LOCAL COHOMOLOGY

0DWN

Contents

1. Introduction

0DWP This chapter continues the study of local cohomology. A reference is [Gro68]. The definition of local cohomology can be found in Dualizing Complexes, Section 9. For Noetherian rings taking local cohomology is the same as deriving a suitable torsion functor as is shown in Dualizing Complexes, Section 10. The relationship with depth can be found in Dualizing Complexes, Section 11.

In the first part of this chapter we discuss finiteness properties of local cohomology leading to a proof of a fairly general version of Grothendieck's finiteness theorem, see Theorem 7.4 and Lemma 8.1 (higher direct images of coherent modules under open immersions). Our methods incorporate a few very slick arguments the reader can find in papers of Faltings, see [Fal78] and [Fal81].

The second part of this chapter is devoted to theorems on formal functions and algebraization of formal functions, mainly in the local Noetherian case (we discuss the global case elsewhere – insert future reference here). Section 10 discusses some of the tricks one has in the case of formal functions along an effective Cartier divisor cut out by a global regular function. Section 11 discusses derived completion with respect to a finite type sheaf of ideals in complete generality. We show how this

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material relates to the usual theorem on formal functions in Section 12. In Sections 13 and 14 we algebraize formal sections and coherent formal modules.

2. Generalities

0DWQ The following lemma tells us that the functor $R\Gamma_Z$ is related to cohomology with supports.

0A6T **Lemma 2.1.** *Let A be a ring and let I be a finitely generated ideal. Set $Z = V(I) \subset X = \mathrm{Spec}(A)$. For $K \in D(A)$ corresponding to $\widetilde{K} \in D_{QCoh}(\mathcal{O}_X)$ via Derived Categories of Schemes, Lemma 3.5 there is a functorial isomorphism*

$$R\Gamma_Z(K) = R\Gamma_Z(X, \widetilde{K})$$

where on the left we have Dualizing Complexes, Equation (9.0.1) and on the right we have the functor of Cohomology, Section 22.

Proof. By Cohomology, Section 22 there exists a distinguished triangle

$$R\Gamma_Z(X, \widetilde{K}) \to R\Gamma(X, \widetilde{K}) \to R\Gamma(U, \widetilde{K}) \to R\Gamma_Z(X, \widetilde{K})[1]$$

where $U = X \setminus Z$. We know that $R\Gamma(X, \widetilde{K}) = K$ by Derived Categories of Schemes, Lemma 3.5. Say $I = (f_1, \ldots, f_r)$. Then we obtain a finite affine open covering $\mathcal{U} : U = D(f_1) \cup \ldots \cup D(f_r)$. By Derived Categories of Schemes, Lemma 8.4 the alternating Čech complex $\mathrm{Tot}(\check{\mathcal{C}}^\bullet_{alt}(\mathcal{U}, \widetilde{K^\bullet}))$ computes $R\Gamma(U, \widetilde{K})$ where $K^\bullet$ is any complex of A-modules representing K. Working through the definitions we find

$$R\Gamma(U, \widetilde{K}) = \mathrm{Tot}\left(K^\bullet \otimes_A (\prod\nolimits_{i_0} A_{f_{i_0}} \to \prod\nolimits_{i_0 < i_1} A_{f_{i_0} f_{i_1}} \to \ldots \to A_{f_1 \ldots f_r})\right)$$

It is clear that $K^\bullet = R\Gamma(X, \widetilde{K^\bullet}) \to R\Gamma(U, \widetilde{K}^\bullet)$ is induced by the diagonal map from A into $\prod A_{f_i}$. Hence we conclude that

$$R\Gamma_Z(X, \mathcal{F}^\bullet) = \mathrm{Tot}\left(K^\bullet \otimes_A (A \to \prod\nolimits_{i_0} A_{f_{i_0}} \to \prod\nolimits_{i_0 < i_1} A_{f_{i_0} f_{i_1}} \to \ldots \to A_{f_1 \ldots f_r})\right)$$

By Dualizing Complexes, Lemma 9.1 this complex computes $R\Gamma_Z(K)$ and we see the lemma holds. $\square$

0DWR **Lemma 2.2.** *Let A be a ring and let $I \subset A$ be a finitely generated ideal. Set $X = \mathrm{Spec}(A)$, $Z = V(I)$, $U = X \setminus Z$, and $j : U \to X$ the inclusion morphism. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_U$-module. Then*

1. *there exists an A-module M such that $\mathcal{F}$ is the restriction of $\widetilde{M}$ to U,*
2. *given M there is an exact sequence*

$$0 \to H^0_Z(M) \to M \to H^0(U, \mathcal{F}) \to H^1_Z(M) \to 0$$

and isomorphisms $H^p(U, \mathcal{F}) = H^{p+1}_Z(M)$ for $p \geq 1$,

3. *we may take $M = H^0(U, \mathcal{F})$ in which case we have $H^0_Z(M) = H^1_Z(M) = 0$.*

Proof. The existence of M follows from Properties, Lemma 22.1 and the fact that quasi-coherent sheaves on X correspond to A-modules (Schemes, Lemma 7.5). Then we look at the distinguished triangle

$$R\Gamma_Z(X, \widetilde{M}) \to R\Gamma(X, \widetilde{M}) \to R\Gamma(U, \widetilde{M}|_U) \to R\Gamma_Z(X, \widetilde{M})[1]$$

of Cohomology, Section 22. Since X is affine we have $R\Gamma(X, \widetilde{M}) = M$ by Cohomology of Schemes, Lemma 2.2. By our choice of M we have $\mathcal{F} = \widetilde{M}|_U$ and hence this produces an exact sequence

$$0 \to H^0_Z(X, \widetilde{M}) \to M \to H^0(U, \mathcal{F}) \to H^1_Z(X, \widetilde{M}) \to 0$$

and isomorphisms $H^p(U, \mathcal{F}) = H^{p+1}_Z(X, \widetilde{M})$ for $p \geq 1$. By Lemma 2.1 we have $H^i_Z(M) = H^i_Z(X, \widetilde{M})$ for all i. Thus (1) and (2) do hold. Finally, setting $M' = H^0(U, \mathcal{F})$ we see that the kernel and cokernel of $M \to M'$ are I-power torsion. Therefore $\widetilde{M}|_U \to \widetilde{M'}|_U$ is an isomorphism and we can indeed use M' as predicted in (3). It goes without saying that we obtain zero for both $H^0_Z(M')$ and $H^0_Z(M')$. $\square$

0DWS **Lemma 2.3.** *Let $I, J \subset A$ be finitely generated ideals of a ring A. If M is an I-power torsion module, then the canonical map*

$$H^i_{V(I) \cap V(J)}(M) \to H^i_{V(J)}(M)$$

is an isomorphism for all i.

Proof. Use the spectral sequence of Dualizing Complexes, Lemma 9.6 to reduce to the statement $R\Gamma_I(M) = M$ which is immediate from the construction of local cohomology in Dualizing Complexes, Section 9. $\square$

0DWT **Lemma 2.4.** *Let $S \subset A$ be a multiplicative set of a ring A. Let M be an A-module with $S^{-1}M = 0$. Then $\text{colim}_{f \in S} H^0_{V(f)}(M) = M$ and $\text{colim}_{f \in S} H^1_{V(f)}(M) = 0$.*

Proof. The statement on H^0 follows directly from the definitions. To see the statement on H^1 observe that $R\Gamma_{V(f)}$ and $H^1_{V(f)}$ commute with colimits. Hence we may assume M is annihilated by some $f \in S$. Then $H^1_{V(ff')}(M) = 0$ for all $f' \in S$ (for example by Lemma 2.3). $\square$

0DWU **Lemma 2.5.** *Let $I \subset A$ be a finitely generated ideal of a ring A. Let $\mathfrak{p}$ be a prime ideal. Let M be an A-module. Let $i \geq 0$ be an integer and consider the map*

$$\Psi : \text{colim}_{f \in A, f \not\in \mathfrak{p}} H^i_{V((I,f))}(M) \longrightarrow H^i_{V(I)}(M)$$

Then

1. $\text{Im}(\Psi)$ *is the set of elements which map to zero in $H^i_{V(I)}(M)_\mathfrak{p}$,*
2. *if $H^{i-1}_{V(I)}(M)_\mathfrak{p} = 0$, then Ψ is injective,*
3. *if $H^{i-1}_{V(I)}(M)_\mathfrak{p} = H^i_{V(I)}(M)_\mathfrak{p} = 0$, then Ψ is an isomorphism.*

Proof. For $f \in A$, $f \not\in \mathfrak{p}$ the spectral sequence of Dualizing Complexes, Lemma 9.6 degenerates to give short exact sequences

$$0 \to H^1_{V(f)}(H^{i-1}_{V(I)}(M)) \to H^i_{V((I,f))}(M) \to H^0_{V(f)}(H^i_{V(I)}(M)) \to 0$$

This proves (1) and part (2) follows from this and Lemma 2.4. Part (3) is a formal consequence. $\square$

0DWV **Lemma 2.6.** *Let $I \subset I' \subset A$ be finitely generated ideals of a Noetherian ring A. Let M be an A-module. Let $i \geq 0$ be an integer. Consider the map*

$$\Psi : H^i_{V(I')}(M) \to H^i_{V(I)}(M)$$

The following are true:

1. *if $H^i_{\mathfrak{p}A_\mathfrak{p}}(M_\mathfrak{p}) = 0$ for all $\mathfrak{p} \in V(I) \setminus V(I')$, then Ψ is surjective,*

(2) *if* $H^{i-1}_{\mathfrak{p}A_\mathfrak{p}}(M_\mathfrak{p}) = 0$ *for all* $\mathfrak{p} \in V(I) \setminus V(I')$*, then* Ψ *is injective,*
(3) *if* $H^{i}_{\mathfrak{p}A_\mathfrak{p}}(M_\mathfrak{p}) = H^{i-1}_{\mathfrak{p}A_\mathfrak{p}}(M_\mathfrak{p}) = 0$ *for all* $\mathfrak{p} \in V(I) \setminus V(I')$*, then* Ψ *is an isomorphism.*

Proof. Proof of (1). Let $\xi \in H^i_{V(I)}(M)$. Since A is Noetherian, there exists a largest ideal $I \subset I'' \subset I'$ such that ξ is the image of some $\xi'' \in H^i_{V(I'')}(M)$. If $V(I'') = V(I')$, then we are done. If not, choose a generic point $\mathfrak{p} \in V(I'')$ not in $V(I')$. Then we have $H^i_{V(I'')}(M)_\mathfrak{p} = H^i_{\mathfrak{p}A_\mathfrak{p}}(M_\mathfrak{p}) = 0$ by assumption. By Lemma 2.5 we can increase I'' which contradicts maximality.

Proof of (2). Let $\xi' \in H^i_{V(I')}(M)$ be in the kernel of Ψ. Since A is Noetherian, there exists a largest ideal $I \subset I'' \subset I'$ such that ξ' maps to zero in $H^i_{V(I'')}(M)$. If $V(I'') = V(I')$, then we are done. If not, then choose a generic point $\mathfrak{p} \in V(I'')$ not in $V(I')$. Then we have $H^{i-1}_{V(I'')}(M)_\mathfrak{p} = H^{i-1}_{\mathfrak{p}A_\mathfrak{p}}(M_\mathfrak{p}) = 0$ by assumption. By Lemma 2.5 we can increase I'' which contradicts maximality.

Part (3) is formal from parts (1) and (2). □

3. Finiteness of local cohomology, I

0AW7 We will follow Faltings approach to finiteness of local cohomology modules, see [Fal78] and [Fal81]. Here is a lemma which shows that it suffices to prove local cohomology modules have an annihilator in order to prove that they are finite modules.

0AW8 **Lemma 3.1.** *Let A be a Noetherian ring, $I \subset A$ an ideal, M a finite A-module, and $n \geq 0$ an integer. Let $Z = V(I)$. The following are equivalent*

This is a special case of [Fal78, Lemma 3].

(1) $H^i_Z(M)$ *is finite for* $i \leq n$,
(2) *there exists an* $e \geq 0$ *such that* I^e *annihilates* $H^i_Z(M)$ *for* $i \leq n$*, and*
(3) *there exists an ideal* $J \subset A$ *with* $V(J) \subset Z$ *such that* J *annihilates* $H^i_Z(M)$ *for* $i \leq n$.

Proof. We prove the lemma by induction on n. For $n = 0$ we have $H^0_Z(M) \subset M$ is finite, hence (1), (2), and (3) are true. Assume that $n > 0$.

If (1) is true, then, since $H^i_Z(M) = H^i_I(M)$ (Dualizing Complexes, Lemma 10.1) is I-power torsion, we see that (2) holds. It is clear that (2) implies (3).

Assume (3) is true. Let $N = H^0_Z(M)$ and $M' = M/N$. By Dualizing Complexes, Lemma 11.6 we may replace M by M'. Thus we may assume that $H^0_Z(M) = 0$. This means that $\text{depth}_I(M) > 0$ (Dualizing Complexes, Lemma 11.1). Pick $f \in I$ a nonzerodivisor on M. After raising f to a suitable power, we may assume $f \in J$ as $V(J) \subset V(I)$. Then the long exact local cohomology sequence associated to the short exact sequence

$$0 \to M \to M \to M/fM \to 0$$

turns into short exact sequences

$$0 \to H^i_Z(M) \to H^i_Z(M/fM) \to H^{i+1}_Z(M) \to 0$$

for $i < n$. We conclude that J^2 annihilates $H^i_Z(M/fM)$ for $i < n$. By induction hypothesis we see that $H^i_Z(M/fM)$ is finite for $i < n$. Using the short exact sequence once more we see that $H^{i+1}_Z(M)$ is finite for $i < n$ as desired. □

The following result of Faltings allows us to prove finiteness of local cohomology at the level of local rings.

0AW9 **Lemma 3.2.** *Let A be a Noetherian ring, $I \subset A$ an ideal, M a finite A-module, and $n \geq 0$ an integer. Let $Z = V(I)$. The following are equivalent* This is a special case of [Fal81, Satz 1].

1. *the modules $H^i_Z(M)$ are finite for $i \leq n$, and*
2. *for all $\mathfrak p \in \operatorname{Spec}(A)$ the modules $H^i_Z(M)_\mathfrak p$, $i \leq n$ are finite $A_\mathfrak p$-modules.*

Proof. The implication (1) $\Rightarrow$ (2) is immediate. We prove the converse by induction on n. The case $n = 0$ is clear because both (1) and (2) are always true in that case.

Assume $n > 0$ and that (2) is true. Let $N = H^0_Z(M)$ and $M' = M/N$. By Dualizing Complexes, Lemma 11.6 we may replace M by M'. Thus we may assume that $H^0_Z(M) = 0$. This means that $\text{depth}_I(M) > 0$ (Dualizing Complexes, Lemma 11.1). Pick $f \in I$ a nonzerodivisor on M and consider the short exact sequence

$$0 \to M \to M \to M/fM \to 0$$

which produces a long exact sequence

$$0 \to H^0_Z(M/fM) \to H^1_Z(M) \to H^1_Z(M) \to H^1_Z(M/fM) \to H^2_Z(M) \to \ldots$$

and similarly after localization. Thus assumption (2) implies that the modules $H^i_Z(M/fM)_\mathfrak p$ are finite for $i < n$. Hence by induction assumption $H^i_Z(M/fM)$ are finite for $i < n$.

Let $\mathfrak p$ be a prime of A which is associated to $H^i_Z(M)$ for some $i \leq n$. Say $\mathfrak p$ is the annihilator of the element $x \in H^i_Z(M)$. Then $\mathfrak p \in Z$, hence $f \in \mathfrak p$. Thus $fx = 0$ and hence x comes from an element of $H^{i-1}_Z(M/fM)$ by the boundary map δ in the long exact sequence above. It follows that $\mathfrak p$ is an associated prime of the finite module $\operatorname{Im}(\delta)$. We conclude that $\operatorname{Ass}(H^i_Z(M))$ is finite for $i \leq n$, see Algebra, Lemma 62.5.

Recall that

$$H^i_Z(M) \subset \prod\nolimits_{\mathfrak p \in \operatorname{Ass}(H^i_Z(M))} H^i_Z(M)_\mathfrak p$$

by Algebra, Lemma 62.19. Since by assumption the modules on the right hand side are finite and I-power torsion, we can find integers $e_{\mathfrak p, i} \geq 0$, $i \leq n$, $\mathfrak p \in \operatorname{Ass}(H^i_Z(M))$ such that $I^{e_{\mathfrak p, i}}$ annihilates $H^i_Z(M)_\mathfrak p$. We conclude that I^e with $e = \max\{e_{\mathfrak p, i}\}$ annihilates $H^i_Z(M)$ for $i \leq n$. By Lemma 3.1 we see that $H^i_Z(M)$ is finite for $i \leq n$. $\square$

0BPX **Lemma 3.3.** *Let A be a ring and let $J \subset I \subset A$ be finitely generated ideals. Let $i \geq 0$ be an integer. Set $Z = V(I)$. If $H^i_Z(A)$ is annihilated by J^n for some n, then $H^i_Z(M)$ annihilated by J^m for some $m = m(M)$ for every finitely presented A-module M such that M_f is a finite locally free A_f-module for all $f \in I$.*

Proof. Consider the annihilator $\mathfrak a$ of $H^i_Z(M)$. Let $\mathfrak p \subset A$ with $\mathfrak p \not\in Z$. By assumption there exists an $f \in I$, $f \not\in \mathfrak p$ and an isomorphism $\varphi : A_f^{\oplus r} \to M_f$ of A_f-modules. Clearing denominators (and using that M is of finite presentation) we find maps

$$a : A^{\oplus r} \longrightarrow M \quad \text{and} \quad b : M \longrightarrow A^{\oplus r}$$

with $a_f = f^N\varphi$ and $b_f = f^N\varphi^{-1}$ for some N. Moreover we may assume that $a \circ b$ and $b \circ a$ are equal to multiplication by f^{2N}. Thus we see that $H^i_Z(M)$ is annihilated by $f^{2N}J^n$, i.e., $f^{2N}J^n \subset \mathfrak a$.

As $U = \mathrm{Spec}(A) \setminus Z$ is quasi-compact we can find finitely many $f_1, \ldots, f_t$ and $N_1, \ldots, N_t$ such that $U = \bigcup D(f_j)$ and $f_j^{2N_j} J^n \subset \mathfrak{a}$. Then $V(I) = V(f_1, \ldots, f_t)$ and since I is finitely generated we conclude $I^M \subset (f_1, \ldots, f_t)$ for some M. All in all we see that $J^m \subset \mathfrak{a}$ for $m \gg 0$, for example $m = M(2N_1 + \ldots + 2N_t)n$ will do. $\square$

0BPY **Lemma 3.4.** *Let A be a Noetherian ring. Let $I \subset A$ be an ideal. Set $Z = V(I)$. Let $n \geq 0$ be an integer. If $H^i_Z(A)$ is finite for $0 \leq i \leq n$, then the same is true for $H^i_Z(M)$, $0 \leq i \leq n$ for any finite A-module M such that M_f is a finite locally free A_f-module for all $f \in I$.*

Proof. The assumption that $H^i_Z(A)$ is finite for $0 \leq i \leq n$ implies there exists an $e \geq 0$ such that I^e annihilates $H^i_Z(A)$ for $0 \leq i \leq n$, see Lemma 3.1. Then Lemma 3.3 implies that $H^i_Z(M)$, $0 \leq i \leq n$ is annihilated by I^m for some $m = m(M, i)$. We may take the same m for all $0 \leq i \leq n$. Then Lemma 3.1 implies that $H^i_Z(M)$ is finite for $0 \leq i \leq n$ as desired. $\square$

4. Finiteness of pushforwards, I

0BL8 In this section we discuss the easiest nontrivial case of the finiteness theorem, namely, the finiteness of the first local cohomology or what is equivalent, finiteness of $j_*\mathcal{F}$ where $j : U \to X$ is an open immersion, X is locally Noetherian, and $\mathcal{F}$ is a coherent sheaf on U. Following a method of Kollár ([Kol16] and [Kol15]) we find a necessary and sufficient condition, see Proposition 4.7. The reader who is interested in higher direct images or higher local cohomology groups should skip ahead to Section 8 or Section 7 (which are developed independently of the rest of this section).

0BJZ **Lemma 4.1.** *Let X be a locally Noetherian scheme. Let $j : U \to X$ be the inclusion of an open subscheme with complement Z. For $x \in U$ let $i_x : W_x \to U$ be the integral closed subscheme with generic point x. Let $\mathcal{F}$ be a coherent $\mathcal{O}_U$-module. The following are equivalent*

1. *for all $x \in Ass(\mathcal{F})$ the $\mathcal{O}_X$-module $j_*i_{x,*}\mathcal{O}_{W_x}$ is coherent,*
2. *$j_*\mathcal{F}$ is coherent.*

Proof. We first prove that (1) implies (2). Assume (1) holds. The statement is local on X, hence we may assume X is affine. Then U is quasi-compact, hence $\mathrm{Ass}(\mathcal{F})$ is finite (Divisors, Lemma 2.5). Thus we may argue by induction on the number of associated points. Let $x \in U$ be a generic point of an irreducible component of the support of $\mathcal{F}$. By Divisors, Lemma 2.5 we have $x \in \mathrm{Ass}(\mathcal{F})$. By our choice of x we have $\dim(\mathcal{F}_x) = 0$ as $\mathcal{O}_{X,x}$-module. Hence $\mathcal{F}_x$ has finite length as an $\mathcal{O}_{X,x}$-module (Algebra, Lemma 61.3). Thus we may use induction on this length.

Set $\mathcal{G} = j_*i_{x,*}\mathcal{O}_{W_x}$. This is a coherent $\mathcal{O}_X$-module by assumption. We have $\mathcal{G}_x = \kappa(x)$. Choose a nonzero map $\varphi_x : \mathcal{F}_x \to \kappa(x) = \mathcal{G}_x$. By Cohomology of Schemes, Lemma 9.6 there is an open $x \in V \subset U$ and a map $\varphi_V : \mathcal{F}|_V \to \mathcal{G}|_V$ whose stalk at x is φ_x. Choose $f \in \Gamma(X, \mathcal{O}_X)$ which does not vanish at x such that $D(f) \subset V$. By Cohomology of Schemes, Lemma 10.4 (for example) we see that φ_V extends to $f^n\mathcal{F} \to \mathcal{G}|_U$ for some n. Precomposing with multiplication by f^n we obtain a map $\mathcal{F} \to \mathcal{G}|_U$ whose stalk at x is nonzero. Let $\mathcal{F}' \subset \mathcal{F}$ be the kernel. Note that $\mathrm{Ass}(\mathcal{F}') \subset \mathrm{Ass}(\mathcal{F})$, see Divisors, Lemma 2.4. Since

$\text{length}_{\mathcal{O}_{X,x}}(\mathcal{F}') = \text{length}_{\mathcal{O}_{X,x}}(\mathcal{F}) - 1$ we may apply the induction hypothesis to conclude $j_*\mathcal{F}'$ is coherent. Since $\mathcal{G} = j_*(\mathcal{G}|_U) = j_*i_{x,*}\mathcal{O}_{W_x}$ is coherent, we can consider the exact sequence

$$0 \to j_*\mathcal{F}' \to j_*\mathcal{F} \to \mathcal{G}$$

By Schemes, Lemma 24.1 the sheaf $j_*\mathcal{F}$ is quasi-coherent. Hence the image of $j_*\mathcal{F}$ in $j_*(\mathcal{G}|_U)$ is coherent by Cohomology of Schemes, Lemma 9.3. Finally, $j_*\mathcal{F}$ is coherent by Cohomology of Schemes, Lemma 9.2.

