ÉTALE MORPHISMS OF SCHEMES

024J

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1. Introduction

024K In this Chapter, we discuss étale morphisms of schemes. We illustrate some of the more important concepts by working with the Noetherian case. Our principal goal is to collect for the reader enough commutative algebra results to start reading a treatise on étale cohomology. An auxiliary goal is to provide enough evidence to ensure that the reader stops calling the phrase "the étale topology of schemes" an exercise in general nonsense, if (s)he does indulge in such blasphemy.

We will refer to the other chapters of the Stacks project for standard results in algebraic geometry (on schemes and commutative algebra). We will provide detailed proofs of the new results that we state here.

2. Conventions

039F In this chapter, frequently schemes will be assumed locally Noetherian and frequently rings will be assumed Noetherian. But in all the statements we will reiterate this when necessary, and make sure we list all the hypotheses! On the other hand, here are some general facts that we will use often and are useful to keep in mind:

(1) A ring homomorphism $A \to B$ of finite type with A Noetherian is of finite presentation. See Algebra, Lemma 31.4.
(2) A morphism (locally) of finite type between locally Noetherian schemes is automatically (locally) of finite presentation. See Morphisms, Lemma 22.9.
(3) Add more like this here.

3. Unramified morphisms

024L We first define "unramified homomorphisms of local rings" for Noetherian local rings. We cannot use the term "unramified" as there already is a notion of an unramified ring map (Algebra, Section 151) and it is different. After discussing the notion a bit we globalize it to describe unramified morphisms of locally Noetherian schemes.

024M **Definition 3.1.** Let A, B be Noetherian local rings. A local homomorphism $A \to B$ is said to be *unramified homomorphism of local rings* if

(1) $\mathfrak{m}_A B = \mathfrak{m}_B$,
(2) $\kappa(\mathfrak{m}_B)$ is a finite separable extension of $\kappa(\mathfrak{m}_A)$, and
(3) B is essentially of finite type over A (this means that B is the localization of a finite type A-algebra at a prime).

This is the local version of the definition in Algebra, Section 151. In that section a ring map $R \to S$ is defined to be unramified if and only if it is of finite type, and $\Omega_{S/R} = 0$. We say $R \to S$ is unramified at a prime $\mathfrak{q} \subset S$ if there exists a $g \in S$, $g \notin \mathfrak{q}$ such that $R \to S_g$ is an unramified ring map. It is shown in Algebra, Lemmas 151.5 and 151.7 that given a ring map $R \to S$ of finite type, and a prime $\mathfrak{q}$ of S lying over $\mathfrak{p} \subset R$, then we have

$$R \to S \text{ is unramified at } \mathfrak{q} \Leftrightarrow \mathfrak{p}S_\mathfrak{q} = \mathfrak{q}S_\mathfrak{q} \text{ and } \kappa(\mathfrak{p}) \subset \kappa(\mathfrak{q}) \text{ finite separable}$$

Thus we see that for a local homomorphism of local rings the properties of our definition above are closely related to the question of being unramified. In fact, we have proved the following lemma.

039G **Lemma 3.2.** *Let $A \to B$ be of finite type with A a Noetherian ring. Let $\mathfrak{q}$ be a prime of B lying over $\mathfrak{p} \subset A$. Then $A \to B$ is unramified at $\mathfrak{q}$ if and only if $A_\mathfrak{p} \to B_\mathfrak{q}$ is an unramified homomorphism of local rings.*

Proof. See discussion above. □

We will characterize the property of being unramified in terms of completions. For a Noetherian local ring A we denote $A^\wedge$ the completion of A with respect to the maximal ideal. It is also a Noetherian local ring, see Algebra, Lemma 97.6.

039H **Lemma 3.3.** *Let A, B be Noetherian local rings. Let $A \to B$ be a local homomorphism.*

(1) *if $A \to B$ is an unramified homomorphism of local rings, then $B^\wedge$ is a finite $A^\wedge$ module,*
(2) *if $A \to B$ is an unramified homomorphism of local rings and $\kappa(\mathfrak{m}_A) = \kappa(\mathfrak{m}_B)$, then $A^\wedge \to B^\wedge$ is surjective,*
(3) *if $A \to B$ is an unramified homomorphism of local rings and $\kappa(\mathfrak{m}_A)$ is separably closed, then $A^\wedge \to B^\wedge$ is surjective,*
(4) *if A and B are complete discrete valuation rings, then $A \to B$ is an unramified homomorphism of local rings if and only if the uniformizer for A maps to a uniformizer for B, and the residue field extension is finite separable (and B is essentially of finite type over A).*

Proof. Part (1) is a special case of Algebra, Lemma 97.7. For part (2), note that the $\kappa(\mathfrak{m}_A)$-vector space $B^\wedge/\mathfrak{m}_{A^\wedge}B^\wedge$ is generated by 1. Hence by Nakayama's lemma (Algebra, Lemma 20.1) the map $A^\wedge \to B^\wedge$ is surjective. Part (3) is a special case of part (2). Part (4) is immediate from the definitions. □

039I **Lemma 3.4.** *Let A, B be Noetherian local rings. Let $A \to B$ be a local homomorphism such that B is essentially of finite type over A. The following are equivalent*

(1) *$A \to B$ is an unramified homomorphism of local rings*
(2) *$A^\wedge \to B^\wedge$ is an unramified homomorphism of local rings, and*
(3) *$A^\wedge \to B^\wedge$ is unramified.*

Proof. The equivalence of (1) and (2) follows from the fact that $\mathfrak{m}_A A^\wedge$ is the maximal ideal of $A^\wedge$ (and similarly for B) and faithful flatness of $B \to B^\wedge$. For example if $A^\wedge \to B^\wedge$ is unramified, then $\mathfrak{m}_A B^\wedge = (\mathfrak{m}_A B)B^\wedge = \mathfrak{m}_B B^\wedge$ and hence $\mathfrak{m}_A B = \mathfrak{m}_B$.

Assume the equivalent conditions (1) and (2). By Lemma 3.3 we see that $A^\wedge \to B^\wedge$ is finite. Hence $A^\wedge \to B^\wedge$ is of finite presentation, and by Algebra, Lemma 151.7 we conclude that $A^\wedge \to B^\wedge$ is unramified at $\mathfrak{m}_{B^\wedge}$. Since $B^\wedge$ is local we conclude that $A^\wedge \to B^\wedge$ is unramified.

Assume (3). By Algebra, Lemma 151.5 we conclude that $A^\wedge \to B^\wedge$ is an unramified homomorphism of local rings, i.e., (2) holds. □

024N **Definition 3.5.** (See Morphisms, Definition 36.1 for the definition in the general case.) Let Y be a locally Noetherian scheme. Let $f : X \to Y$ be locally of finite type. Let $x \in X$.

(1) We say f is *unramified at x* if $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is an unramified homomorphism of local rings.
(2) The morphism $f : X \to Y$ is said to be *unramified* if it is unramified at all points of X.

Let us prove that this definition agrees with the definition in the chapter on morphisms of schemes. This in particular guarantees that the set of points where a morphism is unramified is open.

039J **Lemma 3.6.** *Let Y be a locally Noetherian scheme. Let $f : X \to Y$ be locally of finite type. Let $x \in X$. The morphism f is unramified at x in the sense of Definition 3.5 if and only if it is unramified in the sense of Morphisms, Definition 36.1.*

Proof. This follows from Lemma 3.2 and the definitions. □

Here are some results on unramified morphisms. The formulations as given in this list apply only to morphisms locally of finite type between locally Noetherian schemes. In each case we give a reference to the general result as proved earlier in the project, but in some cases one can prove the result more easily in the Noetherian case. Here is the list:

(1) Unramifiedness is local on the source and the target in the Zariski topology.
(2) Unramified morphisms are stable under base change and composition. See Morphisms, Lemmas 36.5 and 36.4.
(3) Unramified morphisms of schemes are locally quasi-finite and quasi-compact unramified morphisms are quasi-finite. See Morphisms, Lemma 36.10
(4) Unramified morphisms have relative dimension 0. See Morphisms, Definition 30.1 and Morphisms, Lemma 30.5.
(5) A morphism is unramified if and only if all its fibres are unramified. That is, unramifiedness can be checked on the scheme theoretic fibres. See Morphisms, Lemma 36.12.
(6) Let X and Y be unramified over a base scheme S. Any S-morphism from X to Y is unramified. See Morphisms, Lemma 36.16.

4. Three other characterizations of unramified morphisms

024O The following theorem gives three equivalent notions of being unramified at a point. See Morphisms, Lemma 36.14 for (part of) the statement for general schemes.

024P **Theorem 4.1.** *Let Y be a locally Noetherian scheme. Let $f : X \to Y$ be a morphism of schemes which is locally of finite type. Let x be a point of X. The following are equivalent*

(1) *f is unramified at x,*
(2) *the stalk $\Omega_{X/Y,x}$ of the module of relative differentials at x is trivial,*
(3) *there exist open neighbourhoods U of x and V of $f(x)$, and a commutative diagram*

$$\begin{array}{ccc} U & \xrightarrow{\quad i \quad} & \mathbf{A}^n_V \\ & \searrow \quad \swarrow & \\ & V & \end{array}$$

where i is a closed immersion defined by a quasi-coherent sheaf of ideals $\mathcal{I}$ such that the differentials dg for $g \in \mathcal{I}_{i(x)}$ generate $\Omega_{\mathbf{A}^n_V/V,i(x)}$, and
(4) *the diagonal $\Delta_{X/Y} : X \to X \times_Y X$ is a local isomorphism at x.*

Proof. The equivalence of (1) and (2) is proved in Morphisms, Lemma 36.14.

If f is unramified at x, then f is unramified in an open neighbourhood of x; this does not follow immediately from Definition 3.5 of this chapter but it does follow from Morphisms, Definition 36.1 which we proved to be equivalent in Lemma 3.6. Choose affine opens $V \subset Y$, $U \subset X$ with $f(U) \subset V$ and $x \in U$, such that f is unramified on U, i.e., $f|_U : U \to V$ is unramified. By Morphisms, Lemma 36.13 the morphism $U \to U \times_V U$ is an open immersion. This proves that (1) implies (4).

If $\Delta_{X/Y}$ is a local isomorphism at x, then $\Omega_{X/Y,x} = 0$ by Morphisms, Lemma 33.7. Hence we see that (4) implies (2). At this point we know that (1), (2) and (4) are all equivalent.

Assume (3). The assumption on the diagram combined with Morphisms, Lemma 33.15 show that $\Omega_{U/V,x} = 0$. Since $\Omega_{U/V,x} = \Omega_{X/Y,x}$ we conclude (2) holds.

Finally, assume that (2) holds. To prove (3) we may localize on X and Y and assume that X and Y are affine. Say $X = \mathrm{Spec}(B)$ and $Y = \mathrm{Spec}(A)$. The point $x \in X$ corresponds to a prime $\mathfrak{q} \subset B$. Our assumption is that $\Omega_{B/A,\mathfrak{q}} = 0$ (see Morphisms, Lemma 33.5 for the relationship between differentials on schemes and modules of differentials in commutative algebra). Since Y is locally Noetherian and f locally of finite type we see that A is Noetherian and $B \cong A[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$, see Properties, Lemma 5.2 and Morphisms, Lemma 15.2. In particular, $\Omega_{B/A}$ is a finite B-module. Hence we can find a single $g \in B$, $g \notin \mathfrak{q}$ such that the principal localization $(\Omega_{B/A})_g$ is zero. Hence after replacing B by B_g we see that $\Omega_{B/A} = 0$ (formation of modules of differentials commutes with localization, see Algebra, Lemma 131.8). This means that $\mathrm{d}(f_j)$ generate the kernel of the canonical map $\Omega_{A[x_1,\ldots,x_n]/A} \otimes_A B \to \Omega_{B/A}$. Thus the surjection $A[x_1, \ldots, x_n] \to B$ of A-algebras gives the commutative diagram of (3), and the theorem is proved. □

How can we use this theorem? Well, here are a few remarks:

(1) Suppose that $f : X \to Y$ and $g : Y \to Z$ are two morphisms locally of finite type between locally Noetherian schemes. There is a canonical short exact sequence
$$f^*(\Omega_{Y/Z}) \to \Omega_{X/Z} \to \Omega_{X/Y} \to 0$$
see Morphisms, Lemma 33.9. The theorem therefore implies that if $g \circ f$ is unramified, then so is f. This is Morphisms, Lemma 36.16.

