a presheaf of sets which satisfies the sheaf condition for the fpqc topology. On the other hand, it is clearly a presheaf of rings as well. Hence we can think of this as a functor

$$\begin{array}{llll} \mathcal{O}: & (Sch/S)^{opp} & \longrightarrow & Rings \\ & T/S & \longmapsto & \Gamma(T, \mathcal{O}_T) \end{array}$$

which satisfies the sheaf condition for the fpqc topology. Correspondingly there is a notion of $\mathcal{O}$-module, and so on and so forth.

16. Faithfully flat descent

03O6

03O7 **Definition 16.1.** Let $\mathcal{U} = \{t_i : T_i \to T\}_{i \in I}$ be a family of morphisms of schemes with fixed target. A *descent datum* for quasi-coherent sheaves with respect to $\mathcal{U}$ is a family $(\mathcal{F}_i, \varphi_{ij})_{i,j \in I}$ where

(1) for all i, $\mathcal{F}_i$ is a quasi-coherent sheaf on T_i, and

(2) for all $i, j \in I$ the map $\varphi_{ij} : \mathrm{pr}_0^*\mathcal{F}_i \cong \mathrm{pr}_1^*\mathcal{F}_j$ is an isomorphism on $T_i \times_T T_j$ such that the diagrams

$$\begin{tikzcd} \mathrm{pr}_0^*\mathcal{F}_i \arrow[rr, "\mathrm{pr}_{01}^*\varphi_{ij}"] \arrow[dr, "\mathrm{pr}_{02}^*\varphi_{ik}"'] & & \mathrm{pr}_1^*\mathcal{F}_j \arrow[dl, "\mathrm{pr}_{12}^*\varphi_{jk}"] \\ & \mathrm{pr}_2^*\mathcal{F}_k & \end{tikzcd}$$

commute on $T_i \times_T T_j \times_T T_k$.

This descent datum is called *effective* if there exist a quasi-coherent sheaf $\mathcal{F}$ over T and $\mathcal{O}_{T_i}$-module isomorphisms $\varphi_i : t_i^*\mathcal{F} \cong \mathcal{F}_i$ satisfying the cocycle condition, namely

$$\varphi_{ij} = \mathrm{pr}_1^*(\varphi_j) \circ \mathrm{pr}_0^*(\varphi_i)^{-1}.$$

In this and the next section we discuss some ingredients of the proof of the following theorem, as well as some related material.

03O8 **Theorem 16.2.** *If $\mathcal{V} = \{T_i \to T\}_{i \in I}$ is an fpqc covering, then all descent data for quasi-coherent sheaves with respect to $\mathcal{V}$ are effective.*

Proof. See Descent, Proposition 5.2. □

In other words, the fibered category of quasi-coherent sheaves is a stack on the fpqc site. The proof of the theorem is in two steps. The first one is to realize that for Zariski coverings this is easy (or well-known) using standard glueing of sheaves (see Sheaves, Section 33) and the locality of quasi-coherence. The second step is the case of an fpqc covering of the form $\{\mathrm{Spec}(B) \to \mathrm{Spec}(A)\}$ where $A \to B$ is a faithfully flat ring map. This is a lemma in algebra, which we now present.

Descent of modules. If $A \to B$ is a ring map, we consider the complex

$$(B/A)_\bullet : B \to B \otimes_A B \to B \otimes_A B \otimes_A B \to \ldots$$

where B is in degree 0, $B \otimes_A B$ in degree 1, etc, and the maps are given by

$$\begin{array}{rcl} b & \mapsto & 1 \otimes b - b \otimes 1, \\ b_0 \otimes b_1 & \mapsto & 1 \otimes b_0 \otimes b_1 - b_0 \otimes 1 \otimes b_1 + b_0 \otimes b_1 \otimes 1, \\ & \text{etc.} & \end{array}$$

03O9 **Lemma 16.3.** *If $A \to B$ is faithfully flat, then the complex $(B/A)_\bullet$ is exact in positive degrees, and $H^0((B/A)_\bullet) = A$.*

Proof. See Descent, Lemma 3.6. □

Grothendieck proves this in three steps. Firstly, he assumes that the map $A \to B$ has a section, and constructs an explicit homotopy to the complex where A is the only nonzero term, in degree 0. Secondly, he observes that to prove the result, it suffices to do so after a faithfully flat base change $A \to A'$, replacing B with $B' = B \otimes_A A'$. Thirdly, he applies the faithfully flat base change $A \to A' = B$ and remark that the map $A' = B \to B' = B \otimes_A B$ has a natural section.

The same strategy proves the following lemma.

03OA **Lemma 16.4.** *If $A \to B$ is faithfully flat and M is an A-module, then the complex $(B/A)_\bullet \otimes_A M$ is exact in positive degrees, and $H^0((B/A)_\bullet \otimes_A M) = M$.*

Proof. See Descent, Lemma 3.6. □

03OB **Definition 16.5.** Let $A \to B$ be a ring map and N a B-module. A *descent datum* for N with respect to $A \to B$ is an isomorphism $\varphi : N \otimes_A B \cong B \otimes_A N$ of $B \otimes_A B$-modules such that the diagram of $B \otimes_A B \otimes_A B$-modules

$$\begin{tikzcd} N \otimes_A B \otimes_A B \arrow[rr, "\varphi_{01}"] \arrow[dr, "\varphi_{02}"'] & & B \otimes_A N \otimes_A B \arrow[dl, "\varphi_{12}"] \\ & B \otimes_A B \otimes_A N & \end{tikzcd}$$

commutes where $\varphi_{01} = \varphi \otimes \mathrm{id}_B$ and similarly for φ_{12} and φ_{02}.

If $N' = B \otimes_A M$ for some A-module M, then it has a canonical descent datum given by the map

$$\begin{array}{rccc} \varphi_{\text{can}}: & N' \otimes_A B & \to & B \otimes_A N' \\ & b_0 \otimes m \otimes b_1 & \mapsto & b_0 \otimes b_1 \otimes m. \end{array}$$

03OC **Definition 16.6.** A descent datum (N, φ) is called *effective* if there exists an A-module M such that $(N, \varphi) \cong (B \otimes_A M, \varphi_{\text{can}})$, with the obvious notion of isomorphism of descent data.

Theorem 16.2 is a consequence the following result.

03OD **Theorem 16.7.** *If $A \to B$ is faithfully flat then descent data with respect to $A \to B$ are effective.*

Proof. See Descent, Proposition 3.9. See also Descent, Remark 3.11 for an alternative view of the proof. □

03OE **Remarks 16.8.** The results on descent of modules have several applications:

(1) The exactness of the Čech complex in positive degrees for the covering $\{\mathrm{Spec}(B) \to \mathrm{Spec}(A)\}$ where $A \to B$ is faithfully flat. This will give some vanishing of cohomology.
(2) If (N, φ) is a descent datum with respect to a faithfully flat map $A \to B$, then the corresponding A-module is given by
$$M = \mathrm{Ker}\begin{pmatrix} N & \longrightarrow & B \otimes_A N \\ n & \longmapsto & 1 \otimes n - \varphi(n \otimes 1) \end{pmatrix}.$$
See Descent, Proposition 3.9.

17. Quasi-coherent sheaves

03OF We can apply the descent of modules to study quasi-coherent sheaves.

03OG **Proposition 17.1.** *For any quasi-coherent sheaf $\mathcal{F}$ on S the presheaf*

$$\begin{array}{rccc} \mathcal{F}^a : & Sch/S & \to & Ab \\ & (f : T \to S) & \mapsto & \Gamma(T, f^*\mathcal{F}) \end{array}$$

is an $\mathcal{O}$-module which satisfies the sheaf condition for the fpqc topology.

Proof. This is proved in Descent, Lemma 8.1. We indicate the proof here. As established in Lemma 15.6, it is enough to check the sheaf property on Zariski coverings and faithfully flat morphisms of affine schemes. The sheaf property for

Zariski coverings is standard scheme theory, since $\Gamma(U, i^*\mathcal{F}) = \mathcal{F}(U)$ when $i : U \hookrightarrow S$ is an open immersion.

For $\{\text{Spec}(B) \to \text{Spec}(A)\}$ with $A \to B$ faithfully flat and $\mathcal{F}|_{\text{Spec}(A)} = \widetilde{M}$ this corresponds to the fact that $M = H^0\left((B/A)_\bullet \otimes_A M\right)$, i.e., that

$$0 \to M \to B \otimes_A M \to B \otimes_A B \otimes_A M$$

is exact by Lemma 16.4. $\square$

There is an abstract notion of a quasi-coherent sheaf on a ringed site. We briefly introduce this here. For more information please consult Modules on Sites, Section 23. Let $\mathcal{C}$ be a category, and let U be an object of $\mathcal{C}$. Then $\mathcal{C}/U$ indicates the category of objects over U, see Categories, Example 2.13. If $\mathcal{C}$ is a site, then $\mathcal{C}/U$ is a site as well, namely the coverings of V/U are families $\{V_i/U \to V/U\}$ of morphisms of $\mathcal{C}/U$ with fixed target such that $\{V_i \to V\}$ is a covering of $\mathcal{C}$. Moreover, given any sheaf $\mathcal{F}$ on $\mathcal{C}$ the *restriction* $\mathcal{F}|_{\mathcal{C}/U}$ (defined in the obvious manner) is a sheaf as well. See Sites, Section 25 for details.

03OH **Definition 17.2.** Let $\mathcal{C}$ be a *ringed site*, i.e., a site endowed with a sheaf of rings $\mathcal{O}$. A sheaf of $\mathcal{O}$-modules $\mathcal{F}$ on $\mathcal{C}$ is called *quasi-coherent* if for all $U \in \text{Ob}(\mathcal{C})$ there exists a covering $\{U_i \to U\}_{i\in I}$ of $\mathcal{C}$ such that the restriction $\mathcal{F}|_{\mathcal{C}/U_i}$ is isomorphic to the cokernel of an $\mathcal{O}$-linear map of free $\mathcal{O}$-modules

$$\bigoplus\nolimits_{k\in K} \mathcal{O}|_{\mathcal{C}/U_i} \longrightarrow \bigoplus\nolimits_{l\in L} \mathcal{O}|_{\mathcal{C}/U_i}.$$

The direct sum over K is the sheaf associated to the presheaf $V \mapsto \bigoplus_{k\in K} \mathcal{O}(V)$ and similarly for the other.

Although it is useful to be able to give a general definition as above this notion is not well behaved in general.

03OI **Remark 17.3.** In the case where $\mathcal{C}$ has a final object, e.g. S, it suffices to check the condition of the definition for $U = S$ in the above statement. See Modules on Sites, Lemma 23.3.

03OJ **Theorem 17.4** (Meta theorem on quasi-coherent sheaves)**.** *Let S be a scheme. Let $\mathcal{C}$ be a site. Assume that*

1. *the underlying category $\mathcal{C}$ is a full subcategory of Sch/S,*
2. *any Zariski covering of $T \in \text{Ob}(\mathcal{C})$ can be refined by a covering of $\mathcal{C}$,*
3. *S/S is an object of $\mathcal{C}$,*
4. *every covering of $\mathcal{C}$ is an fpqc covering of schemes.*

Then the presheaf $\mathcal{O}$ is a sheaf on $\mathcal{C}$ and any quasi-coherent $\mathcal{O}$-module on $(\mathcal{C}, \mathcal{O})$ is of the form $\mathcal{F}^a$ for some quasi-coherent sheaf $\mathcal{F}$ on S.

Proof. After some formal arguments this is exactly Theorem 16.2. Details omitted. In Descent, Proposition 8.11 we prove a more precise version of the theorem for the big Zariski, fppf, étale, smooth, and syntomic sites of S, as well as the small Zariski and étale sites of S. $\square$

In other words, there is no difference between quasi-coherent modules on the scheme S and quasi-coherent $\mathcal{O}$-modules on sites $\mathcal{C}$ as in the theorem. More precise statements for the big and small sites $(Sch/S)_{fppf}$, $S_{étale}$, etc can be found in Descent, Section 8. In this chapter we will sometimes refer to a "site as in Theorem 17.4" in order to conveniently state results which hold in any of those situations.

18. Čech cohomology

03OK Our next goal is to use descent theory to show that $H^i(\mathcal{C}, \mathcal{F}^a) = H^i_{Zar}(S, \mathcal{F})$ for all quasi-coherent sheaves $\mathcal{F}$ on S, and any site $\mathcal{C}$ as in Theorem 17.4. To this end, we introduce Čech cohomology on sites. See [Art62] and Cohomology on Sites, Sections 8, 9 and 10 for more details.

03OL **Definition 18.1.** Let $\mathcal{C}$ be a category, $\mathcal{U} = \{U_i \to U\}_{i \in I}$ a family of morphisms of $\mathcal{C}$ with fixed target, and $\mathcal{F} \in PAb(\mathcal{C})$ an abelian presheaf. We define the *Čech complex* $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})$ by

$$\prod_{i_0 \in I} \mathcal{F}(U_{i_0}) \to \prod_{i_0, i_1 \in I} \mathcal{F}(U_{i_0} \times_U U_{i_1}) \to \prod_{i_0, i_1, i_2 \in I} \mathcal{F}(U_{i_0} \times_U U_{i_1} \times_U U_{i_2}) \to \ldots$$

where the first term is in degree 0, and the maps are the usual ones. Again, it is essential to allow the case $i_0 = i_1$ etc. The *Čech cohomology groups* are defined by

$$\check{H}^p(\mathcal{U}, \mathcal{F}) = H^p(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})).$$

03OM **Lemma 18.2.** *The functor $\check{\mathcal{C}}^\bullet(\mathcal{U}, -)$ is exact on the category $PAb(\mathcal{C})$.*

In other words, if $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ is a short exact sequence of presheaves of abelian groups, then

$$0 \to \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}_1) \to \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}_2) \to \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}_3) \to 0$$

is a short exact sequence of complexes.

Proof. This follows at once from the definition of a short exact sequence of presheaves. Namely, as the category of abelian presheaves is the category of functors on some category with values in *Ab*, it is automatically an abelian category: a sequence $\mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3$ is exact in *PAb* if and only if for all $U \in \mathrm{Ob}(\mathcal{C})$, the sequence $\mathcal{F}_1(U) \to \mathcal{F}_2(U) \to \mathcal{F}_3(U)$ is exact in *Ab*. So the complex above is merely a product of short exact sequences in each degree. See also Cohomology on Sites, Lemma 9.1. □

This shows that $\check{H}^\bullet(\mathcal{U}, -)$ is a δ-functor. We now proceed to show that it is a universal δ-functor. We thus need to show that it is an *effaceable* functor. We start by recalling the Yoneda lemma.

03ON **Lemma 18.3** (Yoneda Lemma)**.** *For any presheaf $\mathcal{F}$ on a category $\mathcal{C}$ there is a functorial isomorphism*

$$\mathrm{Hom}_{PSh(\mathcal{C})}(h_U, \mathcal{F}) = \mathcal{F}(U).$$

Proof. See Categories, Lemma 3.5. □

Given a set E we denote (in this section) $\mathbf{Z}[E]$ the free abelian group on E. In a formula $\mathbf{Z}[E] = \bigoplus_{e \in E} \mathbf{Z}$, i.e., $\mathbf{Z}[E]$ is a free $\mathbf{Z}$-module having a basis consisting of the elements of E. Using this notation we introduce the free abelian presheaf on a presheaf of sets.

03OO **Definition 18.4.** Let $\mathcal{C}$ be a category. Given a presheaf of sets $\mathcal{G}$, we define the *free abelian presheaf on $\mathcal{G}$*, denoted $\mathbf{Z}_\mathcal{G}$, by the rule

$$\mathbf{Z}_\mathcal{G}(U) = \mathbf{Z}[\mathcal{G}(U)]$$

for $U \in \mathrm{Ob}(\mathcal{C})$ with restriction maps induced by the restriction maps of $\mathcal{G}$. In the special case $\mathcal{G} = h_U$ we write simply $\mathbf{Z}_U = \mathbf{Z}_{h_U}$.

The functor $\mathcal{G} \mapsto \mathbf{Z}_\mathcal{G}$ is left adjoint to the forgetful functor $PAb(\mathcal{C}) \to PSh(\mathcal{C})$. Thus, for any presheaf $\mathcal{F}$, there is a canonical isomorphism

$$\mathrm{Hom}_{PAb(\mathcal{C})}(\mathbf{Z}_U, \mathcal{F}) = \mathrm{Hom}_{PSh(\mathcal{C})}(h_U, \mathcal{F}) = \mathcal{F}(U)$$

the last equality by the Yoneda lemma. In particular, we have the following result.

03OP **Lemma 18.5.** *The Čech complex $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})$ can be described explicitly as follows*

$$\begin{aligned}
\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}) &= \left(\prod_{i_0 \in I} \mathrm{Hom}_{PAb(\mathcal{C})}(\mathbf{Z}_{U_{i_0}}, \mathcal{F}) \to \prod_{i_0, i_1 \in I} \mathrm{Hom}_{PAb(\mathcal{C})}(\mathbf{Z}_{U_{i_0} \times_U U_{i_1}}, \mathcal{F}) \to \ldots\right) \\
&= \mathrm{Hom}_{PAb(\mathcal{C})}\left(\left(\bigoplus_{i_0 \in I} \mathbf{Z}_{U_{i_0}} \leftarrow \bigoplus_{i_0, i_1 \in I} \mathbf{Z}_{U_{i_0} \times_U U_{i_1}} \leftarrow \ldots\right), \mathcal{F}\right)
\end{aligned}$$

Proof. This follows from the formula above. See Cohomology on Sites, Lemma 9.3. □

This reduces us to studying only the complex in the first argument of the last Hom.

03OQ **Lemma 18.6.** *The complex of abelian presheaves*

$$\mathbf{Z}_\mathcal{U}^\bullet \quad : \quad \bigoplus_{i_0 \in I} \mathbf{Z}_{U_{i_0}} \leftarrow \bigoplus_{i_0, i_1 \in I} \mathbf{Z}_{U_{i_0} \times_U U_{i_1}} \leftarrow \bigoplus_{i_0, i_1, i_2 \in I} \mathbf{Z}_{U_{i_0} \times_U U_{i_1} \times_U U_{i_2}} \leftarrow \ldots$$

is exact in all degrees except 0 *in* $PAb(\mathcal{C})$*.*

Proof. For any $V \in \mathrm{Ob}(\mathcal{C})$ the complex of abelian groups $\mathbf{Z}_\mathcal{U}^\bullet(V)$ is

$$\begin{aligned}
&\mathbf{Z}\left[\textstyle\coprod_{i_0 \in I} Mor_\mathcal{C}(V, U_{i_0})\right] \leftarrow \mathbf{Z}\left[\textstyle\coprod_{i_0, i_1 \in I} Mor_\mathcal{C}(V, U_{i_0} \times_U U_{i_1})\right] \leftarrow \ldots = \\
&\textstyle\bigoplus_{\varphi : V \to U} \left(\mathbf{Z}\left[\coprod_{i_0 \in I} Mor_\varphi(V, U_{i_0})\right] \leftarrow \mathbf{Z}\left[\coprod_{i_0, i_1 \in I} Mor_\varphi(V, U_{i_0}) \times Mor_\varphi(V, U_{i_1})\right] \leftarrow \ldots\right)
\end{aligned}$$

where

$$Mor_\varphi(V, U_i) = \{V \to U_i \text{ such that } V \to U_i \to U \text{ equals } \varphi\}.$$

Set $S_\varphi = \coprod_{i \in I} Mor_\varphi(V, U_i)$, so that

$$\mathbf{Z}_\mathcal{U}^\bullet(V) = \bigoplus_{\varphi : V \to U} \left(\mathbf{Z}[S_\varphi] \leftarrow \mathbf{Z}[S_\varphi \times S_\varphi] \leftarrow \mathbf{Z}[S_\varphi \times S_\varphi \times S_\varphi] \leftarrow \ldots\right).$$

Thus it suffices to show that for each $S = S_\varphi$, the complex

$$\mathbf{Z}[S] \leftarrow \mathbf{Z}[S \times S] \leftarrow \mathbf{Z}[S \times S \times S] \leftarrow \ldots$$

is exact in negative degrees. To see this, we can give an explicit homotopy. Fix $s \in S$ and define $K : n_{(s_0, \ldots, s_p)} \mapsto n_{(s, s_0, \ldots, s_p)}$. One easily checks that K is a nullhomotopy for the operator

$$\delta : \eta_{(s_0, \ldots, s_p)} \mapsto \sum\nolimits_{i=0}^{p} (-1)^p \eta_{(s_0, \ldots, \hat{s}_i, \ldots, s_p)}.$$

See Cohomology on Sites, Lemma 9.4 for more details. □

03OR **Lemma 18.7.** *Let $\mathcal{C}$ be a category. If $\mathcal{I}$ is an injective object of $PAb(\mathcal{C})$ and $\mathcal{U}$ is a family of morphisms with fixed target in $\mathcal{C}$, then $\check{H}^p(\mathcal{U}, \mathcal{I}) = 0$ for all $p > 0$.*

Proof. The Čech complex is the result of applying the functor $\mathrm{Hom}_{PAb(\mathcal{C})}(-, \mathcal{I})$ to the complex $\mathbf{Z}^\bullet_\mathcal{U}$, i.e.,

$$\check{H}^p(\mathcal{U}, \mathcal{I}) = H^p(\mathrm{Hom}_{PAb(\mathcal{C})}(\mathbf{Z}^\bullet_\mathcal{U}, \mathcal{I})).$$

But we have just seen that $\mathbf{Z}^\bullet_\mathcal{U}$ is exact in negative degrees, and the functor $\mathrm{Hom}_{PAb(\mathcal{C})}(-, \mathcal{I})$ is exact, hence $\mathrm{Hom}_{PAb(\mathcal{C})}(\mathbf{Z}^\bullet_\mathcal{U}, \mathcal{I})$ is exact in positive degrees. □

03OS **Theorem 18.8.** *On $PAb(\mathcal{C})$ the functors $\check{H}^p(\mathcal{U}, -)$ are the right derived functors of $\check{H}^0(\mathcal{U}, -)$.*

Proof. By the Lemma 18.7, the functors $\check{H}^p(\mathcal{U}, -)$ are universal δ-functors since they are effaceable. So are the right derived functors of $\check{H}^0(\mathcal{U}, -)$. Since they agree in degree 0, they agree by the universal property of universal δ-functors. For more details see Cohomology on Sites, Lemma 9.6. □

03OT **Remark 18.9.** Observe that all of the preceding statements are about presheaves so we haven't made use of the topology yet.

19. The Čech-to-cohomology spectral sequence

03OU This spectral sequence is fundamental in proving foundational results on cohomology of sheaves.

03OV **Lemma 19.1.** *The forgetful functor $Ab(\mathcal{C}) \to PAb(\mathcal{C})$ transforms injectives into injectives.*

Proof. This is formal using the fact that the forgetful functor has a left adjoint, namely sheafification, which is an exact functor. For more details see Cohomology on Sites, Lemma 10.1. □

03OW **Theorem 19.2.** *Let $\mathcal{C}$ be a site. For any covering $\mathcal{U} = \{U_i \to U\}_{i \in I}$ of $U \in \mathrm{Ob}(\mathcal{C})$ and any abelian sheaf $\mathcal{F}$ on $\mathcal{C}$ there is a spectral sequence*

$$E_2^{p,q} = \check{H}^p(\mathcal{U}, \underline{H}^q(\mathcal{F})) \Rightarrow H^{p+q}(U, \mathcal{F}),$$

where $\underline{H}^q(\mathcal{F})$ is the abelian presheaf $V \mapsto H^q(V, \mathcal{F})$.

Proof. Choose an injective resolution $\mathcal{F} \to \mathcal{I}^\bullet$ in $Ab(\mathcal{C})$, and consider the double complex $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}^\bullet)$ and the maps

$$\begin{tikzcd} \Gamma(U, I^\bullet) \arrow[r] & \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}^\bullet) \\ & \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}) \arrow[u] \end{tikzcd}$$

Here the horizontal map is the natural map $\Gamma(U, I^\bullet) \to \check{\mathcal{C}}^0(\mathcal{U}, \mathcal{I}^\bullet)$ to the left column, and the vertical map is induced by $\mathcal{F} \to \mathcal{I}^0$ and lands in the bottom row. By assumption, $\mathcal{I}^\bullet$ is a complex of injectives in $Ab(\mathcal{C})$, hence by Lemma 19.1, it is a complex of injectives in $PAb(\mathcal{C})$. Thus, the rows of the double complex are exact in positive degrees (Lemma 18.7), and the kernel of $\check{\mathcal{C}}^0(\mathcal{U}, \mathcal{I}^\bullet) \to \check{\mathcal{C}}^1(\mathcal{U}, \mathcal{I}^\bullet)$ is equal to $\Gamma(U, \mathcal{I}^\bullet)$, since $\mathcal{I}^\bullet$ is a complex of sheaves. In particular, the cohomology of the total complex is the standard cohomology of the global sections functor $H^0(U, \mathcal{F})$.

For the vertical direction, the qth cohomology group of the pth column is

$$\prod_{i_0,\ldots,i_p} H^q(U_{i_0} \times_U \ldots \times_U U_{i_p}, \mathcal{F}) = \prod_{i_0,\ldots,i_p} \underline{H}^q(\mathcal{F})(U_{i_0} \times_U \ldots \times_U U_{i_p})$$

in the entry $E_1^{p,q}$. So this is a standard double complex spectral sequence, and the E_2-page is as prescribed. For more details see Cohomology on Sites, Lemma 10.6. □

03OX **Remark 19.3.** This is a Grothendieck spectral sequence for the composition of functors

$$Ab(\mathcal{C}) \longrightarrow PAb(\mathcal{C}) \xrightarrow{\check{H}^0} Ab.$$

20. Big and small sites of schemes

03X7 Let S be a scheme. Let τ be one of the topologies we will be discussing. Thus $\tau \in \{fppf, syntomic, smooth, \acute{e}tale, Zariski\}$. Of course if you are only interested in the étale topology, then you can simply assume $\tau = \acute{e}tale$ throughout. Moreover, we will discuss étale morphisms, étale coverings, and étale sites in more detail starting in Section 25. In order to proceed with the discussion of cohomology of quasi-coherent sheaves it is convenient to introduce the big τ-site and in case $\tau \in \{\acute{e}tale, Zariski\}$, the small τ-site of S. In order to do this we first introduce the notion of a τ-covering.

03X8 **Definition 20.1.** (See Topologies, Definitions 7.1, 6.1, 5.1, 4.1, and 3.1.) Let $\tau \in \{fppf, syntomic, smooth, \acute{e}tale, Zariski\}$. A family of morphisms of schemes $\{f_i : T_i \to T\}_{i\in I}$ with fixed target is called a *τ-covering* if and only if each f_i is flat of finite presentation, syntomic, smooth, étale, resp. an open immersion, and we have $\bigcup f_i(T_i) = T$.

It turns out that the class of all τ-coverings satisfies the axioms (1), (2) and (3) of Definition 10.2 (our definition of a site), see Topologies, Lemmas 7.3, 6.3, 5.3, 4.3, and 3.2. In order to be able to compare any of these new topologies to the fpqc topology and to use the preceding results on descent on modules we single out a special class of τ-coverings of affine schemes called standard coverings.

03X9 **Definition 20.2.** (See Topologies, Definitions 7.5, 6.5, 5.5, 4.5, and 3.4.) Let $\tau \in \{fppf, syntomic, smooth, \acute{e}tale, Zariski\}$. Let T be an affine scheme. A *standard τ-covering* of T is a family $\{f_j : U_j \to T\}_{j=1,\ldots,m}$ with each U_j is affine, and each f_j flat and of finite presentation, standard syntomic, standard smooth, étale, resp. the immersion of a standard principal open in T and $T = \bigcup f_j(U_j)$.

03XA **Lemma 20.3.** *Let $\tau \in \{fppf, syntomic, smooth, \acute{e}tale, Zariski\}$. Any τ-covering of an affine scheme can be refined by a standard τ-covering.*

Proof. See Topologies, Lemmas 7.4, 6.4, 5.4, 4.4, and 3.3. □

Finally, we come to our definition of the sites we will be working with. This is actually somewhat involved since, contrary to what happens in [AGV71], we do not want to choose a universe. Instead we pick a "partial universe" (which is a suitably large set as in Sets, Section 5), and consider all schemes contained in this set. Of course we make sure that our favorite base scheme S is contained in the partial universe. Having picked the underlying category we pick a suitably large set of τ-coverings which turns this into a site. The details are in the chapter on

topologies on schemes; there is a lot of freedom in the choices made, but in the end the actual choices made will not affect the étale (or other) cohomology of S (just as in [AGV71] the actual choice of universe doesn't matter at the end). Moreover, the way the material is written the reader who is happy using strongly inaccessible cardinals (i.e., universes) can do so as a substitute.

03XB **Definition 20.4.** Let S be a scheme. Let $\tau \in \{fppf, syntomic, smooth, étale, Zariski\}$.

(1) A *big τ-site of S* is any of the sites $(Sch/S)_\tau$ constructed as explained above and in more detail in Topologies, Definitions 7.8, 6.8, 5.8, 4.8, and 3.7.
(2) If $\tau \in \{étale, Zariski\}$, then the *small τ-site of S* is the full subcategory S_τ of $(Sch/S)_\tau$ whose objects are schemes T over S whose structure morphism $T \to S$ is étale, resp. an open immersion. A covering in S_τ is a covering $\{U_i \to U\}$ in $(Sch/S)_\tau$ such that U is an object of S_τ.

The underlying category of the site $(Sch/S)_\tau$ has reasonable "closure" properties, i.e., given a scheme T in it any locally closed subscheme of T is isomorphic to an object of $(Sch/S)_\tau$. Other such closure properties are: closed under fibre products of schemes, taking countable disjoint unions, taking finite type schemes over a given scheme, given an affine scheme $\text{Spec}(R)$ one can complete, localize, or take the quotient of R by an ideal while staying inside the category, etc. On the other hand, for example arbitrary disjoint unions of schemes in $(Sch/S)_\tau$ will take you outside of it. Also note that, given an object T of $(Sch/S)_\tau$ there will exist τ-coverings $\{T_i \to T\}_{i\in I}$ (as in Definition 20.1) which are not coverings in $(Sch/S)_\tau$ for example because the schemes T_i are not objects of the category $(Sch/S)_\tau$. But our choice of the sites $(Sch/S)_\tau$ is such that there always does exist a covering $\{U_j \to T\}_{j\in J}$ of $(Sch/S)_\tau$ which refines the covering $\{T_i \to T\}_{i\in I}$, see Topologies, Lemmas 7.7, 6.7, 5.7, 4.7, and 3.6. We will mostly ignore these issues in this chapter.

If $\mathcal{F}$ is a sheaf on $(Sch/S)_\tau$ or S_τ, then we denote

$$H^p_\tau(U, \mathcal{F}), \text{ in particular } H^p_\tau(S, \mathcal{F})$$

the cohomology groups of $\mathcal{F}$ over the object U of the site, see Section 14. Thus we have $H^p_{fppf}(S, \mathcal{F})$, $H^p_{syntomic}(S, \mathcal{F})$, $H^p_{smooth}(S, \mathcal{F})$, $H^p_{étale}(S, \mathcal{F})$, and $H^p_{Zar}(S, \mathcal{F})$. The last two are potentially ambiguous since they might refer to either the big or small étale or Zariski site. However, this ambiguity is harmless by the following lemma.

03YX **Lemma 20.5.** *Let $\tau \in \{étale, Zariski\}$. If $\mathcal{F}$ is an abelian sheaf defined on $(Sch/S)_\tau$, then the cohomology groups of $\mathcal{F}$ over S agree with the cohomology groups of $\mathcal{F}|_{S_\tau}$ over S.*

Proof. By Topologies, Lemmas 3.13 and 4.13 the functors $S_\tau \to (Sch/S)_\tau$ satisfy the hypotheses of Sites, Lemma 21.8. Hence our lemma follows from Cohomology on Sites, Lemma 7.2. $\square$

For completeness we state and prove the invariance under choice of partial universe of the cohomology groups we are considering. We will prove invariance of the small étale topos in Lemma 21.3 below. For notation and terminology used in this lemma we refer to Topologies, Section 12.

03YY **Lemma 20.6.** *Let $\tau \in \{fppf, syntomic, smooth, étale, Zariski\}$. Let S be a scheme. Let $(Sch/S)_\tau$ and $(Sch'/S)_\tau$ be two big τ-sites of S, and assume that the first is contained in the second. In this case*

(1) *for any abelian sheaf $\mathcal{F}'$ defined on $(Sch'/S)_\tau$ and any object U of $(Sch/S)_\tau$ we have*
$$H^p_\tau(U, \mathcal{F}'|_{(Sch/S)_\tau}) = H^p_\tau(U, \mathcal{F}')$$
In words: the cohomology of $\mathcal{F}'$ over U computed in the bigger site agrees with the cohomology of $\mathcal{F}'$ restricted to the smaller site over U.

(2) *for any abelian sheaf $\mathcal{F}$ on $(Sch/S)_\tau$ there is an abelian sheaf $\mathcal{F}'$ on $(Sch/S)'_\tau$ whose restriction to $(Sch/S)_\tau$ is isomorphic to $\mathcal{F}$.*

Proof. By Topologies, Lemma 12.2 the inclusion functor $(Sch/S)_\tau \to (Sch'/S)_\tau$ satisfies the assumptions of Sites, Lemma 21.8. This implies (2) and (1) follows from Cohomology on Sites, Lemma 7.2. □

21. The étale topos

04HP A *topos* is the category of sheaves of sets on a site, see Sites, Definition 15.1. Hence it is customary to refer to the use the phrase "étale topos of a scheme" to refer to the category of sheaves on the small étale site of a scheme. Here is the formal definition.

04HQ **Definition 21.1.** Let S be a scheme.

(1) The *étale topos*, or the *small étale topos* of S is the category $Sh(S_{étale})$ of sheaves of sets on the small étale site of S.
(2) The *Zariski topos*, or the *small Zariski topos* of S is the category $Sh(S_{Zar})$ of sheaves of sets on the small Zariski site of S.
(3) For $\tau \in \{fppf, syntomic, smooth, étale, Zariski\}$ a *big τ-topos* is the category of sheaves of set on a big τ-topos of S.

Note that the small Zariski topos of S is simply the category of sheaves of sets on the underlying topological space of S, see Topologies, Lemma 3.11. Whereas the small étale topos does not depend on the choices made in the construction of the small étale site, in general the big topoi do depend on those choices.

Here is a lemma, which is one of many possible lemmas expressing the fact that it doesn't matter too much which site we choose to define the small étale topos of a scheme.

04HR **Lemma 21.2.** *Let S be a scheme. Let $S_{affine,étale}$ denote the full subcategory of $S_{étale}$ whose objects are those $U/S \in \text{Ob}(S_{étale})$ with U affine. A covering of $S_{affine,étale}$ will be a standard étale covering, see Topologies, Definition 4.5. Then restriction*
$$\mathcal{F} \longmapsto \mathcal{F}|_{S_{affine,étale}}$$
defines an equivalence of topoi $Sh(S_{étale}) \cong Sh(S_{affine,étale})$.

Proof. This you can show directly from the definitions, and is a good exercise. But it also follows immediately from Sites, Lemma 29.1 by checking that the inclusion functor $S_{affine,étale} \to S_{étale}$ is a special cocontinuous functor (see Sites, Definition 29.2). □

0958 **Lemma 21.3.** *Let S be a scheme. The étale topos of S is independent (up to canonical equivalence) of the construction of the small étale site in Definition 20.4.*

Proof. We have to show, given two big étale sites $Sch_{étale}$ and $Sch'_{étale}$ containing S, then $Sh(S_{étale}) \cong Sh(S'_{étale})$ with obvious notation. By Topologies, Lemma 12.1 we may assume $Sch_{étale} \subset Sch'_{étale}$. By Sets, Lemma 9.9 any affine scheme étale over S is isomorphic to an object of both $Sch_{étale}$ and $Sch'_{étale}$. Thus the induced functor $S_{affine,étale} \to S'_{affine,étale}$ is an equivalence. Moreover, it is clear that both this functor and a quasi-inverse map transform standard étale coverings into standard étale coverings. Hence the result follows from Lemma 21.2. □

0F1B **Lemma 21.4.** *Let S be a scheme. Let $S_{affine,Zar}$ denote the full subcategory of S_{Zar} consisting of affine objects. A covering of $S_{affine,Zar}$ will be a standard Zariski covering, see Topologies, Definition 3.4. Then restriction*

$$\mathcal{F} \longmapsto \mathcal{F}|_{S_{affine,Zar}}$$

defines an equivalence of topoi $Sh(S_{Zar}) \cong Sh(S_{affine,Zar})$.

Proof. Please skip the proof of this lemma. It follows immediately from Sites, Lemma 29.1 by checking that the inclusion functor $S_{affine,Zar} \to S_{Zar}$ is a special cocontinuous functor (see Sites, Definition 29.2). □

22. Cohomology of quasi-coherent sheaves

03OY We start with a simple lemma (which holds in greater generality than stated). It says that the Čech complex of a standard covering is equal to the Čech complex of an fpqc covering of the form $\{\text{Spec}(B) \to \text{Spec}(A)\}$ with $A \to B$ faithfully flat.

03OZ **Lemma 22.1.** *Let $\tau \in \{fppf, syntomic, smooth, étale, Zariski\}$. Let S be a scheme. Let $\mathcal{F}$ be an abelian sheaf on $(Sch/S)_\tau$, or on S_τ in case $\tau = étale$, and let $\mathcal{U} = \{U_i \to U\}_{i \in I}$ be a standard τ-covering of this site. Let $V = \coprod_{i \in I} U_i$. Then*

1. *V is an affine scheme,*
2. *$\mathcal{V} = \{V \to U\}$ is a τ-covering and an fpqc covering,*
3. *the Čech complexes $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})$ and $\check{\mathcal{C}}^\bullet(\mathcal{V}, \mathcal{F})$ agree.*

Proof. As the covering is a standard τ-covering each of the schemes U_i is affine and I is a finite set. Hence V is an affine scheme. It is clear that $V \to U$ is flat and surjective, hence $\mathcal{V}$ is an fpqc covering, see Example 15.3. Note that $\mathcal{U}$ is a refinement of $\mathcal{V}$ and hence there is a map of Čech complexes $\check{\mathcal{C}}^\bullet(\mathcal{V}, \mathcal{F}) \to \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})$, see Cohomology on Sites, Equation (8.2.1). Next, we observe that if $T = \coprod_{j \in J} T_j$ is a disjoint union of schemes in the site on which $\mathcal{F}$ is defined then the family of morphisms with fixed target $\{T_j \to T\}_{j \in J}$ is a Zariski covering, and so

03XC (22.1.1) $$\mathcal{F}(T) = \mathcal{F}(\coprod\nolimits_{j \in J} T_j) = \prod\nolimits_{j \in J} \mathcal{F}(T_j)$$

by the sheaf condition of $\mathcal{F}$. This implies the map of Čech complexes above is an isomorphism in each degree because

$$V \times_U \ldots \times_U V = \coprod\nolimits_{i_0, \ldots i_p} U_{i_0} \times_U \ldots \times_U U_{i_p}$$

as schemes. □

Note that Equality (22.1.1) is false for a general presheaf. Even for sheaves it does not hold on any site, since coproducts may not lead to coverings, and may not be disjoint. But it does for all the usual ones (at least all the ones we will study).

03P0 **Remark 22.2.** In the statement of Lemma 22.1 the covering $\mathcal{U}$ is a refinement of $\mathcal{V}$ but not the other way around. Coverings of the form $\{V \to U\}$ do not form an initial subcategory of the category of all coverings of U. Yet it is still true that we can compute Čech cohomology $\check{H}^n(U, \mathcal{F})$ (which is defined as the colimit over the opposite of the category of coverings $\mathcal{U}$ of U of the Čech cohomology groups of $\mathcal{F}$ with respect to $\mathcal{U}$) in terms of the coverings $\{V \to U\}$. We will formulate a precise lemma (it only works for sheaves) and add it here if we ever need it.

03P1 **Lemma 22.3** (Locality of cohomology)**.** *Let $\mathcal{C}$ be a site, $\mathcal{F}$ an abelian sheaf on $\mathcal{C}$, U an object of $\mathcal{C}$, $p > 0$ an integer and $\xi \in H^p(U, \mathcal{F})$. Then there exists a covering $\mathcal{U} = \{U_i \to U\}_{i \in I}$ of U in $\mathcal{C}$ such that $\xi|_{U_i} = 0$ for all $i \in I$.*

Proof. Choose an injective resolution $\mathcal{F} \to \mathcal{I}^\bullet$. Then ξ is represented by a cocycle $\tilde{\xi} \in \mathcal{I}^p(U)$ with $d^p(\tilde{\xi}) = 0$. By assumption, the sequence $\mathcal{I}^{p-1} \to \mathcal{I}^p \to \mathcal{I}^{p+1}$ in exact in $Ab(\mathcal{C})$, which means that there exists a covering $\mathcal{U} = \{U_i \to U\}_{i \in I}$ such that $\tilde{\xi}|_{U_i} = d^{p-1}(\xi_i)$ for some $\xi_i \in \mathcal{I}^{p-1}(U_i)$. Since the cohomology class $\xi|_{U_i}$ is represented by the cocycle $\tilde{\xi}|_{U_i}$ which is a coboundary, it vanishes. For more details see Cohomology on Sites, Lemma 7.3. $\square$

03P2 **Theorem 22.4.** *Let S be a scheme and $\mathcal{F}$ a quasi-coherent $\mathcal{O}_S$-module. Let $\mathcal{C}$ be either $(Sch/S)_\tau$ for $\tau \in \{fppf, syntomic, smooth, étale, Zariski\}$ or $S_{étale}$. Then*

$$H^p(S, \mathcal{F}) = H^p_\tau(S, \mathcal{F}^a)$$

for all $p \geq 0$ where

1. *the left hand side indicates the usual cohomology of the sheaf $\mathcal{F}$ on the underlying topological space of the scheme S, and*
2. *the right hand side indicates cohomology of the abelian sheaf $\mathcal{F}^a$ (see Proposition 17.1) on the site $\mathcal{C}$.*

Proof. We are going to show that $H^p(U, f^*\mathcal{F}) = H^p_\tau(U, \mathcal{F}^a)$ for any object $f : U \to S$ of the site $\mathcal{C}$. The result is true for $p = 0$ by the sheaf property.

Assume that U is affine. Then we want to prove that $H^p_\tau(U, \mathcal{F}^a) = 0$ for all $p > 0$. We use induction on p.

- p = 1 Pick $\xi \in H^1_\tau(U, \mathcal{F}^a)$. By Lemma 22.3, there exists an fpqc covering $\mathcal{U} = \{U_i \to U\}_{i \in I}$ such that $\xi|_{U_i} = 0$ for all $i \in I$. Up to refining $\mathcal{U}$, we may assume that $\mathcal{U}$ is a standard τ-covering. Applying the spectral sequence of Theorem 19.2, we see that ξ comes from a cohomology class $\check{\xi} \in \check{H}^1(\mathcal{U}, \mathcal{F}^a)$. Consider the covering $\mathcal{V} = \{\coprod_{i \in I} U_i \to U\}$. By Lemma 22.1, $\check{H}^\bullet(\mathcal{U}, \mathcal{F}^a) = \check{H}^\bullet(\mathcal{V}, \mathcal{F}^a)$. On the other hand, since $\mathcal{V}$ is a covering of the form $\{\text{Spec}(B) \to \text{Spec}(A)\}$ and $f^*\mathcal{F} = \widetilde{M}$ for some A-module M, we see the Čech complex $\check{\mathcal{C}}^\bullet(\mathcal{V}, \mathcal{F})$ is none other than the complex $(B/A)_\bullet \otimes_A M$. Now by Lemma 16.4, $H^p((B/A)_\bullet \otimes_A M) = 0$ for $p > 0$, hence $\check{\xi} = 0$ and so $\xi = 0$.
- p > 1 Pick $\xi \in H^p_\tau(U, \mathcal{F}^a)$. By Lemma 22.3, there exists an fpqc covering $\mathcal{U} = \{U_i \to U\}_{i \in I}$ such that $\xi|_{U_i} = 0$ for all $i \in I$. Up to refining $\mathcal{U}$, we may assume that $\mathcal{U}$ is a standard τ-covering. We apply the spectral sequence of

Theorem 19.2. Observe that the intersections $U_{i_0} \times_U \ldots \times_U U_{i_p}$ are affine, so that by induction hypothesis the cohomology groups

$$E_2^{p,q} = \check{H}^p(\mathcal{U}, \underline{H}^q(\mathcal{F}^a))$$

vanish for all $0 < q < p$. We see that ξ must come from a $\check{\xi} \in \check{H}^p(\mathcal{U}, \mathcal{F}^a)$. Replacing $\mathcal{U}$ with the covering $\mathcal{V}$ containing only one morphism and using Lemma 16.4 again, we see that the Čech cohomology class $\check{\xi}$ must be zero, hence $\xi = 0$.

Next, assume that U is separated. Choose an affine open covering $U = \bigcup_{i \in I} U_i$ of U. The family $\mathcal{U} = \{U_i \to U\}_{i \in I}$ is then an fpqc covering, and all the intersections $U_{i_0} \times_U \ldots \times_U U_{i_p}$ are affine since U is separated. So all rows of the spectral sequence of Theorem 19.2 are zero, except the zeroth row. Therefore

$$H^p_\tau(U, \mathcal{F}^a) = \check{H}^p(\mathcal{U}, \mathcal{F}^a) = \check{H}^p(\mathcal{U}, \mathcal{F}) = H^p(U, \mathcal{F})$$

where the last equality results from standard scheme theory, see Cohomology of Schemes, Lemma 2.6.

The general case is technical and (to extend the proof as given here) requires a discussion about maps of spectral sequences, so we won't treat it. It follows from Descent, Proposition 8.10 (whose proof takes a slightly different approach) combined with Cohomology on Sites, Lemma 7.1. □

03P3 **Remark 22.5.** Comment on Theorem 22.4. Since S is a final object in the category $\mathcal{C}$, the cohomology groups on the right-hand side are merely the right derived functors of the global sections functor. In fact the proof shows that $H^p(U, f^*\mathcal{F}) = H^p_\tau(U, \mathcal{F}^a)$ for any object $f : U \to S$ of the site $\mathcal{C}$.

23. Examples of sheaves

03YZ Let S and τ be as in Section 20. We have already seen that any representable presheaf is a sheaf on $(Sch/S)_\tau$ or S_τ, see Lemma 15.8 and Remark 15.9. Here are some special cases.

03P4 **Definition 23.1.** On any of the sites $(Sch/S)_\tau$ or S_τ of Section 20.

(1) The sheaf $T \mapsto \Gamma(T, \mathcal{O}_T)$ is denoted $\mathcal{O}_S$, or $\mathbf{G}_a$, or $\mathbf{G}_{a,S}$ if we want to indicate the base scheme.
(2) Similarly, the sheaf $T \mapsto \Gamma(T, \mathcal{O}_T^*)$ is denoted $\mathcal{O}_S^*$, or $\mathbf{G}_m$, or $\mathbf{G}_{m,S}$ if we want to indicate the base scheme.
(3) The *constant sheaf* $\underline{\mathbf{Z}/n\mathbf{Z}}$ on any site is the sheafification of the constant presheaf $U \mapsto \mathbf{Z}/n\mathbf{Z}$.

The first is a sheaf by Theorem 17.4 for example. The second is a sub presheaf of the first, which is easily seen to be a sheaf itself. The third is a sheaf by definition. Note that each of these sheaves is representable. The first and second by the schemes $\mathbf{G}_{a,S}$ and $\mathbf{G}_{m,S}$, see Groupoids, Section 4. The third by the finite étale group scheme $\mathbf{Z}/n\mathbf{Z}_S$ sometimes denoted $(\mathbf{Z}/n\mathbf{Z})_S$ which is just n copies of S endowed with the obvious group scheme structure over S, see Groupoids, Example 5.6 and the following remark.

03P5 **Remark 23.2.** Let G be an abstract group. On any of the sites $(Sch/S)_\tau$ or S_τ of Section 20 the sheafification $\underline{G}$ of the constant presheaf associated to G in the *Zariski topology* of the site already gives

$$\Gamma(U, \underline{G}) = \{\text{Zariski locally constant maps } U \to G\}$$

This Zariski sheaf is representable by the group scheme G_S according to Groupoids, Example 5.6. By Lemma 15.8 any representable presheaf satisfies the sheaf condition for the τ-topology as well, and hence we conclude that the Zariski sheafification $\underline{G}$ above is also the τ-sheafification.

04HS **Definition 23.3.** Let S be a scheme. The *structure sheaf* of S is the sheaf of rings $\mathcal{O}_S$ on any of the sites S_{Zar}, $S_{étale}$, or $(Sch/S)_\tau$ discussed above.

If there is some possible confusion as to which site we are working on then we will indicate this by using indices. For example we may use $\mathcal{O}_{S_{étale}}$ to stress the fact that we are working on the small étale site of S.

03P6 **Remark 23.4.** In the terminology introduced above a special case of Theorem 22.4 is

$$H^p_{fppf}(X, \mathbf{G}_a) = H^p_{étale}(X, \mathbf{G}_a) = H^p_{Zar}(X, \mathbf{G}_a) = H^p(X, \mathcal{O}_X)$$

for all $p \geq 0$. Moreover, we could use the notation $H^p_{fppf}(X, \mathcal{O}_X)$ to indicate the cohomology of the structure sheaf on the big fppf site of X.

24. Picard groups

03P7 The following theorem is sometimes called "Hilbert 90".

03P8 **Theorem 24.1.** *For any scheme X we have canonical identifications*

$$\begin{aligned} H^1_{fppf}(X, \mathbf{G}_m) &= H^1_{syntomic}(X, \mathbf{G}_m) \\ &= H^1_{smooth}(X, \mathbf{G}_m) \\ &= H^1_{étale}(X, \mathbf{G}_m) \\ &= H^1_{Zar}(X, \mathbf{G}_m) \\ &= \mathrm{Pic}(X) \\ &= H^1(X, \mathcal{O}_X^*) \end{aligned}$$

Proof. Let τ be one of the topologies considered in Section 20. By Cohomology on Sites, Lemma 6.1 we see that $H^1_\tau(X, \mathbf{G}_m) = H^1_\tau(X, \mathcal{O}_\tau^*) = \mathrm{Pic}(\mathcal{O}_\tau)$ where $\mathcal{O}_\tau$ is the structure sheaf of the site $(Sch/X)_\tau$. Now an invertible $\mathcal{O}_\tau$-module is a quasi-coherent $\mathcal{O}_\tau$-module. By Theorem 17.4 or the more precise Descent, Proposition 8.11 we see that $\mathrm{Pic}(\mathcal{O}_\tau) = \mathrm{Pic}(X)$. The last equality is proved in the same way. $\square$

25. The étale site

03P9 At this point we start exploring the étale site of a scheme in more detail. As a first step we discuss a little the notion of an étale morphism.

26. Étale morphisms

03PA For more details, see Morphisms, Section 34 for the formal definition and Étale Morphisms, Sections 11, 12, 13, 14, 16, and 19 for a survey of interesting properties of étale morphisms.

Recall that an algebra A over an algebraically closed field k is *smooth* if it is of finite type and the module of differentials $\Omega_{A/k}$ is finite locally free of rank equal to the dimension. A scheme X over k is *smooth* over k if it is locally of finite type and each affine open is the spectrum of a smooth k-algebra. If k is not algebraically closed then an A-algebra is said to be a smooth k-algebra if $A \otimes_k \overline{k}$ is a smooth $\overline{k}$-algebra. A ring map $A \to B$ is smooth if it is flat, finitely presented, and for all primes $\mathfrak{p} \subset A$ the fibre ring $\kappa(\mathfrak{p}) \otimes_A B$ is smooth over the residue field $\kappa(\mathfrak{p})$. More generally, a morphism of schemes is *smooth* if it is flat, locally of finite presentation, and the geometric fibers are smooth.

For these facts please see Morphisms, Section 32. Using this we may define an étale morphism as follows.

03PB **Definition 26.1.** A morphism of schemes is *étale* if it is smooth of relative dimension 0.

In particular, a morphism of schemes $X \to S$ is étale if it is smooth and $\Omega_{X/S} = 0$.

03PC **Proposition 26.2.** *Facts on étale morphisms.*

1. *Let k be a field. A morphism of schemes $U \to \mathrm{Spec}(k)$ is étale if and only if $U \cong \coprod_{i \in I} \mathrm{Spec}(k_i)$ such that for each $i \in I$ the ring k_i is a field which is a finite separable extension of k.*
2. *Let $\varphi : U \to S$ be a morphism of schemes. The following conditions are equivalent:*
   (a) *φ is étale,*
   (b) *φ is locally finitely presented, flat, and all its fibres are étale,*
   (c) *φ is flat, unramified and locally of finite presentation.*
3. *A ring map $A \to B$ is étale if and only if $B \cong A[x_1, \ldots, x_n]/(f_1, \ldots, f_n)$ such that $\Delta = \det\left(\frac{\partial f_i}{\partial x_j}\right)$ is invertible in B.*
4. *The base change of an étale morphism is étale.*
5. *Compositions of étale morphisms are étale.*
6. *Fibre products and products of étale morphisms are étale.*
7. *An étale morphism has relative dimension 0.*
8. *Let $Y \to X$ be an étale morphism. If X is reduced (respectively regular) then so is Y.*
9. *Étale morphisms are open.*
10. *If $X \to S$ and $Y \to S$ are étale, then any S-morphism $X \to Y$ is also étale.*

Proof. We have proved these facts (and more) in the preceding chapters. Here is a list of references: (1) Morphisms, Lemma 34.7. (2) Morphisms, Lemmas 34.8 and 34.16. (3) Algebra, Lemma 142.2. (4) Morphisms, Lemma 34.4. (5) Morphisms, Lemma 34.3. (6) Follows formally from (4) and (5). (7) Morphisms, Lemmas 34.6 and 28.5. (8) See Algebra, Lemmas 158.7 and 158.5, see also more results of this kind in Étale Morphisms, Section 19. (9) See Morphisms, Lemma 24.10 and 34.12. (10) See Morphisms, Lemma 34.18. □

03PD **Definition 26.3.** A ring map $A \to B$ is called *standard étale* if $B \cong (A[t]/(f))_g$ with $f, g \in A[t]$, with f monic, and $\text{d}f/\text{d}t$ invertible in B.

It is true that a standard étale ring map is étale. Namely, suppose that $B = (A[t]/(f))_g$ with $f, g \in A[t]$, with f monic, and $\text{d}f/\text{d}t$ invertible in B. Then $A[t]/(f)$ is a finite free A-module of rank equal to the degree of the monic polynomial f. Hence B, as a localization of this free algebra is finitely presented and flat over A. To finish the proof that B is étale it suffices to show that the fibre rings

$$\kappa(\mathfrak p) \otimes_A B \cong \kappa(\mathfrak p) \otimes_A (A[t]/(f))_g \cong \kappa(\mathfrak p)[t, 1/\overline{g}]/(\overline{f})$$

are finite products of finite separable field extensions. Here $\overline{f}, \overline{g} \in \kappa(\mathfrak p)[t]$ are the images of f and g. Let

$$\overline{f} = \overline{f}_1 \ldots \overline{f}_a \overline{f}_{a+1}^{e_1} \ldots \overline{f}_{a+b}^{e_b}$$

be the factorization of $\overline{f}$ into powers of pairwise distinct irreducible monic factors $\overline{f}_i$ with $e_1, \ldots, e_b > 0$. By assumption $\text{d}\overline{f}/\text{d}t$ is invertible in $\kappa(\mathfrak p)[t, 1/\overline{g}]$. Hence we see that at least all the $\overline{f}_i$, $i > a$ are invertible. We conclude that

$$\kappa(\mathfrak p)[t, 1/\overline{g}]/(\overline{f}) \cong \prod\nolimits_{i \in I} \kappa(\mathfrak p)[t]/(\overline{f}_i)$$

where $I \subset \{1, \ldots, a\}$ is the subset of indices i such that $\overline{f}_i$ does not divide $\overline{g}$. Moreover, the image of $\text{d}\overline{f}/\text{d}t$ in the factor $\kappa(\mathfrak p)[t]/(\overline{f}_i)$ is clearly equal to a unit times $\text{d}\overline{f}_i/\text{d}t$. Hence we conclude that $\kappa_i = \kappa(\mathfrak p)[t]/(\overline{f}_i)$ is a finite field extension of $\kappa(\mathfrak p)$ generated by one element whose minimal polynomial is separable, i.e., the field extension $\kappa(\mathfrak p) \subset \kappa_i$ is finite separable as desired.

It turns out that any étale ring map is locally standard étale. To formulate this we introduce the following notation. A ring map $A \to B$ is *étale at a prime* $\mathfrak q$ of B if there exists $h \in B$, $h \notin \mathfrak q$ such that $A \to B_h$ is étale. Here is the result.

03PE **Theorem 26.4.** *A ring map $A \to B$ is étale at a prime $\mathfrak q$ if and only if there exists $g \in B$, $g \notin \mathfrak q$ such that B_g is standard étale over A.*

Proof. See Algebra, Proposition 142.16. □

27. Étale coverings

03PF We recall the definition.

03PG **Definition 27.1.** An *étale covering* of a scheme U is a family of morphisms of schemes $\{\varphi_i : U_i \to U\}_{i \in I}$ such that

1. each φ_i is an étale morphism,
2. the U_i cover U, i.e., $U = \bigcup_{i \in I} \varphi_i(U_i)$.

03PH **Lemma 27.2.** *Any étale covering is an fpqc covering.*

Proof. (See also Topologies, Lemma 9.6.) Let $\{\varphi_i : U_i \to U\}_{i \in I}$ be an étale covering. Since an étale morphism is flat, and the elements of the covering should cover its target, the property fp (faithfully flat) is satisfied. To check the property qc (quasi-compact), let $V \subset U$ be an affine open, and write $\varphi_i^{-1}(V) = \bigcup_{j \in J_i} V_{ij}$ for some affine opens $V_{ij} \subset U_i$. Since φ_i is open (as étale morphisms are open), we see that $V = \bigcup_{i \in I} \bigcup_{j \in J_i} \varphi_i(V_{ij})$ is an open covering of V. Further, since V is quasi-compact, this covering has a finite refinement. □

So any statement which is true for fpqc coverings remains true *a fortiori* for étale coverings. For instance, the étale site is subcanonical.

03PI **Definition 27.3.** (For more details see Section 20, or Topologies, Section 4.) Let S be a scheme. The *big étale site over* S is the site $(Sch/S)_{étale}$, see Definition 20.4. The *small étale site over* S is the site $S_{étale}$, see Definition 20.4. We define similarly the *big* and *small Zariski sites* on S, denoted $(Sch/S)_{Zar}$ and S_{Zar}.

Loosely speaking the big étale site of S is made up out of schemes over S and coverings the étale coverings. The small étale site of S is made up out of schemes étale over S with coverings the étale coverings. Actually any morphism between objects of $S_{étale}$ is étale, in virtue of Proposition 26.2, hence to check that $\{U_i \to U\}_{i\in I}$ in $S_{étale}$ is a covering it suffices to check that $\coprod U_i \to U$ is surjective.

The small étale site has fewer objects than the big étale site, it contains only the "opens" of the étale topology on S. It is a full subcategory of the big étale site, and its topology is induced from the topology on the big site. Hence it is true that the restriction functor from the big étale site to the small one is exact and maps injectives to injectives. This has the following consequence.

03PJ **Proposition 27.4.** *Let S be a scheme and $\mathcal{F}$ an abelian sheaf on $(Sch/S)_{étale}$. Then $\mathcal{F}|_{S_{étale}}$ is a sheaf on $S_{étale}$ and*

$$H^p_{étale}(S, \mathcal{F}|_{S_{étale}}) = H^p_{étale}(S, \mathcal{F})$$

for all $p \geq 0$.

Proof. This is a special case of Lemma 20.5. $\square$

In accordance with the general notation introduced in Section 20 we write $H^p_{étale}(S, \mathcal{F})$ for the above cohomology group.

28. Kummer theory

03PK Let $n \in \mathbf{N}$ and consider the functor μ_n defined by

$$\begin{array}{ccc} Sch^{opp} & \longrightarrow & Ab \\ S & \longmapsto & \mu_n(S) = \{t \in \Gamma(S, \mathcal{O}_S^*) \mid t^n = 1\}. \end{array}$$

By Groupoids, Example 5.2 this is a representable functor, and the scheme representing it is denoted μ_n also. By Lemma 15.8 this functor satisfies the sheaf condition for the fpqc topology (in particular, it also satisfies the sheaf condition for the étale, Zariski, etc topology).

03PL **Lemma 28.1.** *If $n \in \mathcal{O}_S^*$ then*

$$0 \to \mu_{n,S} \to \mathbf{G}_{m,S} \xrightarrow{(\cdot)^n} \mathbf{G}_{m,S} \to 0$$

is a short exact sequence of sheaves on both the small and big étale site of S.

Proof. By definition the sheaf $\mu_{n,S}$ is the kernel of the map $(\cdot)^n$. Hence it suffices to show that the last map is surjective. Let U be a scheme over S. Let $f \in \mathbf{G}_m(U) = \Gamma(U, \mathcal{O}_U^*)$. We need to show that we can find an étale cover of U over the members of which the restriction of f is an nth power. Set

$$U' = \underline{\mathrm{Spec}}_U(\mathcal{O}_U[T]/(T^n - f)) \xrightarrow{\pi} U.$$

(See Constructions, Section 3 or 4 for a discussion of the relative spectrum.) Let $\mathrm{Spec}(A) \subset U$ be an affine open, and say $f|_{\mathrm{Spec}(A)}$ corresponds to the unit $a \in A^*$. Then $\pi^{-1}(\mathrm{Spec}(A)) = \mathrm{Spec}(B)$ with $B = A[T]/(T^n - a)$. The ring map $A \to B$ is finite free of rank n, hence it is faithfully flat, and hence we conclude that $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is surjective. Since this holds for every affine open in U we conclude that π is surjective. In addition, n and T^{n-1} are invertible in B, so $nT^{n-1} \in B^*$ and the ring map $A \to B$ is standard étale, in particular étale. Since this holds for every affine open of U we conclude that π is étale. Hence $\mathcal{U} = \{\pi : U' \to U\}$ is an étale covering. Moreover, $f|_{U'} = (f')^n$ where f' is the class of T in $\Gamma(U', \mathcal{O}^*_{U'})$, so $\mathcal{U}$ has the desired property. □

03PM **Remark 28.2.** Lemma 28.1 is false when "étale" is replaced with "Zariski". Since the étale topology is coarser than the smooth topology, see Topologies, Lemma 5.2 it follows that the sequence is also exact in the smooth topology.

By Theorem 24.1 and Lemma 28.1 and general properties of cohomology we obtain the long exact cohomology sequence

$$\begin{tikzcd} 0 \arrow[r] & H^0_{\acute{e}tale}(S, \mu_{n,S}) \arrow[r] & \Gamma(S, \mathcal{O}^*_S) \arrow[r, "(\cdot)^n"] & \Gamma(S, \mathcal{O}^*_S) \arrow[dll, bend left] \\ & H^1_{\acute{e}tale}(S, \mu_{n,S}) \arrow[r] & \mathrm{Pic}(S) \arrow[r, "(\cdot)^n"] & \mathrm{Pic}(S) \arrow[dll, bend left] \\ & H^2_{\acute{e}tale}(S, \mu_{n,S}) \arrow[r] & \ldots \end{tikzcd}$$

at least if n is invertible on S. When n is not invertible on S we can apply the following lemma.

040N **Lemma 28.3.** *For any $n \in \mathbf{N}$ the sequence*

$$0 \to \mu_{n,S} \to \mathbf{G}_{m,S} \xrightarrow{(\cdot)^n} \mathbf{G}_{m,S} \to 0$$

is a short exact sequence of sheaves on the site $(Sch/S)_{fppf}$ and $(Sch/S)_{syntomic}$.

Proof. By definition the sheaf $\mu_{n,S}$ is the kernel of the map $(\cdot)^n$. Hence it suffices to show that the last map is surjective. Since the syntomic topology is weaker than the fppf topology, see Topologies, Lemma 7.2, it suffices to prove this for the syntomic topology. Let U be a scheme over S. Let $f \in \mathbf{G}_m(U) = \Gamma(U, \mathcal{O}^*_U)$. We need to show that we can find a syntomic cover of U over the members of which the restriction of f is an nth power. Set

$$U' = \underline{\mathrm{Spec}}_U(\mathcal{O}_U[T]/(T^n - f)) \xrightarrow{\pi} U.$$

(See Constructions, Section 3 or 4 for a discussion of the relative spectrum.) Let $\mathrm{Spec}(A) \subset U$ be an affine open, and say $f|_{\mathrm{Spec}(A)}$ corresponds to the unit $a \in A^*$. Then $\pi^{-1}(\mathrm{Spec}(A)) = \mathrm{Spec}(B)$ with $B = A[T]/(T^n - a)$. The ring map $A \to B$ is finite free of rank n, hence it is faithfully flat, and hence we conclude that $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is surjective. Since this holds for every affine open in U we conclude that π is surjective. In addition, B is a global relative complete intersection over A, so the ring map $A \to B$ is standard syntomic, in particular syntomic. Since this holds for every affine open of U we conclude that π is syntomic. Hence

$\mathcal{U} = \{\pi : U' \to U\}$ is a syntomic covering. Moreover, $f|_{U'} = (f')^n$ where f' is the class of T in $\Gamma(U', \mathcal{O}^*_{U'})$, so $\mathcal{U}$ has the desired property. $\square$

040O **Remark 28.4.** Lemma 28.3 is false for the smooth, étale, or Zariski topology.

By Theorem 24.1 and Lemma 28.3 and general properties of cohomology we obtain the long exact cohomology sequence

$$\begin{tikzcd} 0 \arrow[r] & H^0_{fppf}(S, \mu_{n,S}) \arrow[r] & \Gamma(S, \mathcal{O}^*_S) \arrow[r, "(\cdot)^n"] & \Gamma(S, \mathcal{O}^*_S) \arrow[dll, bend left] \\ & H^1_{fppf}(S, \mu_{n,S}) \arrow[r] & \mathrm{Pic}(S) \arrow[r, "(\cdot)^n"] & \mathrm{Pic}(S) \arrow[dll, bend left] \\ & H^2_{fppf}(S, \mu_{n,S}) \arrow[r] & \ldots \end{tikzcd}$$

for any scheme S and any integer n. Of course there is a similar sequence with syntomic cohomology.

Let $n \in \mathbf{N}$ and let S be any scheme. There is another more direct way to describe the first cohomology group with values in μ_n. Consider pairs $(\mathcal{L}, \alpha)$ where $\mathcal{L}$ is an invertible sheaf on S and $\alpha : \mathcal{L}^{\otimes n} \to \mathcal{O}_S$ is a trivialization of the nth tensor power of $\mathcal{L}$. Let $(\mathcal{L}', \alpha')$ be a second such pair. An isomorphism $\varphi : (\mathcal{L}, \alpha) \to (\mathcal{L}', \alpha')$ is an isomorphism $\varphi : \mathcal{L} \to \mathcal{L}'$ of invertible sheaves such that the diagram

$$\begin{tikzcd} \mathcal{L}^{\otimes n} \arrow[r, "\alpha"'] \arrow[d, "\varphi^{\otimes n}"'] & \mathcal{O}_S \arrow[d, "1"] \\ (\mathcal{L}')^{\otimes n} \arrow[r, "\alpha'"] & \mathcal{O}_S \end{tikzcd}$$

commutes. Thus we have

040P (28.4.1) $$\mathit{Isom}_S((\mathcal{L}, \alpha), (\mathcal{L}', \alpha')) = \begin{cases} \emptyset & \text{if} \quad \text{they are not isomorphic} \\ H^0(S, \mu_{n,S}) \cdot \varphi & \text{if} \quad \varphi \text{ isomorphism of pairs} \end{cases}$$

Moreover, given two pairs $(\mathcal{L}, \alpha)$, $(\mathcal{L}', \alpha')$ the tensor product

$$(\mathcal{L}, \alpha) \otimes (\mathcal{L}', \alpha') = (\mathcal{L} \otimes \mathcal{L}', \alpha \otimes \alpha')$$

is another pair. The pair $(\mathcal{O}_S, 1)$ is an identity for this tensor product operation, and an inverse is given by

$$(\mathcal{L}, \alpha)^{-1} = (\mathcal{L}^{\otimes -1}, \alpha^{\otimes -1}).$$

Hence the collection of isomorphism classes of pairs forms an abelian group. Note that

$$(\mathcal{L}, \alpha)^{\otimes n} = (\mathcal{L}^{\otimes n}, \alpha^{\otimes n}) \xrightarrow{\alpha} (\mathcal{O}_S, 1)$$

is an isomorphism hence every element of this group has order dividing n. We warn the reader that this group is in general **not** the n-torsion in $\mathrm{Pic}(S)$.

040Q **Lemma 28.5.** *Let S be a scheme. There is a canonical identification*

$$H^1_{\acute{e}tale}(S, \mu_n) = \textit{group of pairs } (\mathcal{L}, \alpha) \textit{ up to isomorphism as above}$$

if n is invertible on S. In general we have

$$H^1_{fppf}(S, \mu_n) = \textit{group of pairs } (\mathcal{L}, \alpha) \textit{ up to isomorphism as above.}$$

The same result holds with fppf replaced by syntomic.

Proof. We first prove the second isomorphism. Let $(\mathcal{L}, \alpha)$ be a pair as above. Choose an affine open covering $S = \bigcup U_i$ such that $\mathcal{L}|_{U_i} \cong \mathcal{O}_{U_i}$. Say $s_i \in \mathcal{L}(U_i)$ is a generator. Then $\alpha(s_i^{\otimes n}) = f_i \in \mathcal{O}_S^*(U_i)$. Writing $U_i = \mathrm{Spec}(A_i)$ we see there exists a global relative complete intersection $A_i \to B_i = A_i[T]/(T^n - f_i)$ such that f_i maps to an nth power in B_i. In other words, setting $V_i = \mathrm{Spec}(B_i)$ we obtain a syntomic covering $\mathcal{V} = \{V_i \to S\}_{i \in I}$ and trivializations $\varphi_i : (\mathcal{L}, \alpha)|_{V_i} \to (\mathcal{O}_{V_i}, 1)$.

We will use this result (the existence of the covering $\mathcal{V}$) to associate to this pair a cohomology class in $H^1_{syntomic}(S, \mu_{n,S})$. We give two (equivalent) constructions.

First construction: using Čech cohomology. Over the double overlaps $V_i \times_S V_j$ we have the isomorphism

$$(\mathcal{O}_{V_i \times_S V_j}, 1) \xrightarrow{\mathrm{pr}_0^* \varphi_i^{-1}} (\mathcal{L}|_{V_i \times_S V_j}, \alpha|_{V_i \times_S V_j}) \xrightarrow{\mathrm{pr}_1^* \varphi_j} (\mathcal{O}_{V_i \times_S V_j}, 1)$$

of pairs. By (28.4.1) this is given by an element $\zeta_{ij} \in \mu_n(V_i \times_S V_j)$. We omit the verification that these ζ_{ij}'s give a 1-cocycle, i.e., give an element $(\zeta_{i_0 i_1}) \in \check{C}(\mathcal{V}, \mu_n)$ with $d(\zeta_{i_0 i_1}) = 0$. Thus its class is an element in $\check{H}^1(\mathcal{V}, \mu_n)$ and by Theorem 19.2 it maps to a cohomology class in $H^1_{syntomic}(S, \mu_{n,S})$.

Second construction: Using torsors. Consider the presheaf

$$\mu_n(\mathcal{L}, \alpha) : U \longmapsto Isom_U((\mathcal{O}_U, 1), (\mathcal{L}, \alpha)|_U)$$

on $(Sch/S)_{syntomic}$. We may view this as a subpresheaf of $\mathcal{H}om_{\mathcal{O}}(\mathcal{O}, \mathcal{L})$ (internal hom sheaf, see Modules on Sites, Section 27). Since the conditions defining this subpresheaf are local, we see that it is a sheaf. By (28.4.1) this sheaf has a free action of the sheaf $\mu_{n,S}$. Hence the only thing we have to check is that it locally has sections. This is true because of the existence of the trivializing cover $\mathcal{V}$. Hence $\mu_n(\mathcal{L}, \alpha)$ is a $\mu_{n,S}$-torsor and by Cohomology on Sites, Lemma 4.3 we obtain a corresponding element of $H^1_{syntomic}(S, \mu_{n,S})$.

Ok, now we have to still show the following

(1) The two constructions give the same cohomology class.
(2) Isomorphic pairs give rise to the same cohomology class.
(3) The cohomology class of $(\mathcal{L}, \alpha) \otimes (\mathcal{L}', \alpha')$ is the sum of the cohomology classes of $(\mathcal{L}, \alpha)$ and $(\mathcal{L}', \alpha')$.
(4) If the cohomology class is trivial, then the pair is trivial.
(5) Any element of $H^1_{syntomic}(S, \mu_{n,S})$ is the cohomology class of a pair.

We omit the proof of (1). Part (2) is clear from the second construction, since isomorphic torsors give the same cohomology classes. Part (3) is clear from the first construction, since the resulting Čech classes add up. Part (4) is clear from the second construction since a torsor is trivial if and only if it has a global section, see Cohomology on Sites, Lemma 4.2.

Part (5) can be seen as follows (although a direct proof would be preferable). Suppose $\xi \in H^1_{syntomic}(S, \mu_{n,S})$. Then ξ maps to an element $\overline{\xi} \in H^1_{syntomic}(S, \mathbf{G}_{m,S})$ with $n\overline{\xi} = 0$. By Theorem 24.1 we see that $\overline{\xi}$ corresponds to an invertible sheaf $\mathcal{L}$ whose nth tensor power is isomorphic to $\mathcal{O}_S$. Hence there exists a pair $(\mathcal{L}, \alpha')$ whose cohomology class ξ' has the same image $\overline{\xi'}$ in $H^1_{syntomic}(S, \mathbf{G}_{m,S})$. Thus it suffices to show that $\xi - \xi'$ is the class of a pair. By construction, and the long exact

cohomology sequence above, we see that $\xi - \xi' = \partial(f)$ for some $f \in H^0(S, \mathcal{O}_S^*)$. Consider the pair $(\mathcal{O}_S, f)$. We omit the verification that the cohomology class of this pair is $\partial(f)$, which finishes the proof of the first identification (with fppf replaced with syntomic).

To see the first, note that if n is invertible on S, then the covering $\mathcal{V}$ constructed in the first part of the proof is actually an étale covering (compare with the proof of Lemma 28.1). The rest of the proof is independent of the topology, apart from the very last argument which uses that the Kummer sequence is exact, i.e., uses Lemma 28.1. □

29. Neighborhoods, stalks and points

03PN We can associate to any geometric point of S a stalk functor which is exact. A map of sheaves on $S_{étale}$ is an isomorphism if and only if it is an isomorphism on all these stalks. A complex of abelian sheaves is exact if and only if the complex of stalks is exact at all geometric points. Altogether this means that the small étale site of a scheme S has enough points. It also turns out that any point of the small étale topos of S (an abstract notion) is given by a geometric point. Thus in some sense the small étale topos of S can be understood in terms of geometric points and neighbourhoods.

03PO **Definition 29.1.** Let S be a scheme.

(1) A *geometric point* of S is a morphism $\mathrm{Spec}(k) \to S$ where k is algebraically closed. Such a point is usually denoted $\overline{s}$, i.e., by an overlined small case letter. We often use $\overline{s}$ to denote the scheme $\mathrm{Spec}(k)$ as well as the morphism, and we use $\kappa(\overline{s})$ to denote k.
(2) We say $\overline{s}$ *lies over* s to indicate that $s \in S$ is the image of $\overline{s}$.
(3) An *étale neighborhood* of a geometric point $\overline{s}$ of S is a commutative diagram

$$\begin{tikzcd} & U \arrow[d, "\varphi"] \\ \overline{s} \arrow[ur, "\overline{u}"] \arrow[r, "\overline{s}"'] & S \end{tikzcd}$$

where φ is an étale morphism of schemes. We write $(U, \overline{u}) \to (S, \overline{s})$.
(4) A *morphism of étale neighborhoods* $(U, \overline{u}) \to (U', \overline{u}')$ is an S-morphism $h : U \to U'$ such that $\overline{u}' = h \circ \overline{u}$.

03PP **Remark 29.2.** Since U and U' are étale over S, any S-morphism between them is also étale, see Proposition 26.2. In particular all morphisms of étale neighborhoods are étale.

04HT **Remark 29.3.** Let S be a scheme and $s \in S$ a point. In More on Morphisms, Definition 31.1 we defined the notion of an étale neighbourhood $(U, u) \to (S, s)$ of (S, s). If $\overline{s}$ is a geometric point of S lying over s, then any étale neighbourhood $(U, \overline{u}) \to (S, \overline{s})$ gives rise to an étale neighbourhood (U, u) of (S, s) by taking $u \in U$ to be the unique point of U such that $\overline{u}$ lies over u. Conversely, given an étale neighbourhood (U, u) of (S, s) the residue field extension $\kappa(s) \subset \kappa(u)$ is finite separable (see Proposition 26.2) and hence we can find an embedding $\kappa(u) \subset \kappa(\overline{s})$ over $\kappa(s)$. In other words, we can find a geometric point $\overline{u}$ of U lying over u such that $(U, \overline{u})$ is an étale neighbourhood of $(S, \overline{s})$. We will use these observations to go between the two types of étale neighbourhoods.

03PQ **Lemma 29.4.** *Let S be a scheme, and let $\overline{s}$ be a geometric point of S. The category of étale neighborhoods is cofiltered. More precisely:*

1. *Let $(U_i, \overline{u}_i)_{i=1,2}$ be two étale neighborhoods of $\overline{s}$ in S. Then there exists a third étale neighborhood $(U, \overline{u})$ and morphisms $(U, \overline{u}) \to (U_i, \overline{u}_i)$, $i = 1, 2$.*
2. *Let $h_1, h_2 : (U, \overline{u}) \to (U', \overline{u}')$ be two morphisms between étale neighborhoods of $\overline{s}$. Then there exist an étale neighborhood $(U'', \overline{u}'')$ and a morphism $h : (U'', \overline{u}'') \to (U, \overline{u})$ which equalizes h_1 and h_2, i.e., such that $h_1 \circ h = h_2 \circ h$.*

Proof. For part (1), consider the fibre product $U = U_1 \times_S U_2$. It is étale over both U_1 and U_2 because étale morphisms are preserved under base change, see Proposition 26.2. The map $\overline{s} \to U$ defined by $(\overline{u}_1, \overline{u}_2)$ gives it the structure of an étale neighborhood mapping to both U_1 and U_2. For part (2), define U'' as the fibre product

$$\begin{tikzcd} U'' \arrow[r] \arrow[d] & U \arrow[d, "{(h_1,h_2)}"] \\ U' \arrow[r, "\Delta"] & U' \times_S U'. \end{tikzcd}$$

Since $\overline{u}$ and $\overline{u}'$ agree over S with $\overline{s}$, we see that $\overline{u}'' = (\overline{u}, \overline{u}')$ is a geometric point of U''. In particular $U'' \neq \emptyset$. Moreover, since U' is étale over S, so is the fibre product $U' \times_S U'$ (see Proposition 26.2). Hence the vertical arrow (h_1, h_2) is étale by Remark 29.2 above. Therefore U'' is étale over U' by base change, and hence also étale over S (because compositions of étale morphisms are étale). Thus $(U'', \overline{u}'')$ is a solution to the problem. $\square$

03PR **Lemma 29.5.** *Let S be a scheme. Let $\overline{s}$ be a geometric point of S. Let $(U, \overline{u})$ be an étale neighborhood of $\overline{s}$. Let $\mathcal{U} = \{\varphi_i : U_i \to U\}_{i \in I}$ be an étale covering. Then there exist $i \in I$ and $\overline{u}_i : \overline{s} \to U_i$ such that $\varphi_i : (U_i, \overline{u}_i) \to (U, \overline{u})$ is a morphism of étale neighborhoods.*

Proof. As $U = \bigcup_{i \in I} \varphi_i(U_i)$, the fibre product $\overline{s} \times_{\overline{u}, U, \varphi_i} U_i$ is not empty for some i. Then look at the cartesian diagram

$$\begin{tikzcd} \overline{s} \times_{\overline{u}, U, \varphi_i} U_i \arrow[r, "\mathrm{pr}_2"'] \arrow[d, "\mathrm{pr}_1"] & U_i \arrow[d, "\varphi_i"] \\ \mathrm{Spec}(k) = \overline{s} \arrow[r, "\overline{u}"] \arrow[u, bend left, "\sigma"] & U \end{tikzcd}$$

The projection pr_1 is the base change of an étale morphisms so it is étale, see Proposition 26.2. Therefore, $\overline{s} \times_{\overline{u}, U, \varphi_i} U_i$ is a disjoint union of finite separable extensions of k, by Proposition 26.2. Here $\overline{s} = \mathrm{Spec}(k)$. But k is algebraically closed, so all these extensions are trivial, and there exists a section σ of pr_1. The composition $\mathrm{pr}_2 \circ \sigma$ gives a map compatible with $\overline{u}$. $\square$

040R **Definition 29.6.** Let S be a scheme. Let $\mathcal{F}$ be a presheaf on $S_{\acute{e}tale}$. Let $\overline{s}$ be a geometric point of S. The *stalk* of $\mathcal{F}$ at $\overline{s}$ is

$$\mathcal{F}_{\overline{s}} = \mathrm{colim}_{(U, \overline{u})} \mathcal{F}(U)$$

where $(U, \overline{u})$ runs over all étale neighborhoods of $\overline{s}$ in S.

By Lemma 29.4, this colimit is over a filtered index category, namely the opposite of the category of étale neighbourhoods. In other words, an element of $\mathcal{F}_{\overline{s}}$

can be thought of as a triple $(U, \overline{u}, \sigma)$ where $\sigma \in \mathcal{F}(U)$. Two triples $(U, \overline{u}, \sigma)$, $(U', \overline{u}', \sigma')$ define the same element of the stalk if there exists a third étale neighbourhood $(U'', \overline{u}'')$ and morphisms of étale neighbourhoods $h : (U'', \overline{u}'') \to (U, \overline{u})$, $h' : (U'', \overline{u}'') \to (U', \overline{u}')$ such that $h^*\sigma = (h')^*\sigma'$ in $\mathcal{F}(U'')$. See Categories, Section 19.

04FM **Lemma 29.7.** *Let S be a scheme. Let $\overline{s}$ be a geometric point of S. Consider the functor*

$$\begin{aligned} u : S_{\acute{e}tale} &\longrightarrow Sets, \\ U &\longmapsto |U_{\overline{s}}| = \{\overline{u} \text{ such that } (U, \overline{u}) \text{ is an étale neighbourhood of } \overline{s}\}. \end{aligned}$$

Here $|U_{\overline{s}}|$ denotes the underlying set of the geometric fibre. Then u defines a point p of the site $S_{\acute{e}tale}$ (Sites, Definition 32.2) and its associated stalk functor $\mathcal{F} \mapsto \mathcal{F}_p$ (Sites, Equation 32.1.1) is the functor $\mathcal{F} \mapsto \mathcal{F}_{\overline{s}}$ defined above.

Proof. In the proof of Lemma 29.5 we have seen that the scheme $U_{\overline{s}}$ is a disjoint union of schemes isomorphic to $\overline{s}$. Thus we can also think of $|U_{\overline{s}}|$ as the set of geometric points of U lying over $\overline{s}$, i.e., as the collection of morphisms $\overline{u} : \overline{s} \to U$ fitting into the diagram of Definition 29.1. From this it follows that $u(S)$ is a singleton, and that $u(U \times_V W) = u(U) \times_{u(V)} u(W)$ whenever $U \to V$ and $W \to V$ are morphisms in $S_{\acute{e}tale}$. And, given a covering $\{U_i \to U\}_{i \in I}$ in $S_{\acute{e}tale}$ we see that $\coprod u(U_i) \to u(U)$ is surjective by Lemma 29.5. Hence Sites, Proposition 33.3 applies, so p is a point of the site $S_{\acute{e}tale}$. Finally, our functor $\mathcal{F} \mapsto \mathcal{F}_{\overline{s}}$ is given by exactly the same colimit as the functor $\mathcal{F} \mapsto \mathcal{F}_p$ associated to p in Sites, Equation 32.1.1 which proves the final assertion. $\square$

04FN **Remark 29.8.** Let S be a scheme and let $\overline{s} : \mathrm{Spec}(k) \to S$ and $\overline{s}' : \mathrm{Spec}(k') \to S$ be two geometric points of S. A *morphism $a : \overline{s} \to \overline{s}'$ of geometric points* is simply a morphism $a : \mathrm{Spec}(k) \to \mathrm{Spec}(k')$ such that $a \circ \overline{s}' = \overline{s}$. Given such a morphism we obtain a functor from the category of étale neighbourhoods of $\overline{s}'$ to the category of étale neighbourhoods of $\overline{s}$ by the rule $(U, \overline{u}') \mapsto (U, \overline{u}' \circ a)$. Hence we obtain a canonical map

$$\mathcal{F}_{\overline{s}'} = \mathrm{colim}_{(U, \overline{u}')} \mathcal{F}(U) \longrightarrow \mathrm{colim}_{(U, \overline{u})} \mathcal{F}(U) = \mathcal{F}_{\overline{s}}$$

from Categories, Lemma 14.7. Using the description of elements of stalks as triples this maps the element of $\mathcal{F}_{\overline{s}'}$ represented by the triple $(U, \overline{u}', \sigma)$ to the element of $\mathcal{F}_{\overline{s}}$ represented by the triple $(U, \overline{u}' \circ a, \sigma)$. Since the functor above is clearly an equivalence we conclude that this canonical map is an isomorphism of stalk functors.

Let us make sure we have the map of stalks corresponding to a pointing in the correct direction. Note that the above means, according to Sites, Definition 37.2, that a defines a morphism $a : p \to p'$ between the points p, p' of the site $S_{\acute{e}tale}$ associated to $\overline{s}, \overline{s}'$ by Lemma 29.7. There are more general morphisms of points (corresponding to specializations of points of S) which we will describe later, and which will not be isomorphisms (insert future reference here).

03PT **Lemma 29.9.** *Let S be a scheme. Let $\overline{s}$ be a geometric point of S.*

1. *The stalk functor $PAb(S_{\acute{e}tale}) \to Ab$, $\mathcal{F} \mapsto \mathcal{F}_{\overline{s}}$ is exact.*
2. *We have $(\mathcal{F}^\#)_{\overline{s}} = \mathcal{F}_{\overline{s}}$ for any presheaf of sets $\mathcal{F}$ on $S_{\acute{e}tale}$.*
3. *The functor $Ab(S_{\acute{e}tale}) \to Ab$, $\mathcal{F} \mapsto \mathcal{F}_{\overline{s}}$ is exact.*

(4) *Similarly the functors $PSh(S_{étale}) \to Sets$ and $Sh(S_{étale}) \to Sets$ given by the stalk functor $\mathcal{F} \mapsto \mathcal{F}_{\overline{x}}$ are exact (see Categories, Definition 23.1) and commute with arbitrary colimits.*

Proof. Before we indicate how to prove this by direct arguments we note that the result follows from the general material in Modules on Sites, Section 35. This is true because $\mathcal{F} \mapsto \mathcal{F}_{\overline{s}}$ comes from a point of the small étale site of S, see Lemma 29.7. We will only give a direct proof of (1), (2) and (3), and omit a direct proof of (4).

Exactness as a functor on $PAb(S_{étale})$ is formal from the fact that directed colimits commute with all colimits and with finite limits. The identification of the stalks in (2) is via the map

$$\kappa : \mathcal{F}_{\overline{s}} \longrightarrow (\mathcal{F}^\#)_{\overline{s}}$$

induced by the natural morphism $\mathcal{F} \to \mathcal{F}^\#$, see Theorem 13.2. We claim that this map is an isomorphism of abelian groups. We will show injectivity and omit the proof of surjectivity.

Let $\sigma \in \mathcal{F}_{\overline{s}}$. There exists an étale neighborhood $(U, \overline{u}) \to (S, \overline{s})$ such that σ is the image of some section $s \in \mathcal{F}(U)$. If $\kappa(\sigma) = 0$ in $(\mathcal{F}^\#)_{\overline{s}}$ then there exists a morphism of étale neighborhoods $(U', \overline{u}') \to (U, \overline{u})$ such that $s|_{U'}$ is zero in $\mathcal{F}^\#(U')$. It follows there exists an étale covering $\{U'_i \to U'\}_{i \in I}$ such that $s|_{U'_i} = 0$ in $\mathcal{F}(U'_i)$ for all i. By Lemma 29.5 there exist $i \in I$ and a morphism $\overline{u}'_i : \overline{s} \to U'_i$ such that $(U'_i, \overline{u}'_i) \to (U', \overline{u}') \to (U, \overline{u})$ are morphisms of étale neighborhoods. Hence $\sigma = 0$ since $(U'_i, \overline{u}'_i) \to (U, \overline{u})$ is a morphism of étale neighbourhoods such that we have $s|_{U'_i} = 0$. This proves κ is injective.

To show that the functor $Ab(S_{étale}) \to Ab$ is exact, consider any short exact sequence in $Ab(S_{étale})$: $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$. This gives us the exact sequence of presheaves

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to \mathcal{H}/^p\mathcal{G} \to 0,$$

where $/^p$ denotes the quotient in $PAb(S_{étale})$. Taking stalks at $\overline{s}$, we see that $(\mathcal{H}/^p\mathcal{G})_{\overline{s}} = (\mathcal{H}/\mathcal{G})_{\overline{s}} = 0$, since the sheafification of $\mathcal{H}/^p\mathcal{G}$ is 0. Therefore,

$$0 \to \mathcal{F}_{\overline{s}} \to \mathcal{G}_{\overline{s}} \to \mathcal{H}_{\overline{s}} \to 0 = (\mathcal{H}/^p\mathcal{G})_{\overline{s}}$$

is exact, since taking stalks is exact as a functor from presheaves. □

03PU **Theorem 29.10.** *Let S be a scheme. A map $a : \mathcal{F} \to \mathcal{G}$ of sheaves of sets is injective (resp. surjective) if and only if the map on stalks $a_{\overline{s}} : \mathcal{F}_{\overline{s}} \to \mathcal{G}_{\overline{s}}$ is injective (resp. surjective) for all geometric points of S. A sequence of abelian sheaves on $S_{étale}$ is exact if and only if it is exact on all stalks at geometric points of S.*

Proof. The necessity of exactness on stalks follows from Lemma 29.9. For the converse, it suffices to show that a map of sheaves is surjective (respectively injective) if and only if it is surjective (respectively injective) on all stalks. We prove this in the case of surjectivity, and omit the proof in the case of injectivity.

Let $\alpha : \mathcal{F} \to \mathcal{G}$ be a map of sheaves such that $\mathcal{F}_{\overline{s}} \to \mathcal{G}_{\overline{s}}$ is surjective for all geometric points. Fix $U \in \mathrm{Ob}(S_{étale})$ and $s \in \mathcal{G}(U)$. For every $u \in U$ choose some $\overline{u} \to U$ lying over u and an étale neighborhood $(V_u, \overline{v}_u) \to (U, \overline{u})$ such that $s|_{V_u} = \alpha(s_{V_u})$ for some $s_{V_u} \in \mathcal{F}(V_u)$. This is possible since α is surjective on stalks. Then $\{V_u \to U\}_{u \in U}$ is an étale covering on which the restrictions of s are in the image of the map α. Thus, α is surjective, see Sites, Section 11. □

040S **Remarks 29.11.** On points of the geometric sites.

(1) Theorem 29.10 says that the family of points of $S_{étale}$ given by the geometric points of S (Lemma 29.7) is conservative, see Sites, Definition 38.1. In particular $S_{étale}$ has enough points.

04FP (2) Suppose $\mathcal{F}$ is a sheaf on the big étale site of S. Let $T \to S$ be an object of the big étale site of S, and let $\overline{t}$ be a geometric point of T. Then we define $\mathcal{F}_{\overline{t}}$ as the stalk of the restriction $\mathcal{F}|_{T_{étale}}$ of $\mathcal{F}$ to the small étale site of T. In other words, we can define the stalk of $\mathcal{F}$ at any geometric point of any scheme $T/S \in \text{Ob}((Sch/S)_{étale})$.

(3) The big étale site of S also has enough points, by considering all geometric points of all objects of this site, see (2).

The following lemma should be skipped on a first reading.

04HU **Lemma 29.12.** *Let S be a scheme.*

(1) *Let p be a point of the small étale site $S_{étale}$ of S given by a functor $u : S_{étale} \to Sets$. Then there exists a geometric point $\overline{s}$ of S such that p is isomorphic to the point of $S_{étale}$ associated to $\overline{s}$ in Lemma 29.7.*

(2) *Let $p : Sh(pt) \to Sh(S_{étale})$ be a point of the small étale topos of S. Then p comes from a geometric point of S, i.e., the stalk functor $\mathcal{F} \mapsto \mathcal{F}_p$ is isomorphic to a stalk functor as defined in Definition 29.6.*

Proof. By Sites, Lemma 32.7 there is a one to one correspondence between points of the site and points of the associated topos, hence it suffices to prove (1). By Sites, Proposition 33.3 the functor u has the following properties: (a) $u(S) = \{*\}$, (b) $u(U \times_V W) = u(U) \times_{u(V)} u(W)$, and (c) if $\{U_i \to U\}$ is an étale covering, then $\coprod u(U_i) \to u(U)$ is surjective. In particular, if $U' \subset U$ is an open subscheme, then $u(U') \subset u(U)$. Moreover, by Sites, Lemma 32.7 we can write $u(U) = p^{-1}(h_U^\#)$, in other words $u(U)$ is the stalk of the representable sheaf h_U. If $U = V \amalg W$, then we see that $h_U = (h_V \amalg h_W)^\#$ and we get $u(U) = u(V) \amalg u(W)$ since p^{-1} is exact.

Consider the restriction of u to S_{Zar}. By Sites, Examples 33.5 and 33.6 there exists a unique point $s \in S$ such that for $S' \subset S$ open we have $u(S') = \{*\}$ if $s \in S'$ and $u(S') = \emptyset$ if $s \notin S'$. Note that if $\varphi : U \to S$ is an object of $S_{étale}$ then $\varphi(U) \subset S$ is open (see Proposition 26.2) and $\{U \to \varphi(U)\}$ is an étale covering. Hence we conclude that $u(U) = \emptyset \Leftrightarrow s \in \varphi(U)$.

Pick a geometric point $\overline{s} : \overline{s} \to S$ lying over s, see Definition 29.1 for customary abuse of notation. Suppose that $\varphi : U \to S$ is an object of $S_{étale}$ with U affine. Note that φ is separated, and that the fibre U_s of φ over s is an affine scheme over $\text{Spec}(\kappa(s))$ which is the spectrum of a finite product of finite separable extensions k_i of $\kappa(s)$. Hence we may apply Étale Morphisms, Lemma 18.2 to get an étale neighbourhood $(V, \overline{v})$ of $(S, \overline{s})$ such that

$$U \times_S V = U_1 \amalg \ldots \amalg U_n \amalg W$$

with $U_i \to V$ an isomorphism and W having no point lying over $\overline{v}$. Thus we conclude that

$$u(U) \times u(V) = u(U \times_S V) = u(U_1) \amalg \ldots \amalg u(U_n) \amalg u(W)$$

and of course also $u(U_i) = u(V)$. After shrinking V a bit we can assume that V has exactly one point lying over s, and hence W has no point lying over s. By the

above this then gives $u(W) = \emptyset$. Hence we obtain

$$u(U) \times u(V) = u(U_1) \amalg \ldots \amalg u(U_n) = \coprod\nolimits_{i=1,\ldots,n} u(V)$$

Note that $u(V) \neq \emptyset$ as s is in the image of $V \to S$. In particular, we see that in this situation $u(U)$ is a finite set with n elements.

Consider the limit

$$\lim\nolimits_{(V,\overline{v})} u(V)$$

over the category of étale neighbourhoods $(V, \overline{v})$ of $\overline{s}$. It is clear that we get the same value when taking the limit over the subcategory of $(V, \overline{v})$ with V affine. By the previous paragraph (applied with the roles of V and U switched) we see that in this case $u(V)$ is always a finite nonempty set. Moreover, the limit is cofiltered, see Lemma 29.4. Hence by Categories, Section 20 the limit is nonempty. Pick an element x from this limit. This means we obtain a $x_{V,\overline{v}} \in u(V)$ for every étale neighbourhood $(V, \overline{v})$ of $(S, \overline{s})$ such that for every morphism of étale neighbourhoods $\varphi : (V', \overline{v}') \to (V, \overline{v})$ we have $u(\varphi)(x_{V',\overline{v}'}) = x_{V,\overline{v}}$.