Assume (2) holds. Exactly in the same manner as above we reduce to the case X affine. We pick $x \in \text{Ass}(\mathcal{F})$ and we set $\mathcal{G} = j_*i_{x,*}\mathcal{O}_{W_x}$. Then we choose a nonzero map $\varphi_x : \mathcal{G}_x = \kappa(x) \to \mathcal{F}_x$ which exists exactly because x is an associated point of $\mathcal{F}$. Arguing exactly as above we may assume φ_x extends to an $\mathcal{O}_U$-module map $\varphi : \mathcal{G}|_U \to \mathcal{F}$. Then φ is injective (for example by Divisors, Lemma 2.10) and we find and injective map $\mathcal{G} = j_*(\mathcal{G}|_V \to j_*\mathcal{F}$. Thus (1) holds. $\square$

0BK0 **Lemma 4.2.** *Let A be a Noetherian ring and let $I \subset A$ be an ideal. Set $X = \text{Spec}(A)$, $Z = V(I)$, $U = X \setminus Z$, and $j : U \to X$ the inclusion morphism. Let $\mathcal{F}$ be a coherent $\mathcal{O}_U$-module. Then*

1. *there exists a finite A-module M such that $\mathcal{F}$ is the restriction of $\widetilde{M}$ to U,*
2. *given M there is an exact sequence*
$$0 \to H^0_Z(M) \to M \to H^0(U, \mathcal{F}) \to H^1_Z(M) \to 0$$
and isomorphisms $H^p(U, \mathcal{F}) = H^{p+1}_Z(M)$ for $p \geq 1$,
3. *given M and $p \geq 0$ the following are equivalent*
   (a) *$R^pj_*\mathcal{F}$ is coherent,*
   (b) *$H^p(U, \mathcal{F})$ is a finite A-module,*
   (c) *$H^{p+1}_Z(M)$ is a finite A-module,*
4. *if the equivalent conditions in (3) hold for $p = 0$, we may take $M = \Gamma(U, \mathcal{F})$ in which case we have $H^0_Z(M) = H^1_Z(M) = 0$.*

Proof. By Properties, Lemma 22.4 there exists a coherent $\mathcal{O}_X$-module $\mathcal{F}'$ whose restriction to U is isomorphic to $\mathcal{F}$. Say $\mathcal{F}'$ corresponds to the finite A-module M as in (1). Note that $R^pj_*\mathcal{F}$ is quasi-coherent (Cohomology of Schemes, Lemma 4.5) and corresponds to the A-module $H^p(U, \mathcal{F})$. By Lemma 2.1 and the general facts in Cohomology, Section 22 we obtain an exact sequence

$$0 \to H^0_Z(M) \to M \to H^0(U, \mathcal{F}) \to H^1_Z(M) \to 0$$

and isomorphisms $H^p(U, \mathcal{F}) = H^{p+1}_Z(M)$ for $p \geq 1$. Here we use that $H^j(X, \mathcal{F}') = 0$ for $j > 0$ as X is affine and $\mathcal{F}'$ is quasi-coherent (Cohomology of Schemes, Lemma 2.2). This proves (2). Parts (3) and (4) are straightforward from (2); see also Lemma 2.2. $\square$

0AWA **Lemma 4.3.** *Let X be a locally Noetherian scheme. Let $j : U \to X$ be the inclusion of an open subscheme with complement Z. Let $\mathcal{F}$ be a coherent $\mathcal{O}_U$-module. Assume*

1. *X is Nagata,*
2. *X is universally catenary, and*
3. *for $x \in Ass(\mathcal{F})$ and $z \in Z \cap \overline{\{x\}}$ we have $\dim(\mathcal{O}_{\overline{\{x\}},z}) \geq 2$.*

Then $j_*\mathcal{F}$ is coherent.

Proof. By Lemma 4.1 it suffices to prove $j_*i_{x,*}\mathcal{O}_{W_x}$ is coherent for $x \in \text{Ass}(\mathcal{F})$. Let $\pi : Y \to X$ be the normalization of X in $\text{Spec}(\kappa(x))$, see Morphisms, Section 51. By Morphisms, Lemma 50.14 the morphism π is finite. Since π is finite $\mathcal{G} = \pi_*\mathcal{O}_Y$ is a coherent $\mathcal{O}_X$-module by Cohomology of Schemes, Lemma 9.9. Observe that $W_x = U \cap \pi(Y)$. Thus $\pi|_{\pi^{-1}(U)} : \pi^{-1}(U) \to U$ factors through $i_x : W_x \to U$ and we obtain a canonical map

$$i_{x,*}\mathcal{O}_{W_x} \longrightarrow (\pi|_{\pi^{-1}(U)})_*(\mathcal{O}_{\pi^{-1}(U)}) = (\pi_*\mathcal{O}_Y)|_U = \mathcal{G}|_U$$

This map is injective (for example by Divisors, Lemma 2.10). Hence $j_*i_{x,*}\mathcal{O}_{W_x} \subset j_*\mathcal{G}|_U$ and it suffices to show that $j_*\mathcal{G}|_U$ is coherent.

It remains to prove that $j_*(\mathcal{G}|_U)$ is coherent. We claim Divisors, Lemma 2.11 applies to

$$\mathcal{G} \longrightarrow j_*(\mathcal{G}|_U)$$

which finishes the proof. Let $z \in X$. If $z \in U$, then the map is an isomorphism on stalks as $j_*(\mathcal{G}|_U)|_U = \mathcal{G}|_U$. If $z \in Z$, then $z \notin \text{Ass}(j_*(\mathcal{G}|_U))$ (Divisors, Lemmas 5.9 and 5.3). Thus it suffices to show that $\text{depth}(\mathcal{G}_z) \geq 2$. Let $y_1, \ldots, y_n \in Y$ be the points mapping to z. By Algebra, Lemma 71.10 it suffices to show that $\text{depth}(\mathcal{O}_{Y,y_i}) \geq 2$ for $i = 1, \ldots, n$. If not, then by Properties, Lemma 12.5 we see that $\dim(\mathcal{O}_{Y,y_i}) = 1$ for some i. This is impossible by the dimension formula (Morphisms, Lemma 49.1) for $\pi : Y \to \overline{\{x\}}$ and assumption (3). □

0BK1 **Lemma 4.4.** *Let X be an integral locally Noetherian scheme. Let $j : U \to X$ be the inclusion of a nonempty open subscheme with complement Z. Assume that for all $z \in Z$ and any associated prime $\mathfrak{p}$ of the completion $\mathcal{O}_{X,z}^\wedge$ we have $\dim(\mathcal{O}_{X,z}^\wedge/\mathfrak{p}) \geq 2$. Then $j_*\mathcal{O}_U$ is coherent.*

Proof. We may assume X is affine. Using Lemmas 3.2 and 4.2 we reduce to $X = \text{Spec}(A)$ where $(A, \mathfrak{m})$ is a Noetherian local domain and $\mathfrak{m} \in Z$. Then we can use induction on $d = \dim(A)$. (The base case is $d = 0, 1$ which do not happen by our assumption on the local rings.) Set $V = \text{Spec}(A) \setminus \{\mathfrak{m}\}$. Observe that the local rings of V have dimension strictly smaller than d. Repeating the arguments for $j' : U \to V$ we and using induction we conclude that $j'_*\mathcal{O}_U$ is a coherent $\mathcal{O}_V$-module. Pick a nonzero $f \in A$ which vanishes on Z. Since $D(f) \cap V \subset U$ we find an n such that multiplication by f^n on U extends to a map $f^n : j'_*\mathcal{O}_U \to \mathcal{O}_V$ over V (for example by Cohomology of Schemes, Lemma 10.4). This map is injective hence there is an injective map

$$j_*\mathcal{O}_U = j''_*j'_*\mathcal{O}_U \to j''_*\mathcal{O}_V$$

on X where $j'' : V \to X$ is the inclusion morphism. Hence it suffices to show that $j''_*\mathcal{O}_V$ is coherent. In other words, we may assume that X is the spectrum of a local Noetherian domain and that Z consists of the closed point.

Assume $X = \text{Spec}(A)$ with $(A, \mathfrak{m})$ local and $Z = \{\mathfrak{m}\}$. Let $A^\wedge$ be the completion of A. Set $X^\wedge = \text{Spec}(A^\wedge)$, $Z^\wedge = \{\mathfrak{m}^\wedge\}$, $U^\wedge = X^\wedge \setminus Z^\wedge$, and $\mathcal{F}^\wedge = \mathcal{O}_{U^\wedge}$. The ring $A^\wedge$ is universally catenary and Nagata (Algebra, Remark 154.9 and Lemma 156.8). Moreover, condition (3) of Lemma 4.3 for $X^\wedge, Z^\wedge, U^\wedge, \mathcal{F}^\wedge$ holds by assumption! Thus we see that $(U^\wedge \to X^\wedge)_*\mathcal{O}_{U^\wedge}$ is coherent. Since the morphism $c : X^\wedge \to X$ is flat we conclude that the pullback of $j_*\mathcal{O}_U$ is $(U^\wedge \to X^\wedge)_*\mathcal{O}_{U^\wedge}$ (Cohomology of Schemes, Lemma 5.2). Finally, since c is faithfully flat we conclude that $j_*\mathcal{O}_U$ is coherent by Descent, Lemma 7.1. □

0BK2 **Remark 4.5.** Let $j : U \to X$ be an open immersion of locally Noetherian schemes. Let $x \in U$. Let $i_x : W_x \to U$ be the integral closed subscheme with generic point x and let $\overline{\{x\}}$ be the closure in X. Then we have a commutative diagram

$$\begin{tikzcd} W_x \arrow[r, "j'"'] \arrow[d, "i_x"'] & \overline{\{x\}} \arrow[d, "i"] \\ U \arrow[r, "j"] & X \end{tikzcd}$$

We have $j_* i_{x,*}\mathcal{O}_{W_x} = i_* j'_*\mathcal{O}_{W_x}$. As the left vertical arrow is a closed immersion we see that $j_* i_{x,*}\mathcal{O}_{W_x}$ is coherent if and only of $j'_*\mathcal{O}_{W_x}$ is coherent.

0AWC **Remark 4.6.** Let X be a locally Noetherian scheme. Let $j : U \to X$ be the inclusion of an open subscheme with complement Z. Let $\mathcal{F}$ be a coherent $\mathcal{O}_U$-module. If there exists an $x \in \text{Ass}(\mathcal{F})$ and $z \in Z \cap \overline{\{x\}}$ such that $\dim(\mathcal{O}_{\overline{\{x\}},z}) \leq 1$, then $j_*\mathcal{F}$ is not coherent. To prove this we can do a flat base change to the spectrum of $\mathcal{O}_{X,z}$. Let $X' = \overline{\{x\}}$. The assumption implies $\mathcal{O}_{X' \cap U} \subset \mathcal{F}$. Thus it suffices to see that $j_*\mathcal{O}_{X'\cap U}$ is not coherent. This is clear because $X' = \{x, z\}$, hence $j_*\mathcal{O}_{X'\cap U}$ corresponds to $\kappa(x)$ as an $\mathcal{O}_{X,z}$-module which cannot be finite as x is not a closed point.

In fact, the converse of Lemma 4.4 holds true: given an open immersion $j : U \to X$ of integral Noetherian schemes and there exists a $z \in X \setminus U$ and an associated prime $\mathfrak{p}$ of the completion $\mathcal{O}_{X,z}^\wedge$ with $\dim(\mathcal{O}_{X,z}^\wedge/\mathfrak{p}) = 1$, then $j_*\mathcal{O}_U$ is not coherent. Namely, you can pass to the local ring, you can enlarge U to the punctured spectrum, you can pass to the completion, and then the argument above gives the nonfiniteness.

0BK3 **Proposition 4.7** (Kollár). *Let $j : U \to X$ be an open immersion of locally Noetherian schemes with complement Z. Let $\mathcal{F}$ be a coherent $\mathcal{O}_U$-module. The following are equivalent* Theorem of Kollár stated in an email dated Wed, 1 Jul 2015.

(1) *$j_*\mathcal{F}$ is coherent,*
(2) *for $x \in Ass(\mathcal{F})$ and $z \in Z \cap \overline{\{x\}}$ and any associated prime $\mathfrak{p}$ of the completion $\mathcal{O}^\wedge_{\overline{\{x\}},z}$ we have $\dim(\mathcal{O}^\wedge_{\overline{\{x\}},z}/\mathfrak{p}) \geq 2$.*

Proof. If (2) holds we get (1) by a combination of Lemmas 4.1, Remark 4.5, and Lemma 4.4. If (2) does not hold, then $j_* i_{x,*}\mathcal{O}_{W_x}$ is not finite for some $x \in \text{Ass}(\mathcal{F})$ by the discussion in Remark 4.6 (and Remark 4.5). Thus $j_*\mathcal{F}$ is not coherent by Lemma 4.1. $\square$

0BL9 **Lemma 4.8.** *Let A be a Noetherian ring and let $I \subset A$ be an ideal. Set $Z = V(I)$. Let M be a finite A-module. The following are equivalent*

(1) *$H^1_Z(M)$ is a finite A-module, and*
(2) *for all $\mathfrak{p} \in Ass(M)$, $\mathfrak{p} \notin Z$ and all $\mathfrak{q} \in V(\mathfrak{p} + I)$ the completion of $(A/\mathfrak{p})_\mathfrak{q}$ does not have associated primes of dimension 1.*

Proof. Follows immediately from Proposition 4.7 via Lemma 4.2. $\square$

The formulation in the following lemma has the advantage that conditions (1) and (2) are inherited by schemes of finite type over X. Moreover, this is the form of finiteness which we will generalize to higher direct images in Section 8.

0AWB **Lemma 4.9.** *Let X be a locally Noetherian scheme. Let $j : U \to X$ be the inclusion of an open subscheme with complement Z. Let $\mathcal{F}$ be a coherent $\mathcal{O}_U$-module. Assume*

1. *X is universally catenary,*
2. *for every $z \in Z$ the formal fibres of $\mathcal{O}_{X,z}$ are (S_1).*

In this situation the following are equivalent

- (a) *for $x \in Ass(\mathcal{F})$ and $z \in Z \cap \overline{\{x\}}$ we have $\dim(\mathcal{O}_{\overline{\{x\}},z}) \geq 2$, and*
- (b) *$j_*\mathcal{F}$ is coherent.*

Proof. Let $x \in \text{Ass}(\mathcal{F})$. By Proposition 4.7 it suffices to check that $A = \mathcal{O}_{\overline{\{x\}},z}$ satisfies the condition of the proposition on associated primes of its completion if and only if $\dim(A) \geq 2$. Observe that A is universally catenary (this is clear) and that its formal fibres are (S_1) as follows from More on Algebra, Lemma 48.10 and Proposition 48.5. Let $\mathfrak{p}' \subset A^\wedge$ be an associated prime. As $A \to A^\wedge$ is flat, by Algebra, Lemma 64.3, we find that $\mathfrak{p}'$ lies over $(0) \subset A$. Since the formal fibre $A^\wedge \otimes_A f.f.(A)$ is (S_1) we see that $\mathfrak{p}'$ is a minimal prime, see Algebra, Lemma 151.2. Since A is universally catenary it is formally catenary by More on Algebra, Proposition 91.5. Hence $\dim(A^\wedge/\mathfrak{p}') = \dim(A)$ which proves the equivalence. □

5. Depth and dimension

0DWW Some helper lemmas.

0DWX **Lemma 5.1.** *Let A be a Noetherian ring. Let $I \subset A$ be an ideal. Let M be a finite A-module. Let $\mathfrak{p} \in V(I)$ be a prime ideal. Assume $e = depth_{IA_\mathfrak{p}}(M_\mathfrak{p}) < \infty$. Then there exists a nonempty open $U \subset V(\mathfrak{p})$ such that $depth_{IA_\mathfrak{q}}(M_\mathfrak{q}) \geq e$ for all $\mathfrak{q} \in U$.*

Proof. By definition of depth we have $IM_\mathfrak{p} \neq M_\mathfrak{p}$ and there exists an $M_\mathfrak{p}$-regular sequence $f_1, \ldots, f_e \in IA_\mathfrak{p}$. After replacing A by a principal localization we may assume $f_1, \ldots, f_e \in I$ form an M-regular sequence, see Algebra, Lemma 67.6. Consider the module $M' = M/IM$. Since $\mathfrak{p} \in \text{Supp}(M')$ and since the support of a finite module is closed, we find $V(\mathfrak{p}) \subset \text{Supp}(M')$. Thus for $\mathfrak{q} \in V(\mathfrak{p})$ we get $IM_\mathfrak{q} \neq M_\mathfrak{q}$. Hence, using that localization is exact, we see that $\text{depth}_{IA_\mathfrak{q}}(M_\mathfrak{q}) \geq e$ for any $\mathfrak{q} \in V(I)$ by definition of depth. □

0DWY **Lemma 5.2.** *Let A be a Noetherian ring. Let M be a finite A-module. Let $\mathfrak{p}$ be a prime ideal. Assume $e = depth_{A_\mathfrak{p}}(M_\mathfrak{p}) < \infty$. Then there exists a nonempty open $U \subset V(\mathfrak{p})$ such that $depth_{A_\mathfrak{q}}(M_\mathfrak{q}) \geq e$ for all $\mathfrak{q} \in U$ and for all but finitely many $\mathfrak{q} \in U$ we have $depth_{A_\mathfrak{q}}(M_\mathfrak{q}) > e$.*

Proof. By definition of depth we have $\mathfrak{p}M_\mathfrak{p} \neq M_\mathfrak{p}$ and there exists an $M_\mathfrak{p}$-regular sequence $f_1, \ldots, f_e \in \mathfrak{p}A_\mathfrak{p}$. After replacing A by a principal localization we may assume $f_1, \ldots, f_e \in \mathfrak{p}$ form an M-regular sequence, see Algebra, Lemma 67.6. Consider the module $M' = M/(f_1, \ldots, f_e)M$. Since $\mathfrak{p} \in \text{Supp}(M')$ and since the support of a finite module is closed, we find $V(\mathfrak{p}) \subset \text{Supp}(M')$. Thus for $\mathfrak{q} \in V(\mathfrak{p})$ we get $\mathfrak{q}M_\mathfrak{q} \neq M_\mathfrak{q}$. Hence, using that localization is exact, we see that $\text{depth}_{A_\mathfrak{q}}(M_\mathfrak{q}) \geq e$ for any $\mathfrak{q} \in V(I)$ by definition of depth. Moreover, as soon as $\mathfrak{q}$ is not an associated prime of the module M', then the depth goes up. Thus we see that the final statement holds by Algebra, Lemma 62.5. □

0DWZ **Lemma 5.3.** *Let $(A, \mathfrak{m})$ be a Noetherian local ring with normalized dualizing complex $\omega_A^\bullet$. Let M be a finite A-module. Set $E^i = Ext_A^{-i}(M, \omega_A^\bullet)$. Then*

1. *E^i is a finite A-module nonzero only for $0 \leq i \leq \dim(Supp(M))$,*
2. *$\dim(Supp(E^i)) \leq i$,*
3. *$depth(M)$ is the smallest integer $\delta \geq 0$ such that $E^\delta \neq 0$,*
4. *$\mathfrak{p} \in Supp(E^0 \oplus \ldots \oplus E^i) \Leftrightarrow depth_{A_\mathfrak{p}}(M_\mathfrak{p}) + \dim(A/\mathfrak{p}) \leq i$,*
5. *the annihilator of E^i is equal to the annihilator of $H^i_\mathfrak{m}(M)$.*

Proof. Parts (1), (2), and (3) are copies of the statements in Dualizing Complexes, Lemma 16.5. For a prime $\mathfrak{p}$ of A we have that $(\omega_A^\bullet)_\mathfrak{p}[-\dim(A/\mathfrak{p})]$ is a normalized dualzing complex for $A_\mathfrak{p}$. See Dualizing Complexes, Lemma 17.3. Thus

$$E^i_\mathfrak{p} = \text{Ext}^{-i}_A(M, \omega_A^\bullet)_\mathfrak{p} = \text{Ext}^{-i+\dim(A/\mathfrak{p})}_{A_\mathfrak{p}}(M_\mathfrak{p}, (\omega_A^\bullet)_\mathfrak{p}[-\dim(A/\mathfrak{p})])$$

is zero for $i - \dim(A/\mathfrak{p}) < \text{depth}_{A_\mathfrak{p}}(M_\mathfrak{p})$ and nonzero for $i = \dim(A/\mathfrak{p}) + \text{depth}_{A_\mathfrak{p}}(M_\mathfrak{p})$ by part (3) over $A_\mathfrak{p}$. This proves part (4). If E is an injective hull of the residue field of A, then we have

$$\text{Hom}_A(H^i_\mathfrak{m}(M), E) = \text{Ext}^{-i}_A(M, \omega_A^\bullet)^\wedge = (E^i)^\wedge = E^i \otimes_A A^\wedge$$

by the local duality theorem (in the form of Dualizing Complexes, Lemma 18.4). Since $A \to A^\wedge$ is faithfully flat, we find (5) is true by Matlis duality (Dualizing Complexes, Proposition 7.8). □

0DX0 **Lemma 5.4.** *Let $(A, \mathfrak{m})$ be a Noetherian local ring.*

1. *Let M be a finite A-module. Then the A-module $H^i_\mathfrak{m}(M)$ satisfies the descending chain condition for any i.*
2. *Let $U = \text{Spec}(A) \setminus \{\mathfrak{m}\}$ be the punctured spectrum of A. Let $\mathcal{F}$ be a coherent $\mathcal{O}_U$-module. Then the A-module $H^i(U, \mathcal{F})$ satisfies the descending chain condition for $i > 0$.*

Proof. Proof of (1). Let $A^\wedge$ be the completion of A. Observe that $H^i_\mathfrak{m}(M) \otimes_A A^\wedge = H^i_{\mathfrak{m}A^\wedge}(M \otimes_A A^\wedge)$ by Dualizing Complexes, Lemma 9.3. Thus we may assume A is complete; some details omitted. If A is complete, then A has a normalized dualizing complex $\omega_A^\bullet$ and we find that $H^i_\mathfrak{m}(M)$ is Matlis dual to the finite A-module $\text{Ext}^{-i}_A(M, \omega_A^\bullet)$ by the local duality theorem (Dualizing Complexes, Lemma 18.4). We find (1) is true by Matlis duality (Dualizing Complexes, Proposition 7.8). Part (2) follows from (1) via Lemma 4.2. □

0DX1 **Lemma 5.5.** *Let $(A, \mathfrak{m})$ be a Noetherian local ring.*

1. *Let (M_n) be an inverse system of finite A-modules. Then the inverse system $H^i_\mathfrak{m}(M_n)$ satisfies the Mittag-Leffler condition for any i.*
2. *Let $U = \text{Spec}(A) \setminus \{\mathfrak{m}\}$ be the punctured spectrum of A. Let $\mathcal{F}_n$ be an inverse system of coherent $\mathcal{O}_U$-modules. Then the inverse system $H^i(U, \mathcal{F}_n)$ satisfies the Mittag-Leffler condition for $i > 0$.*

Proof. Follows immediately from Lemma 5.4. □

6. Improving coherent modules

0DX2 Similar constructions can be found in [DG67] and more recently in [Kol15] and [Kol16].

0DX3 **Lemma 6.1.** *Let X be a Noetherian scheme. Let $Z \subset X$ be a closed subscheme. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Then there is a canonical surjection $\mathcal{F} \to \mathcal{F}'$ of coherent $\mathcal{O}_X$-modules such that*

1. *$\mathcal{F}|_{X \setminus Z} \to \mathcal{F}'|_{X \setminus Z}$ is an isomorphism,*
2. *for $z \in Z$ we have $depth_{\mathcal{O}_{X,z}}(\mathcal{F}'_z) \geq 1$.*

If $f : Y \to X$ is a flat morphism with Y Noetherian, then $f^*\mathcal{F} \to f^*\mathcal{F}'$ is the corresponding quotient for $f^{-1}(Z) \subset Y$ and $f^*\mathcal{F}$.

