EXAMPLES

026Z

Contents

1. Introduction 2
2. An empty limit 2
3. A zero limit 3
4. Non-quasi-compact inverse limit of quasi-compact spaces 3
5. A nonintegral connected scheme whose local rings are domains 4
6. Noncomplete completion 4
7. Noncomplete quotient 6
8. Completion is not exact 7
9. The category of complete modules is not abelian 7
10. The category of derived complete modules 8
11. Nonflat completions 9
12. Nonabelian category of quasi-coherent modules 11
13. Regular sequences and base change 11
14. A Noetherian ring of infinite dimension 13
15. Local rings with nonreduced completion 13
16. A non catenary Noetherian local ring 14
17. Existence of bad local Noetherian rings 15
18. Non-quasi-affine variety with quasi-affine normalization 16
19. A locally closed subscheme which is not open in closed 18
20. Nonexistence of suitable opens 18
21. Nonexistence of quasi-compact dense open subscheme 19
22. Affines over algebraic spaces 20
23. Pushforward of quasi-coherent modules 20
24. A nonfinite module with finite free rank 1 stalks 21
25. A noninvertible ideal invertible in stalks 21
26. A finite flat module which is not projective 22
27. A projective module which is not locally free 22
28. Zero dimensional local ring with nonzero flat ideal 24
29. An epimorphism of zero-dimensional rings which is not surjective 25
30. Finite type, not finitely presented, flat at prime 25
31. Finite type, flat and not of finite presentation 26
32. Topology of a finite type ring map 27
33. Pure not universally pure 28
34. A formally smooth non-flat ring map 28
35. A formally étale non-flat ring map 29
36. A formally étale ring map with nontrivial cotangent complex 30
37. Ideals generated by sets of idempotents and localization 30
38. A ring map which identifies local rings which is not ind-étale 31

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39. Non flasque quasi-coherent sheaf associated to injective module 32
40. A non-separated flat group scheme 33
41. A non-flat group scheme with flat identity component 33
42. A non-separated group algebraic space over a field 34
43. Specializations between points in fibre étale morphism 34
44. A torsor which is not an fppf torsor 35
45. Stack with quasi-compact flat covering which is not algebraic 36
46. Limit preserving on objects, not limit preserving 37
47. A non-algebraic classifying stack 37
48. Sheaf with quasi-compact flat covering which is not algebraic 38
49. Sheaves and specializations 39
50. Sheaves and constructible functions 40
51. The lisse-étale site is not functorial 41
52. Derived pushforward of quasi-coherent modules 42
53. A big abelian category 44
54. Weakly associated points and scheme theoretic density 44
55. Example of non-additivity of traces 45
56. Being projective is not local on the base 46
57. Descent data for schemes need not be effective, even for a projective morphism 47
58. A family of curves whose total space is not a scheme 49
59. Derived base change 50
60. An interesting compact object 51
61. Two differential graded categories 52
62. The stack of proper algebraic spaces is not algebraic 53
63. An example of a non-algebraic Hom-stack 54
64. An algebraic stack not satisfying strong formal effectiveness 57
65. A counter example to Grothendieck's existence theorem 57
66. Affine formal algebraic spaces 58
67. Flat maps are not directed limits of finitely presented flat maps 59
68. The category of modules modulo torsion modules 61
69. Different colimit topologies 62
70. Other chapters 62
References 64

1. Introduction

0270 This chapter will contain examples which illuminate the theory.

2. An empty limit

0AKK This example is due to Waterhouse, see [Wat72]. Let S be an uncountable set. For every finite subset $T \subset S$ consider the set M_T of injective maps $T \to \mathbf{N}$. For $T \subset T' \subset S$ finite the restriction $M_{T'} \to M_T$ is surjective. Thus we have an inverse system over the directed partially ordered set of finite subsets of S with surjective transition maps. But $\lim M_T = \emptyset$ as an element in the limit would define an injective map $S \to \mathbf{N}$.

3. A zero limit

0ANX Let $(S_i)_{i \in I}$ be a directed inverse system of nonempty sets with surjective transition maps and with $\lim S_i = \emptyset$, see Section 2. Let K be a field and set

$$V_i = \bigoplus\nolimits_{s \in S_i} K$$

Then the transition maps $V_i \to V_j$ are surjective for $i \geq j$. However, $\lim V_i = 0$. Namely, if $v = (v_i)$ is an element of the limit, then the support of v_i would be a finite subset $T_i \subset S_i$ with $\lim T_i \neq \emptyset$, see Categories, Lemma 21.7.

For each i consider the unique K-linear map $V_i \to K$ which sends each basis vector $s \in S_i$ to 1. Let $W_i \subset V_i$ be the kernel. Then

$$0 \to (W_i) \to (V_i) \to (K) \to 0$$

is a nonsplit short exact sequence of inverse systems of vector spaces over the directed set I. Hence W_i is a directed system of K-vector spaces with surjective transition maps, vanishing limit, and nonvanishing $R^1 \lim$.

4. Non-quasi-compact inverse limit of quasi-compact spaces

09ZJ Let $\mathbf{N}$ denote the set of natural numbers. For every integer n, let I_n denote the set of all natural numbers $> n$. Define T_n to be the unique topology on $\mathbf{N}$ with basis $\{1\}, \ldots, \{n\}, I_n$. Denote by X_n the topological space $(\mathbf{N}, T_n)$. For each $m < n$, the identity map,

$$f_{n,m} : X_n \longrightarrow X_m$$

is continuous. Obviously for $m < n < p$, the composition $f_{p,n} \circ f_{n,m}$ equals $f_{p,m}$. So $((X_n), (f_{n,m}))$ is a directed inverse system of quasi-compact topological spaces.

Let T be the discrete topology on $\mathbf{N}$, and let X be $(\mathbf{N}, T)$. Then for every integer n, the identity map,

$$f_n : X \longrightarrow X_n$$

is continuous. We claim that this is the inverse limit of the directed system above. Let (Y, S) be any topological space. For every integer n, let

$$g_n : (Y, S) \longrightarrow (\mathbf{N}, T_n)$$

be a continuous map. Assume that for every $m < n$ we have $f_{n,m} \circ g_n = g_m$, i.e., the system (g_n) is compatible with the directed system above. In particular, all of the set maps g_n are equal to a common set map

$$g : Y \longrightarrow \mathbf{N}.$$

Moreover, for every integer n, since $\{n\}$ is open in X_n, also $g^{-1}(\{n\}) = g_n^{-1}(\{n\})$ is open in Y. Therefore the set map g is continuous for the topology S on Y and the topology T on $\mathbf{N}$. Thus $(X, (f_n))$ is the inverse limit of the directed system above.

However, clearly X is not quasi-compact, since the infinite open covering by singleton sets has no inverse limit.

09ZK **Lemma 4.1.** *There exists an inverse system of quasi-compact topological spaces over $\mathbf{N}$ whose limit is not quasi-compact.*

Proof. See discussion above. ☐

5. A nonintegral connected scheme whose local rings are domains

0568 We give an example of an affine scheme $X = \mathrm{Spec}(A)$ which is connected, all of whose local rings are domains, but which is not integral. Connectedness of X means A has no nontrivial idempotents, see Algebra, Lemma 20.3. The local rings of X are domains if, whenever $fg = 0$ in A, every point of X has a neighborhood where either f or g vanishes. As long as A is not a domain, then X is not integral (Properties, Definition 3.1).

Roughly speaking, the construction is as follows: let X_0 be the cross (the union of coordinate axes) on the affine plane. Then let X_1 be the (reduced) full preimage of X_0 on the blowup of the plane (X_1 has three rational components forming a chain). Then blow up the resulting surface at the two singularities of X_1, and let X_2 be the reduced preimage of X_1 (which has five rational components), etc. Take X to be the inverse limit. The only problem with this construction is that blowups glue in a projective line, so X_1 is not affine. Let us correct this by glueing in an affine line instead (so our scheme will be an open subset in what was described above).

Here is a completely algebraic construction: For every $k \geq 0$, let A_k be the following ring: its elements are collections of polynomials $p_i \in \mathbf{C}[x]$ where $i = 0, \ldots, 2^k$ such that $p_i(1) = p_{i+1}(0)$. Set $X_k = \mathrm{Spec}(A_k)$. Observe that X_k is a union of 2^k+1 affine lines that meet transversally in a chain. Define a ring homomorphism $A_k \to A_{k+1}$ by

$$(p_0, \ldots, p_{2^k}) \longmapsto (p_0, p_0(1), p_1, p_1(1), \ldots, p_{2^k}),$$

in other words, every other polynomial is constant. This identifies A_k with a subring of A_{k+1}. Let A be the direct limit of A_k (basically, their union). Set $X = \mathrm{Spec}(A)$. For every k, we have a natural embedding $A_k \to A$, that is, a map $X \to X_k$. Each A_k is connected but not integral; this implies that A is connected but not integral. It remains to show that the local rings of A are domains.

Take $f, g \in A$ with $fg = 0$ and $x \in X$. Let us construct a neighborhood of x on which one of f and g vanishes. Choose k such that $f, g \in A_{k-1}$ (note the $k-1$ index). Let y be the image of x in X_k. It suffices to prove that y has a neighborhood on which either f or g viewed as sections of $\mathcal{O}_{X_k}$ vanishes. If y is a smooth point of X_k, that is, it lies on only one of the $2^k + 1$ lines, this is obvious. We can therefore assume that y is one of the 2^k singular points, so two components of X_k pass through y. However, on one of these two components (the one with odd index), both f and g are constant, since they are pullbacks of functions on X_{k-1}. Since $fg = 0$ everywhere, either f or g (say, f) vanishes on the other component. This implies that f vanishes on both components, as required.

6. Noncomplete completion

05JA Let R be a ring and let $\mathfrak{m}$ be a maximal ideal. Consider the completion

$$R^\wedge = \lim R/\mathfrak{m}^n.$$

Note that $R^\wedge$ is a local ring with maximal ideal $\mathfrak{m}' = \mathrm{Ker}(R^\wedge \to R/\mathfrak{m})$. Namely, if $x = (x_n) \in R^\wedge$ is not in $\mathfrak{m}'$, then $y = (x_n^{-1}) \in R^\wedge$ satisfies $xy = 1$, whence $R^\wedge$ is local by Algebra, Lemma 17.2. Now it is always true that $R^\wedge$ complete in its limit topology (see the discussion in More on Algebra, Section 33). But beyond that, we have the following questions:

(1) Is it true that $\mathfrak{m}R^\wedge = \mathfrak{m}'$?

(2) Is $R^\wedge$ viewed as an $R^\wedge$-module $\mathfrak m'$-adically complete?
(3) Is $R^\wedge$ viewed as an R-module $\mathfrak m$-adically complete?

It turns out that these questions all have a negative answer. The example below was taken from an unpublished note of Bart de Smit and Hendrik Lenstra. See also [Bou61, Exercise III.2.12] and [Yek11, Example 1.8]

Let k be a field, $R = k[x_1, x_2, x_3, \ldots]$, and $\mathfrak m = (x_1, x_2, x_3, \ldots)$. We will think of an element f of $R^\wedge$ as a (possibly) infinite sum

$$f = \sum a_I x^I$$

(using multi-index notation) such that for each $d \geq 0$ there are only finitely many nonzero a_I for $|I| = d$. The maximal ideal $\mathfrak m' \subset R^\wedge$ is the collection of f with zero constant term. In particular, the element

$$f = x_1 + x_2^2 + x_3^3 + \ldots$$

is in $\mathfrak m'$ but not in $\mathfrak m R^\wedge$ which shows that (1) is false in this example. However, if (1) is false, then (3) is necessarily false because $\mathfrak m' = \mathrm{Ker}(R^\wedge \to R/\mathfrak m)$ and we can apply Algebra, Lemma 95.5 with $n = 1$.

To finish we prove that $R^\wedge$ is not $\mathfrak m'$-adically complete. For $n \geq 1$ let $K_n = \mathrm{Ker}(R^\wedge \to R/\mathfrak m^n)$. Then we have short exact sequences

$$0 \to K_n/(\mathfrak m')^n \to R^\wedge/(\mathfrak m')^n \to R/\mathfrak m^n \to 0$$

The projection map $R^\wedge \to R/\mathfrak m^{n+1}$ sends $(\mathfrak m')^n$ onto $\mathfrak m^n/\mathfrak m^{n+1}$. It follows that $K_{n+1} \to K_n/(\mathfrak m')^n$ is surjective. Hence the inverse system $(K_n/(\mathfrak m')^n)$ has surjective transition maps and taking inverse limits we obtain an exact sequence

$$0 \to \lim K_n/(\mathfrak m')^n \to \lim R^\wedge/(\mathfrak m')^n \to \lim R/\mathfrak m^n \to 0$$

by Algebra, Lemma 86.1. Thus we see that $R^\wedge$ is complete with respect to $\mathfrak m'$ if and only if $K_n = (\mathfrak m')^n$ for all $n \geq 1$.

To show that $R^\wedge$ is not $\mathfrak m'$-adically complete in our example we show that $K_2 = \mathrm{Ker}(R^\wedge \to R/\mathfrak m^2)$ is not equal to $(\mathfrak m')^2$. Note that an element of $(\mathfrak m')^2$ can be written as a finite sum

05JB (6.0.1)
$$\sum\nolimits_{i=1,\ldots,t} f_i g_i$$

with $f_i, g_i \in R^\wedge$ having vanishing constant terms. To get an example we are going to choose an $z \in K_2$ of the form

$$z = z_1 + z_2 + z_3 + \ldots$$

with the following properties

(1) there exist sequences $1 < d_1 < d_2 < d_3 < \ldots$ and $0 < n_1 < n_2 < n_3 < \ldots$ such that $z_i \in k[x_{n_i}, x_{n_i+1}, \ldots, x_{n_{i+1}-1}]$ homogeneous of degree d_i, and
(2) in the ring $k[[x_{n_i}, x_{n_i+1}, \ldots, x_{n_{i+1}-1}]]$ the element z_i cannot be written as a sum (6.0.1) with $t \leq i$.

Clearly this implies that z is not in $(\mathfrak m')^2$ because the image of the relation (6.0.1) in the ring $k[[x_{n_i}, x_{n_i+1}, \ldots, x_{n_{i+1}-1}]]$ for i large enough would produce a contradiction. Hence it suffices to prove that for all $t > 0$ there exists a $d \gg 0$ and an integer n such that we can find an homogeneous element $z \in k[x_1, \ldots, x_n]$ of degree d which cannot be written as a sum (6.0.1) for the given t in $k[[x_1, \ldots, x_n]]$. Take

$n > 2t$ and any $d > 1$ prime to the characteristic of p and set $z = \sum_{i=1,\ldots,n} x_i^d$. Then the vanishing locus of the ideal

$$(\frac{\partial z}{\partial x_1}, \ldots, \frac{\partial z}{\partial x_n}) = (dx_1^{d-1}, \ldots, dx_n^{d-1})$$

consists of one point. On the other hand,

$$\frac{\partial(\sum_{i=1,\ldots,t} f_i g_i)}{\partial x_j} \in (f_1, \ldots, f_t, g_1, \ldots, g_t)$$

by the Leibniz rule and hence the vanishing locus of these derivatives contains at least

$$V(f_1, \ldots, f_t, g_1, \ldots, g_t) \subset \mathrm{Spec}(k[[x_1, \ldots, x_n]]).$$

Hence this is a contradiction as the dimension of $V(f_1, \ldots, f_t, g_1, \ldots, g_t)$ is at least $n - 2t \geq 1$.

05JC **Lemma 6.1.** *There exists a local ring R and a maximal ideal $\mathfrak{m}$ such that the completion $R^\wedge$ of R with respect to $\mathfrak{m}$ has the following properties*

1. *$R^\wedge$ is local, but its maximal ideal is not equal to $\mathfrak{m}R^\wedge$,*
2. *$R^\wedge$ is not a complete local ring, and*
3. *$R^\wedge$ is not $\mathfrak{m}$-adically complete as an R-module.*

Proof. This follows from the discussion above as (with $R = k[x_1, x_2, x_3, \ldots]$) the completion of the localization $R_\mathfrak{m}$ is equal to the completion of R. □

7. Noncomplete quotient

05JD Let k be a field. Let

$$R = k[t, z_1, z_2, z_3, \ldots, w_1, w_2, w_3, \ldots, x]/(z_i t - x^i w_i, z_i w_j)$$

Note that in particular $z_i z_j t = 0$ in this ring. Any element f of R can be uniquely written as a finite sum

$$f = \sum_{i=0,\ldots,d} f_i x^i$$

where each $f_i \in k[t, z_i, w_j]$ has no terms involving the products $z_i t$ or $z_i w_j$. Moreover, if f is written in this way, then $f \in (x^n)$ if and only if $f_i = 0$ for $i < n$. So x is a nonzerodivisor and $\bigcap(x^n) = 0$. Let $R^\wedge$ be the completion of R with respect to the ideal (x). Note that $R^\wedge$ is (x)-adically complete, see Algebra, Lemma 95.3. By the above we see that an element of $R^\wedge$ can be uniquely written as an infinite sum

$$f = \sum_{i=0}^{\infty} f_i x^i$$

where each $f_i \in k[t, z_i, w_j]$ has no terms involving the products $z_i t$ or $z_i w_j$. Consider the element

$$f = \sum_{i=1}^{\infty} x^i w_i = x w_1 + x^2 w_2 + x^3 w_3 + \ldots$$

i.e., we have $f_n = w_n$. Note that $f \in (t, x^n)$ for every n because $x^m w_m \in (t)$ for all m. We claim that $f \notin (t)$. To prove this assume that $tg = f$ where $g = \sum g_l x^l$ in canonical form as above. Since $t z_i z_j = 0$ we may as well assume that none of the g_l have terms involving the products $z_i z_j$. Examining the process to get tg in canonical form we see the following: Given any term cm of g_l where $c \in k$ and m is a monomial in t, z_i, w_j and we make the following replacement

(1) if the monomial m does not involve any z_i, then ctm is a term of f_l, and

(2) if the monomial m does involve a z_i then it is equal to $m = z_i$ and we see that cw_i is term of f_{l+i}.

Since g_0 is a polynomial only finitely many of the variables z_i occur in it. Pick n such that z_n does not occur in g_0. Then the rules above show that w_n does not occur in f_n which is a contradiction. It follows that $R^\wedge/(t)$ is not complete, see Algebra, Lemma 95.10.

05JE **Lemma 7.1.** *There exists a ring R complete with respect to a principal ideal I and a principal ideal J such that R/J is not I-adically complete.*

Proof. See discussion above. □

8. Completion is not exact

05JF A quick example is the following. Suppose that $R = k[t]$. Let $P = K = \bigoplus_{n \in \mathbf{N}} R$ and $M = \bigoplus_{n \in \mathbf{N}} R/(t^n)$. Then there is a short exact sequence $0 \to K \to P \to M \to 0$ where the first map is given by multiplication by t^n on the nth summand. We claim that $0 \to K^\wedge \to P^\wedge \to M^\wedge \to 0$ is not exact in the middle. Namely, $\xi = (t^2, t^3, t^4, \ldots) \in P^\wedge$ maps to zero in $M^\wedge$ but is not in the image of $K^\wedge \to P^\wedge$, because it would be the image of $(t, t, t, \ldots)$ which is not an element of $K^\wedge$.

A "smaller" example is the following. In the situation of Lemma 7.1 the short exact sequence $0 \to J \to R \to R/J \to 0$ does not remain exact after completion. Namely, if $f \in J$ is a generator, then $f : R \to J$ is surjective, hence $R \to J^\wedge$ is surjective, hence the image of $J^\wedge \to R$ is $(f) = J$ but the fact that R/J is noncomplete means that the kernel of the surjection $R \to (R/J)^\wedge$ is strictly bigger than J, see Algebra, Lemmas 95.1 and 95.10. By the same token the sequence $R \to R \to R/(f) \to 0$ does not remain exact on completion.

05JG **Lemma 8.1.** *Completion is not an exact functor in general; it is not even right exact in general. This holds even when I is finitely generated on the category of finitely presented modules.*

Proof. See discussion above. □

9. The category of complete modules is not abelian

07JQ Let R be a ring and let $I \subset R$ be a finitely generated ideal. Consider the category $\mathcal{A}$ of I-adically complete R-modules, see Algebra, Definition 95.2. Let $\varphi : M \to N$ be a morphism of $\mathcal{A}$. The cokernel of φ in $\mathcal{A}$ is the completion $(\text{Coker}(\varphi))^\wedge$ of the usual cokernel (as I is finitely generated this completion is complete, see Algebra, Lemma 95.3). Let $K = \text{Ker}(\varphi)$. We claim that K is complete and hence is the kernel of φ in $\mathcal{A}$. Namely, let $K^\wedge$ be the completion. As M is complete we obtain a factorization

$$K \to K^\wedge \to M \xrightarrow{\varphi} N$$

Since φ is continuous for the I-adic topology, $K \to K^\wedge$ has dense image, and $K = \text{Ker}(\varphi)$ we conclude that $K^\wedge$ maps into K. Thus $K^\wedge = K \oplus C$ and K is a direct summand of a complete module, hence complete.

We will give an example that shows that $\text{Im} \neq \text{Coim}$ in general. We take $R = \mathbf{Z}_p = \lim_n \mathbf{Z}/p^n\mathbf{Z}$ to be the ring of p-adic integers and we take $I = (p)$. Consider the map

$$\text{diag}(1, p, p^2, \ldots) : \left(\bigoplus\nolimits_{n \geq 1} \mathbf{Z}_p\right)^\wedge \longrightarrow \prod\nolimits_{n \geq 1} \mathbf{Z}_p$$

where the left hand side is the p-adic completion of the direct sum. Hence an element of the left hand side is a vector $(x_1, x_2, x_3, \ldots)$ with $x_i \in \mathbf{Z}_p$ with p-adic valuation $v_p(x_i) \to \infty$ as $i \to \infty$. This maps to $(x_1, px_2, p^2x_3, \ldots)$. Hence we see that $(1, p, p^2, \ldots)$ is in the closure of the image but not in the image. By our description of kernels and cokernels above it is clear that $\text{Im} \neq \text{Coim}$ for this map.

07JR **Lemma 9.1.** *Let R be a ring and let $I \subset R$ be a finitely generated ideal. The category of I-adically complete R-modules has kernels and cokernels but is not abelian in general.*

Proof. See above. $\square$

10. The category of derived complete modules

0ARC Let A be a ring and let I be an ideal. Consider the category $\mathcal{C}$ of derived complete modules as defined in More on Algebra, Definition 80.4. By More on Algebra, Lemma 80.6 we see that $\mathcal{C}$ is abelian.

Let T be a set and let M_t, $t \in T$ be a family of derived complete modules. We claim that in general $\bigoplus M_t$ is not a complete module. For a specific example, let $A = \mathbf{Z}_p$ and $I = (p)$ and $\bigoplus_{n \in \mathbf{N}} \mathbf{Z}_p$. The map from $\bigoplus_{n \in \mathbf{N}} \mathbf{Z}_p$ to its p-adic completion isn't surjective. This means that $\bigoplus_{n \in \mathbf{N}} \mathbf{Z}_p$ cannot be derived complete as this would imply otherwise, see More on Algebra, Lemma 80.3.

Assume I is finitely generated. Let $^\wedge : D(A) \to D(A)$ denote the **derived completion** functor, see More on Algebra, Lemma 80.9. We claim that

$$M = H^0((\bigoplus M_t)^\wedge) \in \text{Ob}(\mathcal{C})$$

is a direct sum of M_t in the category $\mathcal{C}$. Note that for E a derived complete object of $D(A)$ we have

$$\text{Hom}_{D(A)}((\bigoplus M_t)^\wedge, E) = \text{Hom}_{D(A)}(\bigoplus M_t, E) = \prod \text{Hom}_{D(A)}(M_t, E)$$

Note that the right hand side is zero if $H^i(E) = 0$ for $i < 1$. In particular, applying this with $E = \tau_{\geq 1}(\bigoplus M_t)^\wedge$ which is derived complete by More on Algebra, Lemma 80.6 we see that the canonical map $(\bigoplus M_t)^\wedge \to \tau_{\geq 1}(\bigoplus M_t)^\wedge$ is zero, in other words, we have $H^i((\bigoplus M_t)^\wedge) = 0$ for $i \geq 1$. Then, for an object $N \in \mathcal{C}$ we see that

$$\begin{aligned} \text{Hom}_{\mathcal{C}}(M, N) &= \text{Hom}_{D(A)}((\bigoplus M_t)^\wedge, N) \\ &= \prod \text{Hom}_A(M_t, N) \\ &= \prod \text{Hom}_{\mathcal{C}}(M_t, N) \end{aligned}$$

as desired. This implies that $\mathcal{C}$ has all colimits, see Categories, Lemma 14.11. In fact, arguing similarly as above we see that given a system M_t in $\mathcal{C}$ over a preordered set T the colimit in $\mathcal{C}$ is equal to $H^0((\text{colim}\, M_t)^\wedge)$ where the inner colimit is the colimit in the category of A-modules.

However, we claim that filtered colimits are not exact in the category $\mathcal{C}$. Namely, suppose that $A = \mathbf{Z}_p$ and $I = (p)$. One has inclusions $f_n : \mathbf{Z}_p/p\mathbf{Z}_p \to \mathbf{Z}_p/p^n\mathbf{Z}_p$ of

p-adically complete A-modules given by multiplication by p^{n-1}. There are commutative diagrams

$$\begin{tikzcd} \mathbf{Z}_p/p\mathbf{Z}_p \arrow[r, "f_n"'] \arrow[d, "1"] & \mathbf{Z}_p/p^n\mathbf{Z}_p \arrow[d, "p"'] \\ \mathbf{Z}_p/p\mathbf{Z}_p \arrow[r, "f_{n+1}"] & \mathbf{Z}_p/p^{n+1}\mathbf{Z}_p \end{tikzcd}$$

Now take the colimit of these inclusions in the category $\mathcal{C}$ derived to get $\mathbf{Z}_p/p\mathbf{Z}_p \to 0$. Namely, the colimit in Mod_A of the system on the right is $\mathbf{Q}_p/\mathbf{Z}_p$. The reader can directly compute that $(\mathbf{Q}_p/\mathbf{Z}_p)^\wedge = \mathbf{Z}_p[1]$ in $D(A)$. Thus $H^0 = 0$ which proves our claim.

0ARD **Lemma 10.1.** *Let A be a ring and let $I \subset A$ be an ideal. The category $\mathcal{C}$ of derived complete modules is abelian and the inclusion functor $F : \mathcal{C} \to Mod_A$ is exact and commutes with arbitrary limits. If I is finitely generated, then $\mathcal{C}$ has arbitrary direct sums and colimits, but F does not commute with these in general. Finally, filtered colimits are not exact in $\mathcal{C}$ in general, hence $\mathcal{C}$ is not a Grothendieck abelian category.*

Proof. See discussion above. □

11. Nonflat completions

0AL8 The completion of a ring with respect to an ideal isn't always flat, contrary to the Noetherian case. We have seen two examples of this phenomenon in More on Algebra, Example 79.10. In this section we give two more examples.

