QUOTIENTS OF GROUPOIDS

048A

Contents

1. Introduction

048B This chapter is devoted to generalities concerning groupoids and their quotients (as far as they exist). There is a lot of literature on this subject, see for example [MFK94], [Ses72], [Kol97], [KM97], [Kol08] and many more.

2. Conventions and notation

048C In this chapter the conventions and notation are those introduced in Groupoids in Spaces, Sections 2 and 3.

3. Invariant morphisms

048D

048E **Definition 3.1.** Let S be a scheme, and let B be an algebraic space over S. Let $j = (t, s) : R \to U \times_B U$ be a pre-relation of algebraic spaces over B. We say a morphism $\phi : U \to X$ of algebraic spaces over B is *R-invariant* if the diagram

$$\begin{array}{ccc} R & \xrightarrow[s]{} & U \\ {\scriptstyle t}\downarrow & & \downarrow{\scriptstyle \phi} \\ U & \xrightarrow{\phi} & X \end{array}$$

is commutative. If $j : R \to U \times_B U$ comes from the action of a group algebraic space G on U over B as in Groupoids in Spaces, Lemma 15.1, then we say that ϕ is *G-invariant*.

In other words, a morphism $U \to X$ is R-invariant if it equalizes s and t. We can reformulate this in terms of associated quotient sheaves as follows.

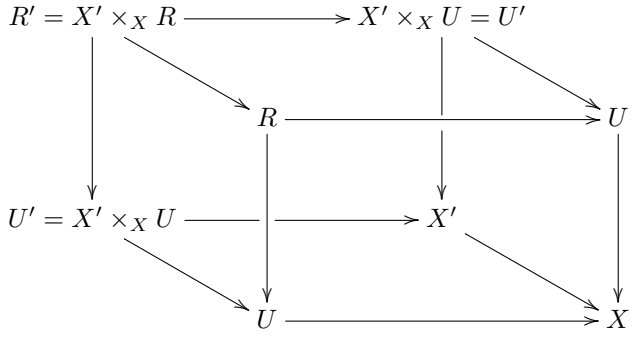
048F **Lemma 3.2.** *Let S be a scheme, and let B be an algebraic space over S. Let $j = (t, s) : R \to U \times_B U$ be a pre-relation of algebraic spaces over B. A morphism of algebraic spaces $\phi : U \to X$ is R-invariant if and only if it factors as $U \to U/R \to X$.*

Proof. This is clear from the definition of the quotient sheaf in Groupoids in Spaces, Section 19. □

048G **Lemma 3.3.** *Let S be a scheme, and let B be an algebraic space over S. Let $j = (t, s) : R \to U \times_B U$ be a pre-relation of algebraic spaces over B. Let $U \to X$ be an R-invariant morphism of algebraic spaces over B. Let $X' \to X$ be any morphism of algebraic spaces.*

1. *Setting $U' = X' \times_X U$, $R' = X' \times_X R$ we obtain a pre-relation $j' : R' \to U' \times_B U'$.*
2. *If j is a relation, then j' is a relation.*
3. *If j is a pre-equivalence relation, then j' is a pre-equivalence relation.*
4. *If j is an equivalence relation, then j' is an equivalence relation.*
5. *If j comes from a groupoid in algebraic spaces (U, R, s, t, c) over B, then*
   (a) *(U, R, s, t, c) is a groupoid in algebraic spaces over X, and*
   (b) *j' comes from the base change (U', R', s', t', c') of this groupoid to X', see Groupoids in Spaces, Lemma 11.6.*
6. *If j comes from the action of a group algebraic space G/B on U as in Groupoids in Spaces, Lemma 15.1 then j' comes from the induced action of G on U'.*

Proof. Omitted. Hint: Functorial point of view combined with the picture:



□

048H **Definition 3.4.** In the situation of Lemma 3.3 we call $j' : R' \to U' \times_B U'$ the *base change* of the pre-relation j to X'. We say it is a *flat base change* if $X' \to X$ is a flat morphism of algebraic spaces.

This kind of base change interacts well with taking quotient sheaves and quotient stacks.

0DTF **Lemma 3.5.** *In the situation of Lemma 3.3 there is an isomorphism of sheaves*

$$U'/R' = X' \times_X U/R$$

For the construction of quotient sheaves, see Groupoids in Spaces, Section 19.

Proof. Since $U \to X$ is R-invariant, it is clear that the map $U \to X$ factors through the quotient sheaf U/R. Recall that by definition

$$R \rightrightarrows U \longrightarrow U/R$$

is a coequalizer diagram in the category Sh of sheaves of sets on $(Sch/S)_{fppf}$. In fact, this is a coequalizer diagram in the comma category Sh/X. Since the base change functor $X' \times_X - : Sh/X \to Sh/X'$ is exact (true in any topos), we conclude. □

0DTG **Lemma 3.6.** *Let S be a scheme. Let B be an algebraic space over S. Let (U, R, s, t, c) be a groupoid in algebraic spaces over B. Let $U \to X$ be an R-invariant morphism of algebraic spaces over B. Let $g : X' \to X$ be a morphism of algebraic spaces over B and let (U', R', s', t', c') be the base change as in Lemma 3.3. Then*

$$\begin{array}{ccc} [U'/R'] & \longrightarrow & [U/R] \\ \downarrow & & \downarrow \\ \mathcal{S}_{X'} & \longrightarrow & \mathcal{S}_X \end{array}$$

is a 2-fibre product of stacks in groupoids over $(Sch/S)_{fppf}$. For the construction of quotient stacks and the morphisms in this diagram, see Groupoids in Spaces, Section 20.

Proof. We will prove this by using the explicit description of the quotient stacks given in Groupoids in Spaces, Lemma 24.1. However, we strongly urge the reader to find their own proof. First, we may view (U, R, s, t, c) as a groupoid in algebraic spaces over X, hence we obtain a map $f : [U/R] \to \mathcal{S}_X$, see Groupoids in Spaces, Lemma 20.2. Similarly, we have $f' : [U'/R'] \to X'$.

