MORE ON FLATNESS

057M

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1. Introduction

057N In this chapter, we discuss some advanced results on flat modules and flat morphisms of schemes and applications. Most of the results on flatness can be found in the paper [GR71] by Raynaud and Gruson.

Before reading this chapter we advise the reader to take a look at the following results (this list also serves as a pointer to previous results):

(1) General discussion on flat modules in Algebra, Section 39.
(2) The relationship between Tor-groups and flatness, see Algebra, Section 75.
(3) Criteria for flatness, see Algebra, Section 99 (Noetherian case), Algebra, Section 101 (Artinian case), Algebra, Section 128 (non-Noetherian case), and finally More on Morphisms, Section 16.
(4) Generic flatness, see Algebra, Section 118 and Morphisms, Section 28.
(5) Openness of the flat locus, see Algebra, Section 129 and More on Morphisms, Section 15.
(6) Flattening, see More on Algebra, Sections 16, 17, 18, 19, and 20.
(7) Additional results in More on Algebra, Sections 21, 22, 26, and 27.

As applications of the material on flatness we discuss the following topics: a non-Noetherian version of Grothendieck's existence theorem, blowing up and flatness, Nagata's theorem on compactifications, the h topology, blow up squares and descent, weak normalization, descent of vector bundles in positive characteristic, and the local structure of perfect complexes in the h topology.

2. Lemmas on étale localization

05FM In this section we list some lemmas on étale localization which will be useful later in this chapter. Please skip this section on a first reading.

057R **Lemma 2.1.** *Let $i : Z \to X$ be a closed immersion of affine schemes. Let $Z' \to Z$ be an étale morphism with Z' affine. Then there exists an étale morphism $X' \to X$ with X' affine such that $Z' \cong Z \times_X X'$ as schemes over Z.*

Proof. See Algebra, Lemma 143.10. □

05H2 **Lemma 2.2.** *Let*

$$\begin{array}{ccc} X & \longleftarrow & X' \\ \downarrow & & \downarrow \\ S & \longleftarrow & S' \end{array}$$

be a commutative diagram of schemes with $X' \to X$ and $S' \to S$ étale. Let $s' \in S'$ be a point. Then

$$X' \times_{S'} \mathrm{Spec}(\mathcal{O}_{S',s'}) \longrightarrow X \times_S \mathrm{Spec}(\mathcal{O}_{S',s'})$$

is étale.

Proof. This is true because $X' \to X_{S'}$ is étale as a morphism of schemes étale over X, see Morphisms, Lemma 37.19 and the base change of an étale morphism is étale, see Morphisms, Lemma 37.4. □

05B9 **Lemma 2.3.** *Let $X \to T \to S$ be morphisms of schemes with $T \to S$ étale. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $x \in X$ be a point. Then*

$$\mathcal{F} \text{ flat over } S \text{ at } x \Leftrightarrow \mathcal{F} \text{ flat over } T \text{ at } x$$

In particular $\mathcal{F}$ is flat over S if and only if $\mathcal{F}$ is flat over T.

Proof. As an étale morphism is a flat morphism (see Morphisms, Lemma 37.12) the implication "$\Leftarrow$" follows from Algebra, Lemma 39.4. For the converse assume that $\mathcal{F}$ is flat at x over S. Denote $\tilde{x} \in X \times_S T$ the point lying over x in X and over the image of x in T in T. Then $(X \times_S T \to X)^*\mathcal{F}$ is flat at $\tilde{x}$ over T via $\mathrm{pr}_2 : X \times_S T \to T$, see Morphisms, Lemma 26.7. The diagonal $\Delta_{T/S} : T \to T \times_S T$ is an open immersion; combine Morphisms, Lemmas 36.13 and 37.5. So X is identified with open subscheme of $X \times_S T$, the restriction of pr_2 to this open is the given morphism $X \to T$, the point $\tilde{x}$ corresponds to the point x in this open, and $(X \times_S T \to X)^*\mathcal{F}$ restricted to this open is $\mathcal{F}$. Whence we see that $\mathcal{F}$ is flat at x over T. □

05BA **Lemma 2.4.** *Let $T \to S$ be an étale morphism. Let $t \in T$ with image $s \in S$. Let M be a $\mathcal{O}_{T,t}$-module. Then*

$$M \text{ flat over } \mathcal{O}_{S,s} \Leftrightarrow M \text{ flat over } \mathcal{O}_{T,t}.$$

Proof. We may replace S by an affine neighbourhood of s and after that T by an affine neighbourhood of t. Set $\mathcal{F} = (\mathrm{Spec}(\mathcal{O}_{T,t}) \to T)_*\widetilde{M}$. This is a quasi-coherent sheaf (see Schemes, Lemma 24.1 or argue directly) on T whose stalk at t is M (details omitted). Apply Lemma 2.3. □

05VL **Lemma 2.5.** *Let S be a scheme and $s \in S$ a point. Denote $\mathcal{O}^h_{S,s}$ (resp. $\mathcal{O}^{sh}_{S,s}$) the henselization (resp. strict henselization), see Algebra, Definition 155.3. Let M^{sh} be a $\mathcal{O}^{sh}_{S,s}$-module. The following are equivalent*

1. *M^{sh} is flat over $\mathcal{O}_{S,s}$,*
2. *M^{sh} is flat over $\mathcal{O}^h_{S,s}$, and*
3. *M^{sh} is flat over $\mathcal{O}^{sh}_{S,s}$.*

If $M^{sh} = M^h \otimes_{\mathcal{O}^h_{S,s}} \mathcal{O}^{sh}_{S,s}$ this is also equivalent to

4. *M^h is flat over $\mathcal{O}_{S,s}$, and*
5. *M^h is flat over $\mathcal{O}^h_{S,s}$.*

If $M^h = M \otimes_{\mathcal{O}_{S,s}} \mathcal{O}^h_{S,s}$ this is also equivalent to

6. *M is flat over $\mathcal{O}_{S,s}$.*

Proof. By More on Algebra, Lemma 46.1 the local ring maps $\mathcal{O}_{S,s} \to \mathcal{O}^h_{S,s} \to \mathcal{O}^{sh}_{S,s}$ are faithfully flat. Hence (3) $\Rightarrow$ (2) $\Rightarrow$ (1) and (5) $\Rightarrow$ (4) follow from Algebra, Lemma 39.4. By faithful flatness the equivalences (6) $\Leftrightarrow$ (5) and (5) $\Leftrightarrow$ (3) follow from Algebra, Lemma 39.8. Thus it suffices to show that (1) $\Rightarrow$ (2) $\Rightarrow$ (3) and (4) $\Rightarrow$ (5). To prove these we may assume S is an affine scheme.

Assume (1). By Lemma 2.4 we see that M^{sh} is flat over $\mathcal{O}_{T,t}$ for any étale neighbourhood $(T,t) \to (S,s)$. Since $\mathcal{O}^h_{S,s}$ and $\mathcal{O}^{sh}_{S,s}$ are directed colimits of local rings of the form $\mathcal{O}_{T,t}$ (see Algebra, Lemmas 155.7 and 155.11) we conclude that M^{sh} is flat over $\mathcal{O}^h_{S,s}$ and $\mathcal{O}^{sh}_{S,s}$ by Algebra, Lemma 39.6. Thus (1) implies (2) and (3). Of course this implies also (2) $\Rightarrow$ (3) by replacing $\mathcal{O}_{S,s}$ by $\mathcal{O}^h_{S,s}$. The same argument applies to prove (4) $\Rightarrow$ (5). □

0DK0 **Lemma 2.6.** *Let S be a scheme and $s \in S$ a point. Denote $\mathcal{O}^h_{S,s}$ (resp. $\mathcal{O}^{sh}_{S,s}$) the henselization (resp. strict henselization), see Algebra, Definition 155.3. Let M^{sh} be an object of $D(\mathcal{O}^{sh}_{S,s})$. Let $a, b \in \mathbf{Z}$. The following are equivalent*

(1) *M^{sh} has tor amplitude in $[a,b]$ over $\mathcal{O}_{S,s}$,*
(2) *M^{sh} has tor amplitude in $[a,b]$ over $\mathcal{O}^h_{S,s}$, and*
(3) *M^{sh} has tor amplitude in $[a,b]$ over $\mathcal{O}^{sh}_{S,s}$.*

If $M^{sh} = M^h \otimes^{\mathbf{L}}_{\mathcal{O}^h_{S,s}} \mathcal{O}^{sh}_{S,s}$ for $M^h \in D(\mathcal{O}^h_{S,s})$ this is also equivalent to

(4) *M^h has tor amplitude in $[a,b]$ over $\mathcal{O}_{S,s}$, and*
(5) *M^h has tor amplitude in $[a,b]$ over $\mathcal{O}^h_{S,s}$.*

If $M^h = M \otimes^{\mathbf{L}}_{\mathcal{O}_{S,s}} \mathcal{O}^h_{S,s}$ for $M \in D(\mathcal{O}_{S,s})$ this is also equivalent to

(6) *M has tor amplitude in $[a,b]$ over $\mathcal{O}_{S,s}$.*

Proof. By More on Algebra, Lemma 46.1 the local ring maps $\mathcal{O}_{S,s} \to \mathcal{O}^h_{S,s} \to \mathcal{O}^{sh}_{S,s}$ are faithfully flat. Hence (3) $\Rightarrow$ (2) $\Rightarrow$ (1) and (5) $\Rightarrow$ (4) follow from More on Algebra, Lemma 68.11. By faithful flatness the equivalences (6) $\Leftrightarrow$ (5) and (5) $\Leftrightarrow$ (3) follow from More on Algebra, Lemma 68.17. Thus it suffices to show that (1) $\Rightarrow$ (3), (2) $\Rightarrow$ (3), and (4) $\Rightarrow$ (5).

Assume (1). In particular M^{sh} has vanishing cohomology in degrees $< a$ and $> b$. Hence we can represent M^{sh} by a complex $P^\bullet$ of free $\mathcal{O}^{sh}_{X,x}$-modules with $P^i = 0$ for $i > b$ (see for example the very general Derived Categories, Lemma 15.4). Note that P^n is flat over $\mathcal{O}_{S,s}$ for all n. Consider $\mathrm{Coker}(d_P^{a-1})$. By More on Algebra, Lemma 68.2 this is a flat $\mathcal{O}_{S,s}$-module. Hence by Lemma 2.5 this is a flat $\mathcal{O}^{sh}_{S,s}$-module. Thus $\tau_{\geq a}P^\bullet$ is a complex of flat $\mathcal{O}^{sh}_{S,s}$-modules representing M^{sh} in $D(\mathcal{O}^{sh}_{S,s}$ and we find that M^{sh} has tor amplitude in $[a,b]$, see More on Algebra, Lemma 68.3. Thus (1) implies (3). Of course this implies also (2) $\Rightarrow$ (3) by replacing $\mathcal{O}_{S,s}$ by $\mathcal{O}^h_{S,s}$. The same argument applies to prove (4) $\Rightarrow$ (5). □

05FN **Lemma 2.7.** *Let $g : T \to S$ be a finite flat morphism of schemes. Let $\mathcal{G}$ be a quasi-coherent $\mathcal{O}_S$-module. Let $t \in T$ be a point with image $s \in S$. Then*

$$t \in \mathit{WeakAss}(g^*\mathcal{G}) \Leftrightarrow s \in \mathit{WeakAss}(\mathcal{G})$$

Proof. The implication "$\Leftarrow$" follows immediately from Divisors, Lemma 6.4. Assume $t \in \mathrm{WeakAss}(g^*\mathcal{G})$. Let $\mathrm{Spec}(A) \subset S$ be an affine open neighbourhood of s. Let $\mathcal{G}$ be the quasi-coherent sheaf associated to the A-module M. Let $\mathfrak{p} \subset A$ be the

prime ideal corresponding to s. As g is finite flat we have $g^{-1}(\mathrm{Spec}(A)) = \mathrm{Spec}(B)$ for some finite flat A-algebra B. Note that $g^*\mathcal{G}$ is the quasi-coherent $\mathcal{O}_{\mathrm{Spec}(B)}$-module associated to the B-module $M \otimes_A B$ and $g_*g^*\mathcal{G}$ is the quasi-coherent $\mathcal{O}_{\mathrm{Spec}(A)}$-module associated to the A-module $M \otimes_A B$. By Algebra, Lemma 78.5 we have $B_{\mathfrak{p}} \cong A_{\mathfrak{p}}^{\oplus n}$ for some integer $n \geq 0$. Note that $n \geq 1$ as we assumed there exists at least one point of T lying over s. Hence we see by looking at stalks that

$$s \in \mathrm{WeakAss}(\mathcal{G}) \Leftrightarrow s \in \mathrm{WeakAss}(g_*g^*\mathcal{G})$$

Now the assumption that $t \in \mathrm{WeakAss}(g^*\mathcal{G})$ implies that $s \in \mathrm{WeakAss}(g_*g^*\mathcal{G})$ by Divisors, Lemma 6.3 and hence by the above $s \in \mathrm{WeakAss}(\mathcal{G})$. □

05FP **Lemma 2.8.** *Let $h : U \to S$ be an étale morphism of schemes. Let $\mathcal{G}$ be a quasi-coherent $\mathcal{O}_S$-module. Let $u \in U$ be a point with image $s \in S$. Then*

$$u \in \mathit{WeakAss}(h^*\mathcal{G}) \Leftrightarrow s \in \mathit{WeakAss}(\mathcal{G})$$

Proof. After replacing S and U by affine neighbourhoods of s and u we may assume that g is a standard étale morphism of affines, see Morphisms, Lemma 37.14. Thus we may assume $S = \mathrm{Spec}(A)$ and $X = \mathrm{Spec}(A[x, 1/g]/(f))$, where f is monic and f' is invertible in $A[x, 1/g]$. Note that $A[x, 1/g]/(f) = (A[x]/(f))_g$ is also the localization of the finite free A-algebra $A[x]/(f)$. Hence we may think of U as an open subscheme of the scheme $T = \mathrm{Spec}(A[x]/(f))$ which is finite locally free over S. This reduces us to Lemma 2.7 above. □

0CTU **Lemma 2.9.** *Let S be a scheme and $s \in S$ a point. Denote $\mathcal{O}_{S,s}^h$ (resp. $\mathcal{O}_{S,s}^{sh}$) the henselization (resp. strict henselization), see Algebra, Definition 155.3. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_S$-module. The following are equivalent*

1. *s is a weakly associated point of $\mathcal{F}$,*
2. *$\mathfrak{m}_s$ is a weakly associated prime of $\mathcal{F}_s$,*
3. *$\mathfrak{m}_s^h$ is a weakly associated prime of $\mathcal{F}_s \otimes_{\mathcal{O}_{S,s}} \mathcal{O}_{S,s}^h$, and*
4. *$\mathfrak{m}_s^{sh}$ is a weakly associated prime of $\mathcal{F}_s \otimes_{\mathcal{O}_{S,s}} \mathcal{O}_{S,s}^{sh}$.*

Proof. The equivalence of (1) and (2) is the definition, see Divisors, Definition 5.1. The implications (2) ⇒ (3) ⇒ (4) follows from Divisors, Lemma 6.4 applied to the flat (More on Algebra, Lemma 46.1) morphisms

$$\mathrm{Spec}(\mathcal{O}_{S,s}^{sh}) \to \mathrm{Spec}(\mathcal{O}_{S,s}^h) \to \mathrm{Spec}(\mathcal{O}_{S,s})$$

and the closed points. To prove (4) ⇒ (2) we may replace S by an affine neighbourhood. Suppose that $x \in \mathcal{F}_s \otimes_{\mathcal{O}_{S,s}} \mathcal{O}_{S,s}^{sh}$ is an element whose annihilator has radical equal to $\mathfrak{m}_s^{sh}$. (See Algebra, Lemma 66.2.) Since $\mathcal{O}_{S,s}^{sh}$ is equal to the limit of $\mathcal{O}_{U,u}$ over étale neighbourhoods $f : (U, u) \to (S, s)$ by Algebra, Lemma 155.11 we may assume that x is the image of some $x' \in \mathcal{F}_s \otimes_{\mathcal{O}_{S,s}} \mathcal{O}_{U,u}$. The local ring map $\mathcal{O}_{U,u} \to \mathcal{O}_{S,s}^{sh}$ is faithfully flat (as it is the strict henselization), hence universally injective (Algebra, Lemma 82.15). It follows that the annihilator of x' is the inverse image of the annihilator of x. Hence the radical of this annihilator is equal to $\mathfrak{m}_u$. Thus u is a weakly associated point of $f^*\mathcal{F}$. By Lemma 2.8 we see that s is a weakly associated point of $\mathcal{F}$. □

3. The local structure of a finite type module

057P The key technical lemma that makes a lot of the arguments in this chapter work is the geometric Lemma 3.2.

057Q **Lemma 3.1.** *Let $f : X \to S$ be a finite type morphism of affine schemes. Let $\mathcal{F}$ be a finite type quasi-coherent $\mathcal{O}_X$-module. Let $x \in X$ with image $s = f(x)$ in S. Set $\mathcal{F}_s = \mathcal{F}|_{X_s}$. Then there exist a closed immersion $i : Z \to X$ of finite presentation, and a quasi-coherent finite type $\mathcal{O}_Z$-module $\mathcal{G}$ such that $i_*\mathcal{G} = \mathcal{F}$ and $Z_s = Supp(\mathcal{F}_s)$.*

Proof. Say the morphism $f : X \to S$ is given by the ring map $A \to B$ and that $\mathcal{F}$ is the quasi-coherent sheaf associated to the B-module M. By Morphisms, Lemma 15.2 we know that $A \to B$ is a finite type ring map, and by Properties, Lemma 17.1 we know that M is a finite B-module. In particular the support of $\mathcal{F}$ is the closed subscheme of $\mathrm{Spec}(B)$ cut out by the annihilator $I = \{x \in B \mid xm = 0 \ \forall m \in M\}$ of M, see Algebra, Lemma 40.5. Let $\mathfrak{q} \subset B$ be the prime ideal corresponding to x and let $\mathfrak{p} \subset A$ be the prime ideal corresponding to s. Note that $X_s = \mathrm{Spec}(B \otimes_A \kappa(\mathfrak{p}))$ and that $\mathcal{F}_s$ is the quasi-coherent sheaf associated to the $B \otimes_A \kappa(\mathfrak{p})$ module $M \otimes_A \kappa(\mathfrak{p})$. By Morphisms, Lemma 5.3 the support of $\mathcal{F}_s$ is equal to $V(I(B \otimes_A \kappa(\mathfrak{p})))$. Since $B \otimes_A \kappa(\mathfrak{p})$ is of finite type over $\kappa(\mathfrak{p})$ there exist finitely many elements $f_1, \ldots, f_m \in I$ such that

$$I(B \otimes_A \kappa(\mathfrak{p})) = (f_1, \ldots, f_n)(B \otimes_A \kappa(\mathfrak{p})).$$

Denote $i : Z \to X$ the closed subscheme cut out by $(f_1, \ldots, f_m)$, in a formula $Z = \mathrm{Spec}(B/(f_1, \ldots, f_m))$. Since M is annihilated by I we can think of M as an $B/(f_1, \ldots, f_m)$-module. In other words, $\mathcal{F}$ is the pushforward of a finite type module on Z. As $Z_s = \mathrm{Supp}(\mathcal{F}_s)$ by construction, this proves the lemma. □

057S **Lemma 3.2.** *Let $f : X \to S$ be morphism of schemes which is locally of finite type. Let $\mathcal{F}$ be a finite type quasi-coherent $\mathcal{O}_X$-module. Let $x \in X$ with image $s = f(x)$ in S. Set $\mathcal{F}_s = \mathcal{F}|_{X_s}$ and $n = \dim_x(Supp(\mathcal{F}_s))$. Then we can construct*

1. *elementary étale neighbourhoods $g : (X', x') \to (X, x)$, $e : (S', s') \to (S, s)$,*
2. *a commutative diagram*

$$\begin{array}{ccccc} X & \xleftarrow{g} & X' & \xleftarrow{i} & Z' \\ \downarrow{\scriptstyle f} & & \downarrow & & \downarrow{\scriptstyle \pi} \\ & & & & Y' \\ & & & & \downarrow{\scriptstyle h} \\ S & \xleftarrow{e} & S' & = & S' \end{array}$$

3. *a point $z' \in Z'$ with $i(z') = x'$, $y' = \pi(z')$, $h(y') = s'$,*
4. *a finite type quasi-coherent $\mathcal{O}_{Z'}$-module $\mathcal{G}$,*

such that the following properties hold

1. *X', Z', Y', S' are affine schemes,*
2. *i is a closed immersion of finite presentation,*
3. *$i_*(\mathcal{G}) \cong g^*\mathcal{F}$,*
4. *π is finite and $\pi^{-1}(\{y'\}) = \{z'\}$,*
5. *the extension $\kappa(y')/\kappa(s')$ is purely transcendental,*

(6) *h is smooth of relative dimension n with geometrically integral fibres.*

Proof. Let $V \subset S$ be an affine neighbourhood of s. Let $U \subset f^{-1}(V)$ be an affine neighbourhood of x. Then it suffices to prove the lemma for $f|_U : U \to V$ and $\mathcal{F}|_U$. Hence in the rest of the proof we assume that X and S are affine.

First, suppose that $X_s = \text{Supp}(\mathcal{F}_s)$, in particular $n = \dim_x(X_s)$. Apply More on Morphisms, Lemmas 47.2 and 47.3. This gives us a commutative diagram

$$\begin{array}{ccc} X & \xleftarrow{g} & X' \\ \downarrow & & \downarrow{\scriptstyle \pi} \\ & & Y' \\ & & \downarrow{\scriptstyle h} \\ S & \xleftarrow{e} & S' \end{array}$$

and point $x' \in X'$. We set $Z' = X'$, $i = \text{id}$, and $\mathcal{G} = g^*\mathcal{F}$ to obtain a solution in this case.

In general choose a closed immersion $Z \to X$ and a sheaf $\mathcal{G}$ on Z as in Lemma 3.1. Applying the result of the previous paragraph to $Z \to S$ and $\mathcal{G}$ we obtain a diagram

$$\begin{array}{ccccc} X & \xleftarrow{} & Z & \xleftarrow{g} & Z' \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f|_Z} & & \downarrow{\scriptstyle \pi} \\ & & & & Y' \\ & & & & \downarrow{\scriptstyle h} \\ S & = & S & \xleftarrow{e} & S' \end{array}$$

and point $z' \in Z'$ satisfying all the required properties. We will use Lemma 2.1 to embed Z' into a scheme étale over X. We cannot apply the lemma directly as we want X' to be a scheme over S'. Instead we consider the morphisms

$$Z' \longrightarrow Z \times_S S' \longrightarrow X \times_S S'$$

The first morphism is étale by Morphisms, Lemma 37.19. The second is a closed immersion as a base change of a closed immersion. Finally, as X, S, S', Z, Z' are all affine we may apply Lemma 2.1 to get an étale morphism of affine schemes $X' \to X \times_S S'$ such that

$$Z' = (Z \times_S S') \times_{(X \times_S S')} X' = Z \times_X X'.$$

As $Z \to X$ is a closed immersion of finite presentation, so is $Z' \to X'$. Let $x' \in X'$ be the point corresponding to $z' \in Z'$. Then the completed diagram

$$\begin{array}{ccccc} X & \xleftarrow{} & X' & \xleftarrow{i} & Z' \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle \pi} \\ & & & & Y' \\ & & & & \downarrow{\scriptstyle h} \\ S & \xleftarrow{e} & S' & = & S' \end{array}$$

is a solution of the original problem. □

057T **Lemma 3.3.** *Assumptions and notation as in Lemma 3.2. If f is locally of finite presentation then π is of finite presentation. In this case the following are equivalent*

(1) *$\mathcal{F}$ is an $\mathcal{O}_X$-module of finite presentation in a neighbourhood of x,*
(2) *$\mathcal{G}$ is an $\mathcal{O}_{Z'}$-module of finite presentation in a neighbourhood of z', and*
(3) *$\pi_*\mathcal{G}$ is an $\mathcal{O}_{Y'}$-module of finite presentation in a neighbourhood of y'.*

Still assuming f locally of finite presentation the following are equivalent to each other

(a) *$\mathcal{F}_x$ is an $\mathcal{O}_{X,x}$-module of finite presentation,*
(b) *$\mathcal{G}_{z'}$ is an $\mathcal{O}_{Z',z'}$-module of finite presentation, and*
(c) *$(\pi_*\mathcal{G})_{y'}$ is an $\mathcal{O}_{Y',y'}$-module of finite presentation.*

Proof. Assume f locally of finite presentation. Then $Z' \to S$ is locally of finite presentation as a composition of such, see Morphisms, Lemma 22.3. Note that $Y' \to S$ is also locally of finite presentation as a composition of a smooth and an étale morphism. Hence Morphisms, Lemma 22.11 implies π is locally of finite presentation. Since π is finite we conclude that it is also separated and quasi-compact, hence π is actually of finite presentation.

To prove the equivalence of (1), (2), and (3) we also consider: (4) $g^*\mathcal{F}$ is a $\mathcal{O}_{X'}$-module of finite presentation in a neighbourhood of x'. The pullback of a module of finite presentation is of finite presentation, see Modules, Lemma 11.5. Hence (1) $\Rightarrow$ (4). The étale morphism g is open, see Morphisms, Lemma 37.13. Hence for any open neighbourhood $U' \subset X'$ of x', the image $g(U')$ is an open neighbourhood of x and the map $\{U' \to g(U')\}$ is an étale covering. Thus (4) $\Rightarrow$ (1) by Descent, Lemma 7.3. Using Descent, Lemma 7.10 and some easy topological arguments (see More on Morphisms, Lemma 47.4) we see that (4) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3).

To prove the equivalence of (a), (b), (c) consider the ring maps

$$\mathcal{O}_{X,x} \to \mathcal{O}_{X',x'} \to \mathcal{O}_{Z',z'} \leftarrow \mathcal{O}_{Y',y'}$$

The first ring map is faithfully flat. Hence $\mathcal{F}_x$ is of finite presentation over $\mathcal{O}_{X,x}$ if and only if $g^*\mathcal{F}_{x'}$ is of finite presentation over $\mathcal{O}_{X',x'}$, see Algebra, Lemma 83.2. The second ring map is surjective (hence finite) and finitely presented by assumption, hence $g^*\mathcal{F}_{x'}$ is of finite presentation over $\mathcal{O}_{X',x'}$ if and only if $\mathcal{G}_{z'}$ is of finite presentation over $\mathcal{O}_{Z',z'}$, see Algebra, Lemma 36.23. Because π is finite, of finite presentation, and $\pi^{-1}(\{y'\}) = \{x'\}$ the ring homomorphism $\mathcal{O}_{Y',y'} \leftarrow \mathcal{O}_{Z',z'}$ is finite and of finite presentation, see More on Morphisms, Lemma 47.4. Hence $\mathcal{G}_{z'}$ is of finite presentation over $\mathcal{O}_{Z',z'}$ if and only if $\pi_*\mathcal{G}_{y'}$ is of finite presentation over $\mathcal{O}_{Y',y'}$, see Algebra, Lemma 36.23. □

057U **Lemma 3.4.** *Assumptions and notation as in Lemma 3.2. The following are equivalent*

(1) *$\mathcal{F}$ is flat over S in a neighbourhood of x,*
(2) *$\mathcal{G}$ is flat over S' in a neighbourhood of z', and*
(3) *$\pi_*\mathcal{G}$ is flat over S' in a neighbourhood of y'.*

The following are equivalent also

(a) *$\mathcal{F}_x$ is flat over $\mathcal{O}_{S,s}$,*
(b) *$\mathcal{G}_{z'}$ is flat over $\mathcal{O}_{S',s'}$, and*

(c) $(\pi_*\mathcal{G})_{y'}$ *is flat over* $\mathcal{O}_{S',s'}$.

Proof. To prove the equivalence of (1), (2), and (3) we also consider: (4) $g^*\mathcal{F}$ is flat over S in a neighbourhood of x'. We will use Lemma 2.3 to equate flatness over S and S' without further mention. The étale morphism g is flat and open, see Morphisms, Lemma 37.13. Hence for any open neighbourhood $U' \subset X'$ of x', the image $g(U')$ is an open neighbourhood of x and the map $U' \to g(U')$ is surjective and flat. Thus (4) $\Leftrightarrow$ (1) by Morphisms, Lemma 26.13. Note that

$$\Gamma(X', g^*\mathcal{F}) = \Gamma(Z', \mathcal{G}) = \Gamma(Y', \pi_*\mathcal{G})$$

Hence the flatness of $g^*\mathcal{F}$, $\mathcal{G}$ and $\pi_*\mathcal{G}$ over S' are all equivalent (this uses that X', Z', Y', and S' are all affine). Some omitted topological arguments (compare More on Morphisms, Lemma 47.4) regarding affine neighbourhoods now show that (4) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3).

To prove the equivalence of (a), (b), (c) consider the commutative diagram of local ring maps

$$\begin{array}{ccccc} \mathcal{O}_{X',x'} & \xrightarrow[\iota]{} & \mathcal{O}_{Z',z'} & \xleftarrow[\alpha]{} & \mathcal{O}_{Y',y'} & \xleftarrow[\beta]{} & \mathcal{O}_{S',s'} \\ \uparrow{\scriptstyle \gamma} & & & & & & \uparrow{\scriptstyle \epsilon} \\ \mathcal{O}_{X,x} & & \xleftarrow{\qquad\qquad \varphi \qquad\qquad} & & & & \mathcal{O}_{S,s} \end{array}$$

We will use Lemma 2.4 to equate flatness over $\mathcal{O}_{S,s}$ and $\mathcal{O}_{S',s'}$ without further mention. The map γ is faithfully flat. Hence $\mathcal{F}_x$ is flat over $\mathcal{O}_{S,s}$ if and only if $g^*\mathcal{F}_{x'}$ is flat over $\mathcal{O}_{S',s'}$, see Algebra, Lemma 39.9. As $\mathcal{O}_{S',s'}$-modules the modules $g^*\mathcal{F}_{x'}$, $\mathcal{G}_{z'}$, and $\pi_*\mathcal{G}_{y'}$ are all isomorphic, see More on Morphisms, Lemma 47.4. This finishes the proof. □

4. One step dévissage

05H3 In this section we explain what is a one step dévissage of a module. A one step dévissage exist étale locally on base and target. We discuss base change, Zariski shrinking and étale localization of a one step dévissage.

05H4 **Definition 4.1.** Let S be a scheme. Let X be locally of finite type over S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module of finite type. Let $s \in S$ be a point. A *one step dévissage of* $\mathcal{F}/X/S$ *over* s is given by morphisms of schemes over S

$$X \xleftarrow{i} Z \xrightarrow{\pi} Y$$

and a quasi-coherent $\mathcal{O}_Z$-module $\mathcal{G}$ of finite type such that

(1) X, S, Z and Y are affine,
(2) i is a closed immersion of finite presentation,
(3) $\mathcal{F} \cong i_*\mathcal{G}$,
(4) π is finite, and
(5) the structure morphism $Y \to S$ is smooth with geometrically irreducible fibres of dimension $\dim(\mathrm{Supp}(\mathcal{F}_s))$.

In this case we say $(Z, Y, i, \pi, \mathcal{G})$ is a one step dévissage of $\mathcal{F}/X/S$ over s.

Note that such a one step dévissage can only exist if X and S are affine. In the definition above we only require X to be (locally) of finite type over S and we continue working in this setting below. In [GR71] the authors use consistently the

setup where $X \to S$ is locally of finite presentation and $\mathcal{F}$ quasi-coherent $\mathcal{O}_X$-module of finite type. The advantage of this choice is that it "makes sense" to ask for $\mathcal{F}$ to be of finite presentation as an $\mathcal{O}_X$-module, whereas in our setting it "does not make sense". Please see More on Morphisms, Section 58 for a discussion; the observations made there show that in our setup we may consider the condition of $\mathcal{F}$ being "locally of finite presentation relative to S", and we could work consistently with this notion. Instead however, we will rely on the results of Lemma 3.3 and the observations in Remark 6.3 to deal with this issue in an ad hoc fashion whenever it comes up.

05H5 **Definition 4.2.** Let S be a scheme. Let X be locally of finite type over S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module of finite type. Let $x \in X$ be a point with image s in S. A *one step dévissage of $\mathcal{F}/X/S$ at x* is a system $(Z, Y, i, \pi, \mathcal{G}, z, y)$, where $(Z, Y, i, \pi, \mathcal{G})$ is a one step dévissage of $\mathcal{F}/X/S$ over s and

(1) $\dim_x(\mathrm{Supp}(\mathcal{F}_s)) = \dim(\mathrm{Supp}(\mathcal{F}_s))$,
(2) $z \in Z$ is a point with $i(z) = x$ and $\pi(z) = y$,
(3) we have $\pi^{-1}(\{y\}) = \{z\}$,
(4) the extension $\kappa(y)/\kappa(s)$ is purely transcendental.

A one step dévissage of $\mathcal{F}/X/S$ at x can only exist if X and S are affine. Condition (1) assures us that $Y \to S$ has relative dimension equal to $\dim_x(\mathrm{Supp}(\mathcal{F}_s))$ via condition (5) of Definition 4.1.

05H6 **Lemma 4.3.** *Let $f : X \to S$ be morphism of schemes which is locally of finite type. Let $\mathcal{F}$ be a finite type quasi-coherent $\mathcal{O}_X$-module. Let $x \in X$ with image $s = f(x)$ in S. Then there exists a commutative diagram of pointed schemes*

$$\begin{array}{ccc} (X, x) & \xleftarrow{g} & (X', x') \\ {\scriptstyle f}\downarrow & & \downarrow \\ (S, s) & \longleftarrow & (S', s') \end{array}$$

such that $(S', s') \to (S, s)$ and $(X', x') \to (X, x)$ are elementary étale neighbourhoods, and such that $g^*\mathcal{F}/X'/S'$ has a one step dévissage at x'.

Proof. This is immediate from Definition 4.2 and Lemma 3.2. □

05H7 **Lemma 4.4.** *Let S, X, $\mathcal{F}$, s be as in Definition 4.1. Let $(Z, Y, i, \pi, \mathcal{G})$ be a one step dévissage of $\mathcal{F}/X/S$ over s. Let $(S', s') \to (S, s)$ be any morphism of pointed schemes. Given this data let X', Z', Y', i', π' be the base changes of X, Z, Y, i, π via $S' \to S$. Let $\mathcal{F}'$ be the pullback of $\mathcal{F}$ to X' and let $\mathcal{G}'$ be the pullback of $\mathcal{G}$ to Z'. If S' is affine, then $(Z', Y', i', \pi', \mathcal{G}')$ is a one step dévissage of $\mathcal{F}'/X'/S'$ over s'.*

Proof. Fibre products of affines are affine, see Schemes, Lemma 17.2. Base change preserves closed immersions, morphisms of finite presentation, finite morphisms, smooth morphisms, morphisms with geometrically irreducible fibres, and morphisms of relative dimension n, see Morphisms, Lemmas 2.4, 22.4, 45.6, 35.5, 30.2, and More on Morphisms, Lemma 27.2. We have $i'_*\mathcal{G}' \cong \mathcal{F}'$ because push-forward along the finite morphism i commutes with base change, see Cohomology of Schemes, Lemma 5.1. We have $\dim(\mathrm{Supp}(\mathcal{F}_s)) = \dim(\mathrm{Supp}(\mathcal{F}'_{s'}))$ by Morphisms, Lemma 29.3 because

$$\mathrm{Supp}(\mathcal{F}_s) \times_s s' = \mathrm{Supp}(\mathcal{F}'_{s'}).$$

This proves the lemma. □

05H8 **Lemma 4.5.** *Let S, X, $\mathcal{F}$, x, s be as in Definition 4.2. Let $(Z, Y, i, \pi, \mathcal{G}, z, y)$ be a one step dévissage of $\mathcal{F}/X/S$ at x. Let $(S', s') \to (S, s)$ be a morphism of pointed schemes which induces an isomorphism $\kappa(s) = \kappa(s')$. Let $(Z', Y', i', \pi', \mathcal{G}')$ be as constructed in Lemma 4.4 and let $x' \in X'$ (resp. $z' \in Z'$, $y' \in Y'$) be the unique point mapping to both $x \in X$ (resp. $z \in Z$, $y \in Y$) and $s' \in S'$. If S' is affine, then $(Z', Y', i', \pi', \mathcal{G}', z', y')$ is a one step dévissage of $\mathcal{F}'/X'/S'$ at x'.*

Proof. By Lemma 4.4 $(Z', Y', i', \pi', \mathcal{G}')$ is a one step dévissage of $\mathcal{F}'/X'/S'$ over s'. Properties (1) – (4) of Definition 4.2 hold for $(Z', Y', i', \pi', \mathcal{G}', z', y')$ as the assumption that $\kappa(s) = \kappa(s')$ insures that the fibres $X'_{s'}$, $Z'_{s'}$, and $Y'_{s'}$ are isomorphic to X_s, Z_s, and Y_s. □

05H9 **Definition 4.6.** Let S, X, $\mathcal{F}$, x, s be as in Definition 4.2. Let $(Z, Y, i, \pi, \mathcal{G}, z, y)$ be a one step dévissage of $\mathcal{F}/X/S$ at x. Let us define a *standard shrinking* of this situation to be given by standard opens $S' \subset S$, $X' \subset X$, $Z' \subset Z$, and $Y' \subset Y$ such that $s \in S'$, $x \in X'$, $z \in Z'$, and $y \in Y'$ and such that

$$(Z', Y', i|_{Z'}, \pi|_{Z'}, \mathcal{G}|_{Z'}, z, y)$$

is a one step dévissage of $\mathcal{F}|_{X'}/X'/S'$ at x.

05HA **Lemma 4.7.** *With assumption and notation as in Definition 4.6 we have:*

05HB (1) *If $S' \subset S$ is a standard open neighbourhood of s, then setting $X' = X_{S'}$, $Z' = Z_{S'}$ and $Y' = Y_{S'}$ we obtain a standard shrinking.*

05HC (2) *Let $W \subset Y$ be a standard open neighbourhood of y. Then there exists a standard shrinking with $Y' = W \times_S S'$.*

05HD (3) *Let $U \subset X$ be an open neighbourhood of x. Then there exists a standard shrinking with $X' \subset U$.*

Proof. Part (1) is immediate from Lemma 4.5 and the fact that the inverse image of a standard open under a morphism of affine schemes is a standard open, see Algebra, Lemma 17.4.

Let $W \subset Y$ as in (2). Because $Y \to S$ is smooth it is open, see Morphisms, Lemma 35.10. Hence we can find a standard open neighbourhood S' of s contained in the image of W. Then the fibres of $W_{S'} \to S'$ are nonempty open subschemes of the fibres of $Y \to S$ over S' and hence geometrically irreducible too. Setting $Y' = W_{S'}$ and $Z' = \pi^{-1}(Y')$ we see that $Z' \subset Z$ is a standard open neighbourhood of z. Let $\overline{h} \in \Gamma(Z, \mathcal{O}_Z)$ be a function such that $Z' = D(\overline{h})$. As $i : Z \to X$ is a closed immersion, we can find a function $h \in \Gamma(X, \mathcal{O}_X)$ such that $i^\sharp(h) = \overline{h}$. Take $X' = D(h) \subset X$. In this way we obtain a standard shrinking as in (2).

Let $U \subset X$ be as in (3). We may after shrinking U assume that U is a standard open. By More on Morphisms, Lemma 47.4 there exists a standard open $W \subset Y$ neighbourhood of y such that $\pi^{-1}(W) \subset i^{-1}(U)$. Apply (2) to get a standard shrinking X', S', Z', Y' with $Y' = W_{S'}$. Since $Z' \subset \pi^{-1}(W) \subset i^{-1}(U)$ we may replace X' by $X' \cap U$ (still a standard open as U is also standard open) without violating any of the conditions defining a standard shrinking. Hence we win. □

05HE **Lemma 4.8.** *Let S, X, $\mathcal{F}$, x, s be as in Definition 4.2. Let $(Z, Y, i, \pi, \mathcal{G}, z, y)$ be a one step dévissage of $\mathcal{F}/X/S$ at x. Let*

$$\begin{array}{ccc} (Y, y) & \longleftarrow & (Y', y') \\ \downarrow & & \downarrow \\ (S, s) & \longleftarrow & (S', s') \end{array}$$

be a commutative diagram of pointed schemes such that the horizontal arrows are elementary étale neighbourhoods. Then there exists a commutative diagram

$$\begin{array}{ccccc} & & (X'', x'') & \longleftarrow & (Z'', z'') \\ & \swarrow & \downarrow & \swarrow & \downarrow \\ (X, x) & \longleftarrow & (Z, z) & (S'', s'') & \longleftarrow & (Y'', y'') \\ \downarrow & & \downarrow & \swarrow & \swarrow \\ (S, s) & \longleftarrow & (Y, y) & & \end{array}$$

of pointed schemes with the following properties:

1. *$(S'', s'') \to (S', s')$ is an elementary étale neighbourhood and the morphism $S'' \to S$ is the composition $S'' \to S' \to S$,*
2. *Y'' is an open subscheme of $Y' \times_{S'} S''$,*
3. *$Z'' = Z \times_Y Y''$,*
4. *$(X'', x'') \to (X, x)$ is an elementary étale neighbourhood, and*
5. *$(Z'', Y'', i'', \pi'', \mathcal{G}'', z'', y'')$ is a one step dévissage at x'' of the sheaf $\mathcal{F}''$.*

Here $\mathcal{F}''$ (resp. $\mathcal{G}''$) is the pullback of $\mathcal{F}$ (resp. $\mathcal{G}$) via the morphism $X'' \to X$ (resp. $Z'' \to Z$) and $i'' : Z'' \to X''$ and $\pi'' : Z'' \to Y''$ are as in the diagram.

Proof. Let $(S'', s'') \to (S', s')$ be any elementary étale neighbourhood with S'' affine. Let $Y'' \subset Y' \times_{S'} S''$ be any affine open neighbourhood containing the point $y'' = (y', s'')$. Then we obtain an affine (Z'', z'') by (3). Moreover $Z_{S''} \to X_{S''}$ is a closed immersion and $Z'' \to Z_{S''}$ is an étale morphism. Hence Lemma 2.1 applies and we can find an étale morphism $X'' \to X_{S'}$ of affines such that $Z'' \cong X'' \times_{X_{S'}} Z_{S'}$. Denote $i'' : Z'' \to X''$ the corresponding closed immersion. Setting $x'' = i''(z'')$ we obtain a commutative diagram as in the lemma. Properties (1), (2), (3), and (4) hold by construction. Thus it suffices to show that (5) holds for a suitable choice of $(S'', s'') \to (S', s')$ and Y''.

We first list those properties which hold for any choice of $(S'', s'') \to (S', s')$ and Y'' as in the first paragraph. As we have $Z'' = X'' \times_X Z$ by construction we see that $i''_* \mathcal{G}'' = \mathcal{F}''$ (with notation as in the statement of the lemma), see Cohomology of Schemes, Lemma 5.1. Set $n = \dim(\mathrm{Supp}(\mathcal{F}_s)) = \dim_x(\mathrm{Supp}(\mathcal{F}_s))$. The morphism $Y'' \to S''$ is smooth of relative dimension n (because $Y' \to S'$ is smooth of relative dimension n as the composition $Y' \to Y_{S'} \to S'$ of an étale and smooth morphism of relative dimension n and because base change preserves smooth morphisms of relative dimension n). We have $\kappa(y'') = \kappa(y)$ and $\kappa(s) = \kappa(s'')$ hence $\kappa(y'')$ is a purely transcendental extension of $\kappa(s'')$. The morphism of fibres $X''_{s''} \to X_s$ is an étale morphism of affine schemes over $\kappa(s) = \kappa(s'')$ mapping the point x'' to the

point x and pulling back $\mathcal{F}_s$ to $\mathcal{F}''_{s''}$. Hence

$$\dim(\mathrm{Supp}(\mathcal{F}''_{s''})) = \dim(\mathrm{Supp}(\mathcal{F}_s)) = n = \dim_x(\mathrm{Supp}(\mathcal{F}_s)) = \dim_{x''}(\mathrm{Supp}(\mathcal{F}''_{s''}))$$

because dimension is invariant under étale localization, see Descent, Lemma 21.2. As $\pi'' : Z'' \to Y''$ is the base change of π we see that π'' is finite and as $\kappa(y) = \kappa(y'')$ we see that $\pi^{-1}(\{y''\}) = \{z''\}$.

At this point we have verified all the conditions of Definition 4.1 except we have not verified that $Y'' \to S''$ has geometrically irreducible fibres. Of course in general this is not going to be true, and it is at this point that we will use that $\kappa(s) \subset \kappa(y)$ is purely transcendental. Namely, let $T \subset Y'_{s'}$ be the irreducible component of $Y'_{s'}$ containing $y' = (y, s')$. Note that T is an open subscheme of $Y'_{s'}$ as this is a smooth scheme over $\kappa(s')$. By Varieties, Lemma 7.14 we see that T is geometrically connected because $\kappa(s') = \kappa(s)$ is algebraically closed in $\kappa(y') = \kappa(y)$. As T is smooth we see that T is geometrically irreducible. Hence More on Morphisms, Lemma 46.4 applies and we can find an elementary étale morphism $(S'', s'') \to (S', s')$ and an affine open $Y'' \subset Y'_{S''}$ such that all fibres of $Y'' \to S''$ are geometrically irreducible and such that $T = Y''_{s''}$. After shrinking (first Y'' and then S'') we may assume that both Y'' and S'' are affine. This finishes the proof of the lemma. □

05HF **Lemma 4.9.** *Let S, X, $\mathcal{F}$, s be as in Definition 4.1. Let $(Z, Y, i, \pi, \mathcal{G})$ be a one step dévissage of $\mathcal{F}/X/S$ over s. Let $\xi \in Y_s$ be the (unique) generic point. Then there exists an integer $r > 0$ and an $\mathcal{O}_Y$-module map*

$$\alpha : \mathcal{O}_Y^{\oplus r} \longrightarrow \pi_*\mathcal{G}$$

such that

$$\alpha : \kappa(\xi)^{\oplus r} \longrightarrow (\pi_*\mathcal{G})_\xi \otimes_{\mathcal{O}_{Y,\xi}} \kappa(\xi)$$

is an isomorphism. Moreover, in this case we have

$$\dim(\mathit{Supp}(\mathrm{Coker}(\alpha)_s)) < \dim(\mathit{Supp}(\mathcal{F}_s)).$$

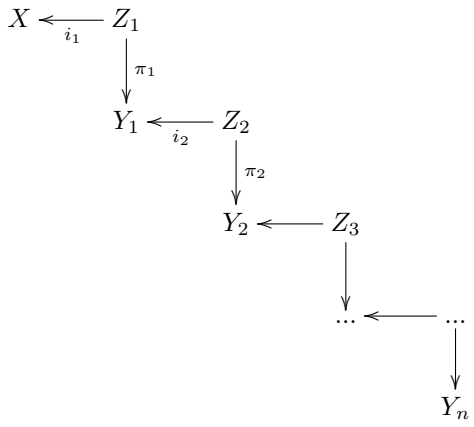
Proof. By assumption the schemes S and Y are affine. Write $S = \mathrm{Spec}(A)$ and $Y = \mathrm{Spec}(B)$. As π is finite the $\mathcal{O}_Y$-module $\pi_*\mathcal{G}$ is a finite type quasi-coherent $\mathcal{O}_Y$-module. Hence $\pi_*\mathcal{G} = \widetilde{N}$ for some finite B-module N. Let $\mathfrak{p} \subset B$ be the prime ideal corresponding to ξ. To obtain α set $r = \dim_{\kappa(\mathfrak{p})} N \otimes_B \kappa(\mathfrak{p})$ and pick $x_1, \ldots, x_r \in N$ which form a basis of $N \otimes_B \kappa(\mathfrak{p})$. Take $\alpha : B^{\oplus r} \to N$ to be the map given by the formula $\alpha(b_1, \ldots, b_r) = \sum b_i x_i$. It is clear that $\alpha : \kappa(\mathfrak{p})^{\oplus r} \to N \otimes_B \kappa(\mathfrak{p})$ is an isomorphism as desired. Finally, suppose α is any map with this property. Then $N' = \mathrm{Coker}(\alpha)$ is a finite B-module such that $N' \otimes \kappa(\mathfrak{p}) = 0$. By Nakayama's lemma (Algebra, Lemma 20.1) we see that $N'_\mathfrak{p} = 0$. Since the fibre Y_s is geometrically irreducible of dimension n with generic point ξ and since we have just seen that ξ is not in the support of $\mathrm{Coker}(\alpha)$ the last assertion of the lemma holds. □

5. Complete dévissage

05HG In this section we explain what is a complete dévissage of a module and prove that such exist. The material in this section is mainly bookkeeping.

05HH **Definition 5.1.** Let S be a scheme. Let X be locally of finite type over S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module of finite type. Let $s \in S$ be a point. A *complete*

dévissage of $\mathcal{F}/X/S$ over s is given by a diagram



of schemes over S, finite type quasi-coherent $\mathcal{O}_{Z_k}$-modules $\mathcal{G}_k$, and $\mathcal{O}_{Y_k}$-module maps

$$\alpha_k : \mathcal{O}_{Y_k}^{\oplus r_k} \longrightarrow \pi_{k,*}\mathcal{G}_k, \quad k = 1, \ldots, n$$

satisfying the following properties:

(1) $(Z_1, Y_1, i_1, \pi_1, \mathcal{G}_1)$ is a one step dévissage of $\mathcal{F}/X/S$ over s,
(2) the map α_k induces an isomorphism

$$\kappa(\xi_k)^{\oplus r_k} \longrightarrow (\pi_{k,*}\mathcal{G}_k)_{\xi_k} \otimes_{\mathcal{O}_{Y_k,\xi_k}} \kappa(\xi_k)$$

where $\xi_k \in (Y_k)_s$ is the unique generic point,
(3) for $k = 2, \ldots, n$ the system $(Z_k, Y_k, i_k, \pi_k, \mathcal{G}_k)$ is a one step dévissage of $\mathrm{Coker}(\alpha_{k-1})/Y_{k-1}/S$ over s,
(4) $\mathrm{Coker}(\alpha_n) = 0$.

In this case we say that $(Z_k, Y_k, i_k, \pi_k, \mathcal{G}_k, \alpha_k)_{k=1,\ldots,n}$ is a complete dévissage of $\mathcal{F}/X/S$ over s.

05HI **Definition 5.2.** Let S be a scheme. Let X be locally of finite type over S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module of finite type. Let $x \in X$ be a point with image $s \in S$. A *complete dévissage of $\mathcal{F}/X/S$ at x* is given by a system

$$(Z_k, Y_k, i_k, \pi_k, \mathcal{G}_k, \alpha_k, z_k, y_k)_{k=1,\ldots,n}$$

such that $(Z_k, Y_k, i_k, \pi_k, \mathcal{G}_k, \alpha_k)$ is a complete dévissage of $\mathcal{F}/X/S$ over s, and such that

(1) $(Z_1, Y_1, i_1, \pi_1, \mathcal{G}_1, z_1, y_1)$ is a one step dévissage of $\mathcal{F}/X/S$ at x,
(2) for $k = 2, \ldots, n$ the system $(Z_k, Y_k, i_k, \pi_k, \mathcal{G}_k, z_k, y_k)$ is a one step dévissage of $\mathrm{Coker}(\alpha_{k-1})/Y_{k-1}/S$ at y_{k-1}.

Again we remark that a complete dévissage can only exist if X and S are affine.

05HJ **Lemma 5.3.** *Let S, X, $\mathcal{F}$, s be as in Definition 5.1. Let $(S', s') \to (S, s)$ be any morphism of pointed schemes. Let $(Z_k, Y_k, i_k, \pi_k, \mathcal{G}_k, \alpha_k)_{k=1,\ldots,n}$ be a complete dévissage of $\mathcal{F}/X/S$ over s. Given this data let $X', Z'_k, Y'_k, i'_k, \pi'_k$ be the base changes of X, Z_k, Y_k, i_k, π_k via $S' \to S$. Let $\mathcal{F}'$ be the pullback of $\mathcal{F}$ to X' and let $\mathcal{G}'_k$ be the pullback of $\mathcal{G}_k$ to Z'_k. Let α'_k be the pullback of α_k to Y'_k. If S' is affine, then $(Z'_k, Y'_k, i'_k, \pi'_k, \mathcal{G}'_k, \alpha'_k)_{k=1,\ldots,n}$ is a complete dévissage of $\mathcal{F}'/X'/S'$ over s'.*

Proof. By Lemma 4.4 we know that the base change of a one step dévissage is a one step dévissage. Hence it suffices to prove that formation of $\mathrm{Coker}(\alpha_k)$ commutes with base change and that condition (2) of Definition 5.1 is preserved by base change. The first is true as $\pi'_{k,*}\mathcal{G}'_k$ is the pullback of $\pi_{k,*}\mathcal{G}_k$ (by Cohomology of Schemes, Lemma 5.1) and because $\otimes$ is right exact. The second because by the same token we have

$$(\pi_{k,*}\mathcal{G}_k)_{\xi_k} \otimes_{\mathcal{O}_{Y_k,\xi_k}} \kappa(\xi_k) \otimes_{\kappa(\xi_k)} \kappa(\xi'_k) \cong (\pi'_{k,*}\mathcal{G}'_k)_{\xi'_k} \otimes_{\mathcal{O}_{Y'_k,\xi'_k}} \kappa(\xi'_k)$$

with obvious notation. □

05HK **Lemma 5.4.** *Let S, X, $\mathcal{F}$, x, s be as in Definition 5.2. Let $(S', s') \to (S, s)$ be a morphism of pointed schemes which induces an isomorphism $\kappa(s) = \kappa(s')$. Let $(Z_k, Y_k, i_k, \pi_k, \mathcal{G}_k, \alpha_k, z_k, y_k)_{k=1,\ldots,n}$ be a complete dévissage of $\mathcal{F}/X/S$ at x. Let $(Z'_k, Y'_k, i'_k, \pi'_k, \mathcal{G}'_k, \alpha'_k)_{k=1,\ldots,n}$ be as constructed in Lemma 5.3 and let $x' \in X'$ (resp. $z'_k \in Z'$, $y'_k \in Y'$) be the unique point mapping to both $x \in X$ (resp. $z_k \in Z_k$, $y_k \in Y_k$) and $s' \in S'$. If S' is affine, then $(Z'_k, Y'_k, i'_k, \pi'_k, \mathcal{G}'_k, \alpha'_k, z'_k, y'_k)_{k=1,\ldots,n}$ is a complete dévissage of $\mathcal{F}'/X'/S'$ at x'.*

Proof. Combine Lemma 5.3 and Lemma 4.5. □

05HL **Definition 5.5.** Let S, X, $\mathcal{F}$, x, s be as in Definition 5.2. Consider a complete dévissage $(Z_k, Y_k, i_k, \pi_k, \mathcal{G}_k, \alpha_k, z_k, y_k)_{k=1,\ldots,n}$ of $\mathcal{F}/X/S$ at x. Let us define a *standard shrinking* of this situation to be given by standard opens $S' \subset S$, $X' \subset X$, $Z'_k \subset Z_k$, and $Y'_k \subset Y_k$ such that $s_k \in S'$, $x_k \in X'$, $z_k \in Z'$, and $y_k \in Y'$ and such that

$$(Z'_k, Y'_k, i'_k, \pi'_k, \mathcal{G}'_k, \alpha'_k, z_k, y_k)_{k=1,\ldots,n}$$

is a one step dévissage of $\mathcal{F}'/X'/S'$ at x where $\mathcal{G}'_k = \mathcal{G}_k|_{Z'_k}$ and $\mathcal{F}' = \mathcal{F}|_{X'}$.

05HM **Lemma 5.6.** *With assumption and notation as in Definition 5.5 we have:*

05HN (1) *If $S' \subset S$ is a standard open neighbourhood of s, then setting $X' = X_{S'}$, $Z'_k = Z_{S'}$ and $Y'_k = Y_{S'}$ we obtain a standard shrinking.*

05HP (2) *Let $W \subset Y_n$ be a standard open neighbourhood of y. Then there exists a standard shrinking with $Y'_n = W \times_S S'$.*

05HQ (3) *Let $U \subset X$ be an open neighbourhood of x. Then there exists a standard shrinking with $X' \subset U$.*

Proof. Part (1) is immediate from Lemmas 5.4 and 4.7.

Proof of (2). For convenience denote $X = Y_0$. We apply Lemma 4.7 (2) to find a standard shrinking S', Y'_{n-1}, Z'_n, Y'_n of the one step dévissage of $\mathrm{Coker}(\alpha_{n-1})/Y_{n-1}/S$ at y_{n-1} with $Y'_n = W \times_S S'$. We may repeat this procedure and find a standard shrinking $S'', Y''_{n-2}, Z''_{n-1}, Y''_{n-1}$ of the one step dévissage of $\mathrm{Coker}(\alpha_{n-2})/Y_{n-2}/S$ at y_{n-2} with $Y''_{n-1} = Y'_{n-1} \times_S S''$. We may continue in this manner until we obtain $S^{(n)}, Y_0^{(n)}, Z_1^{(n)}, Y_1^{(n)}$. At this point it is clear that we obtain our desired standard shrinking by taking $S^{(n)}$, $X^{(n)}$, $Z_k^{(n-k)} \times_S S^{(n)}$, and $Y_k^{(n-k)} \times_S S^{(n)}$ with the desired property.

Proof of (3). We use induction on the length of the complete dévissage. First we apply Lemma 4.7 (3) to find a standard shrinking S', X', Z'_1, Y'_1 of the one step dévissage of $\mathcal{F}/X/S$ at x with $X' \subset U$. If $n = 1$, then we are done. If $n > 1$, then by induction we can find a standard shrinking S'', Y''_1, Z''_k, and Y''_k of

the complete dévissage $(Z_k, Y_k, i_k, \pi_k, \mathcal{G}_k, \alpha_k, z_k, y_k)_{k=2,\ldots,n}$ of $\mathrm{Coker}(\alpha_1)/Y_1/S$ at x such that $Y_1'' \subset Y_1'$. Using Lemma 4.7 (2) we can find $S''' \subset S'$, $X''' \subset X'$, Z_1''' and $Y_1''' = Y_1'' \times_S S'''$ which is a standard shrinking. The solution to our problem is to take

$$S''', X''', Z_1''', Y_1''', Z_2'' \times_S S''', Y_2'' \times_S S''', \ldots, Z_n'' \times_S S''', Y_n'' \times_S S'''$$

This ends the proof of the lemma. □

05HR **Proposition 5.7.** *Let S be a scheme. Let X be locally of finite type over S. Let $x \in X$ be a point with image $s \in S$. There exists a commutative diagram*

$$\begin{array}{ccc} (X, x) & \xleftarrow[g]{} & (X', x') \\ \downarrow & & \downarrow \\ (S, s) & \longleftarrow & (S', s') \end{array}$$

of pointed schemes such that the horizontal arrows are elementary étale neighbourhoods and such that $g^*\mathcal{F}/X'/S'$ has a complete dévissage at x.

Proof. We prove this by induction on the integer $d = \dim_x(\mathrm{Supp}(\mathcal{F}_s))$. By Lemma 4.3 there exists a diagram

$$\begin{array}{ccc} (X, x) & \xleftarrow[g]{} & (X', x') \\ \downarrow & & \downarrow \\ (S, s) & \longleftarrow & (S', s') \end{array}$$

of pointed schemes such that the horizontal arrows are elementary étale neighbourhoods and such that $g^*\mathcal{F}/X'/S'$ has a one step dévissage at x'. The local nature of the problem implies that we may replace $(X, x) \to (S, s)$ by $(X', x') \to (S', s')$. Thus after doing so we may assume that there exists a one step dévissage $(Z_1, Y_1, i_1, \pi_1, \mathcal{G}_1)$ of $\mathcal{F}/X/S$ at x.

We apply Lemma 4.9 to find a map

$$\alpha_1 : \mathcal{O}_{Y_1}^{\oplus r_1} \longrightarrow \pi_{1,*}\mathcal{G}_1$$

which induces an isomorphism of vector spaces over $\kappa(\xi_1)$ where $\xi_1 \in Y_1$ is the unique generic point of the fibre of Y_1 over s. Moreover $\dim_{y_1}(\mathrm{Supp}(\mathrm{Coker}(\alpha_1)_s)) < d$. It may happen that the stalk of $\mathrm{Coker}(\alpha_1)_s$ at y_1 is zero. In this case we may shrink Y_1 by Lemma 4.7 (2) and assume that $\mathrm{Coker}(\alpha_1) = 0$ so we obtain a complete dévissage of length zero.

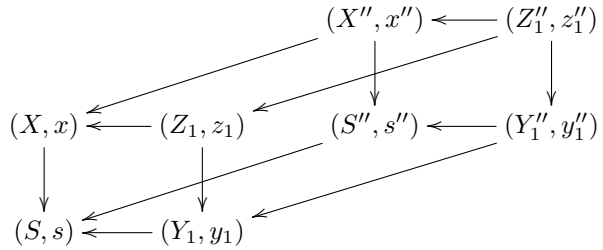
Assume now that the stalk of $\mathrm{Coker}(\alpha_1)_s$ at y_1 is not zero. In this case, by induction, there exists a commutative diagram

05HS (5.7.1)
$$\begin{array}{ccc} (Y_1, y_1) & \xleftarrow[h]{} & (Y_1', y_1') \\ \downarrow & & \downarrow \\ (S, s) & \longleftarrow & (S', s') \end{array}$$

of pointed schemes such that the horizontal arrows are elementary étale neighbourhoods and such that $h^* \mathrm{Coker}(\alpha_1)/Y_1'/S'$ has a complete dévissage

$$(Z_k, Y_k, i_k, \pi_k, \mathcal{G}_k, \alpha_k, z_k, y_k)_{k=2,\ldots,n}$$

at y_1'. (In particular $i_2 : Z_2 \to Y_1'$ is a closed immersion into Y_2'.) At this point we apply Lemma 4.8 to $S, X, \mathcal{F}, x, s$, the system $(Z_1, Y_1, i_1, \pi_1, \mathcal{G}_1)$ and diagram (5.7.1). We obtain a diagram



with all the properties as listed in the referenced lemma. In particular $Y_1'' \subset Y_1' \times_{S'} S''$. Set $X_1 = Y_1' \times_{S'} S''$ and let $\mathcal{F}_1$ denote the pullback of $\mathrm{Coker}(\alpha_1)$. By Lemma 5.4 the system

05HT (5.7.2) $$(Z_k \times_{S'} S'', Y_k \times_{S'} S'', i_k'', \pi_k'', \mathcal{G}_k'', \alpha_k'', z_k'', y_k'')_{k=2,\ldots,n}$$

is a complete dévissage of $\mathcal{F}_1$ to X_1. Again, the nature of the problem allows us to replace $(X, x) \to (S, s)$ by $(X'', x'') \to (S'', s'')$. In this we see that we may assume:

(a) There exists a one step dévissage $(Z_1, Y_1, i_1, \pi_1, \mathcal{G}_1)$ of $\mathcal{F}/X/S$ at x,
(b) there exists an $\alpha_1 : \mathcal{O}_{Y_1}^{\oplus r_1} \to \pi_{1,*}\mathcal{G}_1$ such that $\alpha \otimes \kappa(\xi_1)$ is an isomorphism,
(c) $Y_1 \subset X_1$ is open, $y_1 = x_1$, and $\mathcal{F}_1|_{Y_1} \cong \mathrm{Coker}(\alpha_1)$, and
(d) there exists a complete dévissage $(Z_k, Y_k, i_k, \pi_k, \mathcal{G}_k, \alpha_k, z_k, y_k)_{k=2,\ldots,n}$ of $\mathcal{F}_1/X_1/S$ at x_1.

To finish the proof all we have to do is shrink the one step dévissage and the complete dévissage such that they fit together to a complete dévissage. (We suggest the reader do this on their own using Lemmas 4.7 and 5.6 instead of reading the proof that follows.) Since $Y_1 \subset X_1$ is an open neighbourhood of x_1 we may apply Lemma 5.6 (3) to find a standard shrinking $S', X_1', Z_2', Y_2', \ldots, Y_n'$ of the datum (d) so that $X_1' \subset Y_1$. Note that X_1' is also a standard open of the affine scheme Y_1. Next, we shrink the datum (a) as follows: first we shrink the base S to S', see Lemma 4.7 (1) and then we shrink the result to S'', X'', Z_1'', Y_1'' using Lemma 4.7 (2) such that eventually $Y_1'' = X_1' \times_S S''$ and $S'' \subset S'$. Then we see that

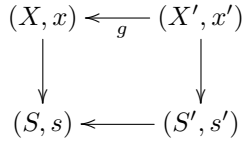
$$Z_1'', Y_1'', Z_2' \times_{S'} S'', Y_2' \times_{S'} S'', \ldots, Y_n' \times_{S'} S''$$

gives the complete dévissage we were looking for. □

Some more bookkeeping gives the following consequence.

05HU **Lemma 5.8.** *Let $X \to S$ be a finite type morphism of schemes. Let $\mathcal{F}$ be a finite type quasi-coherent $\mathcal{O}_X$-module. Let $s \in S$ be a point. There exists an elementary étale neighbourhood $(S', s') \to (S, s)$ and étale morphisms $h_i : Y_i \to X_{S'}$, $i = 1, \ldots, n$ such that for each i there exists a complete dévissage of $\mathcal{F}_i/Y_i/S'$ over s', where $\mathcal{F}_i$ is the pullback of $\mathcal{F}$ to Y_i and such that $X_s = (X_{S'})_{s'} \subset \bigcup h_i(Y_i)$.*

Proof. For every point $x \in X_s$ we can find a diagram



of pointed schemes such that the horizontal arrows are elementary étale neighbourhoods and such that $g^*\mathcal{F}/X'/S'$ has a complete dévissage at x'. As $X \to S$ is of finite type the fibre X_s is quasi-compact, and since each $g : X' \to X$ as above is open we can cover X_s by a finite union of $g(X'_{s'})$. Thus we can find a finite family of such diagrams

$$\begin{array}{ccc} (X, x) & \xleftarrow[g_i]{} & (X'_i, x'_i) \\ \downarrow & & \downarrow \\ (S, s) & \longleftarrow & (S'_i, s'_i) \end{array} \qquad i = 1, \ldots, n$$

such that $X_s = \bigcup g_i(X'_i)$. Set $S' = S'_1 \times_S \ldots \times_S S'_n$ and let $Y_i = X_i \times_{S'_i} S'$ be the base change of X'_i to S'. By Lemma 5.3 we see that the pullback of $\mathcal{F}$ to Y_i has a complete dévissage over s and we win. □

6. Translation into algebra

05HV It may be useful to spell out algebraically what it means to have a complete dévissage. We introduce the following notion (which is not that useful so we give it an impossibly long name).

05HW **Definition 6.1.** Let $R \to S$ be a ring map. Let $\mathfrak{q}$ be a prime of S lying over the prime $\mathfrak{p}$ of R. A *elementary étale localization of the ring map $R \to S$ at $\mathfrak{q}$* is given by a commutative diagram of rings and accompanying primes

$$\begin{array}{ccc} S & \longrightarrow & S' \\ \uparrow & & \uparrow \\ R & \longrightarrow & R' \end{array} \qquad \begin{array}{ccc} \mathfrak{q} & \longrightarrow & \mathfrak{q}' \\ \downarrow & & \downarrow \\ \mathfrak{p} & \longrightarrow & \mathfrak{p}' \end{array}$$

such that $R \to R'$ and $S \to S'$ are étale ring maps and $\kappa(\mathfrak{p}) = \kappa(\mathfrak{p}')$ and $\kappa(\mathfrak{q}) = \kappa(\mathfrak{q}')$.

05HX **Definition 6.2.** Let $R \to S$ be a finite type ring map. Let $\mathfrak{r}$ be a prime of R. Let N be a finite S-module. A *complete dévissage of $N/S/R$ over $\mathfrak{r}$* is given by R-algebra maps

$$\begin{array}{ccccccccccc} & A_1 & & A_2 & & \ldots & & A_n & \\ \nearrow & & \nwarrow & & \nearrow & & \nwarrow & & \nearrow & & \nwarrow \\ S & & B_1 & & \ldots & & \ldots & & B_n \end{array}$$

finite A_i-modules M_i and B_i-module maps $\alpha_i : B_i^{\oplus r_i} \to M_i$ such that

(1) $S \to A_1$ is surjective and of finite presentation,
(2) $B_i \to A_{i+1}$ is surjective and of finite presentation,
(3) $B_i \to A_i$ is finite,
(4) $R \to B_i$ is smooth with geometrically irreducible fibres,
(5) $N \cong M_1$ as S-modules,
(6) $\mathrm{Coker}(\alpha_i) \cong M_{i+1}$ as B_i-modules,
(7) $\alpha_i : \kappa(\mathfrak{p}_i)^{\oplus r_i} \to M_i \otimes_{B_i} \kappa(\mathfrak{p}_i)$ is an isomorphism where $\mathfrak{p}_i = \mathfrak{r}B_i$, and
(8) $\mathrm{Coker}(\alpha_n) = 0$.

In this situation we say that $(A_i, B_i, M_i, \alpha_i)_{i=1,\ldots,n}$ is a complete dévissage of $N/S/R$ over $\mathfrak{r}$.

05HY **Remark 6.3.** Note that the R-algebras B_i for all i and A_i for $i \geq 2$ are of finite presentation over R. If S is of finite presentation over R, then it is also the case that A_1 is of finite presentation over R. In this case all the ring maps in the complete dévissage are of finite presentation. See Algebra, Lemma 6.2. Still assuming S of finite presentation over R the following are equivalent

(1) M is of finite presentation over S,
(2) M_1 is of finite presentation over A_1,
(3) M_1 is of finite presentation over B_1,
(4) each M_i is of finite presentation both as an A_i-module and as a B_i-module.

The equivalences (1) $\Leftrightarrow$ (2) and (2) $\Leftrightarrow$ (3) follow from Algebra, Lemma 36.23. If M_1 is finitely presented, so is $\mathrm{Coker}(\alpha_1)$ (see Algebra, Lemma 5.3) and hence M_2, etc.

05HZ **Definition 6.4.** Let $R \to S$ be a finite type ring map. Let $\mathfrak{q}$ be a prime of S lying over the prime $\mathfrak{r}$ of R. Let N be a finite S-module. A *complete dévissage of $N/S/R$ at $\mathfrak{q}$* is given by a complete dévissage $(A_i, B_i, M_i, \alpha_i)_{i=1,\ldots,n}$ of $N/S/R$ over $\mathfrak{r}$ and prime ideals $\mathfrak{q}_i \subset B_i$ lying over $\mathfrak{r}$ such that

(1) $\kappa(\mathfrak{r}) \subset \kappa(\mathfrak{q}_i)$ is purely transcendental,
(2) there is a unique prime $\mathfrak{q}'_i \subset A_i$ lying over $\mathfrak{q}_i \subset B_i$,
(3) $\mathfrak{q} = \mathfrak{q}'_1 \cap S$ and $\mathfrak{q}_i = \mathfrak{q}'_{i+1} \cap A_i$,
(4) $R \to B_i$ has relative dimension $\dim_{\mathfrak{q}_i}(\mathrm{Supp}(M_i \otimes_R \kappa(\mathfrak{r})))$.