0AL9 **Lemma 11.1.** *Let R be a ring. Let M be an R-module which is countable. Then M is a finite R-module if and only if $M \otimes_R R^{\mathbf{N}} \to M^{\mathbf{N}}$ is surjective.*

Proof. If M is a finite module, then the map is surjective by Algebra, Proposition 88.2. Conversely, assume the map is surjective. Let $m_1, m_2, m_3, \ldots$ be an enumeration of the elements of M. Let $\sum_{j=1,\ldots,m} x_j \otimes a_j$ be an element of the tensor product mapping to the element $(m_n) \in M^{\mathbf{N}}$. Then we see that $x_1, \ldots, x_m$ generate M over R as in the proof of Algebra, Proposition 88.2. □

0ALA **Lemma 11.2.** *Let R be a countable ring. Let M be a countable R-module. Then M is finitely presented if and only if the canonical map $M \otimes_R R^{\mathbf{N}} \to M^{\mathbf{N}}$ is an isomorphism.*

Proof. If M is a finitely presented module, then the map is an isomorphism by Algebra, Proposition 88.3. Conversely, assume the map is an isomorphism. By Lemma 11.1 the module M is finite. Choose a surjection $R^{\oplus m} \to M$ with kernel K. Then K is countable as a submodule of $R^{\oplus m}$. Arguing as in the proof of Algebra, Proposition 88.3 we see that $K \otimes_R R^{\mathbf{N}} \to K^{\mathbf{N}}$ is surjective. Hence we conclude that K is a finite R-module by Lemma 11.1. Thus M is finitely presented. □

0ALB **Lemma 11.3.** *Let R be a countable ring. Then R is coherent if and only if $R^{\mathbf{N}}$ is a flat R-module.*

Proof. If R is coherent, then $R^{\mathbf{N}}$ is a flat module by Algebra, Proposition 89.5. Assume $R^{\mathbf{N}}$ is flat. Let $I \subset R$ be a finitely generated ideal. To prove the lemma we show that I is finitely presented as an R-module. Namely, the map $I \otimes_R R^{\mathbf{N}} \to R^{\mathbf{N}}$

is injective as $R^{\mathbf{N}}$ is flat and its image is $I^{\mathbf{N}}$ by Lemma 11.1. Thus we conclude by Lemma 11.2. $\square$

Let R be a countable ring. Observe that $R[[x]]$ is isomorphic to $R^{\mathbf{N}}$ as an R-module. By Lemma 11.3 we see that $R \to R[[x]]$ is flat if and only if R is coherent. There are plenty of noncoherent countable rings, for example

$$R = k[y, z, a_1, b_1, a_2, b_2, a_3, b_3, \ldots]/(a_1y + b_1z, a_2y + b_2z, a_3y + b_3z, \ldots)$$

where k is a countable field. This ring is not coherent because the ideal (y, z) of R is not a finitely presented R-module. Note that $R[[x]]$ is the completion of $R[x]$ by the principal ideal (x).

0ALC **Lemma 11.4.** *There exists a ring such that the completion $R[[x]]$ of $R[x]$ at (x) is not flat over R and a fortiori not flat over $R[x]$.*

Proof. See discussion above. $\square$

Next, we will construct an example where the completion of a localization is nonflat. To do this consider the ring

$$R = k[y, z, a_1, a_2, a_3, \ldots]/(ya_i, a_ia_j)$$

Denote $f \in R$ the residue class of z. We claim the ring map

0ALD
$$R[[x]] \longrightarrow R_f[[x]] \tag{11.4.1}$$

isn't flat. Let I be the kernel of $y : R[[x]] \to R[[x]]$. A typical element g of I looks like $g = \sum g_{n,m}a_mx^n$ where $g_{n,m} \in k[z]$ and for a given n only a finite number of nonzero $g_{n,m}$. Let J be the kernel of $y : R_f[[x]] \to R_f[[x]]$. We claim that $J \neq IR_f[[x]]$. Namely, if this were true then we would have

$$\sum z^{-n}a_nx^n = \sum\nolimits_{i=1,\ldots,m} h_ig_i$$

for some $m \geq 1$, $g_i \in I$, and $h_i \in R_f[[x]]$. Say $h_i = \bar{h}_i \bmod (y, a_1, a_2, a_3, \ldots)$ with $\bar{h}_i \in k[z, 1/z][[x]]$. Looking at the coefficient of a_n and using the description of the elements g_i above we would get

$$z^{-n}x^n = \sum \bar{h}_i\bar{g}_{i,n}$$

for some $\bar{g}_{i,n} \in k[z][[x]]$. This would mean that all $z^{-n}x^n$ are contained in the finite $k[z][[x]]$-module generated by the elements $\bar{h}_i$. Since $k[z][[x]]$ is Noetherian this implies that the $R[z][[x]]$-submodule of $k[z, 1/z][[x]]$ generated by $1, z^{-1}x, z^{-2}x^2, \ldots$ is finite. By Algebra, Lemma 35.2 we would conclude that $z^{-1}x$ is integral over $k[z][[x]]$ which is absurd. On the other hand, if (11.4.1) were flat, then we would get $J = IR_f[[x]]$ by tensoring the exact sequence $0 \to I \to R[[x]] \xrightarrow{y} R[[x]]$ with $R_f[[x]]$.

0ALE **Lemma 11.5.** *There exists a ring A complete with respect to a principal ideal I and an element $f \in A$ such that the I-adic completion $A_f^\wedge$ of A_f is not flat over A.*

Proof. Set $A = R[[x]]$ and $I = (x)$ and observe that $R_f[[x]]$ is the completion of $R[[x]]_f$. $\square$

12. Nonabelian category of quasi-coherent modules

0ALF In Sheaves on Stacks, Section 11 we defined the category of quasi-coherent modules on a category fibred in groupoids over Sch. Although we show in Sheaves on Stacks, Section 14 that this category is abelian for algebraic stacks, in this section we show that this is not the case for formal algebraic spaces.

Namely, consider $\mathbf{Z}_p$ viewed as topological ring using the p-adic topology. Let $X = \text{Spf}(\mathbf{Z}_p)$, see Formal Spaces, Definition 5.9. Then X is a sheaf in sets on $(Sch/\mathbf{Z})_{fppf}$ and gives rise to a stack in setoids $\mathcal{X}$, see Stacks, Lemma 6.2. Thus the discussion of Sheaves on Stacks, Section 14 applies.

Let $\mathcal{F}$ be a quasi-coherent module on $\mathcal{X}$. Since $X = \text{colim}\,\text{Spec}(\mathbf{Z}/p^n\mathbf{Z})$ it is clear from Sheaves on Stacks, Lemma 11.5 that $\mathcal{F}$ is given by a sequence $(\mathcal{F}_n)$ where

(1) $\mathcal{F}_n$ is a quasi-coherent module on $\text{Spec}(\mathbf{Z}/p^n\mathbf{Z})$, and
(2) the transition maps give isomorphisms $\mathcal{F}_n = \mathcal{F}_{n+1}/p^n\mathcal{F}_{n+1}$.

Converting into modules we see that $\mathcal{F}$ corresponds to a system (M_n) where each M_n is an abelian group annihilated by p^n and the transition maps induce isomorphisms $M_n = M_{n+1}/p^nM_{n+1}$. In this situation the module $M = \lim M_n$ is a p-adically complete module and $M_n = M/p^nM$, see Algebra, Lemma 97.1. We conclude that the category of quasi-coherent modules on X is equivalent to the category of p-adically complete abelian groups. This category is not abelian, see Section 9.

0ALG **Lemma 12.1.** *The category of quasi-coherent[1] modules on a formal algebraic space X is not abelian in general, even if X is a Noetherian affine formal algebraic space.*

Proof. See discussion above. $\square$

13. Regular sequences and base change

063Z We are going to construct a ring R with a regular sequence (x, y, z) such that there exists a nonzero element $\delta \in R/zR$ with $x\delta = y\delta = 0$.

To construct our example we first construct a peculiar module E over the ring $k[x, y, z]$ where k is any field. Namely, E will be a push-out as in the following diagram

$$\begin{tikzcd}
\frac{xk[x,y,z,y^{-1}]}{xyk[x,y,z]} \arrow[r] \arrow[d, "z/x"] & \frac{k[x,y,z,x^{-1},y^{-1}]}{yk[x,y,z,x^{-1}]} \arrow[r] \arrow[d] & \frac{k[x,y,z,x^{-1},y^{-1}]}{yk[x,y,z,x^{-1}]+xk[x,y,z,y^{-1}]} \arrow[d] \\
\frac{k[x,y,z,y^{-1}]}{yzk[x,y,z]} \arrow[r] & E \arrow[r] & \frac{k[x,y,z,x^{-1},y^{-1}]}{yk[x,y,z,x^{-1}]+xk[x,y,z,y^{-1}]}
\end{tikzcd}$$

where the rows are short exact sequences (we dropped the outer zeros due to typesetting problems). Another way to describe E is as

$$E = \{(f, g) \mid f \in k[x, y, z, x^{-1}, y^{-1}], g \in k[x, y, z, y^{-1}]\}/\sim$$

where $(f, g) \sim (f', g')$ if and only if there exists a $h \in k[x, y, z, y^{-1}]$ such that

$$f = f' + xh \bmod yk[x, y, z, x^{-1}], \quad g = g' - zh \bmod yzk[x, y, z]$$

[1] With quasi-coherent modules as defined above. Due to how things are setup in the Stacks project, this is really the correct definition; as seen above our definition agrees with what one would naively have defined to be quasi-coherent modules on $\text{Spf}(A)$, namely complete A-modules.

We claim: (a) $x : E \to E$ is injective, (b) $y : E/xE \to E/xE$ is injective, (c) $E/(x, y)E = 0$, (d) there exists a nonzero element $\delta \in E/zE$ such that $x\delta = y\delta = 0$.

To prove (a) suppose that (f, g) is a pair that gives rise to an element of E and that $(xf, xg) \sim 0$. Then there exists a $h \in k[x, y, z, y^{-1}]$ such that $xf + xh \in yk[x, y, z, x^{-1}]$ and $xg - zh \in yzk[x, y, z]$. We may assume that $h = \sum a_{i,j,k} x^i y^j z^k$ is a sum of monomials where only $j \leq 0$ occurs. Then $xg - zh \in yzk[x, y, z]$ implies that only $i > 0$ occurs, i.e., $h = xh'$ for some $h' \in k[x, y, z, y^{-1}]$. Then $(f, g) \sim (f + xh', g - zh')$ and we see that we may assume that $g = 0$ and $h = 0$. In this case $xf \in yk[x, y, z, x^{-1}]$ implies $f \in yk[x, y, z, x^{-1}]$ and we see that $(f, g) \sim 0$. Thus $x : E \to E$ is injective.

Since multiplication by x is an isomorphism on $\frac{k[x,y,z,x^{-1},y^{-1}]}{yk[x,y,z,x^{-1}]}$ we see that E/xE is isomorphic to

$$\frac{k[x, y, z, y^{-1}]}{yzk[x, y, z] + xk[x, y, z, y^{-1}] + zk[x, y, z, y^{-1}]} = \frac{k[x, y, z, y^{-1}]}{xk[x, y, z, y^{-1}] + zk[x, y, z, y^{-1}]}$$

and hence multiplication by y is an isomorphism on E/xE. This clearly implies (b) and (c).

Let $e \in E$ be the equivalence class of $(1, 0)$. Suppose that $e \in zE$. Then there exist $f \in k[x, y, z, x^{-1}, y^{-1}]$, $g \in k[x, y, z, y^{-1}]$, and $h \in k[x, y, z, y^{-1}]$ such that

$$1 + zf + xh \in yk[x, y, z, x^{-1}], \quad 0 + zg - zh \in yzk[x, y, z].$$

This is impossible: the monomial 1 cannot occur in zf, nor in xh. On the other hand, we have $ye = 0$ and $xe = (x, 0) \sim (0, -z) = z(0, -1)$. Hence setting δ equal to the congruence class of e in E/zE we obtain (d).

0640 **Lemma 13.1.** *There exists a local ring R and a regular sequence x, y, z (in the maximal ideal) such that there exists a nonzero element $\delta \in R/zR$ with $x\delta = y\delta = 0$.*

Proof. Let $R = k[x, y, z] \oplus E$ where E is the module above considered as a square zero ideal. Then it is clear that x, y, z is a regular sequence in R, and that the element $\delta \in E/zE \subset R/zR$ gives an element with the desired properties. To get a local example we may localize R at the maximal ideal $\mathfrak{m} = (x, y, z, E)$. The sequence x, y, z remains a regular sequence (as localization is exact), and the element δ remains nonzero as it is supported at $\mathfrak{m}$. $\square$

0641 **Lemma 13.2.** *There exists a local homomorphism of local rings $A \to B$ and a regular sequence x, y in the maximal ideal of B such that $B/(x, y)$ is flat over A, but such that the images $\overline{x}, \overline{y}$ of x, y in $B/\mathfrak{m}_A B$ do not form a regular sequence, nor even a Koszul-regular sequence.*

Proof. Set $A = k[z]_{(z)}$ and let $B = (k[x, y, z] \oplus E)_{(x,y,z,E)}$. Since x, y, z is a regular sequence in B, see proof of Lemma 13.1, we see that x, y is a regular sequence in B and that $B/(x, y)$ is a torsion free A-module, hence flat. On the other hand, there exists a nonzero element $\delta \in B/\mathfrak{m}_A B = B/zB$ which is annihilated by $\overline{x}, \overline{y}$. Hence $H_2(K_\bullet(B/\mathfrak{m}_A B, \overline{x}, \overline{y})) \neq 0$. Thus $\overline{x}, \overline{y}$ is not Koszul-regular, in particular it is not a regular sequence, see More on Algebra, Lemma 27.2. $\square$

14. A Noetherian ring of infinite dimension

02JC A Noetherian local ring has finite dimension as we saw in Algebra, Proposition 59.8. But there exist Noetherian rings of infinite dimension. See [Nag62, Appendix, Example 1].

Namely, let k be a field, and consider the ring

$$R = k[x_1, x_2, x_3, \ldots].$$

Let $\mathfrak{p}_i = (x_{2^{i-1}}, x_{2^{i-1}+1}, \ldots, x_{2^i-1})$ for $i = 1, 2, \ldots$ which are prime ideals of R. Let S be the multiplicative subset

$$S = \bigcap\nolimits_{i \geq 1} (R \setminus \mathfrak{p}_i).$$

Consider the ring $A = S^{-1}R$. We claim that

(1) The maximal ideals of the ring A are the ideals $\mathfrak{m}_i = \mathfrak{p}_i A$.
(2) We have $A_{\mathfrak{m}_i} = R_{\mathfrak{p}_i}$ which is a Noetherian local ring of dimension 2^i.
(3) The ring A is Noetherian.

Hence it is clear that this is the example we are looking for. Details omitted.

15. Local rings with nonreduced completion

02JD In Algebra, Example 118.5 we gave an example of a characteristic p Noetherian local domain R of dimension 1 whose completion is nonreduced. In this section we present the example of [FR70, Proposition 3.1] which gives a similar ring in characteristic zero.

Let $\mathbf{C}\{x\}$ be the ring of convergent power series over the field $\mathbf{C}$ of complex numbers. The ring of all power series $\mathbf{C}[[x]]$ is its completion. Let $K = \mathbf{C}\{x\}[1/x]$ be the field of convergent Laurent series. The K-module $\Omega_{K/\mathbf{C}}$ of algebraic differentials of K over $\mathbf{C}$ is an infinite dimensional K-vector space (proof omitted). We may choose $f_n \in x\mathbf{C}\{x\}$, $n \geq 1$ such that $\mathrm{d}x, \mathrm{d}f_1, \mathrm{d}f_2, \ldots$ are part of a basis of $\Omega_{K/\mathbf{C}}$. Thus we can find a $\mathbf{C}$-derivation

$$D : \mathbf{C}\{x\} \longrightarrow \mathbf{C}((x))$$

such that $D(x) = 0$ and $D(f_i) = x^{-n}$. Let

$$A = \{f \in \mathbf{C}\{x\} \mid D(f) \in \mathbf{C}[[x]]\}$$

We claim that

(1) $\mathbf{C}\{x\}$ is integral over A,
(2) A is a local domain,
(3) $\dim(A) = 1$,
(4) the maximal ideal of A is generated by x and xf_1,
(5) A is Noetherian, and
(6) the completion of A is equal to the ring of dual numbers over $\mathbf{C}[[x]]$.

Since the dual numbers are nonreduced the ring A gives the example.

Note that if $0 \neq f \in x\mathbf{C}\{x\}$ then we may write $D(f) = h/f^n$ for some $n \geq 0$ and $h \in \mathbf{C}[[x]]$. Hence $D(f^{n+1}/(n+1)) \in \mathbf{C}[[x]]$ and $D(f^{n+2}/(n+2)) \in \mathbf{C}[[x]]$. Thus we see $f^{n+1}, f^{n+2} \in A$! In particular we see (1) holds. We also conclude that the fraction field of A is equal to the fraction field of $\mathbf{C}\{x\}$. It also follows immediately that $A \cap x\mathbf{C}\{x\}$ is the set of nonunits of A, hence A is a local domain of dimension 1. If we can show (4) then it will follow that A is Noetherian (proof omitted). Suppose that $f \in A \cap x\mathbf{C}\{x\}$. Write $D(f) = h$, $h \in \mathbf{C}[[x]]$. Write $h = c + xh'$ with $c \in \mathbf{C}$, $h' \in \mathbf{C}[[x]]$. Then $D(f - cxf_1) = c + xh' - c = xh'$. On the other hand $f - cxf_1 = xg$ with $g \in \mathbf{C}\{x\}$, but by the computation above we have $D(g) = h' \in \mathbf{C}[[x]]$ and hence $g \in A$. Thus $f = cxf_1 + xg \in (x, xf_1)$ as desired.

Finally, why is the completion of A nonreduced? Denote $\hat{A}$ the completion of A. Of course this maps surjectively to the completion $\mathbf{C}[[x]]$ of $\mathbf{C}\{x\}$ because $x \in A$. Denote this map $\psi : \hat{A} \to \mathbf{C}[[x]]$. Above we saw that $\mathfrak{m}_A = (x, xf_1)$ and hence $D(\mathfrak{m}_A^n) \subset (x^{n-1})$ by an easy computation. Thus $D : A \to \mathbf{C}[[x]]$ is continuous and gives rise to a continuous derivation $\hat{D} : \hat{A} \to \mathbf{C}[[x]]$ over ψ. Hence we get a ring map

$$\psi + \epsilon\hat{D} : \hat{A} \longrightarrow \mathbf{C}[[x]][\epsilon].$$

Since $\hat{A}$ is a one dimensional Noetherian complete local ring, if we can show this arrow is surjective then it will follow that $\hat{A}$ is nonreduced. Actually the map is an isomorphism but we omit the verification of this. The subring $\mathbf{C}[x]_{(x)} \subset A$ gives rise to a map $i : \mathbf{C}[[x]] \to \hat{A}$ on completions such that $i \circ \psi = \mathrm{id}$ and such that $D \circ i = 0$ (as $D(x) = 0$ by construction). Consider the elements $x^n f_n \in A$. We have

$$(\psi + \epsilon D)(x^n f_n) = x^n f_n + \epsilon$$

for all $n \geq 1$. Surjectivity easily follows from these remarks.

16. A non catenary Noetherian local ring

02JE Even though there is a succesful dimension theory of Noetherian local rings there are non-catenary Noetherian local rings. An example may be found in [Nag62, Appendix, Example 2]. In fact, we will present this example in the simplest case. Namely, we will construct a local Noetherian domain A of dimension 2 which is not universally catenary. (Note that A is automatically catenary, see Exercises, Exercise 17.3.) The existence of a Noetherian local ring which is not universally catenary implies the existence of a Noetherian local ring which is not catenary – and we spell this out at the end of this section in the particular example at hand.

Let k be a field, and consider the formal power series ring $k[[x]]$ in one variable over k. Let

$$z = \sum\nolimits_{i=1}^{\infty} a_i x^i$$

be a formal power series. We assume z as an element of the Laurent series field $k((x)) = k[[x]][1/x]$ is transcendental over $k(x)$. Put

$$z_j = x^{-j}(z - \sum\nolimits_{i=1,\ldots,j-1} a_i x^i) = \sum\nolimits_{i=j}^{\infty} a_i x^{i-j} \in k[[x]].$$

Note that $z = z_1$. Let R be the subring of $k[[x]]$ generated by x, z and all of the z_j, in other words

$$R = k[x, z_1, z_2, z_3, \ldots] \subset k[[x]].$$

Consider the ideals $\mathfrak{m} = (x)$ and $\mathfrak{n} = (x - 1, z_1, z_2, \ldots)$ of R.

We have $x(z_{j+1} + a_j) = z_j$. Hence $R/\mathfrak{m} = k$ and $\mathfrak{m}$ is a maximal ideal. Moreover, any element of R not in $\mathfrak{m}$ maps to a unit in $k[[x]]$ and hence $R_\mathfrak{m} \subset k[[x]]$. In fact it is easy to deduce that $R_\mathfrak{m}$ is a discrete valuation ring and residue field k.

We claim that

$$R/(x - 1) = k[x, z_1, z_2, z_3, \ldots]/(x - 1) \cong k[z].$$

Namely, the relation above implies that $(x-1)(z_{j+1}+a_j) = -z_{j+1} - a_j + z_j$, and hence we may express the class of z_{j+1} in terms of z_j in the quotient $R/(x-1)$. Since the fraction field of R has transcendence degree 2 over k by construction we see that z is transcendental over k in $R/(x-1)$, whence the desired isomorphism. Hence $\mathfrak n = (x-1, z)$ and is a maximal ideal. In fact the map

$$k[x, x^{-1}, z]_{(x-1,z)} \longrightarrow R_{\mathfrak n}$$

is an isomorphism (since x^{-1} is invertible in $R_{\mathfrak n}$ and since $z_{j+1} = x^{-1}z_j - a_j = \ldots = f_j(x, x^{-1}, z)$). This shows that $R_{\mathfrak n}$ is a regular local ring of dimension 2 and residue field k.

Let S be the multiplicative subset

$$S = (R \setminus \mathfrak m) \cap (R \setminus \mathfrak n) = R \setminus (\mathfrak m \cup \mathfrak n)$$

and set $B = S^{-1}R$. We claim that

(1) The ring B is a k-algebra.
(2) The maximal ideals of the ring B are the two ideals $\mathfrak mB$ and $\mathfrak nB$.
(3) The residue fields at these maximal ideals is k.
(4) We have $B_{\mathfrak mB} = R_{\mathfrak m}$ and $B_{\mathfrak nB} = R_{\mathfrak n}$ which are Noetherian regular local rings of dimensions 1 and 2.
(5) The ring B is Noetherian.

We omit the details of the verifications.

Whenever given a k-algebra B with the properties listed above we get an example as follows. Take $A = k + \text{rad}(B) \subset B$, in our case $\text{rad}(B) = \mathfrak mB + \mathfrak nB$. It is easy to see that B is finite over A and hence A is Noetherian by Eakin's theorem (see [Eak68], or [Nag62, Appendix A1], or insert future reference here). Also A is a local domain with the same fraction field as B and residue field k. Since the dimension of B is 2 we see that A has dimension 2 as well, by Algebra, Lemma 111.4.

If A were universally catenary then the dimension formula, Algebra, Lemma 112.1 would give $\dim(B_{\mathfrak mB}) = 2$ contradiction.

Note that B is generated by one element over A. Hence $B = A[x]/\mathfrak p$ for some prime $\mathfrak p$ of $A[x]$. Let $\mathfrak m' \subset A[x]$ be the maximal ideal corresponding to $\mathfrak mB$. Then on the one hand $\dim(A[x]_{\mathfrak m'}) = 3$ and on the other hand

$$(0) \subset \mathfrak pA[x]_{\mathfrak m'} \subset \mathfrak m'A[x]_{\mathfrak m'}$$

is a maximal chain of primes. Hence $A[x]_{\mathfrak m'}$ is an example of a non catenary Noetherian local ring.

17. Existence of bad local Noetherian rings

0AL7 Let $(A, \mathfrak m, \kappa)$ be a Noetherian complete local ring. In [Lec86a] it was shown that A is the completion of a Noetherian local domain if $\text{depth}(A) \geq 1$ and A contains either $\mathbf{Q}$ or $\mathbf{F}_p$ as a subring, or contains $\mathbf{Z}$ as a subring and A is torsion free as a $\mathbf{Z}$-module. This produces many examples of Noetherian local domains with "bizarre" properties.

Applying this for example to $A = \mathbf{C}[[x, y]]/(y^2)$ we find a Noetherian local domain whose completion is nonreduced. Please compare with Section 15.

In [LLPY01] conditions were found that characterize when A is the completion of a reduced local Noetherian ring.

In [Hei93] it was shown that A is the completion of a local Noetherian UFD R if $\text{depth}(A) \geq 2$ and A contains either $\mathbf{Q}$ or $\mathbf{F}_p$ as a subring, or contains $\mathbf{Z}$ as a subring and A is torsion free as a $\mathbf{Z}$-module. In particular R is normal (Algebra, Lemma 119.11) hence the henselization of R is a normal domain too (More on Algebra, Lemma 42.6). Thus A as above is the completion of a henselian Noetherian local normal domain (because the completion of R and its henselization agree, see More on Algebra, Lemma 42.3).

Apply this to find a Noetherian local UFD R such that $R^\wedge \cong \mathbf{C}[[x, y, z, w]]/(wx, wy)$. Note that $\text{Spec}(R^\wedge)$ is the union of a regular 2-dimensional and a regular 3-dimensional component. The ring R cannot be universally catenary: Let

$$X \longrightarrow \text{Spec}(R)$$

be the blowing up of the maximal ideal. Then X is an integral scheme. There is a closed point $x \in X$ such that $\dim(\mathcal{O}_{X,x}) = 2$, namely, on the level of the complete local ring we pick x to lie on the strict transform of the 2-dimensional component and not on the strict transform of the 3-dimensional component. By Morphisms, Lemma 49.1 we see that R is not universally catenary. Please compare with Section 16.

The ring above is catenary (being a 3-dimensional local Noetherian UFD). However, in [Ogo80] the author constructs a normal local Noetherian domain R with $R^\wedge \cong \mathbf{C}[[x, y, z, w]]/(wx, wy)$ such that R is not catenary. See also [Hei82] and [Lec86b].

In [Hei94] it was shown that A is the completion of a local Noetherian ring R with an isolated singularity provided A contains either $\mathbf{Q}$ or $\mathbf{F}_p$ as a subring or A has residue characteristic $p > 0$ and p cannot map to a nonzero zerodivisor in any proper localization of A. Here we say a Noetherian local ring R has an isolated singularity if $R_\mathfrak{p}$ is a regular local ring for all nonmaximal primes $\mathfrak{p} \subset R$.

The paper [Nis12] contains a long list of "bad" Noetherian local rings with given completions. In particular it constructs an example of a 2-dimensional Nagata local normal domain whose completion is $\mathbf{C}[[x, y, z]]/(yz)$ and one whose completion is $\mathbf{C}[[x, y, z]]/(y^2 - z^3)$.

As an aside, in [Loe03] it was shown that A is the completion of an excellent Noetherian local domain if A is reduced, equidimensional, and no integer in A is a zero divisor. However, this doesn't lead to "bad" Noetherian local rings as we obtain excellent ones!

18. Non-quasi-affine variety with quasi-affine normalization

0271 The existence of an example of this kind is mentioned in [DG67, II Remark 6.6.13]. They refer to the fifth volume of EGA for such an example, but the fifth volume did not appear.

Let k be a field. Let $Y = \mathbf{A}^2_k \setminus \{(0, 0)\}$. We are going to construct a finite surjective birational morphism $\pi : Y \longrightarrow X$ with X a variety over k such that X is not quasi-affine. Namely, consider the following curves in Y:

$$\begin{aligned} C_1 &: \quad x = 0 \\ C_2 &: \quad y = 0 \end{aligned}$$

Note that $C_1 \cap C_2 = \emptyset$. We choose the isomorphism $\varphi : C_1 \to C_2$, $(0, y) \mapsto (y^{-1}, 0)$. We claim there is a unique morphism $\pi : Y \to X$ as above such that

$$\begin{tikzcd} C_1 \arrow[r, shift left, "\mathrm{id}"] \arrow[r, shift right, "\varphi"'] & Y \arrow[r, "\pi"] & X \end{tikzcd}$$

is a coequalizer diagram in the category of varieties (and even in the category of schemes). Accepting this for the moment let us show that such an X cannot be quasi-affine. Namely, it is clear that we would get

$$\Gamma(X, \mathcal{O}_X) = \{f \in k[x, y] \mid f(0, y) = f(y^{-1}, 0)\} = k \oplus (xy) \subset k[x, y].$$

In particular these functions do not separate the points $(1, 0)$ and $(-1, 0)$ whose images in X (we will see below) are distinct (if the characteristic of k is not 2).