(2) Since $\Omega_{X/Y}$ is isomorphic to the conormal sheaf of the diagonal morphism (Morphisms, Lemma 33.7) we see that if $X \to Y$ is a monomorphism of locally Noetherian schemes and locally of finite type, then $X \to Y$ is unramified. In particular, open and closed immersions of locally Noetherian schemes are unramified. See Morphisms, Lemmas 36.7 and 36.8.

(3) The theorem also implies that the set of points where a morphism $f : X \to Y$ (locally of finite type of locally Noetherian schemes) is not unramified is the support of the coherent sheaf $\Omega_{X/Y}$. This allows one to give a scheme theoretic definition to the "ramification locus".

5. The functorial characterization of unramified morphisms

024Q In basic algebraic geometry we learn that some classes of morphisms can be characterized functorially, and that such descriptions are quite useful. Unramified morphisms too have such a characterization.

024R **Theorem 5.1.** *Let $f : X \to S$ be a morphism of schemes. Assume S is a locally Noetherian scheme, and f is locally of finite type. Then the following are equivalent:*

(1) *f is unramified,*
(2) *the morphism f is formally unramified: for any affine S-scheme T and subscheme T_0 of T defined by a square-zero ideal, the natural map*
$$\mathrm{Hom}_S(T, X) \longrightarrow \mathrm{Hom}_S(T_0, X)$$
is injective.

Proof. See More on Morphisms, Lemma 6.9 for a more general statement and proof. What follows is a sketch of the proof in the current case.

Firstly, one checks both properties are local on the source and the target. This we may assume that S and X are affine. Say $X = \mathrm{Spec}(B)$ and $S = \mathrm{Spec}(R)$. Say $T = \mathrm{Spec}(C)$. Let J be the square-zero ideal of C with $T_0 = \mathrm{Spec}(C/J)$. Assume that we are given the diagram

$$\begin{array}{ccccc} & & B & & \\ & \nearrow & \downarrow{\scriptstyle \phi} & \searrow{\scriptstyle \bar{\phi}} & \\ R & \longrightarrow & C & \longrightarrow & C/J \end{array}$$

Secondly, one checks that the association $\phi' \mapsto \phi' - \phi$ gives a bijection between the set of liftings of $\bar{\phi}$ and the module $\mathrm{Der}_R(B, J)$. Thus, we obtain the implication (1) $\Rightarrow$ (2) via the description of unramified morphisms having trivial module of differentials, see Theorem 4.1.

To obtain the reverse implication, consider the surjection $q : C = (B \otimes_R B)/I^2 \to B = C/J$ defined by the square zero ideal $J = I/I^2$ where I is the kernel of the multiplication map $B \otimes_R B \to B$. We already have a lifting $B \to C$ defined by, say, $b \mapsto b \otimes 1$. Thus, by the same reasoning as above, we obtain a bijective correspondence between liftings of $\mathrm{id} : B \to C/J$ and $\mathrm{Der}_R(B, J)$. The hypothesis therefore implies that the latter module is trivial. But we know that $J \cong \Omega_{B/R}$. Thus, B/R is unramified. □

6. Topological properties of unramified morphisms

024S The first topological result that will be of utility to us is one which says that unramified and separated morphisms have "nice" sections. The material in this section does not require any Noetherian hypotheses.

024T **Proposition 6.1.** *Sections of unramified morphisms.*

(1) *Any section of an unramified morphism is an open immersion.*
(2) *Any section of a separated morphism is a closed immersion.*
(3) *Any section of an unramified separated morphism is open and closed.*

Proof. Fix a base scheme S. If $f : X' \to X$ is any S-morphism, then the graph $\Gamma_f : X' \to X' \times_S X$ is obtained as the base change of the diagonal $\Delta_{X/S} : X \to X \times_S X$ via the projection $X' \times_S X \to X \times_S X$. If $g : X \to S$ is separated (resp. unramified) then the diagonal is a closed immersion (resp. open immersion) by Schemes, Definition 21.3 (resp. Morphisms, Lemma 36.13). Hence so is the graph as a base change (by Schemes, Lemma 18.2). In the special case $X' = S$, we obtain (1), resp. (2). Part (3) follows on combining (1) and (2). □

We can now explicitly describe the sections of unramified morphisms.

024U **Theorem 6.2.** *Let Y be a connected scheme. Let $f : X \to Y$ be unramified and separated. Every section of f is an isomorphism onto a connected component. There exists a bijective correspondence*

$$\text{sections of } f \leftrightarrow \left\{ \begin{array}{c} \text{connected components } X' \text{ of } X \text{ such that} \\ \text{the induced map } X' \to Y \text{ is an isomorphism} \end{array} \right\}$$

In particular, given $x \in X$ there is at most one section passing through x.

Proof. Direct from Proposition 6.1 part (3). □

The preceding theorem gives us some idea of the "rigidity" of unramified morphisms. Further indication is provided by the following proposition which, besides being intrinsically interesting, is also useful in the theory of the algebraic fundamental group (see [Gro71, Exposé V]). See also the more general Morphisms, Lemma 36.17.

024V **Proposition 6.3.** *Let S is be a scheme. Let $\pi : X \to S$ be unramified and separated. Let Y be an S-scheme and $y \in Y$ a point. Let $f, g : Y \to X$ be two S-morphisms. Assume*

(1) *Y is connected*
(2) *$x = f(y) = g(y)$, and*
(3) *the induced maps $f^\sharp, g^\sharp : \kappa(x) \to \kappa(y)$ on residue fields are equal.*

Then $f = g$.

Proof. The maps $f, g : Y \to X$ define maps $f', g' : Y \to X_Y = Y \times_S X$ which are sections of the structure map $X_Y \to Y$. Note that $f = g$ if and only if $f' = g'$. The structure map $X_Y \to Y$ is the base change of π and hence unramified and separated also (see Morphisms, Lemmas 36.5 and Schemes, Lemma 21.12). Thus according to Theorem 6.2 it suffices to prove that f' and g' pass through the same point of X_Y. And this is exactly what the hypotheses (2) and (3) guarantee, namely $f'(y) = g'(y) \in X_Y$. □

0AKI **Lemma 6.4.** *Let S be a Noetherian scheme. Let $X \to S$ be a quasi-compact unramified morphism. Let $Y \to S$ be a morphism with Y Noetherian. Then $\mathrm{Mor}_S(Y, X)$ is a finite set.*

Proof. Assume first $X \to S$ is separated (which is often the case in practice). Since Y is Noetherian it has finitely many connected components. Thus we may assume Y is connected. Choose a point $y \in Y$ with image $s \in S$. Since $X \to S$ is unramified and quasi-compact then fibre X_s is finite, say $X_s = \{x_1, \ldots, x_n\}$ and $\kappa(x_i)/\kappa(s)$ is a finite field extension. See Morphisms, Lemma 36.10, 21.5, and 21.10. For each i there are at most finitely many $\kappa(s)$-algebra maps $\kappa(x_i) \to \kappa(y)$ (by elementary field theory). Thus $\mathrm{Mor}_S(Y, X)$ is finite by Proposition 6.3.

General case. There exists a nonempty open $U \subset S$ such that $X_U \to U$ is finite (in particular separated), see Morphisms, Lemma 52.1 (the lemma applies since we've already seen above that a quasi-compact unramified morphism is quasi-finite and since $X \to S$ is quasi-separated by Morphisms, Lemma 15.7). Let $Z \subset S$ be the reduced closed subscheme supported on the complement of U. By Noetherian induction, we see that $\mathrm{Mor}_Z(Y_Z, X_Z)$ is finite (details omitted). By the result of the first paragraph the set $\mathrm{Mor}_U(Y_U, X_U)$ is finite. Thus it suffices to show that

$$\mathrm{Mor}_S(Y, X) \longrightarrow \mathrm{Mor}_Z(Y_Z, X_Z) \times \mathrm{Mor}_U(Y_U, X_U)$$

is injective. This follows from the fact that the set of points where two morphisms $a, b : Y \to X$ agree is open in Y, due to the fact that $\Delta : X \to X \times_S X$ is open, see Morphisms, Lemma 36.13. □

7. Universally injective, unramified morphisms

06ND Recall that a morphism of schemes $f : X \to Y$ is universally injective if any base change of f is injective (on underlying topological spaces), see Morphisms, Definition 10.1. Universally injective and unramified morphisms can be characterized as follows.

05VH **Lemma 7.1.** *Let $f : X \to S$ be a morphism of schemes. The following are equivalent:*

(1) *f is unramified and a monomorphism,*
(2) *f is unramified and universally injective,*
(3) *f is locally of finite type and a monomorphism,*
(4) *f is universally injective, locally of finite type, and formally unramified,*
(5) *f is locally of finite type and X_s is either empty or $X_s \to s$ is an isomorphism for all $s \in S$.*

Proof. We have seen in More on Morphisms, Lemma 6.9 that being formally unramified and locally of finite type is the same thing as being unramified. Hence (4) is equivalent to (2). A monomorphism is certainly universally injective and formally unramified hence (3) implies (4). It is clear that (1) implies (3). Finally, if (2) holds, then $\Delta : X \to X \times_S X$ is both an open immersion (Morphisms, Lemma 36.13) and surjective (Morphisms, Lemma 10.2) hence an isomorphism, i.e., f is a monomorphism. In this way we see that (2) implies (1).

Condition (3) implies (5) because monomorphisms are preserved under base change (Schemes, Lemma 23.5) and because of the description of monomorphisms towards the spectra of fields in Schemes, Lemma 23.11. Condition (5) implies (4) by Morphisms, Lemmas 10.2 and 36.12. □

This leads to the following useful characterization of closed immersions.

04XV **Lemma 7.2.** *Let $f : X \to S$ be a morphism of schemes. The following are equivalent:*

(1) *f is a closed immersion,*
(2) *f is a proper monomorphism,*
(3) *f is proper, unramified, and universally injective,*
(4) *f is universally closed, unramified, and a monomorphism,*
(5) *f is universally closed, unramified, and universally injective,*
(6) *f is universally closed, locally of finite type, and a monomorphism,*
(7) *f is universally closed, universally injective, locally of finite type, and formally unramified.*

Proof. The equivalence of (4) – (7) follows immediately from Lemma 7.1.

Let $f : X \to S$ satisfy (6). Then f is separated, see Schemes, Lemma 23.3 and has finite fibres. Hence More on Morphisms, Lemma 44.1 shows f is finite. Then Morphisms, Lemma 45.15 implies f is a closed immersion, i.e., (1) holds.

Note that (1) $\Rightarrow$ (2) because a closed immersion is proper and a monomorphism (Morphisms, Lemma 42.6 and Schemes, Lemma 23.8). By Lemma 7.1 we see that (2) implies (3). It is clear that (3) implies (5). □

Here is another result of a similar flavor.

04DG **Lemma 7.3.** *Let $\pi : X \to S$ be a morphism of schemes. Let $s \in S$. Assume that*

(1) *π is finite,*
(2) *π is unramified,*
(3) *$\pi^{-1}(\{s\}) = \{x\}$, and*
(4) *$\kappa(s) \subset \kappa(x)$ is purely inseparable*[1].

[1] In view of condition (2) this is equivalent to $\kappa(s) = \kappa(x)$.

Then there exists an open neighbourhood U of s such that $\pi|_{\pi^{-1}(U)} : \pi^{-1}(U) \to U$ is a closed immersion.