05I0 **Remark 6.5.** Let $A \to B$ be a finite type ring map and let N be a finite B-module. Let $\mathfrak{q}$ be a prime of B lying over the prime $\mathfrak{r}$ of A. Set $X = \mathrm{Spec}(B)$, $S = \mathrm{Spec}(A)$ and $\mathcal{F} = \widetilde{N}$ on X. Let x be the point corresponding to $\mathfrak{q}$ and let $s \in S$ be the point corresponding to $\mathfrak{p}$. Then

(1) if there exists a complete dévissage of $\mathcal{F}/X/S$ over s then there exists a complete dévissage of $N/B/A$ over $\mathfrak{p}$, and
(2) there exists a complete dévissage of $\mathcal{F}/X/S$ at x if and only if there exists a complete dévissage of $N/B/A$ at $\mathfrak{q}$.

There is just a small twist in that we omitted the condition on the relative dimension in the formulation of "a complete dévissage of $N/B/A$ over $\mathfrak{p}$" which is why the implication in (1) only goes in one direction. The notion of a complete dévissage at $\mathfrak{q}$ does have this condition built in. In any case we will only use that existence for $\mathcal{F}/X/S$ implies the existence for $N/B/A$.

05I1 **Lemma 6.6.** *Let $R \to S$ be a finite type ring map. Let M be a finite S-module. Let $\mathfrak{q}$ be a prime ideal of S. There exists an elementary étale localization $R' \to S', \mathfrak{q}', \mathfrak{p}'$ of the ring map $R \to S$ at $\mathfrak{q}$ such that there exists a complete dévissage of $(M \otimes_S S')/S'/R'$ at $\mathfrak{q}'$.*

Proof. This is a reformulation of Proposition 5.7 via Remark 6.5 □

7. Localization and universally injective maps

05DD

05DE **Lemma 7.1.** *Let $R \to S$ be a ring map. Let N be a S-module. Assume*

(1) *R is a local ring with maximal ideal $\mathfrak{m}$,*
(2) *$\overline{S} = S/\mathfrak{m}S$ is Noetherian, and*
(3) *$\overline{N} = N/\mathfrak{m}_R N$ is a finite $\overline{S}$-module.*

Let $\Sigma \subset S$ be the multiplicative subset of elements which are not a zerodivisor on $\overline{N}$. Then $\Sigma^{-1}S$ is a semi-local ring whose spectrum consists of primes $\mathfrak{q} \subset S$ contained in an element of $Ass_S(\overline{N})$. Moreover, any maximal ideal of $\Sigma^{-1}S$ corresponds to an associated prime of $\overline{N}$ over $\overline{S}$.

Proof. Note that $\mathrm{Ass}_S(\overline{N}) = \mathrm{Ass}_{\overline{S}}(\overline{N})$, see Algebra, Lemma 63.14. This is a finite set by Algebra, Lemma 63.5. Say $\{\mathfrak{q}_1, \ldots, \mathfrak{q}_r\} = \mathrm{Ass}_S(\overline{N})$. We have $\Sigma = S \setminus (\bigcup \mathfrak{q}_i)$ by Algebra, Lemma 63.9. By the description of $\mathrm{Spec}(\Sigma^{-1}S)$ in Algebra, Lemma 17.5 and by Algebra, Lemma 15.2 we see that the primes of $\Sigma^{-1}S$ correspond to the primes of S contained in one of the $\mathfrak{q}_i$. Hence the maximal ideals of $\Sigma^{-1}S$ correspond one-to-one with the maximal (w.r.t. inclusion) elements of the set $\{\mathfrak{q}_1, \ldots, \mathfrak{q}_r\}$. This proves the lemma. □

05DF **Lemma 7.2.** *Assumption and notation as in Lemma 7.1. Assume moreover that*

(1) *S is local and $R \to S$ is a local homomorphism,*
(2) *S is essentially of finite presentation over R,*
(3) *N is finitely presented over S, and*
(4) *N is flat over R.*

Then each $s \in \Sigma$ defines a universally injective R-module map $s : N \to N$, and the map $N \to \Sigma^{-1}N$ is R-universally injective.

Proof. By Algebra, Lemma 128.4 the sequence $0 \to N \to N \to N/sN \to 0$ is exact and N/sN is flat over R. This implies that $s : N \to N$ is universally injective, see Algebra, Lemma 39.12. The map $N \to \Sigma^{-1}N$ is universally injective as the directed colimit of the maps $s : N \to N$. □

05DG **Lemma 7.3.** *Let $R \to S$ be a ring map. Let N be an S-module. Let $S \to S'$ be a ring map. Assume*

(1) *$R \to S$ is a local homomorphism of local rings*
(2) *S is essentially of finite presentation over R,*
(3) *N is of finite presentation over S,*
(4) *N is flat over R,*
(5) *$S \to S'$ is flat, and*
(6) *the image of $\mathrm{Spec}(S') \to \mathrm{Spec}(S)$ contains all primes $\mathfrak{q}$ of S lying over $\mathfrak{m}_R$ such that $\mathfrak{q}$ is an associated prime of $N/\mathfrak{m}_R N$.*

Then $N \to N \otimes_S S'$ is R-universally injective.

Proof. Set $N' = N \otimes_R S'$. Consider the commutative diagram

$$\begin{array}{ccc} N & \longrightarrow & N' \\ \downarrow & & \downarrow \\ \Sigma^{-1}N & \longrightarrow & \Sigma^{-1}N' \end{array}$$

where $\Sigma \subset S$ is the set of elements which are not a zerodivisor on $N/\mathfrak{m}_R N$. If we can show that the map $N \to \Sigma^{-1}N'$ is universally injective, then $N \to N'$ is too (see Algebra, Lemma 82.10).

By Lemma 7.1 the ring $\Sigma^{-1}S$ is a semi-local ring whose maximal ideals correspond to associated primes of $N/\mathfrak{m}_R N$. Hence the image of $\mathrm{Spec}(\Sigma^{-1}S') \to \mathrm{Spec}(\Sigma^{-1}S)$ contains all these maximal ideals by assumption. By Algebra, Lemma 39.16 the

ring map $\Sigma^{-1}S \to \Sigma^{-1}S'$ is faithfully flat. Hence $\Sigma^{-1}N \to \Sigma^{-1}N'$, which is the map

$$N \otimes_S \Sigma^{-1}S \longrightarrow N \otimes_S \Sigma^{-1}S'$$

is universally injective, see Algebra, Lemmas 82.15 and 82.8. Finally, we apply Lemma 7.2 to see that $N \to \Sigma^{-1}N$ is universally injective. As the composition of universally injective module maps is universally injective (see Algebra, Lemma 82.9) we conclude that $N \to \Sigma^{-1}N'$ is universally injective and we win. □

05DH **Lemma 7.4.** *Let $R \to S$ be a ring map. Let N be an S-module. Let $S \to S'$ be a ring map. Assume*

(1) *$R \to S$ is of finite presentation and N is of finite presentation over S,*
(2) *N is flat over R,*
(3) *$S \to S'$ is flat, and*
(4) *the image of $\mathrm{Spec}(S') \to \mathrm{Spec}(S)$ contains all primes $\mathfrak{q}$ such that $\mathfrak{q}$ is an associated prime of $N \otimes_R \kappa(\mathfrak{p})$ where $\mathfrak{p}$ is the inverse image of $\mathfrak{q}$ in R.*

Then $N \to N \otimes_S S'$ is R-universally injective.

Proof. By Algebra, Lemma 82.11 it suffices to show that $N_\mathfrak{q} \to (N \otimes_R S')_\mathfrak{q}$ is a $R_\mathfrak{p}$-universally injective for any prime $\mathfrak{q}$ of S lying over $\mathfrak{p}$ in R. Thus we may apply Lemma 7.3 to the ring maps $R_\mathfrak{p} \to S_\mathfrak{q} \to S'_\mathfrak{q}$ and the module $N_\mathfrak{q}$. □

The reader may want to compare the following lemma to Algebra, Lemmas 99.1 and 128.4 and the results of Section 25. In each case the conclusion is that the map $u : M \to N$ is universally injective with flat cokernel.

05FQ **Lemma 7.5.** *Let $(R, \mathfrak{m})$ be a local ring. Let $u : M \to N$ be an R-module map. If M is a projective R-module, N is a flat R-module, and $\overline{u} : M/\mathfrak{m}M \to N/\mathfrak{m}N$ is injective then u is universally injective.*

Proof. By Algebra, Theorem 85.4 the module M is free. If we show the result holds for every finitely generated direct summand of M, then the lemma follows. Hence we may assume that M is finite free. Write $N = \mathrm{colim}_i N_i$ as a directed colimit of finite free modules, see Algebra, Theorem 81.4. Note that $u : M \to N$ factors through N_i for some i (as M is finite free). Denote $u_i : M \to N_i$ the corresponding R-module map. As $\overline{u}$ is injective we see that $\overline{u_i} : M/\mathfrak{m}M \to N_i/\mathfrak{m}N_i$ is injective and remains injective on composing with the maps $N_i/\mathfrak{m}N_i \to N_{i'}/\mathfrak{m}N_{i'}$ for all $i' \geq i$. As M and $N_{i'}$ are finite free over the local ring R this implies that $M \to N_{i'}$ is a split injection for all $i' \geq i$. Hence for any R-module Q we see that $M \otimes_R Q \to N_{i'} \otimes_R Q$ is injective for all $i' \geq i$. As $- \otimes_R Q$ commutes with colimits we conclude that $M \otimes_R Q \to N_{i'} \otimes_R Q$ is injective as desired. □

05FR **Lemma 7.6.** *Assumption and notation as in Lemma 7.1. Assume moreover that N is projective as an R-module. Then each $s \in \Sigma$ defines a universally injective R-module map $s : N \to N$, and the map $N \to \Sigma^{-1}N$ is R-universally injective.*

Proof. Pick $s \in \Sigma$. By Lemma 7.5 the map $s : N \to N$ is universally injective. The map $N \to \Sigma^{-1}N$ is universally injective as the directed colimit of the maps $s : N \to N$. □

8. Completion and Mittag-Leffler modules

05DI

05DJ **Lemma 8.1.** *Let R be a ring. Let $I \subset R$ be an ideal. Let A be a set. Assume R is Noetherian and complete with respect to I. The completion $(\bigoplus_{\alpha\in A} R)^\wedge$ is flat and Mittag-Leffler.*

Proof. By More on Algebra, Lemma 28.1 the map $(\bigoplus_{\alpha\in A} R)^\wedge \to \prod_{\alpha\in A} R$ is universally injective. Thus, by Algebra, Lemmas 82.7 and 89.7 it suffices to show that $\prod_{\alpha\in A} R$ is flat and Mittag-Leffler. By Algebra, Proposition 90.6 (and Algebra, Lemma 90.5) we see that $\prod_{\alpha\in A} R$ is flat. Thus we conclude because a product of copies of R is Mittag-Leffler, see Algebra, Lemma 91.3. □

05DK **Lemma 8.2.** *Let R be a ring. Let $I \subset R$ be an ideal. Let M be an R-module. Assume*

1. *R is Noetherian and I-adically complete,*
2. *M is flat over R, and*
3. *M/IM is a projective R/I-module.*

Then the I-adic completion $M^\wedge$ is a flat Mittag-Leffler R-module.

Proof. Choose a surjection $F \to M$ where F is a free R-module. By Algebra, Lemma 97.9 the module $M^\wedge$ is a direct summand of the module $F^\wedge$. Hence it suffices to prove the lemma for F. In this case the lemma follows from Lemma 8.1. □

In Lemmas 8.3 and 8.4 the assumption that S be Noetherian holds if $R \to S$ is of finite type, see Algebra, Lemma 31.1.

05DL **Lemma 8.3.** *Let R be a ring. Let $I \subset R$ be an ideal. Let $R \to S$ be a ring map, and N an S-module. Assume*

1. *R is a Noetherian ring,*
2. *S is a Noetherian ring,*
3. *N is a finite S-module, and*
4. *for any finite R-module Q, any $\mathfrak{q} \in Ass_S(Q \otimes_R N)$ satisfies $IS + \mathfrak{q} \neq S$.*

Then the map $N \to N^\wedge$ of N into the I-adic completion of N is universally injective as a map of R-modules.

Proof. We have to show that for any finite R-module Q the map $Q \otimes_R N \to Q \otimes_R N^\wedge$ is injective, see Algebra, Theorem 82.3. As there is a canonical map $Q \otimes_R N^\wedge \to (Q \otimes_R N)^\wedge$ it suffices to prove that the canonical map $Q \otimes_R N \to (Q \otimes_R N)^\wedge$ is injective. Hence we may replace N by $Q \otimes_R N$ and it suffices to prove the injectivity for the map $N \to N^\wedge$.

Let $K = \mathrm{Ker}(N \to N^\wedge)$. It suffices to show that $K_\mathfrak{q} = 0$ for $\mathfrak{q} \in \mathrm{Ass}(N)$ as N is a submodule of $\prod_{\mathfrak{q}\in \mathrm{Ass}(N)} N_\mathfrak{q}$, see Algebra, Lemma 63.19. Pick $\mathfrak{q} \in \mathrm{Ass}(N)$. By the last assumption we see that there exists a prime $\mathfrak{q}' \supset IS + \mathfrak{q}$. Since $K_\mathfrak{q}$ is a localization of $K_{\mathfrak{q}'}$ it suffices to prove the vanishing of $K_{\mathfrak{q}'}$. Note that $K = \bigcap I^n N$, hence $K_{\mathfrak{q}'} \subset \bigcap I^n N_{\mathfrak{q}'}$. Hence $K_{\mathfrak{q}'} = 0$ by Algebra, Lemma 51.4. □

05DM **Lemma 8.4.** *Let R be a ring. Let $I \subset R$ be an ideal. Let $R \to S$ be a ring map, and N an S-module. Assume*

1. *R is a Noetherian ring,*

(2) S *is a Noetherian ring,*
(3) N *is a finite* S*-module,*
(4) N *is flat over* R*, and*
(5) *for any prime* $\mathfrak{q} \subset S$ *which is an associated prime of* $N \otimes_R \kappa(\mathfrak{p})$ *where* $\mathfrak{p} = R \cap \mathfrak{q}$ *we have* $IS + \mathfrak{q} \neq S$.

Then the map $N \to N^\wedge$ *of* N *into the* I*-adic completion of* N *is universally injective as a map of* R*-modules.*

Proof. This follows from Lemma 8.3 because Algebra, Lemma 65.5 and Remark 65.6 guarantee that the set of associated primes of tensor products $N \otimes_R Q$ are contained in the set of associated primes of the modules $N \otimes_R \kappa(\mathfrak{p})$. □

9. Projective modules

05DN The following lemma can be used to prove projectivity by Noetherian induction on the base, see Lemma 9.2.

05DP **Lemma 9.1.** *Let* R *be a ring. Let* $I \subset R$ *be an ideal. Let* $R \to S$ *be a ring map, and* N *an* S*-module. Assume*

(1) R *is Noetherian and* I*-adically complete,*
(2) $R \to S$ *is of finite type,*
(3) N *is a finite* S*-module,*
(4) N *is flat over* R*,*
(5) N/IN *is projective as a* R/I*-module, and*
(6) *for any prime* $\mathfrak{q} \subset S$ *which is an associated prime of* $N \otimes_R \kappa(\mathfrak{p})$ *where* $\mathfrak{p} = R \cap \mathfrak{q}$ *we have* $IS + \mathfrak{q} \neq S$.

Then N *is projective as an* R*-module.*

Proof. By Lemma 8.4 the map $N \to N^\wedge$ is universally injective. By Lemma 8.2 the module $N^\wedge$ is Mittag-Leffler. By Algebra, Lemma 89.7 we conclude that N is Mittag-Leffler. Hence N is countably generated, flat and Mittag-Leffler as an R-module, whence projective by Algebra, Lemma 93.1. □

05FS **Lemma 9.2.** *Let* R *be a ring. Let* $R \to S$ *be a ring map. Assume*

(1) R *is Noetherian,*
(2) $R \to S$ *is of finite type and flat, and*
(3) *every fibre ring* $S \otimes_R \kappa(\mathfrak{p})$ *is geometrically integral over* $\kappa(\mathfrak{p})$.

Then S *is projective as an* R*-module.*

Proof. Consider the set

$$\{I \subset R \mid S/IS \text{ not projective as } R/I\text{-module}\}$$

We have to show this set is empty. To get a contradiction assume it is nonempty. Then it contains a maximal element I. Let $J = \sqrt{I}$ be its radical. If $I \neq J$, then S/JS is projective as a R/J-module, and S/IS is flat over R/I and J/I is a nilpotent ideal in R/I. Applying Algebra, Lemma 77.7 we see that S/IS is a projective R/I-module, which is a contradiction. Hence we may assume that I is a radical ideal. In other words we are reduced to proving the lemma in case R is a reduced ring and S/IS is a projective R/I-module for every nonzero ideal I of R.

Assume R is a reduced ring and S/IS is a projective R/I-module for every nonzero ideal I of R. By generic flatness, Algebra, Lemma 118.1 (applied to a localization

R_g which is a domain) or the more general Algebra, Lemma 118.7 there exists a nonzero $f \in R$ such that S_f is free as an R_f-module. Denote $R^\wedge = \lim R/(f^n)$ the (f)-adic completion of R. Note that the ring map

$$R \longrightarrow R_f \times R^\wedge$$

is a faithfully flat ring map, see Algebra, Lemma 97.2. Hence by faithfully flat descent of projectivity, see Algebra, Theorem 95.6 it suffices to prove that $S \otimes_R R^\wedge$ is a projective $R^\wedge$-module. To see this we will use the criterion of Lemma 9.1. First of all, note that $S/fS = (S \otimes_R R^\wedge)/f(S \otimes_R R^\wedge)$ is a projective $R/(f)$-module and that $S \otimes_R R^\wedge$ is flat and of finite type over $R^\wedge$ as a base change of such. Next, suppose that $\mathfrak{p}^\wedge$ is a prime ideal of $R^\wedge$. Let $\mathfrak{p} \subset R$ be the corresponding prime of R. As $R \to S$ has geometrically integral fibre rings, the same is true for the fibre rings of any base change. Hence $\mathfrak{q}^\wedge = \mathfrak{p}^\wedge(S \otimes_R R^\wedge)$, is a prime ideals lying over $\mathfrak{p}^\wedge$ and it is the unique associated prime of $S \otimes_R \kappa(\mathfrak{p}^\wedge)$. Thus we win if $f(S \otimes_R R^\wedge) + \mathfrak{q}^\wedge \neq S \otimes_R R^\wedge$. This is true because $\mathfrak{p}^\wedge + fR^\wedge \neq R^\wedge$ as f lies in the Jacobson radical of the f-adically complete ring $R^\wedge$ and because $R^\wedge \to S \otimes_R R^\wedge$ is surjective on spectra as its fibres are nonempty (irreducible spaces are nonempty). □

05FT **Lemma 9.3.** *Let R be a ring. Let $R \to S$ be a ring map. Assume*

(1) *$R \to S$ is of finite presentation and flat, and*
(2) *every fibre ring $S \otimes_R \kappa(\mathfrak{p})$ is geometrically integral over $\kappa(\mathfrak{p})$.*

Then S is projective as an R-module.

Proof. We can find a cocartesian diagram of rings

$$\begin{array}{ccc} S_0 & \longrightarrow & S \\ \uparrow & & \uparrow \\ R_0 & \longrightarrow & R \end{array}$$

such that R_0 is of finite type over $\mathbf{Z}$, the map $R_0 \to S_0$ is of finite type and flat with geometrically integral fibres, see More on Morphisms, Lemmas 34.4, 34.6, 34.7, and 34.11. By Lemma 9.2 we see that S_0 is a projective R_0-module. Hence $S = S_0 \otimes_{R_0} R$ is a projective R-module, see Algebra, Lemma 94.1. □

05FU **Remark 9.4.** Lemma 9.3 is a key step in the development of results in this chapter. The analogue of this lemma in [GR71] is [GR71, I Proposition 3.3.1]: If $R \to S$ is smooth with geometrically integral fibres, then S is projective as an R-module. This is a special case of Lemma 9.3, but as we will later improve on this lemma anyway, we do not gain much from having a stronger result at this point. We briefly sketch the proof of this as it is given in [GR71].

(1) First reduce to the case where R is Noetherian as above.
(2) Since projectivity descends through faithfully flat ring maps, see Algebra, Theorem 95.6 we may work locally in the fppf topology on R, hence we may assume that $R \to S$ has a section $\sigma : S \to R$. (Just by the usual trick of base changing to S.) Set $I = \mathrm{Ker}(S \to R)$.
(3) Localizing a bit more on R we may assume that I/I^2 is a free R-module and that the completion $S^\wedge$ of S with respect to I is isomorphic to $R[[t_1, \ldots, t_n]]$, see Morphisms, Lemma 35.20. Here we are using that $R \to S$ is smooth.

(4) To prove that S is projective as an R-module, it suffices to prove that S is flat, countably generated and Mittag-Leffler as an R-module, see Algebra, Lemma 93.1. The first two properties are evident. Thus it suffices to prove that S is Mittag-Leffler as an R-module. By Algebra, Lemma 91.4 the module $R[[t_1, \ldots, t_n]]$ is Mittag-Leffler over R. Hence Algebra, Lemma 89.7 shows that it suffices to show that the $S \to S^\wedge$ is universally injective as a map of R-modules.
(5) Apply Lemma 7.4 to see that $S \to S^\wedge$ is R-universally injective. Namely, as $R \to S$ has geometrically integral fibres, any associated point of any fibre ring is just the generic point of the fibre ring which is in the image of $\mathrm{Spec}(S^\wedge) \to \mathrm{Spec}(S)$.

There is an analogy between the proof as sketched just now, and the development of the arguments leading to the proof of Lemma 9.3. In both a completion plays an essential role, and both times the assumption of having geometrically integral fibres assures one that the map from S to the completion of S is R-universally injective.

10. Flat finite type modules, Part I

05I2 In some cases given a ring map $R \to S$ of finite presentation and a finite S-module N the flatness of N over R implies that N is of finite presentation. In this section we prove this is true "pointwise". We remark that the first proof of Proposition 10.3 uses the geometric results of Section 3 but not the existence of a complete dévissage.

05I3 **Lemma 10.1.** *Let $(R, \mathfrak{m})$ be a local ring. Let $R \to S$ be a finitely presented flat ring map with geometrically integral fibres. Write $\mathfrak{p} = \mathfrak{m}S$. Let $\mathfrak{q} \subset S$ be a prime ideal lying over $\mathfrak{m}$. Let N be a finite S-module. There exist $r \geq 0$ and an S-module map*

$$\alpha : S^{\oplus r} \longrightarrow N$$

such that $\alpha : \kappa(\mathfrak{p})^{\oplus r} \to N \otimes_S \kappa(\mathfrak{p})$ is an isomorphism. For any such α the following are equivalent:

(1) *$N_\mathfrak{q}$ is R-flat,*
(2) *α is R-universally injective and $\mathrm{Coker}(\alpha)_\mathfrak{q}$ is R-flat,*
(3) *α is injective and $\mathrm{Coker}(\alpha)_\mathfrak{q}$ is R-flat,*
(4) *$\alpha_\mathfrak{p}$ is an isomorphism and $\mathrm{Coker}(\alpha)_\mathfrak{q}$ is R-flat, and*
(5) *$\alpha_\mathfrak{q}$ is injective and $\mathrm{Coker}(\alpha)_\mathfrak{q}$ is R-flat.*

Proof. To obtain α set $r = \dim_{\kappa(\mathfrak{p})} N \otimes_S \kappa(\mathfrak{p})$ and pick $x_1, \ldots, x_r \in N$ which form a basis of $N \otimes_S \kappa(\mathfrak{p})$. Define $\alpha(s_1, \ldots, s_r) = \sum s_i x_i$. This proves the existence.

Fix an α. The most interesting implication is (1) $\Rightarrow$ (2) which we prove first. Assume (1). Because $S/\mathfrak{m}S$ is a domain with fraction field $\kappa(\mathfrak{p})$ we see that $(S/\mathfrak{m}S)^{\oplus r} \to N_\mathfrak{p}/\mathfrak{m}N_\mathfrak{p} = N \otimes_S \kappa(\mathfrak{p})$ is injective. Hence by Lemmas 7.5 and 9.3. the map $S^{\oplus r} \to N_\mathfrak{p}$ is R-universally injective. It follows that $S^{\oplus r} \to N$ is R-universally injective, see Algebra, Lemma 82.10. Then also the localization $\alpha_\mathfrak{q}$ is R-universally injective, see Algebra, Lemma 82.12. We conclude that $\mathrm{Coker}(\alpha)_\mathfrak{q}$ is R-flat by Algebra, Lemma 82.7.

The implication (2) $\Rightarrow$ (3) is immediate. If (3) holds, then $\alpha_\mathfrak{p}$ is injective as a localization of an injective module map. By Nakayama's lemma (Algebra, Lemma 20.1) $\alpha_\mathfrak{p}$ is surjective too. Hence (3) $\Rightarrow$ (4). If (4) holds, then $\alpha_\mathfrak{p}$ is an isomorphism,

so α is injective as $S_{\mathfrak{q}} \to S_{\mathfrak{p}}$ is injective. Namely, elements of $S \setminus \mathfrak{p}$ are nonzerodivisors on S by a combination of Lemmas 7.6 and 9.3. Hence (4) $\Rightarrow$ (5). Finally, if (5) holds, then $N_{\mathfrak{q}}$ is R-flat as an extension of flat modules, see Algebra, Lemma 39.13. Hence (5) $\Rightarrow$ (1) and the proof is finished. □

05I4 **Lemma 10.2.** *Let $(R, \mathfrak{m})$ be a local ring. Let $R \to S$ be a ring map of finite presentation. Let N be a finite S-module. Let $\mathfrak{q}$ be a prime of S lying over $\mathfrak{m}$. Assume that $N_{\mathfrak{q}}$ is flat over R, and assume there exists a complete dévissage of $N/S/R$ at $\mathfrak{q}$. Then N is a finitely presented S-module, free as an R-module, and there exists an isomorphism*

$$N \cong B_1^{\oplus r_1} \oplus \ldots \oplus B_n^{\oplus r_n}$$

as R-modules where each B_i is a smooth R-algebra with geometrically irreducible fibres.

Proof. Let $(A_i, B_i, M_i, \alpha_i, \mathfrak{q}_i)_{i=1,\ldots,n}$ be the given complete dévissage. We prove the lemma by induction on n. Note that N is finitely presented as an S-module if and only if M_1 is finitely presented as an B_1-module, see Remark 6.3. Note that $N_{\mathfrak{q}} \cong (M_1)_{\mathfrak{q}_1}$ as R-modules because (a) $N_{\mathfrak{q}} \cong (M_1)_{\mathfrak{q}_1'}$ where $\mathfrak{q}_1'$ is the unique prime in A_1 lying over $\mathfrak{q}_1$ and (b) $(A_1)_{\mathfrak{q}_1'} = (A_1)_{\mathfrak{q}_1}$ by Algebra, Lemma 41.11, so (c) $(M_1)_{\mathfrak{q}_1'} \cong (M_1)_{\mathfrak{q}_1}$. Hence $(M_1)_{\mathfrak{q}_1}$ is a flat R-module. Thus we may replace (S, N) by (B_1, M_1) in order to prove the lemma. By Lemma 10.1 the map $\alpha_1 : B_1^{\oplus r_1} \to M_1$ is R-universally injective and $\mathrm{Coker}(\alpha_1)_{\mathfrak{q}}$ is R-flat. Note that $(A_i, B_i, M_i, \alpha_i, \mathfrak{q}_i)_{i=2,\ldots,n}$ is a complete dévissage of $\mathrm{Coker}(\alpha_1)/B_1/R$ at $\mathfrak{q}_1$. Hence the induction hypothesis implies that $\mathrm{Coker}(\alpha_1)$ is finitely presented as a B_1-module, free as an R-module, and has a decomposition as in the lemma. This implies that M_1 is finitely presented as a B_1-module, see Algebra, Lemma 5.3. It further implies that $M_1 \cong B_1^{\oplus r_1} \oplus \mathrm{Coker}(\alpha_1)$ as R-modules, hence a decomposition as in the lemma. Finally, B_1 is projective as an R-module by Lemma 9.3 hence free as an R-module by Algebra, Theorem 85.4. This finishes the proof. □

05I5 **Proposition 10.3.** *Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Let $x \in X$ with image $s \in S$. Assume that*

1. *f is locally of finite presentation,*
2. *$\mathcal{F}$ is of finite type, and*
3. *$\mathcal{F}$ is flat at x over S.*

Then there exists an elementary étale neighbourhood $(S', s') \to (S, s)$ and an open subscheme

$$V \subset X \times_S \mathrm{Spec}(\mathcal{O}_{S',s'})$$

which contains the unique point of $X \times_S \mathrm{Spec}(\mathcal{O}_{S',s'})$ mapping to x such that the pullback of $\mathcal{F}$ to V is an $\mathcal{O}_V$-module of finite presentation and flat over $\mathcal{O}_{S',s'}$.

First proof. This proof is longer but does not use the existence of a complete dévissage. The problem is local around x and s, hence we may assume that X and S are affine. During the proof we will finitely many times replace S by an elementary étale neighbourhood of (S, s). The goal is then to find (after such a replacement) an open $V \subset X \times_S \mathrm{Spec}(\mathcal{O}_{S,s})$ containing x such that $\mathcal{F}|_V$ is flat over S and finitely presented. Of course we may also replace S by $\mathrm{Spec}(\mathcal{O}_{S,s})$ at any point of the proof, i.e., we may assume S is a local scheme. We will prove the proposition by induction on the integer $n = \dim_x(\mathrm{Supp}(\mathcal{F}_s))$.

We can choose

(1) elementary étale neighbourhoods $g : (X', x') \to (X, x)$, $e : (S', s') \to (S, s)$,
(2) a commutative diagram

$$\begin{array}{ccccc} X & \xleftarrow{g} & X' & \xleftarrow{i} & Z' \\ \downarrow{\scriptstyle f} & & \downarrow & & \downarrow{\scriptstyle \pi} \\ & & & & Y' \\ & & & & \downarrow{\scriptstyle h} \\ S & \xleftarrow{e} & S' & = & S' \end{array}$$

(3) a point $z' \in Z'$ with $i(z') = x'$, $y' = \pi(z')$, $h(y') = s'$,
(4) a finite type quasi-coherent $\mathcal{O}_{Z'}$-module $\mathcal{G}$,

as in Lemma 3.2. We are going to replace S by $\mathrm{Spec}(\mathcal{O}_{S',s'})$, see remarks in first paragraph of the proof. Consider the diagram

$$\begin{array}{ccccc} X_{\mathcal{O}_{S',s'}} & \xleftarrow{g} & X'_{\mathcal{O}_{S',s'}} & \xleftarrow{i} & Z'_{\mathcal{O}_{S',s'}} \\ & \searrow^{f} & \downarrow & & \downarrow{\scriptstyle \pi} \\ & & & & Y'_{\mathcal{O}_{S',s'}} \\ & & & \swarrow_{h} & \\ & & \mathrm{Spec}(\mathcal{O}_{S',s'}) & & \end{array}$$

Here we have base changed the schemes X', Z', Y' over S' via $\mathrm{Spec}(\mathcal{O}_{S',s'}) \to S'$ and the scheme X over S via $\mathrm{Spec}(\mathcal{O}_{S',s'}) \to S$. It is still the case that g is étale, see Lemma 2.2. After replacing X by $X_{\mathcal{O}_{S',s'}}$, X' by $X'_{\mathcal{O}_{S',s'}}$, Z' by $Z'_{\mathcal{O}_{S',s'}}$, and Y' by $Y'_{\mathcal{O}_{S',s'}}$ we may assume we have a diagram as Lemma 3.2 where in addition $S = S'$ is a local scheme with closed point s. By Lemmas 3.3 and 3.4 the result for $Y' \to S$, the sheaf $\pi_*\mathcal{G}$, and the point y' implies the result for $X \to S$, $\mathcal{F}$ and x. Hence we may assume that S is local and $X \to S$ is a smooth morphism of affines with geometrically irreducible fibres of dimension n.

The base case of the induction: $n = 0$. As $X \to S$ is smooth with geometrically irreducible fibres of dimension 0 we see that $X \to S$ is an open immersion, see Descent, Lemma 25.2. As S is local and the closed point is in the image of $X \to S$ we conclude that $X = S$. Thus we see that $\mathcal{F}$ corresponds to a finite flat $\mathcal{O}_{S,s}$ module. In this case the result follows from Algebra, Lemma 78.5 which tells us that $\mathcal{F}$ is in fact finite free.

The induction step. Assume the result holds whenever the dimension of the support in the closed fibre is $< n$. Write $S = \mathrm{Spec}(A)$, $X = \mathrm{Spec}(B)$ and $\mathcal{F} = \widetilde{N}$ for some B-module N. Note that A is a local ring; denote its maximal ideal $\mathfrak{m}$. Then $\mathfrak{p} = \mathfrak{m}B$ is the unique minimal prime lying over $\mathfrak{m}$ as $X \to S$ has geometrically irreducible fibres. Finally, let $\mathfrak{q} \subset B$ be the prime corresponding to x. By Lemma 10.1 we can choose a map

$$\alpha : B^{\oplus r} \to N$$

such that $\kappa(\mathfrak{p})^{\oplus r} \to N \otimes_B \kappa(\mathfrak{p})$ is an isomorphism. Moreover, as $N_\mathfrak{q}$ is A-flat the lemma also shows that α is injective and that $\mathrm{Coker}(\alpha)_\mathfrak{q}$ is A-flat. Set $Q =$

$\mathrm{Coker}(\alpha)$. Note that the support of $Q/\mathfrak{m}Q$ does not contain $\mathfrak{p}$. Hence it is certainly the case that $\dim_{\mathfrak{q}}(\mathrm{Supp}(Q/\mathfrak{m}Q)) < n$. Combining everything we know about Q we see that the induction hypothesis applies to Q. It follows that there exists an elementary étale morphism $(S', s) \to (S, s)$ such that the conclusion holds for $Q \otimes_A A'$ over $B \otimes_A A'$ where $A' = \mathcal{O}_{S',s'}$. After replacing A by A' we have an exact sequence

$$0 \to B^{\oplus r} \to N \to Q \to 0$$

(here we use that α is injective as mentioned above) of finite B-modules and we also get an element $g \in B$, $g \notin \mathfrak{q}$ such that Q_g is finitely presented over B_g and flat over A. Since localization is exact we see that

$$0 \to B_g^{\oplus r} \to N_g \to Q_g \to 0$$

is still exact. As B_g and Q_g are flat over A we conclude that N_g is flat over A, see Algebra, Lemma 39.13, and as B_g and Q_g are finitely presented over B_g the same holds for N_g, see Algebra, Lemma 5.3. □

Second proof. We apply Proposition 5.7 to find a commutative diagram

$$\begin{array}{ccc} (X, x) & \xleftarrow{g} & (X', x') \\ \downarrow & & \downarrow \\ (S, s) & \longleftarrow & (S', s') \end{array}$$

of pointed schemes such that the horizontal arrows are elementary étale neighbourhoods and such that $g^*\mathcal{F}/X'/S'$ has a complete dévissage at x. (In particular S' and X' are affine.) By Morphisms, Lemma 26.13 we see that $g^*\mathcal{F}$ is flat at x' over S and by Lemma 2.3 we see that it is flat at x' over S'. Via Remark 6.5 we deduce that

$$\Gamma(X', g^*\mathcal{F})/\Gamma(X', \mathcal{O}_{X'})/\Gamma(S', \mathcal{O}_{S'})$$

has a complete dévissage at the prime of $\Gamma(X', \mathcal{O}_{X'})$ corresponding to x'. We may base change this complete dévissage to the local ring $\mathcal{O}_{S',s'}$ of $\Gamma(S', \mathcal{O}_{S'})$ at the prime corresponding to s'. Thus Lemma 10.2 implies that

$$\Gamma(X', \mathcal{F}') \otimes_{\Gamma(S', \mathcal{O}_{S'})} \mathcal{O}_{S',s'}$$

is flat over $\mathcal{O}_{S',s'}$ and of finite presentation over $\Gamma(X', \mathcal{O}_{X'}) \otimes_{\Gamma(S', \mathcal{O}_{S'})} \mathcal{O}_{S',s'}$. In other words, the restriction of $\mathcal{F}$ to $X' \times_{S'} \mathrm{Spec}(\mathcal{O}_{S',s'})$ is of finite presentation and flat over $\mathcal{O}_{S',s'}$. Since the morphism $X' \times_{S'} \mathrm{Spec}(\mathcal{O}_{S',s'}) \to X \times_S \mathrm{Spec}(\mathcal{O}_{S',s'})$ is étale (Lemma 2.2) its image $V \subset X \times_S \mathrm{Spec}(\mathcal{O}_{S',s'})$ is an open subscheme, and by étale descent the restriction of $\mathcal{F}$ to V is of finite presentation and flat over $\mathcal{O}_{S',s'}$. (Results used: Morphisms, Lemma 37.13, Descent, Lemma 7.3, and Morphisms, Lemma 26.13.) □

05M9 **Lemma 10.4.** *Let $f : X \to S$ be a morphism of schemes which is locally of finite type. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module of finite type. Let $s \in S$. Then the set*

$$\{x \in X_s \mid \mathcal{F} \text{ flat over } S \text{ at } x\}$$

is open in the fibre X_s.

Proof. Suppose $x \in U$. Choose an elementary étale neighbourhood $(S', s') \to (S, s)$ and open $V \subset X \times_S \mathrm{Spec}(\mathcal{O}_{S',s'})$ as in Proposition 10.3. Note that $X_{s'} = X_s$ as $\kappa(s) = \kappa(s')$. If $x' \in V \cap X_{s'}$, then the pullback of $\mathcal{F}$ to $X \times_S S'$ is flat over S' at x'. Hence $\mathcal{F}$ is flat at x' over S, see Morphisms, Lemma 26.13. In other words $X_s \cap V \subset U$ is an open neighbourhood of x in U. □

05KT **Lemma 10.5.** *Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Let $x \in X$ with image $s \in S$. Assume that*

(1) *f is locally of finite type,*
(2) *$\mathcal{F}$ is of finite type, and*
(3) *$\mathcal{F}$ is flat at x over S.*

Then there exists an elementary étale neighbourhood $(S', s') \to (S, s)$ and an open subscheme

$$V \subset X \times_S \mathrm{Spec}(\mathcal{O}_{S',s'})$$

which contains the unique point of $X \times_S \mathrm{Spec}(\mathcal{O}_{S',s'})$ mapping to x such that the pullback of $\mathcal{F}$ to V is flat over $\mathcal{O}_{S',s'}$.

Proof. (The only difference between this and Proposition 10.3 is that we do not assume f is of finite presentation.) The question is local on X and S, hence we may assume X and S are affine. Write $X = \mathrm{Spec}(B)$, $S = \mathrm{Spec}(A)$ and write $B = A[x_1, \ldots, x_n]/I$. In other words we obtain a closed immersion $i : X \to \mathbf{A}^n_S$. Denote $t = i(x) \in \mathbf{A}^n_S$. We may apply Proposition 10.3 to $\mathbf{A}^n_S \to S$, the sheaf $i_*\mathcal{F}$ and the point t. We obtain an elementary étale neighbourhood $(S', s') \to (S, s)$ and an open subscheme

$$W \subset \mathbf{A}^n_{\mathcal{O}_{S',s'}}$$

such that the pullback of $i_*\mathcal{F}$ to W is flat over $\mathcal{O}_{S',s'}$. This means that $V := W \cap \big(X \times_S \mathrm{Spec}(\mathcal{O}_{S',s'})\big)$ is the desired open subscheme. □

05KU **Lemma 10.6.** *Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Let $s \in S$. Assume that*

(1) *f is of finite presentation,*
(2) *$\mathcal{F}$ is of finite type, and*
(3) *$\mathcal{F}$ is flat over S at every point of the fibre X_s.*

Then there exists an elementary étale neighbourhood $(S', s') \to (S, s)$ and an open subscheme

$$V \subset X \times_S \mathrm{Spec}(\mathcal{O}_{S',s'})$$

which contains the fibre $X_s = X \times_S s'$ such that the pullback of $\mathcal{F}$ to V is an $\mathcal{O}_V$-module of finite presentation and flat over $\mathcal{O}_{S',s'}$.

Proof. For every point $x \in X_s$ we can use Proposition 10.3 to find an elementary étale neighbourhood $(S_x, s_x) \to (S, s)$ and an open $V_x \subset X \times_S \mathrm{Spec}(\mathcal{O}_{S_x,s_x})$ such that $x \in X_s = X \times_S s_x$ is contained in V_x and such that the pullback of $\mathcal{F}$ to V_x is an $\mathcal{O}_{V_x}$-module of finite presentation and flat over $\mathcal{O}_{S_x,s_x}$. In particular we may view the fibre $(V_x)_{s_x}$ as an open neighbourhood of x in X_s. Because X_s is quasi-compact we can find a finite number of points $x_1, \ldots, x_n \in X_s$ such that X_s is the union of the $(V_{x_i})_{s_{x_i}}$. Choose an elementary étale neighbourhood $(S', s') \to (S, s)$ which dominates each of the neighbourhoods (S_{x_i}, s_{x_i}), see More on Morphisms,

Lemma 35.4. Set $V = \bigcup V_i$ where V_i is the inverse images of the open V_{x_i} via the morphism

$$X \times_S \operatorname{Spec}(\mathcal{O}_{S',s'}) \longrightarrow X \times_S \operatorname{Spec}(\mathcal{O}_{S_{x_i},s_{x_i}})$$

By construction V contains X_s and by construction the pullback of $\mathcal{F}$ to V is an $\mathcal{O}_V$-module of finite presentation and flat over $\mathcal{O}_{S',s'}$. □

05KV **Lemma 10.7.** *Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Let $s \in S$. Assume that*

(1) *f is of finite type,*
(2) *$\mathcal{F}$ is of finite type, and*
(3) *$\mathcal{F}$ is flat over S at every point of the fibre X_s.*

Then there exists an elementary étale neighbourhood $(S', s') \to (S, s)$ and an open subscheme

$$V \subset X \times_S \operatorname{Spec}(\mathcal{O}_{S',s'})$$

which contains the fibre $X_s = X \times_S s'$ such that the pullback of $\mathcal{F}$ to V is flat over $\mathcal{O}_{S',s'}$.

Proof. (The only difference between this and Lemma 10.6 is that we do not assume f is of finite presentation.) For every point $x \in X_s$ we can use Lemma 10.5 to find an elementary étale neighbourhood $(S_x, s_x) \to (S, s)$ and an open $V_x \subset X \times_S \operatorname{Spec}(\mathcal{O}_{S_x,s_x})$ such that $x \in X_s = X \times_S s_x$ is contained in V_x and such that the pullback of $\mathcal{F}$ to V_x is flat over $\mathcal{O}_{S_x,s_x}$. In particular we may view the fibre $(V_x)_{s_x}$ as an open neighbourhood of x in X_s. Because X_s is quasi-compact we can find a finite number of points $x_1, \ldots, x_n \in X_s$ such that X_s is the union of the $(V_{x_i})_{s_{x_i}}$. Choose an elementary étale neighbourhood $(S', s') \to (S, s)$ which dominates each of the neighbourhoods (S_{x_i}, s_{x_i}), see More on Morphisms, Lemma 35.4. Set $V = \bigcup V_i$ where V_i is the inverse images of the open V_{x_i} via the morphism

$$X \times_S \operatorname{Spec}(\mathcal{O}_{S',s'}) \longrightarrow X \times_S \operatorname{Spec}(\mathcal{O}_{S_{x_i},s_{x_i}})$$

By construction V contains X_s and by construction the pullback of $\mathcal{F}$ to V is flat over $\mathcal{O}_{S',s'}$. □

05I6 **Lemma 10.8.** *Let S be a scheme. Let X be locally of finite type over S. Let $x \in X$ with image $s \in S$. If X is flat at x over S, then there exists an elementary étale neighbourhood $(S', s') \to (S, s)$ and an open subscheme*

$$V \subset X \times_S \operatorname{Spec}(\mathcal{O}_{S',s'})$$

which contains the unique point of $X \times_S \operatorname{Spec}(\mathcal{O}_{S',s'})$ mapping to x such that $V \to \operatorname{Spec}(\mathcal{O}_{S',s'})$ is flat and of finite presentation.

Proof. The question is local on X and S, hence we may assume X and S are affine. Write $X = \operatorname{Spec}(B)$, $S = \operatorname{Spec}(A)$ and write $B = A[x_1, \ldots, x_n]/I$. In other words we obtain a closed immersion $i : X \to \mathbf{A}^n_S$. Denote $t = i(x) \in \mathbf{A}^n_S$. We may apply Proposition 10.3 to $\mathbf{A}^n_S \to S$, the sheaf $\mathcal{F} = i_*\mathcal{O}_X$ and the point t. We obtain an elementary étale neighbourhood $(S', s') \to (S, s)$ and an open subscheme

$$W \subset \mathbf{A}^n_{\mathcal{O}_{S',s'}}$$

such that the pullback of $i_*\mathcal{O}_X$ is flat and of finite presentation. This means that $V := W \cap \big(X \times_S \operatorname{Spec}(\mathcal{O}_{S',s'})\big)$ is the desired open subscheme. □

05I7 **Lemma 10.9.** *Let $f : X \to S$ be a morphism which is locally of finite presentation. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module of finite type. If $x \in X$ and $\mathcal{F}$ is flat at x over S, then $\mathcal{F}_x$ is an $\mathcal{O}_{X,x}$-module of finite presentation.*

Proof. Let $s = f(x)$. By Proposition 10.3 there exists an elementary étale neighbourhood $(S', s') \to (S, s)$ such that the pullback of $\mathcal{F}$ to $X \times_S \mathrm{Spec}(\mathcal{O}_{S',s'})$ is of finite presentation in a neighbourhood of the point $x' \in X_{s'} = X_s$ corresponding to x. The ring map

$$\mathcal{O}_{X,x} \longrightarrow \mathcal{O}_{X \times_S \mathrm{Spec}(\mathcal{O}_{S',s'}),x'} = \mathcal{O}_{X \times_S S',x'}$$

is flat and local as a localization of an étale ring map. Hence $\mathcal{F}_x$ is of finite presentation over $\mathcal{O}_{X,x}$ by descent, see Algebra, Lemma 83.2 (and also that a flat local ring map is faithfully flat, see Algebra, Lemma 39.17). □

05I8 **Lemma 10.10.** *Let $f : X \to S$ be a morphism which is locally of finite type. Let $x \in X$ with image $s \in S$. If f is flat at x over S, then $\mathcal{O}_{X,x}$ is essentially of finite presentation over $\mathcal{O}_{S,s}$.*

Proof. We may assume X and S affine. Write $X = \mathrm{Spec}(B)$, $S = \mathrm{Spec}(A)$ and write $B = A[x_1, \ldots, x_n]/I$. In other words we obtain a closed immersion $i : X \to \mathbf{A}^n_S$. Denote $t = i(x) \in \mathbf{A}^n_S$. We may apply Lemma 10.9 to $\mathbf{A}^n_S \to S$, the sheaf $\mathcal{F} = i_*\mathcal{O}_X$ and the point t. We conclude that $\mathcal{O}_{X,x}$ is of finite presentation over $\mathcal{O}_{\mathbf{A}^n_S,t}$ which implies what we want. □

11. Extending properties from an open

0B47 In this section we collect a number of results of the form: If $f : X \to S$ is a flat morphism of schemes and f satisfies some property over a dense open of S, then f satisfies the same property over all of S.

081N **Lemma 11.1.** *Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $U \subset S$ be open. Assume*

1. *f is locally of finite presentation,*
2. *$\mathcal{F}$ is of finite type and flat over S,*
3. *$U \subset S$ is retrocompact and scheme theoretically dense,*
4. *$\mathcal{F}|_{f^{-1}U}$ is of finite presentation.*

Then $\mathcal{F}$ is of finite presentation.

Proof. The problem is local on X and S, hence we may assume X and S affine. Write $S = \mathrm{Spec}(A)$ and $X = \mathrm{Spec}(B)$. Let N be a finite B-module such that $\mathcal{F}$ is the quasi-coherent sheaf associated to N. We have $U = D(f_1) \cup \ldots \cup D(f_n)$ for some $f_i \in A$, see Algebra, Lemma 29.1. As U is schematically dense the map $A \to A_{f_1} \times \ldots \times A_{f_n}$ is injective. Pick a prime $\mathfrak{q} \subset B$ lying over $\mathfrak{p} \subset A$ corresponding to $x \in X$ mapping to $s \in S$. By Lemma 10.9 the module $N_\mathfrak{q}$ is of finite presentation over $B_\mathfrak{q}$. Choose a surjection $\varphi : B^{\oplus m} \to N$ of B-modules. Choose $k_1, \ldots, k_t \in \mathrm{Ker}(\varphi)$ and set $N' = B^{\oplus m}/\sum Bk_j$. There is a canonical surjection $N' \to N$ and N is the filtered colimit of the B-modules N' constructed in this manner. Thus we see that we can choose $k_1, \ldots, k_t$ such that (a) $N'_{f_i} \cong N_{f_i}$, $i = 1, \ldots, n$ and (b) $N'_\mathfrak{q} \cong N_\mathfrak{q}$. This in particular implies that $N'_\mathfrak{q}$ is flat over A.

By openness of flatness, see Algebra, Theorem 129.4 we conclude that there exists a $g \in B$, $g \notin \mathfrak{q}$ such that N'_g is flat over A. Consider the commutative diagram

$$\begin{array}{ccc} N'_g & \longrightarrow & N_g \\ \downarrow & & \downarrow \\ \prod N'_{gf_i} & \longrightarrow & \prod N_{gf_i} \end{array}$$

The bottom arrow is an isomorphism by choice of $k_1, \ldots, k_t$. The left vertical arrow is an injective map as $A \to \prod A_{f_i}$ is injective and N'_g is flat over A. Hence the top horizontal arrow is injective, hence an isomorphism. This proves that N_g is of finite presentation over B_g. We conclude by applying Algebra, Lemma 23.2. □

081P **Lemma 11.2.** *Let $f : X \to S$ be a morphism of schemes. Let $U \subset S$ be open. Assume*

(1) *f is locally of finite type and flat,*
(2) *$U \subset S$ is retrocompact and scheme theoretically dense,*
(3) *$f|_{f^{-1}U} : f^{-1}U \to U$ is locally of finite presentation.*

Then f is of locally of finite presentation.

Proof. The question is local on X and S, hence we may assume X and S affine. Choose a closed immersion $i : X \to \mathbf{A}^n_S$ and apply Lemma 11.1 to $i_*\mathcal{O}_X$. Some details omitted. □

081L **Lemma 11.3.** *Let $f : X \to S$ be a morphism of schemes which is flat and locally of finite type. Let $U \subset S$ be a dense open such that $X_U \to U$ has relative dimension $\leq e$, see Morphisms, Definition 30.1. If also either*

(1) *f is locally of finite presentation, or*
(2) *$U \subset S$ is retrocompact,*

then f has relative dimension $\leq e$.

Proof. Proof in case (1). Let $W \subset X$ be the open subscheme constructed and studied in More on Morphisms, Lemmas 22.7 and 22.9. Note that every generic point of every fibre is contained in W, hence it suffices to prove the result for W. Since $W = \bigcup_{d \geq 0} U_d$, it suffices to prove that $U_d = \emptyset$ for $d > e$. Since f is flat and locally of finite presentation it is open hence $f(U_d)$ is open (Morphisms, Lemma 26.10). Thus if U_d is not empty, then $f(U_d) \cap U \neq \emptyset$ as desired.

Proof in case (2). We may replace S by its reduction. Then U is scheme theoretically dense. Hence f is locally of finite presentation by Lemma 11.2. In this way we reduce to case (1). □

0B48 **Lemma 11.4.** *Let $f : X \to S$ be a morphism of schemes which is flat and proper. Let $U \subset S$ be a dense open such that $X_U \to U$ is finite. If also either f is locally of finite presentation or $U \subset S$ is retrocompact, then f is finite.*

Proof. By Lemma 11.3 the fibres of f have dimension zero. Hence f is quasi-finite (Morphisms, Lemma 30.5) whence has finite fibres (Morphisms, Lemma 21.10). Hence f is finite by More on Morphisms, Lemma 44.1. □

081M **Lemma 11.5.** *Let $f : X \to S$ be a morphism of schemes and $U \subset S$ an open. If*

(1) *f is separated, locally of finite type, and flat,*

(2) $f^{-1}(U) \to U$ *is an isomorphism, and*
(3) $U \subset S$ *is retrocompact and scheme theoretically dense,*

then f *is an open immersion.*

Proof. By Lemma 11.2 the morphism f is locally of finite presentation. The image $f(X) \subset S$ is open (Morphisms, Lemma 26.10) hence we may replace S by $f(X)$. Thus we have to prove that f is an isomorphism. We may assume S is affine. We can reduce to the case that X is quasi-compact because it suffices to show that any quasi-compact open $X' \subset X$ whose image is S maps isomorphically to S. Thus we may assume f is quasi-compact. All the fibers of f have dimension 0, see Lemma 11.3. Hence f is quasi-finite, see Morphisms, Lemma 30.5. Let $s \in S$. Choose an elementary étale neighbourhood $g : (T, t) \to (S, s)$ such that $X \times_S T = V \amalg W$ with $V \to T$ finite and $W_t = \emptyset$, see More on Morphisms, Lemma 41.6. Denote $\pi : V \amalg W \to T$ the given morphism. Since π is flat and locally of finite presentation, we see that $\pi(V)$ is open in T (Morphisms, Lemma 26.10). After shrinking T we may assume that $T = \pi(V)$. Since f is an isomorphism over U we see that π is an isomorphism over $g^{-1}U$. Since $\pi(V) = T$ this implies that $\pi^{-1}g^{-1}U$ is contained in V. By Morphisms, Lemma 26.15 we see that $\pi^{-1}g^{-1}U \subset V \amalg W$ is scheme theoretically dense. Hence we deduce that $W = \emptyset$. Thus $X \times_S T = V$ is finite over T. This implies that f is finite (after replacing S by an open neighbourhood of s), for example by Descent, Lemma 23.25. Then f is finite locally free (Morphisms, Lemma 49.2) and after shrinking S to a smaller open neighbourhood of s we see that f is finite locally free of some degree d (Morphisms, Lemma 49.5). But $d = 1$ as is clear from the fact that the degree is 1 over the dense open U. Hence f is an isomorphism. □

12. Flat finitely presented modules

05I9 In some cases given a ring map $R \to S$ of finite presentation and a finitely presented S-module N the flatness of N over R implies that N is projective as an R-module, at least after replacing S by an étale extension. In this section we collect a some results of this nature.

05IA **Lemma 12.1.** *Let R be a ring. Let $R \to S$ be a finitely presented flat ring map with geometrically integral fibres. Let $\mathfrak{q} \subset S$ be a prime ideal lying over the prime $\mathfrak{r} \subset R$. Set $\mathfrak{p} = \mathfrak{r}S$. Let N be a finitely presented S-module. There exists $r \geq 0$ and an S-module map*

$$\alpha : S^{\oplus r} \longrightarrow N$$

such that $\alpha : \kappa(\mathfrak{p})^{\oplus r} \to N \otimes_S \kappa(\mathfrak{p})$ is an isomorphism. For any such α the following are equivalent:

(1) $N_{\mathfrak{q}}$ *is R-flat,*
(2) *there exists an $f \in R$, $f \notin \mathfrak{r}$ such that $\alpha_f : S_f^{\oplus r} \to N_f$ is R_f-universally injective and a $g \in S$, $g \notin \mathfrak{q}$ such that* $\mathrm{Coker}(\alpha)_g$ *is R-flat,*
(3) $\alpha_{\mathfrak{r}}$ *is $R_{\mathfrak{r}}$-universally injective and* $\mathrm{Coker}(\alpha)_{\mathfrak{q}}$ *is R-flat*
(4) $\alpha_{\mathfrak{r}}$ *is injective and* $\mathrm{Coker}(\alpha)_{\mathfrak{q}}$ *is R-flat,*
(5) $\alpha_{\mathfrak{p}}$ *is an isomorphism and* $\mathrm{Coker}(\alpha)_{\mathfrak{q}}$ *is R-flat, and*
(6) $\alpha_{\mathfrak{q}}$ *is injective and* $\mathrm{Coker}(\alpha)_{\mathfrak{q}}$ *is R-flat.*

Proof. To obtain α set $r = \dim_{\kappa(\mathfrak{p})} N \otimes_S \kappa(\mathfrak{p})$ and pick $x_1, \ldots, x_r \in N$ which form a basis of $N \otimes_S \kappa(\mathfrak{p})$. Define $\alpha(s_1, \ldots, s_r) = \sum s_i x_i$. This proves the existence.

Fix a choice of α. We may apply Lemma 10.1 to the map $\alpha_{\mathfrak{r}} : S_{\mathfrak{r}}^{\oplus r} \to N_{\mathfrak{r}}$. Hence we see that (1), (3), (4), (5), and (6) are all equivalent. Since it is also clear that (2) implies (3) we see that all we have to do is show that (1) implies (2).

Assume (1). By openness of flatness, see Algebra, Theorem 129.4, the set

$$U_1 = \{\mathfrak{q}' \subset S \mid N_{\mathfrak{q}'} \text{ is flat over } R\}$$

is open in $\mathrm{Spec}(S)$. It contains $\mathfrak{q}$ by assumption and hence $\mathfrak{p}$. Because $S^{\oplus r}$ and N are finitely presented S-modules the set

$$U_2 = \{\mathfrak{q}' \subset S \mid \alpha_{\mathfrak{q}'} \text{ is an isomorphism}\}$$

is open in $\mathrm{Spec}(S)$, see Algebra, Lemma 79.2. It contains $\mathfrak{p}$ by (5). As $R \to S$ is finitely presented and flat the map $\Phi : \mathrm{Spec}(S) \to \mathrm{Spec}(R)$ is open, see Algebra, Proposition 41.8. For any prime $\mathfrak{r}' \in \Phi(U_1 \cap U_2)$ we see that there exists a prime $\mathfrak{q}'$ lying over $\mathfrak{r}'$ such that $N_{\mathfrak{q}'}$ is flat and such that $\alpha_{\mathfrak{q}'}$ is an isomorphism, which implies that $\alpha \otimes \kappa(\mathfrak{p}')$ is an isomorphism where $\mathfrak{p}' = \mathfrak{r}'S$. Thus $\alpha_{\mathfrak{r}'}$ is $R_{\mathfrak{r}'}$-universally injective by the implication (1) $\Rightarrow$ (3). Hence if we pick $f \in R$, $f \notin \mathfrak{r}$ such that $D(f) \subset \Phi(U_1 \cap U_2)$ then we conclude that α_f is R_f-universally injective, see Algebra, Lemma 82.11. The same reasoning also shows that for any $\mathfrak{q}' \in U_1 \cap \Phi^{-1}(\Phi(U_1 \cap U_2))$ the module $\mathrm{Coker}(\alpha)_{\mathfrak{q}'}$ is R-flat. Note that $\mathfrak{q} \in U_1 \cap \Phi^{-1}(\Phi(U_1 \cap U_2))$. Hence we can find a $g \in S$, $g \notin \mathfrak{q}$ such that $D(g) \subset U_1 \cap \Phi^{-1}(\Phi(U_1 \cap U_2))$ and we win. □

05IB **Lemma 12.2.** *Let $R \to S$ be a ring map of finite presentation. Let N be a finitely presented S-module flat over R. Let $\mathfrak{r} \subset R$ be a prime ideal. Assume there exists a complete dévissage of $N/S/R$ over $\mathfrak{r}$. Then there exists an $f \in R$, $f \notin \mathfrak{r}$ such that*

$$N_f \cong B_1^{\oplus r_1} \oplus \ldots \oplus B_n^{\oplus r_n}$$

as R-modules where each B_i is a smooth R_f-algebra with geometrically irreducible fibres. Moreover, N_f is projective as an R_f-module.

Proof. Let $(A_i, B_i, M_i, \alpha_i)_{i=1,\ldots,n}$ be the given complete dévissage. We prove the lemma by induction on n. Note that the assertions of the lemma are entirely about the structure of N as an R-module. Hence we may replace N by M_1, and we may think of M_1 as a B_1-module. See Remark 6.3 in order to see why M_1 is of finite presentation as a B_1-module. By Lemma 12.1 we may, after replacing R by R_f for some $f \in R$, $f \notin \mathfrak{r}$, assume the map $\alpha_1 : B_1^{\oplus r_1} \to M_1$ is R-universally injective. Since M_1 and $B_1^{\oplus r_1}$ are R-flat and finitely presented as B_1-modules we see that $\mathrm{Coker}(\alpha_1)$ is R-flat (Algebra, Lemma 82.7) and finitely presented as a B_1-module. Note that $(A_i, B_i, M_i, \alpha_i)_{i=2,\ldots,n}$ is a complete dévissage of $\mathrm{Coker}(\alpha_1)$. Hence the induction hypothesis implies that, after replacing R by R_f for some $f \in R$, $f \notin \mathfrak{r}$, we may assume that $\mathrm{Coker}(\alpha_1)$ has a decomposition as in the lemma and is projective. In particular $M_1 = B_1^{\oplus r_1} \oplus \mathrm{Coker}(\alpha_1)$. This proves the statement regarding the decomposition. The statement on projectivity follows as B_1 is projective as an R-module by Lemma 9.3. □

05IC **Remark 12.3.** There is a variant of Lemma 12.2 where we weaken the flatness condition by assuming only that N is flat at some given prime $\mathfrak{q}$ lying over $\mathfrak{r}$ but where we strengthen the dévissage condition by assuming the existence of a complete dévissage *at* $\mathfrak{q}$. Compare with Lemma 10.2.

The following is the main result of this section.

05ID **Proposition 12.4.** *Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Let $x \in X$ with image $s \in S$. Assume that*

(1) *f is locally of finite presentation,*
(2) *$\mathcal{F}$ is of finite presentation, and*
(3) *$\mathcal{F}$ is flat at x over S.*

Then there exists a commutative diagram of pointed schemes

$$\begin{array}{ccc} (X, x) & \xleftarrow{g} & (X', x') \\ \downarrow & & \downarrow \\ (S, s) & \longleftarrow & (S', s') \end{array}$$

whose horizontal arrows are elementary étale neighbourhoods such that X', S' are affine and such that $\Gamma(X', g^*\mathcal{F})$ is a projective $\Gamma(S', \mathcal{O}_{S'})$-module.

Proof. By openness of flatness, see More on Morphisms, Theorem 15.1 we may replace X by an open neighbourhood of x and assume that $\mathcal{F}$ is flat over S. Next, we apply Proposition 5.7 to find a diagram as in the statement of the proposition such that $g^*\mathcal{F}/X'/S'$ has a complete dévissage over s'. (In particular S' and X' are affine.) By Morphisms, Lemma 26.13 we see that $g^*\mathcal{F}$ is flat over S and by Lemma 2.3 we see that it is flat over S'. Via Remark 6.5 we deduce that

$$\Gamma(X', g^*\mathcal{F})/\Gamma(X', \mathcal{O}_{X'})/\Gamma(S', \mathcal{O}_{S'})$$

has a complete dévissage over the prime of $\Gamma(S', \mathcal{O}_{S'})$ corresponding to s'. Thus Lemma 12.2 implies that the result of the proposition holds after replacing S' by a standard open neighbourhood of s'. □

In the rest of this section we prove a number of variants on this result. The first is a "global" version.

05KW **Lemma 12.5.** *Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Let $s \in S$. Assume that*

(1) *f is of finite presentation,*
(2) *$\mathcal{F}$ is of finite presentation, and*
(3) *$\mathcal{F}$ is flat over S at every point of the fibre X_s.*

Then there exists an elementary étale neighbourhood $(S', s') \to (S, s)$ and a commutative diagram of schemes

$$\begin{array}{ccc} X & \xleftarrow{g} & X' \\ \downarrow & & \downarrow \\ S & \longleftarrow & S' \end{array}$$

such that g is étale, $X_s \subset g(X')$, the schemes X', S' are affine, and such that $\Gamma(X', g^*\mathcal{F})$ is a projective $\Gamma(S', \mathcal{O}_{S'})$-module.

Proof. For every point $x \in X_s$ we can use Proposition 12.4 to find a commutative diagram

$$\begin{array}{ccc} (X, x) & \xleftarrow{g_x} & (Y_x, y_x) \\ \downarrow & & \downarrow \\ (S, s) & \longleftarrow & (S_x, s_x) \end{array}$$

whose horizontal arrows are elementary étale neighbourhoods such that Y_x, S_x are affine and such that $\Gamma(Y_x, g_x^*\mathcal{F})$ is a projective $\Gamma(S_x, \mathcal{O}_{S_x})$-module. In particular $g_x(Y_x) \cap X_s$ is an open neighbourhood of x in X_s. Because X_s is quasi-compact we can find a finite number of points $x_1, \ldots, x_n \in X_s$ such that X_s is the union of the $g_{x_i}(Y_{x_i}) \cap X_s$. Choose an elementary étale neighbourhood $(S', s') \to (S, s)$ which dominates each of the neighbourhoods (S_{x_i}, s_{x_i}), see More on Morphisms, Lemma 35.4. We may also assume that S' is affine. Set $X' = \coprod Y_{x_i} \times_{S_{x_i}} S'$ and endow it with the obvious morphism $g : X' \to X$. By construction $g(X')$ contains X_s and

$$\Gamma(X', g^*\mathcal{F}) = \bigoplus \Gamma(Y_{x_i}, g_{x_i}^*\mathcal{F}) \otimes_{\Gamma(S_{x_i}, \mathcal{O}_{S_{x_i}})} \Gamma(S', \mathcal{O}_{S'}).$$

This is a projective $\Gamma(S', \mathcal{O}_{S'})$-module, see Algebra, Lemma 94.1. □

The following two lemmas are reformulations of the results above in case $\mathcal{F} = \mathcal{O}_X$.

05IE **Lemma 12.6.** *Let $f : X \to S$ be locally of finite presentation. Let $x \in X$ with image $s \in S$. If f is flat at x over S, then there exists a commutative diagram of pointed schemes*

$$\begin{array}{ccc} (X, x) & \xleftarrow{g} & (X', x') \\ \downarrow & & \downarrow \\ (S, s) & \longleftarrow & (S', s') \end{array}$$

whose horizontal arrows are elementary étale neighbourhoods such that X', S' are affine and such that $\Gamma(X', \mathcal{O}_{X'})$ is a projective $\Gamma(S', \mathcal{O}_{S'})$-module.

Proof. This is a special case of Proposition 12.4. □

05KX **Lemma 12.7.** *Let $f : X \to S$ be of finite presentation. Let $s \in S$. If X is flat over S at all points of X_s, then there exists an elementary étale neighbourhood $(S', s') \to (S, s)$ and a commutative diagram of schemes*

$$\begin{array}{ccc} X & \xleftarrow{g} & X' \\ \downarrow & & \downarrow \\ S & \longleftarrow & S' \end{array}$$

with g étale, $X_s \subset g(X')$, such that X', S' are affine, and such that $\Gamma(X', \mathcal{O}_{X'})$ is a projective $\Gamma(S', \mathcal{O}_{S'})$-module.

Proof. This is a special case of Lemma 12.5. □

The following lemmas explain consequences of Proposition 12.4 in case we only assume the morphism and the sheaf are of finite type (and not necessarily of finite presentation).

05KY **Lemma 12.8.** *Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Let $x \in X$ with image $s \in S$. Assume that*

1. *f is locally of finite presentation,*
2. *$\mathcal{F}$ is of finite type, and*
3. *$\mathcal{F}$ is flat at x over S.*

Then there exists an elementary étale neighbourhood $(S', s') \to (S, s)$ and a commutative diagram of pointed schemes

$$\begin{array}{ccc} (X, x) & \xleftarrow[g]{} & (X', x') \\ \downarrow & & \downarrow \\ (S, s) & \longleftarrow & (\mathrm{Spec}(\mathcal{O}_{S',s'}), s') \end{array}$$

such that $X' \to X \times_S \mathrm{Spec}(\mathcal{O}_{S',s'})$ is étale, $\kappa(x) = \kappa(x')$, the scheme X' is affine of finite presentation over $\mathcal{O}_{S',s'}$, the sheaf $g^*\mathcal{F}$ is of finite presentation over $\mathcal{O}_{X'}$, and such that $\Gamma(X', g^*\mathcal{F})$ is a free $\mathcal{O}_{S',s'}$-module.

Proof. To prove the lemma we may replace (S, s) by any elementary étale neighbourhood, and we may also replace S by $\mathrm{Spec}(\mathcal{O}_{S,s})$. Hence by Proposition 10.3 we may assume that $\mathcal{F}$ is finitely presented and flat over S in a neighbourhood of x. In this case the result follows from Proposition 12.4 because Algebra, Theorem 85.4 assures us that projective = free over a local ring. □

05KZ **Lemma 12.9.** *Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Let $x \in X$ with image $s \in S$. Assume that*

1. *f is locally of finite type,*
2. *$\mathcal{F}$ is of finite type, and*
3. *$\mathcal{F}$ is flat at x over S.*

Then there exists an elementary étale neighbourhood $(S', s') \to (S, s)$ and a commutative diagram of pointed schemes

$$\begin{array}{ccc} (X, x) & \xleftarrow[g]{} & (X', x') \\ \downarrow & & \downarrow \\ (S, s) & \longleftarrow & (\mathrm{Spec}(\mathcal{O}_{S',s'}), s') \end{array}$$

such that $X' \to X \times_S \mathrm{Spec}(\mathcal{O}_{S',s'})$ is étale, $\kappa(x) = \kappa(x')$, the scheme X' is affine, and such that $\Gamma(X', g^*\mathcal{F})$ is a free $\mathcal{O}_{S',s'}$-module.

Proof. (The only difference with Lemma 12.8 is that we do not assume f is of finite presentation.) The problem is local on X and S. Hence we may assume X and S are affine, say $X = \mathrm{Spec}(B)$ and $S = \mathrm{Spec}(A)$. Since B is a finite type A-algebra we can find a surjection $A[x_1, \ldots, x_n] \to B$. In other words, we can choose a closed immersion $i : X \to \mathbf{A}^n_S$. Set $t = i(x)$ and $\mathcal{G} = i_*\mathcal{F}$. Note that $\mathcal{G}_t \cong \mathcal{F}_x$ are $\mathcal{O}_{S,s}$-modules. Hence $\mathcal{G}$ is flat over S at t. We apply Lemma 12.8 to the morphism $\mathbf{A}^n_S \to S$, the point t, and the sheaf $\mathcal{G}$. Thus we can find an elementary étale neighbourhood $(S', s') \to (S, s)$ and a commutative diagram of pointed schemes

$$\begin{array}{ccc} (\mathbf{A}^n_S, t) & \xleftarrow[h]{} & (Y, y) \\ \downarrow & & \downarrow \\ (S, s) & \longleftarrow & (\mathrm{Spec}(\mathcal{O}_{S',s'}), s') \end{array}$$

such that $Y \to \mathbf{A}^n_{\mathcal{O}_{S',s'}}$ is étale, $\kappa(t) = \kappa(y)$, the scheme Y is affine, and such that $\Gamma(Y, h^*\mathcal{G})$ is a projective $\mathcal{O}_{S',s'}$-module. Then a solution to the original problem is given by the closed subscheme $X' = Y \times_{\mathbf{A}^n_S} X$ of Y. □

05L0 **Lemma 12.10.** *Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Let $s \in S$. Assume that*

(1) *f is of finite presentation,*
(2) *$\mathcal{F}$ is of finite type, and*
(3) *$\mathcal{F}$ is flat over S at all points of X_s.*

Then there exists an elementary étale neighbourhood $(S', s') \to (S, s)$ and a commutative diagram of schemes

$$\begin{array}{ccc} X & \xleftarrow{g} & X' \\ \downarrow & & \downarrow \\ S & \longleftarrow & \mathrm{Spec}(\mathcal{O}_{S',s'}) \end{array}$$

such that $X' \to X \times_S \mathrm{Spec}(\mathcal{O}_{S',s'})$ is étale, $X_s = g((X')_{s'})$, the scheme X' is affine of finite presentation over $\mathcal{O}_{S',s'}$, the sheaf $g^*\mathcal{F}$ is of finite presentation over $\mathcal{O}_{X'}$, and such that $\Gamma(X', g^*\mathcal{F})$ is a free $\mathcal{O}_{S',s'}$-module.