To show that X exists consider the Zariski open $D(x + y) \subset Y$ of Y. This is the spectrum of the ring $k[x, y, 1/(x + y)]$ and the curves C_1, C_2 are completely contained in $D(x + y)$. Moreover the morphism

$$C_1 \amalg C_2 \longrightarrow D(x + y) \cap Y = \mathrm{Spec}(k[x, y, 1/(x + y)])$$

is a closed immersion. It follows from More on Algebra, Lemma 5.1 that the ring

$$A = \{f \in k[x, y, 1/(x + y)] \mid f(0, y) = f(y^{-1}, 0)\}$$

is of finite type over k. On the other hand we have the open $D(xy) \subset Y$ of Y which is disjoint from the curves C_1 and C_2. It is the spectrum of the ring

$$B = k[x, y, 1/xy].$$

Note that we have $A_{xy} \cong B_{x+y}$ (since A clearly contains the elements $xyP(x, y)$ any polynomial P and the element $xy/(x + y)$). The scheme X is obtained by glueing the affine schemes $\mathrm{Spec}(A)$ and $\mathrm{Spec}(B)$ using the isomorphism $A_{xy} \cong B_{x+y}$ and hence is clearly of finite type over k. To see that it is separated one has to show that the ring map $A \otimes_k B \to B_{x+y}$ is surjective. To see this use that $A \otimes_k B$ contains the element $xy/(x + y) \otimes 1/xy$ which maps to $1/(x + y)$. The morphism $X \to Y$ is given by the natural maps $D(x + y) \to \mathrm{Spec}(A)$ and $D(xy) \to \mathrm{Spec}(B)$. Since these are both finite we deduce that $X \to Y$ is finite as desired. We omit the verification that X is indeed the coequalizer of the displayed diagram above, however, see (insert future reference for pushouts in the category of schemes here). Note that the morphism $\pi : Y \to X$ does map the points $(1, 0)$ and $(-1, 0)$ to distinct points in X because the function $(x + y^3)/(x + y)^2 \in A$ has value $1/1$, resp. $-1/(-1)^2 = -1$ which are always distinct (unless the characteristic is 2 – please find your own points for characteristic 2). We summarize this discussion in the form of a lemma.

0272 **Lemma 18.1.** *Let k be a field. There exists a variety X whose normalization is quasi-affine but which is itself not quasi-affine.*

Proof. See discussion above and (insert future reference on normalization here). $\square$

19. A locally closed subscheme which is not open in closed

078B This is a copy of Morphisms, Example 3.4. Here is an example of an immersion which is not a composition of an open immersion followed by a closed immersion. Let k be a field. Let $X = \mathrm{Spec}(k[x_1, x_2, x_3, \ldots])$. Let $U = \bigcup_{n=1}^{\infty} D(x_n)$. Then $U \to X$ is an open immersion. Consider the ideals

$$I_n = (x_1^n, x_2^n, \ldots, x_{n-1}^n, x_n - 1, x_{n+1}, x_{n+2}, \ldots) \subset k[x_1, x_2, x_3, \ldots][1/x_n].$$

Note that $I_n k[x_1, x_2, x_3, \ldots][1/x_n x_m] = (1)$ for any $m \neq n$. Hence the quasi-coherent ideals $\widetilde{I_n}$ on $D(x_n)$ agree on $D(x_n x_m)$, namely $\widetilde{I_n}|_{D(x_n x_m)} = \mathcal{O}_{D(x_n x_m)}$ if $n \neq m$. Hence these ideals glue to a quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_U$. Let $Z \subset U$ be the closed subscheme corresponding to $\mathcal{I}$. Thus $Z \to X$ is an immersion.

We claim that we cannot factor $Z \to X$ as $Z \to \overline{Z} \to X$, where $\overline{Z} \to X$ is closed and $Z \to \overline{Z}$ is open. Namely, $\overline{Z}$ would have to be defined by an ideal $I \subset k[x_1, x_2, x_3, \ldots]$ such that $I_n = Ik[x_1, x_2, x_3, \ldots][1/x_n]$. But the only element $f \in k[x_1, x_2, x_3, \ldots]$ which ends up in all I_n is 0! Hence I does not exist.

20. Nonexistence of suitable opens

086G This section complements the results of Properties, Section 29.

Let k be a field and let $A = k[z_1, z_2, z_3, \ldots]/I$ where I is the ideal generated by all pairwise products $z_i z_j$, $i \neq j$, $i, j \in \mathbf{N}$. Set $S = \mathrm{Spec}(A)$. Let $s \in S$ be the closed point corresponding to the maximal ideal (z_i). We claim there is no quasi-compact open $V \subset S \setminus \{s\}$ which is dense in $S \setminus \{s\}$. Note that $S \setminus \{s\} = \bigcup D(z_i)$. Each $D(z_i)$ is open and irreducible with generic point η_i. We conclude that $\eta_i \in V$ for all i. However, a principal affine open of $S \setminus \{s\}$ is of the form $D(f)$ where $f \in (z_1, z_2, \ldots)$. Then $f \in (z_1, \ldots, z_n)$ for some n and we see that $D(f)$ contains only finitely many of the points η_i. Thus V cannot be quasi-compact.

Let k be a field and let $B = k[x, z_1, z_2, z_3, \ldots]/J$ where J is the ideal generated by the products xz_i, $i \in \mathbf{N}$ and by all pairwise products $z_i z_j$, $i \neq j$, $i, j \in \mathbf{N}$. Set $T = \mathrm{Spec}(B)$. Consider the principal open $U = D(x)$. We claim there is no quasi-compact open $V \subset S$ such that $V \cap U = \emptyset$ and $V \cup U$ is dense in S. Let $t \in T$ be the closed point corresponding to the maximal ideal (x, z_i). The closure of U in T is $\overline{U} = U \cup \{t\}$. Hence $V \subset \bigcup_i D(z_i)$ is a quasi-compact open. By the arguments of the previous paragraph we see that V cannot be dense in $\bigcup D(z_i)$.

086H **Lemma 20.1.** *Nonexistence quasi-compact opens of affines:*

1. *There exist an affine scheme S and affine open $U \subset S$ such that there is no quasi-compact open $V \subset S$ with $U \cap V = \emptyset$ and $U \cup V$ dense in S.*
2. *There exists an affine scheme S and a closed point $s \in S$ such that $S \setminus \{s\}$ does not contain a quasi-compact dense open.*

Proof. See discussion above. □

Let X be the glueing of two copies of the affine scheme T (see above) along the affine open U. Thus there is a morphism $\pi : X \to T$ and $X = U_1 \cup U_2$ such that π maps U_i isomorphically to T and $U_1 \cap U_2$ isomorphically to U. Note that X is quasi-separated (by Schemes, Lemma 21.7) and quasi-compact. We claim there does not exist a separated, dense, quasi-compact open $W \subset X$. Namely, consider the two closed points $x_1 \in U_1$, $x_2 \in U_2$ mapping to the closed point $t \in T$ introduced above. Let $\tilde{\eta} \in U_1 \cap U_2$ be the generic point mapping to the (unique) generic point η of U. Note that $\tilde{\eta} \rightsquigarrow x_1$ and $\tilde{\eta} \rightsquigarrow x_2$ lying over the specialization $\eta \rightsquigarrow s$. Since $\pi|_W : W \to T$ is separated we conclude that we cannot have both x_1 and $x_2 \in W$ (by the valuative criterion of separatedness Schemes, Lemma 22.2). Say $x_1 \notin W$. Then $W \cap U_1$ is a quasi-compact (as X is quasi-separated) dense open of U_1 which does not contain x_1. Now observe that there exists an isomorphism $(T, t) \cong (S, s)$ of schemes (by sending x to z_1 and z_i to z_{i+1}). Hence by the first paragraph of this section we arrive at a contradiction.

086I **Lemma 20.2.** *There exists a quasi-compact and quasi-separated scheme X which does not contain a separated quasi-compact dense open.*

Proof. See discussion above. $\square$

21. Nonexistence of quasi-compact dense open subscheme

087H Let X be a quasi-compact and quasi-separated algebraic space over a field k. We know that the schematic locus $X' \subset X$ is a dense open subspace, see Properties of Spaces, Proposition 12.3. In fact, this result holds when X is reasonable, see Decent Spaces, Proposition 10.1. A natural question is whether one can find a quasi-compact dense open subscheme of X. It turns out this is not possible in general.

Assume the characteristic of k is not 2. Let $B = k[x, z_1, z_2, z_3, \ldots]/J$ where J is the ideal generated by the products xz_i, $i \in \mathbf{N}$ and by all pairwise products z_iz_j, $i \neq j$, $i, j \in \mathbf{N}$. Set $U = \text{Spec}(B)$. Denote $0 \in U$ the closed point all of whose coordinates are zero. Set

$$j : R = \Delta \amalg \Gamma \longrightarrow U \times_k U$$

where Δ is the image of the diagonal morphism of U over k and

$$\Gamma = \{((x, 0, 0, 0, \ldots), (-x, 0, 0, 0, \ldots)) \mid x \in \mathbf{A}^1_k, x \neq 0\}.$$

It is clear that $s, t : R \to U$ are étale, and hence j is an étale equivalence relation. The quotient $X = U/R$ is an algebraic space (Spaces, Theorem 10.5). Note that j is not an immersion because $(0, 0) \in \Delta$ is in the closure of Γ. Hence X is not a scheme. On the other hand, X is quasi-separated as R is quasi-compact. Denote 0_X the image of the point $0 \in U$. We claim that $X \setminus \{0_X\}$ is a scheme, namely

$$X \setminus \{0_X\} = \text{Spec}\left(k[x^2, x^{-2}]\right) \amalg \text{Spec}\left(k[z_1, z_2, z_3, \ldots]/(z_iz_j)\right) \setminus \{0\}$$

(details omitted). On the other hand, we have seen in Section 20 that the scheme on the right hand side does not contain a quasi-compact dense open.

087I **Lemma 21.1.** *There exists a quasi-compact and quasi-separated algebraic space which does not contain a quasi-compact dense open subscheme.*

Proof. See discussion above. $\square$

Using the construction of Spaces, Example 14.2 in the same manner as we used the construction of Spaces, Example 14.1 above, one obtains an example of a quasi-compact, quasi-separated, and locally separated algebraic space which does not contain a quasi-compact dense open subscheme.

22. Affines over algebraic spaces

088V Suppose that $f : Y \to X$ is a morphism of schemes with f locally of finite type and Y affine. Then there exists an immersion $Y \to \mathbf{A}^n_X$ of Y into affine n-space over X. See the slightly more general Morphisms, Lemma 37.2.

Now suppose that $f : Y \to X$ is a morphism of algebraic spaces with f locally of finite type and Y an affine scheme. Then it is not true in general that we can find an immersion of Y into affine n-space over X.

A first (nasty) counter example is $Y = \text{Spec}(k)$ and $X = [\mathbf{A}^1_k/\mathbf{Z}]$ where k is a field of characteristic zero and $\mathbf{Z}$ acts on $\mathbf{A}^1_k$ by translation $(n, t) \mapsto t + n$. Namely, for any morphism $Y \to \mathbf{A}^n_X$ over X we can pullback to the covering $\mathbf{A}^1_k$ of X and we get an infinite disjoint union of $\mathbf{A}^1_k$'s mapping into $\mathbf{A}^{n+1}_k$ which is not an immersion.

A second counter example is $Y = \mathbf{A}^1_k \to X = \mathbf{A}^1_k/R$ with $R = \{(t, t)\} \amalg \{(t, -t), t \neq 0\}$. Namely, in this case the morphism $Y \to \mathbf{A}^n_X$ would be given by some regular functions $f_1, \ldots, f_n$ on Y and hence the fibre product of Y with the covering $\mathbf{A}^{n+1}_k \to \mathbf{A}^n_X$ would be the scheme

$$\{(f_1(t), \ldots, f_n(t), t)\} \amalg \{(f_1(t), \ldots, f_n(t), -t), t \neq 0\}$$

with obvious morphism to $\mathbf{A}^{n+1}_k$ which is not an immersion. Note that this gives a counter example with X quasi-separated.

088W **Lemma 22.1.** *There exists a finite type morphism of algebraic spaces $Y \to X$ with Y affine and X quasi-separated, such that there does not exist an immersion $Y \to \mathbf{A}^n_X$ over X.*

Proof. See discussion above. ☐

23. Pushforward of quasi-coherent modules

078C In Schemes, Lemma 24.1 we proved that f_* transforms quasi-coherent modules into quasi-coherent modules when f is quasi-compact and quasi-separated. Here are some examples to show that these conditions are both necessary.

Suppose that $Y = \text{Spec}(A)$ is an affine scheme and that $X = \coprod_{n \in \mathbf{N}} Y$. We claim that $f_*\mathcal{O}_X$ is not quasi-coherent where $f : X \to Y$ is the obvious morphism. Namely, for $a \in A$ we have

$$f_*\mathcal{O}_X(D(a)) = \prod\nolimits_{n \in \mathbf{N}} A_a$$

Hence, in order for $f_*\mathcal{O}_X$ to be quasi-coherent we would need

$$\prod\nolimits_{n \in \mathbf{N}} A_a = \left(\prod\nolimits_{n \in \mathbf{N}} A\right)_a$$

for all $a \in A$. This isn't true in general, for example if $A = \mathbf{Z}$ and $a = 2$, then $(1, 1/2, 1/4, 1/8, \ldots)$ is an element of the left hand side which is not in the right hand side. Note that f is a non-quasi-compact separated morphism.

Let k be a field. Set

$$A = k[t, z, x_1, x_2, x_3, \ldots]/(tx_1z, t^2x_2^2z, t^3x_3^3z, \ldots)$$

Let $Y = \mathrm{Spec}(A)$. Let $V \subset Y$ be the open subscheme $V = D(x_1) \cup D(x_2) \cup \ldots$. Let X be two copies of Y glued along V. Let $f : X \to Y$ be the obvious morphism. Then we have an exact sequence

$$0 \to f_*\mathcal{O}_X \to \mathcal{O}_Y \oplus \mathcal{O}_Y \xrightarrow{(1,-1)} j_*\mathcal{O}_V$$

where $j : V \to Y$ is the inclusion morphism. Since

$$A \longrightarrow \prod A_{x_n}$$

is injective (details omitted) we see that $\Gamma(Y, f_*\mathcal{O}_X) = A$. On the other hand, the kernel of the map

$$A_t \longrightarrow \prod A_{tx_n}$$

is nonzero because it contains the element z. Hence $\Gamma(D(t), f_*\mathcal{O}_X)$ is strictly bigger than A_t because it contains $(z, 0)$. Thus we see that $f_*\mathcal{O}_X$ is not quasi-coherent. Note that f is quasi-compact but non-quasi-separated.

078D **Lemma 23.1.** *Schemes, Lemma 24.1 is sharp in the sense that one can neither drop the assumption of quasi-compactness nor the assumption of quasi-separatedness.*

Proof. See discussion above. □

24. A nonfinite module with finite free rank 1 stalks

065J Let $R = \mathbf{Q}[x]$. Set $M = \sum_{n\in\mathbf{N}} \frac{1}{x-n}R$ as a submodule of the fraction field of R. Then M is not finitely generated, but for every prime $\mathfrak{p}$ of R we have $M_\mathfrak{p} \cong R_\mathfrak{p}$ as an $R_\mathfrak{p}$-module.

25. A noninvertible ideal invertible in stalks

0CBZ Let A be a domain and let $I \subset A$ be a nonzero ideal. Recall that when we say I is invertible, we mean that I is invertible as an A-module. We are going to make an example of this situation where I is not invertible, yet $I_\mathfrak{q} = (f) \subset A_\mathfrak{q}$ is a (nonzero) principal ideal for every prime ideal $\mathfrak{q} \subset A$. In the literature the property that $I_\mathfrak{q}$ is principal for all primes $\mathfrak{q}$ is sometimes expressed by saying "I is a locally principal ideal". We can't use this terminology as our "local" always means "local in the Zariski topology" (or whatever topology we are currently working with).

Let $R = \mathbf{Q}[x]$ and let $M = \sum \frac{1}{x-n}R$ be the module constructed in Section 24. Consider the ring[2]

$$A = \mathrm{Sym}^*_R(M)$$

and the ideal $I = MA = \bigoplus_{d\geq 1} \mathrm{Sym}^d_R(M)$. Since M is not finitely generated as an R-module we see that I cannot be generated by finitely many elements as an ideal in A. Since an invertible module is finitely generated, this means that I is not invertible. On the other hand, let $\mathfrak{p} \subset R$ be a prime ideal. By construction $M_\mathfrak{p} \cong R_\mathfrak{p}$. Hence

$$A_\mathfrak{p} = \mathrm{Sym}^*_{R_\mathfrak{p}}(M_\mathfrak{p}) \cong \mathrm{Sym}^*_{R_\mathfrak{p}}(R_\mathfrak{p}) = R_\mathfrak{p}[T]$$

as a graded $R_\mathfrak{p}$-algebra. It follows that $I_\mathfrak{p} \subset A_\mathfrak{p}$ is generated by the nonzerodivisor T. Thus certainly for any prime ideal $\mathfrak{q} \subset A$ we see that $I_\mathfrak{q}$ is generated by a single element.

[2] The ring A is an example of a non-Noetherian domain whose local rings are Noetherian.

0CC0 **Lemma 25.1.** *There exists a domain A and a nonzero ideal $I \subset A$ such that $I_\mathfrak{q} \subset A_\mathfrak{q}$ is a principal ideal for all primes $\mathfrak{q} \subset A$ but I is not an invertible A-module.*

Proof. See discussion above. □

26. A finite flat module which is not projective

052H This is a copy of Algebra, Remark 77.3. It is not true that a finite R-module which is R-flat is automatically projective. A counter example is where $R = \mathcal{C}^\infty(\mathbf{R})$ is the ring of infinitely differentiable functions on $\mathbf{R}$, and $M = R_\mathfrak{m} = R/I$ where $\mathfrak{m} = \{f \in R \mid f(0) = 0\}$ and $I = \{f \in R \mid \exists \epsilon, \epsilon > 0 : f(x) = 0 \ \forall x, |x| < \epsilon\}$.

The morphism $\text{Spec}(R/I) \to \text{Spec}(R)$ is also an example of a flat closed immersion which is not open.

05FY **Lemma 26.1.** *Strange flat modules.*

(1) *There exists a ring R and a finite flat R-module M which is not projective.*
(2) *There exists a closed immersion which is flat but not open.*

Proof. See discussion above. □

27. A projective module which is not locally free

05WG We give two examples. One where the rank is between 0 and 1 and one where the rank is $\aleph_0$.

05WH **Lemma 27.1.** *Let R be a ring. Let $I \subset R$ be an ideal generated by a countable collection of idempotents. Then I is projective as an R-module.*

Proof. Say $I = (e_1, e_2, e_3, \ldots)$ with e_n an idempotent of R. After inductively replacing e_{n+1} by $e_n + (1 - e_n)e_{n+1}$ we may assume that $(e_1) \subset (e_2) \subset (e_3) \subset \ldots$ and hence $I = \bigcup_{n \geq 1}(e_n) = \text{colim}_n \, e_n R$. In this case

$$\text{Hom}_R(I, M) = \text{Hom}_R(\text{colim}_n \, e_n R, M) = \lim_n \text{Hom}_R(e_n R, M) = \lim_n e_n M$$

Note that the transition maps $e_{n+1}M \to e_n M$ are given by multiplication by e_n and are surjective. Hence by Algebra, Lemma 85.4 the functor $\text{Hom}_R(I, M)$ is exact, i.e., I is a projective R-module. □

05WI **Lemma 27.2.** *Let R be a nonzero ring. Let $n \geq 1$. Let M be an R-module generated by $< n$ elements. Then any R-module map $f : R^{\oplus n} \to M$ has a nonzero kernel.*

Proof. Choose a surjection $R^{\oplus n-1} \to M$. We may lift the map f to a map $f' : R^{\oplus n} \to R^{\oplus n-1}$. It suffices to prove f' has a nonzero kernel. The map $f' : R^{\oplus n} \to R^{\oplus n-1}$ is given by a matrix $A = (a_{ij})$. If one of the a_{ij} is not nilpotent, say $a = a_{ij}$ is not, then we can replace A by the localization A_a and we may assume a_{ij} is a unit. Since if we find a nonzero kernel after localization then there was a nonzero kernel to start with as localization is exact, see Algebra, Proposition 9.12. In this case we can do a base change on both $R^{\oplus n}$ and $R^{\oplus n-1}$ and reduce to the case where

$$A = \begin{pmatrix} 1 & 0 & 0 & \ldots \\ 0 & a_{22} & a_{23} & \ldots \\ 0 & a_{32} & \ldots & \\ \ldots & \ldots & & \end{pmatrix}$$

Hence in this case we win by induction on n. If not then each a_{ij} is nilpotent. Set $I = (a_{ij}) \subset R$. Note that $I^{m+1} = 0$ for some $m \geq 0$. Let m be the largest integer such that $I^m \neq 0$. Then we see that $(I^m)^{\oplus n}$ is contained in the kernel of the map and we win. □

Suppose that $P \subset Q$ is an inclusion of R-modules with Q a finite R-module and P locally free, see Algebra, Definition 77.1. Suppose that Q can be generated by N elements as an R-module. Then it follows from Lemma 27.2 that P is finite locally free (with the free parts having rank at most N). And in this case P is a finite R-module, see Algebra, Lemma 77.2.

Combining this with the above we see that a non-finitely-generated ideal which is generated by a countable collection of idempotents is projective but not locally free. An explicit example is $R = \prod_{n \in \mathbf{N}} \mathbf{F}_2$ and I the ideal generated by the idempotents

$$e_n = (1, 1, \ldots, 1, 0, \ldots)$$

where the sequence of 1's has length n.

05WJ **Lemma 27.3.** *There exists a ring R and an ideal I such that I is projective as an R-module but not locally free as an R-module.*

Proof. See above. □

05WK **Lemma 27.4.** *Let K be a field. Let C_i, $i = 1, \ldots, n$ be smooth, projective, geometrically irreducible curves over K. Let $P_i \in C_i(K)$ be a rational point and let $Q_i \in C_i$ be a point such that $[\kappa(Q_i) : K] = 2$. Then $[P_1 \times \ldots \times P_n]$ is nonzero in $A_0(U_1 \times_K \ldots \times_K U_n)$ where $U_i = C_i \setminus \{Q_i\}$.*

Proof. There is a degree map $\deg : A_0(C_1 \times_K \ldots \times_K C_n) \to \mathbf{Z}$ Because each Q_i has degree 2 over K we see that any zero cycle supported on the "boundary"

$$C_1 \times_K \ldots \times_K C_n \setminus U_1 \times_K \ldots \times_K U_n$$

has degree divisible by 2. □

We can construct another example of a projective but not locally free module using the lemma above as follows. Let C_n, $n = 1, 2, 3, \ldots$ be smooth, projective, geometrically irreducible curves over $\mathbf{Q}$ each with a pair of points $P_n, Q_n \in C_n$ such that $\kappa(P_n) = \mathbf{Q}$ and $\kappa(Q_n)$ is a quadratic extension of $\mathbf{Q}$. Set $U_n = C_n \setminus \{Q_n\}$; this is an affine curve. Let $\mathcal{L}_n$ be the inverse of the ideal sheaf of P_n on U_n. Note that $c_1(\mathcal{L}_n) = [P_n]$ in the group of zero cycles $A_0(U_n)$. Set $A_n = \Gamma(U_n, \mathcal{O}_{U_n})$. Let $L_n = \Gamma(U_n, \mathcal{L}_n)$ which is a locally free module of rank 1 over A_n. Set

$$B_n = A_1 \otimes_{\mathbf{Q}} A_2 \otimes_{\mathbf{Q}} \ldots \otimes_{\mathbf{Q}} A_n$$

so that $\text{Spec}(B_n) = U_1 \times \ldots \times U_n$ all products over $\text{Spec}(\mathbf{Q})$. For $i \leq n$ we set

$$L_{n,i} = A_1 \otimes_{\mathbf{Q}} \ldots \otimes_{\mathbf{Q}} M_i \otimes_{\mathbf{Q}} \ldots \otimes_{\mathbf{Q}} A_n$$

which is a locally free B_n-module of rank 1. Note that this is also the global sections of $\text{pr}_i^*\mathcal{L}_n$. Set

$$B_\infty = \text{colim}_n B_n \quad \text{and} \quad L_{\infty,i} = \text{colim}_n L_{n,i}$$

Finally, set

$$M = \bigoplus\nolimits_{i \geq 1} L_{\infty,i}.$$

This is a direct sum of finite locally free modules, hence projective. We claim that M is not locally free. Namely, suppose that $f \in B_\infty$ is a nonzero function such that

M_f is free over $(B_\infty)_f$. Let $e_1, e_2, \ldots$ be a basis. Choose $n \geq 1$ such that $f \in B_n$. Choose $m \geq n+1$ such that $e_1, \ldots, e_{n+1}$ are in

$$\bigoplus\nolimits_{1 \leq i \leq m} L_{m,i}.$$

Because the elements $e_1, \ldots, e_{n+1}$ are part of a basis after a faithfully flat base change we conclude that the chern classes

$$c_i(\text{pr}_1^*\mathcal{L}_1 \oplus \ldots \oplus \text{pr}_m^*\mathcal{L}_m), \quad i = m, m-1, \ldots, m-n$$

are zero in the chow group of

$$D(f) \subset U_1 \times \ldots \times U_m$$

Since f is the pullback of a function on $U_1 \times \ldots \times U_n$ this implies in particular that

$$c_{m-n}(\mathcal{O}_W^{\oplus n} \oplus \text{pr}_1^*\mathcal{L}_{n+1} \oplus \ldots \oplus \text{pr}_{m-n}^*\mathcal{L}_m) = 0.$$

on the variety

$$W = (C_{n+1} \times \ldots \times C_m)_K$$

over the field $K = \mathbf{Q}(C_1 \times \ldots \times C_n)$. In other words the cycle

$$[(P_{n+1} \times \ldots \times P_m)_K]$$

is zero in the chow group of zero cycles on W. This contradicts Lemma 27.4 above because the points Q_i, $n+1 \leq i \leq m$ induce corresponding points Q_i' on $(C_n)_K$ and as $K/\mathbf{Q}$ is geometrically irreducible we have $[\kappa(Q_i') : K] = 2$.

05WL **Lemma 27.5.** *There exists a countable ring R and a projective module M which is a direct sum of countably many locally free rank 1 modules such that M is not locally free.*

Proof. See above. $\square$

28. Zero dimensional local ring with nonzero flat ideal

05FZ In [Laz67] and [Laz69] there is an example of a zero dimensional local ring with a nonzero flat ideal. Here is the construction. Let k be a field. Let X_i, Y_i, $i \geq 1$ be variables. Take $R = k[X_i, Y_i]/(X_i - Y_iX_{i+1}, Y_i^2)$. Denote x_i, resp. y_i the image of X_i, resp. Y_i in this ring. Note that

$$x_i = y_ix_{i+1} = y_iy_{i+1}x_{i+2} = y_iy_{i+1}y_{i+2}x_{i+3} = \ldots$$

in this ring. The ring R has only one prime ideal, namely $\mathfrak{m} = (x_i, y_i)$. We claim that the ideal $I = (x_i)$ is flat as an R-module.

Note that the annihilator of x_i in R is the ideal $(x_1, x_2, x_3, \ldots, y_i, y_{i+1}, y_{i+2}, \ldots)$. Consider the R-module M generated by elements e_i, $i \geq 1$ and relations $e_i = y_ie_{i+1}$. Then M is flat as it is the colimit $\text{colim}_i R$ of copies of R with transition maps

$$R \xrightarrow{y_1} R \xrightarrow{y_2} R \xrightarrow{y_3} \ldots$$

Note that the annihilator of e_i in M is the ideal $(x_1, x_2, x_3, \ldots, y_i, y_{i+1}, y_{i+2}, \ldots)$. Since every element of M, resp. I can be written as fe_i, resp. hx_i for some $f, h \in R$ we see that the map $M \to I$, $e_i \to x_i$ is an isomorphism and I is flat.

05G0 **Lemma 28.1.** *There exists a local ring R with a unique prime ideal and a nonzero ideal $I \subset R$ which is a flat R-module*

Proof. See discussion above. $\square$

29. An epimorphism of zero-dimensional rings which is not surjective

06RH In [Laz68] and [Laz69] one can find the following example. Let k be a field. Consider the ring homomorphism

$$k[x_1, x_2, \ldots, z_1, z_2, \ldots]/(x_i^{4^i}, z_i^{4^i}) \longrightarrow k[x_1, x_2, \ldots, y_1, y_2, \ldots]/(x_i^{4^i}, y_i - x_{i+1}y_{i+1}^2)$$

which maps x_i to x_i and z_i to $x_i y_i$. Note that $y_i^{4^{i+1}}$ is zero in the right hand side but that y_1 is not zero (details omitted). This map is not surjective: we can think of the above as a map of $\mathbf{Z}$-graded algebras by setting $\deg(x_i) = -1$, $\deg(z_i) = 0$, and $\deg(y_i) = 1$ and then it is clear that y_1 is not in the image. Finally, the map is an epimorphism because

$$y_{i-1} \otimes 1 = x_i y_i^2 \otimes 1 = y_i \otimes x_i y_i = x_i y_i \otimes y_i = 1 \otimes x_i y_i^2.$$

hence the tensor product of the target over the source is isomorphic to the target.