An object of the 2-fibre product $\mathcal{S}_{X'} \times_{\mathcal{S}_X} [U/R]$ over a scheme T over S is the same as a morphism $x' : T \to X'$ and an object y of $[U/R]$ over T such that such that the composition $g \circ x'$ is equal to $f(y)$. This makes sense because objects of $\mathcal{S}_X$ over T are morphisms $T \to X$. By Groupoids in Spaces, Lemma 24.1 we may assume y is given by a $[U/R]$-descent datum (u_i, r_{ij}) relative to an fppf covering $\{T_i \to T\}$. The agreement of $g \circ x' = f(y)$ means that the diagrams

$$\begin{array}{ccc} T_i & \xrightarrow[u_i]{} & U \\ \downarrow & & \downarrow \\ T & \xrightarrow{x'} X' \xrightarrow{g} & X \end{array} \quad \text{and} \quad \begin{array}{ccc} T_i \times_T T_j & \xrightarrow[r_{ij}]{} & R \\ \downarrow & & \downarrow \\ T & \xrightarrow{x'} X' \xrightarrow{g} & X \end{array}$$

are commutative.

On the other hand, an object y' of $[U'/R']$ over a scheme T over S by Groupoids in Spaces, Lemma 24.1 is given by a $[U'/R']$-descent datum (u'_i, r'_{ij}) relative to an fppf covering $\{T_i \to T\}$. Setting $f'(y') = x' : T \to X'$ we see that the diagrams

$$\begin{array}{ccc} T_i & \xrightarrow[u'_i]{} & U' \\ \downarrow & & \downarrow \\ T & \xrightarrow{x'} & X' \end{array} \quad \text{and} \quad \begin{array}{ccc} T_i \times_T T_j & \xrightarrow[r'_{ij}]{} & U' \\ \downarrow & & \downarrow \\ T & \xrightarrow{x'} & X' \end{array}$$

are commutative.

With this notation in place, we define a functor

$$[U'/R'] \longrightarrow \mathcal{S}_{X'} \times_{\mathcal{S}_X} [U/R]$$

by sending $y' = (u'_i, r'_{ij})$ as above to the object $(x', (u_i, r_{ij}))$ where $x' = f'(y')$, where u_i is the composition $T_i \to U' \to U$, and where r_{ij} is the composition $T_i \times_T T_j \to R' \to R$. Conversely, given an object $(x', (u_i, r_{ij})$ of the right hand side we can send this to the object $((x', u_i), (x', r_{ij}))$ of the left hand side. We omit the discussion of what to do with morphisms (works in exactly the same manner). □

4. Categorical quotients

048I This is the most basic kind of quotient one can consider.

048J **Definition 4.1.** Let S be a scheme, and let B be an algebraic space over S. Let $j = (t, s) : R \to U \times_B U$ be pre-relation in algebraic spaces over B.

(1) We say a morphism $\phi : U \to X$ of algebraic spaces over B is a *categorical quotient* if it is R-invariant, and for every R-invariant morphism $\psi : U \to Y$ of algebraic spaces over B there exists a unique morphism $\chi : X \to Y$ such that $\psi = \phi \circ \chi$.
(2) Let $\mathcal{C}$ be a full subcategory of the category of algebraic spaces over B. Assume U, R are objects of $\mathcal{C}$. In this situation we say a morphism $\phi : U \to X$ of algebraic spaces over B is a *categorical quotient in* $\mathcal{C}$ if $X \in \mathrm{Ob}(\mathcal{C})$, and ϕ is R-invariant, and for every R-invariant morphism $\psi : U \to Y$ with $Y \in \mathrm{Ob}(\mathcal{C})$ there exists a unique morphism $\chi : X \to Y$ such that $\psi = \phi \circ \chi$.
(3) If $B = S$ and $\mathcal{C}$ is the category of schemes over S, then we say $U \to X$ is a *categorical quotient in the category of schemes*, or simply a *categorical quotient in schemes*.

We often single out a category $\mathcal{C}$ of algebraic spaces over B by some separation axiom, see Example 4.3 for some standard cases. Note that $\phi : U \to X$ is a categorical quotient if and only if $U \to X$ is a coequalizer for the morphisms $t, s : R \to U$ in the category. Hence we immediately deduce the following lemma.

048K **Lemma 4.2.** *Let S be a scheme, and let B be an algebraic space over S. Let $j : R \to U \times_B U$ be a pre-relation in algebraic spaces over B. If a categorical quotient in the category of algebraic spaces over B exists, then it is unique up to unique isomorphism. Similarly for categorical quotients in full subcategories of Spaces/B.*

Proof. See Categories, Section 11. □

049V **Example 4.3.** Let S be a scheme, and let B be an algebraic space over S. Here are some standard examples of categories $\mathcal{C}$ that we often come up when applying Definition 4.1:

(1) $\mathcal{C}$ is the category of all algebraic spaces over B,
(2) B is separated and $\mathcal{C}$ is the category of all separated algebraic spaces over B,
(3) B is quasi-separated and $\mathcal{C}$ is the category of all quasi-separated algebraic spaces over B,
(4) B is locally separated and $\mathcal{C}$ is the category of all locally separated algebraic spaces over B,

(5) B is decent and $\mathcal{C}$ is the category of all decent algebraic spaces over B, and
(6) $S = B$ and $\mathcal{C}$ is the category of schemes over S.

In this case, if $\phi : U \to X$ is a categorical quotient then we say $U \to X$ is (1) a *categorical quotient*, (2) a *categorical quotient in separated algebraic spaces*, (3) a *categorical quotient in quasi-separated algebraic spaces*, (4) a *categorical quotient in locally separated algebraic spaces*, (5) a *categorical quotient in decent algebraic spaces*, (6) a *categorical quotient in schemes*.

048L **Definition 4.4.** Let S be a scheme, and let B be an algebraic space over S. Let $\mathcal{C}$ be a full subcategory of the category of algebraic spaces over B closed under fibre products. Let $j = (t, s) : R \to U \times_B U$ be pre-relation in $\mathcal{C}$, and let $U \to X$ be an R-invariant morphism with $X \in \mathrm{Ob}(\mathcal{C})$.

(1) We say $U \to X$ is a *universal categorical quotient* in $\mathcal{C}$ if for every morphism $X' \to X$ in $\mathcal{C}$ the morphism $U' = X' \times_X U \to X'$ is the categorical quotient in $\mathcal{C}$ of the base change $j' : R' \to U'$ of j.
(2) We say $U \to X$ is a *uniform categorical quotient* in $\mathcal{C}$ if for every flat morphism $X' \to X$ in $\mathcal{C}$ the morphism $U' = X' \times_X U \to X'$ is the categorical quotient in $\mathcal{C}$ of the base change $j' : R' \to U'$ of j.