We will use the choice of x to construct a functorial bijective map

$$c : |U_{\overline{s}}| \longrightarrow u(U)$$

for $U \in \mathrm{Ob}(S_{\acute{e}tale})$ which will conclude the proof. See Lemma 29.7 and its proof for a description of $|U_{\overline{s}}|$. First we claim that it suffices to construct the map for U affine. We omit the proof of this claim. Assume $U \to S$ in $S_{\acute{e}tale}$ with U affine, and let $\overline{u} : \overline{s} \to U$ be an element of $|U_{\overline{s}}|$. Choose a $(V, \overline{v})$ such that $U \times_S V$ decomposes as in the third paragraph of the proof. Then the pair $(\overline{u}, \overline{v})$ gives a geometric point of $U \times_S V$ lying over $\overline{v}$ and determines one of the components U_i of $U \times_S V$. More precisely, there exists a section $\sigma : V \to U \times_S V$ of the projection pr_U such that $(\overline{u}, \overline{v}) = \sigma \circ \overline{v}$. Set $c(\overline{u}) = u(\mathrm{pr}_U)(u(\sigma)(x_{V,\overline{v}})) \in u(U)$. We have to check this is independent of the choice of $(V, \overline{v})$. By Lemma 29.4 the category of étale neighbourhoods is cofiltered. Hence it suffice to show that given a morphism of étale neighbourhood $\varphi : (V', \overline{v}') \to (V, \overline{v})$ and a choice of a section $\sigma' : V' \to U \times_S V'$ of the projection such that $(\overline{u}, \overline{v'}) = \sigma' \circ \overline{v}'$ we have $u(\sigma')(x_{V',\overline{v}'}) = u(\sigma)(x_{V,\overline{v}})$. Consider the diagram

$$\begin{tikzcd}
V' \arrow[r, "\varphi"'] \arrow[d, "\sigma'"] & V \arrow[d, "\sigma"] \\
U \times_S V' \arrow[r, "1\times\varphi"] & U \times_S V
\end{tikzcd}$$

Now, it may not be the case that this diagram commutes. The reason is that the schemes V' and V may not be connected, and hence the decompositions used to construct σ' and σ above may not be unique. But we do know that $\sigma \circ \varphi \circ \overline{v}' = (1 \times \varphi) \circ \sigma' \circ \overline{v}'$ by construction. Hence, since $U \times_S V$ is étale over S, there exists an open neighbourhood $V'' \subset V'$ of $\overline{v'}$ such that the diagram does commute when restricted to V'', see Morphisms, Lemma 33.17. This means we may extend the diagram above to

$$\begin{tikzcd}
V'' \arrow[r] \arrow[d, "\sigma'|_{V''}"] & V' \arrow[r, "\varphi"'] \arrow[d, "\sigma'"] & V \arrow[d, "\sigma"] \\
U \times_S V'' \arrow[r] & U \times_S V' \arrow[r, "1\times\varphi"] & U \times_S V
\end{tikzcd}$$

such that the left square and the outer rectangle commute. Since u is a functor this implies that $x_{V'',\overline{v}'}$ maps to the same element in $u(U \times_S V)$ no matter which route we take through the diagram. On the other hand, it maps to the elements $x_{V',\overline{v}'}$ and $x_{V,\overline{v}}$ in $u(V')$ and $u(V)$. This implies the desired equality $u(\sigma')(x_{V',\overline{v}'}) = u(\sigma)(x_{V,\overline{v}})$.

In a similar manner one proves that the construction $c : |U_{\overline{s}}| \to u(U)$ is functorial in U; details omitted. And finally, by the results of the third paragraph it is clear that the map c is bijective which ends the proof of the lemma. □

30. Points in other topologies

06VW In this section we briefly discuss the existence of points for some sites other than the étale site of a scheme. We refer to Sites, Section 38 and Topologies, Section 2 ff for the terminology used in this section. All of the geometric sites have enough points.

06VX **Lemma 30.1.** *Let S be a scheme. All of the following sites have enough points S_{Zar}, $S_{étale}$, $(Sch/S)_{Zar}$, $(Aff/S)_{Zar}$, $(Sch/S)_{étale}$, $(Aff/S)_{étale}$, $(Sch/S)_{smooth}$, $(Aff/S)_{smooth}$, $(Sch/S)_{syntomic}$, $(Aff/S)_{syntomic}$, $(Sch/S)_{fppf}$, and $(Aff/S)_{fppf}$.*

Proof. For each of the big sites the associated topos is equivalent to the topos defined by the site $(Aff/S)_\tau$, see Topologies, Lemmas 3.10, 4.11, 5.9, 6.9, and 7.11. The result for the sites $(Aff/S)_\tau$ follows immediately from Deligne's result Sites, Lemma 39.4.

The result for S_{Zar} is clear. The result for $S_{étale}$ either follows from (the proof of) Theorem 29.10 or from Lemma 21.2 and Deligne's result applied to $S_{affine,étale}$. □

The lemma above guarantees the existence of points, but it doesn't tell us what these points look like. We can explicitly construct *some* points as follows. Suppose $\overline{s} : \text{Spec}(k) \to S$ is a geometric point with k algebraically closed. Consider the functor

$$u : (Sch/S)_{fppf} \longrightarrow Sets, \quad u(U) = U(k) = Mor_S(\text{Spec}(k), U).$$

Note that $U \mapsto U(k)$ commutes with finite limits as $S(k) = \{\overline{s}\}$ and $(U_1 \times_U U_2)(k) = U_1(k) \times_{U(k)} U_2(k)$. Moreover, if $\{U_i \to U\}$ is an fppf covering, then $\coprod U_i(k) \to U(k)$ is surjective. By Sites, Proposition 33.3 we see that u defines a point p of $(Sch/S)_{fppf}$ with stalks

$$\mathcal{F}_p = \text{colim}_{(U,x)} \mathcal{F}(U)$$

where the colimit is over pairs $U \to S$, $x \in U(k)$ as usual. But... this category has an initial object, namely $(\text{Spec}(k), \text{id})$, hence we see that

$$\mathcal{F}_p = \mathcal{F}(\text{Spec}(k))$$

which isn't terribly interesting! In fact, in general these points won't form a conservative family of points. A more interesting type of point is described in the following remark.

06VY **Remark 30.2.** Let $S = \text{Spec}(A)$ be an affine scheme. Let (p, u) be a point of the site $(Aff/S)_{fppf}$, see Sites, Sections 32 and 33. Let $B = \mathcal{O}_p$ be the stalk of the structure sheaf at the point p. Recall that

This is discussed in [Sch14].

$$B = \text{colim}_{(U,x)} \mathcal{O}(U) = \text{colim}_{(\text{Spec}(C), x_C)} C$$

where $x_C \in u(\mathrm{Spec}(C))$. It can happen that $\mathrm{Spec}(B)$ is an object of $(\mathit{Aff}/S)_{fppf}$ and that there is an element $x_B \in u(\mathrm{Spec}(B))$ mapping to the compatible system x_C. In this case the system of neighbourhoods has an initial object and it follows that $\mathcal{F}_p = \mathcal{F}(\mathrm{Spec}(B))$ for any sheaf $\mathcal{F}$ on $(\mathit{Aff}/S)_{fppf}$. It is straightforward to see that if $\mathcal{F} \mapsto \mathcal{F}(\mathrm{Spec}(B))$ defines a point of $\mathit{Sh}((\mathit{Aff}/S)_{fppf})$, then B has to be a local A-algebra such that for every faithfully flat, finitely presented ring map $B \to B'$ there is a section $B' \to B$. Conversely, for any such A-algebra B the functor $\mathcal{F} \mapsto \mathcal{F}(\mathrm{Spec}(B))$ is the stalk functor of a point. Details omitted. It is not clear what a general point of the site $(\mathit{Aff}/S)_{fppf}$ looks like.

31. Supports of abelian sheaves

04FQ First we talk about supports of local sections.

04HV **Lemma 31.1.** *Let S be a scheme. Let $\mathcal{F}$ be a subsheaf of the final object of the étale topos of S (see Sites, Example 10.2). Then there exists a unique open $W \subset S$ such that $\mathcal{F} = h_W$.*

Proof. The condition means that $\mathcal{F}(U)$ is a singleton or empty for all $\varphi : U \to S$ in $\mathrm{Ob}(S_{étale})$. In particular local sections always glue. If $\mathcal{F}(U) \neq \emptyset$, then $\mathcal{F}(\varphi(U)) \neq \emptyset$ because $\{\varphi : U \to \varphi(U)\}$ is a covering. Hence we can take $W = \bigcup_{\varphi : U \to S, \mathcal{F}(U) \neq \emptyset} \varphi(U)$. □

04FR **Lemma 31.2.** *Let S be a scheme. Let $\mathcal{F}$ be an abelian sheaf on $S_{étale}$. Let $\sigma \in \mathcal{F}(U)$ be a local section. There exists an open subset $W \subset U$ such that*

1. *$W \subset U$ is the largest Zariski open subset of U such that $\sigma|_W = 0$,*
2. *for every $\varphi : V \to U$ in $S_{étale}$ we have*
$$\sigma|_V = 0 \Leftrightarrow \varphi(V) \subset W,$$
3. *for every geometric point $\overline{u}$ of U we have*
$$(U, \overline{u}, \sigma) = 0 \text{ in } \mathcal{F}_{\overline{s}} \Leftrightarrow \overline{u} \in W$$
where $\overline{s} = (U \to S) \circ \overline{u}$.

Proof. Since $\mathcal{F}$ is a sheaf in the étale topology the restriction of $\mathcal{F}$ to U_{Zar} is a sheaf on U in the Zariski topology. Hence there exists a Zariski open W having property (1), see Modules, Lemma 5.2. Let $\varphi : V \to U$ be an arrow of $S_{étale}$. Note that $\varphi(V) \subset U$ is an open subset and that $\{V \to \varphi(V)\}$ is an étale covering. Hence if $\sigma|_V = 0$, then by the sheaf condition for $\mathcal{F}$ we see that $\sigma|_{\varphi(V)} = 0$. This proves (2). To prove (3) we have to show that if $(U, \overline{u}, \sigma)$ defines the zero element of $\mathcal{F}_{\overline{s}}$, then $\overline{u} \in W$. This is true because the assumption means there exists a morphism of étale neighbourhoods $(V, \overline{v}) \to (U, \overline{u})$ such that $\sigma|_V = 0$. Hence by (2) we see that $V \to U$ maps into W, and hence $\overline{u} \in W$. □

Let S be a scheme. Let $s \in S$. Let $\mathcal{F}$ be a sheaf on $S_{étale}$. By Remark 29.8 the isomorphism class of the stalk of the sheaf $\mathcal{F}$ at a geometric points lying over s is well defined.

04FS **Definition 31.3.** Let S be a scheme. Let $\mathcal{F}$ be an abelian sheaf on $S_{étale}$.

1. The *support of $\mathcal{F}$* is the set of points $s \in S$ such that $\mathcal{F}_{\overline{s}} \neq 0$ for any (some) geometric point $\overline{s}$ lying over s.

(2) Let $\sigma \in \mathcal{F}(U)$ be a section. The *support of* σ is the closed subset $U \setminus W$, where $W \subset U$ is the largest open subset of U on which σ restricts to zero (see Lemma 31.2).

In general the support of an abelian sheaf is not closed. For example, suppose that $S = \mathrm{Spec}(\mathbf{A}^1_{\mathbf{C}})$. Let $i_t : \mathrm{Spec}(\mathbf{C}) \to S$ be the inclusion of the point $t \in \mathbf{C}$. We will see later that $\mathbf{F}_t = i_{t,*}(\mathbf{Z}/2\mathbf{Z})$ is an abelian sheaf whose support is exactly $\{t\}$, see Section 46. Then

$$\bigoplus\nolimits_{n \in \mathbf{N}} \mathcal{F}_n$$

is an abelian sheaf with support $\{1, 2, 3, \ldots\} \subset S$. This is true because taking stalks commutes with colimits, see Lemma 29.9. Thus an example of an abelian sheaf whose support is not closed. Here are some basic facts on supports of sheaves and sections.

04FT **Lemma 31.4.** *Let S be a scheme. Let $\mathcal{F}$ be an abelian sheaf on $S_{étale}$. Let $U \in \mathrm{Ob}(S_{étale})$ and $\sigma \in \mathcal{F}(U)$.*

(1) *The support of σ is closed in U.*
(2) *The support of $\sigma + \sigma'$ is contained in the union of the supports of $\sigma, \sigma' \in \mathcal{F}(U)$.*
(3) *If $\varphi : \mathcal{F} \to \mathcal{G}$ is a map of abelian sheaves on $S_{étale}$, then the support of $\varphi(\sigma)$ is contained in the support of $\sigma \in \mathcal{F}(U)$.*
(4) *The support of $\mathcal{F}$ is the union of the images of the supports of all local sections of $\mathcal{F}$.*
(5) *If $\mathcal{F} \to \mathcal{G}$ is surjective then the support of $\mathcal{G}$ is a subset of the support of $\mathcal{F}$.*
(6) *If $\mathcal{F} \to \mathcal{G}$ is injective then the support of $\mathcal{F}$ is a subset of the support of $\mathcal{G}$.*

Proof. Part (1) holds by definition. Parts (2) and (3) hold because they holds for the restriction of $\mathcal{F}$ and $\mathcal{G}$ to U_{Zar}, see Modules, Lemma 5.2. Part (4) is a direct consequence of Lemma 31.2 part (3). Parts (5) and (6) follow from the other parts. $\square$

04FU **Lemma 31.5.** *The support of a sheaf of rings on $S_{étale}$ is closed.*

Proof. This is true because (according to our conventions) a ring is 0 if and only if $1 = 0$, and hence the support of a sheaf of rings is the support of the unit section. $\square$

32. Henselian rings

03QD We begin by stating a theorem which has already been used many times in the Stacks project. There are many versions of this result; here we just state the algebraic version.

03QE **Theorem 32.1.** *Let $A \to B$ be finite type ring map and $\mathfrak{p} \subset A$ a prime ideal. Then there exist an étale ring map $A \to A'$ and a prime $\mathfrak{p}' \subset A'$ lying over $\mathfrak{p}$ such that*

(1) $\kappa(\mathfrak{p}) = \kappa(\mathfrak{p}')$,
(2) $B \otimes_A A' = B_1 \times \ldots \times B_r \times C$,
(3) *$A' \to B_i$ is finite and there exists a unique prime $q_i \subset B_i$ lying over $\mathfrak{p}'$, and*
(4) *all irreducible components of the fibre $\mathrm{Spec}(C \otimes_{A'} \kappa(\mathfrak{p}'))$ of C over $\mathfrak{p}'$ have dimension at least 1.*

Proof. See Algebra, Lemma 142.22, or see [GD67, Théorème 18.12.1]. For a slew of versions in terms of morphisms of schemes, see More on Morphisms, Section 36. □

Recall Hensel's lemma. There are many versions of this lemma. Here are two:

(f) if $f \in \mathbf{Z}_p[T]$ monic and $f \bmod p = g_0 h_0$ with $gcd(g_0, h_0) = 1$ then f factors as $f = gh$ with $\bar{g} = g_0$ and $\bar{h} = h_0$,

(r) if $f \in \mathbf{Z}_p[T]$, monic $a_0 \in \mathbf{F}_p$, $\bar{f}(a_0) = 0$ but $\bar{f}'(a_0) \neq 0$ then there exists $a \in \mathbf{Z}_p$ with $f(a) = 0$ and $\bar{a} = a_0$.

Both versions are true (we will see this later). The first version asks for lifts of factorizations into coprime parts, and the second version asks for lifts of simple roots modulo the maximal ideal. It turns out that requiring these conditions for a general local ring are equivalent, and are equivalent to many other conditions. We use the root lifting property as the definition of a henselian local ring as it is often the easiest one to check.

03QF **Definition 32.2.** (See Algebra, Definition 149.1.) A local ring $(R, \mathfrak{m}, \kappa)$ is called *henselian* if for all $f \in R[T]$ monic, for all $a_0 \in \kappa$ such that $\bar{f}(a_0) = 0$ and $\bar{f}'(a_0) \neq 0$, there exists an $a \in R$ such that $f(a) = 0$ and $a \bmod \mathfrak{m} = a_0$.

A good example of henselian local rings to keep in mind is complete local rings. Recall (Algebra, Definition 155.1) that a complete local ring is a local ring $(R, \mathfrak{m})$ such that $R \cong \lim_n R/\mathfrak{m}^n$, i.e., it is complete and separated for the $\mathfrak{m}$-adic topology.

03QG **Theorem 32.3.** *Complete local rings are henselian.*

Proof. Newton's method. See Algebra, Lemma 149.9. □

03QH **Theorem 32.4.** *Let $(R, \mathfrak{m}, \kappa)$ be a local ring. The following are equivalent:*

1. *R is henselian,*
2. *for any $f \in R[T]$ and any factorization $\bar{f} = g_0 h_0$ in $\kappa[T]$ with $\gcd(g_0, h_0) = 1$, there exists a factorization $f = gh$ in $R[T]$ with $\bar{g} = g_0$ and $\bar{h} = h_0$,*
3. *any finite R-algebra S is isomorphic to a finite product of local rings finite over R,*
4. *any finite type R-algebra A is isomorphic to a product $A \cong A' \times C$ where $A' \cong A_1 \times \ldots \times A_r$ is a product of finite local R-algebras and all the irreducible components of $C \otimes_R \kappa$ have dimension at least 1,*
5. *if A is an étale R-algebra and $\mathfrak{n}$ is a maximal ideal of A lying over $\mathfrak{m}$ such that $\kappa \cong A/\mathfrak{n}$, then there exists an isomorphism $\varphi : A \cong R \times A'$ such that $\varphi(\mathfrak{n}) = \mathfrak{m} \times A' \subset R \times A'$.*

Proof. This is just a subset of the results from Algebra, Lemma 149.3. Note that part (5) above corresponds to part (8) of Algebra, Lemma 149.3 but is formulated slightly differently. □

03QJ **Lemma 32.5.** *If R is henselian and A is a finite R-algebra, then A is a finite product of henselian local rings.*

Proof. See Algebra, Lemma 149.4. □

03QK **Definition 32.6.** A local ring R is called *strictly henselian* if it is henselian and its residue field is separably closed.

03QI **Example 32.7.** In the case $R = \mathbf{C}[[t]]$, the étale R-algebras are finite products of the trivial extension $R \to R$ and the extensions $R \to R[X, X^{-1}]/(X^n - t)$. The latter ones factor through the open $D(t) \subset \mathrm{Spec}(R)$, so any étale covering can be refined by the covering $\{\mathrm{id} : \mathrm{Spec}(R) \to \mathrm{Spec}(R)\}$. We will see below that this is a somewhat general fact on étale coverings of spectra of henselian rings. This will show that higher étale cohomology of the spectrum of a strictly henselian ring is zero.

03QL **Theorem 32.8.** *Let $(R, \mathfrak{m}, \kappa)$ be a local ring and $\kappa \subset \kappa^{sep}$ a separable algebraic closure. There exist canonical flat local ring maps $R \to R^h \to R^{sh}$ where*

1. *R^h, R^{sh} are filtered colimits of étale R-algebras,*
2. *R^h is henselian, R^{sh} is strictly henselian,*
3. *$\mathfrak{m}R^h$ (resp. $\mathfrak{m}R^{sh}$) is the maximal ideal of R^h (resp. R^{sh}), and*
4. *$\kappa = R^h/\mathfrak{m}R^h$, and $\kappa^{sep} = R^{sh}/\mathfrak{m}R^{sh}$ as extensions of κ.*

Proof. The structure of R^h and R^{sh} is described in Algebra, Lemmas 151.1 and 151.2. □

The rings constructed in Theorem 32.8 are called respectively the *henselization* and the *strict henselization* of the local ring R, see Algebra, Definition 151.3. Many of the properties of R are reflected in its (strict) henselization, see More on Algebra, Section 44.

33. Stalks of the structure sheaf

04HW In this section we identify the stalk of the structure sheaf at a geometric point with the strict henselization of the local ring at the corresponding "usual" point.

04HX **Lemma 33.1.** *Let S be a scheme. Let $\overline{s}$ be a geometric point of S lying over $s \in S$. Let $\kappa = \kappa(s)$ and let $\kappa \subset \kappa^{sep} \subset \kappa(\overline{s})$ denote the separable algebraic closure of κ in $\kappa(\overline{s})$. Then there is a canonical identification*

$$(\mathcal{O}_{S,s})^{sh} \cong \mathcal{O}_{S,\overline{s}}$$

where the left hand side is the strict henselization of the local ring $\mathcal{O}_{S,s}$ as described in Theorem 32.8 and right hand side is the stalk of the structure sheaf $\mathcal{O}_S$ on $S_{étale}$ at the geometric point $\overline{s}$.

Proof. Let $\mathrm{Spec}(A) \subset S$ be an affine neighbourhood of s. Let $\mathfrak{p} \subset A$ be the prime ideal corresponding to s. With these choices we have canonical isomorphisms $\mathcal{O}_{S,s} = A_{\mathfrak{p}}$ and $\kappa(s) = \kappa(\mathfrak{p})$. Thus we have $\kappa(\mathfrak{p}) \subset \kappa^{sep} \subset \kappa(\overline{s})$. Recall that

$$\mathcal{O}_{S,\overline{s}} = \mathrm{colim}_{(U,\overline{u})} \mathcal{O}(U)$$

where the limit is over the étale neighbourhoods of $(S, \overline{s})$. A cofinal system is given by those étale neighbourhoods $(U, \overline{u})$ such that U is affine and $U \to S$ factors through $\mathrm{Spec}(A)$. In other words, we see that

$$\mathcal{O}_{S,\overline{s}} = \mathrm{colim}_{(B,\mathfrak{q},\phi)} B$$

where the colimit is over étale A-algebras B endowed with a prime $\mathfrak{q}$ lying over $\mathfrak{p}$ and a $\kappa(\mathfrak{p})$-algebra map $\phi : \kappa(\mathfrak{q}) \to \kappa(\overline{s})$. Note that since $\kappa(\mathfrak{q})$ is finite separable over $\kappa(\mathfrak{p})$ the image of ϕ is contained in κ^{sep}. Via these translations the result of the lemma is equivalent to the result of Algebra, Lemma 151.13. □

03PS **Definition 33.2.** Let S be a scheme. Let $\overline{s}$ be a geometric point of S lying over the point $s \in S$.

(1) The *étale local ring of S at $\overline{s}$* is the stalk of the structure sheaf $\mathcal{O}_S$ on $S_{étale}$ at $\overline{s}$. We sometimes call this the *strict henselization of $\mathcal{O}_{S,s}$* relative to the geometric point $\overline{s}$. Notation used: $\mathcal{O}_{S,\overline{s}} = \mathcal{O}^{sh}_{S,s}$.
(2) The *henselization of $\mathcal{O}_{S,s}$* is the henselization of the local ring of S at s. See Algebra, Definition 151.3, and Theorem 32.8. Notation: $\mathcal{O}^h_{S,s}$.
(3) The *strict henselization of S at $\overline{s}$* is the scheme $\text{Spec}(\mathcal{O}^{sh}_{S,s})$.
(4) The *henselization of S at s* is the scheme $\text{Spec}(\mathcal{O}^h_{S,s})$.

04HY **Lemma 33.3.** *Let S be a scheme. Let $s \in S$. Then we have*

$$\mathcal{O}^h_{S,s} = \text{colim}_{(U,u)} \mathcal{O}(U)$$

where the colimit is over the filtered category of étale neighbourhoods (U, u) of (S, s) such that $\kappa(s) = \kappa(u)$.

Proof. This lemma is a copy of More on Morphisms, Lemma 31.5. □

03QM **Remark 33.4.** Let S be a scheme. Let $s \in S$. If S is locally noetherian then $\mathcal{O}^h_{S,s}$ is also noetherian and it has the same completion:

$$\widehat{\mathcal{O}_{S,s}} \cong \widehat{\mathcal{O}^h_{S,s}}.$$

In particular, $\mathcal{O}_{S,s} \subset \mathcal{O}^h_{S,s} \subset \widehat{\mathcal{O}_{S,s}}$. The henselization of $\mathcal{O}_{S,s}$ is in general much smaller than its completion and inherits many of its properties. For example, if $\mathcal{O}_{S,s}$ is reduced, then so is $\mathcal{O}^h_{S,s}$, but this is not true for the completion in general. Insert future references here.

04HZ **Lemma 33.5.** *Let S be a scheme. The small étale site $S_{étale}$ endowed with its structure sheaf $\mathcal{O}_S$ is a locally ringed site, see Modules on Sites, Definition 39.4.*

Proof. This follows because the stalks $\mathcal{O}^{sh}_{S,s} = \mathcal{O}_{S,\overline{s}}$ are local, and because $S_{étale}$ has enough points, see Lemma 33.1, Theorem 29.10, and Remarks 29.11. See Modules on Sites, Lemmas 39.2 and 39.3 for the fact that this implies the small étale site is locally ringed. □

34. Functoriality of small étale topos

04I0 So far we haven't yet discussed the functoriality of the étale site, in other words what happens when given a morphism of schemes. A precise formal discussion can be found in Topologies, Section 4. In this and the next sections we discuss this material briefly specifically in the setting of small étale sites.

Let $f : X \to Y$ be a morphism of schemes. We obtain a functor

04I1 (34.0.1) $$u : Y_{étale} \longrightarrow X_{étale}, \quad V/Y \longmapsto X \times_Y V/X.$$

This functor has the following important properties

(1) u(final object) = final object,
(2) u preserves fibre products,
(3) if $\{V_j \to V\}$ is a covering in $Y_{étale}$, then $\{u(V_j) \to u(V)\}$ is a covering in $X_{étale}$.

Each of these is easy to check (omitted). As a consequence we obtain what is called a *morphism of sites*

$$f_{small} : X_{étale} \longrightarrow Y_{étale},$$

see Sites, Definition 14.1 and Sites, Proposition 14.7. It is not necessary to know about the abstract notion in detail in order to work with étale sheaves and étale cohomology. It usually suffices to know that there are functors $f_{small,*}$ (pushforward) and f_{small}^{-1} (pullback) on étale sheaves, and to know some of their simple properties. We will discuss these properties in the next sections, but we will sometimes refer to the more abstract material for proofs since that is often the natural setting to prove them.

35. Direct images

03PV Let us define the pushforward of a presheaf.

03PW **Definition 35.1.** Let $f : X \to Y$ be a morphism of schemes. Let $\mathcal{F}$ a presheaf of sets on $X_{étale}$. The *direct image*, or *pushforward* of $\mathcal{F}$ (under f) is

$$f_*\mathcal{F} : Y_{étale}^{opp} \longrightarrow Sets, \quad (V/Y) \longmapsto \mathcal{F}(X \times_Y V/X).$$

We sometimes write $f_* = f_{small,*}$ to distinguish from other direct image functors (such as usual Zariski pushforward or $f_{big,*}$).

This is a well-defined étale presheaf since the base change of an étale morphism is again étale. A more categorical way of saying this is that $f_*\mathcal{F}$ is the composition of functors $\mathcal{F} \circ u$ where u is as in Equation (34.0.1). This makes it clear that the construction is functorial in the presheaf $\mathcal{F}$ and hence we obtain a functor

$$f_* = f_{small,*} : PSh(X_{étale}) \longrightarrow PSh(Y_{étale})$$

Note that if $\mathcal{F}$ is a presheaf of abelian groups, then $f_*\mathcal{F}$ is also a presheaf of abelian groups and we obtain

$$f_* = f_{small,*} : PAb(X_{étale}) \longrightarrow PAb(Y_{étale})$$

as before (i.e., defined by exactly the same rule).

03PX **Remark 35.2.** We claim that the direct image of a sheaf is a sheaf. Namely, if $\{V_j \to V\}$ is an étale covering in $Y_{étale}$ then $\{X \times_Y V_j \to X \times_Y V\}$ is an étale covering in $X_{étale}$. Hence the sheaf condition for $\mathcal{F}$ with respect to $\{X \times_Y V_i \to X \times_Y V\}$ is equivalent to the sheaf condition for $f_*\mathcal{F}$ with respect to $\{V_i \to V\}$. Thus if $\mathcal{F}$ is a sheaf, so is $f_*\mathcal{F}$.

03PY **Definition 35.3.** Let $f : X \to Y$ be a morphism of schemes. Let $\mathcal{F}$ a sheaf of sets on $X_{étale}$. The *direct image*, or *pushforward* of $\mathcal{F}$ (under f) is

$$f_*\mathcal{F} : Y_{étale}^{opp} \longrightarrow Sets, \quad (V/Y) \longmapsto \mathcal{F}(X \times_Y V/X)$$

which is a sheaf by Remark 35.2. We sometimes write $f_* = f_{small,*}$ to distinguish from other direct image functors (such as usual Zariski pushforward or $f_{big,*}$).

The exact same discussion as above applies and we obtain functors

$$f_* = f_{small,*} : Sh(X_{étale}) \longrightarrow Sh(Y_{étale})$$

and

$$f_* = f_{small,*} : Ab(X_{étale}) \longrightarrow Ab(Y_{étale})$$

called *direct image* again.

The functor f_* on abelian sheaves is left exact. (See Homology, Section 7 for what it means for a functor between abelian categories to be left exact.) Namely, if $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3$ is exact on $X_{étale}$, then for every $U/X \in \text{Ob}(X_{étale})$ the sequence of abelian groups $0 \to \mathcal{F}_1(U) \to \mathcal{F}_2(U) \to \mathcal{F}_3(U)$ is exact. Hence for every $V/Y \in \text{Ob}(Y_{étale})$ the sequence of abelian groups $0 \to f_*\mathcal{F}_1(V) \to f_*\mathcal{F}_2(V) \to f_*\mathcal{F}_3(V)$ is exact, because this is the previous sequence with $U = X \times_Y V$.

04I2 **Definition 35.4.** Let $f : X \to Y$ be a morphism of schemes. The right derived functors $\{R^p f_*\}_{p \geq 1}$ of $f_* : Ab(X_{étale}) \to Ab(Y_{étale})$ are called *higher direct images*.

The higher direct images and their derived category variants are discussed in more detail in (insert future reference here).

36. Inverse image

03PZ In this section we briefly discuss pullback of sheaves on the small étale sites. The precise construction of this is in Topologies, Section 4.

03Q0 **Definition 36.1.** Let $f : X \to Y$ be a morphism of schemes. The *inverse image*, or *pullback*[2] functors are the functors

$$f^{-1} = f^{-1}_{small} : Sh(Y_{étale}) \longrightarrow Sh(X_{étale})$$

and

$$f^{-1} = f^{-1}_{small} : Ab(Y_{étale}) \longrightarrow Ab(X_{étale})$$

which are left adjoint to $f_* = f_{small,*}$. Thus f^{-1} thus characterized by the fact that

$$\text{Hom}_{Sh(X_{étale})}(f^{-1}\mathcal{G}, \mathcal{F}) = \text{Hom}_{Sh(Y_{étale})}(\mathcal{G}, f_*\mathcal{F})$$

functorially, for any $\mathcal{F} \in Sh(X_{étale})$ and $\mathcal{G} \in Sh(Y_{étale})$. We similarly have

$$\text{Hom}_{Ab(X_{étale})}(f^{-1}\mathcal{G}, \mathcal{F}) = \text{Hom}_{Ab(Y_{étale})}(\mathcal{G}, f_*\mathcal{F})$$

for $\mathcal{F} \in Ab(X_{étale})$ and $\mathcal{G} \in Ab(Y_{étale})$.

It is not trivial that such an adjoint exists. On the other hand, it exists in a fairly general setting, see Remark 36.3 below. The general machinery shows that $f^{-1}\mathcal{G}$ is the sheaf associated to the presheaf

04I3 (36.1.1) $$U/X \longmapsto \text{colim}_{U \to X \times_Y V}\, \mathcal{G}(V/Y)$$

where the colimit is over the category of pairs $(V/Y, \varphi : U/X \to X \times_Y V/X)$. To see this apply Sites, Proposition 14.7 to the functor u of Equation (34.0.1) and use the description of $u_s = (u_p\ )^\#$ in Sites, Sections 13 and 5. We will occasionally use this formula for the pullback in order to prove some of its basic properties.

03Q1 **Lemma 36.2.** *Let $f : X \to Y$ be a morphism of schemes.*

(1) *The functor $f^{-1} : Ab(Y_{étale}) \to Ab(X_{étale})$ is exact.*
(2) *The functor $f^{-1} : Sh(Y_{étale}) \to Sh(X_{étale})$ is exact, i.e., it commutes with finite limits and colimits, see Categories, Definition 23.1.*

[2]We use the notation f^{-1} for pullbacks of sheaves of sets or sheaves of abelian groups, and we reserve f^* for pullbacks of sheaves of modules via a morphism of ringed sites/topoi.

(3) *Let $\overline{x} \to X$ be a geometric point. Let $\mathcal{G}$ be a sheaf on $Y_{étale}$. Then there is a canonical identification*
$$(f^{-1}\mathcal{G})_{\overline{x}} = \mathcal{G}_{\overline{y}}.$$
where $\overline{y} = f \circ \overline{x}$.
(4) *For any $V \to Y$ étale we have $f^{-1}h_V = h_{X \times_Y V}$.*

Proof. The exactness of f^{-1} on sheaves of sets is a consequence of Sites, Proposition 14.7 applied to our functor u of Equation (34.0.1). In fact the exactness of pullback is part of the definition of a morphism of topoi (or sites if you like). Thus we see (2) holds. It implies part (1) since given an abelian sheaf $\mathcal{G}$ on $Y_{étale}$ the underlying sheaf of sets of $f^{-1}\mathcal{F}$ is the same as f^{-1} of the underlying sheaf of sets of $\mathcal{F}$, see Sites, Section 44. See also Modules on Sites, Lemma 30.2. In the literature (1) and (2) are sometimes deduced from (3) via Theorem 29.10.

Part (3) is a general fact about stalks of pullbacks, see Sites, Lemma 34.2. We will also prove (3) directly as follows. Note that by Lemma 29.9 taking stalks commutes with sheafification. Now recall that $f^{-1}\mathcal{G}$ is the sheaf associated to the presheaf
$$U \longrightarrow \mathrm{colim}_{U \to X \times_Y V} \mathcal{G}(V),$$
see Equation (36.1.1). Thus we have
$$\begin{aligned}(f^{-1}\mathcal{G})_{\overline{x}} &= \mathrm{colim}_{(U,\overline{u})} f^{-1}\mathcal{G}(U) \\ &= \mathrm{colim}_{(U,\overline{u})} \mathrm{colim}_{a:U \to X \times_Y V} \mathcal{G}(V) \\ &= \mathrm{colim}_{(V,\overline{v})} \mathcal{G}(V) \\ &= \mathcal{G}_{\overline{y}}\end{aligned}$$
in the third equality the pair $(U, \overline{u})$ and the map $a : U \to X \times_Y V$ corresponds to the pair $(V, a \circ \overline{u})$.

Part (4) can be proved in a similar manner by identifying the colimits which define $f^{-1}h_V$. Or you can use Yoneda's lemma (Categories, Lemma 3.5) and the functorial equalities
$$Mor_{Sh(X_{étale})}(f^{-1}h_V, \mathcal{F}) = Mor_{Sh(Y_{étale})}(h_V, f_*\mathcal{F}) = f_*\mathcal{F}(V) = \mathcal{F}(X \times_Y V)$$
combined with the fact that representable presheaves are sheaves. See also Sites, Lemma 13.5 for a completely general result. $\square$

The pair of functors (f_*, f^{-1}) define a morphism of small étale topoi
$$f_{small} : Sh(X_{étale}) \longrightarrow Sh(Y_{étale})$$
Many generalities on cohomology of sheaves hold for topoi and morphisms of topoi. We will try to point out when results are general and when they are specific to the étale topos.

03Q2 **Remark 36.3.** More generally, let $\mathcal{C}_1, \mathcal{C}_2$ be sites, and assume they have final objects and fibre products. Let $u : \mathcal{C}_2 \to \mathcal{C}_1$ be a functor satisfying:

(1) if $\{V_i \to V\}$ is a covering of $\mathcal{C}_2$, then $\{u(V_i) \to u(V)\}$ is a covering of $\mathcal{C}_1$ (we say that u is *continuous*), and
(2) u commutes with finite limits (i.e., u is left exact, i.e., u preserves fibre products and final objects).

Then one can define $f_* : Sh(\mathcal{C}_1) \to Sh(\mathcal{C}_2)$ by $f_*\mathcal{F}(V) = \mathcal{F}(u(V))$. Moreover, there exists an exact functor f^{-1} which is left adjoint to f_*, see Sites, Definition 14.1 and Proposition 14.7. Warning: It is not enough to require simply that u is continuous and commutes with fibre products in order to get a morphism of topoi.

37. Functoriality of big topoi

04DI Given a morphism of schemes $f : X \to Y$ there are a whole host of morphisms of topoi associated to f, see Topologies, Section 11 for a list. Perhaps the most used ones are the morphisms of topoi

$$f_{big} = f_{big,\tau} : Sh((Sch/X)_\tau) \longrightarrow Sh((Sch/Y)_\tau)$$

where $\tau \in \{Zariski, étale, smooth, syntomic, fppf\}$. These each correspond to a continuous functor

$$(Sch/Y)_\tau \longrightarrow (Sch/X)_\tau, \quad V/Y \longmapsto X \times_Y V/X$$

which preserves final objects, fibre products and covering, and hence defines a morphism of sites

$$f_{big} : (Sch/X)_\tau \longrightarrow (Sch/Y)_\tau.$$

See Topologies, Sections 3, 4, 5, 6, and 7. In particular, pushforward along f_{big} is given by the rule

$$(f_{big,*}\mathcal{F})(V/Y) = \mathcal{F}(X \times_Y V/X)$$

It turns out that these morphisms of topoi have an inverse image functor f_{big}^{-1} which is very easy to describe. Namely, we have

$$(f_{big}^{-1}\mathcal{G})(U/X) = \mathcal{G}(U/Y)$$

where the structure morphism of U/Y is the composition of the structure morphism $U \to X$ with f, see Topologies, Lemmas 3.15, 4.15, 5.10, 6.10, and 7.12.

38. Functoriality and sheaves of modules

04I4 In this section we are going to reformulate some of the material explained in Descent, Section 8 in the setting of étale topologies. Let $f : X \to Y$ be a morphism of schemes. We have seen above, see Sections 34, 35, and 36 that this induces a morphism f_{small} of small étale sites. In Descent, Remark 8.4 we have seen that f also induces a natural map

$$f_{small}^\sharp : \mathcal{O}_{Y_{étale}} \longrightarrow f_{small,*}\mathcal{O}_{X_{étale}}$$

of sheaves of rings on $Y_{étale}$ such that $(f_{small}, f_{small}^\sharp)$ is a morphism of ringed sites. See Modules on Sites, Definition 6.1 for the definition of a morphism of ringed sites. Let us just recall here that $f_{small}^\sharp$ is defined by the compatible system of maps

$$\mathrm{pr}_V^\sharp : \mathcal{O}(V) \longrightarrow \mathcal{O}(X \times_Y V)$$

for V varying over the objects of $Y_{étale}$.

It is clear that this construction is compatible with compositions of morphisms of schemes. More precisely, if $f : X \to Y$ and $g : Y \to Z$ are morphisms of schemes, then we have

$$(g_{small}, g_{small}^\sharp) \circ (f_{small}, f_{small}^\sharp) = ((g \circ f)_{small}, (g \circ f)_{small}^\sharp)$$

as morphisms of ringed topoi. Moreover, by Modules on Sites, Definition 13.1 we see that given a morphism $f : X \to Y$ of schemes we get well defined pullback and direct image functors

$$f_{small}^* : Mod(\mathcal{O}_{Y_{\acute{e}tale}}) \longrightarrow Mod(\mathcal{O}_{X_{\acute{e}tale}}),$$
$$f_{small,*} : Mod(\mathcal{O}_{X_{\acute{e}tale}}) \longrightarrow Mod(\mathcal{O}_{Y_{\acute{e}tale}})$$

which are adjoint in the usual way. If $g : Y \to Z$ is another morphism of schemes, then we have $(g \circ f)_{small}^* = f_{small}^* \circ g_{small}^*$ and $(g \circ f)_{small,*} = g_{small,*} \circ f_{small,*}$ because of what we said about compositions.

There is quite a bit of difference between the category of all $\mathcal{O}_X$ modules on X and the category between all $\mathcal{O}_{X_{\acute{e}tale}}$-modules on $X_{\acute{e}tale}$. But the results of Descent, Section 8 tell us that there is not much difference between considering quasi-coherent modules on S and quasi-coherent modules on $S_{\acute{e}tale}$. (We have already seen this in Theorem 17.4 for example.) In particular, if $f : X \to Y$ is any morphism of schemes, then the pullback functors f_{small}^* and f^* match for quasi-coherent sheaves, see Descent, Proposition 8.14. Moreover, the same is true for pushforward provided f is quasi-compact and quasi-separated, see Descent, Lemma 8.15.

A few words about functoriality of the structure sheaf on big sites. Let $f : X \to Y$ be a morphism of schemes. Choose any of the topologies $\tau \in \{Zariski, \acute{e}tale, smooth, syntomic, fppf\}$. Then the morphism $f_{big} : (Sch/X)_\tau \to (Sch/Y)_\tau$ becomes a morphism of ringed sites by a map

$$f_{big}^\sharp : \mathcal{O}_Y \longrightarrow f_{big,*}\mathcal{O}_X$$

see Descent, Remark 8.4. In fact it is given by the same construction as in the case of small sites explained above.

39. Comparing topologies

09XL In this section we start studying what happens when you compare sheaves with respect to different topologies.

09XM **Lemma 39.1.** *Let S be a scheme. Let $\mathcal{F}$ be a sheaf of sets on $S_{\acute{e}tale}$. Let $s, t \in \mathcal{F}(S)$. Then there exists an open $W \subset S$ characterized by the following property: A morphism $f : T \to S$ factors through W if and only if $s|_T = t|_T$ (restriction is pullback by f_{small}).*

Proof. Consider the presheaf which assigns to $U \in \mathrm{Ob}(S_{\acute{e}tale})$ the empty set if $s|_U \neq t|_U$ and a singleton else. It is clear that this is a subsheaf of the final object of $Sh(S_{\acute{e}tale})$. By Lemma 31.1 we find an open $W \subset S$ representing this presheaf. For a geometric point $\overline{x}$ of S we see that $\overline{x} \in W$ if and only if the stalks of s and t at $\overline{x}$ agree. By the description of stalks of pullbacks in Lemma 36.2 we see that W has the desired property. $\square$

09XN **Lemma 39.2.** *Let S be a scheme. Let $\tau \in \{Zariski, \acute{e}tale\}$. Consider the morphism*

$$\pi_S : (Sch/S)_\tau \longrightarrow S_\tau$$

of Topologies, Lemma 3.13 or 4.13. Let $\mathcal{F}$ be a sheaf on S_τ. Then $\pi_S^{-1}\mathcal{F}$ is given by the rule

$$(\pi_S^{-1}\mathcal{F})(T) = \Gamma(T_\tau, f_{small}^{-1}\mathcal{F})$$

where $f : T \to S$. Moreover, $\pi_S^{-1}\mathcal{F}$ satisfies the sheaf condition with respect to fpqc coverings.

Proof. Observe that we have a morphism $i_f : Sh(T_\tau) \to Sh(Sch/S)_\tau)$ such that $\pi_S \circ i_f = f_{small}$ as morphisms $T_\tau \to S_\tau$, see Topologies, Lemmas 3.12, 3.16, 4.12, and 4.16. Since pullback is transitive we see that $i_f^{-1}\pi_S^{-1}\mathcal{F} = f_{small}^{-1}\mathcal{F}$ as desired.

Let $\{g_i : T_i \to T\}_{i \in I}$ be an fpqc covering. The final statement means the following: Given a sheaf $\mathcal{G}$ on T_τ and given sections $s_i \in \Gamma(T_i, g_{i,small}^{-1}\mathcal{G})$ whose pullbacks to $T_i \times_T T_j$ agree, there is a unique section s of $\mathcal{G}$ over T whose pullback to T_i agrees with s_i.

Let $V \to T$ be an object of T_τ and let $t \in \mathcal{G}(V)$. For every i there is a largest open $W_i \subset T_i \times_T V$ such that the pullbacks of s_i and t agree as sections of the pullback of $\mathcal{G}$ to $W_i \subset T_i \times_T V$, see Lemma 39.1. Because s_i and s_j agree over $T_i \times_T T_j$ we find that W_i and W_j pullback to the same open over $T_i \times_T T_j \times_T V$. By Descent, Lemma 10.6 we find an open $W \subset V$ whose inverse image to $T_i \times_T V$ recovers W_i.

By construction of $g_{i,small}^{-1}\mathcal{G}$ there exists a τ-covering $\{T_{ij} \to T_i\}_{j \in J_i}$, for each j an open immersion or étale morphism $V_{ij} \to T$, a section $t_{ij} \in \mathcal{G}(V_{ij})$, and commutative diagrams

$$\begin{tikzcd} T_{ij} \arrow[r] \arrow[d] & V_{ij} \arrow[d] \\ T_i \arrow[r] & T \end{tikzcd}$$

such that $s_i|_{T_{ij}}$ is the pullback of t_{ij}. In other words, after replacing the covering $\{T_i \to T\}$ by $\{T_{ij} \to T\}$ we may assume there are factorizations $T_i \to V_i \to T$ with $V_i \in \mathrm{Ob}(T_\tau)$ and sections $t_i \in \mathcal{G}(V_i)$ pulling back to s_i over T_i. By the result of the previous paragraph we find opens $W_i \subset V_i$ such that $t_i|_{W_i}$ "agrees with" every s_j over $T_j \times_T W_i$. Note that $T_i \to V_i$ factors through W_i. Hence $\{W_i \to T\}$ is a τ-covering and the lemma is proven. $\square$

0A3H **Lemma 39.3.** *Let S be a scheme. Let $f : T \to S$ be a morphism such that*

1. *f is flat and quasi-compact, and*
2. *the geometric fibres of f are connected.*

Let $\mathcal{F}$ be a sheaf on $S_{étale}$. Then $\Gamma(S, \mathcal{F}) = \Gamma(T, f_{small}^{-1}\mathcal{F})$.

Proof. There is a canonical map $\Gamma(S, \mathcal{F}) \to \Gamma(T, f_{small}^{-1}\mathcal{F})$. Since f is surjective (because its fibres are connected) we see that this map is injective.

To show that the map is surjective, let $\alpha \in \Gamma(T, f_{small}^{-1}\mathcal{F})$. Since $\{T \to S\}$ is an fpqc covering we can use Lemma 39.2 to see that suffices to prove that α pulls back to the same section over $T \times_S T$ by the two projections. Let $\overline{s} \to S$ be a geometric point. It suffices to show the agreement holds over $(T \times_S T)_{\overline{s}}$ as every geometric point of $T \times_S T$ is contained in one of these geometric fibres. In other words, we are trying to show that $\alpha|_{T_{\overline{s}}}$ pulls back to the same section over

$$(T \times_S T)_{\overline{s}} = T_{\overline{s}} \times_{\overline{s}} T_{\overline{s}}$$

by the two projections to $T_{\overline{s}}$. However, since $\mathcal{F}|_{T_{\overline{s}}}$ is the pullback of $\mathcal{F}|_{\overline{s}}$ it is a constant sheaf with value $\mathcal{F}_{\overline{s}}$. Since $T_{\overline{s}}$ is connected by assumption, any section of a constant sheaf is constant. Hence $\alpha|_{T_{\overline{s}}}$ corresponds to an element of $\mathcal{F}_{\overline{s}}$. Thus

the two pullbacks to $(T \times_S T)_{\overline{s}}$ both correspond to this same element and we conclude. $\square$

Here is a version of Lemma 39.3 where we do not assume that the morphism is flat.

0EZK **Lemma 39.4.** *Let S be a scheme. Let $f : X \to S$ be a morphism such that*

(1) *f is submersive, and*
(2) *the geometric fibres of f are connected.*

Let $\mathcal{F}$ be a sheaf on $S_{étale}$. Then $\Gamma(S, \mathcal{F}) = \Gamma(X, f_{small}^{-1}\mathcal{F})$.

Proof. There is a canonical map $\Gamma(S, \mathcal{F}) \to \Gamma(X, f_{small}^{-1}\mathcal{F})$. Since f is surjective (because its fibres are connected) we see that this map is injective.

To show that the map is surjective, let $\tau \in \Gamma(X, f_{small}^{-1}\mathcal{F})$. It suffices to find an étale covering $\{U_i \to S\}$ and sections $\sigma_i \in \mathcal{F}(U_i)$ such that σ_i pulls back to $\tau|_{X \times_S U_i}$. Namely, the injectivity shown above guarantees that σ_i and σ_j restrict to the same section of $\mathcal{F}$ over $U_i \times_S U_j$. Thus we obtain a unique section $\sigma \in \mathcal{F}(S)$ which restricts to σ_i over U_i. Then the pullback of σ to X is τ because this is true locally.

Let $\overline{x}$ be a geometric point of X with image $\overline{s}$ in S. Consider the image of τ in the stalk

$$(f_{small}^{-1}\mathcal{F})_{\overline{x}} = \mathcal{F}_{\overline{s}}$$

See Lemma 36.2. We can find an étale neighbourhood $U \to S$ of $\overline{s}$ and a section $\sigma \in \mathcal{F}(U)$ mapping to this image in the stalk. Thus after replacing S by U and X by $X \times_S U$ we may assume there exits a section σ of $\mathcal{F}$ over S whose image in $(f_{small}^{-1}\mathcal{F})_{\overline{x}}$ is the same as τ.

By Lemma 39.1 there exists a maximal open $W \subset X$ such that $f_{small}^{-1}\sigma$ and τ agree over W and the formation of W commutes with further pullback. Observe that the pullback of $\mathcal{F}$ to the geometric fibre $X_{\overline{s}}$ is the pullback of $\mathcal{F}_{\overline{s}}$ viewed as a sheaf on $\overline{s}$ by $X_{\overline{s}} \to \overline{s}$. Hence we see that τ and σ give sections of the constant sheaf with value $\mathcal{F}_{\overline{s}}$ on $X_{\overline{s}}$ which agree in one point. Since $X_{\overline{s}}$ is connected by assumption, we conclude that W contains X_s. The same argument for different geometric fibres shows that W contains every fibre it meets. Since f is submersive, we conclude that W is the inverse image of an open neighbourhood of s in S. This finishes the proof. $\square$

0A3I **Lemma 39.5.** *Let $k \subset K$ be an extension of fields with k separably algebraically closed. Let S be a scheme over k. Denote $p : S_K = S \times_{\mathrm{Spec}(k)} \mathrm{Spec}(K) \to S$ the projection. Let $\mathcal{F}$ be a sheaf on $S_{étale}$. Then $\Gamma(S, \mathcal{F}) = \Gamma(S_K, p_{small}^{-1}\mathcal{F})$.*

Proof. Follows from Lemma 39.3. Namely, it is clear that p is flat and quasi-compact as the base change of $\mathrm{Spec}(K) \to \mathrm{Spec}(k)$. On the other hand, if $\overline{s} : \mathrm{Spec}(L) \to S$ is a geometric point, then the fibre of p over $\overline{s}$ is the spectrum of $K \otimes_k L$ which is irreducible hence connected by Algebra, Lemma 46.2. $\square$

40. Recovering morphisms

04JH In this section we prove that the rule which associates to a scheme its locally ringed small étale topos is fully faithful in a suitable sense, see Theorem 40.5.

04I5 **Lemma 40.1.** *Let $f : X \to Y$ be a morphism of schemes. The morphism of ringed sites $(f_{small}, f_{small}^\sharp)$ associated to f is a morphism of locally ringed sites, see Modules on Sites, Definition 39.9.*

Proof. Note that the assertion makes sense since we have seen that $(X_{étale}, \mathcal{O}_{X_{étale}})$ and $(Y_{étale}, \mathcal{O}_{Y_{étale}})$ are locally ringed sites, see Lemma 33.5. Moreover, we know that $X_{étale}$ has enough points, see Theorem 29.10 and Remarks 29.11. Hence it suffices to prove that $(f_{small}, f_{small}^\sharp)$ satisfies condition (3) of Modules on Sites, Lemma 39.8. To see this take a point p of $X_{étale}$. By Lemma 29.12 p corresponds to a geometric point $\overline{x}$ of X. By Lemma 36.2 the point $q = f_{small} \circ p$ corresponds to the geometric point $\overline{y} = f \circ \overline{x}$ of Y. Hence the assertion we have to prove is that the induced map of stalks

$$\mathcal{O}_{Y,\overline{y}} \longrightarrow \mathcal{O}_{X,\overline{x}}$$

is a local ring map. Suppose that $a \in \mathcal{O}_{Y,\overline{y}}$ is an element of the left hand side which maps to an element of the maximal ideal of the right hand side. Suppose that a is the equivalence class of a triple $(V, \overline{v}, a)$ with $V \to Y$ étale, $\overline{v} : \overline{x} \to V$ over Y, and $a \in \mathcal{O}(V)$. It maps to the equivalence class of $(X \times_Y V, \overline{x} \times \overline{v}, \text{pr}_V^\sharp(a))$ in the local ring $\mathcal{O}_{X,\overline{x}}$. But it is clear that being in the maximal ideal means that pulling back $\text{pr}_V^\sharp(a)$ to an element of $\kappa(\overline{x})$ gives zero. Hence also pulling back a to $\kappa(\overline{x})$ is zero. Which means that a lies in the maximal ideal of $\mathcal{O}_{Y,\overline{y}}$. $\square$

04IJ **Lemma 40.2.** *Let X, Y be schemes. Let $f : X \to Y$ be a morphism of schemes. Let t be a 2-morphism from $(f_{small}, f_{small}^\sharp)$ to itself, see Modules on Sites, Definition 8.1. Then $t = id$.*

Proof. This means that $t : f_{small}^{-1} \to f_{small}^{-1}$ is a transformation of functors such that the diagram

$$\begin{tikzcd} f_{small}^{-1}\mathcal{O}_Y \arrow[dr, "f_{small}^\sharp"'] & & f_{small}^{-1}\mathcal{O}_Y \arrow[ll, "t"'] \arrow[dl, "f_{small}^\sharp"] \\ & \mathcal{O}_X & \end{tikzcd}$$

is commutative. Suppose $V \to Y$ is étale with V affine. By Morphisms, Lemma 37.2 we may choose an immersion $i : V \to \mathbf{A}^n_Y$ over Y. In terms of sheaves this means that i induces an injection $h_i : h_V \to \prod_{j=1,\ldots,n} \mathcal{O}_Y$ of sheaves. The base change i' of i to X is an immersion (Schemes, Lemma 18.2). Hence $i' : X \times_Y V \to \mathbf{A}^n_X$ is an immersion, which in turn means that $h_{i'} : h_{X \times_Y V} \to \prod_{j=1,\ldots,n} \mathcal{O}_X$ is an injection of sheaves. Via the identification $f_{small}^{-1}h_V = h_{X \times_Y V}$ of Lemma 36.2 the map $h_{i'}$ is equal to

$$f_{small}^{-1}h_V \xrightarrow{f^{-1}h_i} \prod\nolimits_{j=1,\ldots,n} f_{small}^{-1}\mathcal{O}_Y \xrightarrow{\prod f^\sharp} \prod\nolimits_{j=1,\ldots,n} \mathcal{O}_X$$

(verification omitted). This means that the map $t : f_{small}^{-1}h_V \to f_{small}^{-1}h_V$ fits into the commutative diagram

$$\begin{tikzcd} f_{small}^{-1}h_V \arrow[r, "f^{-1}h_i"] \arrow[d, "t"] & \prod_{j=1,\ldots,n} f_{small}^{-1}\mathcal{O}_Y \arrow[r, "\prod f^\sharp"] \arrow[d, "\prod t"] & \prod_{j=1,\ldots,n} \mathcal{O}_X \arrow[d, "\text{id}"] \\ f_{small}^{-1}h_V \arrow[r, "f^{-1}h_i"] & \prod_{j=1,\ldots,n} f_{small}^{-1}\mathcal{O}_Y \arrow[r, "\prod f^\sharp"] & \prod_{j=1,\ldots,n} \mathcal{O}_X \end{tikzcd}$$

The commutativity of the right square holds by our assumption on t explained above. Since the composition of the horizontal arrows is injective by the discussion

above we conclude that the left vertical arrow is the identity map as well. Any sheaf of sets on $Y_{\acute{e}tale}$ admits a surjection from a (huge) coproduct of sheaves of the form h_V with V affine (combine Lemma 21.2 with Sites, Lemma 12.5). Thus we conclude that $t : f_{small}^{-1} \to f_{small}^{-1}$ is the identity transformation as desired. $\square$

04LW **Lemma 40.3.** *Let X, Y be schemes. Any two morphisms $a, b : X \to Y$ of schemes for which there exists a 2-isomorphism $(a_{small}, a_{small}^\sharp) \cong (b_{small}, b_{small}^\sharp)$ in the 2-category of ringed topoi are equal.*

Proof. Let us argue this carefuly since it is a bit confusing. Let $t : a_{small}^{-1} \to b_{small}^{-1}$ be the 2-isomorphism. Consider any open $V \subset Y$. Note that h_V is a subsheaf of the final sheaf $*$. Thus both $a_{small}^{-1} h_V = h_{a^{-1}(V)}$ and $b_{small}^{-1} h_V = h_{b^{-1}(V)}$ are subsheaves of the final sheaf. Thus the isomorphism

$$t : a_{small}^{-1} h_V = h_{a^{-1}(V)} \to b_{small}^{-1} h_V = h_{b^{-1}(V)}$$

has to be the identity, and $a^{-1}(V) = b^{-1}(V)$. It follows that a and b are equal on underlying topological spaces. Next, take a section $f \in \mathcal{O}_Y(V)$. This determines and is determined by a map of sheaves of sets $f : h_V \to \mathcal{O}_Y$. Pull this back and apply t to get a commutative diagram

$$\begin{tikzcd}
h_{b^{-1}(V)} \arrow[r, equal] & b_{small}^{-1} h_V \arrow[d, "b_{small}^{-1}(f)"] & & a_{small}^{-1} h_V \arrow[ll, "t"'] \arrow[d, "a_{small}^{-1}(f)"] \arrow[r, equal] & h_{a^{-1}(V)} \\
& b_{small}^{-1} \mathcal{O}_Y \arrow[dr, "b^\sharp"'] & & a_{small}^{-1} \mathcal{O}_Y \arrow[ll, "t"'] \arrow[dl, "a^\sharp"] & \\
& & \mathcal{O}_X & &
\end{tikzcd}$$

where the triangle is commutative by definition of a 2-isomorphism in Modules on Sites, Section 8. Above we have seen that the composition of the top horizontal arrows comes from the identity $a^{-1}(V) = b^{-1}(V)$. Thus the commutativity of the diagram tells us that $a_{small}^\sharp(f) = b_{small}^\sharp(f)$ in $\mathcal{O}_X(a^{-1}(V)) = \mathcal{O}_X(b^{-1}(V))$. Since this holds for every open V and every $f \in \mathcal{O}_Y(V)$ we conclude that $a = b$ as morphisms of schemes. $\square$

04I6 **Lemma 40.4.** *Let X, Y be affine schemes. Let*

$$(g, g^\#) : (Sh(X_{\acute{e}tale}), \mathcal{O}_X) \longrightarrow (Sh(Y_{\acute{e}tale}), \mathcal{O}_Y)$$

be a morphism of locally ringed topoi. Then there exists a unique morphism of schemes $f : X \to Y$ such that $(g, g^\#)$ is 2-isomorphic to $(f_{small}, f_{small}^\sharp)$, see Modules on Sites, Definition 8.1.

Proof. In this proof we write $\mathcal{O}_X$ for the structure sheaf of the small étale site $X_{\acute{e}tale}$, and similarly for $\mathcal{O}_Y$. Say $Y = \mathrm{Spec}(B)$ and $X = \mathrm{Spec}(A)$. Since $B = \Gamma(Y_{\acute{e}tale}, \mathcal{O}_Y)$, $A = \Gamma(X_{\acute{e}tale}, \mathcal{O}_X)$ we see that $g^\sharp$ induces a ring map $\varphi : B \to A$. Let $f = \mathrm{Spec}(\varphi) : X \to Y$ be the corresponding morphism of affine schemes. We will show this f does the job.

Let $V \to Y$ be an affine scheme étale over Y. Thus we may write $V = \mathrm{Spec}(C)$ with C an étale B-algebra. We can write

$$C = B[x_1, \ldots, x_n]/(P_1, \ldots, P_n)$$

with P_i polynomials such that $\Delta = \det(\partial P_i/\partial x_j)$ is invertible in C, see for example Algebra, Lemma 142.2. If T is a scheme over Y, then a T-valued point of V is given by n sections of $\Gamma(T, \mathcal{O}_T)$ which satisfy the polynomial equations $P_1 = 0, \ldots, P_n = 0$. In other words, the sheaf h_V on $Y_{\acute{e}tale}$ is the equalizer of the two maps

$$\prod\nolimits_{i=1,\ldots,n} \mathcal{O}_Y \underset{b}{\overset{a}{\rightrightarrows}} \prod\nolimits_{j=1,\ldots,n} \mathcal{O}_Y$$

where $b(h_1, \ldots, h_n) = 0$ and $a(h_1, \ldots, h_n) = (P_1(h_1, \ldots, h_n), \ldots, P_n(h_1, \ldots, h_n))$. Since g^{-1} is exact we conclude that the top row of the following solid commutative diagram is an equalizer diagram as well:

$$\begin{tikzcd} g^{-1}h_V \arrow[r] \arrow[d, dotted] & \prod_{i=1,\ldots,n} g^{-1}\mathcal{O}_Y \arrow[r, shift left, "g^{-1}a"] \arrow[r, shift right, "g^{-1}b"'] \arrow[d, "\prod g^\sharp"] & \prod_{j=1,\ldots,n} g^{-1}\mathcal{O}_Y \arrow[d, "\prod g^\sharp"] \\ h_{X \times_Y V} \arrow[r] & \prod_{i=1,\ldots,n} \mathcal{O}_X \arrow[r, shift left, "a'"] \arrow[r, shift right, "b'"'] & \prod_{j=1,\ldots,n} \mathcal{O}_X \end{tikzcd}$$

Here b' is the zero map and a' is the map defined by the images $P'_i = \varphi(P_i) \in A[x_1, \ldots, x_n]$ via the same rule $a'(h_1, \ldots, h_n) = (P'_1(h_1, \ldots, h_n), \ldots, P'_n(h_1, \ldots, h_n))$. that a was defined by. The commutativity of the diagram follows from the fact that $\varphi = g^\sharp$ on global sections. The lower row is an equalizer diagram also, by exactly the same arguments as before since $X \times_Y V$ is the affine scheme $\text{Spec}(A \otimes_B C)$ and $A \otimes_B C = A[x_1, \ldots, x_n]/(P'_1, \ldots, P'_n)$. Thus we obtain a unique dotted arrow $g^{-1}h_V \to h_{X \times_Y V}$ fitting into the diagram

We claim that the map of sheaves $g^{-1}h_V \to h_{X \times_Y V}$ is an isomorphism. Since the small étale site of X has enough points (Theorem 29.10) it suffices to prove this on stalks. Hence let $\overline{x}$ be a geometric point of X, and denote p the associate point of the small étale topos of X. Set $q = g \circ p$. This is a point of the small étale topos of Y. By Lemma 29.12 we see that q corresponds to a geometric point $\overline{y}$ of Y. Consider the map of stalks

$$(g^\sharp)_p : \mathcal{O}_{Y,\overline{y}} = \mathcal{O}_{Y,q} = (g^{-1}\mathcal{O}_Y)_p \longrightarrow \mathcal{O}_{X,p} = \mathcal{O}_{X,\overline{x}}$$

Since $(g, g^\sharp)$ is a morphism of *locally* ringed topoi $(g^\sharp)_p$ is a local ring homomorphism of strictly henselian local rings. Applying localization to the big commutative diagram above and Algebra, Lemma 149.12 we conclude that $(g^{-1}h_V)_p \to (h_{X \times_Y V})_p$ is an isomorphism as desired.

We claim that the isomorphisms $g^{-1}h_V \to h_{X \times_Y V}$ are functorial. Namely, suppose that $V_1 \to V_2$ is a morphism of affine schemes étale over Y. Write $V_i = \text{Spec}(C_i)$ with

$$C_i = B[x_{i,1}, \ldots, x_{i,n_i}]/(P_{i,1}, \ldots, P_{i,n_i})$$

The morphism $V_1 \to V_2$ is given by a B-algebra map $C_2 \to C_1$ which in turn is given by some polynomials $Q_j \in B[x_{1,1}, \ldots, x_{1,n_1}]$ for $j = 1, \ldots, n_2$. Then it is an

easy matter to show that the diagram of sheaves

$$\begin{tikzcd}
h_{V_1} \arrow[r] \arrow[d] & \prod_{i=1,\ldots,n_1} \mathcal{O}_Y \arrow[d, "Q_1,\ldots,Q_{n_2}"] \\
h_{V_2} \arrow[r] & \prod_{i=1,\ldots,n_2} \mathcal{O}_Y
\end{tikzcd}$$

is commutative, and pulling back to $X_{étale}$ we obtain the solid commutative diagram

$$\begin{tikzcd}
g^{-1}h_{V_1} \arrow[r] \arrow[drr] \arrow[dd, dotted] & \prod_{i=1,\ldots,n_1} g^{-1}\mathcal{O}_Y \arrow[drr, "Q_1,\ldots,Q_{n_2}"] \arrow[dd, "g^\sharp"] & & \\
& & g^{-1}h_{V_2} \arrow[r] \arrow[dd, dotted] & \prod_{i=1,\ldots,n_2} g^{-1}\mathcal{O}_Y \arrow[dd, "g^\sharp"] \\
h_{X \times_Y V_1} \arrow[r] \arrow[drr] & \prod_{i=1,\ldots,n_1} \mathcal{O}_X \arrow[drr, "Q'_1,\ldots,Q'_{n_2}"] & & \\
& & h_{X \times_Y V_2} \arrow[r] & \prod_{i=1,\ldots,n_2} \mathcal{O}_X
\end{tikzcd}$$

where $Q'_j \in A[x_{1,1}, \ldots, x_{1,n_1}]$ is the image of Q_j via φ. Since the dotted arrows exist, make the two squares commute, and the horizontal arrows are injective we see that the whole diagram commutes. This proves functoriality (and also that the construction of $g^{-1}h_V \to h_{X \times_Y V}$ is independent of the choice of the presentation, although we strictly speaking do not need to show this).

At this point we are able to show that $f_{small,*} \cong g_*$. Namely, let $\mathcal{F}$ be a sheaf on $X_{étale}$. For every $V \in \mathrm{Ob}(X_{étale})$ affine we have

$$\begin{aligned}
(g_*\mathcal{F})(V) &= Mor_{Sh(Y_{étale})}(h_V, g_*\mathcal{F}) \\
&= Mor_{Sh(X_{étale})}(g^{-1}h_V, \mathcal{F}) \\
&= Mor_{Sh(X_{étale})}(h_{X \times_Y V}, \mathcal{F}) \\
&= \mathcal{F}(X \times_Y V) \\
&= f_{small,*}\mathcal{F}(V)
\end{aligned}$$

where in the third equality we use the isomorphism $g^{-1}h_V \cong h_{X \times_Y V}$ constructed above. These isomorphisms are clearly functorial in $\mathcal{F}$ and functorial in V as the isomorphisms $g^{-1}h_V \cong h_{X \times_Y V}$ are functorial. Now any sheaf on $Y_{étale}$ is determined by the restriction to the subcategory of affine schemes (Lemma 21.2), and hence we obtain an isomorphism of functors $f_{small,*} \cong g_*$ as desired.

Finally, we have to check that, via the isomorphism $f_{small,*} \cong g_*$ above, the maps $f_{small}^\sharp$ and $g^\sharp$ agree. By construction this is already the case for the global sections of $\mathcal{O}_Y$, i.e., for the elements of B. We only need to check the result on sections over an affine V étale over Y (by Lemma 21.2 again). Writing $V = \mathrm{Spec}(C)$, $C = B[x_i]/(P_j)$ as before it suffices to check that the coordinate functions x_i are mapped to the same sections of $\mathcal{O}_X$ over $X \times_Y V$. And this is exactly what it

means that the diagram

$$\begin{tikzcd} g^{-1}h_V \arrow[r] \arrow[d, dotted] & \prod_{i=1,\ldots,n} g^{-1}\mathcal{O}_Y \arrow[d, "\prod g^\sharp"] \\ h_{X\times_Y V} \arrow[r] & \prod_{i=1,\ldots,n} \mathcal{O}_X \end{tikzcd}$$

commutes. Thus the lemma is proved. □

Here is a version for general schemes.

04I7 **Theorem 40.5.** *Let X, Y be schemes. Let*

$$(g, g^\#) : (Sh(X_{\acute{e}tale}), \mathcal{O}_X) \longrightarrow (Sh(Y_{\acute{e}tale}), \mathcal{O}_Y)$$

be a morphism of locally ringed topoi. Then there exists a unique morphism of schemes $f : X \to Y$ such that $(g, g^\#)$ is isomorphic to $(f_{small}, f_{small}^\sharp)$. In other words, the construction

$$Sch \longrightarrow \textit{Locally ringed topoi}, \quad X \longrightarrow (X_{\acute{e}tale}, \mathcal{O}_X)$$

is fully faithful (morphisms up to 2-isomorphisms on the right hand side).

Proof. You can prove this theorem by carefuly adjusting the arguments of the proof of Lemma 40.4 to the global setting. However, we want to indicate how we can glue the result of that lemma to get a global morphism due to the rigidity provided by the result of Lemma 40.2. Unfortunately, this is a bit messy.

Let us prove existence when Y is affine. In this case choose an affine open covering $X = \bigcup U_i$. For each i the inclusion morphism $j_i : U_i \to X$ induces a morphism of locally ringed topoi $(j_{i,small}, j_{i,small}^\sharp) : (Sh(U_{i,\acute{e}tale}), \mathcal{O}_{U_i}) \to (Sh(X_{\acute{e}tale}), \mathcal{O}_X)$ by Lemma 40.1. We can compose this with $(g, g^\sharp)$ to obtain a morphism of locally ringed topoi

$$(g, g^\sharp) \circ (j_{i,small}, j_{i,small}^\sharp) : (Sh(U_{i,\acute{e}tale}), \mathcal{O}_{U_i}) \to (Sh(X_{\acute{e}tale}), \mathcal{O}_X)$$

see Modules on Sites, Lemma 39.10. By Lemma 40.4 there exists a unique morphism of schemes $f_i : U_i \to Y$ and a 2-isomorphism

$$t_i : (f_{i,small}, f_{i,small}^\sharp) \longrightarrow (g, g^\sharp) \circ (j_{i,small}, j_{i,small}^\sharp).$$

Set $U_{i,i'} = U_i \cap U_{i'}$, and denote $j_{i,i'} : U_{i,i'} \to U_i$ the inclusion morphism. Since we have $j_i \circ j_{i,i'} = j_{i'} \circ j_{i',i}$ we see that

$$\begin{aligned} (g, g^\sharp) \circ (j_{i,small}, j_{i,small}^\sharp) \circ (j_{i,i',small}, j_{i,i',small}^\sharp) = \\ (g, g^\sharp) \circ (j_{i',small}, j_{i',small}^\sharp) \circ (j_{i',i,small}, j_{i',i,small}^\sharp) \end{aligned}$$

Hence by uniqueness (see Lemma 40.3) we conclude that $f_i \circ j_{i,i'} = f_{i'} \circ j_{i',i}$, in other words the morphisms of schemes $f_i = f \circ j_i$ are the restrictions of a global morphism of schemes $f : X \to Y$. Consider the diagram of 2-isomorphisms (where we drop the components $^\sharp$ to ease the notation)

$$\begin{tikzcd} g \circ j_{i,small} \circ j_{i,i',small} \arrow[r, "t_i \star \mathrm{id}_{j_{i,i',small}}"] \arrow[d, equal] & f_{small} \circ j_{i,small} \circ j_{i,i',small} \arrow[d, equal] \\ g \circ j_{i',small} \circ j_{i',i,small} \arrow[r, "t_{i'} \star \mathrm{id}_{j_{i',i,small}}"] & f_{small} \circ j_{i',small} \circ j_{i',i,small} \end{tikzcd}$$

The notation $\star$ indicates horizontal composition, see Categories, Definition 28.1 in general and Sites, Section 36 for our particular case. By the result of Lemma 40.2 this diagram commutes. Hence for any sheaf $\mathcal{G}$ on $Y_{étale}$ the isomorphisms $t_i : f_{small}^{-1}\mathcal{G}|_{U_i} \to g^{-1}\mathcal{G}|_{U_i}$ agree over $U_{i,i'}$ and we obtain a global isomorphism $t : f_{small}^{-1}\mathcal{G} \to g^{-1}\mathcal{G}$. It is clear that this isomorphism is functorial in $\mathcal{G}$ and is compatible with the maps $f_{small}^\sharp$ and $g^\sharp$ (because it is compatible with these maps locally). This proves the theorem in case Y is affine.

In the general case, let $V \subset Y$ be an affine open. Then h_V is a subsheaf of the final sheaf $*$ on $Y_{étale}$. As g is exact we see that $g^{-1}h_V$ is a subsheaf of the final sheaf on $X_{étale}$. Hence by Lemma 31.1 there exists an open subscheme $W \subset X$ such that $g^{-1}h_V = h_W$. By Modules on Sites, Lemma 39.12 there exists a commutative diagram of morphisms of locally ringed topoi

$$\begin{tikzcd} (Sh(W_{étale}), \mathcal{O}_W) \arrow[r] \arrow[d, "g'"'] & (Sh(X_{étale}), \mathcal{O}_X) \arrow[d, "g"] \\ (Sh(V_{étale}), \mathcal{O}_V) \arrow[r] & (Sh(Y_{étale}), \mathcal{O}_Y) \end{tikzcd}$$

where the horizontal arrows are the localization morphisms (induced by the inclusion morphisms $V \to Y$ and $W \to X$) and where g' is induced from g. By the result of the preceding paragraph we obtain a morphism of schemes $f' : W \to V$ and a 2-isomorphism $t : (f'_{small}, (f'_{small})^\sharp) \to (g', (g')^\sharp)$. Exactly as before these morphisms f' (for varying affine opens $V \subset Y$) agree on overlaps by uniqueness, so we get a morphism $f : X \to Y$. Moreover, the 2-isomorphisms t are compatible on overlaps by Lemma 40.2 again and we obtain a global 2-isomorphism $(f_{small}, (f_{small})^\sharp) \to (g, (g)^\sharp)$. as desired. Some details omitted. $\square$

41. Push and pull

04C6 Let $f : X \to Y$ be a morphism of schemes. Here is a list of conditions we will consider in the following:

(A) For every étale morphism $U \to X$ and $u \in U$ there exist an étale morphism $V \to Y$ and a disjoint union decomposition $X \times_Y V = W \amalg W'$ and a morphism $h : W \to U$ over X with u in the image of h.

(B) For every $V \to Y$ étale, and every étale covering $\{U_i \to X \times_Y V\}$ there exists an étale covering $\{V_j \to V\}$ such that for each j we have $X \times_Y V_j = \coprod W_{ij}$ where $W_{ij} \to X \times_Y V$ factors through $U_i \to X \times_Y V$ for some i.

(C) For every $U \to X$ étale, there exists a $V \to Y$ étale and a surjective morphism $X \times_Y V \to U$ over X.

It turns out that each of these properties has meaning in terms of the behaviour of the functor $f_{small,*}$. We will work this out in the next few sections.

42. Property (A)

04DJ Please see Section 41 for the definition of property (A).

04DK **Lemma 42.1.** *Let $f : X \to Y$ be a morphism of schemes. Assume (A).*

(1) $f_{small,*} : Ab(X_{étale}) \to Ab(Y_{étale})$ *reflects injections and surjections,*
(2) $f_{small}^{-1}f_{small,*}\mathcal{F} \to \mathcal{F}$ *is surjective for any abelian sheaf $\mathcal{F}$ on $X_{étale}$,*
(3) $f_{small,*} : Ab(X_{étale}) \to Ab(Y_{étale})$ *is faithful.*

Proof. Let $\mathcal{F}$ be an abelian sheaf on $X_{étale}$. Let U be an object of $X_{étale}$. By assumption we can find a covering $\{W_i \to U\}$ in $X_{étale}$ such that each W_i is an open and closed subscheme of $X \times_Y V_i$ for some object V_i of $Y_{étale}$. The sheaf condition shows that

$$\mathcal{F}(U) \subset \prod \mathcal{F}(W_i)$$

and that $\mathcal{F}(W_i)$ is a direct summand of $\mathcal{F}(X \times_Y V_i) = f_{small,*}\mathcal{F}(V_i)$. Hence it is clear that $f_{small,*}$ reflects injections.

Next, suppose that $a : \mathcal{G} \to \mathcal{F}$ is a map of abelian sheaves such that $f_{small,*}a$ is surjective. Let $s \in \mathcal{F}(U)$ with U as above. With W_i, V_i as above we see that it suffices to show that $s|_{W_i}$ is étale locally the image of a section of $\mathcal{G}$ under a. Since $\mathcal{F}(W_i)$ is a direct summand of $\mathcal{F}(X \times_Y V_i)$ it suffices to show that for any $V \in \mathrm{Ob}(Y_{étale})$ any element $s \in \mathcal{F}(X \times_Y V)$ is étale locally on $X \times_Y V$ the image of a section of $\mathcal{G}$ under a. Since $\mathcal{F}(X \times_Y V) = f_{small,*}\mathcal{F}(V)$ we see by assumption that there exists a covering $\{V_j \to V\}$ such that s is the image of $s_j \in f_{small,*}\mathcal{G}(V_j) = \mathcal{G}(X \times_Y V_j)$. This proves $f_{small,*}$ reflects surjections.

Parts (2), (3) follow formally from part (1), see Modules on Sites, Lemma 15.1. $\square$

04DL **Lemma 42.2.** *Let $f : X \to Y$ be a separated locally quasi-finite morphism of schemes. Then property (A) above holds.*

Proof. Let $U \to X$ be an étale morphism and $u \in U$. The geometric statement (A) reduces directly to the case where U and Y are affine schemes. Denote $x \in X$ and $y \in Y$ the images of u. Since $X \to Y$ is locally quasi-finite, and $U \to X$ is locally quasi-finite (see Morphisms, Lemma 34.6) we see that $U \to Y$ is locally quasi-finite (see Morphisms, Lemma 19.12). Moreover both $X \to Y$ and $U \to Y$ are separated. Thus More on Morphisms, Lemma 36.5 applies to both morphisms. This means we may pick an étale neighbourhood $(V, v) \to (Y, y)$ such that

$$X \times_Y V = W \amalg R, \quad U \times_Y V = W' \amalg R'$$

and points $w \in W$, $w' \in W'$ such that

(1) W, R are open and closed in $X \times_Y V$,
(2) W', R' are open and closed in $U \times_Y V$,
(3) $W \to V$ and $W' \to V$ are finite,
(4) w, w' map to v,
(5) $\kappa(v) \subset \kappa(w)$ and $\kappa(v) \subset \kappa(w')$ are purely inseparable, and
(6) no other point of W or W' maps to v.

Here is a commutative diagram

$$\begin{tikzcd}
U \arrow[d] & U \times_Y V \arrow[l] \arrow[d] & W' \amalg R' \arrow[l] \arrow[d] \\
X \arrow[d] & X \times_Y V \arrow[l] \arrow[d] & W \amalg R \arrow[l] \\
Y & V \arrow[l] &
\end{tikzcd}$$

After shrinking V we may assume that W' maps into W: just remove the image the inverse image of R in W'; this is a closed set (as $W' \to V$ is finite) not containing v. Then $W' \to W$ is finite because both $W \to V$ and $W' \to V$ are finite. Hence

$W' \to W$ is finite étale, and there is exactly one point in the fibre over w with $\kappa(w) = \kappa(w')$. Hence $W' \to W$ is an isomorphism in an open neighbourhood W° of w, see Étale Morphisms, Lemma 14.2. Since $W \to V$ is finite the image of $W \setminus W^\circ$ is a closed subset T of V not containing v. Thus after replacing V by $V \setminus T$ we may assume that $W' \to W$ is an isomorphism. Now the decomposition $X \times_Y V = W \amalg R$ and the morphism $W \to U$ are as desired and we win. $\square$

04DM **Lemma 42.3.** *Let $f : X \to Y$ be an integral morphism of schemes. Then property (A) holds.*

Proof. Let $U \to X$ be étale, and let $u \in U$ be a point. We have to find $V \to Y$ étale, a disjoint union decomposition $X \times_Y V = W \amalg W'$ and an X-morphism $W \to U$ with u in the image. We may shrink U and Y and assume U and Y are affine. In this case also X is affine, since an integral morphism is affine by definition. Write $Y = \text{Spec}(A)$, $X = \text{Spec}(B)$ and $U = \text{Spec}(C)$. Then $A \to B$ is an integral ring map, and $B \to C$ is an étale ring map. By Algebra, Lemma 142.3 we can find a finite A-subalgebra $B' \subset B$ and an étale ring map $B' \to C'$ such that $C = B \otimes_{B'} C'$. Thus the question reduces to the étale morphism $U' = \text{Spec}(C') \to X' = \text{Spec}(B')$ over the finite morphism $X' \to Y$. In this case the result follows from Lemma 42.2. $\square$

04C9 **Lemma 42.4.** *Let $f : X \to Y$ be a morphism of schemes. Denote $f_{small} : Sh(X_{étale}) \to Sh(Y_{étale})$ the associated morphism of small étale topoi. Assume at least one of the following*

1. *f is integral, or*
2. *f is separated and locally quasi-finite.*

Then the functor $f_{small,*} : Ab(X_{étale}) \to Ab(Y_{étale})$ has the following properties

1. *the map $f_{small}^{-1} f_{small,*}\mathcal{F} \to \mathcal{F}$ is always surjective,*
2. *$f_{small,*}$ is faithful, and*
3. *$f_{small,*}$ reflects injections and surjections.*

Proof. Combine Lemmas 42.2, 42.3, and 42.1. $\square$

43. Property (B)

04DN Please see Section 41 for the definition of property (B).

04DO **Lemma 43.1.** *Let $f : X \to Y$ be a morphism of schemes. Assume (B) holds. Then the functor $f_{small,*} : Sh(X_{étale}) \to Sh(Y_{étale})$ transforms surjections into surjections.*

Proof. This follows from Sites, Lemma 41.2. $\square$

04DP **Lemma 43.2.** *Let $f : X \to Y$ be a morphism of schemes. Suppose*

1. *$V \to Y$ is an étale morphism of schemes,*
2. *$\{U_i \to X \times_Y V\}$ is an étale covering, and*
3. *$v \in V$ is a point.*

Assume that for any such data there exists an étale neighbourhood $(V', v') \to (V, v)$, a disjoint union decomposition $X \times_Y V' = \coprod W'_i$, and morphisms $W'_i \to U_i$ over $X \times_Y V$. Then property (B) holds.

Proof. Omitted. $\square$

04DQ **Lemma 43.3.** *Let $f : X \to Y$ be a finite morphism of schemes. Then property (B) holds.*

Proof. Consider $V \to Y$ étale, $\{U_i \to X \times_Y V\}$ an étale covering, and $v \in V$. We have to find a $V' \to V$ and decomposition and maps as in Lemma 43.2. We may shrink V and Y, hence we may assume that V and Y are affine. Since X is finite over Y, this also implies that X is affine. During the proof we may (finitely often) replace (V, v) by an étale neighbourhood (V', v') and correspondingly the covering $\{U_i \to X \times_Y V\}$ by $\{V' \times_V U_i \to X \times_Y V'\}$.

Since $X \times_Y V \to V$ is finite there exist finitely many (pairwise distinct) points $x_1, \ldots, x_n \in X \times_Y V$ mapping to v. We may apply More on Morphisms, Lemma 36.5 to $X \times_Y V \to V$ and the points $x_1, \ldots, x_n$ lying over v and find an étale neighbourhood $(V', v') \to (V, v)$ such that

$$X \times_Y V' = R \amalg \coprod T_a$$

with $T_a \to V'$ finite with exactly one point p_a lying over v' and moreover $\kappa(v') \subset \kappa(p_a)$ purely inseparable, and such that $R \to V'$ has empty fibre over v'. Because $X \to Y$ is finite, also $R \to V'$ is finite. Hence after shrinking V' we may assume that $R = \emptyset$. Thus we may assume that $X \times_Y V = X_1 \amalg \ldots \amalg X_n$ with exactly one point $x_l \in X_l$ lying over v with moreover $\kappa(v) \subset \kappa(x_l)$ purely inseparable. Note that this property is preserved under refinement of the étale neighbourhood (V, v).

For each l choose an i_l and a point $u_l \in U_{i_l}$ mapping to x_l. Now we apply property (A) for the finite morphism $X \times_Y V \to V$ and the étale morphisms $U_{i_l} \to X \times_Y V$ and the points u_l. This is permissible by Lemma 42.3 This gives produces an étale neighbourhood $(V', v') \to (V, v)$ and decompositions

$$X \times_Y V' = W_l \amalg R_l$$

and X-morphisms $a_l : W_l \to U_{i_l}$ whose image contains u_{i_l}. Here is a picture:

$$\begin{tikzcd}
 & & & U_{i_l} \arrow[d] & \\
W_l \arrow[r] \arrow[urrr] & W_l \amalg R_l \arrow[r, equal] & X \times_Y V' \arrow[r] \arrow[d] & X \times_Y V \arrow[r] \arrow[d] & X \arrow[d] \\
 & & V' \arrow[r] & V \arrow[r] & Y
\end{tikzcd}$$

After replacing (V, v) by (V', v') we conclude that each x_l is contained in an open and closed neighbourhood W_l such that the inclusion morphism $W_l \to X \times_Y V$ factors through $U_i \to X \times_Y V$ for some i. Replacing W_l by $W_l \cap X_l$ we see that these open and closed sets are disjoint and moreover that $\{x_1, \ldots, x_n\} \subset W_1 \cup \ldots \cup W_n$. Since $X \times_Y V \to V$ is finite we may shrink V and assume that $X \times_Y V = W_1 \amalg \ldots \amalg W_n$ as desired. $\square$

04DR **Lemma 43.4.** *Let $f : X \to Y$ be an integral morphism of schemes. Then property (B) holds.*

Proof. Consider $V \to Y$ étale, $\{U_i \to X \times_Y V\}$ an étale covering, and $v \in V$. We have to find a $V' \to V$ and decomposition and maps as in Lemma 43.2. We may shrink V and Y, hence we may assume that V and Y are affine. Since X is

integral over Y, this also implies that X and $X \times_Y V$ are affine. We may refine the covering $\{U_i \to X \times_Y V\}$, and hence we may assume that $\{U_i \to X \times_Y V\}_{i=1,\ldots,n}$ is a standard étale covering. Write $Y = \mathrm{Spec}(A)$, $X = \mathrm{Spec}(B)$, $V = \mathrm{Spec}(C)$, and $U_i = \mathrm{Spec}(B_i)$. Then $A \to B$ is an integral ring map, and $B \otimes_A C \to B_i$ are étale ring maps. By Algebra, Lemma 142.3 we can find a finite A-subalgebra $B' \subset B$ and an étale ring map $B' \otimes_A C \to B'_i$ for $i = 1, \ldots, n$ such that $B_i = B \otimes_{B'} B'_i$. Thus the question reduces to the étale covering $\{\mathrm{Spec}(B'_i) \to X' \times_Y V\}_{i=1,\ldots,n}$ with $X' = \mathrm{Spec}(B')$ finite over Y. In this case the result follows from Lemma 43.3. □

04C2 **Lemma 43.5.** *Let $f : X \to Y$ be a morphism of schemes. Assume f is integral (for example finite). Then*

1. *$f_{small,*}$ transforms surjections into surjections (on sheaves of sets and on abelian sheaves),*
2. *$f_{small}^{-1} f_{small,*}\mathcal{F} \to \mathcal{F}$ is surjective for any abelian sheaf $\mathcal{F}$ on $X_{étale}$,*
3. *$f_{small,*} : Ab(X_{étale}) \to Ab(Y_{étale})$ is faithful and reflects injections and surjections, and*
4. *$f_{small,*} : Ab(X_{étale}) \to Ab(Y_{étale})$ is exact.*

Proof. Parts (2), (3) we have seen in Lemma 42.4. Part (1) follows from Lemmas 43.4 and 43.1. Part (4) is a consequence of part (1), see Modules on Sites, Lemma 15.2. □

44. Property (C)

04DS Please see Section 41 for the definition of property (C).

04DT **Lemma 44.1.** *Let $f : X \to Y$ be a morphism of schemes. Assume (C) holds. Then the functor $f_{small,*} : Sh(X_{étale}) \to Sh(Y_{étale})$ reflects injections and surjections.*

Proof. Follows from Sites, Lemma 41.4. We omit the verification that property (C) implies that the functor $Y_{étale} \to X_{étale}$, $V \mapsto X \times_Y V$ satisfies the assumption of Sites, Lemma 41.4. □

04DU **Remark 44.2.** Property (C) holds if $f : X \to Y$ is an open immersion. Namely, if $U \in \mathrm{Ob}(X_{étale})$, then we can view U also as an object of $Y_{étale}$ and $U \times_Y X = U$. Hence property (C) does not imply that $f_{small,*}$ is exact as this is not the case for open immersions (in general).

04DV **Lemma 44.3.** *Let $f : X \to Y$ be a morphism of schemes. Assume that for any $V \to Y$ étale we have that*

1. *$X \times_Y V \to V$ has property (C), and*
2. *$X \times_Y V \to V$ is closed.*

Then the functor $Y_{étale} \to X_{étale}$, $V \mapsto X \times_Y V$ is almost cocontinuous, see Sites, Definition 42.3.

Proof. Let $V \to Y$ be an object of $Y_{étale}$ and let $\{U_i \to X \times_Y V\}_{i \in I}$ be a covering of $X_{étale}$. By assumption (1) for each i we can find an étale morphism $h_i : V_i \to V$ and a surjective morphism $X \times_Y V_i \to U_i$ over $X \times_Y V$. Note that $\bigcup h_i(V_i) \subset V$ is an open set containing the closed set $Z = \mathrm{Im}(X \times_Y V \to V)$. Let $h_0 : V_0 = V \setminus Z \to V$ be the open immersion. It is clear that $\{V_i \to V\}_{i \in I \cup \{0\}}$ is an étale covering such that for each $i \in I \cup \{0\}$ we have either $V_i \times_Y X = \emptyset$ (namely if $i = 0$), or $V_i \times_Y X \to V \times_Y X$ factors through $U_i \to X \times_Y V$ (if $i \neq 0$). Hence the functor $Y_{étale} \to X_{étale}$ is almost cocontinuous. □

04DW **Lemma 44.4.** *Let $f : X \to Y$ be an integral morphism of schemes which defines a homeomorphism of X with a closed subset of Y. Then property (C) holds.*

Proof. Let $g : U \to X$ be an étale morphism. We need to find an object $V \to Y$ of $Y_{étale}$ and a surjective morphism $X \times_Y V \to U$ over X. Suppose that for every $u \in U$ we can find an object $V_u \to Y$ of $Y_{étale}$ and a morphism $h_u : X \times_Y V_u \to U$ over X with $u \in \mathrm{Im}(h_u)$. Then we can take $V = \coprod V_u$ and $h = \coprod h_u$ and we win. Hence given a point $u \in U$ we find a pair (V_u, h_u) as above. To do this we may shrink U and assume that U is affine. In this case $g : U \to X$ is locally quasi-finite. Let $g^{-1}(g(\{u\})) = \{u, u_2, \ldots, u_n\}$. Since there are no specializations $u_i \rightsquigarrow u$ we may replace U by an affine neighbourhood so that $g^{-1}(g(\{u\})) = \{u\}$.

The image $g(U) \subset X$ is open, hence $f(g(U))$ is locally closed in Y. Choose an open $V \subset Y$ such that $f(g(U)) = f(X) \cap V$. It follows that g factors through $X \times_Y V$ and that the resulting $\{U \to X \times_Y V\}$ is an étale covering. Since f has property (B) , see Lemma 43.4, we see that there exists an étale covering $\{V_j \to V\}$ such that $X \times_Y V_j \to X \times_Y V$ factor through U. This implies that $V' = \coprod V_j$ is étale over Y and that there is a morphism $h : X \times_Y V' \to U$ whose image surjects onto $g(U)$. Since u is the only point in its fibre it must be in the image of h and we win. $\square$

We urge the reader to think of the following lemma as a way station[3] on the journey towards the ultimate truth regarding $f_{small,*}$ for integral universally injective morphisms.

04DX **Lemma 44.5.** *Let $f : X \to Y$ be a morphism of schemes. Assume that f is universally injective and integral (for example a closed immersion). Then*

1. *$f_{small,*} : Sh(X_{étale}) \to Sh(Y_{étale})$ reflects injections and surjections,*
2. *$f_{small,*} : Sh(X_{étale}) \to Sh(Y_{étale})$ commutes with pushouts and coequalizers (and more generally finite connected colimits),*
3. *$f_{small,*}$ transforms surjections into surjections (on sheaves of sets and on abelian sheaves),*
4. *the map $f_{small}^{-1} f_{small,*}\mathcal{F} \to \mathcal{F}$ is surjective for any sheaf (of sets or of abelian groups) $\mathcal{F}$ on $X_{étale}$,*
5. *the functor $f_{small,*}$ is faithful (on sheaves of sets and on abelian sheaves),*
6. *$f_{small,*} : Ab(X_{étale}) \to Ab(Y_{étale})$ is exact, and*
7. *the functor $Y_{étale} \to X_{étale}$, $V \mapsto X \times_Y V$ is almost cocontinuous.*

Proof. By Lemmas 42.3, 43.4 and 44.4 we know that the morphism f has properties (A), (B), and (C). Moreover, by Lemma 44.3 we know that the functor $Y_{étale} \to X_{étale}$ is almost cocontinuous. Now we have

1. property (C) implies (1) by Lemma 44.1,
2. almost continuous implies (2) by Sites, Lemma 42.6,
3. property (B) implies (3) by Lemma 43.1.

Properties (4), (5), and (6) follow formally from the first three, see Sites, Lemma 41.1 and Modules on Sites, Lemma 15.2. Property (7) we saw above. $\square$

[3] A way station is a place where people stop to eat and rest when they are on a long journey.

45. Topological invariance of the small étale site

04DY In the following theorem we show that the small étale site is a topological invariant in the following sense: If $f : X \to Y$ is a morphism of schemes which is a universal homeomorphism, then $X_{\acute{e}tale} \cong Y_{\acute{e}tale}$ as sites. This improves the result of Étale Morphisms, Theorem 15.2. We first prove the result for morphisms and then we state the result for categories.

0BTY **Theorem 45.1.** *Let X and Y be two schemes over a base scheme S. Let $S' \to S$ be a universal homeomorphism. Denote X' (resp. Y') the base change to S'. If X is étale over S, then the map*

$$Mor_S(Y, X) \longrightarrow Mor_{S'}(Y', X')$$

is bijective.

Proof. After base changing via $Y \to S$, we may assume that $Y = S$. Thus we may and do assume both X and Y are étale over S. In other words, the theorem states that the base change functor is a fully faithful functor from the category of schemes étale over S to the category of schemes étale over S'.

Consider the forgetful functor

0BTZ
$$\begin{matrix} \text{descent data } (X', \varphi') \text{ relative to } S'/S \\ \text{with } X' \text{ étale over } S' \end{matrix} \longrightarrow \text{schemes } X' \text{ étale over } S' \quad (45.1.1)$$

We claim this functor is an equivalence. On the other hand, the functor

0BU0
$$\text{schemes } X \text{ étale over } S \longrightarrow \begin{matrix} \text{descent data } (X', \varphi') \text{ relative to } S'/S \\ \text{with } X' \text{ étale over } S' \end{matrix} \quad (45.1.2)$$

is fully faithful by Étale Morphisms, Lemma 20.3. Thus the claim implies the theorem.

Proof of the claim. Recall that a universal homeomorphism is the same thing as an integral, universally injective, surjective morphism, see Morphisms, Lemma 43.5. In particular, the diagonal $\Delta : S' \to S' \times_S S'$ is a thickening by Morphisms, Lemma 10.2. Thus by Étale Morphisms, Theorem 15.1 we see that given $X' \to S'$ étale there is a unique isomorphism

$$\varphi' : X' \times_S S' \to S' \times_S X'$$

of schemes étale over $S' \times_S S'$ which pulls back under Δ to $\text{id} : X' \to X'$ over S'. Since $S' \to S' \times_S S' \times_S S'$ is a thickening as well (it is bijective and a closed immersion) we conclude that (X', φ') is a descent datum relative to S'/S. The canonical nature of the construction of φ' shows that it is compatible with morphisms between schemes étale over S'. In other words, we obtain a quasi-inverse $X' \mapsto (X', \varphi')$ of the functor (45.1.1). This proves the claim and finishes the proof of the theorem. $\square$

04DZ **Theorem 45.2.** *Let $f : X \to Y$ be a morphism of schemes. Assume f is integral, universally injective and surjective (i.e., f is a universal homeomorphism, see Morphisms, Lemma 43.5). The functor* [DG67, IV Theorem 18.1.2]

$$V \longmapsto V_X = X \times_Y V$$

defines an equivalence of categories

$$\{\textit{schemes } V \textit{ étale over } Y\} \leftrightarrow \{\textit{schemes } U \textit{ étale over } X\}$$

We give two proofs. The first uses effectivity of descent for quasi-compact, separated, étale morphisms relative to surjective integral morphisms. The second uses the material on properties (A), (B), and (C) discussed earlier in the chapter.