Proof. Condition (2) on $\mathcal{F}'$ just means that $\mathcal{F}'$ has no associated points in Z. For example if $Z = X$, then $\mathcal{F}' = 0$ is the solution. The statement on pullbacks follows from Divisors, Lemma 3.1.

Existence of $\mathcal{F} \to \mathcal{F}'$. Let $\mathcal{G} \subset \mathcal{F}$ be the quasi-coherent subsheaf of sections supported in Z, see Properties, Definition 24.6. Set $\mathcal{F}' = \mathcal{F}/\mathcal{G}$. Since $\mathcal{F}'$ does not have any nonzero section whose support is contained in Z we see that $\text{Ass}(\mathcal{F}') \cap Z = \emptyset$ and the proof is complete. $\square$

0DX4 **Lemma 6.2.** *Let X be a Noetherian scheme. Let $Z \subset X$ be a closed subscheme. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Denote $j : X \setminus Z \to X$ the inclusion morphism. Assume $\mathcal{F}' = j_*(\mathcal{F}|_{X \setminus Z})$ is coherent (Proposition 4.7 and Lemma 4.9). Then $\mathcal{F} \to \mathcal{F}'$ is the unique map of coherent $\mathcal{O}_X$-modules such that*

1. *$\mathcal{F}|_{X \setminus Z} \to \mathcal{F}'|_{X \setminus Z}$ is an isomorphism,*
2. *for $z \in Z$ we have $depth_{\mathcal{O}_{X,z}}(\mathcal{F}'_z) \geq 2$.*

If $f : Y \to X$ is a flat morphism with Y Noetherian, then $f^*\mathcal{F} \to f^*\mathcal{F}'$ is the corresponding map for $f^{-1}(Z) \subset Y$ and $f^*\mathcal{F}$.

Proof. Let us show that $\text{depth}_{\mathcal{O}_{X,z}}(\mathcal{F}'_z) \geq 2$ for $z \in Z$. Namely, let U be the punctured spectrum of $\mathcal{O}_{X,z}$. Then U contains the inverse image of $X \setminus Z$ along $\text{Spec}(\mathcal{O}_{X,z}) \to X$. Since $\mathcal{F}' = j_*(\mathcal{F}|_{X \setminus Z}) = j_*(\mathcal{F}'|_{X \setminus Z})$, the same is true after base change by the flat morphism $\text{Spec}(\mathcal{O}_{X,z}) \to X$ (Cohomology of Schemes, Lemma 5.2). A fortiori, the canonical map $\mathcal{F}'_z \to H^0(U, \mathcal{F}'|_U)$ is an isomorphism. This means that $H^i_{\mathfrak{m}_z}(\mathcal{F}'_z)$ is zero for $i = 0, 1$, see Lemma 4.2. Thus the depth is at least 2. We omit the proof of the other statements. $\square$

0DX5 **Lemma 6.3.** *Let X be a Noetherian scheme. Let $Z \subset X$ be a closed subscheme. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Assume X is universally catenary and the formal fibres of local rings have (S_1). Then there exists a canonical map $\mathcal{F} \to \mathcal{F}'$ of coherent $\mathcal{O}_X$-modules such that*

1. *$\mathcal{F}|_{X \setminus Z} \to \mathcal{F}'|_{X \setminus Z}$ is an isomorphism,*
2. *for $z \in Z$ we have either*
   - (a) *$depth_{\mathcal{O}_{X,z}}(\mathcal{F}'_z) \geq 2$, or*
   - (b) *there is an $x \in Ass(\mathcal{F}|_{X \setminus Z})$ with $z \in \overline{\{x\}}$ and $\dim(\mathcal{O}_{\overline{\{x\}},z}) = 1$ and in this case $\mathcal{F}_z \to \mathcal{F}'_z$ is the unique surjection with $depth_{\mathcal{O}_{X,z}}(\mathcal{F}'_z) = 1$.*

If $f : Y \to X$ is a Cohen-Macaulay morphism with Y Noetherian, then $f^*\mathcal{F} \to f^*\mathcal{F}'$ satisfies the same properties with respect to $f^{-1}(Z) \subset Y$ and $f^*\mathcal{F}$.

Proof. We first replace $\mathcal{F}$ by the quotient of it constructed in Lemma 6.1. Recall that $\text{Ass}(\mathcal{F}) = \{x_1, \ldots, x_n\}$ is finite (and $x_i \notin Z$ by our choice of $\mathcal{F}$). Let Y_i be the closure of $\{x_i\}$. Let $Z_{i,j}$ be the irreducible components of $Z \cap Y_i$. Observe that $\text{Supp}(\mathcal{F}) \cap Z = \bigcup Z_{i,j}$. Let $z_{i,j} \in Z_{i,j}$ be the generic point. Let

$$d_{i,j} = \dim(\mathcal{O}_{\overline{\{x_i\}}, z_{i,j}})$$

If $d_{i,j} = 1$, then condition (2)(b) holds for $\mathcal{F}$ at $z_{i,j}$. Thus we do not need to modify $\mathcal{F}$ at these points. Furthermore, still assuming $d_{i,j} = 1$, using Lemma 5.2 we can find an open neighbourhood $z_{i,j} \in V_{i,j} \subset X$ such that $\text{depth}_{\mathcal{O}_{X,z}}(\mathcal{F}_z) \geq 2$ for $z \in Z_{i,j} \cap V_{i,j}$, $z \neq z_{i,j}$. Set

$$Z' = X \setminus \left(X \setminus Z \cup \bigcup\nolimits_{d_{i,j}=1} V_{i,j}) \right)$$

Denote $j' : X \setminus Z' \to X$. By our choice of Z' the assumptions of Lemma 4.9 are satisfied. We conclude by setting $\mathcal{F}' = j'_*(\mathcal{F}|_{X \setminus Z'})$ and applying Lemma 6.2.

The final statement follows from the formula for the change in depth along a flat local homomorphism, see Algebra, Lemma 157.1 and the assumption on the fibres of f inherent in f being Cohen-Macaulay. Details omitted. □

7. Finiteness of local cohomology, II

0BJQ We continue the discussion of finiteness of local cohomology started in Section 3. Let A be a Noetherian ring and let $I \subset A$ be an ideal. Set $X = \text{Spec}(A)$ and $Z = V(I) \subset X$. Let M be a finite A-module. We define

0BJR (7.0.1) $$s_{A,I}(M) = \min\{\text{depth}_{A_\mathfrak{p}}(M_\mathfrak{p}) + \dim((A/\mathfrak{p})_\mathfrak{q}) \mid \mathfrak{p} \in X \setminus Z, \mathfrak{q} \in Z, \mathfrak{p} \subset \mathfrak{q}\}$$

Our conventions on depth are that the depth of 0 is ∞ thus we only need to consider primes $\mathfrak{p}$ in the support of M. It will turn out that $s_{A,I}(M)$ is an important invariant of the situation.

0BJS **Lemma 7.1.** *Let $A \to B$ be a finite homomorphism of Noetherian rings. Let $I \subset A$ be an ideal and set $J = IB$. Let M be a finite B-module. If A is universally catenary, then $s_{B,J}(M) = s_{A,I}(M)$.*

Proof. Let $\mathfrak{p} \subset \mathfrak{q} \subset A$ be primes with $I \subset \mathfrak{q}$ and $I \not\subset \mathfrak{p}$. Since $A \to B$ is finite there are finitely many primes $\mathfrak{p}_i$ lying over $\mathfrak{p}$. By Algebra, Lemma 71.10 we have

$$\text{depth}(M_\mathfrak{p}) = \min \text{depth}(M_{\mathfrak{p}_i})$$

Let $\mathfrak{p}_i \subset \mathfrak{q}_{ij}$ be primes lying over $\mathfrak{q}$. By going up for $A \to B$ (Algebra, Lemma 35.22) there is at least one $\mathfrak{q}_{ij}$ for each i. Then we see that

$$\dim((B/\mathfrak{p}_i)_{\mathfrak{q}_{ij}}) = \dim((A/\mathfrak{p})_\mathfrak{q})$$

by the dimension formula, see Algebra, Lemma 112.1. This implies that the minimum of the quantities used to define $s_{B,J}(M)$ for the pairs $(\mathfrak{p}_i, \mathfrak{q}_{ij})$ is equal to the quantity for the pair $(\mathfrak{p}, \mathfrak{q})$. This proves the lemma. □

0BJT **Lemma 7.2.** *Let A be a universally catenary Noetherian local ring. Let $I \subset A$ be an ideal. Let M be a finite A-module. Then*

$$s_{A,I}(M) \geq s_{A^\wedge, I^\wedge}(M^\wedge)$$

If the formal fibres of A are (S_n), then $\min(n+1, s_{A,I}(M)) \leq s_{A^\wedge, I^\wedge}(M^\wedge)$.

Proof. Write $X = \mathrm{Spec}(A)$, $X^\wedge = \mathrm{Spec}(A^\wedge)$, $Z = V(I) \subset X$, and $Z^\wedge = V(I^\wedge)$. Let $\mathfrak p' \subset \mathfrak q' \subset A^\wedge$ be primes with $\mathfrak p' \not\in Z^\wedge$ and $\mathfrak q' \in Z^\wedge$. Let $\mathfrak p \subset \mathfrak q$ be the corresponding primes of A. Then $\mathfrak p \not\in Z$ and $\mathfrak q \in Z$. Picture

$$\begin{tikzcd}
\mathfrak p' \arrow[r] \arrow[d, dash] & \mathfrak q' \arrow[r] \arrow[d, dash] & A^\wedge \\
\mathfrak p \arrow[r] & \mathfrak q \arrow[r] & A \arrow[u]
\end{tikzcd}$$

Let us write

$$\begin{aligned}
a &= \dim(A/\mathfrak p) = \dim(A^\wedge/\mathfrak p A^\wedge),\\
b &= \dim(A/\mathfrak q) = \dim(A^\wedge/\mathfrak q A^\wedge),\\
a' &= \dim(A^\wedge/\mathfrak p'),\\
b' &= \dim(A^\wedge/\mathfrak q')
\end{aligned}$$

Equalities by More on Algebra, Lemma 40.1. We also write

$$\begin{aligned}
p &= \dim(A^\wedge_{\mathfrak p'}/\mathfrak p A^\wedge_{\mathfrak p'}) = \dim((A^\wedge/\mathfrak p A^\wedge)_{\mathfrak p'})\\
q &= \dim(A^\wedge_{\mathfrak q'}/\mathfrak p A^\wedge_{\mathfrak q'}) = \dim((A^\wedge/\mathfrak q A^\wedge)_{\mathfrak q'})
\end{aligned}$$

Since A is universally catenary we see that $A^\wedge/\mathfrak p A^\wedge = (A/\mathfrak p)^\wedge$ is equidimensional of dimension a (More on Algebra, Proposition 91.5). Hence $a = a' + p$. Similarly $b = b' + q$. By Algebra, Lemma 157.1 applied to the flat local ring map $A_\mathfrak p \to A^\wedge_{\mathfrak p'}$ we have

$$\mathrm{depth}(M^\wedge_{\mathfrak p'}) = \mathrm{depth}(M_\mathfrak p) + \mathrm{depth}(A^\wedge_{\mathfrak p'}/\mathfrak p A^\wedge_{\mathfrak p'})$$

The quantity we are minimizing for $s_{A,I}(M)$ is

$$s(\mathfrak p, \mathfrak q) = \mathrm{depth}(M_\mathfrak p) + \dim((A/\mathfrak p)_\mathfrak q) = \mathrm{depth}(M_\mathfrak p) + a - b$$

(last equality as A is catenary). The quantity we are minimizing for $s_{A^\wedge, I^\wedge}(M^\wedge)$ is

$$s(\mathfrak p', \mathfrak q') = \mathrm{depth}(M^\wedge_{\mathfrak p'}) + \dim((A^\wedge/\mathfrak p')_{\mathfrak q'}) = \mathrm{depth}(M^\wedge_{\mathfrak p'}) + a' - b'$$

(last equality as $A^\wedge$ is catenary). Now we have enough notation in place to start the proof.

Let $\mathfrak p \subset \mathfrak q \subset A$ be primes with $\mathfrak p \not\in Z$ and $\mathfrak q \in Z$ such that $s_{A,I}(M) = s(\mathfrak p, \mathfrak q)$. Then we can pick $\mathfrak q'$ minimal over $\mathfrak q A^\wedge$ and $\mathfrak p' \subset \mathfrak q'$ minimal over $\mathfrak p A^\wedge$ (using going down for $A \to A^\wedge$). Then we have four primes as above with $p = 0$ and $q = 0$. Moreover, we have $\mathrm{depth}(A^\wedge_{\mathfrak p'}/\mathfrak p A^\wedge_{\mathfrak p'}) = 0$ also because $p = 0$. This means that $s(\mathfrak p', \mathfrak q') = s(\mathfrak p, \mathfrak q)$. Thus we get the first inequality.

Assume that the formal fibres of A are (S_n). Then $\mathrm{depth}(A^\wedge_{\mathfrak p'}/\mathfrak p A^\wedge_{\mathfrak p'}) \geq \min(n, p)$. Hence

$$s(\mathfrak p', \mathfrak q') \geq s(\mathfrak p, \mathfrak q) + q + \min(n, p) - p \geq s_{A,I}(M) + q + \min(n,p) - p$$

Thus the only way we can get in trouble is if $p > n$. If this happens then

$$\begin{aligned}
s(\mathfrak p', \mathfrak q') &= \mathrm{depth}(M^\wedge_{\mathfrak p'}) + \dim((A^\wedge/\mathfrak p')_{\mathfrak q'})\\
&= \mathrm{depth}(M_\mathfrak p) + \mathrm{depth}(A^\wedge_{\mathfrak p'}/\mathfrak p A^\wedge_{\mathfrak p'}) + \dim((A^\wedge/\mathfrak p')_{\mathfrak q'})\\
&\geq 0 + n + 1
\end{aligned}$$

because $(A^\wedge/\mathfrak p')_{\mathfrak q'}$ has at least two primes. This proves the second inequality. $\square$

The method of proof of the following lemma works more generally, but the stronger results one gets will be subsumed in Theorem 7.4 below.

0BJU **Lemma 7.3.** *Let A be a Gorenstein Noetherian local ring. Let $I \subset A$ be an ideal and set $Z = V(I) \subset \text{Spec}(A)$. Let M be a finite A-module. Let $s = s_{A,I}(M)$ as in (7.0.1). Then $H^i_Z(M)$ is finite for $i < s$, but $H^s_Z(M)$ is not finite.* This is a special case of [Fal78, Satz 1].

Proof. An important role will be played by the finite A-modules

$$E^i = \text{Ext}^i_A(M, A)$$

For $\mathfrak{p} \subset A$ we will write $H^i_\mathfrak{p}$ to denote the local cohomology of a $A_\mathfrak{p}$-module. Then we see that the $\mathfrak{p}A_\mathfrak{p}$-adic completion of

$$(E^i)_\mathfrak{p} = \text{Ext}^i_{A_\mathfrak{p}}(M_\mathfrak{p}, A_\mathfrak{p})$$

is Matlis dual to

$$H^{\dim(A_\mathfrak{p})-i}_\mathfrak{p}(M_\mathfrak{p})$$

by Dualizing Complexes, Lemma 18.4 and the fact that $A_\mathfrak{p}$ is Gorenstein. In particular we deduce from this the following fact: an ideal $J \subset A$ annihilates $(E^i)_\mathfrak{p}$ if and only if J annihilates $H^{\dim(A_\mathfrak{p})-i}_\mathfrak{p}(M_\mathfrak{p})$. Set $Z_n = \{\mathfrak{p} \in Z \mid \dim(A/\mathfrak{p}) \leq n\}$. Observe that $Z_{-1} = \emptyset$ and $Z_n = Z$ for $n = \dim(Z)$.

Proof of finiteness for $i < s$. We will use a double induction to do this. For $i < s$ consider the induction hypothesis IH_i: $H^a_Z(M)$ is finite for $0 \leq a \leq i$. The case IH_0 is trivial because $H^0_Z(M)$ is a submodule of M and hence finite.

Induction step. Assume IH_{i-1} holds for some $0 < i < s$. For $0 \leq a \leq i-1$ let J_a be the annihilator of $H^a_Z(M)$. Observe that $V(J_a) \subset Z$ as the support of the finite A-module $H^a_Z(M)$ is contained in Z. We will show by descending induction on n that there exists an ideal J with $V(J) \subset Z$ such that the associated primes of $JH^i_Z(M)$ are in Z_n. For $n = -1$ this implies $JH^i_Z(M) = 0$ (Algebra, Lemma 62.7) and hence the finiteness of $H^i_Z(M)$ by Lemma 3.1. The base case $n = \dim(Z)$ is trivial.

Thus we assume given J with the property for n. Let $\mathfrak{q} \in Z_n$. With $Z_\mathfrak{q} = V(IA_\mathfrak{q})$ we have $H^j_Z(M)_\mathfrak{q} = H^j_{Z_\mathfrak{q}}(M_\mathfrak{q})$ by Dualizing Complexes, Lemma 9.3. Consider the spectral sequence

$$H^p_\mathfrak{q}(H^q_Z(M)_\mathfrak{q}) \Rightarrow H^{p+q}_\mathfrak{q}(M_\mathfrak{q})$$

of Dualizing Complexes, Lemma 9.6 for the ideals $IA_\mathfrak{q} \subset \mathfrak{q}A_\mathfrak{q} \subset A_\mathfrak{q}$. Below we will find an ideal $J' \subset A$ with $V(J') \subset Z$ such that $H^i_\mathfrak{q}(M_\mathfrak{q})$ is annihilated by J' for all $\mathfrak{q} \in Z_n \setminus Z_{n-1}$. Claim: $JJ'J_0 \ldots J_{i-1}$ will work for $n-1$. Namely, let $\mathfrak{q} \in Z_n \setminus Z_{n-1}$. The spectral sequence above defines a filtration

$$E^{0,i}_\infty = E^{0,i}_{i+2} \subset \ldots \subset E^{0,i}_3 \subset E^{0,i}_2 = H^0_\mathfrak{q}(H^i_Z(M)_\mathfrak{q})$$

The module $E^{0,i}_\infty$ is annihilated by J'. The subquotients $E^{0,i}_j/E^{0,i}_{j+1}$ are annihilated by J_{i-j+1} because the target of $d^{0,i}_j$ is a subquotient of $H^j_\mathfrak{q}(H^{i-j+1}_Z(M))$. Finally, by our choice of J we have $JH^i_Z(M)_\mathfrak{q} \subset H^0_\mathfrak{q}(H^i_Z(M)_\mathfrak{q})$. Thus $\mathfrak{q}$ cannot be an associated prime of $JJ'J_0 \ldots J_{i-1}H^i_Z(M)$ as desired.

By our initial remarks we see that J' should annihilate

$$(E^{\dim(A_\mathfrak{q})-i})_\mathfrak{q} = (E^{\dim(A)-n-i})_\mathfrak{q}$$

for all $\mathfrak{q} \in Z_n \setminus Z_{n-1}$. But if J' works for one $\mathfrak{q}$, then it works for all $\mathfrak{q}$ in an open neighbourhood of $\mathfrak{q}$ as the modules $E^{\dim(A)-n-i}$ are finite. Since every subset of X is Noetherian with the induced topology (Topology, Lemma 9.2), we conclude that it suffices to prove the existence of J' for one $\mathfrak{q}$.

Since the ext modules are finite the existence of J' is equivalent to

$$\mathrm{Supp}(E^{\dim(A)-n-i}) \cap \mathrm{Spec}(A_\mathfrak{q}) \subset Z.$$

This is equivalent to showing the localization at every $\mathfrak{p} \subset \mathfrak{q}$, $\mathfrak{p} \notin Z$ is zero. Using local duality over $A_\mathfrak{p}$ we find that we need to prove that

$$H_\mathfrak{p}^{\dim(A_\mathfrak{p})-\dim(A)+n+i}(M_\mathfrak{p}) = H_\mathfrak{p}^{i-\dim((A/\mathfrak{p})_\mathfrak{q})}(M_\mathfrak{p})$$

is zero (this uses that A is catenary). This vanishes exactly by our definition of $s(M)$ and Dualizing Complexes, Lemma 11.1. This finishes the proof of finiteness for $i < s$.

To prove $H^s_Z(M)$ is not finite we work backwards through the arguments above. First, we pick a $\mathfrak{q} \in Z$, $\mathfrak{p} \subset \mathfrak{q}$ with $\mathfrak{p} \notin Z$ such that $s = \mathrm{depth}_{A_\mathfrak{p}}(M_\mathfrak{p}) + \dim((A/\mathfrak{p})_\mathfrak{q})$. Then $H_\mathfrak{p}^{i-\dim((A/\mathfrak{p})_\mathfrak{q})}(M_\mathfrak{p})$ is nonzero by the nonvanishing in Dualizing Complexes, Lemma 11.1. Set $n = \dim(A/\mathfrak{q})$. Then there does not exist an ideal $J \subset A$ with $V(J) \subset Z$ such that $J(E^{\dim(A)-n-s})_\mathfrak{q} = 0$. Thus $H^s_\mathfrak{q}(M_\mathfrak{q})$ is not annihilated by an ideal $J \subset A$ with $V(J) \subset Z$. It follows from the spectral sequence displayed above that at least one of the modules $H^i_Z(M)_\mathfrak{q}$, $0 \leq i \leq s$ is not annihilated by an ideal $J \subset A$ with $V(J) \subset Z$. Since $H^i_Z(M)$ is finite for $i < s$ and hence are annihilated by such ideals, we conclude that $H^s_Z(M)$ is not finite. $\square$

Observe that the hypotheses of the following theorem are satisfied by excellent Noetherian rings (by definition), by Noetherian rings which have a dualizing complex (Dualizing Complexes, Lemma 17.4 and Dualizing Complexes, Lemma 23.2), and by quotients of regular Noetherian rings.

0BJV **Theorem 7.4.** *Let A be a Noetherian ring and let $I \subset A$ be an ideal. Set $Z = V(I) \subset \mathrm{Spec}(A)$. Let M be a finite A-module. Set $s = s_{A,I}(M)$ as in (7.0.1). Assume that*

This is a special case of [Fal81, Satz 2].

1. *A is universally catenary,*
2. *the formal fibres of the local rings of A are Cohen-Macaulay.*

Then $H^i_Z(M)$ is finite for $0 \leq i < s$ and $H^s_Z(M)$ is not finite.

Proof. By Lemma 3.2 we may assume that A is a local ring.

If A is a Noetherian complete local ring, then we can write A as the quotient of a regular complete local ring B by Cohen's structure theorem (Algebra, Theorem 154.8). Using Lemma 7.1 and Dualizing Complexes, Lemma 9.2 we reduce to the case of a regular local ring which is a consequence of Lemma 7.3 because a regular local ring is Gorenstein (Dualizing Complexes, Lemma 21.3).