Proof. The question is local on S. Hence we may assume that $S = \mathrm{Spec}(A)$. By definition of a finite morphism this implies $X = \mathrm{Spec}(B)$. Note that the ring map $\varphi : A \to B$ defining π is a finite unramified ring map. Let $\mathfrak{p} \subset A$ be the prime corresponding to s. Let $\mathfrak{q} \subset B$ be the prime corresponding to x. Conditions (2), (3) and (4) imply that $B_\mathfrak{q}/\mathfrak{p}B_\mathfrak{q} = \kappa(\mathfrak{p})$. By Algebra, Lemma 41.11 we have $B_\mathfrak{q} = B_\mathfrak{p}$ (note that a finite ring map satisfies going up, see Algebra, Section 41.) Hence we see that $B_\mathfrak{p}/\mathfrak{p}B_\mathfrak{p} = \kappa(\mathfrak{p})$. As B is a finite A-module we see from Nakayama's lemma (see Algebra, Lemma 20.1) that $B_\mathfrak{p} = \varphi(A_\mathfrak{p})$. Hence (using the finiteness of B as an A-module again) there exists a $f \in A$, $f \notin \mathfrak{p}$ such that $B_f = \varphi(A_f)$ as desired. □

The topological results presented above will be used to give a functorial characterization of étale morphisms similar to Theorem 5.1.

8. Examples of unramified morphisms

024W Here are a few examples.

024X **Example 8.1.** Let k be a field. Unramified quasi-compact morphisms $X \to \mathrm{Spec}(k)$ are affine. This is true because X has dimension 0 and is Noetherian, hence is a finite discrete set, and each point gives an affine open, so X is a finite disjoint union of affines hence affine. Noether normalization forces X to be the spectrum of a finite k-algebra A. This algebra is a product of finite separable field extensions of k. Thus, an unramified quasi-compact morphism to $\mathrm{Spec}(k)$ corresponds to a finite number of finite separable field extensions of k. In particular, an unramified morphism with a connected source and a one point target is forced to be a finite separable field extension. As we will see later, $X \to \mathrm{Spec}(k)$ is étale if and only if it is unramified. Thus, in this case at least, we obtain a very easy description of the étale topology of a scheme. Of course, the cohomology of this topology is another story.

024Y **Example 8.2.** Property (3) in Theorem 4.1 gives us a canonical source of examples for unramified morphisms. Fix a ring R and an integer n. Let $I = (g_1, \ldots, g_m)$ be an ideal in $R[x_1, \ldots, x_n]$. Let $\mathfrak{q} \subset R[x_1, \ldots, x_n]$ be a prime. Assume $I \subset \mathfrak{q}$ and that the matrix

$$\left(\frac{\partial g_i}{\partial x_j}\right) \bmod \mathfrak{q} \quad \in \quad \mathrm{Mat}(n \times m, \kappa(\mathfrak{q}))$$

has rank n. Then the morphism $f : Z = \mathrm{Spec}(R[x_1, \ldots, x_n]/I) \to \mathrm{Spec}(R)$ is unramified at the point $x \in Z \subset \mathbf{A}^n_R$ corresponding to $\mathfrak{q}$. Clearly we must have $m \geq n$. In the extreme case $m = n$, i.e., the differential of the map $\mathbf{A}^n_R \to \mathbf{A}^n_R$ defined by the g_i's is an isomorphism of the tangent spaces, then f is also flat x and, hence, is an étale map (see Algebra, Definition 137.5, Lemma 137.6 and Example 137.7).

024Z **Example 8.3.** Fix an extension of number fields L/K with rings of integers $\mathcal{O}_L$ and $\mathcal{O}_K$. The injection $K \to L$ defines a morphism $f : \mathrm{Spec}(\mathcal{O}_L) \to \mathrm{Spec}(\mathcal{O}_K)$. As discussed above, the points where f is unramified in our sense correspond to the set of points where f is unramified in the conventional sense. In the conventional sense, the locus of ramification in $\mathrm{Spec}(\mathcal{O}_L)$ can be defined by vanishing set of the

different; this is an ideal in $\mathcal{O}_L$. In fact, the different is nothing but the annihilator of the module $\Omega_{\mathcal{O}_L/\mathcal{O}_K}$. Similarly, the discriminant is an ideal in $\mathcal{O}_K$, namely it is the norm of the different. The vanishing set of the discriminant is precisely the set of points of K which ramify in L. Thus, denoting by X the complement of the closed subset defined by the different in $\mathrm{Spec}(\mathcal{O}_L)$, we obtain a morphism $X \to \mathrm{Spec}(\mathcal{O}_K)$ which is unramified. Furthermore, this morphism is also flat, as any local homomorphism of discrete valuation rings is flat, and hence this morphism is actually étale. If L/K is finite Galois, then denoting by Y the complement of the closed subset defined by the discriminant in $\mathrm{Spec}(\mathcal{O}_K)$, we see that we get even a finite étale morphism $X \to Y$. Thus, this is an example of a finite étale covering.

9. Flat morphisms

0250 This section simply exists to summarize the properties of flatness that will be useful to us. Thus, we will be content with stating the theorems precisely and giving references for the proofs.

After briefly recalling the necessary facts about flat modules over Noetherian rings, we state a theorem of Grothendieck which gives sufficient conditions for "hyperplane sections" of certain modules to be flat.

0251 **Definition 9.1.** Flatness of modules and rings.

(1) A module N over a ring A is said to be *flat* if the functor $M \mapsto M \otimes_A N$ is exact.
(2) If this functor is also faithful, we say that N is *faithfully flat* over A.
(3) A morphism of rings $f : A \to B$ is said to be *flat (resp. faithfully flat)* if the functor $M \mapsto M \otimes_A B$ is exact (resp. faithful and exact).

Here is a list of facts with references to the algebra chapter.

(1) Free and projective modules are flat. This is clear for free modules and follows for projective modules as they are direct summands of free modules and $\otimes$ commutes with direct sums.
(2) Flatness is a local property, that is, M is flat over A if and only if $M_\mathfrak{p}$ is flat over $A_\mathfrak{p}$ for all $\mathfrak{p} \in \mathrm{Spec}(A)$. See Algebra, Lemma 39.18.
(3) If M is a flat A-module and $A \to B$ is a ring map, then $M \otimes_A B$ is a flat B-module. See Algebra, Lemma 39.7.
(4) Finite flat modules over local rings are free. See Algebra, Lemma 78.5.
(5) If $f : A \to B$ is a morphism of arbitrary rings, f is flat if and only if the induced maps $A_{f^{-1}(\mathfrak{q})} \to B_\mathfrak{q}$ are flat for all $\mathfrak{q} \in \mathrm{Spec}(B)$. See Algebra, Lemma 39.18
(6) If $f : A \to B$ is a local homomorphism of local rings, f is flat if and only if it is faithfully flat. See Algebra, Lemma 39.17.
(7) A map $A \to B$ of rings is faithfully flat if and only if it is flat and the induced map on spectra is surjective. See Algebra, Lemma 39.16.
(8) If A is a Noetherian local ring, the completion $A^\wedge$ is faithfully flat over A. See Algebra, Lemma 97.3.
(9) Let A be a Noetherian local ring and M an A-module. Then M is flat over A if and only if $M \otimes_A A^\wedge$ is flat over $A^\wedge$. (Combine the previous statement with Algebra, Lemma 39.8.)

Before we move on to the geometric category, we present Grothendieck's theorem, which provides a convenient recipe for producing flat modules.

0252 **Theorem 9.2.** *Let A, B be Noetherian local rings. Let $f : A \to B$ be a local homomorphism. If M is a finite B-module that is flat as an A-module, and $t \in \mathfrak{m}_B$ is an element such that multiplication by t is injective on $M/\mathfrak{m}_A M$, then M/tM is also A-flat.*

Proof. See Algebra, Lemma 99.1. See also [Mat70, Section 20]. □

0253 **Definition 9.3.** (See Morphisms, Definition 26.1). Let $f : X \to Y$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module.

(1) Let $x \in X$. We say $\mathcal{F}$ is *flat over Y at $x \in X$* if $\mathcal{F}_x$ is a flat $\mathcal{O}_{Y,f(x)}$-module. This uses the map $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ to think of $\mathcal{F}_x$ as a $\mathcal{O}_{Y,f(x)}$-module.
(2) Let $x \in X$. We say f is *flat at $x \in X$* if $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is flat.
(3) We say f is *flat* if it is flat at all points of X.
(4) A morphism $f : X \to Y$ that is flat and surjective is sometimes said to be *faithfully flat*.

Once again, here is a list of results:

(1) The property (of a morphism) of being flat is, by fiat, local in the Zariski topology on the source and the target.
(2) Open immersions are flat. (This is clear because it induces isomorphisms on local rings.)
(3) Flat morphisms are stable under base change and composition. Morphisms, Lemmas 26.8 and 26.6.
(4) If $f : X \to Y$ is flat, then the pullback functor $QCoh(\mathcal{O}_Y) \to QCoh(\mathcal{O}_X)$ is exact. This is immediate by looking at stalks.
(5) Let $f : X \to Y$ be a morphism of schemes, and assume Y is quasi-compact and quasi-separated. In this case if the functor f^* is exact then f is flat. (Proof omitted. Hint: Use Properties, Lemma 23.1 to see that Y has "enough" ideal sheaves and use the characterization of flatness in Algebra, Lemma 39.5.)

10. Topological properties of flat morphisms

0254 We "recall" below some openness properties that flat morphisms enjoy.

0255 **Theorem 10.1.** *Let Y be a locally Noetherian scheme. Let $f : X \to Y$ be a morphism which is locally of finite type. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. The set of points in X where $\mathcal{F}$ is flat over Y is an open set. In particular the set of points where f is flat is open in X.*

Proof. See More on Morphisms, Theorem 15.1. □

039K **Theorem 10.2.** *Let Y be a locally Noetherian scheme. Let $f : X \to Y$ be a morphism which is flat and locally of finite type. Then f is (universally) open.*

Proof. See Morphisms, Lemma 26.10. □

0256 **Theorem 10.3.** *A faithfully flat quasi-compact morphism is a quotient map for the Zariski topology.*

Proof. See Morphisms, Lemma 26.12. □

An important reason to study flat morphisms is that they provide the adequate framework for capturing the notion of a family of schemes parametrized by the points of another scheme. Naively one may think that any morphism $f : X \to S$ should be thought of as a family parametrized by the points of S. However, without a flatness restriction on f, really bizarre things can happen in this so-called family. For instance, we aren't guaranteed that relative dimension (dimension of the fibres) is constant in a family. Other numerical invariants, such as the Hilbert polynomial, too may change from fibre to fibre. Flatness prevents such things from happening and, therefore, provides some "continuity" to the fibres.

11. Étale morphisms

0257 In this section, we will define étale morphisms and prove a number of important properties about them. The most important one, no doubt, is the functorial characterization presented in Theorem 16.1. Following this, we will also discuss a few properties of rings which are insensitive to an étale extension (properties which hold for a ring if and only if they hold for all its étale extensions) to motivate the basic tenet of étale cohomology – étale morphisms are the algebraic analogue of local isomorphisms.

As the title suggests, we will define the class of étale morphisms – the class of morphisms (whose surjective families) we shall deem to be coverings in the category of schemes over a base scheme S in order to define the étale site $S_{étale}$. Intuitively, an étale morphism is supposed to capture the idea of a covering space and, therefore, should be close to a local isomorphism. If we're working with varieties over algebraically closed fields, this last statement can be made into a definition provided we replace "local isomorphism" with "formal local isomorphism" (isomorphism after completion). One can then give a definition over any base field by asking that the base change to the algebraic closure be étale (in the aforementioned sense). But, rather than proceeding via such aesthetically displeasing constructions, we will adopt a cleaner, albeit slightly more abstract, algebraic approach.