049W **Lemma 4.5.** *In the situation of Definition 4.1. If $\phi : U \to X$ is a categorical quotient and U is reduced, then X is reduced. The same holds for categorical quotients in a category of spaces $\mathcal{C}$ listed in Example 4.3.*

Proof. Let X_{red} be the reduction of the algebraic space X. Since U is reduced the morphism $\phi : U \to X$ factors through $i : X_{red} \to X$ (Properties of Spaces, Lemma 12.4). Denote this morphism by $\phi_{red} : U \to X_{red}$. Since $\phi \circ s = \phi \circ t$ we see that also $\phi_{red} \circ s = \phi_{red} \circ t$ (as $i : X_{red} \to X$ is a monomorphism). Hence by the universal property of ϕ there exists a morphism $\chi : X \to X_{red}$ such that $\phi_{red} = \phi \circ \chi$. By uniqueness we see that $i \circ \chi = \mathrm{id}_X$ and $\chi \circ i = \mathrm{id}_{X_{red}}$. Hence i is an isomorphism and X is reduced.

To show that this argument works in a category $\mathcal{C}$ one just needs to show that the reduction of an object of $\mathcal{C}$ is an object of $\mathcal{C}$. We omit the verification that this holds for each of the standard examples. □

5. Quotients as orbit spaces

048M Let $j = (t, s) : R \to U \times_B U$ be a pre-relation. If j is a pre-equivalence relation, then loosely speaking the "orbits" of R on U are the subsets $t(s^{-1}(\{u\}))$ of U. However, if j is just a pre-relation, then we need to take the equivalence relation generated by R.

048N **Definition 5.1.** Let S be a scheme, and let B be an algebraic space over S. Let $j : R \to U \times_B U$ be a pre-relation over B. If $u \in |U|$, then the *orbit*, or more precisely the *R-orbit* of u is

$$O_u = \left\{ u' \in |U| \ : \begin{array}{c} \exists n \geq 1,\ \exists u_0, \ldots, u_n \in |U| \text{ such that } u_0 = u \text{ and } u_n = u' \\ \text{and for all } i \in \{0, \ldots, n-1\} \text{ either } u_i = u_{i+1} \text{ or} \\ \exists r \in |R|,\ s(r) = u_i, t(r) = u_{i+1} \text{ or} \\ \exists r \in |R|,\ t(r) = u_i, s(r) = u_{i+1} \end{array} \right\}$$

It is clear that these are the equivalence classes of an equivalence relation, i.e., we have $u' \in O_u$ if and only if $u \in O_{u'}$. The following lemma is a reformulation of Groupoids in Spaces, Lemma 4.4.

048O **Lemma 5.2.** *Let $B \to S$ as in Section 2. Let $j : R \to U \times_B U$ be a pre-equivalence relation of algebraic spaces over B. Then*

$$O_u = \{u' \in |U| \text{ such that } \exists r \in |R|,\ s(r) = u,\ t(r) = u'\}.$$

Proof. By the aforementioned Groupoids in Spaces, Lemma 4.4 we see that the orbits O_u as defined in the lemma give a disjoint union decomposition of $|U|$. Thus we see they are equal to the orbits as defined in Definition 5.1. □

048P **Lemma 5.3.** *In the situation of Definition 5.1. Let $\phi : U \to X$ be an R-invariant morphism of algebraic spaces over B. Then $|\phi| : |U| \to |X|$ is constant on the orbits.*

Proof. To see this we just have to show that $\phi(u) = \phi(u')$ for all $u, u' \in |U|$ such that there exists an $r \in |R|$ such that $s(r) = u$ and $t(r) = u'$. And this is clear since ϕ equalizes s and t. □

There are several problems with considering the orbits $O_u \subset |U|$ as a tool for singling out properties of quotient maps. One issue is the following. Suppose that $\mathrm{Spec}(k) \to B$ is a geometric point of B. Consider the canonical map

$$U(k) \longrightarrow |U|.$$

Then it is usually not the case that the equivalence classes of the equivalence relation generated by $j(R(k)) \subset U(k) \times U(k)$ are the inverse images of the orbits $O_u \subset |U|$. A silly example is to take $S = B = \mathrm{Spec}(\mathbf{Z})$, $U = R = \mathrm{Spec}(k)$ with $s = t = \mathrm{id}_k$. Then $|U| = |R|$ is a single point but $U(k)/R(k)$ is enormous. A more interesting example is to take $S = B = \mathrm{Spec}(\mathbf{Q})$, choose some of number fields $K \subset L$, and set $U = \mathrm{Spec}(L)$ and $R = \mathrm{Spec}(L \otimes_K L)$ with obvious maps $s, t : R \to U$. In this case $|U|$ still has just one point, but the quotient

$$U(k)/R(k) = \mathrm{Hom}(K, k)$$

consists of more than one element. We conclude from both examples that if $U \to X$ is an R-invariant map and if we want it to "separate orbits" we get a much stronger and interesting notion by considering the induced maps $U(k) \to X(k)$ and ask that those maps separate orbits.

There is an issue with this too. Namely, suppose that $S = B = \mathrm{Spec}(\mathbf{R})$, $U = \mathrm{Spec}(\mathbf{C})$, and $R = \mathrm{Spec}(\mathbf{C}) \amalg \mathrm{Spec}(K)$ for some field extension $\sigma : \mathbf{C} \to K$. Let the maps s, t be given by the identity on the component $\mathrm{Spec}(\mathbf{C})$, but by $\sigma, \sigma \circ \tau$ on the second component where τ is complex conjugation. If K is a nontrivial extension of $\mathbf{C}$, then the two points $1, \tau \in U(\mathbf{C})$ are not equivalent under $j(R(\mathbf{C}))$. But after choosing an extension $\mathbf{C} \subset \Omega$ of sufficiently large cardinality (for example larger than the cardinality of K) then the images of $1, \tau \in U(\mathbf{C})$ in $U(\Omega)$ do become equivalent! It seems intuitively clear that this happens either because $s, t : R \to U$ are not locally of finite type or because the cardinality of the field k is not large enough.

Keeping this in mind we make the following definition.

048Q **Definition 5.4.** Let S be a scheme, and let B be an algebraic space over S. Let $j : R \to U \times_B U$ be a pre-relation over B. Let $\mathrm{Spec}(k) \to B$ be a geometric point of B.