Let A be a Noetherian local ring. Let $\mathfrak{m}$ be the maximal ideal. We may assume $I \subset \mathfrak{m}$, otherwise the lemma is trivial. Let $A^\wedge$ be the completion of A, let $Z^\wedge = V(IA^\wedge)$, and let $M^\wedge = M \otimes_A A^\wedge$ be the completion of M (Algebra, Lemma 96.1). Then $H^i_Z(M) \otimes_A A^\wedge = H^i_{Z^\wedge}(M^\wedge)$ by Dualizing Complexes, Lemma 9.3 and flatness of $A \to A^\wedge$ (Algebra, Lemma 96.2). Hence it suffices to show that $H^i_{Z^\wedge}(M^\wedge)$ is finite for $i < s$ and not finite for $i = s$, see Algebra, Lemma 82.2. Since we know the result is true for $A^\wedge$ it suffices to show that $s_{A,I}(M) = s_{A^\wedge, I^\wedge}(M^\wedge)$. This follows from Lemma 7.2. $\square$

0BJW **Remark 7.5.** The astute reader will have realized that we can get away with a slightly weaker condition on the formal fibres of the local rings of A. Namely, in the situation of Theorem 7.4 assume A is universally catenary but make no assumptions on the formal fibres. Suppose we have an n and we want to prove that $H^i_Z(M)$ are finite for $i \leq n$. Then the exact same proof shows that it suffices that $s_{A,I}(M) > n$ and that the formal fibres of local rings of A are (S_n). On the other hand, if we want to show that $H^s_Z(M)$ is not finite where $s = s_{A,I}(M)$, then our arguments prove this if the formal fibres are (S_{s-1}).

8. Finiteness of pushforwards, II

0BJX This section is the continuation of Section 4. In this section we reap the fruits of the labor done in Section 7.

0BJY **Lemma 8.1.** *Let X be a locally Noetherian scheme. Let $j : U \to X$ be the inclusion of an open subscheme with complement Z. Let $\mathcal{F}$ be a coherent $\mathcal{O}_U$-module. Let $n \geq 0$ be an integer. Assume*

1. *X is universally catenary,*
2. *for every $z \in Z$ the formal fibres of $\mathcal{O}_{X,z}$ are (S_n).*

In this situation the following are equivalent

(a) *for $x \in Supp(\mathcal{F})$ and $z \in Z \cap \overline{\{x\}}$ we have $depth_{\mathcal{O}_{X,x}}(\mathcal{F}_x) + \dim(\mathcal{O}_{\overline{\{x\}},z}) > n$,*
(b) *$R^pj_*\mathcal{F}$ is coherent for $0 \leq p < n$.*

Proof. The statement is local on X, hence we may assume X is affine. Say $X = \text{Spec}(A)$ and $Z = V(I)$. Let M be a finite A-module whose associated coherent $\mathcal{O}_X$-module restricts to $\mathcal{F}$ over U, see Lemma 4.2. This lemma also tells us that $R^pj_*\mathcal{F}$ is coherent if and only if $H^{p+1}_Z(M)$ is a finite A-module. Observe that the minimum of the expressions $\text{depth}_{\mathcal{O}_{X,x}}(\mathcal{F}_x) + \dim(\mathcal{O}_{\overline{\{x\}},z})$ is the number $s_{A,I}(M)$ of (7.0.1). Having said this the lemma follows from Theorem 7.4 as elucidated by Remark 7.5. $\square$

0BLT **Lemma 8.2.** *Let X be a locally Noetherian scheme. Let $j : U \to X$ be the inclusion of an open subscheme with complement Z. Let $n \geq 0$ be an integer. If $R^pj_*\mathcal{O}_U$ is coherent for $0 \leq p < n$, then the same is true for $R^pj_*\mathcal{F}$, $0 \leq p < n$ for any finite locally free $\mathcal{O}_U$-module $\mathcal{F}$.*

Proof. The question is local on X, hence we may assume X is affine. Say $X = \text{Spec}(A)$ and $Z = V(I)$. Via Lemma 4.2 our lemma follows from Lemma 3.4. $\square$

0BM5 **Lemma 8.3.** *Let A be a ring and let $J \subset I \subset A$ be finitely generated ideals. Let $p \geq 0$ be an integer. Set $U = \text{Spec}(A) \setminus V(I)$. If $H^p(U, \mathcal{O}_U)$ is annihilated by J^n for some n, then $H^p(U, \mathcal{F})$ annihilated by J^m for some $m = m(\mathcal{F})$ for every finite locally free $\mathcal{O}_U$-module $\mathcal{F}$.* [BdJ14, Lemma 1.9]

Proof. Consider the annihilator $\mathfrak{a}$ of $H^p(U, \mathcal{F})$. Let $u \in U$. There exists an open neighbourhood $u \in U' \subset U$ and an isomorphism $\varphi : \mathcal{O}_{U'}^{\oplus r} \to \mathcal{F}|_{U'}$. Pick $f \in A$ such that $u \in D(f) \subset U'$. There exist maps

$$a : \mathcal{O}_U^{\oplus r} \longrightarrow \mathcal{F} \quad \text{and} \quad b : \mathcal{F} \longrightarrow \mathcal{O}_U^{\oplus r}$$

whose restriction to $D(f)$ are equal to $f^N\varphi$ and $f^N\varphi^{-1}$ for some N. Moreover we may assume that $a \circ b$ and $b \circ a$ are equal to multiplication by f^{2N}. This follows from Properties, Lemma 17.3 since U is quasi-compact (I is finitely generated), separated, and $\mathcal{F}$ and $\mathcal{O}_U^{\oplus r}$ are finitely presented. Thus we see that $H^p(U, \mathcal{F})$ is annihilated by $f^{2N}J^n$, i.e., $f^{2N}J^n \subset \mathfrak{a}$.

As U is quasi-compact we can find finitely many $f_1, \ldots, f_t$ and $N_1, \ldots, N_t$ such that $U = \bigcup D(f_i)$ and $f_i^{2N_i}J^n \subset \mathfrak{a}$. Then $V(I) = V(f_1, \ldots, f_t)$ and since I is finitely generated we conclude $I^M \subset (f_1, \ldots, f_t)$ for some M. All in all we see that $J^m \subset \mathfrak{a}$ for $m \gg 0$, for example $m = M(2N_1 + \ldots + 2N_t)n$ will do. $\square$

9. Cohomological dimension

0DX6 A quick section about cohomological dimension.

0DX7 **Lemma 9.1.** *Let $I \subset A$ be a finitely generated ideal of a ring A. Set $Y = V(I) \subset X = \mathrm{Spec}(A)$. Let $d \geq -1$ be an integer. The following are equivalent*

1. *$H^i_Y(A) = 0$ for $i > d$,*
2. *$H^i_Y(M) = 0$ for $i > d$ for every A-module M, and*
3. *if $d = -1$, then $Y = \emptyset$, if $d = 0$, then Y is open and closed in X, and if $d > 0$ then $H^i(X \setminus Y, \mathcal{F}) = 0$ for $i \geq d$ for every quasi-coherent $\mathcal{O}_{X\setminus Y}$-module $\mathcal{F}$.*

Proof. Observe that $R\Gamma_Y(-)$ has finite cohomological dimension by Dualizing Complexes, Lemma 9.1 for example. Hence we can choose a large integer N such that $H^i_Y(M) = 0$ for all A-modules M.

Let us prove that (1) and (2) are equivalent. It is immediate that (2) implies (1). Assume (1). Choose any A-module M and fit it into a short exact sequence $0 \to N \to F \to M \to 0$ where F is a free A-module. Since $R\Gamma_Y$ is a right adjoint, we see that $H^i_Y(-)$ commutes with direct sums. Hence $H^i_Y(F) = 0$ for $i > d$ by assumption (1). Then we see that $H^i_Y(M) = H^{i+1}_Y(N)$ for all $i > d$. Thus if we've shown the vanishing of $H^j_Y(N)$ for some $j > d + 1$ and all A-modules N, then we obtain the vanishing of $H^{j-1}_Y(M)$ for all A-modules M. By induction we find that (2) is true.

Assume $d = -1$ and (2) holds. Then $0 = H^0_Y(A/I) = A/I \Rightarrow A = I \Rightarrow Y = \emptyset$. Thus (3) holds. We omit the proof of the converse.

Assume $d = 0$ and (2) holds. Set $J = H^0_I(A) = \{x \in A \mid I^n x = 0 \text{ for some } n > 0\}$. Then

$$H^1_Y(A) = \mathrm{Coker}(A \to \Gamma(X\setminus Y, \mathcal{O}_{X\setminus Y})) \quad \text{and} \quad H^1_Y(I) = \mathrm{Coker}(I \to \Gamma(X\setminus Y, \mathcal{O}_{X\setminus Y}))$$

and the kernel of the first map is equal to J. See Lemma 2.2. We conclude from (2) that $I(A/J) = A/J$. Thus we may pick $f \in I$ mapping to 1 in A/J. Then $1 - f \in J$ so $I^n(1 - f) = 0$ for some $n > 0$. Hence $f^n = f^{n+1}$. Then $e = f^n \in I$ is an idempotent. Consider the complementary idempotent $e' = 1 - f^n \in J$. For any element $g \in I$ we have $g^m e' = 0$ for some $m > 0$. Thus I is contained in the radical of ideal $(e) \subset I$. This means $Y = V(I) = V(e)$ is open and closed in X as predicted in (3). Conversely, if $Y = V(I)$ is open and closed, then the functor $H^0_Y(-)$ is exact and has vanshing higher derived functors.

If $d > 0$, then we see immediately from Lemma 2.2 that (2) is equivalent to (3). $\square$

0DX8 **Definition 9.2.** Let $I \subset A$ be a finitely generated ideal of a ring A. The smallest integer $d \geq -1$ satisfying the equivalent conditions of Lemma 9.1 is called the *cohomological dimension of I in A* and is denoted $\text{cd}(A, I)$.

Thus we have $\text{cd}(A, I) = -1$ if $I = A$ and $\text{cd}(A, I) = 0$ if I is locally nilpotent or generated by an idempotent. Observe that $\text{cd}(A, I)$ exists by the following lemma.

0DX9 **Lemma 9.3.** *Let $I \subset A$ be a finitely generated ideal of a ring A. Then*

(1) *$cd(A, I)$ is at most equal to the number of generators of I,*
(2) *$cd(A, I) \leq r$ if there exist $f_1, \ldots, f_r \in A$ such that $V(f_1, \ldots, f_r) = V(I)$,*
(3) *$cd(A, I) \leq c$ if $\text{Spec}(A) \setminus V(I)$ can be covered by c affine opens.*

Proof. The explicit description for $R\Gamma_Y(-)$ given in Dualizing Complexes, Lemma 9.1 shows that (1) is true. We can deduce (2) from (1) using the fact that $R\Gamma_Z$ depends only on the closed subset Z and not on the choice of the finitely generated ideal $I \subset A$ with $V(I) = Z$. This follows either from the construction of local cohomology in Dualizing Complexes, Section 9 combined with More on Algebra, Lemma 77.6. or it follows from Lemma 2.1. To see (3) we use Lemma 9.1 and the vanishing result of Cohomology of Schemes, Lemma 4.2. $\square$

0DXA **Lemma 9.4.** *Let $A \to B$ be a ring map. Let $I \subset A$ be a finitely generated ideal. Then $cd(B, IB) \leq cd(A, I)$. If $A \to B$ is faithfully flat, then equality holds.*

Proof. Use the definition and Dualizing Complexes, Lemma 9.3. $\square$

0DXB **Lemma 9.5.** *Let $I \subset A$ be a finitely generated ideal of a ring A. Then $cd(A, I) = \max cd(A_\mathfrak{p}, I_\mathfrak{p})$.*

Proof. Let $Y = V(I)$ and $Y' = V(I_\mathfrak{p}) \subset \text{Spec}(A_\mathfrak{p})$. Recall that $R\Gamma_Y(A) \otimes_A A_\mathfrak{p} = R\Gamma_{Y'}(A_\mathfrak{p})$ by Dualizing Complexes, Lemma 9.3. Thus we conclude by Algebra, Lemma 23.1. $\square$

0DXC **Lemma 9.6.** *Let $I \subset A$ be a finitely generated ideal of a ring A. Then $cd(A, I) \leq \dim(A)$.*

Proof. Recall that $\dim(A)$ denotes the Krull dimension. By Lemma 9.5 we may assume A is local. If $V(I) = \emptyset$, then the result is true. If $V(I) \neq \emptyset$, then $\dim(\text{Spec}(A) \setminus V(I)) < \dim(A)$ because the closed point is missing. Observe that $U = \text{Spec}(A) \setminus V(I)$ is a quasi-compact open of the spectral space $\text{Spec}(A)$, hence a spectral space itself. See Algebra, Lemma 25.2 and Topology, Lemma 23.4. Thus Cohomology, Proposition 23.4 implies $H^i(U, \mathcal{F}) = 0$ for $i \geq \dim(A)$ which implies what we want by Lemma 9.1. In the Noetherian case we can use Grothendieck's Cohomology, Proposition 21.7. $\square$

0DXD **Lemma 9.7.** *Let $I \subset A$ be a finitely generated ideal of a ring A. If $cd(A, I) = 1$ then $\text{Spec}(A) \setminus V(I)$ is nonempty affine.*

Proof. This follows from Lemma 9.1 and Cohomology of Schemes, Lemma 3.1. $\square$

0DXE **Lemma 9.8.** *Let $(A, \mathfrak{m})$ be a Noetherian local ring of dimension d. Then $H^d_\mathfrak{m}(A)$ is nonzero and $cd(A, \mathfrak{m}) = d$.*

Proof. By one of the characterizations of dimension, there exists an ideal of definition for A generated by d elements, see Algebra, Proposition 59.8. Hence $\text{cd}(A, \mathfrak{m}) \leq$

d by Lemma 9.3. Thus $H^d_{\mathfrak m}(A)$ is nonzero if and only if $\text{cd}(A, I) = d$ if and only if $\text{cd}(A, I) \geq d$.

Let $A \to A^\wedge$ be the map from A to its completion. Observe that $A^\wedge$ is a Noetherian local ring of the same dimension as A with maximal ideal $\mathfrak m A^\wedge$. See Algebra, Lemmas 96.6, 96.4, and 96.3 and More on Algebra, Lemma 40.1. By Lemma 9.4 it suffices to prove the lemma for $A^\wedge$.

By the previous paragraph we may assume that A is a complete local ring. Then A has a normalized dualizing complex $\omega_A^\bullet$ (Dualizing Complexes, Lemma 22.4). The local duality theorem (in the form of Dualizing Complexes, Lemma 18.4) tells us $H^d_{\mathfrak m}(A)$ is Matlis dual to $\text{Ext}^{-d}(A, \omega_A^\bullet) = H^{-d}(\omega_A^\bullet)$ which is nonzero for example by Dualizing Complexes, Lemma 16.11. $\square$

0DXF **Lemma 9.9.** *Let $(A, \mathfrak m)$ be a Noetherian local ring. Let $I \subset A$ be a proper ideal. Let $\mathfrak p \subset A$ be a prime ideal such that $V(\mathfrak p) \cap V(I) = \{\mathfrak m\}$. Then $\dim(A/\mathfrak p) \leq cd(A, I)$.*

Proof. By Lemma 9.4 we have $\text{cd}(A, I) \geq \text{cd}(A/\mathfrak p, I(A/\mathfrak p))$. Since $V(I) \cap V(\mathfrak p) = \{\mathfrak m\}$ we have $\text{cd}(A/\mathfrak p, I(A/\mathfrak p)) = \text{cd}(A/\mathfrak p, \mathfrak m/\mathfrak p)$. By Lemma 9.8 this is equal to $\dim(A/\mathfrak p)$. $\square$

10. Formal functions for a principal ideal

0BLA In this section we ask if completion and taking cohomology commute for sheaves of modules on schemes over an affine base A when completion is with respect to a principal ideal in A. Of course, we have already discussed the theorem on formal functions in Cohomology of Schemes, Section 20. Moreover, we will see in Section 12 that derived completion commutes with derived cohomology in great generality. In this section we just collect a few simple special cases of this material that will help us with future developments.

0BLB **Lemma 10.1.** *Let A be a Noetherian ring complete with respect to a principal ideal (f). Let X be a scheme over $\text{Spec}(A)$. Let*

$$\ldots \to \mathcal F_2 \to \mathcal F_1 \to \mathcal F_0$$

be an inverse system of $\mathcal O_X$-modules. Assume

(1) $\Gamma(X, \mathcal F_0)$ *is a finite A-module,*
(2) *multiplication by f on $\mathcal F_{n+1}$ factors through $\mathcal F_{n+1} \to \mathcal F_n$ to give a short exact sequence $0 \to \mathcal F_n \to \mathcal F_{n+1} \to \mathcal F_0 \to 0$*

Then

$$M = \lim \Gamma(X, \mathcal F_n)$$

is a finite A-module, f is a nonzerodivisor on M, and M/fM is the image of M in $\Gamma(X, \mathcal F_0)$.

Proof. Assumption (2) implies that $\mathcal F_0$ is annihilated by f and then by induction that $\mathcal F_n$ is annihilated by f^{n+1}. Set $M_n = \Gamma(X, \mathcal F_n)$. Since f^{n+1} annihilates M_n we see that $\bigcap f^n M = 0$. Since the kernel of $f : M_{n+1} \to M_{n+1}$ dies in M_n by (2) we see that $f : M \to M$ is injective. The cokernel of $f : M \to M$ is the image of $M \to M_0$. Namely, if $m = (m_n)$ is an element of M with $m_0 = 0$, then each m_{n+1} is in the image of $M_n \to M_{n+1}$ by assumption (2). If $m'_n \in M_n$ maps to m_{n+1} then $f(m'_n) = (m_n)$ in M. Since A is Noetherian and M_0 is finite, we see that $M/fM \subset M_0$ is a finite module. By Algebra, Lemma 95.12 we conclude that M is finite over A. $\square$

0BLC **Lemma 10.2.** *Let A be a ring. Let $f \in A$. Let X be a scheme over* $\mathrm{Spec}(A)$. *Let*

$$\ldots \to \mathcal{F}_2 \to \mathcal{F}_1 \to \mathcal{F}_0$$

be an inverse system of $\mathcal{O}_X$-modules. Assume

(1) *$H^1(X, \mathcal{F}_0)$ is an A-module of finite length,*
(2) *multiplication by f on $\mathcal{F}_{n+1}$ factors through $\mathcal{F}_{n+1} \to \mathcal{F}_n$ to give a short exact sequence $0 \to \mathcal{F}_n \to \mathcal{F}_{n+1} \to \mathcal{F}_0 \to 0$,*

Then the system $M_n = \Gamma(X, \mathcal{F}_n)$ satisfies the Mittag-Leffler condition.

Proof. By the short exact sequences and induction we see that $H^1_n = H^1(X, \mathcal{F}_n)$ is an A-module of finite length for all n. Fix n. Our goal is to show that

$$Q_m = \mathrm{Coker}(M_m \to M_n), \quad m \geq n$$

stabilizes for $m \gg n$. Note that $Q_m \subset H^1_{m-n}$ has finite length and that we have surjective maps $Q_{m+1} \to Q_m$ for all $m \geq n$. Applying cohomology to the short exact sequence

$$0 \to \mathcal{F}_{m-n} \to \mathcal{F}_m \to \mathcal{F}_n \to 0$$

we get an exact sequence

$$0 \to Q_m \to H^1_{m-n} \to H^1_m \to H^1_n$$

of finite length modules. Set $q_m = \mathrm{length}_A(Q_m)$ and $l_m = \mathrm{length}_A(H^1_m)$. Then we conclude that

$$l_m \leq l_{m-n} - q_m + l_n$$

Above we have seen that $q_{m+1} \geq q_m$ for all n. If the sequence does not stabilize then for some m_0 we have $q_m > l_n$ for all $m \geq m_0$. Then we would get

$$l_m \leq l_{m-n} - q_m + l_n \leq l_{m-n} - 1$$

provided $m \geq m_0$. This would imply that the sequence $l_{m_0}, l_{m_0+n}, l_{m_0+2n}, \ldots$ is strictly decreasing contradicting the fact that $l_m > q_m$ and the sequence q_m is nondecreasing. Thus the sequence stabilizes. $\square$

0DXG **Lemma 10.3.** *Let A be a ring. Let $f \in A$. Let X be a scheme over* $\mathrm{Spec}(A)$. *Let*

$$\ldots \to \mathcal{F}_2 \to \mathcal{F}_1 \to \mathcal{F}_0$$

be an inverse system of $\mathcal{O}_X$-modules. Assume

(1) *for every n there is an $m > n$ such that the image of $H^1(X, \mathcal{F}_m) \to H^1(X, \mathcal{F}_n)$ is an A-module of finite length,*
(2) *multiplication by f on $\mathcal{F}_{n+1}$ factors through $\mathcal{F}_{n+1} \to \mathcal{F}_n$ to give a short exact sequence $0 \to \mathcal{F}_n \to \mathcal{F}_{n+1} \to \mathcal{F}_0 \to 0$,*

Then the system $M_n = \Gamma(X, \mathcal{F}_n)$ satisfies the Mittag-Leffler condition.

Proof. Observe that condition (1) implies that the system $H^1(X, \mathcal{F}_n)$ has the Mittag-Leffler condition. Denote $H^1_n \subset H^1(X, \mathcal{F}_n)$ the stable image which is a finite length A-module by condition (1). For any $m' > m > n$ we have a map of short exact sequences

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{F}_{m'-n} \arrow[r] \arrow[d] & \mathcal{F}_{m'} \arrow[r] \arrow[d] & \mathcal{F}_n \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \mathcal{F}_{m-n} \arrow[r] & \mathcal{F}_m \arrow[r] & \mathcal{F}_n \arrow[r] & 0 \end{tikzcd}$$

In particular, the boundary maps $\delta : M_n \to H^1(X, \mathcal{F}_{m-n})$ have image in H^1_{m-n}. Consider the six term sequence

$$0 \to M_{m-n} \to M_m \to M_n \to H^1_{m-n} \to H^1_m \to H^1_n$$

This is exact, except possibly at H^1_m. However, it is easy to show exactness there as well: let $\xi \in H^1_m$ map to zero in H^1_n. Then choose a very large $m' > m$ and a lift $\xi' \in H^1_{m'}$ mapping to ξ. Then ξ' maps to zero in $H^1_n \subset H^1(X, \mathcal{F}_n)$. Thus we can lift ξ' to an element $\xi'' \in H^1(X, \mathcal{F}_{m'-n})$. Since m' was chosen large enough, the image of ξ'' in $H^1(X, \mathcal{F}_{m-n})$ lies in H^1_{m-n} and maps to ξ as desired.

To finish the proof argue exactly as in the proof of Lemma 10.2. $\square$

0BLD **Lemma 10.4.** *Let A be a ring and $f \in A$. Let X be a scheme over A. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Assume that $\mathcal{F}[f^n] = \mathrm{Ker}(f^n : \mathcal{F} \to \mathcal{F})$ stabilizes. Then* [BdJ14, Lemma 1.6]

$$R\Gamma(X, \lim \mathcal{F}/f^n\mathcal{F}) = R\Gamma(X, \mathcal{F})^\wedge$$

where the right hand side indicates the derived completion with respect to the ideal $(f) \subset A$. Let H^p be the pth cohomology group of this complex. Then there are short exact sequences

$$0 \to R^1 \lim H^{p-1}(X, \mathcal{F}/f^n\mathcal{F}) \to H^p \to \lim H^p(X, \mathcal{F}/f^n\mathcal{F}) \to 0$$

and

$$0 \to H^0(H^p(X, \mathcal{F})^\wedge) \to H^p \to T_f(H^{p+1}(X, \mathcal{F})) \to 0$$

where $T_f(-)$ denote the f-adic Tate module as in More on Algebra, Example 81.4.

