DEFORMATION PROBLEMS

0DVK

Contents

1. Introduction

0DVL The goal of this chapter is to work out examples of the general theory developed in the chapters Formal Deformation Theory, Deformation Theory, The Cotangent Complex.

Section 3 of the paper [Sch68] by Schlessinger discusses some examples as well.

2. Examples of deformation problems

06LA List of things that should go here:

(1) Deformations of schemes:
   (a) The Rim-Schlessinger condition.
   (b) Computing the tangent space.
   (c) Computing the infinitesimal deformations.
   (d) The deformation category of an affine hypersurface.
(2) Deformations of sheaves (for example fix X/S, a finite type point s of S, and a quasi-coherent sheaf $\mathcal{F}_s$ over X_s).

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(3) Deformations of algebraic spaces (very similar to deformations of schemes; maybe even easier?).
(4) Deformations of maps (eg morphisms between schemes; you can fix both or one of the target and/or source).
(5) Add more here.

3. General outline

0DVM This section lays out the procedure for discussing the next few examples.

Step I. For each section we fix a Noetherian ring Λ and we fix a finite ring map $\Lambda \to k$ where k is a field. As usual we let $\mathcal{C}_\Lambda = \mathcal{C}_{\Lambda,k}$ be our base category, see Formal Deformation Theory, Definition 3.1.

Step II. In each section we define a category $\mathcal{F}$ cofibred in groupoids over $\mathcal{C}_\Lambda$. Occassionally we will consider instead a functor $F : \mathcal{C}_\Lambda \to Sets$.

Step III. We explain to what extend $\mathcal{F}$ satisfies the Rim-Schlesssinger condition (RS) discussed in Formal Deformation Theory, Section 16. Similarly, we may discuss to what extend our $\mathcal{F}$ satisfies (S1) and (S2) or to what extend F satisfies the corresponding Schlessinger's conditions (H1) and (H2). See Formal Deformation Theory, Section 10.

Step IV. Let x_0 be an object of $\mathcal{F}(k)$, in other words an object of $\mathcal{F}$ over k. In this chapter we will use the notation

$$\mathcal{D}ef_{x_0} = \mathcal{F}_{x_0}$$

to denote the predeformation category constructed in Formal Deformation Theory, Remark 6.4. If $\mathcal{F}$ satisfies (RS), then $\mathcal{D}ef_{x_0}$ is a deformation category (Formal Deformation Theory, Lemma 16.11) and satisfies (S1) and (S2) (Formal Deformation Theory, Lemma 16.6). If (S1) and (S2) are satisfied, then an important question is whether the tangent space

$$T\mathcal{D}ef_{x_0} = T_{x_0}\mathcal{F} = T\mathcal{F}_{x_0}$$

(see Formal Deformation Theory, Remark 12.5 and Definition 12.1) is finite dimensional. Namely, this insures that $\mathcal{D}ef_{x_0}$ has a versal formal object (Formal Deformation Theory, Lemma 13.4).

Step V. If $\mathcal{F}$ passes Step IV, then the next question is whether the k-vector space

$$\mathrm{Inf}(\mathcal{D}ef_{x_0}) = \mathrm{Inf}_{x_0}(\mathcal{F})$$

of infinitesimal automorphisms of x_0 is finite dimensional. Namely, if true, this implies that $\mathcal{D}ef_{x_0}$ admits a presentation by a smooth prorepresentable groupoid in functors on $\mathcal{C}_\Lambda$, see Formal Deformation Theory, Theorem 26.4.

4. Finite projective modules

0DVN This section is just a warmup. Of course finite projective modules should not have any "moduli".

0D3I **Example 4.1** (Finite projective modules)**.** Let $\mathcal{F}$ be the category defined as follows

(1) an object is a pair (A, M) consisting of an object A of $\mathcal{C}_\Lambda$ and a finite projective A-module M, and

(2) a morphism $(f, g) : (B, N) \to (A, M)$ consists of a morphism $f : B \to A$ in $\mathcal{C}_\Lambda$ together with a map $g : N \to M$ which is f-linear and induces an isomorpism $N \otimes_{B,f} A \cong M$.

The functor $p : \mathcal{F} \to \mathcal{C}_\Lambda$ sends (A, M) to A and (f, g) to f. It is clear that p is cofibred in groupoids. Given a finite dimensional k-vector space V, let $x_0 = (k, V)$ be the corresponding object of $\mathcal{F}(k)$. We set

$$\mathcal{D}ef_V = \mathcal{F}_{x_0}$$

Since every finite projective module over a local ring is finite free (Algebra, Lemma 77.2) we see that

$$\begin{matrix}\text{isomorphism classes} \\ \text{of objects of } \mathcal{F}(A)\end{matrix} = \coprod\nolimits_{n\geq 0} \{*\}$$

Although this means that the deformation theory of $\mathcal{F}$ is essentially trivial, we still work through the steps outlined in Section 3 to provide an easy example.

0DVP **Lemma 4.2.** *Example 4.1 satisfies the Rim-Schlessinger condition (RS). In particular, $\mathcal{D}ef_V$ is a deformation category for any finite dimensional vector space V over k.*

Proof. Let $A_1 \to A$ and $A_2 \to A$ be morphisms of $\mathcal{C}_\Lambda$. Assume $A_2 \to A$ is surjective. According to Formal Deformation Theory, Lemma 16.4 it suffices to show that the functor $\mathcal{F}(A_1 \times_A A_2) \to \mathcal{F}(A_1) \times_{\mathcal{F}(A)} \mathcal{F}(A_2)$ is an equivalence of categories.

Thus we have to show that the category of finite projective modules over $A_1 \times_A A_2$ is equivalent to the fibre product of the categories of finite projective modules over A_1 and A_2 over the category of finite projective modules over A. This is a special case of More on Algebra, Lemma 6.9. We recall that the inverse functor sends the triple (M_1, M_2, φ) where M_1 is a finite projective A_1-module, M_2 is a finite projective A_2-module, and $\varphi : M_1 \otimes_{A_1} A \to M_2 \otimes_{A_2} A$ is an isomorphism of A-module, to the finite projective $A_1 \times_A A_2$-module $M_1 \times_\varphi M_2$. □

0DVQ **Lemma 4.3.** *In Example 4.1 let V be a finite dimensional k-vector space. Then*

$$T\mathcal{D}ef_V = (0) \quad \text{and} \quad Inf(\mathcal{D}ef_V) = End_k(V)$$

are finite dimensional.

Proof. With $\mathcal{F}$ as in Example 4.1 set $x_0 = (k, V) \in \mathrm{Ob}(\mathcal{F}(k))$. Recall that $T\mathcal{D}ef_V = T_{x_0}\mathcal{F}$ is the set of isomorphism classes of pairs (x, α) consisting of an object x of $\mathcal{F}$ over the dual numbers $k[\epsilon]$ and a morphism $\alpha : x \to x_0$ of $\mathcal{F}$ lying over $k[\epsilon] \to k$.

Up to isomorphism, there is a unique pair (M, α) consisting of a finite projective module M over $k[\epsilon]$ and $k[\epsilon]$-linear map $\alpha : M \to V$ which induces an isomorphism $M \otimes_{k[\epsilon]} k \to V$. For example, if $V = k^{\oplus n}$, then we take $M = k[\epsilon]^{\oplus n}$ with the obvious map α.

Similarly, $\mathrm{Inf}(\mathcal{D}ef_V) = \mathrm{Inf}_{x_0}(\mathcal{F})$ is the set of automorphisms of the trivial deformation x_0' of x_0 over $k[\epsilon]$. See Formal Deformation Theory, Definition 19.2 for details.

Given (M, α) as in the second paragraph, we see that an element of $\mathrm{Inf}_{x_0}(\mathcal{F})$ is an automorphism $\gamma : M \to M$ with $\gamma \bmod \epsilon = \mathrm{id}$. Then we can write $\gamma = \mathrm{id}_M + \epsilon\psi$

where $\psi : M/\epsilon M \to M/\epsilon M$ is k-linear. Using α we can think of ψ as an element of $\mathrm{End}_k(V)$ and this finishes the proof. □

5. Representations of a group

0DVR The deformation theory of representations can be very interesting.

0D3J **Example 5.1** (Representations of a group)**.** Let Γ be a group. Let $\mathcal{F}$ be the category defined as follows

(1) an object is a triple (A, M, ρ) consisting of an object A of $\mathcal{C}_\Lambda$, a finite projective A-module M, and a homomorphism $\rho : \Gamma \to \mathrm{GL}_A(M)$, and
(2) a morphism $(f, g) : (B, N, \tau) \to (A, M, \rho)$ consists of a morphism $f : B \to A$ in $\mathcal{C}_\Lambda$ together with a map $g : N \to M$ which is f-linear and Γ-equivariant and induces an isomorpism $N \otimes_{B,f} A \cong M$.

The functor $p : \mathcal{F} \to \mathcal{C}_\Lambda$ sends (A, M, ρ) to A and (f, g) to f. It is clear that p is cofibred in groupoids. Given a finite dimensional k-vector space V and a representation $\rho_0 : \Gamma \to \mathrm{GL}_k(V)$, let $x_0 = (k, V, \rho_0)$ be the corresponding object of $\mathcal{F}(k)$. We set

$$\mathcal{D}ef_{V,\rho_0} = \mathcal{F}_{x_0}$$

Since every finite projective module over a local ring is finite free (Algebra, Lemma 77.2) we see that

$$\begin{array}{c}\text{isomorphism classes}\\ \text{of objects of } \mathcal{F}(A)\end{array} = \coprod_{n\geq 0} \begin{array}{c}\mathrm{GL}_n(A)\text{-conjugacy classes of}\\ \text{homomorphisms } \rho : \Gamma \to \mathrm{GL}_n(A)\end{array}$$

This is already more interesting than the discussion in Section 4.

0DVS **Lemma 5.2.** *Example 5.1 satisfies the Rim-Schlessinger condition (RS). In particular, $\mathcal{D}ef_{V,\rho_0}$ is a deformation category for any finite dimensional representation $\rho_0 : \Gamma \to GL_k(V)$.*

Proof. Let $A_1 \to A$ and $A_2 \to A$ be morphisms of $\mathcal{C}_\Lambda$. Assume $A_2 \to A$ is surjective. According to Formal Deformation Theory, Lemma 16.4 it suffices to show that the functor $\mathcal{F}(A_1 \times_A A_2) \to \mathcal{F}(A_1) \times_{\mathcal{F}(A)} \mathcal{F}(A_2)$ is an equivalence of categories.

Consider an object

$$((A_1, M_1, \rho_1), (A_2, M_2, \rho_2), (\mathrm{id}_A, \varphi))$$

of the category $\mathcal{F}(A_1) \times_{\mathcal{F}(A)} \mathcal{F}(A_2)$. Then, as seen in the proof of Lemma 4.2, we can consider the finite projective $A_1 \times_A A_2$-module $M_1 \times_\varphi M_2$. Since φ is compatible with the given actions we obtain

$$\rho_1 \times \rho_2 : \Gamma \longrightarrow \mathrm{GL}_{A_1 \times_A A_2}(M_1 \times_\varphi M_2)$$

Then $(M_1 \times_\varphi M_2, \rho_1 \times \rho_2)$ is an object of $\mathcal{F}(A_1 \times_A A_2)$. This construction determines a quasi-inverse to our functor. □

0DVT **Lemma 5.3.** *In Example 5.1 let $\rho_0 : \Gamma \to GL_k(V)$ be a finite dimensional representation. Then*

$$T\mathcal{D}ef_{V,\rho_0} = \mathrm{Ext}^1_{k[\Gamma]}(V, V) = H^1(\Gamma, End_k(V)) \quad \text{and} \quad Inf(\mathcal{D}ef_{V,\rho_0}) = H^0(\Gamma, End_k(V))$$

Thus $Inf(\mathcal{D}ef_{V,\rho_0})$ is always finite dimensional and $T\mathcal{D}ef_{V,\rho_0}$ is finite dimensional if Γ is finitely generated.

Proof. We first deal with the infinitesimal automorphisms. Let $M = V \otimes_k k[\epsilon]$ with induced action $\rho_0' : \Gamma \to \mathrm{GL}_n(M)$. Then an infinitesimal automorphism, i.e., an element of $\mathrm{Inf}(\mathcal{D}ef_{V,\rho_0})$, is given by an automorphism $\gamma = \mathrm{id} + \epsilon\psi : M \to M$ as in the proof of Lemma 4.3, where moreover ψ has to commute with the action of Γ (given by ρ_0). Thus we see that

$$\mathrm{Inf}(\mathcal{D}ef_{V,\rho_0}) = H^0(\Gamma, \mathrm{End}_k(V))$$

as predicted in the lemma.

Next, let $(k[\epsilon], M, \rho)$ be an object of $\mathcal{F}$ over $k[\epsilon]$ and let $\alpha : M \to V$ be a Γ-equivariant map inducing an isomorphism $M/\epsilon M \to V$. Since M is free as a $k[\epsilon]$-module we obtain an extension of Γ-modules

$$0 \to V \to M \xrightarrow{\alpha} V \to 0$$

We omit the detailed construction of the map on the left. Conversely, if we have an extension of Γ-modules as above, then we can use this to make a $k[\epsilon]$-module structure on M and get an object of $\mathcal{F}(k[\epsilon])$ together with a map α as above. It follows that

$$T\mathcal{D}ef_{V,\rho_0} = \mathrm{Ext}^1_{k[\Gamma]}(V, V)$$

as predicted in the lemma. This is equal to $H^1(\Gamma, \mathrm{End}_k(V))$ by Étale Cohomology, Lemma 56.4.

The statement on dimensions follows from Étale Cohomology, Lemma 56.5. □

In Example 5.1 if Γ is finitely generated and (V, ρ_0) is a finite dimensional representation of Γ over k, then $\mathcal{D}ef_{V,\rho_0}$ admits a presentation by a smooth prorepresentable groupoid in functors over $\mathcal{C}_\Lambda$ and a fortiori has a (minimal) versal formal object. This follows from Lemmas 5.2 and 5.3 and the general discussion in Section 3.

0ET1 **Lemma 5.4.** *In Example 5.1 assume Γ finitely generated. Let $\rho_0 : \Gamma \to GL_k(V)$ be a finite dimensional representation. Assume Λ is a complete local ring with residue field k (the classical case). Then the functor*

$$F : \mathcal{C}_\Lambda \longrightarrow Sets, \quad A \longmapsto \mathrm{Ob}(\mathcal{D}ef_{V,\rho_0}(A))/\cong$$

of isomorphism classes of objects has a hull. If $H^0(\Gamma, End_k(V)) = k$, then F is prorepresentable.

Proof. The existence of a hull follows from Lemmas 5.2 and 5.3 and Formal Deformation Theory, Lemma 16.6 and Remark 15.7.

Assume $H^0(\Gamma, \mathrm{End}_k(V)) = k$. To see that F is prorepresentable it suffices to show that F is a deformation functor, see Formal Deformation Theory, Theorem 18.2. In other words, we have to show F satisfies (RS). For this we can use the criterion of Formal Deformation Theory, Lemma 16.7. The required surjectivity of automorphism groups will follow if we show that

$$A \cdot \mathrm{id}_M = \mathrm{End}_{A[\Gamma]}(M)$$

for any object (A, M, ρ) of $\mathcal{F}$ such that $M \otimes_A k$ is isomorphic to V as a representation of Γ. Since the left hand side is contained in the right hand side, it suffices to show $\mathrm{length}_A \mathrm{End}_{A[\Gamma]}(M) \leq \mathrm{length}_A A$. Choose pairwise distinct ideals $(0) = I_n \subset \ldots \subset I_1 \subset A$ with $n = \mathrm{length}(A)$. By correspondingly filtering M, we see that it suffices to prove $\mathrm{Hom}_{A[\Gamma]}(M, I_t M/I_{t+1}M)$ has length 1. Since $I_t M/I_{t+1}M \cong M \otimes_A k$ and

since any $A[\Gamma]$-module map $M \to M \otimes_A k$ factors uniquely through the quotient map $M \to M \otimes_A k$ to give an element of

$$\mathrm{End}_{A[\Gamma]}(M \otimes_A k) = \mathrm{End}_{k[\Gamma]}(V) = k$$

we conclude. □

6. Continuous representations

0DVU A very interesting thing one can do is to take an infinite Galois group and study the deformation theory of its representations, see [Maz89].