06RI **Lemma 29.1.** *There exists an epimorphism of local rings of dimension* 0 *which is not a surjection.*

Proof. See discussion above. □

30. Finite type, not finitely presented, flat at prime

05G1 Let k be a field. Consider the local ring $A_0 = k[x, y]_{(x,y)}$. Denote $\mathfrak{p}_{0,n} = (y + x^n + x^{2n+1})$. This is a prime ideal. Set

$$A = A_0[z_1, z_2, z_3, \ldots]/(z_n z_m, z_n(y + x^n + x^{2n+1}))$$

Note that $A \to A_0$ is a surjection whose kernel is an ideal of square zero. Hence A is also a local ring and the prime ideals of A are in one-to-one correspondence with the prime ideals of A_0. Denote $\mathfrak{p}_n$ the prime ideal of A corresponding to $\mathfrak{p}_{0,n}$. Observe that $\mathfrak{p}_n$ is the annihilator of z_n in A. Let

$$C = A[z]/(xz^2 + z + y)[\frac{1}{2zx + 1}]$$

Note that $A \to C$ is an étale ring map, see Algebra, Example 135.8. Let $\mathfrak{q} \subset C$ be the maximal ideal generated by x, y, z and all z_n. As $A \to C$ is flat we see that the annihilator of z_n in C is $\mathfrak{p}_n C$. We compute

$$\begin{aligned} C/\mathfrak{p}_n C &= A_0[z]/(xz^2 + z + y, y + x^n + x^{2n+1})[1/(2zx + 1)] \\ &= k[x]_{(x)}[z]/(xz^2 + z - x^n - x^{2n+1})[1/(2zx + 1)] \\ &= k[x]_{(x)}[z]/(z - x^n) \times k[x]_{(x)}[z]/(xz + x^{n+1} + 1)[1/(2zx + 1)] \\ &= k[x]_{(x)} \times k(x) \end{aligned}$$

because $(z - x^n)(xz + x^{n+1} + 1) = xz^2 + z - x^n - x^{2n+1}$. Hence we see that $\mathfrak{p}_n C = \mathfrak{r}_n \cap \mathfrak{q}_n$ with $\mathfrak{r}_n = \mathfrak{p}_n C + (z - x^n)C$ and $\mathfrak{q}_n = \mathfrak{p}_n C + (xz + x^{n+1} + 1)C$. Since $\mathfrak{q}_n + \mathfrak{r}_n = C$ we also get $\mathfrak{p}_n C = \mathfrak{r}_n \mathfrak{q}_n$. It follows that $\mathfrak{q}_n$ is the annihilator of $\xi_n = (z - x^n)z_n$. Observe that on the one hand $\mathfrak{r}_n \subset \mathfrak{q}$, and on the other hand $\mathfrak{q}_n + \mathfrak{q} = C$. This follows for example because $\mathfrak{q}_n$ is a maximal ideal of C distinct from $\mathfrak{q}$. Similarly we have $\mathfrak{q}_n + \mathfrak{q}_m = C$ for $n \neq m$. At this point we let

$$B = \mathrm{Im}(C \longrightarrow C_{\mathfrak{q}})$$

We observe that the elements ξ_n map to zero in B as $xz + x^{n+1} + 1$ is not in $\mathfrak q$. Denote $\mathfrak q' \subset B$ the image of $\mathfrak q$. By construction B is a finite type A-algebra, with $B_{\mathfrak q'} \cong C_{\mathfrak q}$. In particular we see that $B_{\mathfrak q'}$ is flat over A.

We claim there does not exist an element $g' \in B$, $g' \not\in \mathfrak q'$ such that $B_{g'}$ is of finite presentation over A. We sketch a proof of this claim. Choose an element $g \in C$ which maps to $g' \in B$. Consider the map $C_g \to B_{g'}$. By Algebra, Lemma 6.3 we see that B_g is finitely presented over A if and only if the kernel of $C_g \to B_{g'}$ is finitely generated. But the element $g \in C$ is not contained in $\mathfrak q$, hence maps to a nonzero element of $A_0[z]/(xz^2 + z + y)$. Hence g can only be contained in finitely many of the prime ideals $\mathfrak q_n$, because the primes $(y + x^n + x^{2n+1}, xz + x^{n+1} + 1)$ are an infinite collection of codimension 1 points of the 2-dimensional irreducible Noetherian space $\mathrm{Spec}(k[x, y, z]/(xz^2 + z + y))$. The map

$$\bigoplus\nolimits_{g \not\in \mathfrak q_n} C/\mathfrak q_n \longrightarrow C_g, \quad (c_n) \longrightarrow \sum c_n \xi_n$$

is injective and its image is the kernel of $C_g \to B_{g'}$. We omit the proof of this statement. (Hint: Write $A = A_0 \oplus I$ as an A_0-module where I is the kernel of $A \to A_0$. Similarly, write $C = C_0 \oplus IC$. Write $IC = \bigoplus Cz_n \cong \bigoplus (C/\mathfrak r_n \oplus C/\mathfrak q_n)$ and study the effect of multiplication by g on the summands.) This concludes the sketch of the proof of the claim. This also proves that $B_{g'}$ is not flat over A for any g' as above. Namely, if it were flat, then the annihilator of the image of z_n in $B_{g'}$ would be $\mathfrak p_n B_{g'}$, and would not contain $z - x^n$.

As a consequence we can answer (negatively) a question posed in [GR71, Part I, Remarques (3.4.7) (v)]. Here is a precise statement.

05G2 **Lemma 30.1.** *There exists a local ring A, a finite type ring map $A \to B$ and a prime $\mathfrak q$ lying over $\mathfrak m_A$ such that $B_{\mathfrak q}$ is flat over A, and for any element $g \in B$, $g \not\in \mathfrak q$ the ring B_g is neither finitely presented over A nor flat over A.*

Proof. See discussion above. ☐

31. Finite type, flat and not of finite presentation

05LB In this section we give some examples of ring maps and morphisms which are of finite type and flat but not of finite presentation.

Let R be a ring which has an ideal I such that R/I is a finite flat module but not projective, see Section 26 for an explicit example. Note that this means that I is not finitely generated, see Algebra, Lemma 107.5. Note that $I = I^2$, see Algebra, Lemma 107.2. The base ring in our examples will be R and correspondingly the base scheme $S = \mathrm{Spec}(R)$.

Consider the ring map $R \to R \oplus R/I\epsilon$ where $\epsilon^2 = 0$ by convention. This is a finite, flat ring map which is not of finite presentation. All the fibre rings are complete intersections and geometrically irreducible.

Let $A = R[x, y]/(xy, ay; a \in I)$. Note that as an R-module we have $A = \bigoplus_{i \geq 0} Ry^i \oplus \bigoplus_{j > 0} R/Ix^j$. Hence $R \to A$ is a flat finite type ring map which is not of finite presentation. Each fibre ring is isomorphic to either $\kappa(\mathfrak p)[x, y]/(xy)$ or $\kappa(\mathfrak p)[x]$.

We can turn the previous example into a projective morphism by taking $B = R[X_0, X_1, X_2]/(X_1X_2, aX_2; a \in I)$. In this case $X = \mathrm{Proj}(B) \to S$ is a proper flat morphism which is not of finite presentation such that for each $s \in S$ the fibre X_s is isomorphic either to $\mathbf{P}^1_s$ or to the closed subscheme of $\mathbf{P}^2_s$ defined by the vanishing of X_1X_2 (this is a projective nodal curve of arithmetic genus 0).

Let $M = R \oplus R \oplus R/I$. Set $B = \text{Sym}_R(M)$ the symmetric algebra on M. Set $X = \text{Proj}(B)$. Then $X \to S$ is a proper flat morphism, not of finite presentation such that for $s \in S$ the geometric fibre is isomorphic to either $\mathbf{P}^1_s$ or $\mathbf{P}^2_s$. In particular these fibres are smooth and geometrically irreducible.

05LC **Lemma 31.1.** *There exist examples of*

1. *a flat finite type ring map with geometrically irreducible complete intersection fibre rings which is not of finite presentation,*
2. *a flat finite type ring map with geometrically connected, geometrically reduced, dimension 1, complete intersection fibre rings which is not of finite presentation,*
3. *a proper flat morphism of schemes $X \to S$ each of whose fibres is isomorphic to either $\mathbf{P}^1_s$ or to the vanishing locus of X_1X_2 in $\mathbf{P}^2_s$ which is not of finite presentation, and*
4. *a proper flat morphism of schemes $X \to S$ each of whose fibres is isomorphic to either $\mathbf{P}^1_s$ or $\mathbf{P}^2_s$ which is not of finite presentation.*

Proof. See discussion above. $\square$

32. Topology of a finite type ring map

05JH Let $A \to B$ be a local map of local domains. If A is Noetherian, $A \to B$ is essentially of finite type, and $A/\mathfrak{m}_A \subset B/\mathfrak{m}_B$ is finite then there exists a prime $\mathfrak{q} \subset B$, $\mathfrak{q} \neq \mathfrak{m}_B$ such that $A \to B/\mathfrak{q}$ is the localization of a quasi-finite ring map. See More on Morphisms, Lemma 44.6.

In this section we give an example that shows this result is false A is no longer Noetherian. Namely, let k be a field and set

$$A = \{a_0 + a_1x + a_2x^2 + \ldots \mid a_0 \in k, a_i \in k((y)) \text{ for } i \geq 1\}$$

and

$$C = \{a_0 + a_1x + a_2x^2 + \ldots \mid a_0 \in k[y], a_i \in k((y)) \text{ for } i \geq 1\}.$$

The inclusion $A \to C$ is of finite type as C is generated by y over A. We claim that A is a local ring with maximal ideal $\mathfrak{m} = \{a_1x + a_2x^2 + \ldots \in A\}$ and no prime ideals besides (0) and $\mathfrak{m}$. Namely, an element $f = a_0 + a_1x + a_2x^2 + \ldots$ of A is invertible as soon as $a_0 \neq 0$. If $\mathfrak{q} \subset A$ is a nonzero prime ideal, and $f = a_ix^i + \ldots \in \mathfrak{q}$, then using properties of power series one sees that for any $g \in k((y))$ the element $g^{i+1}x^{i+1} \in \mathfrak{q}$, i.e., $gx \in \mathfrak{q}$. This proves that $\mathfrak{q} = \mathfrak{m}$.

As to the spectrum of the ring C, arguing in the same way as above we see that any nonzero prime ideal contains the prime $\mathfrak{p} = \{a_1x + a_2x^2 + \ldots \in C\}$ which lies over $\mathfrak{m}$. Thus the only prime of C which lies over (0) is (0). Set $\mathfrak{m}_C = yC + \mathfrak{p}$ and $B = C_{\mathfrak{m}_C}$. Then $A \to B$ is the desired example.

05JI **Lemma 32.1.** *There exists a local homomorphism $A \to B$ of local domains which is essentially of finite type and such that $A/\mathfrak{m}_A \to B/\mathfrak{m}_B$ is finite such that for every prime $\mathfrak{q} \neq \mathfrak{m}_B$ of B the ring map $A \to B/\mathfrak{q}$ is not the localization of a quasi-finite ring map.*

Proof. See the discussion above. $\square$

33. Pure not universally pure

05JJ Let k be a field. Let

$$R = k[[x, xy, xy^2, \ldots]] \subset k[[x, y]].$$

In other words, a power series $f \in k[[x, y]]$ is in R if and only if $f(0, y)$ is a constant. In particular $R[1/x] = k[[x, y]][1/x]$ and R/xR is a local ring with a maximal ideal whose square is zero. Denote $R[y] \subset k[[x, y]]$ the set of power series $f \in k[[x, y]]$ such that $f(0, y)$ is a polynomial in y. Then $R \to R[y]$ is a finite type but not finitely presented ring map which induces an isomorphism after inverting x. Also there is a surjection $R[y]/xR[y] \to k[y]$ whose kernel has square zero. Consider the finitely presented ring map $R \to S = R[t]/(xt - xy)$. Again $R[1/x] \to S[1/x]$ is an isomorphism and in this case $S/xS \cong (R/xR)[t]/(xy)$ maps onto $k[t]$ with nilpotent kernel. There is a surjection $S \to R[y]$, $t \longmapsto y$ which induces an isomorphism on inverting x and a surjection with nilpotent kernel modulo x. Hence the kernel of $S \to R[y]$ is locally nilpotent. In particular $S \to R[y]$ is a universal homeomorphism.

First we claim that S is an S-module which is relatively pure over R. Since on inverting x we obtain an isomorphism we only need to check this at the maximal ideal $\mathfrak{m} \subset R$. Since R is complete with respect to its maximal ideal it is henselian hence we need only check that every prime $\mathfrak{p} \subset R$, $\mathfrak{p} \neq \mathfrak{m}$, the unique prime $\mathfrak{q}$ of S lying over $\mathfrak{p}$ satisfies $\mathfrak{m}S + \mathfrak{q} \neq S$. Since $\mathfrak{p} \neq \mathfrak{m}$ it corresponds to a unique prime ideal of $k[[x, y]][1/x]$. Hence either $\mathfrak{p} = (0)$ or $\mathfrak{p} = (f)$ for some irreducible element $f \in k[[x, y]]$ which is not associated to x (here we use that $k[[x, y]]$ is a UFD – insert future reference here). In the first case $\mathfrak{q} = (0)$ and the result is clear. In the second case we may multiply f by a unit so that $f \in R[y]$ (Weierstrass preparation; details omitted). Then it is easy to see that $R[y]/fR[y] \cong k[[x, y]]/(f)$ hence f defines a prime ideal of $R[y]$ and $\mathfrak{m}R[y] + fR[y] \neq R[y]$. Since $S \to R[y]$ is a universal homeomorphism we deduce the desired result for S also.

Second we claim that S is not universally relatively pure over R. Namely, to see this it suffices to find a valuation ring $\mathcal{O}$ and a local ring map $R \to \mathcal{O}$ such that $\mathrm{Spec}(R[y] \otimes_R \mathcal{O}) \to \mathrm{Spec}(\mathcal{O})$ does not hit the closed point of $\mathrm{Spec}(\mathcal{O})$. Equivalently, we have to find $\varphi : R \to \mathcal{O}$ such that $\varphi(x) \neq 0$ and $v(\varphi(x)) > v(\varphi(xy))$ where v is the valuation of $\mathcal{O}$. (Because this means that the valuation of y is negative.) To do this consider the ring map

$$R \longrightarrow \{a_0 + a_1 x + a_2 x^2 + \ldots \mid a_0 \in k[y^{-1}], a_i \in k((y))\}$$

defined in the obvious way. We can find a valuation ring $\mathcal{O}$ dominating the localization of the right hand side at the maximal ideal (y^{-1}, x) and we win.

05JK **Lemma 33.1.** *There exists a morphism of affine schemes of finite presentation $X \to S$ and an $\mathcal{O}_X$-module $\mathcal{F}$ of finite presentation such that $\mathcal{F}$ is pure relative to S, but not universally pure relative to S.*

Proof. See discussion above. □

34. A formally smooth non-flat ring map

057V

Let k be a field. Consider the k-algebra $k[\mathbf{Q}]$. This is the k-algebra with basis $x_\alpha, \alpha \in \mathbf{Q}$ and multiplication determined by $x_\alpha x_\beta = x_{\alpha+\beta}$. (In particular $x_0 = 1$.) Consider the k-algebra homomorphism

$$k[\mathbf{Q}] \longrightarrow k, \quad x_\alpha \longmapsto 1.$$

It is surjective with kernel J generated by the elements $x_\alpha - 1$. Let us compute J/J^2. Note that multiplication by x_α on J/J^2 is the identity map. Denote z_α the class of $x_\alpha - 1$ modulo J^2. These classes generate J/J^2. Since

$$(x_\alpha - 1)(x_\beta - 1) = x_{\alpha+\beta} - x_\alpha - x_\beta + 1 = (x_{\alpha+\beta} - 1) - (x_\alpha - 1) - (x_\beta - 1)$$

we see that $z_{\alpha+\beta} = z_\alpha + z_\beta$ in J/J^2. A general element of J/J^2 is of the form $\sum \lambda_\alpha z_\alpha$ with $\lambda_\alpha \in k$ (only finitely many nonzero). Note that if the characteristic of k is $p > 0$ then

$$0 = p z_{\alpha/p} = z_{\alpha/p} + \ldots + z_{\alpha/p} = z_\alpha$$

and we see that $J/J^2 = 0$. If the characteristic of k is zero, then

$$J/J^2 = \mathbf{Q} \otimes_{\mathbf{Z}} k \cong k$$

(details omitted) is not zero.

We claim that $k[\mathbf{Q}] \to k$ is a formally smooth ring map if the characteristic of k is positive. Namely, suppose given a solid commutative diagram

$$\begin{tikzcd} k \arrow[r] \arrow[rd, dotted] & A \\ k[\mathbf{Q}] \arrow[u] \arrow[r, "\varphi"] & A' \arrow[u] \end{tikzcd}$$

with $A' \to A$ a surjection whose kernel I has square zero. To show that $k[\mathbf{Q}] \to k$ is formally smooth we have to prove that φ factors through k. Since $\varphi(x_\alpha - 1)$ maps to zero in A we see that φ induces a map $\overline{\varphi} : J/J^2 \to I$ whose vanishing is the obstruction to the desired factorization. Since $J/J^2 = 0$ if the characteristic is $p > 0$ we get the result we want, i.e., $k[\mathbf{Q}] \to k$ is formally smooth in this case. Finally, this ring map is not flat, for example as the nonzerodivisor $x_2 - 1$ is mapped to zero.

057W **Lemma 34.1.** *There exists a formally smooth ring map which is not flat.*

Proof. See discussion above. □

35. A formally étale non-flat ring map

060H In this section we give a counterexample to the final sentence in [DG67, 0, Example 19.10.3(i)] (this was not one of the items caught in their later errata lists). Consider $A \to A/J$ for a local ring A and a nonzero proper ideal J such that $J^2 = J$ (so J isn't finitely generated); the valuation ring of an algebraically closed non-archimedean field with J its maximal ideal is a source of such (A, J). These non-flat quotient maps are formally étale. Namely, suppose given a commutative diagram

$$\begin{tikzcd} A/J \arrow[r] & R/I \\ A \arrow[u] \arrow[r, "\varphi"] & R \arrow[u] \end{tikzcd}$$

where I is an ideal of the ring R with $I^2 = 0$. Then $A \to R$ factors uniquely through A/J because

$$\varphi(J) = \varphi(J^2) \subset (\varphi(J)A)^2 \subset I^2 = 0.$$

Hence this also provides a counterexample to the formally étale case of the "structure theorem" for locally finite type and formally étale morphisms in [DG67, IV, Theorem 18.4.6(i)] (but not a counterexample to part (ii), which is what people actually use in practice). The error in the proof of the latter is that the very last step of the proof is to invoke the incorrect [DG67, 0, Example 19.3.10(i)], which is how the counterexample just mentioned creeps in.

060I **Lemma 35.1.** *There exist formally étale nonflat ring maps.*

Proof. See discussion above. □

36. A formally étale ring map with nontrivial cotangent complex

06E5 Let k be a field. Consider the ring

$$R = k[\{x_n\}_{n \geq 1}, \{y_n\}_{n \geq 1}]/(x_1 y_1, x_{nm}^m - x_n, y_{nm}^m - y_n)$$

Let A be the localization at the maximal ideal generated by all x_n, y_n and denote $J \subset A$ the maximal ideal. Set $B = A/J$. By construction $J^2 = J$ and hence $A \to B$ is formally étale (see Section 35). We claim that the element $x_1 \otimes y_1$ is a nonzero element in the kernel of

$$J \otimes_A J \longrightarrow J.$$

Namely, (A, J) is the colimit of the localizations (A_n, J_n) of the rings

$$R_n = k[x_n, y_n]/(x_n^n y_n^n)$$

at their corresponding maximal ideals. Then $x_1 \otimes y_1$ corresponds to the element $x_n^n \otimes y_n^n \in J_n \otimes_{A_n} J_n$ and is nonzero (by an explicit computation which we omit). Since $\otimes$ commutes with colimits we conclude. By [Ill72, III Section 3.3] we see that J is not weakly regular. Hence by [Ill72, III Proposition 3.3.3] we see that the cotangent complex $L_{B/A}$ is not zero. In fact, we can be more precise. We have $H_0(L_{B/A}) = \Omega_{B/A}$ and $H_1(L_{B/A}) = 0$ because $J/J^2 = 0$. But from the five-term exact sequence of Quillen's fundamental spectral sequence (see Cotangent, Remark 11.5 or [Rei, Corollary 8.2.6]) and the nonvanishing of $\mathrm{Tor}_2^A(B, B) = \mathrm{Ker}(J \otimes_A J \to J)$ we conclude that $H_2(L_{B/A})$ is nonzero.

06E6 **Lemma 36.1.** *There exists a formally étale surjective ring map $A \to B$ with $L_{B/A}$ not equal to zero.*

Proof. See discussion above. □

37. Ideals generated by sets of idempotents and localization

04QK Let R be a ring. Consider the ring

$$B(R) = R[x_n; n \in \mathbf{Z}]/(x_n(x_n - 1), x_n x_m; n \neq m)$$

It is easy to show that every prime $\mathfrak{q} \subset B(R)$ is either of the form

$$\mathfrak{q} = \mathfrak{p}B(R) + (x_n; n \in \mathbf{Z})$$

or of the form

$$\mathfrak{q} = \mathfrak{p}B(R) + (x_n - 1) + (x_m; n \neq m, m \in \mathbf{Z}).$$

Hence we see that

$$\mathrm{Spec}(B(R)) = \mathrm{Spec}(R) \amalg \coprod\nolimits_{n \in \mathbf{Z}} \mathrm{Spec}(R)$$

where the topology is not just the disjoint union topology. It has the following properties: Each of the copies indexed by $n \in \mathbf{Z}$ is an open subscheme, namely it is the standard open $D(x_n)$. The "central" copy of $\mathrm{Spec}(R)$ is in the closure of the union of any infinitely many of the other copies of $\mathrm{Spec}(R)$. Note that this last copy of $\mathrm{Spec}(R)$ is cut out by the ideal $(x_n, n \in \mathbf{Z})$ which is generated by the idempotents x_n. Hence we see that if $\mathrm{Spec}(R)$ is connected, then the decomposition above is exactly the decomposition of $\mathrm{Spec}(B(R))$ into connected components.

Next, let $A = \mathbf{C}[x, y]/((y - x^2 + 1)(y + x^2 - 1))$. The spectrum of A consists of two irreducible components $C_1 = \mathrm{Spec}(A_1)$, $C_2 = \mathrm{Spec}(A_2)$ with $A_1 = \mathbf{C}[x, y]/(y - x^2 + 1)$ and $A_2 = \mathbf{C}[x, y]/(y + x^2 - 1)$. Note that these are parametrized by $(x, y) = (t, t^2 - 1)$ and $(x, y) = (t, -t^2 + 1)$ which meet in $P = (-1, 0)$ and $Q = (1, 0)$. We can make a twisted version of $B(A)$ where we glue $B(A_1)$ to $B(A_2)$ in the following way: Above P we let $x_n \in B(A_1) \otimes \kappa(P)$ correspond to $x_n \in B(A_2) \otimes \kappa(P)$, but above Q we let $x_n \in B(A_1) \otimes \kappa(Q)$ correspond to $x_{n+1} \in B(A_2) \otimes \kappa(Q)$. Let $B^{twist}(A)$ denote the resulting A-algebra. Details omitted. By construction $B^{twist}(A)$ is Zariski locally over A isomorphic to the untwisted version. Namely, this happens over both the principal open $\mathrm{Spec}(A) \setminus \{P\}$ and the principal open $\mathrm{Spec}(A) \setminus \{Q\}$. However, our choice of glueing produces enough "monodromy" such that $\mathrm{Spec}(B^{twist}(A))$ is connected (details omitted). Finally, there is a central copy of $\mathrm{Spec}(A) \to \mathrm{Spec}(B^{twist}(A))$ which gives a closed subscheme whose ideal is Zariski locally on $B^{twist}(A)$ cut out by ideals generated by idempotents, but not globally (as $B^{twist}(A)$ has no nontrivial idempotents).

04QL **Lemma 37.1.** *There exists an affine scheme $X = \mathrm{Spec}(A)$ and a closed subscheme $T \subset X$ such that T is Zariski locally on X cut out by ideals generated by idempotents, but T is not cut out by an ideal generated by idempotents.*

Proof. See above. □

38. A ring map which identifies local rings which is not ind-étale

09AN Note that the ring map $R \to B(R)$ constructed in Section 37 is a colimit of finite products of copies of R. Hence $R \to B(R)$ is ind-Zariski, see Pro-étale Cohomology, Definition 4.1. Next, consider the ring map $A \to B^{twist}(A)$ constructed in Section 37. Since this ring map is Zariski locally on $\mathrm{Spec}(A)$ isomorphic to an ind-Zariski ring map $R \to B(R)$ we conclude that it identifies local rings (see Pro-étale Cohomology, Lemma 4.6). The discussion in Section 37 shows there is a section $B^{twist}(A) \to A$ whose kernel is not generated by idempotents. Now, if $A \to B^{twist}(A)$ were ind-étale, i.e., $B^{twist}(A) = \mathrm{colim}\, A_i$ with $A \to A_i$ étale, then the kernel of $A_i \to A$ would be generated by an idempotent (Algebra, Lemmas 141.8 and 141.9). This would contradict the result mentioned above.

09AP **Lemma 38.1.** *There is a ring map $A \to B$ which identifies local rings but which is not ind-étale. A fortiori it is not ind-Zariski.*

Proof. See discussion above. □

39. Non flasque quasi-coherent sheaf associated to injective module

0273 For more examples of this type see [BGI71, Exposé II, Appendix I] where Illusie explains some examples due to Verdier.

Consider the affine scheme $X = \mathrm{Spec}(A)$ where

$$A = k[x, y, z_1, z_2, \ldots]/(x^n z_n)$$

is the ring from Properties, Example 25.2. Set $I = (x) \subset A$. Consider the quasi-compact open $U = D(x)$ of X. We have seen in loc. cit. that there is a section $s \in \mathcal{O}_X(U)$ which does not come from an A-module map $I^n \to A$ for any $n \geq 0$.

Let $\alpha : A \to J$ be the embedding of A into an injective A-module. Let $Q = J/\alpha(A)$ and denote $\beta : J \to Q$ the quotient map. We claim that the map

$$\Gamma(X, \widetilde{J}) \longrightarrow \Gamma(U, \widetilde{J})$$

is not surjective. Namely, we claim that $\alpha(s)$ is not in the image. To see this, we argue by contradiction. So assume that $x \in J$ is an element which restricts to $\alpha(s)$ over U. Then $\beta(x) \in Q$ is an element which restricts to 0 over U. Hence we know that $I^n\beta(x) = 0$ for some n, see Properties, Lemma 25.1. This implies that we get a morphism $\varphi : I^n \to A$, $h \mapsto \alpha^{-1}(hx)$. It is easy to see that this morphism φ gives rise to the section s via the map of Properties, Lemma 25.1 which is a contradiction.

0274 **Lemma 39.1.** *There exists an affine scheme $X = \mathrm{Spec}(A)$ and an injective A-module J such that $\widetilde{J}$ is not a flasque sheaf on X. Even the restriction $\Gamma(X, \widetilde{J}) \to \Gamma(U, \widetilde{J})$ with U a standard open need not be surjective.*

Proof. See above. □

In fact, we can use a similar construction to get an example of an injective module whose associated quasi-coherent sheaf has nonzero cohomology over a quasi-compact open. Namely, we start with the ring

$$A = k[x, y, w_1, u_1, w_2, u_2, \ldots]/(x^n w_n, y^n u_n, u_n^2, w_n^2)$$

where k is a field. Choose an injective map $A \to I$ where I is an injective A-module. We claim that the element $1/xy$ in $A_{xy} \subset I_{xy}$ is not in the image of $I_x \oplus I_y \to I_{xy}$. Arguing by contradiction, suppose that

$$\frac{1}{xy} = \frac{i}{x^n} + \frac{j}{y^n}$$

for some $n \geq 1$ and $i, j \in I$. Clearing denominators we obtain

$$(xy)^{n+m-1} = x^m y^{n+m} i + x^{n+m} y^m j$$

for some $m \geq 0$. Multiplying with $u_{n+m}w_{n+m}$ we see that $u_{n+m}w_{n+m}(xy)^{n+m-1} = 0$ in A which is the desired contradiction. Let $U = D(x) \cup D(y) \subset X = \mathrm{Spec}(A)$. For any A-module M we have an exact sequence

$$0 \to H^0(U, \widetilde{M}) \to M_x \oplus M_y \to M_{xy} \to H^1(U, \widetilde{M}) \to 0$$

by Mayer-Vietoris. We conclude that $H^1(U, \widetilde{I})$ is nonzero.