Proof. We start with the canonical identifications

$$\begin{aligned} R\Gamma(X, \mathcal{F})^\wedge &= R\lim R\Gamma(X, \mathcal{F}) \otimes_A^{\mathbf{L}} (A \xrightarrow{f^n} A) \\ &= R\lim R\Gamma(X, \mathcal{F} \xrightarrow{f^n} \mathcal{F}) \\ &= R\Gamma(X, R\lim(\mathcal{F} \xrightarrow{f^n} \mathcal{F})) \end{aligned}$$

The first equality holds by More on Algebra, Lemma 80.16. The second by the projection formula, see Cohomology, Lemma 45.3. The third by Cohomology, Lemma 32.2. Note that by Derived Categories of Schemes, Lemma 3.2 we have $\lim \mathcal{F}/f^n\mathcal{F} = R\lim \mathcal{F}/f^n\mathcal{F}$. Thus to finish the proof of the first statement of the lemma it suffices to show that the pro-objects $(f^n : \mathcal{F} \to \mathcal{F})$ and $(\mathcal{F}/f^n\mathcal{F})$ are isomorphic. There is clearly a map from the first system to the second. Suppose that $\mathcal{F}[f^c] = \mathcal{F}[f^{c+1}] = \mathcal{F}[f^{c+2}] = \ldots$. Then we can define an arrow of systems in $D(\mathcal{O}_X)$ in the other direction by the diagrams

$$\begin{tikzcd} \mathcal{F}/\mathcal{F}[f^c] \arrow[r, "f^{n+c}"'] \arrow[d, "f^c"'] & \mathcal{F} \arrow[d, "1"] \\ \mathcal{F} \arrow[r, "f^n"] & \mathcal{F} \end{tikzcd}$$

Since the top horizontal arrow is injective the complex in the top row is quasi-isomorphic to $\mathcal{F}/f^{n+c}\mathcal{F}$. Some details omitted.

Since $R\Gamma(X, -)$ commutes with derived limits (Injectives, Lemma 13.6) we see that

$$R\Gamma(X, \lim \mathcal{F}/f^n\mathcal{F}) = R\Gamma(X, R\lim \mathcal{F}/f^n\mathcal{F}) = R\lim R\Gamma(X, \mathcal{F}/f^n\mathcal{F})$$

(for first equality see first paragraph of proof). By More on Algebra, Remark 75.9 we obtain exact sequences

$$0 \to R^1 \lim H^{p-1}(X, \mathcal{F}/f^n\mathcal{F}) \to H^p(X, \lim \mathcal{F}/I^n\mathcal{F}) \to \lim H^p(X, \mathcal{F}/I^n\mathcal{F}) \to 0$$

of A-modules. The second set of short exact sequences follow immediately from the discussion in More on Algebra, Example 81.4. $\square$

11. Generalities on derived completion

0995 We urge the reader to skip this section on a first reading.

The algebra version of this material can be found in More on Algebra, Section 80. Let $\mathcal{O}$ be a sheaf of rings on a site $\mathcal{C}$. Let f be a global section of $\mathcal{O}$. We denote $\mathcal{O}_f$ the sheaf associated to the presheaf of localizations $U \mapsto \mathcal{O}(U)_f$.

0996 **Lemma 11.1.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let f be a global section of $\mathcal{O}$.*

1. *For $L, N \in D(\mathcal{O}_f)$ we have $R\mathcal{H}om_\mathcal{O}(L, N) = R\mathcal{H}om_{\mathcal{O}_f}(L, N)$. In particular the two $\mathcal{O}_f$-structures on $R\mathcal{H}om_\mathcal{O}(L, N)$ agree.*
2. *For $K \in D(\mathcal{O})$ and $L \in D(\mathcal{O}_f)$ we have*
$$R\mathcal{H}om_\mathcal{O}(L, K) = R\mathcal{H}om_{\mathcal{O}_f}(L, R\mathcal{H}om_\mathcal{O}(\mathcal{O}_f, K))$$
In particular $R\mathcal{H}om_\mathcal{O}(\mathcal{O}_f, R\mathcal{H}om_\mathcal{O}(\mathcal{O}_f, K)) = R\mathcal{H}om_\mathcal{O}(\mathcal{O}_f, K)$.
3. *If g is a second global section of $\mathcal{O}$, then*
$$R\mathcal{H}om_\mathcal{O}(\mathcal{O}_f, R\mathcal{H}om_\mathcal{O}(\mathcal{O}_g, K)) = R\mathcal{H}om_\mathcal{O}(\mathcal{O}_{gf}, K).$$

Proof. Proof of (1). Let $\mathcal{J}^\bullet$ be a K-injective complex of $\mathcal{O}_f$-modules representing N. By Cohomology on Sites, Lemma 21.9 it follows that $\mathcal{J}^\bullet$ is a K-injective complex of $\mathcal{O}$-modules as well. Let $\mathcal{F}^\bullet$ be a complex of $\mathcal{O}_f$-modules representing L. Then

$$R\mathcal{H}om_\mathcal{O}(L, N) = R\mathcal{H}om_\mathcal{O}(\mathcal{F}^\bullet, \mathcal{J}^\bullet) = R\mathcal{H}om_{\mathcal{O}_f}(\mathcal{F}^\bullet, \mathcal{J}^\bullet)$$

by Modules on Sites, Lemma 11.4 because $\mathcal{J}^\bullet$ is a K-injective complex of $\mathcal{O}$ and of $\mathcal{O}_f$-modules.

Proof of (2). Let $\mathcal{I}^\bullet$ be a K-injective complex of $\mathcal{O}$-modules representing K. Then $R\mathcal{H}om_\mathcal{O}(\mathcal{O}_f, K)$ is represented by $\mathcal{H}om_\mathcal{O}(\mathcal{O}_f, \mathcal{I}^\bullet)$ which is a K-injective complex of $\mathcal{O}_f$-modules and of $\mathcal{O}$-modules by Cohomology on Sites, Lemmas 21.10 and 21.9. Let $\mathcal{F}^\bullet$ be a complex of $\mathcal{O}_f$-modules representing L. Then

$$R\mathcal{H}om_\mathcal{O}(L, K) = R\mathcal{H}om_\mathcal{O}(\mathcal{F}^\bullet, \mathcal{I}^\bullet) = R\mathcal{H}om_{\mathcal{O}_f}(\mathcal{F}^\bullet, \mathcal{H}om_\mathcal{O}(\mathcal{O}_f, \mathcal{I}^\bullet))$$

by Modules on Sites, Lemma 27.6 and because $\mathcal{H}om_\mathcal{O}(\mathcal{O}_f, \mathcal{I}^\bullet)$ is a K-injective complex of $\mathcal{O}_f$-modules.

Proof of (3). This follows from the fact that $R\mathcal{H}om_\mathcal{O}(\mathcal{O}_g, \mathcal{I}^\bullet)$ is K-injective as a complex of $\mathcal{O}$-modules and the fact that $\mathcal{H}om_\mathcal{O}(\mathcal{O}_f, \mathcal{H}om_\mathcal{O}(\mathcal{O}_g, \mathcal{H})) = \mathcal{H}om_\mathcal{O}(\mathcal{O}_{gf}, \mathcal{H})$ for all sheaves of $\mathcal{O}$-modules $\mathcal{H}$. $\square$

Let $K \in D(\mathcal{O})$. We denote $T(K, f)$ a derived limit (Derived Categories, Definition 32.1) of the system

$$\ldots \to K \xrightarrow{f} K \xrightarrow{f} K$$

in $D(\mathcal{O})$.

0997 **Lemma 11.2.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let f be a global section of $\mathcal{O}$. Let $K \in D(\mathcal{O})$. The following are equivalent*

(1) $R\mathcal{H}om_{\mathcal{O}}(\mathcal{O}_f, K) = 0$,
(2) $R\mathcal{H}om_{\mathcal{O}}(L, K) = 0$ *for all* L *in* $D(\mathcal{O}_f)$,
(3) $T(K, f) = 0$.

Proof. It is clear that (2) implies (1). The implication (1) $\Rightarrow$ (2) follows from Lemma 11.1. A free resolution of the $\mathcal{O}$-module $\mathcal{O}_f$ is given by

$$0 \to \bigoplus\nolimits_{n \in \mathbf{N}} \mathcal{O} \to \bigoplus\nolimits_{n \in \mathbf{N}} \mathcal{O} \to \mathcal{O}_f \to 0$$

where the first map sends a local section $(x_0, x_1, \ldots)$ to $(fx_0 - x_1, fx_1 - x_2, \ldots)$ and the second map sends $(x_0, x_1, \ldots)$ to $x_0 + x_1/f + x_2/f^2 + \ldots$. Applying $\mathcal{H}om_{\mathcal{O}}(-, \mathcal{I}^\bullet)$ where $\mathcal{I}^\bullet$ is a K-injective complex of $\mathcal{O}$-modules representing K we get a short exact sequence of complexes

$$0 \to \mathcal{H}om_{\mathcal{O}}(\mathcal{O}_f, \mathcal{I}^\bullet) \to \prod \mathcal{I}^\bullet \to \prod \mathcal{I}^\bullet \to 0$$

because $\mathcal{I}^n$ is an injective $\mathcal{O}$-module. The products are products in $D(\mathcal{O})$, see Injectives, Lemma 13.4. This means that the object $T(K, f)$ is a representative of $R\mathcal{H}om_{\mathcal{O}}(\mathcal{O}_f, K)$ in $D(\mathcal{O})$. Thus the equivalence of (1) and (3). $\square$

0998 **Lemma 11.3.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $K \in D(\mathcal{O})$. The rule which associates to U the set $\mathcal{I}(U)$ of sections $f \in \mathcal{O}(U)$ such that $T(K|_U, f) = 0$ is a sheaf of ideals in $\mathcal{O}$.*

Proof. We will use the results of Lemma 11.2 without further mention. If $f \in \mathcal{I}(U)$, and $g \in \mathcal{O}(U)$, then $\mathcal{O}_{U,gf}$ is an $\mathcal{O}_{U,f}$-module hence $R\mathcal{H}om_{\mathcal{O}}(\mathcal{O}_{U,gf}, K|_U) = 0$, hence $gf \in \mathcal{I}(U)$. Suppose $f, g \in \mathcal{O}(U)$. Then there is a short exact sequence

$$0 \to \mathcal{O}_{U,f+g} \to \mathcal{O}_{U,f(f+g)} \oplus \mathcal{O}_{U,g(f+g)} \to \mathcal{O}_{U,gf(f+g)} \to 0$$

because f, g generate the unit ideal in $\mathcal{O}(U)_{f+g}$. This follows from Algebra, Lemma 22.1 and the easy fact that the last arrow is surjective. Because $R\mathcal{H}om_{\mathcal{O}}(-, K|_U)$ is an exact functor of triangulated categories the vanishing of $R\mathcal{H}om_{\mathcal{O}_U}(\mathcal{O}_{U,f(f+g)}, K|_U)$, $R\mathcal{H}om_{\mathcal{O}_U}(\mathcal{O}_{U,g(f+g)}, K|_U)$, and $R\mathcal{H}om_{\mathcal{O}_U}(\mathcal{O}_{U,gf(f+g)}, K|_U)$, implies the vanishing of $R\mathcal{H}om_{\mathcal{O}_U}(\mathcal{O}_{U,f+g}, K|_U)$. We omit the verification of the sheaf condition. $\square$

We can make the following definition for any ringed site.

0999 **Definition 11.4.** Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{I} \subset \mathcal{O}$ be a sheaf of ideals. Let $K \in D(\mathcal{O})$. We say that K is *derived complete with respect to $\mathcal{I}$* if for every object U of $\mathcal{C}$ and $f \in \mathcal{I}(U)$ the object $T(K|_U, f)$ of $D(\mathcal{O}_U)$ is zero.

It is clear that the full subcategory $D_{comp}(\mathcal{O}) = D_{comp}(\mathcal{O}, \mathcal{I}) \subset D(\mathcal{O})$ consisting of derived complete objects is a saturated triangulated subcategory, see Derived Categories, Definitions 3.4 and 6.1. This subcategory is preserved under products and homotopy limits in $D(\mathcal{O})$. But it is not preserved under countable direct sums in general.

099A **Lemma 11.5.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{I} \subset \mathcal{O}$ be a sheaf of ideals. If $K \in D(\mathcal{O})$ and $L \in D_{comp}(\mathcal{O})$, then $R\mathcal{H}om_{\mathcal{O}}(K, L) \in D_{comp}(\mathcal{O})$.*

Proof. Let U be an object of $\mathcal{C}$ and let $f \in \mathcal{I}(U)$. Recall that

$$\mathrm{Hom}_{D(\mathcal{O}_U)}(\mathcal{O}_{U,f}, R\mathcal{H}om_{\mathcal{O}}(K, L)|_U) = \mathrm{Hom}_{D(\mathcal{O}_U)}(K|_U \otimes^{\mathbf{L}}_{\mathcal{O}_U} \mathcal{O}_{U,f}, L|_U)$$

by Cohomology on Sites, Lemma 28.2. The right hand side is zero by Lemma 11.2 and the relationship between internal hom and actual hom, see Cohomology on Sites, Lemma 28.1. The same vanishing holds for all U'/U. Thus the object $R\mathcal{H}om_{\mathcal{O}_U}(\mathcal{O}_{U,f}, R\mathcal{H}om_{\mathcal{O}}(K, L)|_U)$ of $D(\mathcal{O}_U)$ has vanishing 0th cohomology sheaf (by locus citatus). Similarly for the other cohomology sheaves, i.e., $R\mathcal{H}om_{\mathcal{O}_U}(\mathcal{O}_{U,f}, R\mathcal{H}om_{\mathcal{O}}(K, L)|_U)$ is zero in $D(\mathcal{O}_U)$. By Lemma 11.2 we conclude. $\square$

099C **Lemma 11.6.** *Let $\mathcal{C}$ be a site. Let $\mathcal{O} \to \mathcal{O}'$ be a homomorphism of sheaves of rings. Let $\mathcal{I} \subset \mathcal{O}$ be a sheaf of ideals. The inverse image of $D_{comp}(\mathcal{O}, \mathcal{I})$ under the restriction functor $D(\mathcal{O}') \to D(\mathcal{O})$ is $D_{comp}(\mathcal{O}', \mathcal{I}\mathcal{O}')$.*

Proof. Using Lemma 11.3 we see that $K' \in D(\mathcal{O}')$ is in $D_{comp}(\mathcal{O}', \mathcal{I}\mathcal{O}')$ if and only if $T(K'|_U, f)$ is zero for every local section $f \in \mathcal{I}(U)$. Observe that the cohomology sheaves of $T(K'|_U, f)$ are computed in the category of abelian sheaves, so it doesn't matter whether we think of f as a section of $\mathcal{O}$ or take the image of f as a section of $\mathcal{O}'$. The lemma follows immediately from this and the definition of derived complete objects. $\square$

099J **Lemma 11.7.** *Let $f : (Sh(\mathcal{D}), \mathcal{O}') \to (Sh(\mathcal{C}), \mathcal{O})$ be a morphism of ringed topoi. Let $\mathcal{I} \subset \mathcal{O}$ and $\mathcal{I}' \subset \mathcal{O}'$ be sheaves of ideals such that $f^\sharp$ sends $f^{-1}\mathcal{I}$ into $\mathcal{I}'$. Then Rf_* sends $D_{comp}(\mathcal{O}', \mathcal{I}')$ into $D_{comp}(\mathcal{O}, \mathcal{I})$.*

Proof. We may assume f is given by a morphism of ringed sites corresponding to a continuous functor $\mathcal{C} \to \mathcal{D}$ (Modules on Sites, Lemma 7.2). Let U be an object of $\mathcal{C}$ and let g be a section of $\mathcal{I}$ over U. We have to show that $\mathrm{Hom}_{D(\mathcal{O}_U)}(\mathcal{O}_{U,g}, Rf_*K|_U) = 0$ whenever K is derived complete with respect to $\mathcal{I}'$. Namely, by Cohomology on Sites, Lemma 28.1 this, applied to all objects over U and all shifts of K, will imply that $R\mathcal{H}om_{\mathcal{O}_U}(\mathcal{O}_{U,g}, Rf_*K|_U)$ is zero, which implies that $T(Rf_*K|_U, g)$ is zero (Lemma 11.2) which is what we have to show (Definition 11.4). Let V in $\mathcal{D}$ be the image of U. Then

$$\mathrm{Hom}_{D(\mathcal{O}_U)}(\mathcal{O}_{U,g}, Rf_*K|_U) = \mathrm{Hom}_{D(\mathcal{O}'_V)}(\mathcal{O}'_{V,g'}, K|_V) = 0$$

where $g' = f^\sharp(g) \in \mathcal{I}'(V)$. The second equality because K is derived complete and the first equality because the derived pullback of $\mathcal{O}_{U,g}$ is $\mathcal{O}'_{V,g'}$ and Cohomology on Sites, Lemma 20.1. $\square$

The following lemma is the simplest case where one has derived completion.

099B **Lemma 11.8.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed on a site. Let $f_1, \ldots, f_r$ be global sections of $\mathcal{O}$. Let $\mathcal{I} \subset \mathcal{O}$ be the ideal sheaf generated by $f_1, \ldots, f_r$. Then the inclusion functor $D_{comp}(\mathcal{O}) \to D(\mathcal{O})$ has a left adjoint, i.e., given any object K of $D(\mathcal{O})$ there exists a map $K \to K^\wedge$ with $K^\wedge$ in $D_{comp}(\mathcal{O})$ such that the map*

$$\mathrm{Hom}_{D(\mathcal{O})}(K^\wedge, E) \longrightarrow \mathrm{Hom}_{D(\mathcal{O})}(K, E)$$

is bijective whenever E is in $D_{comp}(\mathcal{O})$. In fact we have

$$K^\wedge = R\mathcal{H}om_{\mathcal{O}}(\mathcal{O} \to \prod\nolimits_{i_0} \mathcal{O}_{f_{i_0}} \to \prod\nolimits_{i_0 < i_1} \mathcal{O}_{f_{i_0} f_{i_1}} \to \ldots \to \mathcal{O}_{f_1 \ldots f_r}, K)$$

functorially in K.

Proof. Define $K^\wedge$ by the last displayed formula of the lemma. There is a map of complexes

$$(\mathcal{O} \to \prod\nolimits_{i_0} \mathcal{O}_{f_{i_0}} \to \prod\nolimits_{i_0<i_1} \mathcal{O}_{f_{i_0}f_{i_1}} \to \ldots \to \mathcal{O}_{f_1\ldots f_r}) \longrightarrow \mathcal{O}$$

which induces a map $K \to K^\wedge$. It suffices to prove that $K^\wedge$ is derived complete and that $K \to K^\wedge$ is an isomorphism if K is derived complete.

Let f be a global section of $\mathcal{O}$. By Lemma 11.1 the object $R\mathcal{H}om_\mathcal{O}(\mathcal{O}_f, K^\wedge)$ is equal to

$$R\mathcal{H}om_\mathcal{O}((\mathcal{O}_f \to \prod\nolimits_{i_0} \mathcal{O}_{ff_{i_0}} \to \prod\nolimits_{i_0<i_1} \mathcal{O}_{ff_{i_0}f_{i_1}} \to \ldots \to \mathcal{O}_{ff_1\ldots f_r}), K)$$

If $f = f_i$ for some i, then $f_1, \ldots, f_r$ generate the unit ideal in $\mathcal{O}_f$, hence the extended alternating Čech complex

$$\mathcal{O}_f \to \prod\nolimits_{i_0} \mathcal{O}_{ff_{i_0}} \to \prod\nolimits_{i_0<i_1} \mathcal{O}_{ff_{i_0}f_{i_1}} \to \ldots \to \mathcal{O}_{ff_1\ldots f_r}$$

is zero (even homotopic to zero). In this way we see that $K^\wedge$ is derived complete.

If K is derived complete, then $R\mathcal{H}om_\mathcal{O}(\mathcal{O}_f, K)$ is zero for all $f = f_{i_0} \ldots f_{i_p}$, $p \geq 0$. Thus $K \to K^\wedge$ is an isomorphism in $D(\mathcal{O})$. □

Next we explain why derived completion is a completion.

0A0E **Lemma 11.9.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed on a site. Let $f_1, \ldots, f_r$ be global sections of $\mathcal{O}$. Let $\mathcal{I} \subset \mathcal{O}$ be the ideal sheaf generated by $f_1, \ldots, f_r$. Let $K \in D(\mathcal{O})$. The derived completion $K^\wedge$ of Lemma 11.8 is given by the formula*

$$K^\wedge = R\lim K \otimes_\mathcal{O}^\mathbf{L} K_n$$

where $K_n = K(\mathcal{O}, f_1^n, \ldots, f_r^n)$ is the Koszul complex on $f_1^n, \ldots, f_r^n$ over $\mathcal{O}$.

Proof. In More on Algebra, Lemma 26.13 we have seen that the extended alternating Čech complex

$$\mathcal{O} \to \prod\nolimits_{i_0} \mathcal{O}_{f_{i_0}} \to \prod\nolimits_{i_0<i_1} \mathcal{O}_{f_{i_0}f_{i_1}} \to \ldots \to \mathcal{O}_{f_1\ldots f_r}$$

is a colimit of the Koszul complexes $K^n = K(\mathcal{O}, f_1^n, \ldots, f_r^n)$ sitting in degrees $0, \ldots, r$. Note that K^n is a finite chain complex of finite free $\mathcal{O}$-modules with dual $\mathcal{H}om_\mathcal{O}(K^n, \mathcal{O}) = K_n$ where K_n is the Koszul cochain complex sitting in degrees $-r, \ldots, 0$ (as usual). By Lemma 11.8 the functor $E \mapsto E^\wedge$ is gotten by taking $R\mathcal{H}om$ from the extended alternating Čech complex into E:

$$E^\wedge = R\mathcal{H}om(\mathrm{colim}\, K^n, E)$$

This is equal to $R\lim(E \otimes_\mathcal{O}^\mathbf{L} K_n)$ by Cohomology on Sites, Lemma 39.10. □

099D **Lemma 11.10.** *There exist a way to construct*

1. *for every pair (A, I) consisting of a ring A and a finitely generated ideal $I \subset A$ a complex $K(A, I)$ of A-modules,*
2. *a map $K(A, I) \to A$ of complexes of A-modules,*
3. *for every ring map $A \to B$ and finitely generated ideal $I \subset A$ a map of complexes $K(A, I) \to K(B, IB)$,*

such that

(a) *for $A \to B$ and $I \subset A$ finitely generated the diagram*

$$\begin{tikzcd} K(A,I) \arrow[r] \arrow[d] & A \arrow[d] \\ K(B,IB) \arrow[r] & B \end{tikzcd}$$

commutes,

(b) *for $A \to B \to C$ and $I \subset A$ finitely generated the composition of the maps $K(A,I) \to K(B,IB) \to K(C,IC)$ is the map $K(A,I) \to K(C,IC)$.*

(c) *for $A \to B$ and a finitely generated ideal $I \subset A$ the induced map $K(A,I) \otimes_A^{\mathbf{L}} B \to K(B,IB)$ is an isomorphism in $D(B)$, and*

(d) *if $I = (f_1, \ldots, f_r) \subset A$ then there is a commutative diagram*

$$\begin{tikzcd} (A \to \prod_{i_0} A_{f_{i_0}} \to \prod_{i_0 < i_1} A_{f_{i_0} f_{i_1}} \to \ldots \to A_{f_1 \ldots f_r}) \arrow[r] \arrow[d] & K(A,I) \arrow[d] \\ A \arrow[r, "1"] & A \end{tikzcd}$$

in $D(A)$ whose horizontal arrows are isomorphisms.

Proof. Let S be the set of rings A_0 of the form $A_0 = \mathbf{Z}[x_1, \ldots, x_n]/J$. Every finite type $\mathbf{Z}$-algebra is isomorphic to an element of S. Let $\mathcal{A}_0$ be the category whose objects are pairs (A_0, I_0) where $A_0 \in S$ and $I_0 \subset A_0$ is an ideal and whose morphisms $(A_0, I_0) \to (B_0, J_0)$ are ring maps $\varphi : A_0 \to B_0$ such that $J_0 = \varphi(I_0)B_0$.