(1) We say $\overline{u}, \overline{u}' \in U(k)$ are *weakly R-equivalent* if they are in the same equivalence class for the equivalence relation generated by the relation $j(R(k)) \subset U(k) \times U(k)$.
(2) We say $\overline{u}, \overline{u}' \in U(k)$ are *R-equivalent* if for some overfield $k \subset \Omega$ the images in $U(\Omega)$ are weakly R-equivalent.
(3) The *weak orbit*, or more precisely the *weak R-orbit* of $\overline{u} \in U(k)$ is set of all elements of $U(k)$ which are weakly R-equivalent to $\overline{u}$.
(4) The *orbit*, or more precisely the *R-orbit* of $\overline{u} \in U(k)$ is set of all elements of $U(k)$ which are R-equivalent to $\overline{u}$.

It turns out that in good cases orbits and weak orbits agree, see Lemma 5.7. The following lemma illustrates the difference in the special case of a pre-equivalence relation.

048R **Lemma 5.5.** *Let S be a scheme, and let B be an algebraic space over S. Let $\mathrm{Spec}(k) \to B$ be a geometric point of B. Let $j : R \to U \times_B U$ be a pre-equivalence relation over B. In this case the weak orbit of $\overline{u} \in U(k)$ is simply*

$$\{\overline{u}' \in U(k) \text{ such that } \exists \overline{r} \in R(k),\ s(\overline{r}) = \overline{u},\ t(\overline{r}) = \overline{u}'\}$$

and the orbit of $\overline{u} \in U(k)$ is

$$\{\overline{u}' \in U(k) : \exists \text{ field extension } K/k,\ \exists\ r \in R(K),\ s(r) = \overline{u},\ t(r) = \overline{u}'\}$$

Proof. This is true because by definition of a pre-equivalence relation the image $j(R(k)) \subset U(k) \times U(k)$ is an equivalence relation. □

Let us describe the recipe for turning any pre-relation into a pre-equivalence relation. We will use the morphisms

048S (5.5.1)
$$\begin{array}{ccccccc} j_{diag} & : & U & \longrightarrow & U \times_B U, & u & \longmapsto & (u, u) \\ j_{flip} & : & R & \longrightarrow & U \times_B U, & r & \longmapsto & (s(r), t(r)) \\ j_{comp} & : & R \times_{s,U,t} R & \longrightarrow & U \times_B U, & (r, r') & \longmapsto & (t(r), s(r')) \end{array}$$

We define $j_1 = (t_1, s_1) : R_1 \to U \times_B U$ to be the morphism

$$j \amalg j_{diag} \amalg j_{flip} : R \amalg U \amalg R \longrightarrow U \times_B U$$

with notation as in Equation (5.5.1). For $n > 1$ we set

$$j_n = (t_n, s_n) : R_n = R_1 \times_{s_1,U,t_{n-1}} R_{n-1} \longrightarrow U \times_B U$$

where t_n comes from t_1 precomposed with projection onto R_1 and s_n comes from s_{n-1} precomposed with projection onto R_{n-1}. Finally, we denote

$$j_\infty = (t_\infty, s_\infty) : R_\infty = \coprod\nolimits_{n \geq 1} R_n \longrightarrow U \times_B U.$$

048T **Lemma 5.6.** *Let S be a scheme, and let B be an algebraic space over S. Let $j : R \to U \times_B U$ be a pre-relation over B. Then $j_\infty : R_\infty \to U \times_B U$ is a pre-equivalence relation over B. Moreover*

(1) *$\phi : U \to X$ is R-invariant if and only if it is R_∞-invariant,*
(2) *the canonical map of quotient sheaves $U/R \to U/R_\infty$ (see Groupoids in Spaces, Section 19) is an isomorphism,*
(3) *weak R-orbits agree with weak R_∞-orbits,*
(4) *R-orbits agree with R_∞-orbits,*
(5) *if s, t are locally of finite type, then s_∞, t_∞ are locally of finite type,*
(6) *add more here as needed.*

Proof. Omitted. Hint for (5): Any property of s, t which is stable under composition and stable under base change, and Zariski local on the source will be inherited by s_∞, t_∞. □

048U **Lemma 5.7.** *Let S be a scheme, and let B be an algebraic space over S. Let $j : R \to U \times_B U$ be a pre-relation over B. Let $\mathrm{Spec}(k) \to B$ be a geometric point of B.*

(1) *If $s, t : R \to U$ are locally of finite type then weak R-equivalence on $U(k)$ agrees with R-equivalence, and weak R-orbits agree with R-orbits on $U(k)$.*
(2) *If k has sufficiently large cardinality then weak R-equivalence on $U(k)$ agrees with R-equivalence, and weak R-orbits agree with R-orbits on $U(k)$.*

Proof. We first prove (1). Assume s, t locally of finite type. By Lemma 5.6 we may assume that R is a pre-equivalence relation. Let k be an algebraically closed field over B. Suppose $\overline{u}, \overline{u}' \in U(k)$ are R-equivalent. Then for some extension field Ω/k there exists a point $\overline{r} \in R(\Omega)$ mapping to $(\overline{u}, \overline{u}') \in (U \times_B U)(\Omega)$, see Lemma 5.5. Hence

$$Z = R \times_{j, U \times_B U, (\overline{u}, \overline{u}')} \mathrm{Spec}(k)$$

is nonempty. As s is locally of finite type we see that also j is locally of finite type, see Morphisms of Spaces, Lemma 23.6. This implies Z is a nonempty algebraic space locally of finite type over the algebraically closed field k (use Morphisms of Spaces, Lemma 23.3). Thus Z has a k-valued point, see Morphisms of Spaces, Lemma 24.1. Hence we conclude there exists a $\overline{r} \in R(k)$ with $j(\overline{r}) = (\overline{u}, \overline{u}')$, and we conclude that $\overline{u}, \overline{u}'$ are R-equivalent as desired.

The proof of part (2) is the same, except that it uses Morphisms of Spaces, Lemma 24.2 instead of Morphisms of Spaces, Lemma 24.1. This shows that the assertion holds as soon as $|k| > \lambda(R)$ with $\lambda(R)$ as introduced just above Morphisms of Spaces, Lemma 24.1. □

In the following definition we use the terminology "k is a field over B" to mean that $\mathrm{Spec}(k)$ comes equipped with a morphism $\mathrm{Spec}(k) \to B$.

048V **Definition 5.8.** Let S be a scheme, and let B be an algebraic space over S. Let $j : R \to U \times_B U$ be a pre-relation over B.