Proof. For every point $x \in X_s$ we can use Lemma 12.8 to find an elementary étale neighbourhood $(S_x, s_x) \to (S, s)$ and a commutative diagram

$$\begin{array}{ccc} (X, x) & \xleftarrow{g_x} & (Y_x, y_x) \\ \downarrow & & \downarrow \\ (S, s) & \longleftarrow & (\mathrm{Spec}(\mathcal{O}_{S_x,s_x}), s_x) \end{array}$$

such that $Y_x \to X \times_S \mathrm{Spec}(\mathcal{O}_{S_x,s_x})$ is étale, $\kappa(x) = \kappa(y_x)$, the scheme Y_x is affine of finite presentation over $\mathcal{O}_{S_x,s_x}$, the sheaf $g_x^*\mathcal{F}$ is of finite presentation over $\mathcal{O}_{Y_x}$, and such that $\Gamma(Y_x, g_x^*\mathcal{F})$ is a free $\mathcal{O}_{S_x,s_x}$-module. In particular $g_x((Y_x)_{s_x})$ is an open neighbourhood of x in X_s. Because X_s is quasi-compact we can find a finite number of points $x_1, \ldots, x_n \in X_s$ such that X_s is the union of the $g_{x_i}((Y_{x_i})_{s_{x_i}})$. Choose an elementary étale neighbourhood $(S', s') \to (S, s)$ which dominates each of the neighbourhoods (S_{x_i}, s_{x_i}), see More on Morphisms, Lemma 35.4. Set

$$X' = \coprod Y_{x_i} \times_{\mathrm{Spec}(\mathcal{O}_{S_{x_i},s_{x_i}})} \mathrm{Spec}(\mathcal{O}_{S',s'})$$

and endow it with the obvious morphism $g : X' \to X$. By construction $X_s = g(X'_{s'})$ and

$$\Gamma(X', g^*\mathcal{F}) = \bigoplus \Gamma(Y_{x_i}, g_{x_i}^*\mathcal{F}) \otimes_{\mathcal{O}_{S_{x_i},s_{x_i}}} \mathcal{O}_{S',s'}.$$

This is a free $\mathcal{O}_{S',s'}$-module as a direct sum of base changes of free modules. Some minor details omitted. □

05L1 **Lemma 12.11.** *Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Let $s \in S$. Assume that*

(1) *f is of finite type,*
(2) *$\mathcal{F}$ is of finite type, and*
(3) *$\mathcal{F}$ is flat over S at all points of X_s.*

Then there exists an elementary étale neighbourhood $(S', s') \to (S, s)$ and a commutative diagram of schemes

$$\begin{array}{ccc} X & \xleftarrow{g} & X' \\ \downarrow & & \downarrow \\ S & \longleftarrow & \mathrm{Spec}(\mathcal{O}_{S',s'}) \end{array}$$

such that $X' \to X \times_S \mathrm{Spec}(\mathcal{O}_{S',s'})$ is étale, $X_s = g((X')_{s'})$, the scheme X' is affine, and such that $\Gamma(X', g^*\mathcal{F})$ is a free $\mathcal{O}_{S',s'}$-module.

Proof. (The only difference with Lemma 12.10 is that we do not assume f is of finite presentation.) For every point $x \in X_s$ we can use Lemma 12.9 to find an elementary étale neighbourhood $(S_x, s_x) \to (S, s)$ and a commutative diagram

$$\begin{array}{ccc} (X, x) & \xleftarrow{g_x} & (Y_x, y_x) \\ \downarrow & & \downarrow \\ (S, s) & \longleftarrow & (\mathrm{Spec}(\mathcal{O}_{S_x,s_x}), s_x) \end{array}$$

such that $Y_x \to X \times_S \mathrm{Spec}(\mathcal{O}_{S_x,s_x})$ is étale, $\kappa(x) = \kappa(y_x)$, the scheme Y_x is affine, and such that $\Gamma(Y_x, g_x^*\mathcal{F})$ is a free $\mathcal{O}_{S_x,s_x}$-module. In particular $g_x((Y_x)_{s_x})$ is an open neighbourhood of x in X_s. Because X_s is quasi-compact we can find a finite number of points $x_1, \ldots, x_n \in X_s$ such that X_s is the union of the $g_{x_i}((Y_{x_i})_{s_{x_i}})$. Choose an elementary étale neighbourhood $(S', s') \to (S, s)$ which dominates each of the neighbourhoods (S_{x_i}, s_{x_i}), see More on Morphisms, Lemma 35.4. Set

$$X' = \coprod Y_{x_i} \times_{\mathrm{Spec}(\mathcal{O}_{S_{x_i},s_{x_i}})} \mathrm{Spec}(\mathcal{O}_{S',s'})$$

and endow it with the obvious morphism $g : X' \to X$. By construction $X_s = g(X'_{s'})$ and

$$\Gamma(X', g^*\mathcal{F}) = \bigoplus \Gamma(Y_{x_i}, g_{x_i}^*\mathcal{F}) \otimes_{\mathcal{O}_{S_{x_i},s_{x_i}}} \mathcal{O}_{S',s'}.$$

This is a free $\mathcal{O}_{S',s'}$-module as a direct sum of base changes of free modules. □

13. Flat finite type modules, Part II

05IF We will need the following lemma.

0CU6 **Lemma 13.1.** *Let $R \to S$ be a ring map of finite presentation. Let N be a finitely presented S-module. Let $\mathfrak{q} \subset S$ be a prime ideal lying over $\mathfrak{p} \subset R$. Set $\overline{S} = S \otimes_R \kappa(\mathfrak{p})$, $\overline{\mathfrak{q}} = \mathfrak{q}\overline{S}$, and $\overline{N} = N \otimes_R \kappa(\mathfrak{p})$. Then we can find a $g \in S$ with $g \notin \mathfrak{q}$ such that $\overline{g} \in \mathfrak{r}$ for all $\mathfrak{r} \in Ass_{\overline{S}}(\overline{N})$ such that $\mathfrak{r} \not\subset \overline{\mathfrak{q}}$.*

Proof. Namely, if $\mathrm{Ass}_{\overline{S}}(\overline{N}) = \{\mathfrak{r}_1, \ldots, \mathfrak{r}_n\}$ (finiteness by Algebra, Lemma 63.5), then after renumbering we may assume that

$$\mathfrak{r}_1 \subset \overline{\mathfrak{q}}, \ldots, \mathfrak{r}_r \subset \overline{\mathfrak{q}}, \quad \mathfrak{r}_{r+1} \not\subset \overline{\mathfrak{q}}, \ldots, \mathfrak{r}_n \not\subset \overline{\mathfrak{q}}$$

Since $\overline{\mathfrak{q}}$ is a prime ideal we see that the product $\mathfrak{r}_{r+1} \ldots \mathfrak{r}_n$ is not contained in $\overline{\mathfrak{q}}$ and hence we can pick an element a of $\overline{S}$ contained in $\mathfrak{r}_{r+1}, \ldots, \mathfrak{r}_n$ but not in $\overline{\mathfrak{q}}$. If there exists $g \in S$ mapping to a, then g works. In general we can find a nonzero element $\lambda \in \kappa(\mathfrak{p})$ such that λa is the image of a $g \in S$. □

The following lemma has a slightly stronger variant Lemma 13.4 below.

05IG **Lemma 13.2.** *Let $R \to S$ be a ring map of finite presentation. Let N be a finitely presented S-module which is flat as an R-module. Let M be an R-module. Let $\mathfrak{q}$ be a prime of S lying over $\mathfrak{p} \subset R$. Then*

$$\mathfrak{q} \in \mathit{WeakAss}_S(M \otimes_R N) \Leftrightarrow \Big(\mathfrak{p} \in \mathit{WeakAss}_R(M) \text{ and } \overline{\mathfrak{q}} \in \mathit{Ass}_{\overline{S}}(\overline{N})\Big)$$

Here $\overline{S} = S \otimes_R \kappa(\mathfrak{p})$, $\overline{\mathfrak{q}} = \mathfrak{q}\overline{S}$, and $\overline{N} = N \otimes_R \kappa(\mathfrak{p})$.

Proof. Pick $g \in S$ as in Lemma 13.1. Apply Proposition 12.4 to the morphism of schemes $\mathrm{Spec}(S_g) \to \mathrm{Spec}(R)$, the quasi-coherent module associated to N_g, and the points corresponding to the primes $\mathfrak{q}S_g$ and $\mathfrak{p}$. Translating into algebra we obtain a commutative diagram of rings

$$\begin{array}{ccccc} S & \longrightarrow & S_g & \longrightarrow & S' \\ & \nwarrow & \uparrow & & \uparrow \\ & & R & \longrightarrow & R' \end{array} \qquad \begin{array}{ccccc} \mathfrak{q} & \longrightarrow & \mathfrak{q}S_g & \longrightarrow & \mathfrak{q}' \\ & \searrow & \downarrow & & \downarrow \\ & & \mathfrak{p} & \longrightarrow & \mathfrak{p}' \end{array}$$

endowed with primes as shown, the horizontal arrows are étale, and $N \otimes_S S'$ is projective as an R'-module. Set $N' = N \otimes_S S'$, $M' = M \otimes_R R'$, $\overline{S}' = S' \otimes_{R'} \kappa(\mathfrak{q}')$, $\overline{\mathfrak{q}}' = \mathfrak{q}'\overline{S}'$, and

$$\overline{N}' = N' \otimes_{R'} \kappa(\mathfrak{p}') = \overline{N} \otimes_{\overline{S}} \overline{S}'$$

By Lemma 2.8 we have

$$\begin{aligned} \mathrm{WeakAss}_{S'}(M' \otimes_{R'} N') &= (\mathrm{Spec}(S') \to \mathrm{Spec}(S))^{-1}\mathrm{WeakAss}_S(M \otimes_R N) \\ \mathrm{WeakAss}_{R'}(M') &= (\mathrm{Spec}(R') \to \mathrm{Spec}(R))^{-1}\mathrm{WeakAss}_R(M) \\ \mathrm{Ass}_{\overline{S}'}(\overline{N}') &= (\mathrm{Spec}(\overline{S}') \to \mathrm{Spec}(\overline{S}))^{-1}\mathrm{Ass}_{\overline{S}}(\overline{N}) \end{aligned}$$

Use Algebra, Lemma 66.9 for $\overline{N}$ and $\overline{N}'$. In particular we have

$$\begin{aligned} \mathfrak{q} \in \mathrm{WeakAss}_S(M \otimes_R N) &\Leftrightarrow \mathfrak{q}' \in \mathrm{WeakAss}_{S'}(M' \otimes_{R'} N') \\ \mathfrak{p} \in \mathrm{WeakAss}_R(M) &\Leftrightarrow \mathfrak{p}' \in \mathrm{WeakAss}_{R'}(M') \\ \overline{\mathfrak{q}} \in \mathrm{Ass}_{\overline{S}}(\overline{N}) &\Leftrightarrow \overline{\mathfrak{q}}' \in \mathrm{WeakAss}_{\overline{S}'}(\overline{N}') \end{aligned}$$

Our careful choice of g and the formula for $\mathrm{Ass}_{\overline{S}'}(\overline{N}')$ above shows that

0CU7 (13.2.1) $$\text{if } \mathfrak{r}' \in \mathrm{Ass}_{\overline{S}'}(\overline{N}') \text{ lies over } \mathfrak{r} \subset \overline{S} \text{ then } \mathfrak{r} \subset \overline{\mathfrak{q}}$$

This will be a key observation later in the proof. We will use the characterization of weakly associated primes given in Algebra, Lemma 66.2 without further mention.

Suppose that $\overline{\mathfrak{q}} \notin \mathrm{Ass}_{\overline{S}}(\overline{N})$. Then $\overline{\mathfrak{q}}' \notin \mathrm{Ass}_{\overline{S}'}(\overline{N}')$. By Algebra, Lemmas 63.9, 63.5, and 15.2 there exists an element $\overline{a}' \in \overline{\mathfrak{q}}'$ which is not a zerodivisor on $\overline{N}'$. After replacing $\overline{a}'$ by $\lambda\overline{a}'$ for some nonzero $\lambda \in \kappa(\mathfrak{p})$ we can find $a' \in \mathfrak{q}'$ mapping to $\overline{a}'$. By Lemma 7.6 the map $a' : N'_{\mathfrak{p}'} \to N'_{\mathfrak{p}'}$ is $R'_{\mathfrak{p}'}$-universally injective. In particular we see that $a' : M' \otimes_{R'} N' \to M' \otimes_{R'} N'$ is injective after localizing at $\mathfrak{p}'$ and hence after localizing at $\mathfrak{q}'$. Clearly this implies that $\mathfrak{q}' \notin \mathrm{WeakAss}_{S'}(M' \otimes_{R'} N')$. We conclude that $\mathfrak{q} \in \mathrm{WeakAss}_S(M \otimes_R N)$ implies $\overline{\mathfrak{q}} \in \mathrm{Ass}_{\overline{S}}(\overline{N})$.

Assume $\mathfrak{q} \in \mathrm{WeakAss}_S(M \otimes_R N)$. We want to show $\mathfrak{p} \in \mathrm{WeakAss}_S(M)$. Let $z \in M \otimes_R N$ be an element such that $\mathfrak{q}$ is minimal over $J = \mathrm{Ann}_S(z)$. Let $f_i \in \mathfrak{p}$, $i \in I$ be a set of generators of the ideal $\mathfrak{p}$. Since $\mathfrak{q}$ lies over $\mathfrak{p}$, for every i we

can choose an $n_i \geq 1$ and $g_i \in S$, $g_i \notin \mathfrak{q}$ with $g_i f_i^{n_i} \in J$, i.e., $g_i f_i^{n_i} z = 0$. Let $z' \in (M' \otimes_{R'} N')_{\mathfrak{p}'}$ be the image of z. Observe that z' is nonzero because z has nonzero image in $(M \otimes_R N)_{\mathfrak{q}}$ and because $S_{\mathfrak{q}} \to S'_{\mathfrak{q}'}$ is faithfully flat. We claim that $f_i^{n_i} z' = 0$.

Proof of the claim: Let $g_i' \in S'$ be the image of g_i. By the key observation (13.2.1) we find that the image $\overline{g}_i' \in \overline{S}'$ is not contained in $\mathfrak{r}'$ for any $\mathfrak{r}' \in \mathrm{Ass}_{\overline{S}'}(\overline{N})$. Hence by Lemma 7.6 we see that $g_i' : N'_{\mathfrak{p}'} \to N'_{\mathfrak{p}'}$ is $R'_{\mathfrak{p}'}$-universally injective. In particular we see that $g_i' : M' \otimes_{R'} N' \to M' \otimes_{R'} N'$ is injective after localizating at $\mathfrak{p}'$. The claim follows because $g_i f_i^{n_i} z' = 0$.

Our claim shows that the annihilator of z' in $R'_{\mathfrak{p}'}$ contains the elements $f_i^{n_i}$. As $R \to R'$ is étale we have $\mathfrak{p}' R'_{\mathfrak{p}'} = \mathfrak{p} R'_{\mathfrak{p}'}$ by Algebra, Lemma 143.5. Hence the annihilator of z' in $R'_{\mathfrak{p}'}$ has radical equal to $\mathfrak{p}' R_{\mathfrak{p}'}$ (here we use z' is not zero). On the other hand

$$z' \in (M' \otimes_{R'} N')_{\mathfrak{p}'} = M'_{\mathfrak{p}'} \otimes_{R'_{\mathfrak{p}'}} N'_{\mathfrak{p}'}$$

The module $N'_{\mathfrak{p}'}$ is projective over the local ring $R'_{\mathfrak{p}'}$ and hence free (Algebra, Theorem 85.4). Thus we can find a finite free direct summand $F' \subset N'_{\mathfrak{p}'}$ such that $z' \in M'_{\mathfrak{p}'} \otimes_{R'_{\mathfrak{p}'}} F'$. If F' has rank n, then we deduce that $\mathfrak{p}' R'_{\mathfrak{p}'} \in \mathrm{WeakAss}_{R'_{\mathfrak{p}'}}(M'^{\oplus n}_{\mathfrak{p}'})$. This implies $\mathfrak{p}' R'_{\mathfrak{p}'} \in \mathrm{WeakAss}(M'_{\mathfrak{p}'})$ for example by Algebra, Lemma 66.4. Then $\mathfrak{p}' \in \mathrm{WeakAss}_{R'}(M')$ which in turn gives $\mathfrak{p} \in \mathrm{WeakAss}_R(M)$. This finishes the proof of the implication "$\Rightarrow$" of the equivalence of the lemma.

Assume that $\mathfrak{p} \in \mathrm{WeakAss}_R(M)$ and $\overline{\mathfrak{q}} \in \mathrm{Ass}_{\overline{S}}(\overline{N})$. We want to show that $\mathfrak{q}$ is weakly associated to $M \otimes_R N$. Note that $\overline{\mathfrak{q}}'$ is a maximal element of $\mathrm{Ass}_{\overline{S}'}(\overline{N}')$. This is a consequence of (13.2.1) and the fact that there are no inclusions among the primes of $\overline{S}'$ lying over $\overline{\mathfrak{q}}$ (as fibres of étale morphisms are discrete Morphisms, Lemma 37.7). Thus, after replacing $R, S, \mathfrak{p}, \mathfrak{q}, M, N$ by $R', S', \mathfrak{p}', \mathfrak{q}', M', N'$ we may assume, in addition to the assumptions of the lemma, that

(1) $\mathfrak{p} \in \mathrm{WeakAss}_R(M)$,
(2) $\overline{\mathfrak{q}} \in \mathrm{Ass}_{\overline{S}}(\overline{N})$,
(3) N is projective as an R-module, and
(4) $\overline{\mathfrak{q}}$ is maximal in $\mathrm{Ass}_{\overline{S}}(\overline{N})$.

There is one more reduction, namely, we may replace R, S, M, N by their localizations at $\mathfrak{p}$. This leads to one more condition, namely,

(5) R is a local ring with maximal ideal $\mathfrak{p}$.

We will finish by showing that (1) – (5) imply $\mathfrak{q} \in \mathrm{WeakAss}(M \otimes_R N)$.

Since R is local and $\mathfrak{p} \in \mathrm{WeakAss}_R(M)$ we can pick a $y \in M$ whose annihilator I has radical equal to $\mathfrak{p}$. Write $\overline{\mathfrak{q}} = (\overline{g}_1, \ldots, \overline{g}_n)$ for some $\overline{g}_i \in \overline{S}$. Choose $g_i \in S$ mapping to $\overline{g}_i$. Then $\mathfrak{q} = \mathfrak{p}S + g_1 S + \ldots + g_n S$. Consider the map

$$\Psi : N/IN \longrightarrow (N/IN)^{\oplus n}, \quad z \longmapsto (g_1 z, \ldots, g_n z).$$

This is a homomorphism of projective R/I-modules. The local ring R/I is autoassociated (More on Algebra, Definition 15.1) as $\mathfrak{p}/I$ is locally nilpotent. The map $\Psi \otimes \kappa(\mathfrak{p})$ is not injective, because $\overline{\mathfrak{q}} \in \mathrm{Ass}_{\overline{S}}(\overline{N})$. Hence More on Algebra, Lemma 15.4 implies Ψ is not injective. Pick $z \in N/IN$ nonzero in the kernel of Ψ. The annihilator $J = \mathrm{Ann}_S(z)$ contains IS and g_i by construction. Thus $\sqrt{J} \subset S$ contains $\mathfrak{q}$. Let $\mathfrak{s} \subset S$ be a prime minimal over J. Then $\mathfrak{q} \subset \mathfrak{s}$, $\mathfrak{s}$ lies over $\mathfrak{p}$, and

$\mathfrak{s} \in \text{WeakAss}_S(N/IN)$. The last fact by definition of weakly associated primes. Apply the "$\Rightarrow$" part of the lemma (which we've already proven) to the ring map $R \to S$ and the modules R/I and N to conclude that $\overline{\mathfrak{s}} \in \text{Ass}_{\overline{S}}(\overline{N})$. Since $\overline{\mathfrak{q}} \subset \overline{\mathfrak{s}}$ the maximality of $\overline{\mathfrak{q}}$, see condition (4) above, implies that $\overline{\mathfrak{q}} = \overline{\mathfrak{s}}$. This shows that $\mathfrak{q} = \mathfrak{s}$ and we conlude what we want. □

05IH **Lemma 13.3.** *Let S be a scheme. Let $f : X \to S$ be locally of finite type. Let $x \in X$ with image $s \in S$. Let $\mathcal{F}$ be a finite type quasi-coherent sheaf on X. Let $\mathcal{G}$ be a quasi-coherent sheaf on S. If $\mathcal{F}$ is flat at x over S, then*

$$x \in \mathit{WeakAss}_X(\mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{G}) \Leftrightarrow s \in \mathit{WeakAss}_S(\mathcal{G}) \text{ and } x \in \mathit{Ass}_{X_s}(\mathcal{F}_s).$$

Proof. In this paragraph we reduce to f being of finite presentation. The question is local on X and S, hence we may assume X and S are affine. Write $X = \text{Spec}(B)$, $S = \text{Spec}(A)$ and write $B = A[x_1, \ldots, x_n]/I$. In other words we obtain a closed immersion $i : X \to \mathbf{A}^n_S$ over S. Denote $t = i(x) \in \mathbf{A}^n_S$. Note that $i_*\mathcal{F}$ is a finite type quasi-coherent sheaf on $\mathbf{A}^n_S$ which is flat at t over S and note that

$$i_*(\mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{G}) = i_*\mathcal{F} \otimes_{\mathcal{O}_{\mathbf{A}^n_S}} p^*\mathcal{G}$$

where $p : \mathbf{A}^n_S \to S$ is the projection. Note that t is a weakly associated point of $i_*(\mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{G})$ if and only if x is a weakly associated point of $\mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{G}$, see Divisors, Lemma 6.3. Similarly $x \in \text{Ass}_{X_s}(\mathcal{F}_s)$ if and only if $t \in \text{Ass}_{\mathbf{A}^n_s}((i_*\mathcal{F})_s)$ (see Algebra, Lemma 63.14). Hence it suffices to prove the lemma in case $X = \mathbf{A}^n_S$. Thus we may assume that $X \to S$ is of finite presentation.

In this paragraph we reduce to $\mathcal{F}$ being of finite presentation and flat over S. Choose an elementary étale neighbourhood $e : (S', s') \to (S, s)$ and an open $V \subset X \times_S \text{Spec}(\mathcal{O}_{S',s'})$ as in Proposition 10.3. Let $x' \in X' = X \times_S S'$ be the unique point mapping to x and s'. Then it suffices to prove the statement for $X' \to S'$, x', s', $(X' \to X)^*\mathcal{F}$, and $e^*\mathcal{G}$, see Lemma 2.8. Let $v \in V$ the unique point mapping to x' and let $s' \in \text{Spec}(\mathcal{O}_{S',s'})$ be the closed point. Then $\mathcal{O}_{V,v} = \mathcal{O}_{X',x'}$ and $\mathcal{O}_{\text{Spec}(\mathcal{O}_{S',s'}),s'} = \mathcal{O}_{S',s'}$ and similarly for the stalks of pullbacks of $\mathcal{F}$ and $\mathcal{G}$. Also $V_{s'} \subset X'_{s'}$ is an open subscheme. Since the condition of being a weakly associated point depend only on the stalk of the sheaf, we may replace $X' \to S'$, x', s', $(X' \to X)^*\mathcal{F}$, and $e^*\mathcal{G}$ by $V \to \text{Spec}(\mathcal{O}_{S',s'})$, v, s', $(V \to X)^*\mathcal{F}$, and $(\text{Spec}(\mathcal{O}_{S',s'}) \to S)^*\mathcal{G}$. Thus we may assume that f is of finite presentation and $\mathcal{F}$ of finite presentation and flat over S.

Assume f is of finite presentation and $\mathcal{F}$ of finite presentation and flat over S. After shrinking X and S to affine neighbourhoods of x and s, this case is handled by Lemma 13.2. □

05II **Lemma 13.4.** *Let $R \to S$ be a ring map which is essentially of finite type. Let N be a localization of a finite S-module flat over R. Let M be an R-module. Then*

$$\mathit{WeakAss}_S(M \otimes_R N) = \bigcup\nolimits_{\mathfrak{p} \in \mathit{WeakAss}_R(M)} \mathit{Ass}_{S \otimes_R \kappa(\mathfrak{p})}(N \otimes_R \kappa(\mathfrak{p}))$$

Proof. This lemma is a translation of Lemma 13.3 into algebra. Details of translation omitted. □

05IJ **Lemma 13.5.** *Let $f : X \to S$ be a morphism which is locally of finite type. Let $\mathcal{F}$ be a finite type quasi-coherent sheaf on X which is flat over S. Let $\mathcal{G}$ be a*

quasi-coherent sheaf on S. Then we have

$$WeakAss_X(\mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{G}) = \bigcup\nolimits_{s \in WeakAss_S(\mathcal{G})} Ass_{X_s}(\mathcal{F}_s)$$

Proof. Immediate consequence of Lemma 13.3. □

05IK **Theorem 13.6.** *Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Assume*

(1) *$X \to S$ is locally of finite presentation,*
(2) *$\mathcal{F}$ is an $\mathcal{O}_X$-module of finite type, and*
(3) *the set of weakly associated points of S is locally finite in S.*

Then $U = \{x \in X \mid \mathcal{F} \text{ flat at } x \text{ over } S\}$ is open in X and $\mathcal{F}|_U$ is an $\mathcal{O}_U$-module of finite presentation and flat over S.

Proof. Let $x \in X$ be such that $\mathcal{F}$ is flat at x over S. We have to find an open neighbourhood of x such that $\mathcal{F}$ restricts to a S-flat finitely presented module on this neighbourhood. The problem is local on X and S, hence we may assume that X and S are affine. As $\mathcal{F}_x$ is a finitely presented $\mathcal{O}_{X,x}$-module by Lemma 10.9 we conclude from Algebra, Lemma 126.5 there exists a finitely presented $\mathcal{O}_X$-module $\mathcal{F}'$ and a map $\varphi : \mathcal{F}' \to \mathcal{F}$ which induces an isomorphism $\varphi_x : \mathcal{F}'_x \to \mathcal{F}_x$. In particular we see that $\mathcal{F}'$ is flat over S at x, hence by openness of flatness More on Morphisms, Theorem 15.1 we see that after shrinking X we may assume that $\mathcal{F}'$ is flat over S. As $\mathcal{F}$ is of finite type after shrinking X we may assume that φ is surjective, see Modules, Lemma 9.4 or alternatively use Nakayama's lemma (Algebra, Lemma 20.1). By Lemma 13.5 we have

$$\mathrm{WeakAss}_X(\mathcal{F}') \subset \bigcup\nolimits_{s \in \mathrm{WeakAss}(S)} \mathrm{Ass}_{X_s}(\mathcal{F}'_s)$$

As $\mathrm{WeakAss}(S)$ is finite by assumption and since $\mathrm{Ass}_{X_s}(\mathcal{F}'_s)$ is finite by Divisors, Lemma 2.5 we conclude that $\mathrm{WeakAss}_X(\mathcal{F}')$ is finite. Using Algebra, Lemma 15.2 we may, after shrinking X once more, assume that $\mathrm{WeakAss}_X(\mathcal{F}')$ is contained in the generalization of x. Now consider $\mathcal{K} = \mathrm{Ker}(\varphi)$. We have $\mathrm{WeakAss}_X(\mathcal{K}) \subset \mathrm{WeakAss}_X(\mathcal{F}')$ (by Divisors, Lemma 5.4) but on the other hand, φ_x is an isomorphism, also $\varphi_{x'}$ is an isomorphism for all $x' \rightsquigarrow x$. We conclude that $\mathrm{WeakAss}_X(\mathcal{K}) = \emptyset$ whence $\mathcal{K} = 0$ by Divisors, Lemma 5.5. □

05IL **Lemma 13.7.** *Let $R \to S$ be a ring map of finite presentation. Let M be a finite S-module. Assume $WeakAss_R(R)$ is finite. Then*

$$U = \{\mathfrak{q} \subset S \mid M_{\mathfrak{q}} \text{ flat over } R\}$$

is open in $\mathrm{Spec}(S)$ *and for every $g \in S$ such that $D(g) \subset U$ the localization M_g is a finitely presented S_g-module flat over R.*

Proof. Follows immediately from Theorem 13.6. □

05IM **Lemma 13.8.** *Let $f : X \to S$ be a morphism of schemes which is locally of finite type. Assume the set of weakly associated points of S is locally finite in S. Then the set of points $x \in X$ where f is flat is an open subscheme $U \subset X$ and $U \to S$ is flat and locally of finite presentation.*

Proof. The problem is local on X and S, hence we may assume that X and S are affine. Then $X \to S$ corresponds to a finite type ring map $A \to B$. Choose

a surjection $A[x_1, \ldots, x_n] \to B$ and consider B as an $A[x_1, \ldots, x_n]$-module. An application of Lemma 13.7 finishes the proof. □

05IN **Lemma 13.9.** *Let $f : X \to S$ be a morphism of schemes which is locally of finite type and flat. If S is integral, then f is locally of finite presentation.*

Proof. Special case of Lemma 13.8. □

053G **Proposition 13.10.** *Let R be a domain. Let $R \to S$ be a ring map of finite type. Let M be a finite S-module.*

(1) *If S is flat over R, then S is a finitely presented R-algebra.*
(2) *If M is flat as an R-module, then M is finitely presented as an S-module.*

Proof. Part (1) is a special case of Lemma 13.9. For Part (2) choose a surjection $R[x_1, \ldots, x_n] \to S$. By Lemma 13.7 we find that M is finitely presented as an $R[x_1, \ldots, x_n]$-module. We conclude by Algebra, Lemma 6.4. □

05IQ **Lemma 13.11** (Finite type version of Theorem 13.6)**.** *Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Assume*

(1) *$X \to S$ is locally of finite type,*
(2) *$\mathcal{F}$ is an $\mathcal{O}_X$-module of finite type, and*
(3) *the set of weakly associated points of S is locally finite in S.*

Then $U = \{x \in X \mid \mathcal{F}$ flat at x over $S\}$ is open in X and $\mathcal{F}|_U$ is flat over S and locally finitely presented relative to S (see More on Morphisms, Definition 58.1).

Proof. The question is local on X and S. Thus we may assume X and S are affine. Then we may choose a closed immersion $i : X \to \mathbf{A}^n_S$. We apply Theorem 13.6 to $X' = \mathbf{A}^n_S \to S$ and the quasi-coherent module $\mathcal{F}' = i_*\mathcal{F}$ of finite type and we find that

$$U' = \{x' \in X' \mid \mathcal{F}' \text{ flat at } x' \text{ over } S\}$$

is open in X' and that $\mathcal{F}'|_{U'}$ is of finite presentation. Since $\mathcal{F}'$ restricts to zero on $X' \setminus i(X)$ and since $\mathcal{F}'_{i(x)} \cong \mathcal{F}_x$ for all $x \in X$ we see that

$$U' = i(U) \amalg (X' \setminus i(X))$$

Hence $U = i^{-1}(U')$ is open. Moreover, it is clear that $\mathcal{F}'|_{U'} = (i|_U)_*(\mathcal{F}|_U)$. Hence we conclude that $\mathcal{F}|_U$ is finitely presented relative to S by More on Morphisms, Lemmas 58.3 and 58.4. □

05IR **Lemma 13.12.** *Let $R \to S$ be a ring map of finite type. Let M be a finite S-module. Assume $\mathrm{WeakAss}_R(R)$ is finite. Then*

$$U = \{\mathfrak{q} \subset S \mid M_{\mathfrak{q}} \text{ flat over } R\}$$

is open in $\mathrm{Spec}(S)$ *and for every $g \in S$ such that $D(g) \subset U$ the localization M_g is flat over R and an S_g-module finitely presented relative to R (see More on Algebra, Definition 82.2).*

Proof. This is Lemma 13.11 translated into algebra. □

14. Examples of relatively pure modules

05IS In the short section we discuss some examples of results that will serve as motivation for the notion of a *relatively pure module* and the concept of an *impurity* which we will introduce later. Each of the examples is stated as a lemma. Note the similarity with the condition on associated primes to the conditions appearing in Lemmas 7.4, 8.3, 8.4, and 9.1. See also Algebra, Lemma 65.1 for a discussion.

05FV **Lemma 14.1.** *Let R be a local ring with maximal ideal $\mathfrak{m}$. Let $R \to S$ be a ring map. Let N be an S-module. Assume*

(1) *N is projective as an R-module, and*
(2) *$S/\mathfrak{m}S$ is Noetherian and $N/\mathfrak{m}N$ is a finite $S/\mathfrak{m}S$-module.*

Then for any prime $\mathfrak{q} \subset S$ which is an associated prime of $N \otimes_R \kappa(\mathfrak{p})$ where $\mathfrak{p} = R \cap \mathfrak{q}$ we have $\mathfrak{q} + \mathfrak{m}S \neq S$.

Proof. Note that the hypotheses of Lemmas 7.1 and 7.6 are satisfied. We will use the conclusions of these lemmas without further mention. Let $\Sigma \subset S$ be the multiplicative set of elements which are not zerodivisors on $N/\mathfrak{m}N$. The map $N \to \Sigma^{-1}N$ is R-universally injective. Hence we see that any $\mathfrak{q} \subset S$ which is an associated prime of $N \otimes_R \kappa(\mathfrak{p})$ is also an associated prime of $\Sigma^{-1}N \otimes_R \kappa(\mathfrak{p})$. Clearly this implies that $\mathfrak{q}$ corresponds to a prime of $\Sigma^{-1}S$. Thus $\mathfrak{q} \subset \mathfrak{q}'$ where $\mathfrak{q}'$ corresponds to an associated prime of $N/\mathfrak{m}N$ and we win. □

The following lemma gives another (slightly silly) example of this phenomenon.

05IT **Lemma 14.2.** *Let R be a ring. Let $I \subset R$ be an ideal. Let $R \to S$ be a ring map. Let N be an S-module. If N is I-adically complete, then for any R-module M and for any prime $\mathfrak{q} \subset S$ which is an associated prime of $N \otimes_R M$ we have $\mathfrak{q} + IS \neq S$.*

Proof. Let $S^\wedge$ denote the I-adic completion of S. Note that N is an $S^\wedge$-module, hence also $N \otimes_R M$ is an $S^\wedge$-module. Let $z \in N \otimes_R M$ be an element such that $\mathfrak{q} = \mathrm{Ann}_S(z)$. Since $z \neq 0$ we see that $\mathrm{Ann}_{S^\wedge}(z) \neq S^\wedge$. Hence $\mathfrak{q}S^\wedge \neq S^\wedge$. Hence there exists a maximal ideal $\mathfrak{m} \subset S^\wedge$ with $\mathfrak{q}S^\wedge \subset \mathfrak{m}$. Since $IS^\wedge \subset \mathfrak{m}$ by Algebra, Lemma 96.6 we win. □

Note that the following lemma gives an alternative proof of Lemma 14.1 as a projective module over a local ring is free, see Algebra, Theorem 85.4.

05IU **Lemma 14.3.** *Let R be a local ring with maximal ideal $\mathfrak{m}$. Let $R \to S$ be a ring map. Let N be an S-module. Assume N is isomorphic as an R-module to a direct sum of finite R-modules. Then for any R-module M and for any prime $\mathfrak{q} \subset S$ which is an associated prime of $N \otimes_R M$ we have $\mathfrak{q} + \mathfrak{m}S \neq S$.*

Proof. Write $N = \bigoplus_{i \in I} M_i$ with each M_i a finite R-module. Let M be an R-module and let $\mathfrak{q} \subset S$ be an associated prime of $N \otimes_R M$ such that $\mathfrak{q} + \mathfrak{m}S = S$. Let $z \in N \otimes_R M$ be an element with $\mathfrak{q} = \mathrm{Ann}_S(z)$. After modifying the direct sum decomposition a little bit we may assume that $z \in M_1 \otimes_R M$ for some element $1 \in I$. Write $1 = f + \sum x_j g_j$ for some $f \in \mathfrak{q}$, $x_j \in \mathfrak{m}$, and $g_j \in S$. For any $g \in S$ denote g' the R-linear map

$$M_1 \to N \xrightarrow{g} N \to M_1$$

where the first arrow is the inclusion map, the second arrow is multiplication by g and the third arrow is the projection map. Because each $x_j \in R$ we obtain the

equality

$$f' + \sum x_j g'_j = \mathrm{id}_{M_1} \in \mathrm{End}_R(M_1)$$

By Nakayama's lemma (Algebra, Lemma 20.1) we see that f' is surjective, hence by Algebra, Lemma 16.4 we see that f' is an isomorphism. In particular the map

$$M_1 \otimes_R M \to N \otimes_R M \xrightarrow{f} N \otimes_R M \to M_1 \otimes_R M$$

is an isomorphism. This contradicts the assumption that $fz = 0$. □

05IV **Lemma 14.4.** *Let R be a henselian local ring with maximal ideal $\mathfrak{m}$. Let $R \to S$ be a ring map. Let N be an S-module. Assume N is countably generated and Mittag-Leffler as an R-module. Then for any R-module M and for any prime $\mathfrak{q} \subset S$ which is an associated prime of $N \otimes_R M$ we have $\mathfrak{q} + \mathfrak{m}S \neq S$.*

Proof. This lemma reduces to Lemma 14.3 by Algebra, Lemma 153.13. □

Suppose $f : X \to S$ is a morphism of schemes and $\mathcal{F}$ is a quasi-coherent module on X. Let $\xi \in \mathrm{Ass}_{X/S}(\mathcal{F})$ and let $Z = \overline{\{\xi\}}$. Picture

$$\begin{array}{ccc} \xi & & Z \longrightarrow X \\ \downarrow & & \searrow \quad \downarrow f \\ f(\xi) & & S \end{array}$$

Note that $f(Z) \subset \overline{\{f(\xi)\}}$ and that $f(Z)$ is closed if and only if equality holds, i.e., $f(Z) = \overline{\{f(\xi)\}}$. It follows from Lemma 14.1 that if S, X are affine, the fibres X_s are Noetherian, $\mathcal{F}$ is of finite type, and $\Gamma(X, \mathcal{F})$ is a projective $\Gamma(S, \mathcal{O}_S)$-module, then $f(Z) = \overline{\{f(\xi)\}}$ is a closed subset. Slightly different analogous statements holds for the cases described in Lemmas 14.2, 14.3, and 14.4.

15. Impurities

05IW We want to formalize the phenomenon of which we gave examples in Section 14 in terms of specializations of points of $\mathrm{Ass}_{X/S}(\mathcal{F})$. We also want to work locally around a point $s \in S$. In order to do so we make the following definitions.

05FW **Situation 15.1.** Here S, X are schemes and $f : X \to S$ is a finite type morphism. Also, $\mathcal{F}$ is a finite type quasi-coherent $\mathcal{O}_X$-module. Finally s is a point of S.

In this situation consider a morphism $g : T \to S$, a point $t \in T$ with $g(t) = s$, a specialization $t' \rightsquigarrow t$, and a point $\xi \in X_T$ in the base change of X lying over t'. Picture

05IX (15.1.1)
$$\begin{array}{ccc} \xi & & X_T \longrightarrow X \\ \downarrow & & \downarrow \quad \downarrow \\ t' \rightsquigarrow t \longmapsto s & & T \xrightarrow{g} S \end{array}$$

Moreover, denote $\mathcal{F}_T$ the pullback of $\mathcal{F}$ to X_T.

05IY **Definition 15.2.** In Situation 15.1 we say a diagram (15.1.1) defines an *impurity of $\mathcal{F}$ above s* if $\xi \in \mathrm{Ass}_{X_T/T}(\mathcal{F}_T)$ and $\overline{\{\xi\}} \cap X_t = \emptyset$. We will indicate this by saying "let $(g : T \to S, t' \rightsquigarrow t, \xi)$ be an impurity of $\mathcal{F}$ above s".

05FX **Lemma 15.3.** *In Situation 15.1. If there exists an impurity of $\mathcal{F}$ above s, then there exists an impurity $(g : T \to S, t' \rightsquigarrow t, \xi)$ of $\mathcal{F}$ above s such that g is locally of finite presentation and t a closed point of the fibre of g above s.*

Proof. Let $(g : T \to S, t' \rightsquigarrow t, \xi)$ be any impurity of $\mathcal{F}$ above s. We apply Limits, Lemma 14.1 to $t \in T$ and $Z = \overline{\{\xi\}}$ to obtain an open neighbourhood $V \subset T$ of t, a commutative diagram

$$\begin{array}{ccc} V & \xrightarrow{a} & T' \\ \downarrow & & \downarrow b \\ T & \xrightarrow{g} & S, \end{array}$$

and a closed subscheme $Z' \subset X_{T'}$ such that

(1) the morphism $b : T' \to S$ is locally of finite presentation,
(2) we have $Z' \cap X_{a(t)} = \emptyset$, and
(3) $Z \cap X_V$ maps into Z' via the morphism $X_V \to X_{T'}$.

As t' specializes to t we may replace T by the open neighbourhood V of t. Thus we have a commutative diagram

$$\begin{array}{ccccc} X_T & \longrightarrow & X_{T'} & \longrightarrow & X \\ \downarrow & & \downarrow & & \downarrow \\ T & \xrightarrow{a} & T' & \xrightarrow{b} & S \end{array}$$

where $b \circ a = g$. Let $\xi' \in X_{T'}$ denote the image of ξ. By Divisors, Lemma 7.3 we see that $\xi' \in \mathrm{Ass}_{X_{T'}/T'}(\mathcal{F}_{T'})$. Moreover, by construction the closure of $\overline{\{\xi'\}}$ is contained in the closed subset Z' which avoids the fibre $X_{a(t)}$. In this way we see that $(T' \to S, a(t') \rightsquigarrow a(t), \xi')$ is an impurity of $\mathcal{F}$ above s.

Thus we may assume that $g : T \to S$ is locally of finite presentation. Let $Z = \overline{\{\xi\}}$. By assumption $Z_t = \emptyset$. By More on Morphisms, Lemma 24.1 this means that $Z_{t''} = \emptyset$ for t'' in an open subset of $\overline{\{t\}}$. Since the fibre of $T \to S$ over s is a Jacobson scheme, see Morphisms, Lemma 16.10 we find that there exist a closed point $t'' \in \overline{\{t\}}$ such that $Z_{t''} = \emptyset$. Then $(g : T \to S, t' \rightsquigarrow t'', \xi)$ is the desired impurity. □

05IZ **Lemma 15.4.** *In Situation 15.1. Let $(g : T \to S, t' \rightsquigarrow t, \xi)$ be an impurity of $\mathcal{F}$ above s. Assume $T = \lim_{i \in I} T_i$ is a directed limit of affine schemes over S. Then for some i the triple $(T_i \to S, t'_i \rightsquigarrow t_i, \xi_i)$ is an impurity of $\mathcal{F}$ above s.*

Proof. The notation in the statement means this: Let $p_i : T \to T_i$ be the projection morphisms, let $t_i = p_i(t)$ and $t'_i = p_i(t')$. Finally $\xi_i \in X_{T_i}$ is the image of ξ. By Divisors, Lemma 7.3 it is true that ξ_i is a point of the relative assassin of $\mathcal{F}_{T_i}$ over T_i. Thus the only point is to show that $\overline{\{\xi_i\}} \cap X_{t_i} = \emptyset$ for some i.

First proof. Let $Z_i = \overline{\{\xi_i\}} \subset X_{T_i}$ and $Z = \overline{\{\xi\}} \subset X_T$ endowed with the reduced induced scheme structure. Then $Z = \lim Z_i$ by Limits, Lemma 4.4. Choose a field k and a morphism $\mathrm{Spec}(k) \to T$ whose image is t. Then

$$\emptyset = Z \times_T \mathrm{Spec}(k) = (\lim Z_i) \times_{(\lim T_i)} \mathrm{Spec}(k) = \lim Z_i \times_{T_i} \mathrm{Spec}(k)$$

because limits commute with fibred products (limits commute with limits). Each $Z_i \times_{T_i} \mathrm{Spec}(k)$ is quasi-compact because $X_{T_i} \to T_i$ is of finite type and hence

$Z_i \to T_i$ is of finite type. Hence $Z_i \times_{T_i} \mathrm{Spec}(k)$ is empty for some i by Limits, Lemma 4.3. Since the image of the composition $\mathrm{Spec}(k) \to T \to T_i$ is t_i we obtain what we want.

Second proof. Set $Z = \overline{\{\xi\}}$. Apply Limits, Lemma 14.1 to this situation to obtain an open neighbourhood $V \subset T$ of t, a commutative diagram

$$\begin{array}{ccc} V & \xrightarrow{a} & T' \\ \downarrow & & \downarrow b \\ T & \xrightarrow{g} & S, \end{array}$$

and a closed subscheme $Z' \subset X_{T'}$ such that

(1) the morphism $b : T' \to S$ is locally of finite presentation,
(2) we have $Z' \cap X_{a(t)} = \emptyset$, and
(3) $Z \cap X_V$ maps into Z' via the morphism $X_V \to X_{T'}$.

We may assume V is an affine open of T, hence by Limits, Lemmas 4.11 and 4.13 we can find an i and an affine open $V_i \subset T_i$ with $V = f_i^{-1}(V_i)$. By Limits, Proposition 6.1 after possibly increasing i a bit we can find a morphism $a_i : V_i \to T'$ such that $a = a_i \circ f_i|_V$. The induced morphism $X_{V_i} \to X_{T'}$ maps ξ_i into Z'. As $Z' \cap X_{a(t)} = \emptyset$ we conclude that $(T_i \to S, t_i' \rightsquigarrow t_i, \xi_i)$ is an impurity of $\mathcal{F}$ above s. □

05J0 **Lemma 15.5.** *In Situation 15.1. If there exists an impurity $(g : T \to S, t' \rightsquigarrow t, \xi)$ of $\mathcal{F}$ above s with g quasi-finite at t, then there exists an impurity $(g : T \to S, t' \rightsquigarrow t, \xi)$ such that $(T, t) \to (S, s)$ is an elementary étale neighbourhood.*

Proof. Let $(g : T \to S, t' \rightsquigarrow t, \xi)$ be an impurity of $\mathcal{F}$ above s such that g is quasi-finite at t. After shrinking T we may assume that g is locally of finite type. Apply More on Morphisms, Lemma 41.1 to $T \to S$ and $t \mapsto s$. This gives us a diagram

$$\begin{array}{ccccc} T & \longleftarrow & T \times_S U & \longleftarrow & V \\ \downarrow & & \downarrow & \swarrow & \\ S & \longleftarrow & U & & \end{array}$$

where $(U, u) \to (S, s)$ is an elementary étale neighbourhood and $V \subset T \times_S U$ is an open neighbourhood of $v = (t, u)$ such that $V \to U$ is finite and such that v is the unique point of V lying over u. Since the morphism $V \to T$ is étale hence flat we see that there exists a specialization $v' \rightsquigarrow v$ such that $v' \mapsto t'$. Note that $\kappa(t') \subset \kappa(v')$ is finite separable. Pick any point $\zeta \in X_{v'}$ mapping to $\xi \in X_{t'}$. By Divisors, Lemma 7.3 we see that $\zeta \in \mathrm{Ass}_{X_V/V}(\mathcal{F}_V)$. Moreover, the closure $\overline{\{\zeta\}}$ does not meet the fibre X_v as by assumption the closure $\overline{\{\xi\}}$ does not meet X_t. In other words $(V \to S, v' \rightsquigarrow v, \zeta)$ is an impurity of $\mathcal{F}$ above S.

Next, let $u' \in U'$ be the image of v' and let $\theta \in X_U$ be the image of ζ. Then $\theta \mapsto u'$ and $u' \rightsquigarrow u$. By Divisors, Lemma 7.3 we see that $\theta \in \mathrm{Ass}_{X_U/U}(\mathcal{F})$. Moreover, as $\pi : X_V \to X_U$ is finite we see that $\pi\big(\overline{\{\zeta\}}\big) = \overline{\{\pi(\zeta)\}}$. Since v is the unique point of V lying over u we see that $X_u \cap \overline{\{\pi(\zeta)\}} = \emptyset$ because $X_v \cap \overline{\{\zeta\}} = \emptyset$. In this way we conclude that $(U \to S, u' \rightsquigarrow u, \theta)$ is an impurity of $\mathcal{F}$ above s and we win. □

05J1 **Lemma 15.6.** *In Situation 15.1. Assume that S is locally Noetherian. If there exists an impurity of $\mathcal{F}$ above s, then there exists an impurity $(g : T \to S, t' \rightsquigarrow t, \xi)$ of $\mathcal{F}$ above s such that g is quasi-finite at t.*

Proof. We may replace S by an affine neighbourhood of s. By Lemma 15.3 we may assume that we have an impurity $(g : T \to S, t' \rightsquigarrow t, \xi)$ of such that g is locally of finite type and t a closed point of the fibre of g above s. We may replace T by the reduced induced scheme structure on $\overline{\{t'\}}$. Let $Z = \overline{\{\xi\}} \subset X_T$. By assumption $Z_t = \emptyset$ and the image of $Z \to T$ contains t'. By More on Morphisms, Lemma 25.1 there exists a nonempty open $V \subset Z$ such that for any $w \in f(V)$ any generic point ξ' of V_w is in $\mathrm{Ass}_{X_T/T}(\mathcal{F}_T)$. By More on Morphisms, Lemma 24.2 there exists a nonempty open $W \subset T$ with $W \subset f(V)$. By More on Morphisms, Lemma 52.7 there exists a closed subscheme $T' \subset T$ such that $t \in T'$, $T' \to S$ is quasi-finite at t, and there exists a point $z \in T' \cap W$, $z \rightsquigarrow t$ which does not map to s. Choose any generic point ξ' of the nonempty scheme V_z. Then $(T' \to S, z \rightsquigarrow t, \xi')$ is the desired impurity. □

In the following we will use the henselization $S^h = \mathrm{Spec}(\mathcal{O}^h_{S,s})$ of S at s, see Étale Cohomology, Definition 33.2. Since $S^h \to S$ maps to closed point of S^h to s and induces an isomorphism of residue fields, we will indicate $s \in S^h$ this closed point also. Thus $(S^h, s) \to (S, s)$ is a morphism of pointed schemes.

05J2 **Lemma 15.7.** *In Situation 15.1. If there exists an impurity $(S^h \to S, s' \rightsquigarrow s, \xi)$ of $\mathcal{F}$ above s then there exists an impurity $(T \to S, t' \rightsquigarrow t, \xi)$ of $\mathcal{F}$ above s where $(T, t) \to (S, s)$ is an elementary étale neighbourhood.*

Proof. We may replace S by an affine neighbourhood of s. Say $S = \mathrm{Spec}(A)$ and s corresponds to the prime $\mathfrak{p} \subset A$. Then $\mathcal{O}^h_{S,s} = \mathrm{colim}_{(T,t)} \Gamma(T, \mathcal{O}_T)$ where the limit is over the opposite of the cofiltered category of affine elementary étale neighbourhoods (T, t) of (S, s), see More on Morphisms, Lemma 35.5 and its proof. Hence $S^h = \lim_i T_i$ and we win by Lemma 15.4. □

05J3 **Lemma 15.8.** *In Situation 15.1 the following are equivalent*

1. *there exists an impurity $(S^h \to S, s' \rightsquigarrow s, \xi)$ of $\mathcal{F}$ above s where S^h is the henselization of S at s,*
2. *there exists an impurity $(T \to S, t' \rightsquigarrow t, \xi)$ of $\mathcal{F}$ above s such that $(T, t) \to (S, s)$ is an elementary étale neighbourhood, and*
3. *there exists an impurity $(T \to S, t' \rightsquigarrow t, \xi)$ of $\mathcal{F}$ above s such that $T \to S$ is quasi-finite at t.*

Proof. As an étale morphism is locally quasi-finite it is clear that (2) implies (3). We have seen that (3) implies (2) in Lemma 15.5. We have seen that (1) implies (2) in Lemma 15.7. Finally, if $(T \to S, t' \rightsquigarrow t, \xi)$ is an impurity of $\mathcal{F}$ above s such that $(T, t) \to (S, s)$ is an elementary étale neighbourhood, then we can choose a factorization $S^h \to T \to S$ of the structure morphism $S^h \to S$. Choose any point $s' \in S^h$ mapping to t' and choose any $\xi' \in X_{s'}$ mapping to $\xi \in X_{t'}$. Then $(S^h \to S, s' \rightsquigarrow s, \xi')$ is an impurity of $\mathcal{F}$ above s. We omit the details. □

16. Relatively pure modules

05BB The notion of a module pure relative to a base was introduced in [GR71].

05J4 **Definition 16.1.** Let $f : X \to S$ be a morphism of schemes which is of finite type. Let $\mathcal{F}$ be a finite type quasi-coherent $\mathcal{O}_X$-module.

(1) Let $s \in S$. We say $\mathcal{F}$ is *pure along* X_s if there is no impurity $(g : T \to S, t' \rightsquigarrow t, \xi)$ of $\mathcal{F}$ above s with $(T, t) \to (S, s)$ an elementary étale neighbourhood.
(2) We say $\mathcal{F}$ is *universally pure along* X_s if there does not exist any impurity of $\mathcal{F}$ above s.
(3) We say that X is *pure along* X_s if $\mathcal{O}_X$ is pure along X_s.
(4) We say $\mathcal{F}$ is *universally S-pure*, or *universally pure relative to* S if $\mathcal{F}$ is universally pure along X_s for every $s \in S$.
(5) We say $\mathcal{F}$ is *S-pure*, or *pure relative to* S if $\mathcal{F}$ is pure along X_s for every $s \in S$.
(6) We say that X is *S-pure* or *pure relative to* S if $\mathcal{O}_X$ is pure relative to S.

We intentionally restrict ourselves here to morphisms which are of finite type and not just morphisms which are locally of finite type, see Remark 16.2 for a discussion. In the situation of the definition Lemma 15.8 tells us that the following are equivalent

(1) $\mathcal{F}$ is pure along X_s,
(2) there is no impurity $(g : T \to S, t' \rightsquigarrow t, \xi)$ with g quasi-finite at t,
(3) there does not exist any impurity of the form $(S^h \to S, s' \rightsquigarrow s, \xi)$, where S^h is the henselization of S at s.

If we denote $X^h = X \times_S S^h$ and $\mathcal{F}^h$ the pullback of $\mathcal{F}$ to X^h, then we can formulate the last condition in the following more positive way:

(4) All points of $\mathrm{Ass}_{X^h/S^h}(\mathcal{F}^h)$ specialize to points of X_s.

In particular, it is clear that $\mathcal{F}$ is pure along X_s if and only if the pullback of $\mathcal{F}$ to $X \times_S \mathrm{Spec}(\mathcal{O}_{S,s})$ is pure along X_s.

05J5 **Remark 16.2.** Let $f : X \to S$ be a morphism which is locally of finite type and $\mathcal{F}$ a quasi-coherent finite type $\mathcal{O}_X$-module. In this case it is still true that (1) and (2) above are equivalent because the proof of Lemma 15.5 does not use that f is quasi-compact. It is also clear that (3) and (4) are equivalent. However, we don't know if (1) and (3) are equivalent. In this case it may sometimes be more convenient to define purity using the equivalent conditions (3) and (4) as is done in [GR71]. On the other hand, for many applications it seems that the correct notion is really that of being universally pure.

A natural question to ask is if the property of being pure relative to the base is preserved by base change, i.e., if being pure is the same thing as being universally pure. It turns out that this is true over Noetherian base schemes (see Lemma 16.5), or if the sheaf is flat (see Lemmas 18.3 and 18.4). It is not true in general, even if the morphism and the sheaf are of finite presentation, see Examples, Section 40 for a counter example. First we match our usage of "universally" to the usual notion.

05J6 **Lemma 16.3.** *Let $f : X \to S$ be a morphism of schemes which is of finite type. Let $\mathcal{F}$ be a finite type quasi-coherent $\mathcal{O}_X$-module. Let $s \in S$. The following are equivalent*

(1) *$\mathcal{F}$ is universally pure along X_s, and*
(2) *for every morphism of pointed schemes $(S', s') \to (S, s)$ the pullback $\mathcal{F}_{S'}$ is pure along $X_{s'}$.*

In particular, $\mathcal{F}$ is universally pure relative to S if and only if every base change $\mathcal{F}_{S'}$ of $\mathcal{F}$ is pure relative to S'.

Proof. This is formal. □

05J7 **Lemma 16.4.** *Let $f : X \to S$ be a morphism of schemes which is of finite type. Let $\mathcal{F}$ be a finite type quasi-coherent $\mathcal{O}_X$-module. Let $s \in S$. Let $(S', s') \to (S, s)$ be a morphism of pointed schemes. If $S' \to S$ is quasi-finite at s' and $\mathcal{F}$ is pure along X_s, then $\mathcal{F}_{S'}$ is pure along $X_{s'}$.*

Proof. It $(T \to S', t' \rightsquigarrow t, \xi)$ is an impurity of $\mathcal{F}_{S'}$ above s' with $T \to S'$ quasi-finite at t, then $(T \to S, t' \to t, \xi)$ is an impurity of $\mathcal{F}$ above s with $T \to S$ quasi-finite at t, see Morphisms, Lemma 21.12. Hence the lemma follows immediately from the characterization (2) of purity given following Definition 16.1. □

05J8 **Lemma 16.5.** *Let $f : X \to S$ be a morphism of schemes which is of finite type. Let $\mathcal{F}$ be a finite type quasi-coherent $\mathcal{O}_X$-module. Let $s \in S$. If $\mathcal{O}_{S,s}$ is Noetherian then $\mathcal{F}$ is pure along X_s if and only if $\mathcal{F}$ is universally pure along X_s.*

Proof. First we may replace S by $\mathrm{Spec}(\mathcal{O}_{S,s})$, i.e., we may assume that S is Noetherian. Next, use Lemma 15.6 and characterization (2) of purity given in discussion following Definition 16.1 to conclude. □

Purity satisfies flat descent.

05J9 **Lemma 16.6.** *Let $f : X \to S$ be a morphism of schemes which is of finite type. Let $\mathcal{F}$ be a finite type quasi-coherent $\mathcal{O}_X$-module. Let $s \in S$. Let $(S', s') \to (S, s)$ be a morphism of pointed schemes. Assume $S' \to S$ is flat at s'.*

(1) *If $\mathcal{F}_{S'}$ is pure along $X_{s'}$, then $\mathcal{F}$ is pure along X_s.*
(2) *If $\mathcal{F}_{S'}$ is universally pure along $X_{s'}$, then $\mathcal{F}$ is universally pure along X_s.*

Proof. Let $(T \to S, t' \rightsquigarrow t, \xi)$ be an impurity of $\mathcal{F}$ above s. Set $T_1 = T \times_S S'$, and let t_1 be the unique point of T_1 mapping to t and s'. Since $T_1 \to T$ is flat at t_1, see Morphisms, Lemma 26.8, there exists a specialization $t_1' \rightsquigarrow t_1$ lying over $t' \rightsquigarrow t$, see Algebra, Section 41. Choose a point $\xi_1 \in X_{t_1'}$ which corresponds to a generic point of $\mathrm{Spec}(\kappa(t_1') \otimes_{\kappa(t')} \kappa(\xi))$, see Schemes, Lemma 17.5. By Divisors, Lemma 7.3 we see that $\xi_1 \in \mathrm{Ass}_{X_{T_1}/T_1}(\mathcal{F}_{T_1})$. As the Zariski closure of $\{\xi_1\}$ in X_{T_1} maps into the Zariski closure of $\{\xi\}$ in X_T we conclude that this closure is disjoint from X_{t_1}. Hence $(T_1 \to S', t_1' \rightsquigarrow t_1, \xi_1)$ is an impurity of $\mathcal{F}_{S'}$ above s'. In other words we have proved the contrapositive to part (2) of the lemma. Finally, if $(T, t) \to (S, s)$ is an elementary étale neighbourhood, then $(T_1, t_1) \to (S', s')$ is an elementary étale neighbourhood too, and in this way we see that (1) holds. □

05K1 **Lemma 16.7.** *Let $i : Z \to X$ be a closed immersion of schemes of finite type over a scheme S. Let $s \in S$. Let $\mathcal{F}$ be a finite type, quasi-coherent sheaf on Z. Then $\mathcal{F}$ is (universally) pure along Z_s if and only if $i_*\mathcal{F}$ is (universally) pure along X_s.*

Proof. This follows from Divisors, Lemma 8.3. □

17. Examples of relatively pure sheaves

05K2 Here are some example cases where it is possible to see what purity means.

05K3 **Lemma 17.1.** *Let $f : X \to S$ be a morphism of schemes which is of finite type. Let $\mathcal{F}$ be a finite type quasi-coherent $\mathcal{O}_X$-module.*

(1) *If the support of $\mathcal{F}$ is proper over S, then $\mathcal{F}$ is universally pure relative to S.*
(2) *If f is proper, then $\mathcal{F}$ is universally pure relative to S.*
(3) *If f is proper, then X is universally pure relative to S.*

Proof. First we reduce (1) to (2). Namely, let $Z \subset X$ be the scheme theoretic support of $\mathcal{F}$. Let $i : Z \to X$ be the corresponding closed immersion and write $\mathcal{F} = i_*\mathcal{G}$ for some finite type quasi-coherent $\mathcal{O}_Z$-module $\mathcal{G}$, see Morphisms, Section 5. In case (1) $Z \to S$ is proper by assumption. Thus by Lemma 16.7 case (1) reduces to case (2).

Assume f is proper. Let $(g : T \to S, t' \rightsquigarrow t, \xi)$ be an impurity of $\mathcal{F}$ above $s \in S$. Since f is proper, it is universally closed. Hence $f_T : X_T \to T$ is closed. Since $f_T(\xi) = t'$ this implies that $t \in f(\overline{\{\xi\}})$ which is a contradiction. □

05K4 **Lemma 17.2.** *Let $f : X \to S$ be a separated, finite type morphism of schemes. Let $\mathcal{F}$ be a finite type, quasi-coherent $\mathcal{O}_X$-module. Assume that Supp$(\mathcal{F}_s)$ is finite for every $s \in S$. Then the following are equivalent*

(1) *$\mathcal{F}$ is pure relative to S,*
(2) *the scheme theoretic support of $\mathcal{F}$ is finite over S, and*
(3) *$\mathcal{F}$ is universally pure relative to S.*

In particular, given a quasi-finite separated morphism $X \to S$ we see that X is pure relative to S if and only if $X \to S$ is finite.

Proof. Let $Z \subset X$ be the scheme theoretic support of $\mathcal{F}$, see Morphisms, Definition 5.5. Then $Z \to S$ is a separated, finite type morphism of schemes with finite fibres. Hence it is separated and quasi-finite, see Morphisms, Lemma 21.10. By Lemma 16.7 it suffices to prove the lemma for $Z \to S$ and the sheaf $\mathcal{F}$ viewed as a finite type quasi-coherent module on Z. Hence we may assume that $X \to S$ is separated and quasi-finite and that $\text{Supp}(\mathcal{F}) = X$.

It follows from Lemma 17.1 and Morphisms, Lemma 45.11 that (2) implies (3). Trivially (3) implies (1). Assume (1) holds. We will prove that (2) holds. It is clear that we may assume S is affine. By More on Morphisms, Lemma 43.3 we can find a diagram

$$\begin{array}{ccc} X & \xrightarrow{\quad j \quad} & T \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle \pi} & \\ & S & \end{array}$$

with π finite and j a quasi-compact open immersion. If we show that j is closed, then j is a closed immersion and we conclude that $f = \pi \circ j$ is finite. To show that j is closed it suffices to show that specializations lift along j, see Schemes, Lemma 19.8. Let $x \in X$, set $t' = j(x)$ and let $t' \rightsquigarrow t$ be a specialization. We have to show $t \in j(X)$. Set $s' = f(x)$ and $s = \pi(t)$ so $s' \rightsquigarrow s$. By More on Morphisms, Lemma 41.4 we can find an elementary étale neighbourhood $(U, u) \to (S, s)$ and a decomposition

$$T_U = T \times_S U = V \amalg W$$

into open and closed subschemes, such that $V \to U$ is finite and there exists a unique point v of V mapping to u, and such that v maps to t in T. As $V \to T$ is étale, we can lift generalizations, see Morphisms, Lemmas 26.9 and 37.12. Hence

there exists a specialization $v' \rightsquigarrow v$ such that v' maps to $t' \in T$. In particular we see that $v' \in X_U \subset T_U$. Denote $u' \in U$ the image of t'. Note that $v' \in \mathrm{Ass}_{X_U/U}(\mathcal{F})$ because $X_{u'}$ is a finite discrete set and $X_{u'} = \mathrm{Supp}(\mathcal{F}_{u'})$. As $\mathcal{F}$ is pure relative to S we see that v' must specialize to a point in X_u. Since v is the only point of V lying over u (and since no point of W can be a specialization of v') we see that $v \in X_u$. Hence $t \in X$. □

05K5 **Lemma 17.3.** *Let $f : X \to S$ be a finite type, flat morphism of schemes with geometrically integral fibres. Then X is universally pure over S.*

Proof. Let $\xi \in X$ with $s' = f(\xi)$ and $s' \rightsquigarrow s$ a specialization of S. If ξ is an associated point of $X_{s'}$, then ξ is the unique generic point because $X_{s'}$ is an integral scheme. Let ξ_0 be the unique generic point of X_s. As $X \to S$ is flat we can lift $s' \rightsquigarrow s$ to a specialization $\xi' \rightsquigarrow \xi_0$ in X, see Morphisms, Lemma 26.9. The $\xi \rightsquigarrow \xi'$ because ξ is the generic point of $X_{s'}$ hence $\xi \rightsquigarrow \xi_0$. This means that $(\mathrm{id}_S, s' \to s, \xi)$ is not an impurity of $\mathcal{O}_X$ above s. Since the assumption that f is finite type, flat with geometrically integral fibres is preserved under base change, we see that there doesn't exist an impurity after any base change. In this way we see that X is universally S-pure. □

05K6 **Lemma 17.4.** *Let $f : X \to S$ be a finite type, affine morphism of schemes. Let $\mathcal{F}$ be a finite type quasi-coherent $\mathcal{O}_X$-module such that $f_*\mathcal{F}$ is locally projective on S, see Properties, Definition 22.1. Then $\mathcal{F}$ is universally pure over S.*

Proof. After reducing to the case where S is the spectrum of a henselian local ring this follows from Lemma 14.1. □

18. A criterion for purity

05L2 We first prove that given a flat family of finite type quasi-coherent sheaves the points in the relative assassin specialize to points in the relative assassins of nearby fibres (if they specialize at all).

05L3 **Lemma 18.1.** *Let $f : X \to S$ be a morphism of schemes of finite type. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module of finite type. Let $s \in S$. Assume that $\mathcal{F}$ is flat over S at all points of X_s. Let $x' \in Ass_{X/S}(\mathcal{F})$ with $f(x') = s'$ such that $s' \rightsquigarrow s$ is a specialization in S. If x' specializes to a point of X_s, then $x' \rightsquigarrow x$ with $x \in Ass_{X_s}(\mathcal{F}_s)$.*

Proof. Say $x' \rightsquigarrow t$ with $t \in X_s$. Then we can find specializations $x' \rightsquigarrow x \rightsquigarrow t$ with x corresponding to a generic point of an irreducible component of $\overline{\{x'\}} \cap f^{-1}(\{s\})$. By assumption $\mathcal{F}$ is flat over S at x. By More on Morphisms, Lemma 18.3 we see that $x \in \mathrm{Ass}_{X/S}(\mathcal{F})$ as desired. □

05L4 **Lemma 18.2.** *Let $f : X \to S$ be a morphism of schemes of finite type. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module of finite type. Let $s \in S$. Let $(S', s') \to (S, s)$ be an elementary étale neighbourhood and let*

$$\begin{array}{ccc} X & \xleftarrow{g} & X' \\ \downarrow & & \downarrow \\ S & \longleftarrow & S' \end{array}$$

be a commutative diagram of morphisms of schemes. Assume

(1) $\mathcal{F}$ *is flat over* S *at all points of* X_s,
(2) $X' \to S'$ *is of finite type,*
(3) $g^*\mathcal{F}$ *is pure along* $X'_{s'}$,
(4) $g : X' \to X$ *is étale, and*
(5) $g(X')$ *contains* $Ass_{X_s}(\mathcal{F}_s)$.

In this situation $\mathcal{F}$ *is pure along* X_s *if and only if the image of* $X' \to X \times_S S'$ *contains the points of* $Ass_{X \times_S S'/S'}(\mathcal{F} \times_S S')$ *lying over points in* S' *which specialize to* s'.

Proof. Since the morphism $S' \to S$ is étale, we see that if $\mathcal{F}$ is pure along X_s, then $\mathcal{F} \times_S S'$ is pure along X_s, see Lemma 16.4. Since purity satisfies flat descent, see Lemma 16.6, we see that if $\mathcal{F} \times_S S'$ is pure along $X_{s'}$, then $\mathcal{F}$ is pure along X_s. Hence we may replace S by S' and assume that $S = S'$ so that $g : X' \to X$ is an étale morphism between schemes of finite type over S. Moreover, we may replace S by $\mathrm{Spec}(\mathcal{O}_{S,s})$ and assume that S is local.

First, assume that $\mathcal{F}$ is pure along X_s. In this case every point of $\mathrm{Ass}_{X/S}(\mathcal{F})$ specializes to a point of X_s by purity. Hence by Lemma 18.1 we see that every point of $\mathrm{Ass}_{X/S}(\mathcal{F})$ specializes to a point of $\mathrm{Ass}_{X_s}(\mathcal{F}_s)$. Thus every point of $\mathrm{Ass}_{X/S}(\mathcal{F})$ is in the image of g (as the image is open and contains $\mathrm{Ass}_{X_s}(\mathcal{F}_s)$).