0D3K **Example 6.1** (Representations of a topological group)**.** Let Γ be a topological group. Let $\mathcal{F}$ be the category defined as follows

(1) an object is a triple (A, M, ρ) consisting of an object A of $\mathcal{C}_\Lambda$, a finite projective A-module M, and a continuous homomorphism $\rho : \Gamma \to \mathrm{GL}_A(M)$ where $\mathrm{GL}_A(M)$ is given the discrete topology[1], and
(2) a morphism $(f, g) : (B, N, \tau) \to (A, M, \rho)$ consists of a morphism $f : B \to A$ in $\mathcal{C}_\Lambda$ together with a map $g : N \to M$ which is f-linear and Γ-equivariant and induces an isomorpism $N \otimes_{B,f} A \cong M$.

The functor $p : \mathcal{F} \to \mathcal{C}_\Lambda$ sends (A, M, ρ) to A and (f, g) to f. It is clear that p is cofibred in groupoids. Given a finite dimensional k-vector space V and a continuous representation $\rho_0 : \Gamma \to \mathrm{GL}_k(V)$, let $x_0 = (k, V, \rho_0)$ be the corresponding object of $\mathcal{F}(k)$. We set

$$\mathcal{D}ef_{V,\rho_0} = \mathcal{F}_{x_0}$$

Since every finite projective module over a local ring is finite free (Algebra, Lemma 77.2) we see that

$$\begin{array}{c}\text{isomorphism classes} \\ \text{of objects of } \mathcal{F}(A)\end{array} = \coprod_{n \geq 0} \begin{array}{c}\mathrm{GL}_n(A)\text{-conjugacy classes of} \\ \text{continuous homomorphisms } \rho : \Gamma \to \mathrm{GL}_n(A)\end{array}$$

0DVV **Lemma 6.2.** *Example 6.1 satisfies the Rim-Schlessinger condition (RS). In particular, $\mathcal{D}ef_{V,\rho_0}$ is a deformation category for any finite dimensional continuous representation $\rho_0 : \Gamma \to GL_k(V)$.*

Proof. The proof is exactly the same as the proof of Lemma 5.2. □

0DVW **Lemma 6.3.** *In Example 6.1 let $\rho_0 : \Gamma \to GL_k(V)$ be a finite dimensional continuous representation. Then*

$$T\mathcal{D}ef_{V,\rho_0} = H^1(\Gamma, End_k(V)) \quad \text{and} \quad Inf(\mathcal{D}ef_{V,\rho_0}) = H^0(\Gamma, End_k(V))$$

Thus $Inf(\mathcal{D}ef_{V,\rho_0})$ is always finite dimensional and $T\mathcal{D}ef_{V,\rho_0}$ is finite dimensional if Γ is topologically finitely generated.

Proof. The proof is exactly the same as the proof of Lemma 5.3. □

In Example 6.1 if Γ is topologically finitely generated and (V, ρ_0) is a finite dimensional continuous representation of Γ over k, then $\mathcal{D}ef_{V,\rho_0}$ admits a presentation by a smooth prorepresentable groupoid in functors over $\mathcal{C}_\Lambda$ and a fortiori has a (minimal) versal formal object. This follows from Lemmas 6.2 and 6.3 and the general discussion in Section 3.

[1] An alternative would be to require the A-module M with G-action given by ρ is an A-G-module as defined in Étale Cohomology, Definition 56.1. However, since M is a finite A-module, this is equivalent.

0ET2 **Lemma 6.4.** *In Example 6.1 assume Γ is topologically finitely generated. Let $\rho_0 : \Gamma \to GL_k(V)$ be a finite dimensional representation. Assume Λ is a complete local ring with residue field k (the classical case). Then the functor*

$$F : \mathcal{C}_\Lambda \longrightarrow Sets, \quad A \longmapsto \mathrm{Ob}(\mathcal{D}ef_{V,\rho_0}(A))/\cong$$

of isomorphism classes of objects has a hull. If $H^0(\Gamma, End_k(V)) = k$, then F is prorepresentable.

Proof. The proof is exactly the same as the proof of Lemma 5.4. □

7. Graded algebras

0DVX We will use the example in this section in the proof that the stack of polarized proper schemes is an algebraic stack. For this reason we will consider commutative graded algebras whose homogeneous parts are finite projective modules (sometimes called "locally finite").

0D3L **Example 7.1** (Graded algebras)**.** Let $\mathcal{F}$ be the category defined as follows

(1) an object is a pair (A, P) consisting of an object A of $\mathcal{C}_\Lambda$ and a graded A-algebra P such that P_d is a finite projective A-module for all $d \geq 0$, and
(2) a morphism $(f, g) : (B, Q) \to (A, P)$ consists of a morphism $f : B \to A$ in $\mathcal{C}_\Lambda$ together with a map $g : Q \to P$ which is f-linear and induces an isomorpism $Q \otimes_{B,f} A \cong P$.

The functor $p : \mathcal{F} \to \mathcal{C}_\Lambda$ sends (A, P) to A and (f, g) to f. It is clear that p is cofibred in groupoids. Given a graded k-algebra P with $\dim_k(P_d) < \infty$ for all $d \geq 0$, let $x_0 = (k, P)$ be the corresponding object of $\mathcal{F}(k)$. We set

$$\mathcal{D}ef_P = \mathcal{F}_{x_0}$$

0DVY **Lemma 7.2.** *Example 7.1 satisfies the Rim-Schlessinger condition (RS). In particular, $\mathcal{D}ef_P$ is a deformation category for any graded k-algebra P.*

Proof. Let $A_1 \to A$ and $A_2 \to A$ be morphisms of $\mathcal{C}_\Lambda$. Assume $A_2 \to A$ is surjective. According to Formal Deformation Theory, Lemma 16.4 it suffices to show that the functor $\mathcal{F}(A_1 \times_A A_2) \to \mathcal{F}(A_1) \times_{\mathcal{F}(A)} \mathcal{F}(A_2)$ is an equivalence of categories.

Consider an object

$$((A_1, P_1), (A_2, P_2), (\mathrm{id}_A, \varphi))$$

of the category $\mathcal{F}(A_1) \times_{\mathcal{F}(A)} \mathcal{F}(A_2)$. Then we consider $P_1 \times_\varphi P_2$. Since $\varphi : P_1 \otimes_{A_1} A \to P_2 \otimes_{A_2} A$ is an isomorphism of graded algebras, we see that the graded pieces of $P_1 \times_\varphi P_2$ are finite projective $A_1 \times_A A_2$-modules, see proof of Lemma 4.2. Thus $P_1 \times_\varphi P_2$ is an object of $\mathcal{F}(A_1 \times_A A_2)$. This construction determines a quasi-inverse to our functor and the proof is complete. □

0DVZ **Lemma 7.3.** *In Example 7.1 let P be a graded k-algebra. Then*

$$T\mathcal{D}ef_P \quad \text{and} \quad Inf(\mathcal{D}ef_P) = Der_k(P, P)$$

are finite dimensional if P is finitely generated over k.

Proof. We first deal with the infinitesimal automorphisms. Let $Q = P \otimes_k k[\epsilon]$. Then an element of $\mathrm{Inf}(\mathcal{D}ef_P)$ is given by an automorphism $\gamma = \mathrm{id} + \epsilon\delta : Q \to Q$ as above where now $\delta : P \to P$. The fact that γ is graded implies that δ is homogeneous of degree 0. The fact that γ is k-linear implies that δ is k-linear.

The fact that γ is multiplicative implies that δ is a k-derivation. Conversely, given a k-derivation $\delta : P \to P$ homogeneous of degree 0, we obtain an automorphism $\gamma = \mathrm{id} + \epsilon\delta$ as above. Thus we see that

$$\mathrm{Inf}(\mathcal{D}ef_P) = \mathrm{Der}_k(P, P)$$

as predicted in the lemma. Clearly, if P is generated in degrees P_i, $0 \leq i \leq N$, then δ is determined by the linear maps $\delta_i : P_i \to P_i$ for $0 \leq i \leq N$ and we see that

$$\dim_k \mathrm{Der}_k(P, P) < \infty$$

as desired.

To finish the proof of the lemma we show that there is a finite dimensional deformation space. To do this we choose a presentation

$$k[X_1, \ldots, X_n]/(F_1, \ldots, F_m) \longrightarrow P$$

of graded k-algebras where $\deg(X_i) = d_i$ and F_j is homogeneous of degree e_j. Let Q be any graded $k[\epsilon]$-algebra finite free in each degree which comes with an isomorphsm $\alpha : Q/\epsilon Q \to P$ so that (Q, α) defines an element of $T\mathcal{D}ef_P$. Choose a homogeneous element $q_i \in Q$ of degree d_i mapping to the image of X_i in P. Then we obtain

$$k[\epsilon][X_1, \ldots, X_n] \longrightarrow Q, \quad X_i \longmapsto q_i$$

and since $P = Q/\epsilon Q$ this map is surjective by Nakayama's lemma. A small diagram chase shows we can choose homogeneous elements $F_{\epsilon,j} \in k[\epsilon][X_1, \ldots, X_n]$ of degree e_j mapping to zero in Q and mapping to F_j in $k[X_1, \ldots, X_n]$. Then

$$k[\epsilon][X_1, \ldots, X_n]/(F_{\epsilon,1}, \ldots, F_{\epsilon,m}) \longrightarrow Q$$

is a presentation of Q by flatness of Q over $k[\epsilon]$. Write

$$F_{\epsilon,j} = F_j + \epsilon G_j$$

There is some ambiguity in the vector $(G_1, \ldots, G_m)$. First, using different choices of $F_{\epsilon,j}$ we can modify G_j by an arbitrary element of degree e_j in the kernel of $k[X_1, \ldots, X_n] \to P$. Hence, instead of $(G_1, \ldots, G_m)$, we remember the element

$$(g_1, \ldots, g_m) \in P_{e_1} \oplus \ldots \oplus P_{e_m}$$

where g_j is the image of G_j in P_{e_j}. Moreover, if we change our choice of q_i into $q_i + \epsilon p_i$ with p_i of degree d_i then a computation (omitted) shows that g_j changes into

$$g_j^{new} = g_j - \sum\nolimits_{i=1}^{n} p_i \partial F_j/\partial X_i$$

We conclude that the isomorphism class of Q is determined by the image of the vector $(G_1, \ldots, G_m)$ in the k-vector space

$$W = \mathrm{Coker}(P_{d_1} \oplus \ldots \oplus P_{d_n} \xrightarrow{(\frac{\partial F_j}{\partial X_i})} P_{e_1} \oplus \ldots \oplus P_{e_m})$$

In this way we see that we obtain an injection

$$T\mathcal{D}ef_P \longrightarrow W$$

Since W visibly has finite dimension, we conclude that the lemma is true. □

In Example 7.1 if P is a finitely generated graded k-algebra, then $\mathcal{D}ef_P$ admits a presentation by a smooth prorepresentable groupoid in functors over $\mathcal{C}_\Lambda$ and a fortiori has a (minimal) versal formal object. This follows from Lemmas 7.2 and 7.3 and the general discussion in Section 3.

0ET3 **Lemma 7.4.** *In Example 7.1 assume P is a finitely generated graded k-algebra. Assume Λ is a complete local ring with residue field k (the classical case). Then the functor*

$$F : \mathcal{C}_\Lambda \longrightarrow Sets, \quad A \longmapsto \mathrm{Ob}(\mathcal{D}ef_P(A))/\cong$$

of isomorphism classes of objects has a hull.

Proof. This follows immediately from Lemmas 7.2 and 7.3 and Formal Deformation Theory, Lemma 16.6 and Remark 15.7. □

8. Rings

0DY0 The deformation theory of rings is the same as the deformation theory of affine schemes. For rings and schemes when we talk about deformations it means we are thinking about *flat* deformations.

0DY1 **Example 8.1** (Rings)**.** Let $\mathcal{F}$ be the category defined as follows

(1) an object is a pair (A, P) consisting of an object A of $\mathcal{C}_\Lambda$ and a flat A-algebra P, and
(2) a morphism $(f, g) : (B, Q) \to (A, P)$ consists of a morphism $f : B \to A$ in $\mathcal{C}_\Lambda$ together with a map $g : Q \to P$ which is f-linear and induces an isomorpism $Q \otimes_{B,f} A \cong P$.

The functor $p : \mathcal{F} \to \mathcal{C}_\Lambda$ sends (A, P) to A and (f, g) to f. It is clear that p is cofibred in groupoids. Given a k-algebra P, let $x_0 = (k, P)$ be the corresponding object of $\mathcal{F}(k)$. We set

$$\mathcal{D}ef_P = \mathcal{F}_{x_0}$$

0DY2 **Lemma 8.2.** *Example 8.1 satisfies the Rim-Schlessinger condition (RS). In particular, $\mathcal{D}ef_P$ is a deformation category for any k-algebra P.*

Proof. Let $A_1 \to A$ and $A_2 \to A$ be morphisms of $\mathcal{C}_\Lambda$. Assume $A_2 \to A$ is surjective. According to Formal Deformation Theory, Lemma 16.4 it suffices to show that the functor $\mathcal{F}(A_1 \times_A A_2) \to \mathcal{F}(A_1) \times_{\mathcal{F}(A)} \mathcal{F}(A_2)$ is an equivalence of categories. This is a special case of More on Algebra, Lemma 7.7. □

0DY3 **Lemma 8.3.** *In Example 8.1 let P be a k-algebra. Then*

$$T\mathcal{D}ef_P = Ext^1_P(NL_{P/k}, P) \quad \text{and} \quad Inf(\mathcal{D}ef_P) = Der_k(P, P)$$

Proof. Recall that $\mathrm{Inf}(\mathcal{D}ef_P)$ is the set of automorphisms of the trivial deformation $P[\epsilon] = P \otimes_k k[\epsilon]$ of P to $k[\epsilon]$ equal to the identity modulo ϵ. By Deformation Theory, Lemma 2.1 this is equal to $\mathrm{Hom}_P(\Omega_{P/k}, P)$ which in turn is equal to $\mathrm{Der}_k(P, P)$ by Algebra, Lemma 130.3.