(1) We say $\phi : U \to X$ is *set-theoretically R-invariant* if and only if the map $U(k) \to X(k)$ equalizes the two maps $s, t : R(k) \to U(k)$ for every algebraically closed field k over B.
(2) We say $\phi : U \to X$ *separates orbits*, or *separates R-orbits* if it is set-theoretically R-invariant and $\phi(\overline{u}) = \phi(\overline{u}')$ in $X(k)$ implies that $\overline{u}, \overline{u}' \in U(k)$ are in the same orbit for every algebraically closed field k over B.

In Example 5.12 we show that being set-theoretically invariant is "too weak" a notion in the category of algebraic spaces. A more geometric reformulation of what it means to be set-theoretically invariant or to separate orbits is in Lemma 5.17.

048W **Lemma 5.9.** *In the situation of Definition 5.8. A morphism $\phi : U \to X$ is set-theoretically R-invariant if and only if for any algebraically closed field k over B the map $U(k) \to X(k)$ is constant on orbits.*

Proof. This is true because the condition is supposed to hold for all algebraically closed fields over B. □

048X **Lemma 5.10.** *In the situation of Definition 5.8. An invariant morphism is set-theoretically invariant.*

Proof. This is immediate from the definitions. □

048Y **Lemma 5.11.** *In the situation of Definition 5.8. Let $\phi : U \to X$ be a morphism of algebraic spaces over B. Assume*

(1) *ϕ is set-theoretically R-invariant,*
(2) *R is reduced, and*
(3) *X is locally separated over B.*

Then ϕ is R-invariant.

Proof. Consider the equalizer

$$Z = R \times_{(\phi,\phi)\circ j, X\times_B X, \Delta_{X/B}} X$$

algebraic space. Then $Z \to R$ is an immersion by assumption (3). By assumption (1) $|Z| \to |R|$ is surjective. This implies that $Z \to R$ is a bijective closed immersion (use Schemes, Lemma 10.4) and by assumption (2) we conclude that $Z = R$. □

048Z **Example 5.12.** There exist reduced quasi-separated algebraic spaces X, Y and a pair of morphisms $a, b : Y \to X$ which agree on all k-valued points but are not equal. To get an example take $Y = \mathrm{Spec}(k[[x]])$ and

$$X = \mathbf{A}^1_k \Big/ \big(\Delta \amalg \{(x, -x) \mid x \neq 0\}\big)$$

the algebraic space of Spaces, Example 14.1. The two morphisms $a, b : Y \to X$ come from the two maps $x \mapsto x$ and $x \mapsto -x$ from Y to $\mathbf{A}^1_k = \mathrm{Spec}(k[x])$. On the generic point the two maps are the same because on the open part $x \neq 0$ of the space X the functions x and $-x$ are equal. On the closed point the maps are obviously the same. It is also true that $a \neq b$. This implies that Lemma 5.11 does not hold with assumption (3) replaced by the assumption that X be quasi-separated. Namely, consider the diagram

$$\begin{array}{ccc} Y & \xrightarrow{1} & Y \\ {\scriptstyle -1}\downarrow & & \downarrow{\scriptstyle a} \\ Y & \xrightarrow{a} & X \end{array}$$

then the composition $a \circ (-1) = b$. Hence we can set $R = Y$, $U = Y$, $s = 1$, $t = -1$, $\phi = a$ to get an example of a set-theoretically invariant morphism which is not invariant.

The example above is instructive because the map $Y \to X$ even separates orbits. It shows that in the category of algebraic spaces there are simply too many set-theoretically invariant morphisms lying around. Next, let us define what it means for R to be a set-theoretic equivalence relation, while remembering that we need to allow for field extensions to make this work correctly.

0490 **Definition 5.13.** Let S be a scheme, and let B be an algebraic space over S. Let $j : R \to U \times_B U$ be a pre-relation over B.

(1) We say j is a *set-theoretic pre-equivalence relation* if for all algebraically closed fields k over B the relation $\sim_R$ on $U(k)$ defined by

$$\overline{u} \sim_R \overline{u}' \Leftrightarrow \begin{array}{c} \exists \text{ field extension } K/k,\ \exists\ r \in R(K), \\ s(r) = \overline{u},\ t(r) = \overline{u}' \end{array}$$

is an equivalence relation.
(2) We say j is a *set-theoretic equivalence relation* if j is universally injective and a set-theoretic pre-equivalence relation.

Let us reformulate this in more geometric terms.

0491 **Lemma 5.14.** *In the situation of Definition 5.13. The following are equivalent:*

(1) *The morphism j is a set-theoretic pre-equivalence relation.*
(2) *The subset $j(|R|) \subset |U \times_B U|$ contains the image of $|j'|$ for any of the morphisms j' as in Equation (5.5.1).*
(3) *For every algebraically closed field k over B of sufficiently large cardinality the subset $j(R(k)) \subset U(k) \times U(k)$ is an equivalence relation.*

If s, t are locally of finite type these are also equivalent to

(4) *For every algebraically closed field k over B the subset $j(R(k)) \subset U(k) \times U(k)$ is an equivalence relation.*

Proof. Assume (2). Let k be an algebraically closed field over B. We are going to show that $\sim_R$ is an equivalence relation. Suppose that $\overline{u}_i : \mathrm{Spec}(k) \to U$, $i = 1, 2$ are k-valued points of U. Suppose that $(\overline{u}_1, \overline{u}_2)$ is the image of a K-valued point $r \in R(K)$. Consider the solid commutative diagram

$$\begin{array}{ccccc} \mathrm{Spec}(K') & \dashrightarrow & \mathrm{Spec}(k) & \longleftarrow & \mathrm{Spec}(K) \\ \downarrow & & \downarrow{\scriptstyle (\overline{u}_2, \overline{u}_1)} & & \downarrow \\ R & \xrightarrow{j} & U \times_B U & \xleftarrow{j_{flip}} & R \end{array}$$

We also denote $r \in |R|$ the image of r. By assumption the image of $|j_{flip}|$ is contained in the image of $|j|$, in other words there exists a $r' \in |R|$ such that $|j|(r') = |j_{flip}|(r)$. But note that $(\overline{u}_2, \overline{u}_1)$ is in the equivalence class that defines $|j|(r')$ (by the commutativity of the solid part of the diagram). This means there exists a field extension K'/k and a morphism $r' : \mathrm{Spec}(K) \to R$ (abusively denoted r' as well) with $j \circ r' = (\overline{u}_2, \overline{u}_1) \circ i$ where $i : \mathrm{Spec}(K') \to \mathrm{Spec}(K)$ is the obvious map. In other words the dotted part of the diagram commutes. This proves that $\sim_R$ is a symmetric relation on $U(k)$. In the similar way, using that the image of $|j_{diag}|$ is contained in the image of $|j|$ we see that $\sim_R$ is reflexive (details omitted).

