1. Introduction

In this chapter we put material related to limits of schemes. We mostly study limits of inverse systems over directed sets (Categories, Definition 21.1) with affine transition maps. We discuss absolute Noetherian approximation. We characterize schemes locally of finite presentation over a base as those whose associated functor of points is limit preserving. As an application of absolute Noetherian approximation we prove that the image of an affine under an integral morphism is affine. Moreover, we prove some very general variants of Chow’s lemma. A basic reference is [DG67].

2. Directed limits of schemes with affine transition maps

In this section we construct the limit.
01YW **Lemma 2.1.** Let $I$ be a directed set. Let $(S_i, f_{ii'})$ be an inverse system of schemes over $I$. If all the schemes $S_i$ are affine, then the limit $S = \lim_i S_i$ exists in the category of schemes. In fact $S$ is affine and $S = \text{Spec}(\text{colim}_i R_i)$ with $R_i = \Gamma(S_i, \mathcal{O})$.

**Proof.** Just define $S = \text{Spec}(\text{colim}_i R_i)$. It follows from Schemes, Lemma 6.4 that $S$ is the limit even in the category of locally ringed spaces. \qed

01YX **Lemma 2.2.** Let $I$ be a directed set. Let $(S_i, f_{ii'})$ be an inverse system of schemes over $I$. If all the morphisms $f_{ii'} : S_i \to S_{i'}$ are affine, then the limit $S = \lim_i S_i$ exists in the category of schemes. Moreover,

1. each of the morphisms $f_i : S \to S_i$ is affine,
2. for an element $0 \in I$ and any open subscheme $U_0 \subset S_0$ we have

$$f_0^{-1}(U_0) = \lim_{i \geq 0} f_{i0}^{-1}(U_0)$$

in the category of schemes.

**Proof.** Choose an element $0 \in I$. Note that $I$ is nonempty as the limit is directed. For every $i \geq 0$ consider the quasi-coherent sheaf of $\mathcal{O}_{S_i}$-algebras $A_i = f_{0i^*}\mathcal{O}_{S_i}$. Recall that $S_i = \text{Spec}_{S_0}(A_i)$, see Morphisms, Lemma 11.3. Set $A = \text{colim}_{i \geq 0} A_i$. This is a quasi-coherent sheaf of $\mathcal{O}_{S_0}$-algebras, see Schemes, Section 24. Set $S = \text{Spec}_{S_0}(A)$. By Morphisms, Lemma 11.5 we get for $i \geq 0$ morphisms $f_i : S \to S_i$ compatible with the transition morphisms. Note that the morphisms $f_i$ are affine by Morphisms, Lemma 11.11 for example. By Lemma 2.1 above we see that for any affine open $U_0 \subset S_0$ the inverse image $U = f_0^{-1}(U_0) \subset S$ is the limit of the system of opens $U_i = f_{i0}^{-1}(U_0)$, $i \geq 0$ in the category of schemes.

Let $T$ be a scheme. Let $g_i : T \to S_i$ be a compatible system of morphisms. To show that $S = \lim_i S_i$ we have to prove there is a unique morphism $g : T \to S$ with $g_i = f_i \circ g$ for all $i \in I$. For every $t \in T$ there exists an affine open $U_0 \subset S_0$ containing $g_0(t)$. Let $V \subset g_0^{-1}(U_0)$ be an affine open neighbourhood containing $t$. By the remarks above we obtain a unique morphism $g_V : V \to U = f_0^{-1}(U_0)$ such that $f_i \circ g_V = g_i|_V$ for all $i$. The open sets $V \subset T$ so constructed form a basis for the topology of $T$. The morphisms $g_V$ glue to a morphism $g : T \to S$ because of the uniqueness property. This gives the desired morphism $g : T \to S$.

The final statement is clear from the construction of the limit above. \qed

01YZ **Lemma 2.3.** Let $I$ be a directed set. Let $(S_i, f_{ii'})$ be an inverse system of schemes over $I$. Assume all the morphisms $f_{ii'} : S_i \to S_{i'}$ are affine, let $S = \lim_i S_i$. Let $0 \in I$. Suppose that $T$ is a scheme over $S_0$. Then

$$T \times_{S_0} S = \lim_{i \geq 0} T \times_{S_0} S_i$$

**Proof.** The right hand side is a scheme by Lemma 2.2. The equality is formal, see Categories, Lemma 14.9. \qed

3. Infinite products

0CNH Infinite products of schemes usually do not exist. For example in Examples, Section 49 it is shown that an infinite product of copies of $\mathbb{P}^1$ is not even an algebraic space.

On the other hand, infinite products of affine schemes do exist and are affine. Using Schemes, Lemma 6.4 this corresponds to the fact that in the category of rings we
have infinite coproducts: if \( I \) is a set and \( R_i \) is a ring for each \( i \), then we can consider the ring

\[
R = \bigotimes R_i = \text{colim}_{\{i_1, \ldots, i_n\} \subseteq I} R_{i_1} \otimes \cdots \otimes R_{i_n}
\]

Given another ring \( A \) a map \( R \to A \) is the same thing as a collection of ring maps \( R_i \to A \) for all \( i \in I \) as follows from the corresponding property of finite tensor products.

**Lemma 3.1.** Let \( S \) be a scheme. Let \( I \) be a set and for each \( i \in I \) let \( f_i : T_i \to S \) be an affine morphism. Then the product \( T = \prod T_i \) exists in the category of schemes over \( S \). In fact, we have

\[
T = \lim_{\{i_1, \ldots, i_n\} \subseteq I} T_{i_1} \times_S \cdots \times_S T_{i_n}
\]

and the projection morphisms \( T \to T_{i_1} \times_S \cdots \times_S T_{i_n} \) are affine.

**Proof.** Omitted. Hint: Argue as in the discussion preceding the lemma and use Lemma 2.2 for existence of the limit.

**Lemma 3.2.** Let \( S \) be a scheme. Let \( I \) be a set and for each \( i \in I \) let \( f_i : T_i \to S \) be a surjective affine morphism. Then the product \( T = \prod T_i \) in the category of schemes over \( S \) (Lemma 3.1) maps surjectively to \( S \).

**Proof.** Let \( s \in S \). Choose \( t_i \in T_i \) mapping to \( s \). Choose a huge field extension \( K/\kappa(s) \) such that \( \kappa(s_i) \) embeds into \( K \) for each \( i \). Then we get morphisms \( \text{Spec}(K) \to T_i \) with image \( s_i \) agreeing as morphisms to \( S \). Whence a morphism \( \text{Spec}(K) \to T \) which proves there is a point of \( T \) mapping to \( s \).

**Lemma 3.3.** Let \( S \) be a scheme. Let \( I \) be a set and for each \( i \in I \) let \( f_i : T_i \to S \) be an integral morphism. Then the product \( T = \prod T_i \) in the category of schemes over \( S \) (Lemma 3.1) is integral over \( S \).

**Proof.** Omitted. Hint: On affine pieces this reduces to the following algebra fact: if \( A \to B_i \) is integral for all \( i \), then \( A \to \bigotimes A B_i \) is integral.

### 4. Descending properties

**Lemma 4.1.** Let \( S = \text{lim} S_i \) be the limit of a directed inverse system of schemes with affine transition morphisms (Lemma 2.2). Then \( S_{\text{set}} = \text{lim}_i S_{i, \text{set}} \) where \( S_{\text{set}} \) indicates the underlying set of the scheme \( S \).

**Proof.** Pick \( i \in I \). Take \( U_i \subset S_i \) an affine open. Denote \( U_{i'} = f_{i'i}^{-1}(U_i) \) and \( U = f_i^{-1}(U_i) \). Here \( f_{i'i} : S_{i'} \to S_i \) is the transition morphism and \( f_i : S \to S_i \) is the projection. By Lemma 2.2, we have \( U = \text{lim}_{i \geq i'} U_{i'} \). Suppose we can show that \( U_{\text{set}} = \text{lim}_{i \geq i'} U_{i', \text{set}} \). Then the lemma follows by a simple argument using an affine covering of \( S_i \). Hence we may assume all \( S_i \) and \( S \) affine. This reduces us to the algebra question considered in the next paragraph.

Suppose given a system of rings \( (A_i, \varphi_{i'i}) \) over \( I \). Set \( A = \text{colim} A_i \) with canonical maps \( \varphi_i : A_i \to A \). Then

\[
\text{Spec}(A) = \text{lim}_i \text{Spec}(A_i)
\]
Namely, suppose that we are given primes $p_i \subset A_i$ such that $p_i = \varphi_i^{-1}(p_{i'})$ for all $i' \geq i$. Then we simply set

$$p = \{ x \in A \mid \exists i, x_i \in p_i \text{ with } \varphi_i(x_i) = x \}$$

It is clear that this is an ideal and has the property that $\varphi_i^{-1}(p) = p_i$. Then it follows easily that it is a prime ideal as well.

**Lemma 4.4.** Let $S = \lim S_i$ be the limit of a directed inverse system of schemes with affine transition morphisms (Lemma 2.2). Then $S_{\text{top}} = \lim S_{i, \text{top}}$ where $S_{\text{top}}$ indicates the underlying topological space of the scheme $S$.

**Proof.** We will use the criterion of Topology, Lemma 14.3. We have seen that $S_{\text{set}} = \lim S_{i, \text{set}}$ in Lemma 4.1. The maps $f_i : S \to S_i$ are morphisms of schemes hence continuous. Thus $f_i^{-1}(U_i)$ is open for each open $U_i \subset S_i$. Finally, let $s \in S$ and let $s \in V \subset S$ be an open neighbourhood. Choose $0 \in I$ and choose an affine open neighbourhood $U_0 \subset S_0$ of the image of $s$. Then $f_{0}^{-1}(U_0) = \lim_{i \geq 0} f_{i0}^{-1}(U_0)$, see Lemma 2.2. Then $f_{0}^{-1}(U_0)$ and $f_{i0}^{-1}(U_0)$ are affine and

$$\mathcal{O}_S(f_{0}^{-1}(U_0)) = \colim_{i \geq 0} \mathcal{O}_{S_i}(f_{i0}^{-1}(U_0))$$

either by the proof of Lemma 2.2 or by Lemma 2.1. Choose $a \in \mathcal{O}_S(f_{0}^{-1}(U_0))$ such that $s \in D(a) \subset V$. This is possible because the principal opens form a basis for the topology on the affine scheme $f_{0}^{-1}(U_0)$. Then we can pick an $i \geq 0$ and $a_i \in \mathcal{O}_{S_i}(f_{i0}^{-1}(U_0))$ mapping to $a$. It follows that $D(a_i) \subset f_{i0}^{-1}(U_0) \subset S_i$ is an open subset whose inverse image in $S$ is $D(a)$. This finishes the proof.

**Lemma 4.3.** Let $S = \lim S_i$ be the limit of a directed inverse system of schemes with affine transition morphisms (Lemma 2.2). If all the schemes $S_i$ are nonempty and quasi-compact, then the limit $S = \lim S_i$ is nonempty.

**Proof.** Choose $0 \in I$. Note that $I$ is nonempty as the limit is directed. Choose an affine open covering $S_0 = \bigcup_{j = 1, \ldots, m} U_j$. Since $I$ is directed there exists a $j \in \{1, \ldots, m\}$ such that $f_{0}^{-1}(U_j) \neq \emptyset$ for all $i \geq 0$. Hence $\lim_{i \geq 0} f_{i0}^{-1}(U_j)$ is not empty since a directed colimit of nonzero rings is nonzero (because $1 \neq 0$). As $\lim_{i \geq 0} f_{i0}^{-1}(U_j)$ is an open subscheme of the limit we win.

**Lemma 4.4.** Let $S = \lim S_i$ be the limit of a directed inverse system of schemes with affine transition morphisms (Lemma 2.2). Let $s \in S$ with images $s_i \in S_i$. Then $\{s\} = \lim \{s_i\}$ as sets and as schemes if endowed with the reduced induced scheme structure.

**Proof.** Choose $0 \in I$ and an affine open covering $U_0 = \bigcup_{j \in J} U_{0,j}$. For $i \geq 0$ let $U_{i,j} = f_{i0}^{-1}(U_{0,j})$ and set $U_i = f_i^{-1}(U_{0,j})$. Here $f_i : S_i \to S$ is the transition morphism and $f_i : S \to S_i$ is the projection. For $j \in J$ the following are equivalent:

(a) $x \in U_j$, (b) $x_0 \in U_{0,j}$, (c) $x_i \in U_{i,j}$ for all $i \geq 0$. Let $J' \subset J$ be the set of indices for which (a), (b), (c) are true. Then $\{s\} = \bigcup_{j \in J'} (\{s\} \cap U_j)$ and similarly for $\{s_i\}$ for $i \geq 0$. Note that $\{s\} \cap U_j$ is the closure of the set $\{s\}$ in the topological space $U_j$. Similarly for $\{s_i\} \cap U_{i,j}$ for $i \geq 0$. Hence it suffices to prove the lemma in the case $S$ and $S_i$ affine for all $i$. This reduces us to the algebra question considered in the next paragraph.
Suppose given a system of rings \((A_i, \varphi_{ii'})\) over \(I\). Set \(A = \text{colim}_i A_i\) with canonical maps \(\varphi_i : A_i \rightarrow A\). Let \(p \subset A\) be a prime and set \(p_i = \varphi_i^{-1}(p)\). Then
\[
V(p) = \text{colim}_i V(p_i)
\]
This follows from Lemma 4.1 because \(A/p = \text{colim} A_i/p_i\). This equality of rings also shows the final statement about reduced induced scheme structures holds true. □

In the rest of this section we work in the following situation.

**Situation 4.5.** Let \(S = \text{lim}_{i \in I} S_i\) be the limit of a directed system of schemes with affine transition morphisms \(f_{i,i'} : S_i' \rightarrow S_i\) (Lemma 2.2). We assume that \(S_i\) is quasi-compact and quasi-separated for all \(i \in I\). We denote \(f_i : S \rightarrow S_i\) the projection. We also choose an element \(0 \in I\).

The type of result we are looking for is the following: If we have an object over \(S\), then for some \(i\) there is a similar object over \(S_i\).

**Lemma 4.6.**

1. We have \(S_{\text{set}} = \text{lim}_{i \in I} S_{i,\text{set}}\) where \(S_{i,\text{set}}\) indicates the underlying set of the scheme \(S_i\).
2. We have \(S_{\text{top}} = \text{lim}_{i \in I} S_{i,\text{top}}\) where \(S_{i,\text{top}}\) indicates the underlying topological space of the scheme \(S_i\).
3. If \(s, s' \in S\) and \(s'\) is not a specialization of \(s\) then for some \(i \in I\) the image \(s_i' \in S_i\) of \(s'\) is not a specialization of the image \(s_i \in S_i\) of \(s\).
4. Add more easy facts on topology of \(S\) here. (Requirement: whatever is added should be easy in the affine case.)

**Proof.** Part (1) is a special case of Lemma 4.1.

Part (2) is a special case of Lemma 4.2.

Part (3) is a special case of Lemma 4.4. □

**Lemma 4.7.**

In Situation 4.5. Suppose that \(\mathcal{F}_0\) is a quasi-coherent sheaf on \(S_0\). Set \(\mathcal{F}_i = f_{0,i}^* \mathcal{F}_0\) for \(i \geq 0\) and set \(\mathcal{F} = f_0^* \mathcal{F}_0\). Then
\[
\Gamma(S, \mathcal{F}) = \text{colim}_{i \geq 0} \Gamma(S_i, \mathcal{F}_i)
\]

**Proof.** Write \(\mathcal{A}_j = f_{0,i}^* \mathcal{O}_{S_i}\). This is a quasi-coherent sheaf of \(\mathcal{O}_{S_0}\)-algebras (see Morphisms, Lemma [11.5] and \(S_i\) is the relative spectrum of \(\mathcal{A}_i\) over \(S_0\). In the proof of Lemma 2.2 we constructed \(S\) as the relative spectrum of \(\mathcal{A} = \text{colim}_{i \geq 0} \mathcal{A}_i\) over \(S_0\). Set
\[
\mathcal{M}_i = \mathcal{F}_0 \otimes_{\mathcal{O}_{S_0}} \mathcal{A}_i
\]
and
\[
\mathcal{M} = \mathcal{F}_0 \otimes_{\mathcal{O}_{S_0}} \mathcal{A}.
\]
Then we have \(f_{0,i}^* \mathcal{F}_i = \mathcal{M}_i\) and \(f_0^* \mathcal{F} = \mathcal{M}\). Since \(\mathcal{A}\) is the colimit of the sheaves \(\mathcal{A}_i\) and since tensor product commutes with directed colimits, we conclude that \(\mathcal{M} = \text{colim}_{i \geq 0} \mathcal{M}_i\). Since \(S_0\) is quasi-compact and quasi-separated we see that
\[
\Gamma(S, \mathcal{F}) = \Gamma(S_0, \mathcal{M})
\]
\[
= \Gamma(S_0, \text{colim}_{i \geq 0} \mathcal{M}_i)
\]
\[
= \text{colim}_{i \geq 0} \Gamma(S_0, \mathcal{M}_i)
\]
\[
= \text{colim}_{i \geq 0} \Gamma(S_i, \mathcal{F}_i)
\]
see Sheaves, Lemma [29.1] and Topology, Lemma [27.1] for the middle equality. □
In Situation 4.5. Suppose for each \( i \) we are given a nonempty closed subset \( Z_i \subset S_i \) with \( f_{ii'}(Z_i) \subset Z_{i'} \). Then there exists a point \( s \in S \) with \( f_i(s) \in Z_i \) for all \( i \).

**Proof.** Let \( Z_i \subset S_i \) also denote the reduced closed subscheme associated to \( Z_i \), see Schemes, Definition 12.5. A closed immersion is affine, and a composition of affine morphisms is affine (see Morphisms, Lemmas 11.9 and 11.7), and hence \( Z_i \to S_{i'} \) is affine when \( i \geq i' \). We conclude that the morphism \( f_{ii'} : Z_i \to Z_{i'} \) is affine by Morphisms, Lemma 11.11. Each of the schemes \( Z_i \) is quasi-compact as a closed subscheme of a quasi-compact scheme. Hence we may apply Lemma 11.3 to see that \( Z = \lim_i Z_i \) is nonempty. Since there is a canonical morphism \( Z \to S \) we win. \( \square \)

**Lemma 4.9.** In Situation 4.5. Suppose we are given an \( i \) and a morphism \( T \to S_i \) such that

1. \( T \times_{S_i} S = \emptyset \), and
2. \( T \) is quasi-compact.

Then \( T \times_{S_i} S_{i'} = \emptyset \) for all sufficiently large \( i' \).

**Proof.** By Lemma 2.3 we see that \( T \times_{S_i} S = \lim_{i' \geq i} T \times_{S_i} S_{i'} \). Hence the result follows from Lemma 4.3. \( \square \)

**Lemma 4.10.** In Situation 4.5. Suppose we are given an \( i \) and a locally constructible subset \( E \subset S_i \) such that \( f_i(S) \subset E \). Then \( f_{ii'}(S_{i'}) \subset E \) for all sufficiently large \( i' \).

**Proof.** Writing \( S_i \) as a finite union of open affine subschemes reduces the question to the case that \( S_i \) is affine and \( E \) is constructible, see Lemma 2.2 and Properties, Lemma 2.11. In this case the complement \( S_i \setminus E \) is constructible too. Hence there exists an affine scheme \( T \) and a morphism \( T \to S_i \) whose image is \( S_i \setminus E \), see Algebra, Lemma 28.3. By Lemma 4.9 we see that \( T \times_{S_i} S_{i'} \) is empty for all sufficiently large \( i' \), and hence \( f_{ii'}(S_{i'}) \subset E \) for all sufficiently large \( i' \). \( \square \)

**Lemma 4.11.** In Situation 4.5 we have the following:

1. Given any quasi-compact open \( V \subset S = \lim_i S_i \) there exists an \( i \in I \) and a quasi-compact open \( V_i \subset S_i \) such that \( f_i^{-1}(V_i) = V \).
2. Given \( V_i \subset S_i \) and \( V_{i'} \subset S_{i'} \) quasi-compact opens such that \( f_i^{-1}(V_i) = f_{ii'}^{-1}(V_{i'}) \) there exists an index \( i'' \geq i, i' \) such that \( f_i^{-1}(V_i) = f_{ii''}^{-1}(V_{i''}) \).
3. If \( V_{i_1}, \ldots, V_{i_n} \subset S_i \) are quasi-compact opens and \( S = f_i^{-1}(V_{i_1}) \cup \cdots \cup f_i^{-1}(V_{i_n}) \) then \( S_{i'} = f_{ii'}^{-1}(V_{i_1} \cap \cdots \cap V_{i_n}) \) for some \( i' \geq i \).

**Proof.** Choose \( i_0 \in I \). Note that \( I \) is nonempty as the limit is directed. For convenience we write \( S_0 = S_{i_0} \) and \( i_0 = 0 \). Choose an affine open covering \( S_0 = U_{1,0} \cup \cdots \cup U_{m,0} \). Denote \( U_{i,j} \subset S_i \) the inverse image of \( U_{j,0} \) under the transition morphism for \( i \geq 0 \). Denote \( U_{j} \) the inverse image of \( U_{j,0} \) in \( S \). Note that \( U_j = \lim_i U_{i,j} \) is a limit of affine schemes.

We first prove the uniqueness statement: Let \( V_i \subset S_i \) and \( V_{i'} \subset S_{i'} \) quasi-compact opens such that \( f_i^{-1}(V_i) = f_{ii'}^{-1}(V_{i'}) \). It suffices to show that \( f_{ii''}^{-1}(V_i \cap U_{i,j}) \) and \( f_{ii''}^{-1}(V_{i'} \cap U_{j,i''}) \) become equal for \( i'' \) large enough. Hence we reduce to the case of a limit of affine schemes. In this case write \( S = \Spec(R) \) and \( S_i = \Spec(R_i) \) for all \( i \in I \). We may write \( V_i = S_i \setminus V(h_1, \ldots, h_m) \) and \( V_{i'} = S_{i'} \setminus V(g_1, \ldots, g_n) \). The assumption means that the ideals \( \sum g_j R \) and \( \sum h_j R \) have the same radical...
in \( R \). This means that \( g_j^N = \sum a_{jj} h^{j'}_j \) and \( h^N_j = \sum b_{jj} g^{j'}_j \) for some \( N \gg 0 \) and \( a_{jj} \) and \( b_{jj} \) in \( R \). Since \( R = \colim V \) we can chose an index \( i'' \geq i \) such that the equations \( g_j^N = \sum a_{jj} h^{j'}_j \) and \( h^N_j = \sum b_{jj} g^{j'}_j \) hold in \( R_{i''} \) for some \( a_{jj} \) and \( b_{jj} \) in \( R_{i'} \). This implies that the ideals \( \sum g_j R_{i''} \) and \( \sum h_j R_{i''} \) have the same radical in \( R_{i''} \) as desired.

We prove existence: If \( S_0 \) is affine, then \( S_i = \Spec(R_i) \) for all \( i \geq 0 \) and \( S = \Spec(R) \) with \( R = \colim V \). Then \( V = S \setminus V(g_1, \ldots, g_n) \) for some \( g_1, \ldots, g_n \in R \). Choose any \( i \) large enough so that each of the \( g_j \) comes from an element \( g_j \in R_i \) and take \( V_i = S_i \setminus V(g_{1,i}, \ldots, g_{n,i}) \). If \( S_0 \) is general, then the opens \( V \cap U_j \) are quasi-compact because \( S \) is quasi-separated. Hence by the affine case we see that for each \( j = 1, \ldots, m \) there exists an \( i_j \in I \) and a quasi-compact open \( V_{ij} \subset U_{ij} \) whose inverse image in \( U_j \) is \( V \cap U_j \). Set \( i = \max(i_1, \ldots, i_m) \) and let \( V_i = \bigcup f_{ij}^{-1}(V_{ij}) \).

The statement on coverings follows from the uniqueness statement for the opens \( V_{i,1} \cup \ldots \cup V_{n,i} \) and \( S_i \) of \( S \).

\[ \square \]

\textbf{Lemma 4.12.} In \textbf{Situation 4.5} if \( S \) is quasi-affine, then for some \( i_0 \in I \) the schemes \( S_i \) for \( i \geq i_0 \) are quasi-affine.

\textbf{Proof.} Choose \( i_0 \in I \). Note that \( I \) is nonempty as the limit is directed. For convenience we write \( S_0 = S_{i_0} \) and \( i_0 = 0 \). Let \( s \in S \). We may choose an affine open \( U_0 \subset S_0 \) containing \( f_0(s) \). Since \( S \) is quasi-affine we may choose an element \( a \in \Gamma(S, \mathcal{O}_S) \) such that \( s \in D(a) \subset f_0^{-1}(U_0) \), and such that \( D(a) \) is affine. By Lemma 4.7 there exists an \( i \geq 0 \) such that \( a \) comes from an element \( a_i \in \Gamma(S_i, \mathcal{O}_{S_i}) \). For any index \( j \geq i \) we denote \( a_j \) the image of \( a_i \) in the global sections of the structure sheaf of \( S_j \). Consider the opens \( D(a_j) \subset S_j \) and \( U_j = f_{j0}^{-1}(U_0) \). Note that \( U_j \) is affine and \( D(a_j) \) is a quasi-compact open of \( S_j \), see Properties, Lemma 26.1 for example. Hence we may apply Lemma 4.11 to the opens \( U_j \) and \( U_j \cup D(a_j) \) to conclude that \( D(a_j) \subset U_j \) for some \( j \geq i \). For such an index \( j \) we see that \( D(a_j) \subset S_j \) is an affine open (because \( D(a_j) \) is a standard affine open of the affine open \( U_j \)) containing the image \( f_j(s) \).