Suppose we can construct $K(A_0, I_0) \to A_0$ functorially for objects of $\mathcal{A}_0$ having properties (a), (b), (c), and (d). Then we take

$$K(A,I) = \mathrm{colim}_{\varphi:(A_0,I_0) \to (A,I)} K(A_0, I_0)$$

where the colimit is over ring maps $\varphi : A_0 \to A$ such that $\varphi(I_0)A = I$ with (A_0, I_0) in $\mathcal{A}_0$. A morphism between $(A_0, I_0) \to (A, I)$ and $(A_0', I_0') \to (A, I)$ are given by maps $(A_0, I_0) \to (A_0', I_0')$ in $\mathcal{A}_0$ commuting with maps to A. The category of these $(A_0, I_0) \to (A, I)$ is filtered (details omitted). Moreover, $\mathrm{colim}_{\varphi:(A_0,I_0) \to (A,I)} A_0 = A$ so that $K(A, I)$ is a complex of A-modules. Finally, given $\varphi : A \to B$ and $I \subset A$ for every $(A_0, I_0) \to (A, I)$ in the colimit, the composition $(A_0, I_0) \to (B, IB)$ lives in the colimit for (B, IB). In this way we get a map on colimits. Properties (a), (b), (c), and (d) follow readily from this and the corresponding properties of the complexes $K(A_0, I_0)$.

Endow $\mathcal{C}_0 = \mathcal{A}_0^{opp}$ with the chaotic topology. We equip $\mathcal{C}_0$ with the sheaf of rings $\mathcal{O} : (A, I) \mapsto A$. The ideals I fit together to give a sheaf of ideals $\mathcal{I} \subset \mathcal{O}$. Choose an injective resolution $\mathcal{O} \to \mathcal{J}^\bullet$. Consider the object

$$\mathcal{F}^\bullet = \bigcup\nolimits_n \mathcal{J}^\bullet[\mathcal{I}^n]$$

Let $U = (A, I) \in \mathrm{Ob}(\mathcal{C}_0)$. Since the topology in $\mathcal{C}_0$ is chaotic, the value $\mathcal{J}^\bullet(U)$ is a resolution of A by injective A-modules. Hence the value $\mathcal{F}^\bullet(U)$ is an object of $D(A)$ representing the image of $R\Gamma_I(A)$ in $D(A)$, see Dualizing Complexes, Section

9. Choose a complex of $\mathcal{O}$-modules $\mathcal{K}^\bullet$ and a commutative diagram

$$\begin{tikzcd} \mathcal{O} \arrow[r] & \mathcal{J}^\bullet \\ \mathcal{K}^\bullet \arrow[u] \arrow[r] & \mathcal{F}^\bullet \arrow[u] \end{tikzcd}$$

where the horizontal arrows are quasi-isomorphisms. This is possible by the construction of the derived category $D(\mathcal{O})$. Set $K(A, I) = \mathcal{K}^\bullet(U)$ where $U = (A, I)$. Properties (a) and (b) are clear and properties (c) and (d) follow from Dualizing Complexes, Lemmas 10.2 and 10.3. $\square$

099E **Lemma 11.11.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{I} \subset \mathcal{O}$ be a finite type sheaf of ideals. There exists a map $K \to \mathcal{O}$ in $D(\mathcal{O})$ such that for every $U \in \mathrm{Ob}(\mathcal{C})$ such that $\mathcal{I}|_U$ is generated by $f_1, \ldots, f_r \in \mathcal{I}(U)$ there is an isomorphism*

$$(\mathcal{O}_U \to \prod\nolimits_{i_0} \mathcal{O}_{U, f_{i_0}} \to \prod\nolimits_{i_0 < i_1} \mathcal{O}_{U, f_{i_0} f_{i_1}} \to \ldots \to \mathcal{O}_{U, f_1 \ldots f_r}) \longrightarrow K|_U$$

compatible with maps to $\mathcal{O}_U$.

Proof. Let $\mathcal{C}' \subset \mathcal{C}$ be the full subcategory of objects U such that $\mathcal{I}|_U$ is generated by finitely many sections. Then $\mathcal{C}' \to \mathcal{C}$ is a special cocontinuous functor (Sites, Definition 28.2). Hence it suffices to work with $\mathcal{C}'$, see Sites, Lemma 28.1. In other words we may assume that for every object U of $\mathcal{C}$ there exists a finitely generated ideal $I \subset \mathcal{I}(U)$ such that $\mathcal{I}|_U = \mathrm{Im}(I \otimes \mathcal{O}_U \to \mathcal{O}_U)$. We will say that I generates $\mathcal{I}|_U$. Warning: We do not know that $\mathcal{I}(U)$ is a finitely generated ideal in $\mathcal{O}(U)$.

Let U be an object and $I \subset \mathcal{O}(U)$ a finitely generated ideal which generates $\mathcal{I}|_U$. On the category $\mathcal{C}/U$ consider the complex of presheaves

$$K_{U,I}^\bullet : U'/U \longmapsto K(\mathcal{O}(U'), I\mathcal{O}(U'))$$

with $K(-,-)$ as in Lemma 11.10. We claim that the sheafification of this is independent of the choice of I. Indeed, if $I' \subset \mathcal{O}(U)$ is a finitely generated ideal which also generates $\mathcal{I}|_U$, then there exists a covering $\{U_j \to U\}$ such that $I\mathcal{O}(U_j) = I'\mathcal{O}(U_j)$. (Hint: this works because both I and I' are finitely generated and generate $\mathcal{I}|_U$.) Hence $K_{U,I}^\bullet$ and $K_{U,I'}^\bullet$ are the *same* for any object lying over one of the U_j. The statement on sheafifications follows. Denote $K_U^\bullet$ the common value.

The independence of choice of I also shows that $K_U^\bullet|_{\mathcal{C}/U'} = K_{U'}^\bullet$ whenever we are given a morphism $U' \to U$ and hence a localization morphism $\mathcal{C}/U' \to \mathcal{C}/U$. Thus the complexes $K_U^\bullet$ glue to give a single well defined complex $K^\bullet$ of $\mathcal{O}$-modules. The existence of the map $K^\bullet \to \mathcal{O}$ and the quasi-isomorphism of the lemma follow immediately from the corresponding properties of the complexes $K(-,-)$ in Lemma 11.10. $\square$

099F **Proposition 11.12.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{I} \subset \mathcal{O}$ be a finite type sheaf of ideals. There exists a left adjoint to the inclusion functor $D_{comp}(\mathcal{O}) \to D(\mathcal{O})$.*

Proof. Let $K \to \mathcal{O}$ in $D(\mathcal{O})$ be as constructed in Lemma 11.11. Let $E \in D(\mathcal{O})$. Then $E^\wedge = R\mathcal{H}om(K, E)$ together with the map $E \to E^\wedge$ will do the job. Namely, locally on the site $\mathcal{C}$ we recover the adjoint of Lemma 11.8. This shows that $E^\wedge$ is always derived complete and that $E \to E^\wedge$ is an isomorphism if E is derived complete. $\square$

0CQH **Remark 11.13** (Comparison with completion)**.** Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{I} \subset \mathcal{O}$ be a finite type sheaf of ideals. Let $K \mapsto K^\wedge$ be the derived completion functor of Proposition 11.12. For any $n \geq 1$ the object $K \otimes_\mathcal{O}^\mathbf{L} \mathcal{O}/\mathcal{I}^n$ is derived complete as it is annihilated by powers of local sections of $\mathcal{I}$. Hence there is a canonical factorization

$$K \to K^\wedge \to K \otimes_\mathcal{O}^\mathbf{L} \mathcal{O}/\mathcal{I}^n$$

of the canonical map $K \to K \otimes_\mathcal{O}^\mathbf{L} \mathcal{O}/\mathcal{I}^n$. These maps are compatible for varying n and we obtain a comparison map

$$K^\wedge \longrightarrow R\lim \left(K \otimes_\mathcal{O}^\mathbf{L} \mathcal{O}/\mathcal{I}^n\right)$$

The right hand side is more recognizable as a kind of completion. In general this comparison map is not an isomorphism.

0A0F **Remark 11.14** (Localization and derived completion)**.** Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{I} \subset \mathcal{O}$ be a finite type sheaf of ideals. Let $K \mapsto K^\wedge$ be the derived completion functor of Proposition 11.12. It follows from the construction in the proof of the proposition that $K^\wedge|_U$ is the derived completion of $K|_U$ for any $U \in \text{Ob}(\mathcal{C})$. But we can also prove this as follows. From the definition of derived complete objects it follows that $K^\wedge|_U$ is derived complete. Thus we obtain a canonical map $a : (K|_U)^\wedge \to K^\wedge|_U$. On the other hand, if E is a derived complete object of $D(\mathcal{O}_U)$, then Rj_*E is a derived complete object of $D(\mathcal{O})$ by Lemma 11.7. Here j is the localization morphism (Modules on Sites, Section 19). Hence we also obtain a canonical map $b : K^\wedge \to Rj_*((K|_U)^\wedge)$. We omit the (formal) verification that the adjoint of b is the inverse of a.

099G **Remark 11.15** (Completed tensor product)**.** Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{I} \subset \mathcal{O}$ be a finite type sheaf of ideals. Denote $K \mapsto K^\wedge$ the adjoint of Proposition 11.12. Then we set

$$K \otimes_\mathcal{O}^\wedge L = (K \otimes_\mathcal{O}^\mathbf{L} L)^\wedge$$

This *completed tensor product* defines a functor $D_{comp}(\mathcal{O}) \times D_{comp}(\mathcal{O}) \to D_{comp}(\mathcal{O})$ such that we have

$$\text{Hom}_{D_{comp}(\mathcal{O})}(K, R\mathcal{H}om_\mathcal{O}(L, M)) = \text{Hom}_{D_{comp}(\mathcal{O})}(K \otimes_\mathcal{O}^\wedge L, M)$$

for $K, L, M \in D_{comp}(\mathcal{O})$. Note that $R\mathcal{H}om_\mathcal{O}(L, M) \in D_{comp}(\mathcal{O})$ by Lemma 11.5.

099H **Lemma 11.16.** *Let $\mathcal{C}$ be a site. Assume $\varphi : \mathcal{O} \to \mathcal{O}'$ is a flat homomorphism of sheaves of rings. Let $f_1, \ldots, f_r$ be global sections of $\mathcal{O}$ such that $\mathcal{O}/(f_1, \ldots, f_r) \cong \mathcal{O}'/(f_1, \ldots, f_r)$. Then the map of extended alternating Čech complexes*

$$\begin{array}{c} \mathcal{O} \to \prod_{i_0} \mathcal{O}_{f_{i_0}} \to \prod_{i_0 < i_1} \mathcal{O}_{f_{i_0} f_{i_1}} \to \ldots \to \mathcal{O}_{f_1 \ldots f_r} \\ \downarrow \\ \mathcal{O}' \to \prod_{i_0} \mathcal{O}'_{f_{i_0}} \to \prod_{i_0 < i_1} \mathcal{O}'_{f_{i_0} f_{i_1}} \to \ldots \to \mathcal{O}'_{f_1 \ldots f_r} \end{array}$$

is a quasi-isomorphism.

Proof. Observe that the second complex is the tensor product of the first complex with $\mathcal{O}'$. We can write the first extended alternating Čech complex as a colimit of the Koszul complexes $K_n = K(\mathcal{O}, f_1^n, \ldots, f_r^n)$, see More on Algebra, Lemma 26.13. Hence it suffices to prove $K_n \to K_n \otimes_\mathcal{O} \mathcal{O}'$ is a quasi-isomorphism. Since $\mathcal{O} \to \mathcal{O}'$ is flat it suffices to show that $H^i \to H^i \otimes_\mathcal{O} \mathcal{O}'$ is an isomorphism where

H^i is the ith cohomology sheaf $H^i = H^i(K_n)$. These sheaves are annihilated by $f_1^n, \ldots, f_r^n$, see More on Algebra, Lemma 26.6. Thus it suffices to show that $\mathcal{O}/(f_1^n, \ldots, f_r^n) \to \mathcal{O}'/(f_1^n, \ldots, f_r^n)$ is an isomorphism. Equivalently, we will show that $\mathcal{O}/(f_1, \ldots, f_r)^n \to \mathcal{O}'/(f_1, \ldots, f_r)^n$ is an isomorphism for all n. This holds for $n = 1$ by assumption. It follows for all n by induction using Modules on Sites, Lemma 28.14 applied to the ring map $\mathcal{O}/(f_1, \ldots, f_r)^{n+1} \to \mathcal{O}/(f_1, \ldots, f_r)^n$ and the module $\mathcal{O}'/(f_1, \ldots, f_r)^{n+1}$. $\square$

099I **Lemma 11.17.** *Let $\mathcal{C}$ be a site. Let $\mathcal{O} \to \mathcal{O}'$ be a homomorphism of sheaves of rings. Let $\mathcal{I} \subset \mathcal{O}$ be a finite type sheaf of ideals. If $\mathcal{O} \to \mathcal{O}'$ is flat and $\mathcal{O}/\mathcal{I} \cong \mathcal{O}'/\mathcal{I}\mathcal{O}'$, then the restriction functor $D(\mathcal{O}') \to D(\mathcal{O})$ induces an equivalence $D_{comp}(\mathcal{O}', \mathcal{I}\mathcal{O}') \to D_{comp}(\mathcal{O}, \mathcal{I})$.*

Proof. Lemma 11.7 implies restriction $r : D(\mathcal{O}') \to D(\mathcal{O})$ sends $D_{comp}(\mathcal{O}', \mathcal{I}\mathcal{O}')$ into $D_{comp}(\mathcal{O}, \mathcal{I})$. We will construct a quasi-inverse $E \mapsto E'$.

Let $K \to \mathcal{O}$ be the morphism of $D(\mathcal{O})$ constructed in Lemma 11.11. Set $K' = K \otimes_{\mathcal{O}}^{\mathbf{L}} \mathcal{O}'$ in $D(\mathcal{O}')$. Then $K' \to \mathcal{O}'$ is a map in $D(\mathcal{O}')$ which satisfies the conclusions of Lemma 11.11 with respect to $\mathcal{I}' = \mathcal{I}\mathcal{O}'$. The map $K \to r(K')$ is a quasi-isomorphism by Lemma 11.16. Now, for $E \in D_{comp}(\mathcal{O}, \mathcal{I})$ we set

$$E' = R\mathcal{H}om_{\mathcal{O}}(r(K'), E)$$

viewed as an object in $D(\mathcal{O}')$ using the $\mathcal{O}'$-module structure on K'. Since E is derived complete we have $E = R\mathcal{H}om_{\mathcal{O}}(K, E)$, see proof of Proposition 11.12. On the other hand, since $K \to r(K')$ is an isomorphism in we see that there is an isomorphism $E \to r(E')$ in $D(\mathcal{O})$. To finish the proof we have to show that, if $E = r(M')$ for an object M' of $D_{comp}(\mathcal{O}', \mathcal{I}')$, then $E' \cong M'$. To get a map we use

$$M' = R\mathcal{H}om_{\mathcal{O}'}(\mathcal{O}', M') \to R\mathcal{H}om_{\mathcal{O}}(r(\mathcal{O}'), r(M')) \to R\mathcal{H}om_{\mathcal{O}}(r(K'), r(M')) = E'$$

where the second arrow uses the map $K' \to \mathcal{O}'$. To see that this is an isomorphism, one shows that r applied to this arrow is the same as the isomorphism $E \to r(E')$ above. Details omitted. $\square$

099K **Lemma 11.18.** *Let $f : (Sh(\mathcal{D}), \mathcal{O}') \to (Sh(\mathcal{C}), \mathcal{O})$ be a morphism of ringed topoi. Let $\mathcal{I} \subset \mathcal{O}$ and $\mathcal{I}' \subset \mathcal{O}'$ be finite type sheaves of ideals such that $f^\sharp$ sends $f^{-1}\mathcal{I}$ into $\mathcal{I}'$. Then Rf_* sends $D_{comp}(\mathcal{O}', \mathcal{I}')$ into $D_{comp}(\mathcal{O}, \mathcal{I})$ and has a left adjoint Lf^*_{comp} which is Lf^* followed by derived completion.*

Proof. The first statement we have seen in Lemma 11.7. Note that the second statement makes sense as we have a derived completion functor $D(\mathcal{O}') \to D_{comp}(\mathcal{O}', \mathcal{I}')$ by Proposition 11.12. OK, so now let $K \in D_{comp}(\mathcal{O}, \mathcal{I})$ and $M \in D_{comp}(\mathcal{O}', \mathcal{I}')$. Then we have

$$\mathrm{Hom}(K, Rf_*M) = \mathrm{Hom}(Lf^*K, M) = \mathrm{Hom}(Lf^*_{comp}K, M)$$

by the universal property of derived completion. $\square$

0A0G **Lemma 11.19.** *Let $f : (Sh(\mathcal{D}), \mathcal{O}') \to (Sh(\mathcal{C}), \mathcal{O})$ be a morphism of ringed topoi. Let $\mathcal{I} \subset \mathcal{O}$ be a finite type sheaf of ideals. Let $\mathcal{I}' \subset \mathcal{O}'$ be the ideal generated by $f^\sharp(f^{-1}\mathcal{I})$. Then Rf_* commutes with derived completion, i.e., $Rf_*(K^\wedge) = (Rf_*K)^\wedge$.*

Generalization of [BS13, Lemma 6.5.9 (2)]. Compare with [HLP14, Theorem 6.5] in the setting of quasi-coherent modules and morphisms of (derived) algebraic stacks.

Proof. By Proposition 11.12 the derived completion functors exist. By Lemma 11.7 the object $Rf_*(K^\wedge)$ is derived complete, and hence we obtain a canonical map $(Rf_*K)^\wedge \to Rf_*(K^\wedge)$ by the universal property of derived completion. We may check this map is an isomorphism locally on $\mathcal{C}$. Thus, since derived completion commutes with localization (Remark 11.14) we may assume that $\mathcal{I}$ is generated by global sections $f_1, \ldots, f_r$. Then $\mathcal{I}'$ is generated by $g_i = f^\sharp(f_i)$. By Lemma 11.9 we have to prove that

$$R\lim\left(Rf_*K \otimes_{\mathcal{O}}^{\mathbf{L}} K(\mathcal{O}, f_1^n, \ldots, f_r^n)\right) = Rf_*\left(R\lim K \otimes_{\mathcal{O}'}^{\mathbf{L}} K(\mathcal{O}', g_1^n, \ldots, g_r^n)\right)$$

Because Rf_* commutes with $R\lim$ (Cohomology on Sites, Lemma 22.3) it suffices to prove that

$$Rf_*K \otimes_{\mathcal{O}}^{\mathbf{L}} K(\mathcal{O}, f_1^n, \ldots, f_r^n) = Rf_*\left(K \otimes_{\mathcal{O}'}^{\mathbf{L}} K(\mathcal{O}', g_1^n, \ldots, g_r^n)\right)$$

This follows from the projection formula (Cohomology on Sites, Lemma 40.1) and the fact that $Lf^*K(\mathcal{O}, f_1^n, \ldots, f_r^n) = K(\mathcal{O}', g_1^n, \ldots, g_r^n)$. $\square$

0BLX **Lemma 11.20.** *Let A be a ring and let $I \subset A$ be a finitely generated ideal. Let $\mathcal{C}$ be a site and let $\mathcal{O}$ be a sheaf of A-algebras. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}$-modules. Then we have*

$$R\Gamma(\mathcal{C}, \mathcal{F})^\wedge = R\Gamma(\mathcal{C}, \mathcal{F}^\wedge)$$

in $D(A)$ where $\mathcal{F}^\wedge$ is the derived completion of $\mathcal{F}$ with respect to $I\mathcal{O}$ and on the left hand wide we have the derived completion with respect to I. This produces two spectral sequences

$$E_2^{i,j} = H^i(H^j(\mathcal{C}, \mathcal{F})^\wedge) \quad \text{and} \quad E_2^{p,q} = H^p(\mathcal{C}, H^q(\mathcal{F}^\wedge))$$

both converging to $H^*(R\Gamma(\mathcal{C}, \mathcal{F})^\wedge) = H^*(\mathcal{C}, \mathcal{F}^\wedge)$

Proof. Apply Lemma 11.19 to the morphism of ringed topoi $(\mathcal{C}, \mathcal{O}) \to (pt, A)$ and take cohomology to get the first statement. The second spectral sequence is just the Leray spectral sequence for this morphism, see Cohomology on Sites, Lemma 15.5. The first spectral sequence is the spectral sequence of More on Algebra, Example 80.19 applied to $R\Gamma(\mathcal{C}, \mathcal{F})^\wedge$. $\square$

0CQI **Remark 11.21.** Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{I} \subset \mathcal{O}$ be a finite type sheaf of ideals. Let $K \mapsto K^\wedge$ be the derived completion of Proposition 11.12. Let $U \in \text{Ob}(\mathcal{C})$ be an object such that $\mathcal{I}$ is generated as an ideal sheaf by $f_1, \ldots, f_r \in \mathcal{I}(U)$. Set $A = \mathcal{O}(U)$ and $I = (f_1, \ldots, f_r) \subset A$. Warning: it may not be the case that $I = \mathcal{I}(U)$. Then we have

$$R\Gamma(U, K^\wedge) = R\Gamma(U, K)^\wedge$$

where the right hand side is the derived completion of the object $R\Gamma(U, K)$ of $D(A)$ with respect to I. This is true because derived completion commutes with localization (Remark 11.14) and Lemma 11.20.

12. Application to theorem on formal functions

0A0H We interrupt the flow of the exposition to talk a little bit about derived completion in the setting of quasi-coherent modules on schemes and to use this to give a somewhat different proof of the theorem on formal functions. We give some pointers to the literature in Remark 12.4.

Lemma 11.19 is a (very formal) derived version of the theorem on formal functions (Cohomology of Schemes, Theorem 20.5). To make this more explicit, suppose $f : X \to S$ is a morphism of schemes, $\mathcal{I} \subset \mathcal{O}_S$ is a quasi-coherent sheaf of ideals of finite type, and $\mathcal{F}$ is a quasi-coherent sheaf on X. Then the lemma says that

0A0I (12.0.1)
$$Rf_*(\mathcal{F}^\wedge) = (Rf_*\mathcal{F})^\wedge$$

where $\mathcal{F}^\wedge$ is the derived completion of $\mathcal{F}$ with respect to $f^{-1}\mathcal{I} \cdot \mathcal{O}_X$ and the right hand side is the derived completion of $\mathcal{F}$ with respect to $\mathcal{I}$. To see that this gives back the theorem on formal functions we have to do a bit of work.

0A0L **Lemma 12.1.** *Let X be a locally Noetherian scheme. Let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. Let K be a pseudo-coherent object of $D(\mathcal{O}_X)$ with derived completion $K^\wedge$. Then*

$$H^p(U, K^\wedge) = \lim H^p(U, K)/I^n H^p(U, K) = H^p(U, K)^\wedge$$

for any affine open $U \subset X$ where $I = \mathcal{I}(U)$ and where on the right we have the derived completion with respect to I.

