

# LOCAL COHOMOLOGY

0DWN

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## 1. Introduction

0DWP This chapter continues the study of local cohomology. A reference is [Gro68]. The definition of local cohomology can be found in Dualizing Complexes, Section 9. For Noetherian rings taking local cohomology is the same as deriving a suitable torsion functor as is shown in Dualizing Complexes, Section 10. The relationship with depth can be found in Dualizing Complexes, Section 11.

We discuss finiteness properties of local cohomology leading to a proof of a fairly general version of Grothendieck's finiteness theorem, see Theorem 10.6 and Lemma 11.1 (higher direct images of coherent modules under open immersions). Our methods incorporate a few very slick arguments the reader can find in papers of Faltings, see [Fal78] and [Fal81].

As applications we offer a discussion of Hartshorne-Lichtenbaum vanishing. We also discuss the action of Frobenius and of differential operators on local cohomology.

## 2. Generalities

0DWQ The following lemma tells us that the functor  $R\Gamma_Z$  is related to cohomology with supports.

0A6T **Lemma 2.1.** *Let  $A$  be a ring and let  $I$  be a finitely generated ideal. Set  $Z = V(I) \subset X = \text{Spec}(A)$ . For  $K \in D(A)$  corresponding to  $\tilde{K} \in D_{QCoh}(\mathcal{O}_X)$  via Derived Categories of Schemes, Lemma 3.5 there is a functorial isomorphism*

$$R\Gamma_Z(K) = R\Gamma_Z(X, \tilde{K})$$

where on the left we have Dualizing Complexes, Equation (9.0.1) and on the right we have the functor of Cohomology, Section 22.

**Proof.** By Cohomology, Section 22 there exists a distinguished triangle

$$R\Gamma_Z(X, \tilde{K}) \rightarrow R\Gamma(X, \tilde{K}) \rightarrow R\Gamma(U, \tilde{K}) \rightarrow R\Gamma_Z(X, \tilde{K})[1]$$

where  $U = X \setminus Z$ . We know that  $R\Gamma(X, \tilde{K}) = K$  by Derived Categories of Schemes, Lemma 3.5. Say  $I = (f_1, \dots, f_r)$ . Then we obtain a finite affine open covering  $\mathcal{U} : U = D(f_1) \cup \dots \cup D(f_r)$ . By Derived Categories of Schemes, Lemma 8.4 the alternating Čech complex  $\text{Tot}(\tilde{\mathcal{C}}_{alt}^\bullet(\mathcal{U}, \tilde{K}^\bullet))$  computes  $R\Gamma(U, \tilde{K})$  where  $K^\bullet$  is any complex of  $A$ -modules representing  $K$ . Working through the definitions we find

$$R\Gamma(U, \tilde{K}) = \text{Tot} \left( K^\bullet \otimes_A \left( \prod_{i_0} A_{f_{i_0}} \rightarrow \prod_{i_0 < i_1} A_{f_{i_0} f_{i_1}} \rightarrow \dots \rightarrow A_{f_1 \dots f_r} \right) \right)$$

It is clear that  $K^\bullet = R\Gamma(X, \tilde{K}^\bullet) \rightarrow R\Gamma(U, \tilde{K}^\bullet)$  is induced by the diagonal map from  $A$  into  $\prod A_{f_i}$ . Hence we conclude that

$$R\Gamma_Z(X, \mathcal{F}^\bullet) = \text{Tot} \left( K^\bullet \otimes_A \left( A \rightarrow \prod_{i_0} A_{f_{i_0}} \rightarrow \prod_{i_0 < i_1} A_{f_{i_0} f_{i_1}} \rightarrow \dots \rightarrow A_{f_1 \dots f_r} \right) \right)$$

By Dualizing Complexes, Lemma 9.1 this complex computes  $R\Gamma_Z(K)$  and we see the lemma holds.  $\square$

0DWR **Lemma 2.2.** *Let  $A$  be a ring and let  $I \subset A$  be a finitely generated ideal. Set  $X = \text{Spec}(A)$ ,  $Z = V(I)$ ,  $U = X \setminus Z$ , and  $j : U \rightarrow X$  the inclusion morphism. Let  $\mathcal{F}$  be a quasi-coherent  $\mathcal{O}_U$ -module. Then*

- (1) *there exists an  $A$ -module  $M$  such that  $\mathcal{F}$  is the restriction of  $\tilde{M}$  to  $U$ ,*
- (2) *given  $M$  there is an exact sequence*

$$0 \rightarrow H_Z^0(M) \rightarrow M \rightarrow H^0(U, \mathcal{F}) \rightarrow H_Z^1(M) \rightarrow 0$$

*and isomorphisms  $H^p(U, \mathcal{F}) = H_Z^{p+1}(M)$  for  $p \geq 1$ ,*

- (3) *we may take  $M = H^0(U, \mathcal{F})$  in which case we have  $H_Z^0(M) = H_Z^1(M) = 0$ .*

**Proof.** The existence of  $M$  follows from Properties, Lemma 22.1 and the fact that quasi-coherent sheaves on  $X$  correspond to  $A$ -modules (Schemes, Lemma 7.5). Then we look at the distinguished triangle

$$R\Gamma_Z(X, \tilde{M}) \rightarrow R\Gamma(X, \tilde{M}) \rightarrow R\Gamma(U, \tilde{M}|_U) \rightarrow R\Gamma_Z(X, \tilde{M})[1]$$

of Cohomology, Section 22. Since  $X$  is affine we have  $R\Gamma(X, \tilde{M}) = M$  by Cohomology of Schemes, Lemma 2.2. By our choice of  $M$  we have  $\mathcal{F} = \tilde{M}|_U$  and hence this produces an exact sequence

$$0 \rightarrow H_Z^0(X, \tilde{M}) \rightarrow M \rightarrow H^0(U, \mathcal{F}) \rightarrow H_Z^1(X, \tilde{M}) \rightarrow 0$$

and isomorphisms  $H^p(U, \mathcal{F}) = H_Z^{p+1}(X, \widetilde{M})$  for  $p \geq 1$ . By Lemma 2.1 we have  $H_Z^i(M) = H_Z^i(X, \widetilde{M})$  for all  $i$ . Thus (1) and (2) do hold. Finally, setting  $M' = H^0(U, \mathcal{F})$  we see that the kernel and cokernel of  $M \rightarrow M'$  are  $I$ -power torsion. Therefore  $\widetilde{M}|_U \rightarrow \widetilde{M}'|_U$  is an isomorphism and we can indeed use  $M'$  as predicted in (3). It goes without saying that we obtain zero for both  $H_Z^0(M')$  and  $H_Z^0(M')$ .  $\square$

0DWS **Lemma 2.3.** *Let  $I, J \subset A$  be finitely generated ideals of a ring  $A$ . If  $M$  is an  $I$ -power torsion module, then the canonical map*

$$H_{V(I) \cap V(J)}^i(M) \rightarrow H_{V(J)}^i(M)$$

*is an isomorphism for all  $i$ .*

**Proof.** Use the spectral sequence of Dualizing Complexes, Lemma 9.6 to reduce to the statement  $R\Gamma_I(M) = M$  which is immediate from the construction of local cohomology in Dualizing Complexes, Section 9.  $\square$

0DWT **Lemma 2.4.** *Let  $S \subset A$  be a multiplicative set of a ring  $A$ . Let  $M$  be an  $A$ -module with  $S^{-1}M = 0$ . Then  $\operatorname{colim}_{f \in S} H_{V(f)}^0(M) = M$  and  $\operatorname{colim}_{f \in S} H_{V(f)}^1(M) = 0$ .*

**Proof.** The statement on  $H^0$  follows directly from the definitions. To see the statement on  $H^1$  observe that  $R\Gamma_{V(f)}$  and  $H_{V(f)}^1$  commute with colimits. Hence we may assume  $M$  is annihilated by some  $f \in S$ . Then  $H_{V(ff')}^1(M) = 0$  for all  $f' \in S$  (for example by Lemma 2.3).  $\square$

0DWU **Lemma 2.5.** *Let  $I \subset A$  be a finitely generated ideal of a ring  $A$ . Let  $\mathfrak{p}$  be a prime ideal. Let  $M$  be an  $A$ -module. Let  $i \geq 0$  be an integer and consider the map*

$$\Psi : \operatorname{colim}_{f \in A, f \notin \mathfrak{p}} H_{V((I, f))}^i(M) \longrightarrow H_{V(I)}^i(M)$$

*Then*

- (1)  $\operatorname{Im}(\Psi)$  is the set of elements which map to zero in  $H_{V(I)}^i(M)_{\mathfrak{p}}$ ,
- (2) if  $H_{V(I)}^{i-1}(M)_{\mathfrak{p}} = 0$ , then  $\Psi$  is injective,
- (3) if  $H_{V(I)}^{i-1}(M)_{\mathfrak{p}} = H_{V(I)}^i(M)_{\mathfrak{p}} = 0$ , then  $\Psi$  is an isomorphism.

**Proof.** For  $f \in A, f \notin \mathfrak{p}$  the spectral sequence of Dualizing Complexes, Lemma 9.6 degenerates to give short exact sequences

$$0 \rightarrow H_{V(f)}^1(H_{V(I)}^{i-1}(M)) \rightarrow H_{V((I, f))}^i(M) \rightarrow H_{V(f)}^0(H_{V(I)}^i(M)) \rightarrow 0$$

This proves (1) and part (2) follows from this and Lemma 2.4. Part (3) is a formal consequence.  $\square$

0DWV **Lemma 2.6.** *Let  $I \subset I' \subset A$  be finitely generated ideals of a Noetherian ring  $A$ . Let  $M$  be an  $A$ -module. Let  $i \geq 0$  be an integer. Consider the map*

$$\Psi : H_{V(I')}^i(M) \rightarrow H_{V(I)}^i(M)$$

*The following are true:*

- (1) if  $H_{\mathfrak{p}A_{\mathfrak{p}}}^i(M_{\mathfrak{p}}) = 0$  for all  $\mathfrak{p} \in V(I) \setminus V(I')$ , then  $\Psi$  is surjective,
- (2) if  $H_{\mathfrak{p}A_{\mathfrak{p}}}^{i-1}(M_{\mathfrak{p}}) = 0$  for all  $\mathfrak{p} \in V(I) \setminus V(I')$ , then  $\Psi$  is injective,
- (3) if  $H_{\mathfrak{p}A_{\mathfrak{p}}}^i(M_{\mathfrak{p}}) = H_{\mathfrak{p}A_{\mathfrak{p}}}^{i-1}(M_{\mathfrak{p}}) = 0$  for all  $\mathfrak{p} \in V(I) \setminus V(I')$ , then  $\Psi$  is an isomorphism.

**Proof.** Proof of (1). Let  $\xi \in H_{V(I)}^i(M)$ . Since  $A$  is Noetherian, there exists a largest ideal  $I \subset I'' \subset I'$  such that  $\xi$  is the image of some  $\xi'' \in H_{V(I'')}^i(M)$ . If  $V(I'') = V(I')$ , then we are done. If not, choose a generic point  $\mathfrak{p} \in V(I'')$  not in  $V(I')$ . Then we have  $H_{V(I'')}^i(M)_{\mathfrak{p}} = H_{\mathfrak{p}A_{\mathfrak{p}}}^i(M_{\mathfrak{p}}) = 0$  by assumption. By Lemma 2.5 we can increase  $I''$  which contradicts maximality.

Proof of (2). Let  $\xi' \in H_{V(I')}^i(M)$  be in the kernel of  $\Psi$ . Since  $A$  is Noetherian, there exists a largest ideal  $I \subset I'' \subset I'$  such that  $\xi'$  maps to zero in  $H_{V(I'')}^i(M)$ . If  $V(I'') = V(I')$ , then we are done. If not, then choose a generic point  $\mathfrak{p} \in V(I'')$  not in  $V(I')$ . Then we have  $H_{V(I'')}^{i-1}(M)_{\mathfrak{p}} = H_{\mathfrak{p}A_{\mathfrak{p}}}^{i-1}(M_{\mathfrak{p}}) = 0$  by assumption. By Lemma 2.5 we can increase  $I''$  which contradicts maximality.

Part (3) is formal from parts (1) and (2). □

### 3. Cohomological dimension

ODX6 A quick section about cohomological dimension.

ODX7 **Lemma 3.1.** *Let  $I \subset A$  be a finitely generated ideal of a ring  $A$ . Set  $Y = V(I) \subset X = \text{Spec}(A)$ . Let  $d \geq -1$  be an integer. The following are equivalent*

- (1)  $H_Y^i(A) = 0$  for  $i > d$ ,
- (2)  $H_Y^i(M) = 0$  for  $i > d$  for every  $A$ -module  $M$ , and
- (3) if  $d = -1$ , then  $Y = \emptyset$ , if  $d = 0$ , then  $Y$  is open and closed in  $X$ , and if  $d > 0$  then  $H^i(X \setminus Y, \mathcal{F}) = 0$  for  $i \geq d$  for every quasi-coherent  $\mathcal{O}_{X \setminus Y}$ -module  $\mathcal{F}$ .

**Proof.** Observe that  $R\Gamma_Y(-)$  has finite cohomological dimension by Dualizing Complexes, Lemma 9.1 for example. Hence we can choose a large integer  $N$  such that  $H_Y^i(M) = 0$  for all  $A$ -modules  $M$ .