We conclude that for every \( s \in S \) there exist an index \( i \in I \), and a global section \( a \in \Gamma(S_i, \mathcal{O}_{S_i}) \) such that \( D(a) \subset S_i \) is an affine open containing \( f_i(s) \). Because \( S \) is quasi-compact we may choose a single index \( i \in I \) and global sections \( a_1, \ldots, a_m \in \Gamma(S_i, \mathcal{O}_{S_i}) \) such that each \( D(a_j) \subset S_i \) is affine open and such that \( f_i : S \to S_i \) has image contained in the union \( W_i = \bigcup_{j=1,\ldots,m} D(a_j) \). For \( i' \geq i \) set \( W_{i'} = f_{i'0}^{-1}(W_i) \). Since \( f_{i'0}^{-1}(W_i) \) is all of \( S \) we see (by Lemma 4.11 again) that for a suitable \( i' \geq i \) we have \( S_{i'} = W_{i'} \). Thus we may replace \( i \) by \( i' \) and assume that \( S_i = \bigcup_{j=1,\ldots,m} D(a_j) \).

This implies that \( \mathcal{O}_{S_i} \) is an ample invertible sheaf on \( S_i \) (see Properties, Definition 26.1) and hence that \( S_i \) is quasi-affine, see Properties, Lemma 27.1. Hence we win.

\[ \square \]

\textbf{Lemma 4.13.} In \textbf{Situation 4.5} if \( S \) is affine, then for some \( i_0 \in I \) the schemes \( S_i \) for \( i \geq i_0 \) are affine.

\textbf{Proof.} By Lemma 4.12 we may assume that \( S_0 \) is quasi-affine for some \( 0 \in I \). Set \( R_0 = \Gamma(S_0, \mathcal{O}_{S_0}) \). Then \( S_0 \) is a quasi-compact open of \( T_0 = \Spec(R_0) \). Denote \( j_0 : S_0 \to T_0 \) the corresponding quasi-compact open immersion. For \( i \geq 0 \) set \( A_i = f_{0i}^* \mathcal{O}_{S_i} \). Since \( f_{0i}^* \) is affine we see that \( S_i = \Spec S_i(A_i) \). Set \( T_i = \Spec T_0(j_0^* A_i) \).
Then $T_i \to T_0$ is affine, hence $T_i$ is affine. Thus $T_i$ is the spectrum of
\[ R_i = \Gamma(T_0, \mathcal{I}_0 \cdot A_i) = \Gamma(S_0, A_i) = \Gamma(S_i, \mathcal{O}_{S_i}). \]

Write $S = \text{Spec}(R)$. We have $R = \text{colim}_i R_i$ by Lemma [4.7]. Hence also $S = \text{lim}_i T_i$. As formation of the relative spectrum of the open $S_0 \subset T_0$ in $T_i$ is $S_i$. Let $Z_0 = T_0 \setminus S_0$ and let $Z_i \subset T_i$ be the inverse image of $Z_0$. As $S_i = T_i \setminus Z_i$, it suffices to show that $Z_i$ is empty for some $i$. Assume $Z_i$ is nonempty for all $i$ to get a contradiction. By Lemma [4.8] there exists a point $s$ of $S = \text{lim}_i T_i$ which maps to a point of $Z_i$ for every $i$. But $S = \text{lim}_i S_i$, and hence we arrive at a contradiction by Lemma [4.6].

086Q Lemma 4.14. In Situation 4.5 if $S$ is separated, then for some $i_0 \in I$ the schemes $S_i$ for $i \geq i_0$ are separated.

Proof. Choose a finite affine open covering $S_0 = U_{01} \cup \ldots \cup U_{0m}$. Set $U_{ij} \subset S_i$ and $U_j \subset S$ equal to the inverse image of $U_{0j}$. Note that $U_{ij}$ and $U_j$ are affine. As $S$ is separated the intersections $U_{ij} \cap U_{j'}$ are affine. Since $U_{ij} \cap U_{j'} = \text{lim}_{i \geq i_0} U_{ij} \cap U_{j'}$ we see that $U_{ij} \cap U_{j'}$ is affine for large $i$ by Lemma [4.13].

To show that $S_i$ is separated for large $i$ it now suffices to show that
\[ \mathcal{O}_{S_i}(U_{ij}) \otimes_{\mathcal{O}_{S_i}} \mathcal{O}_{S_i}(V_{ij}) \to \mathcal{O}_{S_i}(V_{ij} \cap V_{j'}) \]
is surjective for large $i$ (Schemes, Lemma [21.7]).

To get rid of the annoying indices, assume we have affine opens $U, V \subset S_0$ such that $U \cap V$ is affine too. Let $U_i, V_i \subset S_i$, resp. $U, V \subset S$ be the inverse images. We have to show that $\mathcal{O}(U_i) \otimes \mathcal{O}(V_i) \to \mathcal{O}(U_i \cap V_i)$ is surjective for $i$ large enough and we know that $\mathcal{O}(U_i \cap V_i) \to \mathcal{O}(U \cap V)$ is surjective. Note that $\mathcal{O}(U_i) \otimes \mathcal{O}(V_i) \to \mathcal{O}(U \cap V)$ is of finite type, as the diagonal morphism $S_i \to S_i \times S_i$ is an immersion (Schemes, Lemma [21.2]) hence locally of finite type (Morphisms, Lemmas [14.2] and [14.5]). Thus we can choose elements $f_{01}, \ldots, f_{0n} \in \mathcal{O}(U_0 \cap V_0)$ which generate $\mathcal{O}(U_0 \cap V_0)$ over $\mathcal{O}(U_0) \otimes \mathcal{O}(V_0)$. Observe that for $i \geq 0$ the diagram of schemes
\[
\begin{array}{ccc}
U_i \cap V_i & \rightarrow & U_i \\
\downarrow & & \downarrow \\
U_0 \cap V_0 & \rightarrow & U_0
\end{array}
\]
is cartesian. Thus we see that the images $f_{i,1}, \ldots, f_{i,n} \in \mathcal{O}(U_i \cap V_i)$ generate $\mathcal{O}(U_i \cap V_i)$ over $\mathcal{O}(U_i) \otimes \mathcal{O}(V_i)$ and a fortiori over $\mathcal{O}(U_i) \otimes \mathcal{O}(V_i)$. By assumption the images $f_1, \ldots, f_n \in \mathcal{O}(U \otimes V)$ are in the image of the map $\mathcal{O}(U) \otimes \mathcal{O}(V) \to \mathcal{O}(U \cap V)$. Since $\mathcal{O}(U) \otimes \mathcal{O}(V) = \text{colim} \mathcal{O}(U_i) \otimes \mathcal{O}(V_i)$ we see that they are in the image of the map at some finite level and the lemma is proved.

09MT Lemma 4.15. In Situation 4.5 let $\mathcal{L}_0$ be an invertible sheaf of modules on $S_0$. If the pullback $\mathcal{L}$ to $S$ is ample, then for some $i \in I$ the pullback $\mathcal{L}_i$ to $S_i$ is ample.

Proof. The assumption means there are finitely many sections $s_1, \ldots, s_m \in \Gamma(S, \mathcal{L})$ such that $S_{s_j}$ is affine and such that $S = \bigcup S_{s_j}$, see Properties, Definition [26.1]. By Lemma [4.7] we can find an $i \in I$ and sections $s_{i,1}, \ldots, s_{i,m} \in \Gamma(S_i, \mathcal{L}_i)$ mapping to $s_j$. By Lemma [4.13] we may, after increasing $i$, assume that $(S_{i})_{s_{i,j}}$ is affine for $j = 1, \ldots, m$. By Lemma [4.11] we may, after increasing $i$ a last time, assume that $S_i = \bigcup (S_{i})_{s_{i,j}}$. Then $\mathcal{L}_i$ is ample by definition.
Lemma 4.16. Let $S$ be a scheme. Let $X = \lim X_i$ be a directed limit of schemes over $S$ with affine transition morphisms. Let $Y \to X$ be a morphism of schemes over $S$.

1. If $Y \to X$ is a closed immersion, $X_i$ quasi-compact, and $Y$ locally of finite type over $S$, then $Y \to X_i$ is a closed immersion for $i$ large enough.
2. If $Y \to X$ is an immersion, $X_i$ quasi-separated, $Y \to S$ locally of finite type, and $Y$ quasi-compact, then $Y \to X_i$ is an immersion for $i$ large enough.

Proof. Proof of (1). Choose $0 \in I$ and a finite affine open covering $X_0 = U_{0,1} \cup \cdots \cup U_{0,m}$ with the property that $U_{0,j}$ maps into an affine open $W_j \subset S$. Let $V_j \subset Y$, resp. $U_{i,j} \subset X_i$, $i \geq 0$, resp. $U_j \subset X$ be the inverse image of $U_{0,j}$. It suffices to prove that $V_j \to U_{i,j}$ is a closed immersion for $i$ sufficiently large and we know that $V_j \to U_j$ is a closed immersion. Thus we reduce to the following algebra fact: If $A = \colim A_i$ is a directed colimit of $R$-algebras, $A \to B$ is a surjection of $R$-algebras, and $B$ is a finitely generated $R$-algebra, then $A_i \to B$ is surjective for $i$ sufficiently large. Proof of (2). Choose $0 \in I$. Choose a quasi-compact open $X'_0 \subset X_0$ such that $Y \to X_0$ factors through $X'_0$. After replacing $X_i$ by the inverse image of $X'_0$ for $i \geq 0$ we may assume all $X'_i$ are quasi-compact and quasi-separated. Let $U \subset X$ be a quasi-compact open such that $Y \to X$ factors through a closed immersion $Y \to U$ ($U$ exists as $Y$ is quasi-compact). By Lemma 4.11 we may assume that $U = \lim U_i$ with $U_i \subset X_i$ quasi-compact open. By part (1) we see that $Y \to U_i$ is a closed immersion for some $i$. Thus (2) holds.

Lemma 4.17. Let $S$ be a scheme. Let $X = \lim X_i$ be a directed limit of schemes over $S$ with affine transition morphisms. Assume

1. $S$ quasi-separated,
2. $X_i$ quasi-compact and quasi-separated,
3. $X \to S$ separated.

Then $X_i \to S$ is separated for all $i$ large enough.

Proof. Let $0 \in I$. Note that $I$ is nonempty as the limit is directed. As $X_0$ is quasi-compact we can find finitely many affine opens $U_1, \ldots, U_n \subset S$ such that $X_0 \to S$ maps into $U_1 \cup \cdots \cup U_n$. Denote $h_i : X_i \to S$ the structure morphism. It suffices to check that for some $i \geq 0$ the morphisms $h_i^{-1}(U_j) \to U_j$ are separated for $j = 1, \ldots, n$. Since $S$ is quasi-separated the morphisms $U_j \to S$ are quasi-compact. Hence $h_i^{-1}(U_j)$ is quasi-compact and quasi-separated. In this way we reduce to the case $S$ affine. In this case we have to show that $X_i$ is separated and we know that $X$ is separated. Thus the lemma follows from Lemma 4.11.

Lemma 4.18. Let $S$ be a scheme. Let $X = \lim X_i$ be a directed limit of schemes over $S$ with affine transition morphisms. Assume

1. $S$ quasi-compact and quasi-separated,
2. $X_i$ quasi-compact and quasi-separated,
3. $X \to S$ affine.

Then $X_i \to S$ is affine for $i$ large enough.

Proof. Choose a finite affine open covering $S = \bigcup_{j=1, \ldots, n} V_j$. Denote $f : X \to S$ and $f_i : X_i \to S$ the structure morphisms. For each $j$ the scheme $f^{-1}(V_j) = \lim_i f_i^{-1}(V_j)$ is affine (as a finite morphism is affine by definition). Hence by Lemma
there exists an \( i \in I \) such that each \( f_i^{-1}(V_j) \) is affine. In other words, \( f_i : X_i \to S \) is affine for \( i \) large enough, see Morphisms, Lemma \[4.13\] \( \square \)

**Lemma 4.19.** Let \( S \) be a scheme. Let \( X = \text{lim } X_i \) be a directed limit of schemes over \( S \) with affine transition morphisms. Assume

1. \( S \) quasi-compact and quasi-separated,
2. \( X_i \) quasi-compact and quasi-separated,
3. the transition morphisms \( X_{i'} \to X_i \) are finite,
4. \( X_i \to S \) locally of finite type
5. \( X \to S \) integral.

Then \( X_i \to S \) is finite for \( i \) large enough.

**Proof.** By Lemma \[4.18\] we may assume \( X_i \to S \) is affine for all \( i \). Choose a finite affine open covering \( S = \bigcup_{j=1}^{n} V_j \). Denote \( f : X \to S \) and \( f_i : X_i \to S \) the structure morphisms. It suffices to show that there exists an \( i \) such that \( f_i^{-1}(V_j) \) is finite over \( V_j \) for \( j = 1, \ldots, m \) (Morphisms, Lemma \[42.3\]). Namely, for \( i' \geq i \) the composition \( X_{i'} \to X_i \to S \) will be finite as a composition of finite morphisms (Morphisms, Lemma \[42.5\]). This reduces us to the affine case: Let \( R \) be a ring and \( A = \text{colim } A_i \) with \( R \to A \) integral and \( A_i \to A_{i'} \) finite for all \( i \leq i' \). Moreover \( R \to A_i \) is of finite type for all \( i \). Goal: Show that \( A_i \) is finite over \( R \) for some \( i \). To prove this choose an \( i \in I \) and pick generators \( x_1, \ldots, x_m \in A_i \) of \( A_i \) as an \( R \)-algebra. Since \( A \) is integral over \( R \) we can find monic polynomials \( P_j \in R[T] \) such that \( P_j(x_j) = 0 \) in \( A \). Thus there exists an \( i' \geq i \) such that \( P_j(x_j) \) is finite over \( R \) by Algebra, Lemma \[35.5\]. Since \( A_i' \subset A_{i'} \) is finite too we conclude that \( A_{i'} \) is finite over \( R \) by Algebra, Lemma \[7.3\] \( \square \)

**Lemma 4.20.** Let \( S \) be a scheme. Let \( X = \text{lim } X_i \) be a directed limit of schemes over \( S \) with affine transition morphisms. Assume

1. \( S \) quasi-compact and quasi-separated,
2. \( X_i \) quasi-compact and quasi-separated,
3. the transition morphisms \( X_{i'} \to X_i \) are closed immersions,
4. \( X_i \to S \) locally of finite type
5. \( X \to S \) a closed immersion.

Then \( X_i \to S \) is a closed immersion for \( i \) large enough.

**Proof.** By Lemma \[4.18\] we may assume \( X_i \to S \) is affine for all \( i \). Choose a finite affine open covering \( S = \bigcup_{j=1}^{n} V_j \). Denote \( f : X \to S \) and \( f_i : X_i \to S \) the structure morphisms. It suffices to show that there exists an \( i \) such that \( f_i^{-1}(V_j) \) is a closed subscheme of \( V_j \) for \( j = 1, \ldots, m \) (Morphisms, Lemma \[2.1\]). This reduces us to the affine case: Let \( R \) be a ring and \( A = \text{colim } A_i \) with \( R \to A \) surjective and \( A_i \to A_{i'} \) surjective for all \( i \leq i' \). Moreover \( R \to A_i \) is of finite type for all \( i \). Goal: Show that \( R \to A_i \) is surjective for some \( i \). To prove this choose an \( i \in I \) and pick generators \( x_1, \ldots, x_m \in A_i \) of \( A_i \) as an \( R \)-algebra. Since \( R \to A \) is surjective we can find \( r_j \in R \) such that \( r_j \) maps to \( x_j \) in \( A \). Thus there exists an \( i' \geq i \) such that \( r_j \) maps to the image of \( x_j \) in \( A_{i'} \) for \( j = 1, \ldots, m \). Since \( A_i \to A_{i'} \) is surjective this implies that \( R \to A_{i'} \) is surjective. \( \square \)
5. Absolute Noetherian Approximation

A nice reference for this section is Appendix C of the article by Thomason and Trobaugh \[TT90\]. See Categories, Section 21 for our conventions regarding directed systems. We will use the existence result and properties of the limit from Section 2 without further mention.

Lemma 5.1. Let \( W \) be a quasi-affine scheme of finite type over \( \mathbb{Z} \). Suppose \( W \to \text{Spec}(R) \) is an open immersion into an affine scheme. There exists a finite type \( \mathbb{Z} \)-algebra \( A \subset R \) which induces an open immersion \( W \to \text{Spec}(A) \). Moreover, \( R \) is the directed colimit of such subalgebras.

Proof. Choose an affine open covering \( W = \bigcup_{i=1}^{n} W_i \) such that each \( W_i \) is a standard affine open in \( \text{Spec}(R) \). In other words, if we write \( W_i = \text{Spec}(R_i) \) then \( R_i = R_{f_i} \) for some \( f_i \in R \). Choose finitely many \( x_{ij} \in R_i \) which generate \( R_i \) over \( \mathbb{Z} \).

Pick an \( N \gg 0 \) such that each \( f_i^N x_{ij} \) comes from an element of \( R \), say \( y_{ij} \in R \). Set \( A \) equal to the \( \mathbb{Z} \)-algebra generated by the \( f_i \) and the \( y_{ij} \) and (optionally) finitely many additional elements of \( R \). Then \( A \) works. Details omitted. \( \square \)

Lemma 5.2. Suppose given a cartesian diagram of rings

\[
\begin{array}{ccc}
B & \xrightarrow{s} & R \\
\uparrow & & \uparrow t \\
B' & \xrightarrow{t} & R'
\end{array}
\]

Let \( W' \subset \text{Spec}(R') \) be an open of the form \( W' = D(f_1) \cup \ldots \cup D(f_n) \) such that \( t(f_i) = s(g_i) \) for some \( g_i \in B \) and \( B_{g_i} \cong R_{s(g_i)} \). Then \( B' \to R' \) induces an open immersion of \( W' \) into \( \text{Spec}(B') \).

Proof. Set \( h_i = (g_i, f_i) \in B' \). More on Algebra, Lemma 5.3 shows that \( (B')_{h_i} \cong (R')_{f_i} \), as desired. \( \square \)

The following lemma is a precise statement of Noetherian approximation.

Lemma 5.3. Let \( S \) be a quasi-compact and quasi-separated scheme. Let \( V \subset S \) be a quasi-compact open. Let \( I \) be a directed set and let \( \langle V_i, f_{ij} \rangle \) be an inverse system of schemes over \( I \) with affine transition maps, with each \( V_i \) of finite type over \( \mathbb{Z} \), and with \( V = \lim_i V_i \). Then there exist

1. a directed set \( J \),
2. an inverse system of schemes \( \langle S_j, g_{jj'} \rangle \) over \( J \),
3. an order preserving map \( \alpha : J \to I \),
4. open subschemes \( V'_j \subset S_j \), and
5. isomorphisms \( V'_j \to V_{\alpha(j)} \)

such that

1. the transition morphisms \( g_{jj'} : S_j \to S_{j'} \) are affine,
2. each \( S_j \) is of finite type over \( \mathbb{Z} \),
3. \( g_{jj'}^{-1}(V_{j'}) = V_j \),
4. \( S = \lim_j S_j \) and \( V = \lim_i V_i \), and
(5) the diagrams

\[
\begin{array}{c}
\xymatrix{
V & V \\
\downarrow & \downarrow \\
V' & V' \\
\downarrow & \downarrow \\
V_{\alpha(j)} & V_{\alpha(j)}
}
\end{array}
\quad \text{and} \quad
\begin{array}{c}
\xymatrix{
V & V \\
\downarrow & \downarrow \\
V_{\alpha(j)} & V_{\alpha(j')}
}
\end{array}
\]

are commutative.

**Proof.** Set \( Z = S \setminus V \). Choose affine opens \( U_1, \ldots, U_m \subset S \) such that \( Z \subset \bigcup_{i=1,\ldots,m} U_i \). Consider the opens

\[
V \subset V \cup U_1 \subset V \cup U_1 \cup U_2 \subset \ldots \subset V \cup \bigcup_{i=1,\ldots,m} U_i = S
\]

If we can prove the lemma successively for each of the cases

\[
V \cup U_1 \cup \ldots \cup U_t \subset V \cup U_1 \cup \ldots \cup U_{t+1}
\]

then the lemma will follow for \( V \subset S \). In each case we are adding one affine open. Thus we may assume

(1) \( S = U \cup V \),
(2) \( U \) affine open in \( S \),
(3) \( V \) quasi-compact open in \( S \), and
(4) \( V = \lim_i V_i \) with \((V_i, f_{ii'})\) an inverse system over a directed set \( I \), each \( f_{ii'} \) affine and each \( V_i \) of finite type over \( \mathbb{Z} \).

Set \( W = U \cap V \). As \( S \) is quasi-separated, this is a quasi-compact open of \( V \). By Lemma 4.11 (and after shrinking \( I \)) we may assume that there exist opens \( W_i \subset V_i \) such that \( f_{ij}^{-1}(W_j) = W_i \) and such that \( f_i^{-1}(W_i) = W \). Since \( W \) is a quasi-compact open of \( U \) it is quasi-affine. Hence we may assume (after shrinking \( I \) again) that \( W_i \) is quasi-affine for all \( i \), see Lemma 4.12.

Write \( U = \text{Spec}(B) \). Set \( R = \Gamma(W, \mathcal{O}_W) \), and \( R_i = \Gamma(W_i, \mathcal{O}_{W_i}) \). By Lemma 4.7 we have \( R = \text{colim}_i R_i \). Now we have the maps of rings

\[
\begin{array}{c}
\xymatrix{
B & R \\
\downarrow & \\
R_i & \downarrow \\
\uparrow_{t_i} & t_i \\
B_i & R_i
}
\end{array}
\]

We set \( B_i = \{(b, r) \in B \times R_i \mid s(b) = t_i(t)\} \) so that we have a cartesian diagram

\[
\begin{array}{c}
\xymatrix{
B & R \\
\downarrow & \\
B_i & R_i
}
\end{array}
\]

for each \( i \). The transition maps \( R_i \to R_{i'} \) induce maps \( B_i \to B_{i'} \). It is clear that \( B = \text{colim}_i B_i \). In the next paragraph we show that for all sufficiently large \( i \) the composition \( W_i \to \text{Spec}(R_i) \to \text{Spec}(B_i) \) is an open immersion.

As \( W \) is a quasi-compact open of \( U = \text{Spec}(B) \) we can find a finitely many elements \( g_l \in B, l = 1, \ldots, m \) such that \( D(g_l) \subset W \) and such that \( W = \bigcup_{l=1,\ldots,m} D(g_l) \).
Note that this implies $D(g_i) = W_{s(g_i)}$ as open subsets of $U$, where $W_{s(g_i)}$ denotes the largest open subset of $W$ on which $s(g_i)$ is invertible. Hence

$$B_{g_i} = \Gamma(D(g_i), \mathcal{O}_U) = \Gamma(W_{s(g_i)}, \mathcal{O}_W) = R_{s(g_i)},$$

where the last equality is Properties, Lemma 17.1. Since $W_{s(g_i)}$ is affine this also implies that $D(s(g_i)) = W_{s(g_i)}$ as open subsets of $\text{Spec}(R)$. Since $R = \text{colim}_i R_i$ we can (after shrinking $I$) assume there exist $g_{i,i} \in R_i$ for all $i \in I$ such that $s(g_i) = t_i(g_{i,i})$. Of course we choose the $g_{i,i}$ such that $g_{i,i}$ maps to $g_{i,i'}$, under the transition maps $R_i \to R_{i'}$. Then, by Lemma 14.1 we can (after shrinking $I$ again) assume the corresponding opens $D(g_{i,i}) \subset \text{Spec}(R_i)$ are contained in $W_i$ for $i = 1, \ldots, m$ and cover $W$. We conclude that the morphism $W_i \to \text{Spec}(R_i) \to \text{Spec}(R_i)$ is an open immersion, see Lemma 5.2.

By Lemma 5.1 we can write $B_i$ as a directed colimit of subalgebras $A_{i,p} \subset B_i$, $p \in P$, each of finite type over $\mathbf{Z}$ and such that $W_i$ is identified with an open subscheme of $\text{Spec}(A_{i,p})$. Let $S_{i,p}$ be the scheme obtained by glueing $V_i$ and $\text{Spec}(A_{i,p})$ along the open $W_i$, see Schemes, Section 14. Here is the resulting commutative diagram of schemes:

\[
\begin{array}{ccc}
S & \leftarrow & W \\
\downarrow & & \downarrow \\
V_i & \leftarrow & W_i \\
\downarrow & & \downarrow \\
S_{i,p} & \leftarrow & \text{Spec}(A_{i,p})
\end{array}
\]

The morphism $S \to S_{i,p}$ arises because the upper right square is a pushout in the category of schemes. Note that $S_{i,p}$ is of finite type over $\mathbf{Z}$ since it has a finite affine open covering whose members are spectra of finite type $\mathbf{Z}$-algebras. We define a preorder on $J = \displaystyle \bigsqcup_{i \in I} P_i$ by the rule $(i', p') \geq (i, p)$ if and only if $i' \geq i$ and the map $B_i \to B_{i'}$ maps $A_{i,p}$ into $A_{i',p'}$. This is exactly the condition needed to define a morphism $S_{i',p'} \to S_{i,p}$: namely make a commutative diagram as above using the transition morphisms $V_{i'} \to V_i$ and $W_{i'} \to W_i$ and the morphism $\text{Spec}(A_{i',p'}) \to \text{Spec}(A_{i,p})$ induced by the ring map $A_{i,p} \to A_{i',p'}$. The relevant commutativities have been built into the constructions. We claim that $S$ is the directed limit of the schemes $S_{i,p}$. Since by construction the schemes $V_i$ have limit $V$ this boils down to the fact that $B$ is the limit of the rings $A_{i,p}$ which is true by construction. The map $\alpha : J \to I$ is given by the rule $j = (i, p) \mapsto i$. The open subscheme $V'_i$ is just the image of $V_i \to S_{i,p}$ above. The commutativity of the diagrams in (5) is clear from the construction. This finishes the proof of the lemma. \hfill \square

01ZA **Proposition 5.4.** Let $S$ be a quasi-compact and quasi-separated scheme. There exist a directed set $I$ and an inverse system of schemes $(S_i, f_{ii'})$ over $I$ such that

1. the transition morphisms $f_{ii'}$ are affine
2. each $S_i$ is of finite type over $\mathbf{Z}$, and
3. $S = \text{lim}_i S_i$.

