LIMITS OF SCHEMES

01YT

Contents

1. Introduction

01YU In this chapter we put material related to limits of schemes. We mostly study limits of inverse systems over directed sets (Categories, Definition 21.1) with affine transition maps. We discuss absolute Noetherian approximation. We characterize schemes locally of finite presentation over a base as those whose associated functor of points is limit preserving. As an application of absolute Noetherian approximation we prove that the image of an affine under an integral morphism is affine. Moreover, we prove some very general variants of Chow's lemma. A basic reference is [DG67].

2. Directed limits of schemes with affine transition maps

01YV In this section we construct the limit.

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01YW **Lemma 2.1.** *Let I be a directed set. Let $(S_i, f_{ii'})$ be an inverse system of schemes over I. If all the schemes S_i are affine, then the limit $S = \lim_i S_i$ exists in the category of schemes. In fact S is affine and $S = \text{Spec}(\text{colim}_i R_i)$ with $R_i = \Gamma(S_i, \mathcal{O})$.*

Proof. Just define $S = \text{Spec}(\text{colim}_i R_i)$. It follows from Schemes, Lemma 6.4 that S is the limit even in the category of locally ringed spaces. $\square$

01YX **Lemma 2.2.** *Let I be a directed set. Let $(S_i, f_{ii'})$ be an inverse system of schemes over I. If all the morphisms $f_{ii'} : S_i \to S_{i'}$ are affine, then the limit $S = \lim_i S_i$ exists in the category of schemes. Moreover,*

1. *each of the morphisms $f_i : S \to S_i$ is affine,*
2. *for an element $0 \in I$ and any open subscheme $U_0 \subset S_0$ we have*
$$f_0^{-1}(U_0) = \lim_{i \geq 0} f_{i0}^{-1}(U_0)$$
in the category of schemes.

Proof. Choose an element $0 \in I$. Note that I is nonempty as the limit is directed. For every $i \geq 0$ consider the quasi-coherent sheaf of $\mathcal{O}_{S_0}$-algebras $\mathcal{A}_i = f_{i0,*}\mathcal{O}_{S_i}$. Recall that $S_i = \underline{\text{Spec}}_{S_0}(\mathcal{A}_i)$, see Morphisms, Lemma 11.3. Set $\mathcal{A} = \text{colim}_{i \geq 0} \mathcal{A}_i$. This is a quasi-coherent sheaf of $\mathcal{O}_{S_0}$-algebras, see Schemes, Section 24. Set $S = \underline{\text{Spec}}_{S_0}(\mathcal{A})$. By Morphisms, Lemma 11.5 we get for $i \geq 0$ morphisms $f_i : S \to S_i$ compatible with the transition morphisms. Note that the morphisms f_i are affine by Morphisms, Lemma 11.11 for example. By Lemma 2.1 above we see that for any affine open $U_0 \subset S_0$ the inverse image $U = f_0^{-1}(U_0) \subset S$ is the limit of the system of opens $U_i = f_{i0}^{-1}(U_0)$, $i \geq 0$ in the category of schemes.

Let T be a scheme. Let $g_i : T \to S_i$ be a compatible system of morphisms. To show that $S = \lim_i S_i$ we have to prove there is a unique morphism $g : T \to S$ with $g_i = f_i \circ g$ for all $i \in I$. For every $t \in T$ there exists an affine open $U_0 \subset S_0$ containing $g_0(t)$. Let $V \subset g_0^{-1}(U_0)$ be an affine open neighbourhood containing t. By the remarks above we obtain a unique morphism $g_V : V \to U = f_0^{-1}(U_0)$ such that $f_i \circ g_V = g_i|_{U_i}$ for all i. The open sets $V \subset T$ so constructed form a basis for the topology of T. The morphisms g_V glue to a morphism $g : T \to S$ because of the uniqueness property. This gives the desired morphism $g : T \to S$.

The final statement is clear from the construction of the limit above. $\square$

01YZ **Lemma 2.3.** *Let I be a directed set. Let $(S_i, f_{ii'})$ be an inverse system of schemes over I. Assume all the morphisms $f_{ii'} : S_i \to S_{i'}$ are affine, Let $S = \lim_i S_i$. Let $0 \in I$. Suppose that T is a scheme over S_0. Then*
$$T \times_{S_0} S = \lim_{i \geq 0} T \times_{S_0} S_i$$

Proof. The right hand side is a scheme by Lemma 2.2. The equality is formal, see Categories, Lemma 14.9. $\square$

3. Infinite products

0CNH Infinite products of schemes usually do not exist. For example in Examples, Section 49 it is shown that an infinite product of copies of $\mathbf{P}^1$ is not even an algebraic space.

On the other hand, infinite products of affine schemes do exist and are affine. Using Schemes, Lemma 6.4 this corresponds to the fact that in the category of rings we have infinite coproducts: if I is a set and R_i is a ring for each i, then we can consider the ring

$$R = \otimes R_i = \mathrm{colim}_{\{i_1,\ldots,i_n\}\subset I} R_{i_1} \otimes_{\mathbf{Z}} \ldots \otimes_{\mathbf{Z}} R_{i_n}$$

Given another ring A a map $R \to A$ is the same thing as a collection of ring maps $R_i \to A$ for all $i \in I$ as follows from the corresponding property of finite tensor products.

0CNI **Lemma 3.1.** *Let S be a scheme. Let I be a set and for each $i \in I$ let $f_i : T_i \to S$ be an affine morphism. Then the product $T = \prod T_i$ exists in the category of schemes over S. In fact, we have*

$$T = \lim_{\{i_1,\ldots,i_n\}\subset I} T_{i_1} \times_S \ldots \times_S T_{i_n}$$

and the projection morphisms $T \to T_{i_1} \times_S \ldots \times_S T_{i_n}$ are affine.

Proof. Omitted. Hint: Argue as in the discussion preceding the lemma and use Lemma 2.2 for existence of the limit. □

0CNJ **Lemma 3.2.** *Let S be a scheme. Let I be a set and for each $i \in I$ let $f_i : T_i \to S$ be a surjective affine morphism. Then the product $T = \prod T_i$ in the category of schemes over S (Lemma 3.1) maps surjectively to S.*

Proof. Let $s \in S$. Choose $t_i \in T_i$ mapping to s. Choose a huge field extension $K/\kappa(s)$ such that $\kappa(s_i)$ embeds into K for each i. Then we get morphisms $\mathrm{Spec}(K) \to T_i$ with image s_i agreeing as morphisms to S. Whence a morphism $\mathrm{Spec}(K) \to T$ which proves there is a point of T mapping to s. □

0CNK **Lemma 3.3.** *Let S be a scheme. Let I be a set and for each $i \in I$ let $f_i : T_i \to S$ be an integral morphism. Then the product $T = \prod T_i$ in the category of schemes over S (Lemma 3.1) is integral over S.*

Proof. Omitted. Hint: On affine pieces this reduces to the following algebra fact: if $A \to B_i$ is integral for all i, then $A \to \otimes_A B_i$ is integral. □

4. Descending properties

081A First some basic lemmas describing the topology of a limit.

0CUE **Lemma 4.1.** *Let $S = \lim S_i$ be the limit of a directed inverse system of schemes with affine transition morphisms (Lemma 2.2). Then $S_{set} = \lim_i S_{i,set}$ where S_{set} indicates the underlying set of the scheme S.*

Proof. Pick $i \in I$. Take $U_i \subset S_i$ an affine open. Denote $U_{i'} = f_{i'i}^{-1}(U_i)$ and $U = f_i^{-1}(U_i)$. Here $f_{i'i} : S_{i'} \to S_i$ is the transtion morphism and $f_i : S \to S_i$ is the projection. By Lemma 2.2 we have $U = \lim_{i'\geq i} U_{i'}$. Suppose we can show that $U_{set} = \lim_{i'\geq i} U_{i',set}$. Then the lemma follows by a simple argument using an affine covering of S_i. Hence we may assume all S_i and S affine. This reduces us to the algebra question considered in the next paragraph.

Suppose given a system of rings $(A_i, \varphi_{ii'})$ over I. Set $A = \mathrm{colim}_i A_i$ with canonical maps $\varphi_i : A_i \to A$. Then

$$\mathrm{Spec}(A) = \lim_i \mathrm{Spec}(A_i)$$

Namely, suppose that we are given primes $\mathfrak{p}_i \subset A_i$ such that $\mathfrak{p}_i = \varphi_{ii'}^{-1}(\mathfrak{p}_{i'})$ for all $i' \geq i$. Then we simply set

$$\mathfrak{p} = \{x \in A \mid \exists i, x_i \in \mathfrak{p}_i \text{ with } \varphi_i(x_i) = x\}$$

It is clear that this is an ideal and has the property that $\varphi_i^{-1}(\mathfrak{p}) = \mathfrak{p}_i$. Then it follows easily that it is a prime ideal as well. □

0CUF **Lemma 4.2.** *Let $S = \lim S_i$ be the limit of a directed inverse system of schemes with affine transition morphisms (Lemma 2.2). Then $S_{top} = \lim_i S_{i,top}$ where S_{top} indicates the underlying topological space of the scheme S.* [DG67, IV, Proposition 8.2.9]

Proof. We will use the criterion of Topology, Lemma 14.3. We have seen that $S_{set} = \lim_i S_{i,set}$ in Lemma 4.1. The maps $f_i : S \to S_i$ are morphisms of schemes hence continuous. Thus $f_i^{-1}(U_i)$ is open for each open $U_i \subset S_i$. Finally, let $s \in S$ and let $s \in V \subset S$ be an open neighbourhood. Choose $0 \in I$ and choose an affine open neighbourhood $U_0 \subset S_0$ of the image of s. Then $f_0^{-1}(U_0) = \lim_{i \geq 0} f_{i0}^{-1}(U_0)$, see Lemma 2.2. Then $f_0^{-1}(U_0)$ and $f_{i0}^{-1}(U_0)$ are affine and

$$\mathcal{O}_S(f_0^{-1}(U_0)) = \mathrm{colim}_{i \geq 0}\, \mathcal{O}_{S_i}(f_{i0}^{-1}(U_0))$$

either by the proof of Lemma 2.2 or by Lemma 2.1. Choose $a \in \mathcal{O}_S(f_0^{-1}(U_0))$ such that $s \in D(a) \subset V$. This is possible because the principal opens form a basis for the topology on the affine scheme $f_0^{-1}(U_0)$. Then we can pick an $i \geq 0$ and $a_i \in \mathcal{O}_{S_i}(f_{i0}^{-1}(U_0))$ mapping to a. It follows that $D(a_i) \subset f_{i0}^{-1}(U_0) \subset S_i$ is an open subset whose inverse image in S is $D(a)$. This finishes the proof. □

01Z2 **Lemma 4.3.** *Let $S = \lim S_i$ be the limit of a directed inverse system of schemes with affine transition morphisms (Lemma 2.2). If all the schemes S_i are nonempty and quasi-compact, then the limit $S = \lim_i S_i$ is nonempty.*

Proof. Choose $0 \in I$. Note that I is nonempty as the limit is directed. Choose an affine open covering $S_0 = \bigcup_{j=1,\ldots,m} U_j$. Since I is directed there exists a $j \in \{1, \ldots, m\}$ such that $f_{i0}^{-1}(U_j) \neq \emptyset$ for all $i \geq 0$. Hence $\lim_{i \geq 0} f_{i0}^{-1}(U_j)$ is not empty since a directed colimit of nonzero rings is nonzero (because $1 \neq 0$). As $\lim_{i \geq 0} f_{i0}^{-1}(U_j)$ is an open subscheme of the limit we win. □

0CUG **Lemma 4.4.** *Let $S = \lim S_i$ be the limit of a directed inverse system of schemes with affine transition morphisms (Lemma 2.2). Let $s \in S$ with images $s_i \in S_i$. Then $\overline{\{s\}} = \lim_i \overline{\{s_i\}}$ as sets and as schemes if endowed with the reduced induced scheme structure.*

Proof. Choose $0 \in I$ and an affine open covering $U_0 = \bigcup_{j \in J} U_{0,j}$. For $i \geq 0$ let $U_{i,j} = f_{i,0}^{-1}(U_{0,j})$ and set $U_j = f_0^{-1}(U_{0,j})$. Here $f_{i'i} : S_{i'} \to S_i$ is the transtion morphism and $f_i : S \to S_i$ is the projection. For $j \in J$ the following are equivalent: (a) $x \in U_j$, (b) $x_0 \in U_{0,j}$, (c) $x_i \in U_{i,j}$ for all $i \geq 0$. Let $J' \subset J$ be the set of indices for which (a), (b), (c) are true. Then $\overline{\{s\}} = \bigcup_{j \in J'} (\overline{\{s\}} \cap U_j)$ and simiarly for $\overline{\{s_i\}}$ for $i \geq 0$. Note that $\overline{\{s\}} \cap U_j$ is the closure of the set $\{s\}$ in the topological space U_j. Similarly for $\overline{\{s_i\}} \cap U_{i,j}$ for $i \geq 0$. Hence it suffices to prove the lemma in the case S and S_i affine for all i. This reduces us to the algebra question considered in the next paragraph.

Suppose given a system of rings $(A_i, \varphi_{ii'})$ over I. Set $A = \text{colim}_i A_i$ with canonical maps $\varphi_i : A_i \to A$. Let $\mathfrak{p} \subset A$ be a prime and set $\mathfrak{p}_i = \varphi_i^{-1}(\mathfrak{p})$. Then

$$V(\mathfrak{p}) = \lim_i V(\mathfrak{p}_i)$$

This follows from Lemma 4.1 because $A/\mathfrak{p} = \text{colim}\, A_i/\mathfrak{p}_i$. This equality of rings also shows the final statement about reduced induced scheme structures holds true. □

In the rest of this section we work in the following situation.

086P **Situation 4.5.** Let $S = \lim_{i \in I} S_i$ be the limit of a directed system of schemes with affine transition morphisms $f_{i'i} : S_{i'} \to S_i$ (Lemma 2.2). We assume that S_i is quasi-compact and quasi-separated for all $i \in I$. We denote $f_i : S \to S_i$ the projection. We also choose an element $0 \in I$.

The type of result we are looking for is the following: If we have an object over S, then for some i there is a similar object over S_i.

01YY **Lemma 4.6.** *In Situation 4.5.*

1. *We have $S_{set} = \lim_i S_{i,set}$ where S_{set} indicates the underlying set of the scheme S.*
2. *We have $S_{top} = \lim_i S_{i,top}$ where S_{top} indicates the underlying topological space of the scheme S.*
3. *If $s, s' \in S$ and s' is not a specialization of s then for some $i \in I$ the image $s'_i \in S_i$ of s' is not a specialization of the image $s_i \in S_i$ of s.*
4. *Add more easy facts on topology of S here. (Requirement: whatever is added should be easy in the affine case.)*

Proof. Part (1) is a special case of Lemma 4.1.

Part (2) is a special case of Lemma 4.2.

Part (3) is a special case of Lemma 4.4. □

01Z0 **Lemma 4.7.** *In Situation 4.5. Suppose that $\mathcal{F}_0$ is a quasi-coherent sheaf on S_0. Set $\mathcal{F}_i = f_{i0}^*\mathcal{F}_0$ for $i \geq 0$ and set $\mathcal{F} = f_0^*\mathcal{F}_0$. Then*

$$\Gamma(S, \mathcal{F}) = \text{colim}_{i \geq 0} \Gamma(S_i, \mathcal{F}_i)$$

Proof. Write $\mathcal{A}_j = f_{i0,*}\mathcal{O}_{S_i}$. This is a quasi-coherent sheaf of $\mathcal{O}_{S_0}$-algebras (see Morphisms, Lemma 11.5) and S_i is the relative spectrum of $\mathcal{A}_i$ over S_0. In the proof of Lemma 2.2 we constructed S as the relative spectrum of $\mathcal{A} = \text{colim}_{i \geq 0} \mathcal{A}_i$ over S_0. Set

$$\mathcal{M}_i = \mathcal{F}_0 \otimes_{\mathcal{O}_{S_0}} \mathcal{A}_i$$

and

$$\mathcal{M} = \mathcal{F}_0 \otimes_{\mathcal{O}_{S_0}} \mathcal{A}.$$

Then we have $f_{i0,*}\mathcal{F}_i = \mathcal{M}_i$ and $f_{0,*}\mathcal{F} = \mathcal{M}$. Since $\mathcal{A}$ is the colimit of the sheaves $\mathcal{A}_i$ and since tensor product commutes with directed colimits, we conclude that $\mathcal{M} = \text{colim}_{i \geq 0} \mathcal{M}_i$. Since S_0 is quasi-compact and quasi-separated we see that

$$\begin{aligned} \Gamma(S, \mathcal{F}) &= \Gamma(S_0, \mathcal{M}) \\ &= \Gamma(S_0, \text{colim}_{i \geq 0} \mathcal{M}_i) \\ &= \text{colim}_{i \geq 0} \Gamma(S_0, \mathcal{M}_i) \\ &= \text{colim}_{i \geq 0} \Gamma(S_i, \mathcal{F}_i) \end{aligned}$$

see Sheaves, Lemma 29.1 and Topology, Lemma 27.1 for the middle equality. □

01Z3 **Lemma 4.8.** *In Situation 4.5. Suppose for each i we are given a nonempty closed subset $Z_i \subset S_i$ with $f_{ii'}(Z_i) \subset Z_{i'}$. Then there exists a point $s \in S$ with $f_i(s) \in Z_i$ for all i.*

Proof. Let $Z_i \subset S_i$ also denote the reduced closed subscheme associated to Z_i, see Schemes, Definition 12.5. A closed immersion is affine, and a composition of affine morphisms is affine (see Morphisms, Lemmas 11.9 and 11.7), and hence $Z_i \to S_{i'}$ is affine when $i \geq i'$. We conclude that the morphism $f_{ii'} : Z_i \to Z_{i'}$ is affine by Morphisms, Lemma 11.11. Each of the schemes Z_i is quasi-compact as a closed subscheme of a quasi-compact scheme. Hence we may apply Lemma 4.3 to see that $Z = \lim_i Z_i$ is nonempty. Since there is a canonical morphism $Z \to S$ we win. $\square$

05F3 **Lemma 4.9.** *In Situation 4.5. Suppose we are given an i and a morphism $T \to S_i$ such that*

1. *$T \times_{S_i} S = \emptyset$, and*
2. *T is quasi-compact.*

Then $T \times_{S_i} S_{i'} = \emptyset$ for all sufficiently large i'.

Proof. By Lemma 2.3 we see that $T \times_{S_i} S = \lim_{i' \geq i} T \times_{S_i} S_{i'}$. Hence the result follows from Lemma 4.3. $\square$

05F4 **Lemma 4.10.** *In Situation 4.5. Suppose we are given an i and a locally constructible subset $E \subset S_i$ such that $f_i(S) \subset E$. Then $f_{ii'}(S_{i'}) \subset E$ for all sufficiently large i'.*

Proof. Writing S_i as a finite union of open affine subschemes reduces the question to the case that S_i is affine and E is constructible, see Lemma 2.2 and Properties, Lemma 2.1. In this case the complement $S_i \setminus E$ is constructible too. Hence there exists an affine scheme T and a morphism $T \to S_i$ whose image is $S_i \setminus E$, see Algebra, Lemma 28.3. By Lemma 4.9 we see that $T \times_{S_i} S_{i'}$ is empty for all sufficiently large i', and hence $f_{ii'}(S_{i'}) \subset E$ for all sufficiently large i'. $\square$

01Z4 **Lemma 4.11.** *In Situation 4.5 we have the following:*

1. *Given any quasi-compact open $V \subset S = \lim_i S_i$ there exists an $i \in I$ and a quasi-compact open $V_i \subset S_i$ such that $f_i^{-1}(V_i) = V$.*
2. *Given $V_i \subset S_i$ and $V_{i'} \subset S_{i'}$ quasi-compact opens such that $f_i^{-1}(V_i) = f_{i'}^{-1}(V_{i'})$ there exists an index $i'' \geq i, i'$ such that $f_{i''i}^{-1}(V_i) = f_{i''i'}^{-1}(V_{i'})$.*
3. *If $V_{1,i}, \ldots, V_{n,i} \subset S_i$ are quasi-compact opens and $S = f_i^{-1}(V_{1,i}) \cup \ldots \cup f_i^{-1}(V_{n,i})$ then $S_{i'} = f_{i'i}^{-1}(V_{1,i}) \cup \ldots \cup f_{i'i}^{-1}(V_{n,i})$ for some $i' \geq i$.*

Proof. Choose $i_0 \in I$. Note that I is nonempty as the limit is directed. For convenience we write $S_0 = S_{i_0}$ and $i_0 = 0$. Choose an affine open covering $S_0 = U_{1,0} \cup \ldots \cup U_{m,0}$. Denote $U_{j,i} \subset S_i$ the inverse image of $U_{j,0}$ under the transition morphism for $i \geq 0$. Denote U_j the inverse image of $U_{j,0}$ in S. Note that $U_j = \lim_i U_{j,i}$ is a limit of affine schemes.

We first prove the uniqueness statement: Let $V_i \subset S_i$ and $V_{i'} \subset S_{i'}$ quasi-compact opens such that $f_i^{-1}(V_i) = f_{i'}^{-1}(V_{i'})$. It suffices to show that $f_{i''i}^{-1}(V_i \cap U_{j,i''})$ and $f_{i''i'}^{-1}(V_{i'} \cap U_{j,i''})$ become equal for i'' large enough. Hence we reduce to the case of a limit of affine schemes. In this case write $S = \text{Spec}(R)$ and $S_i = \text{Spec}(R_i)$ for all $i \in I$. We may write $V_i = S_i \setminus V(h_1, \ldots, h_m)$ and $V_{i'} = S_{i'} \setminus V(g_1, \ldots, g_n)$. The assumption means that the ideals $\sum g_j R$ and $\sum h_j R$ have the same radical in R. This means that $g_j^N = \sum a_{jj'}h_{j'}$ and $h_j^N = \sum b_{jj'}g_{j'}$ for some $N \gg 0$ and $a_{jj'}$ and $b_{jj'}$ in R. Since $R = \mathrm{colim}_i R_i$ we can chose an index $i'' \geq i$ such that the equations $g_j^N = \sum a_{jj'}h_{j'}$ and $h_j^N = \sum b_{jj'}g_{j'}$ hold in $R_{i''}$ for some $a_{jj'}$ and $b_{jj'}$ in $R_{i''}$. This implies that the ideals $\sum g_j R_{i''}$ and $\sum h_j R_{i''}$ have the same radical in $R_{i''}$ as desired.