Proof. Write $U = \text{Spec}(A)$. The ring A is Noetherian and hence $I \subset A$ is finitely generated. Then we have

$$R\Gamma(U, K^\wedge) = R\Gamma(U, K)^\wedge$$

by Remark 11.21. Now $R\Gamma(U, K)$ is a pseudo-coherent complex of A-modules (Derived Categories of Schemes, Lemma 9.2). By More on Algebra, Lemma 82.3 we conclude that the pth cohomology module of $R\Gamma(U, K^\wedge)$ is equal to the I-adic completion of $H^p(U, K)$. This proves the first equality. The second (less important) equality follows immediately from a second application of the lemma just used. □

0A0K **Lemma 12.2.** *Let X be a locally Noetherian scheme. Let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. Let K be an object of $D(\mathcal{O}_X)$. Then*

(1) *the derived completion $K^\wedge$ is equal to $R\lim(K \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{O}_X/\mathcal{I}^n)$.*

Let K is a pseudo-coherent object of $D(\mathcal{O}_X)$. Then

(2) *the cohomology sheaf $H^q(K^\wedge)$ is equal to $\lim H^q(K)/\mathcal{I}^n H^q(K)$.*

Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module[1]*. Then*

(3) *the derived completion $\mathcal{F}^\wedge$ is equal to $\lim \mathcal{F}/\mathcal{I}^n\mathcal{F}$,*
(4) $\lim \mathcal{F}/I^n\mathcal{F} = R\lim \mathcal{F}/I^n\mathcal{F}$,
(5) $H^p(U, \mathcal{F}^\wedge) = 0$ *for $p \neq 0$ for all affine opens $U \subset X$.*

Proof. Proof of (1). There is a canonical map

$$K \longrightarrow R\lim(K \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{O}_X/\mathcal{I}^n),$$

see Remark 11.13. Derived completion commutes with passing to open subschemes (Remark 11.14). Formation of $R\lim$ commutes with passsing to open subschemes. It follows that to check our map is an isomorphism, we may work locally. Thus we may assume $X = U = \text{Spec}(A)$. Say Say $I = (f_1, \ldots, f_r)$. Let $K_n = K(A, f_1^n, \ldots, f_r^n)$ be the Koszul complex. By More on Algebra, Lemma 82.1 we have seen that the pro-systems $\{K_n\}$ and $\{A/I^n\}$ of $D(A)$ are isomorphic. Using the equivalence $D(A) = D_{QCoh}(\mathcal{O}_X)$ of Derived Categories of Schemes, Lemma 3.5

[1] For example $H^q(K)$ for K pseudo-coherent on our locally Noetherian X.

we see that the pro-systems $\{K(\mathcal{O}_X, f_1^n, \ldots, f_r^n)\}$ and $\{\mathcal{O}_X/\mathcal{I}^n\}$ are isomorphic in $D(\mathcal{O}_X)$. This proves the second equality in

$$K^\wedge = R\lim\left(K \otimes^{\mathbf{L}}_{\mathcal{O}_X} K(\mathcal{O}_X, f_1^n, \ldots, f_r^n)\right) = R\lim(K \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{O}_X/\mathcal{I}^n)$$

The first equality is Lemma 11.9.

Assume K is pseudo-coherent. For $U \subset X$ affine open we have $H^q(U, K^\wedge) = \lim H^q(U, K)/\mathcal{I}^n(U)H^q(U, K)$ by Lemma 12.1. As this is true for every U we see that $H^q(K^\wedge) = \lim H^q(K)/\mathcal{I}^n H^q(K)$ as sheaves. This proves (2).

Part (3) is a special case of (2). Parts (4) and (5) follow from Derived Categories of Schemes, Lemma 3.2. □

0A0M **Lemma 12.3.** *Let A be a Noetherian ring and let $I \subset A$ be an ideal. Let X be a Noetherian scheme over A. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Assume that $H^p(X, \mathcal{F})$ is a finite A-module for all p. Then there are short exact sequences*

$$0 \to R^1\lim H^{p-1}(X, \mathcal{F}/I^n\mathcal{F}) \to H^p(X, \mathcal{F})^\wedge \to \lim H^p(X, \mathcal{F}/I^n\mathcal{F}) \to 0$$

of A-modules where $H^p(X, \mathcal{F})^\wedge$ is the usual I-adic completion. If f is proper, then the $R^1\lim$ term is zero.

Proof. Consider the two spectral sequences of Lemma 11.20. The first degenerates by More on Algebra, Lemma 82.3. We obtain $H^p(X, \mathcal{F})^\wedge$ in degree p. This is where we use the assumption that $H^p(X, \mathcal{F})$ is a finite A-module. The second degenerates because

$$\mathcal{F}^\wedge = \lim \mathcal{F}/I^n\mathcal{F} = R\lim \mathcal{F}/I^n\mathcal{F}$$

is a sheaf by Lemma 12.2. We obtain $H^p(X, \lim \mathcal{F}/I^n\mathcal{F})$ in degree p. Since $R\Gamma(X, -)$ commutes with derived limits (Injectives, Lemma 13.6) we also get

$$R\Gamma(X, \lim \mathcal{F}/I^n\mathcal{F}) = R\Gamma(X, R\lim \mathcal{F}/I^n\mathcal{F}) = R\lim R\Gamma(X, \mathcal{F}/I^n\mathcal{F})$$

By More on Algebra, Remark 76.6 we obtain exact sequences

$$0 \to R^1\lim H^{p-1}(X, \mathcal{F}/I^n\mathcal{F}) \to H^p(X, \lim \mathcal{F}/I^n\mathcal{F}) \to \lim H^p(X, \mathcal{F}/I^n\mathcal{F}) \to 0$$

of A-modules. Combining the above we get the first statement of the lemma. The vanishing of the $R^1\lim$ term follows from Cohomology of Schemes, Lemma 20.4. □

0AKL **Remark 12.4.** Here are some references to discussions of related material the literature. It seems that a "derived formal functions theorem" for proper maps goes back to [Lur04, Theorem 6.3.1]. There is the discussion in [Lur11], especially Chapter 4 which discusses the affine story, see More on Algebra, Section 80. In [GR13, Section 2.9] one finds a discussion of proper base change and derived completion using (ind) coherent modules. An analogue of (12.0.1) for complexes of quasi-coherent modules can be found as [HLP14, Theorem 6.5]

13. Algebraization of formal sections

0DXH Let $(A, \mathfrak{m})$ be a Noetherian local ring. Let $I \subset A$ be an ideal. Let

$$X = \mathrm{Spec}(A) \supset U = \mathrm{Spec}(A) \setminus \{\mathfrak{m}\}$$

and denote $Y = V(I)$ the closed subscheme corresponding to I. Let $\mathcal{F}$ be a coherent $\mathcal{O}_U$-module. In this section we consider the limits

$$\lim_n H^i(U, \mathcal{F}/I^n\mathcal{F})$$

This is closely related to the cohomology of the pullback of $\mathcal{F}$ to the formal completion of U along Y; however, since we have not yet introduced formal schemes, we cannot use this terminology here.

It turns out the modules we are interested in compute the cohomology of the derived completion of the cohomology of $\mathcal{F}$ over U.

0DXI **Lemma 13.1.** *In the situation above the inverse systems $H^i(U, \mathcal{F}/I^n\mathcal{F})$ satisfy the Mittag-Leffler condition for all i and moreover*

$$H^i(R\Gamma(U, \mathcal{F})^\wedge) = \lim H^i(U, \mathcal{F}/I^n\mathcal{F})$$

for all i where $R\Gamma(U, \mathcal{F})^\wedge$ denotes the derived I-adic completion.

Proof. Choose a finite A-module M such that $\mathcal{F}$ is equal to the restriction of $\widetilde{M}$ to U. Then we find short exact sequences

$$0 \to H^0_\mathfrak{m}(M/I^nM) \to M/I^nM \to H^0(U, \mathcal{F}/I^n\mathcal{F}) \to H^1_\mathfrak{m}(M/I^nM) \to 0$$

and isomorphisms $H^i(U, \mathcal{F}/I^n\mathcal{F}) = H^{i+1}_\mathfrak{m}(M/I^nM)$ for $i \geq 1$. See Lemma 4.2. We have the Mittag-Leffler conditions for $H^i_\mathfrak{m}(M/I^nM)$ by Lemma 5.5 and trivially for M/I^nM. Thus we get the Mittag-Leffler condition for $H^i(U, \mathcal{F}/I^n\mathcal{F})$ by More on Algebra, Lemma 75.12. By Lemmas 11.20 and 12.2 we have

$$R\Gamma(U, \mathcal{F})^\wedge = R\Gamma(U, \mathcal{F}^\wedge) = R\Gamma(U, R\lim \mathcal{F}/I^n\mathcal{F})$$

Thus we obtain short exact sequences

$$0 \to R^1 \lim H^{i-1}(U, \mathcal{F}/I^n\mathcal{F}) \to H^i(R\Gamma(U, \mathcal{F})^\wedge) \to \lim H^i(U, \mathcal{F}/I^n\mathcal{F}) \to 0$$

by Cohomology, Lemma 32.1. The $R^1 \lim$ terms vanish by the Mittag-Leffler condition we just established. $\square$

0DXJ **Lemma 13.2.** *Let $(A, \mathfrak{m})$ be a Noetherian local ring complete with respect to an ideal $I \subset A$. Let M be a finite A-module. Then*

$$H^i(R\Gamma_\mathfrak{m}(M)^\wedge) = \lim H^i_\mathfrak{m}(M/I^nM)$$

for all i where $R\Gamma_\mathfrak{m}(M)^\wedge$ denotes the derived I-adic completion.

Proof. Set $U = \text{Spec}(A) \setminus \{\mathfrak{m}\}$ and denote $\mathcal{F}$ the coherent $\mathcal{O}_U$-module corresponding to M. We will use the results of Lemma 13.1 without further mention. The distinguished triangle

$$R\Gamma_\mathfrak{m}(M) \to M \to R\Gamma(U, \mathcal{F}) \to R\Gamma_\mathfrak{m}(M)[1]$$

is transformed into another distinguished triangle by the exact functor of derived completion. Since A is I-adically complete, so is M, and hence the derived completion $M^\wedge$ is equal to M. Thus we find a short exact sequence

$$0 \to H^0(R\Gamma_\mathfrak{m}(M)^\wedge) \to M \to \lim H^0(U, \mathcal{F}/I^n\mathcal{F}) \to H^1(R\Gamma_\mathfrak{m}(M)^\wedge) \to 0$$

and isomorphisms $\lim H^{i-1}(U, \mathcal{F}/I^n\mathcal{F}) = H^i(R\Gamma_\mathfrak{m}(M)^\wedge)$ for $i \geq 2$. This proves the lemma holds for $i \geq 2$ since we have $H^{i-1}(U, \mathcal{F}/I^n\mathcal{F}) = H^i_\mathfrak{m}(M)$ in this case.

For $i = 0, 1$ we compare the sequence above with the exact sequences

$$0 \to H^0_\mathfrak{m}(M/I^nM) \to M/I^nM \to H^0(U, \mathcal{F}/I^n\mathcal{F}) \to H^1_\mathfrak{m}(M/I^nM) \to 0$$

where we have the Mittag-Leffler condition in each spot. Since we have that the limit of the middle two systems gives the middle two modules in the sequence of the previous paragraph we conclude. $\square$

0DXK **Lemma 13.3.** *Let $(A, \mathfrak m)$ be a Noetherian local ring. Let $I \subset A$ be an ideal. Let M be a finite A-module and let $\mathfrak p \subset A$ be a prime. Let s and d be integers. Assume*

1. *A has a dualizing complex,*
2. *$cd(A, I) \leq d$, and*
3. *$depth_{A_\mathfrak p}(M_\mathfrak p) + \dim(A/\mathfrak p) > d + s$.*

Then there exists an $f \in A \setminus \mathfrak p$ which annihilates $H^i(R\Gamma_\mathfrak m(M)^\wedge)$ for $i \leq s$ where $^\wedge$ indicates I-adic completion.

Proof. Recall that

$$R\Gamma_\mathfrak m(M)^\wedge = R\operatorname{Hom}_A(R\Gamma_I(A), R\Gamma_\mathfrak m(M))$$

by the description of derived completion in More on Algebra, Lemma 80.9 combined with the description of local cohomology in Dualizing Complexes, Lemma 10.2. Assumption (2) means that $R\Gamma_I(A)$ has nonzero cohomology only in degrees $\leq d$. Using the canonical truncations of $R\Gamma_I(A)$ we find it suffices to show that

$$\operatorname{Ext}^i(N, R\Gamma_\mathfrak m(M))$$

is annihilated by an $f \in A \setminus \mathfrak p$ for $i \leq s + d$ and any A-module N. In turn using the canonical truncations for $R\Gamma_\mathfrak m(M)$ we see that it suffices to show $H^i_\mathfrak m(M)$ is annihilated by an $f \in A \setminus \mathfrak p$ for $i \leq s + d$. This follows from Lemma 5.3. $\square$

0DXL **Lemma 13.4.** *Let $(A, \mathfrak m)$ be a Noetherian local ring. Let $I \subset A$ be an ideal. Let M be a finite A-module. Let s be an integer. Assume*

1. *A has a dualizing complex,*
2. *if $\mathfrak p \not\in V(I)$ and $V(\mathfrak p) \cap V(I) \neq \{\mathfrak m\}$, then $depth_{A_\mathfrak p}(M_\mathfrak p) + \dim(A/\mathfrak p) > s$.*

Then there exists an $n > 0$ and an ideal $J \subset A$ with $V(J) \cap V(I) = \{\mathfrak m\}$ such that JI^n annihilates $H^i_\mathfrak m(M)$ for $i \leq s$.

Proof. According to Lemma 5.3 we have to show this for the finite A-module $E^i = \operatorname{Ext}^{-i}_A(M, \omega_A^\bullet)$ for $i \leq s$. The support Z of $E^0 \oplus \ldots \oplus E^s$ is closed in $\operatorname{Spec}(A)$ and does not contain any prime as in (4). Hence it is contained in $V(JI^n)$ for some J as in the statement of the lemma. $\square$

0DXM **Lemma 13.5.** *Let $(A, \mathfrak m)$ be a Noetherian local ring. Let $I \subset A$ be an ideal. Let M be a finite A-module. Let s and d be integers. Assume*

1. *A has a dualizing complex,*
2. *if $\mathfrak p \in V(I) \setminus \{\mathfrak m\}$, then no condition,*
3. *if $\mathfrak p \not\in V(I)$ and $V(\mathfrak p) \cap V(I) = \{\mathfrak m\}$, then $\dim(A/\mathfrak p) \leq d$,*
4. *if $\mathfrak p \not\in V(I)$ and $V(\mathfrak p) \cap V(I) \neq \{\mathfrak m\}$, then*

$$depth_{A_\mathfrak p}(M_\mathfrak p) \geq s \quad or \quad depth_{A_\mathfrak p}(M_\mathfrak p) + \dim(A/\mathfrak p) > d + s$$

Then there exists an ideal $J_0 \subset A$ with $V(J_0) \cap V(I) = \{\mathfrak m\}$ such that for any $J \subset J_0$ with $V(J) \cap V(I) = \{\mathfrak m\}$ the map

$$R\Gamma_J(M) \longrightarrow R\Gamma_{J_0}(M)$$

induces an isomorphism in cohomology in degrees $\leq s$ and moreover these modules are annihilated by a power of $J_0 I$.

Proof. Let us consider the set

$$B = \{\mathfrak{p} \notin V(I) \text{ with } V(\mathfrak{p}) \cap V(I) = \{\mathfrak{m}\} \text{ and } \mathrm{depth}(M_\mathfrak{p}) \leq s\}$$

Let $V(J_0) \subset \mathrm{Spec}(A)$ be the closure of B, except if $B = \emptyset$ we take $J_0 = \mathfrak{m}$.

Claim I: $V(J_0) \cap V(I) = \{\mathfrak{m}\}$.

Proof of Claim I. Let $\mathfrak{p}$ be a minimal prime of $V(J_0)$. If $\mathfrak{p} \in B$, then $V(\mathfrak{p}) \cap V(I) = \{\mathfrak{m}\}$ as desired. If $\mathfrak{p} \notin B$, then $B \cap V(\mathfrak{p})$ is dense, hence infinite, and we conclude that $\mathrm{depth}(M_\mathfrak{p}) < s$ by Lemma 5.2. On the other hand, every $\mathfrak{p}' \in B$ satisfies $\mathrm{depth}(M_{\mathfrak{p}'}) + \dim(A/\mathfrak{p}') \leq d + s$ by (3) and the definition of B. Since the set of such primes is closed by Lemma 5.3 we conclude that $\mathrm{depth}(M_\mathfrak{p}) + \dim(A/\mathfrak{p}) \leq d + s$. Also $\mathfrak{p} \notin V(I)$. Hence by (4) we see that $V(\mathfrak{p}) \cap V(I) = \{\mathfrak{m}\}$. This finishes the proof of Claim I.

Claim II: $H^i_{J_0}(M) \to H^i_J(M)$ is an isomorphism for $i \leq s$ and $J \subset J_0$ with $V(J) \cap V(I) = \{\mathfrak{m}\}$.

Proof of claim II. Choose $\mathfrak{p} \in V(J)$ not in $V(J_0)$. It suffices to show that $H^i_{\mathfrak{p}A_\mathfrak{p}}(M_\mathfrak{p}) = 0$, see Lemma 2.6. Since $\mathfrak{p}$ is not in B we see that either $\mathrm{depth}(M_\mathfrak{p}) > s$ and then the group vanishes or $\mathrm{depth}(M_\mathfrak{p}) + \dim(A/\mathfrak{p}) > d + s$. However, by condition (3) we find this also implies $\mathrm{depth}(M_\mathfrak{p}) > s$ and we conclude again.

Claim III. $H^i_J(M)_f$ is a finite A_f-module for $f \in I$ and $i \leq s$.

Proof of Claim III. We will check the conditions of Theorem 7.4 for $J_f \subset A_f$ and the module M_f. Let $\mathfrak{p} \subset \mathfrak{q}$ be primes such that $\mathfrak{p} \notin V(J)$ and $\mathfrak{q} \in V(J)$ but $f \notin \mathfrak{q}$. We have to show that

$$\mathrm{depth}(M_\mathfrak{p}) + \dim((A/\mathfrak{p})_\mathfrak{q}) > s$$

Note that $\mathfrak{p} \notin V(I)$ since $f \notin \mathfrak{q}$. If $V(\mathfrak{p}) \cap V(I) = \{\mathfrak{m}\}$ then $\mathrm{depth}(M_\mathfrak{p}) > s$ because $\mathfrak{p} \notin B$. Hence the desired inequality. If $\mathfrak{p} \notin V(I)$ and $V(\mathfrak{p}) \cap V(I) \neq \{\mathfrak{m}\}$, then there are two cases: either $\mathrm{depth}(M_\mathfrak{p}) \geq s$ and then the desired inequality follows as $\dim((A/\mathfrak{p})_\mathfrak{q}) \geq 1$ or $\mathrm{depth}_{A_\mathfrak{p}}(M_\mathfrak{p}) + \dim(A/\mathfrak{p}) > d + s$ which combined with $\dim(A/\mathfrak{q}) \leq d$ assumed in (3) gives the desired inequality as A is catenary: $\dim((A/\mathfrak{p})_\mathfrak{q}) = \dim(A/\mathfrak{p}) - \dim(A/\mathfrak{q}) \geq \dim(A/\mathfrak{p}) - d$.

Claim IV: The final statement of the lemma is true.

Proof of Claim IV. Set $J = J_0$. Write $I = (f_1, \ldots, f_r)$. By Claim III and Lemma 3.1 the modules $H^i_J(M)_{f_j}$ are finite and annihilated by J^t for some $t > 0$ (pick the same t for all j). Hence there exists a finite A-submodule $N^i \subset H^i_J(M)$ annihilated by J^t inducing isomorphisms $N^i_{f_j} = H^i_J(M)_{f_j}$ for $j = 1, \ldots, r$. Then $H^i_J(M)/N^i$ is I-power torsion. Hence $J^t H^i_J(M)$ is I-power torsion. To finish the proof it suffices to show that $H^0_I(H^i_J(M))$ is annihilated by $J'I^n$ for some $n > 0$ and some ideal $J' \subset J = J_0$ with $V(J') \cap V(I) = \{\mathfrak{m}\}$. Namely, then the final statement holds with J_0 replaced by J'. We will use without further mention that the collection of ideals $J' \subset J$ with $V(J') \cap V(I) = \{\mathfrak{m}\}$ ordered by inclusion is a directed set. Consider the spectral sequence

$$E_2^{p,q} = H^p_I(H^q_J(M)) \Rightarrow H^{a+b}_\mathfrak{m}(M)$$

By Lemma 13.4 we find $J' \subset J$ with $V(J') \cap V(I) = \{\mathfrak{m}\}$ and an integer $n > 0$ such that a power of $J'I$ annihilates $H^i_\mathfrak{m}(M)$ for $i \leq s$. By induction on i we may assume that a power of $J'I$ annihilates $H^q_J(M)$ for $q < i$. The spectral sequence shows there is a map

$$H^i_{\mathfrak m} \to H^0_I(H^i_J(M))$$

whose image the intersection of the kernels of the differentials (each defined on the kernel of the previous one). These differentials map into subquotients of $H^2_I(H^{i-1}_J(M)), H^3_I(H^{i-2}_J(M)), \ldots$ which are killed by a power of $J''I$ for a suitable J'' by induction. Putting everything together we conclude. $\square$

0DXN **Lemma 13.6.** *In Lemma 13.5 if in stead of the empty condition (2) we assume*

(2') *if $\mathfrak p \in V(I) \setminus \{\mathfrak m\}$, then $depth_{A_\mathfrak p}(M_\mathfrak p) + \dim(A/\mathfrak p) > s$,*

then the conditions also imply that $H^i_{J_0}(M)$ is a finite A-module for $i \leq s$.

Proof. We will use the construction of J_0 using the set B from the proof of Lemma 13.5. We will check the conditions of Theorem 7.4. Let $\mathfrak p \subset \mathfrak q$ be primes such that $\mathfrak p \notin V(J_0)$ and $\mathfrak q \in V(J_0)$. We have to show that

$$\text{depth}(M_\mathfrak p) + \dim((A/\mathfrak p)_\mathfrak q) > s$$

If $\mathfrak p \in V(I)$, then $\mathfrak q = \mathfrak m$ because $V(J_0) \cap V(I) = \{\mathfrak m\}$. Hence we see that the desired inequality follows from (2). If $\mathfrak p \notin V(I)$ and $V(\mathfrak p) \cap V(I) = \{\mathfrak m\}$ then $\text{depth}(M_\mathfrak p) > s$ because $\mathfrak p \notin B$. Hence the desired inequality. Finally, if $\mathfrak p \notin V(I)$ and $V(\mathfrak p) \cap V(I) \neq \{\mathfrak m\}$, then there are two cases: either $\text{depth}(M_\mathfrak p) \geq s$ and then the desired inequality follows as $\dim((A/\mathfrak p)_\mathfrak q) \geq 1$ or $\text{depth}_{A_\mathfrak p}(M_\mathfrak p) + \dim(A/\mathfrak p) > d + s$ which combined with $\dim(A/\mathfrak q) \leq d$ assumed in (3) and gives the desired inequality as A is catenary: $\dim((A/\mathfrak p)_\mathfrak q) = \dim(A/\mathfrak p) - \dim(A/\mathfrak q) \geq \dim(A/\mathfrak p) - d$. $\square$

0DXP **Lemma 13.7.** *Let $(A, \mathfrak m)$ be a Noetherian local ring. Let $I \subset A$ be an ideal. Let M be a finite A-module. Let s and d be integers. Assume*

1. *A is I-adically complete and has a dualizing complex,*
2. *if $\mathfrak p \in V(I) \setminus \{\mathfrak m\}$, no condition,*
3. *$cd(A, I) \leq d$,*
4. *if $\mathfrak p \notin V(I)$ and $V(\mathfrak p) \cap V(I) \neq \{\mathfrak m\}$ then*

$$depth_{A_\mathfrak p}(M_\mathfrak p) \geq s \quad or \quad depth_{A_\mathfrak p}(M_\mathfrak p) + \dim(A/\mathfrak p) > d + s$$

Then there exists an ideal $J_0 \subset A$ with $V(J_0) \cap V(I) = \{\mathfrak m\}$ such that for any $J \subset J_0$ with $V(J) \cap V(I) = \{\mathfrak m\}$ the map

$$R\Gamma_J(M) \longrightarrow R\Gamma_J(M)^\wedge = R\Gamma_\mathfrak m(M)^\wedge$$

induces an isomorphism in cohomology in degrees $\leq s$. Here $^\wedge$ denotes derived I-adic completion.