First proof. By Theorem 45.1 we see that the functor is fully faithful. It remains to show that the functor is essentially surjective. Let $U \to X$ be an étale morphism of schemes.

Suppose that the result holds if U and Y are affine. In that case, we choose an affine open covering $U = \bigcup U_i$ such that each U_i maps into an affine open of Y. By assumption (affine case) we can find étale morphisms $V_i \to Y$ such that $X \times_Y V_i \cong U_i$ as schemes over X. Let $V_{i,i'} \subset V_i$ be the open subscheme whose underlying topological space corresponds to $U_i \cap U_{i'}$. Because we have isomorphisms

$$X \times_Y V_{i,i'} \cong U_i \cap U_{i'} \cong X \times_Y V_{i',i}$$

as schemes over X we see by fully faithfulness that we obtain isomorphisms $\theta_{i,i'} : V_{i,i'} \to V_{i',i}$ of schemes over Y. We omit the verification that these isomorphisms satisfy the cocycle condition of Schemes, Section 14. Applying Schemes, Lemma 14.2 we obtain a scheme $V \to Y$ by glueing the schemes V_i along the identifications $\theta_{i,i'}$. It is clear that $V \to Y$ is étale and $X \times_Y V \cong U$ by construction.

Thus it suffices to show the lemma in case U and Y are affine. Recall that in the proof of Theorem 45.1 we showed that U comes with a unique descent datum (U, φ) relative to X/Y. By Étale Morphisms, Proposition 20.6 (which applies because $U \to X$ is quasi-compact and separated as well as étale by our reduction to the affine case) there exists an étale morphism $V \to Y$ such that $X \times_Y V \cong U$ and the proof is complete. □

Second proof. By Theorem 45.1 we see that the functor is fully faithful. It remains to show that the functor is essentially surjective. Let $U \to X$ be an étale morphism of schemes.

Suppose that the result holds if U and Y are affine. In that case, we choose an affine open covering $U = \bigcup U_i$ such that each U_i maps into an affine open of Y. By assumption (affine case) we can find étale morphisms $V_i \to Y$ such that $X \times_Y V_i \cong U_i$ as schemes over X. Let $V_{i,i'} \subset V_i$ be the open subscheme whose underlying topological space corresponds to $U_i \cap U_{i'}$. Because we have isomorphisms

$$X \times_Y V_{i,i'} \cong U_i \cap U_{i'} \cong X \times_Y V_{i',i}$$

as schemes over X we see by fully faithfulness that we obtain isomorphisms $\theta_{i,i'} : V_{i,i'} \to V_{i',i}$ of schemes over Y. We omit the verification that these isomorphisms satisfy the cocycle condition of Schemes, Section 14. Applying Schemes, Lemma 14.2 we obtain a scheme $V \to Y$ by glueing the schemes V_i along the identifications $\theta_{i,i'}$. It is clear that $V \to Y$ is étale and $X \times_Y V \cong U$ by construction.

Thus it suffices to prove that the functor

04E0 (45.2.1) $$\{\text{affine schemes } V \text{ étale over } Y\} \leftrightarrow \{\text{affine schemes } U \text{ étale over } X\}$$

is essentially surjective when X and Y are affine.

Let $U \to X$ be an affine scheme étale over X. We have to find $V \to Y$ étale (and affine) such that $X \times_Y V$ is isomorphic to U over X. Note that an étale morphism of affines has universally bounded fibres, see Morphisms, Lemmas 34.6 and 54.10.

Hence we can do induction on the integer n bounding the degree of the fibres of $U \to X$. See Morphisms, Lemma 54.9 for a description of this integer in the case of an étale morphism. If $n = 1$, then $U \to X$ is an open immersion (see Étale Morphisms, Theorem 14.1), and the result is clear. Assume $n > 1$.

By Lemma 44.4 there exists an étale morphism of schemes $W \to Y$ and a surjective morphism $W_X \to U$ over X. As U is quasi-compact we may replace W by a disjoint union of finitely many affine opens of W, hence we may assume that W is affine as well. Here is a diagram

$$\begin{tikzcd} U \arrow[d] & U \times_Y W \arrow[l] \arrow[d] \arrow[r, equal] & W_X \amalg R \\ X \arrow[d] & W_X \arrow[l] \arrow[d] & \\ Y & W \arrow[l] & \end{tikzcd}$$

The disjoint union decomposition arises because by construction the étale morphism of affine schemes $U \times_Y W \to W_X$ has a section. OK, and now we see that the morphism $R \to X \times_Y W$ is an étale morphism of affine schemes whose fibres have degree universally bounded by $n - 1$. Hence by induction assumption there exists a scheme $V' \to W$ étale such that $R \cong W_X \times_W V'$. Taking $V'' = W \amalg V'$ we find a scheme V'' étale over W whose base change to W_X is isomorphic to $U \times_Y W$ over $X \times_Y W$.

At this point we can use descent to find V over Y whose base change to X is isomorphic to U over X. Namely, by the fully faithfulness of the functor (45.2.1) corresponding to the universal homeomorphism $X \times_Y (W \times_Y W) \to (W \times_Y W)$ there exists a unique isomorphism $\varphi : V'' \times_Y W \to W \times_Y V''$ whose base change to $X \times_Y (W \times_Y W)$ is the canonical descent datum for $U \times_Y W$ over $X \times_Y W$. In particular φ satisfies the cocycle condition. Hence by Descent, Lemma 34.1 we see that φ is effective (recall that all schemes above are affine). Thus we obtain $V \to Y$ and an isomorphism $V'' \cong W \times_Y V$ such that the canonical descent datum on $W \times_Y V/W/Y$ agrees with φ. Note that $V \to Y$ is étale, by Descent, Lemma 20.29. Moreover, there is an isomorphism $V_X \cong U$ which comes from descending the isomorphism

$$V_X \times_X W_X = X \times_Y V \times_Y W = (X \times_Y W) \times_W (W \times_Y V) \cong W_X \times_W V'' \cong U \times_Y W$$

which we have by construction. Some details omitted. $\square$

05YX **Remark 45.3.** In the situation of Theorem 45.2 it is also true that $V \mapsto V_X$ induces an equivalence between those étale morphisms $V \to Y$ with V affine and those étale morphisms $U \to X$ with U affine. This follows for example from Limits, Proposition 11.2.

03SI **Proposition 45.4** (Topological invariance of étale cohomology)**.** *Let $X_0 \to X$ be a universal homeomorphism of schemes (for example the closed immersion defined by a nilpotent sheaf of ideals). Then*

(1) *the étale sites $X_{étale}$ and $(X_0)_{étale}$ are isomorphic,*
(2) *the étale topoi $Sh(X_{étale})$ and $Sh((X_0)_{étale})$ are equivalent, and*

(3) $H^q_{étale}(X, \mathcal{F}) = H^q_{étale}(X_0, \mathcal{F}|_{X_0})$ *for all* q *and for any abelian sheaf* $\mathcal{F}$ *on* $X_{étale}$.

Proof. The equivalence of categories $X_{étale} \to (X_0)_{étale}$ is given by Theorem 45.2. We omit the proof that under this equivalence the étale coverings correspond. Hence (1) holds. Parts (2) and (3) follow formally from (1). $\square$

46. Closed immersions and pushforward

04E1 Before stating and proving Proposition 46.4 in its correct generality we briefly state and prove it for closed immersions. Namely, some of the preceding arguments are quite a bit easier to follow in the case of a closed immersion and so we repeat them here in their simplified form.

In the rest of this section $i : Z \to X$ is a closed immersion. The functor

$$Sch/X \longrightarrow Sch/Z, \quad U \longmapsto U_Z = Z \times_X U$$

will be denoted $U \mapsto U_Z$ as indicated. Since being a closed immersion is preserved under arbitrary base change the scheme U_Z is a closed subscheme of U.

04FV **Lemma 46.1.** *Let* $i : Z \to X$ *be a closed immersion of schemes. Let* U, U' *be schemes étale over* X. *Let* $h : U_Z \to U'_Z$ *be a morphism over* Z. *Then there exists a diagram*

$$U \xleftarrow{a} W \xrightarrow{b} U'$$

such that $a_Z : W_Z \to U_Z$ *is an isomorphism and* $h = b_Z \circ (a_Z)^{-1}$.

Proof. Consider the scheme $M = U \times_Y U'$. The graph $\Gamma_h \subset M_Z$ of h is open. This is true for example as Γ_h is the image of a section of the étale morphism $\text{pr}_{1,Z} : M_Z \to U_Z$, see Étale Morphisms, Proposition 6.1. Hence there exists an open subscheme $W \subset M$ whose intersection with the closed subset M_Z is Γ_h. Set $a = \text{pr}_1|_W$ and $b = \text{pr}_2|_W$. $\square$

04FW **Lemma 46.2.** *Let* $i : Z \to X$ *be a closed immersion of schemes. Let* $V \to Z$ *be an étale morphism of schemes. There exist étale morphisms* $U_i \to X$ *and morphisms* $U_{i,Z} \to V$ *such that* $\{U_{i,Z} \to V\}$ *is a Zariski covering of* V.

Proof. Since we only have to find a Zariski covering of V consisting of schemes of the form U_Z with U étale over X, we may Zariski localize on X and V. Hence we may assume X and V affine. In the affine case this is Algebra, Lemma 142.10. $\square$

If $\overline{x} : \text{Spec}(k) \to X$ is a geometric point of X, then either $\overline{x}$ factors (uniquely) through the closed subscheme Z, or $Z_{\overline{x}} = \emptyset$. If $\overline{x}$ factors through Z we say that $\overline{x}$ is a geometric point of Z (because it is) and we use the notation "$\overline{x} \in Z$" to indicate this.

04FX **Lemma 46.3.** *Let* $i : Z \to X$ *be a closed immersion of schemes. Let* $\mathcal{G}$ *be a sheaf of sets on* $Z_{étale}$. *Let* $\overline{x}$ *be a geometric point of* X. *Then*

$$(i_{small,*}\mathcal{G})_{\overline{x}} = \begin{cases} * & \text{if } \overline{x} \notin Z \\ \mathcal{G}_{\overline{x}} & \text{if } \overline{x} \in Z \end{cases}$$

where $*$ *denotes a singleton set.*

Proof. Note that $i_{small,*}\mathcal{G}|_{U_{étale}} = *$ is the final object in the category of étale sheaves on U, i.e., the sheaf which associates a singleton set to each scheme étale over U. This explains the value of $(i_{small,*}\mathcal{G})_{\overline{x}}$ if $\overline{x} \notin Z$.

Next, suppose that $\overline{x} \in Z$. Note that

$$(i_{small,*}\mathcal{G})_{\overline{x}} = \mathrm{colim}_{(U,\overline{u})}\,\mathcal{G}(U_Z)$$

and on the other hand

$$\mathcal{G}_{\overline{x}} = \mathrm{colim}_{(V,\overline{v})}\,\mathcal{G}(V).$$

Let $\mathcal{C}_1 = \{(U,\overline{u})\}^{opp}$ be the opposite of the category of étale neighbourhoods of $\overline{x}$ in X, and let $\mathcal{C}_2 = \{(V,\overline{v})\}^{opp}$ be the opposite of the category of étale neighbourhoods of $\overline{x}$ in Z. The canonical map

$$\mathcal{G}_{\overline{x}} \longrightarrow (i_{small,*}\mathcal{G})_{\overline{x}}$$

corresponds to the functor $F : \mathcal{C}_1 \to \mathcal{C}_2$, $F(U,\overline{u}) = (U_Z,\overline{x})$. Now Lemmas 46.2 and 46.1 imply that $\mathcal{C}_1$ is cofinal in $\mathcal{C}_2$, see Categories, Definition 17.1. Hence it follows that the displayed arrow is an isomorphism, see Categories, Lemma 17.2. □

04CA **Proposition 46.4.** *Let $i : Z \to X$ be a closed immersion of schemes.*

(1) *The functor*

$$i_{small,*} : Sh(Z_{étale}) \longrightarrow Sh(X_{étale})$$

is fully faithful and its essential image is those sheaves of sets $\mathcal{F}$ on $X_{étale}$ whose restriction to $X \setminus Z$ is isomorphic to $*$, and

(2) *the functor*

$$i_{small,*} : Ab(Z_{étale}) \longrightarrow Ab(X_{étale})$$

is fully faithful and its essential image is those abelian sheaves on $X_{étale}$ whose support is contained in Z.

In both cases i_{small}^{-1} is a left inverse to the functor $i_{small,*}$.

Proof. Let's discuss the case of sheaves of sets. For any sheaf $\mathcal{G}$ on Z the morphism $i_{small}^{-1}i_{small,*}\mathcal{G} \to \mathcal{G}$ is an isomorphism by Lemma 46.3 (and Theorem 29.10). This implies formally that $i_{small,*}$ is fully faithful, see Sites, Lemma 41.1. It is clear that $i_{small,*}\mathcal{G}|_{U_{étale}} \cong *$ where $U = X \setminus Z$. Conversely, suppose that $\mathcal{F}$ is a sheaf of sets on X such that $\mathcal{F}|_{U_{étale}} \cong *$. Consider the adjunction mapping

$$\mathcal{F} \longrightarrow i_{small,*}i_{small}^{-1}\mathcal{F}$$

Combining Lemmas 46.3 and 36.2 we see that it is an isomorphism. This finishes the proof of (1). The proof of (2) is identical. □

47. Integral universally injective morphisms

04FY Here is the general version of Proposition 46.4.

04FZ **Proposition 47.1.** *Let $f : X \to Y$ be a morphism of schemes which is integral and universally injective.*

(1) *The functor*

$$f_{small,*} : Sh(X_{étale}) \longrightarrow Sh(Y_{étale})$$

is fully faithful and its essential image is those sheaves of sets $\mathcal{F}$ on $Y_{étale}$ whose restriction to $Y \setminus f(X)$ is isomorphic to $*$, and

(2) *the functor*

$$f_{small,*} : Ab(X_{étale}) \longrightarrow Ab(Y_{étale})$$

is fully faithful and its essential image is those abelian sheaves on $Y_{étale}$ whose support is contained in $f(X)$.

In both cases f_{small}^{-1} is a left inverse to the functor $f_{small,*}$.

Proof. We may factor f as

$$X \xrightarrow{h} Z \xrightarrow{i} Y$$

where h is integral, universally injective and surjective and $i : Z \to Y$ is a closed immersion. Apply Proposition 46.4 to i and apply Theorem 45.2 to h. $\square$

48. Big sites and pushforward

04E2 In this section we prove some technical results on $f_{big,*}$ for certain types of morphisms of schemes.

04C7 **Lemma 48.1.** *Let $\tau \in \{Zariski, étale, smooth, syntomic, fppf\}$. Let $f : X \to Y$ be a monomorphism of schemes. Then the canonical map $f_{big}^{-1} f_{big,*}\mathcal{F} \to \mathcal{F}$ is an isomorphism for any sheaf $\mathcal{F}$ on $(Sch/X)_\tau$.*

Proof. In this case the functor $(Sch/X)_\tau \to (Sch/Y)_\tau$ is continuous, cocontinuous and fully faithful. Hence the result follows from Sites, Lemma 21.7. $\square$

04C8 **Remark 48.2.** In the situation of Lemma 48.1 it is true that the canonical map $\mathcal{F} \to f_{big}^{-1} f_{big!}\mathcal{F}$ is an isomorphism for any sheaf of sets $\mathcal{F}$ on $(Sch/X)_\tau$. The proof is the same. This also holds for sheaves of abelian groups. However, note that the functor $f_{big!}$ for sheaves of abelian groups is defined in Modules on Sites, Section 16 and is in general different from $f_{big!}$ on sheaves of sets. The result for sheaves of abelian groups follows from Modules on Sites, Lemma 16.4.

04E3 **Lemma 48.3.** *Let $f : X \to Y$ be a closed immersion of schemes. Let $U \to X$ be a syntomic (resp. smooth, resp. étale) morphism. Then there exist syntomic (resp. smooth, resp. étale) morphisms $V_i \to Y$ and morphisms $V_i \times_Y X \to U$ such that $\{V_i \times_Y X \to U\}$ is a Zariski covering of U.*

Proof. Let us prove the lemma when $\tau = syntomic$. The question is local on U. Thus we may assume that U is an affine scheme mapping into an affine of Y. Hence we reduce to proving the following case: $Y = \text{Spec}(A)$, $X = \text{Spec}(A/I)$, and $U = \text{Spec}(\overline{B})$, where $A/I \to \overline{B}$ be a syntomic ring map. By Algebra, Lemma 135.18 we can find elements $\overline{g}_i \in \overline{B}$ such that $\overline{B}_{\overline{g}_i} = A_i/IA_i$ for certain syntomic ring maps $A \to A_i$. This proves the lemma in the syntomic case. The proof of the smooth case is the same except it uses Algebra, Lemma 136.19. In the étale case use Algebra, Lemma 142.10. $\square$

04E4 **Lemma 48.4.** *Let $f : X \to Y$ be a closed immersion of schemes. Let $\{U_i \to X\}$ be a syntomic (resp. smooth, resp. étale) covering. There exists a syntomic (resp. smooth, resp. étale) covering $\{V_j \to Y\}$ such that for each j, either $V_j \times_Y X = \emptyset$, or the morphism $V_j \times_Y X \to X$ factors through U_i for some i.*

Proof. For each i we can choose syntomic (resp. smooth, resp. étale) morphisms $g_{ij} : V_{ij} \to Y$ and morphisms $V_{ij} \times_Y X \to U_i$ over X, such that $\{V_{ij} \times_Y X \to U_i\}$ are Zariski coverings, see Lemma 48.3. This in particular implies that $\bigcup_{ij} g_{ij}(V_{ij})$ contains the closed subset $f(X)$. Hence the family of syntomic (resp. smooth, resp. étale) maps g_{ij} together with the open immersion $Y \setminus f(X) \to Y$ forms the desired syntomic (resp. smooth, resp. étale) covering of Y. □

04C3 **Lemma 48.5.** *Let $f : X \to Y$ be a closed immersion of schemes. Let $\tau \in \{syntomic, smooth, étale\}$. The functor $V \mapsto X \times_Y V$ defines an almost cocontinuous functor (see Sites, Definition 42.3) $(Sch/Y)_\tau \to (Sch/X)_\tau$ between big τ sites.*

Proof. We have to show the following: given a morphism $V \to Y$ and any syntomic (resp. smooth, resp. étale) covering $\{U_i \to X \times_Y V\}$, there exists a smooth (resp. smooth, resp. étale) covering $\{V_j \to V\}$ such that for each j, either $X \times_Y V_j$ is empty, or $X \times_Y V_j \to Z \times_Y V$ factors through one of the U_i. This follows on applying Lemma 48.4 above to the closed immersion $X \times_Y V \to V$. □

04C4 **Lemma 48.6.** *Let $f : X \to Y$ be a closed immersion of schemes. Let $\tau \in \{syntomic, smooth, étale\}$.*

(1) *The pushforward $f_{big,*} : Sh((Sch/X)_\tau) \to Sh((Sch/Y)_\tau)$ commutes with coequalizers and pushouts.*
(2) *The pushforward $f_{big,*} : Ab((Sch/X)_\tau) \to Ab((Sch/Y)_\tau)$ is exact.*

Proof. This follows from Sites, Lemma 42.6, Modules on Sites, Lemma 15.3, and Lemma 48.5 above. □

04C5 **Remark 48.7.** In Lemma 48.6 the case $\tau = fppf$ is missing. The reason is that given a ring A, an ideal I and a faithfully flat, finitely presented ring map $A/I \to \overline{B}$, there is no reason to think that one can find *any* flat finitely presented ring map $A \to B$ with $B/IB \neq 0$ such that $A/I \to B/IB$ factors through $\overline{B}$. Hence the proof of Lemma 48.5 does not work for the fppf topology. In fact it is likely false that $f_{big,*} : Ab((Sch/X)_{fppf}) \to Ab((Sch/Y)_{fppf})$ is exact when f is a closed immersion. If you know an example, please email stacks.project@gmail.com.

49. Exactness of big lower shriek

04CB This is just the following technical result. Note that the functor $f_{big!}$ has nothing whatsoever to do with cohomology with compact support in general.

04CC **Lemma 49.1.** *Let $\tau \in \{Zariski, étale, smooth, syntomic, fppf\}$. Let $f : X \to Y$ be a morphism of schemes. Let*

$$f_{big} : Sh((Sch/X)_\tau) \longrightarrow Sh((Sch/Y)_\tau)$$

be the corresponding morphism of topoi as in Topologies, Lemma 3.15, 4.15, 5.10, 6.10, or 7.12.

(1) *The functor $f_{big}^{-1} : Ab((Sch/Y)_\tau) \to Ab((Sch/X)_\tau)$ has a left adjoint*

$$f_{big!} : Ab((Sch/X)_\tau) \to Ab((Sch/Y)_\tau)$$

which is exact.

(2) *The functor* $f_{big}^* : Mod((Sch/Y)_\tau, \mathcal{O}) \to Mod((Sch/X)_\tau, \mathcal{O})$ *has a left adjoint*
$$f_{big!} : Mod((Sch/X)_\tau, \mathcal{O}) \to Mod((Sch/Y)_\tau, \mathcal{O})$$
which is exact.

Moreover, the two functors $f_{big!}$ *agree on underlying sheaves of abelian groups.*

Proof. Recall that f_{big} is the morphism of topoi associated to the continuous and cocontinuous functor $u : (Sch/X)_\tau \to (Sch/Y)_\tau$, $U/X \mapsto U/Y$. Moreover, we have $f_{big}^{-1}\mathcal{O} = \mathcal{O}$. Hence the existence of $f_{big!}$ follows from Modules on Sites, Lemma 16.2, respectively Modules on Sites, Lemma 40.1. Note that if U is an object of $(Sch/X)_\tau$ then the functor u induces an equivalence of categories
$$u' : (Sch/X)_\tau/U \longrightarrow (Sch/Y)_\tau/U$$
because both sides of the arrow are equal to $(Sch/U)_\tau$. Hence the agreement of $f_{big!}$ on underlying abelian sheaves follows from the discussion in Modules on Sites, Remark 40.2. The exactness of $f_{big!}$ follows from Modules on Sites, Lemma 16.3 as the functor u above which commutes with fibre products and equalizers. $\square$

Next, we prove a technical lemma that will be useful later when comparing sheaves of modules on different sites associated to algebraic stacks.

07AJ **Lemma 49.2.** *Let X be a scheme. Let $\tau \in \{Zariski, étale, smooth, syntomic, fppf\}$. Let $\mathcal{C}_1 \subset \mathcal{C}_2 \subset (Sch/X)_\tau$ be full subcategories with the following properties:*

(1) *For an object U/X of $\mathcal{C}_t$,*
   (a) *if $\{U_i \to U\}$ is a covering of $(Sch/X)_\tau$, then U_i/X is an object of $\mathcal{C}_t$,*
   (b) *$U \times \mathbf{A}^1/X$ is an object of $\mathcal{C}_t$.*
(2) *X/X is an object of $\mathcal{C}_t$.*

We endow $\mathcal{C}_t$ with the structure of a site whose coverings are exactly those coverings $\{U_i \to U\}$ of $(Sch/X)_\tau$ with $U \in \mathrm{Ob}(\mathcal{C}_t)$. Then

(a) *The functor $\mathcal{C}_1 \to \mathcal{C}_2$ is fully faithful, continuous, and cocontinuous.*

Denote $g : Sh(\mathcal{C}_1) \to Sh(\mathcal{C}_2)$ the corresponding morphism of topoi. Denote $\mathcal{O}_t$ the restriction of $\mathcal{O}$ to $\mathcal{C}_t$. Denote $g_!$ the functor of Modules on Sites, Definition 16.1.

(b) *The canonical map $g_!\mathcal{O}_1 \to \mathcal{O}_2$ is an isomorphism.*

Proof. Assertion (a) is immediate from the definitions. In this proof all schemes are schemes over X and all morphisms of schemes are morphisms of schemes over X. Note that g^{-1} is given by restriction, so that for an object U of $\mathcal{C}_1$ we have $\mathcal{O}_1(U) = \mathcal{O}_2(U) = \mathcal{O}(U)$. Recall that $g_!\mathcal{O}_1$ is the sheaf associated to the presheaf $g_{p!}\mathcal{O}_1$ which associates to V in $\mathcal{C}_2$ the group
$$\mathrm{colim}_{V \to U} \mathcal{O}(U)$$
where U runs over the objects of $\mathcal{C}_1$ and the colimit is taken in the category of abelian groups. Below we will use frequently that if
$$V \to U \to U'$$
are morphisms with $U, U' \in \mathrm{Ob}(\mathcal{C}_1)$ and if $f' \in \mathcal{O}(U')$ restricts to $f \in \mathcal{O}(U)$, then $(V \to U, f)$ and $(V \to U', f')$ define the same element of the colimit. Also, $g_!\mathcal{O}_1 \to \mathcal{O}_2$ maps the element $(V \to U, f)$ simply to the pullback of f to V.

Surjectivity. Let V be a scheme and let $h \in \mathcal{O}(V)$. Then we obtain a morphism $V \to X \times \mathbf{A}^1$ induced by h and the structure morphism $V \to X$. Writing $\mathbf{A}^1 =$

Spec($\mathbf{Z}[x]$) we see the element $x \in \mathcal{O}(X \times \mathbf{A}^1)$ pulls back to h. Since $X \times \mathbf{A}^1$ is an object of $\mathcal{C}_1$ by assumptions (1)(b) and (2) we obtain the desired surjectivity.

Injectivity. Let V be a scheme. Let $s = \sum_{i=1,\ldots,n}(V \to U_i, f_i)$ be an element of the colimit displayed above. For any i we can use the morphism $f_i : U_i \to X \times \mathbf{A}^1$ to see that $(V \to U_i, f_i)$ defines the same element of the colimit as $(f_i : V \to X \times \mathbf{A}^1, x)$. Then we can consider

$$f_1 \times \ldots \times f_n : V \to X \times \mathbf{A}^n$$

and we see that s is equivalent in the colimit to

$$\sum\nolimits_{i=1,\ldots,n}(f_1 \times \ldots \times f_n : V \to X \times \mathbf{A}^n, x_i) = (f_1 \times \ldots \times f_n : V \to X \times \mathbf{A}^n, x_1 + \ldots + x_n)$$

Now, if $x_1 + \ldots + x_n$ restricts to zero on V, then we see that $f_1 \times \ldots \times f_n$ factors through $X \times \mathbf{A}^{n-1} = V(x_1 + \ldots + x_n)$. Hence we see that s is equivalent to zero in the colimit. $\square$

50. Étale cohomology

03Q3 In the following sections we prove some basic results on étale cohomology. Here is an example of something we know for cohomology of topological spaces which also holds for étale cohomology.

0A50 **Lemma 50.1** (Mayer-Vietoris for étale cohomology)**.** *Let X be a scheme. Suppose that $X = U \cup V$ is a union of two opens. For any abelian sheaf $\mathcal{F}$ on $X_{étale}$ there exists a long exact cohomology sequence*

$$\begin{aligned} 0 \to H^0_{étale}(X, \mathcal{F}) \to H^0_{étale}(U, \mathcal{F}) \oplus H^0_{étale}(V, \mathcal{F}) \to H^0_{étale}(U \cap V, \mathcal{F}) \\ \to H^1_{étale}(X, \mathcal{F}) \to H^1_{étale}(U, \mathcal{F}) \oplus H^1_{étale}(V, \mathcal{F}) \to H^1_{étale}(U \cap V, \mathcal{F}) \to \ldots \end{aligned}$$

This long exact sequence is functorial in $\mathcal{F}$.

Proof. Observe that if $\mathcal{I}$ is an injective abelian sheaf, then

$$0 \to \mathcal{I}(X) \to \mathcal{I}(U) \oplus \mathcal{I}(V) \to \mathcal{I}(U \cap V) \to 0$$

is exact. This is true in the first and middle spots as $\mathcal{I}$ is a sheaf. It is true on the right, because $\mathcal{I}(U) \to \mathcal{I}(U \cap V)$ is surjective by Cohomology on Sites, Lemma 12.6. Another way to prove it would be to show that the cokernel of the map $\mathcal{I}(U) \oplus \mathcal{I}(V) \to \mathcal{I}(U \cap V)$ is the first Čech cohomology group of $\mathcal{I}$ with respect to the covering $X = U \cup V$ which vanishes by Lemmas 18.7 and 19.1. Thus, if $\mathcal{F} \to \mathcal{I}^\bullet$ is an injective resolution, then

$$0 \to \mathcal{I}^\bullet(X) \to \mathcal{I}^\bullet(U) \oplus \mathcal{I}^\bullet(V) \to \mathcal{I}^\bullet(U \cap V) \to 0$$

is a short exact sequence of complexes and the associated long exact cohomology sequence is the sequence of the statement of the lemma. $\square$

0EYK **Lemma 50.2** (Relative Mayer-Vietoris)**.** *Let $f : X \to Y$ be a morphism of schemes. Suppose that $X = U \cup V$ is a union of two open subschemes. Denote $a = f|_U : U \to Y$, $b = f|_V : V \to Y$, and $c = f|_{U \cap V} : U \cap V \to Y$. For every abelian sheaf $\mathcal{F}$ on $X_{étale}$ there exists a long exact sequence*

$$0 \to f_*\mathcal{F} \to a_*(\mathcal{F}|_U) \oplus b_*(\mathcal{F}|_V) \to c_*(\mathcal{F}|_{U \cap V}) \to R^1 f_*\mathcal{F} \to \ldots$$

on $Y_{étale}$. This long exact sequence is functorial in $\mathcal{F}$.

Proof. Let $\mathcal{F} \to \mathcal{I}^\bullet$ be an injective resolution of $\mathcal{F}$ on $X_{\acute{e}tale}$. We claim that we get a short exact sequence of complexes

$$0 \to f_*\mathcal{I}^\bullet \to a_*\mathcal{I}^\bullet|_U \oplus b_*\mathcal{I}^\bullet|_V \to c_*\mathcal{I}^\bullet|_{U\cap V} \to 0.$$

Namely, for any W in $Y_{\acute{e}tale}$, and for any $n \geq 0$ the corresponding sequence of groups of sections over W

$$0 \to \mathcal{I}^n(W \times_Y X) \to \mathcal{I}^n(W \times_Y U) \oplus \mathcal{I}^n(W \times_Y V) \to \mathcal{I}^n(W \times_Y (U \cap V)) \to 0$$

was shown to be short exact in the proof of Lemma 50.1. The lemma follows by taking cohomology sheaves and using the fact that $\mathcal{I}^\bullet|_U$ is an injective resolution of $\mathcal{F}|_U$ and similarly for $\mathcal{I}^\bullet|_V$, $\mathcal{I}^\bullet|_{U\cap V}$. □

51. Colimits

03Q4 We recall that if $(\mathcal{F}_i, \varphi_{ii'})$ is a diagram of sheaves on a site $\mathcal{C}$ its colimit (in the category of sheaves) is the sheafification of the presheaf $U \mapsto \mathrm{colim}_i\, \mathcal{F}_i(U)$. See Sites, Lemma 10.13. If the system is directed, U is a quasi-compact object of $\mathcal{C}$ which has a cofinal system of coverings by quasi-compact objects, then $\mathcal{F}(U) = \mathrm{colim}\, \mathcal{F}_i(U)$, see Sites, Lemma 17.5. See Cohomology on Sites, Lemma 16.1 for a result dealing with higher cohomology groups of colimits of abelian sheaves.

In Cohomology on Sites, Lemma 16.3 we generalize this result to a system of sheaves on an inverse system of sites. Here is the corresponding notion in the case of a system of étale sheaves living on an inverse system of schemes.

0EZL **Definition 51.1.** Let I be a preordered set. Let $(X_i, f_{i'i})$ be an inverse system of schemes over I. A *system* $(\mathcal{F}_i, \varphi_{i'i})$ *of sheaves on* $(X_i, f_{i'i})$ is given by

(1) a sheaf $\mathcal{F}_i$ on $(X_i)_{\acute{e}tale}$ for all $i \in I$,
(2) for $i' \geq i$ a map $\varphi_{i'i} : f_{i'i}^{-1}\mathcal{F}_i \to \mathcal{F}_{i'}$ of sheaves on $(X_{i'})_{\acute{e}tale}$

such that $\varphi_{i''i} = \varphi_{i''i'} \circ f_{i''i'}^{-1}\varphi_{i'i}$ whenever $i'' \geq i' \geq i$.

In the situation of Definition 51.1, assume I is a directed set and the transition morphisms $f_{i'i}$ affine. Let $X = \lim X_i$ be the limit in the category of schemes, see Limits, Section 2. Denote $f_i : X \to X_i$ the projection morphisms and consider the maps

$$f_i^{-1}\mathcal{F}_i = f_{i'}^{-1}f_{i'i}^{-1}\mathcal{F}_i \xrightarrow{f_{i'}^{-1}\varphi_{i'i}} f_{i'}^{-1}\mathcal{F}_{i'}$$

This turns $f_i^{-1}\mathcal{F}_i$ into a system of sheaves on $X_{\acute{e}tale}$ over I (it is a good exercise to check this). We often want to know whether there is an isomorphism

$$H^q_{\acute{e}tale}(X, \mathrm{colim}\, f_i^{-1}\mathcal{F}_i) = \mathrm{colim}\, H^q_{\acute{e}tale}(X_i, \mathcal{F}_i)$$

It will turn out this is true if X_i is quasi-compact and quasi-separated for all i, see Theorem 51.3.

0EYL **Lemma 51.2.** *Let I be a directed set. Let $(X_i, f_{i'i})$ be an inverse system of schemes over I with affine transition morphisms. Let $X = \lim_{i\in I} X_i$. With notation as in Lemma 21.2 we have*

$$X_{affine,\acute{e}tale} = \mathrm{colim} (X_i)_{affine,\acute{e}tale}$$

as sites in the sense of Sites, Lemma 18.2.

Proof. Let us first prove this when X and X_i are quasi-compact and quasi-separated for all i (as this is true in all cases of interest). In this case any object of $X_{affine,\acute{e}tale}$, resp. $(X_i)_{affine,\acute{e}tale}$ is of finite presentation over X. Moreover, the category of schemes of finite presentation over X is the colimit of the categories of schemes of finite presentation over X_i, see Limits, Lemma 10.1. The same holds for the subcategories of affine objects étale over X by Limits, Lemmas 4.13 and 8.10. Finally, if $\{U^j \to U\}$ is a covering of $X_{affine,\acute{e}tale}$ and if $U_i^j \to U_i$ is morphism of affine schemes étale over X_i whose base change to X is $U^j \to U$, then we see that the base change of $\{U_i^j \to U_i\}$ to some $X_{i'}$ is a covering for i' large enough, see Limits, Lemma 8.14.

In the general case, let U be an object of $X_{affine,\acute{e}tale}$. Then $U \to X$ is étale and separated (as U is separated) but in general not quasi-compact. Still, $U \to X$ is locally of finite presentation and hence by Limits, Lemma 10.5 there exists an i, a quasi-compact and quasi-separated scheme U_i, and a morphism $U_i \to X_i$ which is locally of finite presentation whose base change to X is $U \to X$. Then $U = \lim_{i' \geq i} U_{i'}$ where $U_{i'} = U_i \times_{X_i} X_{i'}$. After increasing i we may assume U_i is affine, see Limits, Lemma 4.13. To check that $U_i \to X_i$ is étale for i sufficiently large, choose a finite affine open covering $U_i = U_{i,1} \cup \ldots \cup U_{i,m}$ such that $U_{i,j} \to U_i \to X_i$ maps into an affine open $W_{i,j} \subset X_i$. Then we can apply Limits, Lemma 8.10 to see that $U_{i,j} \to W_{i,j}$ is étale after possibly increasing i. In this way we see that the functor $\text{colim}(X_i)_{affine,\acute{e}tale} \to X_{affine,\acute{e}tale}$ is essentially surjective. Fully faithfulness follows directly from the already used Limits, Lemma 10.5. The statement on coverings is proved in exactly the same manner as done in the first paragraph of the proof. $\square$

Using the above we get the following general result on colimits and cohomology.

09YQ **Theorem 51.3.** *Let $X = \lim_{i \in I} X_i$ be a limit of a directed system of schemes with affine transition morphisms $f_{i'i} : X_{i'} \to X_i$. We assume that X_i is quasi-compact and quasi-separated for all $i \in I$. Let $(\mathcal{F}_i, \varphi_{i'i})$ be a system of abelian sheaves on $(X_i, f_{i'i})$. Denote $f_i : X \to X_i$ the projection and set $\mathcal{F} = \text{colim}\, f_i^{-1}\mathcal{F}_i$. Then*

$$\text{colim}_{i \in I} H^p_{\acute{e}tale}(X_i, \mathcal{F}_i) = H^p_{\acute{e}tale}(X, \mathcal{F}).$$

for all $p \geq 0$.

Proof. By Lemma 21.2 we can compute the cohomology of $\mathcal{F}$ on $X_{affine,\acute{e}tale}$. Thus the result by a combination of Lemma 51.2 and Cohomology on Sites, Lemma 16.3. $\square$

The following two results are special cases of the theorem above.

03Q5 **Lemma 51.4.** *Let X be a quasi-compact and quasi-separated scheme. Let I be a directed set. Let $(\mathcal{F}_i, \varphi_{ij})$ be a system of abelian sheaves on $X_{\acute{e}tale}$ over I. Then*

$$\text{colim}_{i \in I} H^p_{\acute{e}tale}(X, \mathcal{F}_i) = H^p_{\acute{e}tale}(X, \text{colim}_{i \in I} \mathcal{F}_i).$$

Proof. This is a special case of Theorem 51.3. We also sketch a direct proof. We prove it for all X at the same time, by induction on p.

(1) For any quasi-compact and quasi-separated scheme X and any étale covering $\mathcal{U}$ of X, show that there exists a refinement $\mathcal{V} = \{V_j \to X\}_{j \in J}$ with J finite and each V_j quasi-compact and quasi-separated such that all $V_{j_0} \times_X \ldots \times_X V_{j_p}$ are also quasi-compact and quasi-separated.

(2) Using the previous step and the definition of colimits in the category of sheaves, show that the theorem holds for $p = 0$ and all X.
(3) Using the locality of cohomology (Lemma 22.3), the Čech-to-cohomology spectral sequence (Theorem 19.2) and the fact that the induction hypothesis applies to all $V_{j_0} \times_X \ldots \times_X V_{j_p}$ in the above situation, prove the induction step $p \to p + 1$.

□

03Q6 **Lemma 51.5.** *Let A be a ring, $(I, \leq)$ a directed set and (B_i, φ_{ij}) a system of A-algebras. Set $B = \text{colim}_{i \in I} B_i$. Let $X \to \text{Spec}(A)$ be a quasi-compact and quasi-separated morphism of schemes. Let $\mathcal{F}$ an abelian sheaf on $X_{\acute{e}tale}$. Denote $Y_i = X \times_{\text{Spec}(A)} \text{Spec}(B_i)$, $Y = X \times_{\text{Spec}(A)} \text{Spec}(B)$, $\mathcal{G}_i = (Y_i \to X)^{-1}\mathcal{F}$ and $\mathcal{G} = (Y \to X)^{-1}\mathcal{F}$. Then*

$$H^p_{\acute{e}tale}(Y, \mathcal{G}) = \text{colim}_{i \in I} H^p_{\acute{e}tale}(Y_i, \mathcal{G}_i).$$

Proof. This is a special case of Theorem 51.3. We also outline a direct proof as follows.

(1) Given $V \to Y$ étale with V quasi-compact and quasi-separated, there exist $i \in I$ and $V_i \to Y_i$ such that $V = V_i \times_{Y_i} Y$. If all the schemes considered were affine, this would correspond to the following algebra statement: if $B = \text{colim}\, B_i$ and $B \to C$ is étale, then there exist $i \in I$ and $B_i \to C_i$ étale such that $C \cong B \otimes_{B_i} C_i$. This is proved in Algebra, Lemma 142.3.
(2) In the situation of (1) show that $\mathcal{G}(V) = \text{colim}_{i' \geq i} \mathcal{G}_{i'}(V_{i'})$ where $V_{i'}$ is the base change of V_i to $Y_{i'}$.
(3) By (1), we see that for every étale covering $\mathcal{V} = \{V_j \to Y\}_{j \in J}$ with J finite and the V_js quasi-compact and quasi-separated, there exists $i \in I$ and an étale covering $\mathcal{V}_i = \{V_{ij} \to Y_i\}_{j \in J}$ such that $\mathcal{V} \cong \mathcal{V}_i \times_{Y_i} Y$.
(4) Show that (2) and (3) imply

$$\check{H}^*(\mathcal{V}, \mathcal{G}) = \text{colim}_{i \in I} \check{H}^*(\mathcal{V}_i, \mathcal{G}_i).$$

(5) Cleverly use the Čech-to-cohomology spectral sequence (Theorem 19.2).

□

03Q8 **Lemma 51.6.** *Let $f : X \to Y$ be a morphism of schemes and $\mathcal{F} \in Ab(X_{\acute{e}tale})$. Then $R^p f_*\mathcal{F}$ is the sheaf associated to the presheaf*

$$(V \to Y) \longmapsto H^p_{\acute{e}tale}(X \times_Y V, \mathcal{F}|_{X \times_Y V}).$$

Proof. This lemma is valid for topological spaces, and the proof in this case is the same. See Cohomology on Sites, Lemma 7.4 for details. □

09Z1 **Lemma 51.7.** *Let S be a scheme. Let $X = \lim_{i \in I} X_i$ be a limit of a directed system of schemes over S with affine transition morphisms $f_{i'i} : X_{i'} \to X_i$. We assume the structure morphisms $g_i : X_i \to S$ and $g : X \to S$ are quasi-compact and quasi-separated. Let $(\mathcal{F}_i, \varphi_{i'i})$ be a system of abelian sheaves on $(X_i, f_{i'i})$. Denote $f_i : X \to X_i$ the projection and set $\mathcal{F} = \text{colim}\, f_i^{-1}\mathcal{F}_i$. Then*

$$\text{colim}_{i \in I} R^p g_{i,*}\mathcal{F}_i = R^p g_* \mathcal{F}$$

for all $p \geq 0$.

Proof. Recall (Lemma 51.6) that $R^p g_{i,*}\mathcal{F}_i$ is the sheaf associated to the presheaf $U \mapsto H^p_{étale}(U \times_S X_i, \mathcal{F}_i)$ and similarly for $R^p g_*\mathcal{F}$. Moreover, the colimit of a system of sheaves is the sheafification of the colimit on the level of presheaves. Note that every object of $S_{étale}$ has a covering by quasi-compact and quasi-separated objects (e.g., affine schemes). Moreover, if U is a quasi-compact and quasi-separated object, then we have

$$\operatorname{colim} H^p_{étale}(U \times_S X_i, \mathcal{F}_i) = H^p_{étale}(U \times_S X, \mathcal{F})$$

by Theorem 51.3. Thus the lemma follows. □

0EYM **Lemma 51.8.** *Let I be a directed set. Let $g_i : X_i \to S_i$ be an inverse system of morphisms of schemes over I. Assume g_i is quasi-compact and quasi-separated and for $i' \geq i$ the transition morphisms $f_{i'i} : X_{i'} \to X_i$ and $h_{i'i} : S_{i'} \to S_i$ are affine. Let $g : X \to S$ be the limit of the morphisms g_i, see Limits, Section 2. Denote $f_i : X \to X_i$ and $h_i : S \to S_i$ the projections. Let $(\mathcal{F}_i, \varphi_{i'i})$ be a system of sheaves on $(X_i, f_{i'i})$. Set $\mathcal{F} = \operatorname{colim} f_i^{-1}\mathcal{F}_i$. Then*

$$R^p g_*\mathcal{F} = \operatorname{colim}_{i \in I} h_i^{-1} R^p g_{i,*}\mathcal{F}_i$$

for all $p \geq 0$.

Proof. How is the map of the lemma constructed? For $i' \geq i$ we have a commutative diagram

$$\begin{tikzcd}
X \arrow[r, "f_{i'}"'] \arrow[d, "g"'] & X_{i'} \arrow[r, "f_{i'i}"'] \arrow[d, "g_{i'}"'] & X_i \arrow[d, "g_i"] \\
S \arrow[r, "h_{i'}"] & S_{i'} \arrow[r, "h_{i'i}"] & S_i
\end{tikzcd}$$

If we combine the base change map $h_{i'i}^{-1} Rg_{i,*}\mathcal{F}_i \to Rg_{i',*}f_{i'i}^{-1}\mathcal{F}_i$ (Cohomology on Sites, Lemma 15.1 or Remark 19.3) with the map $Rg_{i',*}\varphi_{i'i}$, then we obtain $\psi_{i'i} : h_{i'i}^{-1} R^p g_{i,*}\mathcal{F}_i \to R^p g_{i',*}\mathcal{F}_{i'}$. Similarly, using the left square in the diagram we obtain maps $\psi_i : h_i^{-1} R^p g_{i,*}\mathcal{F}_i \to R^p g_*\mathcal{F}$. The maps $h_{i'}^{-1}\psi_{i'i}$ and ψ_i are the maps used in the statement of the lemma. For this to make sense, we have to check that $\psi_{i''i} = \psi_{i''i'} \circ h_{i''i'}^{-1}\psi_{i'i}$ and $\psi_{i'} \circ h_{i'}^{-1}\psi_{i'i} = \psi_i$; this follows from Cohomology on Sites, Remark 19.5.

Proof of the equality. First proof using dimension shifting[4]. For any U affine and étale over X by Theorem 51.3 we have

$$g_*\mathcal{F}(U) = H^0(U \times_S X, \mathcal{F}) = \operatorname{colim} H^0(U_i \times_{S_i} X_i, \mathcal{F}_i) = \operatorname{colim} g_{i,*}\mathcal{F}_i(U_i)$$

where the colimit is over i large enough such that there exists an i and U_i affine étale over S_i whose base change is U over S (see Lemma 51.2). The right hand side is equal to $(\operatorname{colim} h_i^{-1} g_{i,*}\mathcal{F}_i)(U)$ by Sites, Lemma 18.4. This proves the lemma for $p = 0$. If $(\mathcal{G}_i, \varphi_{i'i})$ is a system with $\mathcal{G} = \operatorname{colim} f_i^{-1}\mathcal{G}_i$ such that $\mathcal{G}_i$ is an injective abelian sheaf on X_i for all i, then for any U affine and étale over X by Theorem 51.3 we have

$$H^p(U \times_S X, \mathcal{G}) = \operatorname{colim} H^p(U_i \times_{S_i} X_i, \mathcal{G}_i) = 0$$

for $p > 0$ (same colimit as before). Hence $R^p g_*\mathcal{G} = 0$ and we get the result for $p > 0$ for such a system. In general we may choose a short exact sequence of systems

$$0 \to (\mathcal{F}_i, \varphi_{i'i}) \to (\mathcal{G}_i, \varphi_{i'i}) \to (\mathcal{Q}_i, \varphi_{i'i}) \to 0$$

[4]You can also use this method to produce the maps in the lemma.

where $(\mathcal{G}_i, \varphi_{i'i})$ is as above, see Cohomology on Sites, Lemma 16.2. By induction the lemma holds for $p-1$ and by the above we have vanishing for p and $(\mathcal{G}_i, \varphi_{i'i})$. Hence the result for p and $(\mathcal{F}_i, \varphi_{i'i})$ by the long exact sequence of cohomology.

Second proof. Recall that $S_{affine,étale} = \text{colim}(S_i)_{affine,étale}$, see Lemma 51.2. Thus if U is an object of $S_{affine,étale}$, then we can write $U = U_i \times_{S_i} S$ for some i and some U_i in $(S_i)_{affine,étale}$ and

$$(\text{colim}_{i \in I} h_i^{-1} R^p g_{i,*}\mathcal{F}_i)(U) = \text{colim}_{i' \geq i}(R^p g_{i',*}\mathcal{F}_{i'})(U_i \times_{S_i} S_{i'})$$

by Sites, Lemma 18.4 and the construction of the transition maps in the system described above. Since $R^p g_{i',*}\mathcal{F}_{i'}$ is the sheaf associated to the presheaf $U_{i'} \mapsto H^p(U_{i'} \times_{S_{i'}} X_{i'}, \mathcal{F}_{i'})$ and since $R^p g_* \mathcal{F}$ is the sheaf associated to the presheaf $U \mapsto H^p(U \times_S X, \mathcal{F})$ (Lemma 51.6) we obtain a canonical commutative diagram

$$\begin{tikzcd}
\text{colim}_{i' \geq i} H^p(U_i \times_{S_i} X_{i'}, \mathcal{F}_{i'}) \arrow[r] \arrow[d] & \text{colim}_{i' \geq i}(R^p g_{i',*}\mathcal{F}_{i'})(U_i \times_{S_i} S_{i'}) \arrow[d] \\
H^p(U \times_S X, \mathcal{F}) \arrow[r] & R^p g_* \mathcal{F}(U)
\end{tikzcd}$$

Observe that the left hand vertical arrow is an isomorphism by Theorem 51.3. We're trying to show that the right hand vertical arrow is an isomorphism. However, we already know that the source and target of this arrow are sheaves on $S_{affine,étale}$. Hence it suffices to show: (1) an element in the target, locally comes from an element in the source and (2) an element in the source which maps to zero in the target locally vanishes. Part (1) follows immediately from the above and the fact that the lower horizontal arrow comes from a map of presheaves which becomes an isomorphism after sheafification. For part (2), say $\xi \in \text{colim}_{i' \geq i}(R^p g_{i',*}\mathcal{F}_{i'})(U_i \times_{S_i} S_{i'})$ is in the kernel. Choose an $i' \geq i$ and $\xi_{i'} \in (R^p g_{i',*}\mathcal{F}_{i'})(U_i \times_{S_i} S_{i'})$ representing ξ. Choose a standard étale covering $\{U_{i',k} \to U_i \times_{S_i} S_{i'}\}_{k=1,\ldots,m}$ such that $\xi_{i'}|_{U_{i',k}}$ comes from $\xi_{i',k} \in H^p(U_{i',k} \times_{S_{i'}} X_{i'}, \mathcal{F}_{i'})$. Since it is enough to prove that ξ dies locally, we may replace U by the members of the étale covering $\{U_{i',k} \times_{S_{i'}} S \to U = U_i \times_{S_i} S\}$. After this replacement we see that ξ is the image of an element ξ' of the group $\text{colim}_{i' \geq i} H^p(U_i \times_{S_i} X_{i'}, \mathcal{F}_{i'})$ in the diagram above. Since ξ' maps to zero in $R^p g_* \mathcal{F}(U)$ we can do another replacement and assume that ξ' maps to zero in $H^p(U \times_S X, \mathcal{F})$. However, since the left vertical arrow is an isomorphism we then conclude $\xi' = 0$ hence $\xi = 0$ as desired. □

0EYN **Lemma 51.9.** *Let $X = \lim_{i \in I} X_i$ be a directed limit of schemes with affine transition morphisms $f_{i'i}$ and projection morphisms $f_i : X \to X_i$. Let $\mathcal{F}$ be a sheaf on $X_{étale}$. Then*

1. *there are canonical maps $\varphi_{i'i} : f_{i'i}^{-1} f_{i,*}\mathcal{F} \to f_{i',*}\mathcal{F}$ such that $(f_{i,*}\mathcal{F}, \varphi_{i'i})$ is a system of sheaves on $(X_i, f_{i'i})$ as in Definition 51.1, and*
2. *$\mathcal{F} = \text{colim}\, f_i^{-1} f_{i,*}\mathcal{F}$.*

Proof. Via Lemmas 21.2 and 51.2 this is a special case of Sites, Lemma 18.5. □

0DV2 **Lemma 51.10.** *Let I be a directed set. Let $g_i : X_i \to S_i$ be an inverse system of morphisms of schemes over I. Assume g_i is quasi-compact and quasi-separated and for $i' \geq i$ the transition morphisms $X_{i'} \to X_i$ and $S_{i'} \to S_i$ are affine. Let $g : X \to$*

S be the limit of the morphisms g_i, see Limits, Section 2. Denote $f_i : X \to X_i$ and $h_i : S \to S_i$ the projections. Let $\mathcal{F}$ be an abelian sheaf on X. Then we have

$$R^p g_* \mathcal{F} = \mathrm{colim}_{i \in I}\, h_i^{-1} R^p g_{i,*}(f_{i,*}\mathcal{F})$$

Proof. Formal combination of Lemmas 51.8 and 51.9. □

0EZM **Remark 51.11.** Many of the results above have variants for bounded below complexes, but one has to be careful that the bounds have to be uniform. We explain this in the simplest case. Let X be a quasi-compact and quasi-separated scheme. Let I be a directed set. Let $\mathcal{F}_i^\bullet$ be a system over I of complexes of sheaves on $X_{\acute{e}tale}$. Assume there is an integer a such that $\mathcal{F}_i^n = 0$ for $n < a$ and all $i \in I$. Then we have

$$H^p_{\acute{e}tale}(X, \mathrm{colim}\, \mathcal{F}_i^\bullet) = \mathrm{colim}\, H^p_{\acute{e}tale}(X, \mathcal{F}_i^\bullet)$$

If we ever need this we will state a precise lemma with full proof here.

52. Stalks of higher direct images

03Q7

03Q9 **Theorem 52.1.** *Let $f : X \to S$ be a quasi-compact and quasi-separated morphism of schemes, $\mathcal{F}$ an abelian sheaf on $X_{\acute{e}tale}$, and $\overline{s}$ a geometric point of S lying over $s \in S$. Then*

$$(R^n f_* \mathcal{F})_{\overline{s}} = H^n_{\acute{e}tale}(X \times_S \mathrm{Spec}(\mathcal{O}^{sh}_{S,s}), p^{-1}\mathcal{F})$$

where $p : X \times_S \mathrm{Spec}(\mathcal{O}^{sh}_{S,s}) \to X$ is the projection.

Proof. Let $\mathcal{I}$ be the category of étale neighborhoods of $\overline{s}$ on S. By Lemma 51.6 we have

$$(R^n f_* \mathcal{F})_{\overline{s}} = \mathrm{colim}_{(V, \overline{v}) \in \mathcal{I}^{opp}}\, H^n_{\acute{e}tale}(X \times_S V, \mathcal{F}|_{X \times_S V}).$$

We may replace $\mathcal{I}$ by the initial subcategory consisting of affine étale neighbourhoods of $\overline{s}$. Observe that

$$\mathrm{Spec}(\mathcal{O}^{sh}_{S,s}) = \lim_{(V, \overline{v}) \in \mathcal{I}} V$$

by Lemma 33.1 and Limits, Lemma 2.1. Since fibre products commute with limits we also obtain

$$X \times_S \mathrm{Spec}(\mathcal{O}^{sh}_{S,s}) = \lim_{(V, \overline{v}) \in \mathcal{I}} X \times_S V$$

We conclude by Lemma 51.5. □

53. The Leray spectral sequence

03QA

03QB **Lemma 53.1.** *Let $f : X \to Y$ be a morphism and $\mathcal{I}$ an injective object of $Ab(X_{\acute{e}tale})$. Let $V \in \mathrm{Ob}(Y_{\acute{e}tale})$. Then*

1. *for any covering $\mathcal{V} = \{V_j \to V\}_{j \in J}$ we have $\check{H}^p(\mathcal{V}, f_*\mathcal{I}) = 0$ for all $p > 0$,*
2. *$f_*\mathcal{I}$ is acyclic for the functor $\Gamma(V, -)$, and*
3. *if $g : Y \to Z$, then $f_*\mathcal{I}$ is acyclic for g_*.*

Proof. Observe that $\check{\mathcal{C}}^\bullet(\mathcal{V}, f_*\mathcal{I}) = \check{\mathcal{C}}^\bullet(\mathcal{V} \times_Y X, \mathcal{I})$ which has vanishing higher cohomology groups by Lemma 18.7. This proves (1). The second statement follows as a sheaf which has vanishing higher Čech cohomology groups for any covering has vanishing higher cohomology groups. This a wonderful exercise in using the Čech-to-cohomology spectral sequence, but see Cohomology on Sites, Lemma 10.9

for details and a more precise and general statement. Part (3) is a consequence of (2) and the description of $R^p g_*$ in Lemma 51.6. ☐

Using the formalism of Grothendieck spectral sequences, this gives the following.

03QC **Proposition 53.2** (Leray spectral sequence)**.** *Let $f : X \to Y$ be a morphism of schemes and $\mathcal{F}$ an étale sheaf on X. Then there is a spectral sequence*

$$E_2^{p,q} = H^p_{\acute{e}tale}(Y, R^q f_* \mathcal{F}) \Rightarrow H^{p+q}_{\acute{e}tale}(X, \mathcal{F}).$$

Proof. See Lemma 53.1 and see Derived Categories, Section 22. ☐

54. Vanishing of finite higher direct images

03QN The next goal is to prove that the higher direct images of a finite morphism of schemes vanish.

03QO **Lemma 54.1.** *Let R be a strictly henselian local ring. Set $S = \mathrm{Spec}(R)$ and let $\overline{s}$ be its closed point. Then the global sections functor $\Gamma(S, -) : Ab(S_{\acute{e}tale}) \to Ab$ is exact. In fact we have $\Gamma(S, \mathcal{F}) = \mathcal{F}_{\overline{s}}$ for any sheaf of sets $\mathcal{F}$. In particular*

$$\forall p \geq 1, \quad H^p_{\acute{e}tale}(S, \mathcal{F}) = 0$$

for all $\mathcal{F} \in Ab(S_{\acute{e}tale})$.

Proof. If we show that $\Gamma(S, \mathcal{F}) = \mathcal{F}_{\overline{s}}$ then $\Gamma(S, -)$ is exact as the stalk functor is exact. Let $(U, \overline{u})$ be an étale neighbourhood of $\overline{s}$. Pick an affine open neighborhood $\mathrm{Spec}(A)$ of $\overline{u}$ in U. Then $R \to A$ is étale and $\kappa(\overline{s}) = \kappa(\overline{u})$. By Theorem 32.4 we see that $A \cong R \times A'$ as an R-algebra compatible with maps to $\kappa(\overline{s}) = \kappa(\overline{u})$. Hence we get a section

$$\begin{tikzcd} \mathrm{Spec}(A) \arrow[r] & U \arrow[d] \\ & S \arrow[ul] \end{tikzcd}$$

It follows that in the system of étale neighbourhoods of $\overline{s}$ the identity map $(S, \overline{s}) \to (S, \overline{s})$ is cofinal. Hence $\Gamma(S, \mathcal{F}) = \mathcal{F}_{\overline{s}}$. The final statement of the lemma follows as the higher derived functors of an exact functor are zero, see Derived Categories, Lemma 16.9. ☐

03QP **Proposition 54.2.** *Let $f : X \to Y$ be a finite morphism of schemes.*

(1) *For any geometric point $\overline{y} : \mathrm{Spec}(k) \to Y$ we have*

$$(f_*\mathcal{F})_{\overline{y}} = \prod\nolimits_{\overline{x}:\mathrm{Spec}(k)\to X,\ f(\overline{x})=\overline{y}} \mathcal{F}_{\overline{x}}.$$

for $\mathcal{F}$ in $Sh(X_{\acute{e}tale})$ and

$$(f_*\mathcal{F})_{\overline{y}} = \bigoplus\nolimits_{\overline{x}:\mathrm{Spec}(k)\to X,\ f(\overline{x})=\overline{y}} \mathcal{F}_{\overline{x}}.$$

for $\mathcal{F}$ in $Ab(X_{\acute{e}tale})$.

(2) *For any $q \geq 1$ we have $R^q f_* \mathcal{F} = 0$.*

Proof. Let $X^{sh}_{\overline{y}}$ denote the fiber product $X \times_Y \mathrm{Spec}(\mathcal{O}^{sh}_{Y,\overline{y}})$. By Theorem 52.1 the stalk of $R^q f_* \mathcal{F}$ at $\overline{y}$ is computed by $H^q_{\acute{e}tale}(X^{sh}_{\overline{y}}, \mathcal{F})$. Since f is finite, $X^{sh}_{\overline{y}}$ is finite over $\mathrm{Spec}(\mathcal{O}^{sh}_{Y,\overline{y}})$, thus $X^{sh}_{\overline{y}} = \mathrm{Spec}(A)$ for some ring A finite over $\mathcal{O}^{sh}_{Y,\overline{y}}$. Since the latter is strictly henselian, Lemma 32.5 implies that A is a finite product of

henselian local rings $A = A_1 \times \ldots \times A_r$. Since the residue field of $\mathcal{O}^{sh}_{Y,\overline{y}}$ is separably closed the same is true for each A_i. Hence A_i is strictly henselian. This implies that $X^{sh}_{\overline{y}} = \coprod_{i=1}^r \mathrm{Spec}(A_i)$. The vanishing of Lemma 54.1 implies that $(R^q f_*\mathcal{F})_{\overline{y}} = 0$ for $q > 0$ which implies (2) by Theorem 29.10. Part (1) follows from the corresponding statement of Lemma 54.1. $\square$

0959 **Lemma 54.3.** *Consider a cartesian square*

$$\begin{tikzcd} X' \arrow[r, "g'"'] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

of schemes with f a finite morphism. For any sheaf of sets $\mathcal{F}$ on $X_{\acute{e}tale}$ we have $f'_*(g')^{-1}\mathcal{F} = g^{-1}f_*\mathcal{F}$.

Proof. In great generality there is a pullback map $g^{-1}f_*\mathcal{F} \to f'_*(g')^{-1}\mathcal{F}$, see Sites, Section 45. It suffices to check on stalks (Theorem 29.10). Let $\overline{y}' : \mathrm{Spec}(k) \to Y'$ be a geometric point. We have

$$\begin{aligned} (f'_*(g')^{-1}\mathcal{F})_{\overline{y}'} &= \prod\nolimits_{\overline{x}':\mathrm{Spec}(k)\to X',\ f'\circ\overline{x}'=\overline{y}'} ((g')^{-1}\mathcal{F})_{\overline{x}'} \\ &= \prod\nolimits_{\overline{x}':\mathrm{Spec}(k)\to X',\ f'\circ\overline{x}'=\overline{y}'} \mathcal{F}_{g'\circ\overline{x}'} \\ &= \prod\nolimits_{\overline{x}:\mathrm{Spec}(k)\to X,\ f\circ\overline{x}=g\circ\overline{y}'} \mathcal{F}_{\overline{x}} \\ &= (f_*\mathcal{F})_{g\circ\overline{y}'} \\ &= (g^{-1}f_*\mathcal{F})_{\overline{y}'} \end{aligned}$$

The first equality by Proposition 54.2. The second equality by Lemma 36.2. The third equality holds because the diagram is a cartesian square and hence the map

$$\{\overline{x}' : \mathrm{Spec}(k) \to X',\ f' \circ \overline{x}' = \overline{y}'\} \longrightarrow \{\overline{x} : \mathrm{Spec}(k) \to X,\ f \circ \overline{x} = g \circ \overline{y}'\}$$

sending $\overline{x}'$ to $g' \circ \overline{x}'$ is a bijection. The fourth equality by Proposition 54.2. The fifth equality by Lemma 36.2. $\square$

0EYP **Lemma 54.4.** *Consider a cartesian square*

$$\begin{tikzcd} X' \arrow[r, "g'"'] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

of schemes with f an integral morphism. For any sheaf of sets $\mathcal{F}$ on $X_{\acute{e}tale}$ we have $f'_*(g')^{-1}\mathcal{F} = g^{-1}f_*\mathcal{F}$.

Proof. The question is local on Y and hence we may assume Y is affine. Then we can write $X = \lim X_i$ with $f_i : X_i \to Y$ finite (this is easy in the affine case, but see Limits, Lemma 7.2 for a reference). Denote $p_{i'i} : X_{i'} \to X_i$ the transition morphisms and $p_i : X \to X_i$ the projections. Setting $\mathcal{F}_i = p_{i,*}\mathcal{F}$ we obtain from Lemma 51.9 a system $(\mathcal{F}_i, \varphi_{i'i})$ with $\mathcal{F} = \mathrm{colim}\, p_i^{-1}\mathcal{F}_i$. We get $f_*\mathcal{F} = \mathrm{colim}\, f_{i,*}\mathcal{F}_i$

from Lemma 51.7. Set $X'_i = Y' \times_Y X_i$ with projections f'_i and g'_i. Then $X' = \lim X'_i$ as limits commute with limits. Denote $p'_i : X' \to X'_i$ the projections. We have

$$\begin{aligned} g^{-1} f_* \mathcal{F} &= g^{-1} \operatorname{colim} f_{i,*} \mathcal{F}_i \\ &= \operatorname{colim} g^{-1} f_{i,*} \mathcal{F}_i \\ &= \operatorname{colim} f'_{i,*} (g'_i)^{-1} \mathcal{F}_i \\ &= f'_*(\operatorname{colim} (p'_i)^{-1} (g'_i)^{-1} \mathcal{F}_i) \\ &= f'_*(\operatorname{colim} (g')^{-1} p_i^{-1} \mathcal{F}_i) \\ &= f'_* (g')^{-1} \operatorname{colim} p_i^{-1} \mathcal{F}_i \\ &= f'_* (g')^{-1} \mathcal{F} \end{aligned}$$

as desired. For the first equality see above. For the second use that pullback commutes with colimits. For the third use the finite case, see Lemma 54.3. For the fourth use Lemma 51.7. For the fifth use that $g'_i \circ p'_i = p_i \circ g'$. For the sixth use that pullback commutes with colimits. For the seventh use $\mathcal{F} = \operatorname{colim} p_i^{-1} \mathcal{F}_i$. ☐

The following lemma is a case of cohomological descent dealing with étale sheaves and finite surjective morphisms. We will significantly generalize this result once we prove the proper base change theorem.

09Z2 **Lemma 54.5.** *Let $f : X \to Y$ be a surjective finite morphism of schemes. Set $f_n : X_n \to Y$ equal to the $(n+1)$-fold fibre product of X over Y. For $\mathcal{F} \in Ab(Y_{étale})$ set $\mathcal{F}_n = f_{n,*} f_n^{-1} \mathcal{F}$. There is an exact sequence*

$$0 \to \mathcal{F} \to \mathcal{F}_0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \ldots$$

on $X_{étale}$. Moreover, there is a spectral sequence

$$E_1^{p,q} = H^q_{étale}(X_p, f_p^{-1}\mathcal{F})$$

converging to $H^{p+q}(Y_{étale}, \mathcal{F})$. This spectral sequence is functorial in $\mathcal{F}$.

Proof. If we prove the first statement of the lemma, then we obtain a spectral sequence with $E_1^{p,q} = H^q_{étale}(Y, \mathcal{F})$ converging to $H^{p+q}(Y_{étale}, \mathcal{F})$, see Derived Categories, Lemma 21.3. On the other hand, since $R^i f_{p,*} f_p^{-1} \mathcal{F} = 0$ for $i > 0$ (Proposition 54.2) we get

$$H^q_{étale}(X_p, f_p^{-1}\mathcal{F}) = H^q_{étale}(Y, f_{p,*} f_p^{-1}\mathcal{F}) = H^q_{étale}(Y, \mathcal{F}_p)$$

by Proposition 53.2 and we get the spectral sequence of the lemma.

To prove the first statement of the lemma, observe that X_n forms a simplicial scheme over Y, see Simplicial, Example 3.5. Observe moreover, that for each of the projections $d_j : X_{n+1} \to X_n$ there is a map $d_j^{-1} f_n^{-1} \mathcal{F} \to f_{n+1}^{-1} \mathcal{F}$. These maps induce maps

$$\delta_j : \mathcal{F}_n \to \mathcal{F}_{n+1}$$

for $j = 0, \ldots, n+1$. We use the alternating sum of these maps to define the differentials $\mathcal{F}_n \to \mathcal{F}_{n+1}$. Similarly, there is a canonical augmentation $\mathcal{F} \to \mathcal{F}_0$, namely this is just the canonical map $\mathcal{F} \to f_* f^{-1} \mathcal{F}$. To check that this sequence of sheaves is an exact complex it suffices to check on stalks at geometric points

(Theorem 29.10). Thus we let $\overline{y} : \mathrm{Spec}(k) \to Y$ be a geometric point. Let $E = \{\overline{x} : \mathrm{Spec}(k) \to X \mid f(\overline{x}) = \overline{y}\}$. Then E is a finite nonempty set and we see that

$$(\mathcal{F}_n)_{\overline{y}} = \bigoplus\nolimits_{e \in E^{n+1}} \mathcal{F}_{\overline{y}}$$

by Proposition 54.2 and Lemma 36.2. Thus we have to see that given an abelian group M the sequence

$$0 \to M \to \bigoplus\nolimits_{e \in E} M \to \bigoplus\nolimits_{e \in E^2} M \to \ldots$$

is exact. Here the first map is the diagonal map and the map $\bigoplus_{e \in E^{n+1}} M \to \bigoplus_{e \in E^{n+2}} M$ is the alternating sum of the maps induced by the $(n+2)$ projections $E^{n+2} \to E^{n+1}$. This can be shown directly or deduced by applying Simplicial, Lemma 26.9 to the map $E \to \{*\}$. □

09Z3 **Remark 54.6.** In the situation of Lemma 54.5 if $\mathcal{G}$ is a sheaf of sets on $Y_{\acute{e}tale}$, then we have

$$\Gamma(Y, \mathcal{G}) = \mathrm{Equalizer}(\ \Gamma(X_0, f_0^{-1}\mathcal{G}) \rightrightarrows \Gamma(X_1, f_1^{-1}\mathcal{G})\ )$$

This is proved in exactly the same way, by showing that the sheaf $\mathcal{G}$ is the equalizer of the two maps $f_{0,*}f_0^{-1}\mathcal{G} \to f_{1,*}f_1^{-1}\mathcal{G}$.

55. Galois action on stalks

03QW In this section we define an action of the absolute Galois group of a residue field of a point s of S on the stalk functor at any geometric point lying over s.

Galois action on stalks. Let S be a scheme. Let $\overline{s}$ be a geometric point of S. Let $\sigma \in \mathrm{Aut}(\kappa(\overline{s})/\kappa(s))$. Define an action of σ on the stalk $\mathcal{F}_{\overline{s}}$ of a sheaf $\mathcal{F}$ as follows

04JK (55.0.1)
$$\begin{array}{ccc} \mathcal{F}_{\overline{s}} & \longrightarrow & \mathcal{F}_{\overline{s}} \\ (U, \overline{u}, t) & \longmapsto & (U, \overline{u} \circ \mathrm{Spec}(\sigma), t). \end{array}$$

where we use the description of elements of the stalk in terms of triples as in the discussion following Definition 29.6. This is a left action, since if $\sigma_i \in \mathrm{Aut}(\kappa(\overline{s})/\kappa(s))$ then

$$\begin{aligned} \sigma_1 \cdot (\sigma_2 \cdot (U, \overline{u}, t)) &= \sigma_1 \cdot (U, \overline{u} \circ \mathrm{Spec}(\sigma_2), t) \\ &= (U, \overline{u} \circ \mathrm{Spec}(\sigma_2) \circ \mathrm{Spec}(\sigma_1), t) \\ &= (U, \overline{u} \circ \mathrm{Spec}(\sigma_1 \circ \sigma_2), t) \\ &= (\sigma_1 \circ \sigma_2) \cdot (U, \overline{u}, t) \end{aligned}$$

It is clear that this action is functorial in the sheaf $\mathcal{F}$. We note that we could have defined this action by referring directly to Remark 29.8.

03QX **Definition 55.1.** Let S be a scheme. Let $\overline{s}$ be a geometric point lying over the point s of S. Let $\kappa(s) \subset \kappa(s)^{sep} \subset \kappa(\overline{s})$ denote the separable algebraic closure of $\kappa(s)$ in the algebraically closed field $\kappa(\overline{s})$.

1. In this situation the *absolute Galois group* of $\kappa(s)$ is $\mathrm{Gal}(\kappa(s)^{sep}/\kappa(s))$. It is sometimes denoted $\mathrm{Gal}_{\kappa(s)}$.
2. The geometric point $\overline{s}$ is called *algebraic* if $\kappa(s) \subset \kappa(\overline{s})$ is an algebraic closure of $\kappa(s)$.

03QY **Example 55.2.** The geometric point $\mathrm{Spec}(\mathbf{C}) \to \mathrm{Spec}(\mathbf{Q})$ is not algebraic.

Let $\kappa(s) \subset \kappa(s)^{sep} \subset \kappa(\overline{s})$ be as in the definition. Note that as $\kappa(\overline{s})$ is algebraically closed the map

$$\text{Aut}(\kappa(\overline{s})/\kappa(s)) \longrightarrow \text{Gal}(\kappa(s)^{sep}/\kappa(s)) = \text{Gal}_{\kappa(s)}$$

is surjective. Suppose $(U, \overline{u})$ is an étale neighbourhood of $\overline{s}$, and say $\overline{u}$ lies over the point u of U. Since $U \to S$ is étale, the residue field extension $\kappa(s) \subset \kappa(u)$ is finite separable. This implies the following

(1) If $\sigma \in \text{Aut}(\kappa(\overline{s})/\kappa(s)^{sep})$ then σ acts trivially on $\mathcal{F}_{\overline{s}}$.
(2) More precisely, the action of $\text{Aut}(\kappa(\overline{s})/\kappa(s))$ determines and is determined by an action of the absolute Galois group $\text{Gal}_{\kappa(s)}$ on $\mathcal{F}_{\overline{s}}$.
(3) Given $(U, \overline{u}, t)$ representing an element ξ of $\mathcal{F}_{\overline{s}}$ any element of $\text{Gal}(\kappa(s)^{sep}/K)$ acts trivially, where $\kappa(s) \subset K \subset \kappa(s)^{sep}$ is the image of $\overline{u}^\sharp : \kappa(u) \to \kappa(\overline{s})$.

Altogether we see that $\mathcal{F}_{\overline{s}}$ becomes a $\text{Gal}_{\kappa(s)}$-set (see Fundamental Groups, Definition 2.1). Hence we may think of the stalk functor as a functor

$$Sh(S_{\acute{e}tale}) \longrightarrow \text{Gal}_{\kappa(s)}\text{-}Sets, \quad \mathcal{F} \longmapsto \mathcal{F}_{\overline{s}}$$

and from now on we usually do think about the stalk functor in this way.

03QT **Theorem 55.3.** *Let $S = \text{Spec}(K)$ with K a field. Let $\overline{s}$ be a geometric point of S. Let $G = Gal_{\kappa(s)}$ denote the absolute Galois group. Taking stalks induces an equivalence of categories*

$$Sh(S_{\acute{e}tale}) \longrightarrow G\text{-}Sets, \quad \mathcal{F} \longmapsto \mathcal{F}_{\overline{s}}.$$

Proof. Let us construct the inverse to this functor. In Fundamental Groups, Lemma 2.2 we have seen that given a G-set M there exists an étale morphism $X \to \text{Spec}(K)$ such that $Mor_K(\text{Spec}(K^{sep}), X)$ is isomorphic to M as a G-set. Consider the sheaf $\mathcal{F}$ on $\text{Spec}(K)_{\acute{e}tale}$ defined by the rule $U \mapsto Mor_K(U, X)$. This is a sheaf as the étale topology is subcanonical. Then we see that $\mathcal{F}_{\overline{s}} = Mor_K(\text{Spec}(K^{sep}), X) = M$ as G-sets (details omitted). This gives the inverse of the functor and we win. ☐

04JL **Remark 55.4.** Another way to state the conclusion of Theorem 55.3 and Fundamental Groups, Lemma 2.2 is to say that every sheaf on $\text{Spec}(K)_{\acute{e}tale}$ is representable by a scheme X étale over $\text{Spec}(K)$. This does not mean that every sheaf is representable in the sense of Sites, Definition 12.3. The reason is that in our construction of $\text{Spec}(K)_{\acute{e}tale}$ we chose a sufficiently large set of schemes étale over $\text{Spec}(K)$, whereas sheaves on $\text{Spec}(K)_{\acute{e}tale}$ form a proper class.

04JM **Lemma 55.5.** *Assumptions and notations as in Theorem 55.3. There is a functorial bijection*

$$\Gamma(S, \mathcal{F}) = (\mathcal{F}_{\overline{s}})^G$$

Proof. We can prove this using formal arguments and the result of Theorem 55.3 as follows. Given a sheaf $\mathcal{F}$ corresponding to the G-set $M = \mathcal{F}_{\overline{s}}$ we have

$$\begin{aligned} \Gamma(S, \mathcal{F}) &= Mor_{Sh(S_{\acute{e}tale})}(h_{\text{Spec}(K)}, \mathcal{F}) \\ &= Mor_{G\text{-}Sets}(\{*\}, M) \\ &= M^G \end{aligned}$$

Here the first identification is explained in Sites, Sections 2 and 12, the second results from Theorem 55.3 and the third is clear. We will also give a direct proof[5].

[5]For the doubting Thomases out there.

Suppose that $t \in \Gamma(S, \mathcal{F})$ is a global section. Then the triple $(S, \overline{s}, t)$ defines an element of $\mathcal{F}_{\overline{s}}$ which is clearly invariant under the action of G. Conversely, suppose that $(U, \overline{u}, t)$ defines an element of $\mathcal{F}_{\overline{s}}$ which is invariant. Then we may shrink U and assume $U = \mathrm{Spec}(L)$ for some finite separable field extension of K, see Proposition 26.2. In this case the map $\mathcal{F}(U) \to \mathcal{F}_{\overline{s}}$ is injective, because for any morphism of étale neighbourhoods $(U', \overline{u}') \to (U, \overline{u})$ the restriction map $\mathcal{F}(U) \to \mathcal{F}(U')$ is injective since $U' \to U$ is a covering of $S_{étale}$. After enlarging L a bit we may assume $K \subset L$ is a finite Galois extension. At this point we use that

$$\mathrm{Spec}(L) \times_{\mathrm{Spec}(K)} \mathrm{Spec}(L) = \coprod\nolimits_{\sigma \in \mathrm{Gal}(L/K)} \mathrm{Spec}(L)$$

where the maps $\mathrm{Spec}(L) \to \mathrm{Spec}(L \otimes_K L)$ come from the ring maps $a \otimes b \mapsto a\sigma(b)$. Hence we see that the condition that $(U, \overline{u}, t)$ is invariant under all of G implies that $t \in \mathcal{F}(\mathrm{Spec}(L))$ maps to the same element of $\mathcal{F}(\mathrm{Spec}(L) \times_{\mathrm{Spec}(K)} \mathrm{Spec}(L))$ via restriction by either projection (this uses the injectivity mentioned above; details omitted). Hence the sheaf condition of $\mathcal{F}$ for the étale covering $\{\mathrm{Spec}(L) \to \mathrm{Spec}(K)\}$ kicks in and we conclude that t comes from a unique section of $\mathcal{F}$ over $\mathrm{Spec}(K)$. □

04JN **Remark 55.6.** Let S be a scheme and let $\overline{s} : \mathrm{Spec}(k) \to S$ be a geometric point of S. By definition this means that k is algebraically closed. In particular the absolute Galois group of k is trivial. Hence by Theorem 55.3 the category of sheaves on $\mathrm{Spec}(k)_{étale}$ is equivalent to the category of sets. The equivalence is given by taking sections over $\mathrm{Spec}(k)$. This finally provides us with an alternative definition of the stalk functor. Namely, the functor

$$Sh(S_{étale}) \longrightarrow Sets, \quad \mathcal{F} \longmapsto \mathcal{F}_{\overline{s}}$$

is isomorphic to the functor

$$Sh(S_{étale}) \longrightarrow Sh(\mathrm{Spec}(k)_{étale}) = Sets, \quad \mathcal{F} \longmapsto \overline{s}^*\mathcal{F}$$

To prove this rigorously one can use Lemma 36.2 part (3) with $f = \overline{s}$. Moreover, having said this the general case of Lemma 36.2 part (3) follows from functoriality of pullbacks.

56. Group cohomology

0A2H In the following, if we write $H^i(G, M)$ we will mean that G is a topological group and M a discrete G-module with continuous G-action and $H^i(G, -)$ is the ith right derived functor on the category Mod_G of such G-modules, see Definitions 56.1 and 56.2. This includes the case of an abstract group G, which simply means that G is viewed as a topological group with the discrete topology.

When the module has a nondiscrete topology, we will use the notation $H^i_{cont}(G, M)$ to indicate the continuous cohomology groups introduced in [Tat76], see Section 57.

04JP **Definition 56.1.** Let G be a topological group.

(1) A *G-module*, sometimes called a *discrete G-module*, is an abelian group M endowed with a left action $a : G \times M \to M$ by group homomorphisms such that a is continuous when M is given the discrete topology.
(2) A *morphism of G-modules* $f : M \to N$ is a G-equivariant homomorphism from M to N.
(3) The category of G-modules is denoted Mod_G.

Let R be a ring.

(1) An *R-G-module* is an R-module M endowed with a left action $a : G \times M \to M$ by R-linear maps such that a is continuous when M is given the discrete topology.
(2) A *morphism of R-G-modules* $f : M \to N$ is a G-equivariant R-module map from M to N.
(3) The category of R-G-modules is denoted $\text{Mod}_{R,G}$.

The condition that $a : G \times M \to M$ is continuous is equivalent with the condition that the stabilizer of any $x \in M$ is open in G. If G is an abstract group then this corresponds to the notion of an abelian group endowed with a G-action provided we endow G with the discrete topology. Observe that $\text{Mod}_{\mathbf{Z},G} = \text{Mod}_G$.

The category Mod_G has enough injectives, see Injectives, Lemma 3.1. Consider the left exact functor

$$\text{Mod}_G \longrightarrow Ab, \quad M \longmapsto M^G = \{x \in M \mid g \cdot x = x \ \forall g \in G\}$$

We sometimes denote $M^G = H^0(G, M)$ and sometimes we write $M^G = \Gamma_G(M)$. This functor has a total right derived functor $R\Gamma_G(M)$ and ith right derived functor $R^i\Gamma_G(M) = H^i(G, M)$ for any $i \geq 0$.

The same construction works for $H^0(G, -) : \text{Mod}_{R,G} \to \text{Mod}_R$. We will see in Lemma 56.3 that this agrees with the cohomology of the underlying G-module.

04JR **Definition 56.2.** Let G be a topological group. Let M be a discrete G-module with continuous G-action. In other words, M is an object of the category Mod_G introduced in Definition 56.1.

(1) The right derived functors $H^i(G, M)$ of $H^0(G, M)$ on the category Mod_G are called the *continuous group cohomology groups* of M.
(2) If G is an abstract group endowed with the discrete topology then the $H^i(G, M)$ are called the *group cohomology groups* of M.
(3) If G is a Galois group, then the groups $H^i(G, M)$ are called the *Galois cohomology groups* of M.
(4) If G is the absolute Galois group of a field K, then the groups $H^i(G, M)$ are sometimes called the *Galois cohomology groups of K with coefficients in M*. In this case we sometimes write $H^i(K, M)$ instead of $H^i(G, M)$.

0DVD **Lemma 56.3.** *Let G be a topological group. Let R be a ring. For every $i \geq 0$ the diagram*

$$\begin{tikzcd} Mod_{R,G} \arrow[r, "H^i(G,-)"'] \arrow[d] & Mod_R \arrow[d] \\ Mod_G \arrow[r, "H^i(G,-)"] & Ab \end{tikzcd}$$

whose vertical arrows are the forgetful functors is commutative.

Proof. Let us denote the forgetful functor $F : \text{Mod}_{R,G} \to \text{Mod}_G$. Then F has a left adjoint $H : \text{Mod}_G \to \text{Mod}_{R,G}$ given by $H(M) = M \otimes_{\mathbf{Z}} R$. Observe that every object of Mod_G is a quotient of a direct sum of modules of the form $\mathbf{Z}[G/U]$ where $U \subset G$ is an open subgroup. Here $\mathbf{Z}[G/U]$ denotes the G-modules of finite $\mathbf{Z}$-linear combinations of right U congruence classes in G endowed with left G-action. Thus every bounded above complex in Mod_G is quasi-isomorphic to a bounded above

complex in Mod_G whose underlying terms are flat $\mathbf{Z}$-modules (Derived Categories, Lemma 15.4). Thus it is clear that LH exists on $D^-(\text{Mod}_G)$ and is computed by evaluating H on any complex whose terms are flat $\mathbf{Z}$-modules; this follows from Derived Categories, Lemma 15.7 and Proposition 16.8. We conclude from Derived Categories, Lemma 29.4 that

$$\text{Ext}^i(\mathbf{Z}, F(M)) = \text{Ext}^i(R, M)$$

for M in $Mod_{R,G}$. Observe that $H^0(G, -) = \text{Hom}(\mathbf{Z}, -)$ on Mod_G where $\mathbf{Z}$ denotes the G-module with trivial action. Hence $H^i(G, -) = \text{Ext}^i(\mathbf{Z}, -)$ on Mod_G. Similarly we have $H^i(G, -) = \text{Ext}^i(R, -)$ on $\text{Mod}_{R,G}$. Combining everything we see that the lemma is true. □

0DVE **Lemma 56.4.** *Let G be a topological group. Let R be a ring. Let M, N be R-G-modules. If M is finite projective as an R-module, then $Ext^i(M, N) = H^i(G, M^\vee \otimes_R N)$ (for notation see proof).*

Proof. The module $M^\vee = \text{Hom}_R(M, R)$ endowed with the contragredient action of G. Namely $(g \cdot \lambda)(m) = \lambda(g^{-1} \cdot m)$ for $g \in G$, $\lambda \in M^\vee$, $m \in M$. The action of G on $M^\vee \otimes_R N$ is the diagonal one, i.e., given by $g \cdot (\lambda \otimes n) = g \cdot \lambda \otimes g \cdot n$. Note that for a third R-G-module E we have $\text{Hom}(E, M^\vee \otimes_R N) = \text{Hom}(M \otimes_R E, N)$. Namely, this is true on the level of R-modules by Algebra, Lemmas 11.8 and 77.8 and the definitions of G-actions are chosen such that it remains true for R-G-modules. It follows that $M^\vee \otimes_R N$ is an injective R-G-module if N is an injective R-G-module. Hence if $N \to N^\bullet$ is an injective resolution, then $M^\vee \otimes_R N \to M^\vee \otimes_R N^\bullet$ is an injective resolution. Then

$$\text{Hom}(M, N^\bullet) = \text{Hom}(R, M^\vee \otimes_R N^\bullet) = (M^\vee \otimes_R N^\bullet)^G$$

Since the left hand side computes $\text{Ext}^i(M, N)$ and the right hand side computes $H^i(G, M^\vee \otimes_R N)$ the proof is complete. □

0DVF **Lemma 56.5.** *Let G be a topological group. Let k be a field. Let V be a k-G-module. If G is topologically finitely generated and $\dim_k(V) < \infty$, then $\dim_k H^1(G, V) < \infty$.*

Proof. Let $g_1, \ldots, g_r \in G$ be elements which topologically generate G, i.e., this means that the subgroup generated by $g_1, \ldots, g_r$ is dense. By Lemma 56.4 we see that $H^1(G, V)$ is the k-vector space of extensions

$$0 \to V \to E \to k \to 0$$

of k-G-modules. Choose $e \in E$ mapping to $1 \in k$. Write

$$g_i \cdot e = v_i + e$$

for some $v_i \in V$. This is possible because $g_i \cdot 1 = 1$. We claim that the list of elements $v_1, \ldots, v_r \in V$ determine the isomorphism class of the extension E. Once we prove this the lemma follows as this means that our Ext vector space is isomorphic to a subquotient of the k-vector space $V^{\oplus r}$; some details omitted. Since E is an object of the category defined in Definition 56.1 we know there is an open subgroup U such that $u \cdot e = e$ for all $u \in U$. Now pick any $g \in G$. Then gU contains a word w in the elements $g_1, \ldots, g_r$. Say $gu = w$. Since the element $w \cdot e$ is determined by $v_1, \ldots, v_r$, we see that $g \cdot e = (gu) \cdot e = w \cdot e$ is too. □

0DV3 **Lemma 56.6.** *Let G be a profinite topological group. Then*

(1) $H^i(G, M)$ *is torsion for* $i > 0$ *and any* G*-module* M*, and*
(2) $H^i(G, M) = 0$ *if* M *is a* $\mathbf{Q}$*-vector space.*

Proof. Proof of (1). By dimension shifting we see that it suffices to show that $H^1(G, M)$ is torsion for every G-module M. Choose an exact sequence $0 \to M \to I \to N \to 0$ with I an injective object of the category of G-modules. Then any element of $H^1(G, M)$ is the image of an element $y \in N^G$. Choose $x \in I$ mapping to y. The stabilizer $U \subset G$ of x is open, hence has finite index r. Let $g_1, \ldots, g_r \in G$ be a system of representatives for G/U. Then $\sum g_i(x)$ is an invariant element of I which maps to ry. Thus r kills the element of $H^1(G, M)$ we started with. Part (2) follows as then $H^i(G, M)$ is both a $\mathbf{Q}$-vector space and torsion. □

57. Tate's continuous cohomology

0DVG Tate's continuous cohomology ([Tat76]) is defined by the complex of continuous inhomogeneous cochains. We can define this when M is an arbitrary topological abelian group endowed with a continuous G-action. Namely, we consider the complex

$$C^\bullet_{cont}(G, M) : M \to \text{Maps}_{cont}(G, M) \to \text{Maps}_{cont}(G \times G, M) \to \ldots$$

where the boundary map is defined for $n \geq 1$ by the rule

$$\begin{aligned} \mathrm{d}(f)(g_1, \ldots, g_{n+1}) = g_1(f(g_2, \ldots, g_{n+1})) \\ + \sum\nolimits_{j=1,\ldots,n} (-1)^j f(g_1, \ldots, g_j g_{j+1}, \ldots, g_{n+1}) \\ + (-1)^{n+1} f(g_1, \ldots, g_n) \end{aligned}$$

and for $n = 0$ sends $m \in M$ to the map $g \mapsto g(m) - m$. We define

$$H^i_{cont}(G, M) = H^i(C^\bullet_{cont}(G, M))$$

Since the terms of the complex involve continuous maps from G and self products of G into the topological module M, it is not clear that this turns a short exact sequence of topological modules into a long exact cohomology sequence. Another difficulty is that the category of topological abelian groups isn't an abelian category!

However, a short exact sequence of discrete G-modules does give rise to a short exact sequence of complexes of continuous cochains and hence a long exact cohomology sequence of continuous cohomology groups $H^i_{cont}(G, -)$. Therefore, on the category Mod_G of Definition 56.1 the functors $H^i_{cont}(G, M)$ form a cohomological δ-functor as defined in Homology, Section 12. Since the cohomology $H^i(G, M)$ of Definition 56.2 is a universal δ-functor (Derived Categories, Lemma 16.6) we obtain canonical maps

$$H^i(G, M) \longrightarrow H^i_{cont}(G, M)$$

for $M \in \text{Mod}_G$. It is known that these maps are isomorphisms when G is an abstract group (i.e., G has the discrete topology) or when G is a profinite group (insert future reference here). If you know an example showing this map is not an isomorphism for a topological group G and $M \in \text{Ob}(\text{Mod}_G)$ please email stacks.project@gmail.com.

58. Cohomology of a point

03QQ As a consequence of the discussion in the preceding sections we obtain the equivalence of étale cohomology of the spectrum of a field with Galois cohomology.

04JQ **Lemma 58.1.** *Let $S = \text{Spec}(K)$ with K a field. Let $\overline{s}$ be a geometric point of S. Let $G = Gal_{\kappa(s)}$ denote the absolute Galois group. The stalk functor induces an equivalence of categories*

$$Ab(S_{\acute{e}tale}) \longrightarrow Mod_G, \quad \mathcal{F} \longmapsto \mathcal{F}_{\overline{s}}.$$

Proof. In Theorem 55.3 we have seen the equivalence between sheaves of sets and G-sets. The current lemma follows formally from this as an abelian sheaf is just a sheaf of sets endowed with a commutative group law, and a G-module is just a G-set endowed with a commutative group law. □

03QU **Lemma 58.2.** *Notation and assumptions as in Lemma 58.1. Let $\mathcal{F}$ be an abelian sheaf on $\text{Spec}(K)_{\acute{e}tale}$ which corresponds to the G-module M. Then*

(1) *in $D(Ab)$ we have a canonical isomorphism $R\Gamma(S, \mathcal{F}) = R\Gamma_G(M)$,*
(2) *$H^0_{\acute{e}tale}(S, \mathcal{F}) = M^G$, and*
(3) *$H^q_{\acute{e}tale}(S, \mathcal{F}) = H^q(G, M)$.*

Proof. Combine Lemma 58.1 with Lemma 55.5. □

03QV **Example 58.3.** Sheaves on $\text{Spec}(K)_{\acute{e}tale}$. Let $G = \text{Gal}(K^{sep}/K)$ be the absolute Galois group of K.

(1) The constant sheaf $\underline{\mathbf{Z}/n\mathbf{Z}}$ corresponds to the module $\mathbf{Z}/n\mathbf{Z}$ with trivial G-action,
(2) the sheaf $\mathbf{G}_m|_{\text{Spec}(K)_{\acute{e}tale}}$ corresponds to $(K^{sep})^*$ with its G-action,
(3) the sheaf $\mathbf{G}_a|_{\text{Spec}(K^{sep})}$ corresponds to $(K^{sep}, +)$ with its G-action, and
(4) the sheaf $\mu_n|_{\text{Spec}(K^{sep})}$ corresponds to $\mu_n(K^{sep})$ with its G-action.

By Remark 23.4 and Theorem 24.1 we have the following identifications for cohomology groups:

$$\begin{aligned} H^0_{\acute{e}tale}(S_{\acute{e}tale}, \mathbf{G}_m) &= \Gamma(S, \mathcal{O}_S^*) \\ H^1_{\acute{e}tale}(S_{\acute{e}tale}, \mathbf{G}_m) &= H^1_{Zar}(S, \mathcal{O}_S^*) = \text{Pic}(S) \\ H^i_{\acute{e}tale}(S_{\acute{e}tale}, \mathbf{G}_a) &= H^i_{Zar}(S, \mathcal{O}_S) \end{aligned}$$

Also, for any quasi-coherent sheaf $\mathcal{F}$ on $S_{\acute{e}tale}$ we have

$$H^i(S_{\acute{e}tale}, \mathcal{F}) = H^i_{Zar}(S, \mathcal{F}),$$

see Theorem 22.4. In particular, this gives the following sequence of equalities

$$0 = \text{Pic}(\text{Spec}(K)) = H^1_{\acute{e}tale}(\text{Spec}(K)_{\acute{e}tale}, \mathbf{G}_m) = H^1(G, (K^{sep})^*)$$

which is none other than Hilbert's 90 theorem. Similarly, for $i \geq 1$,

$$0 = H^i(\text{Spec}(K), \mathcal{O}) = H^i_{\acute{e}tale}(\text{Spec}(K)_{\acute{e}tale}, \mathbf{G}_a) = H^i(G, K^{sep})$$

where the K^{sep} indicates K^{sep} as a Galois module with addition as group law. In this way we may consider the work we have done so far as a complicated way of computing Galois cohomology groups.

The following result is a curiosity and should be skipped on a first reading.

0D1W **Lemma 58.4.** *Let R be a local ring of dimension* 0*. Let $S = \mathrm{Spec}(R)$. Then every $\mathcal{O}_S$-module on $S_{\acute{e}tale}$ is quasi-coherent.*

Proof. Let $\mathcal{F}$ be an $\mathcal{O}_S$-module on $S_{\acute{e}tale}$. We have to show that $\mathcal{F}$ is determined by the R-module $M = \Gamma(S, \mathcal{F})$. More precisely, if $\pi : X \to S$ is étale we have to show that $\Gamma(X, \mathcal{F}) = \Gamma(X, \pi^*\widetilde{M})$.

Let $\mathfrak{m} \subset R$ be the maximal ideal and let κ be the residue field. By Algebra, Lemma 149.10 the local ring R is henselian. If $X \to S$ is étale, then the underlying topological space of X is discrete by Morphisms, Lemma 34.7 and hence X is a disjoint union of affine schemes each having one point. Moreover, if $X = \mathrm{Spec}(A)$ is affine and has one point, then $R \to A$ is finite étale by Algebra, Lemma 149.5. We have to show that $\Gamma(X, \mathcal{F}) = M \otimes_R A$ in this case.

The functor $A \mapsto A/\mathfrak{m}A$ defines an equivalence of the category of finite étale R-algebras with the category of finite separable κ-algebras by Algebra, Lemma 149.7. Let us first consider the case where $A/\mathfrak{m}A$ is a Galois extension of κ with Galois group G. For each $\sigma \in G$ let $\sigma : A \to A$ denote the corresponding automorphism of A over R. Let $N = \Gamma(X, \mathcal{F})$. Then $\mathrm{Spec}(\sigma) : X \to X$ is an automorphism over S and hence pullback by this defines a map $\sigma : N \to N$ which is a σ-linear map: $\sigma(an) = \sigma(a)\sigma(n)$ for $a \in A$ and $n \in N$. We will apply Galois descent to the quasi-coherent module $\widetilde{N}$ on X endowed with the isomorphisms coming from the action on σ on N. See Descent, Lemma 6.2. This lemma tells us there is an isomorphism $N = N^G \otimes_R A$. On the other hand, it is clear that $N^G = M$ by the sheaf property for $\mathcal{F}$. Thus the required isomorphism holds.