Let us prove that (1) and (2) are equivalent. It is immediate that (2) implies (1). Assume (1). Choose any  $A$ -module  $M$  and fit it into a short exact sequence  $0 \rightarrow N \rightarrow F \rightarrow M \rightarrow 0$  where  $F$  is a free  $A$ -module. Since  $R\Gamma_Y$  is a right adjoint, we see that  $H_Y^i(-)$  commutes with direct sums. Hence  $H_Y^i(F) = 0$  for  $i > d$  by assumption (1). Then we see that  $H_Y^i(M) = H_Y^{i+1}(N)$  for all  $i > d$ . Thus if we've shown the vanishing of  $H_Y^j(N)$  for some  $j > d + 1$  and all  $A$ -modules  $N$ , then we obtain the vanishing of  $H_Y^{j-1}(M)$  for all  $A$ -modules  $M$ . By induction we find that (2) is true.

Assume  $d = -1$  and (2) holds. Then  $0 = H_Y^0(A/I) = A/I \Rightarrow A = I \Rightarrow Y = \emptyset$ . Thus (3) holds. We omit the proof of the converse.

Assume  $d = 0$  and (2) holds. Set  $J = H_I^0(A) = \{x \in A \mid I^n x = 0 \text{ for some } n > 0\}$ . Then

$$H_Y^1(A) = \text{Coker}(A \rightarrow \Gamma(X \setminus Y, \mathcal{O}_{X \setminus Y})) \quad \text{and} \quad H_Y^1(I) = \text{Coker}(I \rightarrow \Gamma(X \setminus Y, \mathcal{O}_{X \setminus Y}))$$

and the kernel of the first map is equal to  $J$ . See Lemma 2.2. We conclude from (2) that  $I(A/J) = A/J$ . Thus we may pick  $f \in I$  mapping to 1 in  $A/J$ . Then  $1 - f \in J$  so  $I^n(1 - f) = 0$  for some  $n > 0$ . Hence  $f^n = f^{n+1}$ . Then  $e = f^n \in I$  is an idempotent. Consider the complementary idempotent  $e' = 1 - f^n \in J$ . For any element  $g \in I$  we have  $g^m e' = 0$  for some  $m > 0$ . Thus  $I$  is contained in the radical of ideal  $(e) \subset I$ . This means  $Y = V(I) = V(e)$  is open and closed in  $X$

as predicted in (3). Conversely, if  $Y = V(I)$  is open and closed, then the functor  $H_Y^0(-)$  is exact and has vanishing higher derived functors.

If  $d > 0$ , then we see immediately from Lemma 2.2 that (2) is equivalent to (3).  $\square$

0DX8 **Definition 3.2.** Let  $I \subset A$  be a finitely generated ideal of a ring  $A$ . The smallest integer  $d \geq -1$  satisfying the equivalent conditions of Lemma 3.1 is called the *cohomological dimension of  $I$  in  $A$*  and is denoted  $\text{cd}(A, I)$ .

Thus we have  $\text{cd}(A, I) = -1$  if  $I = A$  and  $\text{cd}(A, I) = 0$  if  $I$  is locally nilpotent or generated by an idempotent. Observe that  $\text{cd}(A, I)$  exists by the following lemma.

0DX9 **Lemma 3.3.** Let  $I \subset A$  be a finitely generated ideal of a ring  $A$ . Then

- (1)  $\text{cd}(A, I)$  is at most equal to the number of generators of  $I$ ,
- (2)  $\text{cd}(A, I) \leq r$  if there exist  $f_1, \dots, f_r \in A$  such that  $V(f_1, \dots, f_r) = V(I)$ ,
- (3)  $\text{cd}(A, I) \leq c$  if  $\text{Spec}(A) \setminus V(I)$  can be covered by  $c$  affine opens.

**Proof.** The explicit description for  $R\Gamma_Y(-)$  given in Dualizing Complexes, Lemma 9.1 shows that (1) is true. We can deduce (2) from (1) using the fact that  $R\Gamma_Z$  depends only on the closed subset  $Z$  and not on the choice of the finitely generated ideal  $I \subset A$  with  $V(I) = Z$ . This follows either from the construction of local cohomology in Dualizing Complexes, Section 9 combined with More on Algebra, Lemma 79.6. or it follows from Lemma 2.1. To see (3) we use Lemma 3.1 and the vanishing result of Cohomology of Schemes, Lemma 4.2.  $\square$

0ECP **Lemma 3.4.** Let  $I, J \subset A$  be finitely generated ideals of a ring  $A$ . Then  $\text{cd}(A, I + J) \leq \text{cd}(A, I) + \text{cd}(A, J)$ .

**Proof.** Use the definition and Dualizing Complexes, Lemma 9.6.  $\square$

0DXA **Lemma 3.5.** Let  $A \rightarrow B$  be a ring map. Let  $I \subset A$  be a finitely generated ideal. Then  $\text{cd}(B, IB) \leq \text{cd}(A, I)$ . If  $A \rightarrow B$  is faithfully flat, then equality holds.

**Proof.** Use the definition and Dualizing Complexes, Lemma 9.3.  $\square$

0DXB **Lemma 3.6.** Let  $I \subset A$  be a finitely generated ideal of a ring  $A$ . Then  $\text{cd}(A, I) = \max \text{cd}(A_{\mathfrak{p}}, I_{\mathfrak{p}})$ .

**Proof.** Let  $Y = V(I)$  and  $Y' = V(I_{\mathfrak{p}}) \subset \text{Spec}(A_{\mathfrak{p}})$ . Recall that  $R\Gamma_Y(A) \otimes_A A_{\mathfrak{p}} = R\Gamma_{Y'}(A_{\mathfrak{p}})$  by Dualizing Complexes, Lemma 9.3. Thus we conclude by Algebra, Lemma 22.1.  $\square$

0DXC **Lemma 3.7.** Let  $I \subset A$  be a finitely generated ideal of a ring  $A$ . If  $M$  is a finite  $A$ -module, then  $H_{V(I)}^i(M) = 0$  for  $i > \dim(\text{Supp}(M))$ . In particular, we have  $\text{cd}(A, I) \leq \dim(A)$ .

**Proof.** We first prove the second statement. Recall that  $\dim(A)$  denotes the Krull dimension. By Lemma 3.6 we may assume  $A$  is local. If  $V(I) = \emptyset$ , then the result is true. If  $V(I) \neq \emptyset$ , then  $\dim(\text{Spec}(A) \setminus V(I)) < \dim(A)$  because the closed point is missing. Observe that  $U = \text{Spec}(A) \setminus V(I)$  is a quasi-compact open of the spectral space  $\text{Spec}(A)$ , hence a spectral space itself. See Algebra, Lemma 25.2 and Topology, Lemma 23.4. Thus Cohomology, Proposition 23.4 implies  $H^i(U, \mathcal{F}) = 0$  for  $i \geq \dim(A)$  which implies what we want by Lemma 3.1. In the Noetherian case the reader may use Grothendieck's Cohomology, Proposition 21.7.

We will deduce the first statement from the second. Let  $\mathfrak{a}$  be the annihilator of the finite  $A$ -module  $M$ . Set  $B = A/\mathfrak{a}$ . Recall that  $\text{Spec}(B) = \text{Supp}(M)$ , see Algebra, Lemma 39.5. Set  $J = IB$ . Then  $M$  is a  $B$ -module and  $H_{V(I)}^i(M) = H_{V(J)}^i(M)$ , see Dualizing Complexes, Lemma 9.2. Since  $\text{cd}(B, J) \leq \dim(B) = \dim(\text{Supp}(M))$  by the first part we conclude.  $\square$

0DXD **Lemma 3.8.** *Let  $I \subset A$  be a finitely generated ideal of a ring  $A$ . If  $\text{cd}(A, I) = 1$  then  $\text{Spec}(A) \setminus V(I)$  is nonempty affine.*

**Proof.** This follows from Lemma 3.1 and Cohomology of Schemes, Lemma 3.1.  $\square$

0DXE **Lemma 3.9.** *Let  $(A, \mathfrak{m})$  be a Noetherian local ring of dimension  $d$ . Then  $H_{\mathfrak{m}}^d(A)$  is nonzero and  $\text{cd}(A, \mathfrak{m}) = d$ .*

**Proof.** By one of the characterizations of dimension, there exists an ideal of definition for  $A$  generated by  $d$  elements, see Algebra, Proposition 59.8. Hence  $\text{cd}(A, \mathfrak{m}) \leq d$  by Lemma 3.3. Thus  $H_{\mathfrak{m}}^d(A)$  is nonzero if and only if  $\text{cd}(A, \mathfrak{m}) = d$  if and only if  $\text{cd}(A, \mathfrak{m}) \geq d$ .

Let  $A \rightarrow A^\wedge$  be the map from  $A$  to its completion. Observe that  $A^\wedge$  is a Noetherian local ring of the same dimension as  $A$  with maximal ideal  $\mathfrak{m}A^\wedge$ . See Algebra, Lemmas 96.6, 96.4, and 96.3 and More on Algebra, Lemma 42.1. By Lemma 3.5 it suffices to prove the lemma for  $A^\wedge$ .

By the previous paragraph we may assume that  $A$  is a complete local ring. Then  $A$  has a normalized dualizing complex  $\omega_A^\bullet$  (Dualizing Complexes, Lemma 22.4). The local duality theorem (in the form of Dualizing Complexes, Lemma 18.4) tells us  $H_{\mathfrak{m}}^d(A)$  is Matlis dual to  $\text{Ext}^{-d}(A, \omega_A^\bullet) = H^{-d}(\omega_A^\bullet)$  which is nonzero for example by Dualizing Complexes, Lemma 16.11.  $\square$

0DXF **Lemma 3.10.** *Let  $(A, \mathfrak{m})$  be a Noetherian local ring. Let  $I \subset A$  be a proper ideal. Let  $\mathfrak{p} \subset A$  be a prime ideal such that  $V(\mathfrak{p}) \cap V(I) = \{\mathfrak{m}\}$ . Then  $\dim(A/\mathfrak{p}) \leq \text{cd}(A, I)$ .*

**Proof.** By Lemma 3.5 we have  $\text{cd}(A, I) \geq \text{cd}(A/\mathfrak{p}, I(A/\mathfrak{p}))$ . Since  $V(I) \cap V(\mathfrak{p}) = \{\mathfrak{m}\}$  we have  $\text{cd}(A/\mathfrak{p}, I(A/\mathfrak{p})) = \text{cd}(A/\mathfrak{p}, \mathfrak{m}/\mathfrak{p})$ . By Lemma 3.9 this is equal to  $\dim(A/\mathfrak{p})$ .  $\square$

0EH1 **Lemma 3.11.** *Let  $I$  be an ideal of a Noetherian ring  $A$ . Set  $d = \text{cd}(A, I)$ . For  $n \geq 1$  choose a finite free resolution*

$$\dots \rightarrow P_n^{-2} \rightarrow P_n^{-1} \rightarrow P_n^0 \rightarrow A/I^n \rightarrow 0$$

*For every  $n \geq 1$  there exists an  $m > n$  such that  $A/I^m \rightarrow A/I^n$  can be represented by a map of complexes  $\alpha^\bullet : P_m^\bullet \rightarrow P_n^\bullet$  with  $\alpha^i = 0$  for  $i < -d$ .*

**Proof.** We assume  $d \geq 1$ ; the proof in the cases  $d = -1, 0$  is omitted. Denote  $d_n^i : P^i \rightarrow P^{i+1}$  the differentials. Set  $M = \text{Im}(d_n^{-d-1} : P_n^{-d-1} \rightarrow P_n^{-d})$ . Denote  $\xi = d_n^{-d-1} \in \text{Hom}_A(P_n^{-d-1}, M)$ . Since  $\xi \circ d_n^{-d-2} = 0$  we see that  $\xi$  defines an element  $\bar{\xi}$  of  $\text{Ext}_A^{d+1}(A/I^n, M)$ . Since  $\text{cd}(A, I) = d$  we have  $0 = H_Z^{d+1}(M) = H_I^{d+1}(M) = \text{colim Ext}_A^{d+1}(A/I^m, M)$ . Here we have used Dualizing Complexes, Lemmas 10.1 and 8.2. Thus we may pick  $m$  such that  $\bar{\xi}$  maps to zero in  $\text{Ext}_A^{d+1}(A/I^m, M)$ . Choose any map of complexes  $\alpha^\bullet : P_m^\bullet \rightarrow P_n^\bullet$  representing  $A/I^m \rightarrow A/I^n$ . The choice of  $m$  implies we can find  $\varphi : P_m^{-d} \rightarrow M$  such that

$$\xi \circ \alpha^{-d-1} = \varphi \circ d_m^{-d-1}$$

If we think of  $\varphi$  as a map  $P_m^{-d} \rightarrow P_n^{-d}$ , then we see that  $d_n^{-d} \circ \varphi = 0$  and  $\alpha^{-d} \circ d_m^{-d-1} = d_n^{-d-1} \circ \alpha^{-d-1} = \varphi \circ d_m^{-d-1}$ . Thus we obtain a map of complexes

$$\begin{array}{ccccccc} \dots & \longrightarrow & P_n^{-d-1} & \longrightarrow & P_n^{-d} & \longrightarrow & P_n^{-d+1} & \longrightarrow & \dots \\ & & \uparrow 0 & & \uparrow \alpha^{-d-\varphi} & & \uparrow \alpha^{-d+1} & & \\ \dots & \longrightarrow & P_m^{-d-1} & \longrightarrow & P_m^{-d} & \longrightarrow & P_m^{-d+1} & \longrightarrow & \dots \end{array}$$

and everything is clear.  $\square$

0EHU **Lemma 3.12.** *Let  $A$  be a Noetherian ring. Let  $I \subset A$  be an ideal. Let  $b : X' \rightarrow X = \text{Spec}(A)$  be the blowing up of  $I$ . If the fibres of  $b$  have dimension  $\leq d-1$ , then  $\text{cd}(A, I) \leq d$ .*

**Proof.** Set  $U = X \setminus V(I)$ . Denote  $j : U \rightarrow X'$  the canonical open immersion, see Divisors, Section 32. Since the exceptional divisor is an effective Cartier divisor (Divisors, Lemma 32.4) we see that  $j$  is affine, see Divisors, Lemma 13.3. Let  $\mathcal{F}$  be a quasi-coherent  $\mathcal{O}_U$ -module. Then  $R^p j_* \mathcal{F} = 0$  for  $p > 0$ , see Cohomology of Schemes, Lemma 2.3. On the other hand, we have  $R^q b_*(j_* \mathcal{F}) = 0$  for  $q \geq d$  by Limits, Lemma 16.4. Thus by the Leray spectral sequence (Cohomology, Lemma 14.8) we conclude that  $R^n(b \circ j)_* \mathcal{F} = 0$  for  $n \geq d$ . Thus  $H^n(U, \mathcal{F}) = 0$  for  $n \geq d$  (by Cohomology, Lemma 14.6). This means that  $\text{cd}(A, I) \leq d$  by definition.  $\square$

#### 4. More general supports

0EEY Let  $A$  be a Noetherian ring. Let  $M$  be an  $A$ -module. Let  $T \subset \text{Spec}(A)$  be a subset stable under specialization (Topology, Definition 19.1). Let us define

$$H_T^0(M) = \text{colim}_{Z \subset T} H_Z^0(M)$$

where the colimit is over the directed partially ordered set of closed subsets  $Z$  of  $\text{Spec}(A)$  contained in  $T^1$ . In other words, an element  $m$  of  $M$  is in  $H_T^0(M) \subset M$  if and only if the support  $V(\text{Ann}_R(m))$  of  $m$  is contained in  $T$ .