We prove existence: If S_0 is affine, then $S_i = \mathrm{Spec}(R_i)$ for all $i \geq 0$ and $S = \mathrm{Spec}(R)$ with $R = \mathrm{colim}\, R_i$. Then $V = S \setminus V(g_1, \ldots, g_n)$ for some $g_1, \ldots, g_n \in R$. Choose any i large enough so that each of the g_j comes from an element $g_{j,i} \in R_i$ and take $V_i = S_i \setminus V(g_{1,i}, \ldots, g_{n,i})$. If S_0 is general, then the opens $V \cap U_j$ are quasi-compact because S is quasi-separated. Hence by the affine case we see that for each $j = 1, \ldots, m$ there exists an $i_j \in I$ and a quasi-compact open $V_{i_j} \subset U_{j,i_j}$ whose inverse image in U_j is $V \cap U_j$. Set $i = \max(i_1, \ldots, i_m)$ and let $V_i = \bigcup f_{ii_j}^{-1}(V_{i_j})$.

The statement on coverings follows from the uniqueness statement for the opens $V_{1,i} \cup \ldots \cup V_{n,i}$ and S_i of S_i. $\square$

01Z5 **Lemma 4.12.** *In Situation 4.5 if S is quasi-affine, then for some $i_0 \in I$ the schemes S_i for $i \geq i_0$ are quasi-affine.*

Proof. Choose $i_0 \in I$. Note that I is nonempty as the limit is directed. For convenience we write $S_0 = S_{i_0}$ and $i_0 = 0$. Let $s \in S$. We may choose an affine open $U_0 \subset S_0$ containing $f_0(s)$. Since S is quasi-affine we may choose an element $a \in \Gamma(S, \mathcal{O}_S)$ such that $s \in D(a) \subset f_0^{-1}(U_0)$, and such that $D(a)$ is affine. By Lemma 4.7 there exists an $i \geq 0$ such that a comes from an element $a_i \in \Gamma(S_i, \mathcal{O}_{S_i})$. For any index $j \geq i$ we denote a_j the image of a_i in the global sections of the structure sheaf of S_j. Consider the opens $D(a_j) \subset S_j$ and $U_j = f_{j0}^{-1}(U_0)$. Note that U_j is affine and $D(a_j)$ is a quasi-compact open of S_j, see Properties, Lemma 26.4 for example. Hence we may apply Lemma 4.11 to the opens U_j and $U_j \cup D(a_j)$ to conclude that $D(a_j) \subset U_j$ for some $j \geq i$. For such an index j we see that $D(a_j) \subset S_j$ is an affine open (because $D(a_j)$ is a standard affine open of the affine open U_j) containing the image $f_j(s)$.

We conclude that for every $s \in S$ there exist an index $i \in I$, and a global section $a \in \Gamma(S_i, \mathcal{O}_{S_i})$ such that $D(a) \subset S_i$ is an affine open containing $f_i(s)$. Because S is quasi-compact we may choose a single index $i \in I$ and global sections $a_1, \ldots, a_m \in \Gamma(S_i, \mathcal{O}_{S_i})$ such that each $D(a_j) \subset S_i$ is affine open and such that $f_i : S \to S_i$ has image contained in the union $W_i = \bigcup_{j=1,\ldots,m} D(a_j)$. For $i' \geq i$ set $W_{i'} = f_{i'i}^{-1}(W_i)$. Since $f_i^{-1}(W_i)$ is all of S we see (by Lemma 4.11 again) that for a suitable $i' \geq i$ we have $S_{i'} = W_{i'}$. Thus we may replace i by i' and assume that $S_i = \bigcup_{j=1,\ldots,m} D(a_j)$. This implies that $\mathcal{O}_{S_i}$ is an ample invertible sheaf on S_i (see Properties, Definition 26.1) and hence that S_i is quasi-affine, see Properties, Lemma 27.1. Hence we win. $\square$

01Z6 **Lemma 4.13.** *In Situation 4.5 if S is affine, then for some $i_0 \in I$ the schemes S_i for $i \geq i_0$ are affine.*

Proof. By Lemma 4.12 we may assume that S_0 is quasi-affine for some $0 \in I$. Set $R_0 = \Gamma(S_0, \mathcal{O}_{S_0})$. Then S_0 is a quasi-compact open of $T_0 = \mathrm{Spec}(R_0)$. Denote $j_0 : S_0 \to T_0$ the corresponding quasi-compact open immersion. For $i \geq 0$ set $\mathcal{A}_i = f_{0i,*}\mathcal{O}_{S_i}$. Since f_{0i} is affine we see that $S_i = \underline{\mathrm{Spec}}_{S_0}(\mathcal{A}_i)$. Set $T_i = \underline{\mathrm{Spec}}_{T_0}(j_{0,*}\mathcal{A}_i)$.

Then $T_i \to T_0$ is affine, hence T_i is affine. Thus T_i is the spectrum of

$$R_i = \Gamma(T_0, j_{0,*}\mathcal{A}_i) = \Gamma(S_0, \mathcal{A}_i) = \Gamma(S_i, \mathcal{O}_{S_i}).$$

Write $S = \mathrm{Spec}(R)$. We have $R = \mathrm{colim}_i R_i$ by Lemma 4.7. Hence also $S = \lim_i T_i$. As formation of the relative spectrum commutes with base change, the inverse image of the open $S_0 \subset T_0$ in T_i is S_i. Let $Z_0 = T_0 \setminus S_0$ and let $Z_i \subset T_i$ be the inverse image of Z_0. As $S_i = T_i \setminus Z_i$, it suffices to show that Z_i is empty for some i. Assume Z_i is nonempty for all i to get a contradiction. By Lemma 4.8 there exists a point s of $S = \lim T_i$ which maps to a point of Z_i for every i. But $S = \lim_i S_i$, and hence we arrive at a contradiction by Lemma 4.6. □

086Q **Lemma 4.14.** *In Situation 4.5 if S is separated, then for some $i_0 \in I$ the schemes S_i for $i \geq i_0$ are separated.*

Proof. Choose a finite affine open covering $S_0 = U_{0,1} \cup \ldots \cup U_{0,m}$. Set $U_{i,j} \subset S_i$ and $U_j \subset S$ equal to the inverse image of $U_{0,j}$. Note that $U_{i,j}$ and U_j are affine. As S is separated the intersections $U_{j_1} \cap U_{j_2}$ are affine. Since $U_{j_1} \cap U_{j_2} = \lim_{i \geq 0} U_{i,j_1} \cap U_{i,j_2}$ we see that $U_{i,j_1} \cap U_{i,j_2}$ is affine for large i by Lemma 4.13. To show that S_i is separated for large i it now suffices to show that

$$\mathcal{O}_{S_i}(V_{i,j_1}) \otimes_{\mathcal{O}_S(S)} \mathcal{O}_{S_i}(V_{i,j_2}) \longrightarrow \mathcal{O}_{S_i}(V_{i,j_1} \cap V_{i,j_2})$$

is surjective for large i (Schemes, Lemma 21.8).

To get rid of the annoying indices, assume we have affine opens $U, V \subset S_0$ such that $U \cap V$ is affine too. Let $U_i, V_i \subset S_i$, resp. $U, V \subset S$ be the inverse images. We have to show that $\mathcal{O}(U_i) \otimes \mathcal{O}(V_i) \to \mathcal{O}(U_i \cap V_i)$ is surjective for i large enough and we know that $\mathcal{O}(U_) \otimes \mathcal{O}(V) \to \mathcal{O}(U \cap V)$ is surjective. Note that $\mathcal{O}(U_0) \otimes \mathcal{O}(V_0) \to \mathcal{O}(U_0 \cap V_0)$ is of finite type, as the diagonal morphism $S_i \to S_i \times S_i$ is an immersion (Schemes, Lemma 21.2) hence locally of finite type (Morphisms, Lemmas 14.2 and 14.5). Thus we can choose elements $f_{0,1}, \ldots, f_{0,n} \in \mathcal{O}(U_0 \cap V_0)$ which generate $\mathcal{O}(U_0 \cap V_0)$ over $\mathcal{O}(U_0) \otimes \mathcal{O}(V_0)$. Observe that for $i \geq 0$ the diagram of schemes

$$\begin{tikzcd} U_i \cap V_i \arrow[r] \arrow[d] & U_i \arrow[d] \\ U_0 \cap V_0 \arrow[r] & U_0 \end{tikzcd}$$

is cartesian. Thus we see that the images $f_{i,1}, \ldots, f_{i,n} \in \mathcal{O}(U_i \cap V_i)$ generate $\mathcal{O}(U_i \cap V_i)$ over $\mathcal{O}(U_i) \otimes \mathcal{O}(V_0)$ and a fortiori over $\mathcal{O}(U_i) \otimes \mathcal{O}(V_i)$. By assumption the images $f_1, \ldots, f_n \in \mathcal{O}(U \otimes V)$ are in the image of the map $\mathcal{O}(U) \otimes \mathcal{O}(V) \to \mathcal{O}(U \cap V)$. Since $\mathcal{O}(U) \otimes \mathcal{O}(V) = \mathrm{colim}\, \mathcal{O}(U_i) \otimes \mathcal{O}(V_i)$ we see that they are in the image of the map at some finite level and the lemma is proved. □

09MT **Lemma 4.15.** *In Situation 4.5 let $\mathcal{L}_0$ be an invertible sheaf of modules on S_0. If the pullback $\mathcal{L}$ to S is ample, then for some $i \in I$ the pullback $\mathcal{L}_i$ to S_i is ample.*

Proof. The assumption means there are finitely many sections $s_1, \ldots, s_m \in \Gamma(S, \mathcal{L})$ such that S_{s_j} is affine and such that $S = \bigcup S_{s_j}$, see Properties, Definition 26.1. By Lemma 4.7 we can find an $i \in I$ and sections $s_{i,j} \in \Gamma(S_i, \mathcal{L}_i)$ mapping to s_j. By Lemma 4.13 we may, after increasing i, assume that $(S_i)_{s_{i,j}}$ is affine for $j = 1, \ldots, m$. By Lemma 4.11 we may, after increasing i a last time, assume that $S_i = \bigcup (S_i)_{s_{i,j}}$. Then $\mathcal{L}_i$ is ample by definition. □

081B **Lemma 4.16.** *Let S be a scheme. Let $X = \lim X_i$ be a directed limit of schemes over S with affine transition morphisms. Let $Y \to X$ be a morphism of schemes over S.*

(1) *If $Y \to X$ is a closed immersion, X_i quasi-compact, and Y locally of finite type over S, then $Y \to X_i$ is a closed immersion for i large enough.*
(2) *If $Y \to X$ is an immersion, X_i quasi-separated, $Y \to S$ locally of finite type, and Y quasi-compact, then $Y \to X_i$ is an immersion for i large enough.*

Proof. Proof of (1). Choose $0 \in I$ and a finite affine open covering $X_0 = U_{0,1} \cup \ldots \cup U_{0,m}$ with the property that $U_{0,j}$ maps into an affine open $W_j \subset S$. Let $V_j \subset Y$, resp. $U_{i,j} \subset X_i$, $i \geq 0$, resp. $U_j \subset X$ be the inverse image of $U_{0,j}$. It suffices to prove that $V_j \to U_{i,j}$ is a closed immersion for i sufficiently large and we know that $V_j \to U_j$ is a closed immersion. Thus we reduce to the following algebra fact: If $A = \text{colim}\, A_i$ is a directed colimit of R-algebras, $A \to B$ is a surjection of R-algebras, and B is a finitely generated R-algebra, then $A_i \to B$ is surjective for i sufficiently large.

Proof of (2). Choose $0 \in I$. Choose a quasi-compact open $X_0' \subset X_0$ such that $Y \to X_0$ factors through X_0'. After replacing X_i by the inverse image of X_0' for $i \geq 0$ we may assume all X_i' are quasi-compact and quasi-separated. Let $U \subset X$ be a quasi-compact open such that $Y \to X$ factors through a closed immersion $Y \to U$ (U exists as Y is quasi-compact). By Lemma 4.11 we may assume that $U = \lim U_i$ with $U_i \subset X_i$ quasi-compact open. By part (1) we see that $Y \to U_i$ is a closed immersion for some i. Thus (2) holds. □

01ZH **Lemma 4.17.** *Let S be a scheme. Let $X = \lim X_i$ be a directed limit of schemes over S with affine transition morphisms. Assume*

(1) *S quasi-separated,*
(2) *X_i quasi-compact and quasi-separated,*
(3) *$X \to S$ separated.*

Then $X_i \to S$ is separated for all i large enough.

Proof. Let $0 \in I$. Note that I is nonempty as the limit is directed. As X_0 is quasi-compact we can find finitely many affine opens $U_1, \ldots, U_n \subset S$ such that $X_0 \to S$ maps into $U_1 \cup \ldots \cup U_n$. Denote $h_i : X_i \to S$ the structure morphism. It suffices to check that for some $i \geq 0$ the morphisms $h_i^{-1}(U_j) \to U_j$ are separated for $j = 1, \ldots, n$. Since S is quasi-separated the morphisms $U_j \to S$ are quasi-compact. Hence $h_i^{-1}(U_j)$ is quasi-compact and quasi-separated. In this way we reduce to the case S affine. In this case we have to show that X_i is separated and we know that X is separated. Thus the lemma follows from Lemma 4.14. □

09ZM **Lemma 4.18.** *Let S be a scheme. Let $X = \lim X_i$ be a directed limit of schemes over S with affine transition morphisms. Assume*

(1) *S quasi-compact and quasi-separated,*
(2) *X_i quasi-compact and quasi-separated,*
(3) *$X \to S$ affine.*

Then $X_i \to S$ is affine for i large enough.

Proof. Choose a finite affine open covering $S = \bigcup_{j=1,\ldots,n} V_j$. Denote $f : X \to S$ and $f_i : X_i \to S$ the structure morphisms. For each j the scheme $f^{-1}(V_j) = \lim_i f_i^{-1}(V_j)$ is affine (as a finite morphism is affine by definition). Hence by Lemma

4.13 there exists an $i \in I$ such that each $f_i^{-1}(V_j)$ is affine. In other words, $f_i : X_i \to S$ is affine for i large enough, see Morphisms, Lemma 11.3. $\square$

09ZN **Lemma 4.19.** *Let S be a scheme. Let $X = \lim X_i$ be a directed limit of schemes over S with affine transition morphisms. Assume*

(1) *S quasi-compact and quasi-separated,*
(2) *X_i quasi-compact and quasi-separated,*
(3) *the transition morphisms $X_{i'} \to X_i$ are finite,*
(4) *$X_i \to S$ locally of finite type*
(5) *$X \to S$ integral.*

Then $X_i \to S$ is finite for i large enough.

Proof. By Lemma 4.18 we may assume $X_i \to S$ is affine for all i. Choose a finite affine open covering $S = \bigcup_{j=1,\ldots,n} V_j$. Denote $f : X \to S$ and $f_i : X_i \to S$ the structure morphisms. It suffices to show that there exists an i such that $f_i^{-1}(V_j)$ is finite over V_j for $j = 1, \ldots, m$ (Morphisms, Lemma 42.3). Namely, for $i' \geq i$ the composition $X_{i'} \to X_i \to S$ will be finite as a composition of finite morphisms (Morphisms, Lemma 42.5). This reduces us to the affine case: Let R be a ring and $A = \mathrm{colim}\, A_i$ with $R \to A$ integral and $A_i \to A_{i'}$ finite for all $i \leq i'$. Moreover $R \to A_i$ is of finite type for all i. Goal: Show that A_i is finite over R for some i. To prove this choose an $i \in I$ and pick generators $x_1, \ldots, x_m \in A_i$ of A_i as an R-algebra. Since A is integral over R we can find monic polynomials $P_j \in R[T]$ such that $P_j(x_j) = 0$ in A. Thus there exists an $i' \geq i$ such that $P_j(x_j) = 0$ in $A_{i'}$ for $j = 1, \ldots, m$. Then the image A_i' of A_i in $A_{i'}$ is finite over R by Algebra, Lemma 35.5. Since $A_i' \subset A_{i'}$ is finite too we conclude that $A_{i'}$ is finite over R by Algebra, Lemma 7.3. $\square$

0A0N **Lemma 4.20.** *Let S be a scheme. Let $X = \lim X_i$ be a directed limit of schemes over S with affine transition morphisms. Assume*

(1) *S quasi-compact and quasi-separated,*
(2) *X_i quasi-compact and quasi-separated,*
(3) *the transition morphisms $X_{i'} \to X_i$ are closed immersions,*
(4) *$X_i \to S$ locally of finite type*
(5) *$X \to S$ a closed immersion.*

Then $X_i \to S$ is a closed immersion for i large enough.

Proof. By Lemma 4.18 we may assume $X_i \to S$ is affine for all i. Choose a finite affine open covering $S = \bigcup_{j=1,\ldots,n} V_j$. Denote $f : X \to S$ and $f_i : X_i \to S$ the structure morphisms. It suffices to show that there exists an i such that $f_i^{-1}(V_j)$ is a closed subscheme of V_j for $j = 1, \ldots, m$ (Morphisms, Lemma 2.1). This reduces us to the affine case: Let R be a ring and $A = \mathrm{colim}\, A_i$ with $R \to A$ surjective and $A_i \to A_{i'}$ surjective for all $i \leq i'$. Moreover $R \to A_i$ is of finite type for all i. Goal: Show that $R \to A_i$ is surjective for some i. To prove this choose an $i \in I$ and pick generators $x_1, \ldots, x_m \in A_i$ of A_i as an R-algebra. Since $R \to A$ is surjective we can find $r_j \in R$ such that r_j maps to x_j in A. Thus there exists an $i' \geq i$ such that r_j maps to the image of x_j in $A_{i'}$ for $j = 1, \ldots, m$. Since $A_i \to A_{i'}$ is surjective this implies that $R \to A_{i'}$ is surjective. $\square$

5. Absolute Noetherian Approximation

01Z1 A nice reference for this section is Appendix C of the article by Thomason and Trobaugh [TT90]. See Categories, Section 21 for our conventions regarding directed systems. We will use the existence result and properties of the limit from Section 2 without further mention.

01Z7 **Lemma 5.1.** *Let W be a quasi-affine scheme of finite type over $\mathbf{Z}$. Suppose $W \to \mathrm{Spec}(R)$ is an open immersion into an affine scheme. There exists a finite type $\mathbf{Z}$-algebra $A \subset R$ which induces an open immersion $W \to \mathrm{Spec}(A)$. Moreover, R is the directed colimit of such subalgebras.*

Proof. Choose an affine open covering $W = \bigcup_{i=1,\ldots,n} W_i$ such that each W_i is a standard affine open in $\mathrm{Spec}(R)$. In other words, if we write $W_i = \mathrm{Spec}(R_i)$ then $R_i = R_{f_i}$ for some $f_i \in R$. Choose finitely many $x_{ij} \in R_i$ which generate R_i over $\mathbf{Z}$. Pick an $N \gg 0$ such that each $f_i^N x_{ij}$ comes from an element of R, say $y_{ij} \in R$. Set A equal to the $\mathbf{Z}$-algebra generated by the f_i and the y_{ij} and (optionally) finitely many additional elements of R. Then A works. Details omitted. $\square$

01Z9 **Lemma 5.2.** *Suppose given a cartesian diagram of rings*

$$\begin{tikzcd} B \arrow[r, "s"] & R \\ B' \arrow[u] \arrow[r] & R' \arrow[u, "t"'] \end{tikzcd}$$

Let $W' \subset \mathrm{Spec}(R')$ be an open of the form $W' = D(f_1) \cup \ldots \cup D(f_n)$ such that $t(f_i) = s(g_i)$ for some $g_i \in B$ and $B_{g_i} \cong R_{s(g_i)}$. Then $B' \to R'$ induces an open immersion of W' into $\mathrm{Spec}(B')$.

Proof. Set $h_i = (g_i, f_i) \in B'$. More on Algebra, Lemma 5.3 shows that $(B')_{h_i} \cong (R')_{f_i}$ as desired. $\square$

The following lemma is a precise statement of Noetherian approximation.

07RN **Lemma 5.3.** *Let S be a quasi-compact and quasi-separated scheme. Let $V \subset S$ be a quasi-compact open. Let I be a directed set and let $(V_i, f_{ii'})$ be an inverse system of schemes over I with affine transition maps, with each V_i of finite type over $\mathbf{Z}$, and with $V = \lim V_i$. Then there exist*

1. *a directed set J,*
2. *an inverse system of schemes $(S_j, g_{jj'})$ over J,*
3. *an order preserving map $\alpha : J \to I$,*
4. *open subschemes $V'_j \subset S_j$, and*
5. *isomorphisms $V'_j \to V_{\alpha(j)}$*

such that

1. *the transition morphisms $g_{jj'} : S_j \to S_{j'}$ are affine,*
2. *each S_j is of finite type over $\mathbf{Z}$,*
3. *$g_{jj'}^{-1}(V_{j'}) = V_j$,*
4. *$S = \lim S_j$ and $V = \lim V_j$, and*

(5) *the diagrams*

$$\begin{tikzcd} V \arrow[d] \arrow[dr] & \\ V_j' \arrow[r] & V_{\alpha(j)} \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} V_j \arrow[r] \arrow[d] & V_{\alpha(j)} \arrow[d] \\ V_{j'} \arrow[r] & V_{\alpha(j')} \end{tikzcd}$$

are commutative.

Proof. Set $Z = S \setminus V$. Choose affine opens $U_1, \ldots, U_m \subset S$ such that $Z \subset \bigcup_{l=1,\ldots,m} U_l$. Consider the opens

$$V \subset V \cup U_1 \subset V \cup U_1 \cup U_2 \subset \ldots \subset V \cup \bigcup\nolimits_{l=1,\ldots,m} U_l = S$$

If we can prove the lemma successively for each of the cases

$$V \cup U_1 \cup \ldots \cup U_l \subset V \cup U_1 \cup \ldots \cup U_{l+1}$$

then the lemma will follow for $V \subset S$. In each case we are adding one affine open. Thus we may assume

(1) $S = U \cup V$,
(2) U affine open in S,
(3) V quasi-compact open in S, and
(4) $V = \lim_i V_i$ with $(V_i, f_{ii'})$ an inverse system over a directed set I, each $f_{ii'}$ affine and each V_i of finite type over $\mathbf{Z}$.

Set $W = U \cap V$. As S is quasi-separated, this is a quasi-compact open of V. By Lemma 4.11 (and after shrinking I) we may assume that there exist opens $W_i \subset V_i$ such that $f_{ij}^{-1}(W_j) = W_i$ and such that $f_i^{-1}(W_i) = W$. Since W is a quasi-compact open of U it is quasi-affine. Hence we may assume (after shrinking I again) that W_i is quasi-affine for all i, see Lemma 4.12.