The general case (with A local and finite étale over R) is deduced from the Galois case as follows. Choose $A \to B$ finite étale such that B is local with residue field Galois over κ. Let $G = \mathrm{Aut}(B/R) = \mathrm{Gal}(\kappa_B/\kappa)$. Let $H \subset G$ be the Galois group corresponding to the Galois extension κ_B/κ_A. Then as above one shows that $\Gamma(X, \mathcal{F}) = \Gamma(\mathrm{Spec}(B), \mathcal{F})^H$. By the result for Galois extensions (used twice) we get

$$\Gamma(X, \mathcal{F}) = (M \otimes_R B)^H = M \otimes_R A$$

as desired. □

59. Cohomology of curves

03R0 The next task at hand is to compute the étale cohomology of a smooth curve over an algebraically closed field with torsion coefficients, and in particular show that it vanishes in degree at least 3. To prove this, we will compute cohomology at the generic point, which amounts to some Galois cohomology.

60. Brauer groups

03R1 Brauer groups of fields are defined using finite central simple algebras. In this section we review the relevant facts about Brauer groups, most of which are discussed in the chapter Brauer Groups, Section 1. For other references, see [Ser62], [Ser97] or [Wei48].

03R2 **Theorem 60.1.** *Let K be a field. For a unital, associative (not necessarily commutative) K-algebra A the following are equivalent*

(1) *A is finite central simple K-algebra,*

(2) *A is a finite dimensional K-vector space, K is the center of A, and A has no nontrivial two-sided ideal,*
(3) *there exists $d \geq 1$ such that $A \otimes_K \bar{K} \cong Mat(d \times d, \bar{K})$,*
(4) *there exists $d \geq 1$ such that $A \otimes_K K^{sep} \cong Mat(d \times d, K^{sep})$,*
(5) *there exist $d \geq 1$ and a finite Galois extension $K \subset K'$ such that $A \otimes_{K'} K' \cong Mat(d \times d, K')$,*
(6) *there exist $n \geq 1$ and a finite central skew field D over K such that $A \cong Mat(n \times n, D)$.*

The integer d is called the degree of A.

Proof. This is a copy of Brauer Groups, Lemma 8.6. □

03R4 **Lemma 60.2.** *Let A be a finite central simple algebra over K. Then*

$$\begin{array}{ccc} A \otimes_K A^{opp} & \longrightarrow & End_K(A) \\ a \otimes a' & \longmapsto & (x \mapsto axa') \end{array}$$

is an isomorphism of algebras over K.

Proof. See Brauer Groups, Lemma 4.10. □

03R3 **Definition 60.3.** Two finite central simple algebras A_1 and A_2 over K are called *similar*, or *equivalent* if there exist $m, n \geq 1$ such that $\text{Mat}(n \times n, A_1) \cong \text{Mat}(m \times m, A_2)$. We write $A_1 \sim A_2$.

By Brauer Groups, Lemma 5.1 this is an equivalence relation.

03R5 **Definition 60.4.** Let K be a field. The *Brauer group* of K is the set $\text{Br}(K)$ of similarity classes of finite central simple algebras over K, endowed with the group law induced by tensor product (over K). The class of A in $\text{Br}(K)$ is denoted by $[A]$. The neutral element is $[K] = [\text{Mat}(d \times d, K)]$ for any $d \geq 1$.

The previous lemma implies that inverses exist and that $-[A] = [A^{opp}]$. The Brauer group of a field is always torsion. In fact, we will see that $[A]$ has order dividing $\deg(A)$ for any finite central simple algebra A (see Lemma 61.2). In general the Brauer group is not finitely generated, for example the Brauer group of a non-Archimedean local field is $\mathbf{Q}/\mathbf{Z}$. The Brauer group of $\mathbf{C}(x, y)$ is uncountable.

03R6 **Lemma 60.5.** *Let K be a field and let K^{sep} be a separable algebraic closure. Then the set of isomorphism classes of central simple algebras of degree d over K is in bijection with the non-abelian cohomology $H^1(Gal(K^{sep}/K), PGL_d(K^{sep}))$.*

Sketch of proof. The Skolem-Noether theorem (see Brauer Groups, Theorem 6.1) implies that for any field L the group $\text{Aut}_{L\text{-Algebras}}(\text{Mat}_d(L))$ equals $\text{PGL}_d(L)$. By Theorem 60.1, we see that central simple algebras of degree d correspond to forms of the K-algebra $\text{Mat}_d(K)$. Combined we see that isomorphism classes of degree d central simple algebras correspond to elements of $H^1(\text{Gal}(K^{sep}/K), \text{PGL}_d(K^{sep}))$. For more details on twisting, see for example [Sil86]. □

If A is a finite central simple algebra of degree d over a field K, we denote ξ_A the corresponding cohomology class in $H^1(\text{Gal}(K^{sep}/K), \text{PGL}_d(K^{sep}))$. Consider the short exact sequence

$$1 \to (K^{sep})^* \to \text{GL}_d(K^{sep}) \to \text{PGL}_d(K^{sep}) \to 1,$$

which gives rise to a long exact cohomology sequence (up to degree 2) with coboundary map

$$\delta_d : H^1(\mathrm{Gal}(K^{sep}/K), \mathrm{PGL}_d(K^{sep})) \longrightarrow H^2(\mathrm{Gal}(K^{sep}/K), (K^{sep})^*).$$

Explicitly, this is given as follows: if ξ is a cohomology class represented by the 1-cocycle (g_σ), then $\delta_d(\xi)$ is the class of the 2-cocycle

0A2I (60.5.1)
$$(\sigma, \tau) \longmapsto \tilde{g}_\sigma^{-1}\tilde{g}_{\sigma\tau}\sigma(\tilde{g}_\tau^{-1}) \in (K^{sep})^*$$

where $\tilde{g}_\sigma \in \mathrm{GL}_d(K^{sep})$ is a lift of g_σ. Using this we can make explicit the map

$$\delta : \mathrm{Br}(K) \longrightarrow H^2(\mathrm{Gal}(K^{sep}/K), (K^{sep})^*), \quad [A] \longmapsto \delta_{\deg A}(\xi_A)$$

as follows. Assume A has degree d over K. Choose an isomorphism $\varphi : \mathrm{Mat}_d(K^{sep}) \to A \otimes_K K^{sep}$. For $\sigma \in \mathrm{Gal}(K^{sep}/K)$ choose an element $\tilde{g}_\sigma \in \mathrm{Gl}_d(K^{sep})$ such that $\varphi^{-1} \circ \sigma(\varphi)$ is equal to the map $x \mapsto \tilde{g}_\sigma x \tilde{g}_\sigma^{-1}$. The class in H^2 is defined by the two cocycle (60.5.1).

03R7 **Theorem 60.6.** *Let K be a field with separable algebraic closure K^{sep}. The map $\delta : Br(K) \to H^2(Gal(K^{sep}/K), (K^{sep})^*)$ defined above is a group isomorphism.*

Sketch of proof. In the abelian case ($d = 1$), one has the identification

$$H^1(\mathrm{Gal}(K^{sep}/K), \mathrm{GL}_d(K^{sep})) = H^1_{\acute{e}tale}(\mathrm{Spec}(K), \mathrm{GL}_d(\mathcal{O}))$$

the latter of which is trivial by fpqc descent. If this were true in the non-abelian case, this would readily imply injectivity of δ. (See [Del77].) Rather, to prove this, one can reinterpret $\delta([A])$ as the obstruction to the existence of a K-vector space V with a left A-module structure and such that $\dim_K V = \deg A$. In the case where V exists, one has $A \cong \mathrm{End}_K(V)$. For surjectivity, pick a cohomology class $\xi \in H^2(\mathrm{Gal}(K^{sep}/K), (K^{sep})^*)$, then there exists a finite Galois extension $K \subset K' \subset K^{sep}$ such that ξ is the image of some $\xi' \in H^2(\mathrm{Gal}(K'|K), (K')^*)$. Then write down an explicit central simple algebra over K using the data K', ξ'. $\square$

61. The Brauer group of a scheme

0A2J Let S be a scheme. An $\mathcal{O}_S$-algebra $\mathcal{A}$ is called *Azumaya* if it is étale locally a matrix algebra, i.e., if there exists an étale covering $\mathcal{U} = \{\varphi_i : U_i \to S\}_{i \in I}$ such that $\varphi_i^*\mathcal{A} \cong \mathrm{Mat}_{d_i}(\mathcal{O}_{U_i})$ for some $d_i \geq 1$. Two such $\mathcal{A}$ and $\mathcal{B}$ are called *equivalent* if there exist finite locally free $\mathcal{O}_S$-modules $\mathcal{F}$ and $\mathcal{G}$ which have positive rank at every $s \in S$ such that

$$\mathcal{A} \otimes_{\mathcal{O}_S} \mathcal{H}om_{\mathcal{O}_S}(\mathcal{F}, \mathcal{F}) \cong \mathcal{B} \otimes_{\mathcal{O}_S} \mathcal{H}om_{\mathcal{O}_S}(\mathcal{G}, \mathcal{G})$$

as $\mathcal{O}_S$-algebras. The *Brauer group* of S is the set $\mathrm{Br}(S)$ of equivalence classes of Azumaya $\mathcal{O}_S$-algebras with the operation induced by tensor product (over $\mathcal{O}_S$).

0A2K **Lemma 61.1.** *Let S be a scheme. Let $\mathcal{F}$ and $\mathcal{G}$ be finite locally free sheaves of $\mathcal{O}_S$-modules of positive rank. If there exists an isomorphism $\mathcal{H}om_{\mathcal{O}_S}(\mathcal{F}, \mathcal{F}) \cong \mathcal{H}om_{\mathcal{O}_S}(\mathcal{G}, \mathcal{G})$ of $\mathcal{O}_S$-algebras, then there exists an invertible sheaf $\mathcal{L}$ on S such that $\mathcal{F} \otimes_{\mathcal{O}_S} \mathcal{L} \cong \mathcal{G}$ and such that this isomorphism induces the given isomorphism of endomorphism algebras.*

Proof. Fix an isomorphism $\mathcal{H}om_{\mathcal{O}_S}(\mathcal{F}, \mathcal{F}) \to \mathcal{H}om_{\mathcal{O}_S}(\mathcal{G}, \mathcal{G})$. Consider the sheaf $\mathcal{L} \subset \mathcal{H}om(\mathcal{F}, \mathcal{G})$ generated as an $\mathcal{O}_S$-module by the local isomorphisms $\varphi : \mathcal{F} \to \mathcal{G}$ such that conjugation by φ is the given isomorphism of endomorphism algebras. A

local calculation (reducing to the case that $\mathcal{F}$ and $\mathcal{G}$ are finite free and S is affine) shows that $\mathcal{L}$ is invertible. Another local calculation shows that the evaluation map

$$\mathcal{F} \otimes_{\mathcal{O}_S} \mathcal{L} \longrightarrow \mathcal{G}$$

is an isomorphism. □

The argument given in the proof of the following lemma can be found in [Sal81].

0A2L **Lemma 61.2.** *Let S be a scheme. Let $\mathcal{A}$ be an Azumaya algebra which is locally free of rank d^2 over S. Then the class of $\mathcal{A}$ in the Brauer group of S is annihilated by d.* Argument taken from [Sal81].

Proof. Choose an étale covering $\{U_i \to S\}$ and choose isomorphisms $\mathcal{A}|_{U_i} \to \mathcal{H}om(\mathcal{F}_i, \mathcal{F}_i)$ for some locally free $\mathcal{O}_{U_i}$-modules $\mathcal{F}_i$ of rank d. (We may assume $\mathcal{F}_i$ is free.) Consider the composition

$$p_i : \mathcal{F}_i^{\otimes d} \to \wedge^d(\mathcal{F}_i) \to \mathcal{F}_i^{\otimes d}$$

The first arrow is the usual projection and the second arrow is the isomorphism of the top exterior power of $\mathcal{F}_i$ with the submodule of sections of $\mathcal{F}_i^{\otimes d}$ which transform according to the sign character under the action of the symmetric group on d letters. Then $p_i^2 = d!p_i$ and the rank of p_i is 1. Using the given isomorphism $\mathcal{A}|_{U_i} \to \mathcal{H}om(\mathcal{F}_i, \mathcal{F}_i)$ and the canonical isomorphism

$$\mathcal{H}om(\mathcal{F}_i, \mathcal{F}_i)^{\otimes d} = \mathcal{H}om(\mathcal{F}_i^{\otimes d}, \mathcal{F}_i^{\otimes d})$$

we may think of p_i as a section of $\mathcal{A}^{\otimes d}$ over U_i. We claim that $p_i|_{U_i \times_S U_j} = p_j|_{U_i \times_S U_j}$ as sections of $\mathcal{A}^{\otimes d}$. Namely, applying Lemma 61.1 we obtain an invertible sheaf $\mathcal{L}_{ij}$ and a canonical isomorphism

$$\mathcal{F}_i|_{U_i \times_S U_j} \otimes \mathcal{L}_{ij} \longrightarrow \mathcal{F}_j|_{U_i \times_S U_j}.$$

Using this isomorphism we see that p_i maps to p_j. Since $\mathcal{A}^{\otimes d}$ is a sheaf on $S_{\acute{e}tale}$ (Proposition 17.1) we find a canonical global section $p \in \Gamma(S, \mathcal{A}^{\otimes d})$. A local calculation shows that

$$\mathcal{H} = \mathrm{Im}(\mathcal{A}^{\otimes d} \to \mathcal{A}^{\otimes d}, f \mapsto fp)$$

is a locally free module of rank d^d and that (left) multiplication by $\mathcal{A}^{\otimes d}$ induces an isomorphism $\mathcal{A}^{\otimes d} \to \mathcal{H}om(\mathcal{H}, \mathcal{H})$. In other words, $\mathcal{A}^{\otimes d}$ is the trivial element of the Brauer group of S as desired. □

In this setting, the analogue of the isomorphism δ of Theorem 60.6 is a map

$$\delta_S : \mathrm{Br}(S) \to H^2_{\acute{e}tale}(S, \mathbf{G}_m).$$

It is true that δ_S is injective. If S is quasi-compact or connected, then $\mathrm{Br}(S)$ is a torsion group, so in this case the image of δ_S is contained in the *cohomological Brauer group* of S

$$\mathrm{Br}'(S) := H^2_{\acute{e}tale}(S, \mathbf{G}_m)_{\text{torsion}}.$$

So if S is quasi-compact or connected, there is an inclusion $\mathrm{Br}(S) \subset \mathrm{Br}'(S)$. This is not always an equality: there exists a nonseparated singular surface S for which $\mathrm{Br}(S) \subset \mathrm{Br}'(S)$ is a strict inclusion. If S is quasi-projective, then $\mathrm{Br}(S) = \mathrm{Br}'(S)$. However, it is not known whether this holds for a smooth proper variety over $\mathbf{C}$, say.

62. The Artin-Schreier sequence

0A3J Let p be a prime number. Let S be a scheme in characteristic p. The *Artin-Schreier* sequence is the short exact sequence

$$0 \longrightarrow \underline{\mathbf{Z}/p\mathbf{Z}}_S \longrightarrow \mathbf{G}_{a,S} \xrightarrow{F-1} \mathbf{G}_{a,S} \longrightarrow 0$$

where $F - 1$ is the map $x \mapsto x^p - x$.

0A3K **Lemma 62.1.** *Let p be a prime. Let S be a scheme of characteristic p.*

(1) *If S is affine, then $H^q_{étale}(S, \underline{\mathbf{Z}/p\mathbf{Z}}) = 0$ for all $q \geq 2$.*
(2) *If S is a quasi-compact and quasi-separated scheme of dimension d, then $H^q_{étale}(S, \underline{\mathbf{Z}/p\mathbf{Z}}) = 0$ for all $q \geq 2 + d$.*

Proof. Recall that the étale cohomology of the structure sheaf is equal to its cohomology on the underlying topological space (Theorem 22.4). The first statement follows from the Artin-Schreier exact sequence and the vanishing of cohomology of the structure sheaf on an affine scheme (Cohomology of Schemes, Lemma 2.2). The second statement follows by the same argument from the vanishing of Cohomology, Proposition 22.4 and the fact that S is a spectral space (Properties, Lemma 2.4). $\square$

0A3L **Lemma 62.2.** *Let k be an algebraically closed field of characteristic $p > 0$. Let V be a finite dimensional k-vector space. Let $F : V \to V$ be a frobenius linear map, i.e., an additive map such that $F(\lambda v) = \lambda^p F(v)$ for all $\lambda \in k$ and $v \in V$. Then $F - 1 : V \to V$ is surjective with kernel a finite dimensional $\mathbf{F}_p$-vector space of dimension $\leq \dim_k(V)$.*

Proof. If $F = 0$, then the statement holds. If we have a filtration of V by F-stable subvector spaces such that the statement holds for each graded piece, then it holds for (V, F). Combining these two remarks we may assume the kernel of F is zero.

Choose a basis $v_1, \ldots, v_n$ of V and write $F(v_i) = \sum a_{ij} v_j$. Observe that $v = \sum \lambda_i v_i$ is in the kernel if and only if $\sum \lambda_i^p a_{ij} v_j = 0$. Since k is algebraically closed this implies the matrix (a_{ij}) is invertible. Let (b_{ij}) be its inverse. Then to see that $F - 1$ is surjective we pick $w = \sum \mu_i v_i \in V$ and we try to solve

$$(F-1)(\sum \lambda_i v_i) = \sum \lambda_i^p a_{ij} v_j - \sum \lambda_j v_j = \sum \mu_j v_j$$

This is equivalent to

$$\sum \lambda_j^p v_j - \sum b_{ij} \lambda_i v_j = \sum b_{ij} \mu_i v_j$$

in other words

$$\lambda_j^p - \sum b_{ij} \lambda_i = \sum b_{ij} \mu_i, \quad j = 1, \ldots, \dim(V).$$

The algebra

$$A = k[x_1, \ldots, x_n]/(x_j^p - \sum b_{ij} x_i - \sum b_{ij} \mu_i)$$

is standard smooth over k (Algebra, Definition 136.6) because the matrix (b_{ij}) is invertible and the partial derivatives of x_j^p are zero. A basis of A over k is the set of monomials $x_1^{e_1} \ldots x_n^{e_n}$ with $e_i < p$, hence $\dim_k(A) = p^n$. Since k is algebraically closed we see that $\mathrm{Spec}(A)$ has exactly p^n points. It follows that $F - 1$ is surjective and every fibre has p^n points, i.e., the kernel of $F - 1$ is a group with p^n elements. $\square$

0A3M **Lemma 62.3.** *Let X be a separated scheme of finite type over a field k. Let $\mathcal{F}$ be a coherent sheaf of $\mathcal{O}_X$-modules. Then $\dim_k H^d(X, \mathcal{F}) < \infty$ where $d = \dim(X)$.*

Proof. We will prove this by induction on d. The case $d = 0$ holds because in that case X is the spectrum of a finite dimensional k-algebra A (Varieties, Lemma 20.2) and every coherent sheaf $\mathcal{F}$ corresponds to a finite A-module $M = H^0(X, \mathcal{F})$ which has $\dim_k M < \infty$.

Assume $d > 0$ and the result has been shown for separated schemes of finite type of dimension $< d$. The scheme X is Noetherian. Consider the property $\mathcal{P}$ of coherent sheaves on X defined by the rule

$$\mathcal{P}(\mathcal{F}) \Leftrightarrow \dim_k H^d(X, \mathcal{F}) < \infty$$

We are going to use the result of Cohomology of Schemes, Lemma 12.4 to prove that $\mathcal{P}$ holds for every coherent sheaf on X.

Let

$$0 \to \mathcal{F}_1 \to \mathcal{F} \to \mathcal{F}_2 \to 0$$

be a short exact sequence of coherent sheaves on X. Consider the long exact sequence of cohomology

$$H^d(X, \mathcal{F}_1) \to H^d(X, \mathcal{F}) \to H^d(X, \mathcal{F}_2)$$

Thus if $\mathcal{P}$ holds for $\mathcal{F}_1$ and $\mathcal{F}_2$, then it hods for $\mathcal{F}$.

Let $Z \subset X$ be an integral closed subscheme. Let $\mathcal{I}$ be a coherent sheaf of ideals on Z. To finish the proof have to show that $H^d(X, i_*\mathcal{I}) = H^d(Z, \mathcal{I})$ is finite dimensional. If $\dim(Z) < d$, then the result holds because the cohomology group will be zero (Cohomology, Proposition 20.7). In this way we reduce to the situation discussed in the following paragraph.

Assume X is a variety of dimension d and $\mathcal{F} = \mathcal{I}$ is a coherent ideal sheaf. In this case we have a short exact sequence

$$0 \to \mathcal{I} \to \mathcal{O}_X \to i_*\mathcal{O}_Z \to 0$$

where $i : Z \to X$ is the closed subscheme defined by $\mathcal{I}$. By induction hypothesis we see that $H^{d-1}(Z, \mathcal{O}_Z) = H^{d-1}(X, i_*\mathcal{O}_Z)$ is finite dimensional. Thus we see that it suffices to prove the result for the structure sheaf.

We can apply Chow's lemma (Cohomology of Schemes, Lemma 18.1) to the morphism $X \to \mathrm{Spec}(k)$. Thus we get a diagram

$$\begin{tikzcd} X \arrow[dr, "g"'] & X' \arrow[l, "\pi"'] \arrow[r, "i"] \arrow[d, "g'"] & \mathbf{P}^n_k \arrow[dl] \\ & \mathrm{Spec}(k) & \end{tikzcd}$$

as in the statement of Chow's lemma. Also, let $U \subset X$ be the dense open subscheme such that $\pi^{-1}(U) \to U$ is an isomorphism. We may assume X' is a variety as well, see Cohomology of Schemes, Remark 18.2. The morphism $i' = (i, \pi) : X' \to \mathbf{P}^n_X$ is a closed immersion (loc. cit.). Hence

$$\mathcal{L} = i^*\mathcal{O}_{\mathbf{P}^n_k}(1) \cong (i')^*\mathcal{O}_{\mathbf{P}^n_X}(1)$$

is π-relatively ample (for example by Morphisms, Lemma 37.7). Hence by Cohomology of Schemes, Lemma 16.2 there exists an $n \geq 0$ such that $R^p\pi_*\mathcal{L}^{\otimes n} = 0$ for

all $p > 0$. Set $\mathcal{G} = \pi_*\mathcal{L}^{\otimes n}$. Choose any nonzero global section s of $\mathcal{L}^{\otimes n}$. Since $\mathcal{G} = \pi_*\mathcal{L}^{\otimes n}$, the section s corresponds to section of $\mathcal{G}$, i.e., a map $\mathcal{O}_X \to \mathcal{G}$. Since $s|_U \neq 0$ as X' is a variety and $\mathcal{L}$ invertible, we see that $\mathcal{O}_X|_U \to \mathcal{G}|_U$ is nonzero. As $\mathcal{G}|_U = \mathcal{K}\mathcal{L}^{\otimes n}|_{\pi^{-1}(U)}$ is invertible we conclude that we have a short exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{G} \to \mathcal{Q} \to 0$$

where $\mathcal{Q}$ is coherent and supported on a proper closed subscheme of X. Arguing as before using our induction hypothesis, we see that it suffices to prove $\dim H^d(X, \mathcal{G}) < \infty$.

By the Leray spectral sequence (Cohomology, Lemma 13.6) we see that $H^d(X, \mathcal{G}) = H^d(X', \mathcal{L}^{\otimes n})$. Let $\overline{X}' \subset \mathbf{P}^n_k$ be the closure of X'. Then $\overline{X}'$ is a projective variety of dimension d over k and $X' \subset \overline{X}'$ is a dense open. The invertible sheaf $\mathcal{L}$ is the restriction of $\mathcal{O}_{\overline{X}'}(n)$ to X. By Cohomology, Proposition 22.4 the map

$$H^d(\overline{X}', \mathcal{O}_{\overline{X}'}(n)) \longrightarrow H^d(X', \mathcal{L}^{\otimes n})$$

is surjective. Since the cohomology group on the left has finite dimension by Cohomology of Schemes, Lemma 14.1 the proof is complete. $\square$

0A3N **Lemma 62.4.** *Let X be separated of finite type over an algebraically closed field k of characteristic $p > 0$. Then $H^q_{étale}(X, \underline{\mathbf{Z}/p\mathbf{Z}}) = 0$ for $q \geq dim(X) + 1$.*

Proof. Let $d = \dim(X)$. By the vanishing established in Lemma 62.1 it suffices to show that $H^{d+1}_{étale}(X, \underline{\mathbf{Z}/p\mathbf{Z}}) = 0$. By Lemma 62.3 we see that $H^d(X, \mathcal{O}_X)$ is a finite dimensional k-vector space. Hence the long exact cohomology sequence associated to the Artin-Schreier sequence ends with

$$H^d(X, \mathcal{O}_X) \xrightarrow{F-1} H^d(X, \mathcal{O}_X) \to H^{d+1}_{étale}(X, \mathbf{Z}/p\mathbf{Z}) \to 0$$

By Lemma 62.2 the map $F - 1$ in this sequence is surjective. This proves the lemma. $\square$

0A3P **Lemma 62.5.** *Let X be a proper scheme over an algebraically closed field k of characteristic $p > 0$. Then*

(1) *$H^q_{étale}(X, \underline{\mathbf{Z}/p\mathbf{Z}})$ is a finite $\mathbf{Z}/p\mathbf{Z}$-module for all q, and*
(2) *$H^q_{étale}(X, \underline{\mathbf{Z}/p\mathbf{Z}}) \to H^q_{étale}(X_{k'}, \underline{\mathbf{Z}/p\mathbf{Z}}))$ is an isomorphism if $k \subset k'$ is an extension of algebraically closed fields.*

Proof. By Cohomology of Schemes, Lemma 19.2) and the comparison of cohomology of Theorem 22.4 the cohomology groups $H^q_{étale}(X, \mathbf{G}_a) = H^q(X, \mathcal{O}_X)$ are finite dimensional k-vector spaces. Hence by Lemma 62.2 the long exact cohomology sequence associated to the Artin-Schreier sequence, splits into short exact sequences

$$0 \to H^q_{étale}(X, \underline{\mathbf{Z}/p\mathbf{Z}}) \to H^q(X, \mathcal{O}_X) \xrightarrow{F-1} H^q(X, \mathcal{O}_X) \to 0$$

and moreover the $\mathbf{F}_p$-dimension of the cohomology groups $H^q_{étale}(X, \underline{\mathbf{Z}/p\mathbf{Z}})$ is equal to the k-dimension of the vector space $H^q(X, \mathcal{O}_X)$. This proves the first statement. The second statement follows as $H^q(X, \mathcal{O}_X) \otimes_k k' \to H^q(X_{k'}, \mathcal{O}_{X_{k'}})$ is an isomorphism by flat base change (Cohomology of Schemes, Lemma 5.2). $\square$

63. Locally constant sheaves

09Y8 This section is the analogue of Modules on Sites, Section 42 for the étale site.

03RU **Definition 63.1.** Let X be a scheme. Let $\mathcal{F}$ be a sheaf of sets on $X_{étale}$.

(1) Let E be a set. We say $\mathcal{F}$ is the *constant sheaf with value* E if $\mathcal{F}$ is the sheafification of the presheaf $U \mapsto E$. Notation: $\underline{E}_X$ or $\underline{E}$.
(2) We say $\mathcal{F}$ is a *constant sheaf* if it is isomorphic to a sheaf as in (1).
(3) We say $\mathcal{F}$ is *locally constant* if there exists a covering $\{U_i \to X\}$ such that $\mathcal{F}|_{U_i}$ is a constant sheaf.
(4) We say that $\mathcal{F}$ is *finite locally constant* if it is locally constant and the values are finite sets.

Let $\mathcal{F}$ be a sheaf of abelian groups on $X_{étale}$.

(1) Let A be an abelian group. We say $\mathcal{F}$ is the *constant sheaf with value* A if $\mathcal{F}$ is the sheafification of the presheaf $U \mapsto A$. Notation: $\underline{A}_X$ or $\underline{A}$.
(2) We say $\mathcal{F}$ is a *constant sheaf* if it is isomorphic as an abelian sheaf to a sheaf as in (1).
(3) We say $\mathcal{F}$ is *locally constant* if there exists a covering $\{U_i \to X\}$ such that $\mathcal{F}|_{U_i}$ is a constant sheaf.
(4) We say that $\mathcal{F}$ is *finite locally constant* if it is locally constant and the values are finite abelian groups.

Let Λ be a ring. Let $\mathcal{F}$ be a sheaf of Λ-modules on $X_{étale}$.

(1) Let M be a Λ-module. We say $\mathcal{F}$ is the *constant sheaf with value* M if $\mathcal{F}$ is the sheafification of the presheaf $U \mapsto M$. Notation: $\underline{M}_X$ or $\underline{M}$.
(2) We say $\mathcal{F}$ is a *constant sheaf* if it is isomorphic as a sheaf of Λ-modules to a sheaf as in (1).
(3) We say $\mathcal{F}$ is *locally constant* if there exists a covering $\{U_i \to X\}$ such that $\mathcal{F}|_{U_i}$ is a constant sheaf.

095A **Lemma 63.2.** *Let $f : X \to Y$ be a morphism of schemes. If $\mathcal{G}$ is a locally constant sheaf of sets, abelian groups, or Λ-modules on $Y_{étale}$, the same is true for $f^{-1}\mathcal{G}$ on $X_{étale}$.*

Proof. Holds for any morphism of topoi, see Modules on Sites, Lemma 42.2. □

095B **Lemma 63.3.** *Let $f : X \to Y$ be a finite étale morphism of schemes. If $\mathcal{F}$ is a (finite) locally constant sheaf of sets, (finite) locally constant sheaf of abelian groups, or (finite type) locally constant sheaf of Λ-modules on $X_{étale}$, the same is true for $f_*\mathcal{F}$ on $Y_{étale}$.*

Proof. The construction of f_* commutes with étale localization. A finite étale morphism is locally isomorphic to a disjoint union of isomorphisms, see Étale Morphisms, Lemma 18.3. Thus the lemma says that if $\mathcal{F}_i$, $i = 1, \ldots, n$ are (finite) locally constant sheaves of sets, then $\prod_{i=1,\ldots,n} \mathcal{F}_i$ is too. This is clear. Similarly for sheaves of abelian groups and modules. □

03RV **Lemma 63.4.** *Let X be a scheme and $\mathcal{F}$ a sheaf of sets on $X_{étale}$. Then the following are equivalent*

(1) *$\mathcal{F}$ is finite locally constant, and*
(2) *$\mathcal{F} = h_U$ for some finite étale morphism $U \to X$.*

Proof. A finite étale morphism is locally isomorphic to a disjoint union of isomorphisms, see Étale Morphisms, Lemma 18.3. Thus (2) implies (1). Conversely, if $\mathcal{F}$ is finite locally constant, then there exists an étale covering $\{X_i \to X\}$ such that $\mathcal{F}|_{X_i}$ is representable by $U_i \to X_i$ finite étale. Arguing exactly as in the proof of Descent, Lemma 36.1 we obtain a descent datum for schemes (U_i, φ_{ij}) relative to $\{X_i \to X\}$ (details omitted). This descent datum is effective for example by Descent, Lemma 34.1 and the resulting morphism of schemes $U \to X$ is finite étale by Descent, Lemmas 20.23 and 20.29. □

095C **Lemma 63.5.** *Let X be a scheme.*

(1) *Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a map of locally constant sheaves of sets on $X_{étale}$. If $\mathcal{F}$ is finite locally constant, there exists an étale covering $\{U_i \to X\}$ such that $\varphi|_{U_i}$ is the map of constant sheaves associated to a map of sets.*
(2) *Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a map of locally constant sheaves of abelian groups on $X_{étale}$. If $\mathcal{F}$ is finite locally constant, there exists an étale covering $\{U_i \to X\}$ such that $\varphi|_{U_i}$ is the map of constant abelian sheaves associated to a map of abelian groups.*
(3) *Let Λ be a ring. Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a map of locally constant sheaves of Λ-modules on $X_{étale}$. If $\mathcal{F}$ is of finite type, then there exists an étale covering $\{U_i \to X\}$ such that $\varphi|_{U_i}$ is the map of constant sheaves of Λ-modules associated to a map of Λ-modules.*

Proof. This holds on any site, see Modules on Sites, Lemma 42.3. □

03RX **Lemma 63.6.** *Let X be a scheme.*

(1) *The category of finite locally constant sheaves of sets is closed under finite limits and colimits inside $Sh(X_{étale})$.*
(2) *The category of finite locally constant abelian sheaves is a weak Serre subcategory of $Ab(X_{étale})$.*
(3) *Let Λ be a Noetherian ring. The category of finite type, locally constant sheaves of Λ-modules on $X_{étale}$ is a weak Serre subcategory of $Mod(X_{étale}, \Lambda)$.*

Proof. This holds on any site, see Modules on Sites, Lemma 42.5. □

095D **Lemma 63.7.** *Let X be a scheme. Let Λ be a ring. The tensor product of two locally constant sheaves of Λ-modules on $X_{étale}$ is a locally constant sheaf of Λ-modules.*

Proof. This holds on any site, see Modules on Sites, Lemma 42.6. □

09BF **Lemma 63.8.** *Let X be a connected scheme. Let Λ be a ring and let $\mathcal{F}$ be a locally constant sheaf of Λ-modules. Then there exists a Λ-module M and an étale covering $\{U_i \to X\}$ such that $\mathcal{F}|_{U_i} \cong \underline{M}|_{U_i}$.*

Proof. Choose an étale covering $\{U_i \to X\}$ such that $\mathcal{F}|_{U_i}$ is constant, say $\mathcal{F}|_{U_i} \cong \underline{M_i}_{U_i}$. Observe that $U_i \times_X U_j$ is empty if M_i is not isomorphic to M_j. For each Λ-module M let $I_M = \{i \in I \mid M_i \cong M\}$. As étale morphisms are open we see that $U_M = \bigcup_{i \in I_M} \text{Im}(U_i \to X)$ is an open subset of X. Then $X = \coprod U_M$ is a disjoint open covering of X. As X is connected only one U_M is nonempty and the lemma follows. □

64. Locally constant sheaves and the fundamental group

0DV4 We can relate locally constant sheaves to the fundamental group of a scheme in some cases.

0DV5 **Lemma 64.1.** *Let X be a connected scheme. Let $\overline{x}$ be a geometric point of X.*

(1) *There is an equivalence of categories*

$$\left\{\begin{matrix}\textit{finite locally constant} \\ \textit{sheaves of sets on } X_{\acute{e}tale}\end{matrix}\right\} \longleftrightarrow \{\textit{finite } \pi_1(X, \overline{x})\textit{-sets}\}$$

(2) *There is an equivalence of categories*

$$\left\{\begin{matrix}\textit{finite locally constant} \\ \textit{sheaves of abelian groups on } X_{\acute{e}tale}\end{matrix}\right\} \longleftrightarrow \{\textit{finite } \pi_1(X, \overline{x})\textit{-modules}\}$$

(3) *Let Λ be a finite ring. There is an equivalence of categories*

$$\left\{\begin{matrix}\textit{finite type, locally constant} \\ \textit{sheaves of } \Lambda\textit{-modules on } X_{\acute{e}tale}\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\textit{finite } \pi_1(X, \overline{x})\textit{-modules endowed} \\ \textit{with commuting } \Lambda\textit{-module structure}\end{matrix}\right\}$$

Proof. We observe that $\pi_1(X, \overline{x})$ is a profinite topological group, see Fundamental Groups, Definition 6.1. The left hand categories are defined in Section 63. The notation used in the right hand categories is taken from Fundamental Groups, Definition 2.1 for sets and Definition 56.1 for abelian groups. This explains the notation.

Assertion (1) follows from Lemma 63.4 and Fundamental Groups, Theorem 6.2. Parts (2) and (3) follow immediately from this by endowing the underlying (sheaves of) sets with additional structure. For example, a finite locally constant sheaf of abelian groups on $X_{\acute{e}tale}$ is the same thing as a finite locally constant sheaf of sets $\mathcal{F}$ together with a map $+ : \mathcal{F} \times \mathcal{F} \to \mathcal{F}$ satisfying the usual axioms. The equivalence in (1) sends products to products and hence sends $+$ to an addition on the corresponding finite $\pi_1(X, \overline{x})$-set. Since $\pi_1(X, \overline{x})$-modules are the same thing as $\pi_1(X, \overline{x})$-sets with a compatible abelian group structure we obtain (2). Part (3) is proved in exactly the same way. $\square$

0DV6 **Remark 64.2.** The equivalences of Lemma 64.1 are compatible with pullbacks. More precisely, suppose $f : Y \to X$ is a morphism of connected schemes. Let $\overline{y}$ be geometric point of Y and set $\overline{x} = f(\overline{y})$. Then the diagram

$$\begin{tikzcd}
\text{finite locally constant sheaves of sets on } Y_{\acute{e}tale} \arrow[r] & \text{finite } \pi_1(Y, \overline{y})\text{-sets} \\
\text{finite locally constant sheaves of sets on } X_{\acute{e}tale} \arrow[r] \arrow[u, "f^{-1}"'] & \text{finite } \pi_1(X, \overline{x})\text{-sets} \arrow[u]
\end{tikzcd}$$

is commutative, where the vertical arrow on the right comes from the continuous homomorphism $\pi_1(Y, \overline{y}) \to \pi_1(X, \overline{x})$ induced by f. This follows immediately from the commutative diagram in Fundamental Groups, Theorem 6.2.

65. Méthode de la trace

03SH A reference for this section is [AGV71, Exposé IX, §5]. The material here will be used in the proof of Lemma 80.8 below.

Let $f : Y \to X$ be an étale morphism of schemes. There is a sequence

$$f_!, f^{-1}, f_*$$

of adjoint functors between $Ab(X_{\acute{e}tale})$ and $Ab(Y_{\acute{e}tale})$. The functor $f_!$ is discussed in Section 69. The adjunction map $\text{id} \to f_*f^{-1}$ is called *restriction*. The adjunction map $f_!f^{-1} \to \text{id}$ is often called the *trace map*. If f is finite étale, then $f_* = f_!$ (Lemma 69.7) and we can view this as a map $f_*f^{-1} \to \text{id}$.

03SE **Definition 65.1.** Let $f : Y \to X$ be a finite étale morphism of schemes. The map $f_*f^{-1} \to \text{id}$ described above and explicitly below is called the *trace*.

Let $f : Y \to X$ be a finite étale morphism of schemes. The trace map is characterized by the following two properties:

(1) it commutes with étale localization on X and
(2) if $Y = \coprod_{i=1}^d X$ then the trace map is the sum map $f_*f^{-1}\mathcal{F} = \mathcal{F}^{\oplus d} \to \mathcal{F}$.

By Étale Morphisms, Lemma 18.3 every finite étale morphism $f : Y \to X$ is étale locally on X of the form given in (2) for some integer $d \geq 0$. Hence we can define the trace map using the characterization given; in particular we do not need to know about the existence of $f_!$ and the agreement of $f_!$ with f_* in order to construct the trace map. This description shows that if f has constant degree d, then the composition

$$\mathcal{F} \xrightarrow{res} f_*f^{-1}\mathcal{F} \xrightarrow{trace} \mathcal{F}$$

is multiplication by d. The "méthode de la trace" is the following observation: if $\mathcal{F}$ is an abelian sheaf on $X_{\acute{e}tale}$ such that multiplication by d on $\mathcal{F}$ is an isomorphism, then the map

$$H^n_{\acute{e}tale}(X, \mathcal{F}) \longrightarrow H^n_{\acute{e}tale}(Y, f^{-1}\mathcal{F})$$

is injective. Namely, we have

$$H^n_{\acute{e}tale}(Y, f^{-1}\mathcal{F}) = H^n_{\acute{e}tale}(X, f_*f^{-1}\mathcal{F})$$

by the vanishing of the higher direct images (Proposition 54.2) and the Leray spectral sequence (Proposition 53.2). Thus we can consider the maps

$$H^n_{\acute{e}tale}(X, \mathcal{F}) \to H^n_{\acute{e}tale}(Y, f^{-1}\mathcal{F}) = H^n_{\acute{e}tale}(X, f_*f^{-1}\mathcal{F}) \xrightarrow{trace} H^n_{\acute{e}tale}(X, \mathcal{F})$$

and the composition is an isomorphism (under our assumption on $\mathcal{F}$ and f). In particular, if $H^q_{\acute{e}tale}(Y, f^{-1}\mathcal{F}) = 0$ then $H^q_{\acute{e}tale}(X, \mathcal{F}) = 0$ as well. Indeed, multiplication by d induces an isomorphism on $H^q_{\acute{e}tale}(X, \mathcal{F})$ which factors through $H^q_{\acute{e}tale}(Y, f^{-1}\mathcal{F}) = 0$.

This is often combined with the following.

0A3R **Lemma 65.2.** *Let S be a connected scheme. Let ℓ be a prime number. Let $\mathcal{F}$ be a finite type, locally constant sheaf of $\mathbf{F}_\ell$-vector spaces on $S_{\acute{e}tale}$. Then there exists a finite étale morphism $f : T \to S$ of degree prime to ℓ such that $f^{-1}\mathcal{F}$ has a finite filtration whose successive quotients are $\underline{\mathbf{Z}/\ell\mathbf{Z}}_T$.*

Proof. Choose a geometric point $\overline{s}$ of S. Via the equivalence of Lemma 64.1 the sheaf $\mathcal{F}$ corresponds to a finite dimensional $\mathbf{F}_\ell$-vector space V with a continuous $\pi_1(S, \overline{s})$-action. Let $G \subset \text{Aut}(V)$ be the image of the homomorphism $\rho : \pi_1(S, \overline{s}) \to \text{Aut}(V)$ giving the action. Observe that G is finite. The surjective continuous homomorphism $\overline{\rho} : \pi_1(S, \overline{s}) \to G$ corresponds to a Galois object $Y \to S$ of $F\acute{E}t_S$ with automorphism group $G = \text{Aut}(Y/S)$, see Fundamental Groups, Section 7. Let

$H \subset G$ be an ℓ-Sylow subgroup. We claim that $T = Y/H \to S$ works. Namely, let $\bar{t} \in T$ be a geometric point over $\bar{s}$. The image of $\pi_1(T, \bar{t}) \to \pi_1(S, \bar{s})$ is $(\overline{\rho})^{-1}(H)$ as follows from the functorial nature of fundamental groups. Hence the action of $\pi_1(T, \bar{t})$ on V corresponding to $f^{-1}\mathcal{F}$ is through the map $\pi_1(T, \bar{t}) \to H$, see Remark 64.2. As H is a finite ℓ-group, the irreducible constituents of the representation $\rho|_{\pi_1(T,\bar{t})}$ are each trivial of rank 1 (this is a simple lemma on representation theory of finite groups; insert future reference here). Via the equivalence of Lemma 64.1 this means $f^{-1}\mathcal{F}$ is a successive extension of constant sheaves with value $\underline{\mathbf{Z}/\ell\mathbf{Z}}_T$. Moreover the degree of $T = Y/H \to S$ is prime to ℓ as it is equal to the index of H in G. $\square$

66. Galois cohomology

0A2M In this section we prove a result on Galois cohomology (Proposition 66.4) using étale cohomology and the trick from Section 65. This will allow us to prove vanishing of higher étale cohomology groups over the spectrum of a field.

0DV7 **Lemma 66.1.** *Let ℓ be a prime number and n an integer > 0. Let S be a quasi-compact and quasi-separated scheme. Let $X = \lim_{i \in I} X_i$ be the limit of a directed system of S-schemes each $X_i \to S$ being finite étale of constant degree relatively prime to ℓ. For any abelian ℓ-power torsion sheaf $\mathcal{G}$ on S such that $H^n_{étale}(S, \mathcal{G}) \neq 0$ there exists an ℓ-power torsion sheaf $\mathcal{F}$ on X such that $H^n_{étale}(X, \mathcal{F}) \neq 0$*

Proof. Let $g : X \to S$ and $g_i : X_i \to S$ denote the structure morphisms. Fix an ℓ-power torsion sheaf $\mathcal{G}$ on S with $H^n_{étale}(S, \mathcal{G}) \neq 0$. The system given by $\mathcal{G}_i = g_i^{-1}\mathcal{G}$ satisify the conditions of Theorem 51.3 with colimit sheaf given by $g^{-1}\mathcal{G}$. This tells us that:

$$\mathrm{colim}_{i \in I} H^p_{étale}(X_i, g_i^{-1}\mathcal{G}) = H^p_{étale}(X, \mathcal{G})$$

By virtue of the g_i being finite étale morphism of degree prime to ℓ we can apply “la méthode de la trace” and we find the maps

$$H^n_{étale}(S, \mathcal{G}) \to H^p_{étale}(X_i, g_i^{-1}\mathcal{G})$$

are all injective (and compatible with the transition maps). See Section 65. Thus, the colimit is non-zero, i.e., $H^n(X, g^{-1}\mathcal{G}) \neq 0$, giving us the desired result with $\mathcal{F} = g^{-1}\mathcal{G}$.

Conversely, suppose given an ℓ-power torsion sheaf $\mathcal{F}$ on X with $H^n_{étale}(X, \mathcal{F}) \neq 0$. We note that since the g_i are finite morphisms the higher direct images vanish (Proposition 54.2). Then, by applying Lemma 51.7 we may also conclude the same for g. The vanishing of the higher direct images tells us that $H^n_{étale}(X, \mathcal{F}) = H^n(S, g_*\mathcal{F}) \neq 0$ by Leray (Proposition 53.2) giving us what we want with $\mathcal{G} = g_*\mathcal{F}$. $\square$

0DV8 **Lemma 66.2.** *Let ℓ be a prime number and n an integer > 0. Let K be a field with $G = Gal(K^{sep}/K)$ and let $H \subset G$ be a maximal pro-ℓ subgroup with L/K being the corresponding field extension. Then $H^n_{étale}(\mathrm{Spec}(K), \mathcal{F}) = 0$ for all ℓ-power torsion $\mathcal{F}$ if and only if $H^n_{étale}(\mathrm{Spec}(L), \underline{\mathbf{Z}/\ell\mathbf{Z}}) = 0$.*

Proof. Write $L = \bigcup L_i$ as the union of its finite subextensions over K. Our choice of H implies that $[L_i : K]$ is prime to ℓ. Thus $\mathrm{Spec}(L) = \lim_{i \in I} \mathrm{Spec}(L_i)$ as in Lemma 66.1. Thus we may replace K by L and assume that the absolute Galois group G of K is a profinite pro-ℓ group.

Assume $H^n(\mathrm{Spec}(K), \underline{\mathbf{Z}/\ell\mathbf{Z}}) = 0$. Let $\mathcal{F}$ be an ℓ-power torsion sheaf on $\mathrm{Spec}(K)_{\acute{e}tale}$. We will show that $H^n_{\acute{e}tale}(\mathrm{Spec}(K), \mathcal{F}) = 0$. By the correspondence specified in Lemma 58.1 our sheaf $\mathcal{F}$ corresponds to an ℓ-power torsion G-module M. Any finite set of elements $x_1, \ldots, x_m \in M$ must be fixed by an open subgroup U by continuity. Let M' be the module spanned by the orbits of $x_1, \ldots, x_m$. This is a finite abelian ℓ-group as each x_i is killed by a power of ℓ and the orbits are finite. Since M is the filtered colimit of these submodules M', we see that $\mathcal{F}$ is the filtered colimit of the corresponding subsheaves $\mathcal{F}' \subset \mathcal{F}$. Applying Theorem 51.3 to this colimit, we reduce to the case where $\mathcal{F}$ is a finite locally constant sheaf.

Let M be a finite abelian ℓ-group with a continuous action of the profinite pro-ℓ group G. Then there is a G-invariant filtration

$$0 = M_0 \subset M_1 \subset \ldots \subset M_r = M$$

such that $M_{i+1}/M_i \cong \mathbf{Z}/\ell\mathbf{Z}$ with trivial G-action (this is a simple lemma on representation theory of finite groups; insert future reference here). Thus the corresponding sheaf $\mathcal{F}$ has a filtration

$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \ldots \subset \mathcal{F}_r = \mathcal{F}$$

with successive quotients isomorphic to $\underline{\mathbf{Z}/\ell\mathbf{Z}}$. Thus by induction and the long exact cohomology sequence we conclude. □

0DV9 **Lemma 66.3.** *Let ℓ be a prime number and n an integer > 0. Let K be a field with $G = Gal(K^{sep}/K)$ and let $H \subset G$ be a maximal pro-ℓ subgroup with L/K being the corresponding field extension. Then $H^q_{\acute{e}tale}(\mathrm{Spec}(K), \mathcal{F}) = 0$ for $q \geq n$ and all ℓ-torsion sheaves $\mathcal{F}$ if and only if $H^n_{\acute{e}tale}(\mathrm{Spec}(L), \underline{\mathbf{Z}/\ell\mathbf{Z}}) = 0$.*

Proof. The forward direction is trivial, so we need only prove the reverse direction. We proceed by induction on q. The case of $q = n$ is Lemma 66.2. Now let $\mathcal{F}$ be an ℓ-power torsion sheaf on $\mathrm{Spec}(K)$. Let $f : \mathrm{Spec}(K^{sep}) \to \mathrm{Spec}(K)$ be the inclusion of a geometric point. Then consider the exact sequence:

$$0 \to \mathcal{F} \xrightarrow{res} f_*f^{-1}\mathcal{F} \to f_*f^{-1}\mathcal{F}/\mathcal{F} \to 0$$

Note that K^{sep} may be written as the filtered colimit of finite separable extensions. Thus f is the limit of a directed system of finite étale morphisms. We may, as was seen in the proof of Lemma 66.1, conclude that f has vanishing higher direct images. Thus, we may express the higher cohomology of $f_*f^{-1}\mathcal{F}$ as the higher cohomology on the geometric point which clearly vanishes. Hence, as everything here is still ℓ-torsion, we may use the inductive hypothesis in conjunction with the long-exact cohomology sequence to conclude the result for $q + 1$. □

03R8 **Proposition 66.4.** *Let K be a field with separable algebraic closure K^{sep}. Assume that for any finite extension K' of K we have $Br(K') = 0$. Then* [Ser97, Chapter II, Section 3, Proposition 5]

1. *$H^q(Gal(K^{sep}/K), (K^{sep})^*) = 0$ for all $q \geq 1$, and*
2. *$H^q(Gal(K^{sep}/K), M) = 0$ for any torsion $Gal(K^{sep}/K)$-module M and any $q \geq 2$,*

Proof. Set $p = \mathrm{char}(K)$. By Lemma 58.2, Theorem 60.6, and Example 58.3 the proposition is equivalent to showing that if $H^2(\mathrm{Spec}(K'), \mathbf{G}_m|_{\mathrm{Spec}(K')_{\acute{e}tale}}) = 0$ for all finite extensions K'/K then:

- $H^q(\mathrm{Spec}(K), \mathbf{G}_m|_{\mathrm{Spec}(K)_{\acute{e}tale}}) = 0$ for all $q \geq 1$, and

- $H^q(\mathrm{Spec}(K), \mathcal{F}) = 0$ for any torsion sheaf $\mathcal{F}$ and any $q \geq 2$.

We prove the second part first. Since $\mathcal{F}$ is a torsion sheaf, we may use the ℓ-primary decomposition as well as the compatibility of cohomology with colimits (i.e, direct sums, see Theorem 51.3) to reduce to showing $H^q(\mathrm{Spec}(K), \mathcal{F}) = 0$, $q \geq 2$ for all ℓ-power torsion sheaves for every prime ℓ. This allows us to analyze each prime individually.

Suppose that $\ell \neq p$. For any extension K'/K consider the Kummer sequence (Lemma 28.1)

$$0 \to \mu_{\ell, \mathrm{Spec}\, K'} \to \mathbf{G}_{m, \mathrm{Spec}\, K'} \xrightarrow{(\cdot)^\ell} \mathbf{G}_{m, \mathrm{Spec}\, K'} \to 0$$

Since $H^q(\mathrm{Spec}\, K', \mathbf{G}_m|_{\mathrm{Spec}(K')_{\acute{e}tale}}) = 0$ for $q = 2$ by assumption and for $q = 1$ by Theorem 24.1 combined with $\mathrm{Pic}(K) = (0)$. Thus, by the long-exact cohomology sequence we may conclude that $H^2(\mathrm{Spec}\, K', \mu_\ell) = 0$ for any separable K'/K. Now let H be a maximal pro-ℓ subgroup of the absolute Galois group of K and let L be the corresponding extension. We can write L as the colimit of finite extensions, applying Theorem 51.3 to this colimit we see that $H^2(\mathrm{Spec}(L), \mu_\ell) = 0$. Now μ_ℓ must be the constant sheaf. If it weren't, that would imply there exists a Galois extension of degree relatively prime to ℓ of L which is not true by definition of L (namely, the extension one gets by adjoining the ℓth roots of unity to L). Hence, via Lemma 66.3, we conclude the result for $\ell \neq p$.

Now suppose that $\ell = p$. We consider the Artin-Schrier exact sequence (Section 62)

$$0 \longrightarrow \underline{\mathbf{Z}/p\mathbf{Z}}_{\mathrm{Spec}\, K} \longrightarrow \mathbf{G}_{a, \mathrm{Spec}\, K} \xrightarrow{F-1} \mathbf{G}_{a, \mathrm{Spec}\, K} \longrightarrow 0$$

where $F - 1$ is the map $x \mapsto x^p - x$. Then note that the higher Cohomology of $\mathbf{G}_{a, \mathrm{Spec}\, K}$ vanishes, by Remark 23.4 and the vanishing of the higher cohomology of the structure sheaf of an affine scheme (Cohomology of Schemes, Lemma 2.2). Note this can be applied to any field of characteristic p. In particular, we can apply it to the field extension L defined by a maximal pro-p subgroup H. This allows us to conclude $H^n(\mathrm{Spec}\, L, \underline{\mathbf{Z}/p\mathbf{Z}}_{\mathrm{Spec}\, L}) = 0$ for $n \geq 2$, from which the result follows for $\ell = p$, by Lemma 66.3.

To finish the proof we still have to show that $H^q(\mathrm{Gal}(K^{sep}/K), (K^{sep})^*) = 0$ for all $q \geq 1$. Set $G = \mathrm{Gal}(K^{sep}/K)$ and set $M = (K^{sep})^*$ viewed as a G-module. We have already shown (above) that $H^1(G, M) = 0$ and $H^2(G, M) = 0$. Consider the exact sequence

$$0 \to A \to M \to M \otimes \mathbf{Q} \to B \to 0$$

of G-modules. By the above we have $H^i(G, A) = 0$ and $H^i(G, B) = 0$ for $i > 1$ since A and B are torsion G-modules. By Lemma 56.6 we have $H^i(G, M \otimes \mathbf{Q}) = 0$ for $i > 0$. It is a pleasant exercise to see that this implies that $H^i(G, M) = 0$ also for $i \geq 3$. $\square$

03R9 **Definition 66.5.** A field K is called C_r if for every $0 < d^r < n$ and every $f \in K[T_1, \ldots, T_n]$ homogeneous of degree d, there exist $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\alpha_i \in K$ not all zero, such that $f(\alpha) = 0$. Such an α is called a *nontrivial solution* of f.

03RA **Example 66.6.** An algebraically closed field is C_r.

In fact, we have the following simple lemma.

03RB **Lemma 66.7.** *Let k be an algebraically closed field. Let $f_1, \ldots, f_s \in k[T_1, \ldots, T_n]$ be homogeneous polynomials of degree $d_1, \ldots, d_s$ with $d_i > 0$. If $s < n$, then $f_1 = \ldots = f_s = 0$ have a common nontrivial solution.*

Proof. This follows from dimension theory, for example in the form of Varieties, Lemma 33.2 applied $s-1$ times. □

The following result computes the Brauer group of C_1 fields.

03RC **Theorem 66.8.** *Let K be a C_1 field. Then $Br(K) = 0$.*

Proof. Let D be a finite dimensional division algebra over K with center K. We have seen that

$$D \otimes_K K^{sep} \cong \text{Mat}_d(K^{sep})$$

uniquely up to inner isomorphism. Hence the determinant $\det : \text{Mat}_d(K^{sep}) \to K^{sep}$ is Galois invariant and descends to a homogeneous degree d map

$$\det = N_{\text{red}} : D \longrightarrow K$$

called the *reduced norm*. Since K is C_1, if $d > 1$, then there exists a nonzero $x \in D$ with $N_{\text{red}}(x) = 0$. This clearly implies that x is not invertible, which is a contradiction. Hence $\text{Br}(K) = 0$. □

03RE **Definition 66.9.** Let k be a field. A *variety* is separated, integral scheme of finite type over k. A *curve* is a variety of dimension 1.

03RD **Theorem 66.10** (Tsen's theorem)**.** *The function field of a variety of dimension r over an algebraically closed field k is C_r.*

Proof. For projective space one can show directly that the field $k(x_1, \ldots, x_r)$ is C_r (exercise).

General case. Without loss of generality, we may assume X to be projective. Let $f \in K[T_1, \ldots, T_n]_d$ with $0 < d^r < n$. Say the coefficients of f are in $\Gamma(X, \mathcal{O}_X(H))$ for some ample $H \subset X$. Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ with $\alpha_i \in \Gamma(X, \mathcal{O}_X(eH))$. Then $f(\alpha) \in \Gamma(X, \mathcal{O}_X((de+1)H))$. Consider the system of equations $f(\alpha) = 0$. Then by asymptotic Riemann-Roch (Varieties, Proposition 44.13) there exists a $c > 0$ such that

- the number of variables is $n \dim_K \Gamma(X, \mathcal{O}_X(eH)) \sim ne^r c$, and
- the number of equations is $\dim_K \Gamma(X, \mathcal{O}_X((de+1)H)) \sim (de+1)^r c$.

Since $n > d^r$, there are more variables than equations. The equations are homogeneous hence there is a solution by Lemma 66.7. □

03RF **Lemma 66.11.** *Let C be a curve over an algebraically closed field k. Then the Brauer group of the function field of C is zero: $Br(k(C)) = 0$.*

Proof. This is clear from Tsen's theorem, Theorem 66.10 and Theorem 66.8. □

03RG **Lemma 66.12.** *Let k be an algebraically closed field and $k \subset K$ a field extension of transcendence degree 1. Then for all $q \geq 1$, $H^q_{\acute{e}tale}(\text{Spec}(K), \mathbf{G}_m) = 0$.*

Proof. Recall that $H^q_{\acute{e}tale}(\text{Spec}(K), \mathbf{G}_m) = H^q(\text{Gal}(K^{sep}/K), (K^{sep})^*)$ by Lemma 58.2. Thus by Proposition 66.4 it suffices to show that if $K \subset K'$ is a finite field extension, then $\text{Br}(K') = 0$. Now observe that $K' = \text{colim}\, K''$, where K'' runs over the finitely generated subextensions of k contained in K' of transcendence degree

1. Note that $\mathrm{Br}(K') = \mathrm{colim}\,\mathrm{Br}(K'')$ which reduces us to a finitely generated field extension K''/k of transcendence degree 1. Such a field is the function field of a curve over k, hence has trivial Brauer group by Lemma 66.11. □

67. Higher vanishing for the multiplicative group

03RH In this section, we fix an algebraically closed field k and a smooth curve X over k. We denote $i_x : x \hookrightarrow X$ the inclusion of a closed point of X and $j : \eta \hookrightarrow X$ the inclusion of the generic point. We also denote X_0 the set of closed points of X.

03RI **Theorem 67.1** (The Fundamental Exact Sequence). *There is a short exact sequence of étale sheaves on X*

$$0 \longrightarrow \mathbf{G}_{m,X} \longrightarrow j_*\mathbf{G}_{m,\eta} \longrightarrow \bigoplus\nolimits_{x\in X_0} i_{x*}\underline{\mathbf{Z}} \longrightarrow 0.$$

Proof. Let $\varphi : U \to X$ be an étale morphism. Then by properties of étale morphisms (Proposition 26.2), $U = \coprod_i U_i$ where each U_i is a smooth curve mapping to X. The above sequence for U is a product of the corresponding sequences for each U_i, so it suffices to treat the case where U is connected, hence irreducible. In this case, there is a well known exact sequence

$$1 \longrightarrow \Gamma(U, \mathcal{O}_U^*) \longrightarrow k(U)^* \longrightarrow \bigoplus\nolimits_{y\in U^0} \mathbf{Z}_y.$$

This amounts to a sequence

$$0 \longrightarrow \Gamma(U, \mathcal{O}_U^*) \longrightarrow \Gamma(\eta \times_X U, \mathcal{O}^*_{\eta\times_X U}) \longrightarrow \bigoplus\nolimits_{x\in X_0} \Gamma(x \times_X U, \underline{\mathbf{Z}})$$

which, unfolding definitions, is nothing but a sequence

$$0 \longrightarrow \mathbf{G}_m(U) \longrightarrow j_*\mathbf{G}_{m,\eta}(U) \longrightarrow \left(\bigoplus\nolimits_{x\in X_0} i_{x*}\underline{\mathbf{Z}}\right)(U).$$

This defines the maps in the Fundamental Exact Sequence and shows it is exact except possibly at the last step. To see surjectivity, let us recall that if U is a nonsingular curve and D is a divisor on U, then there exists a Zariski open covering $\{U_j \to U\}$ of U such that $D|_{U_j} = \mathrm{div}(f_j)$ for some $f_j \in k(U)^*$. □

03RJ **Lemma 67.2.** *For any $q \geq 1$, $R^q j_*\mathbf{G}_{m,\eta} = 0$.*

Proof. We need to show that $(R^q j_*\mathbf{G}_{m,\eta})_{\bar{x}} = 0$ for every geometric point $\bar{x}$ of X.

Assume that $\bar{x}$ lies over a closed point x of X. Let $\mathrm{Spec}(A)$ be an affine open neighbourhood of x in X, and K the fraction field of A. Then

$$\mathrm{Spec}(\mathcal{O}^{sh}_{X,\bar{x}}) \times_X \eta = \mathrm{Spec}(\mathcal{O}^{sh}_{X,\bar{x}} \otimes_A K).$$

The ring $\mathcal{O}^{sh}_{X,\bar{x}} \otimes_A K$ is a localization of the discrete valuation ring $\mathcal{O}^{sh}_{X,\bar{x}}$, so it is either $\mathcal{O}^{sh}_{X,\bar{x}}$ again, or its fraction field $K^{sh}_{\bar{x}}$. But since some local uniformizer gets inverted, it must be the latter. Hence

$$(R^q j_*\mathbf{G}_{m,\eta})_{(X,\bar{x})} = H^q_{\acute{e}tale}(\mathrm{Spec}\, K^{sh}_{\bar{x}}, \mathbf{G}_m).$$

Now recall that $\mathcal{O}^{sh}_{X,\bar{x}} = \mathrm{colim}_{(U,\bar{u})\to\bar{x}}\, \mathcal{O}(U) = \mathrm{colim}_{A\subset B}\, B$ where $A \to B$ is étale, hence $K^{sh}_{\bar{x}}$ is an algebraic extension of $K = k(X)$, and we may apply Lemma 66.12 to get the vanishing.

Assume that $\bar{x} = \bar{\eta}$ lies over the generic point η of X (in fact, this case is superfluous). Then $\mathcal{O}_{X,\bar{\eta}} = \kappa(\eta)^{sep}$ and thus

$$\begin{aligned}(R^q j_* \mathbf{G}_{m,\eta})_{\bar{\eta}} &= H^q_{\acute{e}tale}(\mathrm{Spec}(\kappa(\eta)^{sep}) \times_X \eta, \mathbf{G}_m) \\ &= H^q_{\acute{e}tale}(\mathrm{Spec}(\kappa(\eta)^{sep}), \mathbf{G}_m) \\ &= 0 \quad \text{for } q \geq 1\end{aligned}$$

since the corresponding Galois group is trivial. □

03RK **Lemma 67.3.** *For all $p \geq 1$, $H^p_{\acute{e}tale}(X, j_* \mathbf{G}_{m,\eta}) = 0$.*

Proof. The Leray spectral sequence reads

$$E_2^{p,q} = H^p_{\acute{e}tale}(X, R^q j_* \mathbf{G}_{m,\eta}) \Rightarrow H^{p+q}_{\acute{e}tale}(\eta, \mathbf{G}_{m,\eta}),$$

which vanishes for $p + q \geq 1$ by Lemma 66.12. Taking $q = 0$, we get the desired vanishing. □

03RL **Lemma 67.4.** *For all $q \geq 1$, $H^q_{\acute{e}tale}(X, \bigoplus_{x \in X_0} i_{x*} \underline{\mathbf{Z}}) = 0$.*

Proof. For X quasi-compact and quasi-separated, cohomology commutes with colimits, so it suffices to show the vanishing of $H^q_{\acute{e}tale}(X, i_{x*}\underline{\mathbf{Z}})$. But then the inclusion i_x of a closed point is finite so $R^p i_{x*}\underline{\mathbf{Z}} = 0$ for all $p \geq 1$ by Proposition 54.2. Applying the Leray spectral sequence, we see that $H^q_{\acute{e}tale}(X, i_{x*}\underline{\mathbf{Z}}) = H^q_{\acute{e}tale}(x, \underline{\mathbf{Z}})$. Finally, since x is the spectrum of an algebraically closed field, all higher cohomology on x vanishes. □

Concluding this series of lemmata, we get the following result.

03RM **Theorem 67.5.** *Let X be a smooth curve over an algebraically closed field. Then*

$$H^q_{\acute{e}tale}(X, \mathbf{G}_m) = 0 \quad \text{for all } q \geq 2.$$

Proof. See discussion above. □

We also get the cohomology long exact sequence

$$0 \to H^0_{\acute{e}tale}(X, \mathbf{G}_m) \to H^0_{\acute{e}tale}(X, j_* \mathbf{G}_{m\eta}) \to H^0_{\acute{e}tale}(X, \bigoplus i_{x*}\underline{\mathbf{Z}}) \to H^1_{\acute{e}tale}(X, \mathbf{G}_m) \to 0$$

although this is the familiar

$$0 \to H^0_{Zar}(X, \mathcal{O}_X^*) \to k(X)^* \to \mathrm{Div}(X) \to \mathrm{Pic}(X) \to 0.$$

68. Picard groups of curves

03RN Our next step is to use the Kummer sequence to deduce some information about the cohomology group of a curve with finite coefficients. In order to get vanishing in the long exact sequence, we review some facts about Picard groups.

Let X be a smooth projective curve over an algebraically closed field k. Let $g = \dim_k H^1(X, \mathcal{O}_X)$ be the genus of X. There exists a short exact sequence

$$0 \to \mathrm{Pic}^0(X) \to \mathrm{Pic}(X) \xrightarrow{\deg} \mathbf{Z} \to 0.$$

The abelian group $\mathrm{Pic}^0(X)$ can be identified with $\mathrm{Pic}^0(X) = \underline{\mathrm{Pic}}^0_{X/k}(k)$, i.e., the k-valued points of an abelian variety $\underline{\mathrm{Pic}}^0_{X/k}$ over k of dimension g. Consequently, if $n \in k^*$ then $\mathrm{Pic}^0(X)[n] \cong (\mathbf{Z}/n\mathbf{Z})^{2g}$ as abelian groups. See Picard Schemes of Curves, Section 6 and Groupoids, Section 9. This key fact, namely the description

of the torsion in the Picard group of a smooth projective curve over an algebraically closed field does not appear to have an elementary proof.

03RQ **Lemma 68.1.** *Let X be a smooth projective curve of genus g over an algebraically closed field k and let $n \geq 1$ be invertible in k. Then there are canonical identifications*

$$H^q_{\acute{e}tale}(X, \mu_n) = \begin{cases} \mu_n(k) & \text{if } q = 0, \\ \mathrm{Pic}^0(X)[n] & \text{if } q = 1, \\ \mathbf{Z}/n\mathbf{Z} & \text{if } q = 2, \\ 0 & \text{if } q \geq 3. \end{cases}$$

Since $\mu_n \cong \underline{\mathbf{Z}/n\mathbf{Z}}$, this gives (noncanonical) identifications

$$H^q_{\acute{e}tale}(X, \underline{\mathbf{Z}/n\mathbf{Z}}) \cong \begin{cases} \mathbf{Z}/n\mathbf{Z} & \text{if } q = 0, \\ (\mathbf{Z}/n\mathbf{Z})^{2g} & \text{if } q = 1, \\ \mathbf{Z}/n\mathbf{Z} & \text{if } q = 2, \\ 0 & \text{if } q \geq 3. \end{cases}$$

Proof. Theorems 24.1 and 67.5 determine the étale cohomology of $\mathbf{G}_m$ on X in terms of the Picard group of X. The Kummer sequence $0 \to \mu_{n,X} \to \mathbf{G}_{m,X} \to \mathbf{G}_{m,X} \to 0$ (Lemma 28.1) then gives us the long exact cohomology sequence

$$\begin{tikzcd} 0 \arrow[r] & \mu_n(k) \arrow[r] & k^* \arrow[r, "(\cdot)^n"] & k^* \arrow[dll, bend left] \\ & H^1_{\acute{e}tale}(X, \mu_n) \arrow[r] & \mathrm{Pic}(X) \arrow[r, "(\cdot)^n"] & \mathrm{Pic}(X) \arrow[dll, bend left] \\ & H^2_{\acute{e}tale}(X, \mu_n) \arrow[r] & 0 \arrow[r] & 0 \ldots \end{tikzcd}$$

The nth power map $k^* \to k^*$ is surjective since k is algebraically closed. So we need to compute the kernel and cokernel of the map $n : \mathrm{Pic}(X) \to \mathrm{Pic}(X)$. Consider the commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Pic}^0(X) \arrow[r] \arrow[d, "(\cdot)^n"] & \mathrm{Pic}(X) \arrow[r, "\deg"'] \arrow[d, "(\cdot)^n"] & \mathbf{Z} \arrow[r] \arrow[d, hook, "n"] & 0 \\ 0 \arrow[r] & \mathrm{Pic}^0(X) \arrow[r] & \mathrm{Pic}(X) \arrow[r, "\deg"] & \mathbf{Z} \arrow[r] & 0 \end{tikzcd}$$

The group $\mathrm{Pic}^0(X)$ is the k-points of the group scheme $\underline{\mathrm{Pic}}^0_{X/k}$, see Picard Schemes of Curves, Lemma 6.7. The same lemma tells us that $\underline{\mathrm{Pic}}^0_{X/k}$ is a g-dimensional abelian variety over k as defined in Groupoids, Definition 9.1. Hence the left vertical map is surjective by Groupoids, Proposition 9.11. Applying the snake lemma gives canonical identifications as stated in the lemma.

To get the noncanonical identifications of the lemma we need to show the kernel of $n : \mathrm{Pic}^0(X) \to \mathrm{Pic}^0(X)$ is isomorphic to $(\mathbf{Z}/n\mathbf{Z})^{\oplus 2g}$. This is also part of Groupoids, Proposition 9.11. ☐

0AMB **Lemma 68.2.** *Let $\pi : X \to Y$ be a nonconstant morphism of smooth projective curves over an algebraically closed field k and let $n \geq 1$ be invertible in k. The map*

$$\pi^* : H^2_{\acute{e}tale}(Y, \mu_n) \longrightarrow H^2_{\acute{e}tale}(X, \mu_n)$$

is given by multiplication by the degree of π.

Proof. Observe that the statement makes sense as we have identified both cohomology groups $H^2_{\acute{e}tale}(Y, \mu_n)$ and $H^2_{\acute{e}tale}(X, \mu_n)$ with $\mathbf{Z}/n\mathbf{Z}$ in Lemma 68.1. In fact, if $\mathcal{L}$ is a line bundle of degree 1 on Y with class $[\mathcal{L}] \in H^1_{\acute{e}tale}(Y, \mathbf{G}_m)$, then the coboundary of $[\mathcal{L}]$ is the generator of $H^2_{\acute{e}tale}(Y, \mu_n)$. Here the coboundary is the coboundary of the long exact sequence of cohomology associated to the Kummer sequence. Thus the result of the lemma follows from the fact that the degree of the line bundle $\pi^*\mathcal{L}$ on X is $\deg(\pi)$. Some details omitted. $\square$

03RR **Lemma 68.3.** *Let X be an affine smooth curve over an algebraically closed field k and $n \in k^*$. Then*

1. $H^0_{\acute{e}tale}(X, \mu_n) = \mu_n(k)$;
2. $H^1_{\acute{e}tale}(X, \mu_n) \cong (\mathbf{Z}/n\mathbf{Z})^{2g+r-1}$, *where r is the number of points in $\bar{X} - X$ for some smooth projective compactification $\bar{X}$ of X, and*
3. *for all $q \geq 2$, $H^q_{\acute{e}tale}(X, \mu_n) = 0$.*

Proof. Write $X = \bar{X} - \{x_1, \ldots, x_r\}$. Then $\text{Pic}(X) = \text{Pic}(\bar{X})/R$, where R is the subgroup generated by $\mathcal{O}_{\bar{X}}(x_i)$, $1 \leq i \leq r$. Since $r \geq 1$, we see that $\text{Pic}^0(X) \to \text{Pic}(X)$ is surjective, hence $\text{Pic}(X)$ is divisible. Applying the Kummer sequence, we get (1) and (3). For (2), recall that

$$\begin{aligned} H^1_{\acute{e}tale}(X, \mu_n) &= \{(\mathcal{L}, \alpha) | \mathcal{L} \in \text{Pic}(X), \alpha : \mathcal{L}^{\otimes n} \to \mathcal{O}_X\}/\cong \\ &= \{(\bar{\mathcal{L}},\ D,\ \bar{\alpha})\}/\tilde{R} \end{aligned}$$

where $\bar{\mathcal{L}} \in \text{Pic}^0(\bar{X})$, D is a divisor on $\bar{X}$ supported on $\{x_1, \ldots, x_r\}$ and $\bar{\alpha} : \bar{\mathcal{L}}^{\otimes n} \cong \mathcal{O}_{\bar{X}}(D)$ is an isomorphism. Note that D must have degree 0. Further $\tilde{R}$ is the subgroup of triples of the form $(\mathcal{O}_{\bar{X}}(D'), nD', 1^{\otimes n})$ where D' is supported on $\{x_1, \ldots, x_r\}$ and has degree 0. Thus, we get an exact sequence

$$0 \longrightarrow H^1_{\acute{e}tale}(\bar{X}, \mu_n) \longrightarrow H^1_{\acute{e}tale}(X, \mu_n) \longrightarrow \bigoplus_{i=1}^{r} \mathbf{Z}/n\mathbf{Z} \xrightarrow{\sum} \mathbf{Z}/n\mathbf{Z} \longrightarrow 0$$

where the middle map sends the class of a triple $(\bar{\mathcal{L}}, D, \bar{\alpha})$ with $D = \sum_{i=1}^r a_i(x_i)$ to the r-tuple $(a_i)_{i=1}^r$. It now suffices to use Lemma 68.1 to count ranks. $\square$

03RS **Remark 68.4.** The "natural" way to prove the previous corollary is to excise X from $\bar{X}$. This is possible, we just haven't developed that theory.

0A44 **Remark 68.5.** Let k be an algebraically closed field. Let n be an integer prime to the characteristic of k. Recall that

$$\mathbf{G}_{m,k} = \mathbf{A}^1_k \setminus \{0\} = \mathbf{P}^1_k \setminus \{0, \infty\}$$

We claim there is a canonical isomorphism

$$H^1_{\acute{e}tale}(\mathbf{G}_{m,k}, \mu_n) = \mathbf{Z}/n\mathbf{Z}$$

What does this mean? This means there is an element 1_k in $H^1_{\acute{e}tale}(\mathbf{G}_{m,k}, \mu_n)$ such that for every morphism $\text{Spec}(k') \to \text{Spec}(k)$ the pullback map on étale cohomology for the map $\mathbf{G}_{m,k'} \to \mathbf{G}_{m,k}$ maps 1_k to $1_{k'}$. (In particular this element is fixed under all automorphisms of k.) To see this, consider the $\mu_{n,\mathbf{Z}}$-torsor $\mathbf{G}_{m,\mathbf{Z}} \to \mathbf{G}_{m,\mathbf{Z}}$, $x \mapsto x^n$. By the identification of torsors with first cohomology, this pulls back

to give our canonical elements 1_k. Twisting back we see that there are canonical identifications

$$H^1_{\acute{e}tale}(\mathbf{G}_{m,k}, \mathbf{Z}/n\mathbf{Z}) = \mathrm{Hom}(\mu_n(k), \mathbf{Z}/n\mathbf{Z}),$$

i.e., these isomorphisms are compatible with respect to maps of algebraically closed fields, in particular with respect to automorphisms of k.

69. Extension by zero

03S2 The general material in Modules on Sites, Section 19 allows us to make the following definition.

03S3 **Definition 69.1.** Let $j : U \to X$ be an étale morphism of schemes.

(1) The restriction functor $j^{-1} : Sh(X_{\acute{e}tale}) \to Sh(U_{\acute{e}tale})$ has a left adjoint $j_!^{Sh} : Sh(U_{\acute{e}tale}) \to Sh(X_{\acute{e}tale})$.
(2) The restriction functor $j^{-1} : Ab(X_{\acute{e}tale}) \to Ab(U_{\acute{e}tale})$ has a left adjoint which is denoted $j_! : Ab(U_{\acute{e}tale}) \to Ab(X_{\acute{e}tale})$ and called *extension by zero*.
(3) Let Λ be a ring. The restriction functor $j^{-1} : Mod(X_{\acute{e}tale}, \Lambda) \to Mod(U_{\acute{e}tale}, \Lambda)$ has a left adjoint which is denoted $j_! : Mod(U_{\acute{e}tale}, \Lambda) \to Mod(X_{\acute{e}tale}, \Lambda)$ and called *extension by zero*.

If $\mathcal{F}$ is an abelian sheaf on $X_{\acute{e}tale}$, then $j_!\mathcal{F} \neq j_!^{Sh}\mathcal{F}$ in general. On the other hand $j_!$ for sheaves of Λ-modules agrees with $j_!$ on underlying abelian sheaves (Modules on Sites, Remark 19.6). The functor $j_!$ is characterized by the functorial isomorphism

$$\mathrm{Hom}_X(j_!\mathcal{F}, \mathcal{G}) = \mathrm{Hom}_U(\mathcal{F}, j^{-1}\mathcal{G})$$

for all $\mathcal{F} \in Ab(U_{\acute{e}tale})$ and $\mathcal{G} \in Ab(X_{\acute{e}tale})$. Similarly for sheaves of Λ-modules.

To describe the functors in Definition 69.1 more explicitly, recall that j^{-1} is just the restriction via the functor $U_{\acute{e}tale} \to X_{\acute{e}tale}$. In other words, $j^{-1}\mathcal{G}(U') = \mathcal{G}(U')$ for U' étale over U. On the other hand, for $\mathcal{F} \in Ab(U_{\acute{e}tale})$ we consider the presheaf

0F4K (69.1.1)
$$j_{p!}\mathcal{F} : X_{\acute{e}tale} \longrightarrow Ab, \quad V \longmapsto \bigoplus\nolimits_{V \to U} \mathcal{F}(V \to U)$$

Then $j_!\mathcal{F}$ is the sheafification of $j_{p!}\mathcal{F}$. This is proven in Modules on Sites, Lemma 19.2; more generally see the discussion in Modules on Sites, Sections 19 and 16.

03S4 **Exercise 69.2.** Prove directly that the functor $j_!$ defined as the sheafification of the functor $j_{p!}$ given in (69.1.1) is a left adjoint to j^{-1}.

03S5 **Proposition 69.3.** *Let $j : U \to X$ be an étale morphism of schemes. Let $\mathcal{F}$ in $Ab(U_{\acute{e}tale})$. If $\overline{x} : \mathrm{Spec}(k) \to X$ is a geometric point of X, then*

$$(j_!\mathcal{F})_{\overline{x}} = \bigoplus\nolimits_{\overline{u}:\mathrm{Spec}(k)\to U,\ j(\overline{u})=\overline{x}} \mathcal{F}_{\overline{u}}.$$

In particular, $j_!$ is an exact functor.

Proof. Exactness of $j_!$ is very general, see Modules on Sites, Lemma 19.3. Of course it does also follow from the description of stalks. The formula for the stalk follows from Modules on Sites, Lemma 37.1 and the description of points of the small étale site in terms of geometric points, see Lemma 29.12.

For later use we note that the isomorphism

$$\begin{aligned}(j_!\mathcal{F})_{\overline{x}} &= (j_{p!}\mathcal{F})_{\overline{x}} \\ &= \mathrm{colim}_{(V,\overline{v})}\, j_{p!}\mathcal{F}(V) \\ &= \mathrm{colim}_{(V,\overline{v})} \bigoplus\nolimits_{\varphi : V \to U} \mathcal{F}(V \xrightarrow{\varphi} U) \\ &\to \bigoplus\nolimits_{\overline{u}:\mathrm{Spec}(k)\to U,\ j(\overline{u})=\overline{x}} \mathcal{F}_{\overline{u}}.\end{aligned}$$

constructed in Modules on Sites, Lemma 37.1 sends $(V, \overline{v}, \varphi, s)$ to the class of s in the stalk of $\mathcal{F}$ at $\overline{u} = \varphi(\overline{v})$. □

0F70 **Lemma 69.4.** *Let $j : U \to X$ be an open immersion of schemes. For any abelian sheaf $\mathcal{F}$ on $U_{étale}$, the adjunction mappings $j^{-1}j_*\mathcal{F} \to \mathcal{F}$ and $\mathcal{F} \to j^{-1}j_!\mathcal{F}$ are isomorphisms. In fact, $j_!\mathcal{F}$ is the unique abelian sheaf on $X_{étale}$ whose restriction to U is $\mathcal{F}$ and whose stalks at geometric points of $X \setminus U$ are zero.*

Proof. We encourage the reader to prove the first statement by working through the definitions, but here we just use that it is a special case of the very general Modules on Sites, Lemma 19.8. For the second statement, observe that if $\mathcal{G}$ is an abelian sheaf on $X_{étale}$ whose restriction to U is $\mathcal{F}$, then we obtain by adjointness a map $j_!\mathcal{F} \to \mathcal{G}$. This map is then an isomorphism at stalks of geometric points of U by Proposition 69.3. Thus if $\mathcal{G}$ has vanishing stalks at geometric points of $X \setminus U$, then $j_!\mathcal{F} \to \mathcal{G}$ is an isomorphism by Theorem 29.10. □

03S6 **Lemma 69.5** (Extension by zero commutes with base change)**.** *Let $f : Y \to X$ be a morphism of schemes. Let $j : V \to X$ be an étale morphism. Consider the fibre product*

$$\begin{tikzcd} V' = Y \times_X V \arrow[r, "j'"] \arrow[d, "f'"'] & Y \arrow[d, "f"] \\ V \arrow[r, "j"] & X \end{tikzcd}$$

Then we have $j'_!f'^{-1} = f^{-1}j_!$ on abelian sheaves and on sheaves of modules.

Proof. This is true because $j'_!f'^{-1}$ is left adjoint to $f'_*(j')^{-1}$ and $f^{-1}j_!$ is left adjoint to $j^{-1}f_*$. Further $f'_*(j')^{-1} = j^{-1}f_*$ because f_* commutes with étale localization (by construction). In fact, the lemma holds very generally in the setting of a morphism of sites, see Modules on Sites, Lemma 20.1. □

0F4L **Lemma 69.6.** *Let $j : U \to X$ be separated and étale. Then there is a functorial injective map $j_!\mathcal{F} \to j_*\mathcal{F}$ on abelian sheaves and sheaves of Λ-modules.*

Proof. We prove this in the case of abelian sheaves. Let us construct a canonical map

$$j_{p!}\mathcal{F} \to j_*\mathcal{F}$$

of abelian presheaves on $X_{étale}$ for any abelian sheaf $\mathcal{F}$ on $U_{étale}$ where $j_{p!}$ is as in (69.1.1). Sheafification of this map will be the desired map $j_!\mathcal{F} \to j_*\mathcal{F}$. Evaluating both sides on $V \to X$ étale we obtain

$$j_{p!}\mathcal{F}(V) = \bigoplus\nolimits_{\varphi : V \to U} \mathcal{F}(V \xrightarrow{\varphi} U) \quad \text{and} \quad j_*\mathcal{F}(V) = \mathcal{F}(V \times_X U)$$

For each φ we have an open and closed immersion

$$\Gamma_\varphi = (1, \varphi) : V \longrightarrow V \times_X U$$

over U. It is open as it is a morphism between schemes étale over U and it is closed as it is a section of a scheme separated over V (Schemes, Lemma 21.11). Thus for a section $s_\varphi \in \mathcal{F}(V \xrightarrow{\varphi} U)$ there exists a unique section s'_φ in $\mathcal{F}(V \times_X U)$ which pulls back to s_φ by Γ_φ and which restricts to zero on the complement of the image of Γ_φ.

To show that our map is injective suppose that $\sum_{i=1,\ldots,n} s_{\varphi_i}$ is an element of $j_{p!}\mathcal{F}(V)$ in the formula above maps to zero in $j_*\mathcal{F}(V)$. Our task is to show that $\sum_{i=1,\ldots,n} s_{\varphi_i}$ restricts to zero on the members of an étale covering of V. Looking at all pairwise equalizers (which are open and closed in V) of the morphisms $\varphi_i : V \to U$ and working locally on V, we may assume the images of the morphisms $\Gamma_{\varphi_1}, \ldots, \Gamma_{\varphi_n}$ are pairwise disjoint. Since our assumption is that $\sum_{i=1,\ldots,n} s'_{\varphi_i} = 0$ we then immediately conclude that $s'_{\varphi_i} = 0$ for each i (by the disjointness of the supports of these sections), whence $s_{\varphi_i} = 0$ for all i as desired. □

03S7 **Lemma 69.7.** *Let $j : U \to X$ be finite and étale. Then the map $j_! \to j_*$ of Lemma 69.6 is an isomorphism on abelian sheaves and sheaves of Λ-modules.*

Proof. It suffices to check $j_!\mathcal{F} \to j_*\mathcal{F}$ is an isomorphism étale locally on X. Thus we may assume $U \to X$ is a finite disjoint union of isomorphisms, see Étale Morphisms, Lemma 18.3. We omit the proof in this case. □

095L **Lemma 69.8.** *Let X be a scheme. Let $Z \subset X$ be a closed subscheme and let $U \subset X$ be the complement. Denote $i : Z \to X$ and $j : U \to X$ the inclusion morphisms. For every abelian sheaf $\mathcal{F}$ on $X_{étale}$ there is a canonical short exact sequence*

$$0 \to j_!j^{-1}\mathcal{F} \to \mathcal{F} \to i_*i^{-1}\mathcal{F} \to 0$$

on $X_{étale}$.

Proof. We obtain the maps by the adjointness properties of the functors involved. For a geometric point $\overline{x}$ in X we have either $\overline{x} \in U$ in which case the map on the left hand side is an isomorphism on stalks and the stalk of $i_*i^{-1}\mathcal{F}$ is zero or $\overline{x} \in Z$ in which case the map on the right hand side is an isomorphism on stalks and the stalk of $j_!j^{-1}\mathcal{F}$ is zero. Here we have used the description of stalks of Lemma 46.3 and Proposition 69.3. □

70. Constructible sheaves

05BE Let X be a scheme. A *constructible locally closed subscheme* of X is a locally closed subscheme $T \subset X$ such that the underlying topological space of T is a constructible subset of X. If $T, T' \subset X$ are locally closed subschemes with the same underlying topological space, then $T_{étale} \cong T'_{étale}$ by the topological invariance of the étale site (Theorem 45.2). Thus in the following definition we may assume our locally closed subschemes are reduced.

03RW **Definition 70.1.** Let X be a scheme.

(1) A sheaf of sets on $X_{étale}$ is *constructible* if for every affine open $U \subset X$ there exists a finite decomposition of U into constructible locally closed subschemes $U = \coprod_i U_i$ such that $\mathcal{F}|_{U_i}$ is finite locally constant for all i.

(2) A sheaf of abelian groups on $X_{étale}$ is *constructible* if for every affine open $U \subset X$ there exists a finite decomposition of U into constructible locally closed subschemes $U = \coprod_i U_i$ such that $\mathcal{F}|_{U_i}$ is finite locally constant for all i.
(3) Let Λ be a Noetherian ring. A sheaf of Λ-modules on $X_{étale}$ is *constructible* if for every affine open $U \subset X$ there exists a finite decomposition of U into constructible locally closed subschemes $U = \coprod_i U_i$ such that $\mathcal{F}|_{U_i}$ is of finite type and locally constant for all i.

It seems that this is the accepted definition. An alternative, which lends itself more readily to generalizations beyond the étale site of a scheme, would have been to define constructible sheaves by starting with h_U, $j_{U!}\mathbf{Z}/n\mathbf{Z}$, and $j_{U!}\underline{\Lambda}$ where U runs over all quasi-compact and quasi-separated objects of $X_{étale}$, and then take the smallest full subcategory of $Sh(X_{étale})$, $Ab(X_{étale})$, and $Mod(X_{étale}, \underline{\Lambda})$ containing these and closed under finite limits and colimits. It follows from Lemma 70.6 and Lemmas 72.5, 72.7, and 72.6 that this produces the same category if X is quasi-compact and quasi-separated. In general this does not produce the same category however.

A disjoint union decomposition $U = \coprod U_i$ of a scheme by locally closed subschemes will be called a *partition* of U (compare with Topology, Section 28).

095E **Lemma 70.2.** *Let X be a quasi-compact and quasi-separated scheme. Let $\mathcal{F}$ be a sheaf of sets on $X_{étale}$. The following are equivalent*

(1) *$\mathcal{F}$ is constructible,*
(2) *there exists an open covering $X = \bigcup U_i$ such that $\mathcal{F}|_{U_i}$ is constructible, and*
(3) *there exists a partition $X = \bigcup X_i$ by constructible locally closed subschemes such that $\mathcal{F}|_{X_i}$ is finite locally constant.*

A similar statement holds for abelian sheaves and sheaves of Λ-modules if Λ is Noetherian.

Proof. It is clear that (1) implies (2).

Assume (2). For every $x \in X$ we can find an i and an affine open neighbourhood $V_x \subset U_i$ of x. Hence we can find a finite affine open covering $X = \bigcup V_j$ such that for each j there exists a finite decomposition $V_j = \coprod V_{j,k}$ by locally closed constructible subsets such that $\mathcal{F}|_{V_{j,k}}$ is finite locally constant. By Topology, Lemma 15.5 each $V_{j,k}$ is constructible as a subset of X. By Topology, Lemma 28.7 we can find a finite stratification $X = \coprod X_l$ with constructible locally closed strata such that each $V_{j,k}$ is a union of X_l. Thus (3) holds.

Assume (3) holds. Let $U \subset X$ be an affine open. Then $U \cap X_i$ is a constructible locally closed subset of U (for example by Properties, Lemma 2.1) and $U = \coprod U \cap X_i$ is a partition of U as in Definition 70.1. Thus (1) holds. □

09YR **Lemma 70.3.** *Let X be a quasi-compact and quasi-separated scheme. Let $\mathcal{F}$ be a sheaf of sets, abelian groups, Λ-modules (with Λ Noetherian) on $X_{étale}$. If there exist constructible locally closed subschemes $T_i \subset X$ such that (a) $X = \bigcup T_j$ and (b) $\mathcal{F}|_{T_j}$ is constructible, then $\mathcal{F}$ is constructible.*

Proof. First, we can assume the covering is finite as X is quasi-compact in the spectral topology (Topology, Lemma 23.2 and Properties, Lemma 2.4). Observe

that each T_i is a quasi-compact and quasi-separated scheme in its own right (because it is constructible in X; details omitted). Thus we can find a finite partition $T_i = \coprod T_{i,j}$ into locally closed constructible parts of T_i such that $\mathcal{F}|_{T_{i,j}}$ is finite locally constant (Lemma 70.2). By Topology, Lemma 15.12 we see that $T_{i,j}$ is a constructible locally closed subscheme of X. Then we can apply Topology, Lemma 28.7 to $X = \bigcup T_{i,j}$ to find the desired partition of X. □

095F **Lemma 70.4.** *Let X be a scheme. Checking constructibility of a sheaf of sets, abelian groups, Λ-modules (with Λ Noetherian) can be done Zariski locally on X.*

Proof. The statement means if $X = \bigcup U_i$ is an open covering such that $\mathcal{F}|_{U_i}$ is constructible, then $\mathcal{F}$ is constructible. If $U \subset X$ is affine open, then $U = \bigcup U \cap U_i$ and $\mathcal{F}|_{U \cap U_i}$ is constructible (it is trivial that the restriction of a constructible sheaf to an open is constructible). It follows from Lemma 70.2 that $\mathcal{F}|_U$ is constructible, i.e., a suitable partition of U exists. □

095G **Lemma 70.5.** *Let $f : X \to Y$ be a morphism of schemes. If $\mathcal{F}$ is a constructible sheaf of sets, abelian groups, or Λ-modules (with Λ Noetherian) on $Y_{étale}$, the same is true for $f^{-1}\mathcal{F}$ on $X_{étale}$.*

Proof. By Lemma 70.4 this reduces to the case where X and Y are affine. By Lemma 70.2 it suffices to find a finite partition of X by constructible locally closed subschemes such that $f^{-1}\mathcal{F}$ is finite locally constant on each of them. To find it we just pull back the partition of Y adapted to $\mathcal{F}$ and use Lemma 63.2. □

03RZ **Lemma 70.6.** *Let X be a scheme.*

(1) *The category of constructible sheaves of sets is closed under finite limits and colimits inside $Sh(X_{étale})$.*
(2) *The category of constructible abelian sheaves is a weak Serre subcategory of $Ab(X_{étale})$.*
(3) *Let Λ be a Noetherian ring. The category of constructible sheaves of Λ-modules on $X_{étale}$ is a weak Serre subcategory of $Mod(X_{étale}, \Lambda)$.*

Proof. We prove (3). We will use the criterion of Homology, Lemma 10.3. Suppose that $\varphi : \mathcal{F} \to \mathcal{G}$ is a map of constructible sheaves of Λ-modules. We have to show that $\mathcal{K} = \text{Ker}(\varphi)$ and $\mathcal{Q} = \text{Coker}(\varphi)$ are constructible. Similarly, suppose that $0 \to \mathcal{F} \to \mathcal{E} \to \mathcal{G} \to 0$ is a short exact sequence of sheaves of Λ-modules with $\mathcal{F}$, $\mathcal{G}$ constructible. We have to show that $\mathcal{E}$ is constructible. In both cases we can replace X with the members of an affine open covering. Hence we may assume X is affine. The we may further replace X by the members of a finite partition of X by constructible locally closed subschemes on which $\mathcal{F}$ and $\mathcal{G}$ are of finite type and locally constant. Thus we may apply Lemma 63.6 to conclude.

The proofs of (1) and (2) are very similar and are omitted. □

095I **Lemma 70.7.** *Let X be a scheme. Let Λ be a Noetherian ring. The tensor product of two constructible sheaves of Λ-modules on $X_{étale}$ is a constructible sheaf of Λ-modules.*

Proof. The question immediately reduces to the case where X is affine. Since any two partitions of X with constructible locally closed strata have a common refinement of the same type and since pullbacks commute with tensor product we reduce to Lemma 63.7. □

09YS **Lemma 70.8.** *Let X be a quasi-compact and quasi-separated scheme.*

(1) *Let $\mathcal{F} \to \mathcal{G}$ be a map of constructible sheaves of sets on $X_{étale}$. Then the set of points $x \in X$ where $\mathcal{F}_{\overline{x}} \to \mathcal{G}_{\overline{x}}$ is surjective, resp. injective, resp. is isomorphic to a given map of sets, is constructible in X.*
(2) *Let $\mathcal{F}$ be a constructible abelian sheaf on $X_{étale}$. The support of $\mathcal{F}$ is constructible.*
(3) *Let Λ be a Noetherian ring. Let $\mathcal{F}$ be a constructible sheaf of Λ-modules on $X_{étale}$. The support of $\mathcal{F}$ is constructible.*

Proof. Proof of (1). Let $X = \coprod X_i$ be a partition of X by locally closed constructible subschemes such that both $\mathcal{F}$ and $\mathcal{G}$ are finite locally constant over the parts (use Lemma 70.2 for both $\mathcal{F}$ and $\mathcal{G}$ and choose a common refinement). Then apply Lemma 63.5 to the restriction of the map to each part.

The proof of (2) and (3) is omitted. $\square$

The following lemma will turn out to be very useful later on. It roughly says that the category of constructible sheaves has a kind of weak "Noetherian" property.