0EEZ **Lemma 4.1.** *Let  $A$  be a Noetherian ring. Let  $T \subset \text{Spec}(A)$  be a subset stable under specialization. For an  $A$ -module  $M$  the following are equivalent*

- (1)  $H_T^0(M) = M$ , and
- (2)  $\text{Supp}(M) \subset T$ .

*The category of such  $A$ -modules is a Serre subcategory of the category  $A$ -modules closed under direct sums.*

**Proof.** The equivalence holds because the support of an element of  $M$  is contained in the support of  $M$  and conversely the support of  $M$  is the union of the supports of its elements. The category of these modules is a Serre subcategory (Homology, Definition 9.1) of  $\text{Mod}_A$  by Algebra, Lemma 39.9. We omit the proof of the statement on direct sums.  $\square$

Let  $A$  be a Noetherian ring. Let  $T \subset \text{Spec}(A)$  be a subset stable under specialization. Let us denote  $\text{Mod}_{A,T} \subset \text{Mod}_A$  the Serre subcategory described in Lemma 4.1. Let us denote  $D_T(A) \subset D(A)$  the strictly full saturated triangulated subcategory

<sup>1</sup>Since  $T$  is stable under specialization we have  $T = \bigcup_{Z \subset T} Z$ , see Topology, Lemma 19.3.

of  $D(A)$  (Derived Categories, Lemma 13.1) consisting of complexes of  $A$ -modules whose cohomology modules are in  $\text{Mod}_{A,T}$ . We obtain functors

$$D(\text{Mod}_{A,T}) \rightarrow D_T(A) \rightarrow D(A)$$

See discussion in Derived Categories, Section 13. Denote  $RH_T^0 : D(A) \rightarrow D(\text{Mod}_{A,T})$  the right derived extension of  $H_T^0$ . We will denote

$$R\Gamma_T : D^+(A) \rightarrow D_T^+(A),$$

the composition of  $RH_T^0 : D^+(A) \rightarrow D^+(\text{Mod}_{A,T})$  with  $D^+(\text{Mod}_{A,T}) \rightarrow D_T^+(A)$ . If the dimension of  $A$  is finite<sup>2</sup>, then we will denote

$$R\Gamma_T : D(A) \rightarrow D_T(A)$$

the composition of  $RH_T^0$  with  $D(\text{Mod}_{A,T}) \rightarrow D_T(A)$ .

0EF0 **Lemma 4.2.** *Let  $A$  be a Noetherian ring. Let  $T \subset \text{Spec}(A)$  be a subset stable under specialization. The functor  $RH_T^0$  is the right adjoint to the functor  $D(\text{Mod}_{A,T}) \rightarrow D(A)$ .*

**Proof.** This follows from the fact that the functor  $H_T^0(-)$  is the right adjoint to the inclusion functor  $\text{Mod}_{A,T} \rightarrow \text{Mod}_A$ , see Derived Categories, Lemma 28.5.  $\square$

0EF1 **Lemma 4.3.** *Let  $A$  be a Noetherian ring. Let  $T \subset \text{Spec}(A)$  be a subset stable under specialization. For any object  $K$  of  $D(A)$  we have*

$$H^i(RH_T^0(K)) = \text{colim}_{Z \subset T \text{ closed}} H_Z^i(K)$$

**Proof.** Let  $J^\bullet$  be a K-injective complex representing  $K$ . By definition  $RH_T^0$  is represented by the complex

$$H_T^0(J^\bullet) = \text{colim } H_Z^0(J^\bullet)$$

where the equality follows from our definition of  $H_T^0$ . Since filtered colimits are exact the cohomology of this complex in degree  $i$  is  $\text{colim } H^i(H_Z^0(J^\bullet)) = \text{colim } H_Z^i(K)$  as desired.  $\square$

0EF2 **Lemma 4.4.** *Let  $A$  be a Noetherian ring. Let  $T \subset \text{Spec}(A)$  be a subset stable under specialization. The functor  $D^+(\text{Mod}_{A,T}) \rightarrow D_T^+(A)$  is an equivalence.*

**Proof.** Let  $M$  be an object of  $\text{Mod}_{A,T}$ . Choose an embedding  $M \rightarrow J$  into an injective  $A$ -module. By Dualizing Complexes, Proposition 5.9 the module  $J$  is a direct sum of injective hulls of residue fields. Let  $E$  be an injective hull of the residue field of  $\mathfrak{p}$ . Since  $E$  is  $\mathfrak{p}$ -power torsion we see that  $H_T^0(E) = 0$  if  $\mathfrak{p} \notin T$  and  $H_T^0(E) = E$  if  $\mathfrak{p} \in T$ . Thus  $H_T^0(J)$  is injective as a direct sum of injective hulls (by the proposition) and we have an embedding  $M \rightarrow H_T^0(J)$ . Thus every object  $M$  of  $\text{Mod}_{A,T}$  has an injective resolution  $M \rightarrow J^\bullet$  with  $J^n$  also in  $\text{Mod}_{A,T}$ . It follows that  $RH_T^0(M) = M$ .

Next, suppose that  $K \in D_T^+(A)$ . Then the spectral sequence

$$R^q H_T^0(H^p(K)) \Rightarrow R^{p+q} H_T^0(K)$$

(Derived Categories, Lemma 21.3) converges and above we have seen that only the terms with  $q = 0$  are nonzero. Thus we see that  $RH_T^0(K) \rightarrow K$  is an isomorphism. Thus the functor  $D^+(\text{Mod}_{A,T}) \rightarrow D_T^+(A)$  is an equivalence with quasi-inverse given by  $RH_T^0$ .  $\square$

<sup>2</sup>If  $\dim(A) = \infty$  the construction may have unexpected properties on unbounded complexes.



0EF3 **Lemma 4.5.** *Let  $A$  be a Noetherian ring. Let  $T \subset \text{Spec}(A)$  be a subset stable under specialization. If  $\dim(A) < \infty$ , then functor  $D(\text{Mod}_{A,T}) \rightarrow D_T(A)$  is an equivalence.*

**Proof.** Say  $\dim(A) = d$ . Then we see that  $H_Z^i(M) = 0$  for  $i > d$  for every closed subset  $Z$  of  $\text{Spec}(A)$ , see Lemma 3.7. By Lemma 4.3 we find that  $H_T^0$  has bounded cohomological dimension.

Let  $K \in D_T(A)$ . We claim that  $RH_T^0(K) \rightarrow K$  is an isomorphism. We know this is true when  $K$  is bounded below, see Lemma 4.4. However, since  $H_T^0$  has bounded cohomological dimension, we see that the  $i$ th cohomology of  $RH_T^0(K)$  only depends on  $\tau_{\geq -d+i}K$  and we conclude. Thus  $D(\text{Mod}_{A,T}) \rightarrow D_T(A)$  is an equivalence with quasi-inverse  $RH_T^0$ .  $\square$

0EF4 **Remark 4.6.** Let  $A$  be a Noetherian ring. Let  $T \subset \text{Spec}(A)$  be a subset stable under specialization. The upshot of the discussion above is that  $R\Gamma_T : D^+(A) \rightarrow D_T^+(A)$  is the right adjoint to the inclusion functor  $D_T^+(A) \rightarrow D^+(A)$ . If  $\dim(A) < \infty$ , then  $R\Gamma_T : D(A) \rightarrow D_T(A)$  is the right adjoint to the inclusion functor  $D_T(A) \rightarrow D(A)$ . In both cases we have

$$H_T^i(K) = H^i(R\Gamma_T(K)) = R^i H_T^0(K) = \text{colim}_{Z \subset T \text{ closed}} H_Z^i(K)$$

This follows by combining Lemmas 4.2, 4.3, 4.4, and 4.5.

0EF5 **Lemma 4.7.** *Let  $A \rightarrow B$  be a flat homomorphism of Noetherian rings. Let  $T \subset \text{Spec}(A)$  be a subset stable under specialization. Let  $T' \subset \text{Spec}(B)$  be the inverse image of  $T$ . Then the canonical map*

$$R\Gamma_T(K) \otimes_A^{\mathbf{L}} B \longrightarrow R\Gamma_{T'}(K \otimes_A^{\mathbf{L}} B)$$

*is an isomorphism for  $K \in D^+(A)$ . If  $A$  and  $B$  have finite dimension, then this is true for  $K \in D(A)$ .*

**Proof.** From the map  $R\Gamma_T(K) \rightarrow K$  we get a map  $R\Gamma_T(K) \otimes_A^{\mathbf{L}} B \rightarrow K \otimes_A^{\mathbf{L}} B$ . The cohomology modules of  $R\Gamma_T(K) \otimes_A^{\mathbf{L}} B$  are supported on  $T'$  and hence we get the arrow of the lemma. This arrow is an isomorphism if  $T$  is a closed subset of  $\text{Spec}(A)$  by Dualizing Complexes, Lemma 9.3. Recall that  $H_T^i(K)$  is the colimit of  $H_Z^i(K)$  where  $Z$  runs over the (directed set of) closed subsets of  $T$ , see Lemma 4.3. Correspondingly  $H_{T'}^i(K \otimes_A^{\mathbf{L}} B) = \text{colim}_{Z'} H_{Z'}^i(K \otimes_A^{\mathbf{L}} B)$  where  $Z'$  is the inverse image of  $Z$ . Thus the result because  $\otimes_A B$  commutes with filtered colimits and there are no higher Tors.  $\square$

0EF6 **Lemma 4.8.** *Let  $A$  be a ring and let  $T, T' \subset \text{Spec}(A)$  subsets stable under specialization. For  $K \in D^+(A)$  there is a spectral sequence*

$$E_2^{p,q} = H_T^p(H_{T'}^q(K)) \Rightarrow H_{T \cap T'}^{p+q}(K)$$

*as in Derived Categories, Lemma 22.2.*

**Proof.** Let  $E$  be an object of  $D_{T \cap T'}(A)$ . Then we have

$$\text{Hom}(E, R\Gamma_T(R\Gamma_{T'}(K))) = \text{Hom}(E, R\Gamma_{T'}(K)) = \text{Hom}(E, K)$$

The first equality by the adjointness property of  $R\Gamma_T$  and the second by the adjointness property of  $R\Gamma_{T'}$ . On the other hand, if  $J^\bullet$  is a bounded below complex of injectives representing  $K$ , then  $H_{T'}^0(J^\bullet)$  is a complex of injective  $A$ -modules representing  $R\Gamma_{T'}(K)$  and hence  $H_T^0(H_{T'}^0(J^\bullet))$  is a complex representing  $R\Gamma_T(R\Gamma_{T'}(K))$ .

Thus  $R\Gamma_T(R\Gamma_{T'}(K))$  is an object of  $D_{T \cap T'}^+(A)$ . Combining these two facts we find that  $R\Gamma_{T \cap T'} = R\Gamma_T \circ R\Gamma_{T'}$ . This produces the spectral sequence by the lemma referenced in the statement.  $\square$

0EF7 **Lemma 4.9.** *Let  $A$  be a Noetherian ring. Let  $T \subset \text{Spec}(A)$  be a subset stable under specialization. Assume  $A$  has finite dimension. Then*

$$R\Gamma_T(K) = R\Gamma_T(A) \otimes_A^{\mathbf{L}} K$$

for  $K \in D(A)$ . For  $K, L \in D(A)$  we have

$$R\Gamma_T(K \otimes_A^{\mathbf{L}} L) = K \otimes_A^{\mathbf{L}} R\Gamma_T(L) = R\Gamma_T(K) \otimes_A^{\mathbf{L}} L = R\Gamma_T(K) \otimes_A^{\mathbf{L}} R\Gamma_T(L)$$

If  $K$  or  $L$  is in  $D_T(A)$  then so is  $K \otimes_A^{\mathbf{L}} L$ .