Conversely, assume that $g(X')$ contains $\mathrm{Ass}_{X/S}(\mathcal{F})$. Let $S^h = \mathrm{Spec}(\mathcal{O}^h_{S,s})$ be the henselization of S at s. Denote $g^h : (X')^h \to X^h$ the base change of g by $S^h \to S$, and denote $\mathcal{F}^h$ the pullback of $\mathcal{F}$ to X^h. By Divisors, Lemma 7.3 and Remark 7.4 the relative assassin $\mathrm{Ass}_{X^h/S^h}(\mathcal{F}^h)$ is the inverse image of $\mathrm{Ass}_{X/S}(\mathcal{F})$ via the projection $X^h \to X$. As we have assumed that $g(X')$ contains $\mathrm{Ass}_{X/S}(\mathcal{F})$ we conclude that the base change $g^h((X')^h) = g(X') \times_S S^h$ contains $\mathrm{Ass}_{X^h/S^h}(\mathcal{F}^h)$. In this way we reduce to the case where S is the spectrum of a henselian local ring. Let $x \in \mathrm{Ass}_{X/S}(\mathcal{F})$. To finish the proof of the lemma we have to show that x specializes to a point of X_s, see criterion (4) for purity in discussion following Definition 16.1. By assumption there exists a $x' \in X'$ such that $g(x') = x$. As $g : X' \to X$ is étale, we see that $x' \in \mathrm{Ass}_{X'/S}(g^*\mathcal{F})$, see Lemma 2.8 (applied to the morphism of fibres $X'_w \to X_w$ where $w \in S$ is the image of x'). Since $g^*\mathcal{F}$ is pure along X'_s we see that $x' \rightsquigarrow y$ for some $y \in X'_s$. Hence $x = g(x') \rightsquigarrow g(y)$ and $g(y) \in X_s$ as desired. □

05L5 **Lemma 18.3.** *Let* $f : X \to S$ *be a morphism of schemes. Let* $\mathcal{F}$ *be a quasi-coherent* $\mathcal{O}_X$*-module. Let* $s \in S$. *Assume*

(1) f *is of finite type,*
(2) $\mathcal{F}$ *is of finite type,*
(3) $\mathcal{F}$ *is flat over* S *at all points of* X_s, *and*
(4) $\mathcal{F}$ *is pure along* X_s.

Then $\mathcal{F}$ *is universally pure along* X_s.

Proof. We first make a preliminary remark. Suppose that $(S', s') \to (S, s)$ is an elementary étale neighbourhood. Denote $\mathcal{F}'$ the pullback of $\mathcal{F}$ to $X' = X \times_S S'$. By the discussion following Definition 16.1 we see that $\mathcal{F}'$ is pure along $X'_{s'}$. Moreover, $\mathcal{F}'$ is flat over S' along $X'_{s'}$. Then it suffices to prove that $\mathcal{F}'$ is universally pure along $X'_{s'}$. Namely, given any morphism $(T, t) \to (S, s)$ of pointed schemes the fibre product $(T', t') = (T \times_S S', (t, s'))$ is flat over (T, t) and hence if $\mathcal{F}_{T'}$ is pure

along $X_{t'}$ then $\mathcal{F}_T$ is pure along X_t by Lemma 16.6. Thus during the proof we may always replace (s, S) by an elementary étale neighbourhood. We may also replace S by $\mathrm{Spec}(\mathcal{O}_{S,s})$ due to the local nature of the problem.

Choose an elementary étale neighbourhood $(S', s') \to (S, s)$ and a commutative diagram

$$\begin{array}{ccc} X & \xleftarrow{g} & X' \\ \downarrow & & \downarrow \\ S & \longleftarrow & \mathrm{Spec}(\mathcal{O}_{S',s'}) \end{array}$$

such that $X' \to X \times_S \mathrm{Spec}(\mathcal{O}_{S',s'})$ is étale, $X_s = g((X')_{s'})$, the scheme X' is affine, and such that $\Gamma(X', g^*\mathcal{F})$ is a free $\mathcal{O}_{S',s'}$-module, see Lemma 12.11. Note that $X' \to \mathrm{Spec}(\mathcal{O}_{S',s'})$ is of finite type (as a quasi-compact morphism which is the composition of an étale morphism and the base change of a finite type morphism). By our preliminary remarks in the first paragraph of the proof we may replace S by $\mathrm{Spec}(\mathcal{O}_{S',s'})$. Hence we may assume there exists a commutative diagram

$$\begin{array}{ccc} X & \xleftarrow{g} & X' \\ & \searrow \quad \swarrow & \\ & S & \end{array}$$

of schemes of finite type over S, where g is étale, $X_s \subset g(X')$, with S local with closed point s, with X' affine, and with $\Gamma(X', g^*\mathcal{F})$ a free $\Gamma(S, \mathcal{O}_S)$-module. Note that in this case $g^*\mathcal{F}$ is universally pure over S, see Lemma 17.4.

In this situation we apply Lemma 18.2 to deduce that $\mathrm{Ass}_{X/S}(\mathcal{F}) \subset g(X')$ from our assumption that $\mathcal{F}$ is pure along X_s and flat over S along X_s. By Divisors, Lemma 7.3 and Remark 7.4 we see that for any morphism of pointed schemes $(T, t) \to (S, s)$ we have

$$\mathrm{Ass}_{X_T/T}(\mathcal{F}_T) \subset (X_T \to X)^{-1}(\mathrm{Ass}_{X/S}(\mathcal{F})) \subset g(X') \times_S T = g_T(X'_T).$$

Hence by Lemma 18.2 applied to the base change of our displayed diagram to (T, t) we conclude that $\mathcal{F}_T$ is pure along X_t as desired. □

05L6 **Lemma 18.4.** *Let $f : X \to S$ be a finite type morphism of schemes. Let $\mathcal{F}$ be a finite type quasi-coherent $\mathcal{O}_X$-module. Assume $\mathcal{F}$ is flat over S. In this case $\mathcal{F}$ is pure relative to S if and only if $\mathcal{F}$ is universally pure relative to S.*

Proof. Immediate consequence of Lemma 18.3 and the definitions. □

05MA **Lemma 18.5.** *Let I be a directed set. Let $(S_i, g_{ii'})$ be an inverse system of affine schemes over I. Set $S = \lim_i S_i$ and $s \in S$. Denote $g_i : S \to S_i$ the projections and set $s_i = g_i(s)$. Suppose that $f : X \to S$ is a morphism of finite presentation, $\mathcal{F}$ a quasi-coherent $\mathcal{O}_X$-module of finite presentation which is pure along X_s and flat over S at all points of X_s. Then there exists an $i \in I$, a morphism of finite presentation $X_i \to S_i$, a quasi-coherent $\mathcal{O}_{X_i}$-module $\mathcal{F}_i$ of finite presentation which is pure along $(X_i)_{s_i}$ and flat over S_i at all points of $(X_i)_{s_i}$ such that $X \cong X_i \times_{S_i} S$ and such that the pullback of $\mathcal{F}_i$ to X is isomorphic to $\mathcal{F}$.*

Proof. Let $U \subset X$ be the set of points where $\mathcal{F}$ is flat over S. By More on Morphisms, Theorem 15.1 this is an open subscheme of X. By assumption $X_s \subset U$.

As X_s is quasi-compact, we can find a quasi-compact open $U' \subset U$ with $X_s \subset U'$. By Limits, Lemma 10.1 we can find an $i \in I$ and a morphism of finite presentation $f_i : X_i \to S_i$ whose base change to S is isomorphic to f_i. Fix such a choice and set $X_{i'} = X_i \times_{S_i} S_{i'}$. Then $X = \lim_{i'} X_{i'}$ with affine transition morphisms. By Limits, Lemma 10.2 we can, after possible increasing i assume there exists a quasi-coherent $\mathcal{O}_{X_i}$-module $\mathcal{F}_i$ of finite presentation whose base change to S is isomorphic to $\mathcal{F}$. By Limits, Lemma 4.11 after possibly increasing i we may assume there exists an open $U_i' \subset X_i$ whose inverse image in X is U'. Note that in particular $(X_i)_{s_i} \subset U_i'$. By Limits, Lemma 10.4 (after increasing i once more) we may assume that $\mathcal{F}_i$ is flat on U_i'. In particular we see that $\mathcal{F}_i$ is flat along $(X_i)_{s_i}$.

Next, we use Lemma 12.5 to choose an elementary étale neighbourhood $(S_i', s_i') \to (S_i, s_i)$ and a commutative diagram of schemes

$$\begin{array}{ccc} X_i & \xleftarrow{g_i} & X_i' \\ \downarrow & & \downarrow \\ S_i & \longleftarrow & S_i' \end{array}$$

such that g_i is étale, $(X_i)_{s_i} \subset g_i(X_i')$, the schemes X_i', S_i' are affine, and such that $\Gamma(X_i', g_i^*\mathcal{F}_i)$ is a projective $\Gamma(S_i', \mathcal{O}_{S_i'})$-module. Note that $g_i^*\mathcal{F}_i$ is universally pure over S_i', see Lemma 17.4. We may base change the diagram above to a diagram with morphisms $(S_{i'}', s_{i'}') \to (S_{i'}, s_{i'})$ and $g_{i'} : X_{i'}' \to X_{i'}$ over $S_{i'}$ for any $i' \geq i$ and we may base change the diagram to a diagram with morphisms $(S', s') \to (S, s)$ and $g : X' \to X$ over S.

At this point we can use our criterion for purity. Set $W_i' \subset X_i \times_{S_i} S_i'$ equal to the image of the étale morphism $X_i' \to X_i \times_{S_i} S_i'$. For every $i' \geq i$ we have similarly the image $W_{i'}' \subset X_{i'} \times_{S_{i'}} S_{i'}'$ and we have the image $W' \subset X \times_S S'$. Taking images commutes with base change, hence $W_{i'}' = W_i' \times_{S_i'} S_{i'}'$ and $W' = W_i \times_{S_i'} S'$. Because $\mathcal{F}$ is pure along X_s the Lemma 18.2 implies that

05MB (18.5.1) $$f^{-1}(\mathrm{Spec}(\mathcal{O}_{S',s'})) \cap \mathrm{Ass}_{X \times_S S'/S'}(\mathcal{F} \times_S S') \subset W'$$

By More on Morphisms, Lemma 25.5 we see that

$$E = \{t \in S' \mid \mathrm{Ass}_{X_t}(\mathcal{F}_t) \subset W'\} \quad \text{and} \quad E_{i'} = \{t \in S_{i'}' \mid \mathrm{Ass}_{X_t}(\mathcal{F}_{i',t}) \subset W_{i'}'\}$$

are locally constructible subsets of S' and $S_{i'}'$. By More on Morphisms, Lemma 25.4 we see that $E_{i'}$ is the inverse image of E_i under the morphism $S_{i'}' \to S_i'$ and that E is the inverse image of E_i under the morphism $S' \to S_i'$. Thus Equation (18.5.1) is equivalent to the assertion that $\mathrm{Spec}(\mathcal{O}_{S',s'})$ maps into E_i. As $\mathcal{O}_{S',s'} = \mathrm{colim}_{i' \geq i} \mathcal{O}_{S_{i'}', s_{i'}'}$ we see that $\mathrm{Spec}(\mathcal{O}_{S_{i'}', s_{i'}'})$ maps into E_i for some $i' \geq i$, see Limits, Lemma 4.10. Then, applying Lemma 18.2 to the situation over $S_{i'}$, we conclude that $\mathcal{F}_{i'}$ is pure along $(X_{i'})_{s_{i'}}$. □

05MC **Lemma 18.6.** *Let $f : X \to S$ be a morphism of finite presentation. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module of finite presentation flat over S. Then the set*

$$U = \{s \in S \mid \mathcal{F} \text{ is pure along } X_s\}$$

is open in S.

Proof. Let $s \in U$. Using Lemma 12.5 we can find an elementary étale neighbourhood $(S', s') \to (S, s)$ and a commutative diagram

$$\begin{array}{ccc} X & \xleftarrow{g} & X' \\ \downarrow & & \downarrow \\ S & \longleftarrow & S' \end{array}$$

such that g is étale, $X_s \subset g(X')$, the schemes X', S' are affine, and such that $\Gamma(X', g^*\mathcal{F})$ is a projective $\Gamma(S', \mathcal{O}_{S'})$-module. Note that $g^*\mathcal{F}$ is universally pure over S', see Lemma 17.4. Set $W' \subset X \times_S S'$ equal to the image of the étale morphism $X' \to X \times_S S'$. Note that W is open and quasi-compact over S'. Set

$$E = \{t \in S' \mid \mathrm{Ass}_{X_t}(\mathcal{F}_t) \subset W'\}.$$

By More on Morphisms, Lemma 25.5 E is a constructible subset of S'. By Lemma 18.2 we see that $\mathrm{Spec}(\mathcal{O}_{S',s'}) \subset E$. By Morphisms, Lemma 23.4 we see that E contains an open neighbourhood V' of s'. Applying Lemma 18.2 once more we see that for any point s_1 in the image of V' in S the sheaf $\mathcal{F}$ is pure along X_{s_1}. Since $S' \to S$ is étale the image of V' in S is open and we win. □

19. How purity is used

05L7 Here are some examples of how purity can be used. The first lemma actually uses a slightly weaker form of purity.

05L8 **Lemma 19.1.** *Let $f : X \to S$ be a morphism of finite type. Let $\mathcal{F}$ be a quasi-coherent sheaf of finite type on X. Assume S is local with closed point s. Assume $\mathcal{F}$ is pure along X_s and that $\mathcal{F}$ is flat over S. Let $\varphi : \mathcal{F} \to \mathcal{G}$ of quasi-coherent $\mathcal{O}_X$-modules. Then the following are equivalent*

1. *the map on stalks φ_x is injective for all $x \in Ass_{X_s}(\mathcal{F}_s)$, and*
2. *φ is injective.*

Proof. Let $\mathcal{K} = \mathrm{Ker}(\varphi)$. Our goal is to prove that $\mathcal{K} = 0$. In order to do this it suffices to prove that $\mathrm{WeakAss}_X(\mathcal{K}) = \emptyset$, see Divisors, Lemma 5.5. We have $\mathrm{WeakAss}_X(\mathcal{K}) \subset \mathrm{WeakAss}_X(\mathcal{F})$, see Divisors, Lemma 5.4. As $\mathcal{F}$ is flat we see from Lemma 13.5 that $\mathrm{WeakAss}_X(\mathcal{F}) \subset \mathrm{Ass}_{X/S}(\mathcal{F})$. By purity any point x' of $\mathrm{Ass}_{X/S}(\mathcal{F})$ is a generalization of a point of X_s, and hence is the specialization of a point $x \in \mathrm{Ass}_{X_s}(\mathcal{F}_s)$, by Lemma 18.1. Hence the injectivity of φ_x implies the injectivity of $\varphi_{x'}$, whence $\mathcal{K}_{x'} = 0$. □

05MD **Proposition 19.2.** *Let $f : X \to S$ be an affine, finitely presented morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module of finite presentation, flat over S. Then the following are equivalent*

1. *$f_*\mathcal{F}$ is locally projective on S, and*
2. *$\mathcal{F}$ is pure relative to S.*

In particular, given a ring map $A \to B$ of finite presentation and a finitely presented B-module N flat over A we have: N is projective as an A-module if and only if $\widetilde{N}$ on $\mathrm{Spec}(B)$ is pure relative to $\mathrm{Spec}(A)$.

Proof. The implication (1) ⇒ (2) is Lemma 17.4. Assume $\mathcal{F}$ is pure relative to S. Note that by Lemma 18.3 this implies $\mathcal{F}$ remains pure after any base change. By Descent, Lemma 7.7 it suffices to prove $f_*\mathcal{F}$ is fpqc locally projective on S.

Pick $s \in S$. We will prove that the restriction of $f_*\mathcal{F}$ to an étale neighbourhood of s is locally projective. Namely, by Lemma 12.5, after replacing S by an affine elementary étale neighbourhood of s, we may assume there exists a diagram

$$\begin{array}{ccc} X & \xleftarrow{g} & X' \\ & \searrow \quad \swarrow & \\ & S & \end{array}$$

of schemes affine and of finite presentation over S, where g is étale, $X_s \subset g(X')$, and with $\Gamma(X', g^*\mathcal{F})$ a projective $\Gamma(S, \mathcal{O}_S)$-module. Note that in this case $g^*\mathcal{F}$ is universally pure over S, see Lemma 17.4. Hence by Lemma 18.2 we see that the open $g(X')$ contains the points of $\mathrm{Ass}_{X/S}(\mathcal{F})$ lying over $\mathrm{Spec}(\mathcal{O}_{S,s})$. Set

$$E = \{t \in S \mid \mathrm{Ass}_{X_t}(\mathcal{F}_t) \subset g(X')\}.$$

By More on Morphisms, Lemma 25.5 E is a constructible subset of S. We have seen that $\mathrm{Spec}(\mathcal{O}_{S,s}) \subset E$. By Morphisms, Lemma 23.4 we see that E contains an open neighbourhood of s. Hence after replacing S by an affine neighbourhood of s we may assume that $\mathrm{Ass}_{X/S}(\mathcal{F}) \subset g(X')$. By Lemma 7.4 this means that

$$\Gamma(X, \mathcal{F}) \longrightarrow \Gamma(X', g^*\mathcal{F})$$

is $\Gamma(S, \mathcal{O}_S)$-universally injective. By Algebra, Lemma 89.7 we conclude that $\Gamma(X, \mathcal{F})$ is Mittag-Leffler as an $\Gamma(S, \mathcal{O}_S)$-module. Since $\Gamma(X, \mathcal{F})$ is countably generated and flat as a $\Gamma(S, \mathcal{O}_S)$-module, we conclude it is projective by Algebra, Lemma 93.1. □

We can use the proposition to improve some of our earlier results. The following lemma is an improvement of Proposition 12.4.

05ME **Lemma 19.3.** *Let $f : X \to S$ be a morphism which is locally of finite presentation. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module which is of finite presentation. Let $x \in X$ with $s = f(x) \in S$. If $\mathcal{F}$ is flat at x over S there exists an affine elementary étale neighbourhood $(S', s') \to (S, s)$ and an affine open $U' \subset X \times_S S'$ which contains $x' = (x, s')$ such that $\Gamma(U', \mathcal{F}|_{U'})$ is a projective $\Gamma(S', \mathcal{O}_{S'})$-module.*

Proof. During the proof we may replace X by an open neighbourhood of x and we may replace S by an elementary étale neighbourhood of s. Hence, by openness of flatness (see More on Morphisms, Theorem 15.1) we may assume that $\mathcal{F}$ is flat over S. We may assume S and X are affine. After shrinking X some more we may assume that any point of $\mathrm{Ass}_{X_s}(\mathcal{F}_s)$ is a generalization of x. This property is preserved on replacing (S, s) by an elementary étale neighbourhood. Hence we may apply Lemma 12.5 to arrive at the situation where there exists a diagram

$$\begin{array}{ccc} X & \xleftarrow{g} & X' \\ & \searrow \quad \swarrow & \\ & S & \end{array}$$

of schemes affine and of finite presentation over S, where g is étale, $X_s \subset g(X')$, and with $\Gamma(X', g^*\mathcal{F})$ a projective $\Gamma(S, \mathcal{O}_S)$-module. Note that in this case $g^*\mathcal{F}$ is universally pure over S, see Lemma 17.4.

Let $U \subset g(X')$ be an affine open neighbourhood of x. We claim that $\mathcal{F}|_U$ is pure along U_s. If we prove this, then the lemma follows because $\mathcal{F}|_U$ will be pure relative

to S after shrinking S, see Lemma 18.6, whereupon the projectivity follows from Proposition 19.2. To prove the claim we have to show, after replacing (S, s) by an arbitrary elementary étale neighbourhood, that any point ξ of $\mathrm{Ass}_{U/S}(\mathcal{F}|_U)$ lying over some $s' \in S$, $s' \rightsquigarrow s$ specializes to a point of U_s. Since $U \subset g(X')$ we can find a $\xi' \in X'$ with $g(\xi') = \xi$. Because $g^*\mathcal{F}$ is pure over S, using Lemma 18.1, we see there exists a specialization $\xi' \rightsquigarrow x'$ with $x' \in \mathrm{Ass}_{X'_s}(g^*\mathcal{F}_s)$. Then $g(x') \in \mathrm{Ass}_{X_s}(\mathcal{F}_s)$ (see for example Lemma 2.8 applied to the étale morphism $X'_s \to X_s$ of Noetherian schemes) and hence $g(x') \rightsquigarrow x$ by our choice of X above! Since $x \in U$ we conclude that $g(x') \in U$. Thus $\xi = g(\xi') \rightsquigarrow g(x') \in U_s$ as desired. □

The following lemma is an improvement of Lemma 12.9.

05MF **Lemma 19.4.** *Let $f : X \to S$ be a morphism which is locally of finite type. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module which is of finite type. Let $x \in X$ with $s = f(x) \in S$. If $\mathcal{F}$ is flat at x over S there exists an affine elementary étale neighbourhood $(S', s') \to (S, s)$ and an affine open $U' \subset X \times_S \mathrm{Spec}(\mathcal{O}_{S',s'})$ which contains $x' = (x, s')$ such that $\Gamma(U', \mathcal{F}|_{U'})$ is a free $\mathcal{O}_{S',s'}$-module.*

Proof. The question is Zariski local on X and S. Hence we may assume that X and S are affine. Then we can find a closed immersion $i : X \to \mathbf{A}^n_S$ over S. It is clear that it suffices to prove the lemma for the sheaf $i_*\mathcal{F}$ on $\mathbf{A}^n_S$ and the point $i(x)$. In this way we reduce to the case where $X \to S$ is of finite presentation. After replacing S by $\mathrm{Spec}(\mathcal{O}_{S',s'})$ and X by an open of $X \times_S \mathrm{Spec}(\mathcal{O}_{S',s'})$ we may assume that $\mathcal{F}$ is of finite presentation, see Proposition 10.3. In this case we may appeal to Lemma 19.3 and Algebra, Theorem 85.4 to conclude. □

05U7 **Lemma 19.5.** *Let $A \to B$ be a local ring map of local rings which is essentially of finite type. Let N be a finite B-module which is flat as an A-module. If A is henselian, then N is a filtered colimit*

$$N = \mathrm{colim}_i F_i$$

of free A-modules F_i such that all transition maps $u_i : F_i \to F_{i'}$ of the system induce injective maps $\overline{u}_i : F_i/\mathfrak{m}_A F_i \to F_{i'}/\mathfrak{m}_A F_{i'}$. Also, N is a Mittag-Leffler A-module.

Proof. We can find a morphism of finite type $X \to S = \mathrm{Spec}(A)$ and a point $x \in X$ lying over the closed point s of S and a finite type quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ such that $\mathcal{F}_x \cong N$ as an A-module. After shrinking X we may assume that each point of $\mathrm{Ass}_{X_s}(\mathcal{F}_s)$ specializes to x. By Lemma 19.4 we see that there exists a fundamental system of affine open neighbourhoods $U_i \subset X$ of x such that $\Gamma(U_i, \mathcal{F})$ is a free A-module F_i. Note that if $U_{i'} \subset U_i$, then

$$F_i/\mathfrak{m}_A F_i = \Gamma(U_{i,s}, \mathcal{F}_s) \longrightarrow \Gamma(U_{i',s}, \mathcal{F}_s) = F_{i'}/\mathfrak{m}_A F_{i'}$$

is injective because a section of the kernel would be supported at a closed subset of X_s not meeting x which is a contradiction to our choice of X above. Since the maps $F_i \to F_{i'}$ are A-universally injective (Lemma 7.5) it follows that N is Mittag-Leffler by Algebra, Lemma 89.9. □

The following lemma should be skipped if reading through for the first time.

0ASX **Lemma 19.6.** *Let $A \to B$ be a local ring map of local rings which is essentially of finite type. Let N be a finite B-module which is flat as an A-module. If A is a valuation ring, then any element of N has a content ideal $I \subset A$ (More on Algebra, Definition 25.1). Also, I is a principal ideal.*

Proof. The final statement follows from the fact that I is a finitely generated ideal by More on Algebra, Lemma 25.2 and Algebra, Lemma 50.15.

Proof of existence of I. Let $A \subset A^h$ be the henselization. Let B' be the localization of $B \otimes_A A^h$ at the maximal ideal $\mathfrak{m}_B \otimes A^h + B \otimes \mathfrak{m}_{A^h}$. Then $B \to B'$ is flat, hence faithfully flat. Let $N' = N \otimes_B B'$. Let $x \in N$ and let $x' \in N'$ be the image. We claim that for an ideal $I \subset A$ we have $x \in IN \Leftrightarrow x' \in IN'$. Namely, $N/IN \to N'/IN'$ is the tensor product of $B \to B'$ with N/IN and $B \to B'$ is universally injective by Algebra, Lemma 82.15. By More on Algebra, Lemma 125.6 and Algebra, Lemma 50.17 the map $A \to A^h$ defines an inclusion preserving bijection $I \mapsto IA^h$ on sets of ideals. We conclude that x has a content ideal in A if and only if x' has a content ideal in A^h. The assertion for $x' \in N'$ follows from Lemma 19.5 and Algebra, Lemma 89.6. □

An application is the following.

0H2T **Lemma 19.7.** *Let $X \to \mathrm{Spec}(R)$ be a proper flat morphism where R is a valuation ring. If the special fibre is reduced, then X and every fibre of $X \to \mathrm{Spec}(R)$ is reduced.*

Proof. Assume the special fibre X_s is reduced. Let $x \in X$ be any point, and let us show that $\mathcal{O}_{X,x}$ is reduced; this will prove that X is reduced. Let $x \rightsquigarrow x'$ be a specialization with x' in the special fibre; such a specialization exists as a proper morphism is closed. Consider the local ring $A = \mathcal{O}_{X,x'}$. Then $\mathcal{O}_{X,x}$ is a localization of A, so it suffices to show that A is reduced. Let $a \in A$ be nonzero and let $I = (\pi) \subset R$ be its content ideal, see Lemma 19.6. Then $a = \pi a'$ and a' maps to a nonzero element of $A/\mathfrak{m}A$ where $\mathfrak{m} \subset R$ is the maximal ideal. If a is nilpotent, so is a', because π is a nonzerodivisor by flatness of A over R. But a' maps to a nonzero element of the reduced ring $A/\mathfrak{m}A = \mathcal{O}_{X_s,x'}$. This is a contradiction unless A is reduced, which is what we wanted to show.

Of course, if X is reduced, so is the generic fibre of X over R. If $\mathfrak{p} \subset R$ is a prime ideal, then $R/\mathfrak{p}$ is a valuation ring by Algebra, Lemma 50.9. Hence redoing the argument with the base change of X to $R/\mathfrak{p}$ proves the fibre over $\mathfrak{p}$ is reduced. □

20. Flattening functors

05MG Let S be a scheme. Recall that a functor $F : (Sch/S)^{opp} \to Sets$ is called limit preserving if for every directed inverse system $\{T_i\}_{i \in I}$ of affine schemes with limit T we have $F(T) = \mathrm{colim}_i F(T_i)$.

05MH **Situation 20.1.** Let $f : X \to S$ be a morphism of schemes. Let $u : \mathcal{F} \to \mathcal{G}$ be a homomorphism of quasi-coherent $\mathcal{O}_X$-modules. For any scheme T over S we will denote $u_T : \mathcal{F}_T \to \mathcal{G}_T$ the base change of u to T, in other words, u_T is the pullback of u via the projection morphism $X_T = X \times_S T \to X$. In this situation we can consider the functor

05MI (20.1.1) $$F_{iso} : (Sch/S)^{opp} \longrightarrow Sets, \quad T \longrightarrow \begin{cases} \{*\} & \text{if} \quad u_T \text{ is an isomorphism,} \\ \emptyset & \text{else.} \end{cases}$$

There are variants F_{inj}, F_{surj}, F_{zero} where we ask that u_T is injective, surjective, or zero.

05MJ **Lemma 20.2.** *In Situation 20.1.*

(1) *Each of the functors F_{iso}, F_{inj}, F_{surj}, F_{zero} satisfies the sheaf property for the fpqc topology.*
(2) *If f is quasi-compact and $\mathcal{G}$ is of finite type, then F_{surj} is limit preserving.*
(3) *If f is quasi-compact and $\mathcal{F}$ of finite type, then F_{zero} is limit preserving.*
(4) *If f is quasi-compact, $\mathcal{F}$ is of finite type, and $\mathcal{G}$ is of finite presentation, then F_{iso} is limit preserving.*

Proof. Let $\{T_i \to T\}_{i\in I}$ be an fpqc covering of schemes over S. Set $X_i = X_{T_i} = X \times_S T_i$ and $u_i = u_{T_i}$. Note that $\{X_i \to X_T\}_{i\in I}$ is an fpqc covering of X_T, see Topologies, Lemma 9.8. In particular, for every $x \in X_T$ there exists an $i \in I$ and an $x_i \in X_i$ mapping to x. Since $\mathcal{O}_{X_T,x} \to \mathcal{O}_{X_i,x_i}$ is flat, hence faithfully flat (see Algebra, Lemma 39.17) we conclude that $(u_i)_{x_i}$ is injective, surjective, bijective, or zero if and only if $(u_T)_x$ is injective, surjective, bijective, or zero. Whence part (1) of the lemma.

Proof of (2). Assume f quasi-compact and $\mathcal{G}$ of finite type. Let $T = \lim_{i\in I} T_i$ be a directed limit of affine S-schemes and assume that u_T is surjective. Set $X_i = X_{T_i} = X \times_S T_i$ and $u_i = u_{T_i} : \mathcal{F}_i = \mathcal{F}_{T_i} \to \mathcal{G}_i = \mathcal{G}_{T_i}$. To prove part (2) we have to show that u_i is surjective for some i. Pick $i_0 \in I$ and replace I by $\{i \mid i \geq i_0\}$. Since f is quasi-compact the scheme X_{i_0} is quasi-compact. Hence we may choose affine opens $W_1, \ldots, W_m \subset X$ and an affine open covering $X_{i_0} = U_{1,i_0} \cup \ldots \cup U_{m,i_0}$ such that U_{j,i_0} maps into W_j under the projection morphism $X_{i_0} \to X$. For any $i \in I$ let $U_{j,i}$ be the inverse image of U_{j,i_0}. Setting $U_j = \lim_i U_{j,i}$ we see that $X_T = U_1 \cup \ldots \cup U_m$ is an affine open covering of X_T. Now it suffices to show, for a given $j \in \{1, \ldots, m\}$ that $u_i|_{U_{j,i}}$ is surjective for some $i = i(j) \in I$. Using Properties, Lemma 17.1 this translates into the following algebra problem: Let A be a ring and let $u : M \to N$ be an A-module map. Suppose that $R = \operatorname{colim}_{i\in I} R_i$ is a directed colimit of A-algebras. If N is a finite A-module and if $u \otimes 1 : M \otimes_A R \to N \otimes_A R$ is surjective, then for some i the map $u \otimes 1 : M \otimes_A R_i \to N \otimes_A R_i$ is surjective. This is Algebra, Lemma 127.5 part (2).

Proof of (3). Exactly the same arguments as given in the proof of (2) reduces this to the following algebra problem: Let A be a ring and let $u : M \to N$ be an A-module map. Suppose that $R = \operatorname{colim}_{i\in I} R_i$ is a directed colimit of A-algebras. If M is a finite A-module and if $u \otimes 1 : M \otimes_A R \to N \otimes_A R$ is zero, then for some i the map $u \otimes 1 : M \otimes_A R_i \to N \otimes_A R_i$ is zero. This is Algebra, Lemma 127.5 part (1).

Proof of (4). Assume f quasi-compact and $\mathcal{F}, \mathcal{G}$ of finite presentation. Arguing in exactly the same manner as in the previous paragraph (using in addition also Properties, Lemma 17.2) part (3) translates into the following algebra statement: Let A be a ring and let $u : M \to N$ be an A-module map. Suppose that $R = \operatorname{colim}_{i\in I} R_i$ is a directed colimit of A-algebras. Assume M is a finite A-module, N is a finitely presented A-module, and $u \otimes 1 : M \otimes_A R \to N \otimes_A R$ is an isomorphism. Then for some i the map $u \otimes 1 : M \otimes_A R_i \to N \otimes_A R_i$ is an isomorphism. This is Algebra, Lemma 127.5 part (3). □

05MK **Situation 20.3.** Let $(A, \mathfrak{m}_A)$ be a local ring. Denote $\mathcal{C}$ the category whose objects are A-algebras A' which are local rings such that the algebra structure $A \to A'$ is a local homomorphism of local rings. A morphism between objects A', A'' of $\mathcal{C}$ is a local homomorphism $A' \to A''$ of A-algebras. Let $A \to B$ be a local ring map of local rings and let M be a B-module. If A' is an object of $\mathcal{C}$ we set $B' = B \otimes_A A'$ and

we set $M' = M \otimes_A A'$ as a B'-module. Given $A' \in \mathrm{Ob}(\mathcal{C})$, consider the condition

05ML (20.3.1) $$\forall \mathfrak{q} \in V(\mathfrak{m}_{A'}B' + \mathfrak{m}_B B') \subset \mathrm{Spec}(B') : M'_{\mathfrak{q}} \text{ is flat over } A'.$$

Note the similarity with More on Algebra, Equation (19.1.1). In particular, if $A' \to A''$ is a morphism of $\mathcal{C}$ and (20.3.1) holds for A', then it holds for A'', see More on Algebra, Lemma 19.2. Hence we obtain a functor

05MM (20.3.2) $$F_{lf} : \mathcal{C} \longrightarrow \mathit{Sets}, \quad A' \longrightarrow \begin{cases} \{*\} & \text{if (20.3.1) holds,} \\ \emptyset & \text{else.} \end{cases}$$

05MN **Lemma 20.4.** *In Situation 20.3.*

(1) *If $A' \to A''$ is a flat morphism in $\mathcal{C}$ then $F_{lf}(A') = F_{lf}(A'')$.*
(2) *If $A \to B$ is essentially of finite presentation and M is a B-module of finite presentation, then F_{lf} is limit preserving: If $\{A_i\}_{i \in I}$ is a directed system of objects of $\mathcal{C}$, then $F_{lf}(\mathrm{colim}_i A_i) = \mathrm{colim}_i F_{lf}(A_i)$.*

Proof. Part (1) is a special case of More on Algebra, Lemma 19.3. Part (2) is a special case of More on Algebra, Lemma 19.4. □

05P4 **Lemma 20.5.** *In Situation 20.3. Let $B \to C$ is a local map of local A-algebras and N a C-module. Denote $F'_{lf} : \mathcal{C} \to \mathit{Sets}$ the functor associated to the pair (C, N). If $M \cong N$ as B-modules and $B \to C$ is finite, then $F_{lf} = F'_{lf}$.*

Proof. Let A' be an object of $\mathcal{C}$. Set $C' = C \otimes_A A'$ and $N' = N \otimes_A A'$ similarly to the definitions of B', M' in Situation 20.3. Note that $M' \cong N'$ as B'-modules. The assumption that $B \to C$ is finite has two consequences: (a) $\mathfrak{m}_C = \sqrt{\mathfrak{m}_B C}$ and (b) $B' \to C'$ is finite. Consequence (a) implies that

$$V(\mathfrak{m}_{A'}C' + \mathfrak{m}_C C') = (\mathrm{Spec}(C') \to \mathrm{Spec}(B'))^{-1} V(\mathfrak{m}_{A'}B' + \mathfrak{m}_B B').$$

Suppose $\mathfrak{q} \subset V(\mathfrak{m}_{A'}B' + \mathfrak{m}_B B')$. Then $M'_{\mathfrak{q}}$ is flat over A' if and only if the $C'_{\mathfrak{q}}$-module $N'_{\mathfrak{q}}$ is flat over A' (because these are isomorphic as A'-modules) if and only if for every maximal ideal $\mathfrak{r}$ of $C'_{\mathfrak{q}}$ the module $N'_{\mathfrak{r}}$ is flat over A' (see Algebra, Lemma 39.18). As $B'_{\mathfrak{q}} \to C'_{\mathfrak{q}}$ is finite by (b), the maximal ideals of $C'_{\mathfrak{q}}$ correspond exactly to the primes of C' lying over $\mathfrak{q}$ (see Algebra, Lemma 36.22) and these primes are all contained in $V(\mathfrak{m}_{A'}C' + \mathfrak{m}_C C')$ by the displayed equation above. Thus the result of the lemma holds. □

05P5 **Lemma 20.6.** *In Situation 20.3 suppose that $B \to C$ is a flat local homomorphism of local rings. Set $N = M \otimes_B C$. Denote $F'_{lf} : \mathcal{C} \to \mathit{Sets}$ the functor associated to the pair (C, N). Then $F_{lf} = F'_{lf}$.*

Proof. Let A' be an object of $\mathcal{C}$. Set $C' = C \otimes_A A'$ and $N' = N \otimes_A A' = M' \otimes_{B'} C'$ similarly to the definitions of B', M' in Situation 20.3. Note that

$$V(\mathfrak{m}_{A'}B' + \mathfrak{m}_B B') = \mathrm{Spec}(\kappa(\mathfrak{m}_B) \otimes_A \kappa(\mathfrak{m}_{A'}))$$

and similarly for $V(\mathfrak{m}_{A'}C' + \mathfrak{m}_C C')$. The ring map

$$\kappa(\mathfrak{m}_B) \otimes_A \kappa(\mathfrak{m}_{A'}) \longrightarrow \kappa(\mathfrak{m}_C) \otimes_A \kappa(\mathfrak{m}_{A'})$$

is faithfully flat, hence $V(\mathfrak{m}_{A'}C' + \mathfrak{m}_C C') \to V(\mathfrak{m}_{A'}B' + \mathfrak{m}_B B')$ is surjective. Finally, if $\mathfrak{r} \in V(\mathfrak{m}_{A'}C' + \mathfrak{m}_C C')$ maps to $\mathfrak{q} \in V(\mathfrak{m}_{A'}B' + \mathfrak{m}_B B')$, then $M'_{\mathfrak{q}}$ is flat over A' if and only if $N'_{\mathfrak{r}}$ is flat over A' because $B' \to C'$ is flat, see Algebra, Lemma 39.9. The lemma follows formally from these remarks. □

05MP **Situation 20.7.** Let $f : X \to S$ be a smooth morphism with geometrically irreducible fibres. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module of finite type. For any scheme T over S we will denote $\mathcal{F}_T$ the base change of $\mathcal{F}$ to T, in other words, $\mathcal{F}_T$ is the pullback of $\mathcal{F}$ via the projection morphism $X_T = X \times_S T \to X$. Note that $X_T \to T$ is smooth with geometrically irreducible fibres, see Morphisms, Lemma 35.5 and More on Morphisms, Lemma 27.2. Let $p \geq 0$ be an integer. Given a point $t \in T$ consider the condition

05MQ (20.7.1) $\mathcal{F}_T$ is free of rank p in a neighbourhood of ξ_t

where ξ_t is the generic point of the fibre X_t. This condition for all $t \in T$ is stable under base change, and hence we obtain a functor

05MR (20.7.2) $$H_p : (Sch/S)^{opp} \longrightarrow Sets, \quad T \longrightarrow \begin{cases} \{*\} & \text{if } \mathcal{F}_T \text{ satisfies (20.7.1) } \forall t \in T, \\ \emptyset & \text{else.} \end{cases}$$

05MS **Lemma 20.8.** *In Situation 20.7.*

(1) *The functor H_p satisfies the sheaf property for the fpqc topology.*
(2) *If $\mathcal{F}$ is of finite presentation, then functor H_p is limit preserving.*

Proof. Let $\{T_i \to T\}_{i \in I}$ be an fpqc[1] covering of schemes over S. Set $X_i = X_{T_i} = X \times_S T_i$ and denote $\mathcal{F}_i$ the pullback of $\mathcal{F}$ to X_i. Assume that $\mathcal{F}_i$ satisfies (20.7.1) for all i. Pick $t \in T$ and let $\xi_t \in X_T$ denote the generic point of X_t. We have to show that $\mathcal{F}$ is free in a neighbourhood of ξ_t. For some $i \in I$ we can find a $t_i \in T_i$ mapping to t. Let $\xi_i \in X_i$ denote the generic point of X_{t_i}, so that ξ_i maps to ξ_t. The fact that $\mathcal{F}_i$ is free of rank p in a neighbourhood of ξ_i implies that $(\mathcal{F}_i)_{x_i} \cong \mathcal{O}_{X_i,x_i}^{\oplus p}$ which implies that $\mathcal{F}_{T,\xi_t} \cong \mathcal{O}_{X_T,\xi_t}^{\oplus p}$ as $\mathcal{O}_{X_T,\xi_t} \to \mathcal{O}_{X_i,x_i}$ is flat, see for example Algebra, Lemma 78.6. Thus there exists an affine neighbourhood U of ξ_t in X_T and a surjection $\mathcal{O}_U^{\oplus p} \to \mathcal{F}_U = \mathcal{F}_T|_U$, see Modules, Lemma 9.4. After shrinking T we may assume that $U \to T$ is surjective. Hence $U \to T$ is a smooth morphism of affines with geometrically irreducible fibres. Moreover, for every $t' \in T$ we see that the induced map

$$\alpha : \mathcal{O}_{U,\xi_{t'}}^{\oplus p} \longrightarrow \mathcal{F}_{U,\xi_{t'}}$$

is an isomorphism (since by the same argument as before the module on the right is free of rank p). It follows from Lemma 10.1 that

$$\Gamma(U, \mathcal{O}_U^{\oplus p}) \otimes_{\Gamma(T,\mathcal{O}_T)} \mathcal{O}_{T,t'} \longrightarrow \Gamma(U, \mathcal{F}_U) \otimes_{\Gamma(T,\mathcal{O}_T)} \mathcal{O}_{T,t'}$$

is injective for every $t' \in T$. Hence we see the surjection α is an isomorphism. This finishes the proof of (1).

Assume that $\mathcal{F}$ is of finite presentation. Let $T = \lim_{i \in I} T_i$ be a directed limit of affine S-schemes and assume that $\mathcal{F}_T$ satisfies (20.7.1). Set $X_i = X_{T_i} = X \times_S T_i$ and denote $\mathcal{F}_i$ the pullback of $\mathcal{F}$ to X_i. Let $U \subset X_T$ denote the open subscheme of points where $\mathcal{F}_T$ is flat over T, see More on Morphisms, Theorem 15.1. By assumption every generic point of every fibre is a point of U, i.e., $U \to T$ is a smooth surjective morphism with geometrically irreducible fibres. We may shrink U a bit and assume that U is quasi-compact. Using Limits, Lemma 4.11 we can find an $i \in I$ and a quasi-compact open $U_i \subset X_i$ whose inverse image in X_T is

[1] It is quite easy to show that H_p is a sheaf for the fppf topology using that flat morphisms of finite presentation are open. This is all we really need later on. But it is kind of fun to prove directly that it also satisfies the sheaf condition for the fpqc topology.

U. After increasing i we may assume that $\mathcal{F}_i|_{U_i}$ is flat over T_i, see Limits, Lemma 10.4. In particular, $\mathcal{F}_i|_{U_i}$ is finite locally free hence defines a locally constant rank function $\rho : U_i \to \{0, 1, 2, \ldots\}$. Let $(U_i)_p \subset U_i$ denote the open and closed subset where ρ has value p. Let $V_i \subset T_i$ be the image of $(U_i)_p$; note that V_i is open and quasi-compact. By assumption the image of $T \to T_i$ is contained in V_i. Hence there exists an $i' \geq i$ such that $T_{i'} \to T_i$ factors through V_i by Limits, Lemma 4.11. Then $\mathcal{F}_{i'}$ satisfies (20.7.1) as desired. Some details omitted. □

0CWF **Lemma 20.9.** *Let $f : X \to S$ be a morphism of schemes which is locally of finite type. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module of finite type. Let $n \geq 0$. The following are equivalent*

1. *for $s \in S$ the closed subset $Z \subset X_s$ of points where $\mathcal{F}$ is not flat over S (see Lemma 10.4) satisfies $\dim(Z) < n$, and*
2. *for $x \in X$ such that $\mathcal{F}$ is not flat at x over S we have $\mathrm{trdeg}_{\kappa(f(x))}(\kappa(x)) < n$.*

If this is true, then it remains true after any base change.

Proof. Let $x \in X$ be a point over $s \in S$. Then the dimension of the closure of $\{x\}$ in X_s is $\mathrm{trdeg}_{\kappa(s)}(\kappa(x))$ by Varieties, Lemma 20.3. Conversely, if $Z \subset X_s$ is a closed subset of dimension d, then there exists a point $x \in Z$ with $\mathrm{trdeg}_{\kappa(s)}(\kappa(x)) = d$ (same reference). Therefore the equivalence of (1) and (2) holds (even fibre by fibre). The statement on base change follows from Morphisms, Lemmas 26.7 and 29.3. □

0CWG **Definition 20.10.** Let $f : X \to S$ be a morphism of schemes which is locally of finite type. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module of finite type. Let $n \geq 0$. We say *$\mathcal{F}$ is flat over S in dimensions $\geq n$* if the equivalent conditions of Lemma 20.9 are satisfied.

05MT **Situation 20.11.** Let $f : X \to S$ be a morphism of schemes which is locally of finite type. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module of finite type. For any scheme T over S we will denote $\mathcal{F}_T$ the base change of $\mathcal{F}$ to T, in other words, $\mathcal{F}_T$ is the pullback of $\mathcal{F}$ via the projection morphism $X_T = X \times_S T \to X$. Note that $X_T \to T$ is of finite type and that $\mathcal{F}_T$ is an $\mathcal{O}_{X_T}$-module of finite type (Morphisms, Lemma 15.4 and Modules, Lemma 9.2). Let $n \geq 0$. By Definition 20.10 and Lemma 20.9 we obtain a functor

(20.11.1)

05MU
$$F_n : (Sch/S)^{opp} \longrightarrow Sets, \quad T \longrightarrow \begin{cases} \{*\} & \text{if } \mathcal{F}_T \text{ is flat over } T \text{ in } \dim \geq n, \\ \emptyset & \text{else.} \end{cases}$$

05MV **Lemma 20.12.** *In Situation 20.11.*

1. *The functor F_n satisfies the sheaf property for the fpqc topology.*
2. *If f is quasi-compact and locally of finite presentation and $\mathcal{F}$ is of finite presentation, then the functor F_n is limit preserving.*

Proof. Let $\{T_i \to T\}_{i \in I}$ be an fpqc covering of schemes over S. Set $X_i = X_{T_i} = X \times_S T_i$ and denote $\mathcal{F}_i$ the pullback of $\mathcal{F}$ to X_i. Assume that $\mathcal{F}_i$ is flat over T_i in dimensions $\geq n$ for all i. Let $t \in T$. Choose an index i and a point $t_i \in T_i$ mapping

to t. Consider the cartesian diagram

$$\begin{array}{ccc} X_{\mathrm{Spec}(\mathcal{O}_{T,t})} & \longleftarrow & X_{\mathrm{Spec}(\mathcal{O}_{T_i,t_i})} \\ \downarrow & & \downarrow \\ \mathrm{Spec}(\mathcal{O}_{T,t}) & \longleftarrow & \mathrm{Spec}(\mathcal{O}_{T_i,t_i}) \end{array}$$

As the lower horizontal morphism is flat we see from More on Morphisms, Lemma 15.2 that the set $Z_i \subset X_{t_i}$ where $\mathcal{F}_i$ is not flat over T_i and the set $Z \subset X_t$ where $\mathcal{F}_T$ is not flat over T are related by the rule $Z_i = Z_{\kappa(t_i)}$. Hence we see that $\mathcal{F}_T$ is flat over T in dimensions $\geq n$ by Morphisms, Lemma 29.3.

Assume that f is quasi-compact and locally of finite presentation and that $\mathcal{F}$ is of finite presentation. In this paragraph we first reduce the proof of (2) to the case where f is of finite presentation. Let $T = \lim_{i \in I} T_i$ be a directed limit of affine S-schemes and assume that $\mathcal{F}_T$ is flat in dimensions $\geq n$. Set $X_i = X_{T_i} = X \times_S T_i$ and denote $\mathcal{F}_i$ the pullback of $\mathcal{F}$ to X_i. We have to show that $\mathcal{F}_i$ is flat in dimensions $\geq n$ for some i. Pick $i_0 \in I$ and replace I by $\{i \mid i \geq i_0\}$. Since T_{i_0} is affine (hence quasi-compact) there exist finitely many affine opens $W_j \subset S$, $j = 1, \ldots, m$ and an affine open overing $T_{i_0} = \bigcup_{j=1,\ldots,m} V_{j,i_0}$ such that $T_{i_0} \to S$ maps V_{j,i_0} into W_j. For $i \geq i_0$ denote $V_{j,i}$ the inverse image of V_{j,i_0} in T_i. If we can show, for each j, that there exists an i such that $\mathcal{F}_{V_{j,i_0}}$ is flat in dimensions $\geq n$, then we win. In this way we reduce to the case that S is affine. In this case X is quasi-compact and we can choose a finite affine open covering $X = W_1 \cup \ldots \cup W_m$. In this case the result for $(X \to S, \mathcal{F})$ is equivalent to the result for $(\coprod W_j, \coprod \mathcal{F}|_{W_j})$. Hence we may assume that f is of finite presentation.

Assume f is of finite presentation and $\mathcal{F}$ is of finite presentation. Let $U \subset X_T$ denote the open subscheme of points where $\mathcal{F}_T$ is flat over T, see More on Morphisms, Theorem 15.1. By assumption the dimension of every fibre of $Z = X_T \setminus U$ over T has dimension $< n$. By Limits, Lemma 18.5 we can find a closed subscheme $Z \subset Z' \subset X_T$ such that $\dim(Z'_t) < n$ for all $t \in T$ and such that $Z' \to X_T$ is of finite presentation. By Limits, Lemmas 10.1 and 8.5 there exists an $i \in I$ and a closed subscheme $Z'_i \subset X_i$ of finite presentation whose base change to T is Z'. By Limits, Lemma 18.1 we may assume all fibres of $Z'_i \to T_i$ have dimension $< n$. By Limits, Lemma 10.4 we may assume that $\mathcal{F}_i|_{X_i \setminus T'_i}$ is flat over T_i. This implies that $\mathcal{F}_i$ is flat in dimensions $\geq n$; here we use that $Z' \to X_T$ is of finite presentation, and hence the complement $X_T \setminus Z'$ is quasi-compact! Thus part (2) is proved and the proof of the lemma is complete. □

05MW **Situation 20.13.** Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. For any scheme T over S we will denote $\mathcal{F}_T$ the base change of $\mathcal{F}$ to T, in other words, $\mathcal{F}_T$ is the pullback of $\mathcal{F}$ via the projection morphism $X_T = X \times_S T \to X$. Since the base change of a flat module is flat we obtain a functor

05MX (20.13.1) $$F_{flat} : (Sch/S)^{opp} \longrightarrow Sets, \quad T \longrightarrow \begin{cases} \{*\} & \text{if } \mathcal{F}_T \text{ is flat over } T, \\ \emptyset & \text{else.} \end{cases}$$

05MY **Lemma 20.14.** *In Situation 20.13.*

(1) *The functor F_{flat} satisfies the sheaf property for the fpqc topology.*

(2) *If f is quasi-compact and locally of finite presentation and $\mathcal{F}$ is of finite presentation, then the functor F_{flat} is limit preserving.*

Proof. Part (1) follows from the following statement: If $T' \to T$ is a surjective flat morphism of schemes over S, then $\mathcal{F}_{T'}$ is flat over T' if and only if $\mathcal{F}_T$ is flat over T, see More on Morphisms, Lemma 15.2. Part (2) follows from Limits, Lemma 10.4 after reducing to the case where X and S are affine (compare with the proof of Lemma 20.12). □

21. Flattening stratifications

052F Just the definitions. The reader looking for a "generic flatness stratification", should consult More on Morphisms, Section 54.

05P6 **Definition 21.1.** Let $X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. We say that the *universal flattening of $\mathcal{F}$ exists* if the functor F_{flat} defined in Situation 20.13 is representable by a scheme S' over S. We say that the *universal flattening of X exists* if the universal flattening of $\mathcal{O}_X$ exists.

Note that if the universal flattening S'[2] of $\mathcal{F}$ exists, then the morphism $S' \to S$ is a surjective monomorphism of schemes such that $\mathcal{F}_{S'}$ is flat over S' and such that a morphism $T \to S$ factors through S' if and only if $\mathcal{F}_T$ is flat over T.

0FJ1 **Example 21.2.** Let $X = S = \mathrm{Spec}(k[x, y])$ where k is a field. Let $\mathcal{F} = \widetilde{M}$ where $M = k[x, x^{-1}, y]/(y)$. For a $k[x, y]$-algebra A set $F_{flat}(A) = F_{flat}(\mathrm{Spec}(A))$. Then $F_{flat}(k[x, y]/(x, y)^n) = \{*\}$ for all n, while $F_{flat}(k[[x, y]]) = \emptyset$. This means that F_{flat} isn't representable (even by an algebraic space, see Formal Spaces, Lemma 33.3). Thus the universal flattening does not exist in this case.

We define (compare with Topology, Remark 28.5) a (locally finite, scheme theoretic) *stratification* of a scheme S to be given by closed subschemes $Z_i \subset S$ indexed by a partially ordered set I such that $S = \bigcup Z_i$ (set theoretically), such that every point of S has a neighbourhood meeting only a finite number of Z_i, and such that

$$Z_i \cap Z_j = \bigcup\nolimits_{k \leq i,j} Z_k.$$

Setting $S_i = Z_i \setminus \bigcup_{j<i} Z_j$ the actual stratification is the decomposition $S = \coprod S_i$ into locally closed subschemes. We often only indicate the strata S_i and leave the construction of the closed subschemes Z_i to the reader. Given a stratification we obtain a monomorphism

$$S' = \coprod\nolimits_{i \in I} S_i \longrightarrow S.$$

We will call this the *monomorphism associated to the stratification*. With this terminology we can define what it means to have a flattening stratification.

05P7 **Definition 21.3.** Let $X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. We say that $\mathcal{F}$ has a *flattening stratification* if the functor F_{flat} defined in Situation 20.13 is representable by a monomorphism $S' \to S$ associated to a stratification of S by locally closed subschemes. We say that X has a *flattening stratification* if $\mathcal{O}_X$ has a flattening stratification.

[2]The scheme S' is sometimes called the *universal flatificator*. In [GR71] it is called the *platificateur universel*. Existence of the universal flattening should not be confused with the type of results discussed in More on Algebra, Section 27.

When a flattening stratification exists, it is often important to understand the index set labeling the strata and its partial ordering. This often has to do with ranks of modules. For example if $X = S$ and $\mathcal{F}$ is a finitely presented $\mathcal{O}_S$-module, then the flattening stratification exists and is given by the Fitting ideals of $\mathcal{F}$, see Divisors, Lemma 9.7.

22. Flattening stratification over an Artinian ring

05PA A flatting stratification exists when the base scheme is the spectrum of an Artinian ring.

05PB **Lemma 22.1.** *Let S be the spectrum of an Artinian ring. For any scheme X over S, and any quasi-coherent $\mathcal{O}_X$-module there exists a universal flattening. In fact the universal flattening is given by a closed immersion $S' \to S$, and hence is a flattening stratification for $\mathcal{F}$ as well.*

Proof. Choose an affine open covering $X = \bigcup U_i$. Then F_{flat} is the product of the functors associated to each of the pairs $(U_i, \mathcal{F}|_{U_i})$. Hence it suffices to prove the result for each $(U_i, \mathcal{F}|_{U_i})$. In the affine case the lemma follows immediately from More on Algebra, Lemma 17.2. □

23. Flattening a map

05PC Theorem 23.3 is the key to further flattening statements.

05PD **Lemma 23.1.** *Let S be a scheme. Let $g : X' \to X$ be a flat morphism of schemes over S with X locally of finite type over S. Let $\mathcal{F}$ be a finite type quasi-coherent $\mathcal{O}_X$-module which is flat over S. If $Ass_{X/S}(\mathcal{F}) \subset g(X')$ then the canonical map*

$$\mathcal{F} \longrightarrow g_* g^* \mathcal{F}$$

is injective, and remains injective after any base change.

Proof. The final assertion means that $\mathcal{F}_T \to (g_T)_* g_T^* \mathcal{F}_T$ is injective for any morphism $T \to S$. The assumption $\mathrm{Ass}_{X/S}(\mathcal{F}) \subset g(X')$ is preserved by base change, see Divisors, Lemma 7.3 and Remark 7.4. The same holds for the assumption of flatness and finite type. Hence it suffices to prove the injectivity of the displayed arrow. Let $\mathcal{K} = \mathrm{Ker}(\mathcal{F} \to g_* g^* \mathcal{F})$. Our goal is to prove that $\mathcal{K} = 0$. In order to do this it suffices to prove that $\mathrm{WeakAss}_X(\mathcal{K}) = \emptyset$, see Divisors, Lemma 5.5. We have $\mathrm{WeakAss}_X(\mathcal{K}) \subset \mathrm{WeakAss}_X(\mathcal{F})$, see Divisors, Lemma 5.4. As $\mathcal{F}$ is flat we see from Lemma 13.5 that $\mathrm{WeakAss}_X(\mathcal{F}) \subset \mathrm{Ass}_{X/S}(\mathcal{F})$. By assumption any point x of $\mathrm{Ass}_{X/S}(\mathcal{F})$ is the image of some $x' \in X'$. Since g is flat the local ring map $\mathcal{O}_{X,x} \to \mathcal{O}_{X',x'}$ is faithfully flat, hence the map

$$\mathcal{F}_x \longrightarrow g^* \mathcal{F}_{x'} = \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{X',x'}$$

is injective (see Algebra, Lemma 82.15). This implies that $\mathcal{K}_x = 0$ as desired. □

05PE **Lemma 23.2.** *Let A be a ring. Let $u : M \to N$ be a surjective map of A-modules. If M is projective as an A-module, then there exists an ideal $I \subset A$ such that for any ring map $\varphi : A \to B$ the following are equivalent*

1. *$u \otimes 1 : M \otimes_A B \to N \otimes_A B$ is an isomorphism, and*
2. *$\varphi(I) = 0$.*

Proof. As M is projective we can find a projective A-module C such that $F = M \oplus C$ is a free A-module. By replacing u by $u \oplus 1 : F = M \oplus C \to N \oplus C$ we see that we may assume M is free. In this case let I be the ideal of A generated by coefficients of all the elements of $\mathrm{Ker}(u)$ with respect to some (fixed) basis of M. The reason this works is that, since u is surjective and $\otimes_A B$ is right exact, $\mathrm{Ker}(u \otimes 1)$ is the image of $\mathrm{Ker}(u) \otimes_A B$ in $M \otimes_A B$. □

05PF **Theorem 23.3.** *In Situation 20.1 assume*

(1) *f is of finite presentation,*
(2) *$\mathcal{F}$ is of finite presentation, flat over S, and pure relative to S, and*
(3) *u is surjective.*

Then F_{iso} is representable by a closed immersion $Z \to S$. Moreover $Z \to S$ is of finite presentation if $\mathcal{G}$ is of finite presentation.

Proof. We will use without further mention that $\mathcal{F}$ is universally pure over S, see Lemma 18.3. By Lemma 20.2 and Descent, Lemmas 37.2 and 39.1 the question is local for the étale topology on S. Hence it suffices to prove, given $s \in S$, that there exists an étale neighbourhood of (S, s) so that the theorem holds.

Using Lemma 12.5 and after replacing S by an elementary étale neighbourhood of s we may assume there exists a commutative diagram

$$\begin{array}{ccc} X & \xleftarrow{g} & X' \\ & \searrow \quad \swarrow & \\ & S & \end{array}$$

of schemes of finite presentation over S, where g is étale, $X_s \subset g(X')$, the schemes X' and S are affine, $\Gamma(X', g^*\mathcal{F})$ a projective $\Gamma(S, \mathcal{O}_S)$-module. Note that $g^*\mathcal{F}$ is universally pure over S, see Lemma 17.4. Hence by Lemma 18.2 we see that the open $g(X')$ contains the points of $\mathrm{Ass}_{X/S}(\mathcal{F})$ lying over $\mathrm{Spec}(\mathcal{O}_{S,s})$. Set

$$E = \{t \in S \mid \mathrm{Ass}_{X_t}(\mathcal{F}_t) \subset g(X')\}.$$

By More on Morphisms, Lemma 25.5 E is a constructible subset of S. We have seen that $\mathrm{Spec}(\mathcal{O}_{S,s}) \subset E$. By Morphisms, Lemma 23.4 we see that E contains an open neighbourhood of s. Hence after replacing S by a smaller affine neighbourhood of s we may assume that $\mathrm{Ass}_{X/S}(\mathcal{F}) \subset g(X')$.

Since we have assumed that u is surjective we have $F_{iso} = F_{inj}$. From Lemma 23.1 it follows that $u : \mathcal{F} \to \mathcal{G}$ is injective if and only if $g^*u : g^*\mathcal{F} \to g^*\mathcal{G}$ is injective, and the same remains true after any base change. Hence we have reduced to the case where, in addition to the assumptions in the theorem, $X \to S$ is a morphism of affine schemes and $\Gamma(X, \mathcal{F})$ is a projective $\Gamma(S, \mathcal{O}_S)$-module. This case follows immediately from Lemma 23.2.

To see that Z is of finite presentation if $\mathcal{G}$ is of finite presentation, combine Lemma 20.2 part (4) with Limits, Remark 6.2. □

07AI **Lemma 23.4.** *Let $f : X \to S$ be a morphism of schemes which is of finite presentation, flat, and pure. Let Y be a closed subscheme of X. Let $F = f_*Y$ be the Weil restriction functor of Y along f, defined by*

$$F : (Sch/S)^{opp} \to Sets, \quad T \mapsto \begin{cases} \{*\} & \text{if} \quad Y_T \to X_T \text{ is an isomorphism,} \\ \emptyset & \text{else.} \end{cases}$$

Then F is representable by a closed immersion $Z \to S$. Moreover $Z \to S$ is of finite presentation if $Y \to S$ is.

Proof. Let $\mathcal{I}$ be the ideal sheaf defining Y in X and let $u : \mathcal{O}_X \to \mathcal{O}_X/\mathcal{I}$ be the surjection. Then for an S-scheme T, the closed immersion $Y_T \to X_T$ is an isomorphism if and only if u_T is an isomorphism. Hence the result follows from Theorem 23.3. □

24. Flattening in the local case

05MZ In this section we start applying the earlier material to obtain a shadow of the flattening stratification.

05PG **Theorem 24.1.** *In Situation 20.3 assume A is henselian, B is essentially of finite type over A, and M is a finite B-module. Then there exists an ideal $I \subset A$ such that A/I corepresents the functor F_{lf} on the category $\mathcal{C}$. In other words given a local homomorphism of local rings $\varphi : A \to A'$ with $B' = B \otimes_A A'$ and $M' = M \otimes_A A'$ the following are equivalent:*

(1) $\forall \mathfrak{q} \in V(\mathfrak{m}_{A'}B' + \mathfrak{m}_B B') \subset \mathrm{Spec}(B') : M'_{\mathfrak{q}}$ *is flat over A', and*
(2) $\varphi(I) = 0$.

If B is essentially of finite presentation over A and M of finite presentation over B, then I is a finitely generated ideal.

Proof. Choose a finite type ring map $A \to C$ and a finite C-module N and a prime $\mathfrak{q}$ of C such that $B = C_{\mathfrak{q}}$ and $M = N_{\mathfrak{q}}$. In the following, when we say "the theorem holds for $(N/C/A, \mathfrak{q})$ we mean that it holds for $(A \to B, M)$ where $B = C_{\mathfrak{q}}$ and $M = N_{\mathfrak{q}}$. By Lemma 20.6 the functor F_{lf} is unchanged if we replace B by a local ring flat over B. Hence, since A is henselian, we may apply Lemma 6.6 and assume that there exists a complete dévissage of $N/C/A$ at $\mathfrak{q}$.

Let $(A_i, B_i, M_i, \alpha_i, \mathfrak{q}_i)_{i=1,\ldots,n}$ be such a complete dévissage of $N/C/A$ at $\mathfrak{q}$. Let $\mathfrak{q}'_i \subset A_i$ be the unique prime lying over $\mathfrak{q}_i \subset B_i$ as in Definition 6.4. Since $C \to A_1$ is surjective and $N \cong M_1$ as C-modules, we see by Lemma 20.5 it suffices to prove the theorem holds for $(M_1/A_1/A, \mathfrak{q}'_1)$. Since $B_1 \to A_1$ is finite and $\mathfrak{q}_1$ is the only prime of B_1 over $\mathfrak{q}'_1$ we see that $(A_1)_{\mathfrak{q}'_1} \to (B_1)_{\mathfrak{q}_1}$ is finite (see Algebra, Lemma 41.11 or More on Morphisms, Lemma 47.4). Hence by Lemma 20.5 it suffices to prove the theorem holds for $(M_1/B_1/A, \mathfrak{q}_1)$.

At this point we may assume, by induction on the length n of the dévissage, that the theorem holds for $(M_2/B_2/A, \mathfrak{q}_2)$. (If $n = 1$, then $M_2 = 0$ which is flat over A.) Reversing the last couple of steps of the previous paragraph, using that $M_2 \cong \mathrm{Coker}(\alpha_2)$ as B_1-modules, we see that the theorem holds for $(\mathrm{Coker}(\alpha_1)/B_1/A, \mathfrak{q}_1)$.

Let A' be an object of $\mathcal{C}$. At this point we use Lemma 10.1 to see that if $(M_1 \otimes_A A')_{\mathfrak{q}'}$ is flat over A' for a prime $\mathfrak{q}'$ of $B_1 \otimes_A A'$ lying over $\mathfrak{m}_{A'}$, then $(\mathrm{Coker}(\alpha_1) \otimes_A A')_{\mathfrak{q}'}$ is flat over A'. Hence we conclude that F_{lf} is a subfunctor of the functor F'_{lf} associated to the module $\mathrm{Coker}(\alpha_1)_{\mathfrak{q}_1}$ over $(B_1)_{\mathfrak{q}_1}$. By the previous paragraph we know F'_{lf} is corepresented by A/J for some ideal $J \subset A$. Hence we may replace A by A/J and assume that $\mathrm{Coker}(\alpha_1)_{\mathfrak{q}_1}$ is flat over A.

Since $\mathrm{Coker}(\alpha_1)$ is a B_1-module for which there exist a complete dévissage of $N_1/B_1/A$ at $\mathfrak{q}_1$ and since $\mathrm{Coker}(\alpha_1)_{\mathfrak{q}_1}$ is flat over A by Lemma 10.2 we see that $\mathrm{Coker}(\alpha_1)$ is free as an A-module, in particular flat as an A-module. Hence

Lemma 10.1 implies $F_{lf}(A')$ is nonempty if and only if $\alpha \otimes 1_{A'}$ is injective. Let $N_1 = \mathrm{Im}(\alpha_1) \subset M_1$ so that we have exact sequences

$$0 \to N_1 \to M_1 \to \mathrm{Coker}(\alpha_1) \to 0 \quad \text{and} \quad B_1^{\oplus r_1} \to N_1 \to 0$$

The flatness of $\mathrm{Coker}(\alpha_1)$ implies the first sequence is universally exact (see Algebra, Lemma 82.5). Hence $\alpha \otimes 1_{A'}$ is injective if and only if $B_1^{\oplus r_1} \otimes_A A' \to N_1 \otimes_A A'$ is an isomorphism. Finally, Theorem 23.3 applies to show this functor is corepresentable by A/I for some ideal I and we conclude F_{lf} is corepresentable by A/I also.

To prove the final statement, suppose that $A \to B$ is essentially of finite presentation and M of finite presentation over B. Let $I \subset A$ be the ideal such that F_{lf} is corepresented by A/I. Write $I = \bigcup I_\lambda$ where I_λ ranges over the finitely generated ideals contained in I. Then, since $F_{lf}(A/I) = \{*\}$ we see that $F_{lf}(A/I_\lambda) = \{*\}$ for some λ, see Lemma 20.4 part (2). Clearly this implies that $I = I_\lambda$. □

05PH **Remark 24.2.** Here is a scheme theoretic reformulation of Theorem 24.1. Let $(X, x) \to (S, s)$ be a morphism of pointed schemes which is locally of finite type. Let $\mathcal{F}$ be a finite type quasi-coherent $\mathcal{O}_X$-module. Assume S henselian local with closed point s. There exists a closed subscheme $Z \subset S$ with the following property: for any morphism of pointed schemes $(T, t) \to (S, s)$ the following are equivalent

(1) $\mathcal{F}_T$ is flat over T at all points of the fibre X_t which map to $x \in X_s$, and
(2) $\mathrm{Spec}(\mathcal{O}_{T,t}) \to S$ factors through Z.

Moreover, if $X \to S$ is of finite presentation at x and $\mathcal{F}_x$ of finite presentation over $\mathcal{O}_{X,x}$, then $Z \to S$ is of finite presentation.

At this point we can obtain some very general results completely for free from the result above. Note that perhaps the most interesting case is when $E = X_s$!

05PI **Lemma 24.3.** *Let S be the spectrum of a henselian local ring with closed point s. Let $X \to S$ be a morphism of schemes which is locally of finite type. Let $\mathcal{F}$ be a finite type quasi-coherent $\mathcal{O}_X$-module. Let $E \subset X_s$ be a subset. There exists a closed subscheme $Z \subset S$ with the following property: for any morphism of pointed schemes $(T, t) \to (S, s)$ the following are equivalent*

(1) *$\mathcal{F}_T$ is flat over T at all points of the fibre X_t which map to a point of $E \subset X_s$, and*
(2) *$\mathrm{Spec}(\mathcal{O}_{T,t}) \to S$ factors through Z.*

Moreover, if $X \to S$ is locally of finite presentation, $\mathcal{F}$ is of finite presentation, and $E \subset X_s$ is closed and quasi-compact, then $Z \to S$ is of finite presentation.

Proof. For $x \in X_s$ denote $Z_x \subset S$ the closed subscheme we found in Remark 24.2. Then it is clear that $Z = \bigcap_{x \in E} Z_x$ works!

To prove the final statement assume X locally of finite presentation, $\mathcal{F}$ of finite presentation and Z closed and quasi-compact. First, choose finitely many affine opens $W_j \subset X$ such that $E \subset \bigcup W_j$. It clearly suffices to prove the result for each morphism $W_j \to S$ with sheaf $\mathcal{F}|_{X_j}$ and closed subset $E \cap W_j$. Hence we may assume X is affine. In this case, More on Algebra, Lemma 19.4 shows that the functor defined by (1) is "limit preserving". Hence we can show that $Z \to S$ is of finite presentation exactly as in the last part of the proof of Theorem 24.1. □

052G **Remark 24.4.** Tracing the proof of Lemma 24.3 to its origins we find a long and winding road. But if we assume that

(1) f is of finite type,
(2) $\mathcal{F}$ is a finite type $\mathcal{O}_X$-module,
(3) $E = X_s$, and
(4) S is the spectrum of a Noetherian complete local ring.

then there is a proof relying completely on more elementary algebra as follows: first we reduce to the case where X is affine by taking a finite affine open cover. In this case Z exists by More on Algebra, Lemma 20.3. The key step in this proof is constructing the closed subscheme Z step by step inside the truncations $\mathrm{Spec}(\mathcal{O}_{S,s}/\mathfrak{m}_s^n)$. This relies on the fact that flattening stratifications always exist when the base is Artinian, and the fact that $\mathcal{O}_{S,s} = \lim \mathcal{O}_{S,s}/\mathfrak{m}_s^n$.