095P **Lemma 70.9.** *Let X be a quasi-compact and quasi-separated scheme. Let $\mathcal{F} = \text{colim}_{i\in I} \mathcal{F}_i$ be a filtered colimit of sheaves of sets, abelian sheaves, or sheaves of modules.*

(1) *If $\mathcal{F}$ and $\mathcal{F}_i$ are constructible sheaves of sets, then the ind-object $\mathcal{F}_i$ is essentially constant with value $\mathcal{F}$.*
(2) *If $\mathcal{F}$ and $\mathcal{F}_i$ are constructible sheaves of abelian groups, then the ind-object $\mathcal{F}_i$ is essentially constant with value $\mathcal{F}$.*
(3) *Let Λ be a Noetherian ring. If $\mathcal{F}$ and $\mathcal{F}_i$ are constructible sheaves of Λ-modules, then the ind-object $\mathcal{F}_i$ is essentially constant with value $\mathcal{F}$.*

Proof. Proof of (1). We will use without further mention that finite limits and colimits of constructible sheaves are constructible (Lemma 63.6). For each i let $T_i \subset X$ be the set of points $x \in X$ where $\mathcal{F}_{i,\overline{x}} \to \mathcal{F}_{\overline{x}}$ is not surjective. Because $\mathcal{F}_i$ and $\mathcal{F}$ are constructible T_i is a constructible subset of X (Lemma 70.8). Since the stalks of $\mathcal{F}$ are finite and since $\mathcal{F} = \text{colim}_{i\in I} \mathcal{F}_i$ we see that for all $x \in X$ we have $x \notin T_i$ for i large enough. Since X is a spectral space by Properties, Lemma 2.4 the constructible topology on X is quasi-compact by Topology, Lemma 23.2. Thus $T_i = \emptyset$ for i large enough. Thus $\mathcal{F}_i \to \mathcal{F}$ is surjective for i large enough. Assume now that $\mathcal{F}_i \to \mathcal{F}$ is surjective for all i. Choose $i \in I$. For $i' \geq i$ denote $S_{i'} \subset X$ the set of points x such that the number of elements in $\text{Im}(\mathcal{F}_{i,\overline{x}} \to \mathcal{F}_{\overline{x}})$ is equal to the number of elements in $\text{Im}(\mathcal{F}_{i,\overline{x}} \to \mathcal{F}_{i',\overline{x}})$. Because $\mathcal{F}_i$, $\mathcal{F}_{i'}$ and $\mathcal{F}$ are constructible $S_{i'}$ is a constructible subset of X (details omitted; hint: use Lemma 70.8). Since the stalks of $\mathcal{F}_i$ and $\mathcal{F}$ are finite and since $\mathcal{F} = \text{colim}_{i'\geq i} \mathcal{F}_{i'}$ we see that for all $x \in X$ we have $x \notin S_{i'}$ for i' large enough. By the same argument as above we can find a large i' such that $S_{i'} = \emptyset$. Thus $\mathcal{F}_i \to \mathcal{F}_{i'}$ factors through $\mathcal{F}$ as desired.

Proof of (2). Observe that a constructible abelian sheaf is a constructible sheaf of sets. Thus case (2) follows from (1).

Proof of (3). We will use without further mention that the category of constructible sheaves of Λ-modules is abelian (Lemma 63.6). For each i let $\mathcal{Q}_i$ be the cokernel of the map $\mathcal{F}_i \to \mathcal{F}$. The support T_i of $\mathcal{Q}_i$ is a constructible subset of X as $\mathcal{Q}_i$ is constructible (Lemma 70.8). Since the stalks of $\mathcal{F}$ are finite Λ-modules and since

$\mathcal{F} = \operatorname{colim}_{i \in I} \mathcal{F}_i$ we see that for all $x \in X$ we have $x \notin T_i$ for i large enough. Since X is a spectral space by Properties, Lemma 2.4 the constructible topology on X is quasi-compact by Topology, Lemma 23.2. Thus $T_i = \emptyset$ for i large enough. This proves the first assertion. For the second, assume now that $\mathcal{F}_i \to \mathcal{F}$ is surjective for all i. Choose $i \in I$. For $i' \geq i$ denote $\mathcal{K}_{i'}$ the image of $\operatorname{Ker}(\mathcal{F}_i \to \mathcal{F})$ in $\mathcal{F}_{i'}$. The support $S_{i'}$ of $\mathcal{K}_{i'}$ is a constructible subset of X as $\mathcal{K}_{i'}$ is constructible. Since the stalks of $\operatorname{Ker}(\mathcal{F}_i \to \mathcal{F})$ are finite Λ-modules and since $\mathcal{F} = \operatorname{colim}_{i' \geq i} \mathcal{F}_{i'}$ we see that for all $x \in X$ we have $x \notin S_{i'}$ for i' large enough. By the same argument as above we can find a large i' such that $S_{i'} = \emptyset$. Thus $\mathcal{F}_i \to \mathcal{F}_{i'}$ factors through $\mathcal{F}$ as desired. □

71. Auxiliary lemmas on morphisms

095J Some lemmas that are useful for proving functoriality properties of constructible sheaves.

03S0 **Lemma 71.1.** *Let $U \to X$ be an étale morphism of quasi-compact and quasi-separated schemes (for example an étale morphism of Noetherian schemes). Then there exists a partition $X = \coprod_i X_i$ by constructible locally closed subschemes such that $X_i \times_X U \to X_i$ is finite étale for all i.*

Proof. If $U \to X$ is separated, then this is More on Morphisms, Lemma 40.4. In general, we may assume X is affine. Choose a finite affine open covering $U = \bigcup U_j$. Apply the previous case to all the morphisms $U_j \to X$ and $U_j \cap U_{j'} \to X$ and choose a common refinement $X = \coprod X_i$ of the resulting partitions. After refining the partition further we may assume X_i affine as well. Fix i and set $V = U \times_X X_i$. The morphisms $V_j = U_j \times_X X_i \to X_i$ and $V_{jj'} = (U_j \cap U_{j'}) \times_X X_i \to X_i$ are finite étale. Hence V_j and $V_{jj'}$ are affine schemes and $V_{jj'} \subset V_j$ is closed as well as open (since $V_{jj'} \to X_i$ is proper, so Morphisms, Lemma 39.7 applies). Then $V = \bigcup V_j$ is separated because $\mathcal{O}(V_j) \to \mathcal{O}(V_{jj'})$ is surjective, see Schemes, Lemma 21.7. Thus the previous case applies to $V \to X_i$ and we can further refine the partition if needed (it actually isn't but we don't need this). □

In the Noetherian case one can prove the preceding lemma by Noetherian induction and the following amusing lemma.

03S1 **Lemma 71.2.** *Let $f : X \to Y$ be a morphism of schemes which is quasi-compact, quasi-separated, and locally of finite type. If η is a generic point of on irreducible component of Y such that $f^{-1}(\eta)$ is finite, then there exists an open $V \subset Y$ containing η such that $f^{-1}(V) \to V$ is finite.*

Proof. This is Morphisms, Lemma 49.1. □

The statement of the following lemma can be strengthened a bit.

095K **Lemma 71.3.** *Let $f : Y \to X$ be a quasi-finite and finitely presented morphism of affine schemes.*

(1) *There exists a surjective morphism of affine schemes $X' \to X$ and a closed subscheme $Z' \subset Y' = X' \times_X Y$ such that*
   (a) *$Z' \subset Y'$ is a thickening, and*
   (b) *$Z' \to X'$ is a finite étale morphism.*

(2) *There exists a finite partition $X = \coprod X_i$ by locally closed, constructible, affine strata, and surjective finite locally free morphisms $X'_i \to X_i$ such that the reduction of $Y'_i = X'_i \times_X Y \to X'_i$ is isomorphic to $\coprod_{j=1}^{n_i} (X'_i)_{red} \to (X'_i)_{red}$ for some n_i.*

Proof. Setting $X' = \coprod X'_i$ we see that (2) implies (1). Write $X = \text{Spec}(A)$ and $Y = \text{Spec}(B)$. Write A as a filtered colimit of finite type $\mathbf{Z}$-algebras A_i. Since B is an A-algebra of finite presentation, we see that there exists $0 \in I$ and a finite type ring map $A_0 \to B_0$ such that $B = \text{colim}\, B_i$ with $B_i = A_i \otimes_{A_0} B_0$, see Algebra, Lemma 126.8. For i sufficiently large we see that $A_i \to B_i$ is quasi-finite, see Limits, Lemma 16.2. Thus we reduce to the case of finite type algebras over $\mathbf{Z}$, in particular we reduce to the Noetherian case. (Details omitted.)

Assume X and Y Noetherian. In this case any locally closed subset of X is constructible. By Lemma 71.2 and Noetherian induction we see that there is a finite partition $X = \coprod X_i$ of X by locally closed strata such that $Y \times_X X_i \to X_i$ is finite. We can refine this partition to get affine strata. Thus after replacing X by $X' = \coprod X_i$ we may assume $Y \to X$ is finite.

Assume X and Y Noetherian and $Y \to X$ finite. Suppose that we can prove (2) after base change by a surjective, flat, quasi-finite morphism $U \to X$. Thus we have a partition $U = \coprod U_i$ and finite locally free morphisms $U'_i \to U_i$ such that $U'_i \times_X Y \to U'_i$ is isomorphic to $\coprod_{j=1}^{n_i} (U'_i)_{red} \to (U'_i)_{red}$ for some n_i. Then, by the argument in the previous paragraph, we can find a partition $X = \coprod X_j$ with locally closed affine strata such that $X_j \times_X U_i \to X_j$ is finite for all i, j. By Morphisms, Lemma 46.2 each $X_j \times_X U_i \to X_j$ is finite locally free. Hence $X_j \times_X U'_i \to X_j$ is finite locally free (Morphisms, Lemma 46.3). It follows that $X = \coprod X_j$ and $X'_j = \coprod_i X_j \times_X U'_i$ is a solution for $Y \to X$. Thus it suffices to prove the result (in the Noetherian case) after a surjective flat quasi-finite base change.

Applying Morphisms, Lemma 46.6 we see we may assume that Y is a closed subscheme of an affine scheme Z which is (set theoretically) a finite union $Z = \bigcup_{i \in I} Z_i$ of closed subschemes mapping isomorphically to X. In this case we will find a finite partition of $X = \coprod X_j$ with affine locally closed strata that works (in other words $X'_j = X_j$). Set $T_i = Y \cap Z_i$. This is a closed subscheme of X. As X is Noetherian we can find a finite partition of $X = \coprod X_j$ by affine locally closed subschemes, such that each $X_j \times_X T_i$ is (set theoretically) a union of strata $X_j \times_X Z_i$. Replacing X by X_j we see that we may assume $I = I_1 \amalg I_2$ with $Z_i \subset Y$ for $i \in I_1$ and $Z_i \cap Y = \emptyset$ for $i \in I_2$. Replacing Z by $\bigcup_{i \in I_1} Z_i$ we see that we may assume $Y = Z$. Finally, we can replace X again by the members of a partition as above such that for every $i, i' \subset I$ the intersection $Z_i \cap Z_{i'}$ is either empty or (set theoretically) equal to Z_i and $Z_{i'}$. This clearly means that Y is (set theoretically) equal to a disjoint union of the Z_i which is what we wanted to show. □

72. More on constructible sheaves

095M Let Λ be a Noetherian ring. Let X be a scheme. We often consider $X_{étale}$ as a ringed site with sheaf of rings $\underline{\Lambda}$. In case of abelian sheaves we often take $\Lambda = \mathbf{Z}/n\mathbf{Z}$ for a suitable integer n.

03S8 **Lemma 72.1.** *Let $j : U \to X$ be an étale morphism of quasi-compact and quasi-separated schemes.*

(1) *The sheaf h_U is a constructible sheaf of sets.*
(2) *The sheaf $j_!\underline{M}$ is a constructible abelian sheaf for a finite abelian group M.*
(3) *If Λ is a Noetherian ring and M is a finite Λ-module, then $j_!\underline{M}$ is a constructible sheaf of Λ-modules on $X_{\acute{e}tale}$.*

Proof. By Lemma 71.1 there is a partition $\coprod_i X_i$ such that $\pi_i : j^{-1}(X_i) \to X_i$ is finite étale. The restriction of h_U to X_i is $h_{j^{-1}(X_i)}$ which is finite locally constant by Lemma 63.4. For cases (2) and (3) we note that

$$j_!(\underline{M})|_{X_i} = \pi_{i!}(\underline{M}) = \pi_{i*}(\underline{M})$$

by Lemmas 69.5 and 69.7. Thus it suffices to show the lemma for $\pi : Y \to X$ finite étale. This is Lemma 63.3. □

03SA **Lemma 72.2.** *Let X be a quasi-compact and quasi-separated scheme.*

(1) *Let $\mathcal{F}$ be a sheaf of sets on $X_{\acute{e}tale}$. Then $\mathcal{F}$ is a filtered colimit of constructible sheaves of sets.*
(2) *Let $\mathcal{F}$ be a torsion abelian sheaf on $X_{\acute{e}tale}$. Then $\mathcal{F}$ is a filtered colimit of constructible abelian sheaves.*
(3) *Let Λ be a Noetherian ring and $\mathcal{F}$ a sheaf of Λ-modules on $X_{\acute{e}tale}$. Then $\mathcal{F}$ is a filtered colimit of constructible sheaves of Λ-modules.*

Proof. Let $\mathcal{B}$ be the collection of quasi-compact and quasi-separated objects of $X_{\acute{e}tale}$. By Modules on Sites, Lemma 29.6 any sheaf of sets is a filtered colimit of sheaves of the form

$$\text{Coequalizer}\left(\coprod\nolimits_{j=1,\ldots,m} h_{V_j} \rightrightarrows \coprod\nolimits_{i=1,\ldots,n} j_{U_i} \right)$$

with V_j and U_i quasi-compact and quasi-separated objects of $X_{\acute{e}tale}$. By Lemmas 72.1 and 70.6 these coequalizers are constructible. This proves (1).

Let Λ be a Noetherian ring. By Modules on Sites, Lemma 29.6 Λ-modules $\mathcal{F}$ is a filtered colimit of modules of the form

$$\text{Coker}\left(\bigoplus\nolimits_{j=1,\ldots,m} j_{V_j!}\underline{\Lambda}_{V_j} \longrightarrow \bigoplus\nolimits_{i=1,\ldots,n} j_{U_i!}\underline{\Lambda}_{U_i} \right)$$

with V_j and U_i quasi-compact and quasi-separated objects of $X_{\acute{e}tale}$. By Lemmas 72.1 and 70.6 these cokernels are constructible. This proves (3).

Proof of (2). First write $\mathcal{F} = \bigcup \mathcal{F}[n]$ where $\mathcal{F}[n]$ is the n-torsion subsheaf. Then we can view $\mathcal{F}[n]$ as a sheaf of $\mathbf{Z}/n\mathbf{Z}$-modules and apply (3). □

095Q **Lemma 72.3.** *Let $f : X \to Y$ be a surjective morphism of quasi-compact and quasi-separated schemes.*

(1) *Let $\mathcal{F}$ be a sheaf of sets on $Y_{\acute{e}tale}$. Then $\mathcal{F}$ is constructible if and only if $f^{-1}\mathcal{F}$ is constructible.*
(2) *Let $\mathcal{F}$ be an abelian sheaf on $Y_{\acute{e}tale}$. Then $\mathcal{F}$ is constructible if and only if $f^{-1}\mathcal{F}$ is constructible.*
(3) *Let Λ be a Noetherian ring. Let $\mathcal{F}$ be sheaf of Λ-modules on $Y_{\acute{e}tale}$. Then $\mathcal{F}$ is constructible if and only if $f^{-1}\mathcal{F}$ is constructible.*

Proof. One implication follows from Lemma 70.5. For the converse, assume $f^{-1}\mathcal{F}$ is constructible. Write $\mathcal{F} = \operatorname{colim} \mathcal{F}_i$ as a filtered colimit of constructible sheaves (of sets, abelian groups, or modules) using Lemma 72.2. Since f^{-1} is a left adjoint it commutes with colimits (Categories, Lemma 24.5) and we see that $f^{-1}\mathcal{F} =$

colim $f^{-1}\mathcal{F}_i$. By Lemma 70.9 we see that $f^{-1}\mathcal{F}_i \to f^{-1}\mathcal{F}$ is surjective for all i large enough. Since f is surjective we conclude (by looking at stalks using Lemma 36.2 and Theorem 29.10) that $\mathcal{F}_i \to \mathcal{F}$ is surjective for all i large enough. Thus $\mathcal{F}$ is the quotient of a constructible sheaf $\mathcal{G}$. Applying the argument once more to $\mathcal{G} \times_\mathcal{F} \mathcal{G}$ or the kernel of $\mathcal{G} \to \mathcal{F}$ we conclude using that f^{-1} is exact and that the category of constructible sheaves (of sets, abelian groups, or modules) is preserved under finite (co)limits or (co)kernels inside $Sh(Y_{étale})$, $Sh(X_{étale})$, $Ab(Y_{étale})$, $Ab(X_{étale})$, $Mod(Y_{étale}, \Lambda)$, and $Mod(X_{étale}, \Lambda)$, see Lemma 70.6. $\square$

095H **Lemma 72.4.** *Let $f : X \to Y$ be a finite étale morphism of schemes. Let Λ be a Noetherian ring. If $\mathcal{F}$ is a constructible sheaf of sets, constructible sheaf of abelian groups, or constructible sheaf of Λ-modules on $X_{étale}$, the same is true for $f_*\mathcal{F}$ on $Y_{étale}$.*

Proof. By Lemma 70.4 it suffices to check this Zariski locally on Y and by Lemma 72.3 we may replace Y by an étale cover (the construction of f_* commutes with étale localization). A finite étale morphism is étale locally isomorphic to a disjoint union of isomorphisms, see Étale Morphisms, Lemma 18.3. Thus, in the case of sheaves of sets, the lemma says that if $\mathcal{F}_i$, $i = 1, \ldots, n$ are constructible sheaves of sets, then $\prod_{i=1,\ldots,n} \mathcal{F}_i$ is too. This is clear. Similarly for sheaves of abelian groups and modules. $\square$

09Y9 **Lemma 72.5.** *Let X be a quasi-compact and quasi-separated scheme. The category of constructible sheaves of sets is the full subcategory of $Sh(X_{étale})$ consisting of sheaves $\mathcal{F}$ which are coequalizers*

$$\mathcal{F}_1 \rightrightarrows \mathcal{F}_0 \longrightarrow \mathcal{F}$$

such that $\mathcal{F}_i$, $i = 0, 1$ is a finite coproduct of sheaves of the form h_U with U a quasi-compact and quasi-separated object of $X_{étale}$.

Proof. In the proof of Lemma 72.2 we have seen that sheaves of this form are constructible. For the converse, suppose that for every constructible sheaf of sets $\mathcal{F}$ we can find a surjection $\mathcal{F}_0 \to \mathcal{F}$ with $\mathcal{F}_0$ as in the lemma. Then we find our surjection $\mathcal{F}_1 \to \mathcal{F}_0 \times_\mathcal{F} \mathcal{F}_0$ because the latter is constructible by Lemma 70.6.

By Topology, Lemma 28.7 we may choose a finite stratification $X = \coprod_{i \in I} X_i$ such that $\mathcal{F}$ is finite locally constant on each stratum. We will prove the result by induction on the cardinality of I. Let $i \in I$ be a minimal element in the partial ordering of I. Then $X_i \subset X$ is closed. By induction, there exist finitely many quasi-compact and quasi-separated objects U_α of $(X \setminus X_i)_{étale}$ and a surjective map $\coprod h_{U_\alpha} \to \mathcal{F}|_{X \setminus X_i}$. These determine a map

$$\coprod h_{U_\alpha} \to \mathcal{F}$$

which is surjective after restricting to $X \setminus X_i$. By Lemma 63.4 we see that $\mathcal{F}|_{X_i} = h_V$ for some scheme V finite étale over X_i. Let $\overline{v}$ be a geometric point of V lying over $\overline{x} \in X_i$. We may think of $\overline{v}$ as an element of the stalk $\mathcal{F}_{\overline{x}} = V_{\overline{x}}$. Thus we can find an étale neighbourhood $(U, \overline{u})$ of $\overline{x}$ and a section $s \in \mathcal{F}(U)$ whose stalk at $\overline{x}$ gives $\overline{v}$. Thinking of s as a map $s : h_U \to \mathcal{F}$, restricting to X_i we obtain a morphism $s|_{X_i} : U \times_X X_i \to V$ over X_i which maps $\overline{u}$ to $\overline{v}$. Since V is quasi-compact (finite over the closed subscheme X_i of the quasi-compact scheme X) a finite number

$s^{(1)}, \ldots, s^{(m)}$ of these sections of $\mathcal{F}$ over $U^{(1)}, \ldots, U^{(m)}$ will determine a jointly surjective map

$$\coprod s^{(j)}|_{X_i} : \coprod U^{(j)} \times_X X_i \longrightarrow V$$

Then we obtain the surjection

$$\coprod h_{U_\alpha} \amalg \coprod h_{U^{(j)}} \to \mathcal{F}$$

as desired. □

095N **Lemma 72.6.** *Let X be a quasi-compact and quasi-separated scheme. Let Λ be a Noetherian ring. The category of constructible sheaves of Λ-modules is exactly the category of modules of the form*

$$\mathrm{Coker}\left(\bigoplus\nolimits_{j=1,\ldots,m} j_{V_j!}\underline{\Lambda}_{V_j} \longrightarrow \bigoplus\nolimits_{i=1,\ldots,n} j_{U_i!}\underline{\Lambda}_{U_i}\right)$$

with V_j and U_i quasi-compact and quasi-separated objects of $X_{étale}$. In fact, we can even assume U_i and V_j affine.

Proof. In the proof of Lemma 72.2 we have seen modules of this form are constructible. Since the category of constructible modules is abelian (Lemma 70.6) it suffices to prove that given a constructible module $\mathcal{F}$ there is a surjection

$$\bigoplus\nolimits_{i=1,\ldots,n} j_{U_i!}\underline{\Lambda}_{U_i} \longrightarrow \mathcal{F}$$

for some affine objects U_i in $X_{étale}$. By Modules on Sites, Lemma 29.6 there is a surjection

$$\Psi : \bigoplus\nolimits_{i \in I} j_{U_i!}\underline{\Lambda}_{U_i} \longrightarrow \mathcal{F}$$

with U_i affine and the direct sum over a possibly infinite index set I. For every finite subset $I' \subset I$ set

$$T_{I'} = \mathrm{Supp}(\mathrm{Coker}(\bigoplus\nolimits_{i \in I'} j_{U_i!}\underline{\Lambda}_{U_i} \longrightarrow \mathcal{F}))$$

By the very definition of constructible sheaves, the set $T_{I'}$ is a constructible subset of X. We want to show that $T_{I'} = \emptyset$ for some I'. Since every stalk $\mathcal{F}_{\overline{x}}$ is a finite type Λ-module and since Ψ is surjective, for every $x \in X$ there is an I' such that $x \notin T_{I'}$. In other words we have $\emptyset = \bigcap_{I' \subset I \text{ finite}} T_{I'}$. Since X is a spectral space by Properties, Lemma 2.4 the constructible topology on X is quasi-compact by Topology, Lemma 23.2. Thus $T_{I'} = \emptyset$ for some $I' \subset I$ finite as desired. □

09YT **Lemma 72.7.** *Let X be a quasi-compact and quasi-separated scheme. The category of constructible abelian sheaves is exactly the category of abelian sheaves of the form*

$$\mathrm{Coker}\left(\bigoplus\nolimits_{j=1,\ldots,m} j_{V_j!}\underline{\mathbf{Z}/m_j\mathbf{Z}}_{V_j} \longrightarrow \bigoplus\nolimits_{i=1,\ldots,n} j_{U_i!}\underline{\mathbf{Z}/n_i\mathbf{Z}}_{U_i}\right)$$

with V_j and U_i quasi-compact and quasi-separated objects of $X_{étale}$ and m_j, n_i positive integers. In fact, we can even assume U_i and V_j affine.

Proof. This follows from Lemma 72.6 applied with $\Lambda = \mathbf{Z}/n\mathbf{Z}$ and the fact that, since X is quasi-compact, every constructible abelian sheaf is annihilated by some positive integer n (details omitted). □

09Z4 **Lemma 72.8.** *Let X be a quasi-compact and quasi-separated scheme. Let Λ be a Noetherian ring. Let $\mathcal{F}$ be a constructible sheaf of sets, abelian groups, or Λ-modules on $X_{étale}$. Let $\mathcal{G} = \text{colim}\, \mathcal{G}_i$ be a filtered colimit of sheaves of sets, abelian groups, or Λ-modules. Then*

$$Mor(\mathcal{F}, \mathcal{G}) = \text{colim}\, Mor(\mathcal{F}, \mathcal{G}_i)$$

in the category of sheaves of sets, abelian groups, or Λ-modules on $X_{étale}$.

Proof. The case of sheaves of sets. By Lemma 72.5 it suffices to prove the lemma for h_U where U is a quasi-compact and quasi-separated object of $X_{étale}$. Recall that $Mor(h_U, \mathcal{G}) = \mathcal{G}(U)$. Hence the result follows from Sites, Lemma 17.5.

In the case of abelian sheaves or sheaves of modules, the result follows in the same way using Lemmas 72.7 and 72.6. For the case of abelian sheaves, we add that $Mor(j_{U!}\underline{\mathbf{Z}/n\mathbf{Z}}, \mathcal{G})$ is equal to the n-torsion elements of $\mathcal{G}(U)$. □

095R **Lemma 72.9.** *Let $f : X \to Y$ be a finite and finitely presented morphism of schemes. Let Λ be a Noetherian ring. If $\mathcal{F}$ is a constructible sheaf of sets, abelian groups, or Λ-modules on $X_{étale}$, then $f_*\mathcal{F}$ is too.*

Proof. It suffices to prove this when X and Y are affine by Lemma 70.4. By Lemmas 54.3 and 72.3 we may base change to any affine scheme surjective over X. By Lemma 71.3 this reduces us to the case of a finite étale morphism (because a thickening leads to an equivalence of étale topoi and even small étale sites, see Theorem 45.2). The finite étale case is Lemma 72.4. □

09YU **Lemma 72.10.** *Let $X = \lim_{i \in I} X_i$ be a limit of a directed system of schemes with affine transition morphisms. We assume that X_i is quasi-compact and quasi-separated for all $i \in I$.*

1. *The category of constructible sheaves of sets on $X_{étale}$ is the colimit of the categories of constructible sheaves of sets on $(X_i)_{étale}$.*
2. *The category of constructible abelian sheaves on $X_{étale}$ is the colimit of the categories of constructible abelian sheaves on $(X_i)_{étale}$.*
3. *Let Λ be a Noetherian ring. The category of constructible sheaves of Λ-modules on $X_{étale}$ is the colimit of the categories of constructible sheaves of Λ-modules on $(X_i)_{étale}$.*

Proof. Proof of (1). Denote $f_i : X \to X_i$ the projection maps. There are 3 parts to the proof corresponding to "faithful", "fully faithful", and "essentially surjective".

Faithful. Choose $0 \in I$ and let $\mathcal{F}_0, \mathcal{G}_0$ be constructible sheaves on X_0. Suppose that $a, b : \mathcal{F}_0 \to \mathcal{G}_0$ are maps such that $f_0^{-1}a = f_0^{-1}b$. Let $E \subset X_0$ be the set of points $x \in X_0$ such that $a_{\overline{x}} = b_{\overline{x}}$. By Lemma 70.8 the subset $E \subset X_0$ is constructible. By assumption $X \to X_0$ maps into E. By Limits, Lemma 4.10 we find an $i \geq 0$ such that $X_i \to X_0$ maps into E. Hence $f_{i0}^{-1}a = f_{i0}^{-1}b$.

Fully faithful. Choose $0 \in I$ and let $\mathcal{F}_0, \mathcal{G}_0$ be constructible sheaves on X_0. Suppose that $a : f_0^{-1}\mathcal{F}_0 \to f_0^{-1}\mathcal{G}_0$ is a map. We claim there is an i and a map $a_i : f_{i0}^{-1}\mathcal{F}_0 \to f_{i0}^{-1}\mathcal{G}_0$ which pulls back to a on X. By Lemma 72.5 we can replace $\mathcal{F}_0$ by a finite coproduct of sheaves represented by quasi-compact and quasi-separated objects of $(X_0)_{étale}$. Thus we have to show: If $U_0 \to X_0$ is such an object of $(X_0)_{étale}$, then

$$f_0^{-1}\mathcal{G}(U) = \text{colim}_{i \geq 0}\, f_{i0}^{-1}\mathcal{G}(U_i)$$

where $U = X \times_{X_0} U_0$ and $U_i = X_i \times_{X_0} U_0$. This is a special case of Theorem 51.3.

Essentially surjective. We have to show every constructible $\mathcal{F}$ on X is isomorphic to $f_i^{-1}\mathcal{F}$ for some constructible $\mathcal{F}_i$ on X_i. Applying Lemma 72.5 and using the results of the previous two paragraphs, we see that it suffices to prove this for h_U for some quasi-compact and quasi-separated object U of $X_{étale}$. In this case we have to show that U is the base change of a quasi-compact and quasi-separated scheme étale over X_i for some i. This follows from Limits, Lemmas 10.1 and 8.10.

Proof of (3). The argument is very similar to the argument for sheaves of sets, but using Lemma 72.6 instead of Lemma 72.5. Details omitted. Part (2) follows from part (3) because every constructible abelian sheaf over a quasi-compact scheme is a constructible sheaf of $\mathbf{Z}/n\mathbf{Z}$-modules for some n. □

09BG **Lemma 72.11.** *Let X be an irreducible scheme with generic point η.*

1. *Let $S' \subset S$ be an inclusion of sets. If we have $\underline{S'} \subset \mathcal{G} \subset \underline{S}$ in $Sh(X_{étale})$ and $S' = \mathcal{G}_{\overline{\eta}}$, then $\mathcal{G} = \underline{S'}$.*
2. *Let $A' \subset A$ be an inclusion of abelian groups. If we have $\underline{A'} \subset \mathcal{G} \subset \underline{A}$ in $Ab(X_{étale})$ and $A' = \mathcal{G}_{\overline{\eta}}$, then $\mathcal{G} = \underline{A'}$.*
3. *Let $M' \subset M$ be an inclusion of modules over a ring Λ. If we have $\underline{M'} \subset \mathcal{G} \subset \underline{M}$ in $Mod(X_{étale}, \underline{\Lambda})$ and $M' = \mathcal{G}_{\overline{\eta}}$, then $\mathcal{G} = \underline{M'}$.*

Proof. This is true because for every étale morphism $U \to X$ with $U \neq \emptyset$ the point η is in the image. □

09Z5 **Lemma 72.12.** *Let X be an integral normal scheme with function field K. Let E be a set.*

1. *Let $g : \mathrm{Spec}(K) \to X$ be the inclusion of the generic point. Then $g_*\underline{E} = \underline{E}$.*
2. *Let $j : U \to X$ be the inclusion of a nonempty open. Then $j_*\underline{E} = \underline{E}$.*

Proof. Proof of (1). Let $x \in X$ be a point. Let $\mathcal{O}_{X,\overline{x}}$ be a strict henselization of $\mathcal{O}_{X,x}$. By More on Algebra, Lemma 44.6 we see that $\mathcal{O}_{X,\overline{x}}$ is a normal domain. Hence $\mathrm{Spec}(K) \times_X \mathrm{Spec}(\mathcal{O}_{X,\overline{x}})$ is irreducible. It follows that the stalk $(g_*\underline{E}_{\overline{x}}$ is equal to E, see Theorem 52.1.

Proof of (2). Since g factors through j there is a map $j_*\underline{E} \to g_*\underline{E}$. This map is injective because for every scheme V étale over X the set $\mathrm{Spec}(K) \times_X V$ is dense in $U \times_X V$. On the other hand, we have a map $\underline{E} \to j_*\underline{E}$ and we conclude. □

0F0M **Lemma 72.13.** *Let X be a quasi-compact and quasi-separated scheme. Let $\eta \in X$ be a generic point of an irreducible component of X.*

1. *Let $\mathcal{F}$ be a torsion abelian sheaf on $X_{étale}$ whose stalk $\mathcal{F}_{\overline{\eta}}$ is zero. Then $\mathcal{F} = \mathrm{colim}\, \mathcal{F}_i$ is a filtered colimit of constructible abelian sheaves $\mathcal{F}_i$ such that for each i the support of $\mathcal{F}_i$ is contained in a closed subscheme not containing η.*
2. *Let Λ be a Noetherian ring and $\mathcal{F}$ a sheaf of Λ-modules on $X_{étale}$ whose stalk $\mathcal{F}_{\overline{\eta}}$ is zero. Then $\mathcal{F} = \mathrm{colim}\, \mathcal{F}_i$ is a filtered colimit of constructible sheaves of Λ-modules $\mathcal{F}_i$ such that for each i the support of $\mathcal{F}_i$ is contained in a closed subscheme not containing η.*

Proof. Proof of (1). We can write $\mathcal{F} = \mathrm{colim}_{i \in I}\, \mathcal{F}_i$ with $\mathcal{F}_i$ constructible abelian by Lemma 72.2. Choose $i \in I$. Since $\mathcal{F}|_\eta$ is zero by assumption, we see that there exists an $i'(i) \geq i$ such that $\mathcal{F}_i|_\eta \to \mathcal{F}_{i'(i)}|_\eta$ is zero, see Lemma 70.9. Then

$\mathcal{G}_i = \mathrm{Im}(\mathcal{F}_i \to \mathcal{F}_{i'(i)})$ is a constructible abelian sheaf (Lemma 70.6) whose stalk at η is zero. Hence the support E_i of $\mathcal{G}_i$ is a constructible subset of X not containing η. Since η is a generic point of an irreducible component of X, we see that $\eta \notin Z_i = \overline{E_i}$ by Topology, Lemma 15.15. Define a new directed set I' by using the set I with ordering defined by the rule i_1 is bigger or equal to i_2 if and only if $i_1 \geq i'(i_2)$. Then the sheaves $\mathcal{G}_i$ form a system over I' with colimit $\mathcal{F}$ and the proof is complete.

The proof in case (2) is exactly the same and we omit it. □

73. Constructible sheaves on Noetherian schemes

03RY If X is a Noetherian scheme then any locally closed subset is a constructible locally closed subset (Topology, Lemma 16.1). Hence an abelian sheaf $\mathcal{F}$ on $X_{étale}$ is constructible if and only if there exists a finite partition $X = \coprod X_i$ such that $\mathcal{F}|_{X_i}$ is finite locally constant. (By convention a partition of a topological space has locally closed parts, see Topology, Section 28.) In other words, we can omit the adjective "constructible" in Definition 70.1. Actually, the category of constructible sheaves on Noetherian schemes has some additional properties which we will catalogue in this section.

09BH **Proposition 73.1.** *Let X be a Noetherian scheme. Let Λ be a Noetherian ring.*

1. *Any sub or quotient sheaf of a constructible sheaf of sets is constructible.*
2. *The category of constructible abelian sheaves on $X_{étale}$ is a (strong) Serre subcategory of $Ab(X_{étale})$. In particular, every sub and quotient sheaf of a constructible abelian sheaf on $X_{étale}$ is constructible.*
3. *The category of constructible sheaves of Λ-modules on $X_{étale}$ is a (strong) Serre subcategory of $Mod(X_{étale}, \Lambda)$. In particular, every submodule and quotient module of a constructible sheaf of Λ-modules on $X_{étale}$ is constructible.*

Proof. Proof of (1). Let $\mathcal{G} \subset \mathcal{F}$ with $\mathcal{F}$ a constructible sheaf of sets on $X_{étale}$. Let $\eta \in X$ be a generic point of an irreducible component of X. By Noetherian induction it suffices to find an open neighbourhood U of η such that $\mathcal{G}|_U$ is locally constant. To do this we may replace X by an étale neighbourhood of η. Hence we may assume $\mathcal{F}$ is constant and X is irreducible.

Say $\mathcal{F} = \underline{S}$ for some finite set S. Then $S' = \mathcal{G}_{\overline{\eta}} \subset S$ say $S' = \{s_1, \ldots, s_t\}$. Pick an étale neighbourhood $(U, \overline{u})$ of $\overline{\eta}$ and sections $\sigma_1, \ldots, \sigma_t \in \mathcal{G}(U)$ which map to s_i in $\mathcal{G}_{\overline{\eta}} \subset S$. Since σ_i maps to an element $s_i \in S' \subset S = \Gamma(X, \mathcal{F})$ we see that the two pullbacks of σ_i to $U \times_X U$ are the same as sections of $\mathcal{G}$. By the sheaf condition for $\mathcal{G}$ we find that σ_i comes from a section of $\mathcal{G}$ over the open $\mathrm{Im}(U \to X)$ of X. Shrinking X we may assume $\underline{S'} \subset \mathcal{G} \subset \underline{S}$. Then we see that $\underline{S'} = \mathcal{G}$ by Lemma 72.11.

Let $\mathcal{F} \to \mathcal{Q}$ be a surjection with $\mathcal{F}$ a constructible sheaf of sets on $X_{étale}$. Then set $\mathcal{G} = \mathcal{F} \times_{\mathcal{Q}} \mathcal{F}$. By the first part of the proof we see that $\mathcal{G}$ is constructible as a subsheaf of $\mathcal{F} \times \mathcal{F}$. This in turn implies that $\mathcal{Q}$ is constructible, see Lemma 70.6.

Proof of (3). we already know that constructible sheaves of modules form a weak Serre subcategory, see Lemma 70.6. Thus it suffices to show the statement on submodules.

Let $\mathcal{G} \subset \mathcal{F}$ be a submodule of a constructible sheaf of Λ-modules on $X_{\acute{e}tale}$. Let $\eta \in X$ be a generic point of an irreducible component of X. By Noetherian induction it suffices to find an open neighbourhood U of η such that $\mathcal{G}|_U$ is locally constant. To do this we may replace X by an étale neighbourhood of η. Hence we may assume $\mathcal{F}$ is constant and X is irreducible.

Say $\mathcal{F} = \underline{M}$ for some finite Λ-module M. Then $M' = \mathcal{G}_{\overline{\eta}} \subset M$. Pick finitely many elements $s_1, \ldots, s_t$ generating M' as a Λ-module. (This is possible as Λ is Noetherian and M is finite.) Pick an étale neighbourhood $(U, \overline{u})$ of $\overline{\eta}$ and sections $\sigma_1, \ldots, \sigma_t \in \mathcal{G}(U)$ which map to s_i in $\mathcal{G}_{\overline{\eta}} \subset M$. Since σ_i maps to an element $s_i \in M' \subset M = \Gamma(X, \mathcal{F})$ we see that the two pullbacks of σ_i to $U \times_X U$ are the same as sections of $\mathcal{G}$. By the sheaf condition for $\mathcal{G}$ we find that σ_i comes from a section of $\mathcal{G}$ over the open $\text{Im}(U \to X)$ of X. Shrinking X we may assume $\underline{M'} \subset \mathcal{G} \subset \underline{M}$. Then we see that $\underline{M'} = \mathcal{G}$ by Lemma 72.11.

Proof of (2). This follows in the usual manner from (3). Details omitted. $\square$

The following lemma tells us that every object of the abelian category of constructible sheaves on X is "Noetherian", i.e., satisfies a.c.c. for subobjects.

09YV **Lemma 73.2.** *Let X be a Noetherian scheme. Let Λ be a Noetherian ring. Consider inclusions*

$$\mathcal{F}_1 \subset \mathcal{F}_2 \subset \mathcal{F}_3 \subset \ldots \subset \mathcal{F}$$

in the category of sheaves of sets, abelian groups, or Λ-modules. If $\mathcal{F}$ is constructible, then for some n we have $\mathcal{F}_n = \mathcal{F}_{n+1} = \mathcal{F}_{n+2} = \ldots$.

Proof. By Proposition 73.1 we see that $\mathcal{F}_i$ and $\operatorname{colim} \mathcal{F}_i$ are constructible. Then the lemma follows from Lemma 70.9. $\square$

09Z6 **Lemma 73.3.** *Let X be a Noetherian scheme.*

(1) *Let $\mathcal{F}$ be a constructible sheaf of sets on $X_{\acute{e}tale}$. There exist an injective map of sheaves*
$$\mathcal{F} \longrightarrow \coprod\nolimits_{i=1,\ldots,n} f_{i,*}\underline{E_i}$$
where $f_i : Y_i \to X$ is a finite morphism and E_i is a finite set.

(2) *Let $\mathcal{F}$ be a constructible abelian sheaf on $X_{\acute{e}tale}$. There exist an injective map of abelian sheaves*
$$\mathcal{F} \longrightarrow \bigoplus\nolimits_{i=1,\ldots,n} f_{i,*}\underline{M_i}$$
where $f_i : Y_i \to X$ is a finite morphism and M_i is a finite abelian group.

(3) *Let Λ be a Noetherian ring. Let $\mathcal{F}$ be a constructible sheaf of Λ-modules on $X_{\acute{e}tale}$. There exist an injective map of sheaves of modules*
$$\mathcal{F} \longrightarrow \bigoplus\nolimits_{i=1,\ldots,n} f_{i,*}\underline{M_i}$$
where $f_i : Y_i \to X$ is a finite morphism and M_i is a finite Λ-module.

Moreover, we may assume each Y_i is irreducible, reduced, maps onto an irreducible and reduced closed subscheme $Z_i \subset X$ such that $Y_i \to Z_i$ is finite étale over a nonempty open of Z_i.

Proof. Proof of (1). Because we have the ascending chain condition for subsheaves of $\mathcal{F}$ (Lemma 73.2), it suffices to show that for every point $x \in X$ we can find a map $\varphi : \mathcal{F} \to f_*\underline{E}$ where $f : Y \to X$ is finite and E is a finite set such that $\varphi_{\overline{x}} : \mathcal{F}_{\overline{x}} \to (f_*S)_{\overline{x}}$ is injective. (This argument can be avoided by picking a partition of X as in Lemma 70.2 and constructing a $Y_i \to X$ for each irreducible component of each part.) Let $Z \subset X$ be the induced reduced scheme structure (Schemes, Definition 12.5) on $\overline{\{x\}}$. Since $\mathcal{F}$ is constructible, there is a finite separable extension $\kappa(x) \subset \mathrm{Spec}(K)$ such that $\mathcal{F}|_{\mathrm{Spec}(K)}$ is the constant sheaf with value E for some finite set E. Let $Y \to Z$ be the normalization of Z in $\mathrm{Spec}(K)$. By Morphisms, Lemma 51.13 we see that Y is a normal integral scheme. As $\kappa(x) \subset K$ is finite, it is clear that K is the function field of Y. Denote $g : \mathrm{Spec}(K) \to Y$ the inclusion. The map $\mathcal{F}|_{\mathrm{Spec}(K)} \to \underline{E}$ is adjoint to a map $\mathcal{F}|_Y \to g_*\underline{E} = \underline{E}$ (Lemma 72.12). This in turn is adjoint to a map $\varphi : \mathcal{F} \to f_*\underline{E}$. Observe that the stalk of φ at a geometric point $\overline{x}$ is injective: we may take a lift $\overline{y} \in Y$ of $\overline{x}$ and the commutative diagram

$$\begin{tikzcd}
\mathcal{F}_{\overline{x}} \arrow[r, equal] \arrow[d] & (\mathcal{F}|_Y)_{\overline{y}} \arrow[d, equal] \\
(f_*\underline{E})_{\overline{x}} \arrow[r] & \underline{E}_{\overline{y}}
\end{tikzcd}$$

proves the injectivity. We are not yet done, however, as the morphism $f : Y \to Z$ is integral but in general not finite[6].

To fix the problem stated in the last sentence of the previous paragraph, we write $Y = \lim_{i \in I} Y_i$ with Y_i irreducible, integral, and finite over Z. Namely, apply Properties, Lemma 22.13 to $f_*\mathcal{O}_Y$ viewed as a sheaf of $\mathcal{O}_Z$-algebras and apply the functor $\underline{\mathrm{Spec}}_Z$. Then $f_*\underline{E} = \mathrm{colim}\, f_{i,*}\underline{E}$ by Lemma 51.7. By Lemma 72.8 the map $\mathcal{F} \to f_*\underline{E}$ factors through $f_{i,*}\underline{E}$ for some i. Since $Y_i \to Z$ is a finite morphism of integral schemes and since the function field extension induced by this morphism is finite separable, we see that the morphism is finite étale over a nonempty open of Z (use Algebra, Lemma 139.9; details omitted). This finishes the proof of (1).

The proofs of (2) and (3) are identical to the proof of (1). $\square$

In the following lemma we use a standard trick to reduce a very general statement to the Noetherian case.

09Z7 **Lemma 73.4.** *Let X be a quasi-compact and quasi-separated scheme.*

(1) *Let $\mathcal{F}$ be a constructible sheaf of sets on $X_{étale}$. There exist an injective map of sheaves*
$$\mathcal{F} \longrightarrow \coprod\nolimits_{i=1,\ldots,n} f_{i,*}\underline{E_i}$$
where $f_i : Y_i \to X$ is a finite and finitely presented morphism and E_i is a finite set.

(2) *Let $\mathcal{F}$ be a constructible abelian sheaf on $X_{étale}$. There exist an injective map of abelian sheaves*
$$\mathcal{F} \longrightarrow \bigoplus\nolimits_{i=1,\ldots,n} f_{i,*}\underline{M_i}$$
where $f_i : Y_i \to X$ is a finite and finitely presented morphism and M_i is a finite abelian group.

[6] If X is a Nagata scheme, for example of finite type over a field, then $Y \to Z$ is finite.

(3) *Let Λ be a Noetherian ring. Let $\mathcal{F}$ be a constructible sheaf of Λ-modules on $X_{étale}$. There exist an injective map of sheaves of modules*

$$\mathcal{F} \longrightarrow \bigoplus\nolimits_{i=1,\ldots,n} f_{i,*}\underline{M_i}$$

where $f_i : Y_i \to X$ is a finite and finitely presented morphism and M_i is a finite Λ-module.

Proof. We will reduce this lemma to the Noetherian case by absolute Noetherian approximation. Namely, by Limits, Proposition 5.4 we can write $X = \lim_{t\in T} X_t$ with each X_t of finite type over $\text{Spec}(\mathbf{Z})$ and with affine transition morphisms. By Lemma 72.10 the category of constructible sheaves (of sets, abelian groups, or Λ-modules) on $X_{étale}$ is the colimit of the corresponding categories for X_t. Thus our constructible sheaf $\mathcal{F}$ is the pullback of a similar constructible sheaf $\mathcal{F}_t$ over X_t for some t. Then we apply the Noetherian case (Lemma 73.3) to find an injection

$$\mathcal{F}_t \longrightarrow \coprod\nolimits_{i=1,\ldots,n} f_{i,*}\underline{E_i} \quad \text{or} \quad \mathcal{F}_t \longrightarrow \bigoplus\nolimits_{i=1,\ldots,n} f_{i,*}\underline{M_i}$$

over X_t for some finite morphisms $f_i : Y_i \to X_t$. Since X_t is Noetherian the morphisms f_i are of finite presentation. Since pullback is exact and since formation of $f_{i,*}$ commutes with base change (Lemma 54.3), we conclude. □

0F0N **Lemma 73.5.** *Let X be a Noetherian scheme. Let $E \subset X$ be a subset closed under specialization.*

(1) *Let $\mathcal{F}$ be a torsion abelian sheaf on $X_{étale}$ whose support is contained in E. Then $\mathcal{F} = \text{colim}\, \mathcal{F}_i$ is a filtered colimit of constructible abelian sheaves $\mathcal{F}_i$ such that for each i the support of $\mathcal{F}_i$ is contained in a closed subset contained in E.*
(2) *Let Λ be a Noetherian ring and $\mathcal{F}$ a sheaf of Λ-modules on $X_{étale}$ whose support is contained in E. Then $\mathcal{F} = \text{colim}\, \mathcal{F}_i$ is a filtered colimit of constructible sheaves of Λ-modules $\mathcal{F}_i$ such that for each i the support of $\mathcal{F}_i$ is contained in a closed subset contained in E.*

Proof. Proof of (1). We can write $\mathcal{F} = \text{colim}_{i\in I} \mathcal{F}_i$ with $\mathcal{F}_i$ constructible abelian by Lemma 72.2. By Proposition 73.1 the image $\mathcal{F}'_i \subset \mathcal{F}$ of the map $\mathcal{F}_i \to \mathcal{F}$ is constructible. Thus $\mathcal{F} = \text{colim}\, \mathcal{F}'_i$ and the support of $\mathcal{F}'_i$ is contained in E. Since the support of $\mathcal{F}'_i$ is constructible (by our definition of constructible sheaves), we see that its closure is also contained in E, see for example Topology, Lemma 23.5.

The proof in case (2) is exactly the same and we omit it. □

74. Complexes with constructible cohomology

095V Let Λ be a ring. Denote $D(X_{étale}, \Lambda)$ the derived category of sheaves of Λ-modules on $X_{étale}$. We denote by $D^b(X_{étale}, \Lambda)$ (respectively D^+, D^-) the full subcategory of bounded (resp. above, below) complexes in $D(X_{étale}, \Lambda)$.

095W **Definition 74.1.** Let X be a scheme. Let Λ be a Noetherian ring. We denote $D_c(X_{étale}, \Lambda)$ the full subcategory of $D(X_{étale}, \Lambda)$ of complexes whose cohomology sheaves are constructible sheaves of Λ-modules.

This definition makes sense by Lemma 70.6 and Derived Categories, Section 17. Thus we see that $D_c(X_{étale}, \Lambda)$ is a strictly full, saturated triangulated subcategory of $D_c(X_{étale}, \Lambda)$.

095X **Lemma 74.2.** *Let Λ be a Noetherian ring. If $j : U \to X$ is an étale morphism of schemes, then*

1. *$K|_U \in D_c(U_{étale}, \Lambda)$ if $K \in D_c(X_{étale}, \Lambda)$, and*
2. *$j_!M \in D_c(X_{étale}, \Lambda)$ if $M \in D_c(U_{étale}, \Lambda)$ and the morphism j is quasi-compact and quasi-separated.*

Proof. The first assertion is clear. The second follows from the fact that $j_!$ is exact and Lemma 72.1. □

095Y **Lemma 74.3.** *Let Λ be a Noetherian ring. Let $f : X \to Y$ be a morphism of schemes. If $K \in D_c(Y_{étale}, \Lambda)$ then $Lf^*K \in D_c(X_{étale}, \Lambda)$.*

Proof. This follows as $f^{-1} = f^*$ is exact and Lemma 70.5. □

095Z **Lemma 74.4.** *Let X be a quasi-compact and quasi-separated scheme. Let Λ be a Noetherian ring. Let $K \in D(X_{étale}, \Lambda)$ and $b \in \mathbf{Z}$ such that $H^b(K)$ is constructible. Then there exist a sheaf $\mathcal{F}$ which is a finite direct sum of $j_{U!}\underline{\Lambda}$ with $U \in \mathrm{Ob}(X_{étale})$ affine and a map $\mathcal{F}[-b] \to K$ in $D(X_{étale}, \Lambda)$ inducing a surjection $\mathcal{F} \to H^b(K)$.*

Proof. Represent K by a complex $\mathcal{K}^\bullet$ of sheaves of Λ-modules. Consider the surjection

$$\mathrm{Ker}(\mathcal{K}^b \to \mathcal{K}^{b+1}) \longrightarrow H^b(K)$$

By Modules on Sites, Lemma 29.5 we may choose a surjection $\bigoplus_{i\in I} j_{U_i!}\underline{\Lambda} \to \mathrm{Ker}(\mathcal{K}^b \to \mathcal{K}^{b+1})$ with U_i affine. For $I' \subset I$ finite, denote $\mathcal{H}_{I'} \subset H^b(K)$ the image of $\bigoplus_{i\in I'} j_{U_i!}\underline{\Lambda}$. By Lemma 70.9 we see that $\mathcal{H}_{I'} = H^b(K)$ for some $I' \subset I$ finite. The lemma follows taking $\mathcal{F} = \bigoplus_{i\in I'} j_{U_i!}\underline{\Lambda}$. □

0960 **Lemma 74.5.** *Let X be a quasi-compact and quasi-separated scheme. Let Λ be a Noetherian ring. Let $K \in D^-(X_{étale}, \Lambda)$. Then the following are equivalent*

1. *K is in $D_c(X_{étale}, \Lambda)$,*
2. *K can be represented by a bounded above complex whose terms are finite direct sums of $j_{U!}\underline{\Lambda}$ with $U \in \mathrm{Ob}(X_{étale})$ affine,*
3. *K can be represented by a bounded above complex of flat constructible sheaves of Λ-modules.*

Proof. It is clear that (2) implies (3) and that (3) implies (1). Assume K is in $D_c^-(X_{étale}, \Lambda)$. Say $H^i(K) = 0$ for $i > b$. By induction on a we will construct a complex $\mathcal{F}^a \to \ldots \to \mathcal{F}^b$ such that each $\mathcal{F}^i$ is a finite direct sum of $j_{U!}\underline{\Lambda}$ with $U \in \mathrm{Ob}(X_{étale})$ affine and a map $\mathcal{F}^\bullet \to K$ which induces an isomorphism $H^i(\mathcal{F}^\bullet) \to H^i(K)$ for $i > a$ and a surjection $H^a(\mathcal{F}^\bullet) \to H^a(K)$. For $a = b$ this can be done by Lemma 74.4. Given such a datum choose a distinguished triangle

$$\mathcal{F}^\bullet \to K \to L \to \mathcal{F}^\bullet[1]$$

Then we see that $H^i(L) = 0$ for $i \geq a$. Choose $\mathcal{F}^{a-1}[-a+1] \to L$ as in Lemma 74.4. The composition $\mathcal{F}^{a-1}[-a+1] \to L \to \mathcal{F}^\bullet$ corresponds to a map $\mathcal{F}^{a-1} \to \mathcal{F}^a$ such that the composition with $\mathcal{F}^a \to \mathcal{F}^{a+1}$ is zero. By TR4 we obtain a map

$$(\mathcal{F}^{a-1} \to \ldots \to \mathcal{F}^b) \to K$$

in $D(X_{étale}, \Lambda)$. This finishes the induction step and the proof of the lemma. □

0961 **Lemma 74.6.** *Let X be a scheme. Let Λ be a Noetherian ring. Let $K, L \in D_c^-(X_{étale}, \Lambda)$. Then $K \otimes_\Lambda^{\mathbf{L}} L$ is in $D_c^-(X_{étale}, \Lambda)$.*

Proof. This follows from Lemmas 74.5 and 70.7. □

75. Tor finite with constructible cohomology

0F4M Let X be a scheme and let Λ be a Noetherian ring. An often used subcategory of the derived category $D_c(X_{étale}, \Lambda)$ defined in Section 74 is the full subcategory consisting of objects having (locally) finite tor dimension. Here is the formal definition.

03TQ **Definition 75.1.** Let X be a scheme. Let Λ be a Noetherian ring. We denote $D_{ctf}(X_{étale}, \Lambda)$ the full subcategory of $D_c(X_{étale}, \Lambda)$ consisting of objects having locally finite tor dimension.

This is a strictly full, saturated triangulated subcategory of $D_c(X_{étale}, \Lambda)$ and $D(X_{étale}, \Lambda)$. By our conventions, see Cohomology on Sites, Definition 43.1, we see that

$$D_{ctf}(X_{étale}, \Lambda) \subset D^b_c(X_{étale}, \Lambda) \subset D^b(X_{étale}, \Lambda)$$

if X is quasi-compact. A good way to think about objects of $D_{ctf}(X_{étale}, \Lambda)$ is given in Lemma 75.3.

03TS **Remark 75.2.** Objects in the derived category $D_{ctf}(X_{étale}, \Lambda)$ in some sense have better global properties than the perfect objects in $D(\mathcal{O}_X)$. Namely, it can happen that a complex of $\mathcal{O}_X$-modules is locally quasi-isomorphic to a finite complex of finite locally free $\mathcal{O}_X$-modules, without being globally quasi-isomorphic to a bounded complex of locally free $\mathcal{O}_X$-modules. The following lemma shows this does not happen for D_{ctf} on a Noetherian scheme.

03TT **Lemma 75.3.** *Let Λ be a Noetherian ring. Let X be a quasi-compact and quasi-separated scheme. Let $K \in D(X_{étale}, \Lambda)$. The following are equivalent*

(1) *$K \in D_{ctf}(X_{étale}, \Lambda)$, and*
(2) *K can be represented by a finite complex of constructible flat sheaves of Λ-modules.*

In fact, if K has tor amplitude in $[a, b]$ then we can represent K by a complex $\mathcal{F}^a \to \ldots \to \mathcal{F}^b$ with $\mathcal{F}^p$ a constructible flat sheaf of Λ-modules.

Proof. It is clear that a finite complex of constructible flat sheaves of Λ-modules has finite tor dimension. It is also clear that it is an object of $D_c(X_{étale}, \Lambda)$. Thus we see that (2) implies (1).

Assume (1). Choose $a, b \in \mathbf{Z}$ such that $H^i(K \otimes^{\mathbf{L}}_\Lambda \mathcal{G}) = 0$ if $i \notin [a, b]$ for all sheaves of Λ-modules $\mathcal{G}$. We will prove the final assertion holds by induction on $b - a$. If $a = b$, then $K = H^a(K)[-a]$ is a flat constructible sheaf and the result holds. Next, assume $b > a$. Represent K by a complex $\mathcal{K}^\bullet$ of sheaves of Λ-modules. Consider the surjection

$$\text{Ker}(\mathcal{K}^b \to \mathcal{K}^{b+1}) \longrightarrow H^b(K)$$

By Lemma 72.6 we can find finitely many affine schemes U_i étale over X and a surjection $\bigoplus j_{U_i!}\underline{\Lambda}_{U_i} \to H^b(K)$. After replacing U_i by standard étale coverings $\{U_{ij} \to U_i\}$ we may assume this surjection lifts to a map $\mathcal{F} = \bigoplus j_{U_i!}\underline{\Lambda}_{U_i} \to \text{Ker}(\mathcal{K}^b \to \mathcal{K}^{b+1})$. This map determines a distinguished triangle

$$\mathcal{F}[-b] \to K \to L \to \mathcal{F}[-b+1]$$

in $D(X_{étale}, \Lambda)$. Since $D_{ctf}(X_{étale}, \Lambda)$ is a triangulated subcategory we see that L is in it too. In fact L has tor amplitude in $[a, b-1]$ as $\mathcal{F}$ surjects onto $H^b(K)$ (details omitted). By induction hypothesis we can find a finite complex $\mathcal{F}^a \to \ldots \to \mathcal{F}^{b-1}$

of flat constructible sheaves of Λ-modules representing L. The map $L \to \mathcal{F}[-b+1]$ corresponds to a map $\mathcal{F}^b \to \mathcal{F}$ annihilating the image of $\mathcal{F}^{b-1} \to \mathcal{F}^b$. Then it follows from axiom TR3 that K is represented by the complex

$$\mathcal{F}^a \to \ldots \to \mathcal{F}^{b-1} \to \mathcal{F}^b$$

which finishes the proof. $\square$

03TR **Remark 75.4.** Let Λ be a Noetherian ring. Let X be a scheme. For a bounded complex $K^\bullet$ of constructible flat Λ-modules on $X_{étale}$ each stalk $K^p_{\bar{x}}$ is a finite projective Λ-module. Hence the stalks of the complex are perfect complexes of Λ-modules.

0962 **Lemma 75.5.** *Let Λ be a Noetherian ring. If $j : U \to X$ is an étale morphism of schemes, then*

1. *$K|_U \in D_{ctf}(U_{étale}, \Lambda)$ if $K \in D_{ctf}(X_{étale}, \Lambda)$, and*
2. *$j_!M \in D_{ctf}(X_{étale}, \Lambda)$ if $M \in D_{ctf}(U_{étale}, \Lambda)$ and the morphism j is quasi-compact and quasi-separated.*

Proof. Perhaps the easiest way to prove this lemma is to reduce to the case where X is affine and then apply Lemma 75.3 to translate it into a statement about finite complexes of flat constructible sheaves of Λ-modules where the result follows from Lemma 72.1. $\square$

0963 **Lemma 75.6.** *Let Λ be a Noetherian ring. Let $f : X \to Y$ be a morphism of schemes. If $K \in D_{ctf}(Y_{étale}, \Lambda)$ then $Lf^*K \in D_{ctf}(X_{étale}, \Lambda)$.*

Proof. Apply Lemma 75.3 to reduce this to a question about finite complexes of flat constructible sheaves of Λ-modules. Then the statement follows as $f^{-1} = f^*$ is exact and Lemma 70.5. $\square$

09BI **Lemma 75.7.** *Let X be a connected scheme. Let Λ be a Noetherian ring. Let $K \in D_{ctf}(X_{étale}, \Lambda)$ have locally constant cohomology sheaves. Then there exists a finite complex of finite projective Λ-modules $M^\bullet$ and an étale covering $\{U_i \to X\}$ such that $K|_{U_i} \cong \underline{M^\bullet}|_{U_i}$ in $D(U_{i,étale}, \Lambda)$.*

Proof. Choose an étale covering $\{U_i \to X\}$ such that $K|_{U_i}$ is constant, say $K|_{U_i} \cong \underline{M_i^\bullet}_{U_i}$ for some finite complex of finite Λ-modules $M_i^\bullet$. See Cohomology on Sites, Lemma 48.1. Observe that $U_i \times_X U_j$ is empty if $M_i^\bullet$ is not isomorphic to $M_j^\bullet$ in $D(\Lambda)$. For each complex of Λ-modules $M^\bullet$ let $I_{M^\bullet} = \{i \in I \mid M_i^\bullet \cong M^\bullet \text{ in } D(\Lambda)\}$. As étale morphisms are open we see that $U_{M^\bullet} = \bigcup_{i \in I_{M^\bullet}} \text{Im}(U_i \to X)$ is an open subset of X. Then $X = \coprod U_{M^\bullet}$ is a disjoint open covering of X. As X is connected only one $U_{M^\bullet}$ is nonempty. As K is in $D_{ctf}(X_{étale}, \Lambda)$ we see that $M^\bullet$ is a perfect complex of Λ-modules, see More on Algebra, Lemma 69.2. Hence we may assume $M^\bullet$ is a finite complex of finite projective Λ-modules. $\square$

76. Torsion sheaves

0DDB A brief section on torsion abelian sheaves and their étale cohomology. Let $\mathcal{C}$ be a site. We have shown in Cohomology on Sites, Lemma 19.7 that any object in $D(\mathcal{C})$ whose cohomology sheaves are torsion sheaves, can be represented by a complex all of whose terms are torsion.

0DDC **Lemma 76.1.** *Let X be a quasi-compact and quasi-separated scheme.*

(1) *If $\mathcal{F}$ is a torsion abelian sheaf on $X_{étale}$, then $H^n_{étale}(X, \mathcal{F})$ is a torsion abelian group for all n.*
(2) *If K in $D^+(X_{étale})$ has torsion cohomology sheaves, then $H^n_{étale}(X, K)$ is a torsion abelian group for all n.*

Proof. To prove (1) we write $\mathcal{F} = \bigcup \mathcal{F}[n]$ where $\mathcal{F}[d]$ is the d-torsion subsheaf. By Lemma 51.4 we have $H^n_{étale}(X, \mathcal{F}) = \operatorname{colim} H^n_{étale}(X, \mathcal{F}[d])$. This proves (1) as $H^n_{étale}(X, \mathcal{F}[d])$ is annihilated by d.

To prove (2) we can use the spectral sequence $E_2^{p,q} = H^p_{étale}(X, H^q(K))$ converging to $H^n_{étale}(X, K)$ (Derived Categories, Lemma 21.3) and the result for sheaves. □

0DDD **Lemma 76.2.** *Let $f : X \to Y$ be a quasi-compact and quasi-separated morphism of schemes.*

(1) *If $\mathcal{F}$ is a torsion abelian sheaf on $X_{étale}$, then $R^n f_*\mathcal{F}$ is a torsion abelian sheaf on $Y_{étale}$ for all n.*
(2) *If K in $D^+(X_{étale})$ has torsion cohomology sheaves, then Rf_*K is an object of $D^+(Y_{étale})$ whose cohomology sheaves are torsion abelian sheaves.*

Proof. Proof of (1). Recall that $R^n f_*\mathcal{F}$ is the sheaf associated to the presheaf $V \mapsto H^n_{étale}(X \times_Y V, \mathcal{F})$ on $Y_{étale}$. See Cohomology on Sites, Lemma 7.4. If we choose V affine, then $X \times_Y V$ is quasi-compact and quasi-separated because f is, hence we can apply Lemma 76.1 to see that $H^n_{étale}(X \times_Y V, \mathcal{F})$ is torsion.

Proof of (2). Recall that $R^n f_* K$ is the sheaf associated to the presheaf $V \mapsto H^n_{étale}(X \times_Y V, K)$ on $Y_{étale}$. See Cohomology on Sites, Lemma 20.6. If we choose V affine, then $X \times_Y V$ is quasi-compact and quasi-separated because f is, hence we can apply Lemma 76.1 to see that $H^n_{étale}(X \times_Y V, K)$ is torsion. □

77. Cohomology with support in a closed subscheme

09XP Let X be a scheme and let $Z \subset X$ be a closed subscheme. Let $\mathcal{F}$ be an abelian sheaf on $X_{étale}$. We let

$$\Gamma_Z(X, \mathcal{F}) = \{s \in \mathcal{F}(X) \mid \operatorname{Supp}(s) \subset Z\}$$

be the sections with support in Z (Definition 31.3). This is a left exact functor which is not exact in general. Hence we obtain a derived functor

$$R\Gamma_Z(X, -) : D(X_{étale}) \longrightarrow D(Ab)$$

and cohomology groups with support in Z defined by $H^q_Z(X, \mathcal{F}) = R^q\Gamma_Z(X, \mathcal{F})$.

Let $\mathcal{I}$ be an injective abelian sheaf on $X_{étale}$. Let $U = X \setminus Z$. Then the restriction map $\mathcal{I}(X) \to \mathcal{I}(U)$ is surjective (Cohomology on Sites, Lemma 12.6) with kernel $\Gamma_Z(X, \mathcal{I})$. It immediately follows that for $K \in D(X_{étale})$ there is a distinguished triangle

$$R\Gamma_Z(X, K) \to R\Gamma(X, K) \to R\Gamma(U, K) \to R\Gamma_Z(X, K)[1]$$

in $D(Ab)$. As a consequence we obtain a long exact cohomology sequence

$$\ldots \to H^i_Z(X, K) \to H^i(X, K) \to H^i(U, K) \to H^{i+1}_Z(X, K) \to \ldots$$

for any K in $D(X_{étale})$.

For an abelian sheaf $\mathcal{F}$ on $X_{étale}$ we can consider the *subsheaf of sections with support in Z*, denoted $\mathcal{H}_Z(\mathcal{F})$, defined by the rule

$$\mathcal{H}_Z(\mathcal{F})(U) = \{s \in \mathcal{F}(U) \mid \operatorname{Supp}(s) \subset U \times_X Z\}$$

Here we use the support of a section from Definition 31.3. Using the equivalence of Proposition 46.4 we may view $\mathcal{H}_Z(\mathcal{F})$ as an abelian sheaf on $Z_{étale}$. Thus we obtain a functor

$$Ab(X_{étale}) \longrightarrow Ab(Z_{étale}), \quad \mathcal{F} \longmapsto \mathcal{H}_Z(\mathcal{F})$$

which is left exact, but in general not exact.

09XQ **Lemma 77.1.** *Let $i : Z \to X$ be a closed immersion of schemes. Let $\mathcal{I}$ be an injective abelian sheaf on $X_{étale}$. Then $\mathcal{H}_Z(\mathcal{I})$ is an injective abelian sheaf on $Z_{étale}$.*

Proof. Observe that for any abelian sheaf $\mathcal{G}$ on $Z_{étale}$ we have

$$\mathrm{Hom}_Z(\mathcal{G}, \mathcal{H}_Z(\mathcal{F})) = \mathrm{Hom}_X(i_*\mathcal{G}, \mathcal{F})$$

because after all any section of $i_*\mathcal{G}$ has support in Z. Since i_* is exact (Section 46) and as $\mathcal{I}$ is injective on $X_{étale}$ we conclude that $\mathcal{H}_Z(\mathcal{I})$ is injective on $Z_{étale}$. □

Denote

$$R\mathcal{H}_Z : D(X_{étale}) \longrightarrow D(Z_{étale})$$

the derived functor. We set $\mathcal{H}^q_Z(\mathcal{F}) = R^q\mathcal{H}_Z(\mathcal{F})$ so that $\mathcal{H}^0_Z(\mathcal{F}) = \mathcal{H}_Z(\mathcal{F})$. By the lemma above we have a Grothendieck spectral sequence

$$E_2^{p,q} = H^p(Z, \mathcal{H}^q_Z(\mathcal{F})) \Rightarrow H^{p+q}_Z(X, \mathcal{F})$$

09XR **Lemma 77.2.** *Let $i : Z \to X$ be a closed immersion of schemes. Let $\mathcal{G}$ be an injective abelian sheaf on $Z_{étale}$. Then $\mathcal{H}^p_Z(i_*\mathcal{G}) = 0$ for $p > 0$.*

Proof. This is true because the functor i_* is exact and transforms injective abelian sheaves into injective abelian sheaves (Cohomology on Sites, Lemma 14.2). □

0A45 **Lemma 77.3.** *Let $i : Z \to X$ be a closed immersion of schemes. Let $j : U \to X$ be the inclusion of the complement of Z. Let $\mathcal{F}$ be an abelian sheaf on $X_{étale}$. There is a distinguished triangle*

$$i_*R\mathcal{H}_Z(\mathcal{F}) \to \mathcal{F} \to Rj_*(\mathcal{F}|_U) \to i_*R\mathcal{H}_Z(\mathcal{F})[1]$$

in $D(X_{étale})$. This produces an exact sequence

$$0 \to i_*\mathcal{H}_Z(\mathcal{F}) \to \mathcal{F} \to j_*(\mathcal{F}|_U) \to i_*\mathcal{H}^1_Z(\mathcal{F}) \to 0$$

and isomorphisms $R^pj_*(\mathcal{F}|_U) \cong i_*\mathcal{H}^{p+1}_Z(\mathcal{F})$ for $p \geq 1$.

Proof. To get the distinguished triangle, choose an injective resolution $\mathcal{F} \to \mathcal{I}^\bullet$. Then we obtain a short exact sequence of complexes

$$0 \to i_*\mathcal{H}_Z(\mathcal{I}^\bullet) \to \mathcal{I}^\bullet \to j_*(\mathcal{I}^\bullet|_U) \to 0$$

by the discussion above. Thus the distinguished triangle by Derived Categories, Section 12. □

Let X be a scheme and let $Z \subset X$ be a closed subscheme. We denote $D_Z(X_{étale})$ the strictly full saturated triangulated subcategory of $D(X_{étale})$ consisting of complexes whose cohomology sheaves are supported on Z. Note that $D_Z(X_{étale})$ only depends on the underlying closed subset of X.

0AEG **Lemma 77.4.** *Let $i : Z \to X$ be a closed immersion of schemes. The map $Ri_{small,*} = i_{small,*} : D(Z_{étale}) \to D(X_{étale})$ induces an equivalence $D(Z_{étale}) \to D_Z(X_{étale})$ with quasi-inverse*

$$i^{-1}_{small}|_{D_Z(X_{étale})} = R\mathcal{H}_Z|_{D_Z(X_{étale})}$$

Proof. Recall that i_{small}^{-1} and $i_{small,*}$ is an adjoint pair of exact functors such that $i_{small}^{-1}i_{small,*}$ is isomorphic to the identify functor on abelian sheaves. See Proposition 46.4 and Lemma 36.2. Thus $i_{small,*} : D(Z_{\acute{e}tale}) \to D_Z(X_{\acute{e}tale})$ is fully faithful and i_{small}^{-1} determines a left inverse. On the other hand, suppose that K is an object of $D_Z(X_{\acute{e}tale})$ and consider the adjunction map $K \to i_{small,*}i_{small}^{-1}K$. Using exactness of $i_{small,*}$ and i_{small}^{-1} this induces the adjunction maps $H^n(K) \to i_{small,*}i_{small}^{-1}H^n(K)$ on cohomology sheaves. Since these cohomology sheaves are supported on Z we see these adjunction maps are isomorphisms and we conclude that $D(Z_{\acute{e}tale}) \to D_Z(X_{\acute{e}tale})$ is an equivalence.

To finish the proof we have to show that $R\mathcal{H}_Z(K) = i_{small}^{-1}K$ if K is an object of $D_Z(X_{\acute{e}tale})$. To do this we can use that $K = i_{small,*}i_{small}^{-1}K$ as we've just proved this is the case. Then we can choose a K-injective representative $\mathcal{I}^\bullet$ for $i_{small}^{-1}K$. Since $i_{small,*}$ is the right adjoint to the exact functor i_{small}^{-1}, the complex $i_{small,*}\mathcal{I}^\bullet$ is K-injective (Derived Categories, Lemma 30.9). We see that $R\mathcal{H}_Z(K)$ is computed by $\mathcal{H}_Z(i_{small,*}\mathcal{I}^\bullet) = \mathcal{I}^\bullet$ as desired. □

0A46 **Lemma 77.5.** *Let X be a scheme. Let $Z \subset X$ be a closed subscheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module and denote $\mathcal{F}^a$ the associated quasi-coherent sheaf on the small étale site of X (Proposition 17.1). Then*

1. *$H^q_Z(X, \mathcal{F})$ agrees with $H^q_Z(X_{\acute{e}tale}, \mathcal{F}^a)$,*
2. *if the complement of Z is retrocompact in X, then $i_*\mathcal{H}^q_Z(\mathcal{F}^a)$ is a quasi-coherent sheaf of $\mathcal{O}_X$-modules equal to $(i_*\mathcal{H}^q_Z(\mathcal{F}))^a$.*

Proof. Let $j : U \to X$ be the inclusion of the complement of Z. The statement (1) on cohomology groups follows from the long exact sequences for cohomology with supports and the agreements $H^q(X_{\acute{e}tale}, \mathcal{F}^a) = H^q(X, \mathcal{F})$ and $H^q(U_{\acute{e}tale}, \mathcal{F}^a) = H^q(U, \mathcal{F})$, see Theorem 22.4. If $j : U \to X$ is a quasi-compact morphism, i.e., if $U \subset X$ is retrocompact, then R^qj_* transforms quasi-coherent sheaves into quasi-coherent sheaves (Cohomology of Schemes, Lemma 4.5) and commutes with taking associated sheaf on étale sites (Descent, Lemma 8.15). We conclude by applying Lemma 77.3. □

78. Schemes with strictly henselian local rings

0EZN In this section we collect some results about the étale cohomology of schemes whose local rings are strictly henselian. For example, here is a fun generalization of Lemma 54.1.

09AX **Lemma 78.1.** *Let S be a scheme all of whose local rings are strictly henselian. Then for any abelian sheaf $\mathcal{F}$ on $S_{\acute{e}tale}$ we have $H^i(S_{\acute{e}tale}, \mathcal{F}) = H^i(S_{Zar}, \mathcal{F})$.*

Proof. Let $\epsilon : S_{\acute{e}tale} \to S_{Zar}$ be the morphism of sites given by the inclusion functor. The Zariski sheaf $R^p\epsilon_*\mathcal{F}$ is the sheaf associated to the presheaf $U \mapsto H^p_{\acute{e}tale}(U, \mathcal{F})$. Thus the stalk at $x \in X$ is $\text{colim}\, H^p_{\acute{e}tale}(U, \mathcal{F}) = H^p_{\acute{e}tale}(\text{Spec}(\mathcal{O}_{X,x}), \mathcal{G}_x)$ where $\mathcal{G}_x$ denotes the pullback of $\mathcal{F}$ to $\text{Spec}(\mathcal{O}_{X,x})$, see Lemma 51.5. Thus the higher direct images of $R^p\epsilon_*\mathcal{F}$ are zero by Lemma 54.1 and we conclude by the Leray spectral sequence. □

09AY **Lemma 78.2.** *Let S be an affine scheme such that (1) all points are closed, and (2) all residue fields are separably algebraically closed. Then for any abelian sheaf $\mathcal{F}$ on $S_{\acute{e}tale}$ we have $H^i(S_{\acute{e}tale}, \mathcal{F}) = 0$ for $i > 0$.*

Proof. Condition (1) implies that the underlying topological space of S is profinite, see Algebra, Lemma 25.5. Thus the higher cohomology groups of an abelian sheaf on the topological space S (i.e., Zariski cohomology) is trivial, see Cohomology, Lemma 22.3. The local rings are strictly henselian by Algebra, Lemma 149.10. Thus étale cohomology of S is computed by Zariski cohomology by Lemma 78.1 and the proof is done. $\square$

The spectrum of an absolutely integrally closed ring is an example of a scheme all of whose local rings are strictly henselian, see More on Algebra, Lemma 14.7. It turns out that normal domains with separably closed fraction fields have an even stronger property as explained in the following lemma.

09Z9 **Lemma 78.3.** *Let X be an integral normal scheme with separably closed function field.*

1. *A separated étale morphism $U \to X$ is a disjoint union of open immersions.*
2. *All local rings of X are strictly henselian.*

Proof. Let R be a normal domain whose fraction field is separably algebraically closed. Let $R \to A$ be an étale ring map. Then $A \otimes_R K$ is as a K-algebra a finite product $\prod_{i=1,\ldots,n} K$ of copies of K. Let e_i, $i = 1, \ldots, n$ be the corresponding idempotents of $A \otimes_R K$. Since A is normal (Algebra, Lemma 158.9) the idempotents e_i are in A (Algebra, Lemma 36.12). Hence $A = \prod Ae_i$ and we may assume $A \otimes_R K = K$. Since $A \subset A \otimes_R K = K$ (by flatness of $R \to A$ and since $R \subset K$) we conclude that A is a domain. By the same argument we conclude that $A \otimes_R A \subset (A \otimes_R A) \otimes_R K = K$. It follows that the map $A \otimes_R A \to A$ is injective as well as surjective. Thus $R \to A$ defines an open immersion by Morphisms, Lemma 10.2 and Étale Morphisms, Theorem 14.1.

Let $f : U \to X$ be a separated étale morphism. Let $\eta \in X$ be the generic point and let $f^{-1}(\{\eta\}) = \{\xi_i\}_{i \in I}$. The result of the previous paragraph shows the following: For any affine open $U' \subset U$ whose image in X is contained in an affine we have $U' = \coprod_{i \in I} U'_i$ where U'_i is the set of point of U' which are specializations of ξ_i. Moreover, the morphism $U'_i \to X$ is an open immersion. It follows that $U_i = \overline{\{\xi_i\}}$ is an open and closed subscheme of U and that $U_i \to X$ is locally on the source an isomorphism. By Morphisms, Lemma 47.7 the fact that $U_i \to X$ is separated, implies that $U_i \to X$ is injective and we conclude that $U_i \to X$ is an open immersion, i.e., (1) holds.

Part (2) follows from part (1) and the description of the strict henselization of $\mathcal{O}_{X,x}$ as the local ring at $\overline{x}$ on the étale site of X (Lemma 33.1). It can also be proved directly, see Fundamental Groups, Lemma 12.2. $\square$

0EZP **Lemma 78.4.** *Let $f : X \to Y$ be a morphism of schemes where X is an integral normal scheme with separably closed function field. Then $R^q f_* \underline{M} = 0$ for $q > 0$ and any abelian group M.*

Proof. Recall that $R^q f_* \underline{M}$ is the sheaf associated to the presheaf $V \mapsto H^q_{étale}(V \times_Y X, M)$ on $Y_{étale}$, see Lemma 51.6. If V is affine, then $V \times_Y X \to X$ is separated and étale. Hence $V \times_Y X = \coprod U_i$ is a disjoint union of open subschemes U_i of X, see Lemma 78.3. By Lemma 78.1 we see that $H^q_{étale}(U_i, M)$ is equal to $H^q_{Zar}(U_i, M)$. This vanishes by Cohomology, Lemma 20.2. $\square$

09ZA **Lemma 78.5.** *Let X be an affine integral normal scheme with separably closed function field. Let $Z \subset X$ be a closed subscheme. Let $V \to Z$ be an étale morphism with V affine. Then V is a finite disjoint union of open subschemes of Z. If $V \to Z$ is surjective and finite étale, then $V \to Z$ has a section.*

Proof. By Algebra, Lemma 142.10 we can lift V to an affine scheme U étale over X. Apply Lemma 78.3 to $U \to X$ to get the first statement.

The final statement is a consequence of the first. Let $V = \coprod_{i=1,\ldots,n} V_i$ be a finite decomposition into open and closed subschemes with $V_i \to Z$ an open immersion. As $V \to Z$ is finite we see that $V_i \to Z$ is also closed. Let $U_i \subset Z$ be the image. Then we have a decomposition into open and closed subschemes

$$Z = \coprod\nolimits_{(A,B)} \bigcap\nolimits_{i \in A} U_i \cap \bigcap\nolimits_{i \in B} U_i^c$$

where the disjoint union is over $\{1, \ldots, n\} = A \amalg B$ where A has at least one element. Each of the strata is contained in a single U_i and we find our section. $\square$

09ZB **Lemma 78.6.** *Let X be a normal integral affine scheme with separably closed function field. Let $Z \subset X$ be a closed subscheme. For any finite abelian group M we have $H^1_{\acute{e}tale}(Z, \underline{M}) = 0$.*

Proof. By Cohomology on Sites, Lemma 4.3 an element of $H^1_{\acute{e}tale}(Z, \underline{M})$ corresponds to a $\underline{M}$-torsor $\mathcal{F}$ on $Z_{\acute{e}tale}$. Such a torsor is clearly a finite locally constant sheaf. Hence $\mathcal{F}$ is representable by a scheme V finite étale over Z, Lemma 63.4. Of course $V \to Z$ is surjective as a torsor is locally trivial. Since $V \to Z$ has a section by Lemma 78.5 we are done. $\square$

09ZC **Lemma 78.7.** *Let X be a normal integral affine scheme with separably closed function field. Let $Z \subset X$ be a closed subscheme. For any finite abelian group M we have $H^q_{\acute{e}tale}(Z, \underline{M}) = 0$ for $q \geq 1$.*

Proof. We have seen that the result is true for H^1 in Lemma 78.6. We will prove the result for $q \geq 2$ by induction on q. Let $\xi \in H^q_{\acute{e}tale}(Z, \underline{M})$.

Let $X = \mathrm{Spec}(R)$. Let $I \subset R$ be the set of elements $f \in R$ such that $\xi|_{Z \cap D(f)} = 0$. All local rings of Z are strictly henselian by Lemma 78.3 and Algebra, Lemma 151.16. Hence étale cohomology on Z or open subschemes of Z is equal to Zariski cohomology, see Lemma 78.1. In particular ξ is Zariski locally trivial. It follows that for every prime $\mathfrak{p}$ of R there exists an $f \in I$ with $f \notin \mathfrak{p}$. Thus if we can show that I is an ideal, then $1 \in I$ and we're done. It is clear that $f \in I$, $r \in R$ implies $rf \in I$. Thus we now assume that $f, g \in I$ and we show that $f + g \in I$. Note that

$$D(f+g) \cap Z = D(f(f+g)) \cap Z \cup D(g(f+g)) \cap Z$$

By Mayer-Vietoris (Cohomology, Lemma 8.2 which applies as étale cohomology on open subschemes of Z equals Zariski cohomology) we have an exact sequence

$$\begin{array}{c} H^{q-1}_{\acute{e}tale}(D(fg(f+g)) \cap Z, \underline{M}) \\ \downarrow \\ H^q_{\acute{e}tale}(D(f+g) \cap Z, \underline{M}) \\ \downarrow \\ H^q_{\acute{e}tale}(D(f(f+g)) \cap Z, \underline{M}) \oplus H^q_{\acute{e}tale}(D(g(f+g)) \cap Z, \underline{M}) \end{array}$$

and the result follows as the first group is zero by induction. $\square$

09ZD **Lemma 78.8.** *Let X be an affine scheme.*

(1) *There exists an integral surjective morphism $X' \to X$ such that for every closed subscheme $Z' \subset X'$, every finite abelian group M, and every $q \geq 1$ we have $H^q_{\acute{e}tale}(Z', \underline{M}) = 0$.*
(2) *For any closed subscheme $Z \subset X$, finite abelian group M, $q \geq 1$, and $\xi \in H^q_{\acute{e}tale}(Z, \underline{M})$ there exists a finite surjective morphism $X' \to X$ of finite presentation such that ξ pulls back to zero in $H^q_{\acute{e}tale}(X' \times_X Z, \underline{M})$.*

Proof. Write $X = \mathrm{Spec}(A)$. Write $A = \mathbf{Z}[x_i]/J$ for some ideal J. Let R be the integral closure of $\mathbf{Z}[x_i]$ in an algebraic closure of the fraction field of $\mathbf{Z}[x_i]$. Let $A' = R/JR$ and set $X' = \mathrm{Spec}(A')$. This gives an example as in (1) by Lemma 78.7.

Proof of (2). Let $X' \to X$ be the integral surjective morphism we found above. Certainly, ξ maps to zero in $H^q_{\acute{e}tale}(X' \times_X Z, \underline{M})$. We may write X' as a limit $X' = \lim X'_i$ of schemes finite and of finite presentation over X; this is easy to do in our current affine case, but it is a special case of the more general Limits, Lemma 7.2. By Lemma 51.5 we see that ξ maps to zero in $H^q_{\acute{e}tale}(X'_i \times_X Z, \underline{M})$ for some i large enough. $\square$

79. Affine analog of proper base change

09Z8 In this section we discuss a result by Ofer Gabber, see [Gab94]. This was also proved by Roland Huber, see [Hub93]. We have already done some of the work needed for Gabber's proof in Section 78.

09ZE **Lemma 79.1.** *Let X be an affine scheme. Let $\mathcal{F}$ be a torsion abelian sheaf on $X_{\acute{e}tale}$. Let $Z \subset X$ be a closed subscheme. Let $\xi \in H^q_{\acute{e}tale}(Z, \mathcal{F}|_Z)$ for some $q > 0$. Then there exists an injective map $\mathcal{F} \to \mathcal{F}'$ of torsion abelian sheaves on $X_{\acute{e}tale}$ such that the image of ξ in $H^q_{\acute{e}tale}(Z, \mathcal{F}'|_Z)$ is zero.*

Proof. By Lemmas 72.2 and 51.4 we can find a map $\mathcal{G} \to \mathcal{F}$ with $\mathcal{G}$ a constructible abelian sheaf and ξ coming from an element ζ of $H^q_{\acute{e}tale}(Z, \mathcal{G}|_Z)$. Suppose we can find an injective map $\mathcal{G} \to \mathcal{G}'$ of torsion abelian sheaves on $X_{\acute{e}tale}$ such that the image of ζ in $H^q_{\acute{e}tale}(Z, \mathcal{G}'|_Z)$ is zero. Then we can take $\mathcal{F}'$ to be the pushout

$$\mathcal{F}' = \mathcal{G}' \amalg_{\mathcal{G}} \mathcal{F}$$

and we conclude the result of the lemma holds. (Observe that restriction to Z is exact, so commutes with finite limits and colimits and moreover it commutes with

arbitrary colimits as a left adjoint to pushforward.) Thus we may assume $\mathcal{F}$ is constructible.

Assume $\mathcal{F}$ is constructible. By Lemma 73.4 it suffices to prove the result when $\mathcal{F}$ is of the form $f_*\underline{M}$ where M is a finite abelian group and $f : Y \to X$ is a finite morphism of finite presentation (such sheaves are still constructible by Lemma 72.9 but we won't need this). Since formation of f_* commutes with any base change (Lemma 54.3) we see that the restriction of $f_*\underline{M}$ to Z is equal to the pushforward of $\underline{M}$ via $Y \times_X Z \to Z$. By the Leray spectral sequence (Proposition 53.2) and vanishing of higher direct images (Proposition 54.2), we find

$$H^q_{\acute{e}tale}(Z, f_*\underline{M}|_Z) = H^q_{\acute{e}tale}(Y \times_X Z, \underline{M}).$$

By Lemma 78.8 we can find a finite surjective morphism $Y' \to Y$ of finite presentation such that ξ maps to zero in $H^q(Y' \times_X Z, \underline{M})$. Denoting $f' : Y' \to X$ the composition $Y' \to Y \to X$ we claim the map

$$f_*\underline{M} \longrightarrow f'_*\underline{M}$$

is injective which finishes the proof by what was said above. To see the desired injectivity we can look at stalks. Namely, if $\overline{x} : \mathrm{Spec}(k) \to X$ is a geometric point, then

$$(f_*\underline{M})_{\overline{x}} = \bigoplus\nolimits_{f(\overline{y}) = \overline{x}} M$$

by Proposition 54.2 and similarly for the other sheaf. Since $Y' \to Y$ is surjective and finite we see that the induced map on geometric points lifting $\overline{x}$ is surjective too and we conclude. $\square$

The lemma above will take care of higher cohomology groups in Gabber's result. The following lemma will be used to deal with global sections.

09ZF **Lemma 79.2.** *Let X be a quasi-compact and quasi-separated scheme. Let $i : Z \to X$ be a closed immersion. Assume that*

1. *for any sheaf $\mathcal{F}$ on X_{Zar} the map $\Gamma(X, \mathcal{F}) \to \Gamma(Z, i^{-1}\mathcal{F})$ is bijective, and*
2. *for any finite morphism $X' \to X$ assumption (1) holds for $Z \times_X X' \to X'$.*

Then for any sheaf $\mathcal{F}$ on $X_{\acute{e}tale}$ we have $\Gamma(X, \mathcal{F}) = \Gamma(Z, i^{-1}_{small}\mathcal{F})$.

Proof. Let $\mathcal{F}$ be a sheaf on $X_{\acute{e}tale}$. There is a canonical (base change) map

$$i^{-1}(\mathcal{F}|_{X_{Zar}}) \longrightarrow (i^{-1}_{small}\mathcal{F})|_{Z_{Zar}}$$

of sheaves on Z_{Zar}. This map is injective as can be seen by looking on stalks. The stalk on the left hand side at $z \in Z$ is the stalk of $\mathcal{F}|_{X_{Zar}}$ at z. The stalk on the right hand side is the colimit over all elementary étale neighbourhoods $(U, u) \to (X, z)$ such that $U \times_X Z \to Z$ has a section over a neighbourhood of z. As étale morphisms are open, the image of $U \to X$ is an open neighbourhood of z in X and injectivity follows.

It follows from this and assumption (1) that the map $\Gamma(X, \mathcal{F}) \to \Gamma(Z, i^{-1}_{small}\mathcal{F})$ is injective. By (2) the same thing is true on all X' finite over X.

Let $s \in \Gamma(Z, i^{-1}_{small}\mathcal{F})$. By construction of $i^{-1}_{small}\mathcal{F}$ there exists an étale covering $\{V_j \to Z\}$, étale morphisms $U_j \to X$, sections $s_j \in \mathcal{F}(U_j)$ and morphisms $V_j \to U_j$ over X such that $s|_{V_j}$ is the pullback of s_j. Observe that every nonempty closed subscheme $T \subset X$ meets Z by assumption (1) applied to the sheaf $(T \to X)_*\underline{\mathbf{Z}}$ for example. Thus we see that $\coprod U_j \to X$ is surjective. By More on Morphisms,

Lemma 40.7 we can find a finite surjective morphism $X' \to X$ such that $X' \to X$ Zariski locally factors through $\coprod U_j \to X$. It follows that $s|_{Z'}$ Zariski locally comes from a section of $\mathcal{F}|_{X'}$. In other words, $s|_{Z'}$ comes from $t' \in \Gamma(X', \mathcal{F}|_{X'})$ by assumption (2). By injectivity we conclude that the two pullbacks of t' to $X' \times_X X'$ are the same (after all this is true for the pullbacks of s to $Z' \times_Z Z'$). Hence we conclude t' comes from a section of $\mathcal{F}$ over X by Remark 54.6. $\square$

0CAM **Lemma 79.3.** *Let $Z \subset X$ be a closed subset of a topological space X. Assume*

1. *X is a spectral space (Topology, Definition 23.1), and*
2. *for $x \in X$ the intersection $Z \cap \overline{\{x\}}$ is connected (in particular nonempty).*

If $Z = Z_1 \amalg Z_2$ with Z_i closed in Z, then there exists a decomposition $X = X_1 \amalg X_2$ with X_i closed in X and $Z_i = Z \cap X_i$.

Proof. Observe that Z_i is quasi-compact. Hence the set of points W_i specializing to Z_i is closed in the constructible topology by Topology, Lemma 24.7. Assumption (2) implies that $X = W_1 \amalg W_2$. Let $x \in \overline{W_1}$. By Topology, Lemma 23.5 part (1) there exists a specialization $x_1 \rightsquigarrow x$ with $x_1 \in W_1$. Thus $\overline{\{x\}} \subset \overline{\{x_1\}}$ and we see that $x \in W_1$. In other words, setting $X_i = W_i$ does the job. $\square$

09ZG **Lemma 79.4.** *Let $Z \subset X$ be a closed subset of a topological space X. Assume*

1. *X is a spectral space (Topology, Definition 23.1), and*
2. *for $x \in X$ the intersection $Z \cap \overline{\{x\}}$ is connected (in particular nonempty).*

Then for any sheaf $\mathcal{F}$ on X we have $\Gamma(X, \mathcal{F}) = \Gamma(Z, \mathcal{F}|_Z)$.

Proof. If $x \rightsquigarrow x'$ is a specialization of points, then there is a canonical map $\mathcal{F}_{x'} \to \mathcal{F}_x$ compatible with sections over opens and functorial in $\mathcal{F}$. Since every point of X specializes to a point of Z it follows that $\Gamma(X, \mathcal{F}) \to \Gamma(Z, \mathcal{F}|_Z)$ is injective. The difficult part is to show that it is surjective.

Denote $\mathcal{B}$ be the set of all quasi-compact opens of X. Write $\mathcal{F}$ as a filtered colimit $\mathcal{F} = \operatorname{colim} \mathcal{F}_i$ where each $\mathcal{F}_i$ is as in Modules, Equation (17.2.1). See Modules, Lemma 17.2. Then $\mathcal{F}|_Z = \operatorname{colim} \mathcal{F}_i|_Z$ as restriction to Z is a left adjoint (Categories, Lemma 24.5 and Sheaves, Lemma 21.8). By Sheaves, Lemma 29.1 the functors $\Gamma(X, -)$ and $\Gamma(Z, -)$ commute with filtered colimits. Hence we may assume our sheaf $\mathcal{F}$ is as in Modules, Equation (17.2.1).

Suppose that we have an embedding $\mathcal{F} \subset \mathcal{G}$. Then we have

$$\Gamma(X, \mathcal{F}) = \Gamma(Z, \mathcal{F}|_Z) \cap \Gamma(X, \mathcal{G})$$

where the intersection takes place in $\Gamma(Z, \mathcal{G}|_Z)$. This follows from the first remark of the proof because we can check whether a global section of $\mathcal{G}$ is in $\mathcal{F}$ by looking at the stalks and because every point of X specializes to a point of Z.

By Modules, Lemma 17.4 there is an injection $\mathcal{F} \to \prod (Z_i \to X)_* \underline{S_i}$ where the product is finite, $Z_i \subset X$ is closed, and S_i is finite. Thus it suffices to prove surjectivity for the sheaves $(Z_i \to X)_* \underline{S_i}$. Observe that

$$\Gamma(X, (Z_i \to X)_* \underline{S_i}) = \Gamma(Z_i, \underline{S_i}) \quad \text{and} \quad \Gamma(X, (Z_i \to X)_* \underline{S_i}|_Z) = \Gamma(Z \cap Z_i, \underline{S_i})$$

Moreover, conditions (1) and (2) are inherited by Z_i; this is clear for (2) and follows from Topology, Lemma 23.4 for (1). Thus it suffices to prove the lemma in the case of a (finite) constant sheaf. This case is a restatement of Lemma 79.3 which finishes the proof. $\square$

0CAF **Example 79.5.** Lemma 79.4 is false if X is not spectral. Here is an example: Let Y be a T_1 topological space, and $y \in Y$ a non-open point. Let $X = Y \amalg \{x\}$, endowed with the topology whose closed sets are $\emptyset$, $\{y\}$, and all $F \amalg \{x\}$, where F is a closed subset of Y. Then $Z = \{x, y\}$ is a closed subset of X, which satisfies assumption (2) of Lemma 79.4. But X is connected, while Z is not. The conclusion of the lemma thus fails for the constant sheaf with value $\{0, 1\}$ on X.

09ZH **Lemma 79.6.** *Let (A, I) be a henselian pair. Set $X = \text{Spec}(A)$ and $Z = \text{Spec}(A/I)$. For any sheaf $\mathcal{F}$ on $X_{étale}$ we have $\Gamma(X, \mathcal{F}) = \Gamma(Z, \mathcal{F}|_Z)$.*

Proof. Recall that the spectrum of any ring is a spectral space, see Algebra, Lemma 25.2. By More on Algebra, Lemma 11.12 we see that $\overline{\{x\}} \cap Z$ is connected for every $x \in X$. By Lemma 79.4 we see that the statement is true for sheaves on X_{Zar}. For any finite morphism $X' \to X$ we have $X' = \text{Spec}(A')$ and $Z \times_X X' = \text{Spec}(A'/IA')$ with (A', IA') a henselian pair, see More on Algebra, Lemma 11.8 and we get the same statement for sheaves on $(X')_{Zar}$. Thus we can apply Lemma 79.2 to conclude. $\square$

Finally, we can state and prove Gabber's theorem.