To show that $\sim_R$ is transitive assume given $\overline{u}_i : \mathrm{Spec}(k) \to U$, $i = 1, 2, 3$ and field extensions K_i/k and points $r_i : \mathrm{Spec}(K_i) \to R$, $i = 1, 2$ such that $j(r_1) = (\overline{u}_1, \overline{u}_2)$ and $j(r_1) = (\overline{u}_2, \overline{u}_3)$. Then we may choose a commutative diagram of fields

$$\begin{array}{ccc} K & \longleftarrow & K_2 \\ \uparrow & & \uparrow \\ K_1 & \longleftarrow & k \end{array}$$

and we may think of $r_1, r_2 \in R(K)$. We consider the commutative solid diagram

$$\begin{array}{ccccc} \mathrm{Spec}(K') & \dashrightarrow & \mathrm{Spec}(k) & \longleftarrow & \mathrm{Spec}(K) \\ \downarrow & & \downarrow{\scriptstyle (\overline{u}_1, \overline{u}_3)} & & \downarrow{\scriptstyle (r_1, r_2)} \\ R & \xrightarrow{j} & U \times_B U & \xleftarrow{j_{comp}} & R \times_{s,U,t} R \end{array}$$

By exactly the same reasoning as in the first part of the proof, but this time using that $|j_{comp}|((r_1, r_2))$ is in the image of $|j|$, we conclude that a field K' and dotted arrows exist making the diagram commute. This proves that $\sim_R$ is transitive and concludes the proof that (2) implies (1).

Assume (1) and let k be an algebraically closed field over B whose cardinality is larger than $\lambda(R)$, see Morphisms of Spaces, Lemma 24.2. Suppose that $\overline{u} \sim_R \overline{u}'$ with $\overline{u}, \overline{u}' \in U(k)$. By assumption there exists a point in $|R|$ mapping to $(\overline{u}, \overline{u}') \in |U \times_B U|$. Hence by Morphisms of Spaces, Lemma 24.2 we conclude there exists an $\overline{r} \in R(k)$ with $j(\overline{r}) = (\overline{u}, \overline{u}')$. In this way we see that (1) implies (3).

Assume (3). Let us show that $\mathrm{Im}(|j_{comp}|) \subset \mathrm{Im}(|j|)$. Pick any point $c \in |R \times_{s,U,t} R|$. We may represent this by a morphism $\overline{c} : \mathrm{Spec}(k) \to R \times_{s,U,t} R$, with k over B having sufficiently large cardinality. By assumption we see that $j_{comp}(\overline{c}) \in U(k) \times U(k) = (U \times_B U)(k)$ is also the image $j(\overline{r})$ for some $\overline{r} \in R(k)$. Hence $j_{comp}(c) = j(r)$ in $|U \times_B U|$ as desired (with $r \in |R|$ the equivalence class of $\overline{r}$). The same argument shows also that $\mathrm{Im}(|j_{diag}|) \subset \mathrm{Im}(|j|)$ and $\mathrm{Im}(|j_{flip}|) \subset \mathrm{Im}(|j|)$ (details omitted). In this way we see that (3) implies (2). At this point we have shown that (1), (2) and (3) are all equivalent.

It is clear that (4) implies (3) (without any assumptions on s, t). To finish the proof of the lemma we show that (1) implies (4) if s, t are locally of finite type. Namely, let k be an algebraically closed field over B. Suppose that $\overline{u} \sim_R \overline{u}'$ with $\overline{u}, \overline{u}' \in U(k)$. By assumption the algebraic space $Z = R \times_{j,U \times_B U,(\overline{u},\overline{u}')} \mathrm{Spec}(k)$ is nonempty. On the other hand, since $j = (t, s)$ is locally of finite type the morphism $Z \to \mathrm{Spec}(k)$ is locally of finite type as well (use Morphisms of Spaces, Lemmas 23.6 and 23.3). Hence Z has a k point by Morphisms of Spaces, Lemma 24.1 and we conclude that $(\overline{u}, \overline{u}') \in j(R(k))$ as desired. This finishes the proof of the lemma. □

049X **Lemma 5.15.** *In the situation of Definition 5.13. The following are equivalent:*

(1) *The morphism j is a set-theoretic equivalence relation.*
(2) *The morphism j is universally injective and $j(|R|) \subset |U \times_B U|$ contains the image of $|j'|$ for any of the morphisms j' as in Equation (5.5.1).*
(3) *For every algebraically closed field k over B of sufficiently large cardinality the map $j : R(k) \to U(k) \times U(k)$ is injective and its image is an equivalence relation.*

If j is decent, or locally separated, or quasi-separated these are also equivalent to

(4) *For every algebraically closed field k over B the map $j : R(k) \to U(k) \times U(k)$ is injective and its image is an equivalence relation.*

Proof. The implications (1) ⇒ (2) and (2) ⇒ (3) follow from Lemma 5.14 and the definitions. The same lemma shows that (3) implies j is a set-theoretic pre-equivalence relation. But of course condition (3) also implies that j is universally injective, see Morphisms of Spaces, Lemma 19.2, so that j is indeed a set-theoretic equivalence relation. At this point we know that (1), (2), (3) are all equivalent.