**Proof.** By construction we may represent  $R\Gamma_T(A)$  by a complex  $J^\bullet$  in  $\text{Mod}_{A,T}$ . Thus if we represent  $K$  by a  $K$ -flat complex  $K^\bullet$  then we see that  $R\Gamma_T(A) \otimes_A^{\mathbf{L}} K$  is represented by the complex  $\text{Tot}(J^\bullet \otimes_A K^\bullet)$  in  $\text{Mod}_{A,T}$ . Using the map  $R\Gamma_T(A) \rightarrow A$  we obtain a map  $R\Gamma_T(A) \otimes_A^{\mathbf{L}} K \rightarrow K$ . Thus by the adjointness property of  $R\Gamma_T$  we obtain a canonical map

$$R\Gamma_T(A) \otimes_A^{\mathbf{L}} K \longrightarrow R\Gamma_T(K)$$

factoring the just constructed map. Observe that  $R\Gamma_T$  commutes with direct sums in  $D(A)$  for example by Lemma 4.3, the fact that directed colimits commute with direct sums, and the fact that usual local cohomology commutes with direct sums (for example by Dualizing Complexes, Lemma 9.1). Thus by More on Algebra, Remark 57.13 it suffices to check the map is an isomorphism for  $K = A[k]$  where  $k \in \mathbf{Z}$ . This is clear.

The final statements follow from the result we've just shown by transitivity of derived tensor products.  $\square$

## 5. Filtrations on local cohomology

0EHV Some tricks related to the spectral sequence of Lemma 4.8.

0EF8 **Lemma 5.1.** *Let  $A$  be a Noetherian ring. Let  $T \subset \text{Spec}(A)$  be a subset stable under specialization. Let  $T' \subset T$  be the set of nonminimal primes in  $T$ . Then  $T'$  is a subset of  $\text{Spec}(A)$  stable under specialization and for every  $A$ -module  $M$  there is an exact sequence*

$$0 \rightarrow \text{colim}_{Z,f} H_f^1(H_Z^{i-1}(M)) \rightarrow H_{T'}^i(M) \rightarrow H_T^i(M) \rightarrow \bigoplus_{\mathfrak{p} \in T \setminus T'} H_{\mathfrak{p}A_{\mathfrak{p}}}^i(M_{\mathfrak{p}})$$

where the colimit is over closed subsets  $Z \subset T$  and  $f \in A$  with  $V(f) \cap Z \subset T'$ .

**Proof.** For every  $Z$  and  $f$  the spectral sequence of Dualizing Complexes, Lemma 9.6 degenerates to give short exact sequences

$$0 \rightarrow H_f^1(H_Z^{i-1}(M)) \rightarrow H_{Z \cap V(f)}^i(M) \rightarrow H_f^0(H_Z^i(M)) \rightarrow 0$$

We will use this without further mention below.

Let  $\xi \in H_T^i(M)$  map to zero in the direct sum. Then we first write  $\xi$  as the image of some  $\xi' \in H_Z^i(M)$  for some closed subset  $Z \subset T$ , see Lemma 4.3. Then  $\xi'$  maps to zero in  $H_{\mathfrak{p}A_{\mathfrak{p}}}^i(M_{\mathfrak{p}})$  for every  $\mathfrak{p} \in Z$ ,  $\mathfrak{p} \notin T'$ . Since there are finitely many of these primes, we may choose  $f \in A$  not contained in any of these such that  $f$  annihilates

$\xi'$ . Then  $\xi'$  is the image of some  $\xi'' \in H_{Z'}^i(M)$  where  $Z' = Z \cap V(f)$ . By our choice of  $f$  we have  $Z' \subset T'$  and we get exactness at the penultimate spot.

Let  $\xi \in H_{T'}^i(M)$  map to zero in  $H_T^i(M)$ . Choose closed subsets  $Z' \subset Z$  with  $Z' \subset T'$  and  $Z \subset T$  such that  $\xi$  comes from  $\xi' \in H_{Z'}^i(M)$  and maps to zero in  $H_Z^i(M)$ . Then we can find  $f \in A$  with  $V(f) \cap Z = Z'$  and we conclude.  $\square$

0EF9 **Lemma 5.2.** *Let  $A$  be a Noetherian ring of finite dimension. Let  $T \subset \text{Spec}(A)$  be a subset stable under specialization. Let  $\{M_n\}_{n \geq 0}$  be an inverse system of  $A$ -modules. Let  $i \geq 0$  be an integer. Assume that for every  $m$  there exists an integer  $m'(m) \geq m$  such that for all  $\mathfrak{p} \in T$  the induced map*

$$H_{\mathfrak{p}A_{\mathfrak{p}}}^i(M_{k,\mathfrak{p}}) \longrightarrow H_{\mathfrak{p}A_{\mathfrak{p}}}^i(M_{m,\mathfrak{p}})$$

*is zero for  $k \geq m'(m)$ . Let  $m'' : \mathbf{N} \rightarrow \mathbf{N}$  be the  $2^{\dim(T)}$ -fold self-composition of  $m'$ . Then the map  $H_T^i(M_k) \rightarrow H_T^i(M_m)$  is zero for all  $k \geq m''(m)$ .*

**Proof.** We first make a general remark: suppose we have an exact sequence

$$(A_n) \rightarrow (B_n) \rightarrow (C_n)$$

of inverse systems of abelian groups. Suppose that for every  $m$  there exists an integer  $m'(m) \geq m$  such that

$$A_k \rightarrow A_m \quad \text{and} \quad C_k \rightarrow C_m$$

are zero for  $k \geq m'(m)$ . Then for  $k \geq m'(m'(m))$  the map  $B_k \rightarrow B_m$  is zero.

We will prove the lemma by induction on  $\dim(T)$  which is finite because  $\dim(A)$  is finite. Let  $T' \subset T$  be the set of nonminimal primes in  $T$ . Then  $T'$  is a subset of  $\text{Spec}(A)$  stable under specialization and the hypotheses of the lemma apply to  $T'$ . Since  $\dim(T') < \dim(T)$  we know the lemma holds for  $T'$ . For every  $A$ -module  $M$  there is an exact sequence

$$H_{T'}^i(M) \rightarrow H_T^i(M) \rightarrow \bigoplus_{\mathfrak{p} \in T \setminus T'} H_{\mathfrak{p}A_{\mathfrak{p}}}^i(M_{\mathfrak{p}})$$

by Lemma 5.1. Thus we conclude by the initial remark of the proof.  $\square$

0EFA **Lemma 5.3.** *Let  $A$  be a Noetherian ring. Let  $T \subset \text{Spec}(A)$  be a subset stable under specialization. Let  $\{M_n\}_{n \geq 0}$  be an inverse system of  $A$ -modules. Let  $i \geq 0$  be an integer. Assume the dimension of  $A$  is finite and that for every  $m$  there exists an integer  $m'(m) \geq m$  such that for all  $\mathfrak{p} \in T$  we have*

- (1)  $H_{\mathfrak{p}A_{\mathfrak{p}}}^{i-1}(M_{k,\mathfrak{p}}) \rightarrow H_{\mathfrak{p}A_{\mathfrak{p}}}^{i-1}(M_{m,\mathfrak{p}})$  is zero for  $k \geq m'(m)$ , and
- (2)  $H_{\mathfrak{p}A_{\mathfrak{p}}}^i(M_{k,\mathfrak{p}}) \rightarrow H_{\mathfrak{p}A_{\mathfrak{p}}}^i(M_{m,\mathfrak{p}})$  has image  $G(\mathfrak{p}, m)$  independent of  $k \geq m'(m)$  and moreover  $G(\mathfrak{p}, m)$  maps injectively into  $H_{\mathfrak{p}A_{\mathfrak{p}}}^i(M_{0,\mathfrak{p}})$ .

*Then there exists an integer  $m_0$  such that for every  $m \geq m_0$  there exists an integer  $m''(m) \geq m$  such that for  $k \geq m''(m)$  the image of  $H_T^i(M_k) \rightarrow H_T^i(M_m)$  maps injectively into  $H_T^i(M_{m_0})$ .*

**Proof.** We first make a general remark: suppose we have an exact sequence

$$(A_n) \rightarrow (B_n) \rightarrow (C_n) \rightarrow (D_n)$$

of inverse systems of abelian groups. Suppose that there exists an integer  $m_0$  such that for every  $m \geq m_0$  there exists an integer  $m'(m) \geq m$  such that the maps

$$\text{Im}(B_k \rightarrow B_m) \longrightarrow B_{m_0} \quad \text{and} \quad \text{Im}(D_k \rightarrow D_m) \longrightarrow D_{m_0}$$

are injective for  $k \geq m'(m)$  and  $A_k \rightarrow A_m$  is zero for  $k \geq m'(m)$ . Then for  $m \geq m'(m_0)$  and  $k \geq m'(m'(m))$  the map

$$\text{Im}(C_k \rightarrow C_m) \rightarrow C_{m'(m_0)}$$

is injective. Namely, let  $c_0 \in C_m$  be the image of  $c_3 \in C_k$  and say  $c_0$  maps to zero in  $C_{m'(m_0)}$ . Picture

$$C_k \rightarrow C_{m'(m'(m))} \rightarrow C_{m'(m)} \rightarrow C_m \rightarrow C_{m'(m_0)}, \quad c_3 \mapsto c_2 \mapsto c_1 \mapsto c_0 \mapsto 0$$

We have to show  $c_0 = 0$ . The image  $d_3$  of  $c_3$  maps to zero in  $C_{m_0}$  and hence we see that the image  $d_1 \in D_{m'(m)}$  is zero. Thus we can choose  $b_1 \in B_{m'(m)}$  mapping to the image  $c_1$ . Since  $c_3$  maps to zero in  $C_{m'(m_0)}$  we find an element  $a_{-1} \in A_{m'(m_0)}$  which maps to the image  $b_{-1} \in B_{m'(m_0)}$  of  $b_1$ . Since  $a_{-1}$  maps to zero in  $A_{m_0}$  we conclude that  $b_1$  maps to zero in  $B_{m_0}$ . Thus the image  $b_0 \in B_m$  is zero which of course implies  $c_0 = 0$  as desired.

We will prove the lemma by induction on  $\dim(T)$  which is finite because  $\dim(A)$  is finite. Let  $T' \subset T$  be the set of nonminimal primes in  $T$ . Then  $T'$  is a subset of  $\text{Spec}(A)$  stable under specialization and the hypotheses of the lemma apply to  $T'$ . Since  $\dim(T') < \dim(T)$  we know the lemma holds for  $T'$ . For every  $A$ -module  $M$  there is an exact sequence

$$0 \rightarrow \text{colim}_{Z,f} H_f^1(H_Z^{i-1}(M)) \rightarrow H_{T'}^i(M) \rightarrow H_T^i(M) \rightarrow \bigoplus_{\mathfrak{p} \in T \setminus T'} H_{\mathfrak{p}A_{\mathfrak{p}}}^i(M_{\mathfrak{p}})$$

by Lemma 5.1. Thus we conclude by the initial remark of the proof and the fact that we've seen the system of groups

$$\{\text{colim}_{Z,f} H_f^1(H_Z^{i-1}(M_n))\}_{n \geq 0}$$

is pro-zero in Lemma 5.2; this uses that the function  $m''(m)$  in that lemma for  $H_Z^{i-1}(M)$  is independent of  $Z$ .  $\square$

## 6. Finiteness of local cohomology, I

0AW7 We will follow Faltings approach to finiteness of local cohomology modules, see [Fal78] and [Fal81]. Here is a lemma which shows that it suffices to prove local cohomology modules have an annihilator in order to prove that they are finite modules.

0AW8 **Lemma 6.1.** *Let  $A$  be a Noetherian ring. Let  $T \subset \text{Spec}(A)$  be a subset stable under specialization. Let  $M$  be a finite  $A$ -module. Let  $n \geq 0$ . The following are equivalent* [Fal78, Lemma 3]

- (1)  $H_T^i(M)$  is finite for  $i \leq n$ ,
- (2) there exists an ideal  $J \subset A$  with  $V(J) \subset T$  such that  $J$  annihilates  $H_T^i(M)$  for  $i \leq n$ .

If  $T = V(I) = Z$  for an ideal  $I \subset A$ , then these are also equivalent to

- (3) there exists an  $e \geq 0$  such that  $I^e$  annihilates  $H_Z^i(M)$  for  $i \leq n$ .

**Proof.** We prove the equivalence of (1) and (2) by induction on  $n$ . For  $n = 0$  we have  $H_T^0(M) \subset M$  is finite. Hence (1) is true. Since  $H_T^0(M) = \text{colim} H_{V(J)}^0(M)$  with  $J$  as in (2) we see that (2) is true. Assume that  $n > 0$ .

Assume (1) is true. Recall that  $H_T^i(M) = H_{V(J)}^i(M)$ , see Dualizing Complexes, Lemma 10.1. Thus  $H_T^i(M) = \text{colim} H_J^i(M)$  where the colimit is over ideals  $J \subset A$

with  $V(J) \subset T$ , see Lemma 4.3. Since  $H_T^i(M)$  is finitely generated for  $i \leq n$  we can find a  $J \subset A$  as in (2) such that  $H_J^i(M) \rightarrow H_T^i(M)$  is surjective for  $i \leq n$ . Thus the finite list of generators are  $J$ -power torsion elements and we see that (2) holds with  $J$  replaced by some power.

Assume we have  $J$  as in (2). Let  $N = H_T^0(M)$  and  $M' = M/N$ . By construction of  $R\Gamma_T$  we find that  $H_T^i(N) = 0$  for  $i > 0$  and  $H_T^0(N) = N$ , see Remark 4.6. Thus we find that  $H_T^0(M') = 0$  and  $H_T^i(M') = H_T^i(M)$  for  $i > 0$ . We conclude that we may replace  $M$  by  $M'$ . Thus we may assume that  $H_T^0(M) = 0$ . This means that the finite set of associated primes of  $M$  are not in  $T$ . By prime avoidance (Algebra, Lemma 14.2) we can find  $f \in J$  not contained in any of the associated primes of  $M$ . Then the long exact local cohomology sequence associated to the short exact sequence

$$0 \rightarrow M \rightarrow M \rightarrow M/fM \rightarrow 0$$

turns into short exact sequences

$$0 \rightarrow H_T^i(M) \rightarrow H_T^i(M/fM) \rightarrow H_T^{i+1}(M) \rightarrow 0$$

for  $i < n$ . We conclude that  $J^2$  annihilates  $H_T^i(M/fM)$  for  $i < n$ . By induction hypothesis we see that  $H_T^i(M/fM)$  is finite for  $i < n$ . Using the short exact sequence once more we see that  $H_T^{i+1}(M)$  is finite for  $i < n$  as desired.