09ZI **Theorem 79.7** (Gabber)**.** *Let (A, I) be a henselian pair. Set $X = \text{Spec}(A)$ and $Z = \text{Spec}(A/I)$. For any torsion abelian sheaf $\mathcal{F}$ on $X_{étale}$ we have $H^q_{étale}(X, \mathcal{F}) = H^q_{étale}(Z, \mathcal{F}|_Z)$.*

Proof. The result holds for $q = 0$ by Lemma 79.6. Let $q \geq 1$. Suppose the result has been shown in all degrees $< q$. Let $\mathcal{F}$ be a torsion abelian sheaf. Let $\mathcal{F} \to \mathcal{F}'$ be an injective map of torsion abelian sheaves (to be chosen later) with cokernel $\mathcal{Q}$ so that we have the short exact sequence

$$0 \to \mathcal{F} \to \mathcal{F}' \to \mathcal{Q} \to 0$$

of torsion abelian sheaves on $X_{étale}$. This gives a map of long exact cohomology sequences over X and Z part of which looks like

$$\begin{tikzcd}
H^{q-1}_{étale}(X, \mathcal{F}') \arrow[r] \arrow[d] & H^{q-1}_{étale}(X, \mathcal{Q}) \arrow[r] \arrow[d] & H^q_{étale}(X, \mathcal{F}) \arrow[r] \arrow[d] & H^q_{étale}(X, \mathcal{F}') \arrow[d] \\
H^{q-1}_{étale}(Z, \mathcal{F}'|_Z) \arrow[r] & H^{q-1}_{étale}(Z, \mathcal{Q}|_Z) \arrow[r] & H^q_{étale}(Z, \mathcal{F}|_Z) \arrow[r] & H^q_{étale}(Z, \mathcal{F}'|_Z)
\end{tikzcd}$$

Using this commutative diagram of abelian groups with exact rows we will finish the proof.

Injectivity for $\mathcal{F}$. Let ξ be a nonzero element of $H^q_{étale}(X, \mathcal{F})$. By Lemma 79.1 applied with $Z = X$ (!) we can find $\mathcal{F} \subset \mathcal{F}'$ such that ξ maps to zero to the right. Then ξ is the image of an element of $H^{q-1}_{étale}(X, \mathcal{Q})$ and bijectivity for $q - 1$ implies ξ does not map to zero in $H^q_{étale}(Z, \mathcal{F}|_Z)$.

Surjectivity for $\mathcal{F}$. Let ξ be an element of $H^q_{étale}(Z, \mathcal{F}|_Z)$. By Lemma 79.1 applied with $Z = Z$ we can find $\mathcal{F} \subset \mathcal{F}'$ such that ξ maps to zero to the right. Then ξ is the image of an element of $H^{q-1}_{étale}(Z, \mathcal{Q}|_Z)$ and bijectivity for $q - 1$ implies ξ is in the image of the vertical map. $\square$

0A51 **Lemma 79.8.** *Let X be a scheme with affine diagonal which can be covered by $n + 1$ affine opens. Let $Z \subset X$ be a closed subscheme. Let $\mathcal{A}$ be a torsion sheaf*

of rings on $X_{\acute{e}tale}$ and let $\mathcal{I}$ be an injective sheaf of $\mathcal{A}$-modules on $X_{\acute{e}tale}$. Then $H^q_{\acute{e}tale}(Z, \mathcal{I}|_Z) = 0$ for $q > n$.

Proof. We will prove this by induction on n. If $n = 0$, then X is affine. Say $X = \mathrm{Spec}(A)$ and $Z = \mathrm{Spec}(A/I)$. Let A^h be the filtered colimit of étale A-algebras B such that $A/I \to B/IB$ is an isomorphism. Then (A^h, IA^h) is a henselian pair and $A/I = A^h/IA^h$, see More on Algebra, Lemma 12.1 and its proof. Set $X^h = \mathrm{Spec}(A^h)$. By Theorem 79.7 we see that

$$H^q_{\acute{e}tale}(Z, \mathcal{I}|_Z) = H^q_{\acute{e}tale}(X^h, \mathcal{I}|_{X^h})$$

By Theorem 51.3 we have

$$H^q_{\acute{e}tale}(X^h, \mathcal{I}|_{X^h}) = \mathrm{colim}_{A \to B} H^q_{\acute{e}tale}(\mathrm{Spec}(B), \mathcal{I}|_{\mathrm{Spec}(B)})$$

where the colimit is over the A-algebras B as above. Since the morphisms $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ are étale, the restriction $\mathcal{I}|_{\mathrm{Spec}(B)}$ is an injective sheaf of $\mathcal{A}|_{\mathrm{Spec}(B)}$-modules (Cohomology on Sites, Lemma 7.1). Thus the cohomology groups on the right are zero and we get the result in this case.

Induction step. We can use Mayer-Vietoris to do the induction step. Namely, suppose that $X = U \cup V$ where U is a union of n affine opens and V is affine. Then, using that the diagonal of X is affine, we see that $U \cap V$ is the union of n affine opens. Mayer-Vietoris gives an exact sequence

$$H^{q-1}_{\acute{e}tale}(U \cap V \cap Z, \mathcal{I}|_Z) \to H^q_{\acute{e}tale}(Z, \mathcal{I}|_Z) \to H^q_{\acute{e}tale}(U \cap Z, \mathcal{I}|_Z) \oplus H^q_{\acute{e}tale}(V \cap Z, \mathcal{I}|_Z)$$

and by our induction hypothesis we obtain vanishing for $q > n$ as desired. $\square$

80. Cohomology of torsion sheaves on curves

03SB The goal of this section is to prove Theorem 80.9.

0A52 **Situation 80.1.** Here k is an algebraically closed field, X is a separated, finite type scheme of dimension ≤ 1 over k, and $\mathcal{F}$ is a torsion abelian sheaf on $X_{\acute{e}tale}$.

In Situation 80.1 we want to prove the following statements

0A53 (1) $H^q_{\acute{e}tale}(X, \mathcal{F}) = 0$ for $q > 2$,
0A54 (2) $H^q_{\acute{e}tale}(X, \mathcal{F}) = 0$ for $q > 1$ if X is affine,
0A55 (3) $H^q_{\acute{e}tale}(X, \mathcal{F}) = 0$ for $q > 1$ if $p = \mathrm{char}(k) > 0$ and $\mathcal{F}$ is p-power torsion,
0A56 (4) $H^q_{\acute{e}tale}(X, \mathcal{F})$ is finite if $\mathcal{F}$ is constructible and torsion prime to $\mathrm{char}(k)$,
0A57 (5) $H^q_{\acute{e}tale}(X, \mathcal{F})$ is finite if X is proper and $\mathcal{F}$ constructible,
0A58 (6) $H^q_{\acute{e}tale}(X, \mathcal{F}) \to H^q_{\acute{e}tale}(X_{k'}, \mathcal{F}|_{X_{k'}})$ is an isomorphism for any extension $k \subset k'$ of algebraically closed fields if $\mathcal{F}$ is torsion prime to $\mathrm{char}(k)$,
0A59 (7) $H^q_{\acute{e}tale}(X, \mathcal{F}) \to H^q_{\acute{e}tale}(X_{k'}, \mathcal{F}|_{X_{k'}})$ is an isomorphism for any extension $k \subset k'$ of algebraically closed fields if X is proper,
0A5A (8) $H^2_{\acute{e}tale}(X, \mathcal{F}) \to H^2_{\acute{e}tale}(U, \mathcal{F})$ is surjective for all $U \subset X$ open.

Given any Situation 80.1 we will say that "statements (1) – (8) hold" if those statements that apply to the given situation are true. We start the proof with the following consequence of our computation of cohomology with constant coefficients.

0A5B **Lemma 80.2.** *In Situation 80.1 assume X is smooth and $\mathcal{F} = \underline{\mathbf{Z}/\ell\mathbf{Z}}$ for some prime number ℓ. Then statements (1) – (8) hold for $\mathcal{F}$.*

Proof. Since X is smooth, we see that X is a finite disjoint union of smooth curves. Hence we may assume X is a smooth curve.

Case I: ℓ different from the characteristic of k. This case follows from Lemma 68.1 (projective case) and Lemma 68.3 (affine case). Statement (6) on cohomology and extension of algebraically closed ground field follows from the fact that the genus g and the number of "punctures" r do not change when passing from k to k'. Statement (8) follows as $H^2_{étale}(U, \mathcal{F})$ is zero as soon as $U \neq X$, because then U is affine (Varieties, Lemmas 42.2 and 42.7).

Case II: ℓ is equal to the characteristic of k. Vanishing by Lemma 62.4. Statements (5) and (7) follow from Lemma 62.5. □

0A47 **Remark 80.3** (Invariance under extension of algebraically closed ground field)**.** Let k be an algebraically closed field of characteristic $p > 0$. In Section 62 we have seen that there is an exact sequence

$$k[x] \to k[x] \to H^1_{étale}(\mathbf{A}^1_k, \mathbf{Z}/p\mathbf{Z}) \to 0$$

where the first arrow maps $f(x)$ to $f^p - f$. A set of representatives for the cokernel is formed by the polynomials

$$\sum\nolimits_{p \not| n} \lambda_n x^n$$

with $\lambda_n \in k$. (If k is not algebraically closed you have to add some constants to this as well.) In particular when $k' \supset k$ is an algebraically closed overfield, then the map

$$H^1_{étale}(\mathbf{A}^1_k, \mathbf{Z}/p\mathbf{Z}) \to H^1_{étale}(\mathbf{A}^1_{k'}, \mathbf{Z}/p\mathbf{Z})$$

is not an isomorphism in general. In particular, the map $\pi_1(\mathbf{A}^1_{k'}) \to \pi_1(\mathbf{A}^1_k)$ between étale fundamental groups (insert future reference here) is not an isomorphism either. Thus the étale homotopy type of the affine line depends on the algebraically closed ground field. From Lemma 80.2 above we see that this is a phenomenon which only happens in characteristic p with p-power torsion coefficients.

0A5C **Lemma 80.4.** *Let k be an algebraically closed field. Let X be a separated finite type scheme over k of dimension ≤ 1. Let $0 \to \mathcal{F}_1 \to \mathcal{F} \to \mathcal{F}_2 \to 0$ be a short exact sequence of torsion abelian sheaves on X. If statements (1) – (8) hold for $\mathcal{F}_1$ and $\mathcal{F}_2$, then they hold for $\mathcal{F}$.*

Proof. This is mostly immediate from the definitions and the long exact sequence of cohomology. Also observe that $\mathcal{F}$ is constructible (resp. of torsion prime to the characteristic of k) if and only if both $\mathcal{F}_1$ and $\mathcal{F}_2$ are constructible (resp. of torsion prime to the characteristic of k). See Proposition 73.1. Some details omitted. □

0A5D **Lemma 80.5.** *Let k be an algebraically closed field. Let $f : X \to Y$ be a finite morphism of separated finite type schemes over k of dimension ≤ 1. Let $\mathcal{F}$ be a torsion abelian sheaf on X. If statements (1) – (8) hold for $\mathcal{F}$, then they hold for $f_*\mathcal{F}$.*

Proof. Follows from the vanishing of the higher direct images R^qf_* (Proposition 54.2), the Leray spectral sequence (Proposition 53.2), and the fact that formation of f_* commutes with arbitrary base change (Lemma 54.3). □

03SG **Lemma 80.6.** *In Situation 80.1 assume X is smooth. Let $j : U \to X$ an open immersion. Let ℓ be a prime number. Let $\mathcal{F} = j_!\underline{\mathbf{Z}/\ell\mathbf{Z}}$. Then statements (1) – (8) hold for $\mathcal{F}$.*

Proof. Consider the short exact sequence

$$0 \longrightarrow j_!\underline{\mathbf{Z}/\ell\mathbf{Z}}_U \longrightarrow \underline{\mathbf{Z}/\ell\mathbf{Z}}_X \longrightarrow \bigoplus\nolimits_{x \in X \setminus U} i_{x*}(\underline{\mathbf{Z}/\ell\mathbf{Z}}) \longrightarrow 0.$$

Statements (1) – (8) hold for $\underline{\mathbf{Z}/\ell\mathbf{Z}}$ by Lemma 80.2. Since the inclusion morphisms $i_x : x \to X$ are finite and since x is the spectrum of an algebraically closed field, we see that $H^q_{\acute{e}tale}(X, i_{x*}\underline{\mathbf{Z}/\ell\mathbf{Z}})$ is zero for $q > 0$ and equal to $\mathbf{Z}/\ell\mathbf{Z}$ for $q = 0$. Thus we get from the long exact cohomology sequence

$$\begin{tikzcd} 0 \arrow[r] & H^0_{\acute{e}tale}(X, \mathcal{F}) \arrow[r] & H^0(X, \underline{\mathbf{Z}/\ell\mathbf{Z}}_X) \arrow[r] & \bigoplus_{x \in X \setminus U} \mathbf{Z}/\ell\mathbf{Z} \arrow[lld] \\ & H^1_{\acute{e}tale}(X, \mathcal{F}) \arrow[r] & H^1_{\acute{e}tale}(X, \underline{\mathbf{Z}/\ell\mathbf{Z}}_X) \arrow[r] & 0 \end{tikzcd}$$

and $H^q_{\acute{e}tale}(X, \mathcal{F}) = H^q_{\acute{e}tale}(X, \underline{\mathbf{Z}/\ell\mathbf{Z}}_X)$ for $q \geq 2$. Each of the statements (1) – (8) follows by inspection. □

0A3Q **Lemma 80.7.** *In Situation 80.1 assume X reduced. Let $j : U \to X$ an open immersion. Let ℓ be a prime number and $\mathcal{F} = j_!\underline{\mathbf{Z}/\ell\mathbf{Z}}$. Then statements (1) – (8) hold for $\mathcal{F}$.*

Proof. The difference with Lemma 80.6 is that here we do not assume X is smooth. Let $\nu : X' \to X$ be the normalization morphism which is finite as varieties are Nagata schemes. Let $j' : U' \to X'$ be the inverse image of U. By Lemma 80.6 the result holds for $j'_!\underline{\mathbf{Z}/\ell\mathbf{Z}}$. By Lemma 80.5 the result holds for $\nu_*j'_!\underline{\mathbf{Z}/\ell\mathbf{Z}}$. In general it won't be true that $\nu_*j'_!\underline{\mathbf{Z}/\ell\mathbf{Z}}$ is equal to $j_!\underline{\mathbf{Z}/\ell\mathbf{Z}}$, but there will be a canonical injective map

$$j_!\underline{\mathbf{Z}/\ell\mathbf{Z}} \longrightarrow \nu_*j'_!\underline{\mathbf{Z}/\ell\mathbf{Z}}$$

whose cokernel is of the form $\bigoplus_{x \in Z} i_{x*}M_x$ where $Z \subset X$ is a finite set of closed points and M_x is a finite dimensional $\mathbf{F}_\ell$-vector space for each $x \in Z$. We obtain a short exact sequence

$$0 \to j_!\underline{\mathbf{Z}/\ell\mathbf{Z}} \to \nu_*j'_!\underline{\mathbf{Z}/\ell\mathbf{Z}} \to \bigoplus\nolimits_{x \in Z} i_{x*}M_x \to 0$$

and we can argue exactly as in the proof of Lemma 80.6 to finish the argument. Some details omitted. □

03SD **Lemma 80.8.** *In Situation 80.1 assume X reduced. Let $j : U \to X$ an open immersion with U connected. Let ℓ be a prime number. Let $\mathcal{G}$ a finite locally constant sheaf of $\mathbf{F}_\ell$-vector spaces on U. Let $\mathcal{F} = j_!\mathcal{G}$. Then statements (1) – (8) hold for $\mathcal{F}$.*

Proof. Let $f : V \to U$ be a finite étale morphism of degree prime to ℓ as in Lemma 65.2. The discussion in Section 65 gives maps

$$\mathcal{G} \to f_*f^{-1}\mathcal{G} \to \mathcal{G}$$

whose composition is an isomorphism. Hence it suffices to prove the lemma with $\mathcal{F} = j_! f_* f^{-1}\mathcal{G}$. By Zariski's Main theorem (More on Morphisms, Lemma 38.3) we can choose a diagram

$$\begin{tikzcd} V \arrow[r, "j'"] \arrow[d, "f"'] & Y \arrow[d, "\overline{f}"] \\ U \arrow[r, "j"] & X \end{tikzcd}$$

with $\overline{f} : Y \to X$ finite and j' an open immersion with dense image. We may replace Y by its reduction (this does not change V as V is reduced being étale over U). Since f is finite we have $V = U \times_X Y$. By Lemma 54.3 we have

$$j^{-1}\overline{f}_* j'_! f^{-1}\mathcal{G} = f_*(j')^{-1} j'_! f^{-1}\mathcal{G} = f_* f^{-1}\mathcal{G}$$

because $(j')^{-1} j'_! = \text{id}$ by Lemma 69.4. Hence there is a canonical map $j_! f_* f^{-1}\mathcal{G} \to \overline{f}_* j'_! f^{-1}\mathcal{G}$ which is an isomorphism over U (by the same lemma). However, the stalk of $\overline{f}_* j'_! f^{-1}\mathcal{G}$ at a geometric point $\overline{x}$ of $X \setminus U$ are zero by Proposition 54.2 and the fact that the stalks of $j'_! f^{-1}\mathcal{F}$ are zero at all geometric points $\overline{y}$ of Y lying over $\overline{x}$ (by Lemma 69.4). Hence $j_! f_* f^{-1}\mathcal{G} \to \overline{f}_* j'_! f^{-1}\mathcal{G}$ is an isomorphism, again by Lemma 69.4. By Lemma 80.5 it suffices to prove the lemma for $j'_! f^{-1}\mathcal{G}$. The existence of the filtration given by Lemma 65.2, the fact that $j'_!$ is exact, and Lemma 80.4 reduces us to the case $\mathcal{F} = j'_!\underline{\mathbf{Z}/\ell\mathbf{Z}}$ which is Lemma 80.7. $\square$

03SC **Theorem 80.9.** *If k is an algebraically closed field, X is a separated, finite type scheme of dimension ≤ 1 over k, and $\mathcal{F}$ is a torsion abelian sheaf on $X_{étale}$, then*

1. $H^q_{étale}(X, \mathcal{F}) = 0$ *for $q > 2$,*
2. $H^q_{étale}(X, \mathcal{F}) = 0$ *for $q > 1$ if X is affine,*
3. $H^q_{étale}(X, \mathcal{F}) = 0$ *for $q > 1$ if $p = char(k) > 0$ and $\mathcal{F}$ is p-power torsion,*
4. $H^q_{étale}(X, \mathcal{F})$ *is finite if $\mathcal{F}$ is constructible and torsion prime to $char(k)$,*
5. $H^q_{étale}(X, \mathcal{F})$ *is finite if X is proper and $\mathcal{F}$ constructible,*
6. $H^q_{étale}(X, \mathcal{F}) \to H^q_{étale}(X_{k'}, \mathcal{F}|_{X_{k'}})$ *is an isomorphism for any extension $k \subset k'$ of algebraically closed fields if $\mathcal{F}$ is torsion prime to $char(k)$,*
7. $H^q_{étale}(X, \mathcal{F}) \to H^q_{étale}(X_{k'}, \mathcal{F}|_{X_{k'}})$ *is an isomorphism for any extension $k \subset k'$ of algebraically closed fields if X is proper,*
8. $H^2_{étale}(X, \mathcal{F}) \to H^2_{étale}(U, \mathcal{F})$ *is surjective for all $U \subset X$ open.*

Proof. The theorem says that in Situation 80.1 statements (1) – (8) hold. Our first step is to replace X by its reduction, which is permissible by Proposition 45.4. By Lemma 72.2 we can write $\mathcal{F}$ as a filtered colimit of constructible abelian sheaves. Taking cohomology commutes with colimits, see Lemma 51.4. Moreover, pullback via $X_{k'} \to X$ commutes with colimits as a left adjoint. Thus it suffices to prove the statements for a constructible sheaf.

In this paragraph we use Lemma 80.4 without further mention. Writing $\mathcal{F} = \mathcal{F}_1 \oplus \ldots \oplus \mathcal{F}_r$ where $\mathcal{F}_i$ is ℓ_i-primary for some prime ℓ_i, we may assume that ℓ^n kills $\mathcal{F}$ for some prime ℓ. Now consider the exact sequence

$$0 \to \mathcal{F}[\ell] \to \mathcal{F} \to \mathcal{F}/\mathcal{F}[\ell] \to 0.$$

Thus we see that it suffices to assume that $\mathcal{F}$ is ℓ-torsion. This means that $\mathcal{F}$ is a constructible sheaf of $\mathbf{F}_\ell$-vector spaces for some prime number ℓ.

By definition this means there is a dense open $U \subset X$ such that $\mathcal{F}|_U$ is finite locally constant sheaf of $\mathbf{F}_\ell$-vector spaces. Since $\dim(X) \leq 1$ we may assume, after shrinking U, that $U = U_1 \amalg \ldots \amalg U_n$ is a disjoint union of irreducible schemes (just remove the closed points which lie in the intersections of ≥ 2 components of U). Consider the short exact sequence

$$0 \to j_! j^{-1}\mathcal{F} \to \mathcal{F} \to \bigoplus\nolimits_{x \in Z} i_{x*} M_x \to 0$$

where $Z = X \setminus U$ and M_x is a finite dimensional $\mathbf{F}_\ell$ vector space, see Lemma 69.8. Since the étale cohomology of $i_{x*}M_x$ vanishes in degrees ≥ 1 and is equal to M_x in degree 0 it suffices to prove the theorem for $j_! j^{-1}\mathcal{F}$ (argue exactly as in the proof of Lemma 80.6). Thus we reduce to the case $\mathcal{F} = j_!\mathcal{G}$ where $\mathcal{G}$ is a finite locally constant sheaf of $\mathbf{F}_\ell$-vector spaces on U.

Since we chose $U = U_1 \amalg \ldots \amalg U_n$ with U_i irreducible we have

$$j_!\mathcal{G} = j_{1!}(\mathcal{G}|_{U_1}) \oplus \ldots \oplus j_{n!}(\mathcal{G}|_{U_n})$$

where $j_i : U_i \to X$ is the inclusion morphism. The case of $j_{i!}(\mathcal{G}|_{U_i})$ is handled in Lemma 80.8. ☐

03RT **Theorem 80.10.** *Let X be a finite type, dimension 1 scheme over an algebraically closed field k. Let $\mathcal{F}$ be a torsion sheaf on $X_{\acute{e}tale}$. Then*

$$H^q_{\acute{e}tale}(X, \mathcal{F}) = 0, \quad \forall q \geq 3.$$

If X affine then also $H^2_{\acute{e}tale}(X, \mathcal{F}) = 0$.

Proof. If X is separated, this follows immediately from the more precise Theorem 80.9. If X is nonseparated, choose an affine open covering $X = X_1 \cup \ldots \cup X_n$. By induction on n we may assume the vanishing holds over $U = X_1 \cup \ldots \cup X_{n-1}$. Then Mayer-Vietoris (Lemma 50.1) gives

$$H^2_{\acute{e}tale}(U, \mathcal{F}) \oplus H^2_{\acute{e}tale}(X_n, \mathcal{F}) \to H^2_{\acute{e}tale}(U \cap X_n, \mathcal{F}) \to H^3_{\acute{e}tale}(X, \mathcal{F}) \to 0$$

However, since $U \cap X_n$ is an open of an affine scheme and hence affine by our dimension assumption, the group $H^2_{\acute{e}tale}(U \cap X_n, \mathcal{F})$ vanishes by Theorem 80.9. ☐

0A5E **Lemma 80.11.** *Let $k \subset k'$ be an extension of separably closed fields. Let X be a proper scheme over k of dimension ≤ 1. Let $\mathcal{F}$ be a torsion abelian sheaf on X. Then the map $H^q_{\acute{e}tale}(X, \mathcal{F}) \to H^q_{\acute{e}tale}(X_{k'}, \mathcal{F}|_{X_{k'}})$ is an isomorphism for $q \geq 0$.*

Proof. We have seen this for algebraically closed fields in Theorem 80.9. Given $k \subset k'$ as in the statement of the lemma we can choose a diagram

$$\begin{tikzcd} k' \arrow[r] & \overline{k}' \\ k \arrow[u] \arrow[r] & \overline{k} \arrow[u] \end{tikzcd}$$

where $k \subset \overline{k}$ and $k' \subset \overline{k}'$ are the algebraic closures. Since k and k' are separably closed the field extensions $k \subset \overline{k}$ and $k' \subset \overline{k}'$ are algebraic and purely inseparable. In this case the morphisms $X_{\overline{k}} \to X$ and $X_{\overline{k}'} \to X_{k'}$ are universal homeomorphisms. Thus the cohomology of $\mathcal{F}$ may be computed on $X_{\overline{k}}$ and the cohomology of $\mathcal{F}|_{X_{k'}}$ may be computed on $X_{\overline{k}'}$, see Proposition 45.4. Hence we deduce the general case from the case of algebraically closed fields. ☐

81. First cohomology of proper schemes

0A5F In Fundamental Groups, Section 9 we have seen, in some sense, that taking $R^1 f_* \underline{G}$ commutes with base change if $f : X \to Y$ is a proper morphism and G is a finite group (not necessarily commutative). In this section we deduce a useful consequence of these results.

0A5G **Lemma 81.1.** *Let A be a henselian local ring. Let X be a proper scheme over A with closed fibre X_0. Let M be a finite abelian group. Then $H^1_{étale}(X, \underline{M}) = H^1_{étale}(X_0, \underline{M})$.*

Proof. By Cohomology on Sites, Lemma 4.3 an element of $H^1_{étale}(X, \underline{M})$ corresponds to a $\underline{M}$-torsor $\mathcal{F}$ on $X_{étale}$. Such a torsor is clearly a finite locally constant sheaf. Hence $\mathcal{F}$ is representable by a scheme V finite étale over X, Lemma 63.4. Conversely, a scheme V finite étale over X with an M-action which turns it into an M-torsor over X gives rise to a cohomology class. The same translation between cohomology classes over X_0 and torsors finite étale over X_0 holds. Thus the lemma is a consequence of the equivalence of categories of Fundamental Groups, Lemma 9.1. ☐

The following technical lemma is a key ingredient in the proof of the proper base change theorem. The argument works word for word for any proper scheme over A whose special fibre has dimension ≤ 1, but in fact the conclusion will be a consequence of the proper base change theorem and we only need this particular version in its proof.

0A5H **Lemma 81.2.** *Let A be a henselian local ring. Let $X = \mathbf{P}^1_A$. Let $X_0 \subset X$ be the closed fibre. Let ℓ be a prime number. Let $\mathcal{I}$ be an injective sheaf of $\mathbf{Z}/\ell\mathbf{Z}$-modules on $X_{étale}$. Then $H^q_{étale}(X_0, \mathcal{I}|_{X_0}) = 0$ for $q > 0$.*

Proof. Observe that X is a separated scheme which can be covered by 2 affine opens. Hence for $q > 1$ this follows from Gabber's affine variant of the proper base change theorem, see Lemma 79.8. Thus we may assume $q = 1$. Let $\xi \in H^1_{étale}(X_0, \mathcal{I}|_{X_0})$. Goal: show that ξ is 0. By Lemmas 72.2 and 51.4 we can find a map $\mathcal{F} \to \mathcal{I}$ with $\mathcal{F}$ a constructible sheaf of $\mathbf{Z}/\ell\mathbf{Z}$-modules and ξ coming from an element ζ of $H^1_{étale}(X_0, \mathcal{F}|_{X_0})$. Suppose we have an injective map $\mathcal{F} \to \mathcal{F}'$ of sheaves of $\mathbf{Z}/\ell\mathbf{Z}$-modules on $X_{étale}$. Since $\mathcal{I}$ is injective we can extend the given map $\mathcal{F} \to \mathcal{I}$ to a map $\mathcal{F}' \to \mathcal{I}$. In this situation we may replace $\mathcal{F}$ by $\mathcal{F}'$ and ζ by the image of ζ in $H^1_{étale}(X_0, \mathcal{F}'|_{X_0})$. Also, if $\mathcal{F} = \mathcal{F}_1 \oplus \mathcal{F}_2$ is a direct sum, then we may replace $\mathcal{F}$ by $\mathcal{F}_i$ and ζ by the image of ζ in $H^1_{étale}(X_0, \mathcal{F}_i|_{X_0})$.

By Lemma 73.4 and the remarks above we may assume $\mathcal{F}$ is of the form $f_*\underline{M}$ where M is a finite $\mathbf{Z}/\ell\mathbf{Z}$-module and $f : Y \to X$ is a finite morphism of finite presentation (such sheaves are still constructible by Lemma 72.9 but we won't need this). Since formation of f_* commutes with any base change (Lemma 54.3) we see that the restriction of $f_*\underline{M}$ to X_0 is equal to the pushforward of $\underline{M}$ via the induced morphism $Y_0 \to X_0$ of special fibres. By the Leray spectral sequence (Proposition 53.2) and vanishing of higher direct images (Proposition 54.2), we find

$$H^1_{étale}(X_0, f_*\underline{M}|_{X_0}) = H^1_{étale}(Y_0, \underline{M}).$$

Since $Y \to \mathrm{Spec}(A)$ is proper we can use Lemma 81.1 to see that the $H^1_{\acute{e}tale}(Y_0, \underline{M})$ is equal to $H^1_{\acute{e}tale}(Y, \underline{M})$. Thus we see that our cohomology class ζ lifts to a cohomology class

$$\tilde{\zeta} \in H^1_{\acute{e}tale}(Y, \underline{M}) = H^1_{\acute{e}tale}(X, f_*\underline{M})$$

However, $\tilde{\zeta}$ maps to zero in $H^1_{\acute{e}tale}(X, \mathcal{I})$ as $\mathcal{I}$ is injective and by commutativity of

$$\begin{tikzcd} H^1_{\acute{e}tale}(X, f_*\underline{M}) \arrow[r] \arrow[d] & H^1_{\acute{e}tale}(X, \mathcal{I}) \arrow[d] \\ H^1_{\acute{e}tale}(X_0, (f_*\underline{M})|_{X_0}) \arrow[r] & H^1_{\acute{e}tale}(X_0, \mathcal{I}|_{X_0}) \end{tikzcd}$$

we conclude that the image ξ of ζ is zero as well. ☐

82. Preliminaries on base change

0EZQ If you are interested in either the smooth base change theorem or the proper base change theorem, you should skip directly to the corresponding sections. In this section and the next few sections we consider commutative diagrams

$$\begin{tikzcd} X \arrow[d, "f"'] & Y \arrow[l, "h"] \arrow[d, "e"] \\ S & T \arrow[l, "g"] \end{tikzcd}$$

of schemes; we usually assume this diagram is cartesian, i.e., $Y = X \times_S T$. A commutative diagram as above gives rise to a commutative diagram

$$\begin{tikzcd} X_{\acute{e}tale} \arrow[d, "f_{small}"'] & Y_{\acute{e}tale} \arrow[l, "h_{small}"] \arrow[d, "e_{small}"] \\ S_{\acute{e}tale} & T_{\acute{e}tale} \arrow[l, "g_{small}"'] \end{tikzcd}$$

of small étale sites. Let us use the notation

$$f^{-1} = f^{-1}_{small}, \quad g_* = g_{small,*}, \quad e^{-1} = e^{-1}_{small}, \text{ and } \quad h_* = h_{small,*}.$$

By Sites, Section 45 we get a base change or pullback map

$$f^{-1}g_*\mathcal{F} \longrightarrow h_*e^{-1}\mathcal{F}$$

for a sheaf $\mathcal{F}$ on $T_{\acute{e}tale}$. If $\mathcal{F}$ is an abelian sheaf on $T_{\acute{e}tale}$, then we get a derived base change map

$$f^{-1}Rg_*\mathcal{F} \longrightarrow Rh_*e^{-1}\mathcal{F}$$

see Cohomology on Sites, Lemma 15.1. Finally, if K is an arbitrary object of $D(T_{\acute{e}tale})$ there is a base change map

$$f^{-1}Rg_*K \longrightarrow Rh_*e^{-1}K$$

see Cohomology on Sites, Remark 19.3.

0EZR **Lemma 82.1.** *Consider a cartesian diagram of schemes*

$$\begin{tikzcd} X \arrow[d, "f"'] & Y \arrow[l, "h"] \arrow[d, "e"] \\ S & T \arrow[l, "g"] \end{tikzcd}$$

and a sheaf $\mathcal{F}$ on $T_{étale}$. Let $\{U_i \to X\}$ be an étale covering such that $U_i \to S$ factors as $U_i \to V_i \to S$ with $V_i \to S$ étale and consider the cartesian diagrams

$$\begin{tikzcd} U_i \arrow[d, "f_i"'] & U_i \times_X Y \arrow[l, "h_i"'] \arrow[d, "e_i"] \\ V_i & V_i \times_S T \arrow[l, "g_i"] \end{tikzcd}$$

Set $\mathcal{F}_i = \mathcal{F}|_{V_i \times_S T}$.

1. *If $f_i^{-1}g_{i,*}\mathcal{F}_i = h_{i,*}e_i^{-1}\mathcal{F}_i$ for all i, then $f^{-1}g_*\mathcal{F} = h_*e^{-1}\mathcal{F}$.*
2. *If $\mathcal{F}$ is an abelian sheaf and $f_i^{-1}R^qg_{i,*}\mathcal{F}_i = R^qh_{i,*}e_i^{-1}\mathcal{F}_i$ for all i, then $f^{-1}R^qg_*\mathcal{F} = R^qh_*e^{-1}\mathcal{F}$.*

Proof. We have $f^{-1}R^qg_*\mathcal{F}|_{U_i} = f_i^{-1}R^qg_{i,*}\mathcal{F}_i$ and $R^qh_*e^{-1}\mathcal{F}|_{U_i} = R^qh_{i,*}e_i^{-1}\mathcal{F}_i$ as follows from the compatibility of localization with morphisms of sites, see Sites, Lemma 28.2 and Cohomology on Sites, Lemma 20.4. ☐

0EZS **Lemma 82.2.** *Consider a tower of cartesian diagrams of schemes*

$$\begin{tikzcd} W \arrow[d, "i"'] & Z \arrow[l, "j"'] \arrow[d, "k"] \\ X \arrow[d, "f"'] & Y \arrow[l, "h"'] \arrow[d, "e"] \\ S & T \arrow[l, "g"] \end{tikzcd}$$

Let K in $D(T_{étale})$. If

$$f^{-1}Rg_*K \to Rh_*e^{-1}K \quad \text{and} \quad i^{-1}Rh_*e^{-1}K \to Rj_*k^{-1}e^{-1}K$$

are isomorphisms, then $(f \circ i)^{-1}Rg_*K \to Rj_*(e \circ k)^{-1}K$ is an isomorphism. Similarly, if $\mathcal{F}$ is an abelian sheaf on $T_{étale}$ and if

$$f^{-1}R^qg_*\mathcal{F} \to R^qh_*e^{-1}\mathcal{F} \quad \text{and} \quad i^{-1}R^qh_*e^{-1}\mathcal{F} \to R^qj_*k^{-1}e^{-1}\mathcal{F}$$

are isomorphisms, then $(f \circ i)^{-1}R^qg_*\mathcal{F} \to R^qj_*(e \circ k)^{-1}\mathcal{F}$ is an isomorphism.

Proof. This is formal, provided one checks that the composition of these base change maps is the base change maps for the outer rectangle, see Cohomology on Sites, Remark 19.5. ☐

0EZT **Lemma 82.3.** *Let I be a directed set. Consider an inverse system of cartesian diagrams of schemes*

$$\begin{tikzcd} X_i \arrow[d, "f_i"'] & Y_i \arrow[l, "h_i"'] \arrow[d, "e_i"] \\ S_i & T_i \arrow[l, "g_i"] \end{tikzcd}$$

with affine transition morphisms and with g_i quasi-compact and quasi-separated. Set $X = \lim X_i$, $S = \lim S_i$, $T = \lim T_i$ and $Y = \lim Y_i$ to obtain the cartesian diagram

$$\begin{tikzcd} X \arrow[d, "f"'] & Y \arrow[l, "h"'] \arrow[d, "e"] \\ S & T \arrow[l, "g"] \end{tikzcd}$$

Let $(\mathcal{F}_i, \varphi_{i'i})$ be a system of sheaves on (T_i) as in Definition 51.1. Set $\mathcal{F} = \operatorname{colim} p_i^{-1}\mathcal{F}_i$ on T where $p_i : T \to T_i$ is the projection. Then we have the following

1. *If $f_i^{-1}g_{i,*}\mathcal{F}_i = h_{i,*}e_i^{-1}\mathcal{F}_i$ for all i, then $f^{-1}g_*\mathcal{F} = h_*e^{-1}\mathcal{F}$.*
2. *If $\mathcal{F}_i$ is an abelian sheaf for all i and $f_i^{-1}R^qg_{i,*}\mathcal{F}_i = R^qh_{i,*}e_i^{-1}\mathcal{F}_i$ for all i, then $f^{-1}R^qg_*\mathcal{F} = R^qh_*e^{-1}\mathcal{F}$.*

Proof. We prove (2) and we omit the proof of (1). We will use without further mention that pullback of sheaves commutes with colimits as it is a left adjoint. Observe that h_i is quasi-compact and quasi-separated as a base change of g_i. Denoting $q_i : Y \to Y_i$ the projections, observe that $e^{-1}\mathcal{F} = \operatorname{colim} e^{-1}p_i^{-1}\mathcal{F}_i = \operatorname{colim} q_i^{-1}e_i^{-1}\mathcal{F}_i$. By Lemma 51.8 this gives

$$R^qh_*e^{-1}\mathcal{F} = \operatorname{colim} r_i^{-1}R^qh_{i,*}e_i^{-1}\mathcal{F}_i$$

where $r_i : X \to X_i$ is the projection. Similarly, we have

$$f^{-1}Rg_*\mathcal{F} = f^{-1}\operatorname{colim} s_i^{-1}R^qg_{i,*}\mathcal{F}_i = \operatorname{colim} r_i^{-1}f_i^{-1}R^qg_{i,*}\mathcal{F}_i$$

where $s_i : S \to S_i$ is the projection. The lemma follows. □

0EZU **Lemma 82.4.** *Consider a cartesian diagram of schemes*

$$\begin{tikzcd} X \arrow[d, "f"'] & Y \arrow[l, "h"'] \arrow[d, "e"] \\ S & T \arrow[l, "g"] \end{tikzcd}$$

where $g : T \to S$ is quasi-compact and quasi-separated. Let $\mathcal{F}$ be an abelian sheaf on $T_{étale}$. Let $q \geq 0$. The following are equivalent

1. *For every geometric point $\overline{x}$ of X with image $\overline{s} = f(\overline{x})$ we have*
$$H^q(\operatorname{Spec}(\mathcal{O}^{sh}_{X,\overline{x}}) \times_S T, \mathcal{F}) = H^q(\operatorname{Spec}(\mathcal{O}^{sh}_{S,\overline{s}}) \times_S T, \mathcal{F})$$
2. *$f^{-1}R^qg_*\mathcal{F} \to R^qh_*e^{-1}\mathcal{F}$ is an isomorphism.*

Proof. Since $Y = X \times_S T$ we have $\operatorname{Spec}(\mathcal{O}^{sh}_{X,\overline{x}}) \times_X Y = \operatorname{Spec}(\mathcal{O}^{sh}_{X,\overline{x}}) \times_S T$. Thus the map in (1) is the map of stalks at $\overline{x}$ for the map in (2) by Theorem 52.1 (and Lemma 36.2). Thus the result by Theorem 29.10. □

0EZV **Lemma 82.5.** *Let $f : X \to S$ be a morphism of schemes. Let $\overline{x}$ be a geometric point of X with image $\overline{s}$ in S. Let $\operatorname{Spec}(K) \to \operatorname{Spec}(\mathcal{O}^{sh}_{S,\overline{s}})$ be a morphism with K a separably closed field. Let $\mathcal{F}$ be an abelian sheaf on $\operatorname{Spec}(K)_{étale}$. Let $q \geq 0$. The following are equivalent*

1. *$H^q(\operatorname{Spec}(\mathcal{O}^{sh}_{X,\overline{x}}) \times_S \operatorname{Spec}(K), \mathcal{F}) = H^q(\operatorname{Spec}(\mathcal{O}^{sh}_{S,\overline{s}}) \times_S \operatorname{Spec}(K), \mathcal{F})$*
2. *$H^q(\operatorname{Spec}(\mathcal{O}^{sh}_{X,\overline{x}}) \times_{\operatorname{Spec}(\mathcal{O}^{sh}_{S,\overline{s}})} \operatorname{Spec}(K), \mathcal{F}) = H^q(\operatorname{Spec}(K), \mathcal{F})$*

Proof. Observe that $\operatorname{Spec}(K) \times_S \operatorname{Spec}(\mathcal{O}^{sh}_{S,\overline{s}})$ is the spectrum of a filtered colimit of étale algebras over K. Since K is separably closed, each étale K-algebra is a finite product of copies of K. Thus we can write

$$\operatorname{Spec}(K) \times_S \operatorname{Spec}(\mathcal{O}^{sh}_{S,\overline{s}}) = \lim_{i \in I} \coprod\nolimits_{a \in A_i} \operatorname{Spec}(K)$$

as a cofiltered limit where each term is a disjoint union of copies of $\text{Spec}(K)$ over a finite set A_i. Note that A_i is nonempty as we are given $\text{Spec}(K) \to \text{Spec}(\mathcal{O}^{sh}_{S,\overline{s}})$. It follows that

$$\begin{aligned} \text{Spec}(\mathcal{O}^{sh}_{X,\overline{x}}) \times_S \text{Spec}(K) &= \text{Spec}(\mathcal{O}^{sh}_{X,\overline{x}}) \times_{\text{Spec}(\mathcal{O}^{sh}_{S,\overline{s}})} \left(\text{Spec}(\mathcal{O}^{sh}_{S,\overline{s}}) \times_S \text{Spec}(K)\right) \\ &= \text{colim}_{i \in I} \coprod\nolimits_{a \in A_i} \text{Spec}(\mathcal{O}^{sh}_{X,\overline{x}}) \times_{\text{Spec}(\mathcal{O}^{sh}_{S,\overline{s}})} \text{Spec}(K) \end{aligned}$$

Since taking cohomology in our setting commutes with limits of schemes (Theorem 51.3) we conclude. ☐

83. Base change for pushforward

0EZW This section is preliminary and should be skipped on a first reading. In this section we discuss for what morphisms $f : X \to S$ we have $f^{-1}g_* = h_*e^{-1}$ on all sheaves (of sets) for every cartesian diagram

$$\begin{tikzcd} X \arrow[d, "f"'] & Y \arrow[l, "h"'] \arrow[d, "e"] \\ S & T \arrow[l, "g"'] \end{tikzcd}$$

with g quasi-compact and quasi-separated.

0EZX **Lemma 83.1.** *Consider the cartesian diagram of schemes*

$$\begin{tikzcd} X \arrow[d, "f"'] & Y \arrow[l, "h"'] \arrow[d, "e"] \\ S & T \arrow[l, "g"'] \end{tikzcd}$$

Assume that f is flat and every object U of $X_{étale}$ has a covering $\{U_i \to U\}$ such that $U_i \to S$ factors as $U_i \to V_i \to S$ with $V_i \to S$ étale and $U_i \to V_i$ quasi-compact with geometrically connected fibres. Then for any sheaf $\mathcal{F}$ of sets on $T_{étale}$ we have $f^{-1}g_*\mathcal{F} = h_*e^{-1}\mathcal{F}$.

Proof. Let $U \to X$ be an étale morphism such that $U \to S$ factors as $U \to V \to S$ with $V \to S$ étale and $U \to V$ quasi-compact with geometrically connected fibres. Observe that $U \to V$ is flat (More on Flatness, Lemma 2.3). We claim that

$$\begin{aligned} f^{-1}g_*\mathcal{F}(U) &= g_*\mathcal{F}(V) \\ &= \mathcal{F}(V \times_S T) \\ &= e^{-1}\mathcal{F}(U \times_X Y) \\ &= h_*e^{-1}\mathcal{F}(U) \end{aligned}$$

Namely, thinking of U as an object of $X_{étale}$ and V as an object of $S_{étale}$ we see that the first equality follows from Lemma 39.3[7]. Thinking of $V \times_S T$ as an object of $T_{étale}$ the second equality follows from the definition of g_*. Observe that $U \times_X Y = U \times_S T$ (because $Y = X \times_S T$) and hence $U \times_X Y \to V \times_S T$ has geometrically connected fibres as a base change of $U \to V$. Thinking of $U \times_X Y$ as an object of $Y_{étale}$, we see that the third equality follows from Lemma 39.3 as before. Finally, the fourth equality follows from the definition of h_*.

[7]Strictly speaking, we are also using that the restriction of $f^{-1}g_*\mathcal{F}$ to $U_{étale}$ is the pullback via $U \to V$ of the restriction of $g_*\mathcal{F}$ to $V_{étale}$. See Sites, Lemma 28.2.

Since by assumption every object of $X_{étale}$ has an étale covering to which the argument of the previous paragraph applies we see that the lemma is true. $\square$

0EYS **Lemma 83.2.** *Consider a cartesian diagram of schemes*

$$\begin{tikzcd} X \arrow[d, "f"'] & Y \arrow[l, "h"] \arrow[d, "e"] \\ S & T \arrow[l, "g"] \end{tikzcd}$$

where f is flat and locally of finite presentation with geometrically reduced fibres. Then $f^{-1}g_*\mathcal{F} = h_*e^{-1}\mathcal{F}$ for any sheaf $\mathcal{F}$ on $T_{étale}$.

Proof. Combine Lemma 83.1 with More on Morphisms, Lemma 41.3. $\square$

0EZY **Lemma 83.3.** *Consider the cartesian diagrams of schemes*

$$\begin{tikzcd} X \arrow[d, "f"'] & Y \arrow[l, "h"] \arrow[d, "e"] \\ S & T \arrow[l, "g"] \end{tikzcd}$$

Assume that S is the spectrum of a separably closed field. Then $f^{-1}g_*\mathcal{F} = h_*e^{-1}\mathcal{F}$ for any sheaf $\mathcal{F}$ on $T_{étale}$.

Proof. We may work locally on X. Hence we may assume X is affine. Then we can write X as a cofiltered limit of affine schemes of finite type over S. By Lemma 82.3 we may assume that X is of finite type over S. Then Lemma 83.1 applies because any scheme of finite type over a separably closed field is a finite disjoint union of connected and geometrically connected schemes (see Varieties, Lemma 7.6). $\square$

0EZZ **Lemma 83.4.** *Consider a cartesian diagram of schemes*

$$\begin{tikzcd} X \arrow[d, "f"'] & Y \arrow[l, "h"] \arrow[d, "e"] \\ S & T \arrow[l, "g"] \end{tikzcd}$$

Assume that

1. *f is flat and open,*
2. *the residue fields of S are separably algebraically closed,*
3. *given an étale morphism $U \to X$ with U affine we can write U as a finite disjoint union of open subschemes of X (for example if X is a normal integral scheme with separably closed function field),*
4. *any nonempty open of a fibre X_s of f is connected (for example if X_s is irreducible or empty).*

Then for any sheaf $\mathcal{F}$ of sets on $T_{étale}$ we have $f^{-1}g_*\mathcal{F} = h_*e^{-1}\mathcal{F}$.

Proof. Omitted. Hint: the assumptions almost trivially imply the condition of Lemma 83.1. The for example in part (3) follows from Lemma 78.3. $\square$

The following lemma doesn't really belong here but there does not seem to be a good place for it anywhere.

0EYR **Lemma 83.5.** *Let $f : X \to S$ be a morphism of schemes which is flat and locally of finite presentation with geometrically reduced fibres. Then $f^{-1} : Sh(S_{étale}) \to Sh(X_{étale})$ commutes with products.*

Proof. Let I be a set and let $\mathcal{G}_i$ be a sheaf on $S_{étale}$ for $i \in I$. Let $U \to X$ be an étale morphism such that $U \to S$ factors as $U \to V \to S$ with $V \to S$ étale and $U \to V$ flat of finite presentation with geometrically connected fibres. Then we have

$$\begin{aligned} f^{-1}(\prod \mathcal{G}_i)(U) &= (\prod \mathcal{G}_i)(V) \\ &= \prod \mathcal{G}_i(V) \\ &= \prod f^{-1}\mathcal{G}_i(U) \\ &= (\prod f^{-1}\mathcal{G}_i)(U) \end{aligned}$$

where we have used Lemma 39.3 in the first and third equality (we are also using that the restriction of $f^{-1}\mathcal{G}$ to $U_{étale}$ is the pullback via $U \to V$ of the restriction of $\mathcal{G}$ to $V_{étale}$, see Sites, Lemma 28.2). By More on Morphisms, Lemma 41.3 every object U of $X_{étale}$ has an étale covering $\{U_i \to U\}$ such that the discussion in the previous paragraph applies to U_i. The lemma follows. $\square$

0F00 **Lemma 83.6.** *Let $f : X \to S$ be a flat morphism of schemes such that for every geometric point $\overline{x}$ of X the map*

$$\mathcal{O}^{sh}_{S,f(\overline{x})} \longrightarrow \mathcal{O}^{sh}_{X,\overline{x}}$$

has geometrically connected fibres. Then for every cartesian diagram of schemes

$$\begin{tikzcd} X \arrow[d, "f"'] & Y \arrow[l, "h"'] \arrow[d, "e"] \\ S & T \arrow[l, "g"] \end{tikzcd}$$

with g quasi-compact and quasi-separated we have $f^{-1}g_*\mathcal{F} = h_*e^{-1}\mathcal{F}$ for any sheaf $\mathcal{F}$ of sets on $T_{étale}$.

Proof. It suffices to check equality on stalks, see Theorem 29.10. By Theorem 52.1 we have

$$(h_*e^{-1}\mathcal{F})_{\overline{x}} = \Gamma(\mathrm{Spec}(\mathcal{O}^{sh}_{X,\overline{x}}) \times_X Y, e^{-1}\mathcal{F})$$

and we have similarly

$$(f^{-1}g_*^{-1}\mathcal{F})_{\overline{x}} = (g_*^{-1}\mathcal{F})_{f(\overline{x})} = \Gamma(\mathrm{Spec}(\mathcal{O}^{sh}_{S,f(\overline{x})}) \times_S T, \mathcal{F})$$

These sets are equal by an application of Lemma 39.3 to the morphism

$$\mathrm{Spec}(\mathcal{O}^{sh}_{X,\overline{x}}) \times_X Y \longrightarrow \mathrm{Spec}(\mathcal{O}^{sh}_{S,f(\overline{x})}) \times_S T$$

which is a base change of $\mathrm{Spec}(\mathcal{O}^{sh}_{X,\overline{x}}) \to \mathrm{Spec}(\mathcal{O}^{sh}_{S,f(\overline{x})})$ because $Y = X \times_S T$. $\square$

84. Base change for higher direct images

0F01 This section is the analogue of Section 83 for higher direct images. This section is preliminary and should be skipped on a first reading.

0F02 **Remark 84.1.** Let $f : X \to S$ be a morphism of schemes. Let n be an integer. We will say $BC(f, n, q_0)$ is true if for every commutative diagram

$$\begin{tikzcd} X \arrow[d, "f"'] & X' \arrow[l] \arrow[d, "f'"'] & Y \arrow[l, "h"'] \arrow[d, "e"] \\ S & S' \arrow[l] & T \arrow[l, "g"] \end{tikzcd}$$

with $X' = X \times_S S'$ and $Y = X' \times_{S'} T$ and g quasi-compact and quasi-separated, and every abelian sheaf $\mathcal{F}$ on $T_{étale}$ annihilated by n the base change map

$$(f')^{-1}R^q g_*\mathcal{F} \longrightarrow R^q h_* e^{-1}\mathcal{F}$$

is an isomorphism for $q \leq q_0$.

0F03 **Lemma 84.2.** *With $f : X \to S$ and n as in Remark 84.1 assume for some $q \geq 1$ we have $BC(f, n, q-1)$. Then for every commutative diagram*

$$\begin{tikzcd} X \arrow[d, "f"'] & X' \arrow[l] \arrow[d, "f'"'] & Y \arrow[l, "h"'] \arrow[d, "e"] \\ S & S' \arrow[l] & T \arrow[l, "g"] \end{tikzcd}$$

with $X' = X \times_S S'$ and $Y = X' \times_{S'} T$ and g quasi-compact and quasi-separated, and every abelian sheaf $\mathcal{F}$ on $T_{étale}$ annihilated by n

1. *the base change map $(f')^{-1}R^q g_*\mathcal{F} \to R^q h_* e^{-1}\mathcal{F}$ is injective,*
2. *if $\mathcal{F} \subset \mathcal{G}$ where $\mathcal{G}$ on $T_{étale}$ is annihilated by n, then*

$$\mathrm{Coker}\left((f')^{-1}R^q g_*\mathcal{F} \to R^q h_* e^{-1}\mathcal{F}\right) \subset \mathrm{Coker}\left((f')^{-1}R^q g_*\mathcal{G} \to R^q h_* e^{-1}\mathcal{G}\right)$$

3. *if in (2) the sheaf $\mathcal{G}$ is an injective sheaf of $\mathbf{Z}/n\mathbf{Z}$-modules, then*

$$\mathrm{Coker}\left((f')^{-1}R^q g_*\mathcal{F} \to R^q h_* e^{-1}\mathcal{F}\right) \subset R^q h_* e^{-1}\mathcal{G}$$

Proof. Choose a short exact sequence $0 \to \mathcal{F} \to \mathcal{I} \to \mathcal{Q} \to 0$ where $\mathcal{I}$ is an injective sheaf of $\mathbf{Z}/n\mathbf{Z}$-modules. Consider the induced diagram

$$\begin{tikzcd} (f')^{-1}R^{q-1}g_*\mathcal{I} \arrow[r] \arrow[d, "\cong"'] & (f')^{-1}R^{q-1}g_*\mathcal{Q} \arrow[r] \arrow[d, "\cong"'] & (f')^{-1}R^q g_*\mathcal{F} \arrow[r] \arrow[d] & 0 \arrow[d] \\ R^{q-1}h_*e^{-1}\mathcal{I} \arrow[r] & R^{q-1}h_*e^{-1}\mathcal{Q} \arrow[r] & R^q h_* e^{-1}\mathcal{F} \arrow[r] & R^q h_* e^{-1}\mathcal{I} \end{tikzcd}$$

with exact rows. We have the zero in the right upper corner as $\mathcal{I}$ is injective. The left two vertical arrows are isomorphisms by $BC(f, n, q-1)$. We conclude that part (1) holds. The above also shows that

$$\mathrm{Coker}\left((f')^{-1}R^q g_*\mathcal{F} \to R^q h_* e^{-1}\mathcal{F}\right) \subset R^q h_* e^{-1}\mathcal{I}$$

hence part (3) holds. To prove (2) choose $\mathcal{F} \subset \mathcal{G} \subset \mathcal{I}$. $\square$

0F04 **Lemma 84.3.** *With $f : X \to S$ and n as in Remark 84.1 assume for some $q \geq 1$ we have $BC(f, n, q-1)$. Consider commutative diagrams*

$$\begin{tikzcd} X \arrow[d, "f"'] & X' \arrow[l] \arrow[d, "f'"'] & Y \arrow[l, "h"] \arrow[d, "e"] & Y' \arrow[l, "\pi'"] \arrow[d, "e'"] \\ S & S' \arrow[l] & T \arrow[l, "g"] & T' \arrow[l, "\pi"] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} X' \arrow[d, "f'"'] & Y' \arrow[l, "h'=h\circ\pi'"] \arrow[d, "e'"] \\ S' & T' \arrow[l, "g'=g\circ\pi"] \end{tikzcd}$$

where all squares are cartesian, g quasi-compact and quasi-separated, and π is integral surjective. Let $\mathcal{F}$ be an abelian sheaf on $T_{étale}$ annihilated by n and set $\mathcal{F}' = \pi^{-1}\mathcal{F}$. If the base change map

$$(f')^{-1}R^q g'_*\mathcal{F}' \longrightarrow R^q h'_*(e')^{-1}\mathcal{F}'$$

is an isomorphism, then the base change map $(f')^{-1}R^q g_*\mathcal{F} \to R^q h_* e^{-1}\mathcal{F}$ is an isomorphism.

Proof. Observe that $\mathcal{F} \to \pi_*\pi^{-1}\mathcal{F}'$ is injective as π is surjective (check on stalks). Thus by Lemma 84.2 we see that it suffices to show that the base change map

$$(f')^{-1}R^q g_*\pi_*\mathcal{F}' \longrightarrow R^q h_* e^{-1}\pi_*\mathcal{F}'$$

is an isomorphism. This follows from the assumption because we have $R^q g_*\pi_*\mathcal{F}' = R^q g'_*\mathcal{F}'$, we have $e^{-1}\pi_*\mathcal{F}' = \pi'_*(e')^{-1}\mathcal{F}'$, and we have $R^q h_*\pi'_*(e')^{-1}\mathcal{F}' = R^q h'_*(e')^{-1}\mathcal{F}'$. This follows from Lemmas 54.4 and 43.5 and the relative leray spectral sequence (Cohomology on Sites, Lemma 14.7). $\square$

0F05 **Lemma 84.4.** *With $f : X \to S$ and n as in Remark 84.1 assume for some $q \geq 1$ we have $BC(f, n, q-1)$. Consider commutative diagrams*

$$\begin{tikzcd} X \arrow[d, "f"'] & X' \arrow[l] \arrow[d, "f'"'] & X'' \arrow[l, "\pi'"] \arrow[d, "f''"'] & Y \arrow[l, "h'"] \arrow[d, "e"] \\ S & S' \arrow[l] & S'' \arrow[l, "\pi"] & T \arrow[l, "g'"] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} X' \arrow[d, "f'"'] & Y \arrow[l, "h=h'\circ\pi'"] \arrow[d, "e"] \\ S' & T \arrow[l, "g=g'\circ\pi"] \end{tikzcd}$$

where all squares are cartesian, g' quasi-compact and quasi-separated, and π is integral. Let $\mathcal{F}$ be an abelian sheaf on $T_{étale}$ annihilated by n. If the base change map

$$(f')^{-1}R^q g_*\mathcal{F} \longrightarrow R^q h_* e^{-1}\mathcal{F}$$

is an isomorphism, then the base change map $(f'')^{-1}R^q g'_*\mathcal{F} \to R^q h'_* e^{-1}\mathcal{F}$ is an isomorphism.

Proof. Since π and π' are integral we have $R\pi_* = \pi_*$ and $R\pi'_* = \pi'_*$, see Lemma 43.5. We also have $(f')^{-1}\pi_* = \pi'_*(f'')^{-1}$. Thus we see that $\pi'_*(f'')^{-1}R^q g'_*\mathcal{F} = (f')^{-1}R^q g_*\mathcal{F}$ and $\pi'_* R^q h'_* e^{-1}\mathcal{F} = R^q h_* e^{-1}\mathcal{F}$. Thus the assumption means that our map becomes an isomorphism after applying the functor π'_*. Hence we see that it is an isomorphism by Lemma 43.5. $\square$

0F06 **Lemma 84.5.** *Let T be a quasi-compact and quasi-separated scheme. Let P be a property for quasi-compact and quasi-separated schemes over T. Assume*

1. *If $T'' \to T'$ is a thickening of quasi-compact and quasi-separated schemes over T, then $P(T'')$ if and only if $P(T')$.*
2. *If $T' = \lim T_i$ is a limit of an inverse system of quasi-compact and quasi-separated schemes over T with affine transition morphisms and $P(T_i)$ holds for all i, then $P(T')$ holds.*

(3) *If $Z \subset T'$ is a closed subscheme with quasi-compact complement $V \subset T'$ and $P(T')$ holds, then either $P(V)$ or $P(Z)$ holds.*

Then $P(T)$ implies $P(\mathrm{Spec}(K))$ for some morphism $\mathrm{Spec}(K) \to T$ where K is a field.

Proof. Consider the set $\mathfrak{T}$ of closed subschemes $T' \subset T$ such that $P(T')$. By assumption (2) this set has a minimal element, say T'. By assumption (1) we see that T' is reduced. Let $\eta \in T'$ be the generic point of an irreducible component of T'. Then $\eta = \mathrm{Spec}(K)$ for some field K and $\eta = \lim V$ where the limit is over the affine open subschemes $V \subset T'$ containing η. By assumption (3) and the minimality of T' we see that $P(V)$ holds for all these V. Hence $P(\eta)$ by (2) and the proof is complete. $\square$

0F07 **Lemma 84.6.** *With $f : X \to S$ and n as in Remark 84.1 assume for some $q \geq 1$ we have that $BC(f, n, q-1)$ is true, but $BC(f, n, q)$ is not. Then there exist a commutative diagram*

$$\begin{tikzcd} X \arrow[d, "f"'] & X' \arrow[l] \arrow[d, "f'"'] & Y \arrow[l, "h"] \arrow[d, "e"] \\ S & S' \arrow[l] & \mathrm{Spec}(K) \arrow[l, "g"'] \end{tikzcd}$$

where $X' = X \times_S S'$, $Y = X' \times_{S'} \mathrm{Spec}(K)$, K is a field, and $\mathcal{F}$ is an abelian sheaf on $\mathrm{Spec}(K)$ annihilated by n such that $(f')^{-1}R^q g_*\mathcal{F} \to R^q h_* e^{-1}\mathcal{F}$ is not an isomorphism.

Proof. Choose a commutative diagram

$$\begin{tikzcd} X \arrow[d, "f"'] & X' \arrow[l] \arrow[d, "f'"'] & Y \arrow[l, "h"] \arrow[d, "e"] \\ S & S' \arrow[l] & T \arrow[l, "g"'] \end{tikzcd}$$

with $X' = X \times_S S'$ and $Y = X' \times_{S'} T$ and g quasi-compact and quasi-separated, and an abelian sheaf $\mathcal{F}$ on $T_{\acute{e}tale}$ annihilated by n such that the base change map $(f')^{-1}R^q g_*\mathcal{F} \to R^q h_* e^{-1}\mathcal{F}$ is not an isomorphism. Of course we may and do replace S' by an affine open of S'; this implies that T is quasi-compact and quasi-separated. By Lemma 84.2 we see $(f')^{-1}R^q g_*\mathcal{F} \to R^q h_* e^{-1}\mathcal{F}$ is injective. Pick a geometric point $\overline{x}$ of X' and an element ξ of $(R^q h_* q^{-1}\mathcal{F})_{\overline{x}}$ which is not in the image of the map $((f')^{-1}R^q g_*\mathcal{F})_{\overline{x}} \to (R^q h_* e^{-1}\mathcal{F})_{\overline{x}}$.

Consider a morphism $\pi : T' \to T$ with T' quasi-compact and quasi-separated and denote $\mathcal{F}' = \pi^{-1}\mathcal{F}$. Denote $\pi' : Y' = Y \times_T T' \to Y$ the base change of π and $e' : Y' \to T'$ the base change of e. Picture

$$\begin{tikzcd} X' \arrow[d, "f'"'] & Y \arrow[l, "h"] \arrow[d, "e"] & Y' \arrow[l, "\pi'"] \arrow[d, "e'"] \\ S' & T \arrow[l, "g"'] & T' \arrow[l, "\pi"'] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} X' \arrow[d, "f'"'] & Y' \arrow[l, "h' = h \circ \pi'"] \arrow[d, "e'"] \\ S' & T' \arrow[l, "g' = g \circ \pi"'] \end{tikzcd}$$

Using pullback maps we obtain a canonical commutative diagram

$$\begin{tikzcd}
(f')^{-1}R^q g_*\mathcal{F} \arrow[r] \arrow[d] & (f')^{-1}R^q g'_*\mathcal{F}' \arrow[d] \\
R^q h_* e^{-1}\mathcal{F} \arrow[r] & R^q h'_*(e')^{-1}\mathcal{F}'
\end{tikzcd}$$

of abelian sheaves on X'. Let $P(T')$ be the property

- The image ξ' of ξ in $(Rh'_*(e')^{-1}\mathcal{F}')_{\overline{x}}$ is not in the image of the map $(f^{-1}R^q g'_*\mathcal{F}')_{\overline{x}} \to (R^q h'_*(e')^{-1}\mathcal{F}')_{\overline{x}}$.

We claim that hypotheses (1), (2), and (3) of Lemma 84.5 hold for P which proves our lemma.

Condition (1) of Lemma 84.5 holds for P because the étale topology of a scheme and a thickening of the scheme is the same. See Proposition 45.4.

Suppose that I is a directed set and that T_i is an inverse system over I of quasi-compact and quasi-separated schemes over T with affine transition morphisms. Set $T' = \lim T_i$. Denote $\mathcal{F}'$ and $\mathcal{F}_i$ the pullback of $\mathcal{F}$ to T', resp. T_i. Consider the diagrams

$$\begin{tikzcd}
X \arrow[d, "f'"'] & Y \arrow[l, "h"] \arrow[d, "e"] & Y_i \arrow[l, "\pi'_i"] \arrow[d, "e_i"] \\
S & T \arrow[l, "g"] & T_i \arrow[l, "\pi_i"]
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
X \arrow[d, "f'"'] & & Y_i \arrow[ll, "h_i = h\circ\pi'_i"] \arrow[d, "e_i"] \\
S & & T_i \arrow[ll, "g_i = g\circ\pi_i"]
\end{tikzcd}$$

as in the previous paragraph. It is clear that $\mathcal{F}'$ on T' is the colimit of the pullbacks of $\mathcal{F}_i$ to T' and that $(e')^{-1}\mathcal{F}'$ is the colimit of the pullbacks of $e_i^{-1}\mathcal{F}_i$ to Y'. By Lemma 51.8 we have

$$R^q h'_*(e')^{-1}\mathcal{F}' = \operatorname{colim} R^q h_{i,*}e_i^{-1}\mathcal{F}_i \quad \text{and} \quad (f')^{-1}R^q g'_*\mathcal{F}' = \operatorname{colim}(f')^{-1}R^q g_{i,*}\mathcal{F}_i$$

It follows that if $P(T_i)$ is true for all i, then $P(T')$ holds. Thus condition (2) of Lemma 84.5 holds for P.

The most interesting is condition (3) of Lemma 84.5. Assume T' is a quasi-compact and quasi-separated scheme over T such that $P(T')$ is true. Let $Z \subset T'$ be a closed subscheme with complement $V \subset T'$ quasi-compact. Consider the diagram

$$\begin{tikzcd}
Y' \times_{T'} Z \arrow[r, "i'"] \arrow[d, "e_Z"'] & Y' \arrow[d, "e'"'] & Y' \times_{T'} V \arrow[l, "j'"'] \arrow[d, "e_V"] \\
Z \arrow[r, "i"] & T' & V \arrow[l, "j"']
\end{tikzcd}$$

Choose an injective map $j^{-1}\mathcal{F}' \to \mathcal{J}$ where $\mathcal{J}$ is an injective sheaf of $\mathbf{Z}/n\mathbf{Z}$-modules on V. Looking at stalks we see that the map

$$\mathcal{F}' \to \mathcal{G} = j_*\mathcal{J} \oplus i_*i^{-1}\mathcal{F}'$$

is injective. Thus ξ' maps to a nonzero element of

$$\begin{aligned}
&\operatorname{Coker}\left(((f')^{-1}R^q g'_*\mathcal{G})_{\overline{x}} \to (R^q h'_*(e')^{-1}\mathcal{G})_{\overline{x}}\right) = \\
&\operatorname{Coker}\left(((f')^{-1}R^q g'_*j_*\mathcal{J})_{\overline{x}} \to (R^q h'_*(e')^{-1}j_*\mathcal{J})_{\overline{x}}\right) \oplus \\
&\operatorname{Coker}\left(((f')^{-1}R^q g'_*i_*i^{-1}\mathcal{F}')_{\overline{x}} \to (R^q h'_*(e')^{-1}i_*i^{-1}\mathcal{F}')_{\overline{x}}\right)
\end{aligned}$$

by part (2) of Lemma 84.2. If ξ' does not map to zero in the second summand, then we use

$$(f')^{-1}R^q g'_* i_* i^{-1}\mathcal{F}' = (f')^{-1}R^q(g' \circ i)_* i^{-1}\mathcal{F}'$$

(because $Ri_* = i_*$ by Proposition 54.2) and

$$R^q h'_*(e')^{-1} i_* i^{-1}\mathcal{F} = R^q h'_* i'_* e_Z^{-1} i^{-1}\mathcal{F} = R^q(h' \circ i')_* e_Z^{-1} i^{-1}\mathcal{F}'$$

(first equality by Lemma 54.3 and the second because $Ri'_* = i'_*$ by Proposition 54.2) to we see that we have $P(Z)$. Finally, suppose ξ' does not map to zero in the first summand. We have

$$(e')^{-1} j_* \mathcal{J} = j'_* e_V^{-1}\mathcal{J} \quad \text{and} \quad R^a j'_* e_V^{-1}\mathcal{J} = 0, \quad a = 1, \ldots, q-1$$

by $BC(f, n, q-1)$ applied to the diagram

$$\begin{tikzcd} X & Y' \arrow[l] & Y \arrow[l, "j'"'] \arrow[d, "e_V"] \\ S \arrow[from=u, "f"'] & T' \arrow[l] \arrow[from=u, "e'"'] & V \arrow[l, "j"'] \end{tikzcd}$$

and the fact that $\mathcal{J}$ is injective. By the relative Leray spectral sequence for $h' \circ j'$ (Cohomology on Sites, Lemma 14.7) we deduce that

$$R^q h'_*(e')^{-1} j_* \mathcal{J} = R^q h'_* j'_* e_V^{-1}\mathcal{J} \longrightarrow R^q(h' \circ j')_* e_V^{-1}\mathcal{J}$$

is injective. Thus ξ maps to a nonzero element of $(R^q(h' \circ j')_* e_V^{-1}\mathcal{J})_{\overline{x}}$. Applying part (3) of Lemma 84.2 to the injection $j^{-1}\mathcal{F}' \to \mathcal{J}$ we conclude that $P(V)$ holds. $\square$

0F08 **Lemma 84.7.** *With $f : X \to S$ and n as in Remark 84.1 assume for some $q \geq 1$ we have that $BC(f, n, q-1)$ is true, but $BC(f, n, q)$ is not. Then there exist a commutative diagram*

$$\begin{tikzcd} X \arrow[d, "f"'] & X' \arrow[l] \arrow[d] & Y \arrow[l, "h"'] \arrow[d] \\ S & S' \arrow[l] & \mathrm{Spec}(K) \arrow[l] \end{tikzcd}$$

with both squares cartesian, where

1. *S' is affine, integral, and normal with algebraically closed function field,*
2. *K is algebraically closed and* $\mathrm{Spec}(K) \to S'$ *is dominant (in other words K is an extension of the function field of S')*

and there exists an integer $d | n$ such that $R^q h_*(\mathbf{Z}/d\mathbf{Z})$ is nonzero.

Conversely, nonvanishing of $R^q h_*(\mathbf{Z}/d\mathbf{Z})$ in the lemma implies $BC(f, n, q)$ isn't true as Lemma 78.4 shows that $R^q(\mathrm{Spec}(K) \to S')_* \mathbf{Z}/d\mathbf{Z} = 0$.

Proof. First choose a diagram and $\mathcal{F}$ as in Lemma 84.6. We may and do assume S' is affine (this is obvious, but see proof of the lemma in case of doubt). By Lemma 84.3 we may assume K is algebraically closed. Then $\mathcal{F}$ corresponds to a $\mathbf{Z}/n\mathbf{Z}$-module. Such a modules is a direct sum of copies of $\mathbf{Z}/d\mathbf{Z}$ for varying $d | n$ hence we may assume $\mathcal{F}$ is constant with value $\mathbf{Z}/d\mathbf{Z}$. By Lemma 84.4 we may replace S' by the normalization of S' in $\mathrm{Spec}(K)$ which finishes the proof. $\square$

85. Smooth base change

0EYQ In this section we prove the smooth base change theorem.

0EYT **Lemma 85.1.** *Let K/k be an extension of fields. Let X be a smooth affine curve over k with a rational point $x \in X(k)$. Let $\mathcal{F}$ be an abelian sheaf on* $\text{Spec}(K)$ *annihilated by an integer n invertible in k. Let $q > 0$ and*

$$\xi \in H^q(X_K, (X_K \to \text{Spec}(K))^{-1}\mathcal{F})$$

There exist

1. *finite extensions K'/K and k'/k with $k' \subset K'$,*
2. *a finite étale Galois cover $Z \to X_{k'}$ with group G*

such that the order of G divides a power of n, such that $Z \to X_{k'}$ is split over $x_{k'}$, and such that ξ dies in $H^q(Z_{K'}, (Z_{K'} \to \text{Spec}(K))^{-1}\mathcal{F})$.

Proof. For $q > 1$ we know that ξ dies in $H^q(X_{\overline{K}}, (X_{\overline{K}} \to \text{Spec}(K))^{-1}\mathcal{F})$ (Theorem 80.9). By Lemma 51.5 we see that this means there is a finite extension K'/K such that ξ dies in $H^q(X_{K'}, (X_{K'} \to \text{Spec}(K))^{-1}\mathcal{F})$. Thus we can take $k' = k$ and $Z = X$ in this case.

Assume $q = 1$. Recall that $\mathcal{F}$ corresponds to a discrete module M with continuous Gal_K-action, see Lemma 58.1. Since M is n-torsion, it is the uninon of finite Gal_K-stable subgroups. Thus we reduce to the case where M is a finite abelian group annihilated by n, see Lemma 51.4. After replacing K by a finite extension we may assume that the action of Gal_K on M is trivial. Thus we may assume $\mathcal{F} = \underline{M}$ is the constant sheaf with value a finite abelian group M annihilated by n.

We can write M as a direct sum of cyclic groups. Any two finite étale Galois coverings whose Galois groups have order invertible in k, can be dominated by a third one whose Galois group has order invertible in k (Fundamental Groups, Section 7). Thus it suffices to prove the lemma when $M = \mathbf{Z}/d\mathbf{Z}$ where $d|n$.

Assume $M = \mathbf{Z}/d\mathbf{Z}$ where $d|n$. In this case $\overline{\xi} = \xi|_{X_{\overline{K}}}$ is an element of

$$H^1(X_{\overline{k}}, \mathbf{Z}/d\mathbf{Z}) = H^1(X_{\overline{K}}, \mathbf{Z}/d\mathbf{Z})$$

See Theorem 80.9. This group classifies $\mathbf{Z}/d\mathbf{Z}$-torsors, see Cohomology on Sites, Lemma 4.3. The torsor corresponding to $\overline{\xi}$ (viewed as a sheaf on $X_{\overline{k},\acute{e}tale}$) in turn gives rise to a finite étale morphism $T \to X_{\overline{k}}$ endowed an action of $\mathbf{Z}/d\mathbf{Z}$ transitive on the fibre of T over $x_{\overline{k}}$, see Lemma 63.4. Choose a connected component $T' \subset T$ (if $\overline{\xi}$ has order d, then T is already connected). Then $T' \to X_{\overline{k}}$ is a finite étale Galois cover whose Galois group is a subgroup $G \subset \mathbf{Z}/d\mathbf{Z}$ (small detail omitted). Moreover the element $\overline{\xi}$ maps to zero under the map $H^1(X_{\overline{k}}, \mathbf{Z}/d\mathbf{Z}) \to H^1(T', \mathbf{Z}/d\mathbf{Z})$ as this is one of the defining properties of T.

Next, we use a limit argument to choose a finite extension k'/k contained in $\overline{k}$ such that $T' \to X_{\overline{k}}$ descends to a finite étale Galois cover $Z \to X_{k'}$ with group G. See Limits, Lemmas 10.1, 8.3, and 8.10. After increasing k' we may assume that Z splits over $x_{k'}$. The image of ξ in $H^1(Z_{\overline{K}}, \mathbf{Z}/d\mathbf{Z})$ is zero by construction. Thus by Lemma 51.5 we can find a finite subextension $\overline{K}/K'/K$ containing k' such that ξ dies in $H^1(Z_{K'}, \mathbf{Z}/d\mathbf{Z})$ and this finishes the proof. □

0EYU **Theorem 85.2** (Smooth base change)**.** *Consider a cartesian diagram of schemes*

$$\begin{tikzcd} X \arrow[d, "f"'] & Y \arrow[l, "h"'] \arrow[d, "e"] \\ S & T \arrow[l, "g"] \end{tikzcd}$$

where f is smooth and g quasi-compact and quasi-separated. Then

$$f^{-1}R^qg_*\mathcal{F} = R^qh_*e^{-1}\mathcal{F}$$

for any q and any abelian sheaf $\mathcal{F}$ on $T_{\acute{e}tale}$ all of whose stalks at geometric points are torsion of orders invertible on S.

First proof of smooth base change. This proof is very long but more direct (using less general theory) than the second proof given below.

The theorem is local on $X_{\acute{e}tale}$. More precisely, suppose we have $U \to X$ étale such that $U \to S$ factors as $U \to V \to S$ with $V \to S$ étale. Then we can consider the cartesian square

$$\begin{tikzcd} U \arrow[d, "f'"'] & U \times_X Y \arrow[l, "h'"'] \arrow[d, "e'"] \\ V & V \times_S T \arrow[l, "g'"] \end{tikzcd}$$

and setting $\mathcal{F}' = \mathcal{F}|_{V \times_S T}$ we have $f^{-1}R^qg_*\mathcal{F}|_U = (f')^{-1}R^qg'_*\mathcal{F}'$ and $R^qh_*e^{-1}\mathcal{F}|_U = R^qh'_*(e')^{-1}\mathcal{F}'$ (as follows from the compatibility of localization with morphisms of sites, see Sites, Lemma 28.2 and and Cohomology on Sites, Lemma 20.4). Thus it suffices to produce an étale covering of X by $U \to X$ and factorizations $U \to V \to S$ as above such that the theorem holds for the diagram with f', h', g', e'.

By the local structure of smooth morphisms, see Morphisms, Lemma 34.20, we may assume X and S are affine and $X \to S$ factors through an étale morphism $X \to \mathbf{A}^d_S$. If we have a tower of cartesian diagrams

$$\begin{tikzcd} W \arrow[d, "i"'] & Z \arrow[l, "j"'] \arrow[d, "k"] \\ X \arrow[d, "f"'] & Y \arrow[l, "h"'] \arrow[d, "e"] \\ S & T \arrow[l, "g"] \end{tikzcd}$$

and the theorem holds for the bottom and top squares, then the theorem holds for the outer rectangle; this is formal. Writing $X \to S$ as the composition

$$X \to \mathbf{A}^{d-1}_S \to \mathbf{A}^{d-2}_S \to \ldots \to \mathbf{A}^1_S \to S$$

we conclude that it suffices to prove the theorem when X and S are affine and $X \to S$ has relative dimension 1.

For every $n \geq 1$ invertible on S, let $\mathcal{F}[n]$ be the subsheaf of sections of $\mathcal{F}$ annihilated by n. Then $\mathcal{F} = \operatorname{colim} \mathcal{F}[n]$ by our assumption on the stalks of $\mathcal{F}$. The functors e^{-1} and f^{-1} commute with colimits as they are left adjoints. The functors R^qh_* and R^qg_* commute with filtered colimits by Lemma 51.7. Thus it suffices to prove

the theorem for $\mathcal{F}[n]$. From now on we fix an integer n, we work with sheaves of $\mathbf{Z}/n\mathbf{Z}$-modules and we assume S is a scheme over $\mathrm{Spec}(\mathbf{Z}[1/n])$.

Next, we reduce to the case where T is affine. Since g is quasi-compact and quasi-separate and S is affine, the scheme T is quasi-compact and quasi-separated. Thus we can use the induction principle of Cohomology of Schemes, Lemma 4.1. Hence it suffices to that if $T = W \cup W'$ is an open covering and the theorem holds for the squares

$$\begin{tikzcd} X \arrow[d] & e^{-1}(W) \arrow[l, "i"'] \arrow[d] \\ S & W \arrow[l, "a"] \end{tikzcd} \quad \begin{tikzcd} X \arrow[d] & e^{-1}(W') \arrow[l, "j"'] \arrow[d] \\ S & W' \arrow[l, "b"] \end{tikzcd} \quad \begin{tikzcd} X \arrow[d] & e^{-1}(W \cap W') \arrow[l, "k"'] \arrow[d] \\ S & W \cap W' \arrow[l, "c"] \end{tikzcd}$$

then the theorem holds for the original diagram. To see this we consider the diagram

$$\begin{tikzcd} f^{-1}R^{q-1}c_*\mathcal{F}|_{W \cap W'} \arrow[r] \arrow[d, "\cong"'] & f^{-1}R^q g_*\mathcal{F} \arrow[r] \arrow[d] & f^{-1}R^q a_*\mathcal{F}|_W \oplus f^{-1}R^q b_*\mathcal{F}|_{W'} \arrow[d, "\cong"'] \\ R^q k_* e^{-1}\mathcal{F}|_{e^{-1}(W \cap W')} \arrow[r] & R^q h_* e^{-1}\mathcal{F} \arrow[r] & R^q i_* e^{-1}\mathcal{F}|_{e^{-1}(W)} \oplus R^q j_* e^{-1}\mathcal{F}|_{e^{-1}(W')} \end{tikzcd}$$

whose rows are the long exact sequences of Lemma 50.2. Thus the 5-lemma gives the desired conclusion.

Summarizing, we may assume S, X, T, and Y affine, $\mathcal{F}$ is n torsion, $X \to S$ is smooth of relative dimension 1, and S is a scheme over $\mathbf{Z}[1/n]$. We will prove the theorem by induction on q. The base case $q = 0$ is handled by Lemma 83.2. Assume $q > 0$ and the theorem holds for all smaller degrees. Choose a short exact sequence $0 \to \mathcal{F} \to \mathcal{I} \to \mathcal{Q} \to 0$ where $\mathcal{I}$ is an injective sheaf of $\mathbf{Z}/n\mathbf{Z}$-modules. Consider the induced diagram

$$\begin{tikzcd} f^{-1}R^{q-1}g_*\mathcal{I} \arrow[r] \arrow[d, "\cong"'] & f^{-1}R^{q-1}g_*\mathcal{Q} \arrow[r] \arrow[d, "\cong"'] & f^{-1}R^q g_*\mathcal{F} \arrow[r] \arrow[d] & 0 \arrow[d] \\ R^{q-1}h_*e^{-1}\mathcal{I} \arrow[r] & R^{q-1}h_*e^{-1}\mathcal{Q} \arrow[r] & R^q h_* e^{-1}\mathcal{F} \arrow[r] & R^q h_* e^{-1}\mathcal{I} \end{tikzcd}$$

with exact rows. We have the zero in the right upper corner as $\mathcal{I}$ is injective. The left two vertical arrows are isomorphisms by induction hypothesis. Thus it suffices to prove that $R^q h_* e^{-1}\mathcal{I} = 0$.

Write $S = \mathrm{Spec}(A)$ and $T = \mathrm{Spec}(B)$ and say the morphism $T \to S$ is given by the ring map $A \to B$. We can write $A \to B = \mathrm{colim}_{i \in I}(A_i \to B_i)$ as a filtered colimit of maps of rings of finite type over $\mathbf{Z}[1/n]$ (see Algebra, Lemma 126.14). For $i \in I$ we set $S_i = \mathrm{Spec}(A_i)$ and $T_i = \mathrm{Spec}(B_i)$. For i large enough we can find a smooth morphism $X_i \to S_i$ of relative dimension 1 such that $X = X_i \times_{S_i} S$, see Limits, Lemmas 10.1, 8.9, and 16.3. Set $Y_i = X_i \times_{S_i} T_i$ to get squares

$$\begin{tikzcd} X_i \arrow[d, "f_i"'] & Y_i \arrow[l, "h_i"'] \arrow[d, "e_i"] \\ S_i & T_i \arrow[l, "g_i"] \end{tikzcd}$$

Observe that $\mathcal{I}_i = (T \to T_i)_*\mathcal{I}$ is an injective sheaf of $\mathbf{Z}/n\mathbf{Z}$-modules on T_i, see Cohomology on Sites, Lemma 14.2. We have $\mathcal{I} = \mathrm{colim}(T \to T_i)^{-1}\mathcal{I}_i$ by Lemma 51.9. Pulling back by e we get $e^{-1}\mathcal{I} = \mathrm{colim}(Y \to Y_i)^{-1}e_i^{-1}\mathcal{I}_i$. By Lemma 51.8 applied to the system of morphisms $Y_i \to X_i$ with limit $Y \to X$ we have

$$R^q h_* e^{-1}\mathcal{I} = \mathrm{colim}(X \to X_i)^{-1} R^q h_{i,*} e_i^{-1}\mathcal{I}_i$$

This reduces us to the case where T and S are affine of finite type over $\mathbf{Z}[1/n]$.

Summarizing, we have an integer $q \geq 1$ such that the theorem holds in degrees $< q$, the schemes S and T affine of finite type type over $\mathbf{Z}[1/n]$, we have $X \to S$ smooth of relative dimension 1 with X affine, and $\mathcal{I}$ is an injective sheaf of $\mathbf{Z}/n\mathbf{Z}$-modules and we have to show that $R^q h_* e^{-1}\mathcal{I} = 0$. We will do this by induction on $\dim(T)$.

The base case is $T = \emptyset$, i.e., $\dim(T) < 0$. If you don't like this, you can take as your base case the case $\dim(T) = 0$. In this case $T \to S$ is finite (in fact even $T \to \mathrm{Spec}(\mathbf{Z}[1/n])$ is finite as the target is Jacobson; details omitted), so h is finite too and hence has vanishing higher direct images (see references below).

Assume $\dim(T) = d \geq 0$ and we know the result for all situations where T has lower dimension. Pick U affine and étale over X and a section ξ of $R^q h_* q^{-1}\mathcal{I}$ over U. We have to show that ξ is zero. Of course, we may replace X by U (and correspondingly Y by $U \times_X Y$) and assume $\xi \in H^0(X, R^q h_* e^{-1}\mathcal{I})$. Moreover, since $R^q h_* e^{-1}\mathcal{I}$ is a sheaf, it suffices to prove that ξ is zero locally on X. Hence we may replace X by the members of an étale covering. In particular, using Lemma 51.6 we may assume that ξ is the image of an element $\tilde{\xi} \in H^q(Y, e^{-1}\mathcal{I})$. In terms of $\tilde{\xi}$ our task is to show that $\tilde{\xi}$ dies in $H^q(U_i \times_X Y, e^{-1}\mathcal{I})$ for some étale covering $\{U_i \to X\}$.

By More on Morphisms, Lemma 34.8 we may assume that $X \to S$ factors as $X \to V \to S$ where $V \to S$ is étale and $X \to V$ is a smooth morphism of affine schemes of relative dimension 1, has a section, and has geometrically connected fibres. Observe that $\dim(V \times_S T) \leq \dim(T) = d$ for example by More on Algebra, Lemma 43.2. Hence we may then replace S by V and T by $V \times_S T$ (exactly as in the discussion in the first paragraph of the proof). Thus we may assume $X \to S$ is smooth of relative dimension 1, geometrically connected fibres, and has a section $\sigma : S \to X$.

Let $\pi : T' \to T$ be a finite surjective morphism. We will use below that $\dim(T') \leq \dim(T) = d$, see Algebra, Lemma 111.3. Choose an injective map $\pi^{-1}\mathcal{I} \to \mathcal{I}'$ into an injective sheaf of $\mathbf{Z}/n\mathbf{Z}$-modules. Then $\mathcal{I} \to \pi_*\mathcal{I}'$ is injective and hence has a splitting (as $\mathcal{I}$ is an injective sheaf of $\mathbf{Z}/n\mathbf{Z}$-modules). Denote $\pi' : Y' = Y \times_T T' \to Y$ the base change of π and $e' : Y' \to T'$ the base change of e. Picture

$$\begin{tikzcd} X \arrow[d, "f"'] & Y \arrow[l, "h"'] \arrow[d, "e"] & Y' \arrow[l, "\pi'"'] \arrow[d, "e'"] \\ S & T \arrow[l, "g"] & T' \arrow[l, "\pi"] \end{tikzcd}$$

By Proposition 54.2 and Lemma 54.3 we have $R\pi'_*(e')^{-1}\mathcal{I}' = e^{-1}\pi_*\mathcal{I}'$. Thus by the Leray spectral sequence (Cohomology on Sites, Lemma 14.5) we have

$$H^q(Y', (e')^{-1}\mathcal{I}') = H^q(Y, e^{-1}\pi_*\mathcal{I}') \supset H^q(Y, e^{-1}\mathcal{I})$$

and this remains true after base change by any $U \to X$ étale. Thus we may replace T by T', $\mathcal{I}$ by $\mathcal{I}'$ and $\tilde{\xi}$ by its image in $H^q(Y', (e')^{-1}\mathcal{I}')$.

Suppose we have a factorization $T \to S' \to S$ where $\pi : S' \to S$ is finite. Setting $X' = S' \times_S X$ we can consider the induced diagram

$$\begin{tikzcd} X & X' \arrow[l, "\pi'"'] \arrow[d, "f'"] & Y \arrow[l, "h'"'] \arrow[d, "e"] \\ S \arrow[from=u, "f"'] & S' \arrow[l, "\pi"] & T \arrow[l, "g"] \end{tikzcd}$$

Since π' has vanishing higher direct images we see that $R^q h_* e^{-1}\mathcal{I} = \pi'_* R^q h'_* e^{-1}\mathcal{I}$ by the Leray spectral sequence. Hence $H^0(X, R^q h_* e^{-1}\mathcal{I}) = H^0(X', R^q h'_* e^{-1}\mathcal{I})$. Thus ξ is zero if and only if the corresponding section of $R^q h'_* e^{-1}\mathcal{I}$ is zero[8]. Thus we may replace S by S' and X by X'. Observe that $\sigma : S \to X$ base changes to $\sigma' : S' \to X'$ and hence after this replacement it is still true that $X \to S$ has a section σ and geometrically connected fibres.

We will use that S and T are Nagata schemes, see Algebra, Proposition 157.16 which will guarantee that various normalizations are finite, see Morphisms, Lemmas 51.15 and 52.10. In particular, we may first replace T by its normalization and then replace S by the normalization of S in T. Then $T \to S$ is a disjoint union of dominant morphisms of integral normal schemes, see Morphisms, Lemma 51.13. Clearly we may argue one connnected component at a time, hence we may assume $T \to S$ is a dominant morphism of integral normal schemes.

Let $s \in S$ and $t \in T$ be the generic points. By Lemma 85.1 there exist finite field extensions $K/\kappa(t)$ and $k/\kappa(s)$ such that k is contained in K and a finite étale Galois covering $Z \to X_k$ with Galois group G of order dividing a power of n split over $\sigma(\mathrm{Spec}(k))$ such that $\tilde{\xi}$ maps to zero in $H^q(Z_K, e^{-1}\mathcal{I}|_{Z_K})$. Let $T' \to T$ be the normalization of T in $\mathrm{Spec}(K)$ and let $S' \to S$ be the normalization of S in $\mathrm{Spec}(k)$. Then we obtain a commutative diagram

$$\begin{tikzcd} S' \arrow[d] & T' \arrow[l] \arrow[d] \\ S & T \arrow[l] \end{tikzcd}$$

whose vertical arrows are finite. By the arguments given above we may and do replace S and T by S' and T' (and correspondingly X by $X \times_S S'$ and Y by $Y \times_T T'$). After this replacement we conclude we have a finite étale Galois covering $Z \to X_s$ of the generic fibre of $X \to S$ with Galois group G of order dividing a power of n split over $\sigma(s)$ such that $\tilde{\xi}$ maps to zero in $H^q(Z_t, (Z_t \to Y)^{-1}e^{-1}\mathcal{I})$. Here $Z_t = Z \times_S t = Z \times_s t = Z \times_{X_s} Y_t$. Since n is invertible on S, by Fundamental Groups, Lemma 31.7 we can find a finite étale morphism $U \to X$ whose restriction to X_s is Z.

At this point we replace X by U and Y by $U \times_X Y$. After this replacement it may no longer be the case that the fibres of $X \to S$ are geometrically connected (there still is a section but we won't use this), but what we gain is that after this

[8]This step can also be seen another way. Namely, we have to show that there is an étale covering $\{U_i \to X\}$ such that $\tilde{\xi}$ dies in $H^q(U_i \times_X Y, e^{-1}\mathcal{I})$. However, if we prove there is an étale covering $\{U'_j \to X'\}$ such that $\tilde{\xi}$ dies in $H^q(U'_i \times_{X'} Y, e^{-1}\mathcal{I})$, then by property (B) for $X' \to X$ (Lemma 43.3) there exists an étale covering $\{U_i \to X\}$ such that $U_i \times_X X'$ is a disjoint union of schemes over X' each of which factors through U'_j for some j. Thus we see that $\tilde{\xi}$ dies in $H^q(U_i \times_X Y, e^{-1}\mathcal{I})$ as desired.

replacement $\tilde{\xi}$ maps to zero in $H^q(Y_t, e^{-1}\mathcal{I})$, i.e., $\tilde{\xi}$ restricts to zero on the generic fibre of $Y \to T$.

Recall that t is the spectrum of the function field of T, i.e., as a scheme t is the limit of the nonempty affine open subschemes of T. By Lemma 51.5 we conclude there exists a nonempty open subscheme $V \subset T$ such that $\tilde{\xi}$ maps to zero in $H^q(Y \times_T V, e^{-1}\mathcal{I}|_{Y \times_T V})$.

Denote $Z = T \setminus V$. Consider the diagram

$$\begin{tikzcd} Y \times_T Z \arrow[r, "i'"'] \arrow[d, "e_Z"'] & Y \arrow[d, "e"'] & Y \times_T V \arrow[l, "j'"] \arrow[d, "e_V"] \\ Z \arrow[r, "i"] & T & V \arrow[l, "j"'] \end{tikzcd}$$

Choose an injection $i^{-1}\mathcal{I} \to \mathcal{I}'$ into an injective sheaf of $\mathbf{Z}/n\mathbf{Z}$-modules on Z. Looking at stalks we see that the map

$$\mathcal{I} \to j_*\mathcal{I}|_V \oplus i_*\mathcal{I}'$$

is injective and hence splits as $\mathcal{I}$ is an injective sheaf of $\mathbf{Z}/n\mathbf{Z}$-modules. Thus it suffices to show that $\tilde{\xi}$ maps to zero in

$$H^q(Y, e^{-1}j_*\mathcal{I}|_V) \oplus H^q(Y, e^{-1}i_*\mathcal{I}')$$

at least after replacing X by the members of an étale covering. Observe that

$$e^{-1}j_*\mathcal{I}|_V = j'_*e_V^{-1}\mathcal{I}|_V, \quad e^{-1}i_*\mathcal{I}' = i'_*e_Z^{-1}\mathcal{I}'$$

By induction hypothesis on q we see that

$$R^a j'_* e_V^{-1}\mathcal{I}|_V = 0, \quad a = 1, \ldots, q-1$$

By the Leray spectral sequence for j' and the vanishing above it follows that

$$H^q(Y, j'_*(e_V^{-1}\mathcal{I}|_V)) \longrightarrow H^q(Y \times_T V, e_V^{-1}\mathcal{I}_V) = H^q(Y \times_T V, e^{-1}\mathcal{I}|_{Y \times_T V})$$

is injective. Thus the vanishing of the image of $\tilde{\xi}$ in the first summand above because we know $\tilde{\xi}$ vanishes in $H^q(Y \times_T V, e^{-1}\mathcal{I}|_{Y \times_T V})$. Since $\dim(Z) < \dim(T) = d$ by induction the image of $\tilde{\xi}$ in the second summand

$$H^q(Y, e^{-1}i_*\mathcal{I}') = H^q(Y, i'_*e_Z^{-1}\mathcal{I}') = H^q(Y \times_T Z, e_Z^{-1}\mathcal{I}')$$

dies after replacing X by the members of a suitable étale covering. This finishes the proof of the smooth base change theorem. □

Second proof of smooth base change. This proof is the same as the longer first proof; it is shorter only in that we have split out the arguments used in a number of lemmas.

The case of $q = 0$ is Lemma 83.2. Thus we may assume $q > 0$ and the result is true for all smaller degrees.

For every $n \geq 1$ invertible on S, let $\mathcal{F}[n]$ be the subsheaf of sections of $\mathcal{F}$ annihilated by n. Then $\mathcal{F} = \operatorname{colim} \mathcal{F}[n]$ by our assumption on the stalks of $\mathcal{F}$. The functors e^{-1} and f^{-1} commute with colimits as they are left adjoints. The functors $R^q h_*$ and $R^q g_*$ commute with filtered colimits by Lemma 51.7. Thus it suffices to prove the theorem for $\mathcal{F}[n]$. From now on we fix an integer n invertible on S and we work with sheaves of $\mathbf{Z}/n\mathbf{Z}$-modules.

By Lemma 82.1 the question is étale local on X and S. By the local structure of smooth morphisms, see Morphisms, Lemma 34.20, we may assume X and S are affine and $X \to S$ factors through an étale morphism $X \to \mathbf{A}^d_S$. Writing $X \to S$ as the composition

$$X \to \mathbf{A}^{d-1}_S \to \mathbf{A}^{d-2}_S \to \ldots \to \mathbf{A}^1_S \to S$$

we conclude from Lemma 82.2 that it suffices to prove the theorem when X and S are affine and $X \to S$ has relative dimension 1.

By Lemma 84.7 it suffices to show that $R^q h_* \mathbf{Z}/d\mathbf{Z} = 0$ for $d|n$ whenever we have a cartesian diagram

$$\begin{tikzcd} X \arrow[d] & Y \arrow[l, "h"'] \arrow[d] \\ S & \mathrm{Spec}(K) \arrow[l] \end{tikzcd}$$

where $X \to S$ is affine and smooth of relative dimension 1, S is the spectrum of a normal domain A with algebraically closed fraction field L, and K/L is an extension of algebraically closed fields.

Recall that $R^q h_* \mathbf{Z}/d\mathbf{Z}$ is the sheaf associated to the presheaf

$$U \longmapsto H^q(U \times_X Y, \mathbf{Z}/d\mathbf{Z}) = H^q(U \times_S \mathrm{Spec}(K), \mathbf{Z}/d\mathbf{Z})$$

on $X_{\acute{e}tale}$ (Lemma 51.6). Thus it suffices to show: given U and $\xi \in H^q(U \times_S \mathrm{Spec}(K), \mathbf{Z}/d\mathbf{Z})$ there exists an étale covering $\{U_i \to U\}$ such that ξ dies in $H^q(U_i \times_S \mathrm{Spec}(K), \mathbf{Z}/d\mathbf{Z})$.