0CRZ **Lemma 39.2.** *There exists an affine scheme $X = \mathrm{Spec}(A)$ whose underlying topological space is Noetherian and an injective A-module I such that $\widetilde{I}$ has nonvanishing H^1 on some quasi-compact open U of X.*

Proof. See above. Note that $\mathrm{Spec}(A) = \mathrm{Spec}(k[x, y])$ as topological spaces. $\square$

40. A non-separated flat group scheme

06E7 Every group scheme over a field is separated, see Groupoids, Lemma 7.3. This is not true for group schemes over a base.

Let k be a field. Let $S = \mathrm{Spec}(k[x]) = \mathbf{A}^1_k$. Let G be the affine line with 0 doubled (see Schemes, Example 14.3) seen as a scheme over S. Thus a fibre of $G \to S$ is either a singleton or a set with two elements (one in U and one in V). Thus we can endow these fibres with the structure of a group (by letting the element in U be the zero of the group structure). More precisely, G has two opens U, V which map isomorphically to S such that $U \cap V$ is mapped isomorphically to $S \setminus \{0\}$. Then

$$G \times_S G = U \times_S U \cup V \times_S U \cup U \times_S V \cup V \times_S V$$

where each piece is isomorphic to S. Hence we can define a multiplication $m : G \times_S G \to G$ as the unique S-morphism which maps the first and the last piece into U and the two middle pieces into V. This matches the pointwise description given above. We omit the verification that this defines a group scheme structure.

06E8 **Lemma 40.1.** *There exists a flat group scheme of finite type over the affine line which is not separated.*

Proof. See the discussion above. $\square$

08IX **Lemma 40.2.** *There exists a flat group scheme of finite type over the infinite dimensional affine space which is not quasi-separated.*

Proof. The same construction as above can be carried out with the infinite dimensional affine space $S = \mathbf{A}^\infty_k = \mathrm{Spec}\, k[x_1, x_2, \ldots]$ as the base and the origin $0 \in S$ corresponding to the maximal ideal $(x_1, x_2, \ldots)$ as the closed point which is doubled in G. The resulting group scheme $G \to S$ is not quasi-separated as explained in Schemes, Example 21.4. $\square$

41. A non-flat group scheme with flat identity component

06RJ Let $X \to S$ be a monomorphism of schemes. Let $G = S \amalg X$. Let $m : G \times_S G \to G$ be the S-morphism

$$G \times_S G = X \times_S X \amalg X \amalg X \amalg S \longrightarrow G = X \amalg S$$

which maps the summands $X \times_S X$ and S into S and maps the summands X into X by the identity morphism. This defines a group law. To see this we have to show that $m \circ (m \times \mathrm{id}_G) = m \circ (\mathrm{id}_G \times m)$ as maps $G \times_S G \times_S G \to G$. Decomposing $G \times_S G \times_S G$ into components as above, we see that we need to verify this for the restriction to each of the 8-pieces. Each piece is isomorphic to either S, X, $X \times_S X$, or $X \times_S X \times_S X$. Moreover, both maps map these pieces to S, X, S, X respectively. Having said this, the fact that $X \to S$ is a monomorphism implies that $X \times_S X \cong X$ and $X \times_S X \times_S X \cong X$ and that there is in each case exactly one S-morphism $S \to S$ or $X \to X$. Thus we see that $m \circ (m \times \mathrm{id}_G) = m \circ (\mathrm{id}_G \times m)$. Thus taking $X \to S$ to be any nonflat monomorphism of schemes (e.g., a closed immersion) we get an example of a group scheme over a base S whose identity component is S (hence flat) but which is not flat.

06RK **Lemma 41.1.** *There exists a group scheme G over a base S whose identity component is flat over S but which is not flat over S.*

Proof. See discussion above. □

42. A non-separated group algebraic space over a field

06E9 Every group scheme over a field is separated, see Groupoids, Lemma 7.3. This is not true for group algebraic spaces over a field (but see end of this section for positive results).

Let k be a field of characteristic zero. Consider the algebraic space $G = \mathbf{A}^1_k/\mathbf{Z}$ from Spaces, Example 14.8. By construction G is the fppf sheaf associated to the presheaf

$$T \longmapsto \Gamma(T, \mathcal{O}_T)/\mathbf{Z}$$

on the category of schemes over k. The obvious addition rule on the presheaf induces an addition $m : G \times G \to G$ which turns G into a group algebraic space over $\text{Spec}(k)$. Note that G is not separated (and not even quasi-separated or locally separated). On the other hand $G \to \text{Spec}(k)$ is of finite type!

06EA **Lemma 42.1.** *There exists a group algebraic space of finite type over a field which is not separated (and not even quasi-separated or locally separated).*

Proof. See discussion above. □

Positive results: If the group algebraic space G is either quasi-separated, or locally separated, or more generally a decent algebraic space, then G is in fact separated, see More on Groupoids in Spaces, Lemma 9.4. Moreover, a finite type, separated group algebraic space over a field is in fact a scheme by More on Groupoids in Spaces, Lemma 10.2. The idea of the proof is that the schematic locus is open dense, see Properties of Spaces, Proposition 12.3 or Decent Spaces, Theorem 10.2. By translating this open we see that every point of G has an open neighbourhood which is a scheme.

43. Specializations between points in fibre étale morphism

06UJ If $f : X \to Y$ is an étale, or more generally a locally quasi-finite morphism of schemes, then there are no specializations between points of fibres, see Morphisms, Lemma 19.8. However, for morphisms of algebraic spaces this doesn't hold in general.

To give an example, let k be a field. Set

$$P = k[u, u^{-1}, y, \{x_n\}_{n \in \mathbf{Z}}].$$

Consider the action of $\mathbf{Z}$ on P by k-algebra maps generated by the automorphism τ given by the rules $\tau(u) = u$, $\tau(y) = uy$, and $\tau(x_n) = x_{n+1}$. For $d \geq 1$ set $I_d = ((1 - u^d)y, x_n - x_{n+d}, n \in \mathbf{Z})$. Then $V(I_d) \subset \text{Spec}(P)$ is the fix point locus of τ^d. Let $S \subset P$ be the multiplicative subset generated by y and all $1 - u^d$, $d \in \mathbf{N}$. Then we see that $\mathbf{Z}$ acts freely on $U = \text{Spec}(S^{-1}P)$. Let $X = U/\mathbf{Z}$ be the quotient algebraic space, see Spaces, Definition 14.4.

Consider the prime ideals $\mathfrak{p}_n = (x_n, x_{n+1}, \ldots)$ in $S^{-1}P$. Note that $\tau(\mathfrak{p}_n) = \mathfrak{p}_{n+1}$. Hence each of these define point $\xi_n \in U$ whose image in X is the same point x of X. Moreover we have the specializations

$$\ldots \rightsquigarrow \xi_n \rightsquigarrow \xi_{n-1} \rightsquigarrow \ldots$$

We conclude that $U \to X$ is an example of the promised type.

06UK **Lemma 43.1.** *There exists an étale morphism of algebraic spaces $f : X \to Y$ and a nontrivial specialization of points $x \rightsquigarrow x'$ in $|X|$ with $f(x) = f(x')$ in $|Y|$.*

Proof. See discussion above. □

44. A torsor which is not an fppf torsor

04AF In Groupoids, Remark 11.5 we raise the question whether any G-torsor is a G-torsor for the fppf topology. In this section we show that this is not always the case.

Let k be a field. All schemes and stacks are over k in what follows. Let $G \to \mathrm{Spec}(k)$ be the group scheme

$$G = (\mu_{2,k})^\infty = \mu_{2,k} \times_k \mu_{2,k} \times_k \mu_{2,k} \times_k \ldots = \lim_n (\mu_{2,k})^n$$

where $\mu_{2,k}$ is the group scheme of second roots of unity over $\mathrm{Spec}(k)$, see Groupoids, Example 5.2. As an inverse limit of affine schemes we see that G is an affine group scheme. In fact it is the spectrum of the ring $k[t_1, t_2, t_3, \ldots]/(t_i^2 - 1)$. The multiplication map $m : G \times_k G \to G$ is on the algebra level given by $t_i \mapsto t_i \otimes t_i$.

We claim that any G-torsor over k is of the form

$$P = \mathrm{Spec}(k[x_1, x_2, x_3, \ldots]/(x_i^2 - a_i))$$

for certain $a_i \in k^*$ and with G-action $G \times_k P \to P$ given by $x_i \to t_i \otimes x_i$ on the algebra level. We omit the proof. Actually for the example we only need that P is a G-torsor which is clear since over $k' = k(\sqrt{a_1}, \sqrt{a_2}, \ldots)$ the scheme P becomes isomorphic to G in a G-equivariant manner. Note that P is trivial if and only if $k' = k$ since if P has a k-rational point then all of the a_i are squares.

We claim that P is an fppf torsor if and only if the field extension $k \subset k' = k(\sqrt{a_1}, \sqrt{a_2}, \ldots)$ is finite. If k' is finite over k, then $\{\mathrm{Spec}(k') \to \mathrm{Spec}(k)\}$ is an fppf covering which trivializes P and we see that P is indeed an fppf torsor. Conversely, suppose that P is an fppf G-torsor. This means that there exists an fppf covering $\{S_i \to \mathrm{Spec}(k)\}$ such that each P_{S_i} is trivial. Pick an i such that S_i is not empty. Let $s \in S_i$ be a closed point. By Varieties, Lemma 14.1 the field extension $k \subset \kappa(s)$ is finite, and by construction $P_{\kappa(s)}$ has a $\kappa(s)$-rational point. Thus we see that $k \subset k' \subset \kappa(s)$ and k' is finite over k.

To get an explicit example take $k = \mathbf{Q}$ and $a_i = i$ for example (or a_i is the ith prime if you like).

077B **Lemma 44.1.** *Let S be a scheme. Let G be a group scheme over S. The stack G-Principal classifying principal homogeneous G-spaces (see Examples of Stacks, Subsection 14.5) and the stack G-Torsors classifying fppf G-torsors (see Examples of Stacks, Subsection 14.8) are not equivalent in general.*

Proof. The discussion above shows that the functor *G-Torsors* $\to$ *G-Principal* isn't essentially surjective in general. □

45. Stack with quasi-compact flat covering which is not algebraic

04AG In this section we briefly describe an example due to Brian Conrad. You can find the example online at this location. Our example is slightly different.

Let k be an algebraically closed field. All schemes and stacks are over k in what follows. Let $G \to \mathrm{Spec}(k)$ be an affine group scheme. In Examples of Stacks, Lemma 15.4 we have given several different equivalent ways to view $\mathcal{X} = [\mathrm{Spec}(k)/G]$ as a stack in groupoids over $(Sch/\mathrm{Spec}(k))_{fppf}$. In particular $\mathcal{X}$ classifies fppf G-torsors. More precisely, a 1-morphism $T \to \mathcal{X}$ corresponds to an fppf G_T-torsor P over T and 2-arrows correspond to isomorphisms of torsors. It follows that the diagonal 1-morphism

$$\Delta : \mathcal{X} \longrightarrow \mathcal{X} \times_{\mathrm{Spec}(k)} \mathcal{X}$$

is representable and affine. Namely, given any pair of fppf G_T-torsors P_1, P_2 over a scheme T/k the scheme $Isom(P_1, P_2)$ is affine over T. The trivial G-torsor over $\mathrm{Spec}(k)$ defines a 1-morphism

$$f : \mathrm{Spec}(k) \longrightarrow \mathcal{X}.$$

We claim that this is a surjective 1-morphism. The reason is simply that by definition for any 1-morphism $T \to \mathcal{X}$ there exists a fppf covering $\{T_i \to T\}$ such that P_{T_i} is isomorphic to the trivial G_{T_i}-torsor. Hence the compositions $T_i \to T \to \mathcal{X}$ factor through f. Thus it is clear that the projection $T \times_{\mathcal{X}} \mathrm{Spec}(k) \to T$ is surjective (which is how we define the property that f is surjective, see Algebraic Stacks, Definition 10.1). In a similar way you show that f is quasi-compact and flat (details omitted). We also record here the observation that

$$\mathrm{Spec}(k) \times_{\mathcal{X}} \mathrm{Spec}(k) \cong G$$

as schemes over k.

Suppose there exists a surjective smooth morphism $p : U \to \mathcal{X}$ where U is a scheme. Consider the fibre product

$$\begin{tikzcd} W \arrow[r] \arrow[d] & U \arrow[d] \\ \mathrm{Spec}(k) \arrow[r] & \mathcal{X} \end{tikzcd}$$

Then we see that W is a nonempty smooth scheme over k which hence has a k-point. This means that we can factor f through U. Hence we obtain

$$G \cong \mathrm{Spec}(k) \times_{\mathcal{X}} \mathrm{Spec}(k) \cong (\mathrm{Spec}(k) \times_k \mathrm{Spec}(k)) \times_{(U \times_k U)} (U \times_{\mathcal{X}} U)$$

and since the projections $U \times_{\mathcal{X}} U \to U$ were assumed smooth we conclude that $U \times_{\mathcal{X}} U \to U \times_k U$ is locally of finite type, see Morphisms, Lemma 14.8. It follows that in this case G is locally of finite type over k. Altogether we have proved the following lemma (which can be significantly generalized).

04AH **Lemma 45.1.** *Let k be a field. Let G be an affine group scheme over k. If the stack $[\mathrm{Spec}(k)/G]$ has a smooth covering by a scheme, then G is of finite type over k.*

Proof. See discussion above. $\square$

To get an explicit example as in the title of this section, take for example $G = (\mu_{2,k})^\infty$ the group scheme of Section 44, which is not locally of finite type over k. By the discussion above we see that $\mathcal{X} = [\mathrm{Spec}(k)/G]$ has properties (1) and (2) of Algebraic Stacks, Definition 12.1, but not property (3). Hence $\mathcal{X}$ is not an algebraic stack. On the other hand, there does exist a scheme U and a surjective, flat, quasi-compact morphism $U \to \mathcal{X}$, namely the morphism $f : \mathrm{Spec}(k) \to \mathcal{X}$ we studied above.

46. Limit preserving on objects, not limit preserving

07Z0 Let S be a nonempty scheme. Let $\mathcal{G}$ be an injective abelian sheaf on $(Sch/S)_{fppf}$. We obtain a stack in groupoids

$$\mathcal{G}\text{-}Torsors \longrightarrow (Sch/S)_{fppf}$$

over S, see Examples of Stacks, Lemma 14.2. This stack is limit preserving on objects over $(Sch/S)_{fppf}$ (see Criteria for Representability, Section 5) because every $\mathcal{G}$-torsor is trivial. On the other hand, $\mathcal{G}\text{-}Torsors$ is in general not limit preserving (see Artin's Axioms, Definition 11.1) as $\mathcal{G}$ need not be limit preserving as a sheaf. For example, take any nonzero injective sheaf $\mathcal{I}$ and set $\mathcal{G} = \prod_{n \in \mathbf{Z}} \mathcal{I}$ to get an example.

07Z1 **Lemma 46.1.** *Let S be a nonempty scheme. There exists a stack in groupoids $p : \mathcal{X} \to (Sch/S)_{fppf}$ such that p is limit preserving on objects, but $\mathcal{X}$ is not limit preserving.*

Proof. See discussion above. □

47. A non-algebraic classifying stack

077C Let $S = \mathrm{Spec}(\mathbf{F}_p)$ and let μ_p denote the group scheme of pth roots of unity over S. In Groupoids in Spaces, Section 19 we have introduced the quotient stack $[S/\mu_p]$ and in Examples of Stacks, Section 15 we have shown $[S/\mu_p]$ is the classifying stack for fppf μ_p-torsors: Given a scheme T over S the category $Mor_S(T, [S/\mu_p])$ is canonically equivalent to the category of fppf μ_p-torsors over T. Finally, in Criteria for Representability, Theorem 17.2 we have seen that $[S/\mu_p]$ is an algebraic stack.

Now we can ask the question: "How about the category fibred in groupoids $\mathcal{S}$ classifying étale μ_p-torsors?" (In other words $\mathcal{S}$ is a category over Sch/S whose fibre category over a scheme T is the category of étale μ_p-torsors over T.)

The first objection is that this isn't a stack for the fppf topology, because descent for objects isn't going to hold. For example the μ_p-torsor $\mathrm{Spec}(\mathbf{F}_p(t)[x]/(x^p - t))$ over $T = \mathrm{Spec}(\mathbf{F}_p(T))$ is fppf locally trivial, but not étale locally trivial.

A fix for this first problem is to work with the étale topology and in this case descent for objects does work. Indeed it is true that $\mathcal{S}$ is a stack in groupoids over $(Sch/S)_{étale}$. Moreover, it is also the case that the diagonal $\Delta : \mathcal{S} \to \mathcal{S} \times \mathcal{S}$ is representable (by schemes). This is true because given two μ_p-torsors (whether they be étale locally trivial or not) the sheaf of isomorphisms between them is representable by a scheme.

Thus we can finally ask if there exists a scheme U and a smooth and surjective 1-morphism $U \to \mathcal{S}$. We will show in two ways that this is impossible: by a direct argument (which we advise the reader to skip) and by an argument using a general result.

Direct argument (sketch): Note that the 1-morphism $\mathcal{S} \to \mathrm{Spec}(\mathbf{F}_p)$ satisfies the infinitesimal lifting criterion for formal smoothness. This is true because given a first order infinitesimal thickening of schemes $T \to T'$ the kernel of $\mu_p(T') \to \mu_p(T)$ is isomorphic to the sections of the ideal sheaf of T in T', and hence $H^1_{\acute{e}tale}(T, \mu_p) = H^1_{\acute{e}tale}(T', \mu_p)$. Moreover, $\mathcal{S}$ is a limit preserving stack. Hence if $U \to \mathcal{S}$ is smooth, then $U \to \mathrm{Spec}(\mathbf{F}_p)$ is limit preserving and satisfies the infinitesimal lifting criterion for formal smoothness. This implies that U is smooth over $\mathbf{F}_p$. In particular U is reduced, hence $H^1_{\acute{e}tale}(U, \mu_p) = 0$. Thus $U \to \mathcal{S}$ factors as $U \to \mathrm{Spec}(\mathbf{F}_p) \to \mathcal{S}$ and the first arrow is smooth. By descent of smoothness, we see that $U \to \mathcal{S}$ being smooth would imply $\mathrm{Spec}(\mathbf{F}_p) \to \mathcal{S}$ is smooth. However, this is not the case as $\mathrm{Spec}(\mathbf{F}_p) \times_{\mathcal{S}} \mathrm{Spec}(\mathbf{F}_p)$ is μ_p which is not smooth over $\mathrm{Spec}(\mathbf{F}_p)$.

Structural argument: In Criteria for Representability, Section 19 we have seen that we can think of algebraic stacks as those stacks in groupoids for the étale topology with diagonal representable by algebraic spaces having a smooth covering. Hence if a smooth surjective $U \to \mathcal{S}$ exists then $\mathcal{S}$ is an algebraic stack, and in particular satisfies descent in the fppf topology. But we've seen above that $\mathcal{S}$ does not satisfies descent in the fppf topology.

Loosely speaking the arguments above show that the classifying stack in the étale topology for étale locally trivial torsors for a group scheme G over a base B is algebraic if and only if G is smooth over B. One of the advantages of working with the fppf topology is that it suffices to assume that $G \to B$ is flat and locally of finite presentation. In fact the quotient stack (for the fppf topology) $[B/G]$ is algebraic if and only if $G \to B$ is flat and locally of finite presentation, see Criteria for Representability, Lemma 18.3.

48. Sheaf with quasi-compact flat covering which is not algebraic

078E Consider the functor $F = (\mathbf{P}^1)^\infty$, i.e., for a scheme T the value $F(T)$ is the set of $f = (f_1, f_2, f_3, \ldots)$ where each $f_i : T \to \mathbf{P}^1$ is a morphism of schemes. Note that $\mathbf{P}^1$ satisfies the sheaf property for fpqc coverings, see Descent, Lemma 10.3. A product of sheaves is a sheaf, so F also satisfies the sheaf property for the fpqc topology. The diagonal of F is representable: if $f : T \to F$ and $g : S \to F$ are morphisms, then $T \times_F S$ is the scheme theoretic intersection of the closed subschemes $T \times_{f_i, \mathbf{P}^1, g_i} S$ inside the scheme $T \times S$. Consider the group scheme SL_2 which comes with a surjective smooth affine morphism $\mathrm{SL}_2 \to \mathbf{P}^1$. Next, consider $U = (\mathrm{SL}_2)^\infty$ with its canonical (product) morphism $U \to F$. Note that U is an affine scheme. We claim the morphism $U \to F$ is flat, surjective, and universally open. Namely, suppose $f : T \to F$ is a morphism. Then $Z = T \times_F U$ is the infinite fibre product of the schemes $Z_i = T \times_{f_i, \mathbf{P}^1} \mathrm{SL}_2$ over T. Each of the morphisms $Z_i \to T$ is surjective smooth and affine which implies that

$$Z = Z_1 \times_T Z_2 \times_T Z_3 \times_T \ldots$$

is a scheme flat and affine over Z. A simple limit argument shows that $Z \to T$ is open as well.

On the other hand, we claim that F isn't an algebraic space. Namely, if F where an algebraic space it would be a quasi-compact and separated (by our description of fibre products over F) algebraic space. Hence cohomology of quasi-coherent sheaves would vanish above a certain cutoff (see Cohomology of Spaces, Proposition 7.2 and remarks preceding it). But clearly by taking the pullback of $\mathcal{O}(-2, -2, \ldots, -2)$ under the projection

$$(\mathbf{P}^1)^\infty \longrightarrow (\mathbf{P}^1)^n$$

(which has a section) we can obtain a quasi-coherent sheaf whose cohomology is nonzero in degree n. Altogether we obtain an answer to a question asked by Anton Geraschenko on mathoverflow.

078F **Lemma 48.1.** *There exists a functor $F : Sch^{opp} \to Sets$ which satisfies the sheaf condition for the fpqc topology, has representable diagonal $\Delta : F \to F \times F$, and such that there exists a surjective, flat, universally open, quasi-compact morphism $U \to F$ where U is a scheme, but such that F is not an algebraic space.*

Proof. See discussion above. □

49. Sheaves and specializations

05LD In the following we fix a big étale site $Sch_{étale}$ as constructed in Topologies, Definition 4.6. Moreover, a scheme will be an object of this site. Recall that if x, x' are points of a scheme X we say x is a *specialization* of x' or we write $x' \rightsquigarrow x$ if $x \in \overline{\{x'\}}$. This is true in particular if $x = x'$.

Consider the functor $F : Sch_{étale} \to Ab$ defined by the following rules:

$$F(X) = \prod\nolimits_{x \in X} \prod\nolimits_{x' \in X, x' \rightsquigarrow x, x' \neq x} \mathbf{Z}/2\mathbf{Z}$$

Given a scheme X we denote $|X|$ the underlying set of points. An element $a \in F(X)$ will be viewed as a map of sets $|X| \times |X| \to \mathbf{Z}/2\mathbf{Z}$, $(x, x') \mapsto a(x, x')$ which is zero if $x = x'$ or if x is not a specialization of x'. Given a morphism of schemes $f : X \to Y$ we define

$$F(f) : F(Y) \longrightarrow F(X)$$

by the rule that for $b \in F(Y)$ we set

$$F(f)(b)(x, x') = \begin{cases} 0 & \text{if } x \text{ is not a specialization of } x' \\ b(f(x), f(x')) & \text{else.} \end{cases}$$

Note that this really does define an element of $F(X)$. We claim that if $f : X \to Y$ and $g : Y \to Z$ are composable morphisms then $F(f) \circ F(g) = F(g \circ f)$. Namely, let $c \in F(Z)$ and let $x' \rightsquigarrow x$ be a specialization of points in X, then

$$F(g \circ f)(x, x') = c(g(f(x)), g(f(x'))) = F(g)(F(f)(c))(x, x')$$

because $f(x') \rightsquigarrow f(x)$. (This also works if $f(x) = f(x')$.)

Let G be the sheafification of F in the étale topology.

I claim that if X is a scheme and $x' \rightsquigarrow x$ is a specialization and $x' \neq x$, then $G(X) \neq 0$. Namely, let $a \in F(X)$ be an element such that when we think of a as a function $|X| \times |X| \to \mathbf{Z}/2\mathbf{Z}$ it is nonzero at (x, x'). Let $\{f_i : U_i \to X\}$ be an étale covering of X. Then we can pick an i and a point $u_i \in U_i$ with $f_i(u_i) = x$. Since generalizations lift along flat morphisms (see Morphisms, Lemma 24.8) we can find a specialization $u_i' \rightsquigarrow u_i$ with $f_i(u_i') = x'$. By our construction above we see that $F(f_i)(a) \neq 0$. Hence a determines a nonzero element of $G(X)$.

Note that if $X = \text{Spec}(k)$ where k is a field (or more generally a ring all of whose prime ideals are maximal), then $F(X) = 0$ and for every étale morphism $U \to X$ we have $F(U) = 0$ because there are no specializations between distinct points in fibres of an étale morphism. Hence $G(X) = 0$.

Suppose that $X \subset X'$ is a thickening, see More on Morphisms, Definition 2.1. Then the category of schemes étale over X' is equivalent to the category of schemes étale over X by the base change functor $U' \mapsto U = U' \times_{X'} X$, see Étale Cohomology, Theorem 45.2. Since it is always the case that $F(U) = F(U')$ in this situation we see that also $G(X) = G(X')$.

As a variant we can consider the presheaf F_n which associates to a scheme X the collection of maps $a : |X|^{n+1} \to \mathbf{Z}/2\mathbf{Z}$ where $a(x_0, \ldots, x_n)$ is nonzero only if $x_n \leadsto \ldots \leadsto x_0$ is a sequence of specializations and $x_n \neq x_{n-1} \neq \ldots \neq x_0$. Let G_n be the sheaf associated to F_n. In exactly the same way as above one shows that G_n is nonzero if $\dim(X) \geq n$ and is zero if $\dim(X) < n$.

05LE **Lemma 49.1.** *There exists a sheaf of abelian groups G on $Sch_{étale}$ with the following properties*

1. $G(X) = 0$ *whenever* $\dim(X) < n$,
2. $G(X)$ *is not zero if* $\dim(X) \geq n$, *and*
3. *if* $X \subset X'$ *is a thickening, then* $G(X) = G(X')$.

Proof. See the discussion above. □

05LF **Remark 49.2.** Here are some remarks:

1. The presheaves F and F_n are separated presheaves.
2. It turns out that F, F_n are not sheaves.
3. One can show that G, G_n is actually a sheaf for the fppf topology.

We will prove these results if we need them.

50. Sheaves and constructible functions

05LG In the following we fix a big étale site $Sch_{étale}$ as constructed in Topologies, Definition 4.6. Moreover, a scheme will be an object of this site. In this section we say that a *constructible partition* of a scheme X is a locally finite disjoint union decomposition $X = \coprod_{i \in I} X_i$ such that each $X_i \subset X$ is a locally constructible subset of X. Locally finite means that for any quasi-compact open $U \subset X$ there are only finitely many $i \in I$ such that $X_i \cap U$ is not empty. Note that if $f : X \to Y$ is a morphism of schemes and $Y = \coprod Y_j$ is a constructible partition, then $X = \coprod f^{-1}(Y_j)$ is a constructible partition of X. Given a set S and a scheme X a *constructible function* $f : |X| \to S$ is a map such that $X = \coprod_{s \in S} f^{-1}(s)$ is a constructible partition of X. If G is an (abstract group) and $a, b : |X| \to G$ are constructible functions, then $ab : |X| \to G, x \mapsto a(x)b(x)$ is a constructible function too. The reason is that given any two constructible partitions there is a third one refining both.

Let A be any abelian group. For any scheme X we define

$$F(X) = \frac{\{a : |X| \to A \mid a \text{ is a constructible function}\}}{\{\text{locally constant functions } |X| \to A\}}$$

We think of an element a of $F(X)$ simply as a function well defined up to adding a locally constant one. Given a morphism of schemes $f : X \to Y$ and an element $b \in F(Y)$, then we define $F(f)(b) = b \circ f$. Thus F is a presheaf on $Sch_{étale}$.