25. Variants of a lemma

0ASZ In this section we discuss variants of Algebra, Lemmas 128.4 and 99.1. The most general version is Proposition 25.13; this was stated as [GR71, Lemma 4.2.2] but the proof in loc.cit. only gives the weaker result as stated in Lemma 25.5. The intricate proof of Proposition 25.13 is due to Ofer Gabber. As we currently have no application for the proposition we encourage the reader to skip to the next section after reading the proof of Lemma 25.5; this lemma will be used in the next section to prove Theorem 26.1.

0AT0 **Situation 25.1.** Let $\varphi : A \to B$ be a local ring homomorphism of local rings which is essentially of finite type. Let M be a flat A-module, N a finite B-module and $u : N \to M$ an A-module map such that $\overline{u} : N/\mathfrak{m}_A N \to M/\mathfrak{m}_A M$ is injective.

In this situation it is our goal to show that u is A-universally injective, N is of finite presentation over B, and N is flat as an A-module. If this is true, we will say *the lemma holds* in the given situation.

0AT1 **Lemma 25.2.** *If in Situation 25.1 the ring A is Noetherian then the lemma holds.*

Proof. Applying Algebra, Lemma 99.1 we see that u is injective and that $N/u(M)$ is flat over A. Then u is A-universally injective (Algebra, Lemma 39.12) and N is A-flat (Algebra, Lemma 39.13). Since B is Noetherian in this case we see that N is of finite presentation. □

0AT2 **Lemma 25.3.** *Let A_0 be a local ring. If the lemma holds for every Situation 25.1 with $A = A_0$, with B a localization of a polynomial algebra over A, and N of finite presentation over B, then the lemma holds for every Situation 25.1 with $A = A_0$.*

Proof. Let $A \to B$, $u : N \to M$ be as in Situation 25.1. Write $B = C/I$ where C is the localization of a polynomial algebra over A at a prime. If we can show that N is finitely presented as a C-module, then a fortiori this shows that N is finitely presented as a B-module (see Algebra, Lemma 6.4). Hence we may assume that B is the localization of a polynomial algebra. Next, write $N = B^{\oplus n}/K$ for some submodule $K \subset B^{\oplus n}$. Since $B/\mathfrak{m}_A B$ is Noetherian (as it is essentially of finite type over a field), there exist finitely many elements $k_1, \ldots, k_s \in K$ such that for $K' = \sum Bk_i$ and $N' = B^{\oplus n}/K'$ the canonical surjection $N' \to N$ induces an isomorphism $N'/\mathfrak{m}_A N' \cong N/\mathfrak{m}_A N$. Now, if the lemma holds for the composition $u' : N' \to M$, then u' is injective, hence $N' = N$ and $u' = u$. Thus the lemma holds for the original situation. □

0AT3 **Lemma 25.4.** *If in Situation 25.1 the ring A is henselian then the lemma holds.*

Proof. It suffices to prove this when B is essentially of finite presentation over A and N is of finite presentation over B, see Lemma 25.3. Let us temporarily make the additional assumption that N is flat over A. Then N is a filtered colimit $N = \operatorname{colim}_i F_i$ of free A-modules F_i such that the transition maps $u_{ii'} : F_i \to F_{i'}$ are injective modulo $\mathfrak{m}_A$, see Lemma 19.5. Each of the compositions $u_i : F_i \to M$ is A-universally injective by Lemma 7.5 wherefore $u = \operatorname{colim} u_i$ is A-universally injective as desired.

Assume A is a henselian local ring, B is essentially of finite presentation over A, N of finite presentation over B. By Theorem 24.1 there exists a finitely generated ideal $I \subset A$ such that N/IN is flat over A/I and such that N/I^2N is not flat over A/I^2 unless $I = 0$. The result of the previous paragraph shows that the lemma holds for $u \bmod I : N/IN \to M/IM$ over A/I. Consider the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & M \otimes_A I/I^2 & \longrightarrow & M/I^2M & \longrightarrow & M/IM \longrightarrow 0 \\ & & \uparrow{\scriptstyle u} & & \uparrow{\scriptstyle u} & & \uparrow{\scriptstyle u} \\ & & N \otimes_A I/I^2 & \longrightarrow & N/I^2N & \longrightarrow & N/IN \longrightarrow 0 \end{array}$$

whose rows are exact by right exactness of $\otimes$ and the fact that M is flat over A. Note that the left vertical arrow is the map $N/IN \otimes_{A/I} I/I^2 \to M/IM \otimes_{A/I} I/I^2$, hence is injective. A diagram chase shows that the lower left arrow is injective, i.e., $\operatorname{Tor}^1_{A/I^2}(I/I^2, M/I^2) = 0$ see Algebra, Remark 75.9. Hence N/I^2N is flat over A/I^2 by Algebra, Lemma 99.8 a contradiction unless $I = 0$. □

The following lemma discusses the special case of Situation 25.1 where M has a B-module structure and u is B-linear. This is the case most often used in practice and it is significantly easier to prove than the general case.

0AT4 **Lemma 25.5.** *Let $A \to B$ be a local ring homomorphism of local rings which is essentially of finite type. Let $u : N \to M$ be a B-module map. If N is a finite B-module, M is flat over A, and $\overline{u} : N/\mathfrak{m}_A N \to M/\mathfrak{m}_A M$ is injective, then u is A-universally injective, N is of finite presentation over B, and N is flat over A.*

Proof. Let $A \to A^h$ be the henselization of A. Let B' be the localization of $B \otimes_A A^h$ at the maximal ideal $\mathfrak{m}_B \otimes A^h + B \otimes \mathfrak{m}_{A^h}$. Since $B \to B'$ is flat (hence faithfully flat, see Algebra, Lemma 39.17), we may replace $A \to B$ with $A^h \to B'$, the module M by $M \otimes_B B'$, the module N by $N \otimes_B B'$, and u by $u \otimes \mathrm{id}_{B'}$, see Algebra, Lemmas 83.2 and 39.9. Thus we may assume that A is a henselian local ring. In this case our lemma follows from the more general Lemma 25.4. □

0AT5 **Lemma 25.6.** *If in Situation 25.1 the ring A is a valuation ring then the lemma holds.*

Proof. Recall that an A-module is flat if and only if it is torsion free, see More on Algebra, Lemma 22.10. Let $T \subset N$ be the A-torsion. Then $u(T) = 0$ and N/T is A-flat. Hence N/T is finitely presented over B, see More on Algebra, Lemma 26.6. Thus T is a finite B-module, see Algebra, Lemma 5.3. Since N/T is A-flat we see that $T/\mathfrak{m}_A T \subset N/\mathfrak{m}_A N$, see Algebra, Lemma 39.12. As $\overline{u}$ is injective but $u(T) = 0$, we conclude that $T/\mathfrak{m}_A T = 0$. Hence $T = 0$ by Nakayama's lemma, see Algebra, Lemma 20.1. At this point we have proved two out of the three assertions

(N is A-flat and of finite presentation over B) and what is left is to show that u is universally injective.

By Algebra, Theorem 82.3 it suffices to show that $N \otimes_A Q \to M \otimes_A Q$ is injective for every finitely presented A-module Q. By More on Algebra, Lemma 126.3 we may assume $Q = A/(a)$ with $a \in \mathfrak{m}_A$ nonzero. Thus it suffices to show that $N/aN \to M/aM$ is injective. Let $x \in N$ with $u(x) \in aM$. By Lemma 19.6 we know that x has a content ideal $I \subset A$. Since I is finitely generated (More on Algebra, Lemma 25.2) and A is a valuation ring, we have $I = (b)$ for some b (by Algebra, Lemma 50.15). By More on Algebra, Lemma 25.3 the element $u(x)$ has content ideal I as well. Since $u(x) \in aM$ we see that $(b) \subset (a)$ by More on Algebra, Definition 25.1. Since $x \in bN$ we conclude $x \in aN$ as desired. □

Consider the following situation

0AT6 (25.6.1)
$$\begin{array}{c} A \to B \text{ of finite presentation, } S \subset B \text{ a multiplicative subset, and} \\ N \text{ a finitely presented } S^{-1}B\text{-module} \end{array}$$

In this situation a *pure spreadout* is an affine open $U \subset \mathrm{Spec}(B)$ with $\mathrm{Spec}(S^{-1}B) \subset U$ and a finitely presented $\mathcal{O}(U)$-module N' extending N such that N' is A-projective and $N' \to N = S^{-1}N'$ is A-universally injective.

In (25.6.1) if $A \to A_1$ is a ring map, then we can base change: take $B_1 = B \otimes_A A_1$, let $S_1 \subset B_1$ be the image of S, and let $N_1 = N \otimes_A A_1$. This works because $S_1^{-1}B_1 = S^{-1}B \otimes_A A_1$. We will use this without further mention in the following.

0AT7 **Lemma 25.7.** *In (25.6.1) if there exists a pure spreadout, then*

(1) *elements of N have content ideals in A, and*
(2) *if $u : N \to M$ is a morphism to a flat A-module M such that $N/\mathfrak{m}N \to M/\mathfrak{m}M$ is injective for all maximal ideals $\mathfrak{m}$ of A, then u is A-universally injective.*

Proof. Choose U, N' as in the definition of a pure spreadout. Any element $x' \in N'$ has a content ideal in A because N' is A-projective (this can easily be seen directly, but it also follows from More on Algebra, Lemma 25.4 and Algebra, Example 91.1). Since $N' \to N$ is A-universally injective, we see that the image $x \in N$ of any $x' \in N'$ has a content ideal in A (it is the same as the content ideal of x'). For a general $x \in N$ we choose $s \in S$ such that sx is in the image of $N' \to N$ and we use that x and sx have the same content ideal.

Let $u : N \to M$ be as in (2). To show that u is A-universally injective, we may replace A by a localization at a maximal ideal (small detail omitted). Assume A is local with maximal ideal $\mathfrak{m}$. Pick $s \in S$ and consider the composition

$$N' \to N \xrightarrow{1/s} N \xrightarrow{u} M$$

Each of these maps is injective modulo $\mathfrak{m}$, hence the composition is A-universally injective by Lemma 7.5. Since $N = \mathrm{colim}_{s \in S}(1/s)N'$ we conclude that u is A-inversally injective as a colimit of universally injective maps. □

0AT8 **Lemma 25.8.** *In (25.6.1) for every $\mathfrak{p} \in \mathrm{Spec}(A)$ there is a finitely generated ideal $I \subset \mathfrak{p}A_\mathfrak{p}$ such that over $A_\mathfrak{p}/I$ we have a pure spreadout.*

Proof. We may replace A by $A_{\mathfrak{p}}$. Thus we may assume A is local and $\mathfrak{p}$ is the maximal ideal $\mathfrak{m}$ of A. We may write $N = S^{-1}N'$ for some finitely presented B-module N' by clearing denominators in a presentation of N over $S^{-1}B$. Since $B/\mathfrak{m}B$ is Noetherian, the kernel K of $N'/\mathfrak{m}N' \to N/\mathfrak{m}N$ is finitely generated. Thus we can pick $s \in S$ such that K is annihilated by s. After replacing B by B_s which is allowed as it just means passing to an affine open subscheme of $\mathrm{Spec}(B)$, we find that the elements of S are injective on $N'/\mathfrak{m}N'$. At this point we choose a local subring $A_0 \subset A$ essentially of finite type over $\mathbf{Z}$, a finite type ring map $A_0 \to B_0$ such that $B = A \otimes_{A_0} B_0$, and a finite B_0-module N'_0 such that $N' = B \otimes_{B_0} N'_0 = A \otimes_{A_0} N'_0$. We claim that $I = \mathfrak{m}_{A_0}A$ works. Namely, we have

$$N'/IN' = N'_0/\mathfrak{m}_{A_0}N'_0 \otimes_{\kappa_{A_0}} A/I$$

which is free over A/I. Multiplication by the elements of S is injective after dividing out by the maximal ideal, hence $N'/IN' \to N/IN$ is universally injective for example by Lemma 7.6. □

0AT9 **Lemma 25.9.** *In (25.6.1) assume N is A-flat, M is a flat A-module, and $u : N \to M$ is an A-module map such that $u \otimes id_{\kappa(\mathfrak{p})}$ is injective for all $\mathfrak{p} \in \mathrm{Spec}(A)$. Then u is A-universally injective.*

Proof. By Algebra, Lemma 82.13 it suffices to check that $N/IN \to M/IM$ is injective for every ideal $I \subset A$. After replacing A by A/I we see that it suffices to prove that u is injective.

Proof that u is injective. Let $x \in N$ be a nonzero element of the kernel of u. Then there exists a weakly associated prime $\mathfrak{p}$ of the module Ax, see Algebra, Lemma 66.5. Replacing A by $A_{\mathfrak{p}}$ we may assume A is local and we find a nonzero element $y \in Ax$ whose annihilator has radical equal to $\mathfrak{m}_A$, see Algebra, Lemma 66.2. Thus $\mathrm{Supp}(y) \subset \mathrm{Spec}(S^{-1}B)$ is nonempty and contained in the closed fibre of $\mathrm{Spec}(S^{-1}B) \to \mathrm{Spec}(A)$. Let $I \subset \mathfrak{m}_A$ be a finitely generated ideal so that we have a pure spreadout over A/I, see Lemma 25.8. Then $I^n y = 0$ for some n. Now $y \in \mathrm{Ann}_M(I^n) = \mathrm{Ann}_A(I^n) \otimes_R N$ by flatness. Thus, to get the desired contradiction, it suffices to show that

$$\mathrm{Ann}_A(I^n) \otimes_R N \longrightarrow \mathrm{Ann}_A(I^n) \otimes_R M$$

is injective. Since N and M are flat and since $\mathrm{Ann}_A(I^n)$ is annihilated by I^n, it suffices to show that $Q \otimes_A N \to Q \otimes_A M$ is injective for every A-module Q annihilated by I. This holds by our choice of I and Lemma 25.7 part (2). □

0ATA **Lemma 25.10.** *Let A be a local domain which is not a field. Let S be a set of finitely generated ideals of A. Assume that S is closed under products and such that $\bigcup_{I \in S} V(I)$ is the complement of the generic point of $\mathrm{Spec}(A)$. Then $\bigcap_{I \in S} I = (0)$.*

Proof. Since $\mathfrak{m}_A \subset A$ is not the generic point of $\mathrm{Spec}(A)$ we see that $I \subset \mathfrak{m}_A$ for at least one $I \in S$. Hence $\bigcap_{I \in S} I \subset \mathfrak{m}_A$. Let $f \in \mathfrak{m}_A$ be nonzero. Then $V(f) \subset \bigcup_{I \in S} V(I)$. Since the constructible topology on $V(f)$ is quasi-compact (Topology, Lemma 23.2 and Algebra, Lemma 26.2) we find that $V(f) \subset V(I_1) \cup \ldots \cup V(I_n)$ for some $I_j \in S$. Because $I_1 \ldots I_n \in S$ we see that $V(f) \subset V(I)$ for some I. As I is finitely generated this implies that $I^m \subset (f)$ for some m and since S is closed under products we see that $I \subset (f^2)$ for some $I \in S$. Then it is not possible to have $f \in I$. □

0ATB **Lemma 25.11.** *Let A be a local ring. Let $I, J \subset A$ be ideals. If J is finitely generated and $I \subset J^n$ for all $n \geq 1$, then $V(I)$ contains the closed points of $\mathrm{Spec}(A) \setminus V(J)$.*

Proof. Let $\mathfrak{p} \subset A$ be a closed point of $\mathrm{Spec}(A) \setminus V(J)$. We want to show that $I \subset \mathfrak{p}$. If not, then some $f \in I$ maps to a nonzero element of $A/\mathfrak{p}$. Note that $V(J) \cap \mathrm{Spec}(A/\mathfrak{p})$ is the set of non-generic points. Hence by Lemma 25.10 applied to the collection of ideals $J^n A/\mathfrak{p}$ we conclude that the image of f is zero in $A/\mathfrak{p}$. □

0ATC **Lemma 25.12.** *Let A be a local ring. Let $I \subset A$ be an ideal. Let $U \subset \mathrm{Spec}(A)$ be quasi-compact open. Let M be an A-module. Assume that*

(1) *M/IM is flat over A/I,*
(2) *M is flat over U,*

Then M/I_2M is flat over A/I_2 where $I_2 = \mathrm{Ker}(I \to \Gamma(U, I/I^2))$.

Proof. It suffices to show that $M \otimes_A I/I_2 \to IM/I_2M$ is injective, see Algebra, Lemma 99.9. This is true over U by assumption (2). Thus it suffices to show that $M \otimes_A I/I_2$ injects into its sections over U. We have $M \otimes_A I/I_2 = M/IM \otimes_A I/I_2$ and M/IM is a filtered colimit of finite free A/I-modules (Algebra, Theorem 81.4). Hence it suffices to show that I/I_2 injects into its sections over U, which follows from the construction of I_2. □

05U9 **Proposition 25.13.** *Let $A \to B$ be a local ring homomorphism of local rings which is essentially of finite type. Let M be a flat A-module, N a finite B-module and $u : N \to M$ an A-module map such that $\overline{u} : N/\mathfrak{m}_A N \to M/\mathfrak{m}_A M$ is injective. Then u is A-universally injective, N is of finite presentation over B, and N is flat over A.*

Proof. We may assume that B is the localization of a finitely presented A-algebra B_0 and that N is the localization of a finitely presented B_0-module M_0, see Lemma 25.3. By More on Morphisms, Lemma 54.1 there exists a "generic flatness stratification" for $\widetilde{M_0}$ on $\mathrm{Spec}(B_0)$ over $\mathrm{Spec}(A)$. Translating back to N we find a sequence of closed subschemes

$$S = \mathrm{Spec}(A) \supset S_0 \supset S_1 \supset \ldots \supset S_t = \emptyset$$

with $S_i \subset S$ cut out by a finitely generated ideal of A such that the pullback of $\widetilde{N}$ to $\mathrm{Spec}(B) \times_S (S_i \setminus S_{i+1})$ is flat over $S_i \setminus S_{i+1}$. We will prove the proposition by induction on t (the base case $t = 1$ will be proved in parallel with the other steps). Let $\mathrm{Spec}(A/J_i)$ be the scheme theoretic closure of $S_i \setminus S_{i+1}$.

Claim 1. N/J_iN is flat over A/J_i. This is immediate for $i = t - 1$ and follows from the induction hypothesis for $i > 0$. Thus we may assume $t > 1$, $S_{t-1} \neq \emptyset$, and $J_0 = 0$ and we have to prove that N is flat. Let $J \subset A$ be the ideal defining S_1. By induction on t again, we also have flatness modulo powers of J. Let A^h be the henselization of A and let B' be the localization of $B \otimes_A A^h$ at the maximal ideal $\mathfrak{m}_B \otimes A^h + B \otimes \mathfrak{m}_{A^h}$. Then $B \to B'$ is faithfully flat. Set $N' = N \otimes_B B'$. Note that N' is A^h-flat if and only if N is A-flat. By Theorem 24.1 there is a smallest ideal $I \subset A^h$ such that N'/IN' is flat over A^h/I, and I is finitely generated. By the above $I \subset J^n A^h$ for all $n \geq 1$. Let $S_i^h \subset \mathrm{Spec}(A^h)$ be the inverse image of $S_i \subset \mathrm{Spec}(A)$. By Lemma 25.11 we see that $V(I)$ contains the closed points of $U = \mathrm{Spec}(A^h) - S_1^h$. By construction N' is A^h-flat over U. By Lemma 25.12 we see

that N'/I_2N' is flat over A/I_2, where $I_2 = \mathrm{Ker}(I \to \Gamma(U, I/I^2))$. Hence $I = I_2$ by minimality of I. This implies that $I = I^2$ locally on U, i.e., we have $I\mathcal{O}_{U,u} = (0)$ or $I\mathcal{O}_{U,u} = (1)$ for all $u \in U$. Since $V(I)$ contains the closed points of U we see that $I = 0$ on U. Since $U \subset \mathrm{Spec}(A^h)$ is scheme theoretically dense (because replaced A by A/J_0 in the beginning of this paragraph), we see that $I = 0$. Thus N' is A^h-flat and hence Claim 1 holds.

We return to the situation as laid out before Claim 1. With A^h the henselization of A, with B' the localization of $B \otimes_A A^h$ at the maximal ideal $\mathfrak{m}_B \otimes A^h + B \otimes \mathfrak{m}_{A^h}$, and with $N' = N \otimes_B B'$ we now see that the flattening ideal $I \subset A^h$ of Theorem 24.1 is nilpotent. If $nil(A^h)$ denotes the ideal of nilpotent elements, then $nil(A^h) = nil(A)A^h$ (More on Algebra, Lemma 46.5). Hence there exists a finitely generated nilpotent ideal $I_0 \subset A$ such that N/I_0N is flat over A/I_0.

Claim 2. For every prime ideal $\mathfrak{p} \subset A$ the map $\kappa(\mathfrak{p}) \otimes_A N \to \kappa(\mathfrak{p}) \otimes_A M$ is injective. We say $\mathfrak{p}$ is bad it this is false. Suppose that C is a nonempty chain of bad primes and set $\mathfrak{p}^* = \bigcup_{\mathfrak{p} \in C} \mathfrak{p}$. By Lemma 25.8 there is a finitely generated ideal $\mathfrak{a} \subset \mathfrak{p}^* A_{\mathfrak{p}^*}$ such that there is a pure spreadout over $V(\mathfrak{a})$. If $\mathfrak{p}^*$ were good, then it would follow from Lemma 25.7 that the points of $V(\mathfrak{a})$ are good. However, since $\mathfrak{a}$ is finitely generated and since $\mathfrak{p}^* A_{\mathfrak{p}^*} = \bigcup_{\mathfrak{p} \in C} A_{\mathfrak{p}^*}$ we see that $V(\mathfrak{a})$ contains a $\mathfrak{p} \in C$, contradiction. Hence $\mathfrak{p}^*$ is bad. By Zorn's lemma, if there exists a bad prime, there exists a maximal one, say $\mathfrak{p}$. In other words, we may assume every $\mathfrak{p}' \supset \mathfrak{p}$, $\mathfrak{p}' \neq \mathfrak{p}$ is good. In this case we see that for every $f \in A$, $f \notin \mathfrak{p}$ the map $u \otimes \mathrm{id}_{A/(\mathfrak{p}+f)}$ is universally injective, see Lemma 25.9. Thus it suffices to show that $N/\mathfrak{p}N$ is separated for the topology defined by the submodules $f(N/\mathfrak{p}N)$. Since $B \to B'$ is faithfully flat, it is enough to prove the same for the module $N'/\mathfrak{p}N'$. By Lemma 19.5 and More on Algebra, Lemma 25.4 elements of $N'/\mathfrak{p}N'$ have content ideals in $A^h/\mathfrak{p}A^h$. Thus it suffices to show that $\bigcap_{f \in A, f \notin \mathfrak{p}} f(A^h/\mathfrak{p}A^h) = 0$. Then it suffices to show the same for $A^h/\mathfrak{q}A^h$ for every prime $\mathfrak{q} \subset A^h$ minimal over $\mathfrak{p}A^h$. Because $A \to A^h$ is the henselization, every $\mathfrak{q}$ contracts to $\mathfrak{p}$ and every $\mathfrak{q}' \supset \mathfrak{q}$, $\mathfrak{q}' \neq \mathfrak{q}$ contracts to a prime $\mathfrak{p}'$ which strictly contains $\mathfrak{p}$. Thus we get the vanishing of the intersections from Lemma 25.10.

At this point we can put everything together. Namely, using Claim 1 and Claim 2 we see that $N/I_0N \to M/I_0M$ is A/I_0-universally injective by Lemma 25.9. Then the diagrams

$$\begin{array}{ccc} N \otimes_A (I_0^n/I_0^{n+1}) & \longrightarrow & M \otimes_A (I_0^n/I_0^{n+1}) \\ \downarrow & & \| \\ I_0^n N/I_0^{n+1}N & \longrightarrow & I_0^n M/I_0^{n+1}M \end{array}$$

show that the left vertical arrows are injective. Hence by Algebra, Lemma 99.9 we see that N is flat. In a similar way the universal injectivity of u can be reduced (even without proving flatness of N first) to the one modulo I_0. This finishes the proof. □

26. Flat finite type modules, Part III

05U8 The following result is one of the main results of this chapter.

05UA **Theorem 26.1.** *Let $f : X \to S$ be locally of finite type. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module of finite type. Let $x \in X$ with image $s \in S$. The following are equivalent*

(1) *$\mathcal{F}$ is flat at x over S, and*
(2) *for every $x' \in Ass_{X_s}(\mathcal{F}_s)$ which specializes to x we have that $\mathcal{F}$ is flat at x' over S.*

Proof. It is clear that (1) implies (2) as $\mathcal{F}_{x'}$ is a localization of $\mathcal{F}_x$ for every point which specializes to x. Set $A = \mathcal{O}_{S,s}$, $B = \mathcal{O}_{X,x}$ and $N = \mathcal{F}_x$. Let $\Sigma \subset B$ be the multiplicative subset of B of elements which act as nonzerodivisors on $N/\mathfrak{m}_A N$. Assumption (2) implies that $\Sigma^{-1}N$ is A-flat by the description of $\mathrm{Spec}(\Sigma^{-1}N)$ in Lemma 7.1. On the other hand, the map $N \to \Sigma^{-1}N$ is injective modulo $\mathfrak{m}_A$ by construction. Hence applying Lemma 25.5 we win. □

Now we apply this directly to obtain the following useful results.

05UB **Lemma 26.2.** *Let S be a local scheme with closed point s. Let $f : X \to S$ be locally of finite type. Let $\mathcal{F}$ be a finite type quasi-coherent $\mathcal{O}_X$-module. Assume that*

(1) *every point of $Ass_{X/S}(\mathcal{F})$ specializes to a point of the closed fibre X_s*[3]*,*
(2) *$\mathcal{F}$ is flat over S at every point of X_s.*

Then $\mathcal{F}$ is flat over S.

Proof. This is immediate from the fact that it suffices to check for flatness at points of the relative assassin of $\mathcal{F}$ over S by Theorem 26.1. □

27. Universal flattening

05PS If $f : X \to S$ is a proper, finitely presented morphism of schemes then one can find a universal flattening of f. In this section we discuss this and some of its variants.

05UC **Lemma 27.1.** *In Situation 20.7. For each $p \geq 0$ the functor H_p (20.7.2) is representable by a locally closed immersion $S_p \to S$. If $\mathcal{F}$ is of finite presentation, then $S_p \to S$ is of finite presentation.*

Proof. For each S we will prove the statement for all $p \geq 0$ concurrently. The functor H_p is a sheaf for the fppf topology by Lemma 20.8. Hence combining Descent, Lemma 39.1, More on Morphisms, Lemma 57.1 , and Descent, Lemma 24.1 we see that the question is local for the étale topology on S. In particular, the question is Zariski local on S.

For $s \in S$ denote ξ_s the unique generic point of the fibre X_s. Note that for every $s \in S$ the restriction $\mathcal{F}_s$ of $\mathcal{F}$ is locally free of some rank $p(s) \geq 0$ in some neighbourhood of ξ_s. (As X_s is irreducible and smooth this follows from generic flatness for $\mathcal{F}_s$ over X_s, see Algebra, Lemma 118.1 although this is overkill.) For future reference we note that

$$p(s) = \dim_{\kappa(\xi_s)}(\mathcal{F}_{\xi_s} \otimes_{\mathcal{O}_{X,\xi_s}} \kappa(\xi_s)).$$

In particular $H_{p(s)}(s)$ is nonempty and $H_q(s)$ is empty if $q \neq p(s)$.

Let $U \subset X$ be an open subscheme. As $f : X \to S$ is smooth, it is open. It is immediate from (20.7.2) that the functor H_p for the pair $(f|_U : U \to f(U), \mathcal{F}|_U)$ and the functor H_p for the pair $(f|_{f^{-1}(f(U))}, \mathcal{F}|_{f^{-1}(f(U))})$ are the same. Hence to prove the existence of S_p over $f(U)$ we may always replace X by U.

[3]For example this holds if f is finite type and $\mathcal{F}$ is pure along X_s, or if f is proper.

Pick $s \in S$. There exists an affine open neighbourhood U of ξ_s such that $\mathcal{F}|_U$ can be generated by at most $p(s)$ elements. By the arguments above we see that in order to prove the statement for $H_{p(s)}$ in an neighbourhood of s we may assume that $\mathcal{F}$ is generated by $p(s)$ elements, i.e., that there exists a surjection

$$u : \mathcal{O}_X^{\oplus p(s)} \longrightarrow \mathcal{F}$$

In this case it is clear that $H_{p(s)}$ is equal to F_{iso} (20.1.1) for the map u (this follows immediately from Lemma 19.1 but also from Lemma 12.1 after shrinking a bit more so that both S and X are affine.) Thus we may apply Theorem 23.3 to see that $H_{p(s)}$ is representable by a closed immersion in a neighbourhood of s.

The result follows formally from the above. Namely, the arguments above show that locally on S the function $s \mapsto p(s)$ is bounded. Hence we may use induction on $p = \max_{s \in S} p(s)$. The functor H_p is representable by a closed immersion $S_p \to S$ by the above. Replace S by $S \setminus S_p$ which drops the maximum by at least one and we win by induction hypothesis.

Assume $\mathcal{F}$ is of finite presentation. Then $S_p \to S$ is locally of finite presentation by Lemma 20.8 part (2) combined with Limits, Remark 6.2. Then we redo the induction argument in the paragraph to see that each S_p is quasi-compact when S is affine: first if $p = \max_{s \in S} p(s)$, then $S_p \subset S$ is closed (see above) hence quasi-compact. Then $U = S \setminus S_p$ is quasi-compact open in S because $S_p \to S$ is a closed immersion of finite presentation (see discussion in Morphisms, Section 23 for example). Then $S_{p-1} \to U$ is a closed immersion of finite presentation, and so S_{p-1} is quasi-compact and $U' = S \setminus (S_p \cup S_{p-1})$ is quasi-compact. And so on. □

05UD **Lemma 27.2.** *In Situation 20.11. Let $h : X' \to X$ be an étale morphism. Set $\mathcal{F}' = h^*\mathcal{F}$ and $f' = f \circ h$. Let F'_n be (20.11.1) associated to $(f' : X' \to S, \mathcal{F}')$. Then F_n is a subfunctor of F'_n and if $h(X') \supset Ass_{X/S}(\mathcal{F})$, then $F_n = F'_n$.*

Proof. Let $T \to S$ be any morphism. Then $h_T : X'_T \to X_T$ is étale as a base change of the étale morphism g. For $t \in T$ denote $Z \subset X_t$ the set of points where $\mathcal{F}_T$ is not flat over T, and similarly denote $Z' \subset X'_t$ the set of points where $\mathcal{F}'_T$ is not flat over T. As $\mathcal{F}'_T = h_T^*\mathcal{F}_T$ we see that $Z' = h_t^{-1}(Z)$, see Morphisms, Lemma 26.13. Hence $Z' \to Z$ is an étale morphism, so $\dim(Z') \leq \dim(Z)$ (for example by Descent, Lemma 21.2 or just because an étale morphism is smooth of relative dimension 0). This implies that $F_n \subset F'_n$.

Finally, suppose that $h(X') \supset \mathrm{Ass}_{X/S}(\mathcal{F})$ and that $T \to S$ is a morphism such that $F'_n(T)$ is nonempty, i.e., such that $\mathcal{F}'_T$ is flat in dimensions $\geq n$ over T. Pick a point $t \in T$ and let $Z \subset X_t$ and $Z' \subset X'_t$ be as above. To get a contradiction assume that $\dim(Z) \geq n$. Pick a generic point $\xi \in Z$ corresponding to a component of dimension $\geq n$. Let $x \in \mathrm{Ass}_{X_t}(\mathcal{F}_t)$ be a generalization of ξ. Then x maps to a point of $\mathrm{Ass}_{X/S}(\mathcal{F})$ by Divisors, Lemma 7.3 and Remark 7.4. Thus we see that x is in the image of h_T, say $x = h_T(x')$ for some $x' \in X'_T$. But $x' \notin Z'$ as $x \rightsquigarrow \xi$ and $\dim(Z') < n$. Hence $\mathcal{F}'_T$ is flat over T at x' which implies that $\mathcal{F}_T$ is flat at x over T (by Morphisms, Lemma 26.13). Since this holds for every such x we conclude that $\mathcal{F}_T$ is flat over T at ξ by Theorem 26.1 which is the desired contradiction. □

05UE **Lemma 27.3.** *Assume that $X \to S$ is a smooth morphism of affine schemes with geometrically irreducible fibres of dimension d and that $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_X$-module of finite presentation. Then $F_d = \coprod_{p=0,\ldots,c} H_p$ for some $c \geq 0$ with F_d as in (20.11.1) and H_p as in (20.7.2).*

Proof. As X is affine and $\mathcal{F}$ is quasi-coherent of finite presentation we know that $\mathcal{F}$ can be generated by $c \geq 0$ elements. Then $\dim_{\kappa(x)}(\mathcal{F}_x \otimes \kappa(x))$ in any point $x \in X$ never exceeds c. In particular $H_p = \emptyset$ for $p > c$. Moreover, note that there certainly is an inclusion $\coprod H_p \to F_d$. Having said this the content of the lemma is that, if a base change $\mathcal{F}_T$ is flat in dimensions $\geq d$ over T and if $t \in T$, then $\mathcal{F}_T$ is free of some rank r in an open neighbourhood $U \subset X_T$ of the unique generic point ξ of X_t. Namely, then H_r contains the image of U which is an open neighbourhood of t. The existence of U follows from More on Morphisms, Lemma 16.7. □

05UF **Lemma 27.4.** *In Situation 20.11. Let $s \in S$ let $d \geq 0$. Assume*

(1) *there exists a complete dévissage of $\mathcal{F}/X/S$ over some point $s \in S$,*
(2) *X is of finite presentation over S,*
(3) *$\mathcal{F}$ is an $\mathcal{O}_X$-module of finite presentation, and*
(4) *$\mathcal{F}$ is flat in dimensions $\geq d+1$ over S.*

Then after possibly replacing S by an open neighbourhood of s the functor F_d (20.11.1) is representable by a monomorphism $Z_d \to S$ of finite presentation.

Proof. A preliminary remark is that X, S are affine schemes and that it suffices to prove F_d is representable by a monomorphism of finite presentation $Z_d \to S$ on the category of affine schemes over S. (Of course we do not require Z_d to be affine.) Hence throughout the proof of the lemma we work in the category of affine schemes over S.

Let $(Z_k, Y_k, i_k, \pi_k, \mathcal{G}_k, \alpha_k)_{k=1,\ldots,n}$ be a complete dévissage of $\mathcal{F}/X/S$ over s, see Definition 5.1. We will use induction on the length n of the dévissage. Recall that $Y_k \to S$ is smooth with geometrically irreducible fibres, see Definition 4.1. Let d_k be the relative dimension of Y_k over S. Recall that $i_{k,*}\mathcal{G}_k = \mathrm{Coker}(\alpha_k)$ and that i_k is a closed immersion. By the definitions referenced above we have $d_1 = \dim(\mathrm{Supp}(\mathcal{F}_s))$ and

$$d_k = \dim(\mathrm{Supp}(\mathrm{Coker}(\alpha_{k-1})_s)) = \dim(\mathrm{Supp}(\mathcal{G}_{k,s}))$$

for $k = 2, \ldots, n$. It follows that $d_1 > d_2 > \ldots > d_n \geq 0$ because α_k is an isomorphism in the generic point of $(Y_k)_s$.

Note that i_1 is a closed immersion and $\mathcal{F} = i_{1,*}\mathcal{G}_1$. Hence for any morphism of schemes $T \to S$ with T affine, we have $\mathcal{F}_T = i_{1,T,*}\mathcal{G}_{1,T}$ and $i_{1,T}$ is still a closed immersion of schemes over T. Thus $\mathcal{F}_T$ is flat in dimensions $\geq d$ over T if and only if $\mathcal{G}_{1,T}$ is flat in dimensions $\geq d$ over T. Because $\pi_1 : Z_1 \to Y_1$ is finite we see in the same manner that $\mathcal{G}_{1,T}$ is flat in dimensions $\geq d$ over T if and only if $\pi_{1,T,*}\mathcal{G}_{1,T}$ is flat in dimensions $\geq d$ over T. The same arguments work for "flat in dimensions $\geq d+1$" and we conclude in particular that $\pi_{1,*}\mathcal{G}_1$ is flat over S in dimensions $\geq d+1$ by our assumption on $\mathcal{F}$.

Suppose that $d_1 > d$. It follows from the discussion above that in particular $\pi_{1,*}\mathcal{G}_1$ is flat over S at the generic point of $(Y_1)_s$. By Lemma 12.1 we may replace S by an affine neighbourhood of s and assume that α_1 is S-universally injective. Because

α_1 is S-universally injective, for any morphism $T \to S$ with T affine, we have a short exact sequence

$$0 \to \mathcal{O}_{Y_{1,T}}^{\oplus r_1} \to \pi_{1,T,*}\mathcal{G}_{1,T} \to \mathrm{Coker}(\alpha_1)_T \to 0$$

and still the first arrow is T-universally injective. Hence the set of points of $(Y_1)_T$ where $\pi_{1,T,*}\mathcal{G}_{1,T}$ is flat over T is the same as the set of points of $(Y_1)_T$ where $\mathrm{Coker}(\alpha_1)_T$ is flat over S. In this way the question reduces to the sheaf $\mathrm{Coker}(\alpha_1)$ which has a complete dévissage of length $n-1$ and we win by induction.

If $d_1 < d$ then F_d is represented by S and we win.

The last case is the case $d_1 = d$. This case follows from a combination of Lemma 27.3 and Lemma 27.1. □

05UG **Theorem 27.5.** *In Situation 20.11. Assume moreover that f is of finite presentation, that $\mathcal{F}$ is an $\mathcal{O}_X$-module of finite presentation, and that $\mathcal{F}$ is pure relative to S. Then F_n is representable by a monomorphism $Z_n \to S$ of finite presentation.*

Proof. The functor F_n is a sheaf for the fppf topology by Lemma 20.12. Observe that a monomorphism of finite presentation is separated and quasi-finite (Morphisms, Lemma 21.15). Hence combining Descent, Lemma 39.1, More on Morphisms, Lemma 57.1 , and Descent, Lemmas 23.33 and 23.15 we see that the question is local for the étale topology on S.

In particular the situation is local for the Zariski topology on S and we may assume that S is affine. In this case the dimension of the fibres of f is bounded above, hence we see that F_n is representable for n large enough. Thus we may use descending induction on n. Suppose that we know F_{n+1} is representable by a monomorphism $Z_{n+1} \to S$ of finite presentation. Consider the base change $X_{n+1} = Z_{n+1} \times_S X$ and the pullback $\mathcal{F}_{n+1}$ of $\mathcal{F}$ to X_{n+1}. The morphism $Z_{n+1} \to S$ is quasi-finite as it is a monomorphism of finite presentation, hence Lemma 16.4 implies that $\mathcal{F}_{n+1}$ is pure relative to Z_{n+1}. Since F_n is a subfunctor of F_{n+1} we conclude that in order to prove the result for F_n it suffices to prove the result for the corresponding functor for the situation $\mathcal{F}_{n+1}/X_{n+1}/Z_{n+1}$. In this way we reduce to proving the result for F_n in case $S_{n+1} = S$, i.e., we may assume that $\mathcal{F}$ is flat in dimensions $\geq n+1$ over S.

Fix n and assume $\mathcal{F}$ is flat in dimensions $\geq n+1$ over S. To finish the proof we have to show that F_n is representable by a monomorphism $Z_n \to S$ of finite presentation. Since the question is local in the étale topology on S it suffices to show that for every $s \in S$ there exists an elementary étale neighbourhood $(S', s') \to (S, s)$ such that the result holds after base change to S'. Thus by Lemma 5.8 we may assume there exist étale morphisms $h_j : Y_j \to X$, $j = 1, \ldots, m$ such that for each j there exists a complete dévissage of $\mathcal{F}_j/Y_j/S$ over s, where $\mathcal{F}_j$ is the pullback of $\mathcal{F}$ to Y_j and such that $X_s \subset \bigcup h_j(Y_j)$. Note that by Lemma 27.2 the sheaves $\mathcal{F}_j$ are still flat over in dimensions $\geq n+1$ over S. Set $W = \bigcup h_j(Y_j)$, which is a quasi-compact open of X. As $\mathcal{F}$ is pure along X_s we see that

$$E = \{t \in S \mid \mathrm{Ass}_{X_t}(\mathcal{F}_t) \subset W\}.$$

contains all generalizations of s. By More on Morphisms, Lemma 25.5 E is a constructible subset of S. We have seen that $\mathrm{Spec}(\mathcal{O}_{S,s}) \subset E$. By Morphisms, Lemma 23.4 we see that E contains an open neighbourhood of s. Hence after

shrinking S we may assume that $E = S$. It follows from Lemma 27.2 that it suffices to prove the lemma for the functor F_n associated to $X = \coprod Y_j$ and $\mathcal{F} = \coprod \mathcal{F}_j$. If $F_{j,n}$ denotes the functor for $Y_j \to S$ and the sheaf $\mathcal{F}_i$ we see that $F_n = \prod F_{j,n}$. Hence it suffices to prove each $F_{j,n}$ is representable by some monomorphism $Z_{j,n} \to S$ of finite presentation, since then

$$Z_n = Z_{1,n} \times_S \ldots \times_S Z_{m,n}$$

Thus we have reduced the theorem to the special case handled in Lemma 27.4. □

We make explicit what the theorem means in terms of universal flattenings in the following lemma.

05UH **Lemma 27.6.** *Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module.*

(1) *If f is of finite presentation, $\mathcal{F}$ is an $\mathcal{O}_X$-module of finite presentation, and $\mathcal{F}$ is pure relative to S, then there exists a universal flattening $S' \to S$ of $\mathcal{F}$. Moreover $S' \to S$ is a monomorphism of finite presentation.*
(2) *If f is of finite presentation and X is pure relative to S, then there exists a universal flattening $S' \to S$ of X. Moreover $S' \to S$ is a monomorphism of finite presentation.*
(3) *If f is proper and of finite presentation and $\mathcal{F}$ is an $\mathcal{O}_X$-module of finite presentation, then there exists a universal flattening $S' \to S$ of $\mathcal{F}$. Moreover $S' \to S$ is a monomorphism of finite presentation.*
(4) *If f is proper and of finite presentation then there exists a universal flattening $S' \to S$ of X.*

Proof. These statements follow immediately from Theorem 27.5 applied to $F_0 = F_{flat}$ and the fact that if f is proper then $\mathcal{F}$ is automatically pure over the base, see Lemma 17.1. □

28. Grothendieck's Existence Theorem, IV

0CTB This section continues the discussion in Cohomology of Schemes, Sections 24, 25, and 27. We will work in the following situation.

0CTC **Situation 28.1.** Here we have an inverse system of rings (A_n) with surjective transition maps whose kernels are locally nilpotent. Set $A = \lim A_n$. We have a scheme X separated and of finite presentation over A. We set $X_n = X \times_{\mathrm{Spec}(A)} \mathrm{Spec}(A_n)$ and we view it as a closed subscheme of X. We assume further given a system $(\mathcal{F}_n, \varphi_n)$ where $\mathcal{F}_n$ is a finitely presented $\mathcal{O}_{X_n}$-module, flat over A_n, with support proper over A_n, and

$$\varphi_n : \mathcal{F}_n \otimes_{\mathcal{O}_{X_n}} \mathcal{O}_{X_{n-1}} \longrightarrow \mathcal{F}_{n-1}$$

is an isomorphism (notation using the equivalence of Morphisms, Lemma 4.1).

Our goal is to see if we can find a quasi-coherent sheaf $\mathcal{F}$ on X such that $\mathcal{F}_n = \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_{X_n}$ for all n.

0CTD **Lemma 28.2.** *In Situation 28.1 consider*

$$K = R\lim_{D_{QCoh}(\mathcal{O}_X)}(\mathcal{F}_n) = DQ_X(R\lim_{D(\mathcal{O}_X)} \mathcal{F}_n)$$

Then K is in $D^b_{QCoh}(\mathcal{O}_X)$ and in fact K has nonzero cohomology sheaves only in degrees ≥ 0.

Proof. Special case of Derived Categories of Schemes, Example 21.5. □

0CTE **Lemma 28.3.** *In Situation 28.1 let K be as in Lemma 28.2. For any perfect object E of $D(\mathcal{O}_X)$ we have*

(1) *$M = R\Gamma(X, K \otimes^{\mathbf{L}} E)$ is a perfect object of $D(A)$ and there is a canonical isomorphism $R\Gamma(X_n, \mathcal{F}_n \otimes^{\mathbf{L}} E|_{X_n}) = M \otimes^{\mathbf{L}}_A A_n$ in $D(A_n)$,*
(2) *$N = R\mathrm{Hom}_X(E, K)$ is a perfect object of $D(A)$ and there is a canonical isomorphism $R\mathrm{Hom}_{X_n}(E|_{X_n}, \mathcal{F}_n) = N \otimes^{\mathbf{L}}_A A_n$ in $D(A_n)$.*

In both statements $E|_{X_n}$ denotes the derived pullback of E to X_n.

Proof. Proof of (2). Write $E_n = E|_{X_n}$ and $N_n = R\mathrm{Hom}_{X_n}(E_n, \mathcal{F}_n)$. Recall that $R\mathrm{Hom}_{X_n}(-,-)$ is equal to $R\Gamma(X_n, R\mathcal{H}om(-,-))$, see Cohomology, Section 44. Hence by Derived Categories of Schemes, Lemma 30.7 we see that N_n is a perfect object of $D(A_n)$ whose formation commutes with base change. Thus the maps $N_n \otimes^{\mathbf{L}}_{A_n} A_{n-1} \to N_{n-1}$ coming from φ_n are isomorphisms. By More on Algebra, Lemma 99.3 we find that $R\lim N_n$ is perfect and that its base change back to A_n recovers N_n. On the other hand, the exact functor $R\mathrm{Hom}_X(E, -) : D_{QCoh}(\mathcal{O}_X) \to D(A)$ of triangulated categories commutes with products and hence with derived limits, whence

$$R\mathrm{Hom}_X(E, K) = R\lim R\mathrm{Hom}_X(E, \mathcal{F}_n) = R\lim R\mathrm{Hom}_X(E_n, \mathcal{F}_n) = R\lim N_n$$

This proves (2). To see that (1) holds, translate it into (2) using Cohomology, Lemma 50.5. □

0CTF **Lemma 28.4.** *In Situation 28.1 let K be as in Lemma 28.2. Then K is pseudo-coherent relative to A.*

Proof. Combinging Lemma 28.3 and Derived Categories of Schemes, Lemma 34.3 we see that $R\Gamma(X, K \otimes^{\mathbf{L}} E)$ is pseudo-coherent in $D(A)$ for all pseudo-coherent E in $D(\mathcal{O}_X)$. Thus the lemma follows from More on Morphisms, Lemma 69.4. □

0CTG **Lemma 28.5.** *In Situation 28.1 let K be as in Lemma 28.2. For any quasi-compact open $U \subset X$ we have*

$$R\Gamma(U, K) \otimes^{\mathbf{L}}_A A_n = R\Gamma(U_n, \mathcal{F}_n)$$

in $D(A_n)$ where $U_n = U \cap X_n$.

Proof. Fix n. By Derived Categories of Schemes, Lemma 33.4 there exists a system of perfect complexes E_m on X such that $R\Gamma(U, K) = \mathrm{hocolim} R\Gamma(X, K \otimes^{\mathbf{L}} E_m)$. In fact, this formula holds not just for K but for every object of $D_{QCoh}(\mathcal{O}_X)$. Applying this to $\mathcal{F}_n$ we obtain

$$\begin{aligned} R\Gamma(U_n, \mathcal{F}_n) &= R\Gamma(U, \mathcal{F}_n) \\ &= \mathrm{hocolim}_m R\Gamma(X, \mathcal{F}_n \otimes^{\mathbf{L}} E_m) \\ &= \mathrm{hocolim}_m R\Gamma(X_n, \mathcal{F}_n \otimes^{\mathbf{L}} E_m|_{X_n}) \end{aligned}$$

Using Lemma 28.3 and the fact that $- \otimes^{\mathbf{L}}_A A_n$ commutes with homotopy colimits we obtain the result. □

0CTH **Lemma 28.6.** *In Situation 28.1 let K be as in Lemma 28.2. Denote $X_0 \subset X$ the closed subset consisting of points lying over the closed subset $\mathrm{Spec}(A_1) = \mathrm{Spec}(A_2) = \ldots$ of $\mathrm{Spec}(A)$. There exists an open $W \subset X$ containing X_0 such that*

(1) $H^i(K)|_W$ *is zero unless* $i = 0$,
(2) $\mathcal{F} = H^0(K)|_W$ *is of finite presentation, and*
(3) $\mathcal{F}_n = \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_{X_n}$.

Proof. Fix $n \geq 1$. By construction there is a canonical map $K \to \mathcal{F}_n$ in $D_{QCoh}(\mathcal{O}_X)$ and hence a canonical map $H^0(K) \to \mathcal{F}_n$ of quasi-coherent sheaves. This explains the meaning of part (3).

Let $x \in X_0$ be a point. We will find an open neighbourhood W of x such that (1), (2), and (3) are true. Since X_0 is quasi-compact this will prove the lemma. Let $U \subset X$ be an affine open neighbourhood of x. Say $U = \mathrm{Spec}(B)$. Choose a surjection $P \to B$ with P smooth over A. By Lemma 28.4 and the definition of relative pseudo-coherence there exists a bounded above complex $F^\bullet$ of finite free P-modules representing Ri_*K where $i : U \to \mathrm{Spec}(P)$ is the closed immersion induced by the presentation. Let M_n be the B-module corresponding to $\mathcal{F}_n|_U$. By Lemma 28.5

$$H^i(F^\bullet \otimes_A A_n) = \begin{cases} 0 & \text{if} \quad i \neq 0 \\ M_n & \text{if} \quad i = 0 \end{cases}$$

Let i be the maximal index such that F^i is nonzero. If $i \leq 0$, then (1), (2), and (3) are true. If not, then $i > 0$ and we see that the rank of the map

$$F^{i-1} \to F^i$$

in the point x is maximal. Hence in an open neighbourhood of x inside $\mathrm{Spec}(P)$ the rank is maximal. Thus after replacing P by a principal localization we may assume that the displayed map is surjective. Since F^i is finite free we may choose a splitting $F^{i-1} = F' \oplus F^i$. Then we may replace $F^\bullet$ by the complex

$$\ldots \to F^{i-2} \to F' \to 0 \to \ldots$$

and we win by induction on i. □

0CTI **Lemma 28.7.** *In Situation 28.1 let K be as in Lemma 28.2. Let $W \subset X$ be as in Lemma 28.6. Set $\mathcal{F} = H^0(K)|_W$. Then, after possibly shrinking the open W, the support of $\mathcal{F}$ is proper over A.*

Proof. Fix $n \geq 1$. Let $I_n = \mathrm{Ker}(A \to A_n)$. By More on Algebra, Lemma 11.3 the pair (A, I_n) is henselian. Let $Z \subset W$ be the support of $\mathcal{F}$. This is a closed subset as $\mathcal{F}$ is of finite presentation. By part (3) of Lemma 28.6 we see that $Z \times_{\mathrm{Spec}(A)} \mathrm{Spec}(A_n)$ is equal to the support of $\mathcal{F}_n$ and hence proper over $\mathrm{Spec}(A/I)$. By More on Morphisms, Lemma 53.9 we can write $Z = Z_1 \amalg Z_2$ with Z_1, Z_2 open and closed in Z, with Z_1 proper over A, and with $Z_1 \times_{\mathrm{Spec}(A)} \mathrm{Spec}(A/I_n)$ equal to the support of $\mathcal{F}_n$. In other words, Z_2 does not meet X_0. Hence after replacing W by $W \setminus Z_2$ we obtain the lemma. □

0CTJ **Lemma 28.8.** *Let $A = \lim A_n$ be a limit of a system of rings whose transition maps are surjective and with locally nilpotent kernels. Let $S = \mathrm{Spec}(A)$. Let $T \to S$ be a monomorphism which is locally of finite type. If $\mathrm{Spec}(A_n) \to S$ factors through T for all n, then $T = S$.*

Proof. Set $S_n = \mathrm{Spec}(A_n)$. Let $T_0 \subset T$ be the common image of the factorizations $S_n \to T$. Then T_0 is quasi-compact. Let $T' \subset T$ be a quasi-compact open containing T_0. Then $S_n \to T$ factors through T'. If we can show that $T' = S$, then $T' = T = S$. Hence we may assume T is quasi-compact.

Assume T is quasi-compact. In this case $T \to S$ is separated and quasi-finite (Morphisms, Lemma 21.15). Using Zariski's Main Theorem (in the form of More on Morphisms, Lemma 43.3) we choose a factorization $T \to W \to S$ with $W \to S$ finite and $T \to W$ an open immersion. Write $W = \mathrm{Spec}(B)$. The (unique) factorizations $S_n \to T$ may be viewed as morphisms into W and we obtain

$$A \longrightarrow B \longrightarrow \lim A_n = A$$

Consider the morphism $h : S = \mathrm{Spec}(A) \to \mathrm{Spec}(B) = W$ coming from the arrow on the right. Then

$$T \times_{W,h} S$$

is an open subscheme of S containing the image of $S_n \to S$ for all n. To finish the proof it suffices to show that any open $U \subset S$ containing the image of $S_n \to S$ for some $n \geq 1$ is equal to S. This is true because $(A, \mathrm{Ker}(A \to A_n))$ is a henselian pair (More on Algebra, Lemma 11.3) and hence every closed point of S is contained in the image of $S_n \to S$. □

0CTK **Theorem 28.9** (Grothendieck Existence Theorem)**.** *In Situation 28.1 there exists a finitely presented $\mathcal{O}_X$-module $\mathcal{F}$, flat over A, with support proper over A, such that $\mathcal{F}_n = \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_{X_n}$ for all n compatibly with the maps φ_n.*

Proof. Apply Lemmas 28.2, 28.3, 28.4, 28.5, 28.6, and 28.7 to get an open subscheme $W \subset X$ containing all points lying over $\mathrm{Spec}(A_n)$ and a finitely presented $\mathcal{O}_W$-module $\mathcal{F}$ whose support is proper over A with $\mathcal{F}_n = \mathcal{F} \otimes_{\mathcal{O}_W} \mathcal{O}_{X_n}$ for all $n \geq 1$. (This makes sense as $X_n \subset W$.) By Lemma 17.1 we see that $\mathcal{F}$ is universally pure relative to $\mathrm{Spec}(A)$. By Theorem 27.5 (for explanation, see Lemma 27.6) there exists a universal flattening $S' \to \mathrm{Spec}(A)$ of $\mathcal{F}$ and moreover the morphism $S' \to \mathrm{Spec}(A)$ is a monomorphism of finite presentation. Since the base change of $\mathcal{F}$ to $\mathrm{Spec}(A_n)$ is $\mathcal{F}_n$ we find that $\mathrm{Spec}(A_n) \to \mathrm{Spec}(A)$ factors (uniquely) through S' for each n. By Lemma 28.8 we see that $S' = \mathrm{Spec}(A)$. This means that $\mathcal{F}$ is flat over A. Finally, since the scheme theoretic support Z of $\mathcal{F}$ is proper over $\mathrm{Spec}(A)$, the morphism $Z \to X$ is closed. Hence the pushforward $(W \to X)_*\mathcal{F}$ is supported on W and has all the desired properties. □

29. Grothendieck's Existence Theorem, V

0DIA In this section we prove an analogue for Grothendieck's existence theorem in the derived category, following the method used in Section 28 for quasi-coherent modules. The classical case is discussed in Cohomology of Schemes, Sections 24, 25, and 27. We will work in the following situation.

0DIB **Situation 29.1.** Here we have an inverse system of rings (A_n) with surjective transition maps whose kernels are locally nilpotent. Set $A = \lim A_n$. We have a scheme X proper, flat, and of finite presentation over A. We set $X_n = X \times_{\mathrm{Spec}(A)} \mathrm{Spec}(A_n)$ and we view it as a closed subscheme of X. We assume further given a system (K_n, φ_n) where K_n is a pseudo-coherent object of $D(\mathcal{O}_{X_n})$ and

$$\varphi_n : K_n \longrightarrow K_{n-1}$$

is a map in $D(\mathcal{O}_{X_n})$ which induces an isomorphism $K_n \otimes^{\mathbf{L}}_{\mathcal{O}_{X_n}} \mathcal{O}_{X_{n-1}} \to K_{n-1}$ in $D(\mathcal{O}_{X_{n-1}})$.

More precisely, we should write $\varphi_n : K_n \to Ri_{n-1,*}K_{n-1}$ where $i_{n-1} : X_{n-1} \to X_n$ is the inclusion morphism and in this notation the condition is that the adjoint map $Li_{n-1}^*K_n \to K_{n-1}$ is an isomorphism. Our goal is to find a pseudo-coherent $K \in D(\mathcal{O}_X)$ such that $K_n = K \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{O}_{X_n}$ for all n (with the same abuse of notation).

0DIC **Lemma 29.2.** *In Situation 29.1 consider*

$$K = R\lim\nolimits_{D_{QCoh}(\mathcal{O}_X)}(K_n) = DQ_X(R\lim\nolimits_{D(\mathcal{O}_X)} K_n)$$

Then K is in $D^-_{QCoh}(\mathcal{O}_X)$.

Proof. The functor DQ_X exists because X is quasi-compact and quasi-separated, see Derived Categories of Schemes, Lemma 21.1. Since DQ_X is a right adjoint it commutes with products and therefore with derived limits. Hence the equality in the statement of the lemma.

By Derived Categories of Schemes, Lemma 21.4 the functor DQ_X has bounded cohomological dimension. Hence it suffices to show that $R\lim K_n \in D^-(\mathcal{O}_X)$. To see this, let $U \subset X$ be an affine open. Then there is a canonical exact sequence

$$0 \to R^1 \lim H^{m-1}(U, K_n) \to H^m(U, R\lim K_n) \to \lim H^m(U, K_n) \to 0$$

by Cohomology, Lemma 37.1. Since U is affine and K_n is pseudo-coherent (and hence has quasi-coherent cohomology sheaves by Derived Categories of Schemes, Lemma 10.1) we see that $H^m(U, K_n) = H^m(K_n)(U)$ by Derived Categories of Schemes, Lemma 3.5. Thus we conclude that it suffices to show that K_n is bounded above independent of n.

Since K_n is pseudo-coherent we have $K_n \in D^-(\mathcal{O}_{X_n})$. Suppose that a_n is maximal such that $H^{a_n}(K_n)$ is nonzero. Of course $a_1 \leq a_2 \leq a_3 \leq \ldots$. Note that $H^{a_n}(K_n)$ is an $\mathcal{O}_{X_n}$-module of finite presentation (Cohomology, Lemma 47.9). We have $H^{a_n}(K_{n-1}) = H^{a_n}(K_n) \otimes_{\mathcal{O}_{X_n}} \mathcal{O}_{X_{n-1}}$. Since $X_{n-1} \to X_n$ is a thickening, it follows from Nakayama's lemma (Algebra, Lemma 20.1) that if $H^{a_n}(K_n) \otimes_{\mathcal{O}_{X_n}} \mathcal{O}_{X_{n-1}}$ is zero, then $H^{a_n}(K_n)$ is zero too. Thus $a_n = a_{n-1}$ for all n and we conclude. □

0DID **Lemma 29.3.** *In Situation 29.1 let K be as in Lemma 29.2. For any perfect object E of $D(\mathcal{O}_X)$ the cohomology*

$$M = R\Gamma(X, K \otimes^{\mathbf{L}} E)$$

is a pseudo-coherent object of $D(A)$ and there is a canonical isomorphism

$$R\Gamma(X_n, K_n \otimes^{\mathbf{L}} E|_{X_n}) = M \otimes^{\mathbf{L}}_A A_n$$

in $D(A_n)$. Here $E|_{X_n}$ denotes the derived pullback of E to X_n.

Proof. Write $E_n = E|_{X_n}$ and $M_n = R\Gamma(X_n, K_n \otimes^{\mathbf{L}} E|_{X_n})$. By Derived Categories of Schemes, Lemma 30.5 we see that M_n is a pseudo-coherent object of $D(A_n)$ whose formation commutes with base change. Thus the maps $M_n \otimes^{\mathbf{L}}_{A_n} A_{n-1} \to M_{n-1}$ coming from φ_n are isomorphisms. By More on Algebra, Lemma 99.1 we find that $R\lim M_n$ is pseudo-coherent and that its base change back to A_n recovers M_n. On the other hand, the exact functor $R\Gamma(X, -) : D_{QCoh}(\mathcal{O}_X) \to D(A)$ of triangulated categories commutes with products and hence with derived limits, whence

$$R\Gamma(X, E \otimes^{\mathbf{L}} K) = R\lim R\Gamma(X, E \otimes^{\mathbf{L}} K_n) = R\lim R\Gamma(X_n, E_n \otimes^{\mathbf{L}} K_n) = R\lim M_n$$

as desired. □

0DIE **Lemma 29.4.** *In Situation 29.1 let K be as in Lemma 29.2. Then K is pseudo-coherent on X.*

Proof. Combinging Lemma 29.3 and Derived Categories of Schemes, Lemma 34.3 we see that $R\Gamma(X, K \otimes^{\mathbf{L}} E)$ is pseudo-coherent in $D(A)$ for all pseudo-coherent E in $D(\mathcal{O}_X)$. Thus it follows from More on Morphisms, Lemma 69.4 that K is pseudo-coherent relative to A. Since X is of flat and of finite presentation over A, this is the same as being pseudo-coherent on X, see More on Morphisms, Lemma 59.18. □

0DIF **Lemma 29.5.** *In Situation 29.1 let K be as in Lemma 29.2. For any quasi-compact open $U \subset X$ we have*

$$R\Gamma(U, K) \otimes^{\mathbf{L}}_A A_n = R\Gamma(U_n, K_n)$$

in $D(A_n)$ where $U_n = U \cap X_n$.

Proof. Fix n. By Derived Categories of Schemes, Lemma 33.4 there exists a system of perfect complexes E_m on X such that $R\Gamma(U, K) = \text{hocolim} R\Gamma(X, K \otimes^{\mathbf{L}} E_m)$. In fact, this formula holds not just for K but for every object of $D_{QCoh}(\mathcal{O}_X)$. Applying this to K_n we obtain

$$\begin{aligned} R\Gamma(U_n, K_n) &= R\Gamma(U, K_n) \\ &= \text{hocolim}_m R\Gamma(X, K_n \otimes^{\mathbf{L}} E_m) \\ &= \text{hocolim}_m R\Gamma(X_n, K_n \otimes^{\mathbf{L}} E_m|_{X_n}) \end{aligned}$$

Using Lemma 29.3 and the fact that $- \otimes^{\mathbf{L}}_A A_n$ commutes with homotopy colimits we obtain the result. □

0DIG **Theorem 29.6** (Derived Grothendieck Existence Theorem)**.** *In Situation 29.1 there exists a pseudo-coherent K in $D(\mathcal{O}_X)$ such that $K_n = K \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{O}_{X_n}$ for all n compatibly with the maps φ_n.*

Proof. Apply Lemmas 29.2, 29.3, 29.4 to get a pseudo-coherent object K of $D(\mathcal{O}_X)$. Choosing affine opens in Lemma 29.5 it follows immediately that K restricts to K_n over X_n. □

0DIH **Remark 29.7.** The result in this section can be generalized. It is probably correct if we only assume $X \to \text{Spec}(A)$ to be separated, of finite presentation, and K_n pseudo-coherent relative to A_n supported on a closed subset of X_n proper over A_n. The outcome will be a K which is pseudo-coherent relative to A supported on a closed subset proper over A. If we ever need this, we will formulate a precise statement and prove it here.

30. Blowing up and flatness

080X In this section we continue our discussion of results of the form: "After a blowup the strict transform becomes flat", see More on Algebra, Section 27 and Divisors, Section 36. We will use the following (more or less standard) notation in this section. If $X \to S$ is a morphism of schemes, $\mathcal{F}$ is a quasi-coherent module on X, and $T \to S$ is a morphism of schemes, then we denote $\mathcal{F}_T$ the pullback of $\mathcal{F}$ to the base change $X_T = X \times_S T$.

080Y **Remark 30.1.** Let S be a quasi-compact and quasi-separated scheme. Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent module on X. Let $U \subset S$ be a quasi-compact open subscheme. Given a U-admissible blowup $S' \to S$ we denote X' the strict transform of X and $\mathcal{F}'$ the strict transform of $\mathcal{F}$ which we think of as a quasi-coherent module on X' (via Divisors, Lemma 34.2). Let P be a property of $\mathcal{F}/X/S$ which is stable under strict transform (as above) for U-admissible blowups. The general problem in this section is: Show (under auxiliary conditions on $\mathcal{F}/X/S$) there exists a U-admissible blowup $S' \to S$ such that the strict transform $\mathcal{F}'/X'/S'$ has P.

The general strategy will be to use that a composition of U-admissible blowups is a U-admissible blowup, see Divisors, Lemma 35.2. In fact, we will make use of the more precise Divisors, Lemma 33.14 and combine it with Divisors, Lemma 34.6. The result is that it suffices to find a sequence of U-admissible blowups

$$S = S_0 \leftarrow S_1 \leftarrow \ldots \leftarrow S_n$$

such that, setting $\mathcal{F}_0 = \mathcal{F}$ and $X_0 = X$ and setting $\mathcal{F}_i/X_i$ equal to the strict transform of $\mathcal{F}_{i-1}/X_{i-1}$, we arrive at $\mathcal{F}_n/X_n/S_n$ with property P.

In particular, choose a finite type quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_S$ such that $V(\mathcal{I}) = S \setminus U$, see Properties, Lemma 25.1. Let $S' \to S$ be the blowup in $\mathcal{I}$ and let $E \subset S'$ be the exceptional divisor (Divisors, Lemma 33.4). Then we see that we've reduced the problem to the case where there exists an effective Cartier divisor $D \subset S$ whose support is $X \setminus U$. In particular we may assume U is scheme theoretically dense in S (Divisors, Lemma 14.4).

Suppose that P is local on S: If $S = \bigcup S_i$ is a finite open covering by quasi-compact opens and P holds for $\mathcal{F}_{S_i}/X_{S_i}/S_i$ then P holds for $\mathcal{F}/X/S$. In this case the general problem above is local on S as well, i.e., if given $s \in S$ we can find a quasi-compact open neighbourhood W of s such that the problem for $\mathcal{F}_W/X_W/W$ is solvable, then the problem is solvable for $\mathcal{F}/X/S$. This follows from Divisors, Lemmas 35.3 and 35.4.

0810 **Lemma 30.2.** *Let R be a ring and let $f \in R$. Let $r \geq 0$ be an integer. Let $R \to S$ be a ring map and let M be an S-module. Assume*

(1) *$R \to S$ is of finite presentation and flat,*
(2) *every fibre ring $S \otimes_R \kappa(\mathfrak{p})$ is geometrically integral over R,*
(3) *M is a finite S-module,*
(4) *M_f is a finitely presented S_f-module,*
(5) *for all $\mathfrak{p} \in R$, $f \notin \mathfrak{p}$ with $\mathfrak{q} = \mathfrak{p}S$ the module $M_\mathfrak{q}$ is free of rank r over $S_\mathfrak{q}$.*

Then there exists a finitely generated ideal $I \subset R$ with $V(f) = V(I)$ such that for all $a \in I$ with $R' = R[\frac{I}{a}]$ the quotient

$$M' = (M \otimes_R R')/a\text{-power torsion}$$

over $S' = S \otimes_R R'$ satisfies the following: for every prime $\mathfrak{p}' \subset R'$ there exists a $g \in S'$, $g \notin \mathfrak{p}'S'$ such that M'_g is a free S'_g-module of rank r.

Proof. This lemma is a generalization of More on Algebra, Lemma 27.5; we urge the reader to read that proof first. Choose a surjection $S^{\oplus n} \to M$, which is possible by (1). Choose a finite submodule $K \subset \mathrm{Ker}(S^{\oplus n} \to M)$ such that $S^{\oplus n}/K \to M$ becomes an isomorphism after inverting f. This is possible by (4). Set $M_1 = S^{\oplus n}/K$

and suppose we can prove the lemma for M_1. Say $I \subset R$ is the corresponding ideal. Then for $a \in I$ the map

$$M_1' = (M_1 \otimes_R R')/a\text{-power torsion} \longrightarrow M' = (M \otimes_R R')/a\text{-power torsion}$$

is surjective. It is also an isomorphism after inverting a in R' as $R'_a = R_f$, see Algebra, Lemma 70.7. But a is a nonzerodivisor on M_1', whence the displayed map is an isomorphism. Thus it suffices to prove the lemma in case M is a finitely presented S-module.

Assume M is a finitely presented S-module satisfying (3). Then $J = \mathrm{Fit}_r(M) \subset S$ is a finitely generated ideal. By Lemma 9.3 we can write S as a direct summand of a free R-module: $\bigoplus_{\alpha \in A} R = S \oplus C$. For any element $h \in S$ writing $h = \sum a_\alpha$ in the decomposition above, we say that the a_α are the coefficients of h. Let $I' \subset R$ be the ideal of coefficients of elements of J. Multiplication by an element of S defines an R-linear map $S \to S$, hence I' is generated by the coefficients of the generators of J, i.e., I' is a finitely generated ideal. We claim that $I = fI'$ works.

We first check that $V(f) = V(I)$. The inclusion $V(f) \subset V(I)$ is clear. Conversely, if $f \notin \mathfrak{p}$, then $\mathfrak{q} = \mathfrak{p}S$ is not an element of $V(J)$ by property (5) and More on Algebra, Lemma 8.7. Hence there is an element of J which does not map to zero in $S \otimes_R \kappa(\mathfrak{p})$. Thus there exists an element of I' which is not contained in $\mathfrak{p}$, so $\mathfrak{p} \notin V(fI') = V(I)$.

Let $a \in I$ and set $R' = R[\frac{I}{a}]$. We may write $a = fa'$ for some $a' \in I'$. By Algebra, Lemmas 70.2 and 70.8 we see that $I'R' = a'R'$ and a' is a nonzerodivisor in R'. Set $S' = S \otimes_S R'$. Every element g of $JS' = \mathrm{Fit}_r(M \otimes_S S')$ can be written as $g = \sum_\alpha c_\alpha$ for some $c_\alpha \in I'R'$. Since $I'R' = a'R'$ we can write $c_\alpha = a'c'_\alpha$ for some $c'_\alpha \in R'$ and $g = (\sum c'_\alpha)a' = g'a'$ in S'. Moreover, there is an $g_0 \in J$ such that $a' = c_\alpha$ for some α. For this element we have $g_0 = g'_0 a'$ in S' where g'_0 is a unit in S'. Let $\mathfrak{p}' \subset R'$ be a prime ideal and $\mathfrak{q}' = \mathfrak{p}'S'$. By the above we see that $JS'_{\mathfrak{q}'}$ is the principal ideal generated by the nonzerodivisor a'. It follows from More on Algebra, Lemma 8.9 that $M'_{\mathfrak{q}'}$ can be generated by r elements. Since M' is finite, there exist $m_1, \ldots, m_r \in M'$ and $g \in S'$, $g \notin \mathfrak{q}'$ such that the corresponding map $(S')^{\oplus r} \to M'$ becomes surjective after inverting g.