**Proof.** This is a special case of Lemma 5.3 with $V = \emptyset$. \hfill \square
Proposition 6.1. Let \( f : X \to S \) be a morphism of schemes. The following are equivalent:

1. The morphism \( f \) is locally of finite presentation.
2. For any directed set \( I \), and any inverse system \( (T_i, f_{ii'}) \) of \( S \)-schemes over \( I \) with each \( T_i \) affine, we have
   \[
   \text{Mor}_S(\lim_i T_i, X) = \colim_i \text{Mor}_S(T_i, X)
   \]
3. For any directed set \( I \), and any inverse system \( (T_i, f_{ii'}) \) of \( S \)-schemes over \( I \) with each \( f_{ii'} \) affine and every \( T_i \) quasi-compact and quasi-separated as a scheme, we have
   \[
   \text{Mor}_S(\lim_i T_i, X) = \colim_i \text{Mor}_S(T_i, X)
   \]

Proof. It is clear that (3) implies (2).

Let us prove that (2) implies (1). Assume (2). Choose any affine opens \( U \subset X \) and \( V \subset S \) such that \( f(U) \subset V \). We have to show that \( \mathcal{O}_S(V) \to \mathcal{O}_X(U) \) is of finite presentation. Let \( (A_i, \varphi_{ii'}) \) be a directed system of \( \mathcal{O}_S(V) \)-algebras. Set \( A = \colim_i A_i \). According to Algebra, Lemma 126.3 we have to show that
\[
\text{Hom}_{\mathcal{O}_S(V)}(\mathcal{O}_X(U), A) = \colim_i \text{Hom}_{\mathcal{O}_S(V)}(\mathcal{O}_X(U), A_i)
\]
Consider the schemes \( T_i = \text{Spec}(A_i) \). They form an inverse system of \( V \)-schemes over \( I \) with transition morphisms \( f_{ii'} : T_i \to T_{i'} \) induced by the \( \mathcal{O}_S(V) \)-algebra maps \( \varphi_{ii'} \). Set \( T := \text{Spec}(A) = \lim_i T_i \). The formula above becomes in terms of morphism sets of schemes
\[
\text{Mor}_V(\lim_i T_i, U) = \colim_i \text{Mor}_V(T_i, U)
\]
We first observe that \( \text{Mor}_V(T_i, U) = \text{Mor}_S(T_i, U) \) and \( \text{Mor}_V(T, U) = \text{Mor}_S(T, U) \). Hence we have to show that
\[
\text{Mor}_S(\lim_i T_i, U) = \colim_i \text{Mor}_S(T_i, U)
\]
and we are given that
\[
\text{Mor}_S(\lim_i T_i, X) = \colim_i \text{Mor}_S(T_i, X)
\]
Hence it suffices to prove that given a morphism \( g_i : T_i \to X \) over \( S \) such that the composition \( T \to T_i \to X \) ends up in \( U \) there exists some \( i' \geq i \) such that the composition \( g_{i'} : T_{i'} \to T_i \to X \) ends up in \( U \). Denote \( Z_{i'} = g_{i'}^{-1}(X \setminus U) \). Assume each \( Z_{i'} \) is nonempty to get a contradiction. By Lemma 4.8 there exists a point \( t \) of \( T \) which is mapped into \( Z_{i'} \) for all \( i' \geq i \). Such a point is not mapped into \( U \). A contradiction.

Finally, let us prove that (1) implies (3). Assume (1). Let an inverse directed system \( (T_i, f_{ii'}) \) of \( S \)-schemes be given. Assume the morphisms \( f_{ii'} \) are affine and each \( T_i \) is quasi-compact and quasi-separated as a scheme. Let \( T = \lim_i T_i \). Denote \( f_i : T \to T_i \) the projection morphisms. We have to show:

(a) Given morphisms \( g_i, g_i' : T_i \to X \) over \( S \) such that \( g_i \circ f_i = g_i' \circ f_i \), then there exists an \( i' \geq i \) such that \( g_i \circ f_{i'i} = g_i' \circ f_{i'i} \).
(b) Given any morphism \( g : T \to X \) over \( S \) there exists an \( i \in I \) and a morphism \( g_i : T_i \to X \) such that \( g = f_i \circ g_i \).
First let us prove the uniqueness part (a). Let \( g_i, g'_i : T_i \to X \) be morphisms such that \( g_i \circ f_i = g'_i \circ f_i \). For any \( i' \geq i \) we set \( g_{i'} = g_i \circ f_{i'} \) and \( g'_{i'} = g'_i \circ f_{i'} \). We also set \( g = g_i \circ f_i = g'_i \circ f_i \). Consider the morphism \((g_i, g'_i) : T_i \to X \times_S X\). Set

\[
W = \bigcup_{U \subset X \text{ affine open}, V \subset S \text{ affine open}, f(U) \subset V} U 	imes_V U.
\]

This is an open in \( X \times_S X \), with the property that the morphism \( \Delta_X/S \) factors through a closed immersion into \( W \), see the proof of Schemes, Lemma \[21.2\]. Note that the composition \((g_i, g'_i) \circ f_i : T \to X \times_S X\) is a morphism into \( W \) because it factors through the diagonal by assumption. Set \( Z_{i'} = (g_{i'}, g'_{i'})^{-1}(X \times_S X \setminus W) \). If each \( Z_{i'} \) is nonempty, then by Lemma \[4.13\] there exists a point \( t \in T \) which maps to \( Z_{i'} \) for all \( i' \geq i \). This is a contradiction with the fact that \( T \) maps into \( W \).

Hence we may increase \( i \) and assume that \((g_i, g'_i) : T_i \to X \times_S X\) is a morphism into \( W \). By construction of \( W \), we can find for each \( i \) affine open covering \( T_i = T_{i,1} \cup \ldots \cup T_{i,n} \) such that \((g_i, g'_i)|_{T_{i,j}}\) is a morphism into \( U \times_V U \) for some pair \((U, V)\) as in the definition of \( W \) above. Since it suffices to prove that \( g_{i'} \) and \( g'_{i'} \) agree on each of the \( f_{i'}^{-1}(T_{j,i}) \) this reduces us to the affine case.

The affine case follows from Algebra, Lemma \[126.3\] and the fact that the ring map \( \mathcal{O}_S(V) \to \mathcal{O}_X(U) \) is of finite presentation (see Morphisms, Lemma \[20.2\]).

Finally, we prove the existence part (b). Let \( g : T \to X \) be a morphism of schemes over \( S \). We can find a finite affine open covering \( T = W_1 \cup \ldots \cup W_n \) such that for each \( j \in \{1, \ldots, n\} \) there exist affine opens \( U_j \subset X \) and \( V_j \subset S \) with \( f(U_j) \subset V_j \) and \( g(W_j) \subset U_j \). By Lemmas \[4.11\] and \[4.13\] (after possibly shrinking \( I \)) we may assume that there exist affine open coverings \( T_i = W_{1,i} \cup \ldots \cup W_{n,i} \) compatible with transition maps such that \( W_j = \lim W_{j,i} \). We apply Algebra, Lemma \[126.3\] to the rings corresponding to the affine schemes \( U_j, V_j, W_{j,i} \) and \( W_{j,i} \), using that \( \mathcal{O}_S(V_j) \to \mathcal{O}_X(U_j) \) is of finite presentation (see Morphisms, Lemma \[20.2\]). Thus we can find for each \( j \) an index \( i_j \in I \) and a morphism \( g_{j,i_j} : W_{j,i_j} \to X \) such that \( g_{j,i_j} \circ f_{i_j}|_{W_j} = g|_{W_j} \). By part (a) proved above, using the quasi-compactness of \( W_{j,i} \cap W_{j',i} \) which follows as \( T_i \) is quasi-separated, we can find an index \( i' \in I \) larger than all \( i_j \) such that

\[
g_{j_1,i_{j_1}} \circ f_{i_{j_1}}|_{W_{j_1,i_{j_1} \cap W_{j_2,i'}}} = g_{j_2,i_{j_2}} \circ f_{i_{j_2}}|_{W_{j_1,i_{j_1} \cap W_{j_2,i'}}}
\]

for all \( j_1, j_2 \in \{1, \ldots, n\} \). Hence the morphisms \( g_{j,i_j} \circ f_{i_j}|_{W_{j,i'}} \) glue to given the desired morphism \( T_{i'} \to X \).

Let \( S \) be a scheme. Let us say that a functor \( F : (\text{Sch}/S)^{opp} \to \text{Sets} \) is \textit{limit preserving} if for every directed inverse system \( \{T_i\}_{i \in I} \) of affine schemes with limit \( T \) we have \( F(T) = \lim_i F(T_i) \). Let \( X \) be a scheme over \( S \), and let \( h_X : (\text{Sch}/S)^{opp} \to \text{Sets} \) be its functor of points, see Schemes, Section \[15\]. In this terminology Proposition \[6.1\] says that a scheme \( X \) is locally of finite presentation over \( S \) if and only if \( h_X \) is limit preserving.

\textbf{0CM0 Lemma 6.3.} Let \( f : X \to S \) be a morphism of schemes. If for every directed limit \( T = \lim_{i \in I} T_i \) of affine schemes over \( S \) the map

\[
\text{colim} \text{Mor}_S(T_i, X) \to \text{Mor}_S(T, X)
\]

is surjective, then \( f \) is locally of finite presentation. In other words, in Proposition \[6.1\] parts (2) and (3) it suffices to check surjectivity of the map.
Proof. The proof is exactly the same as the proof of the implication “(2) implies (1)” in Proposition [6.1]. Choose any affine opens $U \subset X$ and $V \subset S$ such that $f(U) \subset V$. We have to show that $\mathcal{O}_S(V) \rightarrow \mathcal{O}_X(U)$ is of finite presentation. Let $(A_i, \varphi_{ii'})$ be a directed system of $\mathcal{O}_S(V)$-algebras. Set $A = \text{colim}_i A_i$. According to Algebra, Lemma [26.3] it suffices to show that
\[
\text{colim}_i \text{Hom}_{\mathcal{O}_S(V)}(\mathcal{O}_X(U), A_i) \rightarrow \text{Hom}_{\mathcal{O}_S(V)}(\mathcal{O}_X(U), A)
\]
is surjective. Consider the schemes $T_i = \text{Spec}(A_i)$. They form an inverse system of $V$-schemes over $I$ with transition morphisms $f_{ii'} : T_i \rightarrow T_{i'}$ induced by the $\mathcal{O}_S(V)$-algebra maps $\varphi_{ii'}$. Set $T := \text{Spec}(A) = \text{lim}_i T_i$. The formula above becomes in terms of morphism sets of schemes
\[
\text{colim}_i \text{Mor}_V(T_i, U) \rightarrow \text{Mor}_V(\text{lim}_i T_i, U)
\]
We first observe that $\text{Mor}_V(T_i, U) = \text{Mor}_S(T_i, U)$ and $\text{Mor}_V(T, U) = \text{Mor}_S(T, U)$. Hence we have to show that
\[
\text{colim}_i \text{Mor}_S(T_i, U) \rightarrow \text{Mor}_S(\text{lim}_i T_i, U)
\]
is surjective and we are given that
\[
\text{colim}_i \text{Mor}_S(T_i, X) \rightarrow \text{Mor}_S(\text{lim}_i T_i, X)
\]
is surjective. Hence it suffices to prove that given a morphism $g_i : T_i \rightarrow X$ over $S$ such that the composition $T \rightarrow T_i \rightarrow X$ ends up in $U$ there exists some $i' \geq i$ such that the composition $g_{i'} : T_{i'} \rightarrow T_i \rightarrow X$ ends up in $U$. Denote $Z_{i'} = g_{i'}^{-1}(X \setminus U)$. Assume each $Z_{i'}$ is nonempty to get a contradiction. By Lemma [2.8] there exists a point $t$ of $T$ which is mapped into $Z_{i'}$ for all $i' \geq i$. Such a point is not mapped into $U$. A contradiction. \hfill \square

7. Relative approximation

09MU The title of this section refers to results of the following type.

09MV Lemma 7.1. Let $f : X \rightarrow S$ be a morphism of schemes. Assume that
\begin{enumerate}
\item $X$ is quasi-compact and quasi-separated, and
\item $S$ is quasi-separated.
\end{enumerate}
Then $X = \text{lim} X_i$ is a limit of a directed system of schemes $X_i$ of finite presentation over $S$ with affine transition morphisms over $S$.

Proof. Since $f(X)$ is quasi-compact we may replace $S$ by a quasi-compact open containing $f(X)$. Hence we may assume $S$ is quasi-compact as well. Write $X = \text{lim} X_a$ and $S = \text{lim} S_b$ as in Proposition [6.4] i.e., with $X_a$ and $S_b$ of finite type over $Z$ and with affine transition morphisms. By Proposition [6.1] we find that for each $b$ there exists an $a$ and a morphism $f_{a,b} : X_a \rightarrow S_b$ making the diagram
\[
\begin{array}{ccc}
X & \rightarrow & S \\
\downarrow & & \downarrow \\
X_a & \rightarrow & S_b
\end{array}
\]
commute. Moreover the same proposition implies that, given a second triple $(a', b', f_{a',b'})$, there exists an $a'' \geq a'$ such that the compositions $X_{a''} \rightarrow X_a \rightarrow S_b$ and $X_{a''} \rightarrow$
Notation and assumptions as in Situation 8.1. If $X_{a'} \to X_{b'} \to X_b$ are equal. Consider the set of triples $(a, b, f_{a,b})$ endowed with the preorder

$$(a, b, f_{a,b}) \geq (a', b', f_{a',b'}) \iff a \geq a', b' \geq b, \text{ and } f_{a,a'} \circ h_{a,a'} = g_{b',b} \circ f_{a,b}$$

where $h_{a,a'} : X_a \to X_{a'}$ and $g_{b',b} : S_b \to S_b$ are the transition morphisms. The remarks above show that this system is directed. It follows formally from the equalities $X = \lim X_a$ and $S = \lim S_b$ that

$$X = \lim(a, b, f_{a,b}) X_a \times_{f_{a,b}, S_b} S,$$

where the limit is over our directed system above. The transition morphisms $X_a \times_{S_b} S \to X_{a'} \times_{S_{b'}} S$ are affine as the composition

$$X_a \times_{S_b} S \to X_a \times_{S_{b'}} S \to X_{a'} \times_{S_{b'}} S$$

where the first morphism is a closed immersion (by Schemes, Lemma 21.9) and the second is a base change of an affine morphism (Morphisms, Lemma 11.8) and the composition of affine morphisms is affine (Morphisms, Lemma 11.7). The morphisms $f_{a,b}$ are of finite presentation (Morphisms, Lemmas 20.9 and 20.11) and hence the base changes $X_a \times_{f_{a,b}, S_b} S \to S$ are of finite presentation (Morphisms, Lemma 20.4). ∎

**Lemma 7.2.** Let $X \to S$ be an integral morphism with $S$ quasi-compact and quasi-separated. Then $X = \lim X_i$ with $X_i \to S$ finite and of finite presentation.

**Proof.** Consider the sheaf $\mathcal{A} = f_* \mathcal{O}_X$. This is a quasi-coherent sheaf of $\mathcal{O}_S$-algebras, see Schemes, Lemma 24.1. Combining Properties, Lemma 22.13 we can write $\mathcal{A} = \colim_i \mathcal{A}_i$ as a filtered colimit of finite and finitely presented $\mathcal{O}_S$-algebras. Then

$$X_i = \text{Spec}_S(\mathcal{A}_i) \longrightarrow S$$

is a finite and finitely presented morphism of schemes. By construction $X = \lim X_i$ which proves the lemma. ∎

### 8. Descending properties of morphisms

This section is the analogue of Section 1 for properties of morphisms over $S$. We will work in the following situation.

**Situation 8.1.** Let $S = \lim S_i$ be a limit of a directed system of schemes with affine transition morphisms (Lemma 2.2). Let $0 \in I$ and let $f_0 : X_0 \to Y_0$ be a morphism of schemes over $S_0$. Assume $S_0, X_0, Y_0$ are quasi-compact and quasi-separated. Let $f_i : X_i \to Y_i$ be the base change of $f_0$ to $S_i$ and let $f : X \to Y$ be the base change of $f_0$ to $S$.

**Lemma 8.2.** Notation and assumptions as in Situation 8.1. If $f$ is affine, then there exists an index $i \geq 0$ such that $f_i$ is affine.

**Proof.** Let $Y_0 = \bigcup_{j=1,\ldots,m} V_j$ be a finite affine open covering. Set $U_{j,0} = f_0^{-1}(V_j)$. For $i \geq 0$ we denote $V_{j,i}$ the inverse image of $V_j$ in $Y_i$ and $U_{j,i} = f_i^{-1}(V_{j,i})$. Similarly we have $U_j = f^{-1}(V_j)$. Then $U_j = \lim_{i \geq 0} U_{j,i}$ (see Lemma 2.2). Since $U_j$ is affine by assumption we see that each $U_{j,i}$ is affine for $i$ large enough, see Lemma 11.13 As there are finitely many $j$ we can pick an $i$ which works for all $j$. Thus $f_i$ is affine for $i$ large enough, see Morphisms, Lemma 11.13 ∎

**Lemma 8.3.** Notation and assumptions as in Situation 8.1. If
(1) \( f \) is a finite morphism, and
(2) \( f_0 \) is locally of finite type,
then there exists an \( i \geq 0 \) such that \( f_i \) is finite.

**Proof.** A finite morphism is affine, see Morphisms, Definition 42.1. Hence by Lemma 8.2 above after increasing 0 we may assume that \( f_0 \) is affine. By writing \( Y_0 \) as a finite union of affines we reduce to proving the result when \( X_0 \) and \( Y_0 \) are affine and map into a common affine \( W \subset S_0 \). The corresponding algebra statement follows from Algebra, Lemma 162.3. \( \square \)

**Lemma 8.4.** Notation and assumptions as in Situation 8.1. If
(1) \( f \) is unramified, and
(2) \( f_0 \) is locally of finite type,
then there exists an \( i \geq 0 \) such that \( f_i \) is unramified.

**Proof.** Choose a finite affine open covering \( Y_0 = \bigcup_{j=1}^{m} Y_{j,0} \) such that each \( Y_{j,0} \) maps into an affine open \( S_{j,0} \subset S_0 \). For each \( j \) let \( f_0^{-1} Y_{j,0} = \bigcup_{k=1}^{n_j} X_{k,0} \) be a finite affine open covering. Since the property of being unramified is local we see that it suffices to prove the lemma for the morphisms of affines \( X_{k,i} \to Y_{j,i} \to S_{i,i} \) which are the base changes of \( X_{k,0} \to Y_{j,0} \to S_{j,0} \) to \( S_i \). Thus we reduce to the case that \( X_0, Y_0, S_0 \) are affine.

In the affine case we reduce to the following algebra result. Suppose that \( R = \colim_{i \in I} R_i \). For some \( 0 \in I \) suppose given an \( R_0 \)-algebra map \( A_i \to B_i \) of finite type. If \( R \otimes_{R_0} A_0 \to R \otimes_{R_0} B_0 \) is unramified, then for some \( i \geq 0 \) the map \( R_i \otimes_{R_0} A_0 \to R_i \otimes_{R_0} B_0 \) is unramified. This follows from Algebra, Lemma 162.5. \( \square \)

**Lemma 8.5.** Notation and assumptions as in Situation 8.1. If
(1) \( f \) is a closed immersion, and
(2) \( f_0 \) is locally of finite type,
then there exists an \( i \geq 0 \) such that \( f_i \) is a closed immersion.

**Proof.** A closed immersion is affine, see Morphisms, Lemma 11.9. Hence by Lemma 8.2 above after increasing 0 we may assume that \( f_0 \) is affine. By writing \( Y_0 \) as a finite union of affines we reduce to proving the result when \( X_0 \) and \( Y_0 \) are affine and map into a common affine \( W \subset S_0 \). The corresponding algebra statement is a consequence of Algebra, Lemma 162.4. \( \square \)

**Lemma 8.6.** Notation and assumptions as in Situation 8.1. If \( f \) is separated, then \( f_i \) is separated for some \( i \geq 0 \).

**Proof.** Apply Lemma 8.5 to the diagonal morphism \( \Delta_{X_0/S_0} : X_0 \to X_0 \times_{S_0} X_0 \). (This is permissible as diagonal morphisms are locally of finite type and the fibre product \( X_0 \times_{S_0} X_0 \) is quasi-compact and quasi-separated, see Schemes, Lemma 21.2 Morphisms, Lemma 14.5, and Schemes, Remark 21.18. \( \square \)

**Lemma 8.7.** Notation and assumptions as in Situation 8.1. If
(1) \( f \) is flat,
(2) \( f_0 \) is locally of finite presentation,
then \( f_i \) is flat for some \( i \geq 0 \).
Proof. Choose a finite affine open covering $Y_0 = \bigcup_{j=1}^m Y_{j,0}$ such that each $Y_{j,0}$ maps into an affine open $S_{j,0} \subset S_0$. For each $j$ let $f_{j,0}^{-1}Y_{j,0} = \bigcup_{k=1}^{n_j} X_{k,0}$ be a finite affine open covering. Since the property of being flat is local we see that it suffices to prove the lemma for the morphisms of affines $X_{k,i} \to Y_{j,i} \to S_{j,i}$ which are the base changes of $X_{k,0} \to Y_{j,0} \to S_{j,0}$ to $S_i$. Thus we reduce to the case that $X_0, Y_0, S_0$ are affine.

In the affine case we reduce to the following algebra result. Suppose that $R = \text{colim}_{i \in I} R_i$. For some $0 \in I$ suppose given an $R_0$-algebra map $A_i \to B_i$ of finite presentation. If $R \otimes_{R_0} A_0 \to R \otimes_{R_0} B_0$ is flat, then for some $i \geq 0$ the map $R_i \otimes_{R_0} A_0 \to R_i \otimes_{R_0} B_0$ is flat. This follows from Algebra, Lemma 162.1 part (3).

\[ \square \]

Lemma 8.8. Notation and assumptions as in Situation 8.1. If

1. $f$ is finite locally free (of degree $d$),
2. $f_0$ is locally of finite presentation,

then $f_i$ is finite locally free (of degree $d$) for some $i \geq 0$.

Proof. By Lemmas 8.7 and 8.3 we find an $i$ such that $f_i$ is flat and finite. On the other hand, $f_i$ is locally of finite presentation. Hence $f_i$ is finite locally free by Morphisms, Lemma 46.2. If moreover $f$ is finite locally free of degree $d$, then the image of $Y \to Y_i$ is contained in the open and closed locus $W_d \subset Y_i$ over which $f_i$ has degree $d$. By Lemma 4.10 we see that for some $i' \geq i$ the image of $Y_{i'} \to Y_i$ is contained in $W_d$. Then $f_{i'}$ will be finite locally free of degree $d$.

\[ \square \]

Lemma 8.9. Notation and assumptions as in Situation 8.1. If

1. $f$ is smooth,
2. $f_0$ is locally of finite presentation,

then $f_i$ is smooth for some $i \geq 0$.

Proof. Being smooth is local on the source and the target (Morphisms, Lemma 32.2) hence we may assume $S_0, X_0, Y_0$ affine (details omitted). The corresponding algebra fact is Algebra, Lemma 162.8.

\[ \square \]

Lemma 8.10. Notation and assumptions as in Situation 8.1. If

1. $f$ is étale,
2. $f_0$ is locally of finite presentation,

then $f_i$ is étale for some $i \geq 0$.

Proof. Being étale is local on the source and the target (Morphisms, Lemma 34.2) hence we may assume $S_0, X_0, Y_0$ affine (details omitted). The corresponding algebra fact is Algebra, Lemma 162.7.

\[ \square \]

Lemma 8.11. Notation and assumptions as in Situation 8.1. If

1. $f$ is an isomorphism, and
2. $f_0$ is locally of finite presentation,

then $f_i$ is an isomorphism for some $i \geq 0$.

Proof. By Lemmas 8.10 and 8.5 we can find an $i$ such that $f_i$ is flat and a closed immersion. Then $f_i$ identifies $X_i$ with an open and closed subscheme of $Y_i$, see Morphisms, Lemma 25.2. By assumption the image of $Y \to Y_i$ maps into $f_i(X_i)$.
Thus by Lemma 4.10, we find that $Y_{i'}$ maps into $f_i(X_i)$ for some $i' \geq i$. It follows that $X_{i'} \to Y_{i'}$ is surjective and we win. □

Lemma 8.12. Notation and assumptions as in Situation 8.1. If

(1) $f$ is an open immersion, and

(2) $f_0$ is locally of finite presentation,

then $f_i$ is an open immersion for some $i \geq 0$.

Proof. By Lemma 8.10 we can find an $i$ such that $f_i$ is étale. Then $V_i = f_i(X_i)$ is a quasi-compact open subscheme of $Y_i$ (Morphisms, Lemma 34.13). Let $V$ and $V_{i'}$ for $i' \geq i$ be the inverse image of $V_i$ in $Y$ and $Y_{i'}$. Then $f : X \to V$ is an isomorphism (namely it is a surjective open immersion). Hence by Lemma 8.11 we see that $X_{i'} \to V_{i'}$ is an isomorphism for some $i' \geq i$ as desired. □

Lemma 8.13. Notation and assumptions as in Situation 8.1. If

(1) $f$ is a monomorphism, and

(2) $f_0$ is locally of finite type,

then $f_i$ is a monomorphism for some $i \geq 0$.