Recall that $T\mathcal{D}ef_P$ is the set of isomorphism classes of flat deformations Q of P to $k[\epsilon]$, more precisely, the set of isomorphism classes of $\mathcal{D}ef_P(k[\epsilon])$. Recall that a $k[\epsilon]$-algebra Q with $Q/\epsilon Q = P$ is flat over $k[\epsilon]$ if and only if

$$0 \to P \xrightarrow{\epsilon} Q \to P \to 0$$

is exact. This is proven in More on Morphisms, Lemma 10.1 and more generally in Deformation Theory, Lemma 5.2. Thus we may apply Deformation Theory, Lemma 2.3 to see that the set of isomorphism classes of such deformations is equal to $\mathrm{Ext}^1_P(NL_{P/k}, P)$. □

0DZL **Lemma 8.4.** *In Example 8.1 let P be a smooth k-algebra. Then $T\mathcal{D}ef_P = (0)$.*

Proof. By Lemma 8.3 we have to show $\mathrm{Ext}^1_P(NL_{P/k}, P) = (0)$. Since $k \to P$ is smooth $NL_{P/k}$ is quasi-isomorphic to the complex consisting of a finite projective P-module placed in degree 0. □

0DY4 **Lemma 8.5.** *In Lemma 8.3 if P is a finite type k-algebra, then*

(1) *$Inf(\mathcal{D}ef_P)$ is finite dimensional if and only if $\dim(P) = 0$, and*
(2) *$T\mathcal{D}ef_P$ is finite dimensional if $\mathrm{Spec}(P) \to \mathrm{Spec}(k)$ is smooth except at a finite number of points.*

Proof. Proof of (1). We view $\mathrm{Der}_k(P, P)$ as a P-module. If it has finite dimension over k, then it has finite length as a P-module, hence it is supported in finitely many closed points of $\mathrm{Spec}(P)$ (Algebra, Lemma 51.11). Since $\mathrm{Der}_k(P, P) = \mathrm{Hom}_P(\Omega_{P/k}, P)$ we see that $\mathrm{Der}_k(P, P)_\mathfrak{p} = \mathrm{Der}_k(P_\mathfrak{p}, P_\mathfrak{p})$ for any prime $\mathfrak{p} \subset P$ (this uses Algebra, Lemmas 130.8, 130.15, and 10.2). Let $\mathfrak{p}$ be a minimal prime ideal of P corresponding to an irreducible component of dimension $d > 0$. Then $P_\mathfrak{p}$ is an Artinian local ring essentially of finite type over k with residue field and $\Omega_{P_\mathfrak{p}/k}$ is nonzero for example by Algebra, Lemma 138.3. Any nonzero finite module over an Artinian local ring has both a sub and a quotient module isomorphic to the residue field. Thus we find that $\mathrm{Der}_k(P_\mathfrak{p}, P_\mathfrak{p}) = \mathrm{Hom}_{P_\mathfrak{p}}(\Omega_{P_\mathfrak{p}/k}, P_\mathfrak{p})$ is nonzero too. Combining all of the above we find that (1) is true.

Proof of (2). For a prime $\mathfrak{p}$ of P we will use that $NL_{P_\mathfrak{p}/k} = (NL_{P/k})_\mathfrak{p}$ (Algebra, Lemma 132.13) and we will use that $\mathrm{Ext}^1_P(NL_{P/k}, P)_\mathfrak{p} = \mathrm{Ext}^1_{P_\mathfrak{p}}(NL_{P_\mathfrak{p}/k}, P_\mathfrak{p})$ (More on Algebra, Remark 62.21). Given a prime $\mathfrak{p} \subset P$ then $k \to P$ is smooth at $\mathfrak{p}$ if and only if $(NL_{P/k})_\mathfrak{p}$ is quasi-isomorphic to a finite projective module placed in degree 0 (this follows immediately from the definition of a smooth ring map but it also follows from the stronger Algebra, Lemma 135.12).

Assume that P is smooth over k at all but finitely many primes. Then these "bad" primes are maximal ideals $\mathfrak{m}_1, \ldots, \mathfrak{m}_n \subset P$ by Algebra, Lemma 60.3 and the fact that the "bad" primes form a closed subset of $\mathrm{Spec}(P)$. For $\mathfrak{p} \notin \{\mathfrak{m}_1, \ldots, \mathfrak{m}_n\}$ we have $\mathrm{Ext}^1_P(NL_{P/k}, P)_\mathfrak{p} = 0$ by the results above. Thus $\mathrm{Ext}^1_P(NL_{P/k}, P)$ is a finite P-module whose support is contained in $\{\mathfrak{m}_1, \ldots, \mathfrak{m}_r\}$. By Algebra, Proposition 62.6 for example, we find that the dimension over k of $\mathrm{Ext}^1_P(NL_{P/k}, P)$ is a finite integer combination of $\dim_k \kappa(\mathfrak{m}_i)$ and hence finite by the Hilbert Nullstellensatz (Algebra, Theorem 33.1). □

In Example 8.1, let P be a finite type k-algebra. Then $\mathcal{D}ef_P$ admits a presentation by a smooth prorepresentable groupoid in functors over $\mathcal{C}_\Lambda$ if and only if $\dim(P) = 0$. Furthermore, $\mathcal{D}ef_P$ has a versal formal object if $\mathrm{Spec}(P) \to \mathrm{Spec}(k)$ has finitely many singular points. This follows from Lemmas 8.2 and 8.5 and the general discussion in Section 3.

0ET4 **Lemma 8.6.** *In Example 8.1 assume P is a finite type k-algebra such that $\mathrm{Spec}(P) \to \mathrm{Spec}(k)$ is smooth except at a finite number of points. Assume Λ is a complete local ring with residue field k (the classical case). Then the functor*

$$F : \mathcal{C}_\Lambda \longrightarrow Sets, \quad A \longmapsto \mathrm{Ob}(\mathcal{D}ef_P(A))/\cong$$

of isomorphism classes of objects has a hull.

Proof. This follows immediately from Lemmas 8.2 and 8.5 and Formal Deformation Theory, Lemma 16.6 and Remark 15.7. □

0DYS **Lemma 8.7.** *In Example 8.1 let P be a k-algebra. Let $S \subset P$ be a multiplicative subset. There is a natural functor*

$$\mathcal{D}ef_P \longrightarrow \mathcal{D}ef_{S^{-1}P}$$

of deformation categories.

Proof. Given a deformation of P we can take the localization of it to get a deformation of the localization; this is clear and we encourage the reader to skip the proof. More precisely, let $(A, Q) \to (k, P)$ be a morphism in $\mathcal{F}$, i.e., an object of $\mathcal{D}ef_P$. Let $S_Q \subset Q$ be the inverse image of S. Then Hence $(A, S_Q^{-1}Q) \to (k, S^{-1}P)$ is the desired object of $\mathcal{D}ef_{S^{-1}P}$. □

0DYT **Lemma 8.8.** *In Example 8.1 let P be a k-algebra. Let $J \subset P$ be an ideal. Denote (P^h, J^h) the henselization of the pair (P, J). There is a natural functor*

$$\mathcal{D}ef_P \longrightarrow \mathcal{D}ef_{P^h}$$

of deformation categories.

Proof. Given a deformation of P we can take the henselization of it to get a deformation of the henselization; this is clear and we encourage the reader to skip the proof. More precisely, let $(A, Q) \to (k, P)$ be a morphism in $\mathcal{F}$, i.e., an object of $\mathcal{D}ef_P$. Denote $J_Q \subset Q$ the inverse image of J in Q. Let (Q^h, J_Q^h) be the henselization of the pair (Q, J_Q). Recall that $Q \to Q^h$ is flat (More on Algebra, Lemma 12.2) and hence Q^h is flat over A. By More on Algebra, Lemma 12.7 we see that the map $Q^h \to P^h$ induces an isomorphism $Q^h \otimes_A k = Q^h \otimes_Q P = P^h$. Hence $(A, Q^h) \to (k, P^h)$ is the desired object of $\mathcal{D}ef_{P^h}$. □

0DYU **Lemma 8.9.** *In Example 8.1 let P be a k-algebra. Assume P is a local ring and let P^{sh} be a strict henselization of P. There is a natural functor*

$$\mathcal{D}ef_P \longrightarrow \mathcal{D}ef_{P^{sh}}$$

of deformation categories.

Proof. Given a deformation of P we can take the strict henselization of it to get a deformation of the strict henselization; this is clear and we encourage the reader to skip the proof. More precisely, let $(A, Q) \to (k, P)$ be a morphism in $\mathcal{F}$, i.e., an object of $\mathcal{D}ef_P$. Since the kernel of the surjection $Q \to P$ is nilpotent, we find that Q is a local ring with the same residue field as P. Let Q^{sh} be the strict henselization of Q. Recall that $Q \to Q^{sh}$ is flat (More on Algebra, Lemma 44.1) and hence Q^{sh} is flat over A. By Algebra, Lemma 150.16 we see that the map $Q^{sh} \to P^{sh}$ induces an isomorphism $Q^{sh} \otimes_A k = Q^{sh} \otimes_Q P = P^{sh}$. Hence $(A, Q^{sh}) \to (k, P^{sh})$ is the desired object of $\mathcal{D}ef_{P^{sh}}$. □

0DYV **Lemma 8.10.** *In Example 8.1 let P be a k-algebra. Assume P Noetherian and let $J \subset P$ be an ideal. Denote $P^\wedge$ the J-adic completion. There is a natural functor*

$$\mathcal{D}ef_P \longrightarrow \mathcal{D}ef_{P^\wedge}$$

of deformation categories.

Proof. Given a deformation of P we can take the completion of it to get a deformation of the completion; this is clear and we encourage the reader to skip the proof. More precisely, let $(A, Q) \to (k, P)$ be a morphism in $\mathcal{F}$, i.e., an object of $\mathcal{D}ef_P$. Observe that Q is a Noetherian ring: the kernel of the surjective ring map $Q \to P$ is

nilpotent and finitely generated and P is Noetherian; apply Algebra, Lemma 96.5. Denote $J_Q \subset Q$ the inverse image of J in Q. Let $Q^\wedge$ be the J_Q-adic completion of Q. Recall that $Q \to Q^\wedge$ is flat (Algebra, Lemma 96.2) and hence $Q^\wedge$ is flat over A. The induced map $Q^\wedge \to P^\wedge$ induces an isomorphism $Q^\wedge \otimes_A k = Q^\wedge \otimes_Q P = P^\wedge$ by Algebra, Lemma 96.1 for example. Hence $(A, Q^\wedge) \to (k, P^\wedge)$ is the desired object of $\mathcal{D}ef_{P^\wedge}$. □

0DY5 **Lemma 8.11.** *In Lemma 8.3 if $P = k[[x_1, \ldots, x_n]]/(f)$ for some nonzero $f \in (x_1, \ldots, x_n)^2$, then*

(1) *$Inf(\mathcal{D}ef_P)$ is finite dimensional if and only if $n = 1$, and*
(2) *$T\mathcal{D}ef_P$ is finite dimensional if*

$$\sqrt{(f, \partial f/\partial x_1, \ldots, \partial f/\partial x_n)} = (x_1, \ldots, x_n)$$

Proof. Proof of (1). Consider the derivations $\partial/\partial x_i$ of $k[[x_1, \ldots, x_n]]$ over k. Write $f_i = \partial f/\partial x_i$. The derivation

$$\theta = \sum h_i \partial/\partial x_i$$

of $k[[x_1, \ldots, x_n]]$ induces a derivation of $P = k[[x_1, \ldots, x_n]]/(f)$ if and only if $\sum h_i f_i \in (f)$. Moreover, the induced derivation of P is zero if and only if $h_i \in (f)$ for $i = 1, \ldots, n$. Thus we find

$$\mathrm{Ker}((f_1, \ldots, f_n) : P^{\oplus n} \longrightarrow P) \subset \mathrm{Der}_k(P, P)$$

The left hand side is a finite dimensional k-vector space only if $n = 1$; we omit the proof. We also leave it to the reader to see that the right hand side has finite dimension if $n = 1$. This proves (1).

Proof of (2). Let Q be a flat deformation of P over $k[\epsilon]$ as in the proof of Lemma 8.3. Choose lifts $q_i \in Q$ of the image of x_i in P. Then Q is a complete local ring with maximal ideal generated by $q_1, \ldots, q_n$ and ϵ (small argument omitted). Thus we get a surjection

$$k[\epsilon][[x_1, \ldots, x_n]] \longrightarrow Q, \quad x_i \longmapsto q_i$$

Choose an element of the form $f + \epsilon g \in k[\epsilon][[x_1, \ldots, x_n]]$ mapping to zero in Q. Observe that g is well defined modulo (f). Since Q is flat over $k[\epsilon]$ we get

$$Q = k[\epsilon][[x_1, \ldots, x_n]]/(f + \epsilon g)$$

Finally, if we changing the choice of q_i amounts to changing the coordinates x_i into $x_i + \epsilon h_i$ for some $h_i \in k[[x_1, \ldots, x_n]]$. Then $f + \epsilon g$ changes into $f + \epsilon(g + \sum h_i f_i)$ where $f_i = \partial f/\partial x_i$. Thus we see that the isomorphism class of the deformation Q is determined by an element of

$$k[[x_1, \ldots, x_n]]/(f, \partial f/\partial x_1, \ldots, \partial f/\partial x_n)$$

This has finite dimension over k if and only if its support is the closed point of $k[[x_1, \ldots, x_n]]$ if and only if $\sqrt{(f, \partial f/\partial x_1, \ldots, \partial f/\partial x_n)} = (x_1, \ldots, x_n)$. □

9. Schemes

0DY6 The deformation theory of schemes.

0DY7 **Example 9.1** (Schemes)**.** Let $\mathcal{F}$ be the category defined as follows

(1) an object is a pair (A, X) consisting of an object A of $\mathcal{C}_\Lambda$ and a scheme X flat over A, and

(2) a morphism $(f, g) : (B, Y) \to (A, X)$ consists of a morphism $f : B \to A$ in $\mathcal{C}_\Lambda$ together with a morphism $g : X \to Y$ such that

$$\begin{array}{ccc} X & \xrightarrow{g} & Y \\ \downarrow & & \downarrow \\ \mathrm{Spec}(A) & \xrightarrow{f} & \mathrm{Spec}(B) \end{array}$$

is a cartesian commutative diagram of schemes.

The functor $p : \mathcal{F} \to \mathcal{C}_\Lambda$ sends (A, X) to A and (f, g) to f. It is clear that p is cofibred in groupoids. Given a scheme X over k, let $x_0 = (k, X)$ be the corresponding object of $\mathcal{F}(k)$. We set

$$\mathcal{D}ef_X = \mathcal{F}_{x_0}$$

0DY8 **Lemma 9.2.** *Example 9.1 satisfies the Rim-Schlessinger condition (RS). In particular, $\mathcal{D}ef_X$ is a deformation category for any scheme X over k.*

Proof. Let $A_1 \to A$ and $A_2 \to A$ be morphisms of $\mathcal{C}_\Lambda$. Assume $A_2 \to A$ is surjective. According to Formal Deformation Theory, Lemma 16.4 it suffices to show that the functor $\mathcal{F}(A_1 \times_A A_2) \to \mathcal{F}(A_1) \times_{\mathcal{F}(A)} \mathcal{F}(A_2)$ is an equivalence of categories. Observe that

$$\begin{array}{ccc} \mathrm{Spec}(A) & \longrightarrow & \mathrm{Spec}(A_2) \\ \downarrow & & \downarrow \\ \mathrm{Spec}(A_1) & \longrightarrow & \mathrm{Spec}(A_1 \times_A A_2) \end{array}$$

is a pushout diagram as in More on Morphisms, Lemma 14.3. Thus the lemma is a special case of More on Morphisms, Lemma 14.6. □

0DY9 **Lemma 9.3.** *In Example 9.1 let X be a scheme over k. Then*

$$\mathit{Inf}(\mathcal{D}ef_X) = \mathit{Ext}^0_{\mathcal{O}_X}(NL_{X/k}, \mathcal{O}_X) = \mathrm{Hom}_{\mathcal{O}_X}(\Omega_{X/k}, \mathcal{O}_X) = \mathit{Der}_k(\mathcal{O}_X, \mathcal{O}_X)$$

and

$$T\mathcal{D}ef_X = \mathit{Ext}^1_{\mathcal{O}_X}(NL_{X/k}, \mathcal{O}_X)$$

Proof. Recall that $\mathrm{Inf}(\mathcal{D}ef_X)$ is the set of automorphisms of the trivial deformation $X' = X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k[\epsilon])$ of X to $k[\epsilon]$ equal to the identity modulo ϵ. By Deformation Theory, Lemma 8.1 this is equal to $\mathrm{Ext}^0_{\mathcal{O}_X}(NL_{X/k}, \mathcal{O}_X)$. The equality $\mathrm{Ext}^0_{\mathcal{O}_X}(NL_{X/k}, \mathcal{O}_X) = \mathrm{Hom}_{\mathcal{O}_X}(\Omega_{X/k}, \mathcal{O}_X)$ follows from More on Morphisms, Lemma 13.3. The equality $\mathrm{Hom}_{\mathcal{O}_X}(\Omega_{X/k}, \mathcal{O}_X) = \mathrm{Der}_k(\mathcal{O}_X, \mathcal{O}_X)$ follows from Morphisms, Lemma 31.2.