We omit the proof of the equivalence of (2) and (3) in case  $T = V(I)$ .  $\square$

The following result of Faltings allows us to prove finiteness of local cohomology at the level of local rings.

0AW9 **Lemma 6.2.** *Let  $A$  be a Noetherian ring,  $I \subset A$  an ideal,  $M$  a finite  $A$ -module, and  $n \geq 0$  an integer. Let  $Z = V(I)$ . The following are equivalent*

- (1) *the modules  $H_Z^i(M)$  are finite for  $i \leq n$ , and*
- (2) *for all  $\mathfrak{p} \in \text{Spec}(A)$  the modules  $H_Z^i(M)_{\mathfrak{p}}$ ,  $i \leq n$  are finite  $A_{\mathfrak{p}}$ -modules.*

This is a special case of [Fal81, Satz 1].

**Proof.** The implication (1)  $\Rightarrow$  (2) is immediate. We prove the converse by induction on  $n$ . The case  $n = 0$  is clear because both (1) and (2) are always true in that case.

Assume  $n > 0$  and that (2) is true. Let  $N = H_Z^0(M)$  and  $M' = M/N$ . By Dualizing Complexes, Lemma 11.6 we may replace  $M$  by  $M'$ . Thus we may assume that  $H_Z^0(M) = 0$ . This means that  $\text{depth}_I(M) > 0$  (Dualizing Complexes, Lemma 11.1). Pick  $f \in I$  a nonzerodivisor on  $M$  and consider the short exact sequence

$$0 \rightarrow M \rightarrow M \rightarrow M/fM \rightarrow 0$$

which produces a long exact sequence

$$0 \rightarrow H_Z^0(M/fM) \rightarrow H_Z^1(M) \rightarrow H_Z^1(M) \rightarrow H_Z^1(M/fM) \rightarrow H_Z^2(M) \rightarrow \dots$$

and similarly after localization. Thus assumption (2) implies that the modules  $H_Z^i(M/fM)_{\mathfrak{p}}$  are finite for  $i < n$ . Hence by induction assumption  $H_Z^i(M/fM)$  are finite for  $i < n$ .

Let  $\mathfrak{p}$  be a prime of  $A$  which is associated to  $H_Z^i(M)$  for some  $i \leq n$ . Say  $\mathfrak{p}$  is the annihilator of the element  $x \in H_Z^i(M)$ . Then  $\mathfrak{p} \in Z$ , hence  $f \in \mathfrak{p}$ . Thus  $fx = 0$  and hence  $x$  comes from an element of  $H_Z^{i-1}(M/fM)$  by the boundary map  $\delta$  in the long exact sequence above. It follows that  $\mathfrak{p}$  is an associated prime of the finite module  $\text{Im}(\delta)$ . We conclude that  $\text{Ass}(H_Z^i(M))$  is finite for  $i \leq n$ , see Algebra, Lemma 62.5.

Recall that

$$H_Z^i(M) \subset \prod_{\mathfrak{p} \in \text{Ass}(H_Z^i(M))} H_Z^i(M)_{\mathfrak{p}}$$

by Algebra, Lemma 62.19. Since by assumption the modules on the right hand side are finite and  $I$ -power torsion, we can find integers  $e_{\mathfrak{p},i} \geq 0$ ,  $i \leq n$ ,  $\mathfrak{p} \in \text{Ass}(H_Z^i(M))$  such that  $I^{e_{\mathfrak{p},i}}$  annihilates  $H_Z^i(M)_{\mathfrak{p}}$ . We conclude that  $I^e$  with  $e = \max\{e_{\mathfrak{p},i}\}$  annihilates  $H_Z^i(M)$  for  $i \leq n$ . By Lemma 6.1 we see that  $H_Z^i(M)$  is finite for  $i \leq n$ .  $\square$

0BPX **Lemma 6.3.** *Let  $A$  be a ring and let  $J \subset I \subset A$  be finitely generated ideals. Let  $i \geq 0$  be an integer. Set  $Z = V(I)$ . If  $H_Z^i(A)$  is annihilated by  $J^n$  for some  $n$ , then  $H_Z^i(M)$  is annihilated by  $J^m$  for some  $m = m(M)$  for every finitely presented  $A$ -module  $M$  such that  $M_f$  is a finite locally free  $A_f$ -module for all  $f \in I$ .*

**Proof.** Consider the annihilator  $\mathfrak{a}$  of  $H_Z^i(M)$ . Let  $\mathfrak{p} \subset A$  with  $\mathfrak{p} \not\subset Z$ . By assumption there exists an  $f \in I$ ,  $f \notin \mathfrak{p}$  and an isomorphism  $\varphi : A_{\mathfrak{p}}^{\oplus r} \rightarrow M_{\mathfrak{p}}$  of  $A_{\mathfrak{p}}$ -modules. Clearing denominators (and using that  $M$  is of finite presentation) we find maps

$$a : A^{\oplus r} \rightarrow M \quad \text{and} \quad b : M \rightarrow A^{\oplus r}$$

with  $a_f = f^N \varphi$  and  $b_f = f^N \varphi^{-1}$  for some  $N$ . Moreover we may assume that  $a \circ b$  and  $b \circ a$  are equal to multiplication by  $f^{2N}$ . Thus we see that  $H_Z^i(M)$  is annihilated by  $f^{2N} J^n$ , i.e.,  $f^{2N} J^n \subset \mathfrak{a}$ .

As  $U = \text{Spec}(A) \setminus Z$  is quasi-compact we can find finitely many  $f_1, \dots, f_t$  and  $N_1, \dots, N_t$  such that  $U = \bigcup D(f_j)$  and  $f_j^{2N_j} J^n \subset \mathfrak{a}$ . Then  $V(I) = V(f_1, \dots, f_t)$  and since  $I$  is finitely generated we conclude  $I^M \subset (f_1, \dots, f_t)$  for some  $M$ . All in all we see that  $J^m \subset \mathfrak{a}$  for  $m \gg 0$ , for example  $m = M(2N_1 + \dots + 2N_t)n$  will do.  $\square$

0BPY **Lemma 6.4.** *Let  $A$  be a Noetherian ring. Let  $I \subset A$  be an ideal. Set  $Z = V(I)$ . Let  $n \geq 0$  be an integer. If  $H_Z^i(A)$  is finite for  $0 \leq i \leq n$ , then the same is true for  $H_Z^i(M)$ ,  $0 \leq i \leq n$  for any finite  $A$ -module  $M$  such that  $M_f$  is a finite locally free  $A_f$ -module for all  $f \in I$ .*

**Proof.** The assumption that  $H_Z^i(A)$  is finite for  $0 \leq i \leq n$  implies there exists an  $e \geq 0$  such that  $I^e$  annihilates  $H_Z^i(A)$  for  $0 \leq i \leq n$ , see Lemma 6.1. Then Lemma 6.3 implies that  $H_Z^i(M)$ ,  $0 \leq i \leq n$  is annihilated by  $I^m$  for some  $m = m(M, i)$ . We may take the same  $m$  for all  $0 \leq i \leq n$ . Then Lemma 6.1 implies that  $H_Z^i(M)$  is finite for  $0 \leq i \leq n$  as desired.  $\square$

## 7. Finiteness of pushforwards, I

0BL8 In this section we discuss the easiest nontrivial case of the finiteness theorem, namely, the finiteness of the first local cohomology or what is equivalent, finiteness of  $j_* \mathcal{F}$  where  $j : U \rightarrow X$  is an open immersion,  $X$  is locally Noetherian, and  $\mathcal{F}$  is a coherent sheaf on  $U$ . Following a method of Kollár ([Kol16] and [Kol15]) we find a necessary and sufficient condition, see Proposition 7.7. The reader who is interested in higher direct images or higher local cohomology groups should skip ahead to Section 11 or Section 10 (which are developed independently of the rest of this section).

0BJZ **Lemma 7.1.** *Let  $X$  be a locally Noetherian scheme. Let  $j : U \rightarrow X$  be the inclusion of an open subscheme with complement  $Z$ . For  $x \in U$  let  $i_x : W_x \rightarrow U$  be the integral closed subscheme with generic point  $x$ . Let  $\mathcal{F}$  be a coherent  $\mathcal{O}_U$ -module. The following are equivalent*

- (1) *for all  $x \in \text{Ass}(\mathcal{F})$  the  $\mathcal{O}_X$ -module  $j_*i_{x,*}\mathcal{O}_{W_x}$  is coherent,*
- (2)  *$j_*\mathcal{F}$  is coherent.*

**Proof.** We first prove that (1) implies (2). Assume (1) holds. The statement is local on  $X$ , hence we may assume  $X$  is affine. Then  $U$  is quasi-compact, hence  $\text{Ass}(\mathcal{F})$  is finite (Divisors, Lemma 2.5). Thus we may argue by induction on the number of associated points. Let  $x \in U$  be a generic point of an irreducible component of the support of  $\mathcal{F}$ . By Divisors, Lemma 2.5 we have  $x \in \text{Ass}(\mathcal{F})$ . By our choice of  $x$  we have  $\dim(\mathcal{F}_x) = 0$  as  $\mathcal{O}_{X,x}$ -module. Hence  $\mathcal{F}_x$  has finite length as an  $\mathcal{O}_{X,x}$ -module (Algebra, Lemma 61.3). Thus we may use induction on this length.

Set  $\mathcal{G} = j_*i_{x,*}\mathcal{O}_{W_x}$ . This is a coherent  $\mathcal{O}_X$ -module by assumption. We have  $\mathcal{G}_x = \kappa(x)$ . Choose a nonzero map  $\varphi_x : \mathcal{F}_x \rightarrow \kappa(x) = \mathcal{G}_x$ . By Cohomology of Schemes, Lemma 9.6 there is an open  $x \in V \subset U$  and a map  $\varphi_V : \mathcal{F}|_V \rightarrow \mathcal{G}|_V$  whose stalk at  $x$  is  $\varphi_x$ . Choose  $f \in \Gamma(X, \mathcal{O}_X)$  which does not vanish at  $x$  such that  $D(f) \subset V$ . By Cohomology of Schemes, Lemma 10.4 (for example) we see that  $\varphi_V$  extends to  $f^n\mathcal{F} \rightarrow \mathcal{G}|_U$  for some  $n$ . Precomposing with multiplication by  $f^n$  we obtain a map  $\mathcal{F} \rightarrow \mathcal{G}|_U$  whose stalk at  $x$  is nonzero. Let  $\mathcal{F}' \subset \mathcal{F}$  be the kernel. Note that  $\text{Ass}(\mathcal{F}') \subset \text{Ass}(\mathcal{F})$ , see Divisors, Lemma 2.4. Since  $\text{length}_{\mathcal{O}_{X,x}}(\mathcal{F}') = \text{length}_{\mathcal{O}_{X,x}}(\mathcal{F}) - 1$  we may apply the induction hypothesis to conclude  $j_*\mathcal{F}'$  is coherent. Since  $\mathcal{G} = j_*(\mathcal{G}|_U) = j_*i_{x,*}\mathcal{O}_{W_x}$  is coherent, we can consider the exact sequence

$$0 \rightarrow j_*\mathcal{F}' \rightarrow j_*\mathcal{F} \rightarrow \mathcal{G}$$

By Schemes, Lemma 24.1 the sheaf  $j_*\mathcal{F}$  is quasi-coherent. Hence the image of  $j_*\mathcal{F}$  in  $j_*(\mathcal{G}|_U)$  is coherent by Cohomology of Schemes, Lemma 9.3. Finally,  $j_*\mathcal{F}$  is coherent by Cohomology of Schemes, Lemma 9.2.

Assume (2) holds. Exactly in the same manner as above we reduce to the case  $X$  affine. We pick  $x \in \text{Ass}(\mathcal{F})$  and we set  $\mathcal{G} = j_*i_{x,*}\mathcal{O}_{W_x}$ . Then we choose a nonzero map  $\varphi_x : \mathcal{G}_x = \kappa(x) \rightarrow \mathcal{F}_x$  which exists exactly because  $x$  is an associated point of  $\mathcal{F}$ . Arguing exactly as above we may assume  $\varphi_x$  extends to an  $\mathcal{O}_U$ -module map  $\varphi : \mathcal{G}|_U \rightarrow \mathcal{F}$ . Then  $\varphi$  is injective (for example by Divisors, Lemma 2.10) and we find an injective map  $\mathcal{G} = j_*(\mathcal{G}|_U) \rightarrow j_*\mathcal{F}$ . Thus (1) holds.  $\square$

0BK0 **Lemma 7.2.** *Let  $A$  be a Noetherian ring and let  $I \subset A$  be an ideal. Set  $X = \text{Spec}(A)$ ,  $Z = V(I)$ ,  $U = X \setminus Z$ , and  $j : U \rightarrow X$  the inclusion morphism. Let  $\mathcal{F}$  be a coherent  $\mathcal{O}_U$ -module. Then*

- (1) *there exists a finite  $A$ -module  $M$  such that  $\mathcal{F}$  is the restriction of  $\widetilde{M}$  to  $U$ ,*
- (2) *given  $M$  there is an exact sequence*

$$0 \rightarrow H_Z^0(M) \rightarrow M \rightarrow H^0(U, \mathcal{F}) \rightarrow H_Z^1(M) \rightarrow 0$$

*and isomorphisms  $H^p(U, \mathcal{F}) = H_Z^{p+1}(M)$  for  $p \geq 1$ ,*

- (3) *given  $M$  and  $p \geq 0$  the following are equivalent*
  - (a)  *$R^p j_*\mathcal{F}$  is coherent,*
  - (b)  *$H^p(U, \mathcal{F})$  is a finite  $A$ -module,*
  - (c)  *$H_Z^{p+1}(M)$  is a finite  $A$ -module,*

- (4) if the equivalent conditions in (3) hold for  $p = 0$ , we may take  $M = \Gamma(U, \mathcal{F})$  in which case we have  $H_Z^0(M) = H_Z^1(M) = 0$ .