Note that if $\{f_i : U_i \to X\}$ is an fppf covering, and $a \in F(X)$ is such that $F(f_i)(a) = 0$ in $F(U_i)$, then $a \circ f_i$ is a locally constant function for each i. This means in turn that a is a locally constant function as the morphisms f_i are open. Hence $a = 0$ in $F(X)$. Thus we see that F is a separated presheaf (in the fppf topology hence a fortiori in the étale topology).

Let G be the sheafification of F in the étale topology. Since F is separated, and since $F(X) \neq 0$ for example when X is the spectrum of a discrete valuation ring, we see that G is not zero.

Let $X = \mathrm{Spec}(k)$ where k is a field. Then any étale covering of X can be dominated by a covering $\{\mathrm{Spec}(k') \to \mathrm{Spec}(k)\}$ with $k \subset k'$ a finite separable extension of fields. Since $F(\mathrm{Spec}(k')) = 0$ we see that $G(X) = 0$.

Suppose that $X \subset X'$ is a thickening, see More on Morphisms, Definition 2.1. Then the category of schemes étale over X' is equivalent to the category of schemes étale over X by the base change functor $U' \mapsto U = U' \times_{X'} X$, see Étale Cohomology, Theorem 45.2. Since $F(U) = F(U')$ in this situation we see that also $G(X) = G(X')$.

The sheaf G is limit preserving, see Limits of Spaces, Definition 3.1. Namely, let R be a ring which is written as a directed colimit $R = \mathrm{colim}_i R_i$ of rings. Set $X = \mathrm{Spec}(R)$ and $X_i = \mathrm{Spec}(R_i)$, so that $X = \lim_i X_i$. Then $G(X) = \mathrm{colim}_i G(X_i)$. To prove this one first proves that a constructible partition of $\mathrm{Spec}(R)$ comes from a constructible partitions of some $\mathrm{Spec}(R_i)$. Hence the result for F. To get the result for the sheafification, use that any étale ring map $R \to R'$ comes from an étale ring map $R_i \to R'_i$ for some i. Details omitted.

05LH **Lemma 50.1.** *There exists a sheaf of abelian groups G on $Sch_{étale}$ with the following properties*

1. $G(\mathrm{Spec}(k)) = 0$ *whenever k is a field,*
2. *G is limit preserving,*
3. *if $X \subset X'$ is a thickening, then $G(X) = G(X')$, and*
4. *G is not zero.*

Proof. See discussion above. ☐

51. The lisse-étale site is not functorial

07BF The *lisse-étale* site $X_{lisse,étale}$ of X is the category of schemes smooth over X endowed with (usual) étale coverings, see Cohomology of Stacks, Section 11. Let $f : X \to Y$ be a morphism of schemes. There is a functor

$$u : Y_{lisse,étale} \longrightarrow X_{lisse,étale}, \quad V/Y \longmapsto V \times_Y X$$

which is continuous. Hence we obtain an adjoint pair of functors

$$u^s : Sh(X_{lisse,étale}) \longrightarrow Sh(Y_{lisse,étale}), \quad u_s : Sh(Y_{lisse,étale}) \longrightarrow Sh(X_{lisse,étale}),$$

see Sites, Section 13. We claim that, in general, u does **not** define a morphism of sites, see Sites, Definition 14.1. In other words, we claim that u_s is not left exact in general. Note that representable presheaves are sheaves on lisse-étale sites. Hence, by Sites, Lemma 13.5 we see that $u_s h_V = h_{V \times_Y X}$. Now consider two morphisms

$$\begin{tikzcd} V_1 \arrow[rr, shift left, "a"] \arrow[rr, shift right, "b"'] \arrow[dr] & & V_2 \arrow[dl] \\ & Y & \end{tikzcd}$$

of schemes V_1, V_2 smooth over Y. Now if u_s is left exact, then we would have

$$u_s \text{Equalizer}(h_a, h_b : h_{V_1} \to h_{V_2}) = \text{Equalizer}(h_{a\times 1}, h_{b \times 1} : h_{V_1 \times_Y X} \to h_{V_2 \times_Y X})$$

We will take the morphisms $a, b : V_1 \to V_2$ such that there exists no morphism from a scheme smooth over Y into $(a = b) \subset V_1$, i.e., such that the left hand side is the empty sheaf, but such that after base change to X the equalizer is nonempty and smooth over X. A silly example is to take $X = \text{Spec}(\mathbf{F}_p)$, $Y = \text{Spec}(\mathbf{Z})$ and $V_1 = V_2 = \mathbf{A}^1_{\mathbf{Z}}$ with morphisms $a(x) = x$ and $b(x) = x + p$. Note that the equalizer of a and b is the fibre of $\mathbf{A}^1_{\mathbf{Z}}$ over (p).

07BG **Lemma 51.1.** *The lisse-étale site is not functorial, even for morphisms of schemes.*

Proof. See discussion above. $\square$

52. Derived pushforward of quasi-coherent modules

07DC Let k be a field of characteristic $p > 0$. Let $S = \text{Spec}(k[x])$. Let $G = \mathbf{Z}/p\mathbf{Z}$ viewed either as an abstract group or as a constant group scheme over S. Consider the algebraic stack $\mathcal{X} = [S/G]$ where G acts trivially on S, see Examples of Stacks, Remark 15.5 and Criteria for Representability, Lemma 18.3. Consider the structure morphism

$$f : \mathcal{X} \longrightarrow S$$

This morphism is quasi-compact and quasi-separated. Hence we get a functor

$$Rf_{QCoh,*} : D^+_{QCoh}(\mathcal{O}_\mathcal{X}) \longrightarrow D^+_{QCoh}(\mathcal{O}_S),$$

see Derived Categories of Stacks, Proposition 5.1. Let's compute $Rf_{QCoh,*}\mathcal{O}_\mathcal{X}$. Since $D_{QCoh}(\mathcal{O}_S)$ is equivalent to the derived category of $k[x]$-modules (see Derived Categories of Schemes, Lemma 3.5) this is equivalent to computing $R\Gamma(\mathcal{X}, \mathcal{O}_\mathcal{X})$. For this we can use the covering $S \to \mathcal{X}$ and the spectral sequence

$$H^q(S \times_\mathcal{X} \ldots \times_\mathcal{X} S, O) \Rightarrow H^{p+q}(\mathcal{X}, \mathcal{O}_\mathcal{X})$$

see Cohomology of Stacks, Proposition 10.4. Note that

$$S \times_\mathcal{X} \ldots \times_\mathcal{X} S = S \times G^p$$

which is affine. Thus the complex

$$k[x] \to \text{Map}(G, k[x]) \to \text{Map}(G^2, k[x]) \to \ldots$$

computes $R\Gamma(\mathcal{X}, \mathcal{O}_\mathcal{X})$. Here for $\varphi \in \text{Map}(G^{p-1}, k[x])$ its differential is the map which sends $(g_1, \ldots, g_p)$ to

$$\varphi(g_2, \ldots, g_p) + \sum\nolimits_{i=1}^{p-1} (-1)^i \varphi(g_1, \ldots, g_i + g_{i+1}, \ldots, g_p) + (-1)^p \varphi(g_1, \ldots, g_{p-1}).$$

This is just the complex computing the group cohomology of G acting trivially on $k[x]$ (insert future reference here). The cohomology of the cyclic group G on $k[x]$ is exactly one copy of $k[x]$ in each cohomological degree ≥ 0 (insert future reference here). We conclude that

$$Rf_*\mathcal{O}_\mathcal{X} = \bigoplus\nolimits_{n \geq 0} \mathcal{O}_S[-n]$$

Now, consider the complex

$$E = \bigoplus\nolimits_{m \geq 0} \mathcal{O}_\mathcal{X}[m]$$

This is an object of $D_{QCoh}(\mathcal{O}_\mathcal{X})$. We interrupt the discussion for a general result.

08IY **Lemma 52.1.** *Let $\mathcal{X}$ be an algebraic stack. Let K be an object of $D(\mathcal{O}_\mathcal{X})$ whose cohomology sheaves are locally quasi-coherent (Sheaves on Stacks, Definition 11.4) and satisfy the flat base change property (Cohomology of Stacks, Definition 7.1). Then there exists a distinguished triangle*

$$K \to \prod\nolimits_{n \geq 0} \tau_{\geq -n} K \to \prod\nolimits_{n \geq 0} \tau_{\geq -n} K \to K[1]$$

in $D(\mathcal{O}_\mathcal{X})$. In other words, K is the derived limit of its canonical truncations.

Proof. Recall that we work on the "big fppf site" $\mathcal{X}_{fppf}$ of $\mathcal{X}$ (by our conventions for sheaves of $\mathcal{O}_\mathcal{X}$-modules in the chapters Sheaves on Stacks and Cohomology on Stacks). Let $\mathcal{B}$ be the set of objects x of $\mathcal{X}_{fppf}$ which lie over an affine scheme U. Combining Sheaves on Stacks, Lemmas 22.2, 15.1, Descent, Lemma 9.4, and Cohomology of Schemes, Lemma 2.2 we see that $H^p(x, \mathcal{F}) = 0$ if $\mathcal{F}$ is locally quasi-coherent and $x \in \mathcal{B}$. Now the claim follows from Cohomology on Sites, Lemma 22.10 with $d = 0$. $\square$

08IZ **Lemma 52.2.** *Let $\mathcal{X}$ be an algebraic stack. If $\mathcal{F}_n$ is a collection of locally quasi-coherent sheaves with the flat base change property on $\mathcal{X}$, then $\oplus_n \mathcal{F}_n[n] \to \prod_n \mathcal{F}_n[n]$ is an isomorphism in $D(\mathcal{O}_\mathcal{X})$.*

Proof. This is true because by Lemma 52.1 we see that the direct sum is isomorphic to the product. $\square$

We continue our discussion. Since a quasi-coherent module is locally quasi-coherent and satisfies the flat base change property (Sheaves on Stacks, Lemma 11.5) we get

$$E = \prod\nolimits_{m \geq 0} \mathcal{O}_\mathcal{X}[m]$$

Since cohomology commutes with limits we see that

$$Rf_*E = \prod\nolimits_{m \geq 0} \left(\bigoplus\nolimits_{n \geq 0} \mathcal{O}_S[m - n] \right)$$

Note that this complex is not an object of $D_{QCoh}(\mathcal{O}_S)$ because the cohomology sheaf in degree 0 is an infinite product of copies of $\mathcal{O}_S$ which is not even a locally quasi-coherent $\mathcal{O}_S$-module.

07DD **Lemma 52.3.** *A quasi-compact and quasi-separated morphism $f : \mathcal{X} \to \mathcal{Y}$ of algebraic stacks need not induce a functor $Rf_* : D_{QCoh}(\mathcal{O}_\mathcal{X}) \to D_{QCoh}(\mathcal{O}_\mathcal{Y})$.*

Proof. See discussion above. $\square$

53. A big abelian category

07JS The purpose of this section is to give an example of a "big" abelian category $\mathcal{A}$ and objects M, N such that the collection of isomorphism classes of extensions $\mathrm{Ext}_{\mathcal{A}}(M, N)$ is not a set. The example is due to Freyd, see [Fre64, page 131, Exercise A].

We define $\mathcal{A}$ as follows. An object of $\mathcal{A}$ consists of a triple (M, α, f) where M is an abelian group and α is an ordinal and $f : \alpha \to \mathrm{End}(M)$ is a map. A morphism $(M, \alpha, f) \to (M', \alpha', f')$ is given by a homomorphism of abelian groups $\varphi : M \to M'$ such that for *any* ordinal β we have

$$\varphi \circ f(\beta) = f'(\beta) \circ \varphi$$

Here the rule is that we set $f(\beta) = 0$ if β is not in α and similarly we set $f'(\beta)$ equal to zero if β is not an element of α'. We omit the verification that the category so defined is abelian.

Consider the object $Z = (\mathbf{Z}, \emptyset, f)$, i.e., all the operators are zero. The observation is that computed in $\mathcal{A}$ the group $\mathrm{Ext}^1_{\mathcal{A}}(Z, Z)$ is a proper class and not a set. Namely, for each ordinal α we can find an extension $(M, \alpha+1, f)$ of Z by Z whose underlying group is $M = \mathbf{Z} \oplus \mathbf{Z}$ and where the value of f is always zero except for

$$f(\alpha) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

This clearly produces a proper class of isomorphism classes of extensions. In particular, the derived category of $\mathcal{A}$ has proper classes for its collections of morphism, see Derived Categories, Lemma 27.6. This means that some care has to be exercised when defining Verdier quotients of triangulated categories.

07JT **Lemma 53.1.** *There exists a "big" abelian category $\mathcal{A}$ whose* Ext*-groups are proper classes.*

Proof. See discussion above. □

54. Weakly associated points and scheme theoretic density

084J Let k be a field. Let $R = k[z, x_i, y_i]/(z^2, zx_iy_i)$ where i runs over the elements of $\mathbf{N}$. Note that $R = R_0 \oplus M_0$ where $R_0 = k[x_i, y_i]$ is a subring and M_0 is an ideal of square zero with $M_0 \cong R_0/(x_iy_i)$ as R_0-module. The prime $\mathfrak{p} = (z, x_i)$ is weakly associated to R as an R-module (Algebra, Definition 65.1). Indeed, the element z in $R_{\mathfrak{p}}$ is nonzero but annihilated by $\mathfrak{p}R_{\mathfrak{p}}$. On the other hand, consider the open subscheme

$$U = \bigcup D(x_i) \subset \mathrm{Spec}(R) = S$$

We claim that $U \subset S$ is scheme theoretically dense (Morphisms, Definition 7.1). To prove this it suffices to show that $\mathcal{O}_S \to j_*\mathcal{O}_U$ is injective where $j : U \to S$ is the inclusion morphism, see Morphisms, Lemma 7.5. Translated back into algebra, we have to show that for all $g \in R$ the map

$$R_g \longrightarrow \prod R_{x_ig}$$

is injective. Write $g = g_0 + m_0$ with $g_0 \in R_0$ and $m_0 \in M_0$. Then $R_g = R_{g_0}$ (details omitted). Hence we may assume $g \in R_0$. We may also assume g is not zero. Now $R_g = (R_0)_g \oplus (M_0)_g$. Since R_0 is a domain, the map $(R_0)_g \to \prod (R_0)_{x_ig}$ is injective.

If $g \in (x_iy_i)$ then $(M_0)_g = 0$ and there is nothing to prove. If $g \not\in (x_iy_i)$ then, since (x_iy_i) is a radical ideal of R_0, we have to show that $M_0 \to \prod (M_0)_{x_ig}$ is injective. The kernel of $R_0 \to M_0 \to (M_0)_{x_n}$ is (x_iy_i, y_n). Since (x_iy_i, y_n) is a radical ideal, if $g \not\in (x_iy_i, y_n)$ then the kernel of $R_0 \to M_0 \to (M_0)_{x_ng}$ is (x_iy_i, y_n). As $g \not\in (x_iy_i, y_n)$ for all $n \gg 0$ we conclude that the kernel is contained in $\bigcap_{n \gg 0}(x_iy_i, y_n) = (x_iy_i)$ as desired.

Second example due to Ofer Gabber. Let k be a field and let R, resp. R' be the ring of functions $\mathbf{N} \to k$, resp. the ring of eventually constant functions $\mathbf{N} \to k$. Then $\text{Spec}(R)$, resp. $\text{Spec}(R')$ is the Stone-Čech compactification[3] $\beta\mathbf{N}$, resp. the one point compactification[4] $\mathbf{N}^* = \mathbf{N} \cup \{\infty\}$. All points are weakly associated since all primes are minimal in the rings R and R'.

084K **Lemma 54.1.** *There exists a reduced scheme X and a schematically dense open $U \subset X$ such that some weakly associated point $x \in X$ is not in U.*

Proof. In the first example we have $\mathfrak p \not\in U$ by construction. In Gabber's examples the schemes $\text{Spec}(R)$ or $\text{Spec}(R')$ are reduced. $\square$

55. Example of non-additivity of traces

087J Let k be a field and let $R = k[\epsilon]$ be the ring of dual numbers over k. In other words, $R = k[x]/(x^2)$ and ϵ is the congruence class of x in R. Consider the short exact sequence of complexes

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & R \arrow[r, "1"'] \arrow[d, "\epsilon"] & R \arrow[d] \\ R \arrow[r, "1"] & R \arrow[r] & 0 \end{tikzcd}$$

Here the columns are the complexes, the first row is placed in degree 0, and the second row in degree 1. Denote the first complex (i.e., the left column) by $A^\bullet$, the second by $B^\bullet$ and the third $C^\bullet$. We claim that the diagram

087K (55.0.1)
$$\begin{tikzcd} A^\bullet \arrow[r] \arrow[d, "1+\epsilon"'] & B^\bullet \arrow[r] \arrow[d, "1"'] & C^\bullet \arrow[d, "1"'] \\ A^\bullet \arrow[r] & B^\bullet \arrow[r] & C^\bullet \end{tikzcd}$$

commutes in $K(R)$, i.e., is a diagram of complexes commuting up to homotopy. Namely, the square on the right commutes and the one on the left is off by the homotopy $1 : A^1 \to B^0$. On the other hand,

$$\text{Tr}_{A^\bullet}(1 + \epsilon) + \text{Tr}_{C^\bullet}(1) \neq \text{Tr}_{B^\bullet}(1).$$

087L **Lemma 55.1.** *There exists a ring R, a distinguished triangle $(K, L, M, \alpha, \beta, \gamma)$ in the homotopy category $K(R)$, and an endomorphism (a, b, c) of this distinguished triangle, such that K, L, M are perfect complexes and $Tr_K(a) + Tr_M(c) \neq Tr_L(b)$.*

[3]Every element $f \in R$ is of the form ue where u is a unit and e is an idempotent. Then Algebra, Lemma 25.5 shows $\text{Spec}(R)$ is Hausdorff. On the other hand, $\mathbf{N}$ with the discrete topology can be viewed as a dense open subset. Given a set map $\mathbf{N} \to X$ to a Hausdorff, quasi-compact topological space X, we obtain a ring map $\mathcal{C}^0(X; k) \to R$ where $\mathcal{C}^0(X; k)$ is the k-algebra of locally constant maps $X \to k$. This gives $\text{Spec}(R) \to \text{Spec}(\mathcal{C}^0(X; k)) = X$ proving the universal property.

[4]Here one argues that there is really only one extra maximal ideal in R'.

Proof. Consider the example above. The map $\gamma : C^\bullet \to A^\bullet[1]$ is given by multiplication by ϵ in degree 0, see Derived Categories, Definition 10.1. Hence it is also true that

$$\begin{tikzcd} C^\bullet \arrow[r, "\gamma"'] \arrow[d] & A^\bullet[1] \arrow[d] \\ C^\bullet \arrow[r, "\gamma"] & A^\bullet[1] \end{tikzcd}$$

commutes in $K(R)$ as $\epsilon(1+\epsilon) = \epsilon$. Thus we indeed have a morphism of distinguished triangles. □

56. Being projective is not local on the base

08J0 In the chapter on descent we have seen that many properties of morphisms are local on the base, even in the fpqc topology. See Descent, Sections 19, 20, and 21. This is not true for projectivity of morphisms.

08J1 **Lemma 56.1.** *The properties*

$\mathcal{P}(f) =$ "*f is projective*", *and*
$\mathcal{P}(f) =$ "*f is quasi-projective*"

are not Zariski local on the base. A fortiori, they are not fpqc local on the base.

Proof. Following Hironaka [Har77, Example B.3.4.1], we define a proper morphism of smooth complex 3-folds $f : V_Y \to Y$ which is Zariski-locally projective, but not projective. Since f is proper and not projective, it is also not quasi-projective.

Let Y be projective 3-space over the complex numbers $\mathbf{C}$. Let C and D be smooth conics in Y such that the closed subscheme $C \cap D$ is reduced and consists of two complex points P and Q. (For example, let $C = \{[x, y, z, w] : xy = z^2, w = 0\}$, $D = \{[x, y, z, w] : xy = w^2, z = 0\}$, $P = [1, 0, 0, 0]$, and $Q = [0, 1, 0, 0]$.) On $Y - Q$, first blow up the curve C, and then blow up the strict transform of the curve D (Divisors, Definition 30.1). On $Y - P$, first blow up the curve D, and then blow up the strict transform of the curve C. Over $Y - P - Q$, the two varieties we have constructed are canonically isomorphic, and so we can glue them over $Y - P - Q$. The result is a smooth proper 3-fold V_Y over $\mathbf{C}$. The morphism $f : V_Y \to Y$ is proper and Zariski-locally projective (since it is a blowup over $Y - P$ and over $Y - Q$), by Divisors, Lemma 29.13. We will show that V_Y is not projective over $\mathbf{C}$. That will imply that f is not projective.

To do this, let L be the inverse image in V_Y of a complex point of $C - P - Q$, and M the inverse image of a complex point of $D - P - Q$. Then L and M are isomorphic to the projective line $\mathbf{P}^1_{\mathbf{C}}$. Next, let E be the inverse image in V_Y of $C \cup D \subset Y$ in V_Y; thus $E \to C \cup D$ is a proper morphism, with fibers isomorphic to $\mathbf{P}^1$ over $(C \cup D) - \{P, Q\}$. The inverse image of P in E is a union of two lines L_0 and M_0, and we have rational equivalences of cycles $L \sim L_0 + M_0$ and $M \sim M_0$ on E (using that C and D are isomorphic to $\mathbf{P}^1$). Note the asymmetry resulting from the order in which we blew up the two curves. Near Q, the opposite happens. So the inverse image of Q is the union of two lines L_0' and M_0', and we have rational equivalences $L \sim L_0'$ and $M \sim L_0' + M_0'$ on E. Combining these equivalences, we find that $L_0 + M_0' \sim 0$ on E and hence on V_Y. If V_Y were projective over $\mathbf{C}$, it would have an ample line bundle H, which would have degree > 0 on all curves in V_Y. In particular H would have positive degree on $L_0 + M_0'$, contradicting that the degree of a line bundle is well-defined on 1-cycles modulo rational equivalence on a proper scheme over a field (Chow Homology, Lemma 20.2 and Lemma 26.2). So V_Y is not projective over $\mathbf{C}$. $\square$

In different terminology, Hironaka's 3-fold V_Y is a small resolution of the blowup Y' of Y along the reduced subscheme $C \cup D$; here Y' has two node singularities. If we define Z by blowing up Y along C and then along the strict transform of D, then Z is a smooth projective 3-fold, and the non-projective 3-fold V_Y differs from Z by a "flop" over $Y - P$.

57. Descent data for schemes need not be effective, even for a projective morphism

08KE In the chapter on descent we have seen that descent data for schemes relative to an fpqc morphism are effective for several classes of morphisms. In particular, affine morphisms and more generally quasi-affine morphisms satisfy descent for fpqc coverings (Descent, Lemma 35.1). This is not true for projective morphisms.

08KF **Lemma 57.1.** *There is an etale covering $X \to S$ of schemes and a descent datum $(V/X, \varphi)$ relative to $X \to S$ such that $V \to X$ is projective, but the descent datum is not effective in the category of schemes.*

Proof. We imitate Hironaka's example of a smooth separated complex algebraic space of dimension 3 which is not a scheme [Har77, Example B.3.4.2].

Consider the action of the group $G = \mathbf{Z}/2 = \{1, g\}$ on projective 3-space $\mathbf{P}^3$ over the complex numbers by

$$g[x, y, z, w] = [y, x, w, z].$$

The action is free outside the two disjoint lines $L_1 = \{[x, x, z, z]\}$ and $L_2 = \{[x, -x, z, -z]\}$ in $\mathbf{P}^3$. Let $Y = \mathbf{P}^3 - (L_1 \cup L_2)$. There is a smooth quasi-projective scheme $S = Y/G$ over $\mathbf{C}$ such that $Y \to S$ is a G-torsor (Groupoids, Definition 11.3). Explicitly, we can define S as the image of the open subset Y in $\mathbf{P}^3$ under the morphism

$$\begin{aligned} \mathbf{P}^3 &\to \text{Proj } \mathbf{C}[x, y, z, w]^G \\ &= \text{Proj } \mathbf{C}[u_0, u_1, v_0, v_1, v_2]/(v_0 v_1 = v_2^2), \end{aligned}$$

where $u_0 = x + y$, $u_1 = z + w$, $v_0 = (x - y)^2$, $v_1 = (z - w)^2$, and $v_2 = (x - y)(z - w)$, and the ring is graded with u_0, u_1 in degree 1 and v_0, v_1, v_2 in degree 2.

Let $C = \{[x, y, z, w] : xy = z^2, w = 0\}$ and $D = \{[x, y, z, w] : xy = w^2, z = 0\}$. These are smooth conic curves in $\mathbf{P}^3$, contained in the G-invariant open subset Y, with $g(C) = D$. Also, $C \cap D$ consists of the two points $P := [1, 0, 0, 0]$ and $Q := [0, 1, 0, 0]$, and these two points are switched by the action of G.

Let $V_Y \to Y$ be the scheme which over $Y - P$ is defined by blowing up D and then the strict transform of C, and over $Y - Q$ is defined by blowing up C and then the strict transform of D. (This is the same construction as in the proof of Lemma 56.1, except that Y here denotes an open subset of $\mathbf{P}^3$ rather than all of $\mathbf{P}^3$.) Then the action of G on Y lifts to an action of G on V_Y, which switches the inverse images of $Y - P$ and $Y - Q$. This action of G on V_Y gives a descent datum $(V_Y/Y, \varphi_Y)$ on V_Y relative to the G-torsor $Y \to S$. The morphism $V_Y \to Y$ is proper but not projective, as shown in the proof of Lemma 56.1.

Let X be the disjoint union of the open subsets $Y - P$ and $Y - Q$; then we have surjective etale morphisms $X \to Y \to S$. Let V be the pullback of $V_Y \to Y$ to X; then the morphism $V \to X$ is projective, since $V_Y \to Y$ is a blowup over each of the open subsets $Y - P$ and $Y - Q$. Moreover, the descent datum $(V_Y/Y, \varphi_Y)$ pulls back to a descent datum $(V/X, \varphi)$ relative to the etale covering $X \to S$.

Suppose that this descent datum is effective in the category of schemes. That is, there is a scheme $U \to S$ which pulls back to the morphism $V \to X$ together with its descent datum. Then U would be the quotient of V_Y by its G-action.

$$\begin{tikzcd} V \arrow[r] \arrow[d] & X \arrow[d] \\ V_Y \arrow[r] \arrow[d] & Y \arrow[d] \\ U \arrow[r] & S \end{tikzcd}$$

Let E be the inverse image of $C \cup D \subset Y$ in V_Y; thus $E \to C \cup D$ is a proper morphism, with fibers isomorphic to $\mathbf{P}^1$ over $(C \cup D) - \{P, Q\}$. The inverse image of P in E is a union of two lines L_0 and M_0. It follows that the inverse image of $Q = g(P)$ in E is the union of two lines $L'_0 = g(M_0)$ and $M'_0 = g(L_0)$. As shown in the proof of Lemma 56.1, we have a rational equivalence $L_0 + M'_0 = L_0 + g(L_0) \sim 0$ on E.

By descent of closed subschemes, there is a curve $L_1 \subset U$ (isomorphic to $\mathbf{P}^1$) whose inverse image in V_Y is $L_0 \cup g(L_0)$. (Use Descent, Lemma 34.1, noting that a closed immersion is an affine morphism.) Let R be a complex point of L_1. Since we assumed that U is a scheme, we can choose a function f in the local ring $O_{U,R}$ that vanishes at R but not on the whole curve L_1. Let D_{loc} be an irreducible component of the closed subset $\{f = 0\}$ in $\operatorname{Spec} O_{U,R}$; then D_{loc} has codimension 1. The closure of D_{loc} in U is an irreducible divisor D_U in U which contains the point R but not the whole curve L_1. The inverse image of D_U in V_Y is an effective divisor D which intersects $L_0 \cup g(L_0)$ but does not contain either curve L_0 or $g(L_0)$.