Recall that $T_{x_0}\mathcal{D}ef_X$ is the set of isomorphism classes of flat deformations X' of X to $k[\epsilon]$, more precisely, the set of isomorphism classes of $\mathcal{D}ef_X(k[\epsilon])$. Thus the second statement of the lemma follows from Deformation Theory, Lemma 8.1. □

0DYA **Lemma 9.4.** *In Lemma 9.3 if X is proper over k, then $\mathit{Inf}(\mathcal{D}ef_X)$ and $T\mathcal{D}ef_X$ are finite dimensional.*

Proof. By the lemma we have to show $\mathrm{Ext}^1_{\mathcal{O}_X}(NL_{X/k}, \mathcal{O}_X)$ and $\mathrm{Ext}^0_{\mathcal{O}_X}(NL_{X/k}, \mathcal{O}_X)$ are finite dimensional. By More on Morphisms, Lemma 13.4 and the fact that X is Noetherian, we see that $NL_{X/k}$ has coherent cohomology sheaves zero except in

degrees 0 and -1. By Derived Categories of Schemes, Lemma 10.4 the displayed Ext-groups are finite k-vector spaces and the proof is complete. □

In Example 9.1 if X is a proper scheme over k, then $\mathcal{D}ef_X$ admits a presentation by a smooth prorepresentable groupoid in functors over $\mathcal{C}_\Lambda$ and a fortiori has a (minimal) versal formal object. This follows from Lemmas 9.2 and 9.4 and the general discussion in Section 3.

0ET5 **Lemma 9.5.** *In Example 9.1 assume X is a proper k-scheme. Assume Λ is a complete local ring with residue field k (the classical case). Then the functor*

$$F : \mathcal{C}_\Lambda \longrightarrow Sets, \quad A \longmapsto \mathrm{Ob}(\mathcal{D}ef_X(A))/\cong$$

of isomorphism classes of objects has a hull. If $Der_k(\mathcal{O}_X, \mathcal{O}_X) = 0$, then F is prorepresentable.

Proof. The existence of a hull follows immediately from Lemmas 9.2 and 9.4 and Formal Deformation Theory, Lemma 16.6 and Remark 15.7.

Assume $\mathrm{Der}_k(\mathcal{O}_X, \mathcal{O}_X) = 0$. Then $\mathcal{D}ef_X$ and F are equivalent by Formal Deformation Theory, Lemma 19.13. Hence F is a deformation functor (because $\mathcal{D}ef_X$ is a deformation category) with finite tangent space and we can apply Formal Deformation Theory, Theorem 18.2. □

0DYW **Lemma 9.6.** *In Example 9.1 let X be a scheme over k. Let $U \subset X$ be an open subscheme. There is a natural functor*

$$\mathcal{D}ef_X \longrightarrow \mathcal{D}ef_U$$

of deformation categories.

Proof. Given a deformation of X we can take the corresponding open of it to get a deformation of U. We omit the details. □

0DYX **Lemma 9.7.** *In Example 9.1 let $X = \mathrm{Spec}(P)$ be an affine scheme over k. With $\mathcal{D}ef_P$ as in Example 8.1 there is a natural equivalence*

$$\mathcal{D}ef_X \longrightarrow \mathcal{D}ef_P$$

of deformation categories.

Proof. The functor sends (A, Y) to $\Gamma(Y, \mathcal{O}_Y)$. This works because any deformation of X is affine by More on Morphisms, Lemma 2.3. □

0DZM **Lemma 9.8.** *In Example 9.1 let X be a scheme over k Let $p \in X$ be a point. With $\mathcal{D}ef_{\mathcal{O}_{X,p}}$ as in Example 8.1 there is a natural functor*

$$\mathcal{D}ef_X \longrightarrow \mathcal{D}ef_{\mathcal{O}_{X,p}}$$

of deformation categories.

Proof. Choose an affine open $U = \mathrm{Spec}(P) \subset X$ containing p. Then $\mathcal{O}_{X,p}$ is a localization of P. We combine the functors from Lemmas 9.6, 9.7, and 8.7. □

0DYY **Situation 9.9.** Let $\Lambda \to k$ be as in Section 3. Let X be a scheme over k which has an affine open covering $X = U_1 \cup U_2$ with $U_{12} = U_1 \cap U_2$ affine too. Write $U_1 = \mathrm{Spec}(P_1)$, $U_2 = \mathrm{Spec}(P_2)$ and $U_{12} = \mathrm{Spec}(P_{12})$. Let $\mathcal{D}ef_X$, $\mathcal{D}ef_{U_1}$, $\mathcal{D}ef_{U_2}$, and $\mathcal{D}ef_{U_{12}}$ be as in Example 9.1 and let $\mathcal{D}ef_{P_1}$, $\mathcal{D}ef_{P_2}$, and $\mathcal{D}ef_{P_{12}}$ be as in Example 8.1.

0DYZ **Lemma 9.10.** *In Situation 9.9 there is an equivalence*

$$\mathcal{D}ef_X = \mathcal{D}ef_{P_1} \times_{\mathcal{D}ef_{P_{12}}} \mathcal{D}ef_{P_2}$$

of deformation categories, see Examples 9.1 and 8.1.

Proof. It suffices to show that the functors of Lemma 9.6 define an equivalence

$$\mathcal{D}ef_X \longrightarrow \mathcal{D}ef_{U_1} \times_{\mathcal{D}ef_{U_{12}}} \mathcal{D}ef_{U_2}$$

because then we can apply Lemma 9.7 to translate into rings. To do this we construct a quasi-inverse. Denote $F_i : \mathcal{D}ef_{U_i} \to \mathcal{D}ef_{U_{12}}$ the functor of Lemma 9.6. An object of the RHS is given by an A in $\mathcal{C}_\Lambda$, objects $(A, V_1) \to (k, U_1)$ and $(A, V_2) \to (k, U_2)$, and a morphism

$$g : F_1(A, V_1) \to F_2(A, V_2)$$

Now $F_i(A, V_i) = (A, V_{i,3-i})$ where $V_{i,3-i} \subset V_i$ is the open subscheme whose base change to k is $U_{12} \subset U_i$. The morphism g defines an isomorphism $V_{1,2} \to V_{2,1}$ of schemes over A compatible with $\mathrm{id} : U_{12} \to U_{12}$ over k. Thus $(\{1, 2\}, V_i, V_{i,3-i}, g, g^{-1})$ is a glueing data as in Schemes, Section 14. Let Y be the glueing, see Schemes, Lemma 14.1. Then Y is a scheme over A and the compatibilities mentioned above show that there is a canonical isomorphism $Y \times_{\mathrm{Spec}(A)} \mathrm{Spec}(k) = X$. Thus $(A, Y) \to (k, X)$ is an object of $\mathcal{D}ef_X$. We omit the verification that this construction is a functor and is quasi-inverse to the given one. □

10. Morphisms of Schemes

0E3S The deformation theory of morphisms of schemes. Of course this is just an example of deformations of diagrams of schemes.

0E3T **Example 10.1** (Morphisms of schemes)**.** Let $\mathcal{F}$ be the category defined as follows

(1) an object is a pair $(A, X \to Y)$ consisting of an object A of $\mathcal{C}_\Lambda$ and a morphism $X \to Y$ of schemes over A with both X and Y flat over A, and
(2) a morphism $(f, g, h) : (A', X' \to Y') \to (A, X \to Y)$ consists of a morphism $f : A' \to A$ in $\mathcal{C}_\Lambda$ together with morphisms of schemes $g : X \to X'$ and $h : Y \to Y'$ such that

$$\begin{array}{ccc} X & \xrightarrow[g]{} & X' \\ \downarrow & & \downarrow \\ Y & \xrightarrow[h]{} & Y' \\ \downarrow & & \downarrow \\ \mathrm{Spec}(A) & \xrightarrow{f} & \mathrm{Spec}(A') \end{array}$$

is a commutative diagram of schemes where both squares are cartesian.

The functor $p : \mathcal{F} \to \mathcal{C}_\Lambda$ sends $(A, X \to Y)$ to A and (f, g, h) to f. It is clear that p is cofibred in groupoids. Given a morphism of schemes $X \to Y$ over k, let $x_0 = (k, X \to Y)$ be the corresponding object of $\mathcal{F}(k)$. We set

$$\mathcal{D}ef_{X \to Y} = \mathcal{F}_{x_0}$$

0E3U **Lemma 10.2.** *Example 10.1 satisfies the Rim-Schlessinger condition (RS). In particular, $\mathcal{D}ef_{X \to Y}$ is a deformation category for any morphism of schemes $X \to Y$ over k.*

Proof. Let $A_1 \to A$ and $A_2 \to A$ be morphisms of $\mathcal{C}_\Lambda$. Assume $A_2 \to A$ is surjective. According to Formal Deformation Theory, Lemma 16.4 it suffices to show that the functor $\mathcal{F}(A_1 \times_A A_2) \to \mathcal{F}(A_1) \times_{\mathcal{F}(A)} \mathcal{F}(A_2)$ is an equivalence of categories. Observe that

$$\begin{array}{ccc} \mathrm{Spec}(A) & \longrightarrow & \mathrm{Spec}(A_2) \\ \downarrow & & \downarrow \\ \mathrm{Spec}(A_1) & \longrightarrow & \mathrm{Spec}(A_1 \times_A A_2) \end{array}$$

is a pushout diagram as in More on Morphisms, Lemma 14.3. Thus the lemma follows immediately from More on Morphisms, Lemma 14.6 as this describes the category of schemes flat over $A_1 \times_A A_2$ as the fibre product of the category of schemes flat over A_1 with the category of schemes flat over A_2 over the category of schemes flat over A. □

0E3V **Lemma 10.3.** *In Example 9.1 let $f : X \to Y$ be a morphism of schemes over k. There is a canonical exact sequence of k-vector spaces*

$$0 \longrightarrow \mathit{Inf}(\mathcal{D}\mathit{ef}_{X\to Y}) \longrightarrow \mathit{Inf}(\mathcal{D}\mathit{ef}_X \times \mathcal{D}\mathit{ef}_Y) \longrightarrow \mathit{Der}_k(\mathcal{O}_Y, f_*\mathcal{O}_X)$$
$$T\mathcal{D}\mathit{ef}_{X\to Y} \longrightarrow T(\mathcal{D}\mathit{ef}_X \times \mathcal{D}\mathit{ef}_Y) \longrightarrow \mathit{Ext}^1_{\mathcal{O}_X}(Lf^*\,NL_{Y/k}, \mathcal{O}_X)$$

Proof. The obvious map of deformation categories $\mathcal{D}\mathit{ef}_{X\to Y} \to \mathcal{D}\mathit{ef}_X \times \mathcal{D}\mathit{ef}_Y$ gives two of the arrows in the exact sequence of the lemma. Recall that $\mathrm{Inf}(\mathcal{D}\mathit{ef}_{X\to Y})$ is the set of automorphisms of the trivial deformation

$$f' : X' = X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k[\epsilon]) \xrightarrow{f\times \mathrm{id}} Y' = Y \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k[\epsilon])$$

of $X \to Y$ to $k[\epsilon]$ equal to the identity modulo ϵ. This is clearly the same thing as pairs $(\alpha, \beta) \in \mathrm{Inf}(\mathcal{D}\mathit{ef}_X \times \mathcal{D}\mathit{ef}_Y)$ of infinitesimal automorphisms of X and Y compatible with f', i.e., such that $f' \circ \alpha = \beta \circ f'$. By Deformation Theory, Lemma 7.1 for an arbitrary pair (α, β) the difference between the morphism $f' : X' \to Y'$ and the morphism $\beta^{-1} \circ f' \circ \alpha : X' \to Y'$ defines an elment in

$$\mathrm{Der}_k(\mathcal{O}_Y, f_*\mathcal{O}_X) = \mathrm{Hom}_{\mathcal{O}_Y}(\Omega_{Y/k}, f_*\mathcal{O}_X)$$

Equality by More on Morphisms, Lemma 13.3. This defines the last top horizontal arrow and shows exactness in the first two places. For the map

$$\mathrm{Der}_k(\mathcal{O}_Y, f_*\mathcal{O}_X) \to T\mathcal{D}\mathit{ef}_{X\to Y}$$

we interpret elements of the source as morphisms $f_\epsilon : X' \to Y'$ over $\mathrm{Spec}(k[\epsilon])$ equal to f modulo ϵ using Deformation Theory, Lemma 7.1. We send f_ϵ to the isomorphism class of $(f_\epsilon : X' \to Y')$ in $T\mathcal{D}\mathit{ef}_{X\to Y}$. Note that $(f_\epsilon : X' \to Y')$ is isomorphic to the trivial deformation $(f' : X' \to Y')$ exactly when $f_\epsilon = \beta^{-1} \circ f \circ \alpha$ for some pair (α, β) which implies exactness in the third spot. Clearly, if some first order deformation $(f_\epsilon : X_\epsilon \to Y_\epsilon)$ maps to zero in $T(\mathcal{D}\mathit{ef}_X \times \mathcal{D}\mathit{ef}_Y)$, then we can choose isomorphisms $X' \to X_\epsilon$ and $Y' \to Y_\epsilon$ and we conclude we are in the image of the south-west arrow. Therefore we have exactness at the fourth spot. Finally, given two first order deformations X_ϵ, Y_ϵ of X, Y there is an obstruction in

$$\mathit{ob}(X_\epsilon, Y_\epsilon) \in \mathrm{Ext}^1_{\mathcal{O}_X}(Lf^*\,NL_{Y/k}, \mathcal{O}_X)$$

which vanishes if and only if $f : X \to Y$ lifts to $X_\epsilon \to Y_\epsilon$, see Deformation Theory, Lemma 7.1. This finishes the proof. □

0E3W **Lemma 10.4.** *In Lemma 10.3 if X and Y are both proper over k, then $\mathrm{Inf}(\mathcal{D}ef_{X\to Y})$ and $T\mathcal{D}ef_{X\to Y}$ are finite dimensional.*

Proof. Omitted. Hint: argue as in Lemma 9.4 and use the exact sequence of the lemma. □

In Example 10.1 if $X \to Y$ is a morphism of proper schemes over k, then $\mathcal{D}ef_{X\to Y}$ admits a presentation by a smooth prorepresentable groupoid in functors over $\mathcal{C}_\Lambda$ and a fortiori has a (minimal) versal formal object. This follows from Lemmas 10.2 and 10.4 and the general discussion in Section 3.

0ET6 **Lemma 10.5.** *In Example 10.1 assume $X \to Y$ is a morphism of proper k-schemes. Assume Λ is a complete local ring with residue field k (the classical case). Then the functor*

$$F : \mathcal{C}_\Lambda \longrightarrow \mathit{Sets}, \quad A \longmapsto \mathrm{Ob}(\mathcal{D}ef_{X\to Y}(A))/\cong$$

of isomorphism classes of objects has a hull. If $\mathrm{Der}_k(\mathcal{O}_X, \mathcal{O}_X) = \mathrm{Der}_k(\mathcal{O}_Y, \mathcal{O}_Y) = 0$, then F is prorepresentable.

Proof. The existence of a hull follows immediately from Lemmas 10.2 and 10.4 and Formal Deformation Theory, Lemma 16.6 and Remark 15.7.

Assume $\mathrm{Der}_k(\mathcal{O}_X, \mathcal{O}_X) = \mathrm{Der}_k(\mathcal{O}_Y, \mathcal{O}_Y) = 0$. Then the exact sequence of Lemma 10.3 combined with Lemma 9.3 shows that $\mathrm{Inf}(\mathcal{D}ef_{X\to Y}) = 0$. Then $\mathcal{D}ef_{X\to Y}$ and F are equivalent by Formal Deformation Theory, Lemma 19.13. Hence F is a deformation functor (because $\mathcal{D}ef_{X\to Y}$ is a deformation category) with finite tangent space and we can apply Formal Deformation Theory, Theorem 18.2. □

0E3X **Lemma 10.6.** *In Example 9.1 let $f : X \to Y$ be a morphism of schemes over k. If $f_*\mathcal{O}_X = \mathcal{O}_Y$ and $R^1 f_*\mathcal{O}_X = 0$, then the morphism of deformation categories*

$$\mathcal{D}ef_{X\to Y} \to \mathcal{D}ef_X$$

is an equivalence.