Proof. There is no difference between $R\Gamma_\mathfrak a$ and $R\Gamma_{V(\mathfrak a)}$ in our current situation, see Dualizing Complexes, Lemma 10.1. Next, we observe that

$$R\Gamma_\mathfrak m(M)^\wedge = R\Gamma_I(R\Gamma_J(M))^\wedge = R\Gamma_J(M)^\wedge$$

by Dualizing Complexes, Lemmas 9.6 and 12.1 which explains the equality sign in the statement of the lemma.

Suppose M is a finite A-module with $\text{Supp}(M) \subset V(J) \cup V(I)$ for some ideal J with $V(J) \cap V(I) = \{\mathfrak m\}$. Then we claim that $R\Gamma_J(M) \to R\Gamma_\mathfrak m(M)^\wedge$ is an isomorphism. Namely, for any such module there is a short exact sequence $0 \to M_1 \oplus M_2 \to M \to N \to 0$ with M_1 annihilated by a power of J, with M_2 annihilated by a power of I and with N annihilated by a power of $\mathfrak{m}$. In the case of M_1 we see that $R\Gamma_J(M_1) = M_1$ and since M_1 is a finite A-module and I-adically complete we have $M_1^\wedge = M_1$. In the case of M_2 we see that $H^i_J(M_2)$ is annihilated by a power of I and hence derived complete. Thus $R\Gamma_J(M_2) = R\Gamma_J(M_2)^\wedge$ as desired.

Next, let M be as in the statement of the lemma. Observe that the lemma holds for $s < 0$. This is not a trivial case because it is not a priori clear that $H^s(R\Gamma_\mathfrak{m}(M)^\wedge)$ is zero for negative s. However, this vanishing was esthablished in Lemma 13.2. We will prove the lemma by induction for $s \geq 0$.

Let $M' \subset M$ be the submodule of elements whose support is condained in $V(I) \cup V(J)$ for some ideal J with $V(J) \cap V(I) = \{\mathfrak{m}\}$. Then M' is finite and the result holds for M' by the second paragraph of the proof. Moreover, condition (4) for M is inherited by M/M'. After replacing M by M/M' we may assume that $\text{Ass}(M)$ consists of primes as in (4).

The assumptions of Lemma 13.5 are satisfied by Lemma 9.9. Thus we may and do choose an ideal J_0 as in the lemma and an integer $t > 0$ such that $(J_0I)^t$ annihilates $H^s_J(M)$. The assumptions of Lemma 13.3 are satisfied for every $\mathfrak{p} \in \text{Ass}(M)$ (by our mangling of M above). Thus the annihilator $\mathfrak{a} \subset A$ of $H^s(R\Gamma_\mathfrak{m}(M)^\wedge)$ is not contained in $\mathfrak{p}$ for $\mathfrak{p} \in \text{Ass}(M)$. Thus we can find an $f \in \mathfrak{a}(J_0I)^t$ not in any associated prime of M which is an annihilator of both $H^s(R\Gamma_\mathfrak{m}(M)^\wedge)$ and $H^s_J(M)$. Then f is a nonzerodivisor on M and we can consider the short exact sequence

$$0 \to M \xrightarrow{f} M \to M/fM \to 0$$

Our choice of f shows that we obtain

$$\begin{tikzcd}
H^{s-1}_J(M) \arrow[r] \arrow[d] & H^{s-1}_J(M/fM) \arrow[r] \arrow[d] & H^s_J(M) \arrow[r] \arrow[d] & 0 \\
H^{s-1}(R\Gamma_\mathfrak{m}(M)^\wedge) \arrow[r] & H^{s-1}(R\Gamma_\mathfrak{m}(M/fM)^\wedge) \arrow[r] & H^s(R\Gamma_\mathfrak{m}(M)^\wedge) \arrow[r] & 0
\end{tikzcd}$$

for any $J \subset J_0$ with $V(J) \cap V(I) = \{\mathfrak{m}\}$. Thus if we choose J such that it works for M and M/fM and $s - 1$ (possible by induction hypothesis), then we conclude that the lemma is true. □

The lemma above is the main result of this section. We can reformulate it in terms of cohomology of the punctured spectrum as follows.

0DXQ **Theorem 13.8.** *Let $(A, \mathfrak{m})$ be a Noetherian local ring which has a dualizing complex and is complete with respect to an ideal I. Set $X = \text{Spec}(A)$, $Y = V(I)$, and $U = X \setminus \{\mathfrak{m}\}$. Let $\mathcal{F}$ be a coherent sheaf on U. Assume*

1. *$cd(A, I) \leq d$, i.e., $H^i(X \setminus Y, \mathcal{G}) = 0$ for $i \geq d$ and quasi-coherent $\mathcal{G}$ on X,*
2. *for any $x \in X \setminus Y$ whose closure $\overline{\{x\}}$ in X meets $Y \cap U$ we have*

$$depth_{\mathcal{O}_{X,x}}(\mathcal{F}_x) \geq s \quad or \quad depth_{\mathcal{O}_{X,x}}(\mathcal{F}_x) + \dim(\overline{\{x\}}) > d + s$$

Then there exists an open $V_0 \subset U$ containing $Y \cap U$ such that for any open $V_0 \subset V \subset U$ containing $Y \cap U$ the map

$$H^i(V, \mathcal{F}) \to \lim H^i(U, \mathcal{F}/I^n\mathcal{F})$$

is an isomorphism for $i < s$. If in addition $depth_{\mathcal{O}_{X,x}}(\mathcal{F}_x) + \dim(\overline{\{x\}}) > s$ for all $x \in Y \cap U$, then these cohomology groups are finite A-modules.

The method of proof follows roughly the method of proof of [Fal79, Theorem 1] and [Fal80, Satz 2]. The result is almost the same as [Ray74, Theorem 1.1] (affine complement case) and [Ray75, Theorem 3.9] (complement is union of few affines).

Proof. Choose a finite A-module M such that $\mathcal{F}$ is the restriction to U of the coherent $\mathcal{O}_X$-module associated to M, see Lemma 4.2. Then the assumptions of Lemma 13.7 are satisfied. Pick J_0 as in that lemma and set $V_0 = X \setminus V(J_0)$. Then opens $V_0 \subset V \subset U$ containing $Y \cap U$ correspond 1-to-1 with ideals $J \subset J_0$ with $V(J) \cap V(I) = \{\mathfrak{m}\}$. Moreover, for such a choice we have a distinguished triangle

$$R\Gamma_J(M) \to M \to R\Gamma(V, \mathcal{F}) \to R\Gamma_J(M)[1]$$

We similarly have a distinguished triangle

$$R\Gamma_{\mathfrak{m}}(M)^\wedge \to M \to R\Gamma(U, \mathcal{F})^\wedge \to R\Gamma_{\mathfrak{m}}(M)^\wedge[1]$$

involving derived I-adic completions, see proof of Lemma 13.2. The cohomology groups of $R\Gamma(U, \mathcal{F})^\wedge$ are equal to the limits in the statement of the theorem by Lemma 13.1. The canonical map between these triangles and some easy arguments show that our theorem follows from the main Lemma 13.7 (note that we have $i < s$ here whereas we have $i \leq s$ in the lemma; this is because of the shift). The finiteness of the cohomology groups (under the additional assumption) follows from Lemma 13.6. □

0DXR **Lemma 13.9.** *Let $(A, \mathfrak{m})$ be a Noetherian local ring which has a dualizing complex and is complete with respect to an ideal I. Set $X = \mathrm{Spec}(A)$, $Y = V(I)$, and $U = X \setminus \{\mathfrak{m}\}$. Let $\mathcal{F}$ be a coherent sheaf on U. Assume*

1. *$cd(A, I) \leq d$,*
2. *for any $x \in U$ which is an associated point of $\mathcal{F}$ we have $\dim(\overline{\{x\}}) > d + 1$.*

Then the map

$$\mathrm{colim}\, H^0(V, \mathcal{F}) \longrightarrow \lim H^0(U, \mathcal{F}/I^n\mathcal{F})$$

is an isomorphism of finite A-modules where the colimit is over opens $V \subset U$ containing $Y \cap U$.

Proof. Apply Theorem 13.8 with $s = 1$ (we get finiteness too). □

14. Algebraization of coherent formal modules

0DXS Let $(A, \mathfrak{m})$ be a Noetherian local ring. Let $I \subset A$ be an ideal. Let

$$X = \mathrm{Spec}(A) \supset U = \mathrm{Spec}(A) \setminus \{\mathfrak{m}\}$$

and denote $Y = V(I)$ the closed subscheme corresponding to I. In this section we consider inverse systems of coherent $\mathcal{O}_U$-modules $(\mathcal{F}_n)$ with $\mathcal{F}_n$ annihilated by I^n such that the transition maps induce isomorphisms $\mathcal{F}_{n+1}/I^n\mathcal{F}_{n+1} \to \mathcal{F}_n$. The category of these systems was denoted

$$\textit{Coh}(U, I\mathcal{O}_U)$$

in Cohomology of Schemes, Section 23. This category is equivalent to the category of coherent modules on the formal completion of U along Y; however, since we have not yet introduced formal schemes or coherent modules on them, we cannot use this terminology here.

0DXT **Lemma 14.1.** *With $A, \mathfrak{m}, I, X, U$ as above. Consider an inverse system (M_n) of finite A-modules such that M_n is annihilated by I^n and the kernel and cokernel of $M_{n+1}/I^nM_{n+1} \to M_n$ have finite length. Then $\widetilde{M_n}|_U$ is in $\textit{Coh}(U, I\mathcal{O}_U)$. Conversely, every object of $\textit{Coh}(U, I\mathcal{O}_U)$ is of this form.*

Proof. Omitted, but see Lemma 4.2. □

If A is I-adically complete, then an important question is whether the completion functor Cohomology of Schemes, Equation (23.4.1)

$$Coh(\mathcal{O}_U) \longrightarrow Coh(U, I\mathcal{O}_U), \quad \mathcal{F} \longmapsto \mathcal{F}^\wedge$$

is essentially surjective. Fully faithfullness of this functor is often a consequence of the results in Section 13 applied to suitable $\mathcal{H}om$'s. The essential surjectivity of the completion functor was studied systematically in [Gro68], [Ray75], and [Ray74]. We will discuss this material (insert future reference here). In this section we discuss only the case where the closed subset Y is cut out by a single nonzerodivisor and we only deal with algebraization of formal vector bundles.

0DXU **Lemma 14.2.** *With $A, \mathfrak{m}, I, X, U$ as above let $(\mathcal{F}_n)$ be an object of $Coh(U, I\mathcal{O}_U)$. Assume*

1. *A has a dualizing complex and is complete with respect to I,*
2. *$I = (f)$ is a principal ideal for a nonzerodivisor $f \in \mathfrak{m}$,*
3. *$\mathcal{F}_n$ is a finite locally free $\mathcal{O}_U/f^n\mathcal{O}_U$-module,*
4. *if $\mathfrak{p} \in V(f) \setminus \{\mathfrak{m}\}$, then depth$((A/f)_\mathfrak{p}) + \dim(A/\mathfrak{p}) > 1$, and*
5. *if $\mathfrak{p} \notin V(f)$ and $V(\mathfrak{p}) \cap V(f) \neq \{\mathfrak{m}\}$, then depth$(A_\mathfrak{p}) + \dim(A/\mathfrak{p}) > 3$.*

Then there exists a coherent $\mathcal{O}_U$-module $\mathcal{F}$ such that $(\mathcal{F}_n)$ is the completion of $\mathcal{F}$.

Proof. By induction on n and the short exact sequences $0 \to A/f^n \to A/f^{n+1} \to A/f \to 0$ we see that the associate primes of A/f^nA agree with the associated primes of A/fA. Since the associated points of $\mathcal{F}_n$ correspond to the associated primes of A/f^nA not equal to $\mathfrak{m}$ by condition (3), we conclude that $M_n = H^0(U, \mathcal{F}_n)$ is a finite A-module by (4) and Proposition 4.7.

We claim that for any $n > 0$ and $m \gg n$ the image of

$$H^1(U, \mathcal{F}_m) \longrightarrow H^1(U, \mathcal{F}_n)$$

has finite length as an A-module. The image is independent of m for m large enough by Lemma 5.5. Let $\omega_A^\bullet$ be a normalized dualizing complex for A. By the local duality theorem and Matlis duality (Dualizing Complexes, Lemma 18.4 and Proposition 7.8) our claim is equivalent to: the image of

$$\mathrm{Ext}^{-2}_A(M_n, \omega_A^\bullet) \to \mathrm{Ext}^{-2}_A(M_m, \omega_A^\bullet)$$

has finite length for $m \gg n$. The modules in question are finite A-modules supported at $V(f)$. Thus it suffices to show that this map is zero after localization at a prime $\mathfrak{q}$ containing f and different from $\mathfrak{m}$. Let $\omega_{A_\mathfrak{q}}^\bullet$ be a normalized dualizing complex on $A_\mathfrak{q}$ and recall that $\omega_{A_\mathfrak{q}}^\bullet = (\omega_A^\bullet)_\mathfrak{q}[\dim(A/\mathfrak{q})]$ by Dualizing Complexes, Lemma 17.3. Using the local structure of $\mathcal{F}_n$ given in (3) we find that it suffices to show the vanishing of

$$\mathrm{Ext}^{-2+\dim(A/\mathfrak{q})}_{A_\mathfrak{q}}(A_\mathfrak{q}/f^n, \omega_{A_\mathfrak{q}}^\bullet) \to \mathrm{Ext}^{-2+\dim(A/\mathfrak{q})}_{A_\mathfrak{q}}(A_\mathfrak{q}/f^m, \omega_{A_\mathfrak{q}}^\bullet)$$

If $\dim(A/\mathfrak{q}) > 3$, then this is immediate from Lemma 5.3. We will use the long exact sequence

$$\ldots \xrightarrow{f^m} H^{-1}(\omega_{A_\mathfrak{q}}^\bullet) \to \mathrm{Ext}^{-1}_{A_\mathfrak{q}}(A_\mathfrak{q}/f^m, \omega_{A_\mathfrak{q}}^\bullet) \to H^0(\omega_{A_\mathfrak{q}}^\bullet) \xrightarrow{f^m} H^0(\omega_{A_\mathfrak{q}}^\bullet) \to \mathrm{Ext}^0_{A_\mathfrak{q}}(A_\mathfrak{q}/f^m, \omega_{A_\mathfrak{q}}^\bullet) \to 0$$

If $\dim(A/\mathfrak{q}) = 2$, then $H^0(\omega_{A_\mathfrak{q}}^\bullet) = 0$ as the depth of $A_\mathfrak{q}$ is zero by dint of f being a nonzerodivisor. Thus the long exact sequence shows the condition is that

$$f^{m-n} : H^{-1}(\omega_{A_\mathfrak{q}}^\bullet)/f^n \to H^{-1}(\omega_{A_\mathfrak{q}}^\bullet)/f^m$$

is zero. Now $H^{-1}(\omega_\mathfrak q^\bullet)$ is a finite module supported in the primes $\mathfrak p \subset A_\mathfrak q$ such that $\text{depth}(A_\mathfrak p) + \dim((A/\mathfrak p)_\mathfrak q) \leq 1$. By condition (5) all of these primes are contained in $V(f)$. Thus the desired vanishing for m large enough. If $\dim(A/\mathfrak q) = 1$, then condition (4) combined with the fact that f is a nonzerodivisor insures that $A_\mathfrak q$ has depth at least 2. Hence $H^0(\omega_{A_\mathfrak q}^\bullet) = H^{-1}(\omega_{A_\mathfrak q}^\bullet) = 0$ and the long exact sequence shows the claim is equivalent to the vanishing of

$$f^{m-n} : H^{-2}(\omega_{A_\mathfrak q}^\bullet)/f^n \to H^{-2}(\omega_{A_\mathfrak q}^\bullet)/f^m$$

Now $H^{-2}(\omega_\mathfrak q^\bullet)$ is a finite module supported in the primes $\mathfrak p \subset A_\mathfrak q$ such that $\text{depth}(A_\mathfrak p)+\dim((A/\mathfrak p)_\mathfrak q) \leq 2$. By condition (5) all of these primes are contained in $V(f)$. Thus the desired vanishing for m large enough proving the claim.

By Lemmas 10.1 and 10.3 the system of modules (M_n) satisfies the Mittag-Leffler condition, $M = \lim M_n$ is a finite A-module, f is a nonzerodivisor on M and that $M/fM \subset M_1$. To finish the proof, we will show that $M/f^nM \to M_n$ is an isomorphism after localizing at any prime $\mathfrak q \in V(f)$, $\mathfrak q \neq \mathfrak m$. Namely, by the Mittag-Leffler condition, we know that $M/fM \subset M_1$ is the image of $M_m \to M_1$ for some $m \gg 1$. Since the cokernel of $M_m \to M_1$ is contained in $H^1(U, \mathcal F_{m-1})$ which is $\mathfrak m$-power torsion, we conclude that $M/fM \to M_1$ becomes an isomorphism after localizing at $\mathfrak q$. Using induction and suitable short exact sequences the reader conlcudes the same is true for $M/f^nM \to M_n$. □

0DXV **Remark 14.3.** Let $(A, \mathfrak m)$ be a complete Noetherian normal local domain of dimension ≥ 4 and let $f \in \mathfrak m$ be nonzero. Then assumptions (1), (2), (4), (5) of Lemma 14.2 are satisfied. Thus vectorbundles on the formal completion of U along $U \cap V(f)$ can be algebraized.

0DXW **Lemma 14.4.** *With $A, \mathfrak m, I, X, U$ as above let $(\mathcal F_n)$ be an object of $Coh(U, I\mathcal O_U)$. Assume*

1. *$I = (f)$ is a principal ideal for a nonzerodivisor $f \in \mathfrak m$,*
2. *A is complete with respect to $I = (f)$,*
3. *$\mathcal F_n$ is a finite locally free $\mathcal O_U/f^n\mathcal O_U$-module,*
4. *$H^1_\mathfrak m(A/fA)$ and $H^2_\mathfrak m(A/fA)$ are finite A-modules.*

Then there exists a coherent $\mathcal O_U$-module $\mathcal F$ such that $(\mathcal F_n)$ is the completion of $\mathcal F$.

Proof. This lemma is a variant of Lemma 14.2 and if A is a complete local ring, then it follows from that lemma[2]. We suggest the reader skip the proof.

As f is a nonzerodivisor we obtain short exact sequences

$$0 \to A/f^nA \xrightarrow{f} A/f^{n+1}A \to A/fA \to 0$$

and we have corresponding short exact sequences $0 \to \mathcal F_n \to \mathcal F_{n+1} \to \mathcal F_1 \to 0$. We will use Lemma 4.2 without further mention. Our assumptions imply that $H^0(U, \mathcal O_U/f\mathcal O_U)$ and $H^1(U, \mathcal O_U/f\mathcal O_U)$ are finite A-modules. Hence the same thing is true for $\mathcal F_1$, see Lemma 8.2. Thus $H^0(U, \mathcal F_1)$ is a finite A-module and $H^1(U, \mathcal F_1)$ has finite length (as a finite A-module which is $\mathfrak m$-power torsion). Thus Lemmas 10.1 and 10.2 apply to the system above. Setting $M_n = \Gamma(U, \mathcal F_n)$ we find the

[2]Namely, the condition that $H^1_\mathfrak m(A/fA)$ and $H^2_\mathfrak m(A/fA)$ are finite A-modules, is equivalent with $\text{depth}((A/f)_\mathfrak q) + \dim(A/\mathfrak q) > 2$ for all $\mathfrak q \in V(f)$, $\mathfrak q \neq \mathfrak m$ by Theorem 7.4. As f is a nonzerodivisor for such a prime $\text{depth}(A_\mathfrak q) + \dim(A/\mathfrak q) > 3$. The locus of these primes is open by Lemma 5.3. Hence assumption (5) of Lemma 14.2 follows from condition (4) of this lemma.

system of modules (M_n) satisfies the Mittag-Leffler condition, $M = \lim M_n$ is a finite A-module, f is a nonzerodivisor on M and that $M/fM \subset M_1$. To finish the proof, we will show that $M/f^nM \to M_n$ is an isomorphism after localizing at any prime $\mathfrak{q} \in V(f)$, $\mathfrak{q} \neq \mathfrak{m}$. Namely, by the Mittag-Leffler condition, we know that $M/fM \subset M_1$ is the image of $M_m \to M_1$ for some $m \gg 1$. Since the cokernel of $M_m \to M_1$ is contained in $H^1(U, \mathcal{F}_{m-1})$ which is $\mathfrak{m}$-power torsion, we conclude that $M/fM \to M_1$ becomes an isomorphism after localizing at $\mathfrak{q}$. Using induction and suitable short exact sequences the reader conlcude the same is true for $M/f^nM \to M_n$. □

15. Other chapters

References

[BdJ14] Bhargav Bhatt and Aise Johan de Jong, *Lefschetz for local Picard groups*, Ann. Sci. Éc. Norm. Supér. (4) **47** (2014), no. 4, 833–849.

[BS13] Bhargav Bhatt and Peter Scholze, *The pro-étale topology for schemes*, preprint, 2013.

[DG67] Jean Dieudonné and Alexander Grothendieck, *Éléments de géométrie algébrique*, Inst. Hautes Études Sci. Publ. Math. **4, 8, 11, 17, 20, 24, 28, 32** (1961–1967).

[Fal78] Gerd Faltings, *Über die Annulatoren lokaler Kohomologiegruppen*, Arch. Math. (Basel) **30** (1978), no. 5, 473–476.

[Fal79] ———, *Algebraisation of some formal vector bundles*, Ann. of Math. (2) **110** (1979), no. 3, 501–514.

[Fal80] ———, *über lokale Kohomologiegruppen hoher Ordnung*, J. Reine Angew. Math. **313** (1980), 43–51.

[Fal81] ———, *Der Endlichkeitssatz in der lokalen Kohomologie*, Math. Ann. **255** (1981), no. 1, 45–56.

[GR13] Dennis Gaitsgory and Nick Rozenblyum, *Dg indschemes*, 2013.

[Gro68] Alexander Grothendieck, *Cohomologie locale des faisceaux cohérents et théorèmes de Lefschetz locaux et globaux* (*SGA* 2), Advanced Studies in Pure Mathematics, vol. 2, North-Holland Publishing Co., 1968, Augmenté d'un exposé par Michèle Raynaud, Séminaire de Géométrie Algébrique du Bois-Marie, 1962.

[HLP14] Daniel Halpern-Leistner and Anatoly Preygel, *Mapping stacks and categorical notions of properness*, 2014.

[Kol15] János Kollár, *Coherence of local and global hulls*, ArXiv e-prints (2015).

[Kol16] ———, *Variants of normality for Noetherian schemes*, Pure Appl. Math. Q. **12** (2016), no. 1, 1–31.

[Lur04] Jacob Lurie, *Derived algebraic geometry*, 2004.

[Lur11] ———, *Derived algebraic geometry XII: Proper morphisms, completions, and the grothendieck existence theorem*.

[Ray74] Michèle Raynaud, *Théoremes de Lefschetz en cohomologie des faisceaux cohérents et en cohomologie étale. Application au groupe fondamental*, Ann. Sci. École Norm. Sup. (4) **7** (1974), 29–52, Collection of articles dedicated to Henri Cartan on the occasion of his 70th birthday, I.

[Ray75] ———, *Théorèmes de Lefschetz en cohomologie cohérente et en cohomologie étale*, Société Mathématique de France, Paris, 1975, Bull. Soc. Math. France, Mém. No. 41, Supplément au Bull. Soc. Math. France, Tome 103.