Finally, consider the ideal $J' = \mathrm{Fit}_{r-1}(M')$. Note that $J'S'_g$ is generated by the coefficients of relations between $m_1, \ldots, m_r$ (compatibility of Fitting ideal with base change). Thus it suffices to show that $J' = 0$, see More on Algebra, Lemma 8.8. Since $R'_a = R_f$ (Algebra, Lemma 70.7) and $M'_a = M_f$ we see from (5) that J'_a maps to zero in $S_{\mathfrak{q}''}$ for any prime $\mathfrak{q}'' \subset S'$ of the form $\mathfrak{q}'' = \mathfrak{p}''S'$ where $\mathfrak{p}'' \subset R'_a$. Since $S'_a \subset \prod_{\mathfrak{q}'' \text{ as above}} S'_{\mathfrak{q}''}$ (as $(S'_a)_{\mathfrak{p}''} \subset S'_{\mathfrak{q}''}$ by Lemma 7.4) we see that $J'R'_a = 0$. Since a is a nonzerodivisor in R' we conclude that $J' = 0$ and we win. □

0811 **Lemma 30.3.** *Let S be a quasi-compact and quasi-separated scheme. Let $X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent module on X. Let $U \subset S$ be a quasi-compact open. Assume*

(1) *$X \to S$ is affine, of finite presentation, flat, geometrically integral fibres,*
(2) *$\mathcal{F}$ is a module of finite type,*
(3) *$\mathcal{F}_U$ is of finite presentation,*
(4) *$\mathcal{F}$ is flat over S at all generic points of fibres lying over points of U.*

Then there exists a U-admissible blowup $S' \to S$ and an open subscheme $V \subset X_{S'}$ such that (a) the strict transform $\mathcal{F}'$ of $\mathcal{F}$ restricts to a finitely locally free $\mathcal{O}_V$-module and (b) $V \to S'$ is surjective.

Proof. Given $\mathcal{F}/X/S$ and $U \subset S$ with hypotheses as in the lemma, denote P the property "$\mathcal{F}$ is flat over S at all generic points of fibres". It is clear that P is preserved under strict transform, see Divisors, Lemma 34.3 and Morphisms, Lemma 26.7. It is also clear that P is local on S. Hence any and all observations of Remark 30.1 apply to the problem posed by the lemma.

Consider the function $r : U \to \mathbf{Z}_{\geq 0}$ which assigns to $u \in U$ the integer

$$r(u) = \dim_{\kappa(\xi_u)}(\mathcal{F}_{\xi_u} \otimes \kappa(\xi_u))$$

where ξ_u is the generic point of the fibre X_u. By More on Morphisms, Lemma 16.7 and the fact that the image of an open in X_S in S is open, we see that $r(u)$ is locally constant. Accordingly $U = U_0 \amalg U_1 \amalg \ldots \amalg U_c$ is a finite disjoint union of open and closed subschemes where r is constant with value i on U_i. By Divisors, Lemma 35.5 we can find a U-admissible blowup to decompose S into the disjoint union of two schemes, the first containing U_0 and the second $U_1 \cup \ldots \cup U_c$. Repeating this $c-1$ more times we may assume that S is a disjoint union $S = S_0 \amalg S_1 \amalg \ldots \amalg S_c$ with $U_i \subset S_i$. Thus we may assume the function r defined above is constant, say with value r.

By Remark 30.1 we see that we may assume that we have an effective Cartier divisor $D \subset S$ whose support is $S \setminus U$. Another application of Remark 30.1 combined with Divisors, Lemma 14.2 tells us we may assume that $S = \mathrm{Spec}(R)$ and $D = \mathrm{Spec}(R/(f))$ for some nonzerodivisor $f \in R$. This case is handled by Lemma 30.2. □

0812 **Lemma 30.4.** *Let $A \to C$ be a finite locally free ring map of rank d. Let $h \in C$ be an element such that C_h is étale over A. Let $J \subset C$ be an ideal. Set $I = Fit_0(C/J)$ where we think of C/J as a finite A-module. Then $IC_h = JJ'$ for some ideal $J' \subset C_h$. If J is finitely generated so are I and J'.*

Proof. We will use basic properties of Fitting ideals, see More on Algebra, Lemma 8.4. Then IC is the Fitting ideal of $C/J \otimes_A C$. Note that $C \to C \otimes_A C$, $c \mapsto 1 \otimes c$ has a section (the multiplication map). By assumption $C \to C \otimes_A C$ is étale at every prime in the image of $\mathrm{Spec}(C_h)$ under this section. Hence the multiplication map $C \otimes_A C_h \to C_h$ is étale in particular flat, see Algebra, Lemma 143.8. Hence there exists a C_h-algebra such that $C \otimes_A C_h \cong C_h \oplus C'$ as C_h-algebras, see Algebra, Lemma 143.9. Thus $(C/J) \otimes_A C_h \cong (C_h/J_h) \oplus C'/I'$ as C_h-modules for some ideal $I' \subset C'$. Hence $IC_h = JJ'$ with $J' = \mathrm{Fit}_0(C'/I')$ where we view C'/J' as a C_h-module. □

0813 **Lemma 30.5.** *Let $A \to B$ be an étale ring map. Let $a \in A$ be a nonzerodivisor. Let $J \subset B$ be a finite type ideal with $V(J) \subset V(aB)$. For every $\mathfrak{q} \subset B$ there exists a finite type ideal $I \subset A$ with $V(I) \subset V(a)$ and $g \in B$, $g \notin \mathfrak{q}$ such that $IB_g = JJ'$ for some finite type ideal $J' \subset B_g$.*

Proof. We may replace B by a principal localization at an element $g \in B$, $g \notin \mathfrak{q}$. Thus we may assume that B is standard étale, see Algebra, Proposition 144.4. Thus we may assume B is a localization of $C = A[x]/(f)$ for some monic $f \in A[x]$ of some degree d. Say $B = C_h$ for some $h \in C$. Choose elements $h_1, \ldots, h_n \in$

C which generate J over B. The condition $V(J) \subset V(aB)$ signifies that $a^m = \sum b_i h_i$ in B for some large m. Set $h_{n+1} = a^m$. As in Lemma 30.4 we take $I = \mathrm{Fit}_0(C/(h_1, \ldots, h_{r+1}))$. Since the module $C/(h_1, \ldots, h_{r+1})$ is annihilated by a^m we see that $a^{dm} \in I$ which implies that $V(I) \subset V(a)$. □

0814 **Lemma 30.6.** *Let S be a quasi-compact and quasi-separated scheme. Let $X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent module on X. Let $U \subset S$ be a quasi-compact open. Assume there exist finitely many commutative diagrams*

$$\begin{array}{ccccc} & & X_i & \xrightarrow[j_i]{} & X \\ & & \downarrow & & \downarrow \\ S_i^* & \longrightarrow & S_i & \xrightarrow{e_i} & S \end{array}$$

where

(1) *$e_i : S_i \to S$ are quasi-compact étale morphisms and $S = \bigcup e_i(S_i)$,*
(2) *$j_i : X_i \to X$ are étale morphisms and $X = \bigcup j_i(X_i)$,*
(3) *$S_i^* \to S_i$ is an $e_i^{-1}(U)$-admissible blowup such that the strict transform $\mathcal{F}_i^*$ of $j_i^*\mathcal{F}$ is flat over S_i^*.*

Then there exists a U-admissible blowup $S' \to S$ such that the strict transform of $\mathcal{F}$ is flat over S'.

Proof. We claim that the hypotheses of the lemma are preserved under U-admissible blowups. Namely, suppose $b : S' \to S$ is a U-admissible blowup in the quasi-coherent sheaf of ideals $\mathcal{I}$. Moreover, let $S_i^* \to S_i$ be the blowup in the quasi-coherent sheaf of ideals $\mathcal{J}_i$. Then the collection of morphisms $e_i' : S_i' = S_i \times_S S' \to S'$ and $j_i' : X_i' = X_i \times_S S' \to X \times_S S'$ satisfy conditions (1), (2), (3) for the strict transform $\mathcal{F}'$ of $\mathcal{F}$ relative to the blowup $S' \to S$. First, observe that S_i' is the blowup of S_i in the pullback of $\mathcal{I}$, see Divisors, Lemma 33.3. Second, consider the blowup $S_i'^* \to S_i'$ of S_i' in the pullback of the ideal $\mathcal{J}_i$. By Divisors, Lemma 33.12 we get a commutative diagram

$$\begin{array}{ccc} S_i'^* & \longrightarrow & S_i' \\ \downarrow & \searrow & \downarrow \\ S_i^* & \longrightarrow & S_i \end{array}$$

and all the morphisms in the diagram above are blowups. Hence by Divisors, Lemmas 34.3 and 34.6 we see

$$\begin{aligned} & \text{the strict transform of } (j_i')^*\mathcal{F}' \text{ under } S_i'^* \to S_i' \\ = {} & \text{the strict transform of } j_i^*\mathcal{F} \text{ under } S_i'^* \to S_i \\ = {} & \text{the strict transform of } \mathcal{F}_i' \text{ under } S_i'^* \to S_i' \\ = {} & \text{the pullback of } \mathcal{F}_i^* \text{ via } X_i \times_{S_i} S_i'^* \to X_i \end{aligned}$$

which is therefore flat over $S_i'^*$ (Morphisms, Lemma 26.7). Having said this, we see that all observations of Remark 30.1 apply to the problem of finding a U-admissible blowup such that the strict transform of $\mathcal{F}$ becomes flat over the base under assumptions as in the lemma. In particular, we may assume that $S \setminus U$ is the support of an effective Cartier divisor $D \subset S$. Another application of Remark

30.1 combined with Divisors, Lemma 14.2 shows we may assume that $S = \mathrm{Spec}(A)$ and $D = \mathrm{Spec}(A/(a))$ for some nonzerodivisor $a \in A$.

Pick an i and $s \in S_i$. Lemma 30.5 implies we can find an open neighbourhood $s \in W_i \subset S_i$ and a finite type quasi-coherent ideal $\mathcal{I} \subset \mathcal{O}_S$ such that $\mathcal{I} \cdot \mathcal{O}_{W_i} = \mathcal{J}_i \mathcal{J}_i'$ for some finite type quasi-coherent ideal $\mathcal{J}_i' \subset \mathcal{O}_{W_i}$ and such that $V(\mathcal{I}) \subset V(a) = S \backslash U$. Since S_i is quasi-compact we can replace S_i by a finite collection $W_1, \ldots, W_n$ of these opens and assume that for each i there exists a quasi-coherent sheaf of ideals $\mathcal{I}_i \subset \mathcal{O}_S$ such that $\mathcal{I}_i \cdot \mathcal{O}_{S_i} = \mathcal{J}_i \mathcal{J}_i'$ for some finite type quasi-coherent ideal $\mathcal{J}_i' \subset \mathcal{O}_{S_i}$. As in the discussion of the first paragraph of the proof, consider the blowup S' of S in the product $\mathcal{I}_1 \ldots \mathcal{I}_n$ (this blowup is U-admissible by construction). The base change of $S' \to S$ to S_i is the blowup in

$$\mathcal{J}_i \cdot \mathcal{J}_i' \mathcal{I}_1 \ldots \hat{\mathcal{I}}_i \ldots \mathcal{I}_n$$

which factors through the given blowup $S_i^* \to S_i$ (Divisors, Lemma 33.12). In the notation of the diagram above this means that $S_i'^* = S_i'$. Hence after replacing S by S' we arrive in the situation that $j_i^* \mathcal{F}$ is flat over S_i. Hence $j_i^* \mathcal{F}$ is flat over S, see Lemma 2.3. By Morphisms, Lemma 26.13 we see that $\mathcal{F}$ is flat over S. □

0815 **Theorem 30.7.** *Let S be a quasi-compact and quasi-separated scheme. Let X be a scheme over S. Let $\mathcal{F}$ be a quasi-coherent module on X. Let $U \subset S$ be a quasi-compact open. Assume*

(1) *X is quasi-compact,*
(2) *X is locally of finite presentation over S,*
(3) *$\mathcal{F}$ is a module of finite type,*
(4) *$\mathcal{F}_U$ is of finite presentation, and*
(5) *$\mathcal{F}_U$ is flat over U.*

Then there exists a U-admissible blowup $S' \to S$ such that the strict transform $\mathcal{F}'$ of $\mathcal{F}$ is an $\mathcal{O}_{X \times_S S'}$-module of finite presentation and flat over S'.

Proof. We first prove that we can find a U-admissible blowup such that the strict transform is flat. The question is étale local on the source and the target, see Lemma 30.6 for a precise statement. In particular, we may assume that $S = \mathrm{Spec}(R)$ and $X = \mathrm{Spec}(A)$ are affine. For $s \in S$ write $\mathcal{F}_s = \mathcal{F}|_{X_s}$ (pullback of $\mathcal{F}$ to the fibre). As $X \to S$ is of finite type $d = \max_{s \in S} \dim(\mathrm{Supp}(\mathcal{F}_s))$ is an integer. We will do induction on d.

Let $x \in X$ be a point of X lying over $s \in S$ with $\dim_x(\mathrm{Supp}(\mathcal{F}_s)) = d$. Apply Lemma 3.2 to get $g : X' \to X$, $e : S' \to S$, $i : Z' \to X'$, and $\pi : Z' \to Y'$. Observe that $Y' \to S'$ is a smooth morphism of affines with geometrically irreducible fibres of dimension d. Because the problem is étale local it suffices to prove the theorem for $g^* \mathcal{F} / X' / S'$. Because $i : Z' \to X'$ is a closed immersion of finite presentation (and since strict transform commutes with affine pushforward, see Divisors, Lemma 34.4) it suffices to prove the flattening result for $\mathcal{G}$. Since π is finite (hence also affine) it suffices to prove the flattening result for $\pi_* \mathcal{G} / Y' / S'$. Thus we may assume that $X \to S$ is a smooth morphism of affines with geometrically irreducible fibres of dimension d.

Next, we apply a blowup as in Lemma 30.3. Doing so we reach the situation where there exists an open $V \subset X$ surjecting onto S such that $\mathcal{F}|_V$ is finite locally free. Let $\xi \in X$ be the generic point of X_s. Let $r = \dim_{\kappa(\xi)} \mathcal{F}_\xi \otimes \kappa(\xi)$. Choose a

map $\alpha : \mathcal{O}_X^{\oplus r} \to \mathcal{F}$ which induces an isomorphism $\kappa(\xi)^{\oplus r} \to \mathcal{F}_\xi \otimes \kappa(\xi)$. Because $\mathcal{F}$ is locally free over V we find an open neighbourhood W of ξ where α is an isomorphism. Shrink S to an affine open neighbourhood of s such that $W \to S$ is surjective. Say $\mathcal{F}$ is the quasi-coherent module associated to the A-module N. Since $\mathcal{F}$ is flat over S at all generic points of fibres (in fact at all points of W), we see that

$$\alpha_{\mathfrak{p}} : A_{\mathfrak{p}}^{\oplus r} \to N_{\mathfrak{p}}$$

is universally injective for all primes $\mathfrak{p}$ of R, see Lemma 10.1. Hence α is universally injective, see Algebra, Lemma 82.11. Set $\mathcal{H} = \text{Coker}(\alpha)$. By Divisors, Lemma 34.7 we see that, given a U-admissible blowup $S' \to S$ the strict transforms of $\mathcal{F}'$ and $\mathcal{H}'$ fit into an exact sequence

$$0 \to \mathcal{O}_{X \times_S S'}^{\oplus r} \to \mathcal{F}' \to \mathcal{H}' \to 0$$

Hence Lemma 10.1 also shows that $\mathcal{F}'$ is flat at a point x' if and only if $\mathcal{H}'$ is flat at that point. In particular $\mathcal{H}_U$ is flat over U and $\mathcal{H}_U$ is a module of finite presentation. We may apply the induction hypothesis to $\mathcal{H}$ to see that there exists a U-admissible blowup such that the strict transform $\mathcal{H}'$ is flat as desired.

To finish the proof of the theorem we still have to show that $\mathcal{F}'$ is a module of finite presentation (after possibly another U-admissible blowup). This follows from Lemma 11.1 as we can assume $U \subset S$ is scheme theoretically dense (see third paragraph of Remark 30.1). This finishes the proof of the theorem. □

31. Applications

081Q In this section we apply some of the results above.

081R **Lemma 31.1.** *Let S be a quasi-compact and quasi-separated scheme. Let X be a scheme over S. Let $U \subset S$ be a quasi-compact open. Assume*

(1) *$X \to S$ is of finite type and quasi-separated, and*
(2) *$X_U \to U$ is flat and locally of finite presentation.*

Then there exists a U-admissible blowup $S' \to S$ such that the strict transform of X is flat and of finite presentation over S'.

Proof. Since $X \to S$ is quasi-compact and quasi-separated by assumption, the strict transform of X with respect to a blowing up $S' \to S$ is also quasi-compact and quasi-separated. Hence to prove the lemma it suffices to find a U-admissible blowup such that the strict transform is flat and locally of finite presentation. Let $X = W_1 \cup \ldots \cup W_n$ be a finite affine open covering. If we can find a U-admissible blowup $S_i \to S$ such that the strict transform of W_i is flat and locally of finite presentation, then there exists a U-admissible blowing up $S' \to S$ dominating all $S_i \to S$ which does the job (see Divisors, Lemma 35.4; see also Remark 30.1). Hence we may assume X is affine.

Assume X is affine. By Morphisms, Lemma 40.2 we can choose an immersion $j : X \to \mathbf{A}_S^n$ over S. Let $V \subset \mathbf{A}_S^n$ be a quasi-compact open subscheme such that j induces a closed immersion $i : X \to V$ over S. Apply Theorem 30.7 to $V \to S$ and the quasi-coherent module $i_*\mathcal{O}_X$ to obtain a U-admissible blowup $S' \to S$ such that the strict transform of $i_*\mathcal{O}_X$ is flat over S' and of finite presentation over $\mathcal{O}_{V \times_S S'}$. Let X' be the strict transform of X with respect to $S' \to S$. Let $i' : X' \to V \times_S S'$ be the induced morphism. Since taking strict transform commutes with pushforward

along affine morphisms (Divisors, Lemma 34.4), we see that $i'_*\mathcal{O}_{X'}$ is flat over S and of finite presentation as a $\mathcal{O}_{V\times_S S'}$-module. This implies the lemma. □

0B49 **Lemma 31.2.** *Let S be a quasi-compact and quasi-separated scheme. Let X be a scheme over S. Let $U \subset S$ be a quasi-compact open. Assume*

(1) *$X \to S$ is proper, and*
(2) *$X_U \to U$ is finite locally free.*

Then there exists a U-admissible blowup $S' \to S$ such that the strict transform of X is finite locally free over S'.

Proof. By Lemma 31.1 we may assume that $X \to S$ is flat and of finite presentation. After replacing S by a U-admissible blowup if necessary, we may assume that $U \subset S$ is scheme theoretically dense. Then f is finite by Lemma 11.4. Hence f is finite locally free by Morphisms, Lemma 49.2. □

081S **Lemma 31.3.** *Let $\varphi : X \to S$ be a separated morphism of finite type with S quasi-compact and quasi-separated. Let $U \subset S$ be a quasi-compact open such that $\varphi^{-1}U \to U$ is an isomorphism. Then there exists a U-admissible blowup $S' \to S$ such that the strict transform X' of X is isomorphic to an open subscheme of S'.*

Proof. The discussion in Remark 30.1 applies. Thus we may do a first U-admissible blowup and assume the complement $S \setminus U$ is the support of an effective Cartier divisor D. In particular U is scheme theoretically dense in S. Next, we do another U-admissible blowup to get to the situation where $X \to S$ is flat and of finite presentation, see Lemma 31.1. In this case the result follows from Lemma 11.5. □

The following lemma says that a proper modification can be dominated by a blowup.

081T **Lemma 31.4.** *Let $\varphi : X \to S$ be a proper morphism with S quasi-compact and quasi-separated. Let $U \subset S$ be a quasi-compact open such that $\varphi^{-1}U \to U$ is an isomorphism. Then there exists a U-admissible blowup $S' \to S$ which dominates X, i.e., such that there exists a factorization $S' \to X \to S$ of the blowup morphism.*

Proof. The discussion in Remark 30.1 applies. Thus we may do a first U-admissible blowup and assume the complement $S \setminus U$ is the support of an effective Cartier divisor D. In particular U is scheme theoretically dense in S. Choose another U-admissible blowup $S' \to S$ such that the strict transform X' of X is an open subscheme of S', see Lemma 31.3. Since $X' \to S'$ is proper, and $U \subset S'$ is dense, we see that $X' = S'$. Some details omitted. □

0CP1 **Lemma 31.5.** *Let S be a scheme. Let $U \subset W \subset S$ be open subschemes. Let $f : X \to W$ be a morphism and let $s : U \to X$ be a morphism such that $f \circ s = id_U$. Assume*

(1) *f is proper,*
(2) *S is quasi-compact and quasi-separated, and*
(3) *U and W are quasi-compact.*

Then there exists a U-admissible blowup $b : S' \to S$ and a morphism $s' : b^{-1}(W) \to X$ extending s with $f \circ s' = b|_{b^{-1}(W)}$.

Proof. We may and do replace X by the scheme theoretic image of s. Then $X \to W$ is an isomorphism over U, see Morphisms, Lemma 6.8. By Lemma 31.4 there exists a U-admissible blowup $W' \to W$ and an extension $W' \to X$ of s.

We finish the proof by applying Divisors, Lemma 35.3 to extend $W' \to W$ to a U-admissible blowup of S. □

32. Compactifications

0ATT Let S be a quasi-compact and quasi-separated scheme. We will say a scheme X over S *has a compactification over* S or *is compactifyable over* S if there exists a quasi-compact open immersion $X \to \overline{X}$ into a scheme $\overline{X}$ proper over S. If X has a compactification over S, then $X \to S$ is separated and of finite type. It is a theorem of Nagata, see [Lüt93], [Con07], [Nag56], [Nag57], [Nag62], and [Nag63], that the converse is true as well. We will prove this theorem in the next section, see Theorem 33.8.

Let S be a quasi-compact and quasi-separated scheme. Let $X \to S$ be a separated finite type morphism of schemes. The category of *compactifications of* X *over* S is the category defined as follows:

(1) Objects are open immersions $j : X \to \overline{X}$ over S with $\overline{X} \to S$ proper.
(2) Morphisms $(j' : X \to \overline{X}') \to (j : X \to \overline{X})$ are morphisms $f : \overline{X}' \to \overline{X}$ of schemes over S such that $f \circ j' = j$.

If $j : X \to \overline{X}$ is a compactification, then j is a quasi-compact open immersion, see Schemes, Remark 21.18.

Warning. We do *not* assume compactifications $j : X \to \overline{X}$ to have dense image. Consequently, if $f : \overline{X}' \to \overline{X}$ is a morphism of compactifications, it may not be the case that $f^{-1}(j(X)) = j'(X)$.

0ATU **Lemma 32.1.** *Let S be a quasi-compact and quasi-separated scheme. Let X be a compactifyable scheme over S.*

(a) *The category of compactifications of X over S is cofiltered.*
(b) *The full subcategory consisting of compactifications $j : X \to \overline{X}$ such that $j(X)$ is dense and scheme theoretically dense in $\overline{X}$ is initial (Categories, Definition 17.3).*
(c) *If $f : \overline{X}' \to \overline{X}$ is a morphism of compactifications of X such that $j'(X)$ is dense in $\overline{X}'$, then $f^{-1}(j(X)) = j'(X)$.*

Proof. To prove part (a) we have to check conditions (1), (2), (3) of Categories, Definition 20.1. Condition (1) holds exactly because we assumed that X is compactifyable. Let $j_i : X \to \overline{X}_i$, $i = 1, 2$ be two compactifications. Then we can consider the scheme theoretic image $\overline{X}$ of $(j_1, j_2) : X \to \overline{X}_1 \times_S \overline{X}_2$. This determines a third compactification $j : X \to \overline{X}$ which dominates both j_i:

$$(X, \overline{X}_1) \longleftarrow (X, \overline{X}) \longrightarrow (X, \overline{X}_2)$$

Thus (2) holds. Let $f_1, f_2 : \overline{X}_1 \to \overline{X}_2$ be two morphisms between compactifications $j_i : X \to \overline{X}_i$, $i = 1, 2$. Let $\overline{X} \subset \overline{X}_1$ be the equalizer of f_1 and f_2. As $\overline{X}_2 \to S$ is separated, we see that $\overline{X}$ is a closed subscheme of $\overline{X}_1$ and hence proper over S. Moreover, we obtain an open immersion $X \to \overline{X}$ because $f_1|_X = f_2|_X = \mathrm{id}_X$. The morphism $(X \to \overline{X}) \to (j_1 : X \to \overline{X}_1)$ given by the closed immersion $\overline{X} \to \overline{X}_1$ equalizes f_1 and f_2 which proves condition (3).

Proof of (b). Let $j : X \to \overline{X}$ be a compactification. If $\overline{X}'$ denotes the scheme theoretic closure of X in $\overline{X}$, then X is dense and scheme theoretically dense in $\overline{X}'$

by Morphisms, Lemma 7.7. This proves the first condition of Categories, Definition 17.3. Since we have already shown the category of compactifications of X is cofiltered, the second condition of Categories, Definition 17.3 follows from the first (we omit the solution to this categorical exercise).

Proof of (c). After replacing $\overline{X}'$ with the scheme theoretic closure of $j'(X)$ (which doesn't change the underlying topological space) this follows from Morphisms, Lemma 6.8. □

We can also consider the category of all compactifications (for varying X). It turns out that this category, localized at the set of morphisms which induce an isomorphism on the interior is equivalent to the category of compactifyable schemes over S.

0A9Z **Lemma 32.2.** *Let S be a quasi-compact and quasi-separated scheme. Let $f : X \to Y$ be a morphism of schemes over S with Y separated and of finite type over S and X compactifyable over S. Then X has a compactification over Y.*

Proof. Let $j : X \to \overline{X}$ be a compactification of X over S. Then we let $\overline{X}'$ be the scheme theoretic image of $(j, f) : X \to \overline{X} \times_S Y$. The morphism $\overline{X}' \to Y$ is proper because $\overline{X} \times_S Y \to Y$ is proper as a base change of $\overline{X} \to S$. On the other hand, since Y is separated over S, the morphism $(1, f) : X \to X \times_S Y$ is a closed immersion (Schemes, Lemma 21.10) and hence $X \to \overline{X}'$ is an open immersion by Morphisms, Lemma 6.8 applied to the "partial section" $s = (j, f)$ to the projection $\overline{X} \times_S Y \to \overline{X}$. □

Let S be a quasi-compact and quasi-separated scheme. We define the *category of compactifications* to be the category whose objects are pairs $(X, \overline{X})$ where $\overline{X}$ is a scheme proper over S and $X \subset \overline{X}$ is a quasi-compact open and whose morphisms are commutative diagrams

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow \\ \overline{X} & \xrightarrow{\overline{f}} & \overline{Y} \end{array}$$

of morphisms of schemes over S.

0ATV **Lemma 32.3.** *Let S be a quasi-compact and quasi-separated scheme. The collection of morphisms $(u, \overline{u}) : (X', \overline{X}') \to (X, \overline{X})$ such that u is an isomorphism forms a right multiplicative system (Categories, Definition 27.1) of arrows in the category of compactifications.*

Proof. Axiom RMS1 is trivial to verify. Let us check RMS2 holds. Suppose given a diagram

$$\begin{array}{ccc} & & (X', \overline{X}') \\ & & \downarrow{\scriptstyle (u,\overline{u})} \\ (Y, \overline{Y}) & \xrightarrow{(f,\overline{f})} & (X, \overline{X}) \end{array}$$

with $u : X' \to X$ an isomorphism. Then we let $Y' = Y \times_X X'$ with the projection map $v : Y' \to Y$ (an isomorphism). We also set $\overline{Y}' = \overline{Y} \times_{\overline{X}} \overline{X}'$ with the projection

map $\overline{v} : \overline{Y}' \to \overline{Y}$ It is clear that $Y' \to \overline{Y}'$ is an open immersion. The diagram

$$\begin{array}{ccc} (Y', \overline{Y}') & \xrightarrow[(g,\overline{g})]{} & (X', \overline{X}') \\ {\scriptstyle (v,\overline{v})}\downarrow & & \downarrow{\scriptstyle (u,\overline{u})} \\ (Y, \overline{Y}) & \xrightarrow{(f,\overline{f})} & (X, \overline{X}) \end{array}$$

shows that axiom RMS2 holds.

Let us check RMS3 holds. Suppose given a pair of morphisms $(f, \overline{f}), (g, \overline{g}) : (X, \overline{X}) \to (Y, \overline{Y})$ of compactifications and a morphism $(v, \overline{v}) : (Y, \overline{Y}) \to (Y', \overline{Y}')$ such that v is an isomorphism and such that $(v, \overline{v}) \circ (f, \overline{f}) = (v, \overline{v}) \circ (g, \overline{g})$. Then $f = g$. Hence if we let $\overline{X}' \subset \overline{X}$ be the equalizer of $\overline{f}$ and $\overline{g}$, then $(u, \overline{u}) : (X, \overline{X}') \to (X, \overline{X})$ will be a morphism of the category of compactifications such that $(f, \overline{f}) \circ (u, \overline{u}) = (g, \overline{g}) \circ (u, \overline{u})$ as desired. □

0ATW **Lemma 32.4.** *Let S be a quasi-compact and quasi-separated scheme. The functor $(X, \overline{X}) \mapsto X$ defines an equivalence from the category of compactifications localized (Categories, Lemma 27.11) at the right multiplicative system of Lemma 32.3 to the category of compactifyable schemes over S.*

Proof. Denote $\mathcal{C}$ the category of compactifications and denote $Q : \mathcal{C} \to \mathcal{C}'$ the localization functor of Categories, Lemma 27.16. Denote $\mathcal{D}$ the category of compactifyable schemes over S. It is clear from the lemma just cited and our choice of multiplicative system that we obtain a functor $\mathcal{C}' \to \mathcal{D}$. This functor is clearly essentially surjective. If $f : X \to Y$ is a morphism of compactifyable schemes, then we choose an open immersion $Y \to \overline{Y}$ into a scheme proper over S, and then we choose an embedding $X \to \overline{X}$ into a scheme $\overline{X}$ proper over $\overline{Y}$ (possible by Lemma 32.2 applied to $X \to \overline{Y}$). This gives a morphism $(X, \overline{X}) \to (Y, \overline{Y})$ of compactifications which produces our given morphism $X \to Y$. Finally, suppose given a pair of morphisms in the localized category with the same source and target: say

$$a = ((f, \overline{f}) : (X', \overline{X}') \to (Y, \overline{Y}), (u, \overline{u}) : (X', \overline{X}') \to (X, \overline{X}))$$

and

$$b = ((g, \overline{g}) : (X'', \overline{X}'') \to (Y, \overline{Y}), (v, \overline{v}) : (X'', \overline{X}'') \to (X, \overline{X}))$$

which produce the same morphism $X \to Y$ over S, in other words $f \circ u^{-1} = g \circ v^{-1}$. By Categories, Lemma 27.13 we may assume that $(X', \overline{X}') = (X'', \overline{X}'')$ and $(u, \overline{u}) = (v, \overline{v})$. In this case we can consider the equalizer $\overline{X}''' \subset \overline{X}'$ of $\overline{f}$ and $\overline{g}$. The morphism $(w, \overline{w}) : (X', \overline{X}''') \to (X', \overline{X}')$ is in the multiplicative subset and we see that $a = b$ in the localized category by precomposing with $(w, \overline{w})$. □

33. Nagata compactification

0F3T In this section we prove the theorem announced in Section 32.

0F3U **Lemma 33.1.** *Let $X \to S$ be a morphism of schemes. If $X = U \cup V$ is an open cover such that $U \to S$ and $V \to S$ are separated and $U \cap V \to U \times_S V$ is closed, then $X \to S$ is separated.*

Proof. Omitted. Hint: check that $\Delta : X \to X \times_S X$ is closed by using the open covering of $X \times_S X$ given by $U \times_S U$, $U \times_S V$, $V \times_S U$, and $V \times_S V$. □

0F3V **Lemma 33.2.** *Let X be a quasi-compact and quasi-separated scheme. Let $U \subset X$ be a quasi-compact open.*

(1) *If $Z_1, Z_2 \subset X$ are closed subschemes of finite presentation such that $Z_1 \cap Z_2 \cap U = \emptyset$, then there exists a U-admissible blowing up $X' \to X$ such that the strict transforms of Z_1 and Z_2 are disjoint.*
(2) *If $T_1, T_2 \subset U$ are disjoint constructible closed subsets, then there is a U-admissible blowing up $X' \to X$ such that the closures of T_1 and T_2 are disjoint.*

Proof. Proof of (1). The assumption that $Z_i \to X$ is of finite presentation signifies that the quasi-coherent ideal sheaf $\mathcal{I}_i$ of Z_i is of finite type, see Morphisms, Lemma 22.7. Denote $Z \subset X$ the closed subscheme cut out by the product $\mathcal{I}_1\mathcal{I}_2$. Observe that $Z \cap U$ is the disjoint union of $Z_1 \cap U$ and $Z_2 \cap U$. By Divisors, Lemma 35.5 there is a $U \cap Z$-admissible blowup $Z' \to Z$ such that the strict transforms of Z_1 and Z_2 are disjoint. Denote $Y \subset Z$ the center of this blowing up. Then $Y \to X$ is a closed immersion of finite presentation as the composition of $Y \to Z$ and $Z \to X$ (Divisors, Definition 35.1 and Morphisms, Lemma 22.3). Thus the blowing up $X' \to X$ of Y is a U-admissible blowing up. By general properties of strict transforms, the strict transform of Z_1, Z_2 with respect to $X' \to X$ is the same as the strict transform of Z_1, Z_2 with respect to $Z' \to Z$, see Divisors, Lemma 34.2. Thus (1) is proved.

Proof of (2). By Properties, Lemma 25.1 there exists a finite type quasi-coherent sheaf of ideals $\mathcal{J}_i \subset \mathcal{O}_U$ such that $T_i = V(\mathcal{J}_i)$ (set theoretically). By Properties, Lemma 23.2 there exists a finite type quasi-coherent sheaf of ideals $\mathcal{I}_i \subset \mathcal{O}_X$ whose restriction to U is $\mathcal{J}_i$. Apply the result of part (1) to the closed subschemes $Z_i = V(\mathcal{I}_i)$ to conclude. □

0F3W **Lemma 33.3.** *Let $f : X \to Y$ be a proper morphism of quasi-compact and quasi-separated schemes. Let $V \subset Y$ be a quasi-compact open and $U = f^{-1}(V)$. Let $T \subset V$ be a closed subset such that $f|_U : U \to V$ is an isomorphism over an open neighbourhood of T in V. Then there exists a V-admissible blowing up $Y' \to Y$ such that the strict transform $f' : X' \to Y'$ of f is an isomorphism over an open neighbourhood of the closure of T in Y'.*

Proof. Let $T' \subset V$ be the complement of the maximal open over which $f|_U$ is an isomorphism. Then T', T are closed in V and $T \cap T' = \emptyset$. Since V is a spectral topological space, we can find constructible closed subsets T_c, T'_c with $T \subset T_c$, $T' \subset T'_c$ such that $T_c \cap T'_c = \emptyset$ (choose a quasi-compact open W of V containing T' not meeting T and set $T_c = V \setminus W$, then choose a quasi-compact open W' of V containing T_c not meeting T' and set $T'_c = V \setminus W'$). By Lemma 33.2 we may, after replacing Y by a V-admissible blowing up, assume that T_c and T'_c have disjoint closures in Y. Set $Y_0 = Y \setminus \overline{T}'_c$, $V_0 = V \setminus T'_c$, $U_0 = U \times_V V_0$, and $X_0 = X \times_Y Y_0$. Since $U_0 \to V_0$ is an isomorphism, we can find a V_0-admissible blowing up $Y'_0 \to Y_0$ such that the strict transform X'_0 of X_0 maps isomorphically to Y'_0, see Lemma 31.3. By Divisors, Lemma 35.3 there exists a V-admissible blow up $Y' \to Y$ whose restriction to Y_0 is $Y'_0 \to Y_0$. If $f' : X' \to Y'$ denotes the strict transform of f, then we see what we want is true because f' restricts to an isomorphism over Y'_0. □

0F3X **Lemma 33.4.** *Let S be a quasi-compact and quasi-separated scheme. Let $U \to X_1$ and $U \to X_2$ be open immersions of schemes over S and assume U, X_1, X_2 of finite*

type and separated over S. Then there exists a commutative diagram

$$\begin{array}{ccccc} X_1' & \longrightarrow & X & \longleftarrow & X_2' \\ \downarrow & \nwarrow & \uparrow & \nearrow & \downarrow \\ X_1 & \longleftarrow & U & \longrightarrow & X_2 \end{array}$$

of schemes over S where $X_i' \to X_i$ is a U-admissible blowup, $X_i' \to X$ is an open immersion, and X is separated and finite type over S.

Proof. Throughout the proof all schemes will be separated of finite type over S. This in particular implies these schemes are quasi-compact and quasi-separated and the morphisms between them are quasi-compact and separated. See Schemes, Sections 19 and 21. We will use that if $U \to W$ is an immersion of such schemes over S, then the scheme theoretic image Z of U in W is a closed subscheme of W and $U \to Z$ is an open immersion, $U \subset Z$ is scheme theoretically dense, and $U \subset Z$ is dense topologically. See Morphisms, Lemma 7.7.

Let $X_{12} \subset X_1 \times_S X_2$ be the scheme theoretic image of $U \to X_1 \times_S X_2$. The projections $p_i : X_{12} \to X_i$ induce isomorphisms $p_i^{-1}(U) \to U$ by Morphisms, Lemma 6.8. Choose a U-admissible blowup $X_i^i \to X_i$ such that the strict transform X_{12}^i of X_{12} is isomorphic to an open subscheme of X_i^i, see Lemma 31.3. Let $\mathcal{I}_i \subset \mathcal{O}_{X_i}$ be the corresponding finite type quasi-coherent sheaf of ideals. Recall that $X_{12}^i \to X_{12}$ is the blowup in $p_i^{-1}\mathcal{I}_i\mathcal{O}_{X_{12}}$, see Divisors, Lemma 34.2. Let X_{12}' be the blowup of X_{12} in $p_1^{-1}\mathcal{I}_1 p_2^{-1}\mathcal{I}_2\mathcal{O}_{X_{12}}$, see Divisors, Lemma 33.12 for what this entails. We obtain in particular a commutative diagram

$$\begin{array}{ccc} X_{12}' & \longrightarrow & X_{12}^2 \\ \downarrow & & \downarrow \\ X_{12}^1 & \longrightarrow & X_{12} \end{array}$$

where all the morphisms are U-admissible blowing ups. Since $X_{12}^i \subset X_i^i$ is an open we may choose a U-admissible blowup $X_i' \to X_i^i$ restricting to $X_{12}' \to X_{12}^i$, see Divisors, Lemma 35.3. Then $X_{12}' \subset X_i'$ is an open subscheme and the diagram

$$\begin{array}{ccc} X_{12}' & \longrightarrow & X_i' \\ \downarrow & & \downarrow \\ X_{12}^i & \longrightarrow & X_i^i \end{array}$$

is commutative with vertical arrows blowing ups and horizontal arrows open immersions. Note that $X_{12}' \to X_1' \times_S X_2'$ is an immersion and proper (use that $X_{12}' \to X_{12}$ is proper and $X_{12} \to X_1 \times_S X_2$ is closed and $X_1' \times_S X_2' \to X_1 \times_S X_2$ is separated and apply Morphisms, Lemma 42.7). Thus $X_{12}' \to X_1' \times_S X_2'$ is a closed immersion. It follows that if we define X by glueing X_1' and X_2' along the common open subscheme X_{12}', then $X \to S$ is of finite type and separated (Lemma 33.1). As compositions of U-admissible blowups are U-admissible blowups (Divisors, Lemma 35.2) the lemma is proved. □

0F3Y **Lemma 33.5.** *Let $X \to S$ and $Y \to S$ be morphisms of schemes. Let $U \subset X$ be an open subscheme. Let $V \to X \times_S Y$ be a quasi-compact morphism whose*

composition with the first projection maps into U. Let $Z \subset X \times_S Y$ be the scheme theoretic image of $V \to X \times_S Y$. Let $X' \to X$ be a U-admissible blowup. Then the scheme theoretic image of $V \to X' \times_S Y$ is the strict transform of Z with respect to the blowing up.

Proof. Denote $Z' \to Z$ the strict transform. The morphism $Z' \to X'$ induces a morphism $Z' \to X' \times_S Y$ which is a closed immersion (as Z' is a closed subscheme of $X' \times_X Z$ by definition). Thus to finish the proof it suffices to show that the scheme theoretic image Z'' of $V \to Z'$ is Z'. Observe that $Z'' \subset Z'$ is a closed subscheme such that $V \to Z'$ factors through Z''. Since both $V \to X \times_S Y$ and $V \to X' \times_S Y$ are quasi-compact (for the latter this follows from Schemes, Lemma 21.14 and the fact that $X' \times_S Y \to X \times_S Y$ is separated as a base change of a proper morphism), by Morphisms, Lemma 6.3 we see that $Z \cap (U \times_S Y) = Z'' \cap (U \times_S Y)$. Thus the inclusion morphism $Z'' \to Z'$ is an isomorphism away from the exceptional divisor E of $Z' \to Z$. However, the structure sheaf of Z' does not have any nonzero sections supported on E (by definition of strict transforms) and we conclude that the surjection $\mathcal{O}_{Z'} \to \mathcal{O}_{Z''}$ must be an isomorphism. □

0F3Z **Lemma 33.6.** *Let S be a quasi-compact and quasi-separated scheme. Let U be a scheme of finite type and separated over S. Let $V \subset U$ be a quasi-compact open. If V has a compactification $V \subset Y$ over S, then there exists a V-admissible blowing up $Y' \to Y$ and an open $V \subset V' \subset Y'$ such that $V \to U$ extends to a proper morphism $V' \to U$.*

Proof. Consider the scheme theoretic image $Z \subset Y \times_S U$ of the "diagonal" morphism $V \to Y \times_S U$. If we replace Y by a V-admissible blowing up, then Z is replaced by the strict transform with respect to this blowing up, see Lemma 33.5. Hence by Lemma 31.3 we may assume $Z \to Y$ is an open immersion. If $V' \subset Y$ denotes the image, then we see that the induced morphism $V' \to U$ is proper because the projection $Y \times_S U \to U$ is proper and $V' \cong Z$ is a closed subscheme of $Y \times_S U$. □

The following lemma is formulated in the Noetherian case only. The version for quasi-compact and quasi-separated schemes is true as well, but will be trivially implied by the main theorem in this section.

0F40 **Lemma 33.7.** *Let S be a Noetherian scheme. Let U be a scheme of finite type and separated over S. Let $U = U_1 \cup U_2$ be opens such that U_1 and U_2 have compactifications over S and such that $U_1 \cap U_2$ is dense in U. Then U has a compactification over S.*

Proof. Choose a compactification $U_i \subset X_i$ for $i = 1, 2$. We may assume U_i is scheme theoretically dense in X_i. We may assume there is an open $V_i \subset X_i$ and a proper morphism $\psi_i : V_i \to U$ extending $\mathrm{id} : U_i \to U_i$, see Lemma 33.6. Picture

$$\begin{array}{ccccc} U_i & \longrightarrow & V_i & \longrightarrow & X_i \\ \downarrow & \swarrow_{\psi_i} & & & \\ U & & & & \end{array}$$

If $\{i, j\} = \{1, 2\}$ denote $Z_i = U \setminus U_j = U_i \setminus (U_1 \cap U_2)$ and $Z_j = U \setminus U_i = U_j \setminus (U_1 \cap U_2)$. Thus we have

$$U = U_1 \amalg Z_2 = Z_1 \amalg U_2 = Z_1 \amalg (U_1 \cap U_2) \amalg Z_2$$

Denote $Z_{i,i} \subset V_i$ the inverse image of Z_i under ψ_i. Observe that ψ_i is an isomorphism over an open neighbourhood of Z_i. Denote $Z_{i,j} \subset V_i$ the inverse image of Z_j under ψ_i. Observe that $\psi_i : Z_{i,j} \to Z_j$ is a proper morphism. Since Z_i and Z_j are disjoint closed subsets of U, we see that $Z_{i,i}$ and $Z_{i,j}$ are disjoint closed subsets of V_i.

Denote $\overline{Z}_{i,i}$ and $\overline{Z}_{i,j}$ the closures of $Z_{i,i}$ and $Z_{i,j}$ in X_i. After replacing X_i by a V_i-admissible blowup we may assume that $\overline{Z}_{i,i}$ and $\overline{Z}_{i,j}$ are disjoint, see Lemma 33.2. We assume this holds for both X_1 and X_2. Observe that this property is preserved if we replace X_i by a further V_i-admissible blowup.

Set $V_{12} = V_1 \times_U V_2$. We have an immersion $V_{12} \to X_1 \times_S X_2$ which is the composition of the closed immersion $V_{12} = V_1 \times_U V_2 \to V_1 \times_S V_2$ (Schemes, Lemma 21.9) and the open immersion $V_1 \times_S V_2 \to X_1 \times_S X_2$. Let $X_{12} \subset X_1 \times_S X_2$ be the scheme theoretic image of $V_{12} \to X_1 \times_S X_2$. The projection morphisms

$$p_1 : X_{12} \to X_1 \quad \text{and} \quad p_2 : X_{12} \to X_2$$

are proper as X_1 and X_2 are proper over S. If we replace X_1 by a V_1-admissible blowing up, then X_{12} is replaced by the strict transform with respect to this blowing up, see Lemma 33.5.

Denote $\psi : V_{12} \to U$ the compositions $\psi = \psi_1 \circ p_1|_{V_{12}} = \psi_2 \circ p_2|_{V_{12}}$. Consider the closed subscheme

$$Z_{12,2} = (p_1|_{V_{12}})^{-1}(Z_{1,2}) = (p_2|_{V_{12}})^{-1}(Z_{2,2}) = \psi^{-1}(Z_2) \subset V_{12}$$

The morphism $p_1|_{V_{12}} : V_{12} \to V_1$ is an isomorphism over an open neighbourhood of $Z_{1,2}$ because $\psi_2 : V_2 \to U$ is an isomorphism over an open neighbourhood of Z_2 and $V_{12} = V_1 \times_U V_2$. By Lemma 33.3 there exists a V_1-admissible blowing up $X_1' \to X_1$ such that the strict transform $p_1' : X_{12}' \to X_1'$ of p_1 is an isomorphism over an open neighbourhood of the closure of $Z_{1,2}$ in X_1'. After replacing X_1 by X_1' and X_{12} by X_{12}' we may assume that p_1 is an isomorphism over an open neighbourhood of $\overline{Z}_{1,2}$.

The reduction of the previous paragraph tells us that

$$X_{12} \cap (\overline{Z}_{1,2} \times_S \overline{Z}_{2,1}) = \emptyset$$

where the intersection taken in $X_1 \times_S X_2$. Namely, the inverse image $p_1^{-1}(\overline{Z}_{1,2})$ in X_{12} maps isomorphically to $\overline{Z}_{1,2}$. In particular, we see that $Z_{12,2}$ is dense in $p_1^{-1}(\overline{Z}_{1,2})$. Thus p_2 maps $p_1^{-1}(\overline{Z}_{1,2})$ into $\overline{Z}_{2,2}$. Since $\overline{Z}_{2,2} \cap \overline{Z}_{2,1} = \emptyset$ we conclude.

Consider the schemes

$$W_i = U \coprod\nolimits_{U_i} (X_i \setminus \overline{Z}_{i,j}), \quad i = 1, 2$$

obtained by glueing. Let us apply Lemma 33.1 to see that $W_i \to S$ is separated. First, $U \to S$ and $X_i \to S$ are separated. The immersion $U_i \to U \times_S (X_i \setminus \overline{Z}_{i,j})$ is closed because any specialization $u_i \rightsquigarrow u$ with $u_i \in U_i$ and $u \in U \setminus U_i$ can be lifted uniquely to a specialization $u_i \rightsquigarrow v_i$ in V_i along the proper morphism $\psi_i : V_i \to U$ and then v_i must be in $Z_{i,j}$. Thus the image of the immersion is closed, whence the immersion is a closed immersion.

On the other hand, for any valuation ring A over S with fraction field K and any morphism $\gamma : \mathrm{Spec}(K) \to (U_1 \cap U_2)$ over S, there is an i and an extension of γ to a morphism $h_i : \mathrm{Spec}(A) \to W_i$. Namely, for both $i = 1, 2$ there is a morphism $g_i : \mathrm{Spec}(A) \to X_i$ extending γ by the valuative criterion of properness for X_i

over S, see Morphisms, Lemma 43.1. Thus we only are in trouble if $g_i(\mathfrak{m}_A) \in \overline{Z}_{i,j}$ for $i = 1, 2$. This is impossible by the emptyness of the intersection of X_{12} and $\overline{Z}_{1,2} \times_S \overline{Z}_{2,1}$ we proved above.

Consider a diagram

$$\begin{array}{ccccc} W_1' & \longrightarrow & W & \longleftarrow & W_2' \\ \downarrow & \nwarrow & \uparrow & \nearrow & \downarrow \\ W_1 & \longleftarrow & U & \longrightarrow & W_2 \end{array}$$

as in Lemma 33.4. By the previous paragraph for every solid diagram

$$\begin{array}{ccc} \mathrm{Spec}(K) & \xrightarrow{\gamma} & W \\ \downarrow & \nearrow & \downarrow \\ \mathrm{Spec}(A) & \longrightarrow & S \end{array}$$

where $\mathrm{Im}(\gamma) \subset U_1 \cap U_2$ there is an i and an extension $h_i : \mathrm{Spec}(A) \to W_i$ of γ. Using the valuative criterion of properness for $W_i' \to W_i$, we can then lift h_i to $h_i' : \mathrm{Spec}(A) \to W_i'$. Hence the dotted arrow in the diagram exists. Since W is separated over S, we see that the arrow is unique as well. This implies that $W \to S$ is universally closed by Morphisms, Lemma 43.2. As $W \to S$ is already of finite type and separated, we win. □

0F41 **Theorem 33.8.** *Let S be a quasi-compact and quasi-separated scheme. Let $X \to S$ be a separated, finite type morphism. Then X has a compactification over S.*

See [Lüt93], [Con07], [Nag56], [Nag57], [Nag62], and [Nag63]

Proof. We first reduce to the Noetherian case. We strongly urge the reader to skip this paragraph. There exists a closed immersion $X \to X'$ with $X' \to S$ of finite presentation and separated. See Limits, Proposition 9.6. If we find a compactification of X' over S, then taking the scheme theoretic image of X in this will give a compactification of X over S. Thus we may assume $X \to S$ is separated and of finite presentation. We may write $S = \lim S_i$ as a directed limit of a system of Noetherian schemes with affine transition morphisms. See Limits, Proposition 5.4. We can choose an i and a morphism $X_i \to S_i$ of finite presentation whose base change to S is $X \to S$, see Limits, Lemma 10.1. After increasing i we may assume $X_i \to S_i$ is separated, see Limits, Lemma 8.6. If we can find a compactification of X_i over S_i, then the base change of this to S will be a compactification of X over S. This reduces us to the case discussed in the next paragraph.

Assume S is Noetherian. We can choose a finite affine open covering $X = \bigcup_{i=1,\ldots,n} U_i$ such that $U_1 \cap \ldots \cap U_n$ is dense in X. This follows from Properties, Lemma 30.4 and the fact that X is quasi-compact with finitely many irreducible components. For each i we can choose an $n_i \geq 0$ and an immersion $U_i \to \mathbf{A}_S^{n_i}$ by Morphisms, Lemma 40.2. Hence U_i has a compactification over S for $i = 1, \ldots, n$ by taking the scheme theoretic image in $\mathbf{P}_S^{n_i}$. Applying Lemma 33.7 $(n-1)$ times we conclude that the theorem is true. □

34. The h topology

0ETQ For us, loosely speaking, an h sheaf is a sheaf for the Zariski topology which satisfies the sheaf property for surjective proper morphisms of finite presentation, see Lemma

34.17. However, it may be worth pointing out that the definition of the h topology on the category of schemes depends on the reference.

Voevodsky initially defined an h covering to be a finite collection of finite type morphisms which are jointly universally submersive (Morphisms, Definition 25.1). See [Voe96, Definition 3.1.2]. This definition works best if the underlying category of schemes is restricted to all schemes of finite type over a fixed Noetherian base scheme. In this setting, Voevodsky relates h coverings to ph coverings. The ph topology is generated by Zariski coverings and proper surjective morphisms. See Topologies, Section 8 for more information.

In Topologies, Section 10 we study the V topology. A quasi-compact morphism $X \to Y$ defines a V covering, if any specialization of points of Y is the image of a specialization of points in X and the same is true after any base change (Topologies, Lemma 10.13). In this case $X \to Y$ is universally submersive (Topologies, Lemma 10.14). It turns out the notion of a V covering is a good replacement for "families of morphisms with fixed target which are jointly universally submersive" when working with non-Noetherian schemes.

Our approach will be to first prove the equivalence between ph covers and V coverings for (possibly infinite) families of morphisms which are locally of finite presentation. We will then use these families as our notion of h coverings in the Stacks project. For Noetherian schemes and finite families these coverings match those in Voevodsky's definition, see Lemma 34.3. On the category of schemes of finite presentation over a fixed quasi-compact and quasi-separated scheme S these coverings determine the same topology as the one in [BS17, Definition 2.7].

0ETR **Lemma 34.1.** *Let $\{f_i : X_i \to X\}_{i \in I}$ be a family of morphisms of schemes with fixed target with f_i locally of finite presentation for all i. The following are equivalent*

(1) *$\{X_i \to X\}$ is a ph covering, and*
(2) *$\{X_i \to X\}$ is a V covering.*

Proof. Let $U \subset X$ be affine open. Looking at Topologies, Definitions 8.4 and 10.7 it suffices to show that the base change $\{X_i \times_X U \to U\}$ can be refined by a standard ph covering if and only if it can be refined by a standard V covering. Thus we may assume X is affine and we have to show $\{X_i \to X\}$ can be refined by a standard ph covering if and only if it can be refined by a standard V covering. Since a standard ph covering is a standard V covering, see Topologies, Lemma 10.3 it suffices to prove the other implication.

Assume X is affine and assume $\{f_i : X_i \to X\}_{i \in I}$ can be refined by a standard V covering $\{g_j : Y_j \to X\}_{j=1,\ldots,m}$. For each j choose an i_j and a morphism $h_j : Y_j \to X_{i_j}$ such that $g_j = f_{i_j} \circ h_j$. Since Y_j is affine hence quasi-compact, for each j we can find finitely many affine opens $U_{j,k} \subset X_{i_j}$ such that $\mathrm{Im}(h_j) \subset \bigcup U_{j,k}$. Then $\{U_{j,k} \to X\}_{j,k}$ refines $\{X_i \to X\}$ and is a standard V covering (as it is a finite family of morphisms of affines and it inherits the lifting property for valuation rings from the corresponding property of $\{Y_j \to X\}$). Thus we reduce to the case discussed in the next paragraph.

Assume $\{f_i : X_i \to X\}_{i=1,\ldots,n}$ is a standard V covering with f_i of finite presentation. We have to show that $\{X_i \to X\}$ can be refined by a standard ph covering. Choose

a generic flatness stratification

$$X = S \supset S_0 \supset S_1 \supset \ldots \supset S_t = \emptyset$$

as in More on Morphisms, Lemma 54.2 for the finitely presented morphism

$$\coprod_{i=1,\ldots,n} f_i : \coprod_{i=1,\ldots,n} X_i \longrightarrow X$$

of affines. We are going to use all the properties of the stratification without further mention. By construction the base change of each f_i to $U_k = S_k \setminus S_{k+1}$ is flat. Denote Y_k the scheme theoretic closure of U_k in S_k. Since $U_k \to S_k$ is a quasi-compact open immersion (see Properties, Lemma 25.1), we see that $U_k \subset Y_k$ is a quasi-compact dense (and scheme theoretically dense) open immersion, see Morphisms, Lemma 6.3. The morphism $\coprod_{k=0,\ldots,t-1} Y_k \to X$ is finite surjective, hence $\{Y_k \to X\}$ is a standard ph covering and hence a standard V covering (see above). By the transitivity property of standard V coverings (Topologies, Lemma 10.5) it suffices to show that the pullback of the covering $\{X_i \to X\}$ to each Y_k can be refined by a standard V covering. This reduces us to the case described in the next paragraph.

Assume $\{f_i : X_i \to X\}_{i=1,\ldots,n}$ is a standard V covering with f_i of finite presentation and there is a dense quasi-compact open $U \subset X$ such that $X_i \times_X U \to U$ is flat. By Theorem 30.7 there is a U-admissible blowup $X' \to X$ such that the strict transform $f_i' : X_i' \to X'$ of f_i is flat. Observe that the projective (hence closed) morphism $X' \to X$ is surjective as $U \subset X$ is dense and as U is identified with an open of X'. After replacing X' by a further U-admissible blowup if necessary, we may also assume $U \subset X'$ is scheme theoretically dense (see Remark 30.1). Hence for every point $x \in X'$ there is a valuation ring V and a morphism $g : \mathrm{Spec}(V) \to X'$ such that the generic point of $\mathrm{Spec}(V)$ maps into U and the closed point of $\mathrm{Spec}(V)$ maps to x, see Morphisms, Lemma 6.5. Since $\{X_i \to X\}$ is a standard V covering, we can choose an extension of valuation rings $V \subset W$, an index i, and a morphism $\mathrm{Spec}(W) \to X_i$ such that the diagram

$$\begin{array}{ccccc} \mathrm{Spec}(W) & & \longrightarrow & & X_i \\ \downarrow & & & & \downarrow \\ \mathrm{Spec}(V) & \longrightarrow & X' & \longrightarrow & X \end{array}$$

is commutative. Since $X_i' \subset X' \times_X X_i$ is a closed subscheme containing the open $U \times_X X_i$, since $\mathrm{Spec}(W)$ is an integral scheme, and since the induced morphism $h : \mathrm{Spec}(W) \to X' \times_X X_i$ maps the generic point of $\mathrm{Spec}(W)$ into $U \times_X X_i$, we conclude that h factors through the closed subscheme $X_i' \subset X' \times_X X_i$. We conclude that $\{f_i' : X_i' \to X'\}$ is a V covering. In particular, $\coprod f_i'$ is surjective. In particular $\{X_i' \to X'\}$ is an fppf covering. Since an fppf covering is a ph covering (More on Morphisms, Lemma 48.7), we can find a standard ph covering $\{Y_j \to X'\}$ refining $\{X_i' \to X\}$. Say this covering is given by a proper surjective morphism $Y \to X'$ and a finite affine open covering $Y = \bigcup Y_j$. Then the composition $Y \to X$ is proper surjective and we conclude that $\{Y_j \to X\}$ is a standard ph covering. This finishes the proof. □

Here is our definition.

0ETS **Definition 34.2.** Let T be a scheme. A *h covering of T* is a family of morphisms $\{f_i : T_i \to T\}_{i\in I}$ such that each f_i is locally of finite presentation and one of the equivalent conditions of Lemma 34.1 is satisfied.

For Noetherian schemes this is the same thing as a ph covering (we record this in Lemma 34.4 below) and we recover Voevodsky's notion.

0ETT **Lemma 34.3.** *Let X be a Noetherian scheme. Let $\{X_i \to X\}_{i\in I}$ be a finite family of finite type morphisms. The following are equivalent*

(1) *$\coprod_{i\in I} X_i \to X$ is universally submersive (Morphisms, Definition 25.1), and*
(2) *$\{X_i \to X\}_{i\in I}$ is an h covering.*

Proof. The implication (2) $\Rightarrow$ (1) follows from the more general Topologies, Lemma 10.14 and our definition of h covers. Assume $\coprod X_i \to X$ is universally submersive. We will show that $\{X_i \to X\}$ can be refined by a ph covering; this will suffice by Topologies, Lemma 8.7 and our definition of h coverings. The argument will be the same as the one used in the proof of Lemma 34.1.

Choose a generic flatness stratification

$$X = S \supset S_0 \supset S_1 \supset \ldots \supset S_t = \emptyset$$

as in More on Morphisms, Lemma 54.2 for the finitely presented morphism

$$\coprod\nolimits_{i=1,\ldots,n} f_i : \coprod\nolimits_{i=1,\ldots,n} X_i \longrightarrow X$$

We are going to use all the properties of the stratification without further mention. By construction the base change of each f_i to $U_k = S_k \setminus S_{k+1}$ is flat. Denote Y_k the scheme theoretic closure of U_k in S_k. Since $U_k \to S_k$ is a quasi-compact open immersion (all schemes in this paragraph are Noetherian), we see that $U_k \subset Y_k$ is a quasi-compact dense (and scheme theoretically dense) open immersion, see Morphisms, Lemma 6.3. The morphism $\coprod_{k=0,\ldots,t-1} Y_k \to X$ is finite surjective, hence $\{Y_k \to X\}$ is a ph covering. By the transitivity property of ph coverings (Topologies, Lemma 8.8) it suffices to show that the pullback of the covering $\{X_i \to X\}$ to each Y_k can be refined by a ph covering. This reduces us to the case described in the next paragraph.

Assume $\coprod X_i \to X$ is universally submersive and there is a dense open $U \subset X$ such that $X_i \times_X U \to U$ is flat for all i. By Theorem 30.7 there is a U-admissible blowup $X' \to X$ such that the strict transform $f_i' : X_i' \to X'$ of f_i is flat for all i. Observe that the projective (hence closed) morphism $X' \to X$ is surjective as $U \subset X$ is dense and as U is identified with an open of X'. After replacing X' by a further U-admissible blowup if necessary, we may also assume $U \subset X'$ is dense (see Remark 30.1). Hence for every point $x \in X'$ there is a discrete valuation ring A and a morphism $g : \mathrm{Spec}(A) \to X'$ such that the generic point of $\mathrm{Spec}(A)$ maps into U and the closed point of $\mathrm{Spec}(A)$ maps to x, see Limits, Lemma 15.1. Set

$$W = \mathrm{Spec}(A) \times_X \coprod X_i = \coprod \mathrm{Spec}(A) \times_X X_i$$

Since $\coprod X_i \to X$ is universally submersive, there is a specialization $w' \rightsquigarrow w$ in W such that w' maps to the generic point of $\mathrm{Spec}(A)$ and w maps to the closed point of $\mathrm{Spec}(A)$. (If not, then the closed fibre of $W \to \mathrm{Spec}(A)$ is stable under generalizations, hence open, which contradicts the fact that $W \to \mathrm{Spec}(A)$ is submersive.) Say $w' \in \mathrm{Spec}(A) \times_X X_i$ so of course $w \in \mathrm{Spec}(A) \times_X X_i$ as well. Let $x_i' \rightsquigarrow x_i$

be the image of $w' \rightsquigarrow w$ in $X' \times_X X_i$. Since $x_i' \in X_i'$ and since $X_i' \subset X' \times_X X_i$ is a closed subscheme we see that $x_i \in X_i'$. Since x_i maps to $x \in X'$ we conclude that $\coprod X_i' \to X'$ is surjective! In particular $\{X_i' \to X'\}$ is an fppf covering. But an fppf covering is a ph covering (More on Morphisms, Lemma 48.7). Since $X' \to X$ is proper surjective, we conclude that $\{X_i' \to X\}$ is a ph covering and the proof is complete. □

0H45 **Lemma 34.4.** *Let X be a locally Noetherian scheme. A family of morphisms $\{f_i : X_i \to X\}_{i \in I}$ with target X is an h covering if and only if it is a ph covering.*

Proof. By Definition 34.2 a h covering is a ph covering. Conversely, if $\{f_i : X_i \to X\}$ is a ph covering, then the morphisms f_i are locally of finite type (Topologies, Definition 8.4). Since X is locally Noetherian, each f_i is locally of finite presentation and we see that we have a h covering by definition. □

The following lemma and [Ryd07, Theorem 8.4] shows our definition agrees with (or at least is closely related to) the definition in the paper [Ryd07] by David Rydh. We restrict to affine base for simplicity.

0ETU **Lemma 34.5.** *Let X be an affine scheme. Let $\{X_i \to X\}_{i \in I}$ be an h covering. Then there exists a surjective proper morphism*

$$Y \longrightarrow X$$

of finite presentation (!) and a finite affine open covering $Y = \bigcup_{j=1,\ldots,m} Y_j$ such that $\{Y_j \to X\}_{j=1,\ldots,m}$ refines $\{X_i \to X\}_{i \in I}$.

Proof. By assumption there exists a proper surjective morphism $Y \to X$ and a finite affine open covering $Y = \bigcup_{j=1,\ldots,m} Y_j$ such that $\{Y_j \to X\}_{j=1,\ldots,m}$ refines $\{X_i \to X\}_{i \in I}$. This means that for each j there is an index $i_j \in I$ and a morphism $h_j : Y_j \to X_{i_j}$ over X. See Definition 34.2 and Topologies, Definition 8.4. The problem is that we don't know that $Y \to X$ is of finite presentation. By Limits, Lemma 13.2 we can write

$$Y = \lim Y_\lambda$$

as a directed limit of schemes Y_λ proper and of finite presentation over X such that the morphisms $Y \to Y_\lambda$ and the the transition morphisms are closed immersions. Observe that each $Y_\lambda \to X$ is surjective. By Limits, Lemma 4.11 we can find a λ and quasi-compact opens $Y_{\lambda,j} \subset Y_\lambda$, $j = 1, \ldots, m$ covering Y_λ and restricting to Y_j in Y. Then $Y_j = \lim Y_{\lambda,j}$. After increasing λ we may assume $Y_{\lambda,j}$ is affine for all j, see Limits, Lemma 4.13. Finally, since $X_i \to X$ is locally of finite presentation we can use the functorial characterization of morphisms which are locally of finite presentation (Limits, Proposition 6.1) to find a λ such that for each j there is a morphism $h_{\lambda,j} : Y_{\lambda,j} \to X_{i_j}$ whose restriction to Y_j is the morphism h_j chosen above. Thus $\{Y_{\lambda,j} \to X\}$ refines $\{X_i \to X\}$ and the proof is complete. □

We return to the development of the general theory of h coverings.

0ETV **Lemma 34.6.** *An fppf covering is a h covering. Hence syntomic, smooth, étale, and Zariski coverings are h coverings as well.*

Proof. This is true because in an fppf covering the morphisms are required to be locally of finite presentation and because fppf coverings are ph covering, see More on Morphisms, Lemma 48.7. The second statement follows from the first and Topologies, Lemma 7.2. □

0ETW **Lemma 34.7.** *Let $f : Y \to X$ be a surjective proper morphism of schemes which is of finite presentation. Then $\{Y \to X\}$ is an h covering.*

Proof. Combine Topologies, Lemmas 10.10 and 8.6. □

0ETX **Lemma 34.8.** *Let T be a scheme. Let $\{f_i : T_i \to T\}_{i \in I}$ be a family of morphisms such that f_i is locally of finite presentation for all i. The following are equivalent*

(1) *$\{T_i \to T\}_{i \in I}$ is an h covering,*
(2) *there is an h covering which refines $\{T_i \to T\}_{i \in I}$, and*
(3) *$\{\coprod_{i \in I} T_i \to T\}$ is an h covering.*

Proof. This follows from the analogous statement for ph coverings (Topologies, Lemma 8.7) or from the analogous statement for V coverings (Topologies, Lemma 10.8). □

Next, we show that our notion of an h covering satisfies the conditions of Sites, Definition 6.2.

0ETY **Lemma 34.9.** *Let T be a scheme.*

(1) *If $T' \to T$ is an isomorphism then $\{T' \to T\}$ is an h covering of T.*
(2) *If $\{T_i \to T\}_{i \in I}$ is an h covering and for each i we have an h covering $\{T_{ij} \to T_i\}_{j \in J_i}$, then $\{T_{ij} \to T\}_{i \in I, j \in J_i}$ is an h covering.*
(3) *If $\{T_i \to T\}_{i \in I}$ is an h covering and $T' \to T$ is a morphism of schemes then $\{T' \times_T T_i \to T'\}_{i \in I}$ is an h covering.*

Proof. Follows immediately from the corresponding statement for either ph or V coverings (Topologies, Lemma 8.8 or 10.9) and the fact that the class of morphisms which are locally of finite presentation is preserved under base change and composition. □

Next, we define the big h sites we will work with in the Stacks project. It makes sense to read the general discussion in Topologies, Section 2 before proceeding.

0ETZ **Definition 34.10.** A *big h site* is any site Sch_h as in Sites, Definition 6.2 constructed as follows:

(1) Choose any set of schemes S_0, and any set of h coverings Cov_0 among these schemes.
(2) As underlying category take any category Sch_α constructed as in Sets, Lemma 9.2 starting with the set S_0.
(3) Choose any set of coverings as in Sets, Lemma 11.1 starting with the category Sch_α and the class of h coverings, and the set Cov_0 chosen above.

See the remarks following Topologies, Definition 3.5 for motivation and explanation regarding the definition of big sites.

0EUY **Definition 34.11.** Let T be an affine scheme. A *standard h covering* of T is a family $\{f_i : T_i \to T\}_{i=1,\ldots,n}$ with each T_i affine, with f_i of finite presentation satisfying either of the following equivalent conditions: (1) $\{T_i \to T\}$ can be refined by a standard ph covering or (2) $\{T_i \to T\}$ is a V covering.

The equivalence of the conditions follows from Lemma 34.1, Topologies, Definition 8.4, and Lemma 8.7.

Before we continue with the introduction of the big h site of a scheme S, let us point out that the topology on a big h site Sch_h is in some sense induced from the h topology on the category of all schemes.

0EU0 **Lemma 34.12.** *Let Sch_h be a big h site as in Definition 34.10. Let $T \in \mathrm{Ob}(Sch_h)$. Let $\{T_i \to T\}_{i\in I}$ be an arbitrary h covering of T.*

1. *There exists a covering $\{U_j \to T\}_{j\in J}$ of T in the site Sch_h which refines $\{T_i \to T\}_{i\in I}$.*
2. *If $\{T_i \to T\}_{i\in I}$ is a standard h covering, then it is tautologically equivalent to a covering of Sch_h.*
3. *If $\{T_i \to T\}_{i\in I}$ is a Zariski covering, then it is tautologically equivalent to a covering of Sch_h.*

Proof. Omitted. Hint: this is exactly the same as the proof of Topologies, Lemma 8.10. □

0EU1 **Definition 34.13.** Let S be a scheme. Let Sch_h be a big h site containing S.

1. The *big h site of S*, denoted $(Sch/S)_h$, is the site Sch_h/S introduced in Sites, Section 25.
2. The *big affine h site of S*, denoted $(Aff/S)_h$, is the full subcategory of $(Sch/S)_h$ whose objects are affine U/S. A covering of $(Aff/S)_h$ is any covering $\{U_i \to U\}$ of $(Sch/S)_h$ which is a standard h covering.

We explicitly state that the big affine h site is a site.