We first define "étale homomorphisms of local rings" for Noetherian local rings. We cannot use the term "étale", as there already is a notion of an étale ring map (Algebra, Section 143) and it is different.

0258 **Definition 11.1.** Let A, B be Noetherian local rings. A local homomorphism $f : A \to B$ is said to be an *étale homomorphism of local rings* if it is flat and an unramified homomorphism of local rings (please see Definition 3.1).

This is the local version of the definition of an étale ring map in Algebra, Section 143. The exact definition given in that section is that it is a smooth ring map of relative dimension 0. It is shown (in Algebra, Lemma 143.2) that an étale R-algebra S always has a presentation

$$S = R[x_1, \ldots, x_n]/(f_1, \ldots, f_n)$$

such that

$$g = \det \begin{pmatrix} \partial f_1/\partial x_1 & \partial f_2/\partial x_1 & \ldots & \partial f_n/\partial x_1 \\ \partial f_1/\partial x_2 & \partial f_2/\partial x_2 & \ldots & \partial f_n/\partial x_2 \\ \ldots & \ldots & \ldots & \ldots \\ \partial f_1/\partial x_n & \partial f_2/\partial x_n & \ldots & \partial f_n/\partial x_n \end{pmatrix}$$

maps to an invertible element in S. The following two lemmas link the two notions.

039L **Lemma 11.2.** *Let $A \to B$ be of finite type with A a Noetherian ring. Let $\mathfrak{q}$ be a prime of B lying over $\mathfrak{p} \subset A$. Then $A \to B$ is étale at $\mathfrak{q}$ if and only if $A_\mathfrak{p} \to B_\mathfrak{q}$ is an étale homomorphism of local rings.*

Proof. See Algebra, Lemmas 143.3 (flatness of étale maps), 143.5 (étale maps are unramified) and 143.7 (flat and unramified maps are étale). □

039M **Lemma 11.3.** *Let A, B be Noetherian local rings. Let $A \to B$ be a local homomorphism such that B is essentially of finite type over A. The following are equivalent*

(1) *$A \to B$ is an étale homomorphism of local rings*
(2) *$A^\wedge \to B^\wedge$ is an étale homomorphism of local rings, and*
(3) *$A^\wedge \to B^\wedge$ is étale.*

Moreover, in this case $B^\wedge \cong (A^\wedge)^{\oplus n}$ as $A^\wedge$-modules for some $n \geq 1$.

Proof. To see the equivalences of (1), (2) and (3), as we have the corresponding results for unramified ring maps (Lemma 3.4) it suffices to prove that $A \to B$ is flat if and only if $A^\wedge \to B^\wedge$ is flat. This is clear from our lists of properties of flat maps since the ring maps $A \to A^\wedge$ and $B \to B^\wedge$ are faithfully flat. For the final statement, by Lemma 3.3 we see that $B^\wedge$ is a finite flat $A^\wedge$ module. Hence it is finite free by our list of properties on flat modules in Section 9. □

The integer n which occurs in the lemma above is nothing other than the degree $[\kappa(\mathfrak{m}_B) : \kappa(\mathfrak{m}_A)]$ of the residue field extension. In particular, if $\kappa(\mathfrak{m}_A)$ is separably closed, we see that $A^\wedge \to B^\wedge$ is an isomorphism, which vindicates our earlier claims.

0259 **Definition 11.4.** (See Morphisms, Definition 37.1.) Let Y be a locally Noetherian scheme. Let $f : X \to Y$ be a morphism of schemes which is locally of finite type.

(1) Let $x \in X$. We say f is *étale at* $x \in X$ if $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is an étale homomorphism of local rings.
(2) The morphism is said to be *étale* if it is étale at all its points.

Let us prove that this definition agrees with the definition in the chapter on morphisms of schemes. This in particular guarantees that the set of points where a morphism is étale is open.

039N **Lemma 11.5.** *Let Y be a locally Noetherian scheme. Let $f : X \to Y$ be locally of finite type. Let $x \in X$. The morphism f is étale at x in the sense of Definition 11.4 if and only if it is étale at x in the sense of Morphisms, Definition 37.1.*

Proof. This follows from Lemma 11.2 and the definitions. □

Here are some results on étale morphisms. The formulations as given in this list apply only to morphisms locally of finite type between locally Noetherian schemes. In each case we give a reference to the general result as proved earlier in the project, but in some cases one can prove the result more easily in the Noetherian case. Here is the list:

(1) An étale morphism is unramified. (Clear from our definitions.)
(2) Étaleness is local on the source and the target in the Zariski topology.
(3) Étale morphisms are stable under base change and composition. See Morphisms, Lemmas 37.4 and 37.3.

(4) Étale morphisms of schemes are locally quasi-finite and quasi-compact étale morphisms are quasi-finite. (This is true because it holds for unramified morphisms as seen earlier.)
(5) Étale morphisms have relative dimension 0. See Morphisms, Definition 30.1 and Morphisms, Lemma 30.5.
(6) A morphism is étale if and only if it is flat and all its fibres are étale. See Morphisms, Lemma 37.8.
(7) Étale morphisms are open. This is true because an étale morphism is flat, and Theorem 10.2.
(8) Let X and Y be étale over a base scheme S. Any S-morphism from X to Y is étale. See Morphisms, Lemma 37.19.

12. The structure theorem

025A We present a theorem which describes the local structure of étale and unramified morphisms. Besides its obvious independent importance, this theorem also allows us to make the transition to another definition of étale morphisms that captures the geometric intuition better than the one we've used so far.

To state it we need the notion of a *standard étale ring map*, see Algebra, Definition 144.1. Namely, suppose that R is a ring and $f, g \in R[t]$ are polynomials such that

(a) f is a monic polynomial, and
(b) $f' = \mathrm{d}f/\mathrm{d}t$ is invertible in the localization $R[t]_g/(f)$.

Then the map

$$R \longrightarrow R[t]_g/(f) = R[t, 1/g]/(f)$$

is a standard étale algebra, and any standard étale algebra is isomorphic to one of these. It is a pleasant exercise to prove that such a ring map is flat, and unramified and hence étale (as expected of course). A special case of a standard étale ring map is any ring map

$$R \longrightarrow R[t]_{f'}/(f) = R[t, 1/f']/(f)$$

with f a monic polynomial, and any standard étale algebra is (isomorphic to) a principal localization of one of these.

025B **Theorem 12.1.** *Let $f : A \to B$ be an étale homomorphism of local rings. Then there exist $f, g \in A[t]$ such that*

(1) *$B' = A[t]_g/(f)$ is standard étale – see (a) and (b) above, and*
(2) *B is isomorphic to a localization of B' at a prime.*

Proof. Write $B = B'_{\mathfrak{q}}$ for some finite type A-algebra B' (we can do this because B is essentially of finite type over A). By Lemma 11.2 we see that $A \to B'$ is étale at $\mathfrak{q}$. Hence we may apply Algebra, Proposition 144.4 to see that a principal localization of B' is standard étale. □

Here is the version for unramified homomorphisms of local rings.

039O **Theorem 12.2.** *Let $f : A \to B$ be an unramified morphism of local rings. Then there exist $f, g \in A[t]$ such that*

(1) *$B' = A[t]_g/(f)$ is standard étale – see (a) and (b) above, and*
(2) *B is isomorphic to a quotient of a localization of B' at a prime.*

Proof. Write $B = B'_{\mathfrak{q}}$ for some finite type A-algebra B' (we can do this because B is essentially of finite type over A). By Lemma 3.2 we see that $A \to B'$ is unramified at $\mathfrak{q}$. Hence we may apply Algebra, Proposition 152.1 to see that a principal localization of B' is a quotient of a standard étale A-algebra. □

Via standard lifting arguments, one then obtains the following geometric statement which will be of essential use to us.

025C **Theorem 12.3.** *Let $\varphi : X \to Y$ be a morphism of schemes. Let $x \in X$. Let $V \subset Y$ be an affine open neighbourhood of $\varphi(x)$. If φ is étale at x, then there exist exists an affine open $U \subset X$ with $x \in U$ and $\varphi(U) \subset V$ such that we have the following diagram*

$$\begin{array}{ccccc} X & \longleftarrow & U & \xrightarrow[j]{} & \mathrm{Spec}(R[t]_{f'}/(f)) \\ \downarrow & & \downarrow & & \downarrow \\ Y & \longleftarrow & V & = & \mathrm{Spec}(R) \end{array}$$

where j is an open immersion, and $f \in R[t]$ is monic.

Proof. This is equivalent to Morphisms, Lemma 37.14 although the statements differ slightly. See also, Varieties, Lemma 18.3 for a variant for unramified morphisms. □

13. Étale and smooth morphisms

039P An étale morphism is smooth of relative dimension zero. The projection $\mathbf{A}^n_S \to S$ is a standard example of a smooth morphism of relative dimension n. It turns out that any smooth morphism is étale locally of this form. Here is the precise statement.

039Q **Theorem 13.1.** *Let $\varphi : X \to Y$ be a morphism of schemes. Let $x \in X$. If φ is smooth at x, then there exist an integer $n \geq 0$ and affine opens $V \subset Y$ and $U \subset X$ with $x \in U$ and $\varphi(U) \subset V$ such that there exists a commutative diagram*

$$\begin{array}{ccccc} X & \longleftarrow & U & \xrightarrow[\pi]{} & \mathbf{A}^n_R = \mathrm{Spec}(R[x_1, \ldots, x_n]) \\ \downarrow & & \downarrow & & \swarrow \\ Y & \longleftarrow & V & = & \mathrm{Spec}(R) \end{array}$$

where π is étale.

Proof. See Morphisms, Lemma 37.21. □

14. Topological properties of étale morphisms

025F We present a few of the topological properties of étale and unramified morphisms. First, we give what Grothendieck calls the *fundamental property of étale morphisms*, see [Gro71, Exposé I.5].

025G **Theorem 14.1.** *Let $f : X \to Y$ be a morphism of schemes. The following are equivalent:*

1. *f is an open immersion,*
2. *f is universally injective and étale, and*
3. *f is a flat monomorphism, locally of finite presentation.*

Proof. An open immersion is universally injective since any base change of an open immersion is an open immersion. Moreover, it is étale by Morphisms, Lemma 37.9. Hence (1) implies (2).

Assume f is universally injective and étale. Since f is étale it is flat and locally of finite presentation, see Morphisms, Lemmas 37.12 and 37.11. By Lemma 7.1 we see that f is a monomorphism. Hence (2) implies (3).

Assume f is flat, locally of finite presentation, and a monomorphism. Then f is open, see Morphisms, Lemma 26.10. Thus we may replace Y by $f(X)$ and we may assume f is surjective. Then f is open and bijective hence a homeomorphism. Hence f is quasi-compact. Hence Descent, Lemma 25.1 shows that f is an isomorphism and we win. □

Here is another result of a similar flavor.

04DH **Lemma 14.2.** *Let $\pi : X \to S$ be a morphism of schemes. Let $s \in S$. Assume that*

(1) *π is finite,*
(2) *π is étale,*
(3) *$\pi^{-1}(\{s\}) = \{x\}$, and*
(4) *$\kappa(s) \subset \kappa(x)$ is purely inseparable*[2].

Then there exists an open neighbourhood U of s such that $\pi|_{\pi^{-1}(U)} : \pi^{-1}(U) \to U$ is an isomorphism.

Proof. By Lemma 7.3 there exists an open neighbourhood U of s such that $\pi|_{\pi^{-1}(U)} : \pi^{-1}(U) \to U$ is a closed immersion. But a morphism which is étale and a closed immersion is an open immersion (for example by Theorem 14.1). Hence after shrinking U we obtain an isomorphism. □

0EBS **Lemma 14.3.** *Let $U \to X$ be an étale morphism of schemes where X is a scheme in characteristic p. Then the relative Frobenius $F_{U/X} : U \to U \times_{X,F_X} X$ is an isomorphism.*

Proof. The morphism $F_{U/X}$ is a universal homeomorphism by Varieties, Lemma 36.6. The morphism $F_{U/X}$ is étale as a morphism between schemes étale over X (Morphisms, Lemma 37.19). Hence $F_{U/X}$ is an isomorphism by Theorem 14.1. □

15. Topological invariance of the étale topology

06NE Next, we present an extremely crucial theorem which, roughly speaking, says that étaleness is a topological property.