Since the complex 3-fold V_Y is smooth, $O(D)$ is a line bundle on V_Y. We use here that a regular local ring is factorial, or in other words is a UFD, see More on Algebra, Lemma 96.7. The restriction of $O(D)$ to the proper surface $E \subset V_Y$ is a line bundle which has positive degree on the 1-cycle $L_0 + g(L_0)$, by our information on D. Since $L_0 + g(L_0) \sim 0$ on E, this contradicts that the degree of a line bundle is well-defined on 1-cycles modulo rational equivalence on a proper scheme over a field (Chow Homology, Lemma 20.2 and Lemma 26.2). Therefore the descent datum $(V/X, \varphi)$ is in fact not effective; that is, U does not exist as a scheme. $\square$

In this example, the descent datum *is* effective in the category of algebraic spaces. More precisely, U exists as a smooth separated algebraic space of dimension 3 over $\mathbf{C}$, for example by Algebraic Spaces, Lemma 14.3. Hironaka's 3-fold U is a small resolution of the blowup S' of the smooth quasi-projective 3-fold S along the irreducible nodal curve $(C \cup D)/G$; the 3-fold S' has a node singularity. The other small resolution of S' (differing from U by a "flop") is again an algebraic space which is not a scheme.

58. A family of curves whose total space is not a scheme

0D5D In Quot, Section 15 we define a family of curves over a scheme S to be a proper, flat, finitely presented morphism of relative dimension ≤ 1 from an algebraic space X to S. If S is the spectrum of a complete Noetherian local ring, then X is a scheme, see More on Morphisms of Spaces, Lemma 41.5. In this section we show this is not true in general.

Let k be a field. We start with a proper flat morphism

$$Y \longrightarrow \mathbf{A}^1_k$$

and a point $y \in Y(k)$ lying over $0 \in \mathbf{A}^1_k(k)$ with the following properties

(1) the fibre Y_0 is a smooth geometrically irreducible curve over k,
(2) for any proper closed subscheme $T \subset Y$ dominating $\mathbf{A}^1_k$ the intersection $T \cap Y_0$ contains at least one point distinct from y.

Given such a surface we construct our example as follows.

$$\begin{tikzcd} Y \arrow[dr] & Z \arrow[l] \arrow[r] \arrow[d] & X \arrow[dl] \\ & \mathbf{A}^1_k & \end{tikzcd}$$

Here $Z \to Y$ is the blowup of Y in y. Let $E \subset Z$ be the exceptional divisor and let $C \subset Z$ be the strict transform of Y_0. We have $Z_0 = E \cup C$ scheme theoretically (to see this use that Y is smooth at y and moreover $Y \to \mathbf{A}^1_k$ is smooth at y). By Artin's results ([Art70]; use Semistable Reduction, Lemma 9.7 to see that the normal bundle of C is negative) we can blow down the curve C in Z to obtain an algebraic space X as in the diagram. Let $x \in X(k)$ be the image of C.

We claim that X is not a scheme. Namely, if it were a scheme, then there would be an affine open neighbourhood $U \subset X$ of x. Set $T = X \setminus U$. Then T dominates $\mathbf{A}^1_k$ (as the fibres of $X \to \mathbf{A}^1_k$ are proper of dimension 1 and the fibres of $U \to \mathbf{A}^1_k$ are affine hence different). Let $T' \subset Z$ be the closed subscheme mapping isomorphically to T (as $x \notin T$). Then the image of T' in X contradicts condition (2) above (as $T' \cap Z_0$ is contained in the exceptional divisor E of the blowing up $Z \to Y$).

To finish the discussion we need to construct our Y. We will assume the characteristic of k is not 3. Write $\mathbf{A}^1_k = \mathrm{Spec}(k[t])$ and take

$$Y \quad : \quad T_0^3 + T_1^3 + T_2^3 - tT_0T_1T_2 = 0$$

in $\mathbf{P}^2_{k[t]}$. The fibre of this for $t = 0$ is a smooth projective genus 1 curve. On the affine piece $V_+(T_0)$ we get the affine equation

$$1 + x^3 + y^3 - txy = 0$$

which defines a smooth surface over k. Since the same is true on the other affine pieces by symmetry we see that Y is a smooth surface. Finally, we see from the affine equation also that the fraction field is $k(x, y)$ hence Y is a rational surface. Now the Picard group of a rational surface is finitely generated (insert future reference here). Hence in order to choose $y \in Y_0(k)$ with property (2) it suffices to choose y such that

0DYB (58.0.1) $$\mathcal{O}_{Y_0}(ny) \notin \mathrm{Im}(\mathrm{Pic}(Y) \to \mathrm{Pic}(Y_0)) \text{ for all } n > 0$$

Namely, the sum of the 1-dimensional irreducible components of a T contradicting (2) would give an effective Cartier divisor intersection Y_0 in the divisor ny for some $n \geq 1$ and we would conclude that $\mathcal{O}_{Y_0}(ny)$ is in the image of the restriction map. Observe that since Y_0 has genus ≥ 1 the map

$$Y_0(k) \to \text{Pic}(Y_0), \quad y \mapsto \mathcal{O}_{Y_0}(y)$$

is injective. Now if k is an uncountable algebraically closed field, then using the countability of $\text{Pic}(Y)$ and the remark just made, we can find a $y \in Y_0(k)$ satisfying (58.0.1) and hence (2).

0D5E **Lemma 58.1.** *There exists a field k and a family of curves $X \to \mathbf{A}^1_k$ such that X is not a scheme.*

Proof. See discussion above. □

59. Derived base change

08J2 Let $R \to R'$ be a ring map. In More on Algebra, Section 56 we construct a derived base change functor $- \otimes_R^{\mathbf{L}} R' : D(R) \to D(R')$. Next, let $R \to A$ be a second ring map. Picture

$$\begin{tikzcd} A \arrow[r] & A \otimes_R R' \arrow[r, equal] & A' \\ R \arrow[u] \arrow[r] & R' \arrow[u] \arrow[ur] & \end{tikzcd}$$

Given an A-module M the tensor product $M \otimes_R R'$ is a $A \otimes_R R'$-module, i.e., an A'-module. For the ring map $A \to A'$ there is a derived functor

$$- \otimes_A^{\mathbf{L}} A' : D(A) \longrightarrow D(A')$$

but this functor does not agree with $- \otimes_R^{\mathbf{L}} R'$ in general. More precisely, for $K \in D(A)$ the canonical map

$$K \otimes_R^{\mathbf{L}} R' \longrightarrow K \otimes_A^{\mathbf{L}} A'$$

in $D(R')$ constructed in More on Algebra, Equation (57.0.1) isn't an isomorphism in general. Thus one may wonder if there exists a "derived base change functor" $T : D(A) \to D(A')$, i.e., a functor such that $T(K)$ maps to $K \otimes_R^{\mathbf{L}} R'$ in $D(R')$. In this section we show it does not exist in general.

Let k be a field. Set $R = k[x, y]$. Set $R' = R/(xy)$ and $A = R/(x^2)$. The object $A \otimes_R^{\mathbf{L}} R'$ in $D(R')$ is represented by

$$x^2 : R' \longrightarrow R'$$

and we have $H^0(A \otimes_R^{\mathbf{L}} R') = A \otimes_R R'$. We claim that there does not exist an object E of $D(A \otimes_R R')$ mapping to $A \otimes_R^{\mathbf{L}} R'$ in $D(R')$. Namely, for such an E the module $H^0(E)$ would be free, hence E would decompose as $H^0(E)[0] \oplus H^{-1}(E)[1]$. But it is easy to see that $A \otimes_R^{\mathbf{L}} R'$ is not isomorphic to the sum of its cohomology groups in $D(R')$.

08J3 **Lemma 59.1.** *Let $R \to R'$ and $R \to A$ be ring maps. In general there does not exist a functor $T : D(A) \to D(A \otimes_R R')$ of triangulated categories such that an A-module M gives an object $T(M)$ of $D(A \otimes_R R')$ which maps to $M \otimes_R^{\mathbf{L}} R'$ under the map $D(A \otimes_R R') \to D(R')$.*

Proof. See discussion above. □

60. An interesting compact object

09R4 Let R be a ring. Let (A, d) be a differential graded R-algebra. If $A = R$, then we know that every compact object of $D(A, \mathrm{d}) = D(R)$ is represented by a finite complex of finite projective modules. In other words, compact objects are perfect, see More on Algebra, Proposition 70.3. The analogue in the language of differential graded modules would be the question: "Is every compact object of $D(A, \mathrm{d})$ represented by a differential graded A-module P which is finite and graded projective?"

For general differential graded algebras, this is not true. Namely, let k be a field of characteristic 2 (so we don't have to worry about signs). Let $A = k[x, y]/(y^2)$ with

(1) x of degree 0
(2) y of degree -1,
(3) $\mathrm{d}(x) = 0$, and
(4) $\mathrm{d}(y) = x^2 + x$.

Then $x : A \to A$ is a projector in $K(A, \mathrm{d})$. Hence we see that

$$A = \mathrm{Ker}(x) \oplus \mathrm{Im}(1 - x)$$

in $K(A, \mathrm{d})$, see Differential Graded Algebra, Lemma 5.4 and Derived Categories, Lemma 4.13. It is clear that A is a compact object of $D(A, \mathrm{d})$. Then $\mathrm{Ker}(x)$ is a compact object of $D(A, \mathrm{d})$ as follows from Derived Categories, Lemma 34.2.

Next, suppose that M is a differential graded (right) A-module representing $\mathrm{Ker}(x)$ and suppose that M is finite and projective as a graded A-module. Because every finite graded projective module over $k[x, y]/(y^2)$ is graded free, we see that M is finite free as a graded $k[x, y]/(y^2)$-module (i.e., when we forget the differential). We set $N = M/M(x^2 + x)$. Consider the exact sequence

$$0 \to M \xrightarrow{x^2+x} M \to N \to 0$$

Since $x^2 + x$ is of degree 0, in the center of A, and $\mathrm{d}(x^2 + x) = 0$ we see that this is a short exact sequence of differential graded A-modules. Moreover, as $\mathrm{d}(y) = x^2 + x$ we see that the differential on N is linear. The maps

$$H^{-1}(N) \to H^0(M) \quad \text{and} \quad H^0(M) \to H^0(N)$$

are isomorphisms as $H^*(M) = H^0(M) = k$ since $M \cong \mathrm{Ker}(x)$ in $D(A, \mathrm{d})$. A computation of the boundary map shows that $H^*(N) = k[x, y]/(x, y^2)$ as a graded module; we omit the details. Since N is a free $k[x, y]/(y^2, x^2 + x)$-module we have a resolution

$$\ldots \to N[2] \xrightarrow{y} N[1] \xrightarrow{y} N \to N/Ny \to 0$$

compatible with differentials. Since N is bounded and since $H^0(N) = k[x, y]/(x, y^2)$ it follows from Homology, Lemma 22.6 that $H^0(N/Ny) = k[x]/(x)$. But as N/Ny is a finite complex of free $k[x]/(x^2 + x) = k \times k$-modules, we see that its cohomology has to have even dimension, a contradiction.

09R5 **Lemma 60.1.** *There exists a differential graded algebra (A, d) and a compact object E of $D(A, d)$ such that E cannot be represented by a finite and graded projective differential graded A-module.*

Proof. See discussion above. □

61. Two differential graded categories

09R6 In this section we construct two differential graded categories satisfying axioms (A), (B), and (C) as in Differential Graded Algebra, Situation 20.2 whose objects do not come with a **Z**-grading.

Example I. Let X be a topological space. Denote $\underline{\mathbf{Z}}$ the constant sheaf with value $\mathbf{Z}$. Let A be an $\underline{\mathbf{Z}}$-torsor. In this setting we say a sheaf of abelian groups $\mathcal{F}$ is *A-graded* if given a local section $a \in A(U)$ there is a projector $p_a : \mathcal{F}|_U \to \mathcal{F}|_U$ such that whenever we have a local isomorphism $\underline{\mathbf{Z}}|_U \to A|_U$ then $\mathcal{F}|_U = \bigoplus_{n \in \mathbf{Z}} p_n(\mathcal{F})$. Another way to say this is that locally on X the abelian sheaf $\mathcal{F}$ has a **Z**-grading, but on overlaps the different choices of gradings differ by a shift in degree given by the transition functions for the torsor A. We say that a pair $(\mathcal{F}, \mathrm{d})$ is an *A-graded complex of abelian sheaves*, if $\mathcal{F}$ is an A-graded abelian sheaf and $\mathrm{d} : \mathcal{F} \to \mathcal{F}$ is a differential, i.e., $\mathrm{d}^2 = 0$ such that $p_{a+1} \circ \mathrm{d} = \mathrm{d} \circ p_a$ for every local section a of A. In other words, $\mathrm{d}(p_a(\mathcal{F}))$ is contained in $p_{a+1}(\mathcal{F})$.

Next, consider the category $\mathcal{A}$ with

(1) objects are A-graded complexes of abelian sheaves, and
(2) for objects $(\mathcal{F}, \mathrm{d})$, $(\mathcal{G}, \mathrm{d})$ we set
$$\mathrm{Hom}_{\mathcal{A}}((\mathcal{F}, \mathrm{d}), (\mathcal{G}, \mathrm{d})) = \bigoplus \mathrm{Hom}^n(\mathcal{F}, \mathcal{G})$$
where $\mathrm{Hom}^n(\mathcal{F}, \mathcal{G})$ is the group of maps of abelian sheaves f such that $f(p_a(\mathcal{F})) \subset p_{a+n}(\mathcal{G})$ for all local sections a of A. As differential we take $\mathrm{d}(f) = \mathrm{d} \circ f - (-1)^n f \circ \mathrm{d}$, see Differential Graded Algebra, Example 19.6.

We omit the verification that this is indeed a differential graded category satisfying (A), (B), and (C). All the properties may be verified locally on X where one just recovers the differential graded category of complexes of abelian sheaves. Thus we obtain a triangulated category $K(\mathcal{A})$.

Twisted derived category of X. Observe that given an object $(\mathcal{F}, \mathrm{d})$ of $\mathcal{A}$, there is a well defined A-graded cohomology sheaf $H(\mathcal{F}, \mathrm{d})$. Hence it is clear what is meant by a quasi-isomorphism in $K(\mathcal{A})$. We can invert quasi-isomorphisms to obtain the *derived category $D(\mathcal{A})$ of complexes of A-graded sheaves*. If A is the trivial torsor, then $D(\mathcal{A})$ is equal to $D(X)$, but for nonzero torsors, one obtains a kind of *twisted* derived category of X.

Example II. Let C be a smooth curve over a perfect field k of characteristic 2. Then $\Omega_{C/k}$ comes endowed with a canonical square root. Namely, we can write $\Omega_{C/k} = \mathcal{L}^{\otimes 2}$ such that for every local function f on C the section $\mathrm{d}(f)$ is equal to $s^{\otimes 2}$ for some local section s of $\mathcal{L}$. The "reason" is that
$$\mathrm{d}(a_0 + a_1 t + \ldots + a_d t^d) = (\sum\nolimits_{i \text{ odd}} a_i^{1/2} t^{(i-1)/2})^2 \mathrm{d}t$$
(insert future reference here). This in particular determines a canonical connection
$$\nabla_{can} : \Omega_{C/k} \longrightarrow \Omega_{C/k} \otimes_{\mathcal{O}_C} \Omega_{C/k}$$
whose 2-curvature is zero (namely, the unique connection such that the squares have derivative equal to zero). Observe that the category of vector bundles with connections is a tensor category, hence we also obtain canonical connections ∇_{can} on the invertible sheaves $\Omega_{C/k}^{\otimes n}$ for all $n \in \mathbf{Z}$.

Let $\mathcal{A}$ be the category with

(1) objects are pairs $(\mathcal{F}, \nabla)$ consisting of a finite locally free sheaf $\mathcal{F}$ endowed with a connection
$$\nabla : \mathcal{F} \longrightarrow \mathcal{F} \otimes_{\mathcal{O}_C} \Omega_{C/k}$$
whose 2-curvature is zero, and
(2) morphisms between $(\mathcal{F}, \nabla_\mathcal{F})$ and $(\mathcal{G}, \nabla_\mathcal{G})$ are given by
$$\mathrm{Hom}_\mathcal{A}((\mathcal{F}, \nabla_\mathcal{F}), (\mathcal{G}, \nabla_\mathcal{G})) = \bigoplus \mathrm{Hom}_{\mathcal{O}_C}(\mathcal{F}, \mathcal{G} \otimes_{\mathcal{O}_C} \Omega_{C/k}^{\otimes n})$$
For an element $f : \mathcal{F} \to \mathcal{G} \otimes \Omega_{C/k}^{\otimes n}$ of degree n we set
$$\mathrm{d}(f) = \nabla_{\mathcal{G} \otimes \Omega_{C/k}^{\otimes n}} \circ f + f \circ \nabla_\mathcal{F}$$
with suitable identifications.

We omit the verification that this forms a differential graded category with properties (A), (B), (C). Thus we obtain a triangulated homotopy category $K(\mathcal{A})$.

If $C = \mathbf{P}^1_k$, then $K(\mathcal{A})$ is the zero category. However, if C is a smooth proper curve of genus > 1, then $K(\mathcal{A})$ is not zero. Namely, suppose that $\mathcal{N}$ is an invertible sheaf of degree $0 \leq d < g - 1$ with a nonzero section σ. Then set $(\mathcal{F}, \nabla_\mathcal{F}) = (\mathcal{O}_C, \mathrm{d})$ and $(\mathcal{G}, \nabla_\mathcal{G}) = (\mathcal{N}^{\otimes 2}, \nabla_{can})$. We see that
$$\mathrm{Hom}^n_\mathcal{A}((\mathcal{F}, \nabla_\mathcal{F}), (\mathcal{G}, \nabla_\mathcal{G})) = \begin{cases} 0 & \text{if} \quad n < 0 \\ \Gamma(C, \mathcal{N}^{\otimes 2}) & \text{if} \quad n = 0 \\ \Gamma(C, \mathcal{N}^{\otimes 2} \otimes \Omega_{C/k}) & \text{if} \quad n = 1 \end{cases}$$
The first 0 because the degree of $\mathcal{N}^{\otimes 2} \otimes \Omega_{C/k}^{\otimes -1}$ is negative by the condition $d < g - 1$. Now, the section $\sigma^{\otimes 2}$ has derivative equal zero, hence the homomorphism group
$$\mathrm{Hom}_{K(\mathcal{A})}((\mathcal{F}, \nabla_\mathcal{F}), (\mathcal{G}, \nabla_\mathcal{G}))$$
is nonzero.

62. The stack of proper algebraic spaces is not algebraic

0D1Q In Quot, Section 13 we introduced and studied the stack in groupoids
$$p'_{fp,flat,proper} : \mathcal{S}paces'_{fp,flat,proper} \longrightarrow Sch_{fppf}$$
the stack whose category of sections over a scheme S is the category of flat, proper, finitely presented algebraic spaces over S. We proved that this satisfies many of Artin's axioms. In this section we why this stack is not algebraic by showing that formal effectiveness fails in general.

The canonical example uses that the universal deformation space of an abelian variety of dimension g has g^2 formal parameters whereas any effective formal deformation can be defined over a complete local ring of dimension $\leq g(g+1)/2$. Our example will be constructed by writing down a suitable non-effective deformation of a K3 surface. We will only sketch the argument and not give all the details.

Let $k = \mathbf{C}$ be the field of complex numbers. Let $X \subset \mathbf{P}^3_k$ be a smooth degree 4 surface over k. We have $\omega_X \cong \Omega^2_{X/k} \cong \mathcal{O}_X$. Finally, we have $\dim_k H^0(X, T_{X/k}) = 0$, $\dim_k H^1(X, T_{X/k}) = 20$, and $\dim_k H^2(X, T_{X/k}) = 0$. Since $L_{X/k} = \Omega_{X/k}$ because X is smooth over k, and since $\mathrm{Ext}^i_{\mathcal{O}_X}(\Omega_{X/k}, \mathcal{O}_X) = H^i(X, T_{X/k})$, and because we have Cotangent, Lemma 22.1 we find that there is a universal deformation of X over

$$k[[x_1, \ldots, x_{20}]]$$

Suppose that this universal deformation is effective (as in Artin's Axioms, Section 9). Then we would get a flat, proper morphism

$$f : Y \longrightarrow \mathrm{Spec}(k[[x_1, \ldots, x_{20}]])$$

where Y is an algebraic space recovering the universal deformation. This is impossible for the following reason. Since Y is separated we can find an affine open subscheme $V \subset Y$. Since the special fibre X of Y is smooth, we see that f is smooth. Hence Y is regular being smooth over regular and it follows that the complement D of V in Y is an effective Cartier divisor. Then $\mathcal{O}_Y(D)$ is a nontrivial element of $\mathrm{Pic}(Y)$ (to prove this you show that the complement of a nonempty affine open in a proper smooth algebraic space over a field is always a nontrivial in the Picard group and you apply this to the generic fibre of f). Finally, to get a contradiction, we show that $\mathrm{Pic}(Y) = 0$. Namely, the map $\mathrm{Pic}(Y) \to \mathrm{Pic}(X)$ is injective, because $H^1(X, \mathcal{O}_X) = 0$ (hence all deformations of $\mathcal{O}_X$ to $Y \times \mathrm{Spec}(k[[x_i]]/\mathfrak{m}^n)$ are trivial) and Grothendieck's existence theorem (which says that coherent modules giving rise to the same sheaves on thickenings are isomorphic). If X is general enough, then $\mathrm{Pic}(X) = \mathbf{Z}$ generated by $\mathcal{O}_X(1)$. Hence it suffices to show that $\mathcal{O}_X(n)$, $n > 0$ does not deform to the first order neighbourhood[5]. Consider the cup-product

$$H^1(X, \Omega_{X/k}) \times H^1(X, T_{X/k}) \longrightarrow H^2(X, \mathcal{O}_X)$$

This is a nondegenerate pairing by coherent duality. A computation shows that the chern class $c_1(\mathcal{O}_X(n)) \in H^1(X, \Omega_{X/k})$ in Hodge cohomology is nonzero. Hence there is a first order deformation whose cup product with $c_1(\mathcal{O}_X(n))$ is nonzero. Then finally, one shows this cup product is the obstruction class to lifting.

0D1R **Lemma 62.1.** *The stack in groupoids*

$$p'_{fp,flat,proper} : \mathcal{S}paces'_{fp,flat,proper} \longrightarrow Sch_{fppf}$$

whose category of sections over a scheme S is the category of flat, proper, finitely presented algebraic spaces over S (see Quot, Section 13) is not an algebraic stack.

Proof. If it was an algebraic stack, then every formal object would be effective, see Artin's Axioms, Lemma 9.5. The discussion above show this is not the case after base change to $\mathrm{Spec}(\mathbf{C})$. Hence the conclusion. □

63. An example of a non-algebraic Hom-stack

0AF8 Let $\mathcal{Y}, \mathcal{Z}$ be algebraic stacks over a scheme S. The *Hom-stack* $\underline{Mor}_S(\mathcal{Y}, \mathcal{Z})$ is the stack in groupoids over S whose category of sections over a scheme T is given by the category

$$Mor_T(\mathcal{Y} \times_S T, \mathcal{Z} \times_S T)$$

whose objects are 1-morphisms and whose morphisms are 2-morphisms. We omit the proof this this is indeed a stack in groupoids over $(Sch/S)_{fppf}$ (insert future reference here). Of course, in general the Hom-stack will not be algebraic. In this

[5]This argument works as long as the map $c_1 : \mathrm{Pic}(X) \to H^1(X, \Omega_{X/k})$ is injective, which is true for k any field of characteristic zero and any smooth hypersurface X of degree 4 in $\mathbf{P}^3_k$.

section we give an example where it is not true and where $\mathcal{Y}$ is representable by a proper flat scheme over S and $\mathcal{Z}$ is smooth and proper over S.

Let k be an algebraically closed field which is not the algebraic closure of a finite field. Let $S = \mathrm{Spec}(k[[t]])$ and $S_n = \mathrm{Spec}(k[t]/(t^n)) \subset S$. Let $f : X \to S$ be a map satisfying the following

(1) f is projective and flat, and its fibres are geometrically connected curves,
(2) the fibre $X_0 = X \times_S S_0$ is a nodal curve with smooth irreducible components whose dual graph has a loop consisting of rational curves,
(3) X is a regular scheme.

To make such a surface X we can take for example

$$X \quad : \quad T_0T_1T_2 - t(T_0^3 + T_1^3 + T_2^3) = 0$$

in $\mathbf{P}^2_{k[[t]]}$. Let A_0 be a non-zero abelian variety over k for example an elliptic curve. Let $A = A_0 \times_{\mathrm{Spec}(k)} S$ be the constant abelian scheme over S associated to A_0. We will show that the stack $\mathcal{X} = \underline{Mor}_S(X, [S/A]))$ is not algebraic.

Recall that $[S/A]$ is on the one hand the quotient stack of A acting trivially on S and on the other hand equal to the stack classifying fppf A-torsors, see Examples of Stacks, Proposition 15.3. Observe that $[S/A] = [\mathrm{Spec}(k)/A_0] \times_{\mathrm{Spec}(k)} S$. This allows us to describe the fibre category over a scheme T as follows

$$\begin{aligned} \mathcal{X}_T &= \underline{Mor}_S(X, [S/A])_T \\ &= Mor_T(X \times_S T, [S/A] \times_S T) \\ &= Mor_S(X \times_S T, [S/A]) \\ &= Mor_{\mathrm{Spec}(k)}(X \times_S T, [\mathrm{Spec}(k)/A_0]) \end{aligned}$$

for any S-scheme T. In other words, the groupoid $\mathcal{X}_T$ is the groupoid of fppf A_0-torsors on $X \times_S T$. Before we discuss why $\mathcal{X}$ is not an algebraic stack, we need a few lemmas.

0AF9 **Lemma 63.1.** *Let W be a two dimensional regular integral Noetherian scheme with function field K. Let $G \to W$ be an abelian scheme. Then then map $H^1_{fppf}(W, G) \to H^1_{fppf}(\mathrm{Spec}(K), G)$ is injective.*

Sketch of proof. Let $P \to W$ be an fppf G-torsor which is trivial in the generic point. Then we have a morphism $\mathrm{Spec}(K) \to P$ over W and we can take its scheme theoretic image $Z \subset P$. Since $P \to W$ is proper (as a torsor for a proper group algebraic space over W) we see that $Z \to W$ is a proper birational morphism. By Spaces over Fields, Lemma 3.2 the morphism $Z \to W$ is finite away from finitely many closed points of W. By (insert future reference on resolving indeterminacies of morphisms by blowing quadratic transformations for surfaces) the irreducible components of the geometric fibres of $Z \to W$ are rational curves. By More on Groupoids in Spaces, Lemma 11.3 there are no nonconstant morphisms from rational curves to group schemes or torsors over such. Hence $Z \to W$ is finite, whence Z is a scheme and $Z \to W$ is an isomorphism by Morphisms, Lemma 51.8. In other words, the torsor P is trivial. $\square$

0AFA **Lemma 63.2.** *Let G be a smooth commutative group algebraic space over a field K. Then $H^1_{fppf}(\mathrm{Spec}(K), G)$ is torsion.*

Proof. Every G-torsor P over $\text{Spec}(K)$ is smooth over K as a form of G. Hence P has a point over a finite separable extension $K \subset L$. Say $[L : K] = n$. Let $[n](P)$ denote the G-torsor whose class is n times the class of P in $H^1_{fppf}(\text{Spec}(K), G)$. There is a canonical morphism

$$P \times_{\text{Spec}(K)} \ldots \times_{\text{Spec}(K)} P \to [n](P)$$

of algebraic spaces over K. This morphism is symmetric as G is abelian. Hence it factors through the quotient

$$(P \times_{\text{Spec}(K)} \ldots \times_{\text{Spec}(K)} P)/S_n$$

On the other hand, the morphism $\text{Spec}(L) \to P$ defines a morphism

$$(\text{Spec}(L) \times_{\text{Spec}(K)} \ldots \times_{\text{Spec}(K)} \text{Spec}(L))/S_n \longrightarrow (P \times_{\text{Spec}(K)} \ldots \times_{\text{Spec}(K)} P)/S_n$$

and the reader can verify that the scheme on the left has a K-rational point. Thus we see that $[n](P)$ is the trivial torsor. $\square$

To prove $\mathcal{X} = \underline{Mor}_S(X, [S/A])$ is not an algebraic stack, by Artin's Axioms, Lemma 9.5, it is enough to show the following.

0AFB **Lemma 63.3.** *The canonical map $\mathcal{X}(S) \to \lim \mathcal{X}(S_n)$ is not essentially surjective.*

Sketch of proof. Unwinding definitions, it is enough to check that $H^1(X, A_0) \to \lim H^1(X_n, A_0)$ is not surjective. As X is regular and projective, by Lemmas 63.2 and 63.1 each A_0-torsor over X is torsion. In particular, the group $H^1(X, A_0)$ is torsion. It is thus enough to show: (a) the group $H^1(X_0, A_0)$ is non-torsion, and (b) the maps $H^1(X_{n+1}, A_0) \to H^1(X_n, A_0)$ are surjective for all n.