0EU2 **Lemma 34.14.** *Let S be a scheme. Let Sch_h be a big h site containing S. Then $(Aff/S)_h$ is a site.*

Proof. Reasoning as in the proof of Topologies, Lemma 4.9 it suffices to show that the collection of standard h coverings satisfies properties (1), (2) and (3) of Sites, Definition 6.2. This is clear since for example, given a standard h covering $\{T_i \to T\}_{i\in I}$ and for each i a standard h covering $\{T_{ij} \to T_i\}_{j\in J_i}$, then $\{T_{ij} \to T\}_{i\in I, j\in J_i}$ is a h covering (Lemma 34.9), $\bigcup_{i\in I} J_i$ is finite and each T_{ij} is affine. Thus $\{T_{ij} \to T\}_{i\in I, j\in J_i}$ is a standard h covering. □

0EU3 **Lemma 34.15.** *Let S be a scheme. Let Sch_h be a big h site containing S. The underlying categories of the sites Sch_h, $(Sch/S)_h$, and $(Aff/S)_h$ have fibre products. In each case the obvious functor into the category Sch of all schemes commutes with taking fibre products. The category $(Sch/S)_h$ has a final object, namely S/S.*

Proof. For Sch_h it is true by construction, see Sets, Lemma 9.9. Suppose we have $U \to S$, $V \to U$, $W \to U$ morphisms of schemes with $U, V, W \in \mathrm{Ob}(Sch_h)$. The fibre product $V \times_U W$ in Sch_h is a fibre product in Sch and is the fibre product of V/S with W/S over U/S in the category of all schemes over S, and hence also a fibre product in $(Sch/S)_h$. This proves the result for $(Sch/S)_h$. If U, V, W are affine, so is $V \times_U W$ and hence the result for $(Aff/S)_h$. □

Next, we check that the big affine site defines the same topos as the big site.

0EU4 **Lemma 34.16.** *Let S be a scheme. Let Sch_h be a big h site containing S. The functor $(Aff/S)_h \to (Sch/S)_h$ is cocontinuous and induces an equivalence of topoi from $Sh((Aff/S)_h)$ to $Sh((Sch/S)_h)$.*

Proof. The notion of a special cocontinuous functor is introduced in Sites, Definition 29.2. Thus we have to verify assumptions (1) – (5) of Sites, Lemma 29.1. Denote the inclusion functor $u : (Aff/S)_h \to (Sch/S)_h$. Being cocontinuous follows because any h covering of T/S, T affine, can be refined by a standard h covering for example by Lemma 34.5. Hence (1) holds. We see u is continuous simply because a standard h covering is a h covering. Hence (2) holds. Parts (3) and (4) follow immediately from the fact that u is fully faithful. And finally condition (5) follows from the fact that every scheme has an affine open covering (which is a h covering). □

0EU5 **Lemma 34.17.** *Let $\mathcal{F}$ be a presheaf on $(Sch/S)_h$. Then $\mathcal{F}$ is a sheaf if and only if*

(1) *$\mathcal{F}$ satisfies the sheaf condition for Zariski coverings, and*
(2) *if $f : V \to U$ is proper, surjective, and of finite presentation, then $\mathcal{F}(U)$ maps bijectively to the equalizer of the two maps $\mathcal{F}(V) \to \mathcal{F}(V \times_U V)$.*

Moreover, in the presence of (1) property (2) is equivalent to property

(2') *the sheaf property for $\{V \to U\}$ as in (2) with U affine.*

Proof. We will show that if (1) and (2) hold, then $\mathcal{F}$ is sheaf. Let $\{T_i \to T\}$ be a covering in $(Sch/S)_h$. We will verify the sheaf condition for this covering. Let $s_i \in \mathcal{F}(T_i)$ be sections which restrict to the same section over $T_i \times_T T_{i'}$. We will show that there exists a unique section $s \in \mathcal{F}(T)$ restricting to s_i over T_i. Let $T = \bigcup U_j$ be an affine open covering. By property (1) it suffices to produce sections $s_j \in \mathcal{F}(U_j)$ which agree on $U_j \cap U_{j'}$ in order to produce s. Consider the coverings $\{T_i \times_T U_j \to U_j\}$. Then $s_{ji} = s_i|_{T_i \times_T U_j}$ are sections agreeing over $(T_i \times_T U_j) \times_{U_j} (T_{i'} \times_T U_j)$. Choose a proper surjective morphism $V_j \to U_j$ of finite presentation and a finite affine open covering $V_j = \bigcup V_{jk}$ such that $\{V_{jk} \to U_j\}$ refines $\{T_i \times_T U_j \to U_j\}$. See Lemma 34.5. If $s_{jk} \in \mathcal{F}(V_{jk})$ denotes the pullback of s_{ji} to V_{jk} by the implied morphisms, then we find that s_{jk} glue to a section $s'_j \in \mathcal{F}(V_j)$. Using the agreement on overlaps once more, we find that s'_j is in the equalizer of the two maps $\mathcal{F}(V_j) \to \mathcal{F}(V_j \times_{U_j} V_j)$. Hence by (2) we find that s'_j comes from a unique section $s_j \in \mathcal{F}(U_j)$. We omit the verification that these sections s_j have all the desired properties.

Proof of the equivalence of (2) and (2') in the presence of (1). Suppose $V \to U$ is a morphism of $(Sch/S)_h$ which is proper, surjective, and of finite presentation. Choose an affine open covering $U = \bigcup U_i$ and set $V_i = V \times_U U_i$. Then we see that $\mathcal{F}(U) \to \mathcal{F}(V)$ is injective because we know $\mathcal{F}(U_i) \to \mathcal{F}(V_i)$ is injective by (2') and we know $\mathcal{F}(U) \to \prod \mathcal{F}(U_i)$ is injective by (1). Finally, suppose that we are given an $t \in \mathcal{F}(V)$ in the equalizer of the two maps $\mathcal{F}(V) \to \mathcal{F}(V \times_U V)$. Then $t|_{V_i}$ is in the equalizer of the two maps $\mathcal{F}(V_i) \to \mathcal{F}(V_i \times_{U_i} V_i)$ for all i. Hence we obtain a unique section $s_i \in \mathcal{F}(U_i)$ mapping to $t|_{V_i}$ for all i by (2'). We omit the verification that $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ for all i, j; this uses the uniqueness property just shown. By the sheaf property for the covering $U = \bigcup U_i$ we obtain a section $s \in \mathcal{F}(U)$. We omit the proof that s maps to t in $\mathcal{F}(V)$. □

Next, we establish some relationships between the topoi associated to these sites.

0EU6 **Lemma 34.18.** *Let Sch_h be a big h site. Let $f : T \to S$ be a morphism in Sch_h. The functor*

$$u : (Sch/T)_h \longrightarrow (Sch/S)_h, \quad V/T \longmapsto V/S$$

is cocontinuous, and has a continuous right adjoint

$$v : (Sch/S)_h \longrightarrow (Sch/T)_h, \quad (U \to S) \longmapsto (U \times_S T \to T).$$

They induce the same morphism of topoi

$$f_{big} : Sh((Sch/T)_h) \longrightarrow Sh((Sch/S)_h)$$

We have $f_{big}^{-1}(\mathcal{G})(U/T) = \mathcal{G}(U/S)$. *We have* $f_{big,*}(\mathcal{F})(U/S) = \mathcal{F}(U \times_S T/T)$. *Also,* f_{big}^{-1} *has a left adjoint* $f_{big!}$ *which commutes with fibre products and equalizers.*

Proof. The functor u is cocontinuous, continuous, and commutes with fibre products and equalizers. Hence Sites, Lemmas 21.5 and 21.6 apply and we deduce the formula for f_{big}^{-1} and the existence of $f_{big!}$. Moreover, the functor v is a right adjoint because given U/T and V/S we have $\mathrm{Mor}_S(u(U), V) = \mathrm{Mor}_T(U, V \times_S T)$ as desired. Thus we may apply Sites, Lemmas 22.1 and 22.2 to get the formula for $f_{big,*}$. □

0EU7 **Lemma 34.19.** *Given schemes* X, Y, Y *in* $(Sch/S)_h$ *and morphisms* $f : X \to Y$, $g : Y \to Z$ *we have* $g_{big} \circ f_{big} = (g \circ f)_{big}$.

Proof. This follows from the simple description of pushforward and pullback for the functors on the big sites from Lemma 34.18. □

35. More on the h topology

0EUZ In this section we prove a few more results on the h topology. First, some non-examples.

0EV0 **Example 35.1.** The "structure sheaf" $\mathcal{O}$ is not a sheaf in the h topology. For example, consider a surjective closed immersion of finite presentation $X \to Y$. Then $\{X \to Y\}$ is an h covering for example by Lemma 34.7. Moreover, note that $X \times_Y X = X$. Thus if $\mathcal{O}$ where a sheaf in the h topology, then $\mathcal{O}_Y(Y) \to \mathcal{O}_X(X)$ would be bijective. This is not the case as soon as X, Y are affine and the morphism $X \to Y$ is not an isomorphism.

0EV1 **Example 35.2.** On any of the sites $(Sch/S)_h$ the topology is not subcanonical, in other words, representable sheaves are not sheaves. Namely, the "structure sheaf" $\mathcal{O}$ is representable because $\mathcal{O}(X) = \mathrm{Mor}_S(X, \mathbf{A}^1_S)$ in $(Sch/S)_h$ and we saw in Example 35.1 that $\mathcal{O}$ is not a sheaf.

0EV2 **Lemma 35.3.** *Let* T *be an affine scheme which is written as a limit* $T = \lim_{i \in I} T_i$ *of a directed inverse system of affine schemes.*

1. *Let* $\mathcal{V} = \{V_j \to T\}_{j=1,\ldots,m}$ *be a standard h covering of* T, *see Definition 34.11. Then there exists an index* i *and a standard h covering* $\mathcal{V}_i = \{V_{i,j} \to T_i\}_{j=1,\ldots,m}$ *whose base change* $T \times_{T_i} \mathcal{V}_i$ *to* T *is isomorphic to* $\mathcal{V}$.
2. *Let* $\mathcal{V}_i$, $\mathcal{V}'_i$ *be a pair of standard h coverings of* T_i. *If* $f : T \times_{T_i} \mathcal{V}_i \to T \times_{T_i} \mathcal{V}'_i$ *is a morphism of coverings of* T, *then there exists an index* $i' \geq i$ *and a morphism* $f_{i'} : T_{i'} \times_{T_i} \mathcal{V} \to T_{i'} \times_{T_i} \mathcal{V}'_i$ *whose base change to* T *is* f.
3. *If* $f, g : \mathcal{V} \to \mathcal{V}'_i$ *are morphisms of standard h coverings of* T_i *whose base changes* f_T, g_T *to* T *are equal then there exists an index* $i' \geq i$ *such that* $f_{T_{i'}} = g_{T_{i'}}$.

In other words, the category of standard h coverings of T *is the colimit over* I *of the categories of standard h coverings of* T_i.

Proof. By Limits, Lemma 10.1 the category of schemes of finite presentation over T is the colimit over I of the categories of finite presentation over T_i. By Limits, Lemma 8.2 the same is true for category of schemes which are affine and of finite presentation over T. To finish the proof of the lemma it suffices to show that if $\{V_{j,i} \to T_i\}_{j=1,\ldots,m}$ is a finite family of finitely presented morphisms with $V_{j,i}$ affine, and the base change family $\{T \times_{T_i} V_{j,i} \to T\}$ is an h covering, then for some $i' \geq i$ the family $\{T_{i'} \times_{T_i} V_{j,i} \to T_{i'}\}$ is an h covering. To see this we use Lemma 34.5 to choose a finitely presented, proper, surjective morphism $Y \to T$ and a finite affine open covering $Y = \bigcup_{k=1,\ldots,n} Y_k$ such that $\{Y_k \to T\}_{k=1,\ldots,n}$ refines $\{T \times_{T_i} V_{j,i} \to T\}$. Using the arguments above and Limits, Lemmas 13.1, 8.15, and 4.11 we can find an $i' \geq i$ and a finitely presented, surjective, proper morphism $Y_{i'} \to T_{i'}$ and an affine open covering $Y_{i'} = \bigcup_{k=1,\ldots,n} Y_{i',k}$ such that moreover $\{Y_{i',k} \to Y_{i'}\}$ refines $\{T_{i'} \times_{T_i} V_{j,i} \to T_{i'}\}$. It follows that this last mentioned family is a h covering and the proof is complete. □

0EV3 **Lemma 35.4.** *Let S be a scheme contained in a big site Sch_h. Let $F : (Sch/S)_h^{opp} \to Sets$ be an h sheaf satisfying property (b) of Topologies, Lemma 13.1 with $\mathcal{C} = (Sch/S)_h$. Then the extension F' of F to the category of all schemes over S satisfies the sheaf condition for all h coverings and is limit preserving (Limits, Remark 6.2).*

Proof. This is proven by the arguments given in the proofs of Topologies, Lemmas 13.3 and 13.4 using Lemmas 35.3 and 34.12. Details omitted. □

36. Blow up squares and the ph topology

0EW0 Let X be a scheme. Let $Z \subset X$ be a closed subscheme such that the inclusion morphism is of finite presentation, i.e., the quasi-coherent sheaf of ideals corresponding to Z is of finite type. Let $b : X' \to X$ be the blowup of X in Z and let $E = b^{-1}(Z)$ be the exceptional divisor. See Divisors, Section 33. In this situation and in this section, let us say

0EV5 (36.0.1)
$$\begin{array}{ccc} E & \longrightarrow & X' \\ \downarrow & & \downarrow b \\ Z & \longrightarrow & X \end{array}$$

is a *blow up square*.

0EW1 **Lemma 36.1.** *Let $\mathcal{F}$ be a sheaf on a site $(Sch/S)_{ph}$, see Topologies, Definition 8.11. Then for any blow up square (36.0.1) in the category $(Sch/S)_{ph}$ the diagram*

$$\begin{array}{ccc} \mathcal{F}(E) & \longleftarrow & \mathcal{F}(X') \\ \uparrow & & \uparrow \\ \mathcal{F}(Z) & \longleftarrow & \mathcal{F}(X) \end{array}$$

is cartesian in the category of sets.

Proof. Since $Z \amalg X' \to X$ is a surjective proper morphism we see that $\{Z \amalg X' \to X\}$ is a ph covering (Topologies, Lemma 8.6). We have

$$(Z \amalg X') \times_X (Z \amalg X') = Z \amalg E \amalg E \amalg X' \times_X X'$$

Since $\mathcal{F}$ is a Zariski sheaf we see that $\mathcal{F}$ sends disjoint unions to products. Thus the sheaf condition for the covering $\{Z \amalg X' \to X\}$ says that $\mathcal{F}(X) \to \mathcal{F}(Z) \times \mathcal{F}(X')$ is injective with image the set of pairs (t, s') such that (a) $t|_E = s'|_E$ and (b) s' is in the equalizer of the two maps $\mathcal{F}(X') \to \mathcal{F}(X' \times_X X')$. Next, observe that the obvious morphism

$$E \times_Z E \amalg X' \longrightarrow X' \times_X X'$$

is a surjective proper morphism as b induces an isomorphism $X' \setminus E \to X \setminus Z$. We conclude that $\mathcal{F}(X' \times_X X') \to \mathcal{F}(E \times_Z E) \times \mathcal{F}(X')$ is injective. It follows that (a) $\Rightarrow$ (b) which means that the lemma is true. □

0EW2 **Lemma 36.2.** *Let $\mathcal{F}$ be a sheaf on a site $(Sch/S)_{ph}$ as in Topologies, Definition 8.11. Let $X \to X'$ be a morphism of $(Sch/S)_{ph}$ which is a thickening. Then $\mathcal{F}(X') \to \mathcal{F}(X)$ is bijective.*

Proof. Observe that $X \to X'$ is a proper surjective morphism of and $X \times_{X'} X = X$. By the sheaf property for the ph covering $\{X \to X'\}$ (Topologies, Lemma 8.6) we conclude. □

37. Almost blow up squares and the h topology

0EV4 Consider a blow up square (36.0.1). Although the morphism $b : X' \to X$ is projective (Divisors, Lemma 33.13) in general there is no simple way to guarantee that b is of finite presentation. Since h coverings are constructed using morphisms of finite presentation, we need a variant. Namely, we will say a commutative diagram

0EV6 (37.0.1)
$$\begin{array}{ccc} E & \longrightarrow & X' \\ \downarrow & & \downarrow b \\ Z & \longrightarrow & X \end{array}$$

of schemes is an *almost blow up square* if the following conditions are satisfied

(1) $Z \to X$ is a closed immersion of finite presentation,
(2) $E = b^{-1}(Z)$ is a locally principal closed subscheme of X',
(3) b is proper and of finite presentation,
(4) the closed subscheme $X'' \subset X'$ cut out by the quasi-coherent ideal of sections of $\mathcal{O}_{X'}$ supported on E (Properties, Lemma 25.5) is the blow up of X in Z.

It follows that the morphism b induces an isomorphism $X' \setminus E \to X \setminus Z$. For some very simple examples of almost blow up squares, see Examples 37.10 and 37.11.

The base change of a blow up usually isn't a blow up, but almost blow ups are compatible with base change.

0EV7 **Lemma 37.1.** *Consider an almost blow up square (37.0.1). Let $Y \to X$ be any morphism. Then the base change*

$$\begin{array}{ccc} Y \times_X E & \longrightarrow & Y \times_X X' \\ \downarrow & & \downarrow \\ Y \times_X Z & \longrightarrow & Y \end{array}$$

is an almost blow up square too.

Proof. The morphism $Y \times_X X' \to Y$ is proper and of finite presentation by Morphisms, Lemmas 42.5 and 22.4. The morphism $Y \times_X Z \to Y$ is a closed immersion (Morphisms, Lemma 2.4) of finite presentation. The inverse image of $Y \times_X Z$ in $Y \times_X X'$ is equal to the inverse image of E in $Y \times_X X'$ and hence is locally principal (Divisors, Lemma 14.11). Let $X'' \subset X'$, resp. $Y'' \subset Y \times_X X'$ be the closed subscheme corresponding to the quasi-coherent ideal of sections of $\mathcal{O}_{X'}$, resp. $\mathcal{O}_{Y \times_Y X'}$ supported on E, resp. $Y \times_X E$. Clearly, $Y'' \subset Y \times_X X''$ is the closed subscheme corresponding to the quasi-coherent ideal of sections of $\mathcal{O}_{Y \times_Y X''}$ supported on $Y \times_X (E \cap X'')$. Thus Y'' is the strict transform of Y relative to the blowing up $X'' \to X$, see Divisors, Definition 34.1. Thus by Divisors, Lemma 34.2 we see that Y'' is the blow up of $Y \times_X Z$ on Y. □

One can shrink almost blow up squares.

0EV8 **Lemma 37.2.** *Consider an almost blow up square (37.0.1). Let $W \to X'$ be a closed immersion of finite presentation. The following are equivalent*

(1) *$X' \setminus E$ is scheme theoretically contained in W,*
(2) *the blowup X'' of X in Z is scheme theoretically contained in W,*
(3) *the diagram*

$$\begin{array}{ccc} E \cap W & \longrightarrow & W \\ \downarrow & & \downarrow \\ Z & \longrightarrow & X \end{array}$$

is an almost blow up square. Here $E \cap W$ is the scheme theoretic intersection.

Proof. Assume (1). Then the surjection $\mathcal{O}_{X'} \to \mathcal{O}_W$ is an isomorphism over the open $X' \subset E$. Since the ideal sheaf of $X'' \subset X'$ is the sections of $\mathcal{O}_{X'}$ supported on E (by our definition of almost blow up squares) we conclude (2) is true. If (2) is true, then (3) holds. If (3) holds, then (1) holds because $X'' \cap (X' \setminus E)$ is isomorphic to $X \setminus Z$ which in turn is isomorphic to $X' \setminus E$. □

The actual blowup is the limit of shrinkings of any given almost blowup.

0EV9 **Lemma 37.3.** *Consider an almost blow up square (37.0.1) with X quasi-compact and quasi-separated. Then the blowup X'' of X in Z can be written as*

$$X'' = \lim X'_i$$

where the limit is over the directed system of closed subschemes $X'_i \subset X'$ of finite presentation satisfying the equivalent conditions of Lemma 37.2.

Proof. Let $\mathcal{I} \subset \mathcal{O}_{X'}$ be the quasi-coherent sheaf of ideals corresponding to X''. By Properties, Lemma 23.3 we can write $\mathcal{I}$ as the filtered colimit $\mathcal{I} = \operatorname{colim} \mathcal{I}_i$ of its quasi-coherent submodules of finite type. Since these modules correspond 1-to-1 to the closed subschemes X'_i the proof is complete. □

Almost blow up squares exist.

0EVA **Lemma 37.4.** *Let X be a quasi-compact and quasi-separated scheme. Let $Z \subset X$ be a closed subscheme cut out by a finite type quasi-coherent sheaf of ideals. Then there exists an almost blow up square as in (37.0.1).*

Proof. We may write $X = \lim X_i$ as a directed limit of an inverse system of Noetherian schemes with affine transition morphisms, see Limits, Proposition 5.4. We can find an index i and a closed immersion $Z_i \to X_i$ whose base change to X is the closed immersion $Z \to X$. See Limits, Lemmas 10.1 and 8.5. Let $b_i : X_i' \to X_i$ be the blowing up with center Z_i. This produces a blow up square

$$\begin{array}{ccc} E_i & \longrightarrow & X_i' \\ \downarrow & & \downarrow{\scriptstyle b_i} \\ Z_i & \longrightarrow & X_i \end{array}$$

where all the morphisms are finite type morphisms of Noetherian schemes and hence of finite presentation. Thus this is an almost blow up square. By Lemma 37.1 the base change of this diagram to X produces the desired almost blow up square. □

Almost blow up squares are unique up to shrinking as in Lemma 37.2.

0EVB **Lemma 37.5.** *Let X be a quasi-compact and quasi-separated scheme and let $Z \subset X$ be a closed subscheme cut out by a finite type quasi-coherent sheaf of ideals. Suppose given almost blow up squares (37.0.1)*

$$\begin{array}{ccc} E_k & \longrightarrow & X_k' \\ \downarrow & & \downarrow \\ Z & \longrightarrow & X \end{array}$$

for $k = 1, 2$, then there exists an almost blow up square

$$\begin{array}{ccc} E & \longrightarrow & X' \\ \downarrow & & \downarrow \\ Z & \longrightarrow & X \end{array}$$

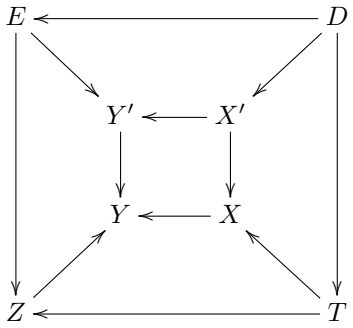
and closed immersions $i_k : X' \to X_k'$ over X with $E = i_k^{-1}(E_k)$.

Proof. Denote $X'' \to X$ the blowing up of Z in X. We view X'' as a closed subscheme of both X_1' and X_2'. Write $X'' = \lim X_{1,i}'$ as in Lemma 37.3. By Limits, Proposition 6.1 there exists an i and a morphism $h : X_{1,i}' \to X_2'$ agreeing with the inclusions $X'' \subset X_{1,i}'$ and $X'' \subset X_2'$. By Limits, Lemma 4.20 the restriction of h to $X_{1,i'}'$ is a closed immersion for some $i' \geq i$. This finishes the proof. □

Our flattening techniques for blowing up are inherited by almost blowups in favorable situations.

0EVC **Lemma 37.6.** *Let Y be a quasi-compact and quasi-separated scheme. Let X be a scheme of finite presentation over Y. Let $V \subset Y$ be a quasi-compact open such*

that $X_V \to V$ is flat. Then there exist a commutative diagram



whose right and left hand squares are almost blow up squares, whose lower and top squares are cartesian, such that $Z \cap V = \emptyset$, and such that $X' \to Y'$ is flat (and of finite presentation).

Proof. If Y is a Noetherian scheme, then this lemma follows immediately from Lemma 31.1 because in this case blow up squares are almost blow up squares (we also use that strict transforms are blow ups). The general case is reduced to the Noetherian case by absolute Noetherian approximation.

We may write $Y = \lim Y_i$ as a directed limit of an inverse system of Noetherian schemes with affine transition morphisms, see Limits, Proposition 5.4. We can find an index i and a morphism $X_i \to Y_i$ of finite presentation whose base change to Y is $X \to Y$. See Limits, Lemmas 10.1. After increasing i we may assume V is the inverse image of an open subscheme $V_i \subset Y_i$, see Limits, Lemma 4.11. Finally, after increasing i we may assume that $X_{i,V_i} \to V_i$ is flat, see Limits, Lemma 8.7. By the Noetherian case, we may construct a diagram as in the lemma for $X_i \to Y_i \supset V_i$. The base change of this diagram by $Y \to Y_i$ provides the solution. Use that base change preserves properties of morphisms, see Morphisms, Lemmas 42.5, 22.4, 2.4, and 26.8 and that base change of an almost blow up square is an almost blow up square, see Lemma 37.1. □

0EVD **Lemma 37.7.** *Let $\mathcal{F}$ be a sheaf on one of the sites $(Sch/S)_h$ constructed in Definition 34.13. Then for any almost blow up square (37.0.1) in the category $(Sch/S)_h$ the diagram*

$$\begin{array}{ccc} \mathcal{F}(E) & \longleftarrow & \mathcal{F}(X') \\ \uparrow & & \uparrow \\ \mathcal{F}(Z) & \longleftarrow & \mathcal{F}(X) \end{array}$$

is cartesian in the category of sets.

Proof. Since $Z \amalg X' \to X$ is a surjective proper morphism of finite presentation we see that $\{Z \amalg X' \to X\}$ is an h covering (Lemma 34.7). We have

$$(Z \amalg X') \times_X (Z \amalg X') = Z \amalg E \amalg E \amalg X' \times_X X'$$

Since $\mathcal{F}$ is a Zariski sheaf we see that $\mathcal{F}$ sends disjoint unions to products. Thus the sheaf condition for the covering $\{Z \amalg X' \to X\}$ says that $\mathcal{F}(X) \to \mathcal{F}(Z) \times \mathcal{F}(X')$ is injective with image the set of pairs (t, s') such that (a) $t|_E = s'|_E$ and (b) s' is

in the equalizer of the two maps $\mathcal{F}(X') \to \mathcal{F}(X' \times_X X')$. Next, observe that the obvious morphism

$$E \times_Z E \amalg X' \longrightarrow X' \times_X X'$$

is a surjective proper morphism of finite presentation as b induces an isomorphism $X' \setminus E \to X \setminus Z$. We conclude that $\mathcal{F}(X' \times_X X') \to \mathcal{F}(E \times_Z E) \times \mathcal{F}(X')$ is injective. It follows that (a) $\Rightarrow$ (b) which means that the lemma is true. □

0EVE **Lemma 37.8.** *Let $\mathcal{F}$ be a sheaf on one of the sites $(Sch/S)_h$ constructed in Definition 34.13. Let $X \to X'$ be a morphism of $(Sch/S)_h$ which is a thickening and of finite presentation. Then $\mathcal{F}(X') \to \mathcal{F}(X)$ is bijective.*

Proof. First proof. Observe that $X \to X'$ is a proper surjective morphism of finite presentation and $X \times_{X'} X = X$. By the sheaf property for the h covering $\{X \to X'\}$ (Lemma 34.7) we conclude.

Second proof (silly). The blow up of X' in X is the empty scheme. The reason is that the affine blowup algebra $A[\frac{I}{a}]$ (Algebra, Section 70) is zero if a is a nilpotent element of A. Details omitted. Hence we get an almost blow up square of the form

$$\begin{array}{ccc} \emptyset & \longrightarrow & \emptyset \\ \downarrow & & \downarrow \\ X & \longrightarrow & X' \end{array}$$

Since $\mathcal{F}$ is a sheaf we have that $\mathcal{F}(\emptyset)$ is a singleton. Applying Lemma 37.7 we get the conclusion. □

0EVF **Proposition 37.9.** *Let $\mathcal{F}$ be a presheaf on one of the sites $(Sch/S)_h$ constructed in Definition 34.13. Then $\mathcal{F}$ is a sheaf if and only if the following conditions are satisfied*

1. *$\mathcal{F}$ is a sheaf for the Zariski topology,*
2. *given a morphism $f : X \to Y$ of $(Sch/S)_h$ with Y affine and f surjective, flat, proper, and of finite presentation, then $\mathcal{F}(Y)$ is the equalizer of the two maps $\mathcal{F}(X) \to \mathcal{F}(X \times_Y X)$,*
3. *given an almost blow up square (37.0.1) with X affine in the category $(Sch/S)_h$ the diagram*

$$\begin{array}{ccc} \mathcal{F}(E) & \longleftarrow & \mathcal{F}(X') \\ \uparrow & & \uparrow \\ \mathcal{F}(Z) & \longleftarrow & \mathcal{F}(X) \end{array}$$

is cartesian in the category of sets.

Proof. Assume $\mathcal{F}$ is a sheaf. Condition (1) holds because a Zariski covering is a h covering, see Lemma 34.6. Condition (2) holds because for f as in (2) we have that $\{X \to Y\}$ is an fppf covering (this is clear) and hence an h covering, see Lemma 34.6. Condition (3) holds by Lemma 37.7.

Conversely, assume $\mathcal{F}$ satisfies (1), (2), and (3). We will prove $\mathcal{F}$ is a sheaf by applying Lemma 34.17. Consider a surjective, finitely presented, proper morphism $f : X \to Y$ in $(Sch/S)_h$ with Y affine. It suffices to show that $\mathcal{F}(Y)$ is the equalizer of the two maps $\mathcal{F}(X) \to \mathcal{F}(X \times_Y X)$.

First, assume that $f : X \to Y$ is in addition a closed immersion (in other words, f is a thickening). Then the blow up of Y in X is the empty scheme and this produces an almost blow up square consisting with $\emptyset, \emptyset, X, Y$ at the vertices (compare with the second proof of Lemma 37.8). Hence we see that condition (3) tells us that

$$\begin{array}{ccc} \mathcal{F}(\emptyset) & \longleftarrow & \mathcal{F}(\emptyset) \\ \uparrow & & \uparrow \\ \mathcal{F}(X) & \longleftarrow & \mathcal{F}(Y) \end{array}$$

is cartesian in the category of sets. Since $\mathcal{F}$ is a sheaf for the Zariski topology, we see that $\mathcal{F}(\emptyset)$ is a singleton. Hence we see that $\mathcal{F}(X) = \mathcal{F}(Y)$.

Interlude A: let $T \to T'$ be a morphism of $(Sch/S)_h$ which is a thickening and of finite presentation. Then $\mathcal{F}(T') \to \mathcal{F}(T)$ is bijective. Namely, choose an affine open covering $T' = \bigcup T'_i$ and let $T_i = T \times_{T'} T'_i$ be the corresponding affine opens of T. Then we have $\mathcal{F}(T'_i) \to \mathcal{F}(T_i)$ is bijective for all i by the result of the previous paragraph. Using the Zariski sheaf property we see that $\mathcal{F}(T') \to \mathcal{F}(T)$ is injective. Repeating the argument we find that it is bijective. Minor details omitted.

Interlude B: consider an almost blow up square (37.0.1) in the category $(Sch/S)_h$. Then we claim the diagram

$$\begin{array}{ccc} \mathcal{F}(E) & \longleftarrow & \mathcal{F}(X') \\ \uparrow & & \uparrow \\ \mathcal{F}(Z) & \longleftarrow & \mathcal{F}(X) \end{array}$$

is cartesian in the category of sets. This is a consequence of condition (3) as follows by choosing an affine open covering of X and arguing as in Interlude A. We omit the details.

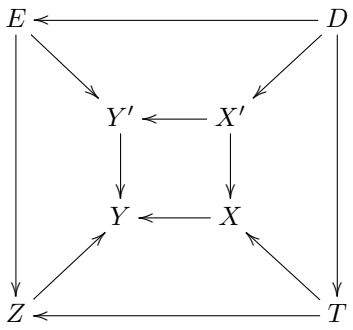
Next, let $f : X \to Y$ be a surjective, finitely presented, proper morphism in $(Sch/S)_h$ with Y affine. Choose a generic flatness stratification

$$Y \supset Y_0 \supset Y_1 \supset \ldots \supset Y_t = \emptyset$$

as in More on Morphisms, Lemma 54.2 for $f : X \to Y$. We are going to use all the properties of the stratification without further mention. Set $X_0 = X \times_Y Y_0$. By the Interlude B we have $\mathcal{F}(Y_0) = \mathcal{F}(Y)$, $\mathcal{F}(X_0) = \mathcal{F}(X)$, and $\mathcal{F}(X_0 \times_{Y_0} X_0) = \mathcal{F}(X \times_Y X)$.

We are going to prove the result by induction on t. If $t = 1$ then $X_0 \to Y_0$ is surjective, proper, flat, and of finite presentation and we see that the result holds by property (2). For $t > 1$ we may replace Y by Y_0 and X by X_0 (see above) and assume $Y = Y_0$.

Consider the quasi-compact open subscheme $V = Y \setminus Y_1 = Y_0 \setminus Y_1$. Choose a diagram



as in Lemma 37.6 for $f : X \to Y \supset V$. Then $f' : X' \to Y'$ is flat and of finite presentation. Also f' is proper (use Morphisms, Lemmas 42.4 and 42.7 to see this). Thus the image $W = f'(X') \subset Y'$ is an open (Morphisms, Lemma 26.10) and closed subscheme of Y'. Observe that $Y' \setminus E$ is contained in W. By Lemma 37.2 this means we may replace Y' by W in the above diagram. In other words, we may and do assume f' is surjective. At this point we know that

$$\begin{array}{ccc} \mathcal{F}(E) & \longleftarrow & \mathcal{F}(Y') \\ \uparrow & & \uparrow \\ \mathcal{F}(Z) & \longleftarrow & \mathcal{F}(Y) \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathcal{F}(D) & \longleftarrow & \mathcal{F}(X') \\ \uparrow & & \uparrow \\ \mathcal{F}(T) & \longleftarrow & \mathcal{F}(X) \end{array}$$

are cartesian by Interlude B. Note that $Z \cap Y_1 \to Z$ is a thickening of finite presentation (as Z is set theoretically contained in Y_1 as a closed subscheme of Y disjoint from V). Thus we obtain a filtration

$$Z \supset Z \cap Y_1 \supset Z \cap Y_2 \subset \ldots \subset Z \cap Y_t = \emptyset$$

as above for the restriction $T = Z \times_Y X \to Z$ of f to T. Thus by induction hypothesis we find that $\mathcal{F}(Z) \to \mathcal{F}(T)$ is an injective map of sets whose image is the equalizer of the two maps $\mathcal{F}(T) \to \mathcal{F}(T \times_Z T)$.

Let $s \in \mathcal{F}(X)$ be in the equalizer of the two maps $\mathcal{F}(X) \to \mathcal{F}(X \times_Y X)$. By the above we see that the restriction $s|_T$ comes from a unique element $t \in \mathcal{F}(Z)$ and similarly that the restriction $s|_{X'}$ comes from a unique element $t' \in \mathcal{F}(Y')$. Chasing sections using the restriction maps for $\mathcal{F}$ corresponding to the arrows in the huge commutative diagram above the reader finds that t and t' restrict to the same element of $\mathcal{F}(E)$ because they restrict to the same element of $\mathcal{F}(D)$ and we have (2); here we use that $D \to E$ is surjective, flat, proper, and of finite presentation as the restriction of $X' \to Y'$. Thus by the first of the two cartesian squares displayed above we get a unique section $u \in \mathcal{F}(Y)$ restricting to t and t' on Z and Y'. To see that u restrict to s on X use the second diagram. $\square$

0EVG **Example 37.10.** Let A be a ring. Let $f \in A$ be an element. Let $J \subset A$ be a finitely generated ideal annihilated by a power of f. Then

$$\begin{array}{ccc} E = \mathrm{Spec}(A/fA + J) & \longrightarrow & \mathrm{Spec}(A/J) = X' \\ \downarrow & & \downarrow \\ Z = \mathrm{Spec}(A/fA) & \longrightarrow & \mathrm{Spec}(A) = X \end{array}$$

is an almost blowup square.

0EVH **Example 37.11.** Let A be a ring. Let $f_1, f_2 \in A$ be elements.

$$\begin{array}{ccc} E = \mathrm{Proj}(A/(f_1, f_2)[T_0, T_1]) & \longrightarrow & \mathrm{Proj}(A[T_0, T_1]/(f_2T_0 - f_1T_1) = X' \\ \downarrow & & \downarrow \\ Z = \mathrm{Spec}(A/(f_1, f_2)) & \longrightarrow & \mathrm{Spec}(A) = X \end{array}$$

is an almost blowup square.

0EVI **Lemma 37.12.** *Let $\mathcal{F}$ be a presheaf on one of the sites $(Sch/S)_h$ constructed in Definition 34.13. Then $\mathcal{F}$ is a sheaf if and only if the following conditions are satisfied*

1. *$\mathcal{F}$ is a sheaf for the Zariski topology,*
2. *given a morphism $f : X \to Y$ of $(Sch/S)_h$ with Y affine and f surjective, flat, proper, and of finite presentation, then $\mathcal{F}(Y)$ is the equalizer of the two maps $\mathcal{F}(X) \to \mathcal{F}(X \times_Y X)$,*
3. *$\mathcal{F}$ turns an almost blow up square as in Example 37.10 in the category $(Sch/S)_h$ into a cartesian diagram of sets, and*
4. *$\mathcal{F}$ turns an almost blow up square as in Example 37.11 in the category $(Sch/S)_h$ into a cartesian diagram of sets.*

Proof. By Proposition 37.9 it suffices to show that given an almost blow up square (37.0.1) with X affine in the category $(Sch/S)_h$ the diagram

$$\begin{array}{ccc} \mathcal{F}(E) & \longleftarrow & \mathcal{F}(X') \\ \uparrow & & \uparrow \\ \mathcal{F}(Z) & \longleftarrow & \mathcal{F}(X) \end{array}$$

is cartesian in the category of sets. The rough idea of the proof is to dominate the morphism by other almost blowup squares to which we can apply assumptions (3) and (4) locally.

Suppose we have an almost blow up square (37.0.1) in the category $(Sch/S)_h$, an open covering $X = \bigcup U_i$, and open coverings $U_i \cap U_j = \bigcup U_{ijk}$ such that the diagrams

$$\begin{array}{ccc} \mathcal{F}(E \cap b^{-1}(U_i)) & \longleftarrow & \mathcal{F}(b^{-1}(U_i)) \\ \uparrow & & \uparrow \\ \mathcal{F}(Z \cap U_i) & \longleftarrow & \mathcal{F}(U_i) \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathcal{F}(E \cap b^{-1}(U_{ijk})) & \longleftarrow & \mathcal{F}(b^{-1}(U_{ijk})) \\ \uparrow & & \uparrow \\ \mathcal{F}(Z \cap U_{ijk}) & \longleftarrow & \mathcal{F}(U_{ijk}) \end{array}$$

are cartesian, then the same is true for

$$\begin{array}{ccc} \mathcal{F}(E) & \longleftarrow & \mathcal{F}(X') \\ \uparrow & & \uparrow \\ \mathcal{F}(Z) & \longleftarrow & \mathcal{F}(X) \end{array}$$

This follows as $\mathcal{F}$ is a sheaf in the Zariski topology.

In particular, if we have a blow up square (37.0.1) such that $b : X' \to X$ is a closed immersion and Z is a locally principal closed subscheme, then we see that $\mathcal{F}(X) = \mathcal{F}(X') \times_{\mathcal{F}(E)} \mathcal{F}(Z)$. Namely, affine locally on X we obtain an almost blow up square as in (3).

Let $Z \subset X$, $E_k \subset X'_k \to X$, $E \subset X' \to X$, and $i_k : X' \to X'_k$ be as in the statement of Lemma 37.5. Then

$$\begin{array}{ccc} E & \longrightarrow & X' \\ \downarrow & & \downarrow \\ E_k & \longrightarrow & X'_k \end{array}$$

is an almost blow up square of the kind discussed in the previous paragraph. Thus

$$\mathcal{F}(X'_k) = \mathcal{F}(X') \times_{\mathcal{F}(E)} \mathcal{F}(E_k)$$

for $k = 1, 2$ by the result of the previous paragraph. It follows that

$$\mathcal{F}(X) \longrightarrow \mathcal{F}(X'_k) \times_{\mathcal{F}(E_k)} \mathcal{F}(Z)$$

is bijective for $k = 1$ if and only if it is bijective for $k = 2$. Thus given a closed immersion $Z \to X$ of finite presentation with X quasi-compact and quasi-separated, whether or not $\mathcal{F}(X) = \mathcal{F}(X') \times_{\mathcal{F}(E)} \mathcal{F}(Z)$ is independent of the choice of the almost blow up square (37.0.1) one chooses. (Moreover, by Lemma 37.4 there does indeed exist an almost blow up square for $Z \subset X$.)

Finally, consider an affine object X of $(Sch/S)_h$ and a closed immersion $Z \to X$ of finite presentation. We will prove the desired property for the pair (X, Z) by induction on the number of generators r for the ideal defining Z in X. If the number of generators is ≤ 2, then we can choose our almost blow up square as in Example 37.11 and we conclude by assumption (4).

Induction step. Suppose $X = \mathrm{Spec}(A)$ and $Z = \mathrm{Spec}(A/(f_1, \ldots, f_r))$ with $r > 2$. Choose a blow up square (37.0.1) for the pair (X, Z). Set $Z_1 = \mathrm{Spec}(A/(f_1, f_2))$ and let

$$\begin{array}{ccc} E_1 & \longrightarrow & Y \\ \downarrow & & \downarrow \\ Z_1 & \longrightarrow & X \end{array}$$

be the almost blow up square constructed in Example 37.11. By Lemma 37.1 the base changes

$$(I) \quad \begin{array}{ccc} Y \times_X E & \longrightarrow & Y \times_X X' \\ \downarrow & & \downarrow \\ Y \times_X Z & \longrightarrow & Y \end{array} \qquad \text{and} \qquad (II) \quad \begin{array}{ccc} E & \longrightarrow & Z_1 \times_X X' \\ \downarrow & & \downarrow \\ Z & \longrightarrow & Z_1 \end{array}$$

are almost blow up squares. The ideal of Z in Z_1 is generated by $r-2$ elements. The ideal of $Y \times_X Z$ is generated by the pullbacks of $f_1, \ldots, f_r$ to Y. Locally on Y the ideal generated by f_1, f_2 can be generated by one element, thus $Y \times_X Z$ is affine locally on Y cut out by at most $r-1$ elements. By induction hypotheses and the discussion above

$$\mathcal{F}(Y) = \mathcal{F}(Y \times_X X') \times_{\mathcal{F}(Y \times_X E)} \mathcal{F}(Y \times_X Z)$$

and

$$\mathcal{F}(Z_1) = \mathcal{F}(Z_1 \times_X X') \times_{\mathcal{F}(E)} \mathcal{F}(Z)$$

By assumption (4) we have

$$\mathcal{F}(X) = \mathcal{F}(Y) \times_{\mathcal{F}(E_1)} \mathcal{F}(Z_1)$$

Now suppose we have a pair (s', t) with $s' \in \mathcal{F}(X')$ and $t \in \mathcal{F}(Z)$ with same restriction in $\mathcal{F}(E)$. Then $(s'|Z_1 \times_X X', t)$ are the image of a unique element $t_1 \in \mathcal{F}(Z_1)$. Similarly, $(s'|_{Y \times_X X'}, t|_{Y \times_X Z})$ are the image of a unique element $s_Y \in \mathcal{F}(Y)$. We claim that s_Y and t_1 restrict to the same element of $\mathcal{F}(E_1)$. This is true because the almost blow up square

$$\begin{array}{ccc} E_1 \times_X E & \longrightarrow & E_1 \times_X X' \\ \downarrow & & \downarrow \\ E_1 \times_X Z & \longrightarrow & E_1 \end{array}$$

is the base change of almost blow up square (I) via $E_1 \to Y$ and the base change of almost blow up square (II) via $E_1 \to Z_1$ and because the pairs of sections used to construct s_Y and t_1 match. Thus by the third fibre product equality we see that there is a unique $s \in \mathcal{F}(X)$ mapping to s_Y in $\mathcal{F}(Y)$ and to t_1 in $\mathcal{F}(Z)$. We omit the verification that s maps to s' in $\mathcal{F}(X')$ and to t in $\mathcal{F}(Z)$; hint: use uniqueness of s just constructed and work affine locally. □

0EX9 **Lemma 37.13.** *Let $p : \mathcal{S} \to (Sch/S)_h$ be a category fibred in groupoids. Then $\mathcal{S}$ is a stack in groupoids if and only if the following conditions are satisfied*

(1) *$\mathcal{S}$ is a stack in groupoids for the Zariski topology,*
(2) *given a morphism $f : X \to Y$ of $(Sch/S)_h$ with Y affine and f surjective, flat, proper, and of finite presentation, then*
$$\mathcal{S}_Y \longrightarrow \mathcal{S}_X \times_{\mathcal{S}_{X \times_Y X}} \mathcal{S}_X$$
is an equivalence of categories,
(3) *for an almost blow up square as in Example 37.10 or 37.11 in the category $(Sch/S)_h$ the functor*
$$\mathcal{S}_X \longrightarrow \mathcal{S}_Z \times_{\mathcal{S}_E} \mathcal{S}_{X'}$$
is an equivalence of categories.

Proof. This lemma is a formal consequence of Lemma 37.12 and our definition of stacks in groupoids. For example, assume (1), (2), (3). To show that $\mathcal{S}$ is a stack, we have to prove descent for morphisms and objects, see Stacks, Definition 5.1.

If x, y are objects of $\mathcal{S}$ over an object U of $(Sch/S)_h$, then our assumptions imply $Isom(x, y)$ is a presheaf on $(Sch/U)_h$ which satisfies (1), (2), (3), and (4) of Lemma 37.12 and therefore is a sheaf. Some details omitted.

Let $\{U_i \to U\}_{i\in I}$ be a covering of $(Sch/S)_h$. Let (x_i, φ_{ij}) be a descent datum in $\mathcal{S}$ relative to the family $\{U_i \to U\}_{i\in I}$, see Stacks, Definition 3.1. Consider the rule F which to V/U in $(Sch/U)_h$ associates the set of pairs (y, ψ_i) where y is an object of $\mathcal{S}_V$ and $\psi_i : y|_{U_i\times_U V} \to x_i|_{U_i\times_U V}$ is a morphism of $\mathcal{S}$ over $U_i \times_U V$ such that

$$\varphi_{ij}|_{U_i\times_U U_j\times_U V} \circ \psi_i|_{U_i\times_U U_j\times_U V} = \psi_j|_{U_i\times_U U_j\times_U V}$$

up to isomorphism. Since we already have descent for morphisms, it is clear that $F(V/U)$ is either empty or a singleton set. On the other hand, we have $F(U_{i_0}/U)$ is nonempty because it contains $(x_{i_0}, \varphi_{i_0 i})$. Since our goal is to prove that $F(U/U)$ is nonempty, it suffices to show that F is a sheaf on $(Sch/U)_h$. To do this we may use the criterion of Lemma 37.12. However, our assumptions (1), (2), (3) imply (by drawing some commutative diagrams which we omit), that properties (1), (2), (3), and (4) of Lemma 37.12 hold for F.

We omit the verification that if $\mathcal{S}$ is a stack in groupoids, then (1), (2), and (3) are satisfied. □

38. Absolute weak normalization and h coverings

0EVS In this section we use the criteria found in Section 37 to exhibit some h sheaves and we relate h sheafification of the structure sheaf to absolute weak normalization. We will need the following elementary lemma to do this.

0EVJ **Lemma 38.1.** *Let Z, X, X', E be an almost blow up square as in Example 37.11. Then $H^p(X', \mathcal{O}_{X'}) = 0$ for $p > 0$ and $\Gamma(X, \mathcal{O}_X) \to \Gamma(X', \mathcal{O}_{X'})$ is a surjective map of rings whose kernel is an ideal of square zero.*

Proof. First assume that $A = \mathbf{Z}[f_1, f_2]$ is the polynomial ring. In this case our almost blow up square is the blowing up of $X = \mathrm{Spec}(A)$ in the closed subscheme Z and in fact $X' \subset \mathbf{P}^1_X$ is an effective Cartier divisor cut out by the global section $f_2T_0 - f_1T_1$ of $\mathcal{O}_{\mathbf{P}^1_X}(1)$. Thus we have a resolution

$$0 \to \mathcal{O}_{\mathbf{P}^1_X}(-1) \to \mathcal{O}_{\mathbf{P}^1_X} \to \mathcal{O}_{X'} \to 0$$

Using the description of the cohomology given in Cohomology of Schemes, Section 8 it follows that in this case $\Gamma(X, \mathcal{O}_X) \to \Gamma(X', \mathcal{O}_{X'})$ is an isomorphism and $H^1(X', \mathcal{O}_{X'}) = 0$.

Next, we observe that any diagram as in Example 37.11 is the base change of the diagram in the previous paragraph by the ring map $\mathbf{Z}[f_1, f_2] \to A$. Hence by More on Morphisms, Lemmas 72.1, 72.2, and 72.4 we conclude that $H^1(X', \mathcal{O}_{X'})$ is zero in general and the surjectivity of the map $H^0(X, \mathcal{O}_X) \to H^0(X', \mathcal{O}_{X'})$ in general.

Next, in the general case, let us study the kernel. If $a \in A$ maps to zero, then looking on affine charts we see that

$$a = (f_1x - f_2)(a_0 + a_1x + \ldots + a_rx^r) \text{ in } A[x]$$

for some $r \geq 0$ and $a_0, \ldots, a_r \in A$ and similarly

$$a = (f_1 - f_2y)(b_0 + b_1y + \ldots + b_sy^s) \text{ in } A[y]$$

for some $s \geq 0$ and $b_0, \ldots, b_s \in A$. This means we have

$$a = f_2a_0,\ f_1a_0 = f_2a_1,\ \ldots,\ f_1a_r = 0,\ a = f_1b_0,\ f_2b_0 = f_1b_1,\ \ldots,\ f_2b_s = 0$$

If (a', r', a'_i, s', b'_j) is a second such system, then we have

$$aa' = f_1 f_2 a_0 b'_0 = f_1 f_2 a_1 b'_1 = f_1 f_2 a_2 b'_2 = \ldots = 0$$

as desired. □

For an $\mathbf{F}_p$-algebra A we set $\operatorname{colim}_F A$ equal to the colimit of the system

$$A \xrightarrow{F} A \xrightarrow{F} A \xrightarrow{F} \ldots$$

where $F : A \to A$, $a \mapsto a^p$ is the Frobenius endomorphism.

0EVK **Lemma 38.2.** *Let p be a prime number. Let S be a scheme over $\mathbf{F}_p$. Let $(Sch/S)_h$ be a site as in Definition 34.13. There is a unique sheaf $\mathcal{F}$ on $(Sch/S)_h$ such that*

$$\mathcal{F}(X) = \operatorname{colim}_F \Gamma(X, \mathcal{O}_X)$$

for any quasi-compact and quasi-separated object X of $(Sch/S)_h$.

Proof. Denote $\mathcal{F}$ the Zariski sheafification of the functor

$$X \longrightarrow \operatorname{colim}_F \Gamma(X, \mathcal{O}_X)$$

For quasi-compact and quasi-separated schemes X we have $\mathcal{F}(X) = \operatorname{colim}_F \Gamma(X, \mathcal{O}_X)$. by Sheaves, Lemma 29.1 and the fact that $\mathcal{O}$ is a sheaf for the Zariski topology. Thus it suffices to show that $\mathcal{F}$ is a h sheaf. To prove this we check conditions (1), (2), (3), and (4) of Lemma 37.12. Condition (1) holds because we performed an (almost unnecessary) Zariski sheafification. Condition (2) holds because $\mathcal{O}$ is an fppf sheaf (Descent, Lemma 8.1) and if A is the equalizer of two maps $B \to C$ of $\mathbf{F}_p$-algebras, then $\operatorname{colim}_F A$ is the equalizer of the two maps $\operatorname{colim}_F B \to \operatorname{colim}_F C$.

We check condition (3). Let A, f, J be as in Example 37.10. We have to show that

$$\operatorname{colim}_F A = \operatorname{colim}_F A/J \times_{\operatorname{colim}_F A/fA+J} \operatorname{colim}_F A/fA$$

This reduces to the following algebra question: suppose $a', a'' \in A$ are such that $F^n(a' - a'') \in fA + J$. Find $a \in A$ and $m \geq 0$ such that $a - F^m(a') \in J$ and $a - F^m(a'') \in fA$ and show that the pair (a, m) is uniquely determined up to a replacement of the form $(a, m) \mapsto (F(a), m+1)$. To do this just write $F^n(a'-a'') = fh+g$ with $h \in A$ and $g \in J$ and set $a = F^n(a') - g = F^n(a'') + fh$ and set $m = n$. To see uniqueness, suppose (a_1, m_1) is a second solution. By a replacement of the form given above we may assume $m = m_1$. Then we see that $a - a_1 \in J$ and $a - a_1 \in fA$. Since J is annihilated by a power of f we see that $a - a_1$ is a nilpotent element. Hence $F^k(a - a_1)$ is zero for some large k. Thus after doing more replacements we get $a = a_1$.

We check condition (4). Let X, X', Z, E be as in Example 37.11. By Lemma 38.1 we see that

$$\mathcal{F}(X) = \operatorname{colim}_F \Gamma(X, \mathcal{O}_X) \longrightarrow \operatorname{colim}_F \Gamma(X', \mathcal{O}_{X'}) = \mathcal{F}(X')$$

is bijective. Since $E = \mathbf{P}^1_Z$ in this case we also see that $\mathcal{F}(Z) \to \mathcal{F}(E)$ is bijective. Thus the conclusion holds in this case as well. □

Let p be a prime number. For an $\mathbf{F}_p$-algebra A we set $\lim_F A$ equal to the limit of the inverse system

$$\ldots \xrightarrow{F} A \xrightarrow{F} A \xrightarrow{F} A$$

where $F : A \to A$, $a \mapsto a^p$ is the Frobenius endomorphism.

0EVL **Lemma 38.3.** *Let p be a prime number. Let S be a scheme over $\mathbf{F}_p$. Let $(Sch/S)_h$ be a site as in Definition 34.13. The rule*

$$\mathcal{F}(X) = \lim_F \Gamma(X, \mathcal{O}_X)$$

defines a sheaf on $(Sch/S)_h$.

Proof. To prove $\mathcal{F}$ is a sheaf, let's check conditions (1), (2), (3), and (4) of Lemma 37.12. Condition (1) holds because limits of sheaves are sheaves and $\mathcal{O}$ is a Zariski sheaf. Condition (2) holds because $\mathcal{O}$ is an fppf sheaf (Descent, Lemma 8.1) and if A is the equalizer of two maps $B \to C$ of $\mathbf{F}_p$-algebras, then $\lim_F A$ is the equalizer of the two maps $\lim_F B \to \lim_F C$.

We check condition (3). Let A, f, J be as in Example 37.10. We have to show that

$$\begin{aligned} \lim_F A &\to \lim_F A/J \times_{\lim_F A/fA+J} \lim_F A/fA \\ &= \lim_F (A/J \times_{A/fA+J} A/fA) \\ &= \lim_F A/(fA \cap J) \end{aligned}$$

is bijective. Since J is annihilated by a power of f we see that $\mathfrak{a} = fA \cap J$ is a nilpotent ideal, i.e., there exists an n such that $\mathfrak{a}^n = 0$. It is straightforward to verify that in this case $\lim_F A \to \lim_F A/\mathfrak{a}$ is bijective.

We check condition (4). Let X, X', Z, E be as in Example 37.11. By Lemma 38.1 and the same argument as above we see that

$$\mathcal{F}(X) = \lim_F \Gamma(X, \mathcal{O}_X) \longrightarrow \lim_F \Gamma(X', \mathcal{O}_{X'}) = \mathcal{F}(X')$$

is bijective. Since $E = \mathbf{P}^1_Z$ in this case we also see that $\mathcal{F}(Z) \to \mathcal{F}(E)$ is bijective. Thus the conclusion holds in this case as well. □

In the following lemma we use the absolute weak normalization X^{awn} of a scheme X, see Morphisms, Section 48.

0EVT **Lemma 38.4.** *Let $(Sch/S)_{ph}$ be a site as in Topologies, Definition 8.11. The rule*

$$X \longmapsto \Gamma(X^{awn}, \mathcal{O}_{X^{awn}})$$

is a sheaf on $(Sch/S)_{ph}$.

Proof. To prove $\mathcal{F}$ is a sheaf, let's check conditions (1) and (2) of Topologies, Lemma 8.15. Condition (1) holds because formation of X^{awn} commutes with open coverings, see Morphisms, Lemma 48.7 and its proof.

Let $\pi : Y \to X$ be a surjective proper morphism. We have to show that the equalizer of the two maps

$$\Gamma(Y^{awn}, \mathcal{O}_{Y^{awn}}) \to \Gamma((Y \times_X Y)^{awn}, \mathcal{O}_{(Y \times_X Y)^{awn}})$$

is equal to $\Gamma(X^{awn}, \mathcal{O}_{X^{awn}})$. Let f be an element of this equalizer. Then we consider the morphism

$$f : Y^{awn} \longrightarrow \mathbf{A}^1_X$$

Since $Y^{awn} \to X$ is universally closed, the scheme theoretic image Z of f is a closed subscheme of $\mathbf{A}^1_X$ proper over X and $f : Y^{awn} \to Z$ is surjective. See Morphisms, Lemma 42.10. Thus $Z \to X$ is finite (Morphisms, Lemma 45.11) and surjective.

Let k be a field and let $z_1, z_2 : \mathrm{Spec}(k) \to Z$ be two morphisms equalized by $Z \to X$. We claim that $z_1 = z_2$. It suffices to show the images $\lambda_i = z_i^* f \in k$ agree (as the structure sheaf of Z is generated by f over the structure sheaf of X). To see this

we choose a field extension K/k and morphisms $y_1, y_2 : \mathrm{Spec}(K) \to Y^{awn}$ such that $z_i \circ (\mathrm{Spec}(K) \to \mathrm{Spec}(k)) = f \circ y_i$. This is possible by the surjectivity of the map $Y^{awn} \to Z$. Choose an algebraically closed extension Ω/k of very large cardinality. For any k-algebra maps $\sigma_i : K \to \Omega$ we obtain

$$\mathrm{Spec}(\Omega) \xrightarrow{\sigma_1, \sigma_2} \mathrm{Spec}(K \otimes_k K) \xrightarrow{y_1, y_2} Y^{awn} \times_X Y^{awn}$$

Since the canonical morphism $(Y \times_X Y)^{awn} \to Y^{awn} \times_X Y^{awn}$ is a universal homeomorphism and since Ω is algebraically closed, we can lift the composition above uniquely to a morphism $\mathrm{Spec}(\Omega) \to (Y \times_X Y)^{awn}$. Since f is in the equalizer above, this proves that $\sigma_1(\lambda_1) = \sigma_2(\lambda_2)$. An easy lemma about field extensions shows that this implies $\lambda_1 = \lambda_2$; details omitted.

We conclude that $Z \to X$ is universally injective, i.e., $Z \to X$ is injective on points and induces purely inseparated residue field extensions (Morphisms, Lemma 10.2). All in all we conclude that $Z \to X$ is a universal homeomorphism, see Morphisms, Lemma 46.5.

Let $g : X^{awn} \to Z$ be the map obtained from the universal property of X^{awn}. Then $Y^{awn} \to X^{awn} \to Z$ and $f : Y^{awn} \to Z$ are two morphisms over X. By the universal property of $Y^{awn} \to Y$ the two corresponding morphisms $Y^{awn} \to Y \times_X Z$ over Y have to be equal. This implies that $g \circ \pi^{wan} = f$ as morphisms into $\mathbf{A}^1_X$ and we conclude that $g \in \Gamma(X^{awn}, \mathcal{O}_{X^{awn}})$ is the element we were looking for. □

0EVU **Lemma 38.5.** *Let S be a scheme. Choose a site $(Sch/S)_h$ as in Definition 34.13. The rule*

$$X \longmapsto \Gamma(X^{awn}, \mathcal{O}_{X^{awn}})$$

is the sheafification of the "structure sheaf" $\mathcal{O}$ on $(Sch/S)_h$. Similarly for the ph topology.

Proof. In Lemma 38.4 we have seen that the rule $\mathcal{F}$ of the lemma defines a sheaf in the ph topology and hence a fortiori a sheaf for the h topology. Clearly, there is a canonical map of presheaves of rings $\mathcal{O} \to \mathcal{F}$. To finish the proof, it suffices to show

(1) if $f \in \mathcal{O}(X)$ maps to zero in $\mathcal{F}(X)$, then there is a h covering $\{X_i \to X\}$ such that $f|_{X_i} = 0$, and
(2) given $f \in \mathcal{F}(X)$ there is a h covering $\{X_i \to X\}$ such that $f|_{X_i}$ is the image of $f_i \in \mathcal{O}(X_i)$.

Let f be as in (1). Then $f|_{X^{awn}} = 0$. This means that f is locally nilpotent. Thus if $X' \subset X$ is the closed subscheme cut out by f, then $X' \to X$ is a surjective closed immersion of finite presentation. Hence $\{X' \to X\}$ is the desired h covering. Let f be as in (2). After replacing X by the members of an affine open covering we may assume $X = \mathrm{Spec}(A)$ is affine. Then $f \in A^{awn}$, see Morphisms, Lemma 48.6. By Morphisms, Lemma 47.11 we can find a ring map $A \to B$ of finite presentation such that $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is a universal homeomorphism and such that f is the image of an element $b \in B$ under the canonical map $B \to A^{awn}$. Then $\{\mathrm{Spec}(B) \to \mathrm{Spec}(A)\}$ is an h covering and we conclude. The statement about the ph topology follows in the same manner (or it can be deduced from the statement for the h topology). □

Let p be a prime number. An $\mathbf{F}_p$-algebra A is called *perfect* if the map $F : A \to A$, $x \mapsto x^p$ is an automorphism of A.

0EVV **Lemma 38.6.** *Let p be a prime number. An $\mathbf{F}_p$-algebra A is absolutely weakly normal if and only if it is perfect.*

Proof. It is immediate from condition (2)(b) in Morphisms, Definition 48.1 that if A is absolutely weakly normal, then it is perfect.

Assume A is perfect. Suppose $x, y \in A$ with $x^3 = y^2$. If $p > 3$ then we can write $p = 2n + 3m$ for some $n, m > 0$. Choose $a, b \in A$ with $a^p = x$ and $b^p = y$. Setting $c = a^n b^m$ we have

$$c^{2p} = x^{2n} y^{2m} = x^{2n+3m} = x^p$$

and hence $c^2 = x$. Similarly $c^3 = y$. If $p = 2$, then write $x = a^2$ to get $a^6 = y^2$ which implies $a^3 = y$. If $p = 3$, then write $y = a^3$ to get $x^3 = a^6$ which implies $x = a^2$.

Suppose $x, y \in A$ with $\ell^\ell x = y^\ell$ for some prime number ℓ. If $\ell \neq p$, then $a = y/\ell$ satisfies $a^\ell = x$ and $\ell a = y$. If $\ell = p$, then $y = 0$ and $x = a^p$ for some a. □

0EVW **Lemma 38.7.** *Let p be a prime number.*

(1) *If A is an $\mathbf{F}_p$-algebra, then $\operatorname{colim}_F A = A^{awn}$.*
(2) *If S is a scheme over $\mathbf{F}_p$, then the h sheafification of $\mathcal{O}$ sends a quasi-compact and quasi-separated X to $\operatorname{colim}_F \Gamma(X, \mathcal{O}_X)$.*

Proof. Proof of (1). Observe that $A \to \operatorname{colim}_F A$ induces a universal homeomorphism on spectra by Algebra, Lemma 46.7. Thus it suffices to show that $B = \operatorname{colim}_F A$ is absolutely weakly normal, see Morphisms, Lemma 48.6. Note that the ring map $F : B \to B$ is an automorphism, in other words, B is a perfect ring. Hence Lemma 38.6 applies.

Proof of (2). This follows from (1) and Lemmas 38.2 and 38.5 by looking affine locally. □

39. Descent vector bundles in positive characteristic

0EXA A reference for this section is [BS17].

For a scheme S let us denote $Vect(S)$ the category of finite locally free $\mathcal{O}_S$-modules. Let p be a prime number. Let S be a quasi-compact and quasi-separated scheme over $\mathbf{F}_p$. In this section we will work with the category

$$\operatorname{colim}_F Vect(S) = \operatorname{colim} \Big(Vect(S) \xrightarrow{F^*} Vect(S) \xrightarrow{F^*} Vect(S) \xrightarrow{F^*} \ldots \Big)$$

where $F : S \to S$ is the absolute Frobenius morphism. In down to earth terms an object of this category is a pair $(\mathcal{E}, n)$ where $\mathcal{E}$ is a finite locally free $\mathcal{O}_S$-module and $n \geq 0$ is an integer. For morphisms we take

$$\operatorname{Hom}_{\operatorname{colim}_F Vect(S)}((\mathcal{E}, n), (\mathcal{G}, m)) = \operatorname{colim}_N \operatorname{Hom}_S(F^{N-n,*}\mathcal{E}, F^{N-m,*}\mathcal{G})$$

where $F : S \to S$ is the absolute Frobenius morphism of S. Thus the object $(\mathcal{E}, n)$ is isomorphic to the object $(F^*\mathcal{E}, n + 1)$.

0EXB **Lemma 39.1.** *Let p be a prime number. Let S be a quasi-compact and quasi-separated scheme over $\mathbf{F}_p$. The category $\operatorname{colim}_F Vect(S)$ is equivalent to the category of finite locally free modules over the sheaf of rings $\operatorname{colim}_F \mathcal{O}_S$ on S.*

Proof. Omitted. □

0EXC **Lemma 39.2.** *Let p be a prime number. Consider an almost blowup square X, X', Z, E in characteristic p as in Example 37.10. Then the functor*

$$\operatorname{colim}_F \mathit{Vect}(X) \longrightarrow \operatorname{colim}_F \mathit{Vect}(Z) \times_{\operatorname{colim}_F \mathit{Vect}(E)} \operatorname{colim}_F \mathit{Vect}(X')$$

is an equivalence.

Proof. Let A, f, J be as in Example 37.10. Since all our schemes are affine and since we have internal Hom in the category of vector bundles, the fully faithfulness of the functor follows if we can show that

$$\operatorname{colim} P \otimes_{A,F^N} A = \operatorname{colim} P \otimes_{A,F^N} A/J \times_{\operatorname{colim} P \otimes_{A,F^N} A/fA+J} \operatorname{colim} P \otimes_{A,F^N} A/fA$$

for a finite projective A-module P. After writing P as a summand of a finite free module, this follows from the case where P is finite free. This case immediately reduces to the case $P = A$. The case $P = A$ follows from Lemma 38.2 (in fact we proved this case directly in the proof of this lemma).

Essential surjectivity. Here we obtain the following algebra problem. Suppose P_1 is a finite projective A/J-module, P_2 is a finite projective A/fA-module, and

$$\varphi : P_1 \otimes_{A/J} A/fA + J \longrightarrow P_2 \otimes_{A/fA} A/fA + J$$

is an isomorphism. Goal: show that there exists an N, a finite projective A-module P, an isomorphism $\varphi_1 : P \otimes_A A/J \to P_1 \otimes_{A/J,F^N} A/J$, and an isomorphism $\varphi_2 : P \otimes_A A/fA \to P_2 \otimes_{A/fA,F^N} A/fA$ compatible with φ in an obvious manner. This can be seen as follows. First, observe that

$$A/(J \cap fA) = A/J \times_{A/fA+J} A/fA$$

Hence by More on Algebra, Lemma 6.9 there is a finite projective module P' over $A/(J \cap fA)$ which comes with isomorphisms $\varphi'_1 : P' \otimes_A A/J \to P_1$ and $\varphi_2 : P' \otimes_A A/fA \to P_2$ compatible with φ. Since J is a finitely generated ideal and f-power torsion we see that $J \cap fA$ is a nilpotent ideal. Hence for some N there is a factorization

$$A \xrightarrow{\alpha} A/(J \cap fA) \xrightarrow{\beta} A$$

of F^N. Setting $P = P' \otimes_{A/(J\cap fA),\beta} A$ we conclude. □

0EXD **Lemma 39.3.** *Let p be a prime number. Consider an almost blowup square X, X', Z, E in characteristic p as in Example 37.11. Then the functor*

$$G : \operatorname{colim}_F \mathit{Vect}(X) \longrightarrow \operatorname{colim}_F \mathit{Vect}(Z) \times_{\operatorname{colim}_F \mathit{Vect}(E)} \operatorname{colim}_F \mathit{Vect}(X')$$

is an equivalence.

Proof. Fully faithfulness. Suppose that $(\mathcal{E}, n)$ and $(\mathcal{F}, m)$ are objects of $\operatorname{colim}_F \mathit{Vect}(X)$. Let $(a, b) : G(\mathcal{E}, n) \to G(\mathcal{F}, m)$ be a morphism in the RHS. We may choose $N \gg 0$ and think of a as a map $a : F^{N-n,*}\mathcal{E}|_Z \to F^{N-m,*}\mathcal{F}|_Z$ and b as a map $b : F^{N-n,*}\mathcal{E}|_{X'} \to F^{N-m,*}\mathcal{F}|_{X'}$ agreeing over E. Choose a finite affine open covering $X = X_1 \cup \ldots \cup X_n$ such that $\mathcal{E}|_{X_i}$ and $\mathcal{F}|_{X_i}$ are finite free $\mathcal{O}_{X_i}$-modules. For each i the base change

$$\begin{array}{ccc} E_i & \longrightarrow & X'_i \\ \downarrow & & \downarrow \\ Z_i & \longrightarrow & X_i \end{array}$$

is another almost blow up square as in Example 37.11. For these squares we know that

$$\operatorname{colim}_F H^0(X_i, \mathcal{O}_{X_i}) = \operatorname{colim}_F H^0(Z_i, \mathcal{O}_{Z_i}) \times_{\operatorname{colim}_F H^0(E_i, \mathcal{O}_{E_i})} \operatorname{colim}_F H^0(X_i', \mathcal{O}_{X_i'})$$

by Lemma 38.2 (see proof of the lemma). Hence after increasing N we may assume the maps $a|_{Z_i}$ and $b|_{X_i'}$ come from maps $c_i : F^{N-n,*}\mathcal{E}|_{X_i} \to F^{N-m,*}\mathcal{F}|_{X_i}$. After possibly increasing N we may assume c_i and c_j agree on $X_i \cap X_j$. Thus these maps glue to give the desired morphism $(\mathcal{E}, n) \to (\mathcal{F}, m)$ in the LHS.

Essential surjectivity. Let $(\mathcal{F}, \mathcal{G}, \varphi)$ be a triple consisting of a finite locally free $\mathcal{O}_Z$-module $\mathcal{F}$, a finite locally free $\mathcal{O}_{X'}$-module $\mathcal{G}$, and an isomorphism $\varphi : \mathcal{F}|_E \to \mathcal{G}|_E$. We have to show that after replacing this triple by a Frobenius power pullback, it comes from a finite locally free $\mathcal{O}_X$-module.