025H **Theorem 15.1.** *Let X and Y be two schemes over a base scheme S. Let S_0 be a closed subscheme of S with the same underlying topological space (for example if the ideal sheaf of S_0 in S has square zero). Denote X_0 (resp. Y_0) the base change $S_0 \times_S X$ (resp. $S_0 \times_S Y$). If X is étale over S, then the map*

$$\mathrm{Mor}_S(Y, X) \longrightarrow \mathrm{Mor}_{S_0}(Y_0, X_0)$$

is bijective.

[2] In view of condition (2) this is equivalent to $\kappa(s) = \kappa(x)$.

Proof. After base changing via $Y \to S$, we may assume that $Y = S$. In this case the theorem states that any S-morphism $\sigma_0 : S_0 \to X$ actually factors uniquely through a section $S \to X$ of the étale structure morphism $f : X \to S$.

Uniqueness. Suppose we have two sections σ, σ' through which σ_0 factors. Because $X \to S$ is étale we see that $\Delta : X \to X \times_S X$ is an open immersion (Morphisms, Lemma 36.13). The morphism $(\sigma, \sigma') : S \to X \times_S X$ factors through this open because for any $s \in S$ we have $(\sigma, \sigma')(s) = (\sigma_0(s), \sigma_0(s))$. Thus $\sigma = \sigma'$.

To prove existence we first reduce to the affine case (we suggest the reader skip this step). Let $X = \bigcup X_i$ be an affine open covering such that each X_i maps into an affine open S_i of S. For every $s \in S$ we can choose an i such that $\sigma_0(s) \in X_i$. Choose an affine open neighbourhood $U \subset S_i$ of s such that $\sigma_0(U_0) \subset X_{i,0}$. Note that $X' = X_i \times_S U = X_i \times_{S_i} U$ is affine. If we can lift $\sigma_0|_{U_0} : U_0 \to X'_0$ to $U \to X'$, then by uniqueness these local lifts will glue to a global morphism $S \to X$. Thus we may assume S and X are affine.

Existence when S and X are affine. Write $S = \mathrm{Spec}(A)$ and $X = \mathrm{Spec}(B)$. Then $A \to B$ is étale and in particular smooth (of relative dimension 0). As $|S_0| = |S|$ we see that $S_0 = \mathrm{Spec}(A/I)$ with $I \subset A$ locally nilpotent. Thus existence follows from Algebra, Lemma 138.17. □

From the proof of preceding theorem, we also obtain one direction of the promised functorial characterization of étale morphisms. The following theorem will be strengthened in Étale Cohomology, Theorem 45.2.

039R **Theorem 15.2** (Une equivalence remarquable de catégories)**.** *Let S be a scheme. Let $S_0 \subset S$ be a closed subscheme with the same underlying topological space (for example if the ideal sheaf of S_0 in S has square zero). The functor* [DG67, IV, Theorem 18.1.2]

$$X \longmapsto X_0 = S_0 \times_S X$$

defines an equivalence of categories

$$\{\text{schemes } X \text{ étale over } S\} \leftrightarrow \{\text{schemes } X_0 \text{ étale over } S_0\}$$

Proof. By Theorem 15.1 we see that this functor is fully faithful. It remains to show that the functor is essentially surjective. Let $Y \to S_0$ be an étale morphism of schemes.

Suppose that the result holds if S and Y are affine. In that case, we choose an affine open covering $Y = \bigcup V_j$ such that each V_j maps into an affine open of S. By assumption (affine case) we can find étale morphisms $W_j \to S$ such that $W_{j,0} \cong V_j$ (as schemes over S_0). Let $W_{j,j'} \subset W_j$ be the open subscheme whose underlying topological space corresponds to $V_j \cap V_{j'}$. Because we have isomorphisms

$$W_{j,j',0} \cong V_j \cap V_{j'} \cong W_{j',j,0}$$

as schemes over S_0 we see by fully faithfulness that we obtain isomorphisms $\theta_{j,j'} : W_{j,j'} \to W_{j',j}$ of schemes over S. We omit the verification that these isomorphisms satisfy the cocycle condition of Schemes, Section 14. Applying Schemes, Lemma 14.2 we obtain a scheme $X \to S$ by glueing the schemes W_j along the identifications $\theta_{j,j'}$. It is clear that $X \to S$ is étale and $X_0 \cong Y$ by construction.

Thus it suffices to show the lemma in case S and Y are affine. Say $S = \mathrm{Spec}(R)$ and $S_0 = \mathrm{Spec}(R/I)$ with I locally nilpotent. By Algebra, Lemma 143.2 we know that Y is the spectrum of a ring $\overline{A}$ with

$$\overline{A} = (R/I)[x_1, \ldots, x_n]/(\overline{f}_1, \ldots, \overline{f}_n)$$

such that

$$\overline{g} = \det \begin{pmatrix} \partial \overline{f}_1/\partial x_1 & \partial \overline{f}_2/\partial x_1 & \ldots & \partial \overline{f}_n/\partial x_1 \\ \partial \overline{f}_1/\partial x_2 & \partial \overline{f}_2/\partial x_2 & \ldots & \partial \overline{f}_n/\partial x_2 \\ \ldots & \ldots & \ldots & \ldots \\ \partial \overline{f}_1/\partial x_n & \partial \overline{f}_2/\partial x_n & \ldots & \partial \overline{f}_n/\partial x_n \end{pmatrix}$$

maps to an invertible element in $\overline{A}$. Choose any lifts $f_i \in R[x_1, \ldots, x_n]$. Set

$$A = R[x_1, \ldots, x_n]/(f_1, \ldots, f_n)$$

Since I is locally nilpotent the ideal IA is locally nilpotent (Algebra, Lemma 32.3). Observe that $\overline{A} = A/IA$. It follows that the determinant of the matrix of partials of the f_i is invertible in the algebra A by Algebra, Lemma 32.4. Hence $R \to A$ is étale and the proof is complete. □

16. The functorial characterization

025J We finally present the promised functorial characterization. Thus there are four ways to think about étale morphisms of schemes:

(1) as a smooth morphism of relative dimension 0,
(2) as locally finitely presented, flat, and unramified morphisms,
(3) using the structure theorem, and
(4) using the functorial characterization.

025K **Theorem 16.1.** *Let $f : X \to S$ be a morphism that is locally of finite presentation. The following are equivalent*

(1) *f is étale,*
(2) *for all affine S-schemes Y, and closed subschemes $Y_0 \subset Y$ defined by square-zero ideals, the natural map*

$$\mathrm{Mor}_S(Y, X) \longrightarrow \mathrm{Mor}_S(Y_0, X)$$

is bijective.

Proof. This is More on Morphisms, Lemma 8.10. □

This characterization says that solutions to the equations defining X can be lifted uniquely through nilpotent thickenings.

17. Étale local structure of unramified morphisms

04HG In the chapter More on Morphisms, Section 41 the reader can find some results on the étale local structure of quasi-finite morphisms. In this section we want to combine this with the topological properties of unramified morphisms we have seen in this chapter. The basic overall picture to keep in mind is

$$\begin{array}{ccccc} V & \longrightarrow & X_U & \longrightarrow & X \\ & \searrow & \downarrow & & \downarrow f \\ & & U & \longrightarrow & S \end{array}$$

see More on Morphisms, Equation (41.0.1). We start with a very general case.

04HH **Lemma 17.1.** *Let $f : X \to S$ be a morphism of schemes. Let $x_1, \ldots, x_n \in X$ be points having the same image s in S. Assume f is unramified at each x_i. Then there exists an étale neighbourhood $(U, u) \to (S, s)$ and opens $V_{i,j} \subset X_U$, $i = 1, \ldots, n$, $j = 1, \ldots, m_i$ such that*

(1) *$V_{i,j} \to U$ is a closed immersion passing through u,*
(2) *u is not in the image of $V_{i,j} \cap V_{i',j'}$ unless $i = i'$ and $j = j'$, and*
(3) *any point of $(X_U)_u$ mapping to x_i is in some $V_{i,j}$.*

Proof. By Morphisms, Definition 36.1 there exists an open neighbourhood of each x_i which is locally of finite type over S. Replacing X by an open neighbourhood of $\{x_1, \ldots, x_n\}$ we may assume f is locally of finite type. Apply More on Morphisms, Lemma 41.3 to get the étale neighbourhood (U, u) and the opens $V_{i,j}$ finite over U. By Lemma 7.3 after possibly shrinking U we get that $V_{i,j} \to U$ is a closed immersion. □

04HI **Lemma 17.2.** *Let $f : X \to S$ be a morphism of schemes. Let $x_1, \ldots, x_n \in X$ be points having the same image s in S. Assume f is separated and f is unramified at each x_i. Then there exists an étale neighbourhood $(U, u) \to (S, s)$ and a disjoint union decomposition*

$$X_U = W \amalg \coprod\nolimits_{i,j} V_{i,j}$$

such that

(1) *$V_{i,j} \to U$ is a closed immersion passing through u,*
(2) *the fibre W_u contains no point mapping to any x_i.*

In particular, if $f^{-1}(\{s\}) = \{x_1, \ldots, x_n\}$, then the fibre W_u is empty.

Proof. Apply Lemma 17.1. We may assume U is affine, so X_U is separated. Then $V_{i,j} \to X_U$ is a closed map, see Morphisms, Lemma 42.7. Suppose $(i, j) \neq (i', j')$. Then $V_{i,j} \cap V_{i',j'}$ is closed in $V_{i,j}$ and its image in U does not contain u. Hence after shrinking U we may assume that $V_{i,j} \cap V_{i',j'} = \emptyset$. Moreover, $\bigcup V_{i,j}$ is a closed and open subscheme of X_U and hence has an open and closed complement W. This finishes the proof. □

The following lemma is in some sense much weaker than the preceding one but it may be useful to state it explicitly here. It says that a finite unramified morphism is étale locally on the base a closed immersion.

04HJ **Lemma 17.3.** *Let $f : X \to S$ be a finite unramified morphism of schemes. Let $s \in S$. There exists an étale neighbourhood $(U, u) \to (S, s)$ and a finite disjoint union decomposition*

$$X_U = \coprod\nolimits_j V_j$$

such that each $V_j \to U$ is a closed immersion.

Proof. Since $X \to S$ is finite the fibre over s is a finite set $\{x_1, \ldots, x_n\}$ of points of X. Apply Lemma 17.2 to this set (a finite morphism is separated, see Morphisms, Section 45). The image of W in U is a closed subset (as $X_U \to U$ is finite, hence proper) which does not contain u. After removing this from U we see that $W = \emptyset$ as desired. □

18. Étale local structure of étale morphisms

04HK This is a bit silly, but perhaps helps form intuition about étale morphisms. We simply copy over the results of Section 17 and change "closed immersion" into "isomorphism".