This is discussed in [Vak06, Section 5.3] and [Ran89, Theorem 3.3].

Proof. We construct a quasi-inverse to the forgetful functor of the lemma. Namely, suppose that (A, U) is an object of $\mathcal{D}ef_X$. The given map $X \to U$ is a finite order thickening and we can use it to identify the underlying topological spaces of U and X, see More on Morphisms, Section 2. Thus we may and do think of $\mathcal{O}_U$ as a sheaf of A-algebras on X; moreover the fact that $U \to \mathrm{Spec}(A)$ is flat, means that $\mathcal{O}_U$ is flat as a sheaf of A-modules. In particular, we have a filtration

$$0 = \mathfrak{m}_A^n\mathcal{O}_U \subset \mathfrak{m}_A^{n-1}\mathcal{O}_U \subset \ldots \subset \mathfrak{m}_A^2\mathcal{O}_U \subset \mathfrak{m}_A\mathcal{O}_U \subset \mathcal{O}_U$$

with subquotients equal to $\mathcal{O}_X \otimes_k \mathfrak{m}_A^i/\mathfrak{m}_A^{i+1}$ by flatness, see More on Morphisms, Lemma 10.1 or the more general Deformation Theory, Lemma 5.2. Set

$$\mathcal{O}_V = f_*\mathcal{O}_U$$

viewed as sheaf of A-algebras on Y. Since $R^1 f_*\mathcal{O}_X = 0$ we find by the description above that $R^1 f_*(\mathfrak{m}_A^i\mathcal{O}_U/\mathfrak{m}_A^{i+1}\mathcal{O}_U) = 0$ for all i. This implies that the sequences

$$0 \to (f_*\mathcal{O}_X) \otimes_k \mathfrak{m}_A^i/\mathfrak{m}_A^{i+1} \to f_*(\mathcal{O}_U/\mathfrak{m}_A^{i+1}\mathcal{O}_U) \to f_*(\mathcal{O}_U/\mathfrak{m}_A^i\mathcal{O}_U) \to 0$$

are exact for all i. Reading the references given above backwards (and using induction) we find that $\mathcal{O}_V$ is a flat sheaf of A-algebras with $\mathcal{O}_V/\mathfrak{m}_A\mathcal{O}_V = \mathcal{O}_Y$. Using More on Morphisms, Lemma 2.2 we find that $(Y, \mathcal{O}_V)$ is a scheme, call it V. The equality $\mathcal{O}_V = f_*\mathcal{O}_U$ defines a morphism of ringed spaces $U \to V$ which is easily seen to be a morphism of schemes. This finishes the proof by the flatness already esthablished. □

11. Algebraic spaces

0E3Y The deformation theory of algebraic spaces.

0E3Z **Example 11.1** (Algebraic spaces)**.** Let $\mathcal{F}$ be the category defined as follows

(1) an object is a pair (A, X) consisting of an object A of $\mathcal{C}_\Lambda$ and an algebraic space X flat over A, and
(2) a morphism $(f, g) : (B, Y) \to (A, X)$ consists of a morphism $f : B \to A$ in $\mathcal{C}_\Lambda$ together with a morphism $g : X \to Y$ of algebraic spaces over Λ such that

$$\begin{array}{ccc} X & \xrightarrow{g} & Y \\ \downarrow & & \downarrow \\ \mathrm{Spec}(A) & \xrightarrow{f} & \mathrm{Spec}(B) \end{array}$$

is a cartesian commutative diagram of algebraic spaces.

The functor $p : \mathcal{F} \to \mathcal{C}_\Lambda$ sends (A, X) to A and (f, g) to f. It is clear that p is cofibred in groupoids. Given an algebraic space X over k, let $x_0 = (k, X)$ be the corresponding object of $\mathcal{F}(k)$. We set

$$\mathcal{D}ef_X = \mathcal{F}_{x_0}$$

0E40 **Lemma 11.2.** *Example 11.1 satisfies the Rim-Schlessinger condition (RS). In particular, $\mathcal{D}ef_X$ is a deformation category for any algebraic space X over k.*

Proof. Let $A_1 \to A$ and $A_2 \to A$ be morphisms of $\mathcal{C}_\Lambda$. Assume $A_2 \to A$ is surjective. According to Formal Deformation Theory, Lemma 16.4 it suffices to show that the functor $\mathcal{F}(A_1 \times_A A_2) \to \mathcal{F}(A_1) \times_{\mathcal{F}(A)} \mathcal{F}(A_2)$ is an equivalence of categories. Observe that

$$\begin{array}{ccc} \mathrm{Spec}(A) & \longrightarrow & \mathrm{Spec}(A_2) \\ \downarrow & & \downarrow \\ \mathrm{Spec}(A_1) & \longrightarrow & \mathrm{Spec}(A_1 \times_A A_2) \end{array}$$

is a pushout diagram as in Pushouts of Spaces, Lemma 2.4. Thus the lemma is a special case of Pushouts of Spaces, Lemma 2.9. □

0E41 **Lemma 11.3.** *In Example 11.1 let X be an algebraic space over k. Then*

$$\mathit{Inf}(\mathcal{D}ef_X) = \mathit{Ext}^0_{\mathcal{O}_X}(NL_{X/k}, \mathcal{O}_X) = \mathrm{Hom}_{\mathcal{O}_X}(\Omega_{X/k}, \mathcal{O}_X) = \mathit{Der}_k(\mathcal{O}_X, \mathcal{O}_X)$$

and

$$T\mathcal{D}ef_X = \mathit{Ext}^1_{\mathcal{O}_X}(NL_{X/k}, \mathcal{O}_X)$$

Proof. Recall that $\mathrm{Inf}(\mathcal{D}ef_X)$ is the set of automorphisms of the trivial deformation $X' = X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k[\epsilon])$ of X to $k[\epsilon]$ equal to the identity modulo ϵ. By Deformation Theory, Lemma 14.2 this is equal to $\mathrm{Ext}^0_{\mathcal{O}_X}(NL_{X/k}, \mathcal{O}_X)$. The equality $\mathrm{Ext}^0_{\mathcal{O}_X}(NL_{X/k}, \mathcal{O}_X) = \mathrm{Hom}_{\mathcal{O}_X}(\Omega_{X/k}, \mathcal{O}_X)$ follows from More on Morphisms of Spaces, Lemma 21.4. The equality $\mathrm{Hom}_{\mathcal{O}_X}(\Omega_{X/k}, \mathcal{O}_X) = \mathrm{Der}_k(\mathcal{O}_X, \mathcal{O}_X)$ follows from More on Morphisms of Spaces, Definition 7.2 and Modules on Sites, Definition 32.3.

Recall that $T_{x_0}\mathcal{D}ef_X$ is the set of isomorphism classes of flat deformations X' of X to $k[\epsilon]$, more precisely, the set of isomorphism classes of $\mathcal{D}ef_X(k[\epsilon])$. Thus the second statement of the lemma follows from Deformation Theory, Lemma 14.2. □

0E42 **Lemma 11.4.** *In Lemma 11.3 if X is proper over k, then $\mathit{Inf}(\mathcal{D}ef_X)$ and $T\mathcal{D}ef_X$ are finite dimensional.*

Proof. By the lemma we have to show $\mathrm{Ext}^1_{\mathcal{O}_X}(NL_{X/k}, \mathcal{O}_X)$ and $\mathrm{Ext}^0_{\mathcal{O}_X}(NL_{X/k}, \mathcal{O}_X)$ are finite dimensional. By More on Morphisms of Spaces, Lemma 21.5 and the fact that X is Noetherian, we see that $NL_{X/k}$ has coherent cohomology sheaves zero except in degrees 0 and -1. By Derived Categories of Spaces, Lemma 8.4 the displayed Ext-groups are finite k-vector spaces and the proof is complete. □

In Example 11.1 if X is a proper algebraic space over k, then $\mathcal{D}ef_X$ admits a presentation by a smooth prorepresentable groupoid in functors over $\mathcal{C}_\Lambda$ and a fortiori has a (minimal) versal formal object. This follows from Lemmas 11.2 and 11.4 and the general discussion in Section 3.

0ET7 **Lemma 11.5.** *In Example 11.1 assume X is a proper algebraic space over k. Assume Λ is a complete local ring with residue field k (the classical case). Then the functor*

$$F : \mathcal{C}_\Lambda \longrightarrow \mathit{Sets}, \quad A \longmapsto \mathrm{Ob}(\mathcal{D}ef_X(A))/\cong$$

of isomorphism classes of objects has a hull. If $\mathit{Der}_k(\mathcal{O}_X, \mathcal{O}_X) = 0$, then F is prorepresentable.

Proof. The existence of a hull follows immediately from Lemmas 11.2 and 11.4 and Formal Deformation Theory, Lemma 16.6 and Remark 15.7.

Assume $\mathrm{Der}_k(\mathcal{O}_X, \mathcal{O}_X) = 0$. Then $\mathcal{D}ef_X$ and F are equivalent by Formal Deformation Theory, Lemma 19.13. Hence F is a deformation functor (because $\mathcal{D}ef_X$ is a deformation category) with finite tangent space and we can apply Formal Deformation Theory, Theorem 18.2. □

12. Deformations of completions

0DZ0 In this section we compare the deformation problem posed by an algebra and its completion. We first discuss "liftability".

0DZ1 **Lemma 12.1.** *Let $A' \to A$ be a surjection of rings with nilpotent kernel. Let $A' \to P'$ be a flat ring map. Set $P = P' \otimes_{A'} A$. Let M be an A-flat P-module. Then the following are equivalent*

1. *there is an A'-flat P'-module M' with $M' \otimes_{P'} P = M$, and*
2. *there is an object $K' \in D^-(P')$ with $K' \otimes^{\mathbf{L}}_{P'} P = M$.*

Proof. Suppose that M' is as in (1). Then

$$M = M' \otimes_{P'} P' = M' \otimes_{A'} A = M' \otimes^{\mathbf{L}}_{A} A' = M' \otimes^{\mathbf{L}}_{P'} P$$

The first two equalities are clear, the third holds because M' is flat over A', and the fourth holds by More on Algebra, Lemma 59.2. Thus (2) holds. Conversely, suppose K' is as in (2). We may and do assume M is nonzero. Let t be the largest integer such that $H^t(K')$ is nonzero (exists because M is nonzero). Then $H^t(K') \otimes_{P'} P = H^t(K' \otimes^{\mathbf{L}}_{P'} P)$ is zero if $t > 0$. Since the kernel of $P' \to P$ is nilpotent this implies $H^t(K') = 0$ by Nakayama's lemma a contradiction. Hence $t = 0$ (the case $t < 0$ is absurd as well). Then $M' = H^0(K')$ is a P'-module such that $M = M' \otimes_{P'} P$ and the spectral sequence for Tor gives an injective map

$$\mathrm{Tor}_1^{P'}(M', P) \to H^{-1}(M' \otimes^{\mathbf{L}}_{P'} P) = 0$$

By the reference on derived base change above $0 = \mathrm{Tor}_1^{P'}(M', P) = \mathrm{Tor}_1^{A'}(M', A)$. We conclude that M' is A'-flat by Algebra, Lemma 98.8. □

0DZ2 **Lemma 12.2.** *Consider a commutative diagram of Noetherian rings*

$$\begin{array}{ccccc} A' & \longrightarrow & P' & \longrightarrow & Q' \\ \downarrow & & \downarrow & & \downarrow \\ A & \longrightarrow & P & \longrightarrow & Q \end{array}$$

with cartesian squares, with flat horizontal arrows, and with surjective vertial arrows whose kernels are nilpotent. Let $J' \subset P'$ be an ideal such that $P'/J' = Q'/J'Q'$. Let M be an A-flat P-module. Assume for all $g \in J'$ there exists an A'-flat $(P')_g$-module lifting M_g. Then the following are equivalent

1. *M has an A'-flat lift to a P'-module, and*
2. *$M \otimes_P Q$ has an A'-flat lift to a Q'-module.*

Proof. Let $I = \mathrm{Ker}(A' \to A)$. By induction on the integer $n > 1$ such that $I^n = 0$ we reduce to the case where I is an ideal of square zero; details omitted. We translate the condition of liftability of M into the problem of finding an object of $D^-(P')$ as in Lemma 12.1. The obstruction to doing this is the element

$$\omega(M) \in \mathrm{Ext}^2_P(M, M \otimes^{\mathbf{L}}_P IP) = \mathrm{Ext}^2_P(M, M \otimes_P IP)$$

constructed in Deformation Theory, Lemma 15.1. The equality in the displayed formula holds as $M \otimes^{\mathbf{L}}_P IP = M \otimes_P IP$ since M and P are A-flat[2]. The obstruction for lifting $M \otimes_P Q$ is similarly the element

$$\omega(M \otimes_P Q) \in \mathrm{Ext}^2_Q(M \otimes_P Q, (M \otimes_P Q) \otimes_Q IQ)$$

which is the image of $\omega(M)$ by the functoriality of the construction $\omega(-)$ of Deformation Theory, Lemma 15.1. By More on Algebra, Lemma 87.2 we have

$$\mathrm{Ext}^2_Q(M \otimes_P Q, (M \otimes_P Q) \otimes_Q IQ) = \mathrm{Ext}^2_P(M, M \otimes_P IP) \otimes_P Q$$

here we use that P is Noetherian and M finite. Our assumption on $P' \to Q'$ guarantees that for an P-module E the map $E \to E \otimes_P Q$ is bijective on J'-power torsion, see More on Algebra, Lemma 80.3. Thus we conclude that it suffices to

[2]Choose a resolution $F_\bullet \to I$ by free A-modules. Since $A \to P$ is flat, $P \otimes_A F_\bullet$ is a free resolution of IP. Hence $M \otimes^{\mathbf{L}}_P IP$ is represented by $M \otimes_P P \otimes_A F_\bullet = M \otimes_A F_\bullet$. This only has cohomology in degree 0 as M is A-flat.

show $\omega(M)$ is J'-power torsion. In other words, it suffices to show that $\omega(M)$ dies in

$$\mathrm{Ext}^2_P(M, M \otimes_P IP)_g = \mathrm{Ext}^2_{P_g}(M_g, M_g \otimes_{P_g} IP_g)$$

for all $g \in J'$. Howeover, by the compatibility of formation of $\omega(M)$ with base change again, we conclude that this is true as M_g is assumed to have a lift (of course you have to use the whole string of equivalences again). □

0DZ3 **Lemma 12.3.** *Let $A' \to A$ be a surjective map of Noetherian rings with nilpotent kernel. Let $A \to B$ be a finite type flat ring map. Let $\mathfrak{b} \subset B$ be an ideal such that $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is syntomic on the complement of $V(\mathfrak{b})$. Then B has a flat lift to A' if and only if the $\mathfrak{b}$-adic completion $B^\wedge$ has a flat lift to A'.*

Proof. Choose an A-algebra surjection $P = A[x_1, \ldots, x_n] \to B$. Let $\mathfrak{p} \subset P$ be the inverse image of $\mathfrak{b}$. Set $P' = A'[x_1, \ldots, x_n]$ and denote $\mathfrak{p}' \subset P'$ the inverse image of $\mathfrak{p}$. (Of course $\mathfrak{p}$ and $\mathfrak{p}'$ do not designate prime ideals here.) We will denote $P^\wedge$ and $(P')^\wedge$ the respective completions.