**Proof.** By Properties, Lemma 22.4 there exists a coherent  $\mathcal{O}_X$ -module  $\mathcal{F}'$  whose restriction to  $U$  is isomorphic to  $\mathcal{F}$ . Say  $\mathcal{F}'$  corresponds to the finite  $A$ -module  $M$  as in (1). Note that  $R^p j_* \mathcal{F}$  is quasi-coherent (Cohomology of Schemes, Lemma 4.5) and corresponds to the  $A$ -module  $H^p(U, \mathcal{F})$ . By Lemma 2.1 and the general facts in Cohomology, Section 22 we obtain an exact sequence

$$0 \rightarrow H_Z^0(M) \rightarrow M \rightarrow H^0(U, \mathcal{F}) \rightarrow H_Z^1(M) \rightarrow 0$$

and isomorphisms  $H^p(U, \mathcal{F}) = H_Z^{p+1}(M)$  for  $p \geq 1$ . Here we use that  $H^j(X, \mathcal{F}') = 0$  for  $j > 0$  as  $X$  is affine and  $\mathcal{F}'$  is quasi-coherent (Cohomology of Schemes, Lemma 2.2). This proves (2). Parts (3) and (4) are straightforward from (2); see also Lemma 2.2.  $\square$

0AWA **Lemma 7.3.** *Let  $X$  be a locally Noetherian scheme. Let  $j : U \rightarrow X$  be the inclusion of an open subscheme with complement  $Z$ . Let  $\mathcal{F}$  be a coherent  $\mathcal{O}_U$ -module. Assume*

- (1)  $X$  is Nagata,
- (2)  $X$  is universally catenary, and
- (3) for  $x \in \text{Ass}(\mathcal{F})$  and  $z \in Z \cap \overline{\{x\}}$  we have  $\dim(\mathcal{O}_{\overline{\{x\}}, z}) \geq 2$ .

Then  $j_* \mathcal{F}$  is coherent.

**Proof.** By Lemma 7.1 it suffices to prove  $j_* i_{x,*} \mathcal{O}_{W_x}$  is coherent for  $x \in \text{Ass}(\mathcal{F})$ . Let  $\pi : Y \rightarrow X$  be the normalization of  $X$  in  $\text{Spec}(\kappa(x))$ , see Morphisms, Section 51. By Morphisms, Lemma 50.14 the morphism  $\pi$  is finite. Since  $\pi$  is finite  $\mathcal{G} = \pi_* \mathcal{O}_Y$  is a coherent  $\mathcal{O}_X$ -module by Cohomology of Schemes, Lemma 9.9. Observe that  $W_x = U \cap \pi(Y)$ . Thus  $\pi|_{\pi^{-1}(U)} : \pi^{-1}(U) \rightarrow U$  factors through  $i_x : W_x \rightarrow U$  and we obtain a canonical map

$$i_{x,*} \mathcal{O}_{W_x} \longrightarrow (\pi|_{\pi^{-1}(U)})_*(\mathcal{O}_{\pi^{-1}(U)}) = (\pi_* \mathcal{O}_Y)|_U = \mathcal{G}|_U$$

This map is injective (for example by Divisors, Lemma 2.10). Hence  $j_* i_{x,*} \mathcal{O}_{W_x} \subset j_* \mathcal{G}|_U$  and it suffices to show that  $j_* \mathcal{G}|_U$  is coherent.

It remains to prove that  $j_*(\mathcal{G}|_U)$  is coherent. We claim Divisors, Lemma 5.11 applies to

$$\mathcal{G} \longrightarrow j_*(\mathcal{G}|_U)$$

which finishes the proof. It suffices to show that  $\text{depth}(\mathcal{G}_z) \geq 2$  for  $z \in Z$ . Let  $y_1, \dots, y_n \in Y$  be the points mapping to  $z$ . By Algebra, Lemma 71.10 it suffices to show that  $\text{depth}(\mathcal{O}_{Y, y_i}) \geq 2$  for  $i = 1, \dots, n$ . If not, then by Properties, Lemma 12.5 we see that  $\dim(\mathcal{O}_{Y, y_i}) = 1$  for some  $i$ . This is impossible by the dimension formula (Morphisms, Lemma 49.1) for  $\pi : Y \rightarrow \overline{\{x\}}$  and assumption (3).  $\square$

0BK1 **Lemma 7.4.** *Let  $X$  be an integral locally Noetherian scheme. Let  $j : U \rightarrow X$  be the inclusion of a nonempty open subscheme with complement  $Z$ . Assume that for all  $z \in Z$  and any associated prime  $\mathfrak{p}$  of the completion  $\mathcal{O}_{X, z}^\wedge$  we have  $\dim(\mathcal{O}_{X, z}^\wedge / \mathfrak{p}) \geq 2$ . Then  $j_* \mathcal{O}_U$  is coherent.*

**Proof.** We may assume  $X$  is affine. Using Lemmas 6.2 and 7.2 we reduce to  $X = \text{Spec}(A)$  where  $(A, \mathfrak{m})$  is a Noetherian local domain and  $\mathfrak{m} \in Z$ . Then we can use induction on  $d = \dim(A)$ . (The base case is  $d = 0, 1$  which do not happen by our assumption on the local rings.) Set  $V = \text{Spec}(A) \setminus \{\mathfrak{m}\}$ . Observe that the



local rings of  $V$  have dimension strictly smaller than  $d$ . Repeating the arguments for  $j' : U \rightarrow V$  we and using induction we conclude that  $j'_* \mathcal{O}_U$  is a coherent  $\mathcal{O}_V$ -module. Pick a nonzero  $f \in A$  which vanishes on  $Z$ . Since  $D(f) \cap V \subset U$  we find an  $n$  such that multiplication by  $f^n$  on  $U$  extends to a map  $f^n : j'_* \mathcal{O}_U \rightarrow \mathcal{O}_V$  over  $V$  (for example by Cohomology of Schemes, Lemma 10.4). This map is injective hence there is an injective map

$$j_* \mathcal{O}_U = j''_* j'_* \mathcal{O}_U \rightarrow j''_* \mathcal{O}_V$$

on  $X$  where  $j'' : V \rightarrow X$  is the inclusion morphism. Hence it suffices to show that  $j''_* \mathcal{O}_V$  is coherent. In other words, we may assume that  $X$  is the spectrum of a local Noetherian domain and that  $Z$  consists of the closed point.

Assume  $X = \text{Spec}(A)$  with  $(A, \mathfrak{m})$  local and  $Z = \{\mathfrak{m}\}$ . Let  $A^\wedge$  be the completion of  $A$ . Set  $X^\wedge = \text{Spec}(A^\wedge)$ ,  $Z^\wedge = \{\mathfrak{m}^\wedge\}$ ,  $U^\wedge = X^\wedge \setminus Z^\wedge$ , and  $\mathcal{F}^\wedge = \mathcal{O}_{U^\wedge}$ . The ring  $A^\wedge$  is universally catenary and Nagata (Algebra, Remark 154.9 and Lemma 156.8). Moreover, condition (3) of Lemma 7.3 for  $X^\wedge, Z^\wedge, U^\wedge, \mathcal{F}^\wedge$  holds by assumption! Thus we see that  $(U^\wedge \rightarrow X^\wedge)_* \mathcal{O}_{U^\wedge}$  is coherent. Since the morphism  $c : X^\wedge \rightarrow X$  is flat we conclude that the pullback of  $j_* \mathcal{O}_U$  is  $(U^\wedge \rightarrow X^\wedge)_* \mathcal{O}_{U^\wedge}$  (Cohomology of Schemes, Lemma 5.2). Finally, since  $c$  is faithfully flat we conclude that  $j_* \mathcal{O}_U$  is coherent by Descent, Lemma 7.1.  $\square$

0BK2 **Remark 7.5.** Let  $j : U \rightarrow X$  be an open immersion of locally Noetherian schemes. Let  $x \in U$ . Let  $i_x : W_x \rightarrow U$  be the integral closed subscheme with generic point  $x$  and let  $\overline{\{x\}}$  be the closure in  $X$ . Then we have a commutative diagram

$$\begin{array}{ccc} W_x & \xrightarrow{j'} & \overline{\{x\}} \\ i_x \downarrow & & \downarrow i \\ U & \xrightarrow{j} & X \end{array}$$

We have  $j_* i_{x,*} \mathcal{O}_{W_x} = i_* j'_* \mathcal{O}_{W_x}$ . As the left vertical arrow is a closed immersion we see that  $j_* i_{x,*} \mathcal{O}_{W_x}$  is coherent if and only of  $j'_* \mathcal{O}_{W_x}$  is coherent.

0AWC **Remark 7.6.** Let  $X$  be a locally Noetherian scheme. Let  $j : U \rightarrow X$  be the inclusion of an open subscheme with complement  $Z$ . Let  $\mathcal{F}$  be a coherent  $\mathcal{O}_U$ -module. If there exists an  $x \in \text{Ass}(\mathcal{F})$  and  $z \in Z \cap \overline{\{x\}}$  such that  $\dim(\mathcal{O}_{\overline{\{x\}}, z}) \leq 1$ , then  $j_* \mathcal{F}$  is not coherent. To prove this we can do a flat base change to the spectrum of  $\mathcal{O}_{X, z}$ . Let  $X' = \overline{\{x\}}$ . The assumption implies  $\mathcal{O}_{X' \cap U} \subset \mathcal{F}$ . Thus it suffices to see that  $j_* \mathcal{O}_{X' \cap U}$  is not coherent. This is clear because  $X' = \{x, z\}$ , hence  $j_* \mathcal{O}_{X' \cap U}$  corresponds to  $\kappa(x)$  as an  $\mathcal{O}_{X, z}$ -module which cannot be finite as  $x$  is not a closed point.

In fact, the converse of Lemma 7.4 holds true: given an open immersion  $j : U \rightarrow X$  of integral Noetherian schemes and there exists a  $z \in X \setminus U$  and an associated prime  $\mathfrak{p}$  of the completion  $\mathcal{O}_{X, z}^\wedge$  with  $\dim(\mathcal{O}_{X, z}^\wedge / \mathfrak{p}) = 1$ , then  $j_* \mathcal{O}_U$  is not coherent. Namely, you can pass to the local ring, you can enlarge  $U$  to the punctured spectrum, you can pass to the completion, and then the argument above gives the nonfiniteness.

0BK3 **Proposition 7.7** (Kollár). *Let  $j : U \rightarrow X$  be an open immersion of locally Noetherian schemes with complement  $Z$ . Let  $\mathcal{F}$  be a coherent  $\mathcal{O}_U$ -module. The following are equivalent*

Theorem of Kollár stated in an email dated Wed, 1 Jul 2015.

- (1)  $j_*\mathcal{F}$  is coherent,
- (2) for  $x \in \text{Ass}(\mathcal{F})$  and  $z \in Z \cap \overline{\{x\}}$  and any associated prime  $\mathfrak{p}$  of the completion  $\mathcal{O}_{\{x\},z}^\wedge$  we have  $\dim(\mathcal{O}_{\{x\},z}^\wedge/\mathfrak{p}) \geq 2$ .

**Proof.** If (2) holds we get (1) by a combination of Lemmas 7.1, Remark 7.5, and Lemma 7.4. If (2) does not hold, then  $j_*i_{x,*}\mathcal{O}_{W_x}$  is not finite for some  $x \in \text{Ass}(\mathcal{F})$  by the discussion in Remark 7.6 (and Remark 7.5). Thus  $j_*\mathcal{F}$  is not coherent by Lemma 7.1.  $\square$

0BL9 **Lemma 7.8.** *Let  $A$  be a Noetherian ring and let  $I \subset A$  be an ideal. Set  $Z = V(I)$ . Let  $M$  be a finite  $A$ -module. The following are equivalent*

- (1)  $H_Z^1(M)$  is a finite  $A$ -module, and
- (2) for all  $\mathfrak{p} \in \text{Ass}(M)$ ,  $\mathfrak{p} \notin Z$  and all  $\mathfrak{q} \in V(\mathfrak{p} + I)$  the completion of  $(A/\mathfrak{p})_{\mathfrak{q}}$  does not have associated primes of dimension 1.

**Proof.** Follows immediately from Proposition 7.7 via Lemma 7.2.  $\square$

The formulation in the following lemma has the advantage that conditions (1) and (2) are inherited by schemes of finite type over  $X$ . Moreover, this is the form of finiteness which we will generalize to higher direct images in Section 11.

0AWB **Lemma 7.9.** *Let  $X$  be a locally Noetherian scheme. Let  $j : U \rightarrow X$  be the inclusion of an open subscheme with complement  $Z$ . Let  $\mathcal{F}$  be a coherent  $\mathcal{O}_U$ -module. Assume*

- (1)  $X$  is universally catenary,
- (2) for every  $z \in Z$  the formal fibres of  $\mathcal{O}_{X,z}$  are  $(S_1)$ .

*In this situation the following are equivalent*

- (a) for  $x \in \text{Ass}(\mathcal{F})$  and  $z \in Z \cap \overline{\{x\}}$  we have  $\dim(\mathcal{O}_{\{x\},z}^\wedge) \geq 2$ , and
- (b)  $j_*\mathcal{F}$  is coherent.