Ad (a). One constructs a nontorsion A_0-torsor P_0 on the nodal curve X_0 by glueing trivial A_0-torsors on each component of X_0 using non-torsion points on A_0 as the isomorphisms over the nodes. More precisely, let $x \in X_0$ be a node which occurs in a loop consisting of rational curves. Let $X_0' \to X_0$ be the normalization of X_0 in $X_0 \setminus \{x\}$. Let $x', x'' \in X_0'$ be the two points mapping to x_0. Then we take $A_0 \times_{\text{Spec}(k)} X_0'$ and we identify $A_0 \times x'$ with $A_0 \times \{x''\}$ using translation $A_0 \to A_0$ by a nontorsion point $a_0 \in A_0(k)$ (there is such a nontorsion point as k is algebraically closed and not the algebraic closure of a finite field – this is actually not trivial to prove). One can show that the glueing is an algebraic space (in fact one can show it is a scheme) and that it is an nontorsion A_0-torsor over X_0. The reason that it is nontorsion is that if $[n](P_0)$ has a section, then that section produces a morphism $s : X_0' \to A_0$ such that $[n](a_0) = s(x') - s(x'')$ in the group law on $A_0(k)$. However, since the irreducible components of the loop are rational to section s is constant on them (More on Groupoids in Spaces, Lemma 11.3). Hence $s(x') = s(x'')$ and we obtain a contradiction.

Ad (b). Deformation theory shows that the obstruction to deforming an A_0-torsor $P_n \to X_n$ to an A_0-torsor $P_{n+1} \to X_{n+1}$ lies in $H^2(X_0, \omega)$ for a suitable vector bundle ω on X_0. The latter vanishes as X_0 is a curve, proving the claim. $\square$

0AFC **Proposition 63.4.** *The stack $\mathcal{X} = \underline{Mor}_S(X, [S/A])$ is not algebraic.*

Proof. See discussion above. $\square$

0AFD **Remark 63.5.** Proposition 63.4 contradicts [Aok06b, Theorem 1.1]. The problem is the non-effectivity of formal objects for $\underline{Mor}_S(X, [S/A])$. The same problem is mentioned in the Erratum [Aok06a] to [Aok06b]. Unfortunately, the Erratum goes on the assert that $\underline{Mor}_S(\mathcal{Y}, \mathcal{Z})$ is algebraic if $\mathcal{Z}$ is separated, which also contradicts Proposition 63.4 as $[S/A]$ is separated.

64. An algebraic stack not satisfying strong formal effectiveness

0CXW This is [Bha14, Example 4.12]. Let k be an algebraically closed field. Let A be an abelian variety over k. Assume that $A(k)$ is not torsion (this always holds if k is not the algebraic closure of a finite field). Let $\mathcal{X} = [\mathrm{Spec}(k)/A]$. We claim there exists an ideal $I \subset k[x, y]$ such that

$$\mathcal{X}_{\mathrm{Spec}(k[x,y]^\wedge)} \longrightarrow \lim \mathcal{X}_{\mathrm{Spec}(k[x,y]/I^n)}$$

is not essentially surjective. Namely, let I be the ideal generated by $xy(x + y - 1)$. Then $X_0 = V(I)$ consists of three copies of $\mathbf{A}^1_k$ glued into a triangle at three points. Hence we can make an infinite order torsor P_0 for A over X_0 by taking the trivial torsor over the irreducible components of X_0 and glueing using translation by nontorsion points. Exactly as in the proof of Lemma 63.3 we can lift P_0 to a torsor P_n over $X_n = \mathrm{Spec}(k[x, y]/I^n)$. Since $k[x, y]^\wedge$ is a two dimensional regular domain we see that any torsor P for A over $\mathrm{Spec}(k[x, y]^\wedge)$ is torsion (Lemmas 63.1 and 63.2). Hence the system of torsors is not in the image of the displayed functor.

0CXX **Lemma 64.1.** *Let k be an algebraically closed field which is not the closure of a finite field. Let A be an abelian variety over k. Let $\mathcal{X} = [\mathrm{Spec}(k)/A]$. There exists an inverse system of k-algebras R_n with surjective transition maps whose kernels are locally nilpotent and a system (ξ_n) of $\mathcal{X}$ lying over the system $(\mathrm{Spec}(R_n))$ such that this system is not effective in the sense of Artin's Axioms, Remark 19.2.*

Proof. See discussion above. □

65. A counter example to Grothendieck's existence theorem

0ARE Let k be a field and let $A = k[[t]]$. Let X be the glueing of $U = \mathrm{Spec}(A[x])$ and $V = \mathrm{Spec}(A[y])$ by the identification

$$U \setminus \{0_U\} \longrightarrow V \setminus \{0_V\}$$

sending x to y where $0_U \in U$ and $O_V \in V$ are the points corresponding to the maximal ideals (x, t) and (y, t). Set $A_n = A/(t^n)$ and set $X_n = X \times_{\mathrm{Spec}(A)} \mathrm{Spec}(A_n)$. Let $\mathcal{F}_n$ be the coherent sheaf on X_n corresponding to the $A_n[x]$-module $A_n[x]/(x) \cong A_n$ and the $A_n[y]$ module 0 with obvious glueing. Let $\mathcal{I} \subset \mathcal{O}_X$ be the sheaf of ideals generate by t. Then $(\mathcal{F}_n)$ is an object of the category $Coh_{\text{support proper over } A}(X, \mathcal{I})$ defined in Cohomology of Schemes, Section 23. On the other hand, this object is not in the image of the functor Cohomology of Schemes, Equation (26.0.1). Namely, if it where there would be a finite $A[x]$-module M, a finite $A[y]$-module N and an isomorphism $M[1/t] \cong N[1/t]$ such that $M/t^nM \cong A_n[x]/(x)$ and $N/t^nN = 0$ for all n. It is easy to see that this is impossible.

0ARF **Lemma 65.1.** *Counter examples to algebraization of coherent sheaves.*

1. *Grothendieck's existence theorem as stated in Cohomology of Schemes, Theorem 26.1 is false if we drop the assumption that $X \to \mathrm{Spec}(A)$ is separated.*

(2) *The stack of coherent sheaves $\mathcal{C}oh_{X/B}$ of Quot, Theorems 6.1 and 5.12 is in general not algebraic if we drop the assumption that $X \to S$ is separated*
(3) *The functor $\mathrm{Quot}_{\mathcal{F}/X/B}$ of Quot, Proposition 8.4 is not an algebraic space in general if we drop the assumption that $X \to B$ is separated.*

Proof. Part (1) we saw above. This shows that $Coh_{X/A}$ fails axiom [4] of Artin's Axioms, Section 14. Hence it cannot be an algebraic stack by Artin's Axioms, Lemma 9.5. In this way we see that (2) is true. To see (3), note that there are compatible surjections $\mathcal{O}_{X_n} \to \mathcal{F}_n$ for all n. Thus we see that $\mathrm{Quot}_{\mathcal{O}_X/X/A}$ fails axiom [4] and we see that (3) is true as before. □

66. Affine formal algebraic spaces

0ANY Let K be a field and let $(V_i)_{i \in I}$ be a directed inverse system of nonzero vector spaces over K with surjective transition maps and with $\lim V_i = 0$, see Section 3. Let $R_i = K \oplus V_i$ as K-algebra where V_i is an ideal of square zero. Then R_i is an inverse system of K-algebras with surjective transition maps with nilpotent kernels and with $\lim R_i = K$. The affine formal algebraic space $X = \mathrm{colim}\,\mathrm{Spec}(R_i)$ is an example of an affine formal algebraic space which is not McQuillan.

0CBC **Lemma 66.1.** *There exists an affine formal algebraic space which is not McQuillan.*

Proof. See discussion above. □

Let $0 \to W_i \to V_i \to K \to 0$ be a system of exact sequences as in Section 3. Let $A_i = K[V_i]/(ww'; w, w' \in W_i)$. Then there is a compatible system of surjections $A_i \to K[t]$ with nilpotent kernels and the transition maps $A_i \to A_j$ are surjective with nilpotent kernels as well. Recall that V_i is free over K with basis given by $s \in S_i$. Then, if the characteristic of K is zero, the degree d part of A_i is free over K with basis given by s^d, $s \in S_i$ each of which map to t^d. Hence the inverse system of the degree d parts of the A_i is isomorphic to the inverse system of the vector spaces V_i. As $\lim V_i = 0$ we conclude that $\lim A_i = K$, at least when the characteristic of K is zero. This gives an example of an affine formal algebraic space whose "regular functions" do not separate points.

0CBD **Lemma 66.2.** *There exists an affine formal algebraic space X whose regular functions do not separate points, in the following sense: If we write $X = \mathrm{colim}\, X_\lambda$ as in Formal Spaces, Definition 5.1 then $\lim \Gamma(X_\lambda, \mathcal{O}_{X_\lambda})$ is a field, but X_{red} has infinitely many points.*

Proof. See discussion above. □

Let K, I, and (V_i) be as above. Consider systems

$$\Phi = (\Lambda, J_i \subset \Lambda, (M_i) \to (V_i))$$

where Λ is an augmented K-algebra, $J_i \subset \Lambda$ for $i \in I$ is an ideal of square zero, $(M_i) \to (V_i)$ is a map of inverse systems of K-vector spaces such that $M_i \to V_i$ is surjective for each i, such that M_i has a Λ-module structure, such that the transition maps $M_i \to M_j$, $i > j$ are Λ-linear, and such that $J_j M_i \subset \mathrm{Ker}(M_i \to M_j)$ for $i > j$. Claim: There exists a system as above such that $M_j = M_i/J_j M_i$ for all $i > j$.

If the claim is true, then we obtain a representable morphism

$$\mathrm{colim}_{i \in I}\, \mathrm{Spec}(\Lambda/J_i \oplus M_i) \longrightarrow \mathrm{Spf}(\lim \Lambda/J_i)$$

of affine formal algebraic spaces whose source is not McQuillan but the target is. Here $\Lambda/J_i \oplus M_i$ has the usual Λ/J_i-algebra structure where M_i is an ideal of square zero. Representability translates exactly into the condition that $M_i/J_jM_i = M_j$ for $i > j$. The source of the morphism is not McQuillan as the projections $\lim_{i\in I} M_i \to M_i$ are not be surjective. This is true because the maps $\lim V_i \to V_i$ are not surjective and we have the surjection $M_i \to V_i$. Some details omitted.

Proof of the claim. First, note that there exists at least one system, namely

$$\Phi_0 = (K, J_i = (0), (V_i) \xrightarrow{\text{id}} (V_i))$$

Given a system Φ we will prove there exists a morphism of systems $\Phi \to \Phi'$ (morphisms of systems defined in the obvious manner) such that $\text{Ker}(M_i/J_jM_i \to M_j)$ maps to zero in $M'_i/J'_jM'_i$. Once this is done we can do the usual trick of setting $\Phi_n = (\Phi_{n-1})'$ inductively for $n \geq 1$ and taking $\Phi = \text{colim}\, \Phi_n$ to get a system with the desired properties. Details omitted.

Construction of Φ' given Φ. Consider the set U of triples $u = (i, j, \xi)$ where $i > j$ and $\xi \in \text{Ker}(M_i \to M_j)$. We will let $s, t : U \to I$ denote the maps $s(i, j, \xi) = i$ and $t(i, j, \xi) = j$. Then we set $\xi_u \in M_{s(u)}$ the third component of u. We take

$$\Lambda' = \Lambda[x_u; u \in U]/(x_u x_{u'}; u, u' \in U)$$

with augmentation $\Lambda' \to K$ given by the augmentation of Λ and sending x_u to zero. We take $J'_k = J_k\Lambda' + (x_{u,\ t(u)\geq k})$. We set

$$M'_i = M_i \oplus \bigoplus\nolimits_{s(u)\geq i} K\epsilon_{i,u}$$

As transition maps $M'_i \to M'_j$ for $i > j$ we use the given map $M_i \to M_j$ and we send $\epsilon_{i,u}$ to $\epsilon_{j,u}$. The map $M'_i \to V_i$ induces the given map $M_i \to V_i$ and sends $\epsilon_{i,u}$ to zero. Finally, we let Λ' act on M'_i as follows: for $\lambda \in \Lambda$ we act by the Λ-module structure on M_i and via the augmentation $\Lambda \to K$ on $\epsilon_{i,u}$. The element x_u acts as 0 on M_i for all i. Finally, we define

$$x_u\epsilon_{i,u} = \text{image of } \xi_u \text{ in } M_i$$

and we set all other products $x_{u'}\epsilon_{i,u}$ equal to zero. The displayed formula makes sense because $s(u) \geq i$ and $\xi_u \in M_{s(u)}$. The main things the check are $J'_jM'_i \subset M'_i$ maps to zero in M'_j for $i > j$ and that $\text{Ker}(M_i \to M_j)$ maps to zero in $M'_i/J_jM'_i$. The reason for the last fact is that $\xi = x_{(i,j,\xi)}\epsilon_{i,(i,j,\xi)} \in J'_jM'_i$ for any $\xi \in \text{Ker}(M_i \to M_j)$. We omit the details.

0CBE **Lemma 66.3.** *There exists a representable morphism $f : X \to Y$ of affine formal algebraic spaces with Y McQuillan, but X not McQuillan.*

Proof. See discussion above. □

67. Flat maps are not directed limits of finitely presented flat maps

0ATE The goal of this section is to give an example of a flat ring map which is not a filtered colimit of flat and finitely presented ring maps. In [Gab96] it is shown that if A is a nonexcellent local ring of dimension 1 and residue characteristic zero, then the (flat) ring map $A \to A^\wedge$ to its completion is not a filtered colimit of finite type flat ring maps. The example in this section will have a source which is an excellent ring. We encourage the reader to submit other examples; please email stacks.project@gmail.com if you have one.

For the construction, fix a prime p, and let $A = \mathbf{F}_p[x_1, \ldots, x_n]$. Choose an absolute integral closure A^+ of A, i.e., A^+ is the integral closure of A in an algebraic closure of its fraction field. In [HH92, §6.7] it is shown that $A \to A^+$ is flat.

We claim that the A-algebra A^+ is not a filtered colimit of finitely presented flat A-algebras if $n \geq 3$.

We sketch the argument in the case $n = 3$, and we leave the generalization to higher n to the reader. It is enough to prove the analogous statement for the map $R \to R^+$, where R is the strict henselization of A at the origin and R^+ is its absolute integral closure. Observe that R is a henselian regular local ring whose residue field k is an algebraic closure of $\mathbf{F}_p$.

Choose an ordinary abelian surface X over k and a very ample line bundle L on X. The section ring $\Gamma_*(X, L) = \bigoplus_n H^0(X, L^n)$ is the coordinate ring of the affine cone over X with respect to L. It is a normal ring for L sufficiently positive. Let S denote the henselization of $\Gamma_*(X, L)$ at vertex of the cone. Then S is a henselian noetherian normal domain of dimension 3. We obtain a finite injective map $R \to S$ as the henselization of a Noether normalization for the finite type k-algebra $\Gamma_*(X, L)$. As R^+ is an absolute integral closure of R, we can also fix an embedding $S \to R^+$. Thus R^+ is also the absolute integral closure of S. To show R^+ is not a filtered colimit of flat R-algebras, it suffices to show:

(1) If there exists a factorization $S \to P \to R^+$ with P flat and finite type over R, then there exists a factorization $S \to T \to R^+$ with T finite flat over R.
(2) For any factorization $S \to T \to R^+$ with $S \to T$ finite, the ring T is not R-flat.

Indeed, since S is finitely presented over R, if one could write $R^+ = \mathrm{colim}_i P_i$ as a filtered colimit of finitely presented flat R-algebras P_i, then $S \to R^+$ would factor as $S \to P_i \to R^+$ for $i \gg 0$, which contradicts the above pair of assertions. Assertion (1) follows from the fact that R is henselian and a slicing argument, see More on Morphisms, Lemma 21.5. Part (2) was proven in [Bha12]; for the convenience of the reader, we recall the argument.

Let $U \subset \mathrm{Spec}(S)$ be the punctured spectrum, so there are natural maps $X \leftarrow U \subset \mathrm{Spec}(S)$. The first map gives an identification $H^1(U, \mathcal{O}_U) \simeq H^1(X, \mathcal{O}_X)$. By passing to the Witt vectors of the perfection and using the Artin-Schreier sequence[6], this gives an identification $H^1_{\acute{e}tale}(U, \mathbf{Z}_p) \simeq H^1_{\acute{e}tale}(X, \mathbf{Z}_p)$. In particular, this group is a finite free $\mathbf{Z}_p$-module of rank 2 (since X is ordinary). To get a contradiction assume there exists an R-flat T as in (2) above. Let $V \subset \mathrm{Spec}(T)$ denote the preimage of U, and write $f : V \to U$ for the induced finite surjective map. Since U is normal, there is a trace map $f_*\mathbf{Z}_p \to \mathbf{Z}_p$ on $U_{\acute{e}tale}$ whose composition with the pullback $\mathbf{Z}_p \to f_*\mathbf{Z}_p$ is multiplication by $d = \deg(f)$. Passing to cohomology, and using that $H^1_{\acute{e}tale}(U, \mathbf{Z}_p)$ is nontorsion, then shows that $H^1_{\acute{e}tale}(V, \mathbf{Z}_p)$ is nonzero. Since $H^1_{\acute{e}tale}(V, \mathbf{Z}_p) \simeq \lim H^1_{\acute{e}tale}(V, \mathbf{Z}/p^n)$ as there is no $R^1 \lim$ interference, the group $H^1(V_{\acute{e}tale}, \mathbf{Z}/p)$ must be non-zero. Since T is R-flat we have $\Gamma(V, \mathcal{O}_V) = T$ which is strictly henselian and the Artin-Schreier sequence shows $H^1(V, \mathcal{O}_V) \neq 0$. This is equivalent to $H^2_{\mathfrak{m}}(T) \neq 0$, where $\mathfrak{m} \subset R$ is the maximal ideal. Thus, we

[6] Here we use that S is a strictly henselian local ring of characteristic p and hence $S \to S$, $f \mapsto f^p - f$ is surjective. Also S is a normal domain and hence $\Gamma(U, \mathcal{O}_U) = S$. Thus $H^1_{\acute{e}tale}(U, \mathbf{Z}/p)$ is the kernel of the map $H^1(U, \mathcal{O}_U) \to H^1(U, \mathcal{O}_U)$ induced by $f \mapsto f^p - f$.

obtain a contradiction since T is finite flat (i.e., finite free) as an R-module and $H^2_{\mathfrak m}(R) = 0$. This contradiction proves (2).

0ATF **Lemma 67.1.** *There exists a commutative ring A and a flat A-algebra B which cannot be written as a filtered colimit of finitely presented flat A-algebras. In fact, we may either choose A to be a finite type $\mathbf{F}_p$-algebra or a 1-dimensional Noetherian local ring with residue field of characteristic 0.*

Proof. See discussion above. □

68. The category of modules modulo torsion modules

0B0J The category of torsion groups is a Serre subcategory (Homology, Definition 9.1) of the category of all abelian groups. More generally, for any ring A, the category of torsion A-modules is a Serre subcategory of the category of all A-modules, see More on Algebra, Section 50. If A is a domain, then the quotient category (Homology, Lemma 9.6) is equivalent to the category of vector spaces over the fraction field. This follows from the following more general proposition.

0EA5 **Proposition 68.1.** *Let A be a ring. Let S be a multiplicative subset of A. Let Mod_A denote the category of A-modules and $\mathcal{T}$ its Serre subcategory of modules for which any element is annihilated by some element of S. Then there is a canonical equivalence $Mod_A/\mathcal{T} \to Mod_{S^{-1}A}$.*

Proof. The functor $\text{Mod}_A \to \text{Mod}_{S^{-1}A}$ given by $M \mapsto M \otimes_A S^{-1}A$ is exact (by Algebra, Proposition 9.12) and maps modules in $\mathcal{T}$ to zero. Thus, by the universal property given in Homology, Lemma 9.6, the functor descends to a functor $\text{Mod}_A/\mathcal{T} \to \text{Mod}_{S^{-1}A}$.

Conversely, any A-module M with $M \otimes_A S^{-1}A = 0$ is an object of $\mathcal{T}$, since $M \otimes_A S^{-1}A \cong S^{-1}M$ (Algebra, Lemma 11.15). Thus Homology, Lemma 9.7 shows that the functor $\text{Mod}_A/\mathcal{T} \to \text{Mod}_{S^{-1}A}$ is faithful.

Furthermore, this embedding is essentially surjective: a preimage to an $S^{-1}A$-module N is N_A, that is N regarded as an A-module, since the canonical map $N_A \otimes_A S^{-1}A \to N$ which maps $x \otimes a/s$ to $(a/s) \cdot x$ is an isomorphism of $S^{-1}A$-modules. □

0B0K **Proposition 68.2.** *Let A be a ring. Let $Q(A)$ denote its total quotient ring (as in Algebra, Example 9.8). Let Mod_A denote the category of A-modules and $\mathcal{T}$ its Serre subcategory of torsion modules. Let $Mod_{Q(A)}$ denote the category of $Q(A)$-modules. Then there is a canonical equivalence $Mod_A/\mathcal{T} \to Mod_{Q(A)}$.*

Proof. Follows immediately from applying Proposition 68.1 to the multiplicative subset $S = \{f \in A \mid f \text{ is not a zerodivisor in } A\}$, since a module is a torsion module if and only if all of its elements are each annihilated by some element of S. □

0B0L **Proposition 68.3.** *Let A be a Noetherian integral domain. Let K denote its field of fractions. Let Mod_A^{fg} denote the category of finitely generated A-modules and $\mathcal{T}^{fg}$ its Serre subcategory of finitely generated torsion modules. Then $Mod_A^{fg}/\mathcal{T}^{fg}$ is canonically equivalent to the category of finite dimensional K-vector spaces.*

Proof. The equivalence given in Proposition 68.2 restricts along the embedding $\text{Mod}_A^{fg}/\mathcal{T}^{fg} \to \text{Mod}_A/\mathcal{T}$ to an equivalence $\text{Mod}_A^{fg}/\mathcal{T}^{fg} \to \text{Vect}_K^{fd}$. The Noetherian assumption guarantees that Mod_A^{fg} is an abelian category (see More on Algebra, Section 50) and that the canonical functor $\text{Mod}_A^{fg}/\mathcal{T}^{fg} \to \text{Mod}_A/\mathcal{T}$ is full (else torsion submodules of finitely generated modules might not be objects of $\mathcal{T}^{fg}$). $\square$

0B0M **Proposition 68.4.** *The quotient of the category of abelian groups modulo its Serre subcategory of torsion groups is the category of* $\mathbf{Q}$*-vector spaces.*

Proof. The claim follows directly from Proposition 68.2. $\square$

69. Different colimit topologies

0B2Y This example is [TSH98, Example 1.2, page 553]. Let $G_n = \mathbf{Q}\times\mathbf{R}^n$, $n \geq 1$ seen as a topological group for addition endowed with the usual (Euclidean) topology. Consider the closed embeddings $G_n \to G_{n+1}$ mapping $(x_0, \ldots, x_n)$ to $(x_0, \ldots, x_n, 0)$. We claim that $G = \text{colim}\, G_n$ endowed with the topology

$$U \subset G \text{ open} \Leftrightarrow G_n \cap U \text{ open } \forall n$$

is not a topological group.

To see this we consider the set

$$U = \{(x_0, x_1, x_2, \ldots) \text{ such that } |x_j| < |\cos(jx_0)| \text{ for } j > 0\}$$

Using that jx_0 is never an integral multiple of $\pi/2$ as π is not rational it is easy to show that $U \cap G_n$ is open. Since $0 \in U$, if the topology above made G into a topological group, then there would be an open neighbourhood $V \subset G$ of 0 such that $V + v \subset U$. Then, for every $j \geq 0$ there would exist $\epsilon_j > 0$ such that $(0, \ldots, 0, x_j, 0, \ldots) \in V$ for $|x_j| < \epsilon_j$. Since $V + V \subset U$ we would have

$$(x_0, 0, \ldots, 0, x_j, 0, \ldots) \in U$$

for $|x_0| < \epsilon_0$ and $|x_j| < \epsilon_j$. However, if we take j large enough such that $j\epsilon_0 > \pi/2$, then we can choose $x_0 \in \mathbf{Q}$ such that $|\cos(jx_0)|$ is smaller than ϵ_j, hence there exists an x_j with $|\cos(jx_0)| < |x_j| < \epsilon_j$. This contradiction proves the claim.

0B2Z **Lemma 69.1.** *There exists a system $G_1 \to G_2 \to G_3 \to \ldots$ of (abelian) topological groups such that* $\text{colim}\, G_n$ *taken in the category of topological spaces is different from* $\text{colim}\, G_n$ *taken in the category of topological groups.*

Proof. See discussion above. $\square$

70. Other chapters

Preliminaries

(1) Introduction
(2) Conventions
(3) Set Theory
(4) Categories
(5) Topology
(6) Sheaves on Spaces
(7) Sites and Sheaves
(8) Stacks
(9) Fields
(10) Commutative Algebra
(11) Brauer Groups
(12) Homological Algebra
(13) Derived Categories
(14) Simplicial Methods
(15) More on Algebra
(16) Smoothing Ring Maps
(17) Sheaves of Modules
(18) Modules on Sites
(19) Injectives
(20) Cohomology of Sheaves
(21) Cohomology on Sites
(22) Differential Graded Algebra
(23) Divided Power Algebra

(24) Hypercoverings

Schemes

(25) Schemes
(26) Constructions of Schemes
(27) Properties of Schemes
(28) Morphisms of Schemes
(29) Cohomology of Schemes
(30) Divisors
(31) Limits of Schemes
(32) Varieties
(33) Topologies on Schemes
(34) Descent
(35) Derived Categories of Schemes
(36) More on Morphisms
(37) More on Flatness
(38) Groupoid Schemes
(39) More on Groupoid Schemes
(40) Étale Morphisms of Schemes

Topics in Scheme Theory

(41) Chow Homology
(42) Intersection Theory
(43) Picard Schemes of Curves
(44) Adequate Modules
(45) Dualizing Complexes
(46) Duality for Schemes
(47) Discriminants and Differents
(48) Local Cohomology
(49) Algebraic Curves
(50) Resolution of Surfaces
(51) Semistable Reduction
(52) Fundamental Groups of Schemes
(53) Étale Cohomology
(54) Crystalline Cohomology
(55) Pro-étale Cohomology

Algebraic Spaces

(56) Algebraic Spaces
(57) Properties of Algebraic Spaces
(58) Morphisms of Algebraic Spaces
(59) Decent Algebraic Spaces
(60) Cohomology of Algebraic Spaces
(61) Limits of Algebraic Spaces
(62) Divisors on Algebraic Spaces
(63) Algebraic Spaces over Fields
(64) Topologies on Algebraic Spaces
(65) Descent and Algebraic Spaces
(66) Derived Categories of Spaces
(67) More on Morphisms of Spaces
(68) Flatness on Algebraic Spaces
(69) Groupoids in Algebraic Spaces
(70) More on Groupoids in Spaces
(71) Bootstrap
(72) Pushouts of Algebraic Spaces

Topics in Geometry

(73) Quotients of Groupoids
(74) More on Cohomology of Spaces
(75) Simplicial Spaces
(76) Duality for Spaces
(77) Formal Algebraic Spaces
(78) Restricted Power Series
(79) Resolution of Surfaces Revisited

Deformation Theory

(80) Formal Deformation Theory
(81) Deformation Theory
(82) The Cotangent Complex
(83) Deformation Problems

Algebraic Stacks

(84) Algebraic Stacks
(85) Examples of Stacks
(86) Sheaves on Algebraic Stacks
(87) Criteria for Representability
(88) Artin's Axioms
(89) Quot and Hilbert Spaces
(90) Properties of Algebraic Stacks
(91) Morphisms of Algebraic Stacks
(92) Limits of Algebraic Stacks
(93) Cohomology of Algebraic Stacks
(94) Derived Categories of Stacks
(95) Introducing Algebraic Stacks
(96) More on Morphisms of Stacks
(97) The Geometry of Stacks

Topics in Moduli Theory

(98) Moduli Stacks
(99) Moduli of Curves

Miscellany

(100) Examples
(101) Exercises
(102) Guide to Literature
(103) Desirables
(104) Coding Style
(105) Obsolete
(106) GNU Free Documentation License
(107) Auto Generated Index

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