Proof. Recall that a morphism of schemes $V \to W$ is a monomorphism if and only if the diagonal $V \to V \times_W V$ is an isomorphism (Schemes, Lemma 23.2). The morphism $X_0 \to X_0 \times_{Y_0} X_0$ is locally of finite presentation by Morphisms, Lemma 20.12. Since $X_0 \times_{Y_0} X_0$ is quasi-compact and quasi-separated (Schemes, Remark 21.18), we conclude from Lemma 8.11 that $\Delta_i : X_i \to X_i \times_{Y_i} X_i$ is an isomorphism for some $i \geq 0$. For this $i$ the morphism $f_i$ is a monomorphism. □

Lemma 8.14. Notation and assumptions as in Situation 8.1. If

(1) $f$ is surjective, and

(2) $f_0$ is locally of finite presentation,

then there exists an $i \geq 0$ such that $f_i$ is surjective.

Proof. The morphism $f_0$ is of finite presentation. Hence $E = f_0(X_0)$ is a constructible subset of $Y_0$, see Morphisms, Lemma 21.2. Since $f_i$ is the base change of $f_0$ by $Y_i \to Y_0$ we see that the image of $f_i$ is the inverse image of $E$ in $Y_i$. Moreover, we know that $Y \to Y_0$ maps into $E$. Hence we win by Lemma 4.10. □

Lemma 8.15. Notation and assumptions as in Situation 8.1. If

(1) $f$ is syntomic, and

(2) $f_0$ is locally of finite presentation,

then there exists an $i \geq 0$ such that $f_i$ is syntomic.

Proof. Choose a finite affine open covering $Y_0 = \bigcup_{j=1,\ldots,m} Y_{j,0}$ such that each $Y_{j,0}$ maps into an affine open $S_{j,0} \subset S_0$. For each $j$ let $f_0^{-1}Y_{j,0} = \bigcup_{k=1,\ldots,n_j} X_{k,0}$ be a finite affine open covering. Since the property of being syntomic is local we see that it suffices to prove the lemma for the morphisms of affines $X_{k,i} \to Y_{j,i} \to S_{j,i}$ which are the base changes of $X_{k,0} \to Y_{j,0} \to S_{j,0}$ to $S_i$. Thus we reduce to the case that $X_0, Y_0, S_0$ are affine. In the affine case we reduce to the following algebra result. Suppose that $R = \operatorname{colim}_{i \in I} R_i$. For some $0 \in I$ suppose given an $R_0$-algebra map $A_i \to B_i$ of finite presentation. If $R \otimes_{R_0} A_0 \to R \otimes_{R_0} B_0$ is syntomic, then for some $i \geq 0$ the map $R_i \otimes_{R_0} A_0 \to R_i \otimes_{R_0} B_0$ is syntomic. This follows from Algebra, Lemma 152.11. □
9. Finite type closed in finite presentation

Lemma 9.1. Let \( f : X \to S \) be a morphism of schemes. Assume:

1. The morphism \( f \) is locally of finite type.
2. The scheme \( X \) is quasi-compact and quasi-separated.

Then there exists a morphism of finite presentation \( f' : X' \to S \) and an immersion \( X \to X' \) of schemes over \( S \).

Proof. By Proposition 5.4 we can write \( X = \lim_i X_i \) with each \( X_i \) of finite type over \( \mathbb{Z} \) and with transition morphisms \( f_{ii'} : X_i \to X_{i'} \) affine. Consider the commutative diagram

\[
\begin{array}{ccc}
X & \longrightarrow & X_{i,S} \\
\downarrow & & \downarrow \\
S & \longrightarrow & \text{Spec}(\mathbb{Z})
\end{array}
\]

Note that \( X_i \) is of finite presentation over \( \text{Spec}(\mathbb{Z}) \), see Morphisms, Lemma 20.9. Hence the base change \( X_{i,S} \to S \) is of finite presentation by Morphisms, Lemma 20.4. Thus it suffices to show that the arrow \( X \to X_{i,S} \) is an immersion for \( i \) sufficiently large.

To do this we choose a finite affine open covering \( X = V_1 \cup \ldots \cup V_n \) such that \( f \) maps each \( V_j \) into an affine open \( U_j \subset S \). Let \( h_{j,a} \in \mathcal{O}_X(V_j) \) be a finite set of elements which generate \( \mathcal{O}_X(V_j) \) as an \( \mathcal{O}_S(U_j) \)-algebra, see Morphisms, Lemma 14.2. By Lemmas 4.11 and 4.13 (after possibly shrinking \( I \)) we may assume that there exist affine open coverings \( X_i = V_{1,i} \cup \ldots \cup V_{n,i} \) compatible with transition maps such that \( V_j = \lim_i V_{j,i} \). By Lemma 4.7, we can choose \( i \) so large that each \( h_{j,a} \) comes from an element \( h_{j,a,i} \in \mathcal{O}_{X_i}(V_{j,i}) \). Thus the arrow in

\[
V_j \longrightarrow U_j \times_{\text{Spec}(\mathbb{Z})} V_{j,i} = (V_{j,i})_{U_j} \subset (V_{j,i})_S \subset X_{i,S}
\]

is a closed immersion. Since \( \bigcup (V_{j,i})_{U_j} \) forms an open of \( X_{i,S} \) and since the inverse image of \( (V_{j,i})_{U_j} \) in \( X \) is \( V_j \) it follows that \( X \to X_{i,S} \) is an immersion. \( \square \)

Remark 9.2. We cannot do better than this if we do not assume more on \( S \) and the morphism \( f : X \to S \). For example, in general it will not be possible to find a closed immersion \( X \to X' \) as in the lemma. The reason is that this would imply that \( f \) is quasi-compact which may not be the case. An example is to take \( S \) to be infinite dimensional affine space with 0 doubled and \( X \) to be one of the two infinite dimensional affine spaces.

Lemma 9.3. Let \( f : X \to S \) be a morphism of schemes. Assume:

1. The morphism \( f \) is of locally of finite type.
2. The scheme \( X \) is quasi-compact and quasi-separated, and
3. The scheme \( S \) is quasi-separated.

Then there exists a morphism of finite presentation \( f' : X' \to S \) and a closed immersion \( X \to X' \) of schemes over \( S \).

Proof. By Lemma 9.1 above there exists a morphism \( Y \to S \) of finite presentation and an immersion \( i : X \to Y \) of schemes over \( S \). For every point \( x \in X \), there exists an affine open \( V_x \subset Y \) such that \( i^{-1}(V_x) \to V_x \) is a closed immersion. Since
Let $X$ be quasi-compact we can find finitely many affine opens $V_1, \ldots, V_n \subset Y$ such that $i(X) \subset V_1 \cup \ldots \cup V_n$ and $i^{-1}(V_j) \to V_j$ is a closed immersion. In other words such that $i : X \to X' = V_1 \cup \ldots \cup V_n$ is a closed immersion of schemes over $S$. Since $S$ is quasi-separated and $Y$ is quasi-separated over $S$ we deduce that $Y$ is quasi-separated, see Schemes, Lemma \[21.12\]. Hence the open immersion $X' = V_1 \cup \ldots \cup V_n \to Y$ is quasi-compact. This implies that $X' \to Y$ is of finite presentation, see Morphisms, Lemma \[20.6\]. We conclude since then $X' \to Y \to S$ is a composition of morphisms of finite presentation, and hence of finite presentation (see Morphisms, Lemma \[20.3\]).

\[\Box\]

**Lemma 9.4.** Let $X \to Y$ be a closed immersion of schemes. Assume $Y$ quasi-compact and quasi-separated. Then $X$ can be written as a directed limit $X = \lim X_i$ of schemes over $Y$ where $X_i \to Y$ is a closed immersion of finite presentation.

**Proof.** Let $I \subset O_Y$ be the quasi-coherent sheaf of ideals defining $X$ as a closed subscheme of $Y$. By Properties, Lemma \[22.3\] we can write $I$ as a directed colimit $I = \colim_{i \in I} I_i$ of its quasi-coherent sheaves of ideals of finite type. Let $X_i \subset Y$ be the closed subscheme defined by $I_i$. These form an inverse system of schemes indexed by $I$. The transition morphisms $X_i \to X_j$ are affine because they are closed immersions. Each $X_i$ is quasi-compact and quasi-separated since it is a closed subscheme of $Y$ and $Y$ is quasi-compact and quasi-separated by our assumptions.

We have $X = \lim_i X_i$ as follows directly from the fact that $I = \colim_{i \in I} I_i$. Each of the morphisms $X_i \to Y$ is of finite presentation, see Morphisms, Lemma \[20.7\].

\[\Box\]

**Lemma 9.5.** Let $f : X \to S$ be a morphism of schemes. Assume

1. The morphism $f$ is of locally of finite type.
2. The scheme $X$ is quasi-compact and quasi-separated, and
3. The scheme $S$ is quasi-separated.

Then $X = \lim X_i$ where the $X_i \to S$ are of finite presentation, the $X_i$ are quasi-compact and quasi-separated, and the transition morphisms $X_{ij} \to X_i$ are closed immersions (which implies that $X \to X_i$ are closed immersions for all $i$).

**Proof.** By Lemma \[9.3\] there is a closed immersion $X \to Y$ with $Y \to S$ of finite presentation. Then $Y$ is quasi-separated by Schemes, Lemma \[21.12\]. Since $X$ is quasi-compact, we may assume $Y$ is quasi-compact by replacing $Y$ with a quasi-compact open containing $X$. We see that $X = \lim X_i$ with $X_i \to Y$ a closed immersion of finite presentation by Lemma \[9.4\]. The morphisms $X_i \to S$ are of finite presentation by Morphisms, Lemma \[20.3\].

\[\Box\]

**Proposition 9.6.** Let $f : X \to S$ be a morphism of schemes. Assume

1. $f$ is of finite type and separated, and
2. $S$ is quasi-compact and quasi-separated.

Then there exists a separated morphism of finite presentation $f' : X' \to S$ and a closed immersion $X \to X'$ of schemes over $S$.

**Proof.** Apply Lemma \[9.5\] and note that $X_i \to S$ is separated for large $i$ by Lemma \[4.17\] as we have assumed that $X \to S$ is separated.

\[\Box\]

**Lemma 9.7.** Let $f : X \to S$ be a morphism of schemes. Assume

1. $f$ is finite, and
2. $S$ is quasi-compact and quasi-separated.
Then there exists a morphism which is finite and of finite presentation \( f' : X' \to S \) and a closed immersion \( X \to X' \) of schemes over \( S \).

**Proof.** We may write \( X = \lim X_i \) as in Lemma \[9.5\] Applying Lemma \[4.19\] we see that \( X_i \to S \) is finite for large enough \( i \). \( \square \)

**Lemma 9.8.** Let \( f : X \to S \) be a morphism of schemes. Assume

1. \( f \) is finite, and
2. \( S \) quasi-compact and quasi-separated.

Then \( X \) is a directed limit \( X = \lim X_i \) where the transition maps are closed immersions and the objects \( X_i \) are finite and of finite presentation over \( S \).

**Proof.** We may write \( X = \lim X_i \) as in Lemma \[9.5\] Applying Lemma \[4.19\] we see that \( X_i \to S \) is finite for large enough \( i \). \( \square \)

## 10. Descending relative objects

**Lemma 10.1.** Let \( I \) be a directed set. Let \( (S_i, f_{ii'}) \) be an inverse system of schemes over \( I \). Assume

1. the morphisms \( f_{ii'} : S_i \to S_{i'} \) are affine,
2. the schemes \( S_i \) are quasi-compact and quasi-separated.

Let \( S = \lim S_i \). Then we have the following:

1. For any morphism of finite presentation \( X \to S \) there exists an index \( i \in I \) and a morphism of finite presentation \( X_i \to S_i \) such that \( X \cong X_i,S \) as schemes over \( S \).
2. Given an index \( i \in I \), schemes \( X_i, Y_i \) of finite presentation over \( S_i \), and a morphism \( \varphi : X_i,S \to Y_i,S \) over \( S \), there exists an index \( i' \geq i \) and a morphism \( \varphi_{ii'} : X_{i',S_{i'}} \to Y_{i',S_{i'}} \) whose base change to \( S \) is \( \varphi \).
3. Given an index \( i \in I \), schemes \( X_i, Y_i \) of finite presentation over \( S_i \) and a pair of morphisms \( \varphi_i, \psi_i : X_i \to Y_i \) whose base changes \( \varphi_i,S = \psi_i,S \) are equal, there exists an index \( i' \geq i \) such that \( \varphi_{i,i'}, \psi_{i,i'} \).

In other words, the category of schemes of finite presentation over \( S \) is the colimit over \( I \) of the categories of schemes of finite presentation over \( S_i \).

**Proof.** In case each of the schemes \( S_i \) is affine, and we consider only affine schemes of finite presentation over \( S_i \), resp. \( S \) this lemma is equivalent to Algebra, Lemma \[126.8\] We claim that the affine case implies the lemma in general.

Let us prove (3). Suppose given an index \( i \in I \), schemes \( X_i, Y_i \) of finite presentation over \( S_i \) and a pair of morphisms \( \varphi_i, \psi_i : X_i \to Y_i \). Assume that the base changes are equal: \( \varphi_i,S = \psi_i,S \). We will use the notation \( X_{i'} = X_{i,S_{i'}} \) and \( Y_{i'} = Y_{i,S_{i'}} \) for \( i' \geq i \). We also set \( X = \lim_{i' \geq i} X_{i'} \) and similarly for \( Y \). Additionally we denote \( \varphi_{i'} \) and \( \psi_{i'} \) (resp. \( \varphi \) and \( \psi \)) the base change of \( \varphi_i \) and \( \psi_i \) to \( S_i \) (resp. \( S \)). So our assumption means that \( \varphi = \psi \). Since \( Y_i \) and \( X_i \) are of finite presentation over \( S_i \), and since \( S_i \) is quasi-compact and quasi-separated, also \( X_i \) and \( Y_i \) are quasi-compact and quasi-separated (see Morphisms, Lemma \[20.10\]). Hence we may choose a finite affine open
covering $Y_i = \bigcup V_{j,i}$ such that each $V_{j,i}$ maps into an affine open of $S$. As above, denote $V_{j,i}'$ the inverse image of $V_{j,i}$ in $Y_i$ and $V_j$ the inverse image in $Y$. The immersions $V_{j,i}' \to Y_{i'}$ are quasi-compact, and the inverse images $U_{j,i}' = \varphi_i^{-1}(V_{j,i}')$ and $U_{j,i}'' = \psi_i^{-1}(V_{j,i}')$ are quasi-compact opens of $X_{i'}$. By assumption the inverse images of $V_j$ under $\varphi$ and $\psi$ in $X$ are equal. Hence by Lemma 4.11 there exists an index $i' \geq i$ such that $U_{j,i'} = U_{j,i}'$, and a finite affine open covering $U_{j,i}'' = \bigcup W_{j,k,i''}$ which induce coverings $U_{j,i'}'' = \bigcup W_{j,k,i''}$ for all $i'' \geq i'$. By the affine case there exists an index $i''$ such that $\varphi_{i''}|_{W_{j,k,i''}} = \psi_{i''}|_{W_{j,k,i''}}$ for all $j,k$. Then $i''$ is an index such that $\varphi_{i''} = \psi_{i''}$ and (3) is proved.

Let us prove (2). Suppose given an index $i \in I$, schemes $X_i$, $Y_i$ of finite presentation over $S_i$ and a morphism $\varphi : X_i \to Y_i$. We will use the notation $X_i = X_{i,S_i}$ and $Y_i = Y_{i,S_i}$ for all $i \geq i$. We also set $X = \lim_{i \geq i} X_i$ and similarly for $Y$. Since $Y_i$ and $X_i$ are of finite presentation over $S_i$, and since $S_i$ is quasi-compact and quasi-separated, also $X_i$ and $Y_i$ are quasi-compact and quasi-separated (see Morphisms, Lemma 20.10).

Hence we may choose a finite affine open covering $X_i = \bigcup V_{j,i}$ such that each $V_{j,i}$ maps into an affine open of $S$. As above, denote $V_{j,i}'$ the inverse image of $V_{j,i}$ in $Y_i$ and $V_j$ the inverse image in $Y$. The immersions $V_j \to Y$ are quasi-compact, and the inverse images $U_j = \varphi^{-1}(V_j)$ are quasi-compact opens of $X$. Hence by Lemma 4.11 there exists an index $i' \geq i$ and quasi-compact opens $U_{j,i'}'$ of $X_{i'}$ whose inverse image in $X$ is $U_j$. Choose an finite affine open covering $U_{j,i'}' = \bigcup W_{j,k,i''}$ which induce affine open coverings $U_{j,i'}'' = \bigcup W_{j,k,i''}$ for all $i'' \geq i'$ and an affine open covering $U_j = \bigcup W_{j,k}$. By the affine case there exists an index $i''$ and morphisms $\varphi_{j,k,i''} : W_{j,k,i''} \to V_{j,i''}$ such that $\varphi|_{W_{j,k}} = \varphi_{j,k,i''}|_{S}$. By part (3) proved above, there is a further index $i''' \geq i''$ such that

$$\varphi_{j,k,i''} \circ \varphi_{j,k,i''}'|_{S_{j,k,i''}} = \varphi_{j,k,i''}' \circ \varphi_{j,k,i''}|_{S_{j,k,i''}}$$

for all $j,k,l$. Then $i'''$ is an index such that there exists a morphism $\varphi_{i'''} : X_{i'''} \to Y_{i'''}$ whose base change to $S$ gives $\varphi$. Hence (2) holds.

Let us prove (1). Suppose given a scheme $X$ of finite presentation over $S$. Since $X$ is of finite presentation over $S$, and since $S$ is quasi-compact and quasi-separated, also $X$ is quasi-compact and quasi-separated (see Morphisms, Lemma 20.10). Choose a finite affine open covering $X = \bigcup U_j$ such that each $U_j$ maps into an affine open $V_j \subset S$. Denote $U_{j1,j2} = U_{j1} \cap U_{j2}$ and $U_{j1,j2,j3} = U_{j1} \cap U_{j2} \cap U_{j3}$. By Lemmas 4.11 and 4.13 we can find an index $i$ and affine opens $V_{j1,i} \subset S_{j1}$ such that each $V_{j}$ is the inverse of this in $S$. Let $V_{j1,i}$ be the inverse image of $V_{j1,i}$ in $S_i$ for $i \geq i_1$. By the affine case we may find an index $i_2 \geq i_1$ and affine schemes $U_{j2,i2} \to V_{j2,i2}$ such that $U_j = S \times_{S_{j1}} U_{j2,i2}$ is the base change. Denote $U_{j,i} = S_i \times_{S_{i1}} U_{j2,i2}$ for $i \geq i_2$. By Lemma 4.11 there exists an index $i_3 \geq i_2$ and open subschemes $W_{j1,j2,i3} \subset U_{j1,i3}$ whose base change to $S$ is equal to $U_{j1,j2}$. Denote $W_{j1,j2,i} = S_i \times_{S_{i1}} W_{j1,j2,i3}$ for $i \geq i_3$. By part (2) shown above there exists an index $i_4 \geq i_3$ and morphisms $\varphi_{j1,j2,i4} : W_{j1,j2,i4} \to W_{j2,j1,i4}$ whose base change to $S$ gives the identity morphism $U_{j1,j2} = U_{j2,j1}$ for all $j_1,j_2$. For all $i \geq i_4$ denote $\varphi_{j1,j2,i4}^{-1} \circ \varphi_{j1,j2,i4}$ the base change. We claim that for some $i_5 \geq i_4$ the system $(U_{j1,i5},j(W_{j1,j2,i5} \cap W_{j2,j1,i5},(\varphi_{j1,j2,i5})_{j,j2,j1})$ forms a glueing datum as in Schemes, Section 1.4. In order to see this we have to verify that for $i$ large enough we have

$$\varphi_{j1,j2,i}^{-1}(W_{j1,j2,i} \cap W_{j2,j1,i}) = W_{j1,j2,i} \cap W_{j1,j3,i}$$
and that for large enough \( i \) the cocycle condition holds. The first condition follows from Lemma 10.2 and the fact that \( U_{j2,j3} = U_{j1,j2,j3} \). The second from part (1) of the lemma proved above and the fact that the cocycle condition holds for the maps \( \text{id} : U_{j1,j2} \to U_{j2,j1} \). Ok, so now we can use Schemes, Lemma 14.2 to glue the system \( \left( (U_{j0,j1,j2,j3}, (W_{j1,j2,i3}, j1,j2), (\varphi_{j1,j2,i3}, j1,j2) \right) \) to get a scheme \( X_{i5} \to S_{i5} \). By construction the base change of \( X_{i5} \) to \( S \) is formed by glueing the open affines \( U_j \)
along the opens \( U_j \leftarrow U_{j1,j2} \to U_{j2} \). Hence \( S \times_{S_{i5}} X_{i5} \cong X \) as desired. 

\[ \square \]

**Lemma 10.2.** Let \( I \) be a directed set. Let \( (S_i, f_{i\alpha}) \) be an inverse system of schemes over \( I \). Assume

1. all the morphisms \( f_{i\alpha} : S_i \to S_\alpha \) are affine,
2. all the schemes \( S_i \) are quasi-compact and quasi-separated.

Let \( S = \lim_i S_i \). Then we have the following:

1. For any sheaf of \( O_S \)-modules \( F \) of finite presentation there exists an index \( i \in I \) and a sheaf of \( O_{S_i} \)-modules of finite presentation \( F_i \) such that \( F \cong f_i^* F_i \).
2. Suppose given an index \( i \in I \), sheaves of \( O_{S_i} \)-modules \( F_i, G_i \) of finite presentation and a morphism \( \varphi : f_i^* F_i \to f_i^* G_i \) over \( S \). Then there exists an index \( i' \geq i \) and a morphism \( \varphi_i : f_{i'}^* F_i \to f_{i'}^* G_i \) whose base change to \( S \) is \( \varphi \).
3. Suppose given an index \( i \in I \), sheaves of \( O_{S_i} \)-modules \( F_i, G_i \) of finite presentation and a pair of morphisms \( \varphi_i, \psi_i : F_i \to G_i \). Assume that the base changes are equal: \( f_i^* \varphi_i = f_i^* \psi_i \). Then there exists an index \( i' \geq i \) such that \( f_{i'}^* \varphi_i = f_{i'}^* \psi_i \).

In other words, the category of modules of finite presentation over \( S \) is the colimit over \( I \) of the categories modules of finite presentation over \( S_i \).

**Proof.** We sketch two proofs, but we omit the details.

First proof. If \( S \) and \( S_i \) are affine schemes, then this lemma is equivalent to Algebra, Lemma 126.6. In the general case, use Zariski glueing to deduce it from the affine case.

Second proof. We use

1. there is an equivalence of categories between quasi-coherent \( O_S \)-modules and vector bundles over \( S \), see Constructions, Section 6 and
2. a vector bundle \( V(F) \to S \) is of finite presentation over \( S \) if and only if \( F \) is an \( O_S \)-module of finite presentation.

Having said this, we can use Lemma 10.1 to show that the category of vector bundles of finite presentation over \( S \) is the colimit over \( I \) of the categories of vector bundles over \( S_i \). 

\[ \square \]

**Lemma 10.3.** Let \( S = \lim S_i \) be the limit of a directed system of quasi-compact and quasi-separated schemes \( S_i \) with affine transition morphisms. Then

1. any finite locally free \( O_S \)-module is the pullback of a finite locally free \( O_{S_i} \)-module for some \( i \), and
2. any invertible \( O_S \)-module is the pullback of an invertible \( O_{S_i} \)-module for some \( i \).
Proof. Let $E$ be a finite locally free $O_S$-module. Since finite locally free modules are of finite presentation, we can find an $i$ and an $O_{S_i}$-module $E_i$ of finite presentation such that $f^*E_i \cong E$, see Lemma 10.2. After increasing $i$ we may assume $E_i$ is a flat $O_{S_i}$-module, see Algebra, Lemma 162.1. (Using this lemma is not necessary, but it is convenient.) Then $E_i$ is finite locally free by Algebra, Lemma 17.2.

If $L$ is an invertible $O_S$-module, then by the above we can find an $i$ and finite locally free $O_{S_i}$-modules $L_i$ and $N_i$ pulling back to $L$ and $L^{\otimes -1}$. After possible increasing $i$ we see that the map $L \otimes_{O_X} L^{\otimes -1} \to O_X$ descends to a map $L_i \otimes_{O_{S_i}} N_i \to O_{S_i}$. And after increasing $i$ further, we may assume it is an isomorphism. It follows that $L_i$ is an invertible module (Modules, Lemma 22.2) and the proof is complete. □

05LY Lemma 10.4. With notation and assumptions as in Lemma 10.1. Let $i \in I$. Suppose that $\varphi_i : X_i \to Y_i$ is a morphism of schemes of finite presentation over $S_i$ and that $F_i$ is a quasi-coherent $O_{X_i}$-module of finite presentation. If the pullback of $F_i$ to $X_i \times_{S_i} S$ is flat over $Y_i \times_{S_i} S$, then there exists an index $i' \geq i$ such that the pullback of $F_i$ to $X_i \times_{S_i} S_{i'}$ is flat over $Y_i \times_{S_i} S_{i'}$.

Proof. (This lemma is the analogue of Lemma 8.7 for modules.) For $i' \geq i$ denote $X_{i'} = S_{i'} \times_{S_i} X_i$, $F_{i'} = (X_{i'} \to X_i)^*F_i$ and similarly for $Y_{i'}$. Denote $\varphi_{i'}$ the base change of $\varphi_i$ to $S_{i'}$. Also set $X = S \times_{S_i} X_i$, $Y = S \times_{S_i} X_i$, $F = (X \to X_i)^*F_i$ and similarly for $\varphi$. Let $Y_i = \bigcup_{j=1,\ldots,m} V_j$ be a finite affine open covering such that each $V_j,i$ maps into some affine open of $S_i$. For each $j = 1, \ldots, m$ let $\varphi^{-1}_{i'}(V_{j,i}) = \bigcup_{k=1,\ldots,m(j)} U_{k,j,i}$ be a finite affine open covering. For $i' \geq i$ we denote $V_{j,i'}$ the inverse image of $V_{j,i}$. Let $Y_{j,i'} = \bigcup_{k=1,\ldots,m(j)} U_{k,j,i}$ the inverse image of $U_{k,j,i}$ in $X_{i'}$. Similarly we have $U_{k,j,i} \subset X$ and $V_{j,i} \subset Y$. Then $U_{k,j} = \lim_{i' \geq i} U_{k,j,i'}$ and $V_{j,i} = \lim_{i' \geq i} V_{j,i}$ (see Lemma 2.2). Since $X_{i'} = \bigcup_{j,i} U_{k,j,i'}$ is a finite open covering it suffices to prove the lemma for each of the morphisms $U_{k,j,i} \to V_{j,i}$ and the sheaf $F_i|_{U_{k,j,i}}$. Hence we see that the lemma reduces to the case that $X_i$ and $Y_i$ are affine and map into an affine open of $S_i$, i.e., we may also assume that $S$ is affine.