**Proof.** Let  $x \in \text{Ass}(\mathcal{F})$ . By Proposition 7.7 it suffices to check that  $A = \mathcal{O}_{\{x\},z}^\wedge$  satisfies the condition of the proposition on associated primes of its completion if and only if  $\dim(A) \geq 2$ . Observe that  $A$  is universally catenary (this is clear) and that its formal fibres are  $(S_1)$  as follows from More on Algebra, Lemma 50.10 and Proposition 50.5. Let  $\mathfrak{p}' \subset A^\wedge$  be an associated prime. As  $A \rightarrow A^\wedge$  is flat, by Algebra, Lemma 64.3, we find that  $\mathfrak{p}'$  lies over  $(0) \subset A$ . The formal fibre  $A^\wedge \otimes_A F$  is  $(S_1)$  where  $F$  is the fraction field of  $A$ . We conclude that  $\mathfrak{p}'$  is a minimal prime, see Algebra, Lemma 151.2. Since  $A$  is universally catenary it is formally catenary by More on Algebra, Proposition 94.5. Hence  $\dim(A^\wedge/\mathfrak{p}') = \dim(A)$  which proves the equivalence.  $\square$

## 8. Depth and dimension

0DWW Some helper lemmas.

0DWX **Lemma 8.1.** *Let  $A$  be a Noetherian ring. Let  $I \subset A$  be an ideal. Let  $M$  be a finite  $A$ -module. Let  $\mathfrak{p} \in V(I)$  be a prime ideal. Assume  $e = \text{depth}_{IA_{\mathfrak{p}}}(M_{\mathfrak{p}}) < \infty$ . Then there exists a nonempty open  $U \subset V(\mathfrak{p})$  such that  $\text{depth}_{IA_{\mathfrak{q}}}(M_{\mathfrak{q}}) \geq e$  for all  $\mathfrak{q} \in U$ .*

**Proof.** By definition of depth we have  $IM_{\mathfrak{p}} \neq M_{\mathfrak{p}}$  and there exists an  $M_{\mathfrak{p}}$ -regular sequence  $f_1, \dots, f_e \in IA_{\mathfrak{p}}$ . After replacing  $A$  by a principal localization we may assume  $f_1, \dots, f_e \in I$  form an  $M$ -regular sequence, see Algebra, Lemma 67.6.

Consider the module  $M' = M/IM$ . Since  $\mathfrak{p} \in \text{Supp}(M')$  and since the support of a finite module is closed, we find  $V(\mathfrak{p}) \subset \text{Supp}(M')$ . Thus for  $\mathfrak{q} \in V(\mathfrak{p})$  we get  $IM_{\mathfrak{q}} \neq M_{\mathfrak{q}}$ . Hence, using that localization is exact, we see that  $\text{depth}_{IA_{\mathfrak{q}}}(M_{\mathfrak{q}}) \geq e$  for any  $\mathfrak{q} \in V(I)$  by definition of depth.  $\square$

ODWY **Lemma 8.2.** *Let  $A$  be a Noetherian ring. Let  $M$  be a finite  $A$ -module. Let  $\mathfrak{p}$  be a prime ideal. Assume  $e = \text{depth}_{A_{\mathfrak{p}}}(M_{\mathfrak{p}}) < \infty$ . Then there exists a nonempty open  $U \subset V(\mathfrak{p})$  such that  $\text{depth}_{A_{\mathfrak{q}}}(M_{\mathfrak{q}}) \geq e$  for all  $\mathfrak{q} \in U$  and for all but finitely many  $\mathfrak{q} \in U$  we have  $\text{depth}_{A_{\mathfrak{q}}}(M_{\mathfrak{q}}) > e$ .*

**Proof.** By definition of depth we have  $\mathfrak{p}M_{\mathfrak{p}} \neq M_{\mathfrak{p}}$  and there exists an  $M_{\mathfrak{p}}$ -regular sequence  $f_1, \dots, f_e \in \mathfrak{p}A_{\mathfrak{p}}$ . After replacing  $A$  by a principal localization we may assume  $f_1, \dots, f_e \in \mathfrak{p}$  form an  $M$ -regular sequence, see Algebra, Lemma 67.6. Consider the module  $M' = M/(f_1, \dots, f_e)M$ . Since  $\mathfrak{p} \in \text{Supp}(M')$  and since the support of a finite module is closed, we find  $V(\mathfrak{p}) \subset \text{Supp}(M')$ . Thus for  $\mathfrak{q} \in V(\mathfrak{p})$  we get  $\mathfrak{q}M_{\mathfrak{q}} \neq M_{\mathfrak{q}}$ . Hence, using that localization is exact, we see that  $\text{depth}_{A_{\mathfrak{q}}}(M_{\mathfrak{q}}) \geq e$  for any  $\mathfrak{q} \in V(I)$  by definition of depth. Moreover, as soon as  $\mathfrak{q}$  is not an associated prime of the module  $M'$ , then the depth goes up. Thus we see that the final statement holds by Algebra, Lemma 62.5.  $\square$

0ECN **Lemma 8.3.** *Let  $X$  be a Noetherian scheme with dualizing complex  $\omega_X^\bullet$ . Let  $\mathcal{F}$  be a coherent  $\mathcal{O}_X$ -module. Let  $k \geq 0$  be an integer. Assume  $\mathcal{F}$  is  $(S_k)$ . Then there is a finite number of points  $x \in X$  such that*

$$\text{depth}(\mathcal{F}_x) = k \quad \text{and} \quad \dim(\text{Supp}(\mathcal{F}_x)) > k$$

**Proof.** We will prove this lemma by induction on  $k$ . The base case  $k = 0$  says that  $\mathcal{F}$  has a finite number of embedded associated points, which follows from Divisors, Lemma 2.5.

Assume  $k > 0$  and the result holds for all smaller  $k$ . We can cover  $X$  by finitely many affine opens, hence we may assume  $X = \text{Spec}(A)$  is affine. Then  $\mathcal{F}$  is the coherent  $\mathcal{O}_X$ -module associated to a finite  $A$ -module  $M$  which satisfies  $(S_k)$ . We will use Algebra, Lemmas 62.10 and 71.7 without further mention.

Let  $f \in A$  be a nonzerodivisor on  $M$ . Then  $M/fM$  has  $(S_{k-1})$ . By induction we see that there are finitely many primes  $\mathfrak{p} \in V(f)$  with  $\text{depth}((M/fM)_{\mathfrak{p}}) = k - 1$  and  $\dim(\text{Supp}((M/fM)_{\mathfrak{p}})) > k - 1$ . These are exactly the primes  $\mathfrak{p} \in V(f)$  with  $\text{depth}(M_{\mathfrak{p}}) = k$  and  $\dim(\text{Supp}(M_{\mathfrak{p}})) > k$ . Thus we may replace  $A$  by  $A_f$  and  $M$  by  $M_f$  in trying to prove the finiteness statement.

Since  $M$  satisfies  $(S_k)$  and  $k > 0$  we see that  $M$  has no embedded associated primes (Algebra, Lemma 151.2). Thus  $\text{Ass}(M)$  is the set of generic points of the support of  $M$ . Thus Dualizing Complexes, Lemma 20.4 shows the set  $U = \{\mathfrak{q} \mid M_{\mathfrak{q}} \text{ is Cohen-Macaulay}\}$  is an open containing  $\text{Ass}(M)$ . By prime avoidance (Algebra, Lemma 14.2) we can pick  $f \in A$  with  $f \notin \mathfrak{p}$  for  $\mathfrak{p} \in \text{Ass}(M)$  such that  $D(f) \subset U$ . Then  $f$  is a nonzerodivisor on  $M$  (Algebra, Lemma 62.9). After replacing  $A$  by  $A_f$  and  $M$  by  $M_f$  (see above) we find that  $M$  is Cohen-Macaulay. Thus for all  $\mathfrak{q} \subset A$  we have  $\dim(M_{\mathfrak{q}}) = \text{depth}(M_{\mathfrak{q}})$  and hence the set described in the lemma is empty and a fortiori finite.  $\square$

0DWZ **Lemma 8.4.** *Let  $(A, \mathfrak{m})$  be a Noetherian local ring with normalized dualizing complex  $\omega_A^\bullet$ . Let  $M$  be a finite  $A$ -module. Set  $E^i = \text{Ext}_A^{-i}(M, \omega_A^\bullet)$ . Then*

- (1)  $E^i$  is a finite  $A$ -module nonzero only for  $0 \leq i \leq \dim(\text{Supp}(M))$ ,
- (2)  $\dim(\text{Supp}(E^i)) \leq i$ ,
- (3)  $\text{depth}(M)$  is the smallest integer  $\delta \geq 0$  such that  $E^\delta \neq 0$ ,
- (4)  $\mathfrak{p} \in \text{Supp}(E^0 \oplus \dots \oplus E^i) \Leftrightarrow \text{depth}_{A_{\mathfrak{p}}}(M_{\mathfrak{p}}) + \dim(A/\mathfrak{p}) \leq i$ ,
- (5) the annihilator of  $E^i$  is equal to the annihilator of  $H_{\mathfrak{m}}^i(M)$ .

**Proof.** Parts (1), (2), and (3) are copies of the statements in Dualizing Complexes, Lemma 16.5. For a prime  $\mathfrak{p}$  of  $A$  we have that  $(\omega_A^\bullet)_{\mathfrak{p}}[-\dim(A/\mathfrak{p})]$  is a normalized dualizing complex for  $A_{\mathfrak{p}}$ . See Dualizing Complexes, Lemma 17.3. Thus

$$E_{\mathfrak{p}}^i = \text{Ext}_A^{-i}(M, \omega_A^\bullet)_{\mathfrak{p}} = \text{Ext}_{A_{\mathfrak{p}}}^{-i+\dim(A/\mathfrak{p})}(M_{\mathfrak{p}}, (\omega_A^\bullet)_{\mathfrak{p}}[-\dim(A/\mathfrak{p})])$$

is zero for  $i - \dim(A/\mathfrak{p}) < \text{depth}_{A_{\mathfrak{p}}}(M_{\mathfrak{p}})$  and nonzero for  $i = \dim(A/\mathfrak{p}) + \text{depth}_{A_{\mathfrak{p}}}(M_{\mathfrak{p}})$  by part (3) over  $A_{\mathfrak{p}}$ . This proves part (4). If  $E$  is an injective hull of the residue field of  $A$ , then we have

$$\text{Hom}_A(H_{\mathfrak{m}}^i(M), E) = \text{Ext}_A^{-i}(M, \omega_A^\bullet)^\wedge = (E^i)^\wedge = E^i \otimes_A A^\wedge$$

by the local duality theorem (in the form of Dualizing Complexes, Lemma 18.4). Since  $A \rightarrow A^\wedge$  is faithfully flat, we find (5) is true by Matlis duality (Dualizing Complexes, Proposition 7.8).  $\square$

### 9. Annihilators of local cohomology, I

0EFB This section discusses a result due to Faltings, see [Fal78].

0EFC **Proposition 9.1.** *Let  $A$  be a Noetherian ring which has a dualizing complex. Let  $T \subset T' \subset \text{Spec}(A)$  be subsets stable under specialization. Let  $s \geq 0$  an integer. Let  $M$  be a finite  $A$ -module. The following are equivalent* [Fal78].

- (1) there exists an ideal  $J \subset A$  with  $V(J) \subset T'$  such that  $J$  annihilates  $H_T^i(M)$  for  $i \leq s$ , and
- (2) for all  $\mathfrak{p} \notin T'$ ,  $\mathfrak{q} \in T$  with  $\mathfrak{p} \subset \mathfrak{q}$  we have

$$\text{depth}_{A_{\mathfrak{p}}}(M_{\mathfrak{p}}) + \dim((A/\mathfrak{p})_{\mathfrak{q}}) > s$$

**Proof.** Let  $\omega_A^\bullet$  be a dualizing complex. Let  $\delta$  be its dimension function, see Dualizing Complexes, Section 17. An important role will be played by the finite  $A$ -modules

$$E^i = \text{Ext}_A^i(M, \omega_A^\bullet)$$

For  $\mathfrak{p} \subset A$  we will write  $H_{\mathfrak{p}}^i$  to denote the local cohomology of an  $A_{\mathfrak{p}}$ -module with respect to  $\mathfrak{p}A_{\mathfrak{p}}$ . Then we see that the  $\mathfrak{p}A_{\mathfrak{p}}$ -adic completion of

$$(E^i)_{\mathfrak{p}} = \text{Ext}_{A_{\mathfrak{p}}}^{\delta(\mathfrak{p})+i}(M_{\mathfrak{p}}, (\omega_A^\bullet)_{\mathfrak{p}}[-\delta(\mathfrak{p})])$$

is Matlis dual to

$$H_{\mathfrak{p}}^{-\delta(\mathfrak{p})-i}(M_{\mathfrak{p}})$$

by Dualizing Complexes, Lemma 18.4. In particular we deduce from this the following fact: an ideal  $J \subset A$  annihilates  $(E^i)_{\mathfrak{p}}$  if and only if  $J$  annihilates  $H_{\mathfrak{p}}^{-\delta(\mathfrak{p})-i}(M_{\mathfrak{p}})$ .

Set  $T_n = \{\mathfrak{p} \in T \mid \delta(\mathfrak{p}) \leq n\}$ . As  $\delta$  is a bounded function, we see that  $T_a = \emptyset$  for  $a \ll 0$  and  $T_b = T$  for  $b \gg 0$ .

Assume (2). Let us prove the existence of  $J$  as in (1). We will use a double induction to do this. For  $i \leq s$  consider the induction hypothesis  $IH_i$ :  $H_T^a(M)$  is annihilated by some  $J \subset A$  with  $V(J) \subset T'$  for  $0 \leq a \leq i$ . The case  $IH_0$  is trivial because

$H_T^0(M)$  is a submodule of  $M$  and hence finite and hence is annihilated by some ideal  $J$  with  $V(J) \subset T$ .