Write $U = \mathrm{Spec}(B)$. Set $R = \Gamma(W, \mathcal{O}_W)$, and $R_i = \Gamma(W_i, \mathcal{O}_{W_i})$. By Lemma 4.7 we have $R = \mathrm{colim}_i R_i$. Now we have the maps of rings

$$\begin{tikzcd} B \arrow[r, "s"'] & R \\ & R_i \arrow[u, "t_i"'] \end{tikzcd}$$

We set $B_i = \{(b, r) \in B \times R_i \mid s(b) = t_i(t)\}$ so that we have a cartesian diagram

$$\begin{tikzcd} B \arrow[r, "s"'] & R \\ B_i \arrow[u] \arrow[r] & R_i \arrow[u, "t_i"'] \end{tikzcd}$$

for each i. The transition maps $R_i \to R_{i'}$ induce maps $B_i \to B_{i'}$. It is clear that $B = \mathrm{colim}_i B_i$. In the next paragraph we show that for all sufficiently large i the composition $W_i \to \mathrm{Spec}(R_i) \to \mathrm{Spec}(B_i)$ is an open immersion.

As W is a quasi-compact open of $U = \mathrm{Spec}(B)$ we can find a finitely many elements $g_l \in B$, $l = 1, \ldots, m$ such that $D(g_l) \subset W$ and such that $W = \bigcup_{l=1,\ldots,m} D(g_l)$.

Note that this implies $D(g_l) = W_{s(g_l)}$ as open subsets of U, where $W_{s(g_l)}$ denotes the largest open subset of W on which $s(g_l)$ is invertible. Hence

$$B_{g_l} = \Gamma(D(g_l), \mathcal{O}_U) = \Gamma(W_{s(g_l)}, \mathcal{O}_W) = R_{s(g_l)},$$

where the last equality is Properties, Lemma 17.1. Since $W_{s(g_l)}$ is affine this also implies that $D(s(g_l)) = W_{s(g_l)}$ as open subsets of $\mathrm{Spec}(R)$. Since $R = \mathrm{colim}_i R_i$ we can (after shrinking I) assume there exist $g_{l,i} \in R_i$ for all $i \in I$ such that $s(g_l) = t_i(g_{l,i})$. Of course we choose the $g_{l,i}$ such that $g_{l,i}$ maps to $g_{l,i'}$ under the transition maps $R_i \to R_{i'}$. Then, by Lemma 4.11 we can (after shrinking I again) assume the corresponding opens $D(g_{l,i}) \subset \mathrm{Spec}(R_i)$ are contained in W_i for $l = 1, \ldots, m$ and cover W_i. We conclude that the morphism $W_i \to \mathrm{Spec}(R_i) \to \mathrm{Spec}(B_i)$ is an open immersion, see Lemma 5.2

By Lemma 5.1 we can write B_i as a directed colimit of subalgebras $A_{i,p} \subset B_i$, $p \in P_i$ each of finite type over $\mathbf{Z}$ and such that W_i is identified with an open subscheme of $\mathrm{Spec}(A_{i,p})$. Let $S_{i,p}$ be the scheme obtained by glueing V_i and $\mathrm{Spec}(A_{i,p})$ along the open W_i, see Schemes, Section 14. Here is the resulting commutative diagram of schemes:

$$\begin{tikzcd}
& & V \arrow[dll] \arrow[d] & W \arrow[l] \arrow[dll] \arrow[d] \\
V_i \arrow[d] & W_i \arrow[l] \arrow[d] & S \arrow[dll] & U \arrow[l] \arrow[dll] \\
S_{i,p} & \mathrm{Spec}(A_{i,p}) \arrow[l] & &
\end{tikzcd}$$

The morphism $S \to S_{i,p}$ arises because the upper right square is a pushout in the category of schemes. Note that $S_{i,p}$ is of finite type over $\mathbf{Z}$ since it has a finite affine open covering whose members are spectra of finite type $\mathbf{Z}$-algebras. We define a preorder on $J = \coprod_{i \in I} P_i$ by the rule $(i', p') \geq (i, p)$ if and only if $i' \geq i$ and the map $B_i \to B_{i'}$ maps $A_{i,p}$ into $A_{i',p'}$. This is exactly the condition needed to define a morphism $S_{i',p'} \to S_{i,p}$: namely make a commutative diagram as above using the transition morphisms $V_{i'} \to V_i$ and $W_{i'} \to W_i$ and the morphism $\mathrm{Spec}(A_{i',p'}) \to \mathrm{Spec}(A_{i,p})$ induced by the ring map $A_{i,p} \to A_{i',p'}$. The relevant commutativities have been built into the constructions. We claim that S is the directed limit of the schemes $S_{i,p}$. Since by construction the schemes V_i have limit V this boils down to the fact that B is the limit of the rings $A_{i,p}$ which is true by construction. The map $\alpha : J \to I$ is given by the rule $j = (i,p) \mapsto i$. The open subscheme V'_j is just the image of $V_i \to S_{i,p}$ above. The commutativity of the diagrams in (5) is clear from the construction. This finishes the proof of the lemma. $\square$

01ZA **Proposition 5.4.** *Let S be a quasi-compact and quasi-separated scheme. There exist a directed set I and an inverse system of schemes $(S_i, f_{ii'})$ over I such that*

1. *the transition morphisms $f_{ii'}$ are affine*
2. *each S_i is of finite type over $\mathbf{Z}$, and*
3. *$S = \lim_i S_i$.*

Proof. This is a special case of Lemma 5.3 with $V = \emptyset$. $\square$

6. Limits and morphisms of finite presentation

01ZB The following is a generalization of Algebra, Lemma 126.3.

01ZC **Proposition 6.1.** *Let $f : X \to S$ be a morphism of schemes. The following are equivalent:* [DG67, IV, Proposition 8.14.2]

(1) *The morphism f is locally of finite presentation.*
(2) *For any directed set I, and any inverse system $(T_i, f_{ii'})$ of S-schemes over I with each T_i affine, we have*
$$Mor_S(\lim_i T_i, X) = \text{colim}_i\, Mor_S(T_i, X)$$
(3) *For any directed set I, and any inverse system $(T_i, f_{ii'})$ of S-schemes over I with each $f_{ii'}$ affine and every T_i quasi-compact and quasi-separated as a scheme, we have*
$$Mor_S(\lim_i T_i, X) = \text{colim}_i\, Mor_S(T_i, X)$$

Proof. It is clear that (3) implies (2).

Let us prove that (2) implies (1). Assume (2). Choose any affine opens $U \subset X$ and $V \subset S$ such that $f(U) \subset V$. We have to show that $\mathcal{O}_S(V) \to \mathcal{O}_X(U)$ is of finite presentation. Let $(A_i, \varphi_{ii'})$ be a directed system of $\mathcal{O}_S(V)$-algebras. Set $A = \text{colim}_i A_i$. According to Algebra, Lemma 126.3 we have to show that
$$\text{Hom}_{\mathcal{O}_S(V)}(\mathcal{O}_X(U), A) = \text{colim}_i\, \text{Hom}_{\mathcal{O}_S(V)}(\mathcal{O}_X(U), A_i)$$
Consider the schemes $T_i = \text{Spec}(A_i)$. They form an inverse system of V-schemes over I with transition morphisms $f_{ii'} : T_i \to T_{i'}$ induced by the $\mathcal{O}_S(V)$-algebra maps $\varphi_{i'i}$. Set $T := \text{Spec}(A) = \lim_i T_i$. The formula above becomes in terms of morphism sets of schemes
$$Mor_V(\lim_i T_i, U) = \text{colim}_i\, Mor_V(T_i, U).$$
We first observe that $Mor_V(T_i, U) = Mor_S(T_i, U)$ and $Mor_V(T, U) = Mor_S(T, U)$. Hence we have to show that
$$Mor_S(\lim_i T_i, U) = \text{colim}_i\, Mor_S(T_i, U)$$
and we are given that
$$Mor_S(\lim_i T_i, X) = \text{colim}_i\, Mor_S(T_i, X).$$
Hence it suffices to prove that given a morphism $g_i : T_i \to X$ over S such that the composition $T \to T_i \to X$ ends up in U there exists some $i' \geq i$ such that the composition $g_{i'} : T_{i'} \to T_i \to X$ ends up in U. Denote $Z_{i'} = g_{i'}^{-1}(X \setminus U)$. Assume each $Z_{i'}$ is nonempty to get a contradiction. By Lemma 4.8 there exists a point t of T which is mapped into $Z_{i'}$ for all $i' \geq i$. Such a point is not mapped into U. A contradiction.

Finally, let us prove that (1) implies (3). Assume (1). Let an inverse directed system $(T_i, f_{ii'})$ of S-schemes be given. Assume the morphisms $f_{ii'}$ are affine and each T_i is quasi-compact and quasi-separated as a scheme. Let $T = \lim_i T_i$. Denote $f_i : T \to T_i$ the projection morphisms. We have to show:

(a) Given morphisms $g_i, g_i' : T_i \to X$ over S such that $g_i \circ f_i = g_i' \circ f_i$, then there exists an $i' \geq i$ such that $g_i \circ f_{i'i} = g_i' \circ f_{i'i}$.
(b) Given any morphism $g : T \to X$ over S there exists an $i \in I$ and a morphism $g_i : T_i \to X$ such that $g = f_i \circ g_i$.

First let us prove the uniqueness part (a). Let $g_i, g_i' : T_i \to X$ be morphisms such that $g_i \circ f_i = g_i' \circ f_i$. For any $i' \geq i$ we set $g_{i'} = g_i \circ f_{i'i}$ and $g'_{i'} = g'_i \circ f_{i'i}$. We also set $g = g_i \circ f_i = g'_i \circ f_i$. Consider the morphism $(g_i, g'_i) : T_i \to X \times_S X$. Set

$$W = \bigcup\nolimits_{U \subset X \text{ affine open}, V \subset S \text{ affine open}, f(U) \subset V} U \times_V U.$$

This is an open in $X \times_S X$, with the property that the morphism $\Delta_{X/S}$ factors through a closed immersion into W, see the proof of Schemes, Lemma 21.2. Note that the composition $(g_i, g'_i) \circ f_i : T \to X \times_S X$ is a morphism into W because it factors through the diagonal by assumption. Set $Z_{i'} = (g_{i'}, g'_{i'})^{-1}(X \times_S X \setminus W)$. If each $Z_{i'}$ is nonempty, then by Lemma 4.8 there exists a point $t \in T$ which maps to $Z_{i'}$ for all $i' \geq i$. This is a contradiction with the fact that T maps into W. Hence we may increase i and assume that $(g_i, g'_i) : T_i \to X \times_S X$ is a morphism into W. By construction of W, and since T_i is quasi-compact we can find a finite affine open covering $T_i = T_{1,i} \cup \ldots \cup T_{n,i}$ such that $(g_i, g'_i)|_{T_{j,i}}$ is a morphism into $U \times_V U$ for some pair (U, V) as in the definition of W above. Since it suffices to prove that $g_{i'}$ and $g'_{i'}$ agree on each of the $f_{i'i}^{-1}(T_{j,i})$ this reduces us to the affine case. The affine case follows from Algebra, Lemma 126.3 and the fact that the ring map $\mathcal{O}_S(V) \to \mathcal{O}_X(U)$ is of finite presentation (see Morphisms, Lemma 20.2).

Finally, we prove the existence part (b). Let $g : T \to X$ be a morphism of schemes over S. We can find a finite affine open covering $T = W_1 \cup \ldots \cup W_n$ such that for each $j \in \{1, \ldots, n\}$ there exist affine opens $U_j \subset X$ and $V_j \subset S$ with $f(U_j) \subset V_j$ and $g(W_j) \subset U_j$. By Lemmas 4.11 and 4.13 (after possibly shrinking I) we may assume that there exist affine open coverings $T_i = W_{1,i} \cup \ldots \cup W_{n,i}$ compatible with transition maps such that $W_j = \lim_i W_{j,i}$. We apply Algebra, Lemma 126.3 to the rings corresponding to the affine schemes U_j, V_j, $W_{j,i}$ and W_j using that $\mathcal{O}_S(V_j) \to \mathcal{O}_X(U_j)$ is of finite presentation (see Morphisms, Lemma 20.2). Thus we can find for each j an index $i_j \in I$ and a morphism $g_{j,i_j} : W_{j,i_j} \to X$ such that $g_{j,i_j} \circ f_i|_{W_j} : W_j \to W_{j,i} \to X$ equals $g|_{W_j}$. By part (a) proved above, using the quasi-compactness of $W_{j_1,i} \cap W_{j_2,i}$ which follows as T_i is quasi-separated, we can find an index $i' \in I$ larger than all i_j such that

$$g_{j_1,i_{j_1}} \circ f_{i'i_{j_1}}|_{W_{j_1,i'} \cap W_{j_2,i'}} = g_{j_2,i_{j_2}} \circ f_{i'i_{j_2}}|_{W_{j_1,i'} \cap W_{j_2,i'}}$$

for all $j_1, j_2 \in \{1, \ldots, n\}$. Hence the morphisms $g_{j,i_j} \circ f_{i'i_j}|_{W_{j,i'}}$ glue to given the desired morphism $T_{i'} \to X$. $\square$

05LX **Remark 6.2.** Let S be a scheme. Let us say that a functor $F : (Sch/S)^{opp} \to Sets$ is *limit preserving* if for every directed inverse system $\{T_i\}_{i \in I}$ of affine schemes with limit T we have $F(T) = \mathrm{colim}_i F(T_i)$. Let X be a scheme over S, and let $h_X : (Sch/S)^{opp} \to Sets$ be its functor of points, see Schemes, Section 15. In this terminology Proposition 6.1 says that a scheme X is locally of finite presentation over S if and only if h_X is limit preserving.

0CM0 **Lemma 6.3.** *Let $f : X \to S$ be a morphism of schemes. If for every directed limit $T = \lim_{i \in I} T_i$ of affine schemes over S the map*

$$\mathrm{colim}\, Mor_S(T_i, X) \longrightarrow Mor_S(T, X)$$

is surjective, then f is locally of finite presentation. In other words, in Proposition 6.1 parts (2) and (3) it suffices to check surjectivity of the map.

Proof. The proof is exactly the same as the proof of the implication "(2) implies (1)" in Proposition 6.1. Choose any affine opens $U \subset X$ and $V \subset S$ such that $f(U) \subset V$. We have to show that $\mathcal{O}_S(V) \to \mathcal{O}_X(U)$ is of finite presentation. Let $(A_i, \varphi_{ii'})$ be a directed system of $\mathcal{O}_S(V)$-algebras. Set $A = \text{colim}_i A_i$. According to Algebra, Lemma 126.3 it suffices to show that

$$\text{colim}_i \text{Hom}_{\mathcal{O}_S(V)}(\mathcal{O}_X(U), A_i) \to \text{Hom}_{\mathcal{O}_S(V)}(\mathcal{O}_X(U), A)$$

is surjective. Consider the schemes $T_i = \text{Spec}(A_i)$. They form an inverse system of V-schemes over I with transition morphisms $f_{ii'} : T_i \to T_{i'}$ induced by the $\mathcal{O}_S(V)$-algebra maps $\varphi_{i'i}$. Set $T := \text{Spec}(A) = \lim_i T_i$. The formula above becomes in terms of morphism sets of schemes

$$\text{colim}_i Mor_V(T_i, U) \to Mor_V(\lim_i T_i, U)$$

We first observe that $Mor_V(T_i, U) = Mor_S(T_i, U)$ and $Mor_V(T, U) = Mor_S(T, U)$. Hence we have to show that

$$\text{colim}_i Mor_S(T_i, U) \to Mor_S(\lim_i T_i, U)$$

is surjective and we are given that

$$\text{colim}_i Mor_S(T_i, X) \to Mor_S(\lim_i T_i, X)$$

is surjective. Hence it suffices to prove that given a morphism $g_i : T_i \to X$ over S such that the composition $T \to T_i \to X$ ends up in U there exists some $i' \geq i$ such that the composition $g_{i'} : T_{i'} \to T_i \to X$ ends up in U. Denote $Z_{i'} = g_{i'}^{-1}(X \setminus U)$. Assume each $Z_{i'}$ is nonempty to get a contradiction. By Lemma 4.8 there exists a point t of T which is mapped into $Z_{i'}$ for all $i' \geq i$. Such a point is not mapped into U. A contradiction. $\square$

7. Relative approximation

09MU The title of this section refers to results of the following type.

09MV **Lemma 7.1.** *Let $f : X \to S$ be a morphism of schemes. Assume that*

1. *X is quasi-compact and quasi-separated, and*
2. *S is quasi-separated.*

Then $X = \lim X_i$ is a limit of a directed system of schemes X_i of finite presentation over S with affine transition morphisms over S.

Proof. Since $f(X)$ is quasi-compact we may replace S by a quasi-compact open containing $f(X)$. Hence we may assume S is quasi-compact as well. Write $X = \lim X_a$ and $S = \lim S_b$ as in Proposition 5.4, i.e., with X_a and S_b of finite type over $\mathbf{Z}$ and with affine transition morphisms. By Proposition 6.1 we find that for each b there exists an a and a morphism $f_{a,b} : X_a \to S_b$ making the diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & S \arrow[d] \\ X_a \arrow[r] & S_b \end{tikzcd}$$

commute. Moreover the same proposition implies that, given a second triple $(a', b', f_{a',b'})$, there exists an $a'' \geq a'$ such that the compositions $X_{a''} \to X_a \to X_b$ and $X_{a''} \to$

$X_{a'} \to X_{b'} \to X_b$ are equal. Consider the set of triples $(a, b, f_{a,b})$ endowed with the preorder

$$(a, b, f_{a,b}) \geq (a', b', f_{a',b'}) \Leftrightarrow a \geq a',\ b' \geq b, \text{ and } f_{a',b'} \circ h_{a,a'} = g_{b',b} \circ f_{a,b}$$

where $h_{a,a'} : X_a \to X_{a'}$ and $g_{b',b} : S_{b'} \to S_b$ are the transition morphisms. The remarks above show that this system is directed. It follows formally from the equalities $X = \lim X_a$ and $S = \lim S_b$ that

$$X = \lim_{(a,b,f_{a,b})} X_a \times_{f_{a,b},S_b} S.$$

where the limit is over our directed system above. The transition morphisms $X_a \times_{S_b} S \to X_{a'} \times_{S_{b'}} S$ are affine as the composition

$$X_a \times_{S_b} S \to X_a \times_{S_{b'}} S \to X_{a'} \times_{S_{b'}} S$$

where the first morphism is a closed immersion (by Schemes, Lemma 21.10) and the second is a base change of an affine morphism (Morphisms, Lemma 11.8) and the composition of affine morphisms is affine (Morphisms, Lemma 11.7). The morphisms $f_{a,b}$ are of finite presentation (Morphisms, Lemmas 20.9 and 20.11) and hence the base changes $X_a \times_{f_{a,b},S_b} S \to S$ are of finite presentation (Morphisms, Lemma 20.4). □

09YZ **Lemma 7.2.** *Let $X \to S$ be an integral morphism with S quasi-compact and quasi-separated. Then $X = \lim X_i$ with $X_i \to S$ finite and of finite presentation.*

Proof. Consider the sheaf $\mathcal{A} = f_*\mathcal{O}_X$. This is a quasi-coherent sheaf of $\mathcal{O}_S$-algebras, see Schemes, Lemma 24.1. Combining Properties, Lemma 22.13 we can write $\mathcal{A} = \mathrm{colim}_i \mathcal{A}_i$ as a filtered colimit of finite and finitely presented $\mathcal{O}_S$-algebras. Then

$$X_i = \underline{\mathrm{Spec}}_S(\mathcal{A}_i) \longrightarrow S$$

is a finite and finitely presented morphism of schemes. By construction $X = \lim_i X_i$ which proves the lemma. □

8. Descending properties of morphisms

081C This section is the analogue of Section 4 for properties of morphisms over S. We will work in the following situation.

081D **Situation 8.1.** Let $S = \lim S_i$ be a limit of a directed system of schemes with affine transition morphisms (Lemma 2.2). Let $0 \in I$ and let $f_0 : X_0 \to Y_0$ be a morphism of schemes over S_0. Assume S_0, X_0, Y_0 are quasi-compact and quasi-separated. Let $f_i : X_i \to Y_i$ be the base change of f_0 to S_i and let $f : X \to Y$ be the base change of f_0 to S.

01ZN **Lemma 8.2.** *Notation and assumptions as in Situation 8.1. If f is affine, then there exists an index $i \geq 0$ such that f_i is affine.*

Proof. Let $Y_0 = \bigcup_{j=1,\ldots,m} V_{j,0}$ be a finite affine open covering. Set $U_{j,0} = f_0^{-1}(V_{j,0})$. For $i \geq 0$ we denote $V_{j,i}$ the inverse image of $V_{j,0}$ in Y_i and $U_{j,i} = f_i^{-1}(V_{j,i})$. Similarly we have $U_j = f^{-1}(V_j)$. Then $U_j = \lim_{i \geq 0} U_{j,i}$ (see Lemma 2.2). Since U_j is affine by assumption we see that each $U_{j,i}$ is affine for i large enough, see Lemma 4.13. As there are finitely many j we can pick an i which works for all j. Thus f_i is affine for i large enough, see Morphisms, Lemma 11.3. □

01ZO **Lemma 8.3.** *Notation and assumptions as in Situation 8.1. If*

(1) *f is a finite morphism, and*
(2) *f_0 is locally of finite type,*

then there exists an $i \geq 0$ such that f_i is finite.

Proof. A finite morphism is affine, see Morphisms, Definition 42.1. Hence by Lemma 8.2 above after increasing 0 we may assume that f_0 is affine. By writing Y_0 as a finite union of affines we reduce to proving the result when X_0 and Y_0 are affine and map into a common affine $W \subset S_0$. The corresponding algebra statement follows from Algebra, Lemma 162.3. $\square$

0C4W **Lemma 8.4.** *Notation and assumptions as in Situation 8.1. If*

(1) *f is unramified, and*
(2) *f_0 is locally of finite type,*

then there exists an $i \geq 0$ such that f_i is unramified.

Proof. Choose a finite affine open covering $Y_0 = \bigcup_{j=1,\ldots,m} Y_{j,0}$ such that each $Y_{j,0}$ maps into an affine open $S_{j,0} \subset S_0$. For each j let $f_0^{-1}Y_{j,0} = \bigcup_{k=1,\ldots,n_j} X_{k,0}$ be a finite affine open covering. Since the property of being unramified is local we see that it suffices to prove the lemma for the morphisms of affines $X_{k,i} \to Y_{j,i} \to S_{j,i}$ which are the base changes of $X_{k,0} \to Y_{j,0} \to S_{j,0}$ to S_i. Thus we reduce to the case that X_0, Y_0, S_0 are affine

In the affine case we reduce to the following algebra result. Suppose that $R = \mathrm{colim}_{i \in I} R_i$. For some $0 \in I$ suppose given an R_0-algebra map $A_i \to B_i$ of finite type. If $R \otimes_{R_0} A_0 \to R \otimes_{R_0} B_0$ is unramified, then for some $i \geq 0$ the map $R_i \otimes_{R_0} A_0 \to R_i \otimes_{R_0} B_0$ is unramified. This follows from Algebra, Lemma 162.5. $\square$

01ZP **Lemma 8.5.** *Notation and assumptions as in Situation 8.1. If*

(1) *f is a closed immersion, and*
(2) *f_0 is locally of finite type,*

then there exists an $i \geq 0$ such that f_i is a closed immersion.