Of course we may take U affine. Then $U \times_S \mathrm{Spec}(K)$ is a (smooth) affine curve over K and hence we have vanishing for $q > 1$ by Theorem 80.9.

Final case: $q = 1$. We may replace U by the members of an étale covering as in More on Morphisms, Lemma 34.8. Then $U \to S$ factors as $U \to V \to S$ where $U \to V$ has geometrically connected fibres, U, V are affine, $V \to S$ is étale, and there is a section $\sigma : V \to U$. By Lemma 78.3 we see that V is isomorphic to a (finite) disjoint union of (affine) open subschemes of S. Clearly we may replace S by one of these and X by the corresponding component of U. Thus we may assume $X \to S$ has geometrically connected fibres, has a section σ, and $\xi \in H^1(Y, \mathbf{Z}/d\mathbf{Z})$. Since K and L are algebraically closed we have

$$H^1(X_L, \mathbf{Z}/d\mathbf{Z}) = H^1(Y, \mathbf{Z}/d\mathbf{Z})$$

See Lemma 80.11. Thus there is a finite étale Galois covering $Z \to X_L$ with Galois group $G \subset \mathbf{Z}/d\mathbf{Z}$ which annihilates ξ. You can either see this by looking at the statement or proof of Lemma 85.1 or by using directly that ξ corresponds to a $\mathbf{Z}/d\mathbf{Z}$-torsor over X_L. Finally, by Fundamental Groups, Lemma 31.8 we find a (necessarily surjective) finite étale morphism $X' \to X$ whose restriction to X_L is $Z \to X_L$. Since ξ dies in X'_K this finishes the proof. $\square$

The following immediate consquence of the smooth base change theorem is what is often used in practice.

0F09 **Lemma 85.3.** *Let S be a scheme. Let $S' = \lim S_i$ be a directed inverse limit of schemes S_i smooth over S with affine transition morphisms. Let $f : X \to S$ be quas-compact and quasi-separated and form the fibre square*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ S' & \xrightarrow{g} & S \end{array}$$

Then

$$g^{-1}Rf_*E = R(f')_*(g')^{-1}E$$

for any $E \in D^+(X_{étale})$ whose cohomology sheaves $H^q(E)$ have stalks which are torsion of orders invertible on S.

Proof. Consider the spectral sequences

$$E_2^{p,q} = R^pf_*H^q(E) \quad \text{and} \quad {E'}_2^{p,q} = R^pf'_*H^q((g')^{-1}E) = R^pf'_*(g')^{-1}H^q(E)$$

converging to R^nf_*E and $R^nf'_*(g')^{-1}E$. These spectral sequences are constructed in Derived Categories, Lemma 21.3. Combining the smooth base change theorem (Theorem 85.2) with Lemma 82.3 we see that

$$g^{-1}R^pf_*H^q(E) = R^p(f')_*(g')^{-1}H^q(E)$$

Combining all of the above we get the lemma. □

86. Applications of smooth base change

0F0A In this section we discuss some more or less immediate consequences of the smooth base change theorem.

0F1C **Lemma 86.1.** *Let L/K be an extension of fields. Let $g : T \to S$ be a quasi-compact and quasi-separated morphism of schemes over K. Denote $g_L : T_L \to S_L$ the base change of g to* $\text{Spec}(L)$. *Let $E \in D^+(T_{étale})$ have cohomology sheaves whose stalks are torsion of orders invertible in K. Let E_L be the pullback of E to $(T_L)_{étale}$. Then $Rg_{L,*}E_L$ is the pullback of Rg_*E to S_L.*

Proof. If L/K is separable, then L is a filtered colimit of smooth K-algebras, see Algebra, Lemma 153.11. Thus the lemma in this case follows immediately from Lemma 85.3. In the general case, let K' and L' be the perfect closures (Algebra, Definition 44.5) of K and L. Then $\text{Spec}(K') \to \text{Spec}(K)$ and $\text{Spec}(L') \to \text{Spec}(L)$ are universal homeomorphisms as K'/K and L'/L are purely inseparable (see Algebra, Lemma 45.7). Thus we have $(T_{K'})_{étale} = T_{étale}$, $(S_{K'})_{étale} = S_{étale}$, $(T_{L'})_{étale} = (T_L)étale$, and $(S_{L'})_{étale} = (S_L)_{étale}$ by the topological invariance of étale cohomology, see Proposition 45.4. This reduces the lemma to the case of the field extension L'/K' which is separable (by definition of perfect fields, see Algebra, Definition 44.1). □

0F0B **Lemma 86.2.** *Let K/k be an extension of separably closed fields. Let X be a quasi-compact and quasi-separated scheme over k. Let $E \in D^+(X_{étale})$ have cohomology sheaves whose stalks are torsion of orders invertible in k. Then*

(1) *the maps $H^q_{étale}(X, E) \to H^q_{étale}(X_K, E|_{X_K})$ are isomorphisms, and*
(2) *$E \to R(X_K \to X)_*E|_{X_K}$ is an isomorphism.*

Proof. Proof of (1). First let $\overline{k}$ and $\overline{K}$ be the algebraic closures of k and K. The morphisms $\mathrm{Spec}(\overline{k}) \to \mathrm{Spec}(k)$ and $\mathrm{Spec}(\overline{K}) \to \mathrm{Spec}(K)$ are universal homeomorphisms as $\overline{k}/k$ and $\overline{K}/K$ are purely inseparable (see Algebra, Lemma 45.7). Thus $H^q_{étale}(X, \mathcal{F}) = H^q_{étale}(X_{\overline{k}}, \mathcal{F}_{X_{\overline{k}}})$ by the topological invariance of étale cohomology, see Proposition 45.4. Similarly for X_K and $X_{\overline{K}}$. Thus we may assume k and K are algebraically closed. In this case K is a limit of smooth k-algebras, see Algebra, Lemma 153.11. We conclude our lemma is a special case of Theorem 85.2 as reformulated in Lemma 85.3.

Proof of (2). For any quasi-compact and quasi-separated U in $X_{étale}$ the above shows that the restriction of the map $E \to R(X_K \to X)_*E|_{X_K}$ determines an isomorphism on cohomology. Since every object of $X_{étale}$ has an étale covering by such U this proves the desired statement. $\square$

0F1D **Lemma 86.3.** *With $f : X \to S$ and n as in Remark 84.1 assume n is invertible on S and that for some $q \geq 1$ we have that $BC(f, n, q-1)$ is true, but $BC(f, n, q)$ is not. Then there exist a commutative diagram*

$$\begin{tikzcd} X \arrow[d, "f"'] & X' \arrow[l] \arrow[d] & Y \arrow[l, "h"'] \arrow[d] \\ S & S' \arrow[l] & \mathrm{Spec}(K) \arrow[l] \end{tikzcd}$$

with both squares cartesian, where S' is affine, integral, and normal with algebraically closed function field K and there exists an integer $d|n$ such that $R^qh_*(\mathbf{Z}/d\mathbf{Z})$ is nonzero.

Proof. First choose a diagram and $\mathcal{F}$ as in Lemma 84.7. We may and do assume S' is affine (this is obvious, but see proof of the lemma in case of doubt). Let K' be the function field of S' and let $Y' = X' \times_{S'} \mathrm{Spec}(K')$ to get the diagram

$$\begin{tikzcd} X \arrow[d, "f"'] & X' \arrow[l] \arrow[d] & Y' \arrow[l, "h'"'] \arrow[d] & Y \arrow[l] \arrow[d] \\ S & S' \arrow[l] & \mathrm{Spec}(K') \arrow[l] & \mathrm{Spec}(K) \arrow[l] \end{tikzcd}$$

By Lemma 86.2 the total direct image $R(Y \to Y')_*\mathbf{Z}/d\mathbf{Z}$ is isomorphic to $\mathbf{Z}/d\mathbf{Z}$ in $D(Y'_{étale})$; here we use that n is invertible on S. Thus $Rh'_*\mathbf{Z}/d\mathbf{Z} = Rh_*\mathbf{Z}/d\mathbf{Z}$ by the relative Leray spectral sequence. This finishes the proof. $\square$

87. The proper base change theorem

095S The proper base change theorem is stated and proved in this section. Our approach follows roughly the proof in [AGV71, XII, Theorem 5.1] using Gabber's ideas (from the affine case) to slightly simplify the arguments.

0A0B **Lemma 87.1.** *Let (A, I) be a henselian pair. Let $f : X \to \mathrm{Spec}(A)$ be a proper morphism of schemes. Let $Z = X \times_{\mathrm{Spec}(A)} \mathrm{Spec}(A/I)$. For any sheaf $\mathcal{F}$ on the topological space associated to X we have $\Gamma(X, \mathcal{F}) = \Gamma(Z, \mathcal{F}|_Z)$.*

Proof. We will use Lemma 79.4 to prove this. First observe that the underlying topological space of X is spectral by Properties, Lemma 2.4. Let $Y \subset X$ be an irreducible closed subscheme. To finish the proof we show that $Y \cap Z = Y \times_{\mathrm{Spec}(A)}$

$\mathrm{Spec}(A/I)$ is connected. Replacing X by Y we may assume that X is irreducible and we have to show that Z is connected. Let $X \to \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ be the Stein factorization of f (More on Morphisms, Theorem 48.5). Then $A \to B$ is integral and (B, IB) is a henselian pair (More on Algebra, Lemma 11.8). Thus we may assume the fibres of $X \to \mathrm{Spec}(A)$ are geometrically connected. On the other hand, the image $T \subset \mathrm{Spec}(A)$ of f is irreducible and closed as X is proper over A. Hence $T \cap V(I)$ is connected by More on Algebra, Lemma 11.12. Now $Y \times_{\mathrm{Spec}(A)} \mathrm{Spec}(A/I) \to T \cap V(I)$ is a surjective closed map with connected fibres. The result now follows from Topology, Lemma 7.5. □

0A0C **Lemma 87.2.** *Let (A, I) be a henselian pair. Let $f : X \to \mathrm{Spec}(A)$ be a proper morphism of schemes. Let $i : Z \to X$ be the closed immersion of $X \times_{\mathrm{Spec}(A)} \mathrm{Spec}(A/I)$ into X. For any sheaf $\mathcal{F}$ on $X_{\acute{e}tale}$ we have $\Gamma(X, \mathcal{F}) = \Gamma(Z, i^{-1}_{small}\mathcal{F})$.*

Proof. This follows from Lemma 79.2 and 87.1 and the fact that any scheme finite over X is proper over $\mathrm{Spec}(A)$. □

0A3S **Lemma 87.3.** *Let A be a henselian local ring. Let $f : X \to \mathrm{Spec}(A)$ be a proper morphism of schemes. Let $X_0 \subset X$ be the fibre of f over the closed point. For any sheaf $\mathcal{F}$ on $X_{\acute{e}tale}$ we have $\Gamma(X, \mathcal{F}) = \Gamma(X_0, \mathcal{F}|_{X_0})$.*

Proof. This is a special case of Lemma 87.2. □

Let $f : X \to S$ be a morphism of schemes. Let $\overline{s} : \mathrm{Spec}(k) \to S$ be a geometric point. The fibre of f at $\overline{s}$ is the scheme $X_{\overline{s}} = \mathrm{Spec}(k) \times_{\overline{s}, S} X$ viewed as a scheme over $\mathrm{Spec}(k)$. If $\mathcal{F}$ is a sheaf on $X_{\acute{e}tale}$, then denote $\mathcal{F}_{\overline{s}} = p^{-1}_{small}\mathcal{F}$ the pullback of $\mathcal{F}$ to $(X_{\overline{s}})_{\acute{e}tale}$. In the following we will consider the set

$$\Gamma(X_{\overline{s}}, \mathcal{F}_{\overline{s}})$$

Let $s \in S$ be the image point of $\overline{s}$. Let $\kappa(s)^{sep}$ be the separable algebraic closure of $\kappa(s)$ in k as in Definition 55.1. By Lemma 39.5 pullback defines a bijection

$$\Gamma(X_{\kappa(s)^{sep}}, p_{sep}^{-1}\mathcal{F}) \longrightarrow \Gamma(X_{\overline{s}}, \mathcal{F}_{\overline{s}})$$

where $p_{sep} : X_{\kappa(s)^{sep}} = \mathrm{Spec}(\kappa(s)^{sep}) \times_S X \to X$ is the projection.

0A3T **Lemma 87.4.** *Let $f : X \to S$ be a proper morphism of schemes. Let $\overline{s} \to S$ be a geometric point. For any sheaf $\mathcal{F}$ on $X_{\acute{e}tale}$ the canonical map*

$$(f_*\mathcal{F})_{\overline{s}} \longrightarrow \Gamma(X_{\overline{s}}, \mathcal{F}_{\overline{s}})$$

is bijective.

Proof. By Theorem 52.1 (for sheaves of sets) we have

$$(f_*\mathcal{F})_{\overline{s}} = \Gamma(X \times_S \mathrm{Spec}(\mathcal{O}^{sh}_{S,\overline{s}}), p^{-1}_{small}\mathcal{F})$$

where $p : X \times_S \mathrm{Spec}(\mathcal{O}^{sh}_{S,\overline{s}}) \to X$ is the projection. Since the residue field of the strictly henselian local ring $\mathcal{O}^{sh}_{S,\overline{s}}$ is $\kappa(s)^{sep}$ we conclude from the discussion above the lemma and Lemma 87.3. □

0A3U **Lemma 87.5.** *Let $f : X \to Y$ be a proper morphism of schemes. Let $g : Y' \to Y$ be a morphism of schemes. Set $X' = Y' \times_Y X$ with projections $f' : X' \to Y'$ and $g' : X' \to X$. Let $\mathcal{F}$ be any sheaf on $X_{\acute{e}tale}$. Then $g^{-1}f_*\mathcal{F} = f'_*(g')^{-1}\mathcal{F}$.*

Proof. There is a canonical map $g^{-1}f_*\mathcal{F} \to f'_*(g')^{-1}\mathcal{F}$. Namely, it is adjoint to the map

$$f_*\mathcal{F} \longrightarrow g_*f'_*(g')^{-1}\mathcal{F} = f_*g'_*(g')^{-1}\mathcal{F}$$

which is f_* applied to the canonical map $\mathcal{F} \to g'_*(g')^{-1}\mathcal{F}$. To check this map is an isomorphism we can compute what happens on stalks. Let $y' : \mathrm{Spec}(k) \to Y'$ be a geometric point with image y in Y. By Lemma 87.4 the stalks are $\Gamma(X'_{y'}, \mathcal{F}_{y'})$ and $\Gamma(X_y, \mathcal{F}_y)$ respectively. Here the sheaves $\mathcal{F}_y$ and $\mathcal{F}_{y'}$ are the pullbacks of $\mathcal{F}$ by the projections $X_y \to X$ and $X'_{y'} \to X$. Thus we see that the groups agree by Lemma 39.5. We omit the verification that this isomorphism is compatible with our map. □

At this point we start discussing the proper base change theorem. To do so we introduce some notation. consider a commutative diagram

0A29 (87.5.1)
$$\begin{tikzcd} X' \arrow[r, "g'"'] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

of morphisms of schemes. Then we obtain a commutative diagram of sites

$$\begin{tikzcd} X'_{étale} \arrow[r, "g'_{small}"'] \arrow[d, "f'_{small}"'] & X_{étale} \arrow[d, "f_{small}"] \\ Y'_{étale} \arrow[r, "g_{small}"] & Y_{étale} \end{tikzcd}$$

For any object E of $D(X_{étale})$ we obtain a canonical base change map

0A2A (87.5.2)
$$g_{small}^{-1}Rf_{small,*}E \longrightarrow Rf'_{small,*}(g'_{small})^{-1}E$$

in $D(Y'_{étale})$. See Cohomology on Sites, Remark 19.3 where we use the constant sheaf $\mathbf{Z}$ as our sheaf of rings. We will usually omit the subscripts $_{small}$ in this formula. For example, if $E = \mathcal{F}[0]$ where $\mathcal{F}$ is an abelian sheaf on $X_{étale}$, the base change map is a map

0A4A (87.5.3)
$$g^{-1}Rf_*\mathcal{F} \longrightarrow Rf'_*(g')^{-1}\mathcal{F}$$

in $D(Y'_{étale})$.

The map (87.5.2) has no chance of being an isomorphism in the generality given above. The goal is to show it is an isomorphism if the diagram (87.5.1) is cartesian, $f : X \to Y$ proper, the cohomology sheaves of E are torsion, and E is bounded below. To study this question we introduce the following terminology. Let us say that *cohomology commutes with base change for $f : X \to Y$* if (87.5.3) is an isomorphism for every diagram (87.5.1) where $X' = Y' \times_Y X$ and every torsion abelian sheaf $\mathcal{F}$.

0A4B **Lemma 87.6.** *Let $f : X \to Y$ be a proper morphism of schemes. The following are equivalent*

1. *cohomology commutes with base change for f (see above),*
2. *for every prime number ℓ and every injective sheaf of $\mathbf{Z}/\ell\mathbf{Z}$-modules $\mathcal{I}$ on $X_{étale}$ and every diagram (87.5.1) where $X' = Y' \times_Y X$ the sheaves $R^qf'_*(g')^{-1}\mathcal{I}$ are zero for $q > 0$.*

Proof. It is clear that (1) implies (2). Conversely, assume (2) and let $\mathcal{F}$ be a torsion abelian sheaf on $X_{étale}$. Let $Y' \to Y$ be a morphism of schemes and let $X' = Y' \times_Y X$ with projections $g' : X' \to X$ and $f' : X' \to Y'$ as in diagram (87.5.1). We want to show the maps of sheaves

$$g^{-1}R^q f_*\mathcal{F} \longrightarrow R^q f'_*(g')^{-1}\mathcal{F}$$

are isomorphisms for all $q \geq 0$.

For every $n \geq 1$, let $\mathcal{F}[n]$ be the subsheaf of sections of $\mathcal{F}$ annihilated by n. Then $\mathcal{F} = \operatorname{colim} \mathcal{F}[n]$. The functors g^{-1} and $(g')^{-1}$ commute with arbitrary colimits (as left adjoints). Taking higher direct images along f or f' commutes with filtered colimits by Lemma 51.7. Hence we see that

$$g^{-1}R^q f_*\mathcal{F} = \operatorname{colim} g^{-1}R^q f_*\mathcal{F}[n] \quad \text{and} \quad R^q f'_*(g')^{-1}\mathcal{F} = \operatorname{colim} R^q f'_*(g')^{-1}\mathcal{F}[n]$$

Thus it suffices to prove the result in case $\mathcal{F}$ is annihilated by a positive integer n.

If $n = \ell n'$ for some prime number ℓ, then we obtain a short exact sequence

$$0 \to \mathcal{F}[\ell] \to \mathcal{F} \to \mathcal{F}/\mathcal{F}[\ell] \to 0$$

Observe that $\mathcal{F}/\mathcal{F}[\ell]$ is annihilated by n'. Moreover, if the result holds for both $\mathcal{F}[\ell]$ and $\mathcal{F}/\mathcal{F}[\ell]$, then the result holds by the long exact sequence of higher direct images (and the 5 lemma). In this way we reduce to the case that $\mathcal{F}$ is annihilated by a prime number ℓ.

Assume $\mathcal{F}$ is annihilated by a prime number ℓ. Choose an injective resolution $\mathcal{F} \to \mathcal{I}^\bullet$ in $D(X_{étale}, \mathbf{Z}/\ell\mathbf{Z})$. Applying assumption (2) and Leray's acyclicity lemma (Derived Categories, Lemma 16.7) we see that

$$f'_*(g')^{-1}\mathcal{I}^\bullet$$

computes $Rf'_*(g')^{-1}\mathcal{F}$. We conclude by applying Lemma 87.5. □

0A4C **Lemma 87.7.** *Let $f : X \to Y$ and $g : Y \to Z$ be proper morphisms of schemes. Assume*

1. *cohomology commutes with base change for f,*
2. *cohomology commutes with base change for $g \circ f$, and*
3. *f is surjective.*

Then cohomology commutes with base change for g.

Proof. We will use the equivalence of Lemma 87.6 without further mention. Let ℓ be a prime number. Let $\mathcal{I}$ be an injective sheaf of $\mathbf{Z}/\ell\mathbf{Z}$-modules on $Y_{étale}$. Choose an injective map of sheaves $f^{-1}\mathcal{I} \to \mathcal{J}$ where $\mathcal{J}$ is an injective sheaf of $\mathbf{Z}/\ell\mathbf{Z}$-modules on $Z_{étale}$. Since f is surjective the map $\mathcal{I} \to f_*\mathcal{J}$ is injective (look at stalks in geometric points). Since $\mathcal{I}$ is injective we see that $\mathcal{I}$ is a direct summand of $f_*\mathcal{J}$. Thus it suffices to prove the desired vanishing for $f_*\mathcal{J}$.

Let $Z' \to Z$ be a morphism of schemes and set $Y' = Z' \times_Z Y$ and $X' = Z' \times_Z X = Y' \times_Y X$. Denote $a : X' \to X$, $b : Y' \to Y$, and $c : Z' \to Z$ the projections. Similarly for $f' : X' \to Y'$ and $g' : Y' \to Z'$. By Lemma 87.5 we have $b^{-1}f_*\mathcal{J} = f'_*a^{-1}\mathcal{J}$. On the other hand, we know that $R^q f'_*a^{-1}\mathcal{J}$ and $R^q(g' \circ f')_*a^{-1}\mathcal{J}$ are zero for $q > 0$. Using the spectral sequence (Cohomology on Sites, Lemma 14.7)

$$R^p g'_* R^q f'_* a^{-1}\mathcal{J} \Rightarrow R^{p+q}(g' \circ f')_* a^{-1}\mathcal{J}$$

we conclude that $R^p g'_*(b^{-1}f_*\mathcal{J}) = R^p g'_*(f'_*a^{-1}\mathcal{J}) = 0$ for $p > 0$ as desired. □

0A4D **Lemma 87.8.** *Let $f : X \to Y$ and $g : Y \to Z$ be proper morphisms of schemes. Assume*

1. *cohomology commutes with base change for f, and*
2. *cohomology commutes with base change for g.*

Then cohomology commutes with base change for $g \circ f$.

Proof. We will use the equivalence of Lemma 87.6 without further mention. Let ℓ be a prime number. Let $\mathcal{I}$ be an injective sheaf of $\mathbf{Z}/\ell\mathbf{Z}$-modules on $X_{étale}$. Then $f_*\mathcal{I}$ is an injective sheaf of $\mathbf{Z}/\ell\mathbf{Z}$-modules on $Y_{étale}$ (Cohomology on Sites, Lemma 14.2). The result follows formally from this, but we will also spell it out.

Let $Z' \to Z$ be a morphism of schemes and set $Y' = Z' \times_Z Y$ and $X' = Z' \times_Z X = Y' \times_Y X$. Denote $a : X' \to X$, $b : Y' \to Y$, and $c : Z' \to Z$ the projections. Similarly for $f' : X' \to Y'$ and $g' : Y' \to Z'$. By Lemma 87.5 we have $b^{-1}f_*\mathcal{I} = f'_*a^{-1}\mathcal{I}$. On the other hand, we know that $R^qf'_*a^{-1}\mathcal{I}$ and $R^q(g')_*b^{-1}f_*\mathcal{I}$ are zero for $q > 0$. Using the spectral sequence (Cohomology on Sites, Lemma 14.7)

$$R^pg'_*R^qf'_*a^{-1}\mathcal{I} \Rightarrow R^{p+q}(g' \circ f')_*a^{-1}\mathcal{I}$$

we conclude that $R^p(g' \circ f')_*a^{-1}\mathcal{I} = 0$ for $p > 0$ as desired. $\square$

0A4E **Lemma 87.9.** *Let $f : X \to Y$ be a finite morphism of schemes. Then cohomology commutes with base change for f.*

Proof. Observe that a finite morphism is proper, see Morphisms, Lemma 42.11. Moreover, the base change of a finite morphism is finite, see Morphisms, Lemma 42.6. Thus the result follows from Lemma 87.6 combined with Proposition 54.2. $\square$

0A4F **Lemma 87.10.** *To prove that cohomology commutes with base change for every proper morphism of schemes it suffices to prove it holds for the morphism $\mathbf{P}^1_S \to S$ for every scheme S.*

Proof. Let $f : X \to Y$ be a proper morphism of schemes. Let $Y = \bigcup Y_i$ be an affine open covering and set $X_i = f^{-1}(Y_i)$. If we can prove cohomology commutes with base change for $X_i \to Y_i$, then cohomology commutes with base change for f. Namely, the formation of the higher direct images commutes with Zariski (and even étale) localization on the base, see Lemma 51.6. Thus we may assume Y is affine.

Let Y be an affine scheme and let $X \to Y$ be a proper morphism. By Chow's lemma there exists a commutative diagram

$$\begin{tikzcd} X \arrow[dr] & X' \arrow[l, "\pi"'] \arrow[r] \arrow[d] & \mathbf{P}^n_Y \arrow[dl] \\ & Y & \end{tikzcd}$$

where $X' \to \mathbf{P}^n_Y$ is an immersion, and $\pi : X' \to X$ is proper and surjective, see Limits, Lemma 12.1. Since $X \to Y$ is proper, we find that $X' \to Y$ is proper (Morphisms, Lemma 39.4). Hence $X' \to \mathbf{P}^n_Y$ is a closed immersion (Morphisms, Lemma 39.7). It follows that $X' \to X \times_Y \mathbf{P}^n_Y = \mathbf{P}^n_X$ is a closed immersion (as an immersion with closed image).

By Lemma 87.7 it suffices to prove cohomology commutes with base change for π and $X' \to Y$. These morphisms both factor as a closed immersion followed by

a projection $\mathbf{P}^n_S \to S$ (for some S). By Lemma 87.9 the result holds for closed immersions (as closed immersions are finite). By Lemma 87.8 it suffices to prove the result for projections $\mathbf{P}^n_S \to S$.

For every $n \geq 1$ there is a finite surjective morphism

$$\mathbf{P}^1_S \times_S \ldots \times_S \mathbf{P}^1_S \longrightarrow \mathbf{P}^n_S$$

given on coordinates by

$$((x_1 : y_1), (x_2 : y_2), \ldots, (x_n : y_n)) \longmapsto (F_0 : \ldots : F_n)$$

where $F_0, \ldots, F_n$ in $x_1, \ldots, y_n$ are the polynomials with integer coefficients such that

$$\prod (x_i t + y_i) = F_0 t^n + F_1 t^{n-1} + \ldots + F_n$$

Applying Lemmas 87.7, 87.9, and 87.8 one more time we conclude that the lemma is true. □

095T **Theorem 87.11.** *Let $f : X \to Y$ be a proper morphism of schemes. Let $g : Y' \to Y$ be a morphism of schemes. Set $X' = Y' \times_Y X$ and consider the cartesian diagram*

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

Let $\mathcal{F}$ be an abelian torsion sheaf on $X_{étale}$. Then the base change map

$$g^{-1} Rf_* \mathcal{F} \longrightarrow Rf'_*(g')^{-1}\mathcal{F}$$

is an isomorphism.

Proof. In the terminology introduced above, this means that cohomology commutes with base change for every proper morphism of schemes. By Lemma 87.10 it suffices to prove that cohomology commutes with base change for the morphism $\mathbf{P}^1_S \to S$ for every scheme S.

Let S be the spectrum of a strictly henselian local ring with closed point s. Set $X = \mathbf{P}^1_S$ and $X_0 = X_s = \mathbf{P}^1_s$. Let $\mathcal{F}$ be a sheaf of $\mathbf{Z}/\ell\mathbf{Z}$-modules on $X_{étale}$. The key to our proof is that

$$H^q_{étale}(X, \mathcal{F}) = H^q_{étale}(X_0, \mathcal{F}|_{X_0}).$$

Namely, choose a resolution $\mathcal{F} \to \mathcal{I}^\bullet$ by injective sheaves of $\mathbf{Z}/\ell\mathbf{Z}$-modules. Then $\mathcal{I}^\bullet|_{X_0}$ is a resolution of $\mathcal{F}|_{X_0}$ by right $H^0_{étale}(X_0, -)$-acyclic objects, see Lemma 81.2. Leray's acyclicity lemma tells us the right hand side is computed by the complex $H^0_{étale}(X_0, \mathcal{I}^\bullet|_{X_0})$ which is equal to $H^0_{étale}(X, \mathcal{I}^\bullet)$ by Lemma 87.3. This complex computes the left hand side.

Assume S is general and $\mathcal{F}$ is a sheaf of $\mathbf{Z}/\ell\mathbf{Z}$-modules on $X_{étale}$. Let $\overline{s} : \mathrm{Spec}(k) \to S$ be a geometric point of S lying over $s \in S$. We have

$$(R^q f_* \mathcal{F})_{\overline{s}} = H^q_{étale}(\mathbf{P}^1_{\mathcal{O}^{sh}_{S,\overline{s}}}, \mathcal{F}|_{\mathbf{P}^1_{\mathcal{O}^{sh}_{S,\overline{s}}}}) = H^q_{étale}(\mathbf{P}^1_{\kappa(s)^{sep}}, \mathcal{F}|_{\mathbf{P}^1_{\kappa(s)^{sep}}})$$

where $\kappa(s)^{sep}$ is the residue field of $\mathcal{O}^{sh}_{S,\overline{s}}$, i.e., the separable algebraic closure of $\kappa(s)$ in k. The first equality by Theorem 52.1 and the second equality by the displayed formula in the previous paragraph.

Finally, consider any morphism of schemes $g : T \to S$ where S and $\mathcal{F}$ are as above. Set $f' : \mathbf{P}^1_T \to T$ the projection and let $g' : \mathbf{P}^1_T \to \mathbf{P}^1_S$ the morphism induced by g. Consider the base change map

$$g^{-1}R^q f_*\mathcal{F} \longrightarrow R^q f'_*(g')^{-1}\mathcal{F}$$

Let $\overline{t}$ be a geometric point of T with image $\overline{s} = g(\overline{t})$. By our discussion above the map on stalks at $\overline{t}$ is the map

$$H^q_{étale}(\mathbf{P}^1_{\kappa(s)^{sep}}, \mathcal{F}|_{\mathbf{P}^1_{\kappa(s)^{sep}}}) \longrightarrow H^q_{étale}(\mathbf{P}^1_{\kappa(t)^{sep}}, \mathcal{F}|_{\mathbf{P}^1_{\kappa(t)^{sep}}})$$

Since $\kappa(s)^{sep} \subset \kappa(t)^{sep}$ this map is an isomorphism by Lemma 80.11.

This proves cohomology commutes with base change for $\mathbf{P}^1_S \to S$ and sheaves of $\mathbf{Z}/\ell\mathbf{Z}$-modules. In particular, for an injective sheaf of $\mathbf{Z}/\ell\mathbf{Z}$-modules the higher direct images of any base change are zero. In other words, condition (2) of Lemma 87.6 holds and the proof is complete. $\square$

0DDE **Lemma 87.12.** *Let $f : X \to Y$ be a proper morphism of schemes. Let $g : Y' \to Y$ be a morphism of schemes. Set $X' = Y' \times_Y X$ and denote $f' : X' \to Y'$ and $g' : X' \to X$ the projections. Let $E \in D^+(X_{étale})$ have torsion cohomology sheaves. Then the base change map (87.5.2) $g^{-1}Rf_*E \to Rf'_*(g')^{-1}E$ is an isomorphism.*

Proof. This is a simple consequence of the proper base change theorem (Theorem 87.11) using the spectral sequences

$$E_2^{p,q} = R^p f_* H^q(E) \quad \text{and} \quad E'^{p,q}_2 = R^p f'_*(g')^{-1}H^q(E)$$

converging to $R^n f_*E$ and $R^n f'_*(g')^{-1}E$. The spectral sequences are constructed in Derived Categories, Lemma 21.3. Some details omitted. $\square$

0DDF **Lemma 87.13.** *Let $f : X \to Y$ be a proper morphism of schemes. Let $\overline{y} \to Y$ be a geometric point.*

1. *For a torsion abelian sheaf $\mathcal{F}$ on $X_{étale}$ we have $(R^n f_*\mathcal{F})_{\overline{y}} = H^n_{étale}(X_{\overline{y}}, \mathcal{F}_{\overline{y}})$.*
2. *For $E \in D^+(X_{étale})$ with torsion cohomology sheaves we have $(R^n f_*E)_{\overline{y}} = H^n_{étale}(X_{\overline{y}}, E_{\overline{y}})$.*

Proof. In the statement, $\mathcal{F}_{\overline{y}}$ denotes the pullback of $\mathcal{F}$ to the scheme theoretic fibre $X_{\overline{y}} = \overline{y} \times_Y X$. Since pulling back by $\overline{y} \to Y$ produces the stalk of $\mathcal{F}$, the first statement of the lemma is a special case of Theorem 87.11. The second one is a special case of Lemma 87.12. $\square$

88. Applications of proper base change

0A5I In this section we discuss some more or less immediate consequences of the proper base change theorem.

0DDG **Lemma 88.1.** *Let K/k be an extension of separably closed fields. Let X be a proper scheme over k. Let $\mathcal{F}$ be a torsion abelian sheaf on $X_{étale}$. Then the map $H^q_{étale}(X, \mathcal{F}) \to H^q_{étale}(X_K, \mathcal{F}|_{X_K})$ is an isomorphism for $q \geq 0$.*

Proof. Looking at stalks we see that this is a special case of Theorem 87.11. $\square$

095U **Lemma 88.2.** *Let $f : X \to Y$ be a proper morphism of schemes all of whose fibres have dimension $\leq n$. Then for any abelian torsion sheaf $\mathcal{F}$ on $X_{étale}$ we have $R^q f_*\mathcal{F} = 0$ for $q > 2n$.*

Proof. We will prove this by induction on n for all proper morphisms.

If $n = 0$, then f is a finite morphism (More on Morphisms, Lemma 39.1) and the result is true by Proposition 54.2.

If $n > 0$, then using Lemma 87.13 we see that it suffices to prove $H^i_{étale}(X, \mathcal{F}) = 0$ for $i > 2n$ and X a proper scheme, $\dim(X) \leq n$ over an algebraically closed field k and $\mathcal{F}$ is a torsion abelian sheaf on X.

If $n = 1$ this follows from Theorem 80.10. Assume $n > 1$. By Proposition 45.4 we may replace X by its reduction. Let $\nu : X^\nu \to X$ be the normalization. This is a surjective birational finite morphism (see Varieties, Lemma 27.1) and hence an isomorphism over a dense open $U \subset X$ (Morphisms, Lemma 48.5). Then we see that $c : \mathcal{F} \to \nu_*\nu^{-1}\mathcal{F}$ is injective (as ν is surjective) and an isomorphism over U. Denote $i : Z \to X$ the inclusion of the complement of U. Since U is dense in X we have $\dim(Z) < \dim(X) = n$. By Proposition 46.4 have $\text{Coker}(c) = i_*\mathcal{G}$ for some abelian torsion sheaf $\mathcal{G}$ on $Z_{étale}$. Then $H^q_{étale}(X, \text{Coker}(c)) = H^q_{étale}(Z, \mathcal{F})$ (by Proposition 54.2 and the Leray spectral sequence) and by induction hypothesis we conclude that the cokernel of c has cohomology in degrees $\leq 2(n-1)$. Thus it suffices to prove the result for $\nu_*\nu^{-1}\mathcal{F}$. As ν is finite this reduces us to showing that $H^i_{étale}(X^\nu, \nu^{-1}\mathcal{F})$ is zero for $i > 2n$. This case is treated in the next paragraph.

Assume X is integral normal proper scheme over k of dimension n. Choose a nonconstant rational function f on X. The graph $X' \subset X \times \mathbf{P}^1_k$ of f sits into a diagram

$$X \xleftarrow{b} X' \xrightarrow{f} \mathbf{P}^1_k$$

Observe that b is an isomorphism over an open subscheme $U \subset X$ whose complement is a closed subscheme $Z \subset X$ of codimension ≥ 2. Namely, U is the domain of definition of f which contains all codimension 1 points of X, see Morphisms, Lemmas 47.9 and 40.5 (combined with Serre's criterion for normality, see Properties, Lemma 12.5). Moreover the fibres of b have dimension ≤ 1 (as closed subschemes of $\mathbf{P}^1$). Hence $R^ib_*b^{-1}\mathcal{F}$ is nonzero only if $i \in \{0, 1, 2\}$ by induction. Choose a distinguished triangle

$$\mathcal{F} \to Rb_*b^{-1}\mathcal{F} \to Q \to \mathcal{F}[1]$$

Using that $\mathcal{F} \to b_*b^{-1}\mathcal{F}$ is injective as before and using what we just said, we see that Q has nonzero cohomology sheaves only in degrees $0, 1, 2$ sitting on Z. Moreover, these cohomology sheaves are torsion by Lemma 76.2. By induction we see that $H^i(X, Q)$ is zero for $i > 2 + 2\dim(Z) \leq 2 + 2(n-2) = 2n - 2$. Thus it suffices to prove that $H^i(X', b^{-1}\mathcal{F}) = 0$ for $i > 2n$. At this point we use the morphism

$$f : X' \to \mathbf{P}^1_k$$

whose fibres have dimension $< n$. Hence by induction we see that $R^if_*b^{-1}\mathcal{F} = 0$ for $i > 2(n-1)$. We conclude by the Leray spectral seqence

$$H^i(\mathbf{P}^1_k, R^jf_*b^{-1}\mathcal{F}) \Rightarrow H^{i+j}(X', b^{-1}\mathcal{F})$$

and the fact that $\dim(\mathbf{P}^1_k) = 1$. $\square$

When working with mod n coefficients we can do proper base change for unbounded complexes.

0A3V **Lemma 88.3.** *Let $f : X \to Y$ be a morphism of finite type with Y quasi-compact. Then the dimension of the fibres of f is bounded.*

Proof. By Morphisms, Lemma 27.4 the set $U_n \subset X$ of points where the dimension of the fibre is $\leq n$ is open. Since f is of finite type, every point is contained in some U_n. Since Y is quasi-compact and f is of finite type, we see that X is quasi-compact. Hence $X = U_n$ for some n. $\square$

0F0C **Lemma 88.4.** *Let $f : X \to Y$ be a proper morphism of schemes. Let $g : Y' \to Y$ be a morphism of schemes. Set $X' = Y' \times_Y X$ and denote $f' : X' \to Y'$ and $g' : X' \to X$ the projections. Let $n \geq 1$ be an integer. Let $E \in D(X_{étale}, \mathbf{Z}/n\mathbf{Z})$. Then the base change map (87.5.2) $g^{-1}Rf_*E \to Rf'_*(g')^{-1}E$ is an isomorphism.*

Proof. It is enough to prove this when Y and Y' are quasi-compact. By Lemma 88.3 we see that the dimension of the fibres of $f : X \to Y$ and $f' : X' \to Y'$ are bounded. Thus Lemma 88.2 implies that

$$f_* : Mod(X_{étale}, \mathbf{Z}/n\mathbf{Z}) \longrightarrow Mod(Y_{étale}, \mathbf{Z}/n\mathbf{Z})$$

and

$$f'_* : Mod(X'_{étale}, \mathbf{Z}/n\mathbf{Z}) \longrightarrow Mod(Y'_{étale}, \mathbf{Z}/n\mathbf{Z})$$

have finite cohomological dimension in the sense of Derived Categories, Lemma 31.2. Choose a K-injective complex $\mathcal{I}^\bullet$ of $\mathbf{Z}/n\mathbf{Z}$-modules each of whose terms $\mathcal{I}^n$ is an injective sheaf of $\mathbf{Z}/n\mathbf{Z}$-modules representing E. See Injectives, Theorem 12.6. By the usual proper base change theorem we find that $R^qf'_*(g')^{-1}\mathcal{I}^n = 0$ for $q > 0$, see Theorem 87.11. Hence we conclude by Derived Categories, Lemma 31.2 that we may compute $Rf'_*(g')^{-1}E$ by the complex $f'_*(g')^{-1}\mathcal{I}^\bullet$. Another application of the usual proper base change theorem shows that this is equal to $g^{-1}f_*\mathcal{I}^\bullet$ as desired. $\square$

0F0E **Lemma 88.5.** *Let X be a quasi-compact and quasi-separated scheme. Let $E \in D^+(X_{étale})$ and $K \in D^+(\mathbf{Z})$. Then*

$$R\Gamma(X, E \otimes^{\mathbf{L}}_{\mathbf{Z}} \underline{K}) = R\Gamma(X, E) \otimes^{\mathbf{L}}_{\mathbf{Z}} K$$

Proof. Say $H^i(E) = 0$ for $i \geq a$ and $H^j(K) = 0$ for $j \geq b$. We may represent K by a bounded below complex $K^\bullet$ of torsion free $\mathbf{Z}$-modules. (Choose a K-flat complex $L^\bullet$ representing K and then take $K^\bullet = \tau_{\geq b-1}L^\bullet$. This works because $\mathbf{Z}$ has global dimension 1. See More on Algebra, Lemma 63.2.) We may represent E by a bounded below complex $\mathcal{E}^\bullet$. Then $E \otimes^{\mathbf{L}}_{\mathbf{Z}} \underline{K}$ is represented by

$$\mathrm{Tot}(\mathcal{E}^\bullet \otimes_{\mathbf{Z}} \underline{K}^\bullet)$$

Using distinguished triangles

$$\sigma_{\geq -b+n+1}K^\bullet \to K^\bullet \to \sigma_{\leq -b+n}K^\bullet$$

and the trivial vanishing

$$H^n(X, \mathrm{Tot}(\mathcal{E}^\bullet \otimes_{\mathbf{Z}} \sigma_{\geq -a+n+1}\underline{K}^\bullet) = 0$$

and

$$H^n(R\Gamma(X, E) \otimes^{\mathbf{L}}_{\mathbf{Z}} \sigma_{\geq -a+n+1}K^\bullet) = 0$$

we reduce to the case where $K^\bullet$ is a bounded complex of flat $\mathbf{Z}$-modules. Repeating the argument we reduce to the case where $K^\bullet$ is equal to a single flat $\mathbf{Z}$-module sitting in some degree. Next, using the stupid trunctions for $\mathcal{E}^\bullet$ we reduce in exactly the same manner to the case where $\mathcal{E}^\bullet$ is a single abelian sheaf sitting in some degree. Thus it suffices to show that

$$H^n(X, \mathcal{E} \otimes_{\mathbf{Z}} \underline{M}) = H^n(X, \mathcal{E}) \otimes_{\mathbf{Z}} M$$

when M is a flat $\mathbf{Z}$-module and $\mathcal{E}$ is an abelian sheaf on X. In this case we write M is a filtered colimit of finite free $\mathbf{Z}$-modules (Lazard's theorem, see Algebra, Theorem 80.4). By Theorem 51.3 this reduces us to the case of finite free $\mathbf{Z}$-module M in which case the result is trivially true. □

0F0F **Lemma 88.6.** *Let $f : X \to Y$ be a proper morphism of schemes. Let $E \in D^+(X_{\acute{e}tale})$ have torsion cohomology sheaves. Let $K \in D^+(Y_{\acute{e}tale})$. Then*

$$Rf_*E \otimes_{\mathbf{Z}}^{\mathbf{L}} K = Rf_*(E \otimes_{\mathbf{Z}}^{\mathbf{L}} f^{-1}K)$$

in $D^+(Y_{\acute{e}tale})$.

Proof. There is a canonical map from left to right by Cohomology on Sites, Section 45. We will check the equality on stalks. Recall that computing derived tensor products commutes with pullbacks. See Cohomology on Sites, Lemma 18.4. Thus we have

$$(E \otimes_{\mathbf{Z}}^{\mathbf{L}} f^{-1}K)_{\overline{x}} = E_{\overline{x}} \otimes_{\mathbf{Z}}^{\mathbf{L}} K_{\overline{y}}$$

where $\overline{y}$ is the image of $\overline{x}$ in Y. Since $\mathbf{Z}$ has global dimension 1 we see that this complex has vanishing cohomology in degree $< -1 + a + b$ if $H^i(E) = 0$ for $i \geq a$ and $H^j(K) = 0$ for $j \geq b$. Moreover, since $H^i(E)$ is a torsion abelian sheaf for each i, the same is true for the cohomology sheaves of the complex $E \otimes_{\mathbf{Z}}^{\mathbf{L}} K$. Namely, we have

$$(E \otimes_{\mathbf{Z}}^{\mathbf{L}} f^{-1}K) \otimes_{\mathbf{Z}}^{\mathbf{L}} \mathbf{Q} = (E \otimes_{\mathbf{Z}}^{\mathbf{L}} \mathbf{Q}) \otimes_{\mathbf{Q}}^{\mathbf{L}} (f^{-1}K \otimes_{\mathbf{Z}}^{\mathbf{L}} \mathbf{Q})$$

which is zero in the derived category. In this way we see that Lemma 87.13 applies to both sides to see that it suffices to show

$$R\Gamma(X_{\overline{y}}, E|_{X_{\overline{y}}} \otimes_{\mathbf{Z}}^{\mathbf{L}} (X_{\overline{y}} \to \overline{y})^{-1}K_{\overline{y}}) = R\Gamma(X_{\overline{y}}, E|_{X_{\overline{y}}}) \otimes_{\mathbf{Z}}^{\mathbf{L}} K_{\overline{y}}$$

This is shown in Lemma 88.5. □

89. Cohomological dimension

0F0P We can deduce some bounds on the cohomological dimension of schemes and on the cohomological dimension of fields using the results in Section 80 and one, seemingly innocuous, application of the proper base change theorem (in the proof of Proposition 89.6).

0F0Q **Definition 89.1.** Let X be a quasi-compact and quasi-separated scheme. The *cohomological dimension of* X is the smallest element

$$\text{cd}(X) \in \{0, 1, 2, \ldots\} \cup \{\infty\}$$

such that for any abelian torsion sheaf $\mathcal{F}$ on $X_{\acute{e}tale}$ we have $H^i_{\acute{e}tale}(X, \mathcal{F}) = 0$ for $i > \text{cd}(X)$. If $X = \text{Spec}(A)$ we sometimes call this the cohomological dimension of A.

If the scheme is in characteristic p, then we often can obtain sharper bounds for the vanishing of cohomology of p-power torsion sheaves. We will address this elsewhere (insert future reference here).

0F0R **Lemma 89.2.** *Let $X = \lim X_i$ be a directed limit of a system of quasi-compact and quasi-separated schemes with affine transition morphisms. Then $cd(X) \leq \max cd(X_i)$.*

Proof. Denote $f_i : X \to X_i$ the projections. Let $\mathcal{F}$ be an abelian torsion sheaf on $X_{\acute{e}tale}$. Then we have $\mathcal{F} = \lim f_i^{-1} f_{i,*}\mathcal{F}$ by Lemma 51.9. Thus $H^q_{\acute{e}tale}(X, \mathcal{F}) = \text{colim}\, H^q_{\acute{e}tale}(X_i, f_{i,*}\mathcal{F})$ by Theorem 51.3. The lemma follows. $\square$

0F0S **Lemma 89.3.** *Let K be a field. Let X be a 1-dimensional affine scheme of finite type over K. Then $cd(X) \leq 1 + cd(K)$.*

Proof. Let $\mathcal{F}$ be an abelian torsion sheaf on $X_{\acute{e}tale}$. Consider the Leray spectral sequence for the morphism $f : X \to \text{Spec}(K)$. We obtain

$$E_2^{p,q} = H^p(\text{Spec}(K), R^q f_*\mathcal{F})$$

converging to $H^{p+q}_{\acute{e}tale}(X, \mathcal{F})$. The stalk of $R^q f_*\mathcal{F}$ at a geometric point $\text{Spec}(\overline{K}) \to \text{Spec}(K)$ is the cohomology of the pullback of $\mathcal{F}$ to $X_{\overline{K}}$. Hence it vanishes in degrees ≥ 2 by Theorem 80.9. $\square$

0F0T **Lemma 89.4.** *Let L/K be a field extension. Then we have $cd(L) \leq cd(K) + trdeg_K(L)$.*

Proof. If $\text{trdeg}_K(L) = \infty$, then this is clear. If not then we can find a sequence of extensions $L = L_r/L_{r-1}/\ldots/L_1/L_0 = K$ such that $\text{trdeg}_{L_i}(L_{i+1}) = 1$ and $r = \text{trdeg}_K(L)$. Hence it suffices to prove the lemma in the case that $r = 1$. In this case we can write $L = \text{colim}\, A_i$ as a filtered colimit of its finite type K-subalgebras. By Lemma 89.2 it suffices to prove that $\text{cd}(A_i) \leq 1 + \text{cd}(K)$. This follows from Lemma 89.3. $\square$

0F0U **Lemma 89.5.** *Let K be a field. Let X be a scheme of finite type over K. Let $x \in X$. Set $a = trdeg_K(\kappa(x))$ and $d = \dim_x(X)$. Then there is a map*

$$K(t_1, \ldots, t_a)^{sep} \longrightarrow \mathcal{O}^{sh}_{X,x}$$

such that

1. *the residue field of $\mathcal{O}^{sh}_{X,x}$ is a purely inseparable extension of $K(t_1, \ldots, t_a)^{sep}$,*
2. *$\mathcal{O}^{sh}_{X,x}$ is a filtered colimit of finite type $K(t_1, \ldots, t_a)^{sep}$-algebras of dimension $\leq d - a$.*

Proof. We may assume X is affine. By Noether normalization, after possibly shrinking X again, we can choose a finite morphism $\pi : X \to \mathbf{A}^d_K$, see Algebra, Lemma 114.5. Since $\kappa(x)$ is a finite extension of the residue field of $\pi(x)$, this residue field has transcendence degree a over K as well. Thus we can find a finite morphism $\pi' : \mathbf{A}^d_K \to \mathbf{A}^d_K$ such that $\pi'(\pi(x))$ corresponds to the generic point of the linear subspace $\mathbf{A}^a_K \subset \mathbf{A}^d_K$ given by setting the last $d - a$ coordinates equal to zero. Hence the composition

$$X \xrightarrow{\pi' \circ \pi} \mathbf{A}^d_K \xrightarrow{p} \mathbf{A}^a_K$$

of $\pi' \circ \pi$ and the projection p onto the first a coordinates maps x to the generic point $\eta \in \mathbf{A}^a_K$. The induced map

$$K(t_1, \ldots, t_a)^{sep} = \mathcal{O}^{sh}_{\mathbf{A}^a_k, \eta} \longrightarrow \mathcal{O}^{sh}_{X,x}$$

on étale local rings satisfies (1) since it is clear that the residue field of $\mathcal{O}^{sh}_{X,x}$ is an algebraic extension of the separably closed field $K(t_1, \ldots, t_a)^{sep}$. On the other hand, if $X = \text{Spec}(B)$, then $\mathcal{O}^{sh}_{X,x} = \text{colim}\, B_j$ is a filtered colimit of étale B-algebras B_j. Observe that B_j is quasi-finite over $K[t_1, \ldots, t_d]$ as B is finite

over $K[t_1, \ldots, t_d]$. We may similarly write $K(t_1, \ldots, t_a)^{sep} = \text{colim}\, A_i$ as a filtered colimit of étale $K[t_1, \ldots, t_a]$-algebras. For every i we can find an j such that $A_i \to K(t_1, \ldots, t_a)^{sep} \to \mathcal{O}_{X,x}^{sh}$ factors through a map $\psi_{i,j} : A_i \to B_j$. Then B_j is quasi-finite over $A_i[t_{a+1}, \ldots, t_d]$. Hence

$$B_{i,j} = B_j \otimes_{\psi_{i,j}, A_i} K(t_1, \ldots, t_a)^{sep}$$

has dimension $\leq d - a$ as it is quasi-finite over $K(t_1, \ldots, t_a)^{sep}[t_{a+1}, \ldots, t_d]$. The proof of (2) is now finished as $\mathcal{O}_{X,x}^{sh}$ is a filtered colimit[9] of the algebras $B_{i,j}$. Some details omitted. □

0F0V **Proposition 89.6.** *Let K be a field. Let X be an affine scheme of finite type over K. Then we have $cd(X) \leq \dim(X) + cd(K)$.*

Proof. We will prove this by induction on $\dim(X)$. Let $\mathcal{F}$ be an abelian torsion sheaf on $X_{\acute{e}tale}$.

The case $\dim(X) = 0$. In this case the structure morphism $f : X \to \text{Spec}(K)$ is finite. Hence we see that $R^i f_* \mathcal{F} = 0$ for $i > 0$, see Proposition 54.2. Thus $H^i_{\acute{e}tale}(X, \mathcal{F}) = H^i_{\acute{e}tale}(\text{Spec}(K), f_*\mathcal{F})$ by the Leray spectral sequence for f (Cohomology on Sites, Lemma 14.5) and the result is clear.

The case $\dim(X) = 1$. This is Lemma 89.3.

Assume $d = \dim(X) > 1$ and the proposition holds for finite type affine schemes of dimension $< d$ over fields. By Noether normalization, see for example Varieties, Lemma 18.2, there exists a finite morphism $f : X \to \mathbf{A}^d_K$. Recall that $R^i f_* \mathcal{F} = 0$ for $i > 0$ by Proposition 54.2. By the Leray spectral sequence for f (Cohomology on Sites, Lemma 14.5) we conclude that it suffices to prove the result for $\pi_* \mathcal{F}$ on $\mathbf{A}^d_K$.

Interlude I. Let $j : X \to Y$ be an open immersion of smooth d-dimensional varieties over K (not necessarily affine) whose complement is the support of an effective Cartier divisor D. The sheaves $R^q j_* \mathcal{F}$ for $q > 0$ are supported on D. We claim that $(R^q j_* \mathcal{F})_{\overline{y}} = 0$ for $a = \text{trdeg}_K(\kappa(y)) > d - q$. Namely, by Theorem 52.1 we have

$$(R^q j_* \mathcal{F})_{\overline{y}} = H^q(\text{Spec}(\mathcal{O}_{Y,y}^{sh}) \times_Y X, \mathcal{F})$$

Choose a local equation $f \in \mathfrak{m}_y = \mathcal{O}_{Y,y}$ for D. Then we have

$$\text{Spec}(\mathcal{O}_{Y,y}^{sh}) \times_Y X = \text{Spec}(\mathcal{O}_{Y,y}^{sh}[1/f])$$

Using Lemma 89.5 we get an embedding

$$K(t_1, \ldots, t_a)^{sep}(x) = K(t_1, \ldots, t_a)^{sep}[x]_{(x)}[1/x] \longrightarrow \mathcal{O}_{Y,y}^{sh}[1/f]$$

Since the transcendence degree over K of the fraction field of $\mathcal{O}_{Y,y}^{sh}$ is d, we see that $\mathcal{O}_{Y,y}^{sh}[1/f]$ is a filtered colimit of $(d-a-1)$-dimensional finite type algebras over the field $K(t_1, \ldots, t_a)^{sep}(x)$ which itself has cohomological dimension 1 by Lemma 89.4. Thus by induction hypothesis and Lemma 89.2 we obtain the desired vanishing.

[9]Let R be a ring. Let $A = \text{colim}_{i \in I} A_i$ be a filtered colimit of finitely presented R-algebras. Let $B = \text{colim}_{j \in J} B_j$ be a filtered colimit of R-algebras. Let $A \to B$ be an R-algebra map. Assume that for all $i \in I$ there is a $j \in J$ and an R-algebra map $\psi_{i,j} : A_i \to B_j$. Say $(i', j', \psi_{i',j'}) \geq (i, j, \psi_{i,j})$ if $i' \geq i$, $j' \geq j$, and $\psi_{i,j}$ and $\psi_{i',j'}$ are compatible. Then the collection of triples forms a directed set and $B = \text{colim}\, B_j \otimes_{\psi_{i,j} A_i} A$.

Interlude II. Let Z be a smooth variety over K of dimension $d-1$. Let $E_a \subset Z$ be the set of points $z \in Z$ with $\text{trdeg}_K(\kappa(z)) \leq a$. Observe that E_a is closed under specialization, see Varieties, Lemma 20.3. Suppose that $\mathcal{G}$ is a torsion abelian sheaf on Z whose support is contained in E_a. Then we claim that $H^b_{\acute{e}tale}(Z, \mathcal{G}) = 0$ for $b > a + \text{cd}(K)$. Namely, we can write $\mathcal{G} = \text{colim}\, \mathcal{G}_i$ with $\mathcal{G}_i$ a torsion abelian sheaf supported on a closed subscheme Z_i contained in E_a, see Lemma 73.5. Then the induction hypothesis kicks in to imply the desired vanishing for $\mathcal{G}_i$[10]. Finally, we conclude by Theorem 51.3.

Consider the commutative diagram

$$\begin{tikzcd} \mathbf{A}^d_K \arrow[rr, "j"] \arrow[dr, "f"'] & & \mathbf{P}^1_K \times_K \mathbf{A}^{d-1}_K \arrow[dl, "g"] \\ & \mathbf{A}^{d-1}_K & \end{tikzcd}$$

Observe that j is an open immersion of smooth d-dimensional varieties whose complement is an effective Cartier divisor D. Thus we may use the results obtained in interlude I. We are going to study the relative Leray spectral sequence

$$E_2^{p,q} = R^p g_* R^q j_* \mathcal{F} \Rightarrow R^{p+q} f_* \mathcal{F}$$

Since $R^q j_* \mathcal{F}$ for $q > 0$ is supported on D and since $g|_D : D \to \mathbf{A}^{d-1}_K$ is an isomorphism, we find $R^p g_* R^q j_* \mathcal{F} = 0$ for $p > 0$ and $q > 0$. Moreover, we have $R^q j_* \mathcal{F} = 0$ for $q > d$. On the other hand, g is a proper morphism of relative dimension 1. Hence by Lemma 88.2 we see that $R^p j_* \mathcal{F} = 0$ for $p > 2$. Thus the E_2-page of the spectral sequence looks like this

$$\begin{matrix} g_* R^d j_* \mathcal{F} & 0 & 0 \\ \ldots & \ldots & \ldots \\ g_* R^2 j_* \mathcal{F} & 0 & 0 \\ g_* R^1 j_* \mathcal{F} & 0 & 0 \\ g_* j_* \mathcal{F} & R^1 g_* j_* \mathcal{F} & R^2 g_* j_* \mathcal{F} \end{matrix}$$

We conclude that $R^q f_* \mathcal{F} = g_* R^q j_* \mathcal{F}$ for $q > 2$. By interlude I we see that the support of $R^q f_* \mathcal{F}$ for $q > 2$ is contained in the set of points of $\mathbf{A}^{d-1}_K$ whose residue field has transcendence degree $\leq d - q$. By interlude II

$$H^p(\mathbf{A}^{d-1}_K, R^q f_* \mathcal{F}) = 0 \text{ for } p > d - q + \text{cd}(K) \text{ and } q > 2$$

On the other hand, by Theorem 52.1 we have $R^2 f_* \mathcal{F}_{\overline{\eta}} = H^2(\mathbf{A}^1_{\overline{\eta}}, \mathcal{F}) = 0$ (vanishing by the case of dimension 1). Hence by interlude II again we see

$$H^p(\mathbf{A}^{d-1}_K, R^2 f_* \mathcal{F}) = 0 \text{ for } p > d - 2 + \text{cd}(K)$$

Finally, we have

$$H^p(\mathbf{A}^{d-1}_K, R^q f_* \mathcal{F}) = 0 \text{ for } p > d - 1 + \text{cd}(K) \text{ and } q = 0, 1$$

by induction hypothesis. Combining everything we just said with the Leray spectral sequence $H^p(\mathbf{A}^{d-1}_K, R^q f_* \mathcal{F}) \Rightarrow H^{p+q}(\mathbf{A}^d_K, \mathcal{F})$ we conclude. □

[10]Here we first use Proposition 46.4 to write $\mathcal{G}_i$ as the pushforward of a sheaf on Z_i, the induction hypothesis gives the vanishing for this sheaf on Z_i, and the Leray spectral sequence for $Z_i \to Z$ gives the vanishing for $\mathcal{G}_i$.

0F0W **Lemma 89.7.** *Let K be a field. Let X be an affine scheme of finite type over K. Let $E_a \subset X$ be the set of points $x \in X$ with $trdeg_K(\kappa(x)) \leq a$. Let $\mathcal{F}$ be an abelian torsion sheaf on $X_{étale}$ whose support is contained in E_a. Then $H^b_{étale}(X, \mathcal{F}) = 0$ for $b > a + cd(K)$.*

Proof. We can write $\mathcal{F} = \text{colim}\, \mathcal{F}_i$ with $\mathcal{F}_i$ a torsion abelian sheaf supported on a closed subscheme Z_i contained in E_a, see Lemma 73.5. Then Proposition 89.6 gives the desired vanishing for $\mathcal{F}_i$. Details omitted; hints: first use Proposition 46.4 to write $\mathcal{F}_i$ as the pushforward of a sheaf on Z_i, use the vanishing for this sheaf on Z_i, and use the Leray spectral sequence for $Z_i \to Z$ to get the vanishing for $\mathcal{F}_i$. Finally, we conclude by Theorem 51.3. $\square$

0F0X **Lemma 89.8.** *Let $f : X \to Y$ be an affine morphism of schemes of finite type over a field K. Let $E_a(X)$ be the set of points $x \in X$ with $trdeg_K(\kappa(x)) \leq a$. Let $\mathcal{F}$ be an abelian torsion sheaf on $X_{étale}$ whose support is contained in E_a. Then $R^q f_* \mathcal{F}$ has support contained in $E_{a-q}(Y)$.*

Proof. The question is local on Y hence we can assume Y is affine. Then X is affine too and we can choose a diagram

$$\begin{tikzcd} X \arrow[r, "i"'] \arrow[d, "f"'] & \mathbf{A}^{n+m}_K \arrow[d, "\text{pr}"] \\ Y \arrow[r, "j"] & \mathbf{A}^n_K \end{tikzcd}$$

where the horizontal arrows are closed immersions and the vertical arrow on the right is the projection (details omitted). Then $j_* R^q f_* \mathcal{F} = R^q \text{pr}_* i_* \mathcal{F}$ by the vanishing of the higher direct images of i and j, see Proposition 54.2. Moreover, the description of the stalks of j_* in the proposition shows that it suffices to prove the vanishing for $j_* R^q f_* \mathcal{F}$. Thus we may assume f is the projection morphism $\text{pr} : \mathbf{A}^{n+m}_K \to \mathbf{A}^n_K$ and an abelian torsion sheaf $\mathcal{F}$ on $\mathbf{A}^{n+m}_K$ satisfying the assumption in the statement of the lemma.

Let y be a point in $\mathbf{A}^n_K$. By Theorem 52.1 we have

$$(R^q \text{pr}_* \mathcal{F})_{\overline{y}} = H^q(\mathbf{A}^{n+m}_K \times_{A^n_K} \text{Spec}(\mathcal{O}^{sh}_{Y,y}), \mathcal{F}) = H^q(\mathbf{A}^m_{\mathcal{O}^{sh}_{Y,y}}, \mathcal{F})$$

Say $b = \text{trdeg}_K(\kappa(y))$. From Lemma 89.5 we get an embedding

$$L = K(t_1, \ldots, t_b)^{sep} \longrightarrow \mathcal{O}^{sh}_{Y,y}$$

Write $\mathcal{O}^{sh}_{Y,y} = \text{colim}\, B_i$ as the filtered colimit of finite type L-subalgebras $B_i \subset \mathcal{O}^{sh}_{Y,y}$ containing the ring $K[T_1, \ldots, T_n]$ of regular functions on $\mathbf{A}^n_K$. Then we get

$$\mathbf{A}^m_{\mathcal{O}^{sh}_{Y,y}} = \lim \mathbf{A}^m_{B_i}$$

If $z \in \mathbf{A}^m_{B_i}$ is a point in the support of $\mathcal{F}$, then the image x of z in $\mathbf{A}^{m+n}_K$ satisfies $\text{trdeg}_K(\kappa(x)) \leq a$ by our assumption on $\mathcal{F}$ in the lemma. Since $\mathcal{O}^{sh}_{Y,y}$ is a filtered colimit of étale algebras over $K[T_1, \ldots, T_n]$ and since $B_i \subset \mathcal{O}^{sh}_{Y,y}$ we see that $\kappa(z)/\kappa(x)$ is algebraic (some details omitted). Then $\text{trdeg}_K(\kappa(z)) \leq a$ and hence $\text{trdeg}_L(\kappa(z)) \leq a - b$. By Lemma 89.7 we see that

$$H^q(\mathbf{A}^m_{B_i}, \mathcal{F}) = 0 \text{ for } q > a - b$$

Thus by Theorem 51.3 we get $(Rf_* \mathcal{F})_{\overline{y}} = 0$ for $q > a - b$ as desired. $\square$

90. Finite cohomological dimension

0F0Y We continue the discussion started in Section 89.

0F0Z **Definition 90.1.** Let $f : X \to Y$ be a quasi-compact and quasi-separated morphism of schemes. The *cohomological dimension of f* is the smallest element

$$\mathrm{cd}(f) \in \{0, 1, 2, \ldots\} \cup \{\infty\}$$

such that for any abelian torsion sheaf $\mathcal{F}$ on $X_{\acute{e}tale}$ we have $R^i f_* \mathcal{F} = 0$ for $i > \mathrm{cd}(f)$.

0F10 **Lemma 90.2.** *Let K be a field.*

(1) *If $f : X \to Y$ is a morphism of finite type schemes over K, then $cd(f) < \infty$.*
(2) *If $cd(K) < \infty$, then $cd(X) < \infty$ for any finite type scheme X over K.*

Proof. Proof of (1). We may assume Y is affine. We will use the induction principle of Cohomology of Schemes, Lemma 4.1 to prove this. If X is affine too, then the result holds by Lemma 89.8. Thus it suffices to show that if $X = U \cup V$ and the result is true for $U \to Y$, $V \to Y$, and $U \cap V \to Y$, then it is true for f. This follows from the relative Mayer-Vietoris sequence, see Lemma 50.2.

Proof of (2). We will use the induction principle of Cohomology of Schemes, Lemma 4.1 to prove this. If X is affine, then the result holds by Proposition 89.6. Thus it suffices to show that if $X = U \cup V$ and the result is true for U, V, and $U \cap V$, then it is true for X. This follows from the Mayer-Vietoris sequence, see Lemma 50.1. $\square$

0F11 **Lemma 90.3.** *Cohomology and direct sums. Let $n \geq 1$ be an integer.*

(1) *Let $f : X \to Y$ be a quasi-compact and quasi-separated morphism of schemes with $cd(f) < \infty$. Then the functor*
$$Rf_* : D(X_{\acute{e}tale}, \mathbf{Z}/n\mathbf{Z}) \longrightarrow D(Y_{\acute{e}tale}, \mathbf{Z}/n\mathbf{Z})$$
commutes with direct sums.
(2) *Let X be a quasi-compact and quasi-separated scheme with $cd(X) < \infty$. Then the functor*
$$R\Gamma(X, -) : D(X_{\acute{e}tale}, \mathbf{Z}/n\mathbf{Z}) \longrightarrow D(\mathbf{Z}/n\mathbf{Z})$$
commutes with direct sums.

Proof. Proof of (1). Since $\mathrm{cd}(f) < \infty$ we see that

$$f_* : Mod(X_{\acute{e}tale}, \mathbf{Z}/n\mathbf{Z}) \longrightarrow Mod(Y_{\acute{e}tale}, \mathbf{Z}/n\mathbf{Z})$$

has finite cohomological dimension in the sense of Derived Categories, Lemma 31.2. Let I be a set and for $i \in I$ let E_i be an object of $D(X_{\acute{e}tale}, \mathbf{Z}/n\mathbf{Z})$. Choose a K-injective complex $\mathcal{I}_i^\bullet$ of $\mathbf{Z}/n\mathbf{Z}$-modules each of whose terms $\mathcal{I}_i^n$ is an injective sheaf of $\mathbf{Z}/n\mathbf{Z}$-modules representing E_i. See Injectives, Theorem 12.6. Then $\bigoplus E_i$ is represented by the complex $\bigoplus \mathcal{I}_i^\bullet$ (termwise direct sum), see Injectives, Lemma 13.4. By Lemma 51.7 we have

$$R^q f_*(\bigoplus \mathcal{I}_i^n) = \bigoplus R^q f_*(\mathcal{I}_i^n) = 0$$

for $q > 0$ and any n. Hence we conclude by Derived Categories, Lemma 31.2 that we may compute $Rf_*(\bigoplus E_i)$ by the complex

$$f_*(\bigoplus \mathcal{I}_i^\bullet) = \bigoplus f_*(\mathcal{I}_i^\bullet)$$

(equality again by Lemma 51.7) which represents $\bigoplus Rf_*E_i$ by the already used Injectives, Lemma 13.4.

Proof of (2). This is identical to the proof of (1) and we omit it. $\square$

0F0D **Lemma 90.4.** *Let $f : X \to Y$ be a proper morphism of schemes. Let $n \geq 1$ be an integer. Then the functor*

$$Rf_* : D(X_{\acute{e}tale}, \mathbf{Z}/n\mathbf{Z}) \longrightarrow D(Y_{\acute{e}tale}, \mathbf{Z}/n\mathbf{Z})$$

commutes with direct sums.

Proof. It is enough to prove this when Y is quasi-compact. By Lemma 88.3 we see that the dimension of the fibres of $f : X \to Y$ is bounded. Thus Lemma 88.2 implies that $\text{cd}(f) < \infty$. Hence the result by Lemma 90.3. $\square$

0F12 **Lemma 90.5.** *Let X be a quasi-compact and quasi-separated scheme such that $cd(X) < \infty$. Let $n \geq 1$ be an integer. Let $E \in D(X_{\acute{e}tale}, \mathbf{Z}/n\mathbf{Z})$ and $K \in D(\mathbf{Z}/n\mathbf{Z})$. Then*

$$R\Gamma(X, E \otimes^{\mathbf{L}}_{\mathbf{Z}/n\mathbf{Z}} \underline{K}) = R\Gamma(X, E) \otimes^{\mathbf{L}}_{\mathbf{Z}/n\mathbf{Z}} K$$

Proof. There is a canonical map from left to right by Cohomology on Sites, Section 45. Let $T(K)$ be the property that the statement of the lemma holds for $K \in D(\mathbf{Z}/n\mathbf{Z})$. We will check conditions (1), (2), and (3) of More on Algebra, Remark 57.13 hold for T to conclude. Property (1) holds because both sides of the equality commute with direct sums, see Lemma 90.3. Property (2) holds because we are comparing exact functors between triangulated categories and we can use Derived Categories, Lemma 4.3. Property (3) says the lemma holds when $K = \mathbf{Z}/n\mathbf{Z}[k]$ for any shift $k \in \mathbf{Z}$ and this is obvious. $\square$

0F0G **Lemma 90.6.** *Let $f : X \to Y$ be a proper morphism of schemes. Let $n \geq 1$ be an integer. Let $E \in D(X_{\acute{e}tale}, \mathbf{Z}/n\mathbf{Z})$ and $K \in D(Y_{\acute{e}tale}, \mathbf{Z}/n\mathbf{Z})$. Then*

$$Rf_*E \otimes^{\mathbf{L}}_{\mathbf{Z}/n\mathbf{Z}} K = Rf_*(E \otimes^{\mathbf{L}}_{\mathbf{Z}/n\mathbf{Z}} f^{-1}K)$$

in $D(Y_{\acute{e}tale}, \mathbf{Z}/n\mathbf{Z})$.

Proof. There is a canonical map from left to right by Cohomology on Sites, Section 45. We will check the equality on stalks at $\overline{y}$. By the proper base change (in the form of Lemma 88.4 where $Y' = \overline{y}$) this reduces to the case where Y is the spectrum of an algebraically closed field. This is shown in Lemma 90.5 where we use that $\text{cd}(X) < \infty$ by Lemma 88.2. $\square$

91. Künneth in étale cohomology

0F13 We first prove a Künneth formula in case one of the factors is proper. Then we use this formula to prove a base change property for open immersions. This then gives a "base change by morphisms towards spectra of fields" (akin to smooth base change). Finally we use this to get a more general Künneth formula.

0F1E **Remark 91.1.** Consider a cartesian diagram in the category of schemes:

$$\begin{tikzcd}
X \times_S Y \arrow[r, "q"] \arrow[d, "p"] \arrow[dr, "c"] & Y \arrow[d, "g"] \\
X \arrow[r, "f"] & S
\end{tikzcd}$$

Let Λ be a ring and let $E \in D(X_{étale}, \Lambda)$ and $K \in D(Y_{étale}, \Lambda)$. Then there is a canonical map

$$Rf_*E \otimes_\Lambda^{\mathbf{L}} Rg_*K \longrightarrow Rc_*(p^{-1}E \otimes_\Lambda^{\mathbf{L}} q^{-1}K)$$

For example we can define this using the canonical maps $Rf_*E \to Rc_*p^{-1}E$ and $Rg_*K \to Rc_*q^{-1}K$ and the relative cup product defined in Cohomology on Sites, Remark 19.6. Or you can use the adjoint to the map

$$c^{-1}(Rf_*E \otimes_\Lambda^{\mathbf{L}} Rg_*K) = p^{-1}f^{-1}Rf_*E \otimes_\Lambda^{\mathbf{L}} q^{-1}g^{-1}Rg_*K \to p^{-1}E \otimes_\Lambda^{\mathbf{L}} q^{-1}K$$

which uses the adjunction maps $f^{-1}Rf_*E \to E$ and $g^{-1}Rg_*K \to K$.

0F14 **Lemma 91.2.** *Let k be a separably closed field. Let X be a proper scheme over k. Let Y be a quasi-compact and quasi-separated scheme over k.*

(1) *If $E \in D^+(X_{étale})$ has torsion cohomology sheaves and $K \in D^+(Y_{étale})$, then*

$$R\Gamma(X \times_{\mathrm{Spec}(k)} Y, pr_1^{-1}E \otimes_{\mathbf{Z}}^{\mathbf{L}} pr_2^{-1}K) = R\Gamma(X, E) \otimes_{\mathbf{Z}}^{\mathbf{L}} R\Gamma(Y, K)$$

(2) *If $n \geq 1$ is an integer, Y is of finite type over k, $E \in D(X_{étale}, \mathbf{Z}/n\mathbf{Z})$, and $K \in D(Y_{étale}, \mathbf{Z}/n\mathbf{Z})$, then*

$$R\Gamma(X \times_{\mathrm{Spec}(k)} Y, pr_1^{-1}E \otimes_{\mathbf{Z}/n\mathbf{Z}}^{\mathbf{L}} pr_2^{-1}K) = R\Gamma(X, E) \otimes_{\mathbf{Z}/n\mathbf{Z}}^{\mathbf{L}} R\Gamma(Y, K)$$

Proof. Proof of (1). By Lemma 88.6 we have

$$R\mathrm{pr}_{2,*}(\mathrm{pr}_1^{-1}E \otimes_{\mathbf{Z}}^{\mathbf{L}} \mathrm{pr}_2^{-1}K) = R\mathrm{pr}_{2,*}(\mathrm{pr}_1^{-1}E) \otimes_{\mathbf{Z}}^{\mathbf{L}} K$$

By proper base change (in the form of Lemma 87.12) this is equal to the object

$$\underline{R\Gamma(X, E)} \otimes_{\mathbf{Z}}^{\mathbf{L}} K$$

of $D(Y_{étale})$. Taking $R\Gamma(Y, -)$ on this object reproduces the left hand side of the equality in (1) by the Leray spectral sequence for pr_2. Thus we conclude by Lemma 88.5.

Proof of (2). This is exactly the same as the proof of (1) except that we use Lemmas 90.6, 88.4, and 90.5 as well as $\mathrm{cd}(Y) < \infty$ by Lemma 90.2. □

0F1F **Lemma 91.3.** *Let K be a separably closed field. Let X be a scheme of finite type over K. Let $\mathcal{F}$ be an abelian sheaf on $X_{étale}$ whose support is contained in the set of closed points of X. Then $H^q(X, \mathcal{F}) = 0$ for $q > 0$ and $\mathcal{F}$ is globally generated.*

Proof. (If $\mathcal{F}$ is torsion, then the vanishing follows immediately from Lemma 89.7.) By Lemma 73.5 we can write $\mathcal{F}$ as a filtered colimit of constructible sheaves $\mathcal{F}_i$ of $\mathbf{Z}$-modules whose supports $Z_i \subset X$ are finite sets of closed points. By Proposition 46.4 such a sheaf is of the form $(Z_i \to X)_*\mathcal{G}_i$ where $\mathcal{G}_i$ is a sheaf on Z_i. As K is separably closed, the scheme Z_i is a finite disjoint union of spectra of separably closed fields. Recall that $H^q(Z_i, \mathcal{G}_i) = H^q(X, \mathcal{F}_i)$ by the Leray spectral sequence for $Z_i \to X$ and vanising of higher direct images for this morphism (Proposition 54.2). By Lemmas 58.1 and 58.2 we see that $H^q(Z_i, \mathcal{G}_i)$ is zero for $q > 0$ and that $H^0(Z_i, \mathcal{G}_i)$ generates $\mathcal{G}_i$. We conclude the vanishing of $H^q(X, \mathcal{F}_i)$ for $q > 0$ and that $\mathcal{F}_i$ is generated by global sections. By Theorem 51.3 we see that $H^q(X, \mathcal{F}) = 0$ for $q > 0$. The proof is now done because a filtered colimit of globally generated sheaves of abelian groups is globally generated (details omitted). □

0F1G **Lemma 91.4.** *Let K be a separably closed field. Let X be a scheme of finite type over K. Let $Q \in D(X_{étale})$. Assume that $Q_{\overline{x}}$ is nonzero only if x is a closed point of X. Then*

$$Q = 0 \Leftrightarrow H^i(X, Q) = 0 \text{ for all } i$$

Proof. The implication from left to right is trivial. Thus we need to prove the reverse implication.

Assume Q is bounded below; this cases suffices for almost all applications. If Q is not zero, then we can look at the smallest i such that the cohomology sheaf $H^i(Q)$ is nonzero. By Lemma 91.3 we have $H^i(X, Q) = H^0(X, H^i(Q)) \neq 0$ and we conclude.

General case. Let $\mathcal{B} \subset \text{Ob}(X_{étale})$ be the quasi-compact objects. By Lemma 91.3 the assumptions of Cohomology on Sites, Lemma 22.11 are satisfied. We conclude that $H^q(U, Q) = H^0(U, H^q(Q))$ for all $U \in \mathcal{B}$. In particular, this holds for $U = X$. Thus the conclusion by Lemma 91.3 as Q is zero in $D(X_{étale})$ if and only if $H^q(Q)$ is zero for all q. $\square$

0F1H **Lemma 91.5.** *Let K be a field. Let $j : U \to X$ be an open immersion of schemes of finite type over K. Let Y be a scheme of finite type over K. Consider the diagram*

$$\begin{tikzcd}
Y \times_{\text{Spec}(K)} X \arrow[d, "q"'] & Y \times_{\text{Spec}(K)} U \arrow[l, "h"] \arrow[d, "p"] \\
X & U \arrow[l, "j"']
\end{tikzcd}$$

Then the base change map $q^{-1}Rj_*\mathcal{F} \to Rh_*p^{-1}\mathcal{F}$ is an isomorphism for $\mathcal{F}$ an abelian sheaf on $U_{étale}$ whose stalks are torsion of orders invertible in K.

Proof. Write $\mathcal{F} = \text{colim}\, \mathcal{F}[n]$ where the colimit is over the multiplicative system of integers invertible in K. Since cohomology commutes with filtered colimits in our situation (for a precise reference see Lemma 82.3), it suffices to prove the lemma for $\mathcal{F}[n]$. Thus we may assume $\mathcal{F}$ is a sheaf of $\mathbf{Z}/n\mathbf{Z}$-modules for some n invertible in K (we will use this at the very end of the proof). In the proof we use the short hand $X \times_K Y$ for the fibre product over $\text{Spec}(K)$. We will prove the lemma by induction on $\dim(X) + \dim(Y)$. The lemma is trivial if $\dim(X) \leq 0$, since in this case U is an open and closed subscheme of X. Choose a point $z \in X \times_K Y$. We will show the stalk at $\overline{z}$ is an isomorphism.

Suppose that $z \mapsto x \in X$ and assume $\text{trdeg}_K(\kappa(x)) > 0$. Set $X' = \text{Spec}(\mathcal{O}_{X,x}^{sh})$ and denote $U' \subset X'$ the inverse image of U. Consider the base change

$$\begin{tikzcd}
Y \times_K X' \arrow[d, "q'"'] & Y \times_K U' \arrow[l, "h'"] \arrow[d, "p'"] \\
X' & U' \arrow[l, "j'"']
\end{tikzcd}$$

of our diagram by $X' \to X$. Observe that $X' \to X$ is a filtered colimit of étale morphisms. By smooth base change in the form of Lemma 85.3 the pullback of $q^{-1}Rj_*\mathcal{F} \to Rh_*p^{-1}\mathcal{F}$ to X' to $Y \times_K X'$ is the map $(q')^{-1}Rj'_*\mathcal{F}' \to Rj'_*(p')^{-1}\mathcal{F}'$ where $\mathcal{F}'$ is the pullback of $\mathcal{F}$ to U'. (In this step it would suffice to use étale base change which is an essentially trivial result.) So it suffices to show that

$(q')^{-1}Rj'_*\mathcal{F}' \to Rj'_*(p')^{-1}\mathcal{F}'$ is an isomorphism in order to prove that our original map is an isomorphism on stalks at $\overline{z}$. By Lemma 89.5 there is a separably closed field L/K such that $X' = \lim X_i$ with X_i affine of finite type over L and $\dim(X_i) < \dim(X)$. For i large enough there exists an open $U_i \subset X_i$ restricting to U' in X'. We may apply the induction hypothesis to the diagram

$$\begin{tikzcd} Y \times_K X_i \arrow[d, "q_i"'] & Y \times_K U_i \arrow[l, "h_i"'] \arrow[d, "p_i"] \\ X_i & U_i \arrow[l, "j_i"'] \end{tikzcd} \quad \text{equal to} \quad \begin{tikzcd} Y_L \times_L X_i \arrow[d, "q_i"'] & Y_L \times_L U_i \arrow[l, "h_i"'] \arrow[d, "p_i"] \\ X_i & U_i \arrow[l, "j_i"'] \end{tikzcd}$$

over the field L and the pullback of $\mathcal{F}$ to these diagrams. By Lemma 82.3 we conclude that the map $(q')^{-1}Rj'_*\mathcal{F}' \to Rj'_*(p')^{-1}\mathcal{F}$ is an isomorphism.

Suppose that $z \mapsto y \in Y$ and assume $\text{trdeg}_K(\kappa(y)) > 0$. Let $Y' = \text{Spec}(\mathcal{O}^{sh}_{X,x})$. By Lemma 89.5 there is a separably closed field L/K such that $Y' = \lim Y_i$ with Y_i affine of finite type over L and $\dim(Y_i) < \dim(Y)$. In particular Y' is a scheme over L. Denote with a subscript L the base change from schemes over K to schemes over L. Consider the commutative diagrams

$$\begin{tikzcd} Y' \times_K X \arrow[d, "f"'] & Y' \times_K U \arrow[l, "h'"'] \arrow[d, "f'"] \\ Y \times_K X \arrow[d, "q"'] & Y \times_K U \arrow[l, "h"'] \arrow[d, "p"] \\ X & U \arrow[l, "j"'] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} Y' \times_L X_L \arrow[d, "q'"'] & Y' \times_L U_L \arrow[l, "h'"'] \arrow[d, "p'"] \\ X_L \arrow[d] & U_L \arrow[l, "j_L"'] \arrow[d] \\ X & U \arrow[l, "j"'] \end{tikzcd}$$

and observe the top and bottom rows are the same on the left and the right. By smooth base change we see that $f^{-1}Rh_*p^{-1}\mathcal{F} = Rh'_*(f')^{-1}p^{-1}\mathcal{F}$ (similarly to the previous paragraph). By smooth base change for $\text{Spec}(L) \to \text{Spec}(K)$ (Lemma 86.1) we see that $Rj_{L,*}\mathcal{F}_L$ is the pullback of $Rj_*\mathcal{F}$ to X_L. Combining these two observations, we conclude that it suffices to prove the base change map for the upper square in the diagram on the right is an isomorphism in order to prove that our original map is an isomorphism on stalks at $\overline{z}$[11]. Then using that $Y' = \lim Y_i$ and argueing exactly as in the previous paragraph we see that the induction hypothesis forces our map over $Y' \times_K X$ to be an isomorphism.

Thus any counter example with $\dim(X) + \dim(Y)$ minimal would only have nonisomorphisms $q^{-1}Rj_*\mathcal{F} \to Rh_*p^{-1}\mathcal{F}$ on stalks at closed points of $X \times_K Y$ (because a point z of $X \times_K Y$ is a closed point if and only if both the image of z in X and in Y are closed). Since it is enough to prove the isomorphism locally, we may assume X and Y are affine. However, then we can choose an open dense immersion $Y \to Y'$ with Y' projective. (Choose a closed immersion $Y \to \mathbf{A}^n_K$ and let Y' be the scheme theoretic closure of Y in $\mathbf{P}^n_K$.) Then $\dim(Y') = \dim(Y)$ and hence we get a "minimal" counter example with Y projective over K. In the next paragraph we show that this can't happen.

Consider a diagram as in the statement of the lemma such that $q^{-1}Rj_*\mathcal{F} \to Rh_*p^{-1}\mathcal{F}$ is an isomorphism at all non-closed points of $X \times_K Y$ and such that

[11]Here we use that a "vertical composition" of base change maps is a base change map as explained in Cohomology on Sites, Remark 19.4.

Y is projective. The restriction of the map to $(X \times_K Y)_{K^{sep}}$ is the corresponding map for the diagram of the lemma base changed to K^{sep}. Thus we may and do assume K is separably algebraically closed. Choose a distinguished triangle

$$q^{-1}Rj_*\mathcal{F} \to Rh_*p^{-1}\mathcal{F} \to Q \to (q^{-1}Rj_*\mathcal{F})[1]$$

in $D((X \times_K Y)_{étale})$. Since Q is supported in closed points we see that it suffices to prove $H^i(X \times_K Y, Q) = 0$ for all i, see Lemma 91.4. Thus it suffices to prove that $q^{-1}Rj_*\mathcal{F} \to Rh_*p^{-1}\mathcal{F}$ induces an isomorphism on cohomology. Recall that $\mathcal{F}$ is annihilated by n invertible in K. By the Künneth formula of Lemma 91.2 we have

$$\begin{aligned} R\Gamma(X \times_K Y, q^{-1}Rj_*\mathcal{F}) &= R\Gamma(X, Rj_*\mathcal{F}) \otimes^{\mathbf{L}}_{\mathbf{Z}/n\mathbf{Z}} R\Gamma(Y, \mathbf{Z}/n\mathbf{Z}) \\ &= R\Gamma(U, \mathcal{F}) \otimes^{\mathbf{L}}_{\mathbf{Z}/n\mathbf{Z}} R\Gamma(Y, \mathbf{Z}/n\mathbf{Z}) \end{aligned}$$

and

$$R\Gamma(X \times_K Y, Rh_*p^{-1}\mathcal{F}) = R\Gamma(U \times_K Y, p^{-1}\mathcal{F}) = R\Gamma(U, \mathcal{F}) \otimes^{\mathbf{L}}_{\mathbf{Z}/n\mathbf{Z}} R\Gamma(Y, \mathbf{Z}/n\mathbf{Z})$$

This finishes the proof. ☐

0F1I **Lemma 91.6.** *Let K be a field. For any commutative diagram*

$$\begin{tikzcd} X & X' \arrow[l] & Y \arrow[l, "h"'] \\ \mathrm{Spec}(K) \arrow[from=u] & S' \arrow[l] \arrow[from=u, "f'"'] & T \arrow[l, "g"'] \arrow[from=u, "e"] \end{tikzcd}$$

of schemes over K with $X' = X \times_{\mathrm{Spec}(K)} S'$ and $Y = X' \times_{S'} T$ and g quasi-compact and quasi-separated, and every abelian sheaf $\mathcal{F}$ on $T_{étale}$ whose stalks are torsion of orders invertible in K the base change map

$$(f')^{-1}Rg_*\mathcal{F} \longrightarrow Rh_*e^{-1}\mathcal{F}$$

is an isomorphism.

Proof. The question is local on X, hence we may assume X is affine. By Limits, Lemma 7.1 we can write $X = \lim X_i$ as a cofiltered limit with affine transition morphisms of schemes X_i of finite type over K. Denote $X'_i = X_i \times_{\mathrm{Spec}(K)} S'$ and $Y_i = X'_i \times_{S'} T$. By Lemma 82.3 it suffices to prove the statement for the squares with corners X_i, Y_i, S_i, T_i. Thus we may assume X is of finite type over K. Similarly, we may write $\mathcal{F} = \mathrm{colim}\, \mathcal{F}[n]$ where the colimit is over the multiplicative system of integers invertible in K. The same lemma used above reduces us to the case where $\mathcal{F}$ is a sheaf of $\mathbf{Z}/n\mathbf{Z}$-modules for some n invertible in K.

We may replace K by its algebraic closure $\overline{K}$. Namely, formation of direct image commutes with base change to $\overline{K}$ according to Lemma 86.1 (works for both g and h). And it suffices to prove the agreement after restriction to $X'_{\overline{K}}$. Next, we may replace X by its reduction as we have the topological invariance of étale cohomology, see Proposition 45.4. After this replacement the morphism $X \to \mathrm{Spec}(K)$ is flat, finite presentation, with geometrically reduced fibres and the same is true for any base change, in particular for $X' \to S'$. Hence $(f')^{-1}g_*\mathcal{F} \to Rh_*e^{-1}\mathcal{F}$ is an isomorphism by Lemma 83.2.

At this point we may apply Lemma 86.3 to see that it suffices to prove: given a commutative diagram

$$\begin{tikzcd} X \arrow[d, "f"'] & X' \arrow[l] \arrow[d] & Y \arrow[l, "h"] \arrow[d] \\ \mathrm{Spec}(K) & S' \arrow[l] & \mathrm{Spec}(L) \arrow[l] \end{tikzcd}$$

with both squares cartesian, where S' is affine, integral, and normal with algebraically closed function field K, then $R^q h_*(\mathbf{Z}/d\mathbf{Z})$ is zero for $q > 0$ and $d|n$. Observe that this vanishing is equivalent to the statement that

$$(f')^{-1} R^q(\mathrm{Spec}(L) \to S')_* \mathbf{Z}/d\mathbf{Z} \longrightarrow R^q h_* \mathbf{Z}/d\mathbf{Z}$$

is an isomorphism, because the left hand side is zero for example by Lemma 78.4.

Write $S' = \mathrm{Spec}(B)$ so that L is the fraction field of B. Write $B = \bigcup_{i \in I} B_i$ as the union of its finite type K-subalgebras B_i. Let J be the set of pairs (i, g) where $i \in I$ and $g \in B_i$ nonzero with ordering $(i', g') \geq (i, g)$ if and only if $i' \geq i$ and g maps to an invertible element of $(B_{i'})_{g'}$. Then $L = \mathrm{colim}_{(i,g) \in J} (B_i)_g$. For $j = (i, g) \in J$ set $S_j = \mathrm{Spec}(B_i)$ and $U_j = \mathrm{Spec}((B_i)_g)$. Then

$$\begin{tikzcd} X' \arrow[d] & Y \arrow[l, "h"] \arrow[d] \\ S' & \mathrm{Spec}(L) \arrow[l] \end{tikzcd} \quad \text{is the colimit of} \quad \begin{tikzcd} X \times_K S_j \arrow[d] & X \times_K U_j \arrow[l, "h_j"] \arrow[d] \\ S_j & U_j \arrow[l] \end{tikzcd}$$

Thus we may apply Lemma 82.3 to see that it suffices to prove base change holds in the diagrams on the right which is what we proved in Lemma 91.5. ☐

0F1J **Lemma 91.7.** *Let K be a field. Let $n \geq 1$ be invertible in K. Consider a commutative diagram*

$$\begin{tikzcd} X \arrow[d] & X' \arrow[l, "p"] \arrow[d, "f'"'] & Y \arrow[l, "h"] \arrow[d, "e"] \\ \mathrm{Spec}(K) & S' \arrow[l] & T \arrow[l, "g"'] \end{tikzcd}$$

of schemes with $X' = X \times_{\mathrm{Spec}(K)} S'$ and $Y = X' \times_{S'} T$ and g quasi-compact and quasi-separated. The canonical map

$$p^{-1}E \otimes^{\mathbf{L}}_{\mathbf{Z}/n\mathbf{Z}} (f')^{-1} Rg_* F \longrightarrow Rh_*(h^{-1}p^{-1}E \otimes^{\mathbf{L}}_{\mathbf{Z}/n\mathbf{Z}} e^{-1}F)$$

is an isomorphism if E in $D^+(X_{\acute{e}tale}, \mathbf{Z}/n\mathbf{Z})$ has tor amplitude in $[a, \infty]$ for some $a \in \mathbf{Z}$ and F in $D^+(T_{\acute{e}tale}, \mathbf{Z}/n\mathbf{Z})$.

Proof. This lemma is a generalization of Lemma 91.6 to objects of the derived category; the assertion of our lemma is true because in Lemma 91.6 the scheme X over K is arbitrary. We strongly urge the reader to skip the laborious proof (alternative: read only the last paragraph).

We may represent E by a bounded below K-flat complex $\mathcal{E}^\bullet$ consisting of flat $\mathbf{Z}/n\mathbf{Z}$-modules. See Cohomology on Sites, Lemma 43.4. Choose an integer b such that $H^i(F) = 0$ for $i < b$. Choose a large integer N and consider the short exact sequence

$$0 \to \sigma_{\geq N+1}\mathcal{E}^\bullet \to \mathcal{E}^\bullet \to \sigma_{\leq N}\mathcal{E}^\bullet \to 0$$

of stupid truncations. This produces a distinguished triangle $E'' \to E \to E' \to E''[1]$ in $D(X_{\acute{e}tale}, \mathbf{Z}/n\mathbf{Z})$. For fixed F both sides of the arrow in the statement of the lemma are exact functors in E. Observe that

$$p^{-1}E'' \otimes^{\mathbf{L}}_{\mathbf{Z}/n\mathbf{Z}} (f')^{-1}Rg_*F \quad \text{and} \quad Rh_*(h^{-1}p^{-1}E'' \otimes^{\mathbf{L}}_{\mathbf{Z}/n\mathbf{Z}} e^{-1}F)$$

are sitting in degrees $\geq N + b$. Hence, if we can prove the lemma for the object E', then we see that the lemma holds in degrees $\leq N + b$ and we will conclude. Some details omitted. Thus we may assume E is represented by a bounded complex of flat $\mathbf{Z}/n\mathbf{Z}$-modules. Doing another argument of the same nature, we may assume E is given by a single flat $\mathbf{Z}/n\mathbf{Z}$-module $\mathcal{E}$.