Suppose $A' \to B'$ is a flat lift of $A \to B$, in other words, $A' \to B'$ is flat and there is an A-algebra isomorphism $B = B' \otimes_{A'} A$. Then we can choose an A'-algebra map $P' \to B'$ lifting the given surjection $P \to B$. By Nakayama's lemma (Algebra, Lemma 19.1) we find that B' is a quotient of P'. In particular, we find that we can endow B' with an A'-flat P'-module structure lifting B as an A-flat P-module. Conversely, if we can lift B to a P'-module M' flat over A', then M' is a cyclic module $M' \cong P'/J'$ (using Nakayama again) and setting $B' = P'/J'$ we find a flat lift of B as an algebra.

Set $C = B^\wedge$ and $\mathfrak{c} = \mathfrak{b}C$. Suppose that $A' \to C'$ is a flat lift of $A \to C$. Then C' is complete with respect to the inverse image $\mathfrak{c}'$ of $\mathfrak{c}$ (Algebra, Lemma 96.10). We choose an A'-algebra map $P' \to C'$ lifting the A-algebra map $P \to C$. These maps pass through completions to give surjections $P^\wedge \to C$ and $(P')^\wedge \to C'$ (for the second again using Nakayama's lemma). In particular, we find that we can endow C' with an A'-flat $(P')^\wedge$-module structure lifting C as an A-flat $P^\wedge$-module. Conversely, if we can lift C to a $(P')^\wedge$-module N' flat over A', then N' is a cyclic module $N' \cong (P')^\wedge/\tilde{J}$ (using Nakayama again) and setting $C' = (P')^\wedge/\tilde{J}$ we find a flat lift of C as an algebra.

Observe that $P' \to (P')^\wedge$ is a flat ring map which induces an isomorphism $P'/\mathfrak{p}' = (P')^\wedge/\mathfrak{p}'(P')^\wedge$. We conclude that our lemma is a consequence of Lemma 12.2 provided we can show that B_g lifts to an A'-flat P'_g-module for $g \in \mathfrak{p}'$. However, the ring map $A \to B_g$ is syntomic and hence lifts to an A'-flat algebra B' by Smoothing Ring Maps, Proposition 3.2. Since $A' \to P'_g$ is smooth, we can lift $P_g \to B_g$ to a surjective map $P'_g \to B'$ as before and we get what we want. □

Notation. Let $A \to B$ be a ring map. Let N be a B-module. We denote $\mathrm{Exal}_A(B, N)$ the set of isomorphism classes of extensions

$$0 \to N \to C \to B \to 0$$

of A-algebras such that N is an ideal of square zero in C. Given a second such $0 \to N \to C' \to B \to 0$ an isomorphism is a A-algebra isomorpism $C \to C'$ such

that the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \longrightarrow & C & \longrightarrow & B & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow & & \downarrow{\scriptstyle \mathrm{id}} & & \\ 0 & \longrightarrow & N & \longrightarrow & C' & \longrightarrow & B & \longrightarrow & 0 \end{array}$$

commutes. The assignment $N \mapsto \mathrm{Exal}_A(B, N)$ is a functor which transforms products into products. Hence this is an additive functor and $\mathrm{Exal}_A(B, N)$ has a natural B-module structure. In fact, by Deformation Theory, Lemma 2.3 we have $\mathrm{Exal}_A(B, N) = \mathrm{Ext}^1_B(NL_{B/A}, N)$.

0DZ4 **Lemma 12.4.** *Let k be a field. Let B be a finite type k-algebra. Let $J \subset B$ be an ideal such that $\mathrm{Spec}(B) \to \mathrm{Spec}(k)$ is smooth on the complement of $V(J)$. Let N be a finite B-module. Then there is a canonical bijection*

$$Exal_k(B, N) \to Exal_k(B^\wedge, N^\wedge)$$

Here $B^\wedge$ and $N^\wedge$ are the J-adic completions.

Proof. The map is given by completion: given $0 \to N \to C \to B \to 0$ in $\mathrm{Exal}_k(B, N)$ we send it to the completion $C^\wedge$ of C with respect to the inverse image of J. Compare with the proof of Lemma 8.10.

Since $k \to B$ is of finite presentation the complex $NL_{B/k}$ can be represented by a complex $N^{-1} \to N^0$ where N^i is a finite B-module, see Algebra, Section 132 and in particular Algebra, Lemma 132.2. As B is Noetherian, this means that $NL_{B/k}$ is pseudo-coherent. For $g \in J$ the k-algebra B_g is smooth and hence $(NL_{B/k})_g = NL_{B_g/k}$ is quasi-isomorphic to a finite projective B-module sitting in degree 0. Thus $\mathrm{Ext}^i_B(NL_{B/k}, N)_g = 0$ for $i \geq 1$ and any B-module N. By More on Algebra, Lemma 89.3 we conclude that

$$\mathrm{Ext}^1_B(NL_{B/k}, N) \longrightarrow \lim_n \mathrm{Ext}^1_B(NL_{B/k}, N/J^nN)$$

is an isomorphism for any finite B-module N.

Injectivity of the map. Suppose that $0 \to N \to C \to B \to 0$ is in $\mathrm{Exal}_k(B, N)$ and maps to zero in $\mathrm{Exal}_k(B^\wedge, N^\wedge)$. Choose a splitting $C^\wedge = B^\wedge \oplus N^\wedge$. Then the induced map $C \to C^\wedge \to N^\wedge$ gives maps $C \to N/J^nN$ for all n. Hence we see that our element is in the kernel of the maps

$$\mathrm{Ext}^1_B(NL_{B/k}, N) \to \mathrm{Ext}^1_B(NL_{B/k}, N/J^nN)$$

for all n. By the previous paragraph we conclude that our element is zero.

Surjectivity of the map. Let $0 \to N^\wedge \to C' \to B^\wedge \to 0$ be an element of $\mathrm{Exal}_k(B^\wedge, N^\wedge)$. Pulling back by $B \to B^\wedge$ we get an element $0 \to N^\wedge \to C'' \to B \to 0$ in $\mathrm{Exal}_k(B, N^\wedge)$. we have

$$\mathrm{Ext}^1_B(NL_{B/k}, N^\wedge) = \mathrm{Ext}^1_B(NL_{B/k}, N) \otimes_B B^\wedge = \mathrm{Ext}^1_B(NL_{B/k}, N)$$

The first equality as $N^\wedge = N \otimes_B B^\wedge$ (Algebra, Lemma 96.1) and More on Algebra, Remark 62.20. The second equality because $\mathrm{Ext}^1_B(NL_{B/k}, N)$ is J-power torsion (see above), $B \to B^\wedge$ is flat and induces an isomorphism $B/J \to B^\wedge/JB^\wedge$, and More on Algebra, Lemma 80.3. Thus we can find a $C \in \mathrm{Exal}_k(B, N)$ mapping to C'' in $\mathrm{Exal}_k(B, N^\wedge)$. Thus

$$0 \to N^\wedge \to C' \to B^\wedge \to 0 \quad \text{and} \quad 0 \to N^\wedge \to C^\wedge \to B^\wedge \to 0$$

are two elements of $\mathrm{Exal}_k(B^\wedge, N^\wedge)$ mapping to the same element of $\mathrm{Exal}_k(B, N^\wedge)$. Taking the difference we get an element $0 \to N^\wedge \to C' \to B^\wedge \to 0$ of $\mathrm{Exal}_k(B^\wedge, N^\wedge)$ whose image in $\mathrm{Exal}_k(B, N^\wedge)$ is zero. This means there exists

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N^\wedge & \longrightarrow & C' & \longrightarrow & B^\wedge & \longrightarrow & 0 \\ & & & & \uparrow{\scriptstyle \sigma} & \nearrow & & & \\ & & & & B & & & & \end{array}$$

Let $J' \subset C'$ be the inverse image of $JB^\wedge \subset B^\wedge$. To finish the proof it suffices to note that σ is continuous for the J-adic topology on B and the J'-adic topology on C' and that C' is J'-adically complete by Algebra, Lemma 96.10 (here we also use that C' is Noetherian; small detail omitted). Namely, this means that σ factors through the completion $B^\wedge$ and $C' = 0$ in $\mathrm{Exal}_k(B^\wedge, N^\wedge)$. □

0DZ5 **Lemma 12.5.** *In Example 8.1 let P be a k-algebra. Let $J \subset P$ be an ideal. Denote $P^\wedge$ the J-adic completion. If*

(1) *$k \to P$ is of finite type, and*
(2) *$\mathrm{Spec}(P) \to \mathrm{Spec}(k)$ is smooth on the complement of $V(J)$.*

then the functor between deformation categories of Lemma 8.10

$$\mathcal{D}ef_P \longrightarrow \mathcal{D}ef_{P^\wedge}$$

is smooth and induces an isomorphism on tangent spaces.

Proof. We know that $\mathcal{D}ef_P$ and $\mathcal{D}ef_{P^\wedge}$ are deformation categories by Lemma 8.2. Thus it suffices to check our functor identifies tangent spaces and a correspondence between liftability, see Formal Deformation Theory, Lemma 20.3. The property on liftability is proven in Lemma 12.3 and the isomorphism on tangent spaces is the special case of Lemma 12.4 where $N = B$. □

13. Deformations of localizations

0DZ6 In this section we compare the deformation problem posed by an algebra and its localization at a multiplicative subset. We first discuss "liftability".

0DZ7 **Lemma 13.1.** *Let $A' \to A$ be a surjective map of Noetherian rings with nilpotent kernel. Let $A \to B$ be a finite type flat ring map. Let $S \subset B$ be a multiplicative subset such that if $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is not syntomic at $\mathfrak{q}$, then $S \cap \mathfrak{q} = \emptyset$. Then B has a flat lift to A' if and only if $S^{-1}B$ has a flat lift to A'.*

Proof. This proof is the same as the proof of Lemma 12.3 but easier. We suggest the reader to skip the proof. Choose an A-algebra surjection $P = A[x_1, \ldots, x_n] \to B$. Let $S_P \subset P$ be the inverse image of S. Set $P' = A'[x_1, \ldots, x_n]$ and denote $S_{P'} \subset P'$ the inverse image of S_P.

Suppose $A' \to B'$ is a flat lift of $A \to B$, in other words, $A' \to B'$ is flat and there is an A-algebra isomorphism $B = B' \otimes_{A'} A$. Then we can choose an A'-algebra map $P' \to B'$ lifting the given surjection $P \to B$. By Nakayama's lemma (Algebra, Lemma 19.1) we find that B' is a quotient of P'. In particular, we find that we can endow B' with an A'-flat P'-module structure lifting B as an A-flat P-module. Conversely, if we can lift B to a P'-module M' flat over A', then M' is a cyclic module $M' \cong P'/J'$ (using Nakayama again) and setting $B' = P'/J'$ we find a flat lift of B as an algebra.

Set $C = S^{-1}B$. Suppose that $A' \to C'$ is a flat lift of $A \to C$. Elements of C' which map to invertible elements of C are invertible. We choose an A'-algebra map $P' \to C'$ lifting the A-algebra map $P \to C$. By the remark above these maps pass through localizations to give surjections $S_P^{-1}P \to C$ and $S_{P'}^{-1}P' \to C'$ (for the second use Nakayama's lemma). In particular, we find that we can endow C' with an A'-flat $S_{P'}^{-1}P'$-module structure lifting C as an A-flat $S_P^{-1}P$-module. Conversely, if we can lift C to a $S_{P'}^{-1}P'$-module N' flat over A', then N' is a cyclic module $N' \cong S_{P'}^{-1}P'/\tilde{J}$ (using Nakayama again) and setting $C' = S_{P'}^{-1}P'/\tilde{J}$ we find a flat lift of C as an algebra.

The syntomic locus of a morphism of schemes is open by definition. Let $J_B \subset B$ be an ideal cutting out the set of points in $\mathrm{Spec}(B)$ where $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is not syntomic. Denote $J_P \subset P$ and $J_{P'} \subset P'$ the corresponding ideals. Observe that $P' \to S_{P'}^{-1}P'$ is a flat ring map which induces an isomorphism $P'/J_{P'} = S_{P'}^{-1}P'/J_{P'}S_{P'}^{-1}P'$ by our assumption on S in the lemma, namely, the assumption in the lemma is exactly that $B/J_B = S^{-1}(B/J_B)$. We conclude that our lemma is a consequence of Lemma 12.2 provided we can show that B_g lifts to an A'-flat P'_g-module for $g \in J_B$. However, the ring map $A \to B_g$ is syntomic and hence lifts to an A'-flat algebra B' by Smoothing Ring Maps, Proposition 3.2. Since $A' \to P'_g$ is smooth, we can lift $P_g \to B_g$ to a surjective map $P'_g \to B'$ as before and we get what we want. □

0DZ8 **Lemma 13.2.** *Let k be a field. Let B be a finite type k-algebra. Let $S \subset B$ be a multiplicative subset ideal such that if $\mathrm{Spec}(B) \to \mathrm{Spec}(k)$ is not smooth at $\mathfrak{q}$ then $S \cap \mathfrak{q} = \emptyset$. Let N be a finite B-module. Then there is a canonical bijection*

$$Exal_k(B, N) \to Exal_k(S^{-1}B, S^{-1}N)$$

Proof. This proof is the same as the proof of Lemma 12.4 but easier. We suggest the reader to skip the proof. The map is given by localization: given $0 \to N \to C \to B \to 0$ in $\mathrm{Exal}_k(B, N)$ we send it to the localization $S_C^{-1}C$ of C with respect to the inverse image $S_C \subset C$ of S. Compare with the proof of Lemma 8.7.

The smooth locus of a morphism of schemes is open by definition. Let $J \subset B$ be an ideal cutting out the set of points in $\mathrm{Spec}(B)$ where $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is not smooth. Since $k \to B$ is of finite presentation the complex $NL_{B/k}$ can be represented by a complex $N^{-1} \to N^0$ where N^i is a finite B-module, see Algebra, Section 132 and in particular Algebra, Lemma 132.2. As B is Noetherian, this means that $NL_{B/k}$ is pseudo-coherent. For $g \in J$ the k-algebra B_g is smooth and hence $(NL_{B/k})_g = NL_{B_g/k}$ is quasi-isomorphic to a finite projective B-module sitting in degree 0. Thus $\mathrm{Ext}^i_B(NL_{B/k}, N)_g = 0$ for $i \geq 1$ and any B-module N. Finally, we have

$$\mathrm{Ext}^1_{S^{-1}B}(NL_{S^{-1}B/k}, S^{-1}N) = \mathrm{Ext}^1_B(NL_{B/k}, N) \otimes_B S^{-1}B = \mathrm{Ext}^1_B(NL_{B/k}, N)$$

The first equality by More on Algebra, Lemma 87.2 and Algebra, Lemma 132.13. The second because $\mathrm{Ext}^1_B(NL_{B/k}, N)$ is J-power torsion and elements of S act invertibly on J-power torsion modules. This concludes the proof by the description of $\mathrm{Exal}_A(B, N)$ as $\mathrm{Ext}^1_B(NL_{B/A}, N)$ given just above Lemma 12.4. □

0DZ9 **Lemma 13.3.** *In Example 8.1 let P be a k-algebra. Let $S \subset P$ be a multiplicative subset. If*

(1) $k \to P$ *is of finite type, and*
(2) $\mathrm{Spec}(P) \to \mathrm{Spec}(k)$ *is smooth at all points of* $V(g)$ *for all* $g \in S$.

then the functor between deformation categories of Lemma 8.7

$$\mathcal{D}ef_P \longrightarrow \mathcal{D}ef_{S^{-1}P}$$

is smooth and induces an isomorphism on tangent spaces.

Proof. We know that $\mathcal{D}ef_P$ and $\mathcal{D}ef_{S^{-1}P}$ are deformation categories by Lemma 8.2. Thus it suffices to check our functor identifies tangent spaces and a correspondence between liftability, see Formal Deformation Theory, Lemma 20.3. The property on liftability is proven in Lemma 13.1 and the isomorphism on tangent spaces is the special case of Lemma 13.2 where $N = B$. □

14. Deformations of henselizations

0DZA In this section we compare the deformation problem posed by an algebra and its completion. We first discuss "liftability".