Noetherian reduction; we urge the reader to skip this paragraph. Recall that $X = \operatorname{Spec}(A)$ and $Z = \operatorname{Spec}(A/(f_1, f_2))$, $X' = \operatorname{Proj}(A[T_0, T_1]/(f_2T_0 - f_1T_1))$, and $E = \mathbf{P}^1_Z$. By Limits, Lemma 10.3 we can find a finitely generated $\mathbf{F}_p$-subalgebra $A_0 \subset A$ containing f_1 and f_2 such that the triple $(\mathcal{F}, \mathcal{G}, \varphi)$ descends to $X_0 = \operatorname{Spec}(A_0)$ and $Z_0 = \operatorname{Spec}(A_0/(f_1, f_2))$, $X_0' = \operatorname{Proj}(A_0[T_0, T_1]/(f_2T_0 - f_1T_1))$, and $E_0 = \mathbf{P}^1_{Z_0}$. Thus we may assume our schemes are Noetherian.

Assume X is Noetherian. We may choose a finite affine open covering $X = X_1 \cup \ldots \cup X_n$ such that $\mathcal{F}|_{Z \cap X_i}$ is free. Since we can glue objects of $\operatorname{colim}_F \mathit{Vect}(X)$ in the Zariski topology (Lemma 39.1), and since we already know fully faithfulness over X_i and $X_i \cap X_j$ (see first paragraph of the proof), it suffices to prove the existence over each X_i. This reduces us to the case discussed in the next paragraph.

Assume X is Noetherian and $\mathcal{F} = \mathcal{O}_Z^{\oplus r}$. Using φ we get an isomorphism $\mathcal{O}_E^{\oplus r} \to \mathcal{G}|_E$. Let $I = (f_1, f_2) \subset A$. Let $\mathcal{I} \subset \mathcal{O}_{X'}$ be the ideal sheaf of E; it is globally generated by f_1 and f_2. For any n there is a surjection

$$(\mathcal{I}^n/\mathcal{I}^{n+1})^{\oplus r} = \mathcal{I}^n/\mathcal{I}^{n+1} \otimes_{\mathcal{O}_E} \mathcal{G}|_E \longrightarrow \mathcal{I}^n\mathcal{G}/\mathcal{I}^{n+1}\mathcal{G}$$

Hence the first cohomology group of this module is zero. Here we use that $E = \mathbf{P}^1_Z$ and hence its structure sheaf and in fact any globally generated quasi-coherent module has vanishing H^1. Compare with More on Morphisms, Lemma 72.3. Then using the short exact sequences

$$0 \to \mathcal{I}^n\mathcal{G}/\mathcal{I}^{n+1}\mathcal{G} \to \mathcal{G}/\mathcal{I}^{n+1}\mathcal{G} \to \mathcal{G}/\mathcal{I}^n\mathcal{G} \to 0$$

and induction, we see that

$$\lim H^0(X', \mathcal{G}/\mathcal{I}^n\mathcal{G}) \to H^0(E, \mathcal{G}|_E) = H^0(E, \mathcal{O}_E^{\oplus r}) = A/I^{\oplus r}$$

is surjective. By the theorem on formal functions (Cohomology of Schemes, Theorem 20.5) this implies that

$$H^0(X', \mathcal{G}) \to H^0(E, \mathcal{G}|_E) = H^0(E, \mathcal{O}_E^{\oplus r}) = A/I^{\oplus r}$$

is surjective. Thus we can choose a map $\alpha : \mathcal{O}_{X'}^{\oplus r} \to \mathcal{G}$ which is compatible with the given trivialization of $\mathcal{G}|_E$. Thus α is an isomorphism over an open neighbourhood of E in X'. Thus every point of Z has an affine open neighbourhood where we can solve the problem. Since $X' \setminus E \to X \setminus Z$ is an isomorphism, the same holds for points of X not in Z. Thus another Zariski glueing argument finishes the proof. $\square$

0EXE **Proposition 39.4.** *Let p be a prime number. Let S be a scheme in characteristic p. Then the category fibred in groupoids*

$$p : \mathcal{S} \longrightarrow (Sch/S)_h$$

whose fibre category over U is the category of finite locally free $\mathrm{colim}_F \mathcal{O}_U$-modules over U is a stack in groupoids. Moreover, if U is quasi-compact and quasi-separated, then $\mathcal{S}_U$ is colim_F $Vect(U)$.

Proof. The final assertion is the content of Lemma 39.1. To prove the proposition we will check conditions (1), (2), and (3) of Lemma 37.13.

Condition (1) holds because by definition we have glueing for the Zariski topology.

To see condition (2), suppose that $f : X \to Y$ is a surjective, flat, proper morphism of finite presentation over S with Y affine. Since $Y, X, X \times_Y X$ are quasi-compact and quasi-separated, we can use the description of fibre categories given in the statement of the proposition. Then it is clearly enough to show that

$$Vect(Y) \longrightarrow Vect(X) \times_{Vect(X \times_Y X)} Vect(X)$$

is an equivalence (as this will imply the same for the colimits). This follows immediately from fppf descent of finite locally free modules, see Descent, Proposition 5.2 and Lemma 7.6.

Condition (3) is the content of Lemmas 39.2 and 39.3. □

0EXF **Lemma 39.5.** *Let $f : X \to S$ be a proper morphism with geometrically connected fibres where S is the spectrum of a discrete valuation ring. Denote $\eta \in S$ the generic point and denote $X_n \subset X$ the closed subscheme cutout by the nth power of a uniformizer on S. Then there exists an integer n such that the following is true: any finite locally free $\mathcal{O}_X$-module $\mathcal{E}$ such that $\mathcal{E}|_{X_\eta}$ and $\mathcal{E}|_{X_n}$ are free, is free.*

Proof. We first reduce to the case where $X \to S$ has a section. Say $S = \mathrm{Spec}(A)$. Choose a closed point ξ of X_η. Choose an extension of discrete valuation rings $A \subset B$ such that the fraction field of B is $\kappa(\xi)$. This is possible by Krull-Akizuki (Algebra, Lemma 120.18) and the fact that $\kappa(\xi)$ is a finite extension of the fraction field of A. By the valuative criterion of properness (Morphisms, Lemma 43.1) we get a B-valued point $\tau : \mathrm{Spec}(B) \to X$ which induces a section $\sigma : \mathrm{Spec}(B) \to X_B$. For a finite locally free $\mathcal{O}_X$-module $\mathcal{E}$ let $\mathcal{E}_B$ be the pullback to the base change X_B. By flat base change (Cohomology of Schemes, Lemma 5.2) we see that $H^0(X_B, \mathcal{E}_B) = H^0(X, \mathcal{E}) \otimes_A B$. Thus if $\mathcal{E}_B$ is free of rank r, then the sections in $H^0(X, \mathcal{E})$ generate the free B-module $\tau^*\mathcal{E} = \sigma^*\mathcal{E}_B$. In particular, we can find r global sections $s_1, \ldots, s_r$ of $\mathcal{E}$ which generate $\tau^*\mathcal{E}$. Then

$$s_1, \ldots, s_r : \mathcal{O}_X^{\oplus r} \longrightarrow \mathcal{E}$$

is a map of finite locally free $\mathcal{O}_X$-modules of rank r and the pullback to X_B is a map of free $\mathcal{O}_{X_B}$-modules which restricts to an isomorphism in one point of each fibre. Taking the determinant we get a function $g \in \Gamma(X_\eta, \mathcal{O}_{X_B})$ which is invertible in one point of each fibre. As the fibres are proper and connected, we see that g must be invertible (details omitted; hint: use Varieties, Lemma 9.3). Thus it suffices to prove the lemma for the base change $X_B \to \mathrm{Spec}(B)$.

Assume we have a section $\sigma : S \to X$. Let $\mathcal{E}$ be a finite locally free $\mathcal{O}_X$-module which is assumed free on the generic fibre and on X_n (we will choose n later). Choose an isomorphism $\sigma^*\mathcal{E} = \mathcal{O}_S^{\oplus r}$. Consider the map

$$K = R\Gamma(X, \mathcal{E}) \longrightarrow R\Gamma(S, \sigma^*\mathcal{E}) = A^{\oplus r}$$

in $D(A)$. Arguing as above, we see $\mathcal{E}$ is free if (and only if) the induced map $H^0(K) = H^0(X, \mathcal{E}) \to A^{\oplus r}$ is surjective.

Set $L = R\Gamma(X, \mathcal{O}_X^{\oplus r})$ and observe that the corresponding map $L \to A^{\oplus r}$ has the desired property. Observe that $K \otimes_A Q(A) \cong L \otimes_A Q(A)$ by flat base change and the assumption that $\mathcal{E}$ is free on the generic fibre. Let $\pi \in A$ be a uniformizer. Observe that

$$K \otimes^{\mathbf{L}}_A A/\pi^m A = R\Gamma(X, \mathcal{E} \xrightarrow{\pi^m} \mathcal{E})$$

and similarly for L. Denote $\mathcal{E}_{tors} \subset \mathcal{E}$ the coherent subsheaf of sections supported on the special fibre and similarly for other $\mathcal{O}_X$-modules. Choose $k > 0$ such that $(\mathcal{O}_X)_{tors} \to \mathcal{O}_X/\pi^k\mathcal{O}_X$ is injective (Cohomology of Schemes, Lemma 10.3). Since $\mathcal{E}$ is locally free, we see that $\mathcal{E}_{tors} \subset \mathcal{E}/\pi^k\mathcal{E}$. Then for $n \geq m+k$ we have isomorphisms

$$\begin{aligned}
(\mathcal{E} \xrightarrow{\pi^m} \mathcal{E}) &\cong (\mathcal{E}/\pi^k\mathcal{E} \xrightarrow{\pi^m} \mathcal{E}/\pi^{k+m}\mathcal{E}) \\
&\cong (\mathcal{O}_X^{\oplus r}/\pi^k\mathcal{O}_X^{\oplus r} \xrightarrow{\pi^m} \mathcal{O}_X^{\oplus r}/\pi^{k+m}\mathcal{O}_X^{\oplus r}) \\
&\cong (\mathcal{O}_X^{\oplus r} \xrightarrow{\pi^m} \mathcal{O}_X^{\oplus r})
\end{aligned}$$

in $D(\mathcal{O}_X)$. This determines an isomorphism

$$K \otimes^{\mathbf{L}}_A A/\pi^m A \cong L \otimes^{\mathbf{L}}_A A/\pi^m A$$

in $D(A)$ (holds when $n \geq m+k$). Observe that these isomorphisms are compatible with pulling back by σ hence in particular we conclude that $K \otimes^{\mathbf{L}}_A A/\pi^m A \to (A/\pi^m A)^{\oplus r}$ defines an surjection on degree 0 cohomology modules (as this is true for L). Since A is a discrete valuation ring, we have

$$K \cong \bigoplus H^i(K)[-i] \quad \text{and} \quad L \cong \bigoplus H^i(L)[-i]$$

in $D(A)$. See More on Algebra, Example 71.3. The cohomology groups $H^i(K) = H^i(X, \mathcal{E})$ and $H^i(L) = H^i(X, \mathcal{O}_X)^{\oplus r}$ are finite A-modules by Cohomology of Schemes, Lemma 19.2. By More on Algebra, Lemma 126.3 these modules are direct sums of cyclic modules. We have seen above that the rank β_i of the free part of $H^i(K)$ and $H^i(L)$ are the same. Next, observe that

$$H^i(L \otimes^{\mathbf{L}}_A A/\pi^m A) = H^i(L)/\pi^m H^i(L) \oplus H^{i+1}(L)[\pi^m]$$

and similarly for K. Let e be the largest integer such that $A/\pi^e A$ occurs as a summand of $H^i(X, \mathcal{O}_X)$, or equivalently $H^i(L)$, for some i. Then taking $m = e+1$ we see that $H^i(L \otimes^{\mathbf{L}}_A A/\pi^m A)$ is a direct sum of β_i copies of $A/\pi^m A$ and some other cyclic modules each annihilated by π^e. By the same reasoning for K and the isomorphism $K \otimes^{\mathbf{L}}_A A/\pi^m A \cong L \otimes^{\mathbf{L}}_A A/\pi^m A$ it follows that $H^i(K)$ cannot have any cyclic summands of the form $A/\pi^l A$ with $l > e$. (It also follows that K is isomorphic to L as an object of $D(A)$, but we won't need this.) Then the only way the map

$$H^0(K \otimes^{\mathbf{L}}_A A/\pi^{e+1}A) = H^0(K)/\pi^{e+1}H^0(K) \oplus H^1(K)[\pi^{e+1}] \longrightarrow (A/\pi^{e+1}A)^{\oplus r}$$

is surjective, is if it is surjective on the first summand. This is what we wanted to show. (To be precise, the integer n in the statement of the lemma, if there is a section σ, should be equal to $k+e+1$ where k and e are as above and depend only on X.) □

0EXG **Lemma 39.6.** *Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{E}$ be a finite locally free $\mathcal{O}_X$-module. Assume*

(1) *f is flat and proper and $\mathcal{O}_S = f_*\mathcal{O}_X$,*
(2) *S is a normal Noetherian scheme,*
(3) *the pullback of $\mathcal{E}$ to $X \times_S \mathrm{Spec}(\mathcal{O}_{S,s})$ is free for every codimension 1 point $s \in S$.*

Then $\mathcal{E}$ is isomorphic to the pullback of a finite locally free $\mathcal{O}_S$-module.

Proof. We will prove the canonical map

$$\Phi : f^*f_*\mathcal{E} \longrightarrow \mathcal{E}$$

is an isomorphism. By flat base change (Cohomology of Schemes, Lemma 5.2) and assumptions (1) and (3) we see that the pullback of this to $X \times_S \mathrm{Spec}(\mathcal{O}_{S,s})$ is an isomorphism for every codimension 1 point $s \in S$. By Divisors, Lemma 2.11 it suffices to prove that $\mathrm{depth}((f^*f_*\mathcal{E})_x) \geq 2$ for any point $x \in X$ mapping to a point $s \in S$ of codimension ≥ 2. Since f is flat and $(f^*f_*\mathcal{E})_x = (f_*\mathcal{E})_s \otimes_{\mathcal{O}_{S,s}} \mathcal{O}_{X,x}$, it suffices to prove that $\mathrm{depth}((f_*\mathcal{E})_s) \geq 2$, see Algebra, Lemma 163.2. Since S is a normal Noetherian scheme and $\dim(\mathcal{O}_{S,s}) \geq 2$ we have $\mathrm{depth}(\mathcal{O}_{S,s}) \geq 2$, see Properties, Lemma 12.5. Thus we get what we want from Divisors, Lemma 6.6. □

We can use the results above to prove the following miraculous statement.

0EXH **Theorem 39.7.** *Let p be a prime number. Let Y be a quasi-compact and quasi-separated scheme over $\mathbf{F}_p$. Let $f : X \to Y$ be a proper, surjective morphism of finite presentation with geometrically connected fibres. Then the functor*

$$\mathrm{colim}_F \mathit{Vect}(Y) \longrightarrow \mathrm{colim}_F \mathit{Vect}(X)$$

is fully faithful with essential image described as follows. Let $\mathcal{E}$ be a finite locally free $\mathcal{O}_X$-module. Assume for all $y \in Y$ there exists integers $n_y, r_y \geq 0$ such that

$$F^{n_y,*}\mathcal{E}|_{X_{y,red}} \cong \mathcal{O}_{X_{y,red}}^{\oplus r_y}$$

Then for some $n \geq 0$ the nth Frobenius power pullback $F^{n,*}\mathcal{E}$ is the pullback of a finite locally free $\mathcal{O}_Y$-module.

Proof. Proof of fully faithfulness. Since vectorbundles on Y are locally trivial, this reduces to the statement that

$$\mathrm{colim}_F \Gamma(Y, \mathcal{O}_Y) \longrightarrow \mathrm{colim}_F \Gamma(X, \mathcal{O}_X)$$

is bijective. Since $\{X \to Y\}$ is an h covering, this will follow from Lemma 38.2 if we can show that the two maps

$$\mathrm{colim}_F \Gamma(X, \mathcal{O}_X) \longrightarrow \mathrm{colim}_F \Gamma(X \times_Y X, \mathcal{O}_{X \times_Y X})$$

are equal. Let $g \in \Gamma(X, \mathcal{O}_X)$ and denote g_1 and g_2 the two pullbacks of g to $X \times_Y X$. Since $X_{y,red}$ is geometrically connected, we see that $H^0(X_{y,red}, \mathcal{O}_{X_{y,red}})$ is a purely inseparable extension of $\kappa(y)$, see Varieties, Lemma 9.3. Thus $g^q|_{X_{y,red}}$ comes from an element of $\kappa(y)$ for some p-power q (which may depend on y). It follows that g_1^q and g_2^q map to the same element of the residue field at any point of

$(X \times_Y X)_y = X_y \times_y X_y$. Hence $g_1 - g_2$ restricts to zero on $(X \times_Y X)_{red}$. Hence $(g_1 - g_2)^n = 0$ for some n which we may take to be a p-power as desired.

Description of essential image. Let $\mathcal{E}$ be as in the statement of the proposition. We first reduce to the Noetherian case.

Let $y \in Y$ be a point and view it as a morphism $y \to Y$ from the spectrum of the residue field into Y. We can write $y \to Y$ as a filtered limit of morphisms $Y_i \to Y$ of finite presentation with Y_i affine. (It is best to prove this yourself, but it also follows formally from Limits, Lemma 7.2 and 4.13.) For each i set $Z_i = Y_i \times_Y X$. Then $X_y = \lim Z_i$ and $X_{y,red} = \lim Z_{i,red}$. By Limits, Lemma 10.2 we can find an i such that $F^{n_y,*}\mathcal{E}|_{Z_{i,red}} \cong \mathcal{O}_{Z_{i,red}}^{\oplus r_y}$. Fix i. We have $Z_{i,red} = \lim Z_{i,j}$ where $Z_{i,j} \to Z_i$ is a thickening of finite presentation (Limits, Lemma 9.4). Using the same lemma as before we can find a j such that $F^{n_y,*}\mathcal{E}|_{Z_{i,j}} \cong \mathcal{O}_{Z_{i,j}}^{\oplus r_y}$. We conclude that for each $y \in Y$ there exists a morphism $Y_y \to Y$ of finite presentation whose image contains y and a thickening $Z_y \to Y_y \times_Y X$ such that $F^{n_y,*}\mathcal{E}|_{Z_y} \cong \mathcal{O}_{Z_y}^{\oplus r_y}$. Observe that the image of $Y_y \to Y$ is constructible (Morphisms, Lemma 23.2). Since Y is quasi-compact in the constructible topology (Topology, Lemma 23.2 and Properties, Lemma 2.4) we conclude that there are a finite number of morphisms

$$Y_1 \to Y,\ Y_2 \to Y,\ \ldots,\ Y_N \to Y$$

of finite presentation such that $Y = \bigcup \mathrm{Im}(Y_a \to Y)$ set theoretically and such that for each $a \in \{1, \ldots, N\}$ there exist integers $n_a, r_a \geq 0$ and there is a thickening $Z_a \subset Y_a \times_Y X$ of finite presentation such that $F^{n_a,*}\mathcal{E}|_{Z_a} \cong \mathcal{O}_{Z_a}^{\oplus r_a}$.

Formulated in this way, the condition descends to an absolute Noetherian approximation. We strongly urge the reader to skip this paragraph. First write $Y = \lim_{i \in I} Y_i$ as a cofiltered limit of schemes of finite type over $\mathbf{F}_p$ with affine transition morphisms (Limits, Lemma 7.2). Next, we can assume we have proper morphisms $f_i : X_i \to Y_i$ whose base change to Y recovers $f : X \to Y$, see Limits, Lemma 10.1. After increasing i we may assume there exists a finite locally free $\mathcal{O}_{X_i}$-module $\mathcal{E}_i$ whose pullback to X is isomorphic to $\mathcal{E}$, see Limits, Lemma 10.3. Pick $0 \in I$ and denote $E \subset Y_0$ the constructible subset where the geometric fibres of f_0 are connected, see More on Morphisms, Lemma 28.6. Then $Y \to Y_0$ maps into E, see More on Morphisms, Lemma 28.2. Thus $Y_i \to Y_0$ maps into E for $i \gg 0$, see Limits, Lemma 4.10. Hence we see that the fibres of f_i are geometrically connected for $i \gg 0$. By Limits, Lemma 10.1 for large enough i we can find morphisms $Y_{i,a} \to Y_i$ of finite type whose base change to Y recovers $Y_a \to Y$, $a \in \{1, \ldots, N\}$. After possibly increasing i we can find thickenings $Z_{i,a} \subset Y_{i,a} \times_{Y_i} X_i$ whose base change to $Y_a \times_Y X$ recovers Z_a (same reference as before combined with Limits, Lemmas 8.5 and 8.15). Since $Z_a = \lim Z_{i,a}$ we find that after increasing i we may assume $F^{n_a,*}\mathcal{E}_i|_{Z_{i,a}} \cong \mathcal{O}_{Z_{i,a}}^{\oplus r_a}$, see Limits, Lemma 10.2. Finally, after increasing i one more time we may assume $\coprod Y_{i,a} \to Y_i$ is surjective by Limits, Lemma 8.15. At this point all the assumptions hold for $X_i \to Y_i$ and $\mathcal{E}_i$ and we see that it suffices to prove result for $X_i \to Y_i$ and $\mathcal{E}_i$.

Assume Y is of finite type over $\mathbf{F}_p$. To prove the result we will use induction on $\dim(Y)$. We are trying to find an object of colim_F *Vect*(Y) which pulls back to the object of colim_F *Vect*(X) determined by $\mathcal{E}$. By the fully faithfulness already proven and because of Proposition 39.4 it suffices to construct a descent of $\mathcal{E}$ after replacing

Y by the members of a h covering and X by the corresponding base change. This means that we may replace Y by a scheme proper and surjective over Y provided this does not increase the dimension of Y. If $T \subset T'$ is a thickening of schemes of finite type over $\mathbf{F}_p$ then $\mathrm{colim}_F\ \mathit{Vect}(T) = \mathrm{colim}_F\ \mathit{Vect}(T')$ as $\{T \to T'\}$ is a h covering such that $T \times_{T'} T = T$. If $T' \to T$ is a universal homeomorphism of schemes of finite type over $\mathbf{F}_p$, then $\mathrm{colim}_F\ \mathit{Vect}(T) = \mathrm{colim}_F\ \mathit{Vect}(T')$ as $\{T \to T'\}$ is a h covering such that the diagonal $T \subset T \times_{T'} T$ is a thickening.

Using the general remarks made above, we may and do replace X by its reduction and we may assume X is reduced. Consider the Stein factorization $X \to Y' \to Y$, see More on Morphisms, Theorem 53.4. Then $Y' \to Y$ is a universal homeomorphism of schemes of finite type over $\mathbf{F}_p$. By the above we may replace Y by Y'. Thus we may assume $f_*\mathcal{O}_X = \mathcal{O}_Y$ and that Y is reduced. This reduces us to the case discussed in the next paragraph.

Assume Y is reduced and $f_*\mathcal{O}_X = \mathcal{O}_Y$ over a dense open subscheme of Y. Then $X \to Y$ is flat over a dense open subscheme $V \subset Y$, see Morphisms, Proposition 28.2. By Lemma 31.1 there is a V-admissible blowing up $Y' \to Y$ such that the strict transform X' of X is flat over Y'. Observe that $\dim(Y') = \dim(Y)$ as Y and Y' have a common dense open subscheme. By More on Morphisms, Lemma 53.7 and the fact that $V \subset Y'$ is dense all fibres of $f' : X' \to Y'$ are geometrically connected. We still have $(f'_*\mathcal{O}_{X'})|_V = \mathcal{O}_V$. Write

$$Y' \times_Y X = X' \cup E \times_Y X$$

where $E \subset Y'$ is the exceptional divisor of the blowing up. By the general remarks above, it suffices to prove existence for $Y' \times_Y X \to Y'$ and the restriction of $\mathcal{E}$ to $Y' \times_Y X$. Suppose that we find some object ξ' in $\mathrm{colim}_F\ \mathit{Vect}(Y')$ pulling back to the restriction of $\mathcal{E}$ to X' (viewed as an object of the colimit category). By induction on $\dim(Y)$ we can find an object ξ'' in $\mathrm{colim}_F\ \mathit{Vect}(E)$ pulling back to the restriction of $\mathcal{E}$ to $E \times_Y X$. Then the fully faithfullness determines a unique isomorphism $\xi'|_E \to \xi''$ compatible with the given identifications with the restriction of $\mathcal{E}$ to $E \times_{Y'} X'$. Since

$$\{E \times_Y X \to Y' \times_Y X, X' \to Y' \times_Y X\}$$

is a h covering given by a pair of closed immersions with

$$(E \times_Y X) \times_{(Y' \times_Y X)} X' = E \times_{Y'} X'$$

we conclude that ξ' pulls back to the restriction of $\mathcal{E}$ to $Y' \times_Y X$. Thus it suffices to find ξ' and we reduce to the case discussed in the next paragraph.

Assume Y is reduced, f is flat, and $f_*\mathcal{O}_X = \mathcal{O}_Y$ over a dense open subscheme of Y. In this case we consider the normalization $Y^\nu \to Y$ (Morphisms, Section 55). This is a finite surjective morphism (Morphisms, Lemma 55.11 and 18.2) which is an isomorphism over a dense open. Hence by our general remarks we may replace Y by Y^ν and X by $Y^\nu \times_Y X$. After this replacement we see that $\mathcal{O}_Y = f_*\mathcal{O}_X$ (because the Stein factorization has to be an isomorphism in this case; small detail omitted).

Assume Y is a normal Noetherian scheme, that f is flat, and that $f_*\mathcal{O}_X = \mathcal{O}_Y$. After replacing $\mathcal{E}$ by a suitable Frobenius power pullback, we may assume $\mathcal{E}$ is trivial on the scheme theoretic fibres of f at the generic points of the irreducible

components of Y (because colim_F *Vect*$(-)$ is an equivalence on universal homeomorphisms, see above). Similarly to the arguments above (in the reduction to the Noetherian case) we conclude there is a dense open subscheme $V \subset Y$ such that $\mathcal{E}|_{f^{-1}(V)}$ is free. Let $Z \subset Y$ be a closed subscheme such that $Y = V \amalg Z$ set theoretically. Let $z_1, \ldots, z_t \in Z$ be the generic points of the irreducible components of Z of codimension 1. Then $A_i = \mathcal{O}_{Y,z_i}$ is a discrete valuation ring. Let n_i be the integer found in Lemma 39.5 for the scheme X_{A_i} over A_i. After replacing $\mathcal{E}$ by a suitable Frobenius power pullback, we may assume $\mathcal{E}$ is free over $X_{A_i/\mathfrak{m}_i^{n_i}}$ (again because the colimit category is invariant under universal homeomorphisms, see above). Then Lemma 39.5 tells us that $\mathcal{E}$ is free on X_{A_i}. Thus finally we conclude by applying Lemma 39.6. □

40. Blowing up complexes

0ESM This section finds normal forms for perfect objects of the derived category after blowups.

0ESP **Lemma 40.1.** *Let X be a scheme. Let $E \in D(\mathcal{O}_X)$ be pseudo-coherent. For every $p, k \in \mathbf{Z}$ there is an finite type quasi-coherent sheaf of ideals $Fit_{p,k}(E) \subset \mathcal{O}_X$ with the following property: for $U \subset X$ open such that $E|_U$ is isomorphic to*

$$\ldots \to \mathcal{O}_U^{\oplus n_{b-2}} \xrightarrow{d_{b-2}} \mathcal{O}_U^{\oplus n_{b-1}} \xrightarrow{d_{b-1}} \mathcal{O}_U^{\oplus n_b} \to 0 \to \ldots$$

the restriction $Fit_{p,k}(E)|_U$ is generated by the minors of the matrix of d_p of size

$$-k + n_{p+1} - n_{p+2} + \ldots + (-1)^{b-p+1} n_b$$

Convention: the ideal generated by $r \times r$-minors is $\mathcal{O}_U$ if $r \leq 0$ and the ideal generated by $r \times r$-minors where $r > \min(n_p, n_{p+1})$ is zero.

Proof. Observe that E locally on X has the shape as stated in the lemma, see More on Algebra, Section 66, Cohomology, Section 47, and Derived Categories of Schemes, Section 10. Thus it suffices to prove that the ideal of minors is independent of the chosen representative. To do this, it suffices to check in local rings. Over a local ring $(R, \mathfrak{m}, \kappa)$ consider a bounded above complex

$$F^\bullet : \ldots \to R^{\oplus n_{b-2}} \xrightarrow{d_{b-2}} R^{\oplus n_{b-1}} \xrightarrow{d_{b-1}} R^{\oplus n_b} \to 0 \to \ldots$$

Denote $\mathrm{Fit}_{k,p}(F^\bullet) \subset R$ the ideal generated by the minors of size $k - n_{p+1} + n_{p+2} - \ldots + (-1)^{b-p} n_b$ in the matrix of d_p. Suppose some matrix coefficient of some differential of $F^\bullet$ is invertible. Then we pick a largest integer i such that d_i has an invertible matrix coefficient. By Algebra, Lemma 102.2 the complex $F^\bullet$ is isomorphic to a direct sum of a trivial complex $\ldots \to 0 \to R \to R \to 0 \to \ldots$ with nonzero terms in degrees i and $i+1$ and a complex $(F')^\bullet$. We leave it to the reader to see that $\mathrm{Fit}_{p,k}(F^\bullet) = \mathrm{Fit}_{p,k}((F')^\bullet)$; this is where the formula for the size of the minors is used. If $(F')^\bullet$ has another differential with an invertible matrix coefficient, we do it again, etc. Continuing in this manner, we eventually reach a complex $(F^\infty)^\bullet$ all of whose differentials have matrices with coefficients in $\mathfrak{m}$. Here you may have to do an infinite number of steps, but for any cutoff only a finite number of these steps affect the complex in degrees $\geq$ the cutoff. Thus the "limit" $(F^\infty)^\bullet$ is a well defined bounded above complex of finite free modules, comes equipped with a quasi-isomorphism $(F^\infty)^\bullet \to F^\bullet$ into the complex we started

with, and $\mathrm{Fit}_{p,k}(F^\bullet) = \mathrm{Fit}_{p,k}((F^\infty)^\bullet)$. Since the complex $(F^\infty)^\bullet$ is unique up to isomorphism by More on Algebra, Lemma 77.6 the proof is complete. □

0ESQ **Lemma 40.2.** *Let X be a scheme. Let $E \in D(\mathcal{O}_X)$ be perfect. Let $U \subset X$ be a scheme theoretically dense open subscheme such that $H^i(E|_U)$ is finite locally free of constant rank r_i for all $i \in \mathbf{Z}$. Then there exists a U-admissible blowup $b : X' \to X$ such that $H^i(Lb^*E)$ is a perfect $\mathcal{O}_{X'}$-module of tor dimension ≤ 1 for all $i \in \mathbf{Z}$.*

Proof. We will construct and study the blowup affine locally. Namely, suppose that $V \subset X$ is an affine open subscheme such that $E|_V$ can be represented by the complex

$$\mathcal{O}_V^{\oplus n_a} \xrightarrow{d_a} \ldots \xrightarrow{d_{b-1}} \mathcal{O}_V^{\oplus n_b}$$

Set $k_i = r_{i+1} - r_{i+2} + \ldots + (-1)^{b-i+1} r_b$. A computation which we omit show that over $U \cap V$ the rank of d_i is

$$\rho_i = -k_i + n_{i+1} - n_{i+2} + \ldots + (-1)^{b-i+1} n_b$$

in the sense that the cokernel of d_i is finite locally free of rank $n_{i+1} - \rho_i$. Let $\mathcal{I}_i \subset \mathcal{O}_V$ be the ideal generated by the minors of size $\rho_i \times \rho_i$ in the matrix of d_i.

On the one hand, comparing with Lemma 40.1 we see the ideal $\mathcal{I}_i$ corresponds to the global ideal $\mathrm{Fit}_{i,k_i}(E)$ which was shown to be independent of the choice of the complex representing $E|_V$. On the other hand, $\mathcal{I}_i$ is the $(n_{i+1} - \rho_i)$th Fitting ideal of $\mathrm{Coker}(d_i)$. Please keep this in mind.

We let $b : X' \to X$ be the blowing up in the product of the ideals $\mathrm{Fit}_{i,k_i}(E)$; this makes sense as locally on X almost all of these ideals are equal to the unit ideal (see above). This blowup dominates the blowups $b_i : X'_i \to X$ in the ideals $\mathrm{Fit}_{i,k_i}(E)$, see Divisors, Lemma 33.12. By Divisors, Lemma 36.3 each b_i is a U-admissible blowup. It follows that b is a U-admissible blowup (tiny detail omitted; compare with the proof of Divisors, Lemma 35.4). Finally, U is still a scheme theoretically dense open subscheme of X'. Thus after replacing X by X' we end up in the situation discussed in the next paragraph.

Assume $\mathrm{Fit}_{i,k_i}(E)$ is an invertible ideal for all i. Choose an affine open V and a complex of finite free modules representing $E|_V$ as above. It follows from Divisors, Lemma 36.3 that $\mathrm{Coker}(d_i)$ has tor dimension ≤ 1. Thus $\mathrm{Im}(d_i)$ is finite locally free as the kernel of a map from a finite locally free module to a finitely presented module of tor dimension ≤ 1. Hence $\mathrm{Ker}(d_i)$ is finite locally free as well (same argument). Thus the short exact sequence

$$0 \to \mathrm{Im}(d_{i-1}) \to \mathrm{Ker}(d_i) \to H^i(E)|_V \to 0$$

shows what we want and the proof is complete. □

0ESR **Lemma 40.3.** *Let X be an integral scheme. Let $E \in D(\mathcal{O}_X)$ be perfect. Then there exists a nonempty open $U \subset X$ such that $H^i(E|_U)$ is finite locally free of constant rank r_i for all $i \in \mathbf{Z}$ and there exists a U-admissible blowup $b : X' \to X$ such that $H^i(Lb^*E)$ is a perfect $\mathcal{O}_{X'}$-module of tor dimension ≤ 1 for all $i \in \mathbf{Z}$.*

Proof. We strongly urge the reader to find their own proof of the existence of U. Let $\eta \in X$ be the generic point. The restriction of E to η is isomorphic in $D(\kappa(\eta))$ to a finite complex $V^\bullet$ of finite dimensional vector spaces with zero differentials. Set $r_i = \dim_{\kappa(\eta)} V^i$. Then the perfect object E' in $D(\mathcal{O}_X)$ represented by the

complex with terms $\mathcal{O}_X^{\oplus r_i}$ and zero differentials becomes isomorphic to E after pulling back to η. Hence by Derived Categories of Schemes, Lemma 35.9 there is an open neighbourhood U of η such that $E|_U$ and $E'|_U$ are isomorphic. This proves the first assertion. The second follows from the first and Lemma 40.2 as any nonempty open is scheme theoretically dense in the integral scheme X. □

0F8J **Remark 40.4.** Let X be a scheme. Let $E \in D(\mathcal{O}_X)$ be a perfect object such that $H^i(E)$ is a perfect $\mathcal{O}_X$-module of tor dimension ≤ 1 for all $i \in \mathbf{Z}$. This property sometimes allows one to reduce questions about E to questions about $H^i(E)$. For example, suppose

$$\mathcal{E}^a \xrightarrow{d^a} \ldots \xrightarrow{d^{b-2}} \mathcal{E}^{b-1} \xrightarrow{d^{b-1}} \mathcal{E}^b$$

is a bounded complex of finite locally free $\mathcal{O}_X$-modules representing E. Then $\mathrm{Im}(d^i)$ and $\mathrm{Ker}(d^i)$ are finite locally free $\mathcal{O}_X$-modules for all i. Namely, suppose by induction we know this for all indices bigger than i. Then we can first use the short exact sequence

$$0 \to \mathrm{Im}(d^i) \to \mathrm{Ker}(d^{i+1}) \to H^{i+1}(E) \to 0$$

and the assumption that $H^{i+1}(E)$ is perfect of tor dimension ≤ 1 to conclude that $\mathrm{Im}(d^i)$ is finite locally free. The same argument used again for the short exact sequence

$$0 \to \mathrm{Ker}(d^i) \to \mathcal{E}^i \to \mathrm{Im}(d^i) \to 0$$

then gives that $\mathrm{Ker}(d^i)$ is finite locally free. It follows that the distinguished triangles

$$\tau_{\leq k-1}E \to \tau_{\leq k}E \to H^k(E)[-k] \to (\tau_{\leq k-1}E)[1]$$

are represented by the following short exact sequences of bounded complexes of finite locally free modules

$$\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
\mathcal{E}^a & \to & \ldots & \to & \mathcal{E}^{k-2} & \to & \mathrm{Ker}(d^{k-1}) & & \\
\downarrow & & & & \downarrow & & \downarrow & & \\
\mathcal{E}^a & \to & \ldots & \to & \mathcal{E}^{k-2} & \to & \mathcal{E}^{k-1} & \to & \mathrm{Ker}(d^k) \\
 & & & & & & \downarrow & & \downarrow \\
 & & & & & & \mathrm{Im}(d^{k-1}) & \to & \mathrm{Ker}(d^k) \\
 & & & & & & \downarrow & & \\
 & & & & & & 0 & &
\end{array}$$

Here the complexes are the rows and the "obvious" zeros are omitted from the display.

41. Blowing up perfect modules

0F8K This section tries to find normal forms for perfect modules of tor dimension ≤ 1 after blowups. We are only partially successful.

0ESS **Lemma 41.1.** *Let X be a scheme. Let $\mathcal{F}$ be a perfect $\mathcal{O}_X$-module of tor dimension ≤ 1. For any blowup $b : X' \to X$ we have $Lb^*\mathcal{F} = b^*\mathcal{F}$ and $b^*\mathcal{F}$ is a perfect $\mathcal{O}_X$-module of tor dimension ≤ 1.*

Proof. We may assume $X = \mathrm{Spec}(A)$ is affine and we may assume the A-module M corresponding to $\mathcal{F}$ has a presentation

$$0 \to A^{\oplus m} \to A^{\oplus n} \to M \to 0$$

Suppose $I \subset A$ is an ideal and $a \in I$. Recall that the affine blowup algebra $A[\frac{I}{a}]$ is a subring of A_a. Since localization is exact we see that $A_a^{\oplus m} \to A_a^{\oplus n}$ is injective. Hence $A[\frac{I}{a}]^{\oplus m} \to A[\frac{I}{a}]^{\oplus n}$ is injective too. This proves the lemma. □

0EST **Lemma 41.2.** *Let X be a scheme. Let $\mathcal{F}$ be a perfect $\mathcal{O}_X$-module of tor dimension ≤ 1. Let $U \subset X$ be a scheme theoretically dense open such that $\mathcal{F}|_U$ is finite locally free of constant rank r. Then there exists a U-admissible blowup $b : X' \to X$ such that there is a canonical short exact sequence*

$$0 \to \mathcal{K} \to b^*\mathcal{F} \to \mathcal{Q} \to 0$$

where $\mathcal{Q}$ is finite locally free of rank r and $\mathcal{K}$ is a perfect $\mathcal{O}_X$-module of tor dimension ≤ 1 whose restriction to U is zero.

Proof. Combine Divisors, Lemma 36.3 and Lemma 41.1. □

0ESU **Lemma 41.3.** *Let X be a scheme. Let $\mathcal{F}$ be a perfect $\mathcal{O}_X$-module of tor dimension ≤ 1. Let $U \subset X$ be an open such that $\mathcal{F}|_U = 0$. Then there is a U-admissible blowup*

$$b : X' \to X$$

such that $\mathcal{F}' = b^*\mathcal{F}$ is equipped with two canonical locally finite filtrations

$$0 = F^0 \subset F^1 \subset F^2 \subset \ldots \subset \mathcal{F}' \quad \text{and} \quad \mathcal{F}' = F_1 \supset F_2 \supset F_3 \supset \ldots \supset 0$$

such that for each $n \geq 1$ there is an effective Cartier divisor $D_n \subset X'$ with the property that

$$F^i/F^{i-1} \quad \text{and} \quad F_i/F_{i+1}$$

are finite locally free of rank i on D_i.

Proof. Choose an affine open $V \subset X$ such that there exists a presentation

$$0 \to \mathcal{O}_V^{\oplus n} \xrightarrow{A} \mathcal{O}_V^{\oplus n} \to \mathcal{F} \to 0$$

for some n and some matrix A. The ideal we are going to blowup in is the product of the Fitting ideals $\mathrm{Fit}_k(\mathcal{F})$ for $k \geq 0$. This makes sense because in the affine situation above we see that $\mathrm{Fit}_k(\mathcal{F})|_V = \mathcal{O}_V$ for $k > n$. It is clear that this is a U-admissible blowing up. By Divisors, Lemma 33.12 we see that on X' the ideals $\mathrm{Fit}_k(\mathcal{F})$ are invertible. Thus we reduce to the case discussed in the next paragraph.

Assume $\mathrm{Fit}_k(\mathcal{F})$ is an invertible ideal for $k \geq 0$. If $E_k \subset X$ is the effective Cartier divisor defined by $\mathrm{Fit}_k(\mathcal{F})$ for $k \geq 0$, then the effective Cartier divisors D_k in the statement of the lemma will satisfy

$$E_k = D_{k+1} + 2D_{k+2} + 3D_{k+3} + \ldots$$

This makes sense as the collection D_k will be locally finite. Moreover, it uniquely determines the effective Cartier divisors D_k hence it suffices to construct D_k locally.

Choose an affine open $V \subset X$ and presentation of $\mathcal{F}|_V$ as above. We will construct the divisors and filtrations by induction on the integer n in the presentation. We set $D_k|_V = \emptyset$ for $k > n$ and we set $D_n|V = E_{n-1}|_V$. After shrinking V we may assume that $\mathrm{Fit}_{n-1}(\mathcal{F})|_V$ is generated by a single nonzerodivisor $f \in \Gamma(V, \mathcal{O}_V)$. Since $\mathrm{Fit}_{n-1}(\mathcal{F})|_V$ is the ideal generated by the entries of A, we see that there is

a matrix A' in $\Gamma(V, \mathcal{O}_V)$ such that $A = fA'$. Define $\mathcal{F}'$ on V by the short exact sequence

$$0 \to \mathcal{O}_V^{\oplus n} \xrightarrow{A'} \mathcal{O}_V^{\oplus n} \to \mathcal{F}' \to 0$$

Since the entries of A' generate the unit ideal in $\Gamma(V, \mathcal{O}_V)$ we see that $\mathcal{F}'$ locally on V has a presentation with n decreased by 1, see Algebra, Lemma 102.2. Further note that $f^{n-k}\mathrm{Fit}_k(\mathcal{F}') = \mathrm{Fit}_k(\mathcal{F})|_V$ for $k = 0, \ldots, n$. Hence $\mathrm{Fit}_k(\mathcal{F}')$ is an invertible ideal for all k. We conclude by induction that there exist effective Cartier divisors $D'_k \subset V$ such that $\mathcal{F}'$ has two canonical filtrations as in the statement of the lemma. Then we set $D_k|_V = D'_k$ for $k = 1, \ldots, n-1$. Observe that the equalities between effective Cartier divisors displayed above hold with these choices. Finally, we come to the construction of the filtrations. Namely, we have short exact sequences

$$0 \to \mathcal{O}_{D_n \cap V}^{\oplus n} \to \mathcal{F} \to \mathcal{F}' \to 0 \quad \text{and} \quad 0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{O}_{D_n \cap V}^{\oplus n} \to 0$$

coming from the two factorizations $A = A'f = fA'$ of A. These sequences are canonical because in the first one the submodule is $\mathrm{Ker}(f : \mathcal{F} \to \mathcal{F})$ and in the second one the quotient module is $\mathrm{Coker}(f : \mathcal{F} \to \mathcal{F})$. □

0ESV **Lemma 41.4.** *Let X be a scheme. Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a homorphism of perfect $\mathcal{O}_X$-modules of tor dimension ≤ 1. Let $U \subset X$ be a scheme theoretically dense open such that $\mathcal{F}|_U = 0$ and $\mathcal{G}|_U = 0$. Then there is a U-admissible blowup $b : X' \to X$ such that the kernel, image, and cokernel of $b^*\varphi$ are perfect $\mathcal{O}_{X'}$-modules of tor dimension ≤ 1.*

Proof. The assumptions tell us that the object $(\mathcal{F} \to \mathcal{G})$ of $D(\mathcal{O}_X)$ is perfect. Thus we get a U-admissible blowup that works for the cokernel and kernel by Lemmas 40.2 and 41.1 (to see what the complex looks like after pullback). The image is the kernel of the cokernel and hence is going to be perfect of tor dimension ≤ 1 as well. □

42. An operator introduced by Berthelot and Ogus

0F8L Please read Cohomology, Section 55 first.

Let X be a scheme. Let $D \subset X$ be an effective Cartier divisor. Let $\mathcal{I} = \mathcal{I}_D \subset \mathcal{O}_X$ be the ideal sheaf of D, see Divisors, Section 15. Clearly we can apply the discussion in Cohomology, Section 55 to X and $\mathcal{I}$.

0F8M **Lemma 42.1.** *Let X be a scheme. Let $D \subset X$ be an effective Cartier divisor with ideal sheaf $\mathcal{I} \subset \mathcal{O}_X$. Let $\mathcal{F}^\bullet$ be a complex of quasi-coherent $\mathcal{O}_X$-modules such that $\mathcal{F}^i$ is $\mathcal{I}$-torsion free for all i. Then $\eta_\mathcal{I}\mathcal{F}^\bullet$ is a complex of quasi-coherent $\mathcal{O}_X$-modules. Moreover, if $U = \mathrm{Spec}(A) \subset X$ is affine open and $D \cap U = V(f)$, then $\eta_f(\mathcal{F}^\bullet(U))$ is canonically isomorphic to $(\eta_\mathcal{I}\mathcal{F}^\bullet)(U)$.*

Proof. Omitted. □

0GTV **Lemma 42.2.** *Let X be a scheme. Let $D \subset X$ be an effective Cartier divisor with ideal sheaf $\mathcal{I} \subset \mathcal{O}_X$. The functor $L\eta_\mathcal{I} : D(\mathcal{O}_X) \to D(\mathcal{O}_X)$ of Cohomology, Lemma 55.7 sends $D_{QCoh}(\mathcal{O}_X)$ into itself. Moreover, if $X = \mathrm{Spec}(A)$ is affine and $D = V(f)$, then the functor $L\eta_f$ on $D(A)$ defined in More on Algebra, Lemma 97.4 and the functor $L\eta_\mathcal{I}$ on $D_{QCoh}(\mathcal{O}_X)$ correspond via the equivalence of Derived Categories of Schemes, Lemma 3.5.*

Proof. Omitted. □

43. Blowing up complexes, II

0F8R The material in this section will be used to construct a version of Macpherson's graph construction in Section 44.

0F8S **Situation 43.1.** Here X is a scheme, $D \subset X$ is an effective Cartier divisor with ideal sheaf $\mathcal{I} \subset \mathcal{O}_X$, and M is a perfect object of $D(\mathcal{O}_X)$.

Let (X, D, M) be a triple as in Situation 43.1. Consider an affine open $U = \mathrm{Spec}(A) \subset X$ such that

(1) $D \cap U = V(f)$ for some nonzerodivisor $f \in A$, and
(2) there exists a bounded complex $M^\bullet$ of finite free A-modules representing $M|_U$ (via the equivalence of Derived Categories of Schemes, Lemma 3.5).

We will say that $(U, A, f, M^\bullet)$ is an *affine chart* for (X, D, M). Consider the ideals $I_i(M^\bullet, f) \subset A$ defined in More on Algebra, Section 98. Let us say (X, S, M) is a *good triple* if for every $x \in D$ there exists an affine chart $(U, A, f, M^\bullet)$ with $x \in U$ and $I_i(M^\bullet, f)$ principal ideals for all $i \in \mathbf{Z}$.

0F8T **Lemma 43.2.** *In Situation 43.1 let $h : Y \to X$ be a morphism of schemes such that the pullback $E = h^{-1}D$ of D is defined (Divisors, Definition 14.12). Let $(U, A, f, M^\bullet)$ is an affine chart for (X, D, M). Let $V = \mathrm{Spec}(B) \subset Y$ is an affine open with $h(V) \subset U$. Denote $g \in B$ the image of $f \in A$. Then*

(1) *$(V, B, g, M^\bullet \otimes_A B)$ is an affine chart for (Y, E, Lh^*M),*
(2) *$I_i(M^\bullet, f)B = I_i(M^\bullet \otimes_A B, g)$ in B, and*
(3) *if (X, D, M) is a good triple, then (Y, E, Lh^*M) is a good triple.*

Proof. The first statement follows from the following observations: g is a nonzerodivisor in B which defines $E \cap V \subset V$ and $M^\bullet \otimes_A B$ represents $M^\bullet \otimes^{\mathbf{L}}_A B$ and hence represents the pullback of M to V by Derived Categories of Schemes, Lemma 3.8. Part (2) follows from part (1) and More on Algebra, Lemma 98.3. Combined with More on Algebra, Lemma 98.3 we conclude that the second statement of the lemma holds. □

0GTW **Lemma 43.3.** *Let $X, D, \mathcal{I}, M$ be as in Situation 43.1. If (X, D, M) is a good triple, then $L\eta_{\mathcal{I}}M$ is a perfect object of $D(\mathcal{O}_X)$.*

Proof. Translation of More on Algebra, Lemma 98.5. To do the translation use Lemma 42.2. □

0GTX **Lemma 43.4.** *Let $X, D, \mathcal{I}, M$ be as in Situation 43.1. Assume (X, D, M) is a good triple. If there exists a locally bounded complex $\mathcal{M}^\bullet$ of finite locally free $\mathcal{O}_X$-modules representing M, then there exists a locally bounded complex $\mathcal{Q}^\bullet$ of finite locally free $\mathcal{O}_{X'}$-modules representing $L\eta_{\mathcal{I}}M$.*

Proof. By Cohomology, Lemma 55.7 the complex $\mathcal{Q}^\bullet = \eta_{\mathcal{I}}\mathcal{M}^\bullet$ represents $L\eta_{\mathcal{I}}M$. To check that this complex is locally bounded and consists of finite locally free, we may work affine locally. Then the boundedness is clear. Choose an affine chart $(U, A, f, M^\bullet)$ for (X, D, M) such that the ideals $I_i(M^\bullet, f)$ are principal and such that $\mathcal{M}^i|_U$ is finite free for each i. By our assumption that (X, D, M) is a good triple we can do this. Writing $N^i = \Gamma(U, \mathcal{M}^i|_U)$ we get a bounded complex $N^\bullet$ of finite free A-modules representing the same object in $D(A)$ as the complex $M^\bullet$ (by Derived Categories of Schemes, Lemma 3.5). Then $I_i(N^\bullet, f)$ is a principal ideal for all i by More on Algebra, Lemma 98.1. Hence the complex $\eta_f N^\bullet$ is a

bounded complex of finite locally free A-modules. Since $\mathcal{Q}^i|_U$ is the quasi-coherent $\mathcal{O}_U$-module corresponding to $\eta_f N^i$ by Lemma 42.1 we conclude. □

0F9X **Lemma 43.5.** *In Situation 43.1 let $h : Y \to X$ be a morphism of schemes such that the pullback $E = h^{-1}D$ is defined. If (X, D, M) is a good triple, then*

$$Lh^*(L\eta_{\mathcal{I}}M) = L\eta_{\mathcal{J}}(Lh^*M)$$

in $D(\mathcal{O}_Y)$ where $\mathcal{J}$ is the ideal sheaf of E.

Proof. Translation of More on Algebra, Lemma 98.6. Use Lemmas 42.1 and 42.2 to do the translation. □

0GTY **Lemma 43.6.** *In Situation 43.1 there is a unique morphism $b : X' \to X$ such that*

1. *the pullback $D' = b^{-1}D$ is defined and (X', D', M') is a good triple where $M' = Lb^*M$, and*
2. *for any morphism of schemes $h : Y \to X$ such that the pullback $E = h^{-1}D$ is defined and (Y, E, Lh^*M) is a good triple, there is a unique factorization of h through b.*

Moreover, for any affine chart $(U, A, f, M^\bullet)$ the restriction $b^{-1}(U) \to U$ is the blowing up in the product of the ideals $I_i(M^\bullet, f)$ and for any quasi-compact open $W \subset X$ the restriction $b|_{b^{-1}(W)} : b^{-1}(W) \to W$ is a $W \setminus D$-admissible blowing up.

Proof. The proof is just that we will locally blow up X in the product ideals $I_i(M^\bullet, f)$ for any affine chart $(U, A, f, M^\bullet)$. The first few lemmas in More on Algebra, Section 98 show that this is well defined. The universal property (2) then follows from the universal property of blowing up. The details can be found below.

Let $U, A, f, M^\bullet$ be an affine chart for (X, D, M). All but a finite number of the ideals $I_i(M^\bullet, f)$ are equal to A hence it makes sense to look at

$$I = \prod\nolimits_i I_i(M^\bullet, f)$$

and this is a finitely generated ideal of A. Denote

$$b_U : U' \to U$$

the blowing up of U in I. Then $b_U^{-1}(U \cap D)$ is defined by Divisors, Lemma 33.11. Recall that $f^{r_i} \in I_i(M^\bullet, f)$ and hence b_U is a $(U \setminus D)$-admissible blowing up. By Divisors, Lemma 33.12 for each i the morphism b_U factors as $U' \to U_i' \to U$ where $U_i' \to U$ is the blowing up in $I_i(M^\bullet, f)$ and $U' \to U_i'$ is another blowing up. It follows that the pullback $I_i(M^\bullet, f)\mathcal{O}_{U'}$ of $I_i(M^\bullet, f)$ to U' is an invertible ideal sheaf, see Divisors, Lemmas 33.11 and 33.4. It follows that $(U', b^{-1}D, Lb^*M|_U)$ is a good triple, see Lemma 43.2 for the behaviour of the ideals $I_i(-,-)$ under pullback. Finally, we claim that $b_U : U' \to U$ has the universal property mentioned in part (2) of the statement of the lemma. Namely, suppose $h : Y \to U$ is a morphism of schemes such that the pullback $E = h^{-1}(D \cap U)$ is defined and (Y, E, Lh^*M) is a good triple. Then Y is covered by affine charts $(V, B, g, N^\bullet)$ such that $I_i(N^\bullet, g)$ is an invertible ideal for each i. Then g and the image of f in B differ by a unit as they both cut out the effective Cartier divisor $E \cap V$. Hence we may assume g is the image of f by More on Algebra, Lemma 98.2. Then $I_i(N^\bullet, g)$ is isomorphic to $I_i(M^\bullet \otimes_A B, g)$ as a B-module by More on Algebra, Lemma 98.1. Thus $I_i(M^\bullet \otimes_A B, g) = I_i(M^\bullet, f)B$ (Lemma 43.2) is an invertible B-module. Hence

the ideal IB is invertible. It follows that $I\mathcal{O}_Y$ is invertible. Hence we obtain a unique factorization of h through b_U by Divisors, Lemma 33.5.

Let $\mathcal{B}$ be the set of affine opens $U \subset X$ such that there exists an affine chart $(U, A, f, M^\bullet)$ for (X, D, M). Then $\mathcal{B}$ is a basis for the topology on X; details omitted. For $U \in \mathcal{B}$ we have the morphism $b_U : U' \to U$ constructed above which satisfies the universal property over U. If $U_1 \subset U_2 \subset X$ are both in $\mathcal{B}$, then $b_{U_1} : U_1' \to U_1$ is canonically isomorphic to

$$b_{U_2}|_{b_{U_2}^{-1}(U_1)} : b_{U_2}^{-1}(U_1) \longrightarrow U_1$$

by the universal property. In other words, we get an isomorphism $U_1' \to b_{U_2}^{-1}(U_1)$ over U_1. These isomorphisms satisfy the cocycle condition (again by the universal property) and hence by Constructions, Lemma 2.1 we get a morphism $b : X' \to X$ whose restriction to each U in $\mathcal{B}$ is isomorphic to $U' \to U$. Then the morphism $b : X' \to X$ satisfies properties (1) and (2) of the statement of the lemma as these properties may be checked locally (details omitted).

We still have to prove the final assertion of the lemma. Let $W \subset X$ be a quasi-compact open. Choose a finite covering $W = U_1 \cup \ldots \cup U_T$ such that for each $1 \leq t \leq T$ there exists an affine chart $(U_t, A_t, f_t, M_t^\bullet)$. We will use below that for any affine open $V = \mathrm{Spec}(B) \subset U_t \cap U_{t'}$ we have (a) the images of f_t and $f_{t'}$ in B differ by a unit, and (b) the complexes $M_t^\bullet \otimes_A B$ and $M_{t'} \otimes_A B$ define isomorphic objects of $D(B)$. For $i \in \mathbf{Z}$, set

$$N_i = \max\nolimits_{t=1,\ldots,T} \left(\sum\nolimits_{j \geq i} (-1)^{j-i} rk(M_t^j) \right)$$

Then $N_t - \sum_{j \geq i} (-1)^{j-i} rk(M_t^j) \geq 0$ and we can consider the ideals

$$I_{t,i} = f_t^{N_i - \sum_{j \geq i} (-1)^{j-i} rk(M_t^j)} I_i(M_t^\bullet, f_t) \subset A_t$$

It follows from More on Algebra, Lemmas 98.2 and 98.1 that the ideals $I_{t,i}$ glue to a quasi-coherent, finite type ideal $\mathcal{I}_i \subset \mathcal{O}_W$. Moreover, all but a finite number of these ideals are equal to $\mathcal{O}_W$. Clearly, the morphism $X' \to X$ constructed above restricts to the blowing up of W in the product of the ideals $\mathcal{I}_i$. This finishes the proof. □

0F8U **Lemma 43.7.** *In Situation 43.1 let $b : X' \to X$ be the morphism of Lemma 43.6. Consider the effective Cartier divisor $D' = b^{-1}D$ with ideal sheaf $\mathcal{I}' \subset \mathcal{O}_{X'}$. Then $Q = L\eta_{\mathcal{I}'} Lb^* M$ is a perfect object of $D(\mathcal{O}_{X'})$.*

Proof. Follows from Lemmas 43.6 and 43.3. □

0F8V **Lemma 43.8.** *In Situation 43.1 let $h : Y \to X$ be a morphism of schemes such that the pullback $E = h^{-1}D$ is defined. Let $b : X' \to X$, resp. $c : Y' \to Y$ be as constructed in Lemma 43.6 for $D \subset X$ and M, resp. $E \subset Y$ and Lh^*M. Then Y' is the strict transform of Y with respect to $b : X' \to X$ (see proof for a precise formulation of this) and*

$$L\eta_{\mathcal{J}'} L(h \circ c)^* M = L(Y' \to X')^* Q$$

where $Q = L\eta_{\mathcal{I}'} Lb^* M$ as in Lemma 43.7. In particular, if (Y, E, Lh^*M) is a good triple and $k : Y \to X'$ is the unique morphism such that $h = b \circ k$, then $L\eta_{\mathcal{J}} Lh^* M = Lk^* Q$.

Proof. Denote $E' = c^{-1}E$. Then $(Y', E', L(h \circ c)^* M)$ is a good triple. Hence by the universal property of Lemma 43.6 there is a unique morphism

$$h' : Y' \longrightarrow X'$$

such that $b \circ h' = h \circ c$. In particular, there is a morphism $(h', c) : Y' \to X' \times_X Y$. We claim that given $W \subset X$ quasi-compact open, such that $b^{-1}(W) \to W$ is a blowing up, this morphism identifies $Y'|_W$ with the strict transform of Y_W with respect to $b^{-1}(W) \to W$. In turn, to see this is true is a local question on W, and we may therefore prove the statement over an affine chart. We do this in the next paragraph.

Let $(U, A, f, M^\bullet)$ be an affine chart for (X, D, M). Recall from the proof of Lemma 43.7 that the restriction of $b : X' \to X$ to U is the blowing up of $U = \mathrm{Spec}(A)$ in the product of the ideals $I_i(M^\bullet, f)$. Now if $V = \mathrm{Spec}(B) \subset Y$ is any affine open with $h(V) \subset U$, then $(V, B, g, M^\bullet \otimes_A B)$ is an affine chart for $(Y, E, Lh^* M)$ where $g \in B$ is the image of f, see Lemma 43.2. Hence the restriction of $c : Y' \to Y$ to V is the blowing up in the product of the ideals $I_i(M^\bullet, f)B$, i.e., the morphism $c : Y' \to Y$ over $h^{-1}(U)$ is the blowing up of $h^{-1}(U)$ in the ideal $\prod I_i(M^\bullet, f)\mathcal{O}_{h^{-1}(U)}$. Since this is also true for the strict transform, we see that our claim on strict transforms is true.

Having said this the equality $L\eta_{\mathcal{J}'} L(h \circ c)^* M = L(Y' \to X')^* Q$ follows from Lemma 43.5. The final statement is a special case of this (namely, the case where $c = \mathrm{id}_Y$ and $k = h'$). □

0F8W **Lemma 43.9.** *In Situation 43.1 let $W \subset X$ be the maximal open subscheme over which the cohomology sheaves of M are locally free. Then the morphism $b : X' \to X$ of Lemma 43.6 is an isomorphism over W.*

Proof. This is true because for any affine chart $(U, A, f, M^\bullet)$ with $U \subset W$ we have that $I_i(M^\bullet, f)$ are locally generated by a power of f by More on Algebra, Lemma 98.4. Since f is a nonzerodivisor, the blowing up $b^{-1}(U) \to U$ is an isomorphism. □

0GTZ **Lemma 43.10.** *Let $X, D, \mathcal{I}, M$ be as in Situation 43.1. If (X, D, M) is a good triple, then there exists a closed immersion*

$$i : T \longrightarrow D$$

of finite presentation with the following properties

1. *T scheme theoretically contains $D \cap W$ where $W \subset X$ is the maximal open over which the cohomology sheaves of M are locally free,*
2. *the cohomology sheaves of $Li^* L\eta_{\mathcal{I}} M$ are locally free, and*
3. *for any point $t \in T$ with image $x = i(t) \in W$ the rank of $H^i(M)_x$ over $\mathcal{O}_{X,x}$ and the rank of $H^i(Li^* L\eta_{\mathcal{I}} M)_t$ over $\mathcal{O}_{T,t}$ agree.*

Proof. Let $(U, A, f, M^\bullet)$ be an affine chart for (X, D, M) such that $I_i(M^\bullet, f)$ is a principal ideal for all $i \in \mathbf{Z}$. Then we define $T \cap U \subset D \cap U$ as the closed subscheme defined by the ideal

$$J(M^\bullet, f) = \sum J_i(M^\bullet, f) \subset A/fA$$

studied in More on Algebra, Lemmas 98.8 and 98.9; in terms of the second lemma we see that $T \cap U \to D \cap U$ is given by the ring map $A/fA \to C$ studied there. Since

(X, D, M) is a good triple we can cover X by affine charts of this form and by the first of the two lemmas, this construction glues. Hence we obtain a closed subscheme $T \subset D$ which on good affine charts as above is given by the ideal $J(M^\bullet, f)$. Then properties (1) and (2) follow from the second lemma. Details omitted. Small observation to help the reader: since $\eta_f M^\bullet$ is a complex of locally free modules by More on Algebra, Lemma 98.5 we see that $Li^* L\eta_{\mathcal{I}} M|_{T \cap U}$ is represented by the complex $\eta_f M^\bullet \otimes_A C$ of C-modules. The statement (3) on ranks follows from Cohomology, Lemma 55.10. □

0F8X **Lemma 43.11.** *In Situation 43.1. Let $b : X' \to X$ and D' be as in Lemma 43.6. Let $Q = L\eta_{\mathcal{I}'} Lb^* M$ be as in Lemma 43.7. Let $W \subset X$ be the maximal open where M has locally free cohomology modules. Then there exists a closed immersion $i : T \to D'$ of finite presentation such that*

(1) *$D' \cap b^{-1}(W) \subset T$ scheme theoretically,*
(2) *$Li^* Q$ has locally free cohomology sheaves, and*
(3) *for $t \in T$ mapping to $w \in W$ the rank of $H^i(Li^* Q)_t$ over $\mathcal{O}_{T,t}$ is equal to the rank of $H^i(M)_x$ over $\mathcal{O}_{X,x}$.*

Proof. Lemma 43.9 tells us that b is an isomorphism over W. Hence $b^{-1}(W) \subset X'$ is contained in the maximal open $W' \subset X'$ where $Lb^* M$ has locally free cohomology sheaves. Then the actual statements in the lemma are an immediate consequence of Lemma 43.10 applied to $(X', D', Lb^* M)$ and the other lemmas mentioned in the statement. □

0F8Y **Lemma 43.12.** *In Situation 43.1. Let $b : X' \to X$, $D' \subset X'$, and Q be as in Lemma 43.7. Let $\rho = (\rho_i)_{i \in \mathbf{Z}}$ be integers. Let $W(\rho) \subset X$ be the maximal open subscheme where $H^i(M)$ is locally free of rank ρ_i for all i. Let $i : T \to D'$ be as in Lemma 43.11. Then there exists an open and closed subscheme $T(\rho) \subset T$ containing $D' \cap b^{-1}(W(\rho))$ scheme theoretically such that $H^i(Li^* Q|_{T(\rho)})$ is locally free of rank ρ_i for all i.*

Proof. Let $T(\rho) \subset T$ be the open and closed subscheme where $H^i(Li^* Q)$ has rank ρ_i for all i. Then the statement is immediate from the assertion in Lemma 43.11 on ranks of the cohomology modules. □

0GU0 **Lemma 43.13.** *In Situation 43.1. Let $b : X' \to X$, $D' \subset X'$, and Q be as in Lemma 43.7. If there exists a locally bounded complex $\mathcal{M}^\bullet$ of finite locally free $\mathcal{O}_X$-modules representing M, then there exists a locally bounded complex $\mathcal{Q}^\bullet$ of finite locally free $\mathcal{O}_{X'}$-modules representing Q.*

Proof. Recall that $Q = L\eta_{\mathcal{I}'} Lb^* M$ where $\mathcal{I}'$ is the ideal sheaf of the effective Cartier divisor D'. The locally bounded complex $(\mathcal{M}')^\bullet = b^* \mathcal{M}^\bullet$ of finite locally free $\mathcal{O}_{X'}$-modules represents $Lb^* M$. Thus the lemma follows from Lemma 43.4. □

0F9Y **Lemma 43.14.** *Let X be a scheme and let $D \subset X$ be an effective Cartier divisor. Let $M \in D(\mathcal{O}_X)$ be a perfect object. Let $W \subset X$ be the maximal open over which the cohomology sheaves $H^i(M)$ are locally free. There exists a proper morphism $b : X' \longrightarrow X$ and an object Q in $D(\mathcal{O}_{X'})$ with the following properties*

(1) *$b : X' \to X$ is an isomorphism over $X \setminus D$,*
(2) *$b : X' \to X$ is an isomorphism over W,*
(3) *$D' = b^{-1} D$ is an effective Cartier divisor,*
(4) *$Q = L\eta_{\mathcal{I}'} Lb^* M$ where $\mathcal{I}'$ is the ideal sheaf of D',*

(5) *Q is a perfect object of $D(\mathcal{O}_{X'})$,*
(6) *there exists a closed immersion $i : T \to D'$ of finite presentation such that*
   (a) *$D' \cap b^{-1}(W) \subset T$ scheme theoretically,*
   (b) *Li^*Q has finite locally free cohomology sheaves,*
   (c) *for $t \in T$ with image $w \in W$ the rank of $H^i(Li^*Q)_t$ over $\mathcal{O}_{T,t}$ is equal to the rank of $H^i(M)_x$ over $\mathcal{O}_{X,x}$,*
(7) *for any affine chart $(U, A, f, M^\bullet)$ for (X, D, M) the restriction of b to U is the blowing up of $U = \mathrm{Spec}(A)$ in the ideal $I = \prod I_i(M^\bullet, f)$, and*
(8) *for any affine chart $(V, B, g, N^\bullet)$ for (X', D', Lb^*N) such that $I_i(N^\bullet, g)$ is principal, we have*
   (a) *$Q|_V$ corresponds to $\eta_g N^\bullet$,*
   (b) *$T \subset V \cap D'$ corresponds to the ideal $J(N^\bullet, g) = \sum J_i(N^\bullet, g) \subset B/gB$ studied in More on Algebra, Lemma 98.9.*
(9) *If M can be represented by a locally bounded complex of finite locally free $\mathcal{O}_X$-modules, then Q can be represented by a bounded complex of finite locally free $\mathcal{O}_{X'}$-modules.*

Proof. This statement collects the information obtained in Lemmas 43.2, 43.3, 43.5, 43.6, 43.7, 43.8, 43.9, 43.10, 43.11, and 43.13. □

44. Blowing up complexes, III

0F8Z In this section we give an "algebra version" of the version of Macpherson's graph construction given in [Ful98, Section 18.1].

Let X be a scheme. Let E be a perfect object of $D(\mathcal{O}_X)$. Let $U \subset X$ be the maximal open subscheme such that $E|_U$ has locally free cohomology sheaves.

Consider the commutative diagram

$$\begin{array}{ccccc} \mathbf{A}^1_X & \longrightarrow & \mathbf{P}^1_X & \longleftarrow & (\mathbf{P}^1_X)_\infty \\ & \searrow & \downarrow{\scriptstyle p} & \swarrow & \uparrow{\scriptstyle \infty} \\ & & X & & \end{array}$$

Here we recall that $\mathbf{A}^1 = D_+(T_0)$ is the first standard affine open of $\mathbf{P}^1$ and that $\infty = V_+(T_0)$ is the complementary effective Cartier divisor and the diagram above is the pullback of these schemes to X. Observe that $\infty : X \to (\mathbf{P}^1_X)_\infty$ is an isomorphism. Then

$$(\mathbf{P}^1_X, (\mathbf{P}^1_X)_\infty, Lp^*E)$$

is a triple as in Situation 43.1 in Section 43. Let

$$b : W \longrightarrow \mathbf{P}^1_X \quad \text{and} \quad W_\infty = b^{-1}((\mathbf{P}^1_X)_\infty)$$

be the blowing up and effective Cartier divisor constructed starting with this triple in Lemma 43.6. We also denote

$$Q = L\eta_{\mathcal{I}} Lb^* M = L\eta_{\mathcal{I}} L(p \circ b)^* E$$

the perfect object of $D(\mathcal{O}_W)$ considered in Lemma 43.7. Here $\mathcal{I} \subset \mathcal{O}_W$ is the ideal sheaf of W_∞.

0F90 **Lemma 44.1.** *The construction above has the following properties:*
(1) *b is an isomorphism over $\mathbf{P}^1_U \cup \mathbf{A}^1_X$,*
(2) *the restriction of Q to $\mathbf{A}^1_X$ is equal to the pullback of E,*

(3) *there exists a closed immersion $i : T \to W_\infty$ of finite presentation such that $(W_\infty \to X)^{-1}U \subset T$ scheme theoretically and such that Li^*Q has locally free cohomology sheaves,*
(4) *for $t \in T$ with image $u \in U$ we have that the rank $H^i(Li^*Q)_t$ over $\mathcal{O}_{T,t}$ is equal to the rank of $H^i(M)_u$ over $\mathcal{O}_{U,u}$,*
(5) *if E can be represented by a locally bounded complex of finite locally free $\mathcal{O}_X$-modules, then Q can be represented by a locally bounded complex of finite locally free $\mathcal{O}_W$-modules.*

Proof. This follows immediately from the results in Section 43; for a statement collecting everything needed, see Lemma 43.14. □

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