04HL **Lemma 18.1.** *Let $f : X \to S$ be a morphism of schemes. Let $x_1, \ldots, x_n \in X$ be points having the same image s in S. Assume f is étale at each x_i. Then there exists an étale neighbourhood $(U, u) \to (S, s)$ and opens $V_{i,j} \subset X_U$, $i = 1, \ldots, n$, $j = 1, \ldots, m_i$ such that*

(1) *$V_{i,j} \to U$ is an isomorphism,*
(2) *u is not in the image of $V_{i,j} \cap V_{i',j'}$ unless $i = i'$ and $j = j'$, and*
(3) *any point of $(X_U)_u$ mapping to x_i is in some $V_{i,j}$.*

Proof. An étale morphism is unramified, hence we may apply Lemma 17.1. Now $V_{i,j} \to U$ is a closed immersion and étale. Hence it is an open immersion, for example by Theorem 14.1. Replace U by the intersection of the images of $V_{i,j} \to U$ to get the lemma. □

04HM **Lemma 18.2.** *Let $f : X \to S$ be a morphism of schemes. Let $x_1, \ldots, x_n \in X$ be points having the same image s in S. Assume f is separated and f is étale at each x_i. Then there exists an étale neighbourhood $(U, u) \to (S, s)$ and a finite disjoint union decomposition*

$$X_U = W \amalg \coprod\nolimits_{i,j} V_{i,j}$$

of schemes such that

(1) *$V_{i,j} \to U$ is an isomorphism,*
(2) *the fibre W_u contains no point mapping to any x_i.*

In particular, if $f^{-1}(\{s\}) = \{x_1, \ldots, x_n\}$, then the fibre W_u is empty.

Proof. An étale morphism is unramified, hence we may apply Lemma 17.2. As in the proof of Lemma 18.1 the morphisms $V_{i,j} \to U$ are open immersions and we win after replacing U by the intersection of their images. □

The following lemma is in some sense much weaker than the preceding one but it may be useful to state it explicitly here. It says that a finite étale morphism is étale locally on the base a "topological covering space", i.e., a finite product of copies of the base.

04HN **Lemma 18.3.** *Let $f : X \to S$ be a finite étale morphism of schemes. Let $s \in S$. There exists an étale neighbourhood $(U, u) \to (S, s)$ and a finite disjoint union decomposition*

$$X_U = \coprod\nolimits_j V_j$$

of schemes such that each $V_j \to U$ is an isomorphism.

Proof. An étale morphism is unramified, hence we may apply Lemma 17.3. As in the proof of Lemma 18.1 we see that $V_{i,j} \to U$ is an open immersion and we win after replacing U by the intersection of their images. □

19. Permanence properties

025L In what follows, we present a few "permanence" properties of étale homomorphisms of Noetherian local rings (as defined in Definition 11.1). See More on Algebra, Sections 44 and 46 for the analogue of this material for the completion and henselization of a Noetherian local ring.

039S **Lemma 19.1.** *Let A, B be Noetherian local rings. Let $A \to B$ be a étale homomorphism of local rings. Then* $\dim(A) = \dim(B)$.

Proof. See for example Algebra, Lemma 112.7. □

039T **Proposition 19.2.** *Let A, B be Noetherian local rings. Let $f : A \to B$ be an étale homomorphism of local rings. Then depth(A) = depth(B)*

Proof. See Algebra, Lemma 163.2. □

025Q **Proposition 19.3.** *Let A, B be Noetherian local rings. Let $f : A \to B$ be an étale homomorphism of local rings. Then A is Cohen-Macaulay if and only if B is so.*

Proof. A local ring A is Cohen-Macaulay if and only if $\dim(A) = \text{depth}(A)$. As both of these invariants is preserved under an étale extension, the claim follows. □

025N **Proposition 19.4.** *Let A, B be Noetherian local rings. Let $f : A \to B$ be an étale homomorphism of local rings. Then A is regular if and only if B is so.*

Proof. If B is regular, then A is regular by Algebra, Lemma 110.9. Assume A is regular. Let $\mathfrak{m}$ be the maximal ideal of A. Then $\dim_{\kappa(\mathfrak{m})} \mathfrak{m}/\mathfrak{m}^2 = \dim(A) = \dim(B)$ (see Lemma 19.1). On the other hand, $\mathfrak{m}B$ is the maximal ideal of B and hence $\mathfrak{m}_B/\mathfrak{m}_B = \mathfrak{m}B/\mathfrak{m}^2B$ is generated by at most $\dim(B)$ elements. Thus B is regular. (You can also use the slightly more general Algebra, Lemma 112.8.) □

025O **Proposition 19.5.** *Let A, B be Noetherian local rings. Let $f : A \to B$ be an étale homomorphism of local rings. Then A is reduced if and only if B is so.*

Proof. It is clear from the faithful flatness of $A \to B$ that if B is reduced, so is A. See also Algebra, Lemma 164.2. Conversely, assume A is reduced. By assumption B is a localization of a finite type A-algebra B' at some prime $\mathfrak{q}$. After replacing B' by a localization we may assume that B' is étale over A, see Lemma 11.2. Then we see that Algebra, Lemma 163.7 applies to $A \to B'$ and B' is reduced. Hence B is reduced. □

039U **Remark 19.6.** The result on "reducedness" does not hold with a weaker definition of étale local ring maps $A \to B$ where one drops the assumption that B is essentially of finite type over A. Namely, it can happen that a Noetherian local domain A has nonreduced completion $A^\wedge$, see Examples, Section 17. But the ring map $A \to A^\wedge$ is flat, and $\mathfrak{m}_A A^\wedge$ is the maximal ideal of $A^\wedge$ and of course A and $A^\wedge$ have the same residue fields. This is why it is important to consider this notion only for ring extensions which are essentially of finite type (or essentially of finite presentation if A is not Noetherian).

025P **Proposition 19.7.** *Let A, B be Noetherian local rings. Let $f : A \to B$ be an étale homomorphism of local rings. Then A is a normal domain if and only if B is so.* [Gro71, Expose I, Theorem 9.5 part (i)]

Proof. See Algebra, Lemma 164.3 for descending normality. Conversely, assume A is normal. By assumption B is a localization of a finite type A-algebra B' at some prime $\mathfrak{q}$. After replacing B' by a localization we may assume that B' is étale over A, see Lemma 11.2. Then we see that Algebra, Lemma 163.9 applies to $A \to B'$ and we conclude that B' is normal. Hence B is a normal domain. □

The preceding propositions give some indication as to why we'd like to think of étale maps as "local isomorphisms". Another property that gives an excellent indication that we have the "right" definition is the fact that for **C**-schemes of finite type, a morphism is étale if and only if the associated morphism on analytic spaces (the **C**-valued points given the complex topology) is a local isomorphism in the analytic sense (open embedding locally on the source). This fact can be proven with the aid of the structure theorem and the fact that the analytification commutes with the formation of the completed local rings – the details are left to the reader.

20. Descending étale morphisms

0BTH In order to understand the language used in this section we encourage the reader to take a look at Descent, Section 34. Let $f : X \to S$ be a morphism of schemes. Consider the pullback functor

0BTI (20.0.1) $$\text{schemes } U \text{ étale over } S \longrightarrow \begin{matrix}\text{descent data } (V, \varphi) \text{ relative to } X/S \\ \text{with } V \text{ étale over } X\end{matrix}$$

sending U to the canonical descent datum $(X \times_S U, can)$.

0BTJ **Lemma 20.1.** *If $f : X \to S$ is surjective, then the functor (20.0.1) is faithful.*

Proof. Let $a, b : U_1 \to U_2$ be two morphisms between schemes étale over S. Assume the base changes of a and b to X agree. We have to show that $a = b$. By Proposition 6.3 it suffices to show that a and b agree on points and residue fields. This is clear because for every $u \in U_1$ we can find a point $v \in X \times_S U_1$ mapping to u. □

0BTK **Lemma 20.2.** *Assume $f : X \to S$ is submersive and any étale base change of f is submersive. Then the functor (20.0.1) is fully faithful.*

Proof. By Lemma 20.1 the functor is faithful. Let $U_1 \to S$ and $U_2 \to S$ be étale morphisms and let $a : X \times_S U_1 \to X \times_S U_2$ be a morphism compatible with canonical descent data. We will prove that a is the base change of a morphism $U_1 \to U_2$.

Let $U_2' \subset U_2$ be an open subscheme. Consider $W = a^{-1}(X \times_S U_2')$. This is an open subscheme of $X \times_S U_1$ which is compatible with the canonical descent datum on $V_1 = X \times_S U_1$. This means that the two inverse images of W by the projections $V_1 \times_{U_1} V_1 \to V_1$ agree. Since $V_1 \to U_1$ is surjective (as the base change of $X \to S$) we conclude that W is the inverse image of some subset $U_1' \subset U_1$. Since W is open, our assumption on f implies that $U_1' \subset U_1$ is open.

Let $U_2 = \bigcup U_{2,i}$ be an affine open covering. By the result of the preceding paragraph we obtain an open covering $U_1 = \bigcup U_{1,i}$ such that $X \times_S U_{1,i} = a^{-1}(X \times_S U_{2,i})$. If we can prove there exists a morphism $U_{1,i} \to U_{2,i}$ whose base change is the morphism $a_i : X \times_S U_{1,i} \to X \times_S U_{2,i}$ then we can glue these morphisms to a morphism $U_1 \to U_2$ (using faithfulness). In this way we reduce to the case that U_2 is affine. In particular $U_2 \to S$ is separated (Schemes, Lemma 21.13).

Assume $U_2 \to S$ is separated. Then the graph Γ_a of a is a closed subscheme of

$$V = (X \times_S U_1) \times_X (X \times_S U_2) = X \times_S U_1 \times_S U_2$$

by Schemes, Lemma 21.10. On the other hand the graph is open for example because it is a section of an étale morphism (Proposition 6.1). Since a is a morphism of descent data, the two inverse images of $\Gamma_a \subset V$ under the projections $V \times_{U_1 \times_S U_2} V \to V$ are the same. Hence arguing as in the second paragraph of the proof we find an open and closed subscheme $\Gamma \subset U_1 \times_S U_2$ whose base change to X gives Γ_a. Then $\Gamma \to U_1$ is an étale morphism whose base change to X is an isomorphism. This means that $\Gamma \to U_1$ is universally bijective, hence an isomorphism by Theorem 14.1. Thus Γ is the graph of a morphism $U_1 \to U_2$ and the base change of this morphism is a as desired. □

0BTL **Lemma 20.3.** *Let $f : X \to S$ be a morphism of schemes. In the following cases the functor (20.0.1) is fully faithful:*

(1) *f is surjective and universally closed (e.g., finite, integral, or proper),*
(2) *f is surjective and universally open (e.g., locally of finite presentation and flat, smooth, or etale),*
(3) *f is surjective, quasi-compact, and flat.*

Proof. This follows from Lemma 20.2. For example a closed surjective map of topological spaces is submersive (Topology, Lemma 6.5). Finite, integral, and proper morphisms are universally closed, see Morphisms, Lemmas 45.7 and 45.11 and Definition 42.1. On the other hand an open surjective map of topological spaces is submersive (Topology, Lemma 6.4). Flat locally finitely presented, smooth, and étale morphisms are universally open, see Morphisms, Lemmas 26.10, 35.10, and 37.13. The case of surjective, quasi-compact, flat morphisms follows from Morphisms, Lemma 26.12. □

0BTM **Lemma 20.4.** *Let $f : X \to S$ be a morphism of schemes. Let (V, φ) be a descent datum relative to X/S with $V \to X$ étale. Let $S = \bigcup S_i$ be an open covering. Assume that*

(1) *the pullback of the descent datum (V, φ) to $X \times_S S_i/S_i$ is effective,*
(2) *the functor (20.0.1) for $X \times_S (S_i \cap S_j) \to (S_i \cap S_j)$ is fully faithful, and*
(3) *the functor (20.0.1) for $X \times_S (S_i \cap S_j \cap S_k) \to (S_i \cap S_j \cap S_k)$ is faithful.*

Then (V, φ) is effective.