Proof. A closed immersion is affine, see Morphisms, Lemma 11.9. Hence by Lemma 8.2 above after increasing 0 we may assume that f_0 is affine. By writing Y_0 as a finite union of affines we reduce to proving the result when X_0 and Y_0 are affine and map into a common affine $W \subset S_0$. The corresponding algebra statement is a consequence of Algebra, Lemma 162.4. $\square$

01ZQ **Lemma 8.6.** *Notation and assumptions as in Situation 8.1. If f is separated, then f_i is separated for some $i \geq 0$.*

Proof. Apply Lemma 8.5 to the diagonal morphism $\Delta_{X_0/S_0} : X_0 \to X_0 \times_{S_0} X_0$. (This is permissible as diagonal morphisms are locally of finite type and the fibre product $X_0 \times_{S_0} X_0$ is quasi-compact and quasi-separated, see Schemes, Lemma 21.2, Morphisms, Lemma 14.5, and Schemes, Remark 21.18. $\square$

04AI **Lemma 8.7.** *Notation and assumptions as in Situation 8.1. If*

(1) *f is flat,*
(2) *f_0 is locally of finite presentation,*

then f_i is flat for some $i \geq 0$.

Proof. Choose a finite affine open covering $Y_0 = \bigcup_{j=1,\ldots,m} Y_{j,0}$ such that each $Y_{j,0}$ maps into an affine open $S_{j,0} \subset S_0$. For each j let $f_0^{-1}Y_{j,0} = \bigcup_{k=1,\ldots,n_j} X_{k,0}$ be a finite affine open covering. Since the property of being flat is local we see that it suffices to prove the lemma for the morphisms of affines $X_{k,i} \to Y_{j,i} \to S_{j,i}$ which are the base changes of $X_{k,0} \to Y_{j,0} \to S_{j,0}$ to S_i. Thus we reduce to the case that X_0, Y_0, S_0 are affine

In the affine case we reduce to the following algebra result. Suppose that $R = \text{colim}_{i\in I} R_i$. For some $0 \in I$ suppose given an R_0-algebra map $A_i \to B_i$ of finite presentation. If $R \otimes_{R_0} A_0 \to R \otimes_{R_0} B_0$ is flat, then for some $i \geq 0$ the map $R_i \otimes_{R_0} A_0 \to R_i \otimes_{R_0} B_0$ is flat. This follows from Algebra, Lemma 162.1 part (3). □

06AC **Lemma 8.8.** *Notation and assumptions as in Situation 8.1. If*

(1) *f is finite locally free (of degree d),*
(2) *f_0 is locally of finite presentation,*

then f_i is finite locally free (of degree d) for some $i \geq 0$.

Proof. By Lemmas 8.7 and 8.3 we find an i such that f_i is flat and finite. On the other hand, f_i is locally of finite presentation. Hence f_i is finite locally free by Morphisms, Lemma 45.2. If moreover f is finite locally free of degree d, then the image of $Y \to Y_i$ is contained in the open and closed locus $W_d \subset Y_i$ over which f_i has degree d. By Lemma 4.10 we see that for some $i' \geq i$ the image of $Y_{i'} \to Y_i$ is contained in W_d. Then $f_{i'}$ will be finite locally free of degree d. □

0C0C **Lemma 8.9.** *Notation and assumptions as in Situation 8.1. If*

(1) *f is smooth,*
(2) *f_0 is locally of finite presentation,*

then f_i is smooth for some $i \geq 0$.

Proof. Being smooth is local on the source and the target (Morphisms, Lemma 32.2) hence we may assume S_0, X_0, Y_0 affine (details omitted). The corresponding algebra fact is Algebra, Lemma 162.8. □

07RP **Lemma 8.10.** *Notation and assumptions as in Situation 8.1. If*

(1) *f is étale,*
(2) *f_0 is locally of finite presentation,*

then f_i is étale for some $i \geq 0$.

Proof. Being étale is local on the source and the target (Morphisms, Lemma 34.2) hence we may assume S_0, X_0, Y_0 affine (details omitted). The corresponding algebra fact is Algebra, Lemma 162.7. □

081E **Lemma 8.11.** *Notation and assumptions as in Situation 8.1. If*

(1) *f is an isomorphism, and*
(2) *f_0 is locally of finite presentation,*

then f_i is an isomorphism for some $i \geq 0$.

Proof. By Lemmas 8.10 and 8.5 we can find an i such that f_i is flat and a closed immersion. Then f_i identifies X_i with an open and closed subscheme of Y_i, see Morphisms, Lemma 25.2. By assumption the image of $Y \to Y_i$ maps into $f_i(X_i)$.

Thus by Lemma 4.10 we find that $Y_{i'}$ maps into $f_i(X_i)$ for some $i' \geq i$. It follows that $X_{i'} \to Y_{i'}$ is surjective and we win. $\square$

07RQ **Lemma 8.12.** *Notation and assumptions as in Situation 8.1. If*

1. *f is a monomorphism, and*
2. *f_0 is locally of finite type,*

then f_i is a monomorphism for some $i \geq 0$.

Proof. Recall that a morphism of schemes $V \to W$ is a monomorphism if and only if the diagonal $V \to V \times_W V$ is an isomorphism (Schemes, Lemma 23.2). The morphism $X_0 \to X_0 \times_{Y_0} X_0$ is locally of finite presentation by Morphisms, Lemma 20.12. Since $X_0 \times_{Y_0} X_0$ is quasi-compact and quasi-separated (Schemes, Remark 21.18) we conclude from Lemma 8.11 that $\Delta_i : X_i \to X_i \times_{Y_i} X_i$ is an isomorphism for some $i \geq 0$. For this i the morphism f_i is a monomorphism. $\square$

07RR **Lemma 8.13.** *Notation and assumptions as in Situation 8.1. If*

1. *f is surjective, and*
2. *f_0 is locally of finite presentation,*

then there exists an $i \geq 0$ such that f_i is surjective.

Proof. The morphism f_0 is of finite presentation. Hence $E = f_0(X_0)$ is a constructible subset of Y_0, see Morphisms, Lemma 21.2. Since f_i is the base change of f_0 by $Y_i \to Y_0$ we see that the image of f_i is the inverse image of E in Y_i. Moreover, we know that $Y \to Y_0$ maps into E. Hence we win by Lemma 4.10. $\square$

0C3L **Lemma 8.14.** *Notation and assumptions as in Situation 8.1. If*

1. *f is syntomic, and*
2. *f_0 is locally of finite presentation,*

then there exists an $i \geq 0$ such that f_i is syntomic.

Proof. Choose a finite affine open covering $Y_0 = \bigcup_{j=1,\ldots,m} Y_{j,0}$ such that each $Y_{j,0}$ maps into an affine open $S_{j,0} \subset S_0$. For each j let $f_0^{-1}Y_{j,0} = \bigcup_{k=1,\ldots,n_j} X_{k,0}$ be a finite affine open covering. Since the property of being syntomic is local we see that it suffices to prove the lemma for the morphisms of affines $X_{k,i} \to Y_{j,i} \to S_{j,i}$ which are the base changes of $X_{k,0} \to Y_{j,0} \to S_{j,0}$ to S_i. Thus we reduce to the case that X_0, Y_0, S_0 are affine

In the affine case we reduce to the following algebra result. Suppose that $R = \operatorname{colim}_{i \in I} R_i$. For some $0 \in I$ suppose given an R_0-algebra map $A_i \to B_i$ of finite presentation. If $R \otimes_{R_0} A_0 \to R \otimes_{R_0} B_0$ is syntomic, then for some $i \geq 0$ the map $R_i \otimes_{R_0} A_0 \to R_i \otimes_{R_0} B_0$ is syntomic. This follows from Algebra, Lemma 152.11. $\square$

9. Finite type closed in finite presentation

01ZD A result of this type is [Kie72, Satz 2.10]. Another reference is [Con07].

01ZE **Lemma 9.1.** *Let $f : X \to S$ be a morphism of schemes. Assume:*

1. *The morphism f is locally of finite type.*
2. *The scheme X is quasi-compact and quasi-separated.*

Then there exists a morphism of finite presentation $f' : X' \to S$ and an immersion $X \to X'$ of schemes over S.

Proof. By Proposition 5.4 we can write $X = \lim_i X_i$ with each X_i of finite type over $\mathbf{Z}$ and with transition morphisms $f_{ii'} : X_i \to X_{i'}$ affine. Consider the commutative diagram

$$\begin{tikzcd} X \arrow[r] \arrow[rd] & X_{i,S} \arrow[r] \arrow[d] & X_i \arrow[d] \\ & S \arrow[r] & \mathrm{Spec}(\mathbf{Z}) \end{tikzcd}$$

Note that X_i is of finite presentation over $\mathrm{Spec}(\mathbf{Z})$, see Morphisms, Lemma 20.9. Hence the base change $X_{i,S} \to S$ is of finite presentation by Morphisms, Lemma 20.4. Thus it suffices to show that the arrow $X \to X_{i,S}$ is an immersion for i sufficiently large.

To do this we choose a finite affine open covering $X = V_1 \cup \ldots \cup V_n$ such that f maps each V_j into an affine open $U_j \subset S$. Let $h_{j,a} \in \mathcal{O}_X(V_j)$ be a finite set of elements which generate $\mathcal{O}_X(V_j)$ as an $\mathcal{O}_S(U_j)$-algebra, see Morphisms, Lemma 14.2. By Lemmas 4.11 and 4.13 (after possibly shrinking I) we may assume that there exist affine open coverings $X_i = V_{1,i} \cup \ldots \cup V_{n,i}$ compatible with transition maps such that $V_j = \lim_i V_{j,i}$. By Lemma 4.7 we can choose i so large that each $h_{j,a}$ comes from an element $h_{j,a,i} \in \mathcal{O}_{X_i}(V_{j,i})$. Thus the arrow in

$$V_j \longrightarrow U_j \times_{\mathrm{Spec}(\mathbf{Z})} V_{j,i} = (V_{j,i})_{U_j} \subset (V_{j,i})_S \subset X_{i,S}$$

is a closed immersion. Since $\bigcup (V_{j,i})_{U_j}$ forms an open of $X_{i,S}$ and since the inverse image of $(V_{j,i})_{U_j}$ in X is V_j it follows that $X \to X_{i,S}$ is an immersion. $\square$

01ZF **Remark 9.2.** We cannot do better than this if we do not assume more on S and the morphism $f : X \to S$. For example, in general it will not be possible to find a *closed* immersion $X \to X'$ as in the lemma. The reason is that this would imply that f is quasi-compact which may not be the case. An example is to take S to be infinite dimensional affine space with 0 doubled and X to be one of the two infinite dimensional affine spaces.

01ZG **Lemma 9.3.** *Let $f : X \to S$ be a morphism of schemes. Assume:*

1. *The morphism f is of locally of finite type.*
2. *The scheme X is quasi-compact and quasi-separated, and*
3. *The scheme S is quasi-separated.*

Then there exists a morphism of finite presentation $f' : X' \to S$ and a closed immersion $X \to X'$ of schemes over S.

Proof. By Lemma 9.1 above there exists a morphism $Y \to S$ of finite presentation and an immersion $i : X \to Y$ of schemes over S. For every point $x \in X$, there exists an affine open $V_x \subset Y$ such that $i^{-1}(V_x) \to V_x$ is a closed immersion. Since X is quasi-compact we can find finitely may affine opens $V_1, \ldots, V_n \subset Y$ such that $i(X) \subset V_1 \cup \ldots \cup V_n$ and $i^{-1}(V_j) \to V_j$ is a closed immersion. In other words such that $i : X \to X' = V_1 \cup \ldots \cup V_n$ is a closed immersion of schemes over S. Since S is quasi-separated and Y is quasi-separated over S we deduce that Y is quasi-separated, see Schemes, Lemma 21.13. Hence the open immersion $X' = V_1 \cup \ldots \cup V_n \to Y$ is quasi-compact. This implies that $X' \to Y$ is of finite presentation, see Morphisms, Lemma 20.6. We conclude since then $X' \to Y \to S$ is a composition of morphisms of finite presentation, and hence of finite presentation (see Morphisms, Lemma 20.3). $\square$

09ZP **Lemma 9.4.** *Let $X \to Y$ be a closed immersion of schemes. Assume Y quasi-compact and quasi-separated. Then X can be written as a directed limit $X = \lim X_i$ of schemes over Y where $X_i \to Y$ is a closed immersion of finite presentation.*

Proof. Let $\mathcal{I} \subset \mathcal{O}_Y$ be the quasi-coherent sheaf of ideals defining X as a closed subscheme of Y. By Properties, Lemma 22.3 we can write $\mathcal{I}$ as a directed colimit $\mathcal{I} = \text{colim}_{i \in I} \mathcal{I}_i$ of its quasi-coherent sheaves of ideals of finite type. Let $X_i \subset Y$ be the closed subscheme defined by $\mathcal{I}_i$. These form an inverse system of schemes indexed by I. The transition morphisms $X_i \to X_{i'}$ are affine because they are closed immersions. Each X_i is quasi-compact and quasi-separated since it is a closed subscheme of Y and Y is quasi-compact and quasi-separated by our assumptions. We have $X = \lim_i X_i$ as follows directly from the fact that $\mathcal{I} = \text{colim}_{i \in I} \mathcal{I}_a$. Each of the morphisms $X_i \to Y$ is of finite presentation, see Morphisms, Lemma 20.7. $\square$

09ZQ **Lemma 9.5.** *Let $f : X \to S$ be a morphism of schemes. Assume*

1. *The morphism f is of locally of finite type.*
2. *The scheme X is quasi-compact and quasi-separated, and*
3. *The scheme S is quasi-separated.*

Then $X = \lim X_i$ where the $X_i \to S$ are of finite presentation, the X_i are quasi-compact and quasi-separated, and the transition morphisms $X_{i'} \to X_i$ are closed immersions (which implies that $X \to X_i$ are closed immersions for all i).

Proof. By Lemma 9.3 there is a closed immersion $X \to Y$ with $Y \to S$ of finite presentation. Then Y is quasi-separated by Schemes, Lemma 21.13. Since X is quasi-compact, we may assume Y is quasi-compact by replacing Y with a quasi-compact open containing X. We see that $X = \lim X_i$ with $X_i \to Y$ a closed immersion of finite presentation by Lemma 9.4. The morphisms $X_i \to S$ are of finite presentation by Morphisms, Lemma 20.3. $\square$

01ZJ **Proposition 9.6.** *Let $f : X \to S$ be a morphism of schemes. Assume*

1. *f is of finite type and separated, and*
2. *S is quasi-compact and quasi-separated.*

Then there exists a separated morphism of finite presentation $f' : X' \to S$ and a closed immersion $X \to X'$ of schemes over S.

Proof. Apply Lemma 9.5 and note that $X_i \to S$ is separated for large i by Lemma 4.17 as we have assumed that $X \to S$ is separated. $\square$

01ZK **Lemma 9.7.** *Let $f : X \to S$ be a morphism of schemes. Assume*

1. *f is finite, and*
2. *S is quasi-compact and quasi-separated.*

Then there exists a morphism which is finite and of finite presentation $f' : X' \to S$ and a closed immersion $X \to X'$ of schemes over S.

Proof. We may write $X = \lim X_i$ as in Lemma 9.5. Applying Lemma 4.19 we see that $X_i \to S$ is finite for large enough i. $\square$

09YY **Lemma 9.8.** *Let $f : X \to S$ be a morphism of schemes. Assume*

1. *f is finite, and*
2. *S quasi-compact and quasi-separated.*

Then X is a directed limit $X = \lim X_i$ where the transition maps are closed immersions and the objects X_i are finite and of finite presentation over S.

Proof. We may write $X = \lim X_i$ as in Lemma 9.5. Applying Lemma 4.19 we see that $X_i \to S$ is finite for large enough i. $\square$

10. Descending relative objects

01ZL The following lemma is typical of the type of results in this section. We write out the "standard" proof completely. It may be faster to convince yourself that the result is true than to read this proof.

01ZM **Lemma 10.1.** *Let I be a directed set. Let $(S_i, f_{ii'})$ be an inverse system of schemes over I. Assume*

1. *the morphisms $f_{ii'} : S_i \to S_{i'}$ are affine,*
2. *the schemes S_i are quasi-compact and quasi-separated.*

Let $S = \lim_i S_i$. Then we have the following:

1. *For any morphism of finite presentation $X \to S$ there exists an index $i \in I$ and a morphism of finite presentation $X_i \to S_i$ such that $X \cong X_{i,S}$ as schemes over S.*
2. *Given an index $i \in I$, schemes X_i, Y_i of finite presentation over S_i, and a morphism $\varphi : X_{i,S} \to Y_{i,S}$ over S, there exists an index $i' \geq i$ and a morphism $\varphi_{i'} : X_{i,S_{i'}} \to Y_{i,S_{i'}}$ whose base change to S is φ.*
3. *Given an index $i \in I$, schemes X_i, Y_i of finite presentation over S_i and a pair of morphisms $\varphi_i, \psi_i : X_i \to Y_i$ whose base changes $\varphi_{i,S} = \psi_{i,S}$ are equal, there exists an index $i' \geq i$ such that $\varphi_{i,S_{i'}} = \psi_{i,S_{i'}}$.*

In other words, the category of schemes of finite presentation over S is the colimit over I of the categories of schemes of finite presentation over S_i.

Proof. In case each of the schemes S_i is affine, and we consider only affine schemes of finite presentation over S_i, resp. S this lemma is equivalent to Algebra, Lemma 126.8. We claim that the affine case implies the lemma in general.

Let us prove (3). Suppose given an index $i \in I$, schemes X_i, Y_i of finite presentation over S_i and a pair of morphisms $\varphi_i, \psi_i : X_i \to Y_i$. Assume that the base changes are equal: $\varphi_{i,S} = \psi_{i,S}$. We will use the notation $X_{i'} = X_{i,S_{i'}}$ and $Y_{i'} = Y_{i,S_{i'}}$ for $i' \geq i$. We also set $X = X_{i,S}$ and $Y = Y_{i,S}$. Note that according to Lemma 2.3 we have $X = \lim_{i' \geq i} X_{i'}$ and similarly for Y. Additionally we denote $\varphi_{i'}$ and $\psi_{i'}$ (resp. φ and ψ) the base change of φ_i and ψ_i to $S_{i'}$ (resp. S). So our assumption means that $\varphi = \psi$. Since Y_i and X_i are of finite presentation over S_i, and since S_i is quasi-compact and quasi-separated, also X_i and Y_i are quasi-compact and quasi-separated (see Morphisms, Lemma 20.10). Hence we may choose a finite affine open covering $Y_i = \bigcup V_{j,i}$ such that each $V_{j,i}$ maps into an affine open of S. As above, denote $V_{j,i'}$ the inverse image of $V_{j,i}$ in $Y_{i'}$ and V_j the inverse image in Y. The immersions $V_{j,i'} \to Y_{i'}$ are quasi-compact, and the inverse images $U_{j,i'} = \varphi_i^{-1}(V_{j,i'})$ and $U'_{j,i'} = \psi_i^{-1}(V_{j,i'})$ are quasi-compact opens of $X_{i'}$. By assumption the inverse images of V_j under φ and ψ in X are equal. Hence by Lemma 4.11 there exists an index $i' \geq i$ such that of $U_{j,i'} = U'_{j,i'}$ in $X_{i'}$. Choose an finite affine open covering $U_{j,i'} = U'_{j,i'} = \bigcup W_{j,k,i'}$ which induce coverings $U_{j,i''} = U'_{j,i''} = \bigcup W_{j,k,i''}$ for all $i'' \geq i'$. By the affine case there exists an index i'' such that $\varphi_{i''}|_{W_{j,k,i''}} = \psi_{i''}|_{W_{j,k,i''}}$ for all j, k. Then i'' is an index such that $\varphi_{i''} = \psi_{i''}$ and (3) is proved.

Let us prove (2). Suppose given an index $i \in I$, schemes X_i, Y_i of finite presentation over S_i and a morphism $\varphi : X_{i,S} \to Y_{i,S}$. We will use the notation $X_{i'} = X_{i,S_{i'}}$ and $Y_{i'} = Y_{i,S_{i'}}$ for $i' \geq i$. We also set $X = X_{i,S}$ and $Y = Y_{i,S}$. Note that according to Lemma 2.3 we have $X = \lim_{i' \geq i} X_{i'}$ and similarly for Y. Since Y_i and X_i are of finite presentation over S_i, and since S_i is quasi-compact and quasi-separated, also X_i and Y_i are quasi-compact and quasi-separated (see Morphisms, Lemma 20.10). Hence we may choose a finite affine open covering $Y_i = \bigcup V_{j,i}$ such that each $V_{j,i}$ maps into an affine open of S. As above, denote $V_{j,i'}$ the inverse image of $V_{j,i}$ in $Y_{i'}$ and V_j the inverse image in Y. The immersions $V_j \to Y$ are quasi-compact, and the inverse images $U_j = \varphi^{-1}(V_j)$ are quasi-compact opens of X. Hence by Lemma 4.11 there exists an index $i' \geq i$ and quasi-compact opens $U_{j,i'}$ of $X_{i'}$ whose inverse image in X is U_j. Choose an finite affine open covering $U_{j,i'} = \bigcup W_{j,k,i'}$ which induce affine open coverings $U_{j,i''} = \bigcup W_{j,k,i''}$ for all $i'' \geq i'$ and an affine open covering $U_j = \bigcup W_{j,k}$. By the affine case there exists an index i'' and morphisms $\varphi_{j,k,i''} : W_{j,k,i''} \to V_{j,i''}$ such that $\varphi|_{W_{j,k}} = \varphi_{j,k,i'',S}$ for all j, k. By part (3) proved above, there is a further index $i''' \geq i''$ such that

$$\varphi_{j_1,k_1,i'',S_{i'''}}|_{W_{j_1,k_1,i'''} \cap W_{j_2,k_2,i'''}} = \varphi_{j_2,k_2,i'',S_{i'''}}|_{W_{j_1,k_1,i'''} \cap W_{j_2,k_2,i'''}}$$

for all j_1, j_2, k_1, k_2. Then i''' is an index such that there exists a morphism $\varphi_{i'''} : X_{i'''} \to Y_{i'''}$ whose base change to S gives φ. Hence (2) holds.