Condition (4) implies (3) without any further hypotheses on j. Assume j is decent, or locally separated, or quasi-separated and the equivalent conditions (1), (2), (3) hold. By More on Morphisms of Spaces, Lemma 3.4 we see that j is radicial. Let k be any algebraically closed field over B. Let $\overline{u}, \overline{u}' \in U(k)$ with $\overline{u} \sim_R \overline{u}'$. We see that $R \times_{U \times_B U,(\overline{u},\overline{u}')} \mathrm{Spec}(k)$ is nonempty. Hence, as j is radicial, its reduction is

the spectrum of a field purely inseparable over k. As $k = \overline{k}$ we see that it is the spectrum of k. Whence a point $\overline{r} \in R(k)$ with $t(\overline{r}) = \overline{u}$ and $s(\overline{r}) = \overline{u}'$ as desired. □

0492 **Lemma 5.16.** *Let S be a scheme, and let B be an algebraic space over S. Let $j : R \to U \times_B U$ be a pre-relation over B.*

(1) *If j is a pre-equivalence relation, then j is a set-theoretic pre-equivalence relation. This holds in particular when j comes from a groupoid in algebraic spaces, or from an action of a group algebraic space on U.*
(2) *If j is an equivalence relation, then j is a set-theoretic equivalence relation.*

Proof. Omitted. □

049Y **Lemma 5.17.** *Let $B \to S$ be as in Section 2. Let $j : R \to U \times_B U$ be a pre-relation. Let $\phi : U \to X$ be a morphism of algebraic spaces over B. Consider the diagram*

$$\begin{array}{ccc} (U \times_X U) \times_{(U \times_B U)} R & \xrightarrow{p} & R \\ \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle j} \\ U \times_X U & \xrightarrow{c} & U \times_B U \end{array}$$

Then we have:

(1) *The morphism ϕ is set-theoretically invariant if and only if p is surjective.*
(2) *If j is a set-theoretic pre-equivalence relation then ϕ separates orbits if and only if p and q are surjective.*
(3) *If p and q are surjective, then j is a set-theoretic pre-equivalence relation (and ϕ separates orbits).*
(4) *If ϕ is R-invariant and j is a set-theoretic pre-equivalence relation, then ϕ separates orbits if and only if the induced morphism $R \to U \times_X U$ is surjective.*

Proof. Assume ϕ is set-theoretically invariant. This means that for any algebraically closed field k over B and any $\overline{r} \in R(k)$ we have $\phi(s(\overline{r})) = \phi(t(\overline{r}))$. Hence $((\phi(t(\overline{r})), \phi(s(\overline{r}))), \overline{r})$ defines a point in the fibre product mapping to $\overline{r}$ via p. This shows that p is surjective. Conversely, assume p is surjective. Pick $\overline{r} \in R(k)$. As p is surjective, we can find a field extension K/k and a K-valued point $\tilde{r}$ of the fibre product with $p(\tilde{r}) = \overline{r}$. Then $q(\tilde{r}) \in U \times_X U$ maps to $(t(\overline{r}), s(\overline{r}))$ in $U \times_B U$ and we conclude that $\phi(s(\overline{r})) = \phi(t(\overline{r}))$. This proves that ϕ is set-theoretically invariant.

The proofs of (2), (3), and (4) are omitted. Hint: Assume k is an algebraically closed field over B of large cardinality. Consider the associated diagram of sets

$$\begin{array}{ccc} (U(k) \times_{X(k)} U(k)) \times_{U(k) \times U(k)} R(k) & \xrightarrow{p} & R(k) \\ \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle j} \\ U(k) \times_{X(k)} U(k) & \xrightarrow{c} & U(k) \times U(k) \end{array}$$

By the lemmas above the equivalences posed in (2), (3), and (4) become set-theoretic questions related to the diagram we just displayed, using that surjectivity translates into surjectivity on k-valued points by Morphisms of Spaces, Lemma 24.2. □

Because we have seen above that the notion of a set-theoretically invariant morphism is a rather weak one in the category of algebraic spaces, we define an orbit space for a pre-relation as follows.

0493 **Definition 5.18.** Let $B \to S$ as in Section 2. Let $j : R \to U \times_B U$ be a pre-relation. We say $\phi : U \to X$ is an *orbit space for* R if

(1) ϕ is R-invariant,
(2) ϕ separates R-orbits, and
(3) ϕ is surjective.

The definition of separating R-orbits involves a discussion of points with values in algebraically closed fields. But as we've seen in many cases this just corresponds to the surjectivity of certain canonically associated morphisms of algebraic spaces. We summarize some of the discussion above in the following characterization of orbit spaces.

049Z **Lemma 5.19.** *Let $B \to S$ as in Section 2. Let $j : R \to U \times_B U$ be a set-theoretic pre-equivalence relation. A morphism $\phi : U \to X$ is an orbit space for R if and only if*

(1) *$\phi \circ s = \phi \circ t$, i.e., ϕ is invariant,*
(2) *the induced morphism $(t, s) : R \to U \times_X U$ is surjective, and*
(3) *the morphism $\phi : U \to X$ is surjective.*

This characterization applies for example if j is a pre-equivalence relation, or comes from a groupoid in algebraic spaces over B, or comes from the action of a group algebraic space over B on U.

Proof. Follows immediately from Lemma 5.17 part (4). □

In the following lemma it is (probably) not good enough to assume just that the morphisms s, t are locally of finite type. The reason is that it may happen that some map $\phi : U \to X$ is an orbit space, yet is not locally of finite type. In that case $U(k) \to X(k)$ may not be surjective for all algebraically closed fields k over B.

04A0 **Lemma 5.20.** *Let $B \to S$ as in Section 2. Let $j = (t, s) : R \to U \times_B U$ be a pre-relation. Assume R, U are locally of finite type over B. Let $\phi : U \to X$ be an R-invariant morphism of algebraic spaces over B. Then ϕ is an orbit space for R if and only if the natural map*

$$U(k)/\big(\text{equivalence relation generated by } j(R(k))\big) \longrightarrow X(k)$$

is bijective for all algebraically closed fields k over B.

Proof. Note that since U, R are locally of finite type over B all of the morphisms s, t, j, ϕ are locally of finite type, see Morphisms of Spaces, Lemma 23.6. We will also use without further mention Morphisms of Spaces, Lemma 24.1. Assume ϕ is an orbit space. Let k be any algebraically closed field over B. Let $\overline{x} \in X(k)$. Consider $U \times_{\phi, X, \overline{x}} \mathrm{Spec}(k)$. This is a nonempty algebraic space which is locally of finite type over k. Hence it has a k-valued point. This shows the displayed map of the lemma is surjective. Suppose that $\overline{u}, \overline{u}' \in U(k)$ map to the same element of $X(k)$. By Definition 5.8 this means that $\overline{u}, \overline{u}'$ are in the same R-orbit. By Lemma 5.7 this means that they are equivalent under the equivalence relation generated by $j(R(k))$. Thus the displayed morphism is injective.

Conversely, assume the displayed map is bijective for all algebraically closed fields k over B. This condition clearly implies that ϕ is surjective. We have already assumed that ϕ is R-invariant. Finally, the injectivity of all the displayed maps implies that ϕ separates orbits. Hence ϕ is an orbit space. □

6. Coarse quotients

04A1 We only add this here so that we can later say that coarse quotients correspond to coarse moduli spaces (or moduli schemes).