Next, we use the same arguments for the variable F to reduce to the case where F is given by a single sheaf of $\mathbf{Z}/n\mathbf{Z}$-modules $\mathcal{F}$. Say $\mathcal{F}$ is annihilated by an integer $m|n$. If ℓ is a prime number dividing m and $m > \ell$, then we can look at the short exact sequence $0 \to \mathcal{F}[\ell] \to \mathcal{F} \to \mathcal{F}/\mathcal{F}[\ell] \to 0$ and reduce to smaller m. This finally reduces us to the case where $\mathcal{F}$ is annihilated by a prime number ℓ dividing n. In this case observe that

$$p^{-1}\mathcal{E} \otimes^{\mathbf{L}}_{\mathbf{Z}/n\mathbf{Z}} (f')^{-1}Rg_*\mathcal{F} = p^{-1}(\mathcal{E}/\ell\mathcal{E}) \otimes^{\mathbf{L}}_{\mathbf{F}_\ell} (f')^{-1}Rg_*\mathcal{F}$$

by the flatness of $\mathcal{E}$. Similarly for the other term. This reduces us to the case where we are working with sheaves of $\mathbf{F}_\ell$-vector spaces which is discussed

Assume ℓ is a prime number invertible in K. Assume $\mathcal{E}$, $\mathcal{F}$ are sheaves of $\mathbf{F}_\ell$-vector spaces on $X_{\acute{e}tale}$ and $T_{\acute{e}tale}$. We want to show that

$$p^{-1}\mathcal{E} \otimes_{\mathbf{F}_\ell} (f')^{-1}R^qg_*\mathcal{F} \longrightarrow R^qh_*(h^{-1}p^{-1}\mathcal{E} \otimes_{\mathbf{F}_\ell} e^{-1}\mathcal{F})$$

is an isomorphism for every $q \geq 0$. This question is local on X hence we may assume X is affine. We can write $\mathcal{E}$ as a filtered colimit of constructible sheaves of $\mathbf{F}_\ell$-vector spaces on $X_{\acute{e}tale}$, see Lemma 72.2. Since tensor products commute with filtered colimits and since higher direct images do too (Lemma 51.7) we may assume $\mathcal{E}$ is a constructible sheaf of $\mathbf{F}_\ell$-vector spaces on $X_{\acute{e}tale}$. Then we can choose an integer m and finite and finitely presented morphisms $\pi_i : X_i \to X$, $i = 1, \ldots, m$ such that there is an injective map

$$\mathcal{E} \to \bigoplus\nolimits_{i=1,\ldots,m} \pi_{i,*}\mathbf{F}_\ell$$

See Lemma 73.4. Observe that the direct sum is a constructible sheaf as well (Lemma 72.9). Thus the cokernel is constructible too (Lemma 70.6). By dimension shifting, i.e., induction on q, on the category of constructible sheaves of $\mathbf{F}_\ell$-vector spaces on $X_{\acute{e}tale}$, it suffices to prove the result for the sheaves $\pi_{i,*}\mathbf{F}_\ell$ (details omitted; hint: start with proving injectivity for $q = 0$ for all constructible $\mathcal{E}$). To prove this case we extend the diagram of the lemma to

$$\begin{tikzcd}
X_i \arrow[d, "\pi_i"] & X'_i \arrow[l, "p_i"'] \arrow[d, "\pi'_i"] & Y_i \arrow[l, "h_i"'] \arrow[d, "\rho_i"] \\
X \arrow[d] & X' \arrow[l, "p"'] \arrow[d, "f'"'] & Y \arrow[l, "h"'] \arrow[d, "e"] \\
\mathrm{Spec}(K) & S' \arrow[l] & T \arrow[l, "g"']
\end{tikzcd}$$

with all squares cartesian. In the equations below we are going to use that $R\pi_{i,*} = \pi_{i,*}$ and similarly for π'_i, ρ_i, we are going to use the Leray spectral sequence, we are going to use Lemma 54.3, and we are going to use Lemma 90.6 (although this lemma is almost trivial for finite morphisms) for π_i, π'_i, ρ_i. Doing so we see that

$$\begin{aligned} p^{-1}\pi_{i,*}\mathbf{F}_\ell \otimes_{\mathbf{F}_\ell} (f')^{-1}R^qg_*\mathcal{F} &= \pi'_{i,*}\mathbf{F}_\ell \otimes_{\mathbf{F}_\ell} (f')^{-1}R^qg_*\mathcal{F} \\ &= \pi'_{i,*}((\pi'_i)^{-1}(f')^{-1}R^qg_*\mathcal{F}) \end{aligned}$$

Similarly, we have

$$\begin{aligned} R^qh_*(h^{-1}p^{-1}\pi_{i,*}\mathbf{F}_\ell \otimes_{\mathbf{F}_\ell} e^{-1}\mathcal{F}) &= R^qh_*(\rho_{i,*}\mathbf{F}_\ell \otimes_{\mathbf{F}_\ell} e^{-1}\mathcal{F}) \\ &= R^qh_*(\rho_i^{-1}e^{-1}\mathcal{F}) \\ &= \pi'_{i,*}R^qh_{i,*}\rho_i^{-1}e^{-1}\mathcal{F}) \end{aligned}$$

Simce $R^qh_{i,*}\rho_i^{-1}e^{-1}\mathcal{F} = (\pi'_i)^{-1}(f')^{-1}R^qg_*\mathcal{F}$ by Lemma 91.6 we conclude. □

0F1N **Lemma 91.8.** *Let K be a field. Let $n \geq 1$ be invertible in K. Consider a commutative diagram*

$$\begin{tikzcd} X \arrow[d] & X' \arrow[l, "p"'] \arrow[d, "f'"'] & Y \arrow[l, "h"'] \arrow[d, "e"] \\ \mathrm{Spec}(K) & S' \arrow[l] & T \arrow[l, "g"] \end{tikzcd}$$

of schemes of finite type over K with $X' = X \times_{\mathrm{Spec}(K)} S'$ and $Y = X' \times_{S'} T$. The canonical map

$$p^{-1}E \otimes^{\mathbf{L}}_{\mathbf{Z}/n\mathbf{Z}} (f')^{-1}Rg_*F \longrightarrow Rh_*(h^{-1}p^{-1}E \otimes^{\mathbf{L}}_{\mathbf{Z}/n\mathbf{Z}} e^{-1}F)$$

is an isomorphism for E in $D(X_{étale}, \mathbf{Z}/n\mathbf{Z})$ and F in $D(T_{étale}, \mathbf{Z}/n\mathbf{Z})$.

Proof. We will reduce this to Lemma 91.7 using that our functors commute with direct sums. We suggest the reader skip the proof. Recall that derived tensor product commutes with direct sums. Recall that (derived) pullback commutes with direct sums. Recall that Rh_* and Rg_* commute with direct sums, see Lemmas 90.2 and 90.3 (this is where we use our schemes are of finite type over K).

To finish the proof we can argue as follows. First we write $E = \text{hocolim}\tau_{\leq N}E$. Since our functors commute with direct sums, they commute with homotopy colimits. Hence it suffices to prove the lemma for E bounded above. Similarly for F we may assume F is bounded above. Then we can represent E by a bounded above complex $\mathcal{E}^\bullet$ of sheaves of $\mathbf{Z}/n\mathbf{Z}$-modules. Then

$$\mathcal{E}^\bullet = \operatorname{colim} \sigma_{\geq -N}\mathcal{E}^\bullet$$

(stupid truncations). Thus we may assume $\mathcal{E}^\bullet$ is a bounded complex of sheaves of $\mathbf{Z}/n\mathbf{Z}$-modules. For F we choose a bounded above complex of flat(!) sheaves of $\mathbf{Z}/n\mathbf{Z}$-modules. Then we reduce to the case where F is represented by a bounded complex of flat sheaves of $\mathbf{Z}/n\mathbf{Z}$-modules. At this point Lemma 91.7 kicks in and we conclude. □

0F1P **Lemma 91.9.** *Let k be a separably closed field. Let X and Y be finite type schemes over k. Let $n \geq 1$ be an integer invertible in k. Then for $E \in D(X_{étale}, \mathbf{Z}/n\mathbf{Z})$ and $K \in D(Y_{étale}, \mathbf{Z}/n\mathbf{Z})$ we have*

$$R\Gamma(X \times_{\mathrm{Spec}(k)} Y, pr_1^{-1}E \otimes^{\mathbf{L}}_{\mathbf{Z}/n\mathbf{Z}} pr_2^{-1}K) = R\Gamma(X, E) \otimes^{\mathbf{L}}_{\mathbf{Z}/n\mathbf{Z}} R\Gamma(Y, K)$$

Proof. By Lemma 91.8 we have

$$R\mathrm{pr}_{1,*}(\mathrm{pr}_1^{-1}E \otimes^{\mathbf{L}}_{\mathbf{Z}/n\mathbf{Z}} \mathrm{pr}_2^{-1}K) = E \otimes^{\mathbf{L}}_{\mathbf{Z}/n\mathbf{Z}} \underline{R\Gamma(Y, K)}$$

We conclude by Lemma 90.5 which we may use because $\mathrm{cd}(X) < \infty$ by Lemma 90.2. $\square$

92. Comparing chaotic and Zariski topologies

0F1K When constructing the structure sheaf of an affine scheme, we first construct the values on affine opens, and then we extend to all opens. A similar construction is often useful for constructing complexes of abelian groups on a scheme X. Recall that $X_{affine,Zar}$ denotes the category of affine opens of X with topology given by standard Zariski coverings, see Lemma 21.4. Let's denote $X_{affine,chaotic}$ the same underlying category with the chaotic topology, i.e., such that sheaves agree with presheaves. We obtain a morphisms of sites

$$\epsilon : X_{affine,Zar} \longrightarrow X_{affine,chaotic}$$

as in Cohomology on Sites, Section 26.

0F1L **Lemma 92.1.** *In the situation above let K be an object of $D^+(X_{affine,chaotic})$. Then K is in the essential image of the (fully faithful) functor $R\epsilon_*; D(X_{affine,Zar}) \to D(X_{affine,chaotic})$ if and only if the following two conditions hold*

1. *$R\Gamma(\emptyset, K)$ is zero in $D(Ab)$, and*
2. *if $U = V \cup W$ with $U, V, W \subset X$ affine open and $V, W \subset U$ standard open (Algebra, Definition 16.3), then the map $c^K_{U,V,W,V\cap W}$ of Cohomology on Sites, Lemma 25.1 is a quasi-isomorphism.*

Proof. Except for a snafu having to do with the empty set, this follows from the very general Cohomology on Sites, Lemma 28.2 whose hypotheses hold by Schemes, Lemma 11.7 and Cohomology on Sites, Lemma 28.3.

To get around the snafu, denote $X_{affine,almost-chaotic}$ the site where the empty object $\emptyset$ has two coverings, namely, $\{\emptyset \to \emptyset\}$ and the empty covering (see Sites, Example 6.4 for a discussion). Then we have morphisms of sites

$$X_{affine,Zar} \to X_{affine,almost-chaotic} \to X_{affine,chaotic}$$

The argument above works for the first arrow. Then we leave it to the reader to see that an object K of $D^+(X_{affine,chaotic})$ is in the essential image of the (fully faithful) functor $D(X_{affine,almost-chaotic}) \to D(X_{affine,chaotic})$ if and only if $R\Gamma(\emptyset, K)$ is zero in $D(Ab)$. $\square$

93. Comparing big and small topoi

0757 Let S be a scheme. In Topologies, Lemma 4.13 we have introduced comparison morphisms $\pi_S : (Sch/S)_{\acute{e}tale} \to S_{\acute{e}tale}$ and $i_S : Sh(S_{\acute{e}tale}) \to Sh((Sch/S)_{\acute{e}tale})$ with $\pi_S \circ i_S = \mathrm{id}$ and $\pi_{S,*} = i_S^{-1}$. More generally, if $f : T \to S$ is an object of $(Sch/S)_{\acute{e}tale}$, then there is a morphism $i_f : Sh(T_{\acute{e}tale}) \to Sh((Sch/S)_{\acute{e}tale})$ such that $f_{small} = \pi_S \circ i_f$, see Topologies, Lemmas 4.12 and 4.16. In Descent, Remark 8.4 we have extended these to a morphism of ringed sites

$$\pi_S : ((Sch/S)_{\acute{e}tale}, \mathcal{O}) \to (S_{\acute{e}tale}, \mathcal{O}_S)$$

and morphisms of ringed topoi

$$i_S : (Sh(S_{étale}), \mathcal{O}_S) \to (Sh((Sch/S)_{étale}), \mathcal{O})$$

and

$$i_f : (Sh(T_{étale}), \mathcal{O}_T) \to (Sh((Sch/S)_{étale}, \mathcal{O}))$$

Note that the restriction $i_S^{-1} = \pi_{S,*}$ (see Topologies, Definition 4.14) transforms $\mathcal{O}$ into $\mathcal{O}_S$. Similarly, i_f^{-1} transforms $\mathcal{O}$ into $\mathcal{O}_T$. See Descent, Remark 8.4. Hence $i_S^*\mathcal{F} = i_S^{-1}\mathcal{F}$ and $i_f^*\mathcal{F} = i_f^{-1}\mathcal{F}$ for any $\mathcal{O}$-module $\mathcal{F}$ on $(Sch/S)_{étale}$. In particular i_S^* and i_f^* are exact functors. The functor i_S^* is often denoted $\mathcal{F} \mapsto \mathcal{F}|_{S_{étale}}$ (and this does not conflict with the notation in Topologies, Definition 4.14).

0758 **Lemma 93.1.** *Let S be a scheme. Let T be an object of $(Sch/S)_{étale}$.*

(1) *If $\mathcal{I}$ is injective in $Ab((Sch/S)_{étale})$, then*
  (a) *$i_f^{-1}\mathcal{I}$ is injective in $Ab(T_{étale})$,*
  (b) *$\mathcal{I}|_{S_{étale}}$ is injective in $Ab(S_{étale})$,*
(2) *If $\mathcal{I}^\bullet$ is a K-injective complex in $Ab((Sch/S)_{étale})$, then*
  (a) *$i_f^{-1}\mathcal{I}^\bullet$ is a K-injective complex in $Ab(T_{étale})$,*
  (b) *$\mathcal{I}^\bullet|_{S_{étale}}$ is a K-injective complex in $Ab(S_{étale})$,*

The corresponding statements for modules do not hold.

Proof. Parts (1)(b) and (2)(b) follow formally from the fact that the restriction functor $\pi_{S,*} = i_S^{-1}$ is a right adjoint of the exact functor π_S^{-1}, see Homology, Lemma 27.1 and Derived Categories, Lemma 30.9.

Parts (1)(a) and (2)(a) can be seen in two ways. First proof: We can use that i_f^{-1} is a right adjoint of the exact functor $i_{f,!}$. This functor is constructed in Topologies, Lemma 4.12 for sheaves of sets and for abelian sheaves in Modules on Sites, Lemma 16.2. It is shown in Modules on Sites, Lemma 16.3 that it is exact. Second proof. We can use that $i_f = i_T \circ f_{big}$ as is shown in Topologies, Lemma 4.16. Since f_{big} is a localization, we see that pullback by it preserves injectives and K-injectives, see Cohomology on Sites, Lemmas 7.1 and 20.1. Then we apply the already proved parts (1)(b) and (2)(b) to the functor i_T^{-1} to conclude.

Let $S = \text{Spec}(\mathbf{Z})$ and consider the map $2 : \mathcal{O}_S \to \mathcal{O}_S$. This is an injective map of $\mathcal{O}_S$-modules on $S_{étale}$. However, the pullback $\pi_S^*(2) : \mathcal{O} \to \mathcal{O}$ is not injective as we see by evaluating on $\text{Spec}(\mathbf{F}_2)$. Now choose an injection $\alpha : \mathcal{O} \to \mathcal{I}$ into an injective $\mathcal{O}$-module $\mathcal{I}$ on $(Sch/S)_{étale}$. Then consider the diagram

$$\begin{tikzcd} \mathcal{O}_S \arrow[r, "\alpha|_{S_{étale}}"'] \arrow[d, "2"'] & \mathcal{I}|_{S_{étale}} \\ \mathcal{O}_S \arrow[ur, dotted] & \end{tikzcd}$$

Then the dotted arrow cannot exist in the category of $\mathcal{O}_S$-modules because it would mean (by adjunction) that the injective map α factors through the noninjective map $\pi_S^*(2)$ which cannot be the case. Thus $\mathcal{I}|_{S_{étale}}$ is not an injective $\mathcal{O}_S$-module. $\square$

Let $f : T \to S$ be a morphism of schemes. The commutative diagram of Topologies, Lemma 4.16 (3) leads to a commutative diagram of ringed sites

$$\begin{tikzcd} (T_{étale}, \mathcal{O}_T) \arrow[d, "f_{small}"'] & ((Sch/T)_{étale}, \mathcal{O}) \arrow[l, "\pi_T"] \arrow[d, "f_{big}"] \\ (S_{étale}, \mathcal{O}_S) & ((Sch/S)_{étale}, \mathcal{O}) \arrow[l, "\pi_S"'] \end{tikzcd}$$

as one easily sees by writing out the definitions of $f_{small}^\sharp$, $f_{big}^\sharp$, $\pi_S^\sharp$, and $\pi_T^\sharp$. In particular this means that

0759 (93.1.1) $$(f_{big,*}\mathcal{F})|_{S_{étale}} = f_{small,*}(\mathcal{F}|_{T_{étale}})$$

for any sheaf $\mathcal{F}$ on $(Sch/T)_{étale}$ and if $\mathcal{F}$ is a sheaf of $\mathcal{O}$-modules, then (93.1.1) is an isomorphism of $\mathcal{O}_S$-modules on $S_{étale}$.

075A **Lemma 93.2.** *Let $f : T \to S$ be a morphism of schemes.*

(1) *For K in $D((Sch/T)_{étale})$ we have $(Rf_{big,*}K)|_{S_{étale}} = Rf_{small,*}(K|_{T_{étale}})$ in $D(S_{étale})$.*
(2) *For K in $D((Sch/T)_{étale}, \mathcal{O})$ we have $(Rf_{big,*}K)|_{S_{étale}} = Rf_{small,*}(K|_{T_{étale}})$ in $D(Mod(S_{étale}, \mathcal{O}_S))$.*

More generally, let $g : S' \to S$ be an object of $(Sch/S)_{étale}$. Consider the fibre product

$$\begin{tikzcd} T' \arrow[r, "g'"] \arrow[d, "f'"'] & T \arrow[d, "f"] \\ S' \arrow[r, "g"] & S \end{tikzcd}$$

Then

(3) *For K in $D((Sch/T)_{étale})$ we have $i_g^{-1}(Rf_{big,*}K) = Rf'_{small,*}(i_{g'}^{-1}K)$ in $D(S'_{étale})$.*
(4) *For K in $D((Sch/T)_{étale}, \mathcal{O})$ we have $i_g^*(Rf_{big,*}K) = Rf'_{small,*}(i_{g'}^*K)$ in $D(Mod(S'_{étale}, \mathcal{O}_{S'}))$.*
(5) *For K in $D((Sch/T)_{étale})$ we have $g_{big}^{-1}(Rf_{big,*}K) = Rf'_{big,*}((g'_{big})^{-1}K)$ in $D((Sch/S')_{étale})$.*
(6) *For K in $D((Sch/T)_{étale}, \mathcal{O})$ we have $g_{big}^*(Rf_{big,*}K) = Rf'_{big,*}((g'_{big})^*K)$ in $D(Mod(S'_{étale}, \mathcal{O}_{S'}))$.*

Proof. Part (1) follows from Lemma 93.1 and (93.1.1) on choosing a K-injective complex of abelian sheaves representing K.

Part (3) follows from Lemma 93.1 and Topologies, Lemma 4.18 on choosing a K-injective complex of abelian sheaves representing K.

Part (5) is Cohomology on Sites, Lemma 21.1.

Part (6) is Cohomology on Sites, Lemma 21.2.

Part (2) can be proved as follows. Above we have seen that $\pi_S \circ f_{big} = f_{small} \circ \pi_T$ as morphisms of ringed sites. Hence we obtain $R\pi_{S,*} \circ Rf_{big,*} = Rf_{small,*} \circ R\pi_{T,*}$ by Cohomology on Sites, Lemma 19.2. Since the restriction functors $\pi_{S,*}$ and $\pi_{T,*}$ are exact, we conclude.

Part (4) follows from part (6) and part (2) applied to $f' : T' \to S'$. $\square$

Let S be a scheme and let $\mathcal{H}$ be an abelian sheaf on $(Sch/S)_{étale}$. Recall that $H^n_{étale}(U, \mathcal{H})$ denotes the cohomology of $\mathcal{H}$ over an object U of $(Sch/S)_{étale}$.

0DDH **Lemma 93.3.** *Let $f : T \to S$ be a morphism of schemes. Then*

1. *For K in $D(S_{étale})$ we have $H^n_{étale}(S, \pi_S^{-1}K) = H^n(S_{étale}, K)$.*
2. *For K in $D(S_{étale}, \mathcal{O}_S)$ we have $H^n_{étale}(S, L\pi_S^*K) = H^n(S_{étale}, K)$.*
3. *For K in $D(S_{étale})$ we have $H^n_{étale}(T, \pi_S^{-1}K) = H^n(T_{étale}, f_{small}^{-1}K)$.*
4. *For K in $D(S_{étale}, \mathcal{O}_S)$ we have $H^n_{étale}(T, L\pi_S^*K) = H^n(T_{étale}, Lf_{small}^*K)$.*
5. *For M in $D((Sch/S)_{étale})$ we have $H^n_{étale}(T, M) = H^n(T_{étale}, i_f^{-1}M)$.*
6. *For M in $D((Sch/S)_{étale}, \mathcal{O})$ we have $H^n_{étale}(T, M) = H^n(T_{étale}, i_f^*M)$.*

Proof. To prove (5) represent M by a K-injective complex of abelian sheaves and apply Lemma 93.1 and work out the definitions. Part (3) follows from this as $i_f^{-1}\pi_S^{-1} = f_{small}^{-1}$. Part (1) is a special case of (3).

Part (6) follows from the very general Cohomology on Sites, Lemma 35.5. Then part (4) follows because $Lf_{small}^* = i_f^* \circ L\pi_S^*$. Part (2) is a special case of (4). $\square$

0DDI **Lemma 93.4.** *Let S be a scheme. For $K \in D(S_{étale})$ the map*

$$K \longrightarrow R\pi_{S,*}\pi_S^{-1}K$$

is an isomorphism.

Proof. This is true because both π_S^{-1} and $\pi_{S,*} = i_S^{-1}$ are exact functors and the composition $\pi_{S,*} \circ \pi_S^{-1}$ is the identity functor. $\square$

0DDJ **Lemma 93.5.** *Let $f : T \to S$ be a proper morphism of schemes. Then we have*

1. *$\pi_S^{-1} \circ f_{small,*} = f_{big,*} \circ \pi_T^{-1}$ as functors $Sh(T_{étale}) \to Sh((Sch/S)_{étale})$,*
2. *$\pi_S^{-1}Rf_{small,*}K = Rf_{big,*}\pi_T^{-1}K$ for K in $D^+(T_{étale})$ whose cohomology sheaves are torsion,*
3. *$\pi_S^{-1}Rf_{small,*}K = Rf_{big,*}\pi_T^{-1}K$ for K in $D(T_{étale}, \mathbf{Z}/n\mathbf{Z})$, and*
4. *$\pi_S^{-1}Rf_{small,*}K = Rf_{big,*}\pi_T^{-1}K$ for all K in $D(T_{étale})$ if f is finite.*

Proof. Proof of (1). Let $\mathcal{F}$ be a sheaf on $T_{étale}$. Let $g : S' \to S$ be an object of $(Sch/S)_{étale}$. Consider the fibre product

$$\begin{tikzcd} T' \arrow[r, "f'"] \arrow[d, "g'"'] & S' \arrow[d, "g"] \\ T \arrow[r, "f"] & S \end{tikzcd}$$

Then we have

$$(f_{big,*}\pi_T^{-1}\mathcal{F})(S') = (\pi_T^{-1}\mathcal{F})(T') = ((g'_{small})^{-1}\mathcal{F})(T') = (f'_{small,*}(g'_{small})^{-1}\mathcal{F})(S')$$

the second equality by Lemma 39.2. On the other hand

$$(\pi_S^{-1}f_{small,*}\mathcal{F})(S') = (g_{small}^{-1}f_{small,*}\mathcal{F})(S')$$

again by Lemma 39.2. Hence by proper base change for sheaves of sets (Lemma 87.5) we conclude the two sets are canonically isomorphic. The isomorphism is compatible with restriction mappings and defines an isomorphism $\pi_S^{-1}f_{small,*}\mathcal{F} = f_{big,*}\pi_T^{-1}\mathcal{F}$. Thus an isomorphism of functors $\pi_S^{-1} \circ f_{small,*} = f_{big,*} \circ \pi_T^{-1}$.

Proof of (2). There is a canonical base change map $\pi_S^{-1}Rf_{small,*}K \to Rf_{big,*}\pi_T^{-1}K$ for any K in $D(T_{\acute{e}tale})$, see Cohomology on Sites, Remark 19.3. To prove it is an isomorphism, it suffices to prove the pull back of the base change map by $i_g : Sh(S'_{\acute{e}tale}) \to Sh((Sch/S)_{\acute{e}tale})$ is an isomorphism for any object $g : S' \to S$ of $(Sch/S)_{\acute{e}tale}$. Let T', g', f' be as in the previous paragraph. The pullback of the base change map is

$$\begin{aligned} g_{small}^{-1}Rf_{small,*}K &= i_g^{-1}\pi_S^{-1}Rf_{small,*}K \\ &\to i_g^{-1}Rf_{big,*}\pi_T^{-1}K \\ &= Rf'_{small,*}(i_{g'}^{-1}\pi_T^{-1}K) \\ &= Rf'_{small,*}((g'_{small})^{-1}K) \end{aligned}$$

where we have used $\pi_S \circ i_g = g_{small}$, $\pi_T \circ i_{g'} = g'_{small}$, and Lemma 93.2. This map is an isomorphism by the proper base change theorem (Lemma 87.12) provided K is bounded below and the cohomology sheaves of K are torsion.

The proof of part (3) is the same as the proof of part (2), except we use Lemma 88.4 instead of Lemma 87.12.

Proof of (4). If f is finite, then the functors $f_{small,*}$ and $f_{big,*}$ are exact. This follows from Proposition 54.2 for f_{small}. Since any base change f' of f is finite too, we conclude from Lemma 93.2 part (3) that $f_{big,*}$ is exact too (as the higher derived functors are zero). Thus this case follows from part (1). □

94. Comparing fppf and étale topologies

0DDK A model for this section is the section on the comparison of the usual topology and the qc topology on locally compact topological spaces as discussed in Cohomology on Sites, Section 30. We first review some material from Topologies, Sections 11 and 4.

Let S be a scheme and let $(Sch/S)_{fppf}$ be an fppf site. On the same underlying category with have a second topology, namely the étale topology, and hence a second site $(Sch/S)_{\acute{e}tale}$. The identity functor $(Sch/S)_{\acute{e}tale} \to (Sch/S)_{fppf}$ is continuous and defines a morphism of sites

$$\epsilon_S : (Sch/S)_{fppf} \longrightarrow (Sch/S)_{\acute{e}tale}$$

See Cohomology on Sites, Section 26. Please note that $\epsilon_{S,*}$ is the identity functor on underlying presheaves and that ϵ_S^{-1} associates to an étale sheaf the fppf sheafification. Let $S_{\acute{e}tale}$ be the small étale site. There is a morphism of sites

$$\pi_S : (Sch/S)_{\acute{e}tale} \longrightarrow S_{\acute{e}tale}$$

given by the continuous functor $S_{\acute{e}tale} \to (Sch/S)_{\acute{e}tale}$, $U \mapsto U$. Namely, $S_{\acute{e}tale}$ has fibre products and a final object and the functor above commutes with these and Sites, Proposition 14.7 applies.

0DDL **Lemma 94.1.** *With notation as above. Let $\mathcal{F}$ be a sheaf on $S_{\acute{e}tale}$. The rule*

$$(Sch/S)_{fppf} \longrightarrow Sets, \quad (f : X \to S) \longmapsto \Gamma(X, f_{small}^{-1}\mathcal{F})$$

is a sheaf and a fortiori a sheaf on $(Sch/S)_{\acute{e}tale}$. In fact this sheaf is equal to $\pi_S^{-1}\mathcal{F}$ on $(Sch/S)_{\acute{e}tale}$ and $\epsilon_S^{-1}\pi_S^{-1}\mathcal{F}$ on $(Sch/S)_{fppf}$.

Proof. The statement about the étale topology is the content of Lemma 39.2. To finish the proof it suffices to show that $\pi_S^{-1}\mathcal{F}$ is a sheaf for the fppf topology. This is shown in Lemma 39.2 as well. □

In the situation of Lemma 94.1 the composition of ϵ_S and π_S and the equality determine a morphism of sites

$$a_S : (Sch/S)_{fppf} \longrightarrow S_{\acute{e}tale}$$

0DDM **Lemma 94.2.** *With notation as above. Let $f : X \to Y$ be a morphism of $(Sch/S)_{fppf}$. Then there are commutative diagrams of topoi*

$$\begin{tikzcd} Sh((Sch/X)_{fppf}) \arrow[r, "f_{big,fppf}"'] \arrow[d, "\epsilon_X"'] & Sh((Sch/Y)_{fppf}) \arrow[d, "\epsilon_Y"] \\ Sh((Sch/X)_{\acute{e}tale}) \arrow[r, "f_{big,\acute{e}tale}"] & Sh((Sch/Y)_{\acute{e}tale}) \end{tikzcd}$$

and

$$\begin{tikzcd} Sh((Sch/X)_{fppf}) \arrow[r, "f_{big,fppf}"'] \arrow[d, "a_X"'] & Sh((Sch/Y)_{fppf}) \arrow[d, "a_Y"] \\ Sh(X_{\acute{e}tale}) \arrow[r, "f_{small}"] & Sh(Y_{\acute{e}tale}) \end{tikzcd}$$

with $a_X = \pi_X \circ \epsilon_X$ and $a_Y = \pi_X \circ \epsilon_X$.

Proof. The commutativity of the diagrams follows from the discussion in Topologies, Section 11. □

0DDN **Lemma 94.3.** *In Lemma 94.2 if f is proper, then we have $a_Y^{-1} \circ f_{small,*} = f_{big,fppf,*} \circ a_X^{-1}$.*

Proof. You can prove this by repeating the proof of Lemma 93.5 part (1); we will instead deduce the result from this. As $\epsilon_{Y,*}$ is the identity functor on underlying presheaves, it reflects isomorphisms. The description in Lemma 94.1 shows that $\epsilon_{Y,*} \circ a_Y^{-1} = \pi_Y^{-1}$ and similarly for X. To show that the canonical map $a_Y^{-1} f_{small,*}\mathcal{F} \to f_{big,fppf,*}a_X^{-1}\mathcal{F}$ is an isomorphism, it suffices to show that

$$\begin{aligned} \pi_Y^{-1} f_{small,*}\mathcal{F} &= \epsilon_{Y,*}a_Y^{-1} f_{small,*}\mathcal{F} \\ &\to \epsilon_{Y,*} f_{big,fppf,*}a_X^{-1}\mathcal{F} \\ &= f_{big,\acute{e}tale,*}\epsilon_{X,*}a_X^{-1}\mathcal{F} \\ &= f_{big,\acute{e}tale,*}\pi_X^{-1}\mathcal{F} \end{aligned}$$

is an isomorphism. This is part (1) of Lemma 93.5. □

0DEU **Lemma 94.4.** *In Lemma 94.2 assume f is flat, locally of finite presentation, and surjective. Then the functor*

$$Sh(Y_{\acute{e}tale}) \longrightarrow \left\{ (\mathcal{G}, \mathcal{H}, \alpha) \middle| \begin{array}{l} \mathcal{G} \in Sh(X_{\acute{e}tale}),\ \mathcal{H} \in Sh((Sch/Y)_{fppf}), \\ \alpha : a_X^{-1}\mathcal{G} \to f_{big,fppf}^{-1}\mathcal{H} \text{ an isomorphism} \end{array} \right\}$$

sending $\mathcal{F}$ to $(f_{small}^{-1}\mathcal{F}, a_Y^{-1}\mathcal{F}, can)$ is an equivalence.

Proof. The functor a_X^{-1} is fully faithful (as $a_{X,*}a_X^{-1} = \text{id}$ by Lemma 94.1). Hence the forgetful functor $(\mathcal{G}, \mathcal{H}, \alpha) \mapsto \mathcal{H}$ identifies the category of triples with a full subcategory of $Sh((Sch/Y)_{fppf})$. Moreover, the functor a_Y^{-1} is fully faithful, hence the functor in the lemma is fully faithful as well.

Suppose that we have an étale covering $\{Y_i \to Y\}$. Let $f_i : X_i \to Y_i$ be the base change of f. Denote $f_{ij} = f_i \times f_j : X_i \times_X X_j \to Y_i \times_Y Y_j$. Claim: if the lemma is true for f_i and f_{ij} for all i, j, then the lemma is true for f. To see this, note that the given étale covering determines an étale covering of the final object in each of the four sites $Y_{étale}, X_{étale}, (Sch/Y)_{fppf}, (Sch/X)_{fppf}$. Thus the category of sheaves is equivalent to the category of glueing data for this covering (Sites, Lemma 26.5) in each of the four cases. A huge commutative diagram of categories then finishes the proof of the claim. We omit the details. The claim shows that we may work étale locally on Y.

Note that $\{X \to Y\}$ is an fppf covering. Working étale locally on Y, we may assume there exists a morphism $s : X' \to X$ such that the composition $f' = f \circ s : X' \to Y$ is surjective finite locally free, see More on Morphisms, Lemma 43.1. Claim: if the lemma is true for f', then it is true for f. Namely, given a triple $(\mathcal{G}, \mathcal{H}, \alpha)$ for f, we can pullback by s to get a triple $(s_{small}^{-1}\mathcal{G}, \mathcal{H}, s_{big,fppf}^{-1}\alpha)$ for f'. A solution for this triple gives a sheaf $\mathcal{F}$ on $Y_{étale}$ with $a_Y^{-1}\mathcal{F} = \mathcal{H}$. By the first paragraph of the proof this means the triple is in the essential image. This reduces us to the case described in the next paragraph.

Assume f is surjective finite locally free. Let $(\mathcal{G}, \mathcal{H}, \alpha)$ be a triple. In this case consider the triple

$$(\mathcal{G}_1, \mathcal{H}_1, \alpha_1) = (f_{small}^{-1}f_{small,*}\mathcal{G}, f_{big,fppf,*}f_{big,fppf}^{-1}\mathcal{H}, \alpha_1)$$

where α_1 comes from the identifications

$$\begin{aligned} a_X^{-1}f_{small}^{-1}f_{small,*}\mathcal{G} &= f_{big,fppf}^{-1}a_Y^{-1}f_{small,*}\mathcal{G} \\ &= f_{big,fppf}^{-1}f_{big,fppf,*}a_X^{-1}\mathcal{G} \\ &\to f_{big,fppf}^{-1}f_{big,fppf,*}f_{big,fppf}^{-1}\mathcal{H} \end{aligned}$$

where the third equality is Lemma 94.3 and the arrow is given by α. This triple is in the image of our functor because $\mathcal{F}_1 = f_{small,*}\mathcal{F}$ is a solution (to see this use Lemma 94.3 again; details omitted). There is a canonical map of triples

$$(\mathcal{G}, \mathcal{H}, \alpha) \to (\mathcal{G}_1, \mathcal{H}_1, \alpha_1)$$

which uses the unit $\text{id} \to f_{big,fppf,*}f_{big,fppf}^{-1}$ on the second entry (it is enough to prescribe morphisms on the second entry by the first paragraph of the proof). Since $\{f : X \to Y\}$ is an fppf covering the map $\mathcal{H} \to \mathcal{H}_1$ is injective (details omitted). Set

$$\mathcal{G}_2 = \mathcal{G}_1 \amalg_{\mathcal{G}} \mathcal{G}_1 \quad \mathcal{H}_2 = \mathcal{H}_1 \amalg_{\mathcal{H}} \mathcal{H}_1$$

and let α_2 be the induced isomorphism (pullback functors are exact, so this makes sense). Then $\mathcal{H}$ is the equalizer of the two maps $\mathcal{H}_1 \to \mathcal{H}_2$. Repeating the arguments above for the triple $(\mathcal{G}_2, \mathcal{H}_2, \alpha_2)$ we find an injective morphism of triples

$$(\mathcal{G}_2, \mathcal{H}_2, \alpha_2) \to (\mathcal{G}_3, \mathcal{H}_3, \alpha_3)$$

such that this last triple is in the image of our functor. Say it corresponds to $\mathcal{F}_3$ in $Sh(Y_{étale})$. By fully faithfulness we obtain two maps $\mathcal{F}_1 \to \mathcal{F}_3$ and we can let $\mathcal{F}$

be the equalizer of these two maps. By exactness of the pullback functors involved we find that $a_Y^{-1}\mathcal{F} = \mathcal{H}$ as desired. □

0F0H **Lemma 94.5.** *Consider the comparison morphism $\epsilon : (Sch/S)_{fppf} \to (Sch/S)_{étale}$. Let $\mathcal{P}$ denote the class of finite morphisms of schemes. For X in $(Sch/S)_{étale}$ denote $\mathcal{A}'_X \subset Ab((Sch/X)_{étale})$ the full subcategory consisting of sheaves of the form $\pi_X^{-1}\mathcal{F}$ with $\mathcal{F}$ in $Ab(X_{étale})$. Then Cohomology on Sites, Properties (1), (2), (3), (4), and (5) of Cohomology on Sites, Situation 29.1 hold.*

Proof. We first show that $\mathcal{A}'_X \subset Ab((Sch/X)_{étale})$ is a weak Serre subcategory by checking conditions (1), (2), (3), and (4) of Homology, Lemma 10.3. Parts (1), (2), (3) are immediate as π_X^{-1} is exact and fully faithful for example by Lemma 93.4. If $0 \to \pi_X^{-1}\mathcal{F} \to \mathcal{G} \to \pi_X^{-1}\mathcal{F}' \to 0$ is a short exact sequence in $Ab((Sch/X)_{étale})$ then $0 \to \mathcal{F} \to \pi_{X,*}\mathcal{G} \to \mathcal{F}' \to 0$ is exact by Lemma 93.4. Hence $\mathcal{G} = \pi_X^{-1}\pi_{X,*}\mathcal{G}$ is in $\mathcal{A}'_X$ which checks the final condition.

Cohomology on Sites, Property (1) holds by the existence of fibre products of schemes and the fact that the base change of a finite morphism of schemes is a finite morphism of schemes, see Morphisms, Lemma 42.6.

Cohomology on Sites, Property (2) follows from the commutative diagram (3) in Topologies, Lemma 4.16.

Cohomology on Sites, Property (3) is Lemma 94.1.

Cohomology on Sites, Property (4) holds by Lemma 93.5 part (4).

Cohomology on Sites, Property (5) is implied by More on Morphisms, Lemma 43.1. □

0DDS **Lemma 94.6.** *With notation as above.*

(1) *For $X \in \mathrm{Ob}((Sch/S)_{fppf})$ and an abelian sheaf $\mathcal{F}$ on $X_{étale}$ we have $\epsilon_{X,*}a_X^{-1}\mathcal{F} = \pi_X^{-1}\mathcal{F}$ and $R^i\epsilon_{X,*}(a_X^{-1}\mathcal{F}) = 0$ for $i > 0$.*
(2) *For a finite morphism $f : X \to Y$ in $(Sch/S)_{fppf}$ and abelian sheaf $\mathcal{F}$ on X we have $a_Y^{-1}(R^if_{small,*}\mathcal{F}) = R^if_{big,fppf,*}(a_X^{-1}\mathcal{F})$ for all i.*
(3) *For a scheme X and K in $D^+(X_{étale})$ the map $\pi_X^{-1}K \to R\epsilon_{X,*}(a_X^{-1}K)$ is an isomorphism.*
(4) *For a finite morphism $f : X \to Y$ of schemes and K in $D^+(X_{étale})$ we have $a_Y^{-1}(Rf_{small,*}K) = Rf_{big,fppf,*}(a_X^{-1}K)$.*
(5) *For a proper morphism $f : X \to Y$ of schemes and K in $D^+(X_{étale})$ with torsion cohomology sheaves we have $a_Y^{-1}(Rf_{small,*}K) = Rf_{big,fppf,*}(a_X^{-1}K)$.*

Proof. By Lemma 94.5 the lemmas in Cohomology on Sites, Section 29 all apply to our current setting. To translate the results observe that the category $\mathcal{A}_X$ of Cohomology on Sites, Lemma 29.2 is the essential image of $a_X^{-1} : Ab(X_{étale}) \to Ab((Sch/X)_{fppf})$.

Part (1) is equivalent to (V_n) for all n which holds by Cohomology on Sites, Lemma 29.8.

Part (2) follows by applying ϵ_Y^{-1} to the conclusion of Cohomology on Sites, Lemma 29.3.

Part (3) follows from Cohomology on Sites, Lemma 29.8 part (1) because $\pi_X^{-1}K$ is in $D^+_{\mathcal{A}'_X}((Sch/X)_{étale})$ and $a_X^{-1} = \epsilon_X^{-1} \circ a_X^{-1}$.

Part (4) follows from Cohomology on Sites, Lemma 29.8 part (2) for the same reason.

Part (5). We use that

$$\begin{aligned} R\epsilon_{Y,*}Rf_{big,fppf,*}a_X^{-1}K &= Rf_{big,étale,*}R\epsilon_{X,*}a_X^{-1}K \\ &= Rf_{big,étale,*}\pi_X^{-1}K \\ &= \pi_Y^{-1}Rf_{small,*}K \\ &= R\epsilon_{Y,*}a_Y^{-1}Rf_{small,*}K \end{aligned}$$

The first equality by the commutative diagram in Lemma 94.2 and Cohomology on Sites, Lemma 19.2. The second equality is (3). The third is Lemma 93.5 part (2). The fourth is (3) again. Thus the base change map $a_Y^{-1}(Rf_{small,*}K) \to Rf_{big,fppf,*}(a_X^{-1}K)$ induces an isomorphism

$$R\epsilon_{Y,*}a_Y^{-1}Rf_{small,*}K \to R\epsilon_{Y,*}Rf_{big,fppf,*}a_X^{-1}K$$

The proof is finished by the following remark: a map $\alpha : a_Y^{-1}L \to M$ with L in $D^+(Y_{étale})$ and M in $D^+((Sch/Y)_{fppf})$ such that $R\epsilon_{Y,*}\alpha$ is an isomorphism, is an isomorphism. Namely, we show by induction on i that $H^i(\alpha)$ is an isomorphism. This is true for all sufficiently small i. If it holds for $i \leq i_0$, then we see that $R^j\epsilon_{Y,*}H^i(M) = 0$ for $j > 0$ and $i \leq i_0$ by (1) because $H^i(M) = a_Y^{-1}H^i(L)$ in this range. Hence $\epsilon_{Y,*}H^{i_0+1}(M) = H^{i_0+1}(R\epsilon_{Y,*}M)$ by a spectral sequence argument. Thus $\epsilon_{Y,*}H^{i_0+1}(M) = \pi_Y^{-1}H^{i_0+1}(L) = \epsilon_{Y,*}a_Y^{-1}H^{i_0+1}(L)$. This implies $H^{i_0+1}(\alpha)$ is an isomorphism (because $\epsilon_{Y,*}$ reflects isomorphisms as it is the identity on underlying presheaves) as desired. □

0DDT **Lemma 94.7.** *Let X be a scheme. For $K \in D^+(X_{étale})$ the map*

$$K \longrightarrow Ra_{X,*}a_X^{-1}K$$

is an isomorphism with $a_X : Sh((Sch/X)_{fppf}) \to Sh(X_{étale})$ as above.

Proof. We first reduce the statement to the case where K is given by a single abelian sheaf. Namely, represent K by a bounded below complex $\mathcal{F}^\bullet$. By the case of a sheaf we see that $\mathcal{F}^n = a_{X,*}a_X^{-1}\mathcal{F}^n$ and that the sheaves $R^qa_{X,*}a_X^{-1}\mathcal{F}^n$ are zero for $q > 0$. By Leray's acyclicity lemma (Derived Categories, Lemma 16.7) applied to $a_X^{-1}\mathcal{F}^\bullet$ and the functor $a_{X,*}$ we conclude. From now on assume $K = \mathcal{F}$.

By Lemma 94.1 we have $a_{X,*}a_X^{-1}\mathcal{F} = \mathcal{F}$. Thus it suffices to show that $R^qa_{X,*}a_X^{-1}\mathcal{F} = 0$ for $q > 0$. For this we can use $a_X = \epsilon_X \circ \pi_X$ and the Leray spectral sequence (Cohomology on Sites, Lemma 14.7). By Lemma 94.6 we have $R^i\epsilon_{X,*}(a_X^{-1}\mathcal{F}) = 0$ for $i > 0$ and $\epsilon_{X,*}a_X^{-1}\mathcal{F} = \pi_X^{-1}\mathcal{F}$. By Lemma 93.4 we have $R^j\pi_{X,*}(\pi_X^{-1}\mathcal{F}) = 0$ for $j > 0$. This concludes the proof. □

0DDU **Lemma 94.8.** *For a scheme X and $a_X : Sh((Sch/X)_{fppf}) \to Sh(X_{étale})$ as above:*

1. *$H^q(X_{étale}, \mathcal{F}) = H^q_{fppf}(X, a_X^{-1}\mathcal{F})$ for an abelian sheaf $\mathcal{F}$ on $X_{étale}$,*
2. *$H^q(X_{étale}, K) = H^q_{fppf}(X, a_X^{-1}K)$ for $K \in D^+(X_{étale})$.*

Example: if A is an abelian group, then $H^q_{étale}(X, \underline{A}) = H^q_{fppf}(X, \underline{A})$.

Proof. This follows from Lemma 94.7 by Cohomology on Sites, Remark 14.4. □

95. Comparing fppf and étale topologies: modules

0DEV We continue the discussion in Section 94 but in this section we briefly discuss what happens for sheaves of modules.

Let S be a scheme. The morphisms of sites ϵ_S, π_S, and their composition a_S introduced in Section 94 have natural enhancements to morphisms of ringed sites. The first is written as

$$\epsilon_S : ((Sch/S)_{fppf}, \mathcal{O}) \longrightarrow ((Sch/S)_{\acute{e}tale}, \mathcal{O})$$

Note that we can use the same symbol for the structure sheaf as indeed the sheaves have the same underlying presheaf. The second is

$$\pi_S : ((Sch/S)_{\acute{e}tale}, \mathcal{O}) \longrightarrow (S_{\acute{e}tale}, \mathcal{O}_S)$$

The third is the morphism

$$a_S : ((Sch/S)_{fppf}, \mathcal{O}) \longrightarrow (S_{\acute{e}tale}, \mathcal{O}_S)$$

We already know that the category of quasi-coherent modules on the scheme S is the same as the category of quasi-coherent modules on $(S_{\acute{e}tale}, \mathcal{O}_S)$, see Descent, Proposition 8.11. Since we are interested in stating a comparison between étale and fppf cohomology, we will in the rest of this section think of quasi-coherent sheaves in terms of the small étale site. Let us review what we already know about quasi-coherent modules on these sites.

0DEW **Lemma 95.1.** *Let S be a scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_S$-module on $S_{\acute{e}tale}$.*

(1) *The rule*
$$\mathcal{F}^a : (Sch/S)_{\acute{e}tale} \longrightarrow Ab, \quad (f : T \to S) \longmapsto \Gamma(T, f^*_{small}\mathcal{F})$$
satisfies the sheaf condition for fppf and a fortiori étale coverings,
(2) $\mathcal{F}^a = \pi_S^*\mathcal{F}$ *on* $(Sch/S)_{\acute{e}tale}$,
(3) $\mathcal{F}^a = a_S^*\mathcal{F}$ *on* $(Sch/S)_{fppf}$,
(4) *the rule $\mathcal{F} \mapsto \mathcal{F}^a$ defines an equivalence between quasi-coherent $\mathcal{O}_S$-modules and quasi-coherent modules on $((Sch/S)_{\acute{e}tale}, \mathcal{O})$,*
(5) *the rule $\mathcal{F} \mapsto \mathcal{F}^a$ defines an equivalence between quasi-coherent $\mathcal{O}_S$-modules and quasi-coherent modules on $((Sch/S)_{fppf}, \mathcal{O})$,*
(6) *we have $\epsilon_{S,*}a_S^*\mathcal{F} = \pi_S^*\mathcal{F}$ and $a_{S,*}a_S^*\mathcal{F} = \mathcal{F}$,*
(7) *we have $R^i\epsilon_{S,*}(a_S^*\mathcal{F}) = 0$ and $R^ia_{S,*}(a_S^*\mathcal{F}) = 0$ for $i > 0$.*

Proof. We urge the reader to find their own proof of these results based on the material in Descent, Section 8.

We first explain why the notation in this lemma is consistent with our earlier use of the notation $\mathcal{F}^a$ in Sections 17 and 22 and in Descent, Section 8. Namely, we know by Descent, Proposition 8.11 that there exists a quasi-coherent module $\mathcal{F}_0$ on the scheme S (in other words on the small Zariski site) such that $\mathcal{F}$ is the restriction of the rule

$$\mathcal{F}_0^a : (Sch/S)_{\acute{e}tale} \longrightarrow Ab, \quad (f : T \to S) \longmapsto \Gamma(T, f^*\mathcal{F})$$

to the subcategory $S_{\acute{e}tale} \subset (Sch/S)_{\acute{e}tale}$ where here f^* denotes usual pullback of sheaves of modules on schemes. Since $\mathcal{F}_0^a$ is pullback by the morphism of ringed sites

$$((Sch/S)_{\acute{e}tale}, \mathcal{O}) \longrightarrow (S_{Zar}, \mathcal{O}_{S_{Zar}})$$

by Descent, Remark 8.6 it follows immediately (from composition of pullbacks) that $\mathcal{F}^a = \mathcal{F}_0^a$. This proves the sheaf property even for fpqc coverings by Descent, Lemma 8.1 (see also Proposition 17.1). Then (2) and (3) follow again by Descent, Remark 8.6 and (4) and (5) follow from Descent, Proposition 8.11 (see also the meta result Theorem 17.4).

Part (6) is immediate from the description of the sheaf $\mathcal{F}^a = \pi_S^*\mathcal{F} = a_S^*\mathcal{F}$.

For any abelian $\mathcal{H}$ on $(Sch/S)_{fppf}$ the higher direct image $R^p\epsilon_{S,*}\mathcal{H}$ is the sheaf associated to the presheaf $U \mapsto H^p_{fppf}(U, \mathcal{H})$ on $(Sch/S)_{étale}$. See Cohomology on Sites, Lemma 7.4. Hence to prove $R^p\epsilon_{S,*}a_S^*\mathcal{F} = R^p\epsilon_{S,*}\mathcal{F}^a = 0$ for $p > 0$ it suffices to show that any scheme U over S has an étale covering $\{U_i \to U\}_{i\in I}$ such that $H^p_{fppf}(U_i, \mathcal{F}^a) = 0$ for $p > 0$. If we take an open covering by affines, then the required vanishing follows from comparison with usual cohomology (Descent, Proposition 8.10 or Theorem 22.4) and the vanishing of cohomology of quasi-coherent sheaves on affine schemes afforded by Cohomology of Schemes, Lemma 2.2.

To show that $R^pa_{S,*}a_S^{-1}\mathcal{F} = R^pa_{S,*}\mathcal{F}^a = 0$ for $p > 0$ we argue in exactly the same manner. This finishes the proof. □

0DEX **Lemma 95.2.** *Let S be a scheme. For $\mathcal{F}$ a quasi-coherent $\mathcal{O}_S$-module on $S_{étale}$ the maps*

$$\pi_S^*\mathcal{F} \longrightarrow R\epsilon_{S,*}(a_S^*\mathcal{F}) \quad \text{and} \quad \mathcal{F} \longrightarrow Ra_{S,*}(a_S^*\mathcal{F})$$

are isomorphisms with $a_S : Sh((Sch/S)_{fppf}) \to Sh(S_{étale})$ as above.

Proof. This is an immediate consequence of parts (6) and (7) of Lemma 95.1. □

96. Comparing ph and étale topologies

0DDV A model for this section is the section on the comparison of the usual topology and the qc topology on locally compact topological spaces as discussed in Cohomology on Sites, Section 30. We first review some material from Topologies, Sections 11 and 4.

Let S be a scheme and let $(Sch/S)_{ph}$ be a ph site. On the same underlying category with have a second topology, namely the étale topology, and hence a second site $(Sch/S)_{étale}$. The identity functor $(Sch/S)_{étale} \to (Sch/S)_{ph}$ is continuous (by More on Morphisms, Lemma 43.7 and Topologies, Lemma 7.2) and defines a morphism of sites

$$\epsilon_S : (Sch/S)_{ph} \longrightarrow (Sch/S)_{étale}$$

See Cohomology on Sites, Section 26. Please note that $\epsilon_{S,*}$ is the identity functor on underlying presheaves and that ϵ_S^{-1} associates to an étale sheaf the ph sheafification. Let $S_{étale}$ be the small étale site. There is a morphism of sites

$$\pi_S : (Sch/S)_{étale} \longrightarrow S_{étale}$$

given by the continuous functor $S_{étale} \to (Sch/S)_{étale}$, $U \mapsto U$. Namely, $S_{étale}$ has fibre products and a final object and the functor above commutes with these and Sites, Proposition 14.7 applies.

0DDW **Lemma 96.1.** *With notation as above. Let $\mathcal{F}$ be a sheaf on $S_{étale}$. The rule*

$$(Sch/S)_{ph} \longrightarrow Sets, \quad (f : X \to S) \longmapsto \Gamma(X, f_{small}^{-1}\mathcal{F})$$

is a sheaf and a fortiori a sheaf on $(Sch/S)_{étale}$. In fact this sheaf is equal to $\pi_S^{-1}\mathcal{F}$ on $(Sch/S)_{étale}$ and $\epsilon_S^{-1}\pi_S^{-1}\mathcal{F}$ on $(Sch/S)_{ph}$.

Proof. The statement about the étale topology is the content of Lemma 39.2. To finish the proof it suffices to show that $\pi_S^{-1}\mathcal{F}$ is a sheaf for the ph topology. By Topologies, Lemma 8.15 it suffices to show that given a proper surjective morphism $V \to U$ of schemes over S we have an equalizer diagram

$$(\pi_S^{-1}\mathcal{F})(U) \longrightarrow (\pi_S^{-1}\mathcal{F})(V) \rightrightarrows (\pi_S^{-1}\mathcal{F})(V \times_U V)$$

Set $\mathcal{G} = \pi_S^{-1}\mathcal{F}|_{U_{étale}}$. Consider the commutative diagram

$$\begin{tikzcd} V \times_U V \arrow[r] \arrow[d] \arrow[dr, "g"] & V \arrow[d, "f"] \\ V \arrow[r, "f"] & U \end{tikzcd}$$

We have

$$(\pi_S^{-1}\mathcal{F})(V) = \Gamma(V, f^{-1}\mathcal{G}) = \Gamma(U, f_*f^{-1}\mathcal{G})$$

where we use f_* and f^{-1} to denote functorialities between small étale sites. Second, we have

$$(\pi_S^{-1}\mathcal{F})(V \times_U V) = \Gamma(V \times_U V, g^{-1}\mathcal{G}) = \Gamma(U, g_*g^{-1}\mathcal{G})$$

The two maps in the equalizer diagram come from the two maps

$$f_*f^{-1}\mathcal{G} \longrightarrow g_*g^{-1}\mathcal{G}$$

Thus it suffices to prove $\mathcal{G}$ is the equalizer of these two maps of sheaves. Let $\overline{u}$ be a geometric point of U. Set $\Omega = \mathcal{G}_{\overline{u}}$. Taking stalks at $\overline{u}$ by Lemma 87.4 we obtain the two maps

$$H^0(V_{\overline{u}}, \underline{\Omega}) \longrightarrow H^0((V \times_U V)_{\overline{u}}, \underline{\Omega}) = H^0(V_{\overline{u}} \times_{\overline{u}} V_{\overline{u}}, \underline{\Omega})$$

where $\underline{\Omega}$ indicates the constant sheaf with value Ω. Of course these maps are the pullback by the projection maps. Then it is clear that the sections coming from pullback by projection onto the first factor are constant on the fibres of the first projection, and sections coming from pullback by projection onto the first factor are constant on the fibres of the first projection. The sections in the intersection of the images of these pullback maps are constant on all of $V_{\overline{u}} \times_{\overline{u}} V_{\overline{u}}$, i.e., these come from elements of Ω as desired. $\square$

In the situation of Lemma 96.1 the composition of ϵ_S and π_S and the equality determine a morphism of sites

$$a_S : (Sch/S)_{ph} \longrightarrow S_{étale}$$

0DDX **Lemma 96.2.** *With notation as above. Let $f : X \to Y$ be a morphism of $(Sch/S)_{ph}$. Then there are commutative diagrams of topoi*

$$\begin{tikzcd} Sh((Sch/X)_{ph}) \arrow[r, "f_{big,ph}"'] \arrow[d, "\epsilon_X"'] & Sh((Sch/Y)_{ph}) \arrow[d, "\epsilon_Y"] \\ Sh((Sch/X)_{étale}) \arrow[r, "f_{big,étale}"] & Sh((Sch/Y)_{étale}) \end{tikzcd}$$

and

$$\begin{tikzcd} Sh((Sch/X)_{ph}) \arrow[r, "f_{big,ph}"'] \arrow[d, "a_X"'] & Sh((Sch/Y)_{ph}) \arrow[d, "a_Y"] \\ Sh(X_{étale}) \arrow[r, "f_{small}"] & Sh(Y_{étale}) \end{tikzcd}$$

with $a_X = \pi_X \circ \epsilon_X$ *and* $a_Y = \pi_X \circ \epsilon_X$.

Proof. The commutativity of the diagrams follows from the discussion in Topologies, Section 11. □

0DDY **Lemma 96.3.** *In Lemma 96.2 if f is proper, then we have $a_Y^{-1} \circ f_{small,*} = f_{big,ph,*} \circ a_X^{-1}$.*

Proof. You can prove this by repeating the proof of Lemma 93.5 part (1); we will instead deduce the result from this. As $\epsilon_{Y,*}$ is the identity functor on underlying presheaves, it reflects isomorphisms. The description in Lemma 96.1 shows that $\epsilon_{Y,*} \circ a_Y^{-1} = \pi_Y^{-1}$ and similarly for X. To show that the canonical map $a_Y^{-1} f_{small,*}\mathcal{F} \to f_{big,ph,*} a_X^{-1}\mathcal{F}$ is an isomorphism, it suffices to show that

$$\begin{aligned} \pi_Y^{-1} f_{small,*}\mathcal{F} &= \epsilon_{Y,*} a_Y^{-1} f_{small,*}\mathcal{F} \\ &\to \epsilon_{Y,*} f_{big,ph,*} a_X^{-1}\mathcal{F} \\ &= f_{big,étale,*} \epsilon_{X,*} a_X^{-1}\mathcal{F} \\ &= f_{big,étale,*} \pi_X^{-1}\mathcal{F} \end{aligned}$$

is an isomorphism. This is part (1) of Lemma 93.5. □

0F0I **Lemma 96.4.** *Consider the comparison morphism $\epsilon : (Sch/S)_{ph} \to (Sch/S)_{étale}$. Let $\mathcal{P}$ denote the class of proper morphisms of schemes. For X in $(Sch/S)_{étale}$ denote $\mathcal{A}'_X \subset Ab((Sch/X)_{étale})$ the full subcategory consisting of sheaves of the form $\pi_X^{-1}\mathcal{F}$ where $\mathcal{F}$ is a torsion abelian sheaf on $X_{étale}$ Then Cohomology on Sites, Properties (1), (2), (3), (4), and (5) of Cohomology on Sites, Situation 29.1 hold.*

Proof. We first show that $\mathcal{A}'_X \subset Ab((Sch/X)_{étale})$ is a weak Serre subcategory by checking conditions (1), (2), (3), and (4) of Homology, Lemma 10.3. Parts (1), (2), (3) are immediate as π_X^{-1} is exact and fully faithful for example by Lemma 93.4. If $0 \to \pi_X^{-1}\mathcal{F} \to \mathcal{G} \to \pi_X^{-1}\mathcal{F}' \to 0$ is a short exact sequence in $Ab((Sch/X)_{étale})$ then $0 \to \mathcal{F} \to \pi_{X,*}\mathcal{G} \to \mathcal{F}' \to 0$ is exact by Lemma 93.4. In particular we see that $\pi_{X,*}\mathcal{G}$ is an abelian torsion sheaf on $X_{étale}$. Hence $\mathcal{G} = \pi_X^{-1}\pi_{X,*}\mathcal{G}$ is in $\mathcal{A}'_X$ which checks the final condition.

Cohomology on Sites, Property (1) holds by the existence of fibre products of schemes and the fact that the base change of a proper morphism of schemes is a proper morphism of schemes, see Morphisms, Lemma 39.5.

Cohomology on Sites, Property (2) follows from the commutative diagram (3) in Topologies, Lemma 4.16.

Cohomology on Sites, Property (3) is Lemma 96.1.

Cohomology on Sites, Property (4) holds by Lemma 93.5 part (2) and the fact that $R^i f_{small}\mathcal{F}$ is torsion if $\mathcal{F}$ is an abelian torsion sheaf on $X_{étale}$, see Lemma 76.2.

Cohomology on Sites, Property (5) follows from More on Morphisms, Lemma 43.1 combined with the fact that a finite morphism is proper and a surjective proper morphism defines a ph covering, see Topologies, Lemma 8.6. □

0DE4 **Lemma 96.5.** *With notation as above.*

(1) *For $X \in \mathrm{Ob}((Sch/S)_{ph})$ and an abelian torsion sheaf $\mathcal{F}$ on $X_{\acute{e}tale}$ we have $\epsilon_{X,*}a_X^{-1}\mathcal{F} = \pi_X^{-1}\mathcal{F}$ and $R^i\epsilon_{X,*}(a_X^{-1}\mathcal{F}) = 0$ for $i > 0$.*
(2) *For a proper morphism $f : X \to Y$ in $(Sch/S)_{ph}$ and abelian torsion sheaf $\mathcal{F}$ on X we have $a_Y^{-1}(R^if_{small,*}\mathcal{F}) = R^if_{big,ph,*}(a_X^{-1}\mathcal{F})$ for all i.*
(3) *For a scheme X and K in $D^+(X_{\acute{e}tale})$ with torsion cohomology sheaves the map $\pi_X^{-1}K \to R\epsilon_{X,*}(a_X^{-1}K)$ is an isomorphism.*
(4) *For a proper morphism $f : X \to Y$ of schemes and K in $D^+(X_{\acute{e}tale})$ with torsion cohomology sheaves we have $a_Y^{-1}(Rf_{small,*}K) = Rf_{big,ph,*}(a_X^{-1}K)$.*

Proof. By Lemma 96.4 the lemmas in Cohomology on Sites, Section 29 all apply to our current setting. To translate the results observe that the category $\mathcal{A}_X$ of Cohomology on Sites, Lemma 29.2 is the full subcategory of $Ab((Sch/X)_{ph})$ consisting of sheaves of the form $a_X^{-1}\mathcal{F}$ where $\mathcal{F}$ is an abelian torsion sheaf on $X_{\acute{e}tale}$.

Part (1) is equivalent to (V_n) for all n which holds by Cohomology on Sites, Lemma 29.8.

Part (2) follows by applying ϵ_Y^{-1} to the conclusion of Cohomology on Sites, Lemma 29.3.

Part (3) follows from Cohomology on Sites, Lemma 29.8 part (1) because $\pi_X^{-1}K$ is in $D^+_{\mathcal{A}'_X}((Sch/X)_{\acute{e}tale})$ and $a_X^{-1} = \epsilon_X^{-1} \circ a_X^{-1}$.

Part (4) follows from Cohomology on Sites, Lemma 29.8 part (2) for the same reason. □

0DE5 **Lemma 96.6.** *Let X be a scheme. For $K \in D^+(X_{\acute{e}tale})$ with torsion cohomology sheaves the map*

$$K \longrightarrow Ra_{X,*}a_X^{-1}K$$

is an isomorphism with $a_X : Sh((Sch/X)_{ph}) \to Sh(X_{\acute{e}tale})$ as above.

Proof. We first reduce the statement to the case where K is given by a single abelian sheaf. Namely, represent K by a bounded below complex $\mathcal{F}^\bullet$ of torsion abelian sheaves. This is possible by Cohomology on Sites, Lemma 19.7. By the case of a sheaf we see that $\mathcal{F}^n = a_{X,*}a_X^{-1}\mathcal{F}^n$ and that the sheaves $R^qa_{X,*}a_X^{-1}\mathcal{F}^n$ are zero for $q > 0$. By Leray's acyclicity lemma (Derived Categories, Lemma 16.7) applied to $a_X^{-1}\mathcal{F}^\bullet$ and the functor $a_{X,*}$ we conclude. From now on assume $K = \mathcal{F}$ where $\mathcal{F}$ is a torsion abelian sheaf.

By Lemma 96.1 we have $a_{X,*}a_X^{-1}\mathcal{F} = \mathcal{F}$. Thus it suffices to show that $R^qa_{X,*}a_X^{-1}\mathcal{F} = 0$ for $q > 0$. For this we can use $a_X = \epsilon_X \circ \pi_X$ and the Leray spectral sequence (Cohomology on Sites, Lemma 14.7). By Lemma 96.5 we have $R^i\epsilon_{X,*}(a_X^{-1}\mathcal{F}) = 0$ for $i > 0$ and $\epsilon_{X,*}a_X^{-1}\mathcal{F} = \pi_X^{-1}\mathcal{F}$. By Lemma 93.4 we have $R^j\pi_{X,*}(\pi_X^{-1}\mathcal{F}) = 0$ for $j > 0$. This concludes the proof. □

0DE6 **Lemma 96.7.** *For a scheme X and $a_X : Sh((Sch/X)_{ph}) \to Sh(X_{\acute{e}tale})$ as above:*

(1) *$H^q(X_{\acute{e}tale}, \mathcal{F}) = H^q_{ph}(X, a_X^{-1}\mathcal{F})$ for a torsion abelian sheaf $\mathcal{F}$ on $X_{\acute{e}tale}$,*

(2) $H^q(X_{\acute{e}tale}, K) = H^q_{ph}(X, a_X^{-1}K)$ *for* $K \in D^+(X_{\acute{e}tale})$ *with torsion cohomology sheaves.*

Example: if A is a torsion abelian group, then $H^q_{\acute{e}tale}(X, \underline{A}) = H^q_{ph}(X, \underline{A})$.

Proof. This follows from Lemma 96.6 by Cohomology on Sites, Remark 14.4. □

97. Comparing h and étale topologies

0EW7 A model for this section is the section on the comparison of the usual topology and the qc topology on locally compact topological spaces as discussed in Cohomology on Sites, Section 30. Moreover, this section is almost word for word the same as the section comparing the ph and étale topologies. We first review some material from Topologies, Sections 11 and 4 and More on Flatness, Section 34.

Let S be a scheme and let $(Sch/S)_h$ be an h site. On the same underlying category with have a second topology, namely the étale topology, and hence a second site $(Sch/S)_{\acute{e}tale}$. The identity functor $(Sch/S)_{\acute{e}tale} \to (Sch/S)_h$ is continuous (by More on Flatness, Lemma 34.5 and Topologies, Lemma 7.2) and defines a morphism of sites

$$\epsilon_S : (Sch/S)_h \longrightarrow (Sch/S)_{\acute{e}tale}$$

See Cohomology on Sites, Section 26. Please note that $\epsilon_{S,*}$ is the identity functor on underlying presheaves and that ϵ_S^{-1} associates to an étale sheaf the h sheafification. Let $S_{\acute{e}tale}$ be the small étale site. There is a morphism of sites

$$\pi_S : (Sch/S)_{\acute{e}tale} \longrightarrow S_{\acute{e}tale}$$

given by the continuous functor $S_{\acute{e}tale} \to (Sch/S)_{\acute{e}tale}$, $U \mapsto U$. Namely, $S_{\acute{e}tale}$ has fibre products and a final object and the functor above commutes with these and Sites, Proposition 14.7 applies.

0EW8 **Lemma 97.1.** *With notation as above. Let $\mathcal{F}$ be a sheaf on $S_{\acute{e}tale}$. The rule*

$$(Sch/S)_h \longrightarrow Sets, \quad (f : X \to S) \longmapsto \Gamma(X, f_{small}^{-1}\mathcal{F})$$

is a sheaf and a fortiori a sheaf on $(Sch/S)_{\acute{e}tale}$. In fact this sheaf is equal to $\pi_S^{-1}\mathcal{F}$ on $(Sch/S)_{\acute{e}tale}$ and $\epsilon_S^{-1}\pi_S^{-1}\mathcal{F}$ on $(Sch/S)_h$.

Proof. The statement about the étale topology is the content of Lemma 39.2. To finish the proof it suffices to show that $\pi_S^{-1}\mathcal{F}$ is a sheaf for the h topology. However, in Lemma 96.1 we have shown that $\pi_S^{-1}\mathcal{F}$ is a sheaf even in the stronger ph topology. □

In the situation of Lemma 97.1 the composition of ϵ_S and π_S and the equality determine a morphism of sites

$$a_S : (Sch/S)_h \longrightarrow S_{\acute{e}tale}$$

0EW9 **Lemma 97.2.** *With notation as above. Let $f : X \to Y$ be a morphism of $(Sch/S)_h$. Then there are commutative diagrams of topoi*

$$\begin{tikzcd} Sh((Sch/X)_h) \arrow[r, "f_{big,h}"'] \arrow[d, "\epsilon_X"'] & Sh((Sch/Y)_h) \arrow[d, "\epsilon_Y"] \\ Sh((Sch/X)_{\acute{e}tale}) \arrow[r, "f_{big,\acute{e}tale}"] & Sh((Sch/Y)_{\acute{e}tale}) \end{tikzcd}$$

and

$$\begin{tikzcd} Sh((Sch/X)_h) \arrow[r, "f_{big,h}"'] \arrow[d, "a_X"'] & Sh((Sch/Y)_h) \arrow[d, "a_Y"] \\ Sh(X_{étale}) \arrow[r, "f_{small}"] & Sh(Y_{étale}) \end{tikzcd}$$

with $a_X = \pi_X \circ \epsilon_X$ *and* $a_Y = \pi_X \circ \epsilon_X$.

Proof. The commutativity of the diagrams follows similarly to what was said in Topologies, Section 11. □

0EWA **Lemma 97.3.** *In Lemma 97.2 if f is proper, then we have $a_Y^{-1} \circ f_{small,*} = f_{big,h,*} \circ a_X^{-1}$.*

Proof. You can prove this by repeating the proof of Lemma 93.5 part (1); we will instead deduce the result from this. As $\epsilon_{Y,*}$ is the identity functor on underlying presheaves, it reflects isomorphisms. The description in Lemma 97.1 shows that $\epsilon_{Y,*} \circ a_Y^{-1} = \pi_Y^{-1}$ and similarly for X. To show that the canonical map $a_Y^{-1} f_{small,*}\mathcal{F} \to f_{big,h,*}a_X^{-1}\mathcal{F}$ is an isomorphism, it suffices to show that

$$\begin{aligned} \pi_Y^{-1} f_{small,*}\mathcal{F} &= \epsilon_{Y,*}a_Y^{-1} f_{small,*}\mathcal{F} \\ &\to \epsilon_{Y,*} f_{big,h,*}a_X^{-1}\mathcal{F} \\ &= f_{big,étale,*}\epsilon_{X,*}a_X^{-1}\mathcal{F} \\ &= f_{big,étale,*}\pi_X^{-1}\mathcal{F} \end{aligned}$$

is an isomorphism. This is part (1) of Lemma 93.5. □

0F0J **Lemma 97.4.** *Consider the comparison morphism $\epsilon : (Sch/S)_h \to (Sch/S)_{étale}$. Let $\mathcal{P}$ denote the class of proper morphisms. For X in $(Sch/S)_{étale}$ denote $\mathcal{A}'_X \subset Ab((Sch/X)_{étale})$ the full subcategory consisting of sheaves of the form $\pi_X^{-1}\mathcal{F}$ where $\mathcal{F}$ is a torsion abelian sheaf on $X_{étale}$ Then Cohomology on Sites, Properties (1), (2), (3), (4), and (5) of Cohomology on Sites, Situation 29.1 hold.*

Proof. We first show that $\mathcal{A}'_X \subset Ab((Sch/X)_{étale})$ is a weak Serre subcategory by checking conditions (1), (2), (3), and (4) of Homology, Lemma 10.3. Parts (1), (2), (3) are immediate as π_X^{-1} is exact and fully faithful for example by Lemma 93.4. If $0 \to \pi_X^{-1}\mathcal{F} \to \mathcal{G} \to \pi_X^{-1}\mathcal{F}' \to 0$ is a short exact sequence in $Ab((Sch/X)_{étale})$ then $0 \to \mathcal{F} \to \pi_{X,*}\mathcal{G} \to \mathcal{F}' \to 0$ is exact by Lemma 93.4. In particular we see that $\pi_{X,*}\mathcal{G}$ is an abelian torsion sheaf on $X_{étale}$. Hence $\mathcal{G} = \pi_X^{-1}\pi_{X,*}\mathcal{G}$ is in $\mathcal{A}'_X$ which checks the final condition.

Cohomology on Sites, Property (1) holds by the existence of fibre products of schemes, the fact that the base change of a proper morphism of schemes is a proper morphism of schemes, see Morphisms, Lemma 39.5, and the fact that the base change of a morphism of finite presentation is a morphism of finite presentation, see Morphisms, Lemma 20.4.

Cohomology on Sites, Property (2) follows from the commutative diagram (3) in Topologies, Lemma 4.16.

Cohomology on Sites, Property (3) is Lemma 97.1.

Cohomology on Sites, Property (4) holds by Lemma 93.5 part (2) and the fact that $R^i f_{small}\mathcal{F}$ is torsion if $\mathcal{F}$ is an abelian torsion sheaf on $X_{étale}$, see Lemma 76.2.

Cohomology on Sites, Property (5) is implied by More on Morphisms, Lemma 43.1 combined with the fact that a surjective finite locally free morphism is surjective, proper, and of finite presentation and hence defines a h-covering by More on Flatness, Lemma 34.6. ☐

0EWF **Lemma 97.5.** *With notation as above.*

(1) *For $X \in \mathrm{Ob}((Sch/S)_h)$ and an abelian torsion sheaf $\mathcal{F}$ on $X_{\acute{e}tale}$ we have $\epsilon_{X,*}a_X^{-1}\mathcal{F} = \pi_X^{-1}\mathcal{F}$ and $R^i\epsilon_{X,*}(a_X^{-1}\mathcal{F}) = 0$ for $i > 0$.*
(2) *For a proper morphism $f : X \to Y$ in $(Sch/S)_h$ and abelian torsion sheaf $\mathcal{F}$ on X we have $a_Y^{-1}(R^i f_{small,*}\mathcal{F}) = R^i f_{big,h,*}(a_X^{-1}\mathcal{F})$ for all i.*
(3) *For a scheme X and K in $D^+(X_{\acute{e}tale})$ with torsion cohomology sheaves the map $\pi_X^{-1}K \to R\epsilon_{X,*}(a_X^{-1}K)$ is an isomorphism.*
(4) *For a proper morphism $f : X \to Y$ of schemes and K in $D^+(X_{\acute{e}tale})$ with torsion cohomology sheaves we have $a_Y^{-1}(Rf_{small,*}K) = Rf_{big,h,*}(a_X^{-1}K)$.*

Proof. By Lemma 97.4 the lemmas in Cohomology on Sites, Section 29 all apply to our current setting. To translate the results observe that the category $\mathcal{A}_X$ of Cohomology on Sites, Lemma 29.2 is the full subcategory of $Ab((Sch/X)_h)$ consisting of sheaves of the form $a_X^{-1}\mathcal{F}$ where $\mathcal{F}$ is an abelian torsion sheaf on $X_{\acute{e}tale}$.

Part (1) is equivalent to (V_n) for all n which holds by Cohomology on Sites, Lemma 29.8.

Part (2) follows by applying ϵ_Y^{-1} to the conclusion of Cohomology on Sites, Lemma 29.3.

Part (3) follows from Cohomology on Sites, Lemma 29.8 part (1) because $\pi_X^{-1}K$ is in $D^+_{\mathcal{A}'_X}((Sch/X)_{\acute{e}tale})$ and $a_X^{-1} = \epsilon_X^{-1} \circ a_X^{-1}$.

Part (4) follows from Cohomology on Sites, Lemma 29.8 part (2) for the same reason. ☐

0EWG **Lemma 97.6.** *Let X be a scheme. For $K \in D^+(X_{\acute{e}tale})$ with torsion cohomology sheaves the map*

$$K \longrightarrow Ra_{X,*}a_X^{-1}K$$

is an isomorphism with $a_X : Sh((Sch/X)_h) \to Sh(X_{\acute{e}tale})$ as above.

Proof. We first reduce the statement to the case where K is given by a single abelian sheaf. Namely, represent K by a bounded below complex $\mathcal{F}^\bullet$ of torsion abelian sheaves. This is possible by Cohomology on Sites, Lemma 19.7. By the case of a sheaf we see that $\mathcal{F}^n = a_{X,*}a_X^{-1}\mathcal{F}^n$ and that the sheaves $R^q a_{X,*}a_X^{-1}\mathcal{F}^n$ are zero for $q > 0$. By Leray's acyclicity lemma (Derived Categories, Lemma 16.7) applied to $a_X^{-1}\mathcal{F}^\bullet$ and the functor $a_{X,*}$ we conclude. From now on assume $K = \mathcal{F}$ where $\mathcal{F}$ is a torsion abelian sheaf.

By Lemma 97.1 we have $a_{X,*}a_X^{-1}\mathcal{F} = \mathcal{F}$. Thus it suffices to show that $R^q a_{X,*}a_X^{-1}\mathcal{F} = 0$ for $q > 0$. For this we can use $a_X = \epsilon_X \circ \pi_X$ and the Leray spectral sequence (Cohomology on Sites, Lemma 14.7). By Lemma 97.5 we have $R^i\epsilon_{X,*}(a_X^{-1}\mathcal{F}) = 0$ for $i > 0$ and $\epsilon_{X,*}a_X^{-1}\mathcal{F} = \pi_X^{-1}\mathcal{F}$. By Lemma 93.4 we have $R^j\pi_{X,*}(\pi_X^{-1}\mathcal{F}) = 0$ for $j > 0$. This concludes the proof. ☐

0EWH **Lemma 97.7.** *For a scheme X and $a_X : Sh((Sch/X)_h) \to Sh(X_{\acute{e}tale})$ as above:*

(1) *$H^q(X_{\acute{e}tale}, \mathcal{F}) = H^q_h(X, a_X^{-1}\mathcal{F})$ for a torsion abelian sheaf $\mathcal{F}$ on $X_{\acute{e}tale}$,*

(2) *$H^q(X_{étale}, K) = H^q_h(X, a_X^{-1}K)$ for $K \in D^+(X_{étale})$ with torsion cohomology sheaves.*

Example: if A is a torsion abelian group, then $H^q_{étale}(X, \underline{A}) = H^q_h(X, \underline{A})$.

Proof. This follows from Lemma 97.6 by Cohomology on Sites, Remark 14.4. $\square$

98. Blow up squares and étale cohomology

0EW4 Blow up squares are introduced in More on Flatness, Section 36. Using the proper base change theorem we can see that we have a Mayer-Vietoris type result for blow up squares.

0EW5 **Lemma 98.1.** *Let X be a scheme and let $Z \subset X$ be a closed subscheme cut out by a quasi-coherent ideal of finite type. Consider the corresponding blow up square*

$$\begin{tikzcd} E \arrow[r, "j"] \arrow[d, "\pi"] & X' \arrow[d, "b"] \\ Z \arrow[r, "i"] & X \end{tikzcd}$$

For $K \in D^+(X_{étale})$ with torsion cohomology sheaves we have a distinguished triangle

$$K \to Ri_*(K|_Z) \oplus Rb_*(K|_{X'}) \to Rc_*(K|_E) \to K[1]$$

in $D(X_{étale})$ where $c = i \circ \pi = b \circ j$.

Proof. The notation $K|_{X'}$ stands for $b_{small}^{-1}K$. Choose a bounded below complex $\mathcal{F}^\bullet$ of abelian sheaves representing K. Observe that $i_*(\mathcal{F}^\bullet|_Z)$ represents $Ri_*(K|_Z)$ because i_* is exact (Proposition 54.2). Choose a quasi-isomorphism $b_{small}^{-1}\mathcal{F}^\bullet \to \mathcal{I}^\bullet$ where $\mathcal{I}^\bullet$ is a bounded below complex of injective abelian sheaves on $X'_{étale}$. This map is adjoint to a map $\mathcal{F}^\bullet \to b_*(\mathcal{I}^\bullet)$ and $b_*(\mathcal{I}^\bullet)$ represents $Rb_*(K|_{X'})$. We have $\pi_*(\mathcal{I}^\bullet|_E) = (b_*\mathcal{I}^\bullet)|_Z$ by Lemma 87.5 and by Lemma 87.12 this complex represents $R\pi_*(K|_E)$. Hence the map

$$Ri_*(K|_Z) \oplus Rb_*(K|_{X'}) \to Rc_*(K|_E)$$

is represented by the surjective map of bounded below complexes

$$i_*(\mathcal{F}^\bullet|_Z) \oplus b_*(\mathcal{I}^\bullet) \to i_*\left(b_*(\mathcal{I}^\bullet)|_Z\right)$$

To get our distinguished triangle it suffices to show that the canonical map $\mathcal{F}^\bullet \to i_*(\mathcal{F}^\bullet|_Z) \oplus b_*(\mathcal{I}^\bullet)$ maps quasi-isomorphically onto the kernel of the map of complexes displayed above (namely a short exact sequence of complexes determines a distinguished triangle in the derived category, see Derived Categories, Section 12). We may check this on stalks at a geometric point $\overline{x}$ of X. If $\overline{x}$ is not in Z, then $X' \to X$ is an isomorphism over an open neighbourhood of $\overline{x}$. Thus, if $\overline{x}'$ denotes the corresponding geometric point of X' in this case, then we have to show that

$$\mathcal{F}^\bullet_{\overline{x}} \to \mathcal{I}^\bullet_{\overline{x}'}$$

is a quasi-isomorphism. This is true by our choice of $\mathcal{I}^\bullet$. If $\overline{x}$ is in Z, then $b_(\mathcal{I}^\bullet)_{\overline{x}} \to i_*\left(b_*(\mathcal{I}^\bullet)|_Z\right)_{\overline{x}}$ is an isomorphism of complexes of abelian groups. Hence the kernel is equal to $i_*(\mathcal{F}^\bullet|_Z)_{\overline{x}} = \mathcal{F}^\bullet_{\overline{x}}$ as desired. $\square$

0EW3 **Lemma 98.2.** *Let X be a scheme and let $K \in D^+(X_{étale})$ have torsion cohomology sheaves. Let $Z \subset X$ be a closed subscheme cut out by a quasi-coherent ideal of finite type. Consider the corresponding blow up square*

$$\begin{tikzcd} E \arrow[r] \arrow[d] & X' \arrow[d, "b"] \\ Z \arrow[r] & X \end{tikzcd}$$

Then there is a canonical long exact sequence

$$H^p_{étale}(X, K) \to H^p_{étale}(X', K|_{X'}) \oplus H^p_{étale}(Z, K|_Z) \to H^p_{étale}(E, K|_E) \to H^{p+1}_{étale}(X, K)$$

First proof. This follows immediately from Lemma 98.1 and the fact that

$$R\Gamma(X, Rb_*(K|_{X'})) = R\Gamma(X', K|_{X'})$$

(see Cohomology on Sites, Section 14) and similarly for the others. □

Second proof. By Lemma 96.7 these cohomology groups are the cohomology of X, X', E, Z with values in some complex of abelian sheaves on the site $(Sch/X)_{ph}$. (Namely, the object $a_X^{-1}K$ of the derived category, see Lemma 96.1 above and recall that $K|_{X'} = b_{small}^{-1}K$.) By More on Flatness, Lemma 36.1 the ph sheafification of the diagram of representable presheaves is cocartesian. Thus the lemma follows from the very general Cohomology on Sites, Lemma 25.3 applied to the site $(Sch/X)_{ph}$ and the commutative diagram of the lemma. □

0EW6 **Lemma 98.3.** *Let X be a scheme and let $Z \subset X$ be a closed subscheme cut out by a quasi-coherent ideal of finite type. Consider the corresponding blow up square*

$$\begin{tikzcd} E \arrow[r, "j"'] \arrow[d, "\pi"'] & X' \arrow[d, "b"] \\ Z \arrow[r, "i"] & X \end{tikzcd}$$

Suppose given

1. *an object K' of $D^+(X'_{étale})$ with torsion cohomology sheaves,*
2. *an object L of $D^+(Z_{étale})$ with torsion cohomology sheaves, and*
3. *an isomorphism $\gamma : K'|_E \to L|_E$.*

Then there exists an object K of $D^+(X_{étale})$ and isomorphisms $f : K|_{X'} \to K'$, $g : K|_Z \to L$ such that $\gamma = g|_E \circ f^{-1}|_E$. Moreover, given

1. *an object M of $D^+(X_{étale})$ with torsion cohomology sheaves,*
2. *a morphism $\alpha : K' \to M|_{X'}$ of $D(X'_{étale})$,*
3. *a morphism $\beta : L \to M|_Z$ of $D(Z_{étale})$,*

such that

$$\alpha|_E = \beta|_E \circ \gamma.$$

Then there exists a morphism $M \to K$ in $D(X_{étale})$ whose restriction to X' is $a \circ f$ and whose restriction to Z is $b \circ g$.

Proof. If K exists, then Lemma 98.1 tells us a distinguished triangle that it fits in. Thus we simply choose a distinguished triangle

$$K \to Ri_*(L) \oplus Rb_*(K') \to Rc_*(L|_E) \to K[1]$$

where $c = i \circ \pi = b \circ j$. Here the map $Ri_*(L) \to Rc_*(L|_E)$ is Ri_* applied to the adjunction mapping $E \to R\pi_*(L|_E)$. The map $Rb_*(K') \to Rc_*(L|_E)$ is the composition of the canonical map $Rb_*(K') \to Rc_*(K'|_E)) = R$ and $Rc_*(\gamma)$. The maps g and f of the statement of the lemma are the adjoints of these maps. If we restrict this distinguished triangle to Z then the map $Rb_*(K) \to Rc_*(L|_E)$ becomes an isomorphism by the base change theorem (Lemma 87.12) and hence the map $g : K|_Z \to L$ is an isomorphism. Looking at the distinguished triangle we see that $f : K|_{X'} \to K'$ is an isomorphism over $X' \setminus E = X \setminus Z$. Moreover, we have $\gamma \circ f|_E = g|_E$ by construction. Then since γ and g are isomorphisms we conclude that f induces isomorphisms on stalks at geometric points of E as well. Thus f is an isomorphism.

For the final statement, we may replace K' by $K|_{X'}$, L by $K|_Z$, and γ by the canonical identification. Observe that α and β induce a commutative square

$$\begin{tikzcd}
K \arrow[r] \arrow[d, dotted] & Ri_*(K|_Z) \oplus Rb_*(K|_{X'}) \arrow[r] \arrow[d, "\beta\oplus\alpha"'] & Rc_*(K|_E) \arrow[r] \arrow[d, "\alpha|_E"'] & K[1] \arrow[d, dotted] \\
M \arrow[r] & Ri_*(M|_Z) \oplus Rb_*(M|_{X'}) \arrow[r] & Rc_*(M|_E) \arrow[r] & M[1]
\end{tikzcd}$$

Thus by the axioms of a derived category we get a dotted arrow producing a morphism of distinguished triangles. ☐

99. Almost blow up squares and the h topology

0EWL In this section we continue the discussion in More on Flatness, Section 37. For the convenience of the reader we recall that an almost blow up square is a commutative diagram

0EWM (99.0.1)
$$\begin{tikzcd}
E \arrow[r] \arrow[d] & X' \arrow[d, "b"] \\
Z \arrow[r] & X
\end{tikzcd}$$

of schemes satisfying the following conditions:

(1) $Z \to X$ is a closed immersion of finite presentation,
(2) $E = b^{-1}(Z)$ is a locally principal closed subscheme of X',
(3) b is proper and of finite presentation,
(4) the closed subscheme $X'' \subset X'$ cut out by the quasi-coherent ideal of sections of $\mathcal{O}_{X'}$ supported on E (Properties, Lemma 24.5) is the blow up of X in Z.

It follows that the morphism b induces an isomorphism $X' \setminus E \to X \setminus Z$.

We are going to give a criterion for "h sheafiness" for objects in the derived category of the big fppf site $(Sch/S)_{fppf}$ of a scheme S. On the same underlying category we have a second topology, namely the h topology (More on Flatness, Section 34). Recall that fppf coverings are h coverings (More on Flatness, Lemma 34.5). Hence we may consider the morphism

$$\epsilon : (Sch/S)_h \longrightarrow (Sch/S)_{fppf}$$

See Cohomology on Sites, Section 26. In particular, we have a fully faithful functor

$$R\epsilon_* : D((Sch/S)_h) \longrightarrow D((Sch/S)_{fppf})$$

and we can ask: what is the essential image of this functor?

0EWN **Lemma 99.1.** *With notation as above, if K is in the essential image of $R\epsilon_*$, then the maps $c^K_{X,Z,X',E}$ of Cohomology on Sites, Lemma 25.1 are quasi-isomorphisms.*

Proof. Denote $^\#$ sheafification in the h topology. We have seen in More on Flatness, Lemma 37.7 that $h_X^\# = h_Z^\# \amalg_{h_E^\#} h_{X'}^\#$. On the other hand, the map $h_E^\# \to h_{X'}^\#$ is injective as $E \to X'$ is a monomorphism. Thus this lemma is a special case of Cohomology on Sites, Lemma 28.3 (which itself is a formal consequence of Cohomology on Sites, Lemma 25.3). □

0EWQ **Proposition 99.2.** *Let K be an object of $D^+((Sch/S)_{fppf})$. Then K is in the essential image of $R\epsilon_* : D((Sch/S)_h) \to D((Sch/S)_{fppf})$ if and only if $c^K_{X,X',Z,E}$ is a quasi-isomorphism for every almost blow up square (99.0.1) in $(Sch/S)_h$ with X affine.*

Proof. We prove this by applying Cohomology on Sites, Lemma 28.2 whose hypotheses hold by Lemma 99.1 and More on Flatness, Proposition 37.9. □

0EWR **Lemma 99.3.** *Let K be an object of $D^+((Sch/S)_{fppf})$. Then K is in the essential image of $R\epsilon_* : D((Sch/S)_h) \to D((Sch/S)_{fppf})$ if and only if $c^K_{X,X',Z,E}$ is a quasi-isomorphism for every almost blow up square as in More on Flatness, Examples 37.10 and 37.11.*

Proof. We prove this by applying Cohomology on Sites, Lemma 28.2 whose hypotheses hold by Lemma 99.1 and More on Flatness, Lemma 37.12 □

100. Cohomology of the structure sheaf in the h topology

0EWS Let p be a prime number. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site with $p\mathcal{O} = 0$. Then we set $\mathrm{colim}_F\, \mathcal{O}$ equal to the colimit in the category of sheaves of rings of the system

$$\mathcal{O} \xrightarrow{F} \mathcal{O} \xrightarrow{F} \mathcal{O} \xrightarrow{F} \ldots$$

where $F : \mathcal{O} \to \mathcal{O}$, $f \mapsto f^p$ is the Frobenius endomorphism.

0EWT **Lemma 100.1.** *Let p be a prime number. Let S be a scheme over $\mathbf{F}_p$. Consider the sheaf $\mathcal{O}^{perf} = \mathrm{colim}_F\, \mathcal{O}$ on $(Sch/S)_{fppf}$. Then $\mathcal{O}^{perf}$ is in the essential image of $R\epsilon_* : D((Sch/S)_h) \to D((Sch/S)_{fppf})$.*

Proof. We prove this using the criterion of Lemma 99.3. Before check the conditions, we note that for a quasi-compact and quasi-separated object X of $(Sch/S)_{fppf}$ we have

$$H^i_{fppf}(X, \mathcal{O}^{perf}) = \mathrm{colim}_F\, H^i_{fppf}(X, \mathcal{O})$$

See Cohomology on Sites, Lemma 16.1. We will also use that $H^i_{fppf}(X, \mathcal{O}) = H^i(X, \mathcal{O})$, see Descent, Proposition 8.10.

Let A, f, J be as in More on Flatness, Example 37.10 and consider the associated almost blow up square. Since X, X', Z, E are affine, we have no higher cohomology of $\mathcal{O}$. Hence we only have to check that

$$0 \to \mathcal{O}^{perf}(X) \to \mathcal{O}^{perf}(X') \oplus \mathcal{O}^{perf}(Z) \to \mathcal{O}^{perf}(E) \to 0$$

is a short exact sequence. This was shown in (the proof of) More on Flatness, Lemma 38.2.

Let X, X', Z, E be as in More on Flatness, Example 37.11. Since X and Z are affine we have $H^p(X, \mathcal{O}_X) = H^p(Z, \mathcal{O}_X) = 0$ for $p > 0$. By More on Flatness, Lemma 38.1 we have $H^p(X', \mathcal{O}_{X'}) = 0$ for $p > 0$. Since $E = \mathbf{P}^1_Z$ and Z is affine we also have $H^p(E, \mathcal{O}_E) = 0$ for $p > 0$. As in the previous paragraph we reduce to checking that

$$0 \to \mathcal{O}^{perf}(X) \to \mathcal{O}^{perf}(X') \oplus \mathcal{O}^{perf}(Z) \to \mathcal{O}^{perf}(E) \to 0$$

is a short exact sequence. This was shown in (the proof of) More on Flatness, Lemma 38.2. □

0EWU **Proposition 100.2.** *Let p be a prime number. Let S be a quasi-compact and quasi-separated scheme over $\mathbf{F}_p$. Then*

$$H^i((Sch/S)_h, \mathcal{O}^h) = \operatorname{colim}_F H^i(S, \mathcal{O})$$

Here on the left hand side by $\mathcal{O}^h$ we mean the h sheafification of the structure sheaf.

Proof. This is just a reformulation of Lemma 100.1. Recall that $\mathcal{O}^h = \mathcal{O}^{perf} = \operatorname{colim}_F \mathcal{O}$, see More on Flatness, Lemma 38.7. By Lemma 100.1 we see that $\mathcal{O}^{perf}$ viewed as an object of $D((Sch/S)_{fppf})$ is of the form $R\epsilon_* K$ for some $K \in D((Sch/S)_h)$. Then $K = \epsilon^{-1}\mathcal{O}^{perf}$ which is actually equal to $\mathcal{O}^{perf}$ because $\mathcal{O}^{perf}$ is an h sheaf. See Cohomology on Sites, Section 26. Hence $R\epsilon_*\mathcal{O}^{perf} = \mathcal{O}^{perf}$ (with apologies for the confusing notation). Thus the lemma now follows from Leray

$$R\Gamma_h(S, \mathcal{O}^{perf}) = R\Gamma_{fppf}(S, R\epsilon_*\mathcal{O}^{perf}) = R\Gamma_{fppf}(S, \mathcal{O}^{perf})$$

and the fact that

$$H^i_{fppf}(S, \mathcal{O}^{perf}) = H^i_{fppf}(S, \operatorname{colim}_F \mathcal{O}) = \operatorname{colim}_F H^i_{fppf}(S, \mathcal{O})$$

as S is quasi-compact and quasi-separated (see proof of Lemma 100.1). □

101. Other chapters

Preliminaries

(1) Introduction
(2) Conventions
(3) Set Theory
(4) Categories
(5) Topology
(6) Sheaves on Spaces
(7) Sites and Sheaves
(8) Stacks
(9) Fields
(10) Commutative Algebra
(11) Brauer Groups
(12) Homological Algebra
(13) Derived Categories
(14) Simplicial Methods
(15) More on Algebra
(16) Smoothing Ring Maps
(17) Sheaves of Modules
(18) Modules on Sites
(19) Injectives
(20) Cohomology of Sheaves
(21) Cohomology on Sites
(22) Differential Graded Algebra
(23) Divided Power Algebra
(24) Hypercoverings

Schemes

(25) Schemes
(26) Constructions of Schemes
(27) Properties of Schemes
(28) Morphisms of Schemes
(29) Cohomology of Schemes
(30) Divisors
(31) Limits of Schemes
(32) Varieties
(33) Topologies on Schemes
(34) Descent
(35) Derived Categories of Schemes
(36) More on Morphisms
(37) More on Flatness
(38) Groupoid Schemes

(39) More on Groupoid Schemes
(40) Étale Morphisms of Schemes

Topics in Scheme Theory

(41) Chow Homology
(42) Intersection Theory
(43) Picard Schemes of Curves
(44) Weil Cohomology Theories
(45) Adequate Modules
(46) Dualizing Complexes
(47) Duality for Schemes
(48) Discriminants and Differents
(49) de Rham Cohomology
(50) Local Cohomology
(51) Algebraic and Formal Geometry
(52) Algebraic Curves
(53) Resolution of Surfaces
(54) Semistable Reduction
(55) Fundamental Groups of Schemes
(56) Étale Cohomology
(57) Crystalline Cohomology
(58) Pro-étale Cohomology
(59) More Étale Cohomology
(60) The Trace Formula

Algebraic Spaces

(61) Algebraic Spaces
(62) Properties of Algebraic Spaces
(63) Morphisms of Algebraic Spaces
(64) Decent Algebraic Spaces
(65) Cohomology of Algebraic Spaces
(66) Limits of Algebraic Spaces
(67) Divisors on Algebraic Spaces
(68) Algebraic Spaces over Fields
(69) Topologies on Algebraic Spaces
(70) Descent and Algebraic Spaces
(71) Derived Categories of Spaces
(72) More on Morphisms of Spaces
(73) Flatness on Algebraic Spaces
(74) Groupoids in Algebraic Spaces
(75) More on Groupoids in Spaces
(76) Bootstrap
(77) Pushouts of Algebraic Spaces

Topics in Geometry

(78) Chow Groups of Spaces
(79) Quotients of Groupoids
(80) More on Cohomology of Spaces
(81) Simplicial Spaces
(82) Duality for Spaces
(83) Formal Algebraic Spaces
(84) Restricted Power Series
(85) Resolution of Surfaces Revisited

Deformation Theory

(86) Formal Deformation Theory
(87) Deformation Theory
(88) The Cotangent Complex
(89) Deformation Problems

Algebraic Stacks

(90) Algebraic Stacks
(91) Examples of Stacks
(92) Sheaves on Algebraic Stacks
(93) Criteria for Representability
(94) Artin's Axioms
(95) Quot and Hilbert Spaces
(96) Properties of Algebraic Stacks
(97) Morphisms of Algebraic Stacks
(98) Limits of Algebraic Stacks
(99) Cohomology of Algebraic Stacks
(100) Derived Categories of Stacks
(101) Introducing Algebraic Stacks
(102) More on Morphisms of Stacks
(103) The Geometry of Stacks

Topics in Moduli Theory

(104) Moduli Stacks
(105) Moduli of Curves

Miscellany

(106) Examples
(107) Exercises
(108) Guide to Literature
(109) Desirables
(110) Coding Style
(111) Obsolete
(112) GNU Free Documentation License
(113) Auto Generated Index

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