Let us prove (1). Suppose given a scheme X of finite presentation over S. Since X is of finite presentation over S, and since S is quasi-compact and quasi-separated, also X is quasi-compact and quasi-separated (see Morphisms, Lemma 20.10). Choose a finite affine open covering $X = \bigcup U_j$ such that each U_j maps into an affine open $V_j \subset S$. Denote $U_{j_1 j_2} = U_{j_1} \cap U_{j_2}$ and $U_{j_1 j_2 j_3} = U_{j_1} \cap U_{j_2} \cap U_{j_3}$. By Lemmas 4.11 and 4.13 we can find an index i_1 and affine opens $V_{j,i_1} \subset S_{i_1}$ such that each V_j is the inverse of this in S. Let $V_{j,i}$ be the inverse image of V_{j,i_1} in S_i for $i \geq i_1$. By the affine case we may find an index $i_2 \geq i_1$ and affine schemes $U_{j,i_2} \to V_{j,i_2}$ such that $U_j = S \times_{S_{i_2}} U_{j,i_2}$ is the base change. Denote $U_{j,i} = S_i \times_{S_{i_2}} U_{j,i_2}$ for $i \geq i_2$. By Lemma 4.11 there exists an index $i_3 \geq i_2$ and open subschemes $W_{j_1,j_2,i_3} \subset U_{j_1,i_3}$ whose base change to S is equal to $U_{j_1 j_2}$. Denote $W_{j_1,j_2,i} = S_i \times_{S_{i_3}} W_{j_1,j_2,i_3}$ for $i \geq i_3$. By part (2) shown above there exists an index $i_4 \geq i_3$ and morphisms $\varphi_{j_1,j_2,i_4} : W_{j_1,j_2,i_4} \to W_{j_2,j_1,i_4}$ whose base change to S gives the identity morphism $U_{j_1 j_2} = U_{j_2 j_1}$ for all j_1, j_2. For all $i \geq i_4$ denote $\varphi_{j_1,j_2,i} = \text{id}_S \times \varphi_{j_1,j_2,i_4}$ the base change. We claim that for some $i_5 \geq i_4$ the system $((U_{j,i_5})_j, (W_{j_1,j_2,i_5})_{j_1,j_2}, (\varphi_{j_1,j_2,i_5})_{j_1,j_2})$ forms a glueing datum as in Schemes, Section 14. In order to see this we have to verify that for i large enough we have

$$\varphi_{j_1,j_2,i}^{-1}(W_{j_1,j_2,i} \cap W_{j_1,j_3,i}) = W_{j_1,j_2,i} \cap W_{j_1,j_3,i}$$

and that for large enough i the cocycle condition holds. The first condition follows from Lemma 4.11 and the fact that $U_{j_2 j_1 j_3} = U_{j_1 j_2 j_3}$. The second from part (1) of the lemma proved above and the fact that the cocycle condition holds for the maps $\text{id} : U_{j_1 j_2} \to U_{j_2 j_1}$. Ok, so now we can use Schemes, Lemma 14.2 to glue the system $((U_{j,i_5})_j, (W_{j_1,j_2,i_5})_{j_1,j_2}, (\varphi_{j_1,j_2,i_5})_{j_1,j_2})$ to get a scheme $X_{i_5} \to S_{i_5}$. By construction the base change of X_{i_5} to S is formed by glueing the open affines U_j along the opens $U_{j_1} \leftarrow U_{j_1 j_2} \to U_{j_2}$. Hence $S \times_{S_{i_5}} X_{i_5} \cong X$ as desired. □

01ZR **Lemma 10.2.** *Let I be a directed set. Let $(S_i, f_{ii'})$ be an inverse system of schemes over I. Assume*

(1) *all the morphisms* $f_{ii'} : S_i \to S_{i'}$ *are affine,*
(2) *all the schemes* S_i *are quasi-compact and quasi-separated.*

Let $S = \lim_i S_i$. *Then we have the following:*

(1) *For any sheaf of* $\mathcal{O}_S$*-modules* $\mathcal{F}$ *of finite presentation there exists an index* $i \in I$ *and a sheaf of* $\mathcal{O}_{S_i}$*-modules of finite presentation* $\mathcal{F}_i$ *such that* $\mathcal{F} \cong f_i^*\mathcal{F}_i$.
(2) *Suppose given an index* $i \in I$, *sheaves of* $\mathcal{O}_{S_i}$*-modules* $\mathcal{F}_i$, $\mathcal{G}_i$ *of finite presentation and a morphism* $\varphi : f_i^*\mathcal{F}_i \to f_i^*\mathcal{G}_i$ *over* S. *Then there exists an index* $i' \geq i$ *and a morphism* $\varphi_{i'} : f_{i'i}^*\mathcal{F}_i \to f_{i'i}^*\mathcal{G}_i$ *whose base change to* S *is* φ.
(3) *Suppose given an index* $i \in I$, *sheaves of* $\mathcal{O}_{S_i}$*-modules* $\mathcal{F}_i$, $\mathcal{G}_i$ *of finite presentation and a pair of morphisms* $\varphi_i, \psi_i : \mathcal{F}_i \to \mathcal{G}_i$. *Assume that the base changes are equal:* $f_i^*\varphi_i = f_i^*\psi_i$. *Then there exists an index* $i' \geq i$ *such that* $f_{i'i}^*\varphi_i = f_{i'i}^*\psi_i$.

In other words, the category of modules of finite presentation over S *is the colimit over* I *of the categories modules of finite presentation over* S_i.

Proof. Omitted. Since we have written out completely the proof of Lemma 10.1 above it seems wise to use this here and not completely write this proof out also. For example we can use:

(1) there is an equivalence of categories between quasi-coherent $\mathcal{O}_S$-modules and vector bundles over S, see Constructions, Section 6.
(2) a vector bundle $\mathbf{V}(\mathcal{F}) \to S$ is of finite presentation over S if and only if $\mathcal{F}$ is an $\mathcal{O}_S$-module of finite presentation.

Then you can descend morphisms in terms of morphisms of the associated vector-bundles. Similarly for objects. $\square$

0B8W **Lemma 10.3.** *Let* $S = \lim S_i$ *be the limit of a directed system of quasi-compact and quasi-separated schemes* S_i *with affine transition morphisms. Then any invertible* $\mathcal{O}_S$*-module is the pullback of an invertible* $\mathcal{O}_{S_i}$*-module for some* i.

Proof. Let $\mathcal{L}$ be an invertible $\mathcal{O}_S$-module. Since invertible modules are of finite presentation we can find an i and modules $\mathcal{L}_i$ and $\mathcal{N}_i$ of finite presentation over S_i such that $f_i^*\mathcal{L}_i \cong \mathcal{L}$ and $f_i^*\mathcal{N}_i \cong \mathcal{L}^{\otimes -1}$, see Lemma 10.2. Since pullback commutes with tensor product we see that $f_i^*(\mathcal{L}_i \otimes_{\mathcal{O}_{S_i}} \mathcal{N}_i)$ is isomorphic to $\mathcal{O}_S$. Since the tensor product of finitely presented modules is finitely presented, the same lemma implies that $f_{i'i}^*\mathcal{L}_i \otimes_{\mathcal{O}_{S_{i'}}} f_{i'i}^*\mathcal{N}_i$ is isomorphic to $\mathcal{O}_{S_{i'}}$ for some $i' \geq i$. It follows that $f_{i'i}^*\mathcal{L}_i$ is invertible (Modules, Lemma 22.2) and the proof is complete. $\square$

05LY **Lemma 10.4.** *With notation and assumptions as in Lemma 10.1. Let* $i \in I$. *Suppose that* $\varphi_i : X_i \to Y_i$ *is a morphism of schemes of finite presentation over* S_i *and that* $\mathcal{F}_i$ *is a quasi-coherent* $\mathcal{O}_{X_i}$*-module of finite presentation. If the pullback of* $\mathcal{F}_i$ *to* $X_i \times_{S_i} S$ *is flat over* $Y_i \times_{S_i} S$, *then there exists an index* $i' \geq i$ *such that the pullback of* $\mathcal{F}_i$ *to* $X_i \times_{S_i} S_{i'}$ *is flat over* $Y_i \times_{S_i} S_{i'}$.

Proof. (This lemma is the analogue of Lemma 8.7 for modules.) For $i' \geq i$ denote $X_{i'} = S_{i'} \times_{S_i} X_i$, $\mathcal{F}_{i'} = (X_{i'} \to X_i)^*\mathcal{F}_i$ and similarly for $Y_{i'}$. Denote $\varphi_{i'}$ the base change of φ_i to $S_{i'}$. Also set $X = S \times_{S_i} X_i$, $Y = S \times_{S_i} X_i$, $\mathcal{F} = (X \to X_i)^*\mathcal{F}_i$ and φ the base change of φ_i to S. Let $Y_i = \bigcup_{j=1,\ldots,m} V_{j,i}$ be a finite affine open covering such that each $V_{j,i}$ maps into some affine open of S_i. For each $j = 1, \ldots m$

let $\varphi_i^{-1}(V_{j,i}) = \bigcup_{k=1,\ldots,m(j)} U_{k,j,i}$ be a finite affine open covering. For $i' \geq i$ we denote $V_{j,i'}$ the inverse image of $V_{j,i}$ in $Y_{i'}$ and $U_{k,j,i'}$ the inverse image of $U_{k,j,i}$ in $X_{i'}$. Similarly we have $U_{k,j} \subset X$ and $V_j \subset Y$. Then $U_{k,j} = \lim_{i' \geq i} U_{k,j,i'}$ and $V_j = \lim_{i' \geq i} V_j$ (see Lemma 2.2). Since $X_{i'} = \bigcup_{k,j} U_{k,j,i'}$ is a finite open covering it suffices to prove the lemma for each of the morphisms $U_{k,j,i} \to V_{j,i}$ and the sheaf $\mathcal{F}_i|_{U_{k,j,i}}$. Hence we see that the lemma reduces to the case that X_i and Y_i are affine and map into an affine open of S_i, i.e., we may also assume that S is affine.

In the affine case we reduce to the following algebra result. Suppose that $R = \text{colim}_{i \in I} R_i$. For some $i \in I$ suppose given a map $A_i \to B_i$ of finitely presented R_i-algebras. Let N_i be a finitely presented B_i-module. Then, if $R \otimes_{R_i} N_i$ is flat over $R \otimes_{R_i} A_i$, then for some $i' \geq i$ the module $R_{i'} \otimes_{R_i} N_i$ is flat over $R_{i'} \otimes_{R_i} A$. This is exactly the result proved in Algebra, Lemma 162.1 part (3). $\square$

11. Characterizing affine schemes

01ZS If $f : X \to S$ is a surjective integral morphism of schemes such that X is an affine scheme then S is affine too. See [Con07, A.2]. Our proof relies on the Noetherian case which we stated and proved in Cohomology of Schemes, Lemma 13.3. See also [DG67, II 6.7.1].

01ZT **Lemma 11.1.** *Let $f : X \to S$ be a morphism of schemes. Assume that f is surjective and finite, and assume that X is affine. Then S is affine.*

Proof. Since f is surjective and X is quasi-compact we see that S is quasi-compact. Since X is separated and f is surjective and universally closed (Morphisms, Lemma 42.7), we see that S is separated (Morphisms, Lemma 39.11).

By Lemma 9.8 we can write $X = \lim_a X_a$ with $X_a \to S$ finite and of finite presentation. By Lemma 4.13 we see that X_a is affine for some $a \in A$. Replacing X by X_a we may assume that $X \to S$ is surjective, finite, of finite presentation and that X is affine.

By Proposition 5.4 we may write $S = \lim_{i \in I} S_i$ as a directed limits as schemes of finite type over $\mathbf{Z}$. By Lemma 10.1 we can after shrinking I assume there exist schemes $X_i \to S_i$ of finite presentation such that $X_{i'} = X_i \times_S S_{i'}$ for $i' \geq i$ and such that $X = \lim_i X_i$. By Lemma 8.3 we may assume that $X_i \to S_i$ is finite for all $i \in I$ as well. By Lemma 4.13 once again we may assume that X_i is affine for all $i \in I$. Hence the result follows from the Noetherian case, see Cohomology of Schemes, Lemma 13.3. $\square$

05YU **Proposition 11.2.** *Let $f : X \to S$ be a morphism of schemes. Assume that f is surjective and integral, and assume that X is affine. Then S is affine.*

Proof. Since f is surjective and X is quasi-compact we see that S is quasi-compact. Since X is separated and f is surjective and universally closed (Morphisms, Lemma 42.7), we see that S is separated (Morphisms, Lemma 39.11).

By Lemma 7.2 we can write $X = \lim_i X_i$ with $X_i \to S$ finite. By Lemma 4.13 we see that for i sufficiently large the scheme X_i is affine. Moreover, since $X \to S$ factors through each X_i we see that $X_i \to S$ is surjective. Hence we conclude that S is affine by Lemma 11.1. $\square$

09NL **Lemma 11.3.** *Let X be a scheme which is set theoretically the union of finitely many affine closed subschemes. Then X is affine.*

Proof. Let $Z_i \subset X$, $i = 1, \ldots, n$ be affine closed subschemes such that $X = \bigcup Z_i$ set theoretically. Then $\coprod Z_i \to X$ is surjective and integral with affine source. Hence X is affine by Proposition 11.2. $\square$

09MW **Lemma 11.4.** *Let $i : Z \to X$ be a closed immersion of schemes inducing a homeomorphism of underlying topological spaces. Let $\mathcal{L}$ be an invertible sheaf on X. Then $i^*\mathcal{L}$ is ample on Z, if and only if $\mathcal{L}$ is ample on X.*

Proof. If $\mathcal{L}$ is ample, then $i^*\mathcal{L}$ is ample for example by Morphisms, Lemma 35.7. Assume $i^*\mathcal{L}$ is ample. Then Z is quasi-compact (Properties, Definition 26.1) and separated (Properties, Lemma 26.8). Since i is surjective, we see that X is quasi-compact. Since i is universally closed and surjective, we see that X is separated (Morphisms, Lemma 39.11).

By Proposition 5.4 we can write $X = \lim X_i$ as a directed limit of finite type schemes over $\mathbf{Z}$ with affine transition morphisms. We can find an i and an invertible sheaf $\mathcal{L}_i$ on X_i whose pullback to X is isomorphic to $\mathcal{L}$, see Lemma 10.2.

For each i let $Z_i \subset X_i$ be the scheme theoretic image of the morphism $Z \to X$. If $\text{Spec}(A_i) \subset X_i$ is an affine open subscheme with inverse image of $\text{Spec}(A)$ in X and if $Z \cap \text{Spec}(A)$ is defined by the ideal $I \subset A$, then $Z_i \cap \text{Spec}(A_i)$ is defined by the ideal $I_i \subset A_i$ which is the inverse image of I in A_i under the ring map $A_i \to A$, see Morphisms, Example 6.4. Since $\text{colim}\, A_i/I_i = A/I$ it follows that $\lim Z_i = Z$. By Lemma 4.15 we see that $\mathcal{L}_i|_{Z_i}$ is ample for some i. Since Z and hence X maps into Z_i set theoretically, we see that $X_{i'} \to X_i$ maps into Z_i set theoretically for some $i' \geq i$, see Lemma 4.10. (Observe that since X_i is Noetherian, every closed subset of X_i is constructible.) Let $T \subset X_{i'}$ be the scheme theoretic inverse image of Z_i in $X_{i'}$. Observe that $\mathcal{L}_{i'}|_T$ is the pullback of $\mathcal{L}_i|_{Z_i}$ and hence ample by Morphisms, Lemma 35.7 and the fact that $T \to Z_i$ is an affine morphism. Thus we see that $\mathcal{L}_{i'}$ is ample on $X_{i'}$ by Cohomology of Schemes, Lemma 17.5. Pulling back to X (using the same lemma as above) we find that $\mathcal{L}$ is ample. $\square$

0B7L **Lemma 11.5.** *Let $i : Z \to X$ be a closed immersion of schemes inducing a homeomorphism of underlying topological spaces. Then X is quasi-affine if and only if Z is quasi-affine.*

Proof. Recall that a scheme is quasi-affine if and only if the structure sheaf is ample, see Properties, Lemma 27.1. Hence if Z is quasi-affine, then $\mathcal{O}_Z$ is ample, hence $\mathcal{O}_X$ is ample by Lemma 11.4, hence X is quasi-affine. A proof of the converse, which can also be seen in an elementary way, is gotten by reading the argument just given backwards. $\square$

The following lemma does not really belong in this section.

0E21 **Lemma 11.6.** *Let X be a scheme. Let $\mathcal{L}$ be an ample invertible sheaf on X. Assume we have morphisms of schemes*

$$\text{Spec}(k) \leftarrow \text{Spec}(A) \to W \subset X$$

where k is a field, A is an integral k-algebra, W is open in X. Then there exists an $n > 0$ and a section $s \in \Gamma(X, \mathcal{L}^{\otimes n})$ such that X_s is affine, $X_s \subset W$, and $\text{Spec}(A) \to W$ factors through X_s

Proof. Since $\mathrm{Spec}(A)$ is quasi-compact, we may replace W by a quasi-compact open still containing the image of $\mathrm{Spec}(A) \to X$. Recall that X is quasi-separated and quasi-compact by dint of having an ample invertible sheaf, see Properties, Definition 26.1 and Lemma 26.7. By Proposition 5.4 we can write $X = \lim X_i$ as a limit of a directed system of schemes of finite type over $\mathbf{Z}$ with affine transition morphisms. For some i the ample invertible sheaf $\mathcal{L}$ on X descends to an ample invertible sheaf $\mathcal{L}_i$ on X_i and the open W is the inverse image of a quasi-compact open $W_i \subset X_i$, see Lemmas 4.15, 10.3, and 4.11. We may replace $X, W, \mathcal{L}$ by $X_i, W_i, \mathcal{L}_i$ and assume X is of finite presentation over $\mathbf{Z}$. Write $A = \mathrm{colim}\, A_j$ as the colimit of its finite k-subalgebras. Then for some j the morphism $\mathrm{Spec}(A) \to X$ factors through a morphism $\mathrm{Spec}(A_j) \to X$, see Proposition 6.1. Since $\mathrm{Spec}(A_j)$ is finite this reduces the lemma to Properties, Lemma 29.6. $\square$

12. Variants of Chow's Lemma

01ZZ In this section we prove a number of variants of Chow's lemma. The most interesting version is probably just the Noetherian case, which we stated and proved in Cohomology of Schemes, Section 18.

0202 **Lemma 12.1.** *Let S be a quasi-compact and quasi-separated scheme. Let $f : X \to S$ be a separated morphism of finite type. Then there exists an $n \geq 0$ and a diagram*

$$\begin{tikzcd} X \arrow[rd] & X' \arrow[l, "\pi"'] \arrow[r] \arrow[d] & \mathbf{P}^n_S \arrow[ld] \\ & S & \end{tikzcd}$$

where $X' \to \mathbf{P}^n_S$ is an immersion, and $\pi : X' \to X$ is proper and surjective.

Proof. By Proposition 9.6 we can find a closed immersion $X \to Y$ where Y is separated and of finite presentation over S. Clearly, if we prove the assertion for Y, then the result follows for X. Hence we may assume that X is of finite presentation over S.

Write $S = \lim_i S_i$ as a directed limit of Noetherian schemes, see Proposition 5.4. By Lemma 10.1 we can find an index $i \in I$ and a scheme $X_i \to S_i$ of finite presentation so that $X = S \times_{S_i} X_i$. By Lemma 8.6 we may assume that $X_i \to S_i$ is separated. Clearly, if we prove the assertion for X_i over S_i, then the assertion holds for X. The case $X_i \to S_i$ is treated by Cohomology of Schemes, Lemma 18.1. $\square$

Here is a variant of Chow's lemma where we assume the scheme on top has finitely many irreducible components.

0203 **Lemma 12.2.** *Let S be a quasi-compact and quasi-separated scheme. Let $f : X \to S$ be a separated morphism of finite type. Assume that X has finitely many irreducible components. Then there exists an $n \geq 0$ and a diagram*

$$\begin{tikzcd} X \arrow[rd] & X' \arrow[l, "\pi"'] \arrow[r] \arrow[d] & \mathbf{P}^n_S \arrow[ld] \\ & S & \end{tikzcd}$$

where $X' \to \mathbf{P}^n_S$ is an immersion, and $\pi : X' \to X$ is proper and surjective. Moreover, there exists an open dense subscheme $U \subset X$ such that $\pi^{-1}(U) \to U$ is an isomorphism of schemes.

Proof. Let $X = Z_1 \cup \ldots \cup Z_n$ be the decomposition of X into irreducible components. Let $\eta_j \in Z_j$ be the generic point.

There are (at least) two ways to proceed with the proof. The first is to redo the proof of Cohomology of Schemes, Lemma 18.1 using the general Properties, Lemma 29.4 to find suitable affine opens in X. (This is the "standard" proof.) The second is to use absolute Noetherian approximation as in the proof of Lemma 12.1 above. This is what we will do here.

By Proposition 9.6 we can find a closed immersion $X \to Y$ where Y is separated and of finite presentation over S. Write $S = \lim_i S_i$ as a directed limit of Noetherian schemes, see Proposition 5.4. By Lemma 10.1 we can find an index $i \in I$ and a scheme $Y_i \to S_i$ of finite presentation so that $Y = S \times_{S_i} Y_i$. By Lemma 8.6 we may assume that $Y_i \to S_i$ is separated. We have the following diagram

$$\begin{tikzcd} \eta_j \in Z_j \arrow[r] & X \arrow[r] \arrow[dr] & Y \arrow[r] \arrow[d] & Y_i \arrow[d] \\ & & S \arrow[r] & S_i \end{tikzcd}$$

Denote $h : X \to Y_i$ the composition.

For $i' \geq i$ write $Y_{i'} = S_{i'} \times_{S_i} Y_i$. Then $Y = \lim_{i' \geq i} Y_{i'}$, see Lemma 2.3. Choose $j, j' \in \{1, \ldots, n\}$, $j \neq j'$. Note that η_j is not a specialization of $\eta_{j'}$. By Lemma 4.6 we can replace i by a bigger index and assume that $h(\eta_j)$ is not a specialization of $h(\eta_{j'})$ for all pairs (j, j') as above. For such an index, let $Y' \subset Y_i$ be the scheme theoretic image of $h : X \to Y_i$, see Morphisms, Definition 6.2. The morphism h is quasi-compact as the composition of the quasi-compact morphisms $X \to Y$ and $Y \to Y_i$ (which is affine). Hence by Morphisms, Lemma 6.3 the morphism $X \to Y'$ is dominant. Thus the generic points of Y' are all contained in the set $\{h(\eta_1), \ldots, h(\eta_n)\}$, see Morphisms, Lemma 8.3. Since none of the $h(\eta_j)$ is the specialization of another we see that the points $h(\eta_1), \ldots, h(\eta_n)$ are pairwise distinct and are each a generic point of Y'.

We apply Cohomology of Schemes, Lemma 18.1 above to the morphism $Y' \to S_i$. This gives a diagram

$$\begin{tikzcd} Y' \arrow[dr] & Y^* \arrow[l, "\pi"'] \arrow[r] \arrow[d] & \mathbf{P}^n_{S_i} \arrow[dl] \\ & S_i & \end{tikzcd}$$

such that π is proper and surjective and an isomorphism over a dense open subscheme $V \subset Y'$. By our choice of i above we know that $h(\eta_1), \ldots, h(\eta_n) \in V$.