04A2 **Definition 6.1.** Let S be a scheme and B an algebraic space over S. Let $j : R \to U \times_B U$ be a pre-relation. A morphism $\phi : U \to X$ of algebraic spaces over B is called a *coarse quotient* if

(1) ϕ is a categorical quotient, and
(2) ϕ is an orbit space.

If $S = B$, U, R are all schemes, then we say a morphism of schemes $\phi : U \to X$ is a *coarse quotient in schemes* if

(1) ϕ is a categorical quotient in schemes, and
(2) ϕ is an orbit space.

In many situations the algebraic spaces R and U are locally of finite type over B and the orbit space condition simply means that

$$U(k)/\big(\text{equivalence relation generated by } j(R(k))\big) \cong X(k)$$

for all algebraically closed fields k. See Lemma 5.20. If j is also a (set-theoretic) pre-equivalence relation, then the condition is simply equivalent to $U(k)/j(R(k)) \to X(k)$ being bijective for all algebraically closed fields k.

7. Topological properties

04A3 Let S be a scheme and B an algebraic space over S. Let $j : R \to U \times_B U$ be a pre-relation. We say a subset $T \subset |U|$ is *R-invariant* if $s^{-1}(T) = t^{-1}(T)$ as subsets of $|R|$. Note that if T is closed, then it may not be the case that the corresponding reduced closed subspace of U is R-invariant (as in Groupoids in Spaces, Definition 18.1) because the pullbacks $s^{-1}(T)$, $t^{-1}(T)$ may not be reduced. Here are some conditions that we can consider for an invariant morphism $\phi : U \to X$.

04A4 **Definition 7.1.** Let S be a scheme and B an algebraic space over S. Let $j : R \to U \times_B U$ be a pre-relation. Let $\phi : U \to X$ be an R-invariant morphism of algebraic spaces over B.

04A5 (1) The morphism ϕ is submersive.
04A6 (2) For any R-invariant closed subset $Z \subset |U|$ the image $\phi(Z)$ is closed in $|X|$.
04A7 (3) Condition (2) holds and for any pair of R-invariant closed subsets $Z_1, Z_2 \subset |U|$ we have

$$\phi(Z_1 \cap Z_2) = \phi(Z_1) \cap \phi(Z_2)$$

04A8 (4) The morphism $(t, s) : R \to U \times_X U$ is universally submersive.

For each of these properties we can also require them to hold after any flat base change, or after any base change, see Definition 3.4. In this case we say condition (1), (2), (3), or (4) holds *uniformly* or *universally*.

8. Invariant functions

04A9 In some cases it is convenient to pin down the structure sheaf of a quotient by requiring any invariant function to be a local section of the structure sheaf of the quotient.

04AA **Definition 8.1.** Let S be a scheme and B an algebraic space over S. Let $j : R \to U \times_B U$ be a pre-relation. Let $\phi : U \to X$ be an R-invariant morphism. Denote $\phi' = \phi \circ s = \phi \circ t : R \to X$.

(1) We denote $(\phi_*\mathcal{O}_U)^R$ the $\mathcal{O}_X$-sub-algebra of $\phi_*\mathcal{O}_U$ which is the equalizer of the two maps
$$\phi_*\mathcal{O}_U \xrightarrow[\phi_* t^\sharp]{\phi_* s^\sharp} \phi'_*\mathcal{O}_R$$
on $X_{étale}$. We sometimes call this the *sheaf of R-invariant functions on X*.

(2) We say *the functions on X are the R-invariant functions on U* if the natural map $\mathcal{O}_X \to (\phi_*\mathcal{O}_U)^R$ is an isomorphism.

Of course we can require this property holds after any (flat or any) base change, leading to a (uniform or) universal notion. This condition is often thrown in with other conditions in order to obtain a (more) unique quotient. And of course a good deal of motivation for the whole subject comes from the following special case: $U = \mathrm{Spec}(A)$ is an affine scheme over a field $S = B = \mathrm{Spec}(k)$ and where $R = G \times U$, with G an affine group scheme over k. In this case you have the option of taking for the quotient:
$$X = \mathrm{Spec}(A^G)$$
so that at least the condition of the definition above is satisfied. Even though this is a nice thing you can do it is often not the right quotient; for example if $U = \mathrm{GL}_{n,k}$ and G is the group of upper triangular matrices, then the above gives $X = \mathrm{Spec}(k)$, whereas a much better quotient (namely the flag variety) exists.

9. Good quotients

04AB Especially when taking quotients by group actions the following definition is useful.

04AC **Definition 9.1.** Let S be a scheme and B an algebraic space over S. Let $j : R \to U \times_B U$ be a pre-relation. A morphism $\phi : U \to X$ of algebraic spaces over B is called a *good quotient* if

(1) ϕ is invariant,
(2) ϕ is affine,
(3) ϕ is surjective,
(4) condition (3) holds universally, and
(5) the functions on X are the R-invariant functions on U.

In [Ses72] Seshadri gives almost the same definition, except that instead of (4) he simply requires the condition (3) to hold – he does not require it to hold universally.

10. Geometric quotients

04AD This is Mumford's definition of a geometric quotient (at least the definition from the first edition of GIT; as far as we can tell later editions changed "universally submersive" to "submersive").

04AE **Definition 10.1.** Let S be a scheme and B an algebraic space over S. Let $j : R \to U \times_B U$ be a pre-relation. A morphism $\phi : U \to X$ of algebraic spaces over B is called a *geometric quotient* if

(1) ϕ is an orbit space,
(2) condition (1) holds universally, i.e., ϕ is universally submersive, and
(3) the functions on X are the R-invariant functions on U.

11. Other chapters

References

[KM97] Sean Keel and Shigefumi Mori, *Quotients by groupoids*, Ann. of Math. (2) **145** (1997), 193–213.

[Kol97] János Kollár, *Quotient spaces modulo algebraic groups*, Ann. of Math. (2) **145** (1997), no. 1, 33–79.

[Kol08] ———, *Quotients by finite equivalence relations*, 2008.

[MFK94] David Mumford, John Fogarty, and Frances Kirwan, *Geometric invariant theory, 3d ed.*, Ergebnisse der Math., vol. 34, Springer-Verlag, 1994.

[Ses72] Conjeevaram Srirangachari Seshadri, *Quotient spaces modulo reductive algebraic groups*, Ann. of Math. (2) **95** (1972), 511–556.