In the affine case we reduce to the following algebra result. Suppose that $R = \text{colim}_{i \in I} R_i$. For some $i \in I$ suppose given a map $A_i \to B_i$ of finitely presented $R_i$-algebras. Let $N_i$ be a finitely presented $B_i$-module. Then, if $R \otimes_R N_i$ is flat over $R \otimes_R A_i$, then for some $i' \geq i$ the module $R_{i'} \otimes_R N_i$ is flat over $R_{i'} \otimes_R A$. This is exactly the result proved in Algebra, Lemma 162.1 part (3). □

07Y1 Lemma 10.5. For a scheme $T$ denote $\mathcal{C}_T$ the full subcategory of schemes $W$ over $T$ such that $W$ is quasi-compact and quasi-separated and such that the structure morphism $W \to T$ is locally of finite presentation. Let $S = \lim S_i$ be a directed limit of schemes with affine transition morphisms. Then there is an equivalence of categories

$$\text{colim} \mathcal{C}_{S_i} \to \mathcal{C}_S$$

given by the base change functors.

Warning: do not use this lemma if you do not understand the difference between this lemma and Lemma 10.1.

Proof. Fully faithfulness. Suppose we have $i \in I$ and objects $X_i$, $Y_i$ of $\mathcal{C}_{S_i}$. Denote $X = X_i \times_{S_i} S$ and $Y = Y_i \times_{S_i} S$. Suppose given a morphism $f : X \to Y$ over $S$. We can choose a finite affine open covering $Y_i = V_{i,1} \cup \ldots \cup V_{i,m}$ such that $V_{i,j} \to Y_i \to S_i$ maps into an affine open $W_{i,j}$ of $S_i$. Denote $Y = V_1 \cup \ldots \cup V_m$ the
induced affine open covering of $Y$. Since $f : X \to Y$ is quasi-compact (Schemes, Lemma 21.14) after increasing $i$ we may assume that there is a finite open covering $X = U_{i_1} \cup \ldots \cup U_{i_m}$ by quasi-compact opens such that the inverse image of $U_{i,j}$ in $Y$ is $f^{-1}(V_j)$, see Lemma 4.11. By Lemma 10.1 applied to $f|_{f^{-1}(V_j)}$ over $W_j$ we may assume, after increasing $i$, that there is a morphism $f_{i,j} : V_{i,j} \to U_{i,j}$ over $S$ whose base change to $S$ is $f|_{f^{-1}(V_j)}$. Increasing $i$ more we may assume $f_{i,j}$ and $f_{i,j'}$ agree on the quasi-compact open $U_{i,j} \cap U_{i,j'}$. Then we can glue these morphisms to get the desired morphism $f_i : X_i \to Y_i$. This morphism is unique (up to increasing $i$) because this is true for the morphisms $f_{i,j}$.

To show that the functor is essentially surjective we argue in exactly the same way. Namely, suppose that $X$ is an object of $\mathcal{C}$, i.e., assume $S$ is affine. Pick $i \in I$. We can choose a finite affine open covering $X = U_1 \cup \ldots \cup U_m$ such that $U_j \to S \to S_i$ factors through an affine open $W_{i,j} \subset S_i$. Set $W_j = W_{i,j} \times_{S_i} S$. This is an affine open of $S$. By Lemma 10.1 after increasing $i$, we may assume there exist $U_{i,j} \to W_{i,j}$ of finite presentation whose base change to $W_j$ is $U_j$. After increasing $i$ we may assume there exist quasi-compact opens $U_{i,j,j'} \subset U_{i,j}$ whose base changes to $S$ are equal to $U_j \cap U_{j'}$. Claim: after increasing $i$ we may assume the image of the morphism $U_{i,j,j'} \to U_{i,j} \to W_{i,j}$ ends up in $W_{i,j} \cap W_{i,j'}$. Namely, because the complement of $W_{i,j} \cap W_{i,j'}$ is closed in the affine scheme $W_{i,j}$ it is affine. Since $U_j \cap U_{j'} = \lim U_{i,j,j'}$ does map into $W_{i,j} \cap W_{i,j'}$ we can apply Lemma 4.11 to get the claim. Thus we can view both

$$U_{i,j,j'} \text{ and } U_{i,j,j'}$$

as schemes over $W_{i,j'}$ whose base changes to $W_{j'}$ recover $U_j \cap U_{j'}$. Hence after increasing $i$, using Lemma 10.1 we may assume there are isomorphisms $U_{i,j,j'} \to U_{i,j,j'}$ over $W_{i,j'}$ and hence over $S_i$.

Increasing $i$ further (details omitted) we may assume these isomorphisms satisfy the cocycle condition mentioned in Schemes, Section 14. Applying Schemes, Lemma 14.1 we obtain an object $X_i$ of $\mathcal{C}_S$ whose base change to $S$ is isomorphic to $X$; we omit some of the verifications.

11. Characterizing affine schemes

01ZS If $f : X \to S$ is a surjective integral morphism of schemes such that $X$ is an affine scheme then $S$ is affine too. See [Con07, A.2]. Our proof relies on the Noetherian case which we stated and proved in Cohomology of Schemes, Lemma 13.3. See also [DG67, II 6.7.1].

01ZT Lemma 11.1. Let $f : X \to S$ be a morphism of schemes. Assume that $f$ is surjective and finite, and assume that $X$ is affine. Then $S$ is affine.

Proof. Since $f$ is surjective and $X$ is quasi-compact we see that $S$ is quasi-compact. Since $X$ is separated and $f$ is surjective and universally closed (Morphisms, Lemma 42.7), we see that $S$ is separated (Morphisms, Lemma 39.11).

By Lemma 9.8 we can write $X = \lim X_\alpha$ with $X_\alpha \to S$ finite and of finite presentation. By Lemma 4.13 we see that $X_\alpha$ is affine for some $\alpha \in A$. Replacing $X$ by $X_\alpha$ we may assume that $X \to S$ is surjective, finite, of finite presentation and that $X$ is affine.

By Proposition 5.4 we may write $S = \lim S_i$ as a directed limits as schemes of finite type over $Z$. By Lemma 10.1 we can after shrinking $I$ assume there exist schemes $X_i \to S_i$ of finite presentation such that $X_{i'} = X_i \times_S S_{i'}$ for $i' \geq i$ and
such that $X = \lim_i X_i$. By Lemma 8.3 we may assume that $X_i \to S_i$ is finite for all $i \in I$ as well. By Lemma 4.13 once again we may assume that $X_i$ is affine for all $i \in I$. Hence the result follows from the Noetherian case, see Cohomology of Schemes, Lemma 13.3.

\begin{proposition}
Let $f : X \to S$ be a morphism of schemes. Assume that $f$ is surjective and integral, and assume that $X$ is affine. Then $S$ is affine.
\end{proposition}

\begin{proof}
Since $f$ is surjective and $X$ is quasi-compact we see that $S$ is quasi-compact. Since $X$ is separated and $f$ is surjective and universally closed (Morphisms, Lemma 42.7), we see that $S$ is separated (Morphisms, Lemma 39.11).

By Lemma 7.2 we can write $X = \lim_i X_i$ with $X_i \to S$ finite. By Lemma 11.3 we see that for $i$ sufficiently large the scheme $X_i$ is affine. Moreover, since $X \to S$ factors through each $X_i$ we see that $X_i \to S$ is surjective. Hence we conclude that $S$ is affine by Lemma 11.1.
\end{proof}

\begin{lemma}
Let $X$ be a scheme which is set theoretically the union of finitely many affine closed subschemes. Then $X$ is affine.
\end{lemma}

\begin{proof}
Let $Z_i \subset X$, $i = 1, \ldots, n$ be affine closed subschemes such that $X = \bigcup Z_i$ set theoretically. Then $\coprod_i Z_i \to X$ is surjective and integral with affine source. Hence $X$ is affine by Proposition 11.2.
\end{proof}

\begin{lemma}
Let $i : Z \to X$ be a closed immersion of schemes inducing a homeomorphism of underlying topological spaces. Let $\mathcal{L}$ be an invertible sheaf on $X$. Then $i^* \mathcal{L}$ is ample on $Z$, if and only if $\mathcal{L}$ is ample on $X$.
\end{lemma}

\begin{proof}
If $\mathcal{L}$ is ample, then $i^* \mathcal{L}$ is ample for example by Morphisms, Lemma 35.7. Assume $i^* \mathcal{L}$ is ample. Then $Z$ is quasi-compact (Properties, Definition 26.1) and separated (Properties, Lemma 26.8). Since $i$ is surjective, we see that $X$ is quasi-compact. Since $i$ is universally closed and surjective, we see that $X$ is separated (Morphisms, Lemma 39.11).

By Proposition 5.4 we can write $X = \lim_i X_i$ as a directed limit of finite type schemes over $Z$ with affine transition morphisms. We can find an $i$ and an invertible sheaf $\mathcal{L}_i$ on $X_i$ whose pullback to $X$ is isomorphic to $\mathcal{L}$, see Lemma 10.2.

For each $i$ let $Z_i \subset X_i$ be the scheme theoretic image of the morphism $Z \to X$. If $\text{Spec}(A_i) \subset X_i$ is an affine open subscheme with inverse image of $\text{Spec}(A)$ in $X$ and if $Z \cap \text{Spec}(A)$ is defined by the ideal $I \subset A$, then $Z_i \cap \text{Spec}(A_i)$ is defined by the ideal $I_i \subset A_i$ which is the inverse image of $I$ in $A_i$ under the ring map $A_i \to A$, see Morphisms, Example 6.4. Since $\text{colim} A_i/I_i = A/I$ it follows that $\lim Z_i = Z$. By Lemma 4.15 we see that $\mathcal{L}_i|_{Z_i}$ is ample for some $i$. Since $Z$ and hence $X$ maps into $Z_i$ set theoretically, we see that $X_i \to X$ maps into $Z_i$ set theoretically for some $i' \geq i$, see Lemma 4.10. (Observe that since $X_i$ is Noetherian, every closed subset of $X_i$ is constructible.) Let $T \subset X_{i'}$ be the scheme theoretic inverse image of $Z_i$ in $X_{i'}$. Observe that $\mathcal{L}_{i'}|_T$ is the pullback of $\mathcal{L}_i|_{Z_i}$ and hence ample by Morphisms, Lemma 35.7 and the fact that $T \to Z_i$ is an affine morphism. Thus we see that $\mathcal{L}_{i'}$ is ample on $X_{i'}$ by Cohomology of Schemes, Lemma 17.5. Pulling back to $X$ (using the same lemma as above) we find that $\mathcal{L}$ is ample.
\end{proof}
Lemma 11.5. Let \( i : Z \rightarrow X \) be a closed immersion of schemes inducing a homeomorphism of underlying topological spaces. Then \( X \) is quasi-affine if and only if \( Z \) is quasi-affine.

Proof. Recall that a scheme is quasi-affine if and only if the structure sheaf is ample, see Properties, Lemma 27.1. Hence if \( Z \) is quasi-affine, then \( \mathcal{O}_Z \) is ample, hence \( \mathcal{O}_X \) is ample by Lemma 11.4, hence \( X \) is quasi-affine. A proof of the converse, which can also be seen in an elementary way, is gotten by reading the argument just given backwards. \( \square \)

The following lemma does not really belong in this section.

Lemma 11.6. Let \( X \) be a scheme. Let \( \mathcal{L} \) be an ample invertible sheaf on \( X \). Assume we have morphisms of schemes

\[
\text{Spec}(k) \leftarrow \text{Spec}(A) \rightarrow W \subset X
\]

where \( k \) is a field, \( A \) is an integral \( k \)-algebra, \( W \) is open in \( X \). Then there exists an \( n > 0 \) and a section \( s \in \Gamma(X, \mathcal{L}^\otimes n) \) such that \( X_s \) is affine, \( X_s \subset W \), and \( \text{Spec}(A) \rightarrow W \) factors through \( X_s \).

Proof. Since \( \text{Spec}(A) \) is quasi-compact, we may replace \( W \) by a quasi-compact open still containing the image of \( \text{Spec}(A) \rightarrow X \). Recall that \( X \) is quasi-separated and quasi-compact by dint of having an ample invertible sheaf, see Properties, Definition 26.1 and Lemma 26.7. By Proposition 5.4 we can write \( X = \lim X_i \) as a limit of a directed system of schemes of finite type over \( Z \) with affine transition morphisms. For some \( i \) the ample invertible sheaf \( \mathcal{L} \) on \( X \) descends to an ample invertible sheaf \( \mathcal{L}_i \) on \( X_i \) and the open \( W \) is the inverse image of a quasi-compact open \( W_i \subset X_i \), see Lemmas 4.15, 10.3, and 4.11. We may replace \( X, W, \mathcal{L} \) by \( X_i, W_i, \mathcal{L}_i \) and assume \( X \) is of finite presentation over \( Z \). Write \( A = \text{colim} A_j \) as the colimit of its finite \( k \)-subalgebras. Then for some \( j \) the morphism \( \text{Spec}(A) \rightarrow X \) factors through a morphism \( \text{Spec}(A_j) \rightarrow X \), see Proposition 6.1. Since \( \text{Spec}(A_j) \) is finite this reduces the lemma to Properties, Lemma 29.6. \( \square \)

12. Variants of Chow’s Lemma

In this section we prove a number of variants of Chow’s lemma. The most interesting version is probably just the Noetherian case, which we stated and proved in Cohomology of Schemes, Section 18.

Lemma 12.1. Let \( S \) be a quasi-compact and quasi-separated scheme. Let \( f : X \rightarrow S \) be a separated morphism of finite type. Then there exists an \( n \geq 0 \) and a diagram

\[
\begin{array}{ccc}
X & \xleftarrow{\pi} & X' \\
\downarrow & & \downarrow \\
S & \xrightarrow{\pi'} & \mathbb{P}^n_S
\end{array}
\]

where \( X' \rightarrow \mathbb{P}^n_S \) is an immersion, and \( \pi : X' \rightarrow X \) is proper and surjective.

Proof. By Proposition 9.6, we can find a closed immersion \( X \rightarrow Y \) where \( Y \) is separated and of finite presentation over \( S \). Clearly, if we prove the assertion for \( Y \), then the result follows for \( X \). Hence we may assume that \( X \) is of finite presentation over \( S \).
Write $S = \lim_{i} S_i$ as a directed limit of Noetherian schemes, see Proposition 5.4. By Lemma 10.1 we can find an index $i \in I$ and a scheme $X_i \to S_i$ of finite presentation so that $X = S \times_{S_i} X_i$. By Lemma 8.6 we may assume that $X_i \to S_i$ is separated. Clearly, if we prove the assertion for $X_i$ over $S_i$, then the assertion holds for $X$. The case $X_i \to S_i$ is treated by Cohomology of Schemes, Lemma 18.1. □

Here is a variant of Chow’s lemma where we assume the scheme on top has finitely many irreducible components.

Lemma 12.2. Let $S$ be a quasi-compact and quasi-separated scheme. Let $f : X \to S$ be a separated morphism of finite type. Assume that $X$ has finitely many irreducible components. Then there exists an $n \geq 0$ and a diagram

$$
\begin{array}{ccc}
X & \xrightarrow{\pi} & X' \\
\downarrow & & \downarrow \\
S & \quad & \mathbf{P}_S^n
\end{array}
$$

where $X' \to \mathbf{P}_S^n$ is an immersion, and $\pi : X' \to X$ is proper and surjective. Moreover, there exists an open dense subscheme $U \subset X$ such that $\pi^{-1}(U) \to U$ is an isomorphism of schemes.

Proof. Let $X = Z_1 \cup \ldots \cup Z_n$ be the decomposition of $X$ into irreducible components. Let $\eta_j \in Z_j$ be the generic point.

There are (at least) two ways to proceed with the proof. The first is to redo the proof of Cohomology of Schemes, Lemma 18.1 using the general Properties, Lemma 29.4 to find suitable affine opens in $X$. (This is the “standard” proof.) The second is to use absolute Noetherian approximation as in the proof of Lemma 12.1 above. This is what we will do here.

By Proposition 9.6 we can find a closed immersion $X \to Y$ where $Y$ is separated and of finite presentation over $S$. Write $S = \lim_{i} S_i$ as a directed limit of Noetherian schemes, see Proposition 5.4. By Lemma 10.1 we can find an index $i \in I$ and a scheme $Y_i \to S_i$ of finite presentation so that $Y = S \times_{S_i} Y_i$. By Lemma 8.6 we may assume that $Y_i \to S_i$ is separated. We have the following diagram

$$
\begin{array}{ccc}
\eta_j \in Z_j & \xrightarrow{\eta_j} & X & \xrightarrow{\pi} & Y & \xrightarrow{Y_i} \\
\downarrow & & \downarrow & & \downarrow \\
S & \quad & S & \quad & S_i
\end{array}
$$

Denote $h : X \to Y_i$ the composition.

For $i' \geq i$ write $Y_i' = S_{i'} \times_{S_i} Y_i$. Then $Y = \lim_{i' \geq i} Y_i'$, see Lemma 2.3. Choose $j, j' \in \{1, \ldots, n\}$, $j \neq j'$. Note that $\eta_j$ is not a specialization of $\eta_{j'}$. By Lemma 4.6 we can replace $i$ by a bigger index and assume that $h(\eta_j)$ is not a specialization of $h(\eta_{j'})$ for all pairs $(j, j')$ as above. For such an index, let $Y' \subset Y_i$ be the scheme theoretic image of $h : X \to Y_i$, see Morphisms, Definition 6.2. The morphism $h$ is quasi-compact as the composition of the quasi-compact morphisms $X \to Y$ and $Y \to Y_i$ (which is affine). Hence by Morphisms, Lemma 6.3 the morphism $X \to Y'$ is dominant. Thus the generic points of $Y'$ are all contained in the set $\{h(\eta_1), \ldots, h(\eta_n)\}$, see Morphisms, Lemma 8.3. Since none of the $h(\eta_j)$ is the
specialization of another we see that the points $h(\eta_1), \ldots, h(\eta_n)$ are pairwise distinct and are each a generic point of $Y'$.

We apply Cohomology of Schemes, Lemma [18.1] above to the morphism $Y' \to S_i$. This gives a diagram

\[
\begin{array}{ccc}
Y' & \xrightarrow{\pi} & Y^* \xrightarrow{\pi} \mathbb{P}^n_{S_i} \\
\downarrow & & \downarrow \\
S_i & \xleftarrow{\pi} & S_i
\end{array}
\]

such that $\pi$ is proper and surjective and an isomorphism over a dense open subscheme $V \subset Y'$. By our choice of $i$ above we know that $h(\eta_1), \ldots, h(\eta_n) \in V$. Consider the commutative diagram

\[
\begin{array}{ccc}
X' & \xrightarrow{X \times_{Y^*}} & Y^* \xrightarrow{\pi} \mathbb{P}^n_{S_i} \\
\downarrow & & \downarrow \\
X & \xrightarrow{\pi} & Y' \\
\downarrow & & \downarrow \\
S & \xleftarrow{\pi} & S_i
\end{array}
\]

Note that $X' \to X$ is an isomorphism over the open subscheme $U = h^{-1}(V)$ which contains each of the $\eta_j$ and hence is dense in $X$. We conclude $X \leftarrow X' \to \mathbb{P}^n_{S_i}$ is a solution to the problem posed in the lemma. □

13. Applications of Chow’s lemma

Here is a first application of Chow’s lemma.

Lemma 13.1. Assumptions and notation as in Situation 8.1. If

(1) $f$ is proper, and

(2) $f_0$ is locally of finite type,

then there exists an $i$ such that $f_i$ is proper.

Proof. By Lemma 8.6 we see that $f_i$ is separated for some $i \geq 0$. Replacing $0$ by $i$ we may assume that $f_0$ is separated. Observe that $f_0$ is quasi-compact, see Schemes, Lemma 21.14. By Lemma 12.1 we can choose a diagram

\[
\begin{array}{ccc}
X_0 \xleftarrow{X_0'} \mathbb{P}^n_{Y_0} \\
\downarrow & \uparrow \\
Y_0 & \xrightarrow{\pi} & X_0'
\end{array}
\]

where $X_0' \to \mathbb{P}^n_{Y_0}$ is an immersion, and $\pi : X_0' \to X_0$ is proper and surjective. Introduce $X' = X_0' \times_{Y_0} Y$ and $X_i' = X_0' \times_{Y_i} Y_i$. By Morphisms, Lemmas 39.4 and 39.5 we see that $X' \to Y$ is proper. Hence $X' \to \mathbb{P}^n_{Y'}$ is a closed immersion (Morphisms, Lemma 39.7). By Morphisms, Lemma 39.8 it suffices to prove that $X_i' \to Y_i$ is proper for some $i$. By Lemma 8.5 we find that $X_i' \to \mathbb{P}^n_{Y_i}$ is a closed immersion for $i$ large enough. Then $X_i' \to Y_i$ is proper and we win. □
Lemma 13.2. Let $f : X \to S$ be a proper morphism with $S$ quasi-compact and quasi-separated. Then $X = \lim X_i$ is a directed limit of schemes $X_i$ proper and of finite presentation over $S$ such that all transition morphisms and the morphisms $X \to X_i$ are closed immersions.

Proof. By Proposition 9.6 we can find a closed immersion $X \to Y$ with $Y$ separated and of finite presentation over $S$. By Lemma 13.1 we can find a diagram

$$
\begin{array}{ccc}
Y & \xrightarrow{\pi} & Y' \\
\downarrow & & \downarrow \\
S & \xrightarrow{\pi'} & \mathbf{P}^n_S
\end{array}
$$

where $Y' \to \mathbf{P}^n_S$ is an immersion, and $\pi : Y' \to Y$ is proper and surjective. By Lemma 9.4 we can write $X = \lim X_i$ with $X_i \to Y$ a closed immersion of finite presentation. Denote $X'_i \subset Y'$, resp. $X' \subset Y'$. Then $\lim X'_i = X'$. Since $X' \to S$ is proper (Morphisms, Lemmas 39.4), we see that $X' \to \mathbf{P}^n_S$ is a closed immersion (Morphisms, Lemma 39.7). Hence for $i$ large enough we find that $X'_i \to \mathbf{P}^n_S$ is a closed immersion by Lemma 13.2. Thus $X'_i$ is proper over $S$. For such $i$ the morphism $X_i \to S$ is proper by Morphisms, Lemma 39.8.

Lemma 13.3. Let $f : X \to S$ be a proper morphism with $S$ quasi-compact and quasi-separated. Then there exists a directed set $I$, an inverse system $(f_i : X_i \to S_i)$ of morphisms of schemes over $I$, such that the transition morphisms $X_i \to X_j$ and $S_i \to S_j$ are affine, such that $f_i$ is proper, such that $S_i$ is of finite type over $\mathbf{Z}$, and such that $(X \to S) = \lim (X_i \to S_i)$.

Proof. By Lemma 13.1 we can write $X = \lim_{k \in K} X_k$ with $X_k \to S$ proper and of finite presentation. Next, by absolute Noetherian approximation (Proposition 5.4) we can write $S = \lim_{j \in J} S_j$ with $S_j$ of finite type over $\mathbf{Z}$. For each $k$ there exists a $j$ and a morphism $X_{k,j} \to S_j$ of finite presentation with $X_k \cong S \times_{S_j} X_{k,j}$ as schemes over $S$, see Lemma 10.1. After increasing $j$ we may assume $X_{k,j} \to S_j$ is proper, see Lemma 13.1. The set $I$ will be consist of these pairs $(k,j)$ and the corresponding morphism is $X_{k,j} \to S_j$. For every $k' \geq k$ we can find a $j' \geq j$ and a morphism $X_{j',k'} \to X_{j,k}$ over $S_{j'} \to S_j$ whose base change to $S$ gives the morphism $X_{k'} \to X_k$ (follows again from Lemma 10.1). These morphisms form the transition morphisms of the system. Some details omitted.

Lemma 13.4. Let $S$ be a scheme. Let $X = \lim_{i \in I} X_i$ be a directed limit of schemes over $S$ with affine transition morphisms. Let $Y \to X$ be a morphism of schemes over $S$. If $Y \to X$ is proper, $X_i$ quasi-compact and quasi-separated, and $Y$ locally of finite type over $S$, then $Y \to X_i$ is proper for $i$ large enough.

Proof. Choose a closed immersion $Y \to Y'$ with $Y'$ proper and of finite presentation over $X$, see Lemma 13.2. Then choose an $i$ and a proper morphism $Y'_i \to X_i$ such that $Y' = X \times_{X_i} Y'_i$. This is possible by Lemmas 10.1 and 13.1. Then after replacing $i$ by a larger index we have that $Y \to Y'_i$ is a closed immersion, see Lemma 4.16.

Recall the scheme theoretic support of a finite type quasi-coherent module, see Morphisms, Definition 5.3.
Lemma 13.5. Assumptions and notation as in Situation 8.1. Let \( F_0 \) be a quasi-coherent \( \mathcal{O}_{X_0} \)-module. Denote \( F \) and \( F_i \) the pullbacks of \( F_0 \) to \( X \) and \( X_i \). Assume

1. \( f_0 \) is locally of finite type,
2. \( F_0 \) is of finite type,
3. the scheme theoretic support of \( F \) is proper over \( Y \).

Then the scheme theoretic support of \( F_i \) is proper over \( Y_i \) for some \( i \).