Proof. (Recall that pullbacks of descent data are defined in Descent, Definition 34.7.) Set $X_i = X \times_S S_i$. Denote (V_i, φ_i) the pullback of (V, φ) to X_i/S_i. By assumption (1) we can find an étale morphism $U_i \to S_i$ which comes with an isomorphism $X_i \times_{S_i} U_i \to V_i$ compatible with *can* and φ_i. By assumption (2) we obtain isomorphisms $\psi_{ij} : U_i \times_{S_i} (S_i \cap S_j) \to U_j \times_{S_j} (S_i \cap S_j)$. By assumption (3) these isomorphisms satisfy the cocycle condition so that (U_i, ψ_{ij}) is a descend datum for the Zariski covering $\{S_i \to S\}$. Then Descent, Lemma 35.10 (which is essentially just a reformulation of Schemes, Section 14) tells us that there exists a morphism of schemes $U \to S$ and isomorphisms $U \times_S S_i \to U_i$ compatible with ψ_{ij}. The isomorphisms $U \times_S S_i \to U_i$ determine corresponding isomorphisms $X_i \times_S U \to V_i$ which glue to a morphism $X \times_S U \to V$ compatible with the canonical descent datum and φ. □

0BTN **Lemma 20.5.** *Let (A, I) be a henselian pair. Let $U \to \mathrm{Spec}(A)$ be a quasi-compact, separated, étale morphism such that $U \times_{\mathrm{Spec}(A)} \mathrm{Spec}(A/I) \to \mathrm{Spec}(A/I)$ is finite. Then*

$$U = U_{fin} \amalg U_{away}$$

where $U_{fin} \to \mathrm{Spec}(A)$ is finite and U_{away} has no points lying over Z.

Proof. By Zariski's main theorem, the scheme U is quasi-affine. In fact, we can find an open immersion $U \to T$ with T affine and $T \to \mathrm{Spec}(A)$ finite, see More on Morphisms, Lemma 43.3. Write $Z = \mathrm{Spec}(A/I)$ and denote $U_Z \to T_Z$ the base change. Since $U_Z \to Z$ is finite, we see that $U_Z \to T_Z$ is closed as well as open. Hence by More on Algebra, Lemma 11.6 we obtain a unique decomposition $T = T' \amalg T''$ with $T'_Z = U_Z$. Set $U_{fin} = U \cap T'$ and $U_{away} = U \cap T''$. Since $T'_Z \subset U_Z$ we see that all closed points of T' are in U hence $T' \subset U$, hence $U_{fin} = T'$, hence $U_{fin} \to \mathrm{Spec}(A)$ is finite. We omit the proof of uniqueness of the decomposition. □

0BTP **Proposition 20.6.** *Let $f : X \to S$ be a surjective integral morphism. The functor (20.0.1) induces an equivalence*

$$\begin{array}{c}\textit{schemes quasi-compact,}\\ \textit{separated, étale over } S\end{array} \longrightarrow \begin{array}{c}\textit{descent data } (V, \varphi) \textit{ relative to } X/S \textit{ with}\\ V \textit{ quasi-compact, separated, étale over } X\end{array}$$

Proof. By Lemma 20.3 the functor (20.0.1) is fully faithful and the same remains the case after any base change $S \to S'$. Let (V, φ) be a descent data relative to X/S with $V \to X$ quasi-compact, separated, and étale. We can use Lemma 20.4 to see that it suffices to prove the effectivity Zariski locally on S. In particular we may and do assume that S is affine.

If S is affine we can find a directed set Λ and an inverse system $X_\lambda \to S_\lambda$ of finite morphisms of affine schemes of finite type over $\mathrm{Spec}(\mathbf{Z})$ such that $(X \to S) = \lim(X_\lambda \to S_\lambda)$. See Algebra, Lemma 127.15. Since limits commute with limits we deduce that $X \times_S X = \lim X_\lambda \times_{S_\lambda} X_\lambda$ and $X \times_S X \times_S X = \lim X_\lambda \times_{S_\lambda} X_\lambda \times_{S_\lambda} X_\lambda$. Observe that $V \to X$ is a morphism of finite presentation. Using Limits, Lemmas 10.1 we can find an λ and a descent datum $(V_\lambda, \varphi_\lambda)$ relative to X_λ/S_λ whose pullback to X/S is (V, φ). Of course it is enough to show that $(V_\lambda, \varphi_\lambda)$ is effective. Note that V_λ is quasi-compact by construction. After possibly increasing λ we may assume that $V_\lambda \to X_\lambda$ is separated and étale, see Limits, Lemma 8.6 and 8.10. Thus we may assume that f is finite surjective and S affine of finite type over $\mathbf{Z}$.

Consider an open $S' \subset S$ such that the pullback (V', φ') of (V, φ) to $X' = X \times_S S'$ is effective. Below we will prove, that $S' \neq S$ implies there is a strictly larger open over which the descent datum is effective. Since S is Noetherian (and hence has a Noetherian underlying topological space) this will finish the proof. Let $\xi \in S$ be a generic point of an irreducible component of the closed subset $Z = S \setminus S'$. If $\xi \in S'' \subset S$ is an open over which the descent datum is effective, then the descent datum is effective over $S' \cup S''$ by the glueing argument of the first paragraph. Thus in the rest of the proof we may replace S by an affine open neighbourhood of ξ.

After a first such replacement we may assume that Z is irreducible with generic point Z. Let us endow Z with the reduced induced closed subscheme structure. After another shrinking we may assume $X_Z = X \times_S Z = f^{-1}(Z) \to Z$ is flat, see Morphisms, Proposition 28.1. Let (V_Z, φ_Z) be the pullback of the descent datum to X_Z/Z. By More on Morphisms, Lemma 57.1 this descent datum is effective and

we obtain an étale morphism $U_Z \to Z$ whose base change is isomorphic to V_Z in a manner compatible with descent data. Of course $U_Z \to Z$ is quasi-compact and separated (Descent, Lemmas 23.1 and 23.6). Thus after shrinking once more we may assume that $U_Z \to Z$ is finite, see Morphisms, Lemma 52.1.

Let $S = \mathrm{Spec}(A)$ and let $I \subset A$ be the prime ideal corresponding to $Z \subset S$. Let (A^h, IA^h) be the henselization of the pair (A, I). Denote $S^h = \mathrm{Spec}(A^h)$ and $Z^h = V(IA^h) \cong Z$. We claim that it suffices to show effectivity after base change to S^h. Namely, $\{S^h \to S, S' \to S\}$ is an fpqc covering ($A \to A^h$ is flat by More on Algebra, Lemma 12.2) and by More on Morphisms, Lemma 57.1 we have fpqc descent for separated étale morphisms. Namely, if $U^h \to S^h$ and $U' \to S'$ are the objects corresponding to the pullbacks (V^h, φ^h) and (V', φ'), then the required isomorphisms

$$U^h \times_S S^h \to S^h \times_S V^h \quad \text{and} \quad U^h \times_S S' \to S^h \times_S U'$$

are obtained by the fully faithfulness pointed out in the first paragraph. In this way we reduce to the situation described in the next paragraph.

Here $S = \mathrm{Spec}(A)$, $Z = V(I)$, $S' = S \setminus Z$ where (A, I) is a henselian pair, we have $U' \to S'$ corresponding to the descent datum (V', φ') and we have a finite étale morphism $U_Z \to Z$ corresponding to the descent datum (V_Z, φ_Z). We no longer have that A is of finite type over $\mathbf{Z}$; but the rest of the argument will not even use that A is Noetherian. By More on Algebra, Lemma 13.2 we can find a finite étale morphism $U_{fin} \to S$ whose restriction to Z is isomorphic to $U_Z \to Z$. Write $X = \mathrm{Spec}(B)$ and $Y = V(IB)$. Since (B, IB) is a henselian pair (More on Algebra, Lemma 11.8) and since the restriction $V \to X$ to Y is finite (as base change of $U_Z \to Z$) we see that there is a canonical disjoint union decomposition

$$V = V_{fin} \amalg V_{away}$$

were $V_{fin} \to X$ is finite and where V_{away} has no points lying over Y. See Lemma 20.5. Using the uniqueness of this decomposition over $X \times_S X$ we see that φ preserves it and we obtain

$$(V, \varphi) = (V_{fin}, \varphi_{fin}) \amalg (V_{away}, \varphi_{away})$$

in the category of descent data. By More on Algebra, Lemma 13.2 there is a unique isomorphism

$$X \times_S U_{fin} \longrightarrow V_{fin}$$

compatible with the given isomorphism $Y \times_Z U_Z \to V \times_X Y$ over Y. By the uniqueness we see that this isomorphism is compatible with descent data, i.e., $(X \times_S U_{fin}, can) \cong (V_{fin}, \varphi_{fin})$. Denote $U'_{fin} = U_{fin} \times_S S'$. By fully faithfulness we obtain a morphism $U'_{fin} \to U'$ which is the inclusion of an open (and closed) subscheme. Then we set $U = U_{fin} \amalg_{U'_{fin}} U'$ (glueing of schemes as in Schemes, Section 14). The morphisms $X \times_S U_{fin} \to V$ and $X \times_S U' \to V$ glue to a morphism $X \times_S U \to V$ which is the desired isomorphism. □

21. Normal crossings divisors

0CBN Here is the definition.

0BI9 **Definition 21.1.** Let X be a locally Noetherian scheme. A *strict normal crossings divisor* on X is an effective Cartier divisor $D \subset X$ such that for every $p \in D$ the local ring $\mathcal{O}_{X,p}$ is regular and there exists a regular system of parameters $x_1, \ldots, x_d \in \mathfrak{m}_p$ and $1 \leq r \leq d$ such that D is cut out by $x_1 \ldots x_r$ in $\mathcal{O}_{X,p}$.

We often encounter effective Cartier divisors E on locally Noetherian schemes X such that there exists a strict normal crossings divisor D with $E \subset D$ set theoretically. In this case we have $E = \sum a_i D_i$ with $a_i \geq 0$ where $D = \bigcup_{i \in I} D_i$ is the decomposition of D into its irreducible components. Observe that $D' = \bigcup_{a_i > 0} D_i$ is a strict normal crossings divisor with $E = D'$ set theoretically. When the above happens we will say that E is *supported on a strict normal crossings divisor*.

0BIA **Lemma 21.2.** *Let X be a locally Noetherian scheme. Let $D \subset X$ be an effective Cartier divisor. Let $D_i \subset D$, $i \in I$ be its irreducible components viewed as reduced closed subschemes of X. The following are equivalent*

1. *D is a strict normal crossings divisor, and*
2. *D is reduced, each D_i is an effective Cartier divisor, and for $J \subset I$ finite the scheme theoretic intersection $D_J = \bigcap_{j \in J} D_j$ is a regular scheme each of whose irreducible components has codimension $|J|$ in X.*

Proof. Assume D is a strict normal crossings divisor. Pick $p \in D$ and choose a regular system of parameters $x_1, \ldots, x_d \in \mathfrak{m}_p$ and $1 \leq r \leq d$ as in Definition 21.1. Since $\mathcal{O}_{X,p}/(x_i)$ is a regular local ring (and in particular a domain) we see that the irreducible components $D_1, \ldots, D_r$ of D passing through p correspond 1-to-1 to the height one primes $(x_1), \ldots, (x_r)$ of $\mathcal{O}_{X,p}$. By Algebra, Lemma 106.3 we find that the intersections $D_{i_1} \cap \ldots \cap D_{i_s}$ have codimension s in an open neighbourhood of p and that this intersection has a regular local ring at p. Since this holds for all $p \in D$ we conclude that (2) holds.