0DZB **Lemma 14.1.** *Let $A' \to A$ be a surjective map of Noetherian rings with nilpotent kernel. Let $A \to B$ be a finite type flat ring map. Let $\mathfrak{b} \subset B$ be an ideal such that $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is syntomic on the complement of $V(\mathfrak{b})$. Let $(B^h, \mathfrak{b}^h)$ be the henselization of the pair $(B, \mathfrak{b})$. Then B has a flat lift to A' if and only if B^h has a flat lift to A'.*

First proof. This proof is a cheat. Namely, if B has a flat lift B', then taking the henselization $(B')^h$ we obtain a flat lift of B^h (compare with the proof of Lemma 8.8). Conversely, suppose that C' is an A'-flat lift of $(B')^h$. Then let $\mathfrak{c}' \subset C'$ be the inverse image of the ideal $\mathfrak{b}^h$. Then the completion $(C')^\wedge$ of C' with respect to $\mathfrak{c}'$ is a lift of $B^\wedge$ (details omitted). Hence we see that B has a flat lift by Lemma 12.3. □

Second proof. Choose an A-algebra surjection $P = A[x_1, \ldots, x_n] \to B$. Let $\mathfrak{p} \subset P$ be the inverse image of $\mathfrak{b}$. Set $P' = A'[x_1, \ldots, x_n]$ and denote $\mathfrak{p}' \subset P'$ the inverse image of $\mathfrak{p}$. (Of course $\mathfrak{p}$ and $\mathfrak{p}'$ do not designate prime ideals here.) We will denote P^h and $(P')^h$ the respective henselizations. We will use that taking henselizations is functorial and that the henselization of a quotient is the corresponding quotient of the henselization, see More on Algebra, Lemmas 11.12 and 12.7.

Suppose $A' \to B'$ is a flat lift of $A \to B$, in other words, $A' \to B'$ is flat and there is an A-algebra isomorphism $B = B' \otimes_{A'} A$. Then we can choose an A'-algebra map $P' \to B'$ lifting the given surjection $P \to B$. By Nakayama's lemma (Algebra, Lemma 19.1) we find that B' is a quotient of P'. In particular, we find that we can endow B' with an A'-flat P'-module structure lifting B as an A-flat P-module. Conversely, if we can lift B to a P'-module M' flat over A', then M' is a cyclic module $M' \cong P'/J'$ (using Nakayama again) and setting $B' = P'/J'$ we find a flat lift of B as an algebra.

Set $C = B^h$ and $\mathfrak{c} = \mathfrak{b}C$. Suppose that $A' \to C'$ is a flat lift of $A \to C$. Then C' is henselian with respect to the inverse image $\mathfrak{c}'$ of $\mathfrak{c}$ (by More on Algebra, Lemma 11.9 and the fact that the kernel of $C' \to C$ is nilpotent). We choose an A'-algebra map $P' \to C'$ lifting the A-algebra map $P \to C$. These maps pass through henselizations to give surjections $P^h \to C$ and $(P')^h \to C'$ (for the second again using Nakayama's lemma). In particular, we find that we can endow C' with an A'-flat $(P')^h$-module

structure lifting C as an A-flat P^h-module. Conversely, if we can lift C to a $(P')^h$-module N' flat over A', then N' is a cyclic module $N' \cong (P')^h/\tilde{J}$ (using Nakayama again) and setting $C' = (P')^h/\tilde{J}$ we find a flat lift of C as an algebra.

Observe that $P' \to (P')^h$ is a flat ring map which induces an isomorphism $P'/\mathfrak{p}' = (P')^h/\mathfrak{p}'(P')^h$ (More on Algebra, Lemma 12.2). We conclude that our lemma is a consequence of Lemma 12.2 provided we can show that B_g lifts to an A'-flat P'_g-module for $g \in \mathfrak{p}'$. However, the ring map $A \to B_g$ is syntomic and hence lifts to an A'-flat algebra B' by Smoothing Ring Maps, Proposition 3.2. Since $A' \to P'_g$ is smooth, we can lift $P_g \to B_g$ to a surjective map $P'_g \to B'$ as before and we get what we want. □

0DZC **Lemma 14.2.** *Let k be a field. Let B be a finite type k-algebra. Let $J \subset B$ be an ideal such that $\mathrm{Spec}(B) \to \mathrm{Spec}(k)$ is smooth on the complement of $V(J)$. Let N be a finite B-module. Then there is a canonical bijection*

$$Exal_k(B, N) \to Exal_k(B^h, N^h)$$

Here (B^h, J^h) is the henselization of (B, J) and $N^h = N \otimes_B B^h$.

Proof. This proof is the same as the proof of Lemma 12.4 but easier. We suggest the reader to skip the proof. The map is given by henselization: given $0 \to N \to C \to B \to 0$ in $\mathrm{Exal}_k(B, N)$ we send it to the henselization C^h of C with respect to the inverse image $J_C \subset C$ of J. Compare with the proof of Lemma 8.8.

Since $k \to B$ is of finite presentation the complex $NL_{B/k}$ can be represented by a complex $N^{-1} \to N^0$ where N^i is a finite B-module, see Algebra, Section 132 and in particular Algebra, Lemma 132.2. As B is Noetherian, this means that $NL_{B/k}$ is pseudo-coherent. For $g \in J$ the k-algebra B_g is smooth and hence $(NL_{B/k})_g = NL_{B_g/k}$ is quasi-isomorphic to a finite projective B-module sitting in degree 0. Thus $\mathrm{Ext}^i_B(NL_{B/k}, N)_g = 0$ for $i \geq 1$ and any B-module N. Finally, we have

$$\begin{aligned} \mathrm{Ext}^1_{B^h}(NL_{B^h/k}, N^h) &= \mathrm{Ext}^1_{B^h}(NL_{B/k} \otimes_B B^h, N \otimes_B B^h) \\ &= \mathrm{Ext}^1_B(NL_{B/k}, N) \otimes_B B^h \\ &= \mathrm{Ext}^1_B(NL_{B/k}, N) \end{aligned}$$

The first equality by More on Algebra, Lemma 32.8 (or rather its analogue for henselizations of pairs). The second by More on Algebra, Lemma 87.2. The third because $\mathrm{Ext}^1_B(NL_{B/k}, N)$ is J-power torsion, the map $B \to B^h$ is flat and induces an isomorphism $B/J \to B^h/JB^h$ (More on Algebra, Lemma 12.2), and More on Algebra, Lemma 80.3. This concludes the proof by the description of $\mathrm{Exal}_A(B, N)$ as $\mathrm{Ext}^1_B(NL_{B/A}, N)$ given just above Lemma 12.4. □

0DZD **Lemma 14.3.** *In Example 8.1 let P be a k-algebra. Let $J \subset P$ be an ideal. Denote (P^h, J^h) the henselization of the pair (P, J). If*

(1) *$k \to P$ is of finite type, and*
(2) *$\mathrm{Spec}(P) \to \mathrm{Spec}(k)$ is smooth on the complement of $V(J)$,*

then the functor between deformation categories of Lemma 8.8

$$\mathcal{D}ef_P \longrightarrow \mathcal{D}ef_{P^h}$$

is smooth and induces an isomorphism on tangent spaces.

Proof. We know that $\mathcal{D}ef_P$ and $\mathcal{D}ef_{P^h}$ are deformation categories by Lemma 8.2. Thus it suffices to check our functor identifies tangent spaces and a correspondence between liftability, see Formal Deformation Theory, Lemma 20.3. The property on liftability is proven in Lemma 14.1 and the isomorphism on tangent spaces is the special case of Lemma 14.2 where $N = B$. □

15. Application to isolated singularities

0DZE We apply the discussion above to study the deformation theory of a finite type algebra with finitely many singular points.

0DZF **Lemma 15.1.** *In Example 8.1 let P be a k-algebra. Assume that $k \to P$ is of finite type and that $\mathrm{Spec}(P) \to \mathrm{Spec}(k)$ is smooth except at the maximal ideals $\mathfrak{m}_1, \ldots, \mathfrak{m}_n$ of P. Let $P_{\mathfrak{m}_i}$, $P^h_{\mathfrak{m}_i}$, $P^\wedge_{\mathfrak{m}_i}$ be the local ring, henselization, completion. Then the maps of deformation categories*

$$\mathcal{D}ef_P \to \prod \mathcal{D}ef_{P_{\mathfrak{m}_i}} \to \prod \mathcal{D}ef_{P^h_{\mathfrak{m}_i}} \to \prod \mathcal{D}ef_{P^\wedge_{\mathfrak{m}_i}}$$

are smooth and induce isomorphisms on their finite dimensional tangent spaces.

Proof. The tangent space is finite dimensional by Lemma 8.5. The functors between the categories are constructed in Lemmas 8.7, 8.8, and 8.10 (we omit some verifications of the form: the completion of the henselization is the completion).

Set $J = \mathfrak{m}_1 \cap \ldots \cap \mathfrak{m}_n$ and apply Lemma 12.5 to get that $\mathcal{D}ef_P \to \mathcal{D}ef_{P^\wedge}$ is smooth and induces an isomorphism on tangent spaces where $P^\wedge$ is the J-adic completion of P. However, since $P^\wedge = \prod P^\wedge_{\mathfrak{m}_i}$ we see that the map $\mathcal{D}ef_P \to \prod \mathcal{D}ef_{P^\wedge_{\mathfrak{m}_i}}$ is smooth and induces an isomorphism on tangent spaces.

Let (P^h, J^h) be the henselization of the pair (P, J). Then $P^h = \prod P^h_{\mathfrak{m}_i}$ (look at idempotents and use More on Algebra, Lemma 11.6). Hence we can apply Lemma 14.3 to conclude as in the case of completion.

To get the final case it suffices to show that $\mathcal{D}ef_{P_{\mathfrak{m}_i}} \to \mathcal{D}ef_{P^\wedge_{\mathfrak{m}_i}}$ is smooth and induce isomorphisms on tangent spaces for each i separately. To do this, we may replace P by a principal localization whose only singular point is a maximal ideal $\mathfrak{m}$ (corresponding to $\mathfrak{m}_i$ in the original P). Then we can apply Lemma 13.3 with multiplicative subset $S = P \setminus \mathfrak{m}$ to conclude. Minor details omitted. □

16. Unobstructed deformation problems

0DZG Let $p : \mathcal{F} \to \mathcal{C}_\Lambda$ be a category cofibred in groupoids. Recall that we say $\mathcal{F}$ is *smooth* or *unobstructed* if p is smooth. This means that given a surjection $\varphi : A' \to A$ in $\mathcal{C}_\Lambda$ and $x \in \mathrm{Ob}(\mathcal{F}(A))$ there exists a morphism $f : x' \to x$ in $\mathcal{F}$ with $p(f) = \varphi$. See Formal Deformation Theory, Section 9. In this section we give some geometrically meaningful examples.

0DZH **Lemma 16.1.** *In Example 8.1 let P be a local complete intersection over k (Algebra, Definition 133.1). Then $\mathcal{D}ef_P$ is unobstructed.*

Proof. Let $(A, Q) \to (k, P)$ be an object of $\mathcal{D}ef_P$. Then we see that $A \to Q$ is a syntomic ring map by Algebra, Definition 134.1. Hence for any surjection $A' \to A$ in $\mathcal{C}_\Lambda$ we see that there is a morphism $(A', Q') \to (A, Q)$ lifting $A' \to A$ by Smoothing Ring Maps, Proposition 3.2. This proves the lemma. □

0DZN **Lemma 16.2.** *In Situation 9.9 if $U_{12} \to \mathrm{Spec}(k)$ is smooth, then the morphism*

$$\mathcal{D}ef_X \longrightarrow \mathcal{D}ef_{U_1} \times \mathcal{D}ef_{U_2} = \mathcal{D}ef_{P_1} \times \mathcal{D}ef_{P_2}$$

is smooth. If in addition U_1 is a local complete intersection over k, then

$$\mathcal{D}ef_X \longrightarrow \mathcal{D}ef_{U_2} = \mathcal{D}ef_{P_2}$$

is smooth.

Proof. The equality signs hold by Lemma 9.7. Let us think of $\mathcal{C}_\Lambda$ as a deformation category over $\mathcal{C}_\Lambda$ as in Formal Deformation Theory, Section 9. Then

$$\mathcal{D}ef_{P_1} \times \mathcal{D}ef_{P_2} = \mathcal{D}ef_{P_1} \times_{\mathcal{C}_\Lambda} \mathcal{D}ef_{P_2},$$

see Formal Deformation Theory, Remarks 5.2 (14). Using Lemma 9.10 the first statement is that the functor

$$\mathcal{D}ef_{P_1} \times_{\mathcal{D}ef_{P_{12}}} \mathcal{D}ef_{P_2} \longrightarrow \mathcal{D}ef_{P_1} \times_{\mathcal{C}_\Lambda} \mathcal{D}ef_{P_2}$$

is smooth. This follows from Formal Deformation Theory, Lemma 20.2 as long as we can show that $T\mathcal{D}ef_{P_{12}} = (0)$. This vanishing follows from Lemma 8.4 as P_{12} is smooth over k. For the second statement it suffices to show that $\mathcal{D}ef_{P_1} \to \mathcal{C}_\Lambda$ is smooth, see Formal Deformation Theory, Lemma 8.7. In other words, we have to show $\mathcal{D}ef_{P_1}$ is unobstructed, which is Lemma 16.1. □

0DZP **Lemma 16.3.** *In Example 9.1 let X be a scheme over k. Assume*

(1) *X is separated, finite type over k and $\dim(X) \leq 1$,*
(2) *$X \to \mathrm{Spec}(k)$ is smooth except at the closed points $p_1, \ldots, p_n \in X$.*

Let $\mathcal{O}_{X,p_1}$, $\mathcal{O}^h_{X,p_1}$, $\mathcal{O}^\wedge_{X,p_1}$ be the local ring, henselization, completion. Consider the maps of deformation categories

$$\mathcal{D}ef_X \longrightarrow \prod \mathcal{D}ef_{\mathcal{O}_{X,p_i}} \longrightarrow \prod \mathcal{D}ef_{\mathcal{O}^h_{X,p_i}} \longrightarrow \prod \mathcal{D}ef_{\mathcal{O}^\wedge_{X,p_i}}$$

The first arrow is smooth and the second and third arrows are smooth and induce isomorphisms on tangent spaces.

Proof. Choose an affine open $U_2 \subset X$ containing $p_1, \ldots, p_n$ and the generic point of every irreducible component of X. This is possible by Varieties, Lemma 42.3 and Properties, Lemma 29.5. Then $X \setminus U_2$ is finite and we can choose an affine open $U_1 \subset X \setminus \{p_1, \ldots, p_n\}$ such that $X = U_1 \cup U_2$. Set $U_{12} = U_1 \cap U_2$. Then U_1 and U_{12} are smooth affine schemes over k. We conclude that

$$\mathcal{D}ef_X \longrightarrow \mathcal{D}ef_{U_2}$$

is smooth by Lemma 16.2. Applying Lemmas 9.7 and 15.1 we win. □

0DZQ **Lemma 16.4.** *In Example 9.1 let X be a scheme over k. Assume*

(1) *X is separated, finite type over k and $\dim(X) \leq 1$,*
(2) *X is a local complete intersection over k, and*
(3) *$X \to \mathrm{Spec}(k)$ is smooth except at finitely many points.*

Then $\mathcal{D}ef_X$ is unobstructed.