Proof. We may replace \( X_0 \) by the scheme theoretic support of \( F_0 \). By Morphisms, Lemma 5.3 this guarantees that \( X_i \) is the support of \( F_i \) and \( X \) is the support of \( F \). Then, if \( Z \subset X \) denotes the scheme theoretic support of \( F_0 \), we see that \( Z \to X \) is a universal homeomorphism. We conclude that \( X \to Y \) is proper as this is true for \( Z \to Y \) by assumption, see Morphisms, Lemma 39.8. By Lemma 13.1 we see that \( X_i \to Y \) is proper for some \( i \). Then it follows that the scheme theoretic support \( Z_i \) of \( F_i \) is proper over \( Y \) by Morphisms, Lemmas 39.6 and 39.4. □

14. Universally closed morphisms

Let \( f : X \to S \) be a quasi-compact morphism of schemes. Let \( g : T \to S \) be a morphism of schemes. Let \( t \in T \) be a point and \( Z \subset X_T \) be a closed subscheme such that \( Z \cap X_t = \emptyset \). Then there exists an open neighbourhood \( V \subset T \) of \( t \), a commutative diagram

\[
\begin{array}{ccc}
V & \xrightarrow{a} & T' \\
\downarrow & & \downarrow b \\
T & \xrightarrow{g} & S
\end{array}
\]

and a closed subscheme \( Z' \subset X_{T'} \) such that

1. the morphism \( b : T' \to S \) is locally of finite presentation,
2. with \( t' = a(t) \) we have \( Z' \cap X_{t'} = \emptyset \), and
3. \( Z \cap X_V \) maps into \( Z' \) via the morphism \( X_V \to X_{T'} \).

Moreover, we may assume \( V \) and \( T' \) are affine.

Proof. Let \( s = g(t) \). During the proof we may always replace \( T \) by an open neighbourhood of \( t \). Hence we may also replace \( S \) by an open neighbourhood of \( s \). Thus we may and do assume that \( T \) and \( S \) are affine. Say \( S = \text{Spec}(A) \), \( T = \text{Spec}(B) \), \( g \) is given by the ring map \( A \to B \), and \( t \) correspond to the prime ideal \( q \subset B \).

As \( X \to S \) is quasi-compact and \( S \) is affine we may write \( X = \bigcup_{i=1, \ldots, n} U_i \) as a finite union of affine opens. Write \( U_i = \text{Spec}(C_i) \). In particular we have \( X_T = \bigcup_{i=1, \ldots, n} U_{i,T} = \bigcup_{i=1, \ldots, n} \text{Spec}(C_i \otimes_A B) \). Let \( I_i \subset C_i \otimes_A B \) be the ideal corresponding to the closed subscheme \( Z \cap U_{i,T} \). The condition that \( Z \cap X_t = \emptyset \) signifies that \( I_i \) generates the unit ideal in the ring

\[
C_i \otimes_A \kappa(q) = (B \setminus q)^{-1}(C_i \otimes_A B/qC_i \otimes_A B)
\]
Since \( I_i(B \setminus q)^{-1}(C_i \otimes_A B) = (B \setminus q)^{-1} I_i \) this means that \( 1 = x_i/g_i \) for some \( x_i \in I_i \) and \( g_i \in B, g_i \notin q \). Thus, clearing denominators we can find a relation of the form
\[
x_i + \sum_j f_{i,j} c_{i,j} = g_i
\]
with \( x_i \in I_i, f_{i,j} \in q, c_{i,j} \in C_i \otimes_A B, \) and \( g_i \in B, g_i \notin q \). After replacing \( B \) by \( B_{g_{i \ldots g_{n_i}}} \), i.e., after replacing \( T \) by a smaller affine neighbourhood of \( t \), we may assume the equations read
\[
x_i + \sum_j f_{i,j} c_{i,j} = 1
\]
with \( x_i \in I_i, f_{i,j} \in q, c_{i,j} \in C_i \otimes_A B \).

To finish the argument write \( B \) as a colimit of finitely presented \( A \)-algebras \( B_{\lambda} \) over a directed set \( \Lambda \). For each \( \lambda \) set \( q_{\lambda} = (B_{\lambda} \to B)^{-1}(q) \). For sufficiently large \( \lambda \in \Lambda \) we can find
\[
(1) \text{ an element } x_{i,\lambda} \in C_i \otimes_A B_{\lambda} \text{ which maps to } x_i,
(2) \text{ elements } f_{i,j,\lambda} \in q_{i,\lambda} \text{ mapping to } f_{i,j}, \text{ and}
(3) \text{ elements } c_{i,j,\lambda} \in C_i \otimes_A B_{\lambda} \text{ mapping to } c_{i,j}.
\]
After increasing \( \lambda \) a bit more the equation
\[
x_{i,\lambda} + \sum_j f_{i,j,\lambda} c_{i,j,\lambda} = 1
\]
will hold. Fix such a \( \lambda \) and set \( T' = \text{Spec}(B_{\lambda}) \). Then \( t' \in T' \) is the point corresponding to the prime \( q_{\lambda} \). Finally, let \( Z' \subset X_{T'} \) be the scheme theoretic image of \( Z \to X_T \to X_{T'} \). As \( X_T \to X_{T'} \) is affine, we can compute \( Z' \) on the affine open pieces \( U_{i,T'} \) as the closed subscheme associated to \( \text{Ker}(C_i \otimes_A B_{\lambda} \to C_i \otimes_A B/I_i) \), see Morphisms, Example 6.4. Hence \( x_{i,\lambda} \) is in the ideal defining \( Z' \). Thus the last displayed equation shows that \( Z' \cap X_{t'} \) is empty. \( \square \)

05JX **Lemma 14.2.** Let \( f : X \to S \) be a quasi-compact morphism of schemes. The following are equivalent
\[
(1) \text{ } f \text{ is universally closed},
(2) \text{ for every morphism } S' \to S \text{ which is locally of finite presentation the base change } X_{S'} \to S' \text{ is closed}, \text{ and}
(3) \text{ for every } n \text{ the morphism } \mathbb{A}^n \times X \to \mathbb{A}^n \times S \text{ is closed}.
\]

**Proof.** It is clear that (1) implies (2). Let us prove that (2) implies (1). Suppose that the base change \( X_T \to T \) is not closed for some scheme \( T \) over \( S \). By Schemes, Lemma 19.8 this means that there exists some specialization \( t_1 \to t \) in \( T \) and a point \( \xi \in X_T \) mapping to \( t_1 \) such that \( \xi \) does not specialize to a point in the fibre over \( t \). Set \( Z = \{ \xi \} \subset X_T \). Then \( Z \cap X_t = \emptyset \). Apply Lemma 14.1. We find an open neighbourhood \( V \subset T \) of \( t \), a commutative diagram
\[
\begin{array}{ccc}
V & \longrightarrow & T' \\
\downarrow & & \downarrow b \\
T & \longrightarrow & S,
\end{array}
\]
and a closed subscheme \( Z' \subset X_{T'} \), such that
\[
(1) \text{ the morphism } b : T' \to S \text{ is locally of finite presentation},
(2) \text{ with } t' = a(t) \text{ we have } Z' \cap X_{t'} = \emptyset, \text{ and}
(3) Z \cap X_V \text{ maps into } Z' \text{ via the morphism } X_V \to X_{T'}.
\]
Clearly this means that $X_{T'} \to T'$ maps the closed subset $Z'$ to a subset of $T'$ which contains $a(t_1)$ but not $t' = a(t)$. Since $a(t_1) \sim a(t) = t'$ we conclude that $X_{T'} \to T'$ is not closed. Hence we have shown that $X \to S$ not universally closed implies that $X_{T'} \to T'$ is not closed for some $T' \to S$ which is locally of finite presentation. In order words (2) implies (1).

Assume that $\mathbb{A}^n \times X \to \mathbb{A}^n \times S$ is closed for every integer $n$. We want to prove that $X_T \to T$ is closed for every scheme $T$ which is locally of finite presentation over $S$. We may of course assume that $T$ is affine and maps into an affine open $V$ of $S$ (since $X_T \to T$ being a closed is local on $T$). In this case there exists a closed immersion $T \to \mathbb{A}^n \times V$ because $\mathcal{O}_T(T)$ is a finitely presented $\mathcal{O}_S(V)$-algebra, see Morphisms, Lemma 20.2. Then $T \to \mathbb{A}^n \times S$ is a locally closed immersion. Hence we get a cartesian diagram

$$
\begin{array}{ccc}
X_T & \longrightarrow & \mathbb{A}^n \times X \\
\downarrow f_T & & \downarrow f_n \\
T & \longrightarrow & \mathbb{A}^n \times S
\end{array}
$$

of schemes where the horizontal arrows are locally closed immersions. Hence any closed subset $Z \subset X_T$ can be written as $X_T \cap Z'$ for some closed subset $Z' \subset \mathbb{A}^n \times X$. Then $f_T(Z) = T \cap f_n(Z')$ and we see that if $f_n$ is closed, then also $f_T$ is closed. \(\square\)

**Lemma 14.3.** Let $S$ be a scheme. Let $f : X \to S$ be a separated morphism of finite type. The following are equivalent:

1. The morphism $f$ is proper.
2. For any morphism $S' \to S$ which is locally of finite type the base change $X_{S'} \to S'$ is closed.
3. For every $n \geq 0$ the morphism $\mathbb{A}^n \times X \to \mathbb{A}^n \times S$ is closed.

**First proof.** In view of the fact that a proper morphism is the same thing as a separated, finite type, and universally closed morphism, this lemma is a special case of Lemma 14.2 \(\square\)

**Second proof.** Clearly (1) implies (2), and (2) implies (3), so we just need to show (3) implies (1). First we reduce to the case when $S$ is affine. Assume that (3) implies (1) when the base is affine. Now let $f : X \to S$ be a separated morphism of finite type. Being proper is local on the base (see Morphisms, Lemma 39.3), so if $S = \bigcup \alpha S_\alpha$ is an open affine cover, and if we denote $X_\alpha := f^{-1}(S_\alpha)$, then it is enough to show that $f|_{X_\alpha} : X_\alpha \to S_\alpha$ is proper for all $\alpha$. Since $S_\alpha$ is affine, if the map $f|_{X_\alpha}$ satisfies (3), then it will satisfy (1) by assumption, and will be proper. To finish the reduction to the case $S$ is affine, we must show that if $f : X \to S$ is separated of finite type satisfying (3), then $f|_{X_\alpha} : X_\alpha \to S_\alpha$ is separated of finite type satisfying (3). Separatedness and finite type are clear. To see (3), notice that $\mathbb{A}^n \times X_\alpha$ is the open preimage of $\mathbb{A}^n \times S_\alpha$ under the map $1 \times f$. Fix a closed set $Z \subset \mathbb{A}^n \times X_\alpha$. Let $\bar{Z}$ denote the closure of $Z$ in $\mathbb{A}^n \times X$. Then for topological reasons,

$$1 \times f(\bar{Z}) \cap \mathbb{A}^n \times S_\alpha = 1 \times f(Z).$$

Hence $1 \times f(Z)$ is closed, and we have reduced the proof of (3) $\Rightarrow$ (1) to the affine case.
Assume $S$ affine, and $f : X \to S$ separated of finite type. We can apply Chow’s Lemma 12.1 to get $\pi : X' \to X$ proper surjective and $X' \to \mathbb{P}^n_S$ an immersion. If $X$ is proper over $S$, then $X' \to S$ is proper (Morphisms, Lemma 39.4). Since $\mathbb{P}^n_S \to S$ is separated, we conclude that $X' \to \mathbb{P}^n_S$ is proper (Morphisms, Lemma 39.7) and hence a closed immersion (Schemes, Lemma 10.4). Conversely, assume $X' \to \mathbb{P}^n_S$ is a closed immersion. Consider the diagram:

$$
\begin{array}{ccc}
X' & \longrightarrow & \mathbb{P}^n_S \\
p \downarrow & & \downarrow \\
X & \longrightarrow & S
\end{array}
$$

All maps are a priori proper except for $X \to S$. Hence we conclude that $X \to S$ is proper by Morphisms, Lemma 39.8. Therefore, we have shown that $X \to S$ is proper if and only if $X' \to \mathbb{P}^n_S$ is a closed immersion.

Assume $S$ is affine and (3) holds, and let $n, X', \pi$ be as above. Since being a closed morphism is local on the base, the map $X \times \mathbb{P}^n \to S \times \mathbb{P}^n$ is closed since by (3) $X \times \mathbb{A}^n \to S \times \mathbb{A}^n$ is closed and since projective space is covered by copies of affine $n$-space, see Constructions, Lemma 13.3. By Morphisms, Lemma 39.5 the morphism

$$X' \times_S \mathbb{P}^n_S \to X \times_S \mathbb{P}^n_S = X \times \mathbb{P}^n$$

is proper. Since $\mathbb{P}^n$ is separated, the projection

$$X' \times_S \mathbb{P}^n_S = \mathbb{P}^n_{X'} \to X'$$

will be separated as it is just a base change of a separated morphism. Therefore, the map $X' \to X' \times_S \mathbb{P}^n_S$ is proper, since it is a section to a separated map (see Schemes, Lemma 21.11). Composing these morphisms

$$X' \to X' \times_S \mathbb{P}^n_S \to X \times_S \mathbb{P}^n_S = X \times \mathbb{P}^n \to S \times \mathbb{P}^n = \mathbb{P}^n_S$$

we find that the immersion $X' \to \mathbb{P}^n_S$ is closed, and hence a closed immersion. □

15. Noetherian valuative criterion

0CM1 If the base is Noetherian we can show that the valuative criterion holds using only discrete valuation rings.

Many of the results in this section can (and perhaps should) be proved by appealing to the following lemma, although we have not always done so.

0CM2 Lemma 15.1. Let $f : X \to Y$ be a morphism of schemes. Assume $f$ finite type and $Y$ locally Noetherian. Let $y \in Y$ be a point in the closure of the image of $f$. Then there exists a commutative diagram

$$
\begin{array}{ccc}
\text{Spec}(K) & \longrightarrow & X \\
\downarrow & & \downarrow f \\
\text{Spec}(A) & \longrightarrow & Y
\end{array}
$$

where $A$ is a discrete valuation ring and $K$ is its field of fractions mapping the closed point of $\text{Spec}(A)$ to $y$. Moreover, we can assume that the image point of $\text{Spec}(K) \to X$ is a generic point $\eta$ of an irreducible component of $X$ and that $K = \kappa(\eta)$. 

Proof. By the non-Noetherian version of this lemma (Morphisms, Lemma 6.5) there exists a point \( x \in X \) such that \( f(x) \) specializes to \( y \). We may replace \( x \) by any point specializing to \( x \), hence we may assume that \( x \) is a generic point of an irreducible component of \( X \). This produces a ring map \( \mathcal{O}_{Y,y} \rightarrow \kappa(x) \) (see Schemes, Section 13). Let \( R \subset \kappa(x) \) be the image. Then \( R \) is Noetherian as a quotient of the Noetherian local ring \( \mathcal{O}_{Y,y} \). On the other hand, the extension \( \kappa(x) \) is a finitely generated extension of the fraction field of \( R \) as \( f \) is of finite type. Thus there exists a discrete valuation ring \( A \subset \kappa(x) \) with fraction field \( \kappa(x) \) dominating \( R \) by Algebra, Lemma 118.13. Then

\[
\begin{array}{c}
\text{Spec}(\kappa(x)) \\
\downarrow \\
\text{Spec}(A) \\
\downarrow \\
\text{Spec}(\mathcal{O}_{Y,y}) \\
\downarrow \\
\text{Spec}(\kappa(x)) \\
\end{array}
\]

\[
\begin{array}{c}
X \\
\downarrow \\
\text{Spec}(A) \\
\downarrow \\
\text{Spec}(\mathcal{O}_{Y,y}) \\
\downarrow \\
Y
\end{array}
\]

gives the desired diagram. \( \square \)

First we state the result concerning separation. We will often use solid commutative diagrams of morphisms of schemes having the following shape

\[
\begin{array}{c}
\text{Spec}(K) \\
\downarrow \\
\text{Spec}(A) \\
\downarrow \\
\text{Spec}(\mathcal{O}_{Y,y}) \\
\downarrow \\
\text{Spec}(\kappa(x)) \\
\end{array}
\]

\[
\begin{array}{c}
X \\
\downarrow \\
\text{Spec}(A) \\
\downarrow \\
\text{Spec}(\mathcal{O}_{Y,y}) \\
\downarrow \\
Y
\end{array}
\]

with \( A \) a valuation ring and \( K \) its field of fractions.

Lemma 15.2. Let \( S \) be a locally Noetherian scheme. Let \( f : X \rightarrow S \) be a morphism of schemes. Assume \( f \) is locally of finite type. The following are equivalent:

1. The morphism \( f \) is separated.
2. For any diagram (15.1.1) there is at most one dotted arrow.
3. For all diagrams (15.1.1) with \( A \) a discrete valuation ring there is at most one dotted arrow.
4. For any irreducible component \( X_0 \) of \( X \) with generic point \( \eta \in X_0 \), for any discrete valuation ring \( A \subset K = \kappa(\eta) \) with fraction field \( K \) and any diagram (15.1.1) such that the morphism \( \text{Spec}(K) \rightarrow X \) is the canonical one (see Schemes, Section 13) there is at most one dotted arrow.

Proof. Clearly (1) implies (2), (2) implies (3), and (3) implies (4). It remains to show (4) implies (1). Assume (4). We begin by reducing to \( S \) affine. Being separated is a local on the base (see Schemes, Lemma 21.7). Hence, if we can show that whenever \( X \rightarrow S \) has (4) that the restriction \( X_\alpha \rightarrow S_\alpha \) has (4) where \( S_\alpha \subset S \) is an (affine) open subset and \( X_\alpha := f^{-1}(S_\alpha) \), then we will be done. The generic points of the irreducible components of \( X_\alpha \) will be the generic points of irreducible components of \( X \), since \( X_\alpha \) is open in \( X \). Therefore, any two distinct dotted arrows in the diagram

\[
\begin{array}{c}
\text{Spec}(K) \\
\downarrow \\
\text{Spec}(A) \\
\end{array}
\]

\[
\begin{array}{c}
X_\alpha \\
\downarrow \\
S_\alpha
\end{array}
\]
would then give two distinct arrows in diagram (15.1.1) via the maps \( X_\alpha \to X \) and \( S_\alpha \to S \), which is a contradiction. Thus we have reduced to the case \( S \) is affine. We remark that in the course of this reduction, we prove that if \( X \to S \) has (4) then the restriction \( U \to V \) has (4) for opens \( U \subset X \) and \( V \subset S \) with \( f(U) \subset V \).

We next wish to reduce to the case \( X \to S \) is finite type. Assume that we know (4) implies (1) when \( X \) is finite type. Since \( S \) is Noetherian and \( X \) is locally of finite type over \( S \) we see \( X \) is locally Noetherian as well (see Morphisms, Lemma [14.6]). Thus, \( X \to S \) is quasi-separated (see Properties, Lemma [5.4], and therefore we may apply the valuative criterion to check whether \( X \) is separated (see Schemes, Lemma [22.2]). Let \( X = \bigcup_\alpha X_\alpha \) be an affine open cover of \( X \). Given any two dotted arrows, in a diagram (15.1.1), the image of the closed points of \( \text{Spec} \ A \) will fall in two sets \( X_\alpha \) and \( X_\beta \). Since \( X_\alpha \cup X_\beta \) is open, for topological reasons it must contain the image of \( \text{Spec} \ A \) under both maps. Therefore, the two dotted arrows factor through \( X_\alpha \cup X_\beta \to X \), which is a scheme of finite type over \( S \). Since \( X_\alpha \cup X_\beta \) is an open subset of \( X \), by our previous remark, \( X_\alpha \cup X_\beta \) satisfies (4), so by assumption, is separated. This implies the two given dotted arrows are the same. Therefore, we have reduced to \( X \to S \) is finite type.

Assume \( X \to S \) of finite type and assume (4). Since \( X \to S \) is finite type, and \( S \) is an affine Noetherian scheme, \( X \) is also Noetherian (see Morphisms, Lemma [14.6]). Therefore, \( X \to X \times_S X \) will be a quasi-compact immersion of Noetherian schemes. We proceed by contradiction. Assume that \( X \to X \times_S X \) is not closed. Then, there is some \( y \in X \times_S X \) in the closure of the image that is not in the image. As \( X \) is Noetherian it has finitely many irreducible components. Therefore, \( y \) is in the closure of the image of one of the irreducible components \( X_0 \subset X \). Give \( X_0 \) the reduced induced structure. The composition \( X_0 \to X \to X \times_S X \) factors through the closed subscheme \( X_0 \times_S X_0 \subset X \times_S X \). Denote the closure of \( \Delta(X_0) \) in \( X_0 \times_S X_0 \) by \( X_0 \) (again as a reduced closed subscheme). Thus \( y \in X_0 \). Since \( X_0 \to X_0 \times_S X_0 \) is an immersion, the image of \( X_0 \) will be open in \( X_0 \). Hence \( X_0 \) and \( X_0 \) are birational. Since \( X_0 \) is a closed subscheme of a Noetherian scheme, it is Noetherian. Thus, the local ring \( \mathcal{O}_{X_0,y} \) is a local Noetherian domain with fraction field \( K \) equal to the function field of \( X_0 \). By the Krull-Akizuki theorem (see Algebra, Lemma [118.13]), there exists a discrete valuation ring \( A \) dominating \( \mathcal{O}_{X_0,y} \) with fraction field \( K \). This allows to construct a diagram:

\[
\begin{array}{ccc}
\text{Spec}(K) & \longrightarrow & X_0 \\
\downarrow & & \downarrow \Delta \\
\text{Spec}(A) & \longrightarrow & X_0 \times_S X_0
\end{array}
\]

which sends \( \text{Spec} \ K \) to the generic point of \( \Delta(X_0) \) and the closed point of \( A \) to \( y \in X_0 \times_S X_0 \) (use the material in Schemes, Section [13] to construct the arrows). There cannot even exist a set theoretic dotted arrow, since \( y \) is not in the image of \( \Delta \) by our choice of \( y \). By categorical means, the existence of the dotted arrow in the above diagram is equivalent to the uniqueness of the dotted arrow in the
Following diagram:

\[
\begin{align*}
\text{Spec}(K) & \longrightarrow X_0 \\
\downarrow & \\
\text{Spec}(A) & \longrightarrow S
\end{align*}
\]

Therefore, we have non-uniqueness in this latter diagram by the nonexistence in the first. Therefore, \(X_0\) does not satisfy uniqueness for discrete valuation rings, and since \(X_0\) is an irreducible component of \(X\), we have that \(X \rightarrow S\) does not satisfy (4). Therefore, we have shown (4) implies (1). \(\square\)

**Lemma 15.3.** Let \(S\) be a locally Noetherian scheme. Let \(f : X \rightarrow S\) be a morphism of finite type. The following are equivalent:

1. The morphism \(f\) is proper.
2. For any diagram (15.1.1) there exists exactly one dotted arrow.
3. For all diagrams (15.1.1) with \(A\) a discrete valuation ring there exists exactly one dotted arrow.
4. For any irreducible component \(X_0\) of \(X\) with generic point \(\eta \in X_0\), for any discrete valuation ring \(A \subset K = \kappa(\eta)\) with fraction field \(K\) and any diagram (15.1.1) such that the morphism \(\text{Spec}(K) \rightarrow X\) is the canonical one (see Schemes, Section [13]) there exists exactly one dotted arrow.

**Proof.** (1) implies (2) implies (3) implies (4). We will now show (4) implies (1). As in the proof of Lemma 15.2, we can reduce to the case \(S\) is affine, since properness is local on the base, and if \(X \rightarrow S\) satisfies (4), then \(X_\alpha \rightarrow S_\alpha\) does as well for open \(S_\alpha \subset S\) and \(X_\alpha = f^{-1}(S_\alpha)\).

Now \(S\) is a Noetherian scheme, and so \(X\) is as well, since \(X \rightarrow S\) is of finite type. Now we may use Chow’s lemma (Cohomology of Schemes, Lemma [18.1]) to get a surjective, proper, birational \(X' \rightarrow X\) and an immersion \(X' \rightarrow \mathbb{P}^n_S\). We wish to show \(X \rightarrow S\) is universally closed. As in the proof of Lemma 15.3, it is enough to check that \(X' \rightarrow \mathbb{P}^n_S\) is a closed immersion. For the sake of contradiction, assume that \(X' \rightarrow \mathbb{P}^n_S\) is not a closed immersion. Then there is some \(y \in \mathbb{P}^n_S\) that is in the closure of the image of \(X'\), but not in the image. So \(y\) is in the closure of the image of an irreducible component \(X'_0\) of \(X'\), but not in the image. Let \(X' \subset \mathbb{P}^n_S\) be the closure of the image of \(X'_0\). As \(X' \rightarrow \mathbb{P}^n_S\) is an immersion of Noetherian schemes, the morphism \(X'_0 \rightarrow X'_0\) is open and dense. By Algebra, Lemma [18.13] or Properties, Lemma [5.10] we can find a discrete valuation ring \(A\) dominating \(\mathcal{O}_{X'_0,y}\) and with identical field of fractions \(K\). It is clear that \(K\) is the residue field at the generic point of \(X'_0\). Thus the solid commutative diagram

\[
\begin{align*}
\text{Spec} K & \longrightarrow X' \longrightarrow \mathbb{P}^n_S \\
\downarrow & & \downarrow \\
\text{Spec} A & \longrightarrow X \longrightarrow S
\end{align*}
\]

Note that the closed point of \(A\) maps to \(y \in \mathbb{P}^n_S\). By construction, there does not exist a set theoretic lift to \(X'\). As \(X' \rightarrow X\) is birational, the image of \(X'_0\) in \(X\) is an irreducible component \(X_0\) of \(X\) and \(K\) is also identified with the function field of \(X_0\). Hence, as \(X \rightarrow S\) is assumed to satisfy (4), the dotted arrow \(\text{Spec}(A) \rightarrow X\) exists.
Since $X' \to X$ is proper, the dotted arrow lifts to the dotted arrow $\text{Spec}(A) \to X'$ (use Schemes, Proposition [20.6]). We can compose this with the immersion $X' \to \mathbf{P}^n_S$ to obtain another morphism (not depicted in the diagram) from $\text{Spec}(A) \to \mathbf{P}^n_S$. Since $\mathbf{P}^n_S$ is proper over $S$, it satisfies (2), and so these two morphisms agree. This is a contradiction, for we have constructed the forbidden lift of our original map $\text{Spec}(A) \to \mathbf{P}^n_S$ to $X'$.