Consider the commutative diagram

$$\begin{tikzcd}
X' \arrow[r, equal] & X \times_{Y'} Y^* \arrow[r] \arrow[d] & Y^* \arrow[r] \arrow[d] & \mathbf{P}^n_{S_i} \arrow[ddl] \\
 & X \arrow[r] \arrow[d] & Y' \arrow[d] & \\
 & S \arrow[r] & S_i &
\end{tikzcd}$$

Note that $X' \to X$ is an isomorphism over the open subscheme $U = h^{-1}(V)$ which contains each of the η_j and hence is dense in X. We conclude $X \leftarrow X' \to \mathbf{P}^n_S$ is a solution to the problem posed in the lemma. □

13. Applications of Chow's lemma

0204 Here is a first application of Chow's lemma.

081F **Lemma 13.1.** *Assumptions and notation as in Situation 8.1. If*

(1) f *is proper, and*
(2) f_0 *is locally of finite type,*

then there exists an i such that f_i is proper.

Proof. By Lemma 8.6 we see that f_i is separated for some $i \geq 0$. Replacing 0 by i we may assume that f_0 is separated. Observe that f_0 is quasi-compact, see Schemes, Lemma 21.15. By Lemma 12.1 we can choose a diagram

$$\begin{tikzcd}
X_0 \arrow[dr] & X_0' \arrow[l, "\pi"'] \arrow[r] \arrow[d] & \mathbf{P}^n_{Y_0} \arrow[dl] \\
 & Y_0 &
\end{tikzcd}$$

where $X_0' \to \mathbf{P}^n_{Y_0}$ is an immersion, and $\pi : X_0' \to X_0$ is proper and surjective. Introduce $X' = X_0' \times_{Y_0} Y$ and $X_i' = X_0' \times_{Y_0} Y_i$. By Morphisms, Lemmas 39.4 and 39.5 we see that $X' \to Y$ is proper. Hence $X' \to \mathbf{P}^n_Y$ is a closed immersion (Morphisms, Lemma 39.7). By Morphisms, Lemma 39.8 it suffices to prove that $X_i' \to Y_i$ is proper for some i. By Lemma 8.5 we find that $X_i' \to \mathbf{P}^n_{Y_i}$ is a closed immersion for i large enough. Then $X_i' \to Y_i$ is proper and we win. □

09ZR **Lemma 13.2.** *Let $f : X \to S$ be a proper morphism with S quasi-compact and quasi-separated. Then $X = \lim X_i$ is a directed limit of schemes X_i proper and of finite presentation over S such that all transition morphisms and the morphisms $X \to X_i$ are closed immersions.*

Proof. By Proposition 9.6 we can find a closed immersion $X \to Y$ with Y separated and of finite presentation over S. By Lemma 12.1 we can find a diagram

$$\begin{tikzcd}
Y \arrow[dr] & Y' \arrow[l, "\pi"'] \arrow[r] \arrow[d] & \mathbf{P}^n_S \arrow[dl] \\
 & S &
\end{tikzcd}$$

where $Y' \to \mathbf{P}^n_S$ is an immersion, and $\pi : Y' \to Y$ is proper and surjective. By Lemma 9.4 we can write $X = \lim X_i$ with $X_i \to Y$ a closed immersion of finite presentation. Denote $X_i' \subset Y'$, resp. $X' \subset Y'$ the scheme theoretic inverse image of $X_i \subset Y$, resp. $X \subset Y$. Then $\lim X_i' = X'$. Since $X' \to S$ is proper (Morphisms, Lemmas 39.4), we see that $X' \to \mathbf{P}^n_S$ is a closed immersion (Morphisms, Lemma 39.7). Hence for i large enough we find that $X_i' \to \mathbf{P}^n_S$ is a closed immersion by Lemma 4.20. Thus X_i' is proper over S. For such i the morphism $X_i \to S$ is proper by Morphisms, Lemma 39.8. □

0A0P **Lemma 13.3.** *Let $f : X \to S$ be a proper morphism with S quasi-compact and quasi-separated. Then there exists a directed set I, an inverse system $(f_i : X_i \to S_i)$ of morphisms of schemes over I, such that the transition morphisms $X_i \to X_{i'}$ and $S_i \to S_{i'}$ are affine, such that f_i is proper, such that S_i is of finite type over $\mathbf{Z}$, and such that $(X \to S) = \lim(X_i \to S_i)$.*

Proof. By Lemma 13.2 we can write $X = \lim_{k \in K} X_k$ with $X_k \to S$ proper and of finite presentation. Next, by absolute Noetherian approximation (Proposition 5.4) we can write $S = \lim_{j \in J} S_j$ with S_j of finite type over $\mathbf{Z}$. For each k there exists a j and a morphism $X_{k,j} \to S_j$ of finite presentation with $X_k \cong S \times_{S_j} X_{k,j}$ as schemes over S, see Lemma 10.1. After increasing j we may assume $X_{k,j} \to S_j$ is proper, see Lemma 13.1. The set I will be consist of these pairs (k, j) and the corresponding morphism is $X_{k,j} \to S_j$. For every $k' \geq k$ we can find a $j' \geq j$ and a morphism $X_{j',k'} \to X_{j,k}$ over $S_{j'} \to S_j$ whose base change to S gives the morphism $X_{k'} \to X_k$ (follows again from Lemma 10.1). These morphisms form the transition morphisms of the system. Some details omitted. □

Recall the scheme theoretic support of a finite type quasi-coherent module, see Morphisms, Definition 5.5.

081G **Lemma 13.4.** *Assumptions and notation as in Situation 8.1. Let $\mathcal{F}_0$ be a quasi-coherent $\mathcal{O}_{X_0}$-module. Denote $\mathcal{F}$ and $\mathcal{F}_i$ the pullbacks of $\mathcal{F}_0$ to X and X_i. Assume*

1. *f_0 is locally of finite type,*
2. *$\mathcal{F}_0$ is of finite type,*
3. *the scheme theoretic support of $\mathcal{F}$ is proper over Y.*

Then the scheme theoretic support of $\mathcal{F}_i$ is proper over Y_i for some i.

Proof. We may replace X_0 by the scheme theoretic support of $\mathcal{F}_0$. By Morphisms, Lemma 5.3 this guarantees that X_i is the support of $\mathcal{F}_i$ and X is the support of $\mathcal{F}$. Then, if $Z \subset X$ denotes the scheme theoretic support of $\mathcal{F}$, we see that $Z \to X$ is a universal homeomorphism. We conclude that $X \to Y$ is proper as this is true for $Z \to Y$ by assumption, see Morphisms, Lemma 39.8. By Lemma 13.1 we see that $X_i \to Y$ is proper for some i. Then it follows that the scheme theoretic support Z_i of $\mathcal{F}_i$ is proper over Y by Morphisms, Lemmas 39.6 and 39.4. □

14. Universally closed morphisms

05JW In this section we discuss when a quasi-compact (but not necessarily separated) morphism is universally closed. We first prove a lemma which will allow us to check universal closedness after a base change which is locally of finite presentation.

05BD **Lemma 14.1.** *Let $f : X \to S$ be a quasi-compact morphism of schemes. Let $g : T \to S$ be a morphism of schemes. Let $t \in T$ be a point and $Z \subset X_T$ be a closed* *subscheme such that $Z \cap X_t = \emptyset$. Then there exists an open neighbourhood $V \subset T$ of t, a commutative diagram*

$$\begin{tikzcd} V \arrow[r, "a"'] \arrow[d] & T' \arrow[d, "b"] \\ T \arrow[r, "g"] & S, \end{tikzcd}$$

and a closed subscheme $Z' \subset X_{T'}$ such that

(1) *the morphism $b : T' \to S$ is locally of finite presentation,*
(2) *with $t' = a(t)$ we have $Z' \cap X_{t'} = \emptyset$, and*
(3) *$Z \cap X_V$ maps into Z' via the morphism $X_V \to X_{T'}$.*

Moreover, we may assume V and T' are affine.

Proof. Let $s = g(t)$. During the proof we may always replace T by an open neighbourhood of t. Hence we may also replace S by an open neighbourhood of s. Thus we may and do assume that T and S are affine. Say $S = \mathrm{Spec}(A)$, $T = \mathrm{Spec}(B)$, g is given by the ring map $A \to B$, and t correspond to the prime ideal $\mathfrak{q} \subset B$.

As $X \to S$ is quasi-compact and S is affine we may write $X = \bigcup_{i=1,\ldots,n} U_i$ as a finite union of affine opens. Write $U_i = \mathrm{Spec}(C_i)$. In particular we have $X_T = \bigcup_{i=1,\ldots,n} U_{i,T} = \bigcup_{i=1,\ldots n} \mathrm{Spec}(C_i \otimes_A B)$. Let $I_i \subset C_i \otimes_A B$ be the ideal corresponding to the closed subscheme $Z \cap U_{i,T}$. The condition that $Z \cap X_t = \emptyset$ signifies that I_i generates the unit ideal in the ring

$$C_i \otimes_A \kappa(\mathfrak{q}) = (B \setminus \mathfrak{q})^{-1} \left(C_i \otimes_A B/\mathfrak{q}C_i \otimes_A B\right)$$

Since $I_i(B \setminus \mathfrak{q})^{-1}(C_i \otimes_A B) = (B \setminus \mathfrak{q})^{-1} I_i$ this means that $1 = x_i/g_i$ for some $x_i \in I_i$ and $g_i \in B$, $g_i \not\in \mathfrak{q}$. Thus, clearing denominators we can find a relation of the form

$$x_i + \sum\nolimits_j f_{i,j} c_{i,j} = g_i$$

with $x_i \in I_i$, $f_{i,j} \in \mathfrak{q}$, $c_{i,j} \in C_i \otimes_A B$, and $g_i \in B$, $g_i \not\in \mathfrak{q}$. After replacing B by $B_{g_1 \ldots g_n}$, i.e., after replacing T by a smaller affine neighbourhood of t, we may assume the equations read

$$x_i + \sum\nolimits_j f_{i,j} c_{i,j} = 1$$

with $x_i \in I_i$, $f_{i,j} \in \mathfrak{q}$, $c_{i,j} \in C_i \otimes_A B$.

To finish the argument write B as a colimit of finitely presented A-algebras B_λ over a directed set Λ. For each λ set $\mathfrak{q}_\lambda = (B_\lambda \to B)^{-1}(\mathfrak{q})$. For sufficiently large $\lambda \in \Lambda$ we can find

(1) an element $x_{i,\lambda} \in C_i \otimes_A B_\lambda$ which maps to x_i,
(2) elements $f_{i,j,\lambda} \in \mathfrak{q}_{i,\lambda}$ mapping to $f_{i,j}$, and
(3) elements $c_{i,j,\lambda} \in C_i \otimes_A B_\lambda$ mapping to $c_{i,j}$.

After increasing λ a bit more the equation

$$x_{i,\lambda} + \sum\nolimits_j f_{i,j,\lambda} c_{i,j,\lambda} = 1$$

will hold. Fix such a λ and set $T' = \mathrm{Spec}(B_\lambda)$. Then $t' \in T'$ is the point corresponding to the prime $\mathfrak{q}_\lambda$. Finally, let $Z' \subset X_{T'}$ be the scheme theoretic image of $Z \to X_T \to X_{T'}$. As $X_T \to X_{T'}$ is affine, we can compute Z' on the affine open pieces $U_{i,T'}$ as the closed subscheme associated to $\mathrm{Ker}(C_i \otimes_A B_\lambda \to C_i \otimes_A B/I_i)$, see Morphisms, Example 6.4. Hence $x_{i,\lambda}$ is in the ideal defining Z'. Thus the last displayed equation shows that $Z' \cap X_{t'}$ is empty. $\square$

05JX **Lemma 14.2.** *Let $f : X \to S$ be a quasi-compact morphism of schemes. The following are equivalent*

(1) *f is universally closed,*
(2) *for every morphism $S' \to S$ which is locally of finite presentation the base change $X_{S'} \to S'$ is closed, and*
(3) *for every n the morphism $\mathbf{A}^n \times X \to \mathbf{A}^n \times S$ is closed.*

Proof. It is clear that (1) implies (2). Let us prove that (2) implies (1). Suppose that the base change $X_T \to T$ is not closed for some scheme T over S. By Schemes, Lemma 19.8 this means that there exists some specialization $t_1 \rightsquigarrow t$ in T and a point $\xi \in X_T$ mapping to t_1 such that ξ does not specialize to a point in the fibre over t. Set $Z = \overline{\{\xi\}} \subset X_T$. Then $Z \cap X_t = \emptyset$. Apply Lemma 14.1. We find an open neighbourhood $V \subset T$ of t, a commutative diagram

$$\begin{tikzcd} V \arrow[r, "a"'] \arrow[d] & T' \arrow[d, "b"] \\ T \arrow[r, "g"] & S, \end{tikzcd}$$

and a closed subscheme $Z' \subset X_{T'}$ such that

(1) the morphism $b : T' \to S$ is locally of finite presentation,
(2) with $t' = a(t)$ we have $Z' \cap X_{t'} = \emptyset$, and
(3) $Z \cap X_V$ maps into Z' via the morphism $X_V \to X_{T'}$.

Clearly this means that $X_{T'} \to T'$ maps the closed subset Z' to a subset of T' which contains $a(t_1)$ but not $t' = a(t)$. Since $a(t_1) \rightsquigarrow a(t) = t'$ we conclude that $X_{T'} \to T'$ is not closed. Hence we have shown that $X \to S$ not universally closed implies that $X_{T'} \to T'$ is not closed for some $T' \to S$ which is locally of finite presentation. In order words (2) implies (1).

Assume that $\mathbf{A}^n \times X \to \mathbf{A}^n \times S$ is closed for every integer n. We want to prove that $X_T \to T$ is closed for every scheme T which is locally of finite presentation over S. We may of course assume that T is affine and maps into an affine open V of S (since $X_T \to T$ being a closed is local on T). In this case there exists a closed immersion $T \to \mathbf{A}^n \times V$ because $\mathcal{O}_T(T)$ is a finitely presented $\mathcal{O}_S(V)$-algebra, see Morphisms, Lemma 20.2. Then $T \to \mathbf{A}^n \times S$ is a locally closed immersion. Hence we get a cartesian diagram

$$\begin{tikzcd} X_T \arrow[r] \arrow[d, "f_T"'] & \mathbf{A}^n \times X \arrow[d, "f_n"] \\ T \arrow[r] & \mathbf{A}^n \times S \end{tikzcd}$$

of schemes where the horizontal arrows are locally closed immersions. Hence any closed subset $Z \subset X_T$ can be written as $X_T \cap Z'$ for some closed subset $Z' \subset \mathbf{A}^n \times X$. Then $f_T(Z) = T \cap f_n(Z')$ and we see that if f_n is closed, then also f_T is closed. $\square$

0205 **Lemma 14.3.** *Let S be a scheme. Let $f : X \to S$ be a separated morphism of finite type. The following are equivalent:*

(1) *The morphism f is proper.*

(2) *For any morphism $S' \to S$ which is locally of finite type the base change $X_{S'} \to S'$ is closed.*
(3) *For every $n \geq 0$ the morphism $\mathbf{A}^n \times X \to \mathbf{A}^n \times S$ is closed.*

First proof. In view of the fact that a proper morphism is the same thing as a separated, finite type, and universally closed morphism, this lemma is a special case of Lemma 14.2. $\square$

Second proof. Clearly (1) implies (2), and (2) implies (3), so we just need to show (3) implies (1). First we reduce to the case when S is affine. Assume that (3) implies (1) when the base is affine. Now let $f : X \to S$ be a separated morphism of finite type. Being proper is local on the base (see Morphisms, Lemma 39.3), so if $S = \bigcup_\alpha S_\alpha$ is an open affine cover, and if we denote $X_\alpha := f^{-1}(S_\alpha)$, then it is enough to show that $f|_{X_\alpha} : X_\alpha \to S_\alpha$ is proper for all α. Since S_α is affine, if the map $f|_{X_\alpha}$ satisfies (3), then it will satisfy (1) by assumption, and will be proper. To finish the reduction to the case S is affine, we must show that if $f : X \to S$ is separated of finite type satisfying (3), then $f|_{X_\alpha} : X_\alpha \to S_\alpha$ is separated of finite type satisfying (3). Separatedness and finite type are clear. To see (3), notice that $\mathbf{A}^n \times X_\alpha$ is the open preimage of $\mathbf{A}^n \times S_\alpha$ under the map $1 \times f$. Fix a closed set $Z \subset \mathbf{A}^n \times X_\alpha$. Let $\bar{Z}$ denote the closure of Z in $\mathbf{A}^n \times X$. Then for topological reasons,

$$1 \times f(\bar{Z}) \cap \mathbf{A}^n \times S_\alpha = 1 \times f(Z).$$

Hence $1 \times f(Z)$ is closed, and we have reduced the proof of (3) $\Rightarrow$ (1) to the affine case.

Assume S affine, and $f : X \to S$ separated of finite type. We can apply Chow's Lemma 12.1 to get $\pi : X' \to X$ proper surjective and $X' \to \mathbf{P}^n_S$ an immersion. If X is proper over S, then $X' \to S$ is proper (Morphisms, Lemma 39.4). Since $\mathbf{P}^n_S \to S$ is separated, we conclude that $X' \to \mathbf{P}^n_S$ is proper (Morphisms, Lemma 39.7) and hence a closed immersion (Schemes, Lemma 10.4). Conversely, assume $X' \to \mathbf{P}^n_S$ is a closed immersion. Consider the diagram:

05LZ (14.3.1)
$$\begin{tikzcd} X' \arrow[r] \arrow[d, "\pi"'] & \mathbf{P}^n_S \arrow[d] \\ X \arrow[r, "f"] & S \end{tikzcd}$$

All maps are a priori proper except for $X \to S$. Hence we conclude that $X \to S$ is proper by Morphisms, Lemma 39.8. Therefore, we have shown that $X \to S$ is proper if and only if $X' \to \mathbf{P}^n_S$ is a closed immersion.

Assume S is affine and (3) holds, and let n, X', π be as above. Since being a closed morphism is local on the base, the map $X \times \mathbf{P}^n \to S \times \mathbf{P}^n$ is closed since by (3) $X \times \mathbf{A}^n \to S \times \mathbf{A}^n$ is closed and since projective space is covered by copies of affine n-space, see Constructions, Lemma 13.3. By Morphisms, Lemma 39.5 the morphism

$$X' \times_S \mathbf{P}^n_S \to X \times_S \mathbf{P}^n_S = X \times \mathbf{P}^n$$

is proper. Since $\mathbf{P}^n$ is separated, the projection

$$X' \times_S \mathbf{P}^n_S = \mathbf{P}^n_{X'} \to X'$$

will be separated as it is just a base change of a separated morphism. Therefore, the map $X' \to X' \times_S \mathbf{P}^n_S$ is proper, since it is a section to a separated map (see Schemes, Lemma 21.12). Composing these morphisms

$$X' \to X' \times_S \mathbf{P}^n_S \to X \times_S \mathbf{P}^n_S = X \times \mathbf{P}^n \to S \times \mathbf{P}^n = \mathbf{P}^n_S$$

we find that the immersion $X' \to \mathbf{P}^n_S$ is closed, and hence a closed immersion. $\square$

15. Noetherian valuative criterion

0CM1 If the base is Noetherian we can show that the valuative criterion holds using only discrete valuation rings.

Many of the results in this section can (and perhaps should) be proved by appealing to the following lemma, although we have not always done so.

0CM2 **Lemma 15.1.** *Let $f : X \to Y$ be a morphism of schemes. Assume f finite type and Y locally Noetherian. Let $y \in Y$ be a point in the closure of the image of f. Then there exists a commutative diagram*

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \mathrm{Spec}(A) \arrow[r] & Y \end{tikzcd}$$

where A is a discrete valuation ring and K is its field of fractions mapping the closed point of $\mathrm{Spec}(A)$ *to y. Moreover, we can assume that the image point of* $\mathrm{Spec}(K) \to X$ *is a generic point η of an irreducible component of X and that* $K = \kappa(\eta)$.

Proof. By the non-Noetherian version of this lemma (Morphisms, Lemma 6.5) there exists a point $x \in X$ such that $f(x)$ specializes to y. We may replace x by any point specializing to x, hence we may assume that x is a generic point of an irreducible component of X. This produces a ring map $\mathcal{O}_{Y,y} \to \kappa(x)$ (see Schemes, Section 13). Let $R \subset \kappa(x)$ be the image. Then R is Noetherian as a quotient of the Noetherian local ring $\mathcal{O}_{Y,y}$. On the other hand, the extension $\kappa(x)$ is a finitely generated extension of the fraction field of R as f is of finite type. Thus there exists a discrete valuation ring $A \subset \kappa(x)$ with fraction field $\kappa(x)$ dominating R by Algebra, Lemma 118.13. Then

$$\begin{tikzcd} \mathrm{Spec}(\kappa(x)) \arrow[rrr] \arrow[d] & & & X \arrow[d] \\ \mathrm{Spec}(A) \arrow[r] & \mathrm{Spec}(R) \arrow[r] & \mathrm{Spec}(\mathcal{O}_{Y,y}) \arrow[r] & Y \end{tikzcd}$$

gives the desired diagram. $\square$

First we state the result concerning separation. We will often use solid commutative diagrams of morphisms of schemes having the following shape

0206 (15.1.1)
$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r] \arrow[d] & X \arrow[d] \\ \mathrm{Spec}(A) \arrow[r] \arrow[ur, dashed] & S \end{tikzcd}$$

with A a valuation ring and K its field of fractions.

0207 **Lemma 15.2.** *Let S be a locally Noetherian scheme. Let $f : X \to S$ be a morphism of schemes. Assume f is locally of finite type. The following are equivalent:*

1. *The morphism f is separated.*
2. *For any diagram (15.1.1) there is at most one dotted arrow.*
3. *For all diagrams (15.1.1) with A a discrete valuation ring there is at most one dotted arrow.*
4. *For any irreducible component X_0 of X with generic point $\eta \in X_0$, for any discrete valuation ring $A \subset K = \kappa(\eta)$ with fraction field K and any diagram (15.1.1) such that the morphism $\mathrm{Spec}(K) \to X$ is the canonical one (see Schemes, Section 13) there is at most one dotted arrow.*

Proof. Clearly (1) implies (2), (2) implies (3), and (3) implies (4). It remains to show (4) implies (1). Assume (4). We begin by reducing to S affine. Being separated is a local on the base (see Schemes, Lemma 21.8). Hence, if we can show that whenever $X \to S$ has (4) that the restriction $X_\alpha \to S_\alpha$ has (4) where $S_\alpha \subset S$ is an (affine) open subset and $X_\alpha := f^{-1}(S_\alpha)$, then we will be done. The generic points of the irreducible components of X_α will be the generic points of irreducible components of X, since X_α is open in X. Therefore, any two distinct dotted arrows in the diagram

05M0 (15.2.1)
$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r] \arrow[d] & X_\alpha \arrow[d] \\ \mathrm{Spec}(A) \arrow[r] \arrow[ur, dashed] & S_\alpha \end{tikzcd}$$

would then give two distinct arrows in diagram (15.1.1) via the maps $X_\alpha \to X$ and $S_\alpha \to S$, which is a contradiction. Thus we have reduced to the case S is affine. We remark that in the course of this reduction, we prove that if $X \to S$ has (4) then the restriction $U \to V$ has (4) for opens $U \subset X$ and $V \subset S$ with $f(U) \subset V$.