Assume (2). Let $p \in D$. Since $\mathcal{O}_{X,p}$ is finite dimensional we see that p can be contained in at most $\dim(\mathcal{O}_{X,p})$ of the components D_i. Say $p \in D_1, \ldots, D_r$ for some $r \geq 1$. Let $x_1, \ldots, x_r \in \mathfrak{m}_p$ be local equations for $D_1, \ldots, D_r$. Then x_1 is a nonzerodivisor in $\mathcal{O}_{X,p}$ and $\mathcal{O}_{X,p}/(x_1) = \mathcal{O}_{D_1,p}$ is regular. Hence $\mathcal{O}_{X,p}$ is regular, see Algebra, Lemma 106.7. Since $D_1 \cap \ldots \cap D_r$ is a regular (hence normal) scheme it is a disjoint union of its irreducible components (Properties, Lemma 7.6). Let $Z \subset D_1 \cap \ldots \cap D_r$ be the irreducible component containing p. Then $\mathcal{O}_{Z,p} = \mathcal{O}_{X,p}/(x_1, \ldots, x_r)$ is regular of codimension r (note that since we already know that $\mathcal{O}_{X,p}$ is regular and hence Cohen-Macaulay, there is no ambiguity about codimension as the ring is catenary, see Algebra, Lemmas 106.3 and 104.4). Hence $\dim(\mathcal{O}_{Z,p}) = \dim(\mathcal{O}_{X,p}) - r$. Choose additional $x_{r+1}, \ldots, x_n \in \mathfrak{m}_p$ which map to a minimal system of generators of $\mathfrak{m}_{Z,p}$. Then $\mathfrak{m}_p = (x_1, \ldots, x_n)$ by Nakayama's lemma and we see that D is a normal crossings divisor. □

0CBP **Lemma 21.3.** *Let X be a locally Noetherian scheme. Let $D \subset X$ be a strict normal crossings divisor. If $f : Y \to X$ is a smooth morphism of schemes, then the pullback f^*D is a strict normal crossings divisor on Y.*

Proof. As f is flat the pullback is defined by Divisors, Lemma 14.13 hence the statement makes sense. Let $q \in f^*D$ map to $p \in D$. Choose a regular system of parameters $x_1, \ldots, x_d \in \mathfrak{m}_p$ and $1 \leq r \leq d$ as in Definition 21.1. Since f is smooth the local ring homomorphism $\mathcal{O}_{X,p} \to \mathcal{O}_{Y,q}$ is flat and the fibre ring

$$\mathcal{O}_{Y,q}/\mathfrak{m}_p\mathcal{O}_{Y,q} = \mathcal{O}_{Y_p,q}$$

is a regular local ring (see for example Algebra, Lemma 140.3). Pick $y_1, \ldots, y_n \in \mathfrak{m}_q$ which map to a regular system of parameters in $\mathcal{O}_{Y_p,q}$. Then $x_1, \ldots, x_d, y_1, \ldots, y_n$ generate the maximal ideal $\mathfrak{m}_q$. Hence $\mathcal{O}_{Y,q}$ is a regular local ring of dimension $d + n$ by Algebra, Lemma 112.7 and $x_1, \ldots, x_d, y_1, \ldots, y_n$ is a regular system of parameters. Since f^*D is cut out by $x_1 \ldots x_r$ in $\mathcal{O}_{Y,q}$ we conclude that the lemma is true. □

Here is the definition of a normal crossings divisor.

0BSF **Definition 21.4.** Let X be a locally Noetherian scheme. A *normal crossings divisor* on X is an effective Cartier divisor $D \subset X$ such that for every $p \in D$ there exists an étale morphism $U \to X$ with p in the image and $D \times_X U$ a strict normal crossings divisor on U.

For example $D = V(x^2 + y^2)$ is a normal crossings divisor (but not a strict one) on $\mathrm{Spec}(\mathbf{R}[x, y])$ because after pulling back to the étale cover $\mathrm{Spec}(\mathbf{C}[x, y])$ we obtain $(x - iy)(x + iy) = 0$.

0CBQ **Lemma 21.5.** *Let X be a locally Noetherian scheme. Let $D \subset X$ be a normal crossings divisor. If $f : Y \to X$ is a smooth morphism of schemes, then the pullback f^*D is a normal crossings divisor on Y.*

Proof. As f is flat the pullback is defined by Divisors, Lemma 14.13 hence the statement makes sense. Let $q \in f^*D$ map to $p \in D$. Choose an étale morphism $U \to X$ whose image contains p such that $D \times_X U \subset U$ is a strict normal crossings divisor as in Definition 21.4. Set $V = Y \times_X U$. Then $V \to Y$ is étale as a base change of $U \to X$ (Morphisms, Lemma 37.4) and the pullback $D \times_X V$ is a strict normal crossings divisor on V by Lemma 21.3. Thus we have checked the condition of Definition 21.4 for $q \in f^*D$ and we conclude. □

0CBR **Lemma 21.6.** *Let X be a locally Noetherian scheme. Let $D \subset X$ be a closed subscheme. The following are equivalent*

(1) *D is a normal crossings divisor in X,*
(2) *D is reduced, the normalization $\nu : D^\nu \to D$ is unramified, and for any $n \geq 1$ the scheme*

$$Z_n = D^\nu \times_D \ldots \times_D D^\nu \setminus \{(p_1, \ldots, p_n) \mid p_i = p_j \text{ for some } i \neq j\}$$

is regular, the morphism $Z_n \to X$ is a local complete intersection morphism whose conormal sheaf is locally free of rank n.

Proof. First we explain how to think about condition (2). The diagonal of an unramified morphism is open (Morphisms, Lemma 36.13). On the other hand $D^\nu \to D$ is separated, hence the diagonal $D^\nu \to D^\nu \times_D D^\nu$ is closed. Thus Z_n is an open and closed subscheme of $D^\nu \times_D \ldots \times_D D^\nu$. On the other hand, $Z_n \to X$ is unramified as it is the composition

$$Z_n \to D^\nu \times_D \ldots \times_D D^\nu \to \ldots \to D^\nu \times_D D^\nu \to D^\nu \to D \to X$$

and each of the arrows is unramified. Since an unramified morphism is formally unramified (More on Morphisms, Lemma 6.9) we have a conormal sheaf $\mathcal{C}_n = \mathcal{C}_{Z_n/X}$ of $Z_n \to X$, see More on Morphisms, Definition 7.2.

Formation of normalization commutes with étale localization by More on Morphisms, Lemma 19.3. Checking that local rings are regular, or that a morphism is

unramified, or that a morphism is a local complete intersection or that a morphism is unramified and has a conormal sheaf which is locally free of a given rank, may be done étale locally (see More on Algebra, Lemma 45.3, Descent, Lemma 23.30, More on Morphisms, Lemma 62.19 and Descent, Lemma 7.6).

By the remark of the preceding paragraph and the definition of normal crossings divisor it suffices to prove that a strict normal crossings divisor $D = \bigcup_{i \in I} D_i$ satisfies (2). In this case $D^\nu = \coprod D_i$ and $D^\nu \to D$ is unramified (being unramified is local on the source and $D_i \to D$ is a closed immersion which is unramified). Similarly, $Z_1 = D^\nu \to X$ is a local complete intersection morphism because we may check this locally on the source and each morphism $D_i \to X$ is a regular immersion as it is the inclusion of a Cartier divisor (see Lemma 21.2 and More on Morphisms, Lemma 62.9). Since an effective Cartier divisor has an invertible conormal sheaf, we conclude that the requirement on the conormal sheaf is satisfied. Similarly, the scheme Z_n for $n \geq 2$ is the disjoint union of the schemes $D_J = \bigcap_{j \in J} D_j$ where $J \subset I$ runs over the subsets of order n. Since $D_J \to X$ is a regular immersion of codimension n (by the definition of strict normal crossings and the fact that we may check this on stalks by Divisors, Lemma 21.8) it follows in the same manner that $Z_n \to X$ has the required properties. Some details omitted.

Assume (2). Let $p \in D$. Since $D^\nu \to D$ is unramified, it is finite (by Morphisms, Lemma 45.4). Hence $D^\nu \to X$ is finite unramified. By Lemma 17.3 and étale localization (permissible by the discussion in the second paragraph and the definition of normal crossings divisors) we reduce to the case where $D^\nu = \coprod_{i \in I} D_i$ with I finite and $D_i \to U$ a closed immersion. After shrinking X if necessary, we may assume $p \in D_i$ for all $i \in I$. The condition that $Z_1 = D^\nu \to X$ is an unramified local complete intersection morphism with conormal sheaf locally free of rank 1 implies that $D_i \subset X$ is an effective Cartier divisor, see More on Morphisms, Lemma 62.3 and Divisors, Lemma 22.3. To finish the proof we may assume $X = \mathrm{Spec}(A)$ is affine and $D_i = V(f_i)$ with $f_i \in A$ a nonzerodivisor. If $I = \{1, \ldots, r\}$, then $p \in Z_r = V(f_1, \ldots, f_r)$. The same reference as above implies that $(f_1, \ldots, f_r)$ is a Koszul regular ideal in A. Since the conormal sheaf has rank r, we see that $f_1, \ldots, f_r$ is a minimal set of generators of the ideal defining Z_r in $\mathcal{O}_{X,p}$. This implies that $f_1, \ldots, f_r$ is a regular sequence in $\mathcal{O}_{X,p}$ such that $\mathcal{O}_{X,p}/(f_1, \ldots, f_r)$ is regular. Thus we conclude by Algebra, Lemma 106.7 that $f_1, \ldots, f_r$ can be extended to a regular system of parameters in $\mathcal{O}_{X,p}$ and this finishes the proof. □

0CBS **Lemma 21.7.** *Let X be a locally Noetherian scheme. Let $D \subset X$ be a closed subscheme. If X is J-2 or Nagata, then the following are equivalent*

(1) *D is a normal crossings divisor in X,*
(2) *for every $p \in D$ the pullback of D to the spectrum of the strict henselization $\mathcal{O}_{X,p}^{sh}$ is a strict normal crossings divisor.*

Proof. The implication (1) ⇒ (2) is straightforward and does not need the assumption that X is J-2 or Nagata. Namely, let $p \in D$ and choose an étale neighbourhood $(U, u) \to (X, p)$ such that the pullback of D is a strict normal crossings divisor on U. Then $\mathcal{O}_{X,p}^{sh} = \mathcal{O}_{U,u}^{sh}$ and we see that the trace of D on $\mathrm{Spec}(\mathcal{O}_{U,u}^{sh})$ is cut out by part of a regular system of parameters as this is already the case in $\mathcal{O}_{U,u}$.

To prove the implication in the other direction we will use the criterion of Lemma 21.6. Observe that formation of the normalization $D^\nu \to D$ commutes with strict

henselization, see More on Morphisms, Lemma 19.4. If we can show that $D^\nu \to D$ is finite, then we see that $D^\nu \to D$ and the schemes Z_n satisfy all desired properties because these can all be checked on the level of local rings (but the finiteness of the morphism $D^\nu \to D$ is not something we can check on local rings). We omit the detailed verifications.

If X is Nagata, then $D^\nu \to D$ is finite by Morphisms, Lemma 55.11.

Assume X is J-2. Choose a point $p \in D$. We will show that $D^\nu \to D$ is finite over a neighbourhood of p. By assumption there exists a regular system of parameters $f_1, \ldots, f_d$ of $\mathcal{O}_{X,p}^{sh}$ and $1 \leq r \leq d$ such that the trace of D on $\mathrm{Spec}(\mathcal{O}_{X,p}^{sh})$ is cut out by $f_1 \ldots f_r$. Then

$$D^\nu \times_X \mathrm{Spec}(\mathcal{O}_{X,p}^{sh}) = \coprod\nolimits_{i=1,\ldots,r} V(f_i)$$

Choose an affine étale neighbourhood $(U, u) \to (X, p)$ such that f_i comes from $f_i \in \mathcal{O}_U(U)$. Set $D_i = V(f_i) \subset U$. The strict henselization of $\mathcal{O}_{D_i,u}$ is $\mathcal{O}_{X,p}^{sh}/(f_i)$ which is regular. Hence $\mathcal{O}_{D_i,u}$ is regular (for example by More on Algebra, Lemma 46.10). Because X is J-2 the regular locus is open in D_i. Thus after replacing U by a Zariski open we may assume that D_i is regular for each i. It follows that

$$\coprod\nolimits_{i=1,\ldots,r} D_i = D^\nu \times_X U \longrightarrow D \times_X U$$

is the normalization morphism and it is clearly finite. In other words, we have found an étale neighbourhood (U, u) of (X, p) such that the base change of $D^\nu \to D$ to this neighbourhood is finite. This implies $D^\nu \to D$ is finite by descent (Descent, Lemma 23.25) and the proof is complete. □

22. Other chapters

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