**Lemma 15.4.** Let $f : X \to S$ be a finite type morphism of schemes. Assume $S$ is locally Noetherian. Then the following are equivalent

1. $f$ is universally closed,
2. for every $n$ the morphism $\mathbf{A}^n \times X \to \mathbf{A}^n \times S$ is closed,
3. for any diagram (15.1.1) there exists some dotted arrow,
4. for all diagrams (15.1.1) with $A$ a discrete valuation ring there exists some dotted arrow.

**Proof.** The equivalence of (1) and (2) is a special case of Lemma [14.2]. The equivalence of (1) and (3) is a special case of Schemes, Proposition [20.6]. Trivially (3) implies (4). Thus all we have to do is prove that (4) implies (2). We will prove that $\mathbf{A}^n \times X \to \mathbf{A}^n \times S$ is closed by the criterion of Schemes, Lemma [19.8]. Pick $n$ and a specialization $z \to z'$ of points in $\mathbf{A}^n \times S$ and a point $y \in \mathbf{A}^n \times X$ lying over $z$. Note that $\kappa(y)$ is a finitely generated field extension of $\kappa(z)$ as $\mathbf{A}^n \times X \to \mathbf{A}^n \times S$ is of finite type. Hence by Properties, Lemma [5.10] or Algebra, Lemma [118.13] implies that there exists a discrete valuation ring $A \subset \kappa(y)$ with fraction field $\kappa(z)$ dominating the image of $\mathcal{O}_{\mathbf{A}^n \times S, z'}$ in $\kappa(z)$. This gives a commutative diagram

$$
\begin{array}{ccc}
\text{Spec}(\kappa(y)) & \longrightarrow & \mathbf{A}^n \times X \\
\downarrow & & \downarrow \\
\text{Spec}(A) & \longrightarrow & \mathbf{A}^n \times S \\
\end{array}
$$

Now property (4) implies that there exists a morphism $\text{Spec}(A) \to X$ which fits into this diagram. Since we already have the morphism $\text{Spec}(A) \to \mathbf{A}^n$ from the left lower horizontal arrow we also get a morphism $\text{Spec}(A) \to \mathbf{A}^n \times X$ fitting into the left square. Thus the image $y' \in \mathbf{A}^n \times X$ of the closed point is a specialization of $y$ lying over $z'$. This proves that specializations lift along $\mathbf{A}^n \times X \to \mathbf{A}^n \times S$ and we win.

In the Noetherian case one usually does not have to consider all possible diagrams with discrete valuation rings when testing the valuative criteria. We have already seen this in Lemmas [15.2] and [15.3]. Here is another variant.

**Lemma 15.5.** Let $f : X \to S$ and $h : U \to X$ be morphisms of schemes. Assume that $S$ is locally Noetherian, that $f$ and $h$ are of finite type, that $f$ is separated, and that $h(U)$ is dense in $X$. If given any commutative solid diagram

$$
\begin{array}{ccc}
\text{Spec}(K) & \longrightarrow & U \\
\downarrow & \searrow & \downarrow f \\
\text{Spec}(A) & \longrightarrow & \text{Spec}(S) \\
\end{array}
$$

where $A$ is a discrete valuation ring with field of fractions $K$, there exists a dotted arrow making the diagram commute, then $f$ is proper.
Proof. There is an immediate reduction to the case where $S$ is affine. Then $U$ is quasi-compact. Let $U = U_1 \cup \ldots \cup U_n$ be an affine open covering. We may replace $U$ by $U_1 \amalg \ldots \amalg U_n$ without changing the assumptions, hence we may assume $U$ is affine. Thus we can find an open immersion $U \to Y$ over $X$ with $Y$ proper over $X$. (First put $U$ inside $\mathbf{A}^n_X$ using Morphisms, Lemma \[37.2\] and then take the closure inside $\mathbf{P}^n_X$, or you can directly use Morphisms, Lemma \[41.12\]). We can assume $U$ is dense in $Y$ (replace $Y$ by the scheme theoretic closure of $U$ if necessary, see Morphisms, Section \[7\]). Note that $g : Y \to X$ is surjective as the image is closed and contains the dense subset $h(U)$. We will show that $Y \to S$ is proper. This will imply that $X \to S$ is proper by Morphisms, Lemma \[39.8\] thereby finishing the proof. To show that $Y \to S$ is proper we will use part (4) of Lemma \[15.3\]. To do this consider a diagram

$$
\begin{array}{ccc}
\text{Spec}(K) & \to & Y \\
\downarrow & & \downarrow f \circ g \\
\text{Spec}(A) & \to & S
\end{array}
$$

where $A$ is a discrete valuation ring with fraction field $K$ and where $y : \text{Spec}(K) \to Y$ is the inclusion of a generic point. We have to show there exists a unique dotted arrow. Uniqueness holds by the converse to the valuative criterion for separatedness (Schemes, Lemma \[22.1\]) since $Y \to S$ is separated as the composition of the separated morphisms $Y \to X$ and $X \to S$ (Schemes, Lemma \[21.12\]). Existence can be seen as follows. As $y$ is a generic point of $Y$, it is contained in $U$. By assumption of the lemma there exists a morphism $a : \text{Spec}(A) \to X$ such that

$$
\begin{array}{ccc}
\text{Spec}(K) & \to & U \\
\downarrow & & \downarrow f \\
\text{Spec}(A) & \to & S
\end{array}
$$

is commutative. Then since $Y \to X$ is proper, we can apply the valuative criterion for properness (Morphisms, Lemma \[40.1\]) to find a morphism $b : \text{Spec}(A) \to Y$ such that

$$
\begin{array}{ccc}
\text{Spec}(K) & \to & Y \\
\downarrow & & \downarrow g \\
\text{Spec}(A) & \to & X
\end{array}
$$

is commutative. This finishes the proof since $b$ can serve as the dotted arrow above. \[\square\]

Lemma 15.6. Let $f : X \to S$ and $h : U \to X$ be morphisms of schemes. Assume that $S$ is locally Noetherian, that $f$ is locally of finite type, that $h$ is of finite type, and that $h(U)$ is dense in $X$. If given any commutative solid diagram

$$
\begin{array}{ccc}
\text{Spec}(K) & \to & U \\
\downarrow & & \downarrow h \\
\text{Spec}(A) & \to & S
\end{array}
$$

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where $A$ is a discrete valuation ring with field of fractions $K$, there exists at most one dotted arrow making the diagram commute, then $f$ is separated.

**Proof.** We will apply Lemma 15.5 to the morphisms $U \to X$ and $\Delta : X \to X \times_S X$. We check the conditions. Observe that $\Delta$ is quasi-compact by Properties, Lemma 5.4 (and Schemes, Lemma 21.13). Of course $\Delta$ is locally of finite type and separated (true for any diagonal morphism). Finally, suppose given a commutative solid diagram

$$\xymatrix{ \text{Spec}(K) \ar[r]^{h} \ar[d] & U \ar[d] \ar[r]^{\Delta} \ar[dl]_{(a,b)} & X \times_S X \ar[d] & \text{Spec}(A) \ar[r]^{f} \ar[d] & S \ar@{-->}[ul] }$$

where $A$ is a discrete valuation ring with field of fractions $K$. Then $a$ and $b$ give two dotted arrows in the diagram of the lemma and have to be equal. Hence as dotted arrow we can use $a = b$ which gives existence. This finishes the proof. □

**Lemma 15.7.** Let $f : X \to S$ and $h : U \to X$ be morphisms of schemes. Assume that $S$ is locally Noetherian, that $f$ and $h$ are of finite type, and that $h(U)$ is dense in $X$. If given any commutative solid diagram

$$\xymatrix{ \text{Spec}(K) \ar[r]^{h} \ar[d] & U \ar[d] \ar[r]^{f} \ar[dl]_{(a,b)} & S \ar@{-->}[ul] }$$

where $A$ is a discrete valuation ring with field of fractions $K$, there exists a unique dotted arrow making the diagram commute, then $f$ is proper.

**Proof.** Combine Lemmas 15.6 and 15.5 □

16. Limits and dimensions of fibres

The following lemma is most often used in the situation of Lemma 10.1 to assure that if the fibres of the limit have dimension $\leq d$, then the fibres at some finite stage have dimension $\leq d$.

**Lemma 16.1.** Let $I$ be a directed set. Let $(f_i : X_i \to S_i)$ be an inverse system of morphisms of schemes over $I$. Assume

1. all the morphisms $S_i \to S_i$ are affine,
2. all the schemes $S_i$ are quasi-compact and quasi-separated,
3. the morphisms $f_i$ are of finite type, and
4. the morphisms $X_i \to X_i \times_S S_i$ are closed immersions.

Let $f : X = \lim_i X_i \to S = \lim_i S_i$ be the limit. Let $d \geq 0$. If every fibre of $f$ has dimension $\leq d$, then for some $i$ every fibre of $f_i$ has dimension $\leq d$.

**Proof.** For each $i$ let $U_i = \{x \in X_i \mid \dim_{X_i}(f_i(x)) \leq d\}$. This is an open subset of $X_i$, see Morphisms, Lemma 27.4. Set $Z_i = X_i \setminus U_i$ (with reduced induced scheme structure). We have to show that $Z_i = \emptyset$ for some $i$. If not, then $Z = \lim_i Z_i \neq \emptyset$, see Lemma 4.3. Say $z \in Z$ is a point. Note that $Z \subset X$ is a closed subscheme. Set $s = f(z)$. For each $i$ let $s_i \in S_i$ be the image of $s$. We remark that $Z_s$ is the limit
of the schemes \((Z_i)_{s_i}\) and \(Z_s\) is also the limit of the schemes \((Z_i)_{s_i}\) base changed to \(\kappa(s)\). Moreover, all the morphisms

\[
Z_i \to (Z_{i'})_{s_{i'}} \times \text{Spec}(\kappa(s)) \to (Z_i)_{s_i} \times \text{Spec}(\kappa(s)) \to \text{Spec}(\kappa(s)) \to X_s
\]

are closed immersions by assumption (4). Hence \(Z_s\) is the scheme theoretic intersection of the closed subschemes \((Z_i)_{s_i} \times \text{Spec}(\kappa(s))\) in \(X_s\). Since all the irreducible components of the schemes \((Z_i)_{s_i} \times \text{Spec}(\kappa(s))\) have dimension \(> d\) and contain \(z\) we conclude that \(Z_s\) contains an irreducible component of dimension \(> d\) passing through \(z\) which contradicts the fact that \(Z_s \subset X_s\) and \(\dim(X_s) \leq d\).

\[\square\]

**Lemma 16.2.** Notation and assumptions as in Situation 8.1. If

1. \(f\) is a quasi-finite morphism, and
2. \(f_0\) is locally of finite type,

then there exists an \(i \geq 0\) such that \(f_i\) is quasi-finite.

**Proof.** Follows immediately from Lemma 16.1.

\[\square\]

**Lemma 16.3.** Notation and assumptions as in Situation 8.1. If

1. \(f\) has relative dimension \(d\), and
2. \(f_0\) is locally of finite presentation,

then there exists an \(i \geq 0\) such that \(f_i\) has relative dimension \(d\).

**Proof.** By Lemma 16.1 we may assume all fibres of \(f_0\) have dimension \(\leq d\). By Morphisms, Lemma 27.5 the set \(U_0 \subset X_0\) of points \(x \in X_0\) such that the dimension of the fibre of \(X_0 \to Y_0\) at \(x\) is \(\leq d - 1\) is open and retrocompact in \(X_0\). Hence the complement \(E = X_0 \setminus U_0\) is constructible. Moreover the image of \(X \to X_0\) is contained in \(E\) by Morphisms, Lemma 27.3. Thus for \(i \geq 0\) we have that the image of \(X_i \to X_0\) is contained in \(E\) (Lemma 10.10). Then all fibres of \(X_i \to Y_i\) have dimension \(d\) by the aforementioned Morphisms, Lemma 27.3.

\[\square\]

**Lemma 16.4.** Let \(S\) be a quasi-compact and quasi-separated scheme. Let \(f : X \to S\) be a morphism of finite presentation. Let \(d \geq 0\) be an integer. If \(Z \subset X\) be a closed subscheme such that \(\dim(Z_s) \leq d\) for all \(s \in S\), then there exists a closed subscheme \(Z' \subset X\) such that

1. \(Z \subset Z'\),
2. \(Z' \to X\) is of finite presentation, and
3. \(\dim(Z'_s) \leq d\) for all \(s \in S\).

**Proof.** By Proposition 5.4 we can write \(S = \lim S_i\) as the limit of a directed inverse system of Noetherian schemes with affine transition maps. By Lemma 10.11 we may assume that there exist a system of morphisms \(f_i : X_i \to S_i\) of finite presentation such that \(X'_i = X_i \times_{S_i} S'_i\) for all \(i' \geq i\) and such that \(X = X_i \times_{S_i} S\). Let \(Z_i \subset X_i\) be the scheme theoretic image of \(Z \to X \to X_i\). Then for \(i' \geq i\) the morphism \(X'_{i'} \to X_i\) maps \(Z_{i'}\) into \(Z_i\) and the induced morphism \(Z'_{i'} \to Z_i \times_{S_i} S'_{i'}\) is a closed immersion. By Lemma 16.1 we see that the dimension of the fibres of \(Z_i \to S_i\) all have dimension \(\leq d\) for a suitable \(i \in I\). Fix such an \(i\) and set \(Z' = Z_i \times_{S_i} S \subset X\). Since \(S_i\) is Noetherian, we see that \(X_i\) is Noetherian, and hence the morphism \(Z_i \to X_i\) is of finite presentation. Therefore also the base change \(Z' \to X\) is of finite presentation. Moreover, the fibres of \(Z' \to S\) are base changes of the fibres of \(Z_i \to S_i\) and hence have dimension \(\leq d\).

\[\square\]
17. Base change in top degree

For a proper morphism and a finite type quasi-coherent module the base change map is an isomorphism in top degree.

**Lemma 17.1.** Let $f : X \to Y$ be a morphism of schemes. Let $d \geq 0$. Assume

1. $X$ and $Y$ are quasi-compact and quasi-separated, and
2. $R^i f_* F = 0$ for $i > d$ and every quasi-coherent $\mathcal{O}_X$-module $F$.

Then we have

(a) for any base change diagram

$$
\begin{array}{ccc}
X' & \to & X \\
f' \downarrow & & \downarrow f \\
Y' & \to & Y
\end{array}
$$

we have $R^i f'_* F' = 0$ for $i > d$ and any quasi-coherent $\mathcal{O}_{X'}$-module $F'$,

(b) $R^d f'_* (F \otimes_{\mathcal{O}_X} (f')^* G') = R^d f_* F \otimes_{\mathcal{O}_Y} G'$ for any quasi-coherent $\mathcal{O}_{Y'}$-module $G'$,

(c) formation of $R^d f'_* F'$ commutes with arbitrary further base change (see proof for explanation).

**Proof.** Before giving the proofs, we explain the meaning of (c). Suppose we have an additional cartesian square

$$
\begin{array}{ccc}
X'' & \to & X' & \to & X \\
f'' \downarrow & & f' \downarrow & & \downarrow f \\
Y'' & \to & Y' & \to & Y
\end{array}
$$

tacked onto our given diagram. If (a) holds, then there is a canonical map $\gamma : h^* R^d f'_* F' \to R^d f''_*(h')^* F'$. Namely, $\gamma$ is the map on degree $d$ cohomology sheaves induced by the composition

$$
Lh^* R^d f''_* \to R^d f''_* (h')^* F' \to R^d f'_* (h')^* F'.
$$

Here the first arrow is the base change map (Cohomology, Remark [29.3]) and the second arrow complex from the canonical map $L(g')^* \mathcal{F} \to (g')^* \mathcal{F}$. Similarly, since $R f''_* \mathcal{F}$ has no nonzero cohomology sheaves in degrees $> d$ by (a) we have $H^d (Lh^* R f_* \mathcal{F}) = h^* R^d f_* \mathcal{F}$. The content of (c) is that $\gamma$ is an isomorphism.

Having said this, we can check (a), (b), and (c) locally on $Y'$ and $Y''$. Suppose that $V \subset Y$ is a quasi-compact open subscheme. Then we claim (1) and (2) hold for $f|_{f^{-1}(V)} : f^{-1}(V) \to V$. Namely, (1) is immediate and (2) follows because any quasi-coherent module on $f^{-1}(V)$ is the restriction of a quasi-coherent module on $X$ (Properties, Lemma [22.1]) and formation of higher direct images commutes with restriction to opens. Thus we may also work locally on $Y$. In other words, we may assume $Y''$, $Y'$, and $Y$ are affine schemes.

Proof of (a) when $Y'$ and $Y$ are affine. In this case the morphisms $g$ and $g'$ are affine. Thus $g_* = R g_*$ and $g'_* = R g'_*$ (Cohomology of Schemes, Lemma [2.3]) and $g_*$ is identified with the restriction functor on modules (Schemes, Lemma 7.3). Then

$$
g_*(R^i f_* \mathcal{F}) = H^i (R g_* R f_* \mathcal{F}) = H^i (R f_* R g'_* \mathcal{F} = H^i (R f_* g'_* \mathcal{F}) = R f'_* g'_* \mathcal{F}.
$$
which is zero by assumption (2). Hence (a) by our description of $g_\ast$.

Proof of (b) when $Y'$ is affine, say $Y' = \text{Spec}(R')$. By part (a) we have $H^{d+1}(X', \mathcal{F}') = 0$ for any quasi-coherent $\mathcal{O}_{X'}$-module $\mathcal{F}'$, see Cohomology of Schemes, Lemma 4.6

Consider the functor $F$ on $R'$-modules defined by the rule

$$F(M) = H^d(X', \mathcal{F}' \otimes_{\mathcal{O}_{X'}} (f')^* \widetilde{M})$$

By Cohomology, Lemma 20.1 this functor commutes with direct sums (this is where we use that $X$ and hence $X'$ is quasi-compact and quasi-separated). On the other hand, if $M_1 \to M_2 \to M_3 \to 0$ is an exact sequence, then

$$(\mathcal{F}' \otimes_{\mathcal{O}_{X'}} (f')^* \widetilde{M_1}) \to (\mathcal{F}' \otimes_{\mathcal{O}_{X'}} (f')^* \widetilde{M_2}) \to (\mathcal{F}' \otimes_{\mathcal{O}_{X'}} (f')^* \widetilde{M_3}) \to 0$$

is an exact sequence of quasi-coherent modules on $X'$ and by the vanishing of higher cohomology given above we get an exact sequence

$$F(M_1) \to F(M_2) \to F(M_3) \to 0$$

In other words, $F$ is right exact. Any right exact $R'$-linear functor $F : \text{Mod}_{R'} \to \text{Mod}_{R'}$ which commutes with direct sums is given by tensoring with an $R'$-module (omitted; left as exercise for the reader). Thus we obtain $F(M) = H^d(X', \mathcal{F}') \otimes_{R'} M$. Since $R^d(f')_* \mathcal{F}$ and $R^d(f')_* (\mathcal{F}' \otimes_{\mathcal{O}_{X'}} (f')^* \widetilde{M})$ are quasi-coherent (Cohomology of Schemes, Lemma 4.5), the fact that $F(M) = H^d(X', \mathcal{F}') \otimes_{R'} M$ translates into the statement given in (b).

Proof of (c) when $Y'' \to Y' \to Y$ are morphisms of affine schemes. Say $Y'' = \text{Spec}(R'')$ and $Y' = \text{Spec}(R')$. Then we see that $R^d f''_*(h')^* \mathcal{F}'$ is the quasi-coherent module on $Y'$ associated to the $R''$-module $H^d(X'', (h')^* \mathcal{F}')$. Now $h' : X'' \to X'$ is affine hence $H^d(X'', (h')^* \mathcal{F}') = H^d(X, h'_*(h')^* \mathcal{F}')$ by the already used Cohomology of Schemes, Lemma 2.4 We have

$$h'_*(h')^* \mathcal{F}' = \mathcal{F}' \otimes_{\mathcal{O}_{X'}} (f')^* \widetilde{R''}$$

as the reader sees by checking on an affine open covering. Thus $H^d(X'', (h')^* \mathcal{F}') = H^d(X', \mathcal{F}') \otimes_{R'} R''$ by part (b) applied to $f'$ and the proof is complete.

**Lemma 17.2.** Let $f : X \to Y$ be a morphism of schemes. Let $y \in Y$. Assume $f$ is proper and $\dim(X_y) = d$. Then

1. for $\mathcal{F} \in \text{QCoh}(\mathcal{O}_X)$ we have $(R^i f_\ast \mathcal{F})_y = 0$ for all $i > d$,
2. there is an affine open neighbourhood $V \subset Y$ of $y$ such that $f^{-1}(V) \to V$ and $d$ satisfy the assumptions and conclusions of Lemma 17.1.

**Proof.** By Morphisms, Lemma 27.4 and the fact that $f$ is closed, we can find an affine open neighbourhood $V$ of $y$ such that the fibres over points of $V$ all have dimension $\leq d$. Thus we may assume $X \to Y$ is a proper morphism all of whose fibres have dimension $\leq d$ with $Y$ affine. We will show that (2) holds, which will immediately imply (1) for all $y \in Y$.

By Lemma 13.2 we can write $X = \lim X_i$ as a cofiltered limit with $X_i \to Y$ proper and of finite presentation and such that both $X \to X_i$ and transition morphisms are closed immersions. For some $i$ we have that $X_i \to Y$ has fibres of dimension $\leq d$, see Lemma 16.1. For a quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ we have $R^p f_* \mathcal{F} = R^p f_{i,*}(X \to X_i)_\ast \mathcal{F}$ by Cohomology of Schemes, Lemma 2.3 and Leray (Cohomology, Lemma 14.8). Thus we may replace $X$ by $X_i$ and reduce to the case discussed in the next paragraph.
Assume $Y$ is affine and $f : X \to Y$ is proper and of finite presentation and all fibres have dimension $\leq d$. It suffices to show that $H^p(X, \mathcal{F}) = 0$ for $p > d$. Namely, by Cohomology of Schemes, Lemma 4.5 we have $H^p(X, \mathcal{F}) = H^0(Y, R^p f_* \mathcal{F})$. On the other hand, $R^p f_* \mathcal{F}$ is quasi-coherent on $Y$ by Cohomology of Schemes, Lemma 4.5 hence vanishing of global sections implies vanishing. Write $Y = \lim_{i \in I} Y_i$ as a cofiltered limit of affine schemes with $Y_i$ the spectrum of a Noetherian ring (for example a finite type $\mathbb{Z}$-algebra). We can choose an element $0 \in I$ and a finite type morphism $X_0 \to Y_0$ such that $X \cong Y \times_{Y_0} X_0$. After increasing $0$ we may assume $X_0 \to Y_0$ is proper (Lemma 13.1) and that the fibres of $X_0 \to Y_0$ have dimension $\leq d$ (Lemma 16.1). Since $X \to X_0$ is affine, we find that $H^p(X, \mathcal{F}) = H^p(X_0, (X \to X_0)_* \mathcal{F})$ by Cohomology of Schemes, Lemma 2.4. This reduces us to the case discussed in the next paragraph.

Assume $Y$ is affine Noetherian and $f : X \to Y$ is proper and all fibres have dimension $\leq d$. In this case we can write $\mathcal{F} = \text{colim} \mathcal{F}_i$ as a filtered colimit of coherent $\mathcal{O}_X$-modules, see Properties, Lemma 22.6. Then $H^p(X, \mathcal{F}) = \text{colim} H^p(Y, \mathcal{F}_i)$ by Cohomology of Schemes, Lemma 20.1. Thus we may assume $\mathcal{F}$ is coherent. In this case we see that $(R^p f_* \mathcal{F})_y = 0$ for all $y \in Y$ by Cohomology of Schemes, Lemma 20.9. Thus $R^p f_* \mathcal{F} = 0$ and therefore $H^p(X, \mathcal{F}) = 0$ (see above) and we win. \hfill $\square$

0EX4 Lemma 17.3. Let $f : X \to Y$ be a morphism of schemes. Let $d \geq 0$. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Assume

1. $f$ is a proper morphism all of whose fibres have dimension $\leq d$,
2. $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_X$-module of finite type.

Then $R^d f_* \mathcal{F}$ is a quasi-coherent $\mathcal{O}_X$-module of finite type.

Proof. The module $R^d f_* \mathcal{F}$ is quasi-coherent by Cohomology of Schemes, Lemma 4.5. The question is local on $Y$ hence we may assume $Y$ is affine. Say $Y = \text{Spec}(\mathcal{R})$. Then it suffices to prove that $H^d(X, \mathcal{F})$ is a finite $\mathcal{R}$-module.

By Lemma 13.2 we can write $X = \text{lim} X_i$ as a cofiltered limit with $X_i \to Y$ proper and of finite presentation and such that both $X \to X_i$ and transition morphisms are closed immersions. For some $i$ we have that $X_i \to Y$ has fibres of dimension $\leq d$, see Lemma 16.1. We have $R^d f_* \mathcal{F} = R^d f_{i,*} (X \to X_i)_* \mathcal{F}$ by Cohomology of Schemes, Lemma 2.3 and Leray (Cohomology, Lemma 14.8). Thus we may replace $X$ by $X_i$ and reduce to the case discussed in the next paragraph.