We next wish to reduce to the case $X \to S$ is finite type. Assume that we know (4) implies (1) when X is finite type. Since S is Noetherian and X is locally of finite type over S we see X is locally Noetherian as well (see Morphisms, Lemma 14.6). Thus, $X \to S$ is quasi-separated (see Properties, Lemma 5.4), and therefore we may apply the valuative criterion to check whether X is separated (see Schemes, Lemma 22.2). Let $X = \bigcup_\alpha X_\alpha$ be an affine open cover of X. Given any two dotted arrows, in a diagram (15.1.1), the image of the closed points of $\mathrm{Spec}\, A$ will fall in two sets X_α and X_β. Since $X_\alpha \cup X_\beta$ is open, for topological reasons it must contain the image of $\mathrm{Spec}(A)$ under both maps. Therefore, the two dotted arrows factor through $X_\alpha \cup X_\beta \to X$, which is a scheme of finite type over S. Since $X_\alpha \cup X_\beta$ is an open subset of X, by our previous remark, $X_\alpha \cup X_\beta$ satisfies (4), so by assumption, is separated. This implies the two given dotted arrows are the same. Therefore, we have reduced to $X \to S$ is finite type.

Assume $X \to S$ of finite type and assume (4). Since $X \to S$ is finite type, and S is an affine Noetherian scheme, X is also Noetherian (see Morphisms, Lemma 14.6). Therefore, $X \to X \times_S X$ will be a quasi-compact immersion of Noetherian schemes. We proceed by contradiction. Assume that $X \to X \times_S X$ is not closed. Then, there is some $y \in X \times_S X$ in the closure of the image that is not in the image. As X is Noetherian it has finitely many irreducible components. Therefore, y is in the closure of the image of one of the irreducible components $X_0 \subset X$. Give X_0 the reduced induced structure. The composition $X_0 \to X \to X \times_S X$ factors through the closed subscheme $X_0 \times_S X_0 \subset X \times_S X$. Denote the closure of $\Delta(X_0)$ in $X_0 \times_S X_0$ by $\bar{X}_0$ (again as a reduced closed subscheme). Thus $y \in \bar{X}_0$. Since $X_0 \to X_0 \times_S X_0$ is an immersion, the image of X_0 will be open in $\bar{X}_0$. Hence X_0 and $\bar{X}_0$ are birational. Since $\bar{X}_0$ is a closed subscheme of a Noetherian scheme, it is Noetherian. Thus, the local ring $\mathcal{O}_{\bar{X}_0,y}$ is a local Noetherian domain with fraction field K equal to the function field of X_0. By the Krull-Akizuki theorem (see Algebra, Lemma 118.13), there exists a discrete valuation ring A dominating $\mathcal{O}_{\bar{X}_0,y}$ with fraction field K. This allows to construct a diagram:

05M1 (15.2.2)
$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r] \arrow[d] & X_0 \arrow[d, "\Delta"] \\ \mathrm{Spec}(A) \arrow[r] \arrow[ur, dashed] & X_0 \times_S X_0 \end{tikzcd}$$

which sends $\mathrm{Spec}\, K$ to the generic point of $\Delta(X_0)$ and the closed point of A to $y \in X_0 \times_S X_0$ (use the material in Schemes, Section 13 to construct the arrows). There cannot even exist a set theoretic dotted arrow, since y is not in the image of Δ by our choice of y. By categorical means, the existence of the dotted arrow in the above diagram is equivalent to the uniqueness of the dotted arrow in the following diagram:

05M2 (15.2.3)
$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r] \arrow[d] & X_0 \arrow[d] \\ \mathrm{Spec}(A) \arrow[r] \arrow[ur, dashed] & S \end{tikzcd}$$

Therefore, we have non-uniqueness in this latter diagram by the nonexistence in the first. Therefore, X_0 does not satisfy uniqueness for discrete valuation rings, and since X_0 is an irreducible component of X, we have that $X \to S$ does not satisfy (4). Therefore, we have shown (4) implies (1). $\square$

0208 **Lemma 15.3.** *Let S be a locally Noetherian scheme. Let $f : X \to S$ be a morphism of finite type. The following are equivalent:*

1. *The morphism f is proper.*
2. *For any diagram (15.1.1) there exists exactly one dotted arrow.*
3. *For all diagrams (15.1.1) with A a discrete valuation ring there exists exactly one dotted arrow.*
4. *For any irreducible component X_0 of X with generic point $\eta \in X_0$, for any discrete valuation ring $A \subset K = \kappa(\eta)$ with fraction field K and any diagram (15.1.1) such that the morphism $\mathrm{Spec}(K) \to X$ is the canonical one (see Schemes, Section 13) there exists exactly one dotted arrow.*

Proof. (1) implies (2) implies (3) implies (4). We will now show (4) implies (1). As in the proof of Lemma 15.2, we can reduce to the case S is affine, since properness is local on the base, and if $X \to S$ satisfies (4), then $X_\alpha \to S_\alpha$ does as well for open $S_\alpha \subset S$ and $X_\alpha = f^{-1}(S_\alpha)$.

Now S is a Noetherian scheme, and so X is as well, since $X \to S$ is of finite type. Now we may use Chow's lemma (Cohomology of Schemes, Lemma 18.1) to get a surjective, proper, birational $X' \to X$ and an immersion $X' \to \mathbf{P}^n_S$. We wish to show $X \to S$ is universally closed. As in the proof of Lemma 14.3, it is enough to check that $X' \to \mathbf{P}^n_S$ is a closed immersion. For the sake of contradiction, assume that $X' \to \mathbf{P}^n_S$ is not a closed immersion. Then there is some $y \in \mathbf{P}^n_S$ that is in the closure of the image of X', but is not in the image. So y is in the closure of the image of an irreducible component X'_0 of X', but not in the image. Let $\bar{X}'_0 \subset \mathbf{P}^n_S$ be the closure of the image of X'_0. As $X' \to \mathbf{P}^n_S$ is an immersion of Noetherian schemes, the morphism $X'_0 \to \bar{X}'_0$ is open and dense. By Algebra, Lemma 118.13 or Properties, Lemma 5.10 we can find a discrete valuation ring A dominating $\mathcal{O}_{\bar{X}'_0,y}$ and with identical field of fractions K. It is clear that K is the residue field at the generic point of X'_0. Thus the solid commutative diagram

05M3 (15.3.1)
$$\begin{tikzcd}
\operatorname{Spec} K \arrow[r] \arrow[d] & X' \arrow[r] \arrow[d] & \mathbf{P}^n_S \arrow[d] \\
\operatorname{Spec} A \arrow[r, dashed] \arrow[ur, dashed] \arrow[urr] & X \arrow[r] & S
\end{tikzcd}$$

Note that the closed point of A maps to $y \in \mathbf{P}^n_S$. By construction, there does not exist a set theoretic lift to X'. As $X' \to X$ is birational, the image of X'_0 in X is an irreducible component X_0 of X and K is also identified with the function field of X_0. Hence, as $X \to S$ is assumed to satisfy (4), the dotted arrow $\text{Spec}(A) \to X$ exists. Since $X' \to X$ is proper, the dotted arrow lifts to the dotted arrow $\text{Spec}(A) \to X'$ (use Schemes, Proposition 20.6). We can compose this with the immersion $X' \to \mathbf{P}^n_S$ to obtain another morphism (not depicted in the diagram) from $\text{Spec}(A) \to \mathbf{P}^n_S$. Since $\mathbf{P}^n_S$ is proper over S, it satisfies (2), and so these two morphisms agree. This is a contradiction, for we have constructed the forbidden lift of our original map $\text{Spec}(A) \to \mathbf{P}^n_S$ to X'. $\square$

05JY **Lemma 15.4.** *Let $f : X \to S$ be a finite type morphism of schemes. Assume S is locally Noetherian. Then the following are equivalent*

1. *f is universally closed,*
2. *for every n the morphism $\mathbf{A}^n \times X \to \mathbf{A}^n \times S$ is closed,*
3. *for any diagram (15.1.1) there exists some dotted arrow,*
4. *for all diagrams (15.1.1) with A a discrete valuation ring there exists some dotted arrow.*

Proof. The equivalence of (1) and (2) is a special case of Lemma 14.2. The equivalence of (1) and (3) is a special case of Schemes, Proposition 20.6. Trivially (3) implies (4). Thus all we have to do is prove that (4) implies (2). We will prove that $\mathbf{A}^n \times X \to \mathbf{A}^n \times S$ is closed by the criterion of Schemes, Lemma 19.8. Pick n and a specialization $z \leadsto z'$ of points in $\mathbf{A}^n \times S$ and a point $y \in \mathbf{A}^n \times X$ lying over z. Note that $\kappa(y)$ is a finitely generated field extension of $\kappa(z)$ as $\mathbf{A}^n \times X \to \mathbf{A}^n \times S$ is of finite type. Hence by Properties, Lemma 5.10 or Algebra, Lemma 118.13 implies that there exists a discrete valuation ring $A \subset \kappa(y)$ with fraction field $\kappa(z)$

dominating the image of $\mathcal{O}_{\mathbf{A}^n\times S,z'}$ in $\kappa(z)$. This gives a commutative diagram

$$\begin{tikzcd}
\mathrm{Spec}(\kappa(y)) \arrow[r] \arrow[d] & \mathbf{A}^n \times X \arrow[r] \arrow[d] & X \arrow[d] \\
\mathrm{Spec}(A) \arrow[r] & \mathbf{A}^n \times S \arrow[r] & S
\end{tikzcd}$$

Now property (4) implies that there exists a morphism $\mathrm{Spec}(A) \to X$ which fits into this diagram. Since we already have the morphism $\mathrm{Spec}(A) \to \mathbf{A}^n$ from the left lower horizontal arrow we also get a morphism $\mathrm{Spec}(A) \to \mathbf{A}^n \times X$ fitting into the left square. Thus the image $y' \in \mathbf{A}^n \times X$ of the closed point is a specialization of y lying over z'. This proves that specializations lift along $\mathbf{A}^n \times X \to \mathbf{A}^n \times S$ and we win. $\square$

In the Noetherian case one usually does not have to consider all possible diagrams with discrete valuation rings when testing the valuative criteria. We have already seen this in Lemmas 15.2 and 15.3. Here is another variant.

0CM3 **Lemma 15.5.** *Let $f : X \to S$ and $h : U \to X$ be morphisms of schemes. Assume that S is locally Noetherian, that f and h are of finite type, that f is separated, and that $h(U)$ is dense in X. If given any commutative solid diagram*

$$\begin{tikzcd}
\mathrm{Spec}(K) \arrow[r] \arrow[d] & U \arrow[r, "h"] & X \arrow[d, "f"] \\
\mathrm{Spec}(A) \arrow[rr] \arrow[urr, dashed] & & S
\end{tikzcd}$$

where A is a discrete valuation ring with field of fractions K, there exists a dotted arrow making the diagram commute, then f is proper.

Proof. There is an immediate reduction to the case where S is affine. Then U is quasi-compact. Let $U = U_1 \cup \ldots \cup U_n$ be an affine open covering. We may replace U by $U_1 \amalg \ldots \amalg U_n$ without changing the assumptions, hence we may assume U is affine. Thus we can find an open immersion $U \to Y$ over X with Y proper over X. (First put U inside $\mathbf{A}^n_X$ using Morphisms, Lemma 37.2 and then take the closure inside $\mathbf{P}^n_X$, or you can directly use Morphisms, Lemma 41.12.) We can assume U is dense in Y (replace Y by the scheme theoretic closure of U if necessary, see Morphisms, Section 7). Note that $g : Y \to X$ is surjective as the image is closed and contains the dense subset $h(U)$. We will show that $Y \to S$ is proper. This will imply that $X \to S$ is proper by Morphisms, Lemma 39.8 thereby finishing the proof. To show that $Y \to S$ is proper we will use part (4) of Lemma 15.3. To do this consider a diagram

$$\begin{tikzcd}
\mathrm{Spec}(K) \arrow[r, "y"'] \arrow[d] & Y \arrow[d, "f\circ g"] \\
\mathrm{Spec}(A) \arrow[r] \arrow[ur, dotted] & S
\end{tikzcd}$$

where A is a discrete valuation ring with fraction field K and where $y : \mathrm{Spec}(K) \to Y$ is the inclusion of a generic point. We have to show there exists a unique dotted arrow. Uniqueness holds by the converse to the valuative criterion for separatedness (Schemes, Lemma 22.1) since $Y \to S$ is separated as the composition of the separated morphisms $Y \to X$ and $X \to S$ (Schemes, Lemma 21.13). Existence can be seen as follows. As y is a generic point of Y, it is contained in U. By assumption of the lemma there exists a morphism $a : \mathrm{Spec}(A) \to X$ such that

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r, "y"'] \arrow[d] & U \arrow[r] & X \arrow[d, "f"] \\ \mathrm{Spec}(A) \arrow[rr] \arrow[urr, "a"] & & S \end{tikzcd}$$

is commutative. Then since $Y \to X$ is proper, we can apply the valuative criterion for properness (Morphisms, Lemma 40.1) to find a morphism $b : \mathrm{Spec}(A) \to Y$ such that

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r, "y"'] \arrow[d] & Y \arrow[d, "g"] \\ \mathrm{Spec}(A) \arrow[r, "a"] \arrow[ur, "b"] & X \end{tikzcd}$$

is commutative. This finishes the proof since b can serve as the dotted arrow above. $\square$

0CM4 **Lemma 15.6.** *Let $f : X \to S$ and $h : U \to X$ be morphisms of schemes. Assume that S is locally Noetherian, that f is locally of finite type, that h is of finite type, and that $h(U)$ is dense in X. If given any commutative solid diagram*

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r] \arrow[d] & U \arrow[r, "h"] & X \arrow[d, "f"] \\ \mathrm{Spec}(A) \arrow[rr] \arrow[urr, dashed] & & S \end{tikzcd}$$

where A is a discrete valuation ring with field of fractions K, there exists at most one dotted arrow making the diagram commute, then f is separated.

Proof. We will apply Lemma 15.5 to the morphisms $U \to X$ and $\Delta : X \to X \times_S X$. We check the conditions. Observe that Δ is quasi-compact by Properties, Lemma 5.4 (and Schemes, Lemma 21.14). Of course Δ is locally of finite type and separated (true for any diagonal morphism). Finally, suppose given a commutative solid diagram

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r] \arrow[d] & U \arrow[r, "h"] & X \arrow[d, "\Delta"] \\ \mathrm{Spec}(A) \arrow[rr, "(a,b)"] \arrow[urr, dashed] & & X \times_S X \end{tikzcd}$$

where A is a discrete valuation ring with field of fractions K. Then a and b give two dotted arrows in the diagram of the lemma and have to be equal. Hence as dotted arrow we can use $a = b$ which gives existence. This finishes the proof. $\square$

0CM5 **Lemma 15.7.** *Let $f : X \to S$ and $h : U \to X$ be morphisms of schemes. Assume that S is locally Noetherian, that f and h are of finite type, and that $h(U)$ is dense* *in X. If given any commutative solid diagram*

$$\begin{tikzcd}
\mathrm{Spec}(K) \arrow[r] \arrow[d] & U \arrow[r, "h"] & X \arrow[d, "f"] \\
\mathrm{Spec}(A) \arrow[rr] \arrow[urr, dashed] & & S
\end{tikzcd}$$

where A is a discrete valuation ring with field of fractions K, there exists a unique dotted arrow making the diagram commute, then f is proper.

Proof. Combine Lemmas 15.6 and 15.5. $\square$

16. Limits and dimensions of fibres

05M4 The following lemma is most often used in the situation of Lemma 10.1 to assure that if the fibres of the limit have dimension $\leq d$, then the fibres at some finite stage have dimension $\leq d$.

05M5 **Lemma 16.1.** *Let I be a directed set. Let $(f_i : X_i \to S_i)$ be an inverse system of morphisms of schemes over I. Assume*

1. *all the morphisms $S_{i'} \to S_i$ are affine,*
2. *all the schemes S_i are quasi-compact and quasi-separated,*
3. *the morphisms f_i are of finite type, and*
4. *the morphisms $X_{i'} \to X_i \times_{S_i} S_{i'}$ are closed immersions.*

Let $f : X = \lim_i X_i \to S = \lim_i S_i$ be the limit. Let $d \geq 0$. If every fibre of f has dimension $\leq d$, then for some i every fibre of f_i has dimension $\leq d$.

Proof. For each i let $U_i = \{x \in X_i \mid \dim_x((X_i)_{f_i(x)}) \leq d\}$. This is an open subset of X_i, see Morphisms, Lemma 27.4. Set $Z_i = X_i \setminus U_i$ (with reduced induced scheme structure). We have to show that $Z_i = \emptyset$ for some i. If not, then $Z = \lim Z_i \neq \emptyset$, see Lemma 4.3. Say $z \in Z$ is a point. Note that $Z \subset X$ is a closed subscheme. Set $s = f(z)$. For each i let $s_i \in S_i$ be the image of s. We remark that Z_s is the limit of the schemes $(Z_i)_{s_i}$ and Z_s is also the limit of the schemes $(Z_i)_{s_i}$ base changed to $\kappa(s)$. Moreover, all the morphisms

$$Z_s \longrightarrow (Z_{i'})_{s_{i'}} \times_{\mathrm{Spec}(\kappa(s_{i'}))} \mathrm{Spec}(\kappa(s)) \longrightarrow (Z_i)_{s_i} \times_{\mathrm{Spec}(\kappa(s_i))} \mathrm{Spec}(\kappa(s)) \longrightarrow X_s$$

are closed immersions by assumption (4). Hence Z_s is the scheme theoretic intersection of the closed subschemes $(Z_i)_{s_i} \times_{\mathrm{Spec}(\kappa(s_i))} \mathrm{Spec}(\kappa(s))$ in X_s. Since all the irreducible components of the schemes $(Z_i)_{s_i} \times_{\mathrm{Spec}(\kappa(s_i))} \mathrm{Spec}(\kappa(s))$ have dimension $> d$ and contain z we conclude that Z_s contains an irreducible component of dimension $> d$ passing through z which contradicts the fact that $Z_s \subset X_s$ and $\dim(X_s) \leq d$. $\square$

094M **Lemma 16.2.** *Notation and assumptions as in Situation 8.1. If*

1. *f is a quasi-finite morphism, and*
2. *f_0 is locally of finite type,*

then there exists an $i \geq 0$ such that f_i is quasi-finite.

Proof. Follows immediately from Lemma 16.1. $\square$

05M6 **Lemma 16.3.** *Let S be a quasi-compact and quasi-separated scheme. Let $f : X \to S$ be a morphism of finite presentation. Let $d \geq 0$ be an integer. If $Z \subset X$ be a closed subscheme such that $\dim(Z_s) \leq d$ for all $s \in S$, then there exists a closed subscheme $Z' \subset X$ such that*

1. *$Z \subset Z'$,*
2. *$Z' \to X$ is of finite presentation, and*
3. *$\dim(Z'_s) \leq d$ for all $s \in S$.*

Proof. By Proposition 5.4 we can write $S = \lim S_i$ as the limit of a directed inverse system of Noetherian schemes with affine transition maps. By Lemma 10.1 we may assume that there exist a system of morphisms $f_i : X_i \to S_i$ of finite presentation such that $X_{i'} = X_i \times_{S_i} S_{i'}$ for all $i' \geq i$ and such that $X = X_i \times_{S_i} S$. Let $Z_i \subset X_i$ be the scheme theoretic image of $Z \to X \to X_i$. Then for $i' \geq i$ the morphism $X_{i'} \to X_i$ maps $Z_{i'}$ into Z_i and the induced morphism $Z_{i'} \to Z_i \times_{S_i} S_{i'}$ is a closed immersion. By Lemma 16.1 we see that the dimension of the fibres of $Z_i \to S_i$ all have dimension $\leq d$ for a suitable $i \in I$. Fix such an i and set $Z' = Z_i \times_{S_i} S \subset X$. Since S_i is Noetherian, we see that X_i is Noetherian, and hence the morphism $Z_i \to X_i$ is of finite presentation. Therefore also the base change $Z' \to X$ is of finite presentation. Moreover, the fibres of $Z' \to S$ are base changes of the fibres of $Z_i \to S_i$ and hence have dimension $\leq d$. □

0E7D **Lemma 16.4.** *Let $f : X \to Y$ be a morphism of schemes. Let $y \in Y$. Assume f is proper and $\dim(X_y) = d$. Then for any quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ we have $(R^pf_*\mathcal{F})_y = 0$ for all $p > d$.*

Proof. By Morphisms, Lemma 27.4 and the fact that f is closed, we can find an open neighbourhood V of y such that the fibres over points of V all have dimension $\leq d$. Since the question is local on Y we may assume Y is affine and all fibres of f have dimension $\leq d$. By Lemma 13.2 we can write $X = \lim X_i$ as a cofiltered limit with $X_i \to Y$ proper and of finite presentation and such that both $X \to X_i$ and transition morphisms are closed immersions. For some i we have that $X_i \to Y$ has fibres of dimension $\leq d$, see Lemma 16.1. For a quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ we have $R^pf_*\mathcal{F} = R^pf_{i,*}(X \to X_i)_*\mathcal{F}$ by Cohomology of Schemes, Lemma 2.3 and Leray (Cohomology, Lemma 14.8). Thus we may replace X by X_i and reduce to the case discussed in the next paragraph.

Assume Y is affine and $f : X \to Y$ is proper and of finite presentation and all fibres have dimension $\leq d$. It suffices to show that $H^p(X, \mathcal{F}) = 0$ for $p > d$. Namely, by Cohomology of Schemes, Lemma 4.6 we have $H^p(X, \mathcal{F}) = H^0(Y, R^pf_*\mathcal{F})$. On the other hand, $R^pf_*\mathcal{F}$ is quasi-coherent on Y by Cohomology of Schemes, Lemma 4.5, hence vanishing of global sections implies vanishing. Write $Y = \lim_{i\in I} Y_i$ as a cofiltered limit of affine schemes with Y_i the spectrum of a Noetherian ring (for example a finite type $\mathbf{Z}$-algebra). We can choose an element $0 \in I$ and a finite type morphism $X_0 \to Y_0$ such that $X \cong Y \times_{Y_0} X_0$, see Lemma 10.1. After increasing 0 we may assume $X_0 \to Y_0$ is proper (Lemma 13.1) and that the fibres of $X_0 \to Y_0$ have dimension $\leq d$ (Lemma 16.1). Since $X \to X_0$ is affine, we find that $H^p(X, \mathcal{F}) = H^p(X_0, (X \to X_0)_*\mathcal{F})$ by Cohomology of Schemes, Lemma 2.4. This reduces us to the case discussed in the next paragraph.