Proof. Let $p_1, \ldots, p_n \in X$ be the points where $X \to \mathrm{Spec}(k)$ isn't smooth. Choose an affine open $U_2 \subset X$ containing $p_1, \ldots, p_n$ and the generic point of every irreducible component of X. This is possible by Varieties, Lemma 42.3 and Properties, Lemma 29.5. Then $X \setminus U_2$ is finite and we can choose an affine open

$U_1 \subset X \setminus \{p_1, \ldots, p_n\}$ such that $X = U_1 \cup U_2$. Set $U_{12} = U_1 \cap U_2$. Then U_1 and U_{12} are smooth affine schemes over k. We conclude that

$$\mathcal{D}ef_X \longrightarrow \mathcal{D}ef_{U_2}$$

is smooth by Lemma 16.2. Applying Lemmas 9.7 and 16.1 we win. □

17. Smoothings

0E7S Suppose given a finite type scheme or algebraic space X over a field k. It is often useful to find a flat morphism of finite type $Y \to \mathrm{Spec}(k[[t]])$ whose generic fibre is smooth and whose special fibre is isomorphic to X. Such a thing is called a smoothing of X. In this section we will find a smoothing for 1-dimensional separated X which have isolated local complete intersection singularities.

0E7T **Lemma 17.1.** *Let k be a field. Set $S = \mathrm{Spec}(k[[t]])$ and $S_n = \mathrm{Spec}(k[t]/(t^n))$. Let $Y \to S$ be a proper, flat morphism of schemes whose special fibre X is Cohen-Macaulay and equidimensional of dimension d. Denote $X_n = Y \times_S S_n$. If for some $n \geq 1$ the dthe Fitting ideal of Ω_{X_n/S_n} contains t^{n-1}, then the generic fibre of $Y \to S$ is smooth.*

Proof. By More on Morphisms, Lemma 20.7 we see that $Y \to S$ is a Cohen-Macaulay morphism. By Morphisms, Lemma 28.4 we see that $Y \to S$ has relative dimension d. By Divisors, Lemma 10.3 the dth Fitting ideal $\mathcal{I} \subset \mathcal{O}_Y$ of $\Omega_{Y/S}$ cuts out the singular locus of the morphism $Y \to S$. In other words, $V(\mathcal{I}) \subset Y$ is the closed subset of points where $Y \to S$ is not smooth. By Divisors, Lemma 10.1 formation of this Fitting ideal commutes with base change. By assumption we see that t^{n-1} is a section of $\mathcal{I} + t^n\mathcal{O}_Y$. Thus for every $x \in X = V(t) \subset Y$ we conclude that $t^{n-1} \in \mathcal{I}_x$ where $\mathcal{I}_x$ is the stalk at x. This implies that $V(\mathcal{I}) \subset V(t)$ in an open neighbourhood of X in Y. Since $Y \to S$ is proper, this implies $V(\mathcal{I}) \subset V(t)$ as desired. □

0E7U **Lemma 17.2.** *Let k be a field. Let $1 \leq c \leq n$ be integers. Let $f_1, \ldots, f_c \in k[x_1, \ldots x_n]$ be elements. Let a_{ij}, $0 \leq i \leq n$, $1 \leq j \leq c$ be variables. Consider*

$$g_j = f_j + a_{0j} + a_{1j}x_1 + \ldots + a_{nj}x_n \in k[a_{ij}][x_1, \ldots, x_n]$$

Denote $Y \subset \mathbf{A}_k^{n+c(n+1)}$ the closed subscheme cut out by $g_1, \ldots, g_c$. Denote $\pi : Y \to \mathbf{A}_k^{c(n+1)}$ the projection onto the affine space with variables a_{ij}. Then there is a nonempty Zariski open of $\mathbf{A}_k^{c(n+1)}$ over which π is smooth.

Proof. Recall that the set of points where π is smooth is open. Thus the complement, i.e., the singular locus, is closed. By Chevalley's theorem (in the form of Morphisms, Lemma 21.2) the image of the singular locus is constructible. Hence if the generic point of $\mathbf{A}_k^{c(n+1)}$ is not in the image of the singular locus, then the lemma follows (by Topology, Lemma 15.14 for example). Thus we have to show there is no point $y \in Y$ where π is not smooth mapping to the generic point of $\mathbf{A}_k^{c(n+1)}$. Consider the matrix of partial derivatives

$$(\frac{\partial g_j}{\partial x_i}) = (\frac{\partial f_j}{\partial x_i} + a_{ij})$$

The image of this matrix in $\kappa(y)$ must have rank $< c$ since otherwise π would be smooth at y, see discussion in Smoothing Ring Maps, Section 2. Thus we can find

$\lambda_1, \ldots, \lambda_c \in \kappa(y)$ not all zero such that the vector $(\lambda_1, \ldots, \lambda_c)$ is in the kernel of this matrix. After renumbering we may assume $\lambda_1 \neq 0$. Dividing by λ_1 we may assume our vector has the form $(1, \lambda_2, \ldots, \lambda_c)$. Then we obtain

$$a_{i1} = -\frac{\partial f_j}{\partial x_1} - \sum\nolimits_{j=2,\ldots,c} \lambda_j (\frac{\partial f_j}{\partial x_i} + a_{ij})$$

in $\kappa(y)$ for $i = 1, \ldots, n$. Moreover, since $y \in Y$ we also have

$$a_{0j} = -f_j - a_{1j}x_1 - \ldots - a_{nj}x_n$$

in $\kappa(y)$. This means that the subfield of $\kappa(y)$ generated by a_{ij} is contained in the subfield of $\kappa(y)$ generated by the images of $x_1, \ldots, x_n, \lambda_2, \ldots, \lambda_c$, and a_{ij} except for a_{i1} and a_{0j}. We count and we see that the transcendence degree of this is at most $c(n+1) - 1$. Hence y cannot map to the generic point as desired. □

0E7V **Lemma 17.3.** *Let k be a field. Let A be a global complete interesection over k. There exists a flat finite type ring map $k[[t]] \to B$ with $B/tB \cong A$ such that $B[1/t]$ is smooth over $k((t))$.*

Proof. Write $A = k[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ as in Algebra, Definition 133.1. We are going to choose $a_{ij} \in (t) \subset k[[t]]$ and set

$$g_j = f_j + a_{0j} + a_{1j}x_1 + \ldots + a_{nj}x_n \in k[[t]][x_1, \ldots, x_n]$$

After doing this we take $B = k[[t]][x_1, \ldots, x_n]/(g_1, \ldots, g_c)$. We claim that $k[[t]] \to B$ is flat at every prime ideal lying over (t). Namely, the elements $f_1, \ldots, f_c$ form a regular sequence in the local ring at any prime ideal $\mathfrak{p}$ of $k[x_1, \ldots, x_n]$ containing $f_1, \ldots, f_c$ (Algebra, Lemma 133.4). Thus $g_1, \ldots, g_c$ is locally a lift of a regular sequence and we can apply Algebra, Lemma 98.3. Flatness at primes lying over $(0) \subset k[[t]]$ is automatic because $k((t)) = k[[t]]_{(0)}$ is a field. Thus B is flat over $k[[t]]$.

All that remains is to show that for suitable choices of a_{ij} the generic fibre $B_{(0)}$ is smooth over $k((t))$. For this we have to show that we can choose our a_{ij} so that the induced morphism

$$(a_{ij}) : \mathrm{Spec}(k[[t]]) \longrightarrow \mathbf{A}_k^{c(n+1)}$$

maps into the nonempty Zariski open of Lemma 17.2. This is clear because there is no nonzero polynomial in the a_{ij} which vanishes on $(t)^{\oplus c(n+1)}$. (We leave this as an exercise to the reader.) □

0E7W **Lemma 17.4.** *Let k be a field. Let A be a finite dimensional k-algebra which is a local complete intersection over k. Then there is a finite flat $k[[t]]$-algebra B with $B/tB \cong A$ and $B[1/t]$ étale over $k((t))$.*

Proof. Since A is Artinian (Algebra, Lemma 52.2), we can write A as a product of local Artinian rings (Algebra, Lemma 52.6). Thus it suffices to prove the lemma if A is local (this uses that being a local complete intersection is preserved under taking principal localizations, see Algebra, Lemma 133.2). In this case A is a global complete intersection. Consider the algebra B constructed in Lemma 17.3. Then $k[[t]] \to B$ is quasi-finite at the unique prime of B lying over (t) (Algebra, Definition 121.3). Observe that $k[[t]]$ is a henselian local ring (Algebra, Lemma 148.9). Thus $B = B' \times C$ where B' is finite over $k[[t]]$ and C has no prime lying over (t), see Algebra, Lemma 148.3. Then B' is the ring we are looking for (recall that étale is the same thing as smooth of relative dimension 0). □

0E7X **Lemma 17.5.** *Let k be a field. Let A be a k-algebra. Assume*

(1) *A is a local ring essentially of finite type over k,*
(2) *A is a complete intersection over k (Algebra, Definition 133.5).*

Set $d = \dim(A) + trdeg_k(\kappa)$ where κ is the residue field of A. Then there exists an integer n and a flat, essentially of finite type ring map $k[[t]] \to B$ with $B/tB \cong A$ such that t^n is in the dth Fitting ideal of $\Omega_{B/k[[t]]}$.

Proof. By Algebra, Lemma 133.7 we can write A as the localization at a prime $\mathfrak{p}$ of a global complete intersection P over k. Observe that $\dim(P) = d$ by Algebra, Lemma 115.3. By Lemma 17.3 we can find a flat, finite type ring map $k[[t]] \to Q$ such that $P \cong Q/tQ$ and such that $k((t)) \to Q[1/t]$ is smooth. It follows from the construction of Q in the lemma that $k[[t]] \to Q$ is a relative global complete intersection of relative dimension d; alternatively, Algebra, Lemma 134.15 tells us that Q or a suitable principal localization of Q is such a global complete intersection. Hence by Divisors, Lemma 10.3 the dth Fitting ideal $I \subset Q$ of $\Omega_{Q/k[[t]]}$ cuts out the singular locus of $\mathrm{Spec}(Q) \to \mathrm{Spec}(k[[t]])$. Thus $t^n \in I$ for some n. Let $\mathfrak{q} \subset Q$ be the inverse image of $\mathfrak{p}$. Set $B = Q_{\mathfrak{q}}$. The lemma is proved. □

0E7Y **Lemma 17.6.** *Let X be a scheme over a field k. Assume*

(1) *X is proper over k,*
(2) *X is a local complete intersection over k,*
(3) *X has dimension ≤ 1, and*
(4) *$X \to \mathrm{Spec}(k)$ is smooth except at finitely many points.*

Then there exists a flat projective morphism $Y \to \mathrm{Spec}(k[[t]])$ whose generic fibre is smooth and whose special fibre is isomorphic to X.

Proof. Observe that X is Cohen-Macaulay, see Algebra, Lemma 133.3. Thus $X = X' \amalg X''$ with $\dim(X') = 0$ and X'' equidimensional of dimension 1, see Morphisms, Lemma 28.4. Since X' is finite over k (Varieties, Lemma 20.2) we can find $Y' \to \mathrm{Spec}(k[[t]])$ with special fibre X' and generic fibre smooth by Lemma 17.4. Thus it suffices to prove the lemma for X''. After replacing X by X'' we have X is Cohen-Macaulay and equidimensional of dimension 1.

We are going to use deformation theory for the situation $\Lambda = k \to k$. Let $p_1, \ldots, p_r \in X$ be the closed singular points of X, i.e., the points where $X \to \mathrm{Spec}(k)$ isn't smooth. For each i we pick an integer n_i and a flat, essentially of finite type ring map

$$k[[t]] \longrightarrow B_i$$

with $B_i/tB_i \cong \mathcal{O}_{X,p_i}$ such that t^{n_i} is in the 1st Fitting ideal of $\Omega_{B_i/k[[t]]}$. This is possible by Lemma 17.5. Observe that the system (B_i/t^nB_i) defines a formal object of $\mathcal{D}ef_{\mathcal{O}_{X,p_i}}$ over $k[[t]]$. By Lemma 16.3 the map

$$\mathcal{D}ef_X \longrightarrow \prod\nolimits_{i=1,\ldots,r} \mathcal{D}ef_{\mathcal{O}_{X,p_i}}$$

is a smooth map between deformation categories. Hence by Formal Deformation Theory, Lemma 8.8 there exists a formal object (X_n) in $\mathcal{D}ef_X$ mapping to the formal object $\prod_i (B_i/t^n)$ by the arrow above. By More on Morphisms of Spaces, Lemma 42.4 there exists a projective scheme Y over $k[[t]]$ and compatible isomorphisms $Y \times_{\mathrm{Spec}(k[[t]])} \mathrm{Spec}(k[t]/(t^n)) \cong X_n$. By More on Morphisms, Lemma 12.4 we see that $Y \to \mathrm{Spec}(k[[t]])$ is flat. Since X is Cohen-Macaulay and equidimensional

of dimension 1 we may apply Lemma 17.1 to check Y has smooth generic fibre[3]. Choose n strictly larger than the maximum of the integers n_i found above. It we can show t^{n-1} is in the first Fitting ideal of Ω_{X_n/S_n} with $S_n = \mathrm{Spec}(k[t]/(t^n))$, then the proof is done. To do this it suffices to prove this is true in each of the local rings of X_n at closed points p. However, if p corresponds to a smooth point for $X \to \mathrm{Spec}(k)$, then $\Omega_{X_n/S_n,p}$ is free of rank 1 and the first Fitting ideal is equal to the local ring. If $p = p_i$ for some i, then

$$\Omega_{X_n/S_n,p_i} = \Omega_{(B_i/t^nB_i)/(k[t]/(t^n))} = \Omega_{B_i/k[[t]]}/t^n\Omega_{B_i/k[[t]]}$$

Since taking Fitting ideals commutes with base change (with already used this but in this algebraic setting it follows from More on Algebra, Lemma 8.4), and since $n - 1 \geq n_i$ we see that t^{n-1} is in the Fitting ideal of this module over B_i/t^nB_i as desired. □

0E7Z **Lemma 17.7.** *Let k be a field and let X be a scheme over k. Assume*

(1) *X is separated, finite type over k and $\dim(X) \leq 1$,*
(2) *X is a local complete intersection over k, and*
(3) *$X \to \mathrm{Spec}(k)$ is smooth except at finitely many points.*

Then there exists a flat, separated, finite type morphism $Y \to \mathrm{Spec}(k[[t]])$ whose generic fibre is smooth and whose special fibre is isomorphic to X.

Proof. If X is reduced, then we can choose an embedding $X \subset \overline{X}$ as in Varieties, Lemma 42.6. Writing $X = \overline{X} \setminus \{x_1, \ldots, x_n\}$ we see that $\mathcal{O}_{\overline{X},x_i}$ is a discrete valuation ring and hence in particular a local complete intersection (Algebra, Definition 133.5). Thus $\overline{X}$ is a local complete intersection over k because this holds over the open X and at the points x_i by Algebra, Lemma 133.7. Thus we may apply Lemma 17.6 to find a projective flat morphism $\overline{Y} \to \mathrm{Spec}(k[[t]])$ whose generic fibre is smooth and whose special fibre is $\overline{X}$. Then we remove $x_1, \ldots, x_n$ from $\overline{Y}$ to obtain Y.

In the general case, write $X = X' \amalg X''$ where with $\dim(X') = 0$ and X'' equidimensional of dimension 1. Then X'' is reduced and the first paragraph applies to it. On the other hand, X' can be dealt with as in the proof of Lemma 17.6. Some details omitted. □

18. Other chapters

Preliminaries

(1) Introduction
(2) Conventions
(3) Set Theory
(4) Categories
(5) Topology
(6) Sheaves on Spaces
(7) Sites and Sheaves
(8) Stacks
(9) Fields
(10) Commutative Algebra
(11) Brauer Groups
(12) Homological Algebra
(13) Derived Categories
(14) Simplicial Methods
(15) More on Algebra
(16) Smoothing Ring Maps
(17) Sheaves of Modules
(18) Modules on Sites
(19) Injectives
(20) Cohomology of Sheaves
(21) Cohomology on Sites

[3]Warning: in general it is **not** true that the local ring of Y at the point p_i is isomorphic to B_i. We only know that this is true after dividing by t^n on both sides!

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