Assume $Y$ is affine and $f : X \to Y$ is proper and of finite presentation and all fibres have dimension $\leq d$. We can write $\mathcal{F}$ as a quotient of a finitely presented $\mathcal{O}_X$-module $\mathcal{F}'$, see Properties, Lemma 22.8. The map $H^d(X, \mathcal{F}') \to H^d(X, \mathcal{F})$ is surjective, as we have $H^{d+1}(X, \text{Ker}(\mathcal{F}' \to \mathcal{F})) = 0$ by the vanishing of higher cohomology seen in Lemma 17.2 (or its proof). Thus we reduce to the case discussed in the next paragraph.

Assume $Y = \text{Spec}(\mathcal{R})$ is affine and $f : X \to Y$ is proper and of finite presentation and all fibres have dimension $\leq d$ and $\mathcal{F}$ is an $\mathcal{O}_X$-module of finite presentation. Write $Y = \lim_{i \in I} Y_i$ as a cofiltered limit of affine schemes with $Y_i = \text{Spec}(\mathcal{R}_i)$ the spectrum of a Noetherian ring (for example a finite type $\mathbb{Z}$-algebra). We can choose an element $0 \in I$ and a finite type morphism $X_0 \to Y_0$ such that $X \cong Y \times_{Y_0} X_0$, see Lemma 10.1. After increasing $0$ we may assume $X_0 \to Y_0$ is proper (Lemma 13.1) and that the fibres of $X_0 \to Y_0$ have dimension $\leq d$ (Lemma 16.1). After
increasing 0 we can assume there is a coherent \( O_{X_0} \)-module \( \mathcal{F}_0 \) which pulls back to \( \mathcal{F} \), see Lemma \ref{lemma-coherent-module}. By Lemma \ref{lemma-pullback-coherent} we have
\[
H^d(X, \mathcal{F}) = H^d(X_0, \mathcal{F}_0) \otimes_{R_0} R
\]
This finishes the proof because the cohomology module \( H^d(X_0, \mathcal{F}_0) \) is finite by Cohomology of Schemes, Lemma \ref{lemma-finite-cohomology}.

\[\text{Lemma 17.4.}\]
Let \( f : X \to Y \) be a morphism of schemes. Let \( d \geq 0 \). Let \( \mathcal{F} \) be an \( O_X \)-module. Assume
\begin{enumerate}
\item \( f \) is a proper morphism of finite presentation all of whose fibres have dimension \( \leq d \),
\item \( \mathcal{F} \) is an \( O_X \)-module of finite presentation.
\end{enumerate}
Then \( R^d f_* \mathcal{F} \) is an \( O_X \)-module of finite presentation.

\textbf{Proof.} The proof is exactly the same as the proof of Lemma \ref{lemma-proper-cohomology} except that the third paragraph can be skipped. We omit the details. \( \square \)

\section{18. Glueing in closed fibres}

Applying our theory above to the spectrum of a local ring we obtain the following pleasing glueing result for relative schemes.

\[\text{Lemma 18.1.}\]
Let \( S \) be a scheme. Let \( s \in S \) be a closed point such that \( U = S \setminus \{ s \} \to S \) is quasi-compact. With \( V = \text{Spec}(O_{S,s}) \setminus \{ s \} \) there is an equivalence of categories
\[
\{ X \to S \text{ of finite presentation} \} \longrightarrow \left\{ \begin{array}{ccc}
X' & \longrightarrow & Y' \\
\downarrow & & \downarrow \\
U & \longrightarrow & \text{Spec}(O_{S,s}) \\
\end{array} \right. \\
\end{array}
\]
where on the right hand side we consider commutative diagrams whose squares are cartesian and whose vertical arrows are of finite presentation.

\textbf{Proof.} Let \( W \subset S \) be an open neighbourhood of \( s \). By glueing of relative schemes, see Constructions, Section 2 the functor
\[
\{ X \to S \text{ of finite presentation} \} \longrightarrow \left\{ \begin{array}{ccc}
X' & \longrightarrow & Y' \\
\downarrow & & \downarrow \\
U & \longrightarrow & W \setminus \{ s \} \\
\end{array} \right. \\
\end{array}
\]
is an equivalence of categories. We have \( O_{S,s} = \text{colim} O_W(W) \) where \( W \) runs over the affine open neighbourhoods of \( s \). Hence \( \text{Spec}(O_{S,s}) = \lim W \) where \( W \) runs over the affine open neighbourhoods of \( s \). Thus the category of schemes of finite presentation over \( \text{Spec}(O_{S,s}) \) is the limit of the category of schemes of finite presentation over \( W \) where \( W \) runs over the affine open neighbourhoods of \( s \), see Lemma \ref{lemma-limit-categories}. For every affine open \( s \in W \) we see that \( U \cap W \) is quasi-compact as \( U \to S \) is quasi-compact. Hence \( V = \lim W \cap U = \lim W \setminus \{ s \} \) is a limit of quasi-compact and quasi-separated schemes (see Lemma \ref{lemma-limits-category}). Thus also the category of schemes of finite presentation over \( V \) is the limit of the categories of schemes of finite presentation over \( W \cap U \) where \( W \) runs over the affine open neighbourhoods of \( s \). The lemma follows formally from a combination of these results. \( \square \)
Lemma 18.2. Let $S$ be a scheme. Let $U \subset S$ be a retrocompact open. Let $s \in S$ be a point in the complement of $U$. With $V = \text{Spec}(\mathcal{O}_{S,s}) \cap U$ there is an equivalence of categories

$$
\colim_{s \in U' \supset U \text{ open}} \left\{ \begin{array}{c}
X \\
U'
\end{array} \right\} \longrightarrow \left\{ \begin{array}{c}
X' \leftarrow Y' \rightarrow Y \\
U \leftarrow V \rightarrow \text{Spec}(\mathcal{O}_{S,s})
\end{array} \right\}
$$

where on the left hand side the vertical arrow is of finite presentation and on the right hand side we consider commutative diagrams whose squares are cartesian and whose vertical arrows are of finite presentation.

Proof. Let $W \subset S$ be an open neighbourhood of $s$. By glueing of relative schemes, see Constructions, Section 2, the functor

$$
\{ X \to U' = U \cup W \text{ of finite presentation} \} \longrightarrow \left\{ \begin{array}{c}
X' \leftarrow Y' \rightarrow Y \\
U \leftarrow W \cap U \rightarrow W
\end{array} \right\}
$$

is an equivalence of categories. We have $\mathcal{O}_{S,s} = \colim \mathcal{O}_W(W)$ where $W$ runs over the affine open neighbourhoods of $s$. Hence $\text{Spec}(\mathcal{O}_{S,s}) = \lim W$ where $W$ runs over the affine open neighbourhoods of $s$. Thus the category of schemes of finite presentation over $\text{Spec}(\mathcal{O}_{S,s})$ is the limit of the category of schemes of finite presentation over $W$ where $W$ runs over the affine open neighbourhoods of $s$, see Lemma 10.1. For every affine open $s \in W$ we see that $U \cap W$ is quasi-compact as $U \to S$ is quasi-compact. Hence $V = \lim W \cap U$ is a limit of quasi-compact and quasi-separated schemes (see Lemma 2.2). Thus also the category of schemes of finite presentation over $V$ is the limit of the categories of schemes of finite presentation over $W \cap U$ where $W$ runs over the affine open neighbourhoods of $s$. The lemma follows formally from a combination of these results. \qed

Lemma 18.3. Notation and assumptions as in Lemma 18.2. Let $U \subset U' \subset X$ be an open containing $s$.

1. Let $f' : X \to U'$ correspond to $f : X' \to U$ and $g : Y \to \text{Spec}(\mathcal{O}_{S,s})$ via the equivalence. If $f$ and $g$ are separated, proper, finite, étale, then after possibly shrinking $U'$ the morphism $f'$ has the same property.

2. Let $a : X_1 \to X_2$ be a morphism of schemes of finite presentation over $U'$ with base change $a' : X'_1 \to X'_2$ over $U$ and $b : Y_1 \to Y_2$ over $\text{Spec}(\mathcal{O}_{S,s})$. If $a'$ and $b$ are separated, proper, finite, étale, then after possibly shrinking $U'$ the morphism $a$ has the same property.

Proof. Proof of (1). Recall that $\text{Spec}(\mathcal{O}_{S,s})$ is the limit of the affine open neighbourhoods of $s$ in $S$. Since $g$ has the property in question, then the restriction of $f'$ to one of these affine open neighbourhoods does too, see Lemmas 8.6 [13.1] 8.3 and 8.10. Since $f'$ has the given property over $U$ as $f$ does, we conclude as one can check the property locally on the base.

Proof of (2). If we write $\text{Spec}(\mathcal{O}_{S,s}) = \lim W$ where $W$ runs over the affine open neighbourhoods of $s$ in $S$, then we have $Y'_i = \lim W \times_S X_i$. Thus we can use exactly the same arguments as in the proof of (1). \qed
Let $S$ be a scheme. Let $s_1, \ldots, s_n \in S$ be pairwise distinct closed points such that $U = S \setminus \{s_1, \ldots, s_n\} \to S$ is quasi-compact. With $S_i = \text{Spec}(\mathcal{O}_{S,s_i})$ and $U_i = S_i \setminus \{s_i\}$ there is an equivalence of categories

$$FP_S \longrightarrow FP_U \times_{(FP_{U_1} \times \ldots \times FP_{U_n})} (FP_{S_1} \times \ldots \times FP_{S_n})$$

where $FP_T$ is the category of schemes of finite presentation over the scheme $T$.

Proof. For $n = 1$ this is Lemma 18.1 For $n > 1$ the lemma can be proved in exactly the same way or it can be deduced from it. For example, suppose that $f_i : X_i \to S_i$ are objects of $FP_{S_i}$ and $f : X \to U$ is an object of $FP_U$ and we’re given isomorphisms $X_i \times_{S_i} U_i = X \times_U U_i$. By Lemma 18.1 we can find a morphism $f' : X' \to U' = S \setminus \{s_1, \ldots, s_{n-1}\}$ which is of finite presentation, which is isomorphic to $X_i$ over $S_i$, which is isomorphic to $X$ over $U$, and these isomorphisms are compatible with the given isomorphism $X_i \times_{S_i} U_n = X \times_U U_n$. Then we can apply induction to $f_i : X_i \to S_i$, $i \leq n - 1$, $f' : X' \to U'$, and the induced isomorphisms $X_i \times_{S_i} U_i = X' \times_{U'} U_i$, $i \leq n - 1$. This shows essential surjectivity. We omit the proof of fully faithfulness. □

19. Application to modifications

Using the results from Section 18 we can describe the category of modifications of a scheme over a closed point in terms of the local ring.

Let $S$ be a scheme. Let $s \in S$ be a closed point such that $U = S \setminus \{s\} \to S$ is quasi-compact. With $V = \text{Spec}(\mathcal{O}_{S,s}) \setminus \{s\}$ the base change functor

$$\{f : X \to S \text{ of finite presentation}\} \longrightarrow \{g : Y \to \text{Spec}(\mathcal{O}_{S,s}) \text{ of finite presentation}\}$$

$$\{f^{-1}(U) \to U \text{ is an isomorphism}\} \longrightarrow \{g^{-1}(V) \to V \text{ is an isomorphism}\}$$

is an equivalence of categories.

Proof. This is a special case of Lemma 18.1 □

Notation and assumptions as in Lemma 19.1. Let $f : X \to S$ correspond to $g : Y \to \text{Spec}(\mathcal{O}_{S,s})$ via the equivalence. Then $f$ is separated, proper, finite, étale and add more here if and only if $g$ is so.

Proof. The property of being separated, proper, integral, finite, etc is stable under base change. See Schemes, Lemma 21.12 and Morphisms, Lemmas 39.5 and 42.6. Hence if $f$ has the property, then so does $g$. The converse follows from Lemma 18.5 but we also give a direct proof here. Namely, if $g$ has to property, then $f$ does in a neighbourhood of $s$ by Lemmas 8.6, 13.1, 8.3 and 8.10. Since $f$ clearly has the given property over $S \setminus \{s\}$ we conclude as one can check the property locally on the base. □

Remark 19.3. The lemma above can be generalized as follows. Let $S$ be a scheme and let $T \subset S$ be a closed subset. Assume there exists a cofinal system of open neighbourhoods $T \subset W_i$ such that (1) $W_i \setminus T$ is quasi-compact and (2) $W_i \subset W_j$ is an affine morphism. Then $W = \lim W_i$ is a scheme which contains $T$ as a closed subscheme. Set $U = X \setminus T$ and $V = W \setminus T$. Then the base change functor

$$\{f : X \to S \text{ of finite presentation}\} \longrightarrow \{g : Y \to W \text{ of finite presentation}\}$$

$$\{f^{-1}(U) \to U \text{ is an isomorphism}\} \longrightarrow \{g^{-1}(V) \to V \text{ is an isomorphism}\}$$

is an equivalence of categories. If we ever need this we will change this remark into a lemma and provide a detailed proof.
20. Descending finite type schemes

This section continues the theme of Section 9 in the spirit of the results discussed in Section 10.

Situation 20.1. Let $S = \lim_{i \in I} S_i$ be the limit of a directed system of Noetherian schemes with affine transition morphisms $S_{i'} \to S_i$ for $i' \geq i$.

Lemma 20.2. In Situation 20.1. Let $X \to S$ be quasi-separated and of finite type. Then there exists an $i \in I$ and a diagram

\[
\begin{array}{ccc}
X & \to & W \\
\downarrow & & \downarrow \\
S & \to & S_i
\end{array}
\]

such that $W \to S_i$ is of finite type and such that the induced morphism $X \to S \times S_i W$ is a closed immersion.

Proof. By Lemma 9.3 we can find a closed immersion $X \to X'$ over $S$ where $X'$ is a scheme of finite presentation over $S$. By Lemma 10.1 we can find an $i$ and a morphism of finite presentation $X'_i \to S_i$ whose pull back is $X'$. Set $W = X'_i$. □

Lemma 20.3. In Situation 20.1. Let $X \to S$ be quasi-separated and of finite type. Given $i \in I$ and a diagram

\[
\begin{array}{ccc}
X & \to & W \\
\downarrow & & \downarrow \\
S & \to & S_i
\end{array}
\]

as in (20.2.1) for $i' \geq i$ let $X_{i'}$ be the scheme theoretic image of $X \to S_{i'} \times S_i W$. Then $X = \lim_{i' \geq i} X_{i'}$.

Proof. Since $X$ is quasi-compact and quasi-separated formation of the scheme theoretic image of $X \to S_{i'} \times S_i W$ commutes with restriction to open subschemes (Morphisms, Lemma 6.3). Hence we may and do assume $W$ is affine and maps into an affine open $U_i$ of $S_i$. Let $U \subset S, U_{i'} \subset S_{i'}$ be the inverse image of $U_i$. Then $U, U_{i'}, S_{i'} \times S_i W = U_{i'} \times U_i W_i$, and $S \times S_i W = U \times U_i W$ are all affine. This implies $X$ is affine because $X \to S \times S_i W$ is a closed immersion. This also shows the ring map

$O(U) \otimes_{O(U_i)} O(W) \to O(X)$

is surjective. Let $I$ be the kernel. Then we see that $X_{i'}$ is the spectrum of the ring

$O(X_{i'}) = O(U_{i'}) \otimes_{O(U_i)} O(W)/I_{i'}$

where $I_{i'}$ is the inverse image of the ideal $I$ (see Morphisms, Example 6.4). Since $O(U) = \colim O(U_{i'})$ we see that $I = \colim I_{i'}$ and we conclude that $\colim O(X_{i'}) = O(X)$. □

Lemma 20.4. In Situation 20.1. Let $f : X \to Y$ be a morphism of schemes quasi-separated and of finite type over $S$. Let

\[
\begin{array}{ccc}
X & \to & W \\
\downarrow & & \downarrow \\
S & \to & S_i
\end{array}
\quad
\begin{array}{ccc}
Y & \to & V \\
\downarrow & & \downarrow \\
S & \to & S_{i_2}
\end{array}
\]

and
be diagrams as in \([20.2.1]\). Let \(X = \lim_{i \geq i_0} X_i\) and \(Y = \lim_{i \geq i_2} Y_i\) be the corresponding limit descriptions as in Lemma \([20.3]\). Then there exists an \(i_0 \geq \max(i_1, i_2)\) and a morphism
\[
(f_i)_{i \geq i_0} : (X_i)_{i \geq i_0} \to (Y_i)_{i \geq i_0}
\]
of inverse systems over \((S_i)_{i \geq i_0}\) such that such that \(f = \lim_{i \geq i_0} f_i\). If \((g_i)_{i \geq i_0} : (X_i)_{i \geq i_0} \to (Y_i)_{i \geq i_0}\) is a second morphism of inverse systems over \((S_i)_{i \geq i_0}\) such that such that \(f = \lim_{i \geq i_0} g_i\), then \(f_i = g_i\) for all \(i \geq i_0\).

**Proof.** Since \(V \to S_{i_2}\) is of finite presentation and \(X = \lim_{i \geq i_1} X_i\) we can appeal to Proposition \([6.1]\) to find an \(i_0 \geq \max(i_1, i_2)\) and a morphism \(h : X_{i_0} \to V\) over \(S_{i_2}\) such that \(X \to X_{i_0} \to V\) is equal to \(X \to Y \to V\). For \(i \geq i_0\) we get a commutative solid diagram
\[
\begin{array}{ccc}
X & \xrightarrow{f_i} & X_{i_0} \\
\downarrow & & \downarrow h \\
Y & \xrightarrow{g_i} & V \\
\downarrow & & \downarrow \\
S & \xrightarrow{\pi_i} & S_{i_0}
\end{array}
\]

Since \(X \to X_i\) has scheme theoretically dense image and since \(Y_i\) is the scheme theoretic image of \(Y \to S_i \times S_{i_2} V\) we find that the morphism \(X_i \to S_i \times S_{i_2} V\) induced by the diagram factors through \(Y_i\) (Morphisms, Lemma \([6.6]\)). This proves existence.

Uniqueness. Let \(E_i \subset X_i\) be the equalizer of \(f_i\) and \(g_i\) for \(i \geq i_0\). By Schemes, Lemma \([21.5]\) \(E_i\) is a locally closed subscheme of \(X_i\). Since \(X_i\) is a closed subscheme of \(S_i \times S_{i_2} X_{i_0}\) and similarly for \(Y_i\) we see that
\[
E_i = X_i \times (S_i \times S_{i_2} X_{i_0}) (S_i \times S_{i_0}) E_{i_0})
\]
Thus to finish the proof it suffices to show that \(X_{i} \to X_{i_0}\) factors through \(E_{i_0}\) for some \(i \geq i_0\). To do this we will use that \(X \to X_{i_0}\) factors through \(E_{i_0}\) as both \(f_{i_0}\) and \(g_{i_0}\) are compatible with \(f\). Since \(X_i\) is Noetherian, we see that the underlying topological space \(|E_{i_0}|\) is a constructible subset of \(|X_{i_0}|\) (Topology, Lemma \([16.1]\)). Hence \(X_i \to X_{i_0}\) factors through \(E_{i_0}\) set theoretically for large enough \(i\) by Lemma \([4.10]\). For such an \(i\) the scheme theoretic inverse image \((X_i \to X_{i_0})^{-1}(E_{i_0})\) is a closed subscheme of \(X_i\) through which \(X\) factors and hence equal to \(X_i\) since \(X \to X_i\) has scheme theoretically dense image by construction. This concludes the proof. \(\square\)

**0CNS** Remark \(20.5\). In Situation \(20.1\) Lemmas \([20.2, 20.3, 20.4]\) tell us that the category of schemes quasi-separated and of finite type over \(S\) is equivalent to certain types of inverse systems of schemes over \((S_i)_{i \in I}\), namely the ones produced by applying Lemma \([20.3]\) to a diagram of the form \([20.2.1]\). For example, given \(X \to S\) finite type and quasi-separated if we choose two different diagrams \(X \to V_1 \to S_i\) and \(X \to V_2 \to S_i\) as in \([20.2.1]\), then applying Lemma \([20.4]\) to \(\text{id}_X\) (in two directions) we see that the corresponding limit descriptions of \(X\) are canonically isomorphic (up to shrinking the directed set \(I\)). And so on and so forth.

**0CNT** Lemma \(20.6\). Notation and assumptions as in Lemma \([20.4]\) If \(f\) is flat and of finite presentation, then there exists an \(i_3 \geq i_0\) such that for \(i \geq i_3\) we have \(f_i\) is flat, \(X_i = Y_i \times_{Y_{i_3}} X_{i_3}\), and \(X = Y \times_{Y_{i_3}} X_{i_3}\).
Proof. By Lemma 20.1, we can choose an \( i \geq i_2 \) and a morphism \( U \rightarrow Y_i \) of finite presentation such that \( X = Y \times Y_i U \) (this is where we use that \( f \) is of finite presentation). After increasing \( i \) we may assume that \( U \rightarrow Y_i \) is flat, see Lemma 8.7. As discussed in Remark 20.5, we may and do replace the initial diagram used to define the system \( (X_i)_{i \geq i_1} \) by the system corresponding to \( X \rightarrow U \rightarrow S_i \). Thus \( X_{i'} \) for \( i' \geq i \) is defined as the scheme theoretic image of \( X \rightarrow S_{i'} \times S_i U \).

Because \( U \rightarrow Y_i \) is flat (this is where we use that \( f \) is flat), because \( X = Y \times Y_i U \), and because the scheme theoretic image of \( Y \rightarrow Y_i \) is \( Y_i \), we see that the scheme theoretic image of \( X \rightarrow U \rightarrow S_i \) is equal to the closed subscheme \( Y_{i'} \times Y_i U \). Thus we see that \( X_{i'} = Y_{i'} \times Y_i U \) for all \( i' \geq i \) and hence the lemma holds with \( i_3 = i \).

\[ \square \]

0CNV Lemma 20.7. Notation and assumptions as in Lemma 20.4. If \( f \) is smooth, then there exists an \( i_3 \geq i_2 \) such that for \( i \geq i_3 \) we have \( f_i \) is smooth.

Proof. Combine Lemmas 20.6 and 8.9.

0CNV Lemma 20.8. Notation and assumptions as in Lemma 20.4. If \( f \) is proper, then there exists an \( i_3 \geq i_2 \) such that for \( i \geq i_3 \) we have \( f_i \) is proper.

Proof. By the discussion in Remark 20.5, the choice of \( i_1 \) and \( W \) fitting into a diagram as in (20.2.1) is immaterial for the truth of the lemma. Thus we choose \( W \) as follows. First we choose a closed immersion \( X \rightarrow X' \) with \( X' \rightarrow S \) proper and of finite presentation, see Lemma 13.2. Then we choose an \( i_3 \geq i_2 \) and a proper morphism \( W \rightarrow Y_{i_3} \) such that \( X' = Y \times Y_{i_3} W \). This is possible because \( Y = \lim_{i \geq i_2} Y_i \) and Lemmas 10.1 and 13.1. With this choice of \( W \) it is immediate from the construction that for \( i \geq i_3 \) the scheme \( X_i \) is a closed subscheme of \( Y_i \times Y_{i_3} W \subset S_i \times S_i \) and hence proper over \( Y_i \).

\[ \square \]

0CNW Lemma 20.9. In Situation 20.1 suppose that we have a cartesian diagram

\[
\begin{align*}
X^1 & \longrightarrow X^3 \\
\downarrow q & \downarrow a \\
X^2 & \longrightarrow X^4 \\
\downarrow b & \\
S & \longrightarrow S_{i_3}
\end{align*}
\]

of schemes quasi-separated and of finite type over \( S \). For each \( j = 1, 2, 3, 4 \) choose \( i_j \in I \) and a diagram

\[
\begin{align*}
X^j & \longrightarrow W^j \\
\downarrow & \\
S & \longrightarrow S_{i_j}
\end{align*}
\]

as in (20.2.1). Let \( X^j_i = \lim_{i \geq i_j} X^j_i \) be the corresponding limit descriptions as in Lemma 20.4. Let \( (a_i)_{i \geq i_j}, (b_i)_{i \geq i_j}, (p_i)_{i \geq i}, (q_i)_{i \geq i} \) be the corresponding morphisms of systems constructed in Lemma 20.4. Then there exists an \( i_0 \geq \max(i_5, i_6, i_7, i_8) \) such that for \( i \geq i_0 \) we have \( a_i \circ p_i = b_i \circ q_i \) and such that

\[
(q_i, p_i) : X^1_i \longrightarrow X^2_i \times_{b_i, X^j_i, a_i} X^3_i
\]
is a closed immersion. If \( a \) and \( b \) are flat and of finite presentation, then there exists an \( i_{10} \geq \max(i_5, i_6, i_7, i_8, i_9) \) such that for \( i \geq i_{10} \) the last displayed morphism is an isomorphism.

**Proof.** According to the discussion in Remark 20.5 the choice of \( W^1 \) fitting into a diagram as in (20.2.1) is immaterial for the truth of the lemma. Thus we may choose \( W^1 = W^2 \times_{W^3} W^4 \). Then it is immediate from the construction of \( X^1_i \) that \( a_i \circ p_i = b_i \circ q_i \) and that

\[
(q_i, p_i) : X^1_i \longrightarrow X^2_i \times_{b_i, X^4_i, a_i} X^3_i
\]

is a closed immersion.

If \( a \) and \( b \) are flat and of finite presentation, then so are \( p \) and \( q \) as base changes of \( a \) and \( b \). Thus we can apply Lemma 20.6 to each of \( a, b, p, q, \) and \( a \circ p = b \circ q \). It follows that there exists an \( i_9 \in I \) such that

\[
(q_i, p_i) : X^1_i \rightarrow X^2_i \times_{X^4_i} X^3_i
\]

is the base change of \((q_{i_9}, p_{i_9})\) by the morphism by the morphism \( X^4_i \rightarrow X^4_{i_9} \) for all \( i \geq i_9 \). We conclude that \((q_i, p_i)\) is an isomorphism for all sufficiently large \( i \) by Lemma 8.11. \( \square \)

### 21. Other chapters

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