Assume Y is affine Noetherian and $f : X \to Y$ is proper and all fibres have dimension $\leq d$. In this case we can write $\mathcal{F} = \operatorname{colim} \mathcal{F}_i$ as a filtered colimit of coherent

$\mathcal{O}_X$-modules, see Properties, Lemma 22.6. Then $H^p(X, \mathcal{F}) = \text{colim}\, H^p(X, \mathcal{F}_i)$ by Cohomology, Lemma 20.1. Thus we may assume $\mathcal{F}$ is coherent. In this case we see that $(R^pf_*\mathcal{F})_y = 0$ for all $y \in Y$ by Cohomology of Schemes, Lemma 20.9. Thus $R^pf_*\mathcal{F} = 0$ and therefore $H^p(X, \mathcal{F}) = 0$ (see above) and we win. □

17. Glueing in closed fibres

0E8P Applying our theory above to the spectrum of a local ring we obtain the following pleasing glueing result for relative schemes.

0BPA **Lemma 17.1.** *Let S be a scheme. Let $s \in S$ be a closed point such that $U = S \setminus \{s\} \to S$ is quasi-compact. With $V = \text{Spec}(\mathcal{O}_{S,s}) \setminus \{s\}$ there is an equivalence of categories*

$$\left\{X \to S \text{ of finite presentation}\right\} \longrightarrow \left\{\begin{tikzcd} X' \arrow[d] & Y' \arrow[l] \arrow[r] \arrow[d] & Y \arrow[d] \\ U & V \arrow[l] \arrow[r] & \text{Spec}(\mathcal{O}_{S,s}) \end{tikzcd}\right\}$$

where on the right hand side we consider commutative diagrams whose squares are cartesian and whose vertical arrows are of finite presentation.

Proof. Let $W \subset S$ be an open neighbourhood of s. By glueing of relative schemes, see Constructions, Section 2, the functor

$$\left\{X \to S \text{ of finite presentation}\right\} \longrightarrow \left\{\begin{tikzcd} X' \arrow[d] & Y' \arrow[l] \arrow[r] \arrow[d] & Y \arrow[d] \\ U & W \setminus \{s\} \arrow[l] \arrow[r] & W \end{tikzcd}\right\}$$

is an equivalence of categories. We have $\mathcal{O}_{S,s} = \text{colim}\, \mathcal{O}_W(W)$ where W runs over the affine open neighbourhoods of s. Hence $\text{Spec}(\mathcal{O}_{S,s}) = \lim W$ where W runs over the affine open neighbourhoods of s. Thus the category of schemes of finite presentation over $\text{Spec}(\mathcal{O}_{S,s})$ is the limit of the category of schemes of finite presentation over W where W runs over the affine open neighbourhoods of s, see Lemma 10.1. For every affine open $s \in W$ we see that $U \cap W$ is quasi-compact as $U \to S$ is quasi-compact. Hence $V = \lim W \cap U = \lim W \setminus \{s\}$ is a limit of quasi-compact and quasi-separated schemes (see Lemma 2.2). Thus also the category of schemes of finite presentation over V is the limit of the categories of schemes of finite presentation over $W \cap U$ where W runs over the affine open neighbourhoods of s. The lemma follows formally from a combination of these results. □

0BQ5 **Lemma 17.2.** *Let S be a scheme. Let $U \subset S$ be a retrocompact open. Let $s \in S$ be a point in the complement of U. With $V = \text{Spec}(\mathcal{O}_{S,s}) \cap U$ there is an equivalence of categories*

$$\text{colim}_{s \in U' \supset U \text{ open}} \left\{\begin{tikzcd} X \arrow[d] \\ U' \end{tikzcd}\right\} \longrightarrow \left\{\begin{tikzcd} X' \arrow[d] & Y' \arrow[l] \arrow[r] \arrow[d] & Y \arrow[d] \\ U & V \arrow[l] \arrow[r] & \text{Spec}(\mathcal{O}_{S,s}) \end{tikzcd}\right\}$$

where on the left hand side the vertical arrow is of finite presentation and on the right hand side we consider commutative diagrams whose squares are cartesian and whose vertical arrows are of finite presentation.

Proof. Let $W \subset S$ be an open neighbourhood of s. By glueing of relative schemes, see Constructions, Section 2, the functor

$$\left\{X \to U' = U \cup W \text{ of finite presentation}\right\} \longrightarrow \left\{\begin{tikzcd} X' \arrow[d] & Y' \arrow[l] \arrow[r] \arrow[d] & Y \arrow[d] \\ U & W \cap U \arrow[l] \arrow[r] & W \end{tikzcd}\right\}$$

is an equivalence of categories. We have $\mathcal{O}_{S,s} = \text{colim}\, \mathcal{O}_W(W)$ where W runs over the affine open neighbourhoods of s. Hence $\text{Spec}(\mathcal{O}_{S,s}) = \lim W$ where W runs over the affine open neighbourhoods of s. Thus the category of schemes of finite presentation over $\text{Spec}(\mathcal{O}_{S,s})$ is the limit of the category of schemes of finite presentation over W where W runs over the affine open neighbourhoods of s, see Lemma 10.1. For every affine open $s \in W$ we see that $U \cap W$ is quasi-compact as $U \to S$ is quasi-compact. Hence $V = \lim W \cap U$ is a limit of quasi-compact and quasi-separated schemes (see Lemma 2.2). Thus also the category of schemes of finite presentation over V is the limit of the categories of schemes of finite presentation over $W \cap U$ where W runs over the affine open neighbourhoods of s. The lemma follows formally from a combination of these results. $\square$

0E8Q **Lemma 17.3.** *Let S be a scheme. Let $s_1, \ldots, s_n \in S$ be pairwise distinct closed points such that $U = S \setminus \{s_1, \ldots, s_n\} \to S$ is quasi-compact. With $S_i = \text{Spec}(\mathcal{O}_{S,s_i})$ and $U_i = S_i \setminus \{s_i\}$ there is an equivalence of categories*

$$FP_S \longrightarrow FP_U \times_{(FP_{U_1} \times \ldots \times FP_{U_n})} (FP_{S_1} \times \ldots \times FP_{S_n})$$

where FP_T is the category of schemes of finite presentation over the scheme T.

Proof. For $n = 1$ this is Lemma 17.1. For $n > 1$ the lemma can be proved in exactly the same way or it can be deduced from it. For example, suppose that $f_i : X_i \to S_i$ are objects of FP_{S_i} and $f : X \to U$ is an object of FP_U and we're given isomorphisms $X_i \times_{S_i} U_i = X \times_U U_i$. By Lemma 17.1 we can find a morphism $f' : X' \to U' = S \setminus \{s_1, \ldots, s_{n-1}\}$ which is of finite presentation, which is isomorphic to X_i over S_i, which is isomorphic to X over U, and these isomorphisms are compatible with the given isomorphism $X_i \times_{S_n} U_n = X \times_U U_n$. Then we can apply induction to $f_i : X_i \to S_i$, $i \leq n-1$, $f' : X' \to U'$, and the induced isomorphisms $X_i \times_{S_i} U_i = X' \times_{U'} U_i$, $i \leq n-1$. This shows essential surjectivity. We omit the proof of fully faithfulness. $\square$

18. Application to modifications

0B3W Using the results from Section 17 we can describe the category of modifications of a scheme over a closed point in terms of the local ring.

0B3X **Lemma 18.1.** *Let S be a scheme. Let $s \in S$ be a closed point such that $U = S \setminus \{s\} \to S$ is quasi-compact. With $V = \text{Spec}(\mathcal{O}_{S,s}) \setminus \{s\}$ the base change functor*

$$\left\{\begin{matrix} f : X \to S \text{ of finite presentation} \\ f^{-1}(U) \to U \text{ is an isomorphism}\end{matrix}\right\} \longrightarrow \left\{\begin{matrix} g : Y \to \text{Spec}(\mathcal{O}_{S,s}) \text{ of finite presentation} \\ g^{-1}(V) \to V \text{ is an isomorphism}\end{matrix}\right\}$$

is an equivalence of categories.

Proof. This is a special case of Lemma 17.1. □

0BFN **Lemma 18.2.** *Notation and assumptions as in Lemma 18.1. Let $f : X \to S$ correspond to $g : Y \to \text{Spec}(\mathcal{O}_{S,s})$ via the equivalence. Then f is separated, proper, finite, and add more here if and only if g is so.*

Proof. The property of being separated, proper, integral, finite, etc is stable under base change. See Schemes, Lemma 21.13 and Morphisms, Lemmas 39.5 and 42.6. Hence if f has the property, then so does g. Conversely, if g does, then f does in a neighbourhood of s by Lemmas 8.6, 13.1, and 8.3. Since f clearly has the given property over $S \setminus \{s\}$ we conclude as one can check the property locally on the base. □

0B3Y **Remark 18.3.** The lemma above can be generalized as follows. Let S be a scheme and let $T \subset S$ be a closed subset. Assume there exists a cofinal system of open neighbourhoods $T \subset W_i$ such that (1) $W_i \setminus T$ is quasi-compact and (2) $W_i \subset W_j$ is an affine morphism. Then $W = \lim W_i$ is a scheme which contains T as a closed subscheme. Set $U = X \setminus T$ and $V = W \setminus T$. Then the base change functor

$$\left\{\begin{matrix} f : X \to S \text{ of finite presentation} \\ f^{-1}(U) \to U \text{ is an isomorphism} \end{matrix}\right\} \longrightarrow \left\{\begin{matrix} g : Y \to W \text{ of finite presentation} \\ g^{-1}(V) \to V \text{ is an isomorphism} \end{matrix}\right\}$$

is an equivalence of categories. If we ever need this we will change this remark into a lemma and provide a detailed proof.

19. Descending finite type schemes

0CNL This section continues the theme of Section 9 in the spirit of the results discussed in Section 10.

0CNM **Situation 19.1.** Let $S = \lim_{i \in I} S_i$ be the limit of a directed system of Noetherian schemes with affine transition morphisms $S_{i'} \to S_i$ for $i' \geq i$.

0CNN **Lemma 19.2.** *In Situation 19.1. Let $X \to S$ be quasi-separated and of finite type. Then there exists an $i \in I$ and a diagram*

0CNP (19.2.1)
$$\begin{tikzcd} X \arrow[r] \arrow[d] & W \arrow[d] \\ S \arrow[r] & S_i \end{tikzcd}$$

such that $W \to S_i$ is of finite type and such that the induced morphism $X \to S \times_{S_i} W$ is a closed immersion.

Proof. By Lemma 9.3 we can find a closed immersion $X \to X'$ over S where X' is a scheme of finite presentation over S. By Lemma 10.1 we can find an i and a morphism of finite presentation $X'_i \to S_i$ whose pull back is X'. Set $W = X'_i$. □

0CNQ **Lemma 19.3.** *In Situation 19.1. Let $X \to S$ be quasi-separated and of finite type. Given $i \in I$ and a diagram*

$$\begin{tikzcd} X \arrow[r] \arrow[d] & W \arrow[d] \\ S \arrow[r] & S_i \end{tikzcd}$$

as in (19.2.1) for $i' \geq i$ let $X_{i'}$ be the scheme theoretic image of $X \to S_{i'} \times_{S_i} W$. Then $X = \lim_{i' \geq i} X_{i'}$.

Proof. Since X is quasi-compact and quasi-separated formation of the scheme theoretic image of $X \to S_{i'} \times_{S_i} W$ commutes with restriction to open subschemes (Morphisms, Lemma 6.3). Hence we may and do assume W is affine and maps into an affine open U_i of S_i. Let $U \subset S$, $U_{i'} \subset S_{i'}$ be the inverse image of U_i. Then U, $U_{i'}$, $S_{i'} \times_{S_i} W = U_{i'} \times_{U_i} W$, and $S \times_{S_i} W = U \times_{U_i} W$ are all affine. This implies X is affine because $X \to S \times_{S_i} W$ is a closed immersion. This also shows the ring map

$$\mathcal{O}(U) \otimes_{\mathcal{O}(U_i)} \mathcal{O}(W) \to \mathcal{O}(X)$$

is surjective. Let I be the kernel. Then we see that $X_{i'}$ is the spectrum of the ring

$$\mathcal{O}(X_{i'}) = \mathcal{O}(U_{i'}) \otimes_{\mathcal{O}(U_i)} \mathcal{O}(W)/I_{i'}$$

where $I_{i'}$ is the inverse image of the ideal I (see Morphisms, Example 6.4). Since $\mathcal{O}(U) = \operatorname{colim} \mathcal{O}(U_{i'})$ we see that $I = \operatorname{colim} I_{i'}$ and we conclude that $\operatorname{colim} \mathcal{O}(X_{i'}) = \mathcal{O}(X)$. $\square$

0CNR **Lemma 19.4.** *In Situation 19.1. Let $f : X \to Y$ be a morphism of schemes quasi-separated and of finite type over S. Let*

$$\begin{tikzcd} X \arrow[r] \arrow[d] & W \arrow[d] \\ S \arrow[r] & S_{i_1} \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} Y \arrow[r] \arrow[d] & V \arrow[d] \\ S \arrow[r] & S_{i_2} \end{tikzcd}$$

be diagrams as in (19.2.1). Let $X = \lim_{i \geq i_1} X_i$ and $Y = \lim_{i \geq i_2} Y_i$ be the corresponding limit descriptions as in Lemma 19.3. Then there exists an $i_0 \geq \max(i_1, i_2)$ and a morphism

$$(f_i)_{i \geq i_0} : (X_i)_{i \geq i_0} \to (Y_i)_{i \geq i_0}$$

of inverse systems over $(S_i)_{i \geq i_0}$ such that such that $f = \lim_{i \geq i_0} f_i$. If $(g_i)_{i \geq i_0} : (X_i)_{i \geq i_0} \to (Y_i)_{i \geq i_0}$ is a second morphism of inverse systems over $(S_i)_{i \geq i_0}$ such that such that $f = \lim_{i \geq i_0} g_i$ then $f_i = g_i$ for all $i \gg i_0$.

Proof. Since $V \to S_{i_2}$ is of finite presentation and $X = \lim_{i \geq i_1} X_i$ we can appeal to Proposition 6.1 to find an $i_0 \geq \max(i_1, i_2)$ and a morphism $h : X_{i_0} \to V$ over S_{i_2} such that $X \to X_{i_0} \to V$ is equal to $X \to Y \to V$. For $i \geq i_0$ we get a commutative solid diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & X_i \arrow[r] \arrow[d, dotted] \arrow[dd, bend right=60] & X_{i_0} \arrow[d, "h"] \\ Y \arrow[r] \arrow[d] & Y_i \arrow[r] \arrow[d] & V \arrow[d] \\ S \arrow[r] & S_i \arrow[r] & S_{i_0} \end{tikzcd}$$

Since $X \to X_i$ has scheme theoretically dense image and since Y_i is the scheme theoretic image of $Y \to S_i \times_{S_{i_2}} V$ we find that the morphism $X_i \to S_i \times_{S_{i_2}} V$ induced by the diagram factors through Y_i (Morphisms, Lemma 6.6). This proves existence.

Uniqueness. Let $E_i \subset X_i$ be the equalizer of f_i and g_i for $i \geq i_0$. By Schemes, Lemma 21.5 E_i is a locally closed subscheme of X_i. Since X_i is a closed subscheme of $S_i \times_{S_{i_0}} X_{i_0}$ and similarly for Y_i we see that

$$E_i = X_i \times_{(S_i \times_{S_{i_0}} X_{i_0})} (S_i \times_{S_{i_0}} E_{i_0})$$

Thus to finish the proof it suffices to show that $X_i \to X_{i_0}$ factors through E_{i_0} for some $i \geq i_0$. To do this we will use that $X \to X_{i_0}$ factors through E_{i_0} as both f_{i_0} and g_{i_0} are compatible with f. Since X_i is Noetherian, we see that the underlying topological space $|E_{i_0}|$ is a constructible subset of $|X_{i_0}|$ (Topology, Lemma 16.1). Hence $X_i \to X_{i_0}$ factors through E_{i_0} set theoretically for large enough i by Lemma 4.10. For such an i the scheme theoretic inverse image $(X_i \to X_{i_0})^{-1}(E_{i_0})$ is a closed subscheme of X_i through which X factors and hence equal to X_i since $X \to X_i$ has scheme theoretically dense image by construction. This concludes the proof. □

0CNS **Remark 19.5.** In Situation 19.1 Lemmas 19.2, 19.3, and 19.4 tell us that the category of schemes quasi-separated and of finite type over S is equivalent to certain types of inverse systems of schemes over $(S_i)_{i \in I}$, namely the ones produced by applying Lemma 19.3 to a diagram of the form (19.2.1). For example, given $X \to S$ finite type and quasi-separated if we choose two different diagrams $X \to V_1 \to S_{i_1}$ and $X \to V_2 \to S_{i_2}$ as in (19.2.1), then applying Lemma 19.4 to id_X (in two directions) we see that the corresponding limit descriptions of X are canonically isomorphic (up to shrinking the directed set I). And so on and so forth.

0CNT **Lemma 19.6.** *Notation and assumptions as in Lemma 19.4. If f is flat and of finite presentation, then there exists an $i_3 \geq i_0$ such that for $i \geq i_3$ we have f_i is flat, $X_i = Y_i \times_{Y_{i_3}} X_{i_3}$, and $X = Y \times_{Y_{i_3}} X_{i_3}$.*

Proof. By Lemma 10.1 we can choose an $i \geq i_2$ and a morphism $U \to Y_i$ of finite presentation such that $X = Y \times_{Y_i} U$ (this is where we use that f is of finite presentation). After increasing i we may assume that $U \to Y_i$ is flat, see Lemma 8.7. As discussed in Remark 19.5 we may and do replace the initial diagram used to define the system $(X_i)_{i \geq i_1}$ by the system corresponding to $X \to U \to S_i$. Thus $X_{i'}$ for $i' \geq i$ is defined as the scheme theoretic image of $X \to S_{i'} \times_{S_i} U$.

Because $U \to Y_i$ is flat (this is where we use that f is flat), because $X = Y \times_{Y_i} U$, and because the scheme theoretic image of $Y \to Y_i$ is Y_i, we see that the scheme theoretic image of $X \to U$ is U (Morphisms, Lemma 24.15). Observe that $Y_{i'} \to S_{i'} \times_{S_i} Y_i$ is a closed immersion for $i' \geq i$ by construction of the system of Y_j. Then the same argument as above shows that the scheme theoretic image of $X \to S_{i'} \times_{S_i} U$ is equal to the closed subscheme $Y_{i'} \times_{Y_i} U$. Thus we see that $X_{i'} = Y_{i'} \times_{Y_i} U$ for all $i' \geq i$ and hence the lemma holds with $i_3 = i$. □

0CNU **Lemma 19.7.** *Notation and assumptions as in Lemma 19.4. If f is smooth, then there exists an $i_3 \geq i_0$ such that for $i \geq i_3$ we have f_i is smooth.*

Proof. Combine Lemmas 19.6 and 8.9. □

0CNV **Lemma 19.8.** *Notation and assumptions as in Lemma 19.4. If f is proper, then there exists an $i_3 \geq i_0$ such that for $i \geq i_3$ we have f_i is proper.*

Proof. By the discussion in Remark 19.5 the choice of i_1 and W fitting into a diagram as in (19.2.1) is immaterial for the truth of the lemma. Thus we choose W as follows. First we choose a closed immersion $X \to X'$ with $X' \to S$ proper and of finite presentation, see Lemma 13.2. Then we choose an $i_3 \geq i_2$ and a proper morphism $W \to Y_{i_3}$ such that $X' = Y \times_{Y_{i_3}} W$. This is possible because $Y = \lim_{i \geq i_2} Y_i$ and Lemmas 10.1 and 13.1. With this choice of W it is immediate from the construction that for $i \geq i_3$ the scheme X_i is a closed subscheme of $Y_i \times_{Y_{i_3}} W \subset S_i \times_{S_{i_3}} W$ and hence proper over Y_i. □

0CNW **Lemma 19.9.** *In Situation 19.1 suppose that we have a cartesian diagram*

$$\begin{tikzcd} X^1 \arrow[r, "p"] \arrow[d, "q"] & X^3 \arrow[d, "a"] \\ X^2 \arrow[r, "b"] & X^4 \end{tikzcd}$$

of schemes quasi-separated and of finite type over S. For each $j = 1, 2, 3, 4$ choose $i_j \in I$ and a diagram

$$\begin{tikzcd} X^j \arrow[r] \arrow[d] & W^j \arrow[d] \\ S \arrow[r] & S_{i_j} \end{tikzcd}$$

as in (19.2.1). Let $X^j = \lim_{i \geq i_j} X_i^j$ be the corresponding limit descriptions as in Lemma 19.4. Let $(a_i)_{i \geq i_5}$, $(b_i)_{i \geq i_6}$, $(p_i)_{i \geq i_7}$, and $(q_i)_{i \geq i_8}$ be the corresponding morphisms of systems contructed in Lemma 19.4. Then there exists an $i_9 \geq \max(i_5, i_6, i_7, i_8)$ such that for $i \geq i_9$ we have $a_i \circ p_i = b_i \circ q_i$ and such that

$$(q_i, p_i) : X_i^1 \longrightarrow X_i^2 \times_{b_i, X_i^4, a_i} X_i^3$$

is a closed immersion. If a and b are flat and of finite presentation, then there exists an $i_{10} \geq \max(i_5, i_6, i_7, i_8, i_9)$ such that for $i \geq i_{10}$ the last displayed morphism is an isomorphism.

Proof. According to the discussion in Remark 19.5 the choice of W^1 fitting into a diagram as in (19.2.1) is immaterial for the truth of the lemma. Thus we may choose $W^1 = W^2 \times_{W^4} W^3$. Then it is immediate from the construction of X_i^1 that $a_i \circ p_i = b_i \circ q_i$ and that

$$(q_i, p_i) : X_i^1 \longrightarrow X_i^2 \times_{b_i, X_i^4, a_i} X_i^3$$

is a closed immersion.

If a and b are flat and of finite presentation, then so are p and q as base changes of a and b. Thus we can apply Lemma 19.6 to each of a, b, p, q, and $a \circ p = b \circ q$. It follows that there exists an $i_9 \in I$ such that

$$(q_i, p_i) : X_i^1 \to X_i^2 \times_{X_i^4} X_i^3$$

is the base change of (q_{i_9}, p_{i_9}) by the morphism by the morphism $X_i^4 \to X_{i_9}^4$ for all $i \geq i_9$. We conclude that (q_i, p_i) is an isomorphism for all sufficiently large i by Lemma 8.11. □

20. Other chapters

References

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