Induction step. Assume  $IH_{i-1}$  holds for some  $0 < i \leq s$ . Pick  $J'$  with  $V(J') \subset T'$  annihilating  $H_T^a(M)$  for  $0 \leq a \leq i-1$  (the induction hypothesis guarantees we can do this). We will show by descending induction on  $n$  that there exists an ideal  $J$  with  $V(J) \subset T'$  such that the associated primes of  $JH_T^i(M)$  are in  $T_n$ . For  $n \ll 0$  this implies  $JH_T^i(M) = 0$  (Algebra, Lemma 62.7) and hence  $IH_i$  will hold. The base case  $n \gg 0$  is trivial because  $T = T_n$  in this case and all associated primes of  $H_T^i(M)$  are in  $T$ .

Thus we assume given  $J$  with the property for  $n$ . Let  $\mathfrak{q} \in T_n$ . Let  $T_{\mathfrak{q}} \subset \text{Spec}(A_{\mathfrak{q}})$  be the inverse image of  $T$ . We have  $H_T^j(M)_{\mathfrak{q}} = H_{T_{\mathfrak{q}}}^j(M_{\mathfrak{q}})$  by Lemma 4.7. Consider the spectral sequence

$$H_{\mathfrak{q}}^p(H_{T_{\mathfrak{q}}}^q(M_{\mathfrak{q}})) \Rightarrow H_{\mathfrak{q}}^{p+q}(M_{\mathfrak{q}})$$

of Lemma 4.8. Below we will find an ideal  $J'' \subset A$  with  $V(J'') \subset T'$  such that  $H_{\mathfrak{q}}^i(M_{\mathfrak{q}})$  is annihilated by  $J''$  for all  $\mathfrak{q} \in T_n \setminus T_{n-1}$ . Claim:  $J(J')^i J''$  will work for  $n-1$ . Namely, let  $\mathfrak{q} \in T_n \setminus T_{n-1}$ . The spectral sequence above defines a filtration

$$E_{\infty}^{0,i} = E_{i+2}^{0,i} \subset \dots \subset E_3^{0,i} \subset E_2^{0,i} = H_{\mathfrak{q}}^0(H_{T_{\mathfrak{q}}}^i(M_{\mathfrak{q}}))$$

The module  $E_{\infty}^{0,i}$  is annihilated by  $J''$ . The subquotients  $E_j^{0,i}/E_{j+1}^{0,i}$  for  $i+1 \geq j \geq 2$  are annihilated by  $J'$  because the target of  $d_j^{0,i}$  is a subquotient of

$$H_{\mathfrak{q}}^j(H_{T_{\mathfrak{q}}}^{i-j+1}(M_{\mathfrak{q}})) = H_{\mathfrak{q}}^j(H_T^{i-j+1}(M)_{\mathfrak{q}})$$

and  $H_T^{i-j+1}(M)_{\mathfrak{q}}$  is annihilated by  $J'$  by choice of  $J'$ . Finally, by our choice of  $J$  we have  $JH_T^i(M)_{\mathfrak{q}} \subset H_{\mathfrak{q}}^0(H_T^i(M)_{\mathfrak{q}})$  since the non-closed points of  $\text{Spec}(A_{\mathfrak{q}})$  have higher  $\delta$  values. Thus  $\mathfrak{q}$  cannot be an associated prime of  $J(J')^i J'' H_T^i(M)$  as desired.

By our initial remarks we see that  $J''$  should annihilate

$$(E^{-\delta(\mathfrak{q})-i})_{\mathfrak{q}} = (E^{-n-i})_{\mathfrak{q}}$$

for all  $\mathfrak{q} \in T_n \setminus T_{n-1}$ . But if  $J''$  works for one  $\mathfrak{q}$ , then it works for all  $\mathfrak{q}$  in an open neighbourhood of  $\mathfrak{q}$  as the modules  $E^{-n-i}$  are finite. Since every subset of  $\text{Spec}(A)$  is Noetherian with the induced topology (Topology, Lemma 9.2), we conclude that it suffices to prove the existence of  $J''$  for one  $\mathfrak{q}$ .

Since the ext modules are finite the existence of  $J''$  is equivalent to

$$\text{Supp}(E^{-n-i}) \cap \text{Spec}(A_{\mathfrak{q}}) \subset T'.$$

This is equivalent to showing the localization of  $E^{-n-i}$  at every  $\mathfrak{p} \subset \mathfrak{q}$ ,  $\mathfrak{p} \notin T'$  is zero. Using local duality over  $A_{\mathfrak{p}}$  we find that we need to prove that

$$H_{\mathfrak{p}}^{i+n-\delta(\mathfrak{p})}(M_{\mathfrak{p}}) = H_{\mathfrak{p}}^{i-\dim((A/\mathfrak{p})_{\mathfrak{q}})}(M_{\mathfrak{p}})$$

is zero (this uses that  $\delta$  is a dimension function). This vanishes by the assumption in the lemma and  $i \leq s$  and Dualizing Complexes, Lemma 11.1.

To prove the converse implication we assume (2) does not hold and we work backwards through the arguments above. First, we pick a  $\mathfrak{q} \in T$ ,  $\mathfrak{p} \subset \mathfrak{q}$  with  $\mathfrak{p} \notin T'$  such that

$$i = \text{depth}_{A_{\mathfrak{p}}}(M_{\mathfrak{p}}) + \dim((A/\mathfrak{p})_{\mathfrak{q}}) \leq s$$

is minimal. Then  $H_{\mathfrak{p}}^{i-\dim((A/\mathfrak{p})_{\mathfrak{q}})}(M_{\mathfrak{p}})$  is nonzero by the nonvanishing in Dualizing Complexes, Lemma 11.1. Set  $n = \delta(\mathfrak{q})$ . Then there does not exist an ideal  $J \subset A$  with  $V(J) \subset T'$  such that  $J(E^{-n-i})_{\mathfrak{q}} = 0$ . Thus  $H_{\mathfrak{q}}^i(M_{\mathfrak{q}})$  is not annihilated by an ideal  $J \subset A$  with  $V(J) \subset T'$ . By minimality of  $i$  it follows from the spectral sequence displayed above that the module  $H_T^i(M)_{\mathfrak{q}}$  is not annihilated by an ideal  $J \subset A$  with  $V(J) \subset T'$ . Thus  $H_T^i(M)$  is not annihilated by an ideal  $J \subset A$  with  $V(J) \subset T'$ . This finishes the proof of the proposition.  $\square$

0EFE **Lemma 9.2.** *Let  $I$  be an ideal of a Noetherian ring  $A$ . Let  $M$  be a finite  $A$ -module, let  $\mathfrak{p} \subset A$  be a prime ideal, and let  $s \geq 0$  be an integer. Assume*

- (1)  $A$  has a dualizing complex,
- (2)  $\mathfrak{p} \notin V(I)$ , and
- (3) for all primes  $\mathfrak{p}' \subset \mathfrak{p}$  and  $\mathfrak{q} \in V(I)$  with  $\mathfrak{p}' \subset \mathfrak{q}$  we have

$$\text{depth}_{A_{\mathfrak{p}'}}(M_{\mathfrak{p}'}) + \dim((A/\mathfrak{p}')_{\mathfrak{q}}) > s$$

Then there exists an  $f \in A$ ,  $f \notin \mathfrak{p}$  which annihilates  $H_{V(I)}^i(M)$  for  $i \leq s$ .

**Proof.** Consider the sets

$$T = V(I) \quad \text{and} \quad T' = \bigcup_{f \in A, f \notin \mathfrak{p}} V(f)$$

These are subsets of  $\text{Spec}(A)$  stable under specialization. Observe that  $T \subset T'$  and  $\mathfrak{p} \notin T'$ . Assumption (3) says that hypothesis (2) of Proposition 9.1 holds. Hence we can find  $J \subset A$  with  $V(J) \subset T'$  such that  $JH_{V(I)}^i(M) = 0$  for  $i \leq s$ . Choose  $f \in A$ ,  $f \notin \mathfrak{p}$  with  $V(J) \subset V(f)$ . A power of  $f$  annihilates  $H_{V(I)}^i(M)$  for  $i \leq s$ .  $\square$

## 10. Finiteness of local cohomology, II

0BJQ We continue the discussion of finiteness of local cohomology started in Section 6. Using Faltings Annihilator Theorem we easily prove the following fundamental result.

0EFD **Proposition 10.1.** *Let  $A$  be a Noetherian ring which has a dualizing complex. [Fal78]. Let  $T \subset \text{Spec}(A)$  be a subset stable under specialization. Let  $s \geq 0$  an integer. Let  $M$  be a finite  $A$ -module. The following are equivalent*

- (1)  $H_T^i(M)$  is a finite  $A$ -module for  $i \leq s$ , and
- (2) for all  $\mathfrak{p} \notin T$ ,  $\mathfrak{q} \in T$  with  $\mathfrak{p} \subset \mathfrak{q}$  we have

$$\text{depth}_{A_{\mathfrak{p}}}(M_{\mathfrak{p}}) + \dim((A/\mathfrak{p})_{\mathfrak{q}}) > s$$

**Proof.** Formal consequence of Proposition 9.1 and Lemma 6.1.  $\square$

Besides some lemmas for later use, the rest of this section is concerned with the question to what extent the condition in Proposition 10.1 that  $A$  has a dualizing complex can be weakened. The answer is roughly that one has to assume the formal fibres of  $A$  are  $(S_n)$  for sufficiently large  $n$ .

Let  $A$  be a Noetherian ring and let  $I \subset A$  be an ideal. Set  $X = \text{Spec}(A)$  and  $Z = V(I) \subset X$ . Let  $M$  be a finite  $A$ -module. We define

0BJR (10.1.1)  $s_{A,I}(M) = \min\{\text{depth}_{A_{\mathfrak{p}}}(M_{\mathfrak{p}}) + \dim((A/\mathfrak{p})_{\mathfrak{q}}) \mid \mathfrak{p} \in X \setminus Z, \mathfrak{q} \in Z, \mathfrak{p} \subset \mathfrak{q}\}$

Our conventions on depth are that the depth of 0 is  $\infty$  thus we only need to consider primes  $\mathfrak{p}$  in the support of  $M$ . It will turn out that  $s_{A,I}(M)$  is an important invariant of the situation.

0BJS **Lemma 10.2.** *Let  $A \rightarrow B$  be a finite homomorphism of Noetherian rings. Let  $I \subset A$  be an ideal and set  $J = IB$ . Let  $M$  be a finite  $B$ -module. If  $A$  is universally catenary, then  $s_{B,J}(M) = s_{A,I}(M)$ .*

**Proof.** Let  $\mathfrak{p} \subset \mathfrak{q} \subset A$  be primes with  $I \subset \mathfrak{q}$  and  $I \not\subset \mathfrak{p}$ . Since  $A \rightarrow B$  is finite there are finitely many primes  $\mathfrak{p}_i$  lying over  $\mathfrak{p}$ . By Algebra, Lemma 71.10 we have

$$\text{depth}(M_{\mathfrak{p}}) = \min \text{depth}(M_{\mathfrak{p}_i})$$

Let  $\mathfrak{p}_i \subset \mathfrak{q}_{ij}$  be primes lying over  $\mathfrak{q}$ . By going up for  $A \rightarrow B$  (Algebra, Lemma 35.22) there is at least one  $\mathfrak{q}_{ij}$  for each  $i$ . Then we see that

$$\dim((B/\mathfrak{p}_i)_{\mathfrak{q}_{ij}}) = \dim((A/\mathfrak{p})_{\mathfrak{q}})$$

by the dimension formula, see Algebra, Lemma 112.1. This implies that the minimum of the quantities used to define  $s_{B,J}(M)$  for the pairs  $(\mathfrak{p}_i, \mathfrak{q}_{ij})$  is equal to the quantity for the pair  $(\mathfrak{p}, \mathfrak{q})$ . This proves the lemma.  $\square$

0EHW **Lemma 10.3.** *Let  $A$  be a Noetherian ring which has a dualizing complex. Let  $I \subset A$  be an ideal. Let  $M$  be a finite  $A$ -module. Let  $A', M'$  be the  $I$ -adic completions of  $A, M$ . Let  $\mathfrak{p}' \subset \mathfrak{q}'$  be prime ideals of  $A'$  with  $\mathfrak{q}' \in V(IA')$  lying over  $\mathfrak{p} \subset \mathfrak{q}$  in  $A$ . Then*

$$\text{depth}_{A_{\mathfrak{p}'}}(M'_{\mathfrak{p}'}) \geq \text{depth}_{A_{\mathfrak{p}}}(M_{\mathfrak{p}})$$

and

$$\text{depth}_{A_{\mathfrak{p}'}}(M'_{\mathfrak{p}'}) + \dim((A'/\mathfrak{p}')_{\mathfrak{q}'}) = \text{depth}_{A_{\mathfrak{p}}}(M_{\mathfrak{p}}) + \dim((A/\mathfrak{p})_{\mathfrak{q}})$$

**Proof.** We have

$$\text{depth}(M'_{\mathfrak{p}'}) = \text{depth}(M_{\mathfrak{p}}) + \text{depth}(A'_{\mathfrak{p}'}/\mathfrak{p}A'_{\mathfrak{p}'}) \geq \text{depth}(M_{\mathfrak{p}})$$

by flatness of  $A \rightarrow A'$ , see Algebra, Lemma 157.1. Since the fibres of  $A \rightarrow A'$  are Cohen-Macaulay (Dualizing Complexes, Lemma 23.2 and More on Algebra, Section 50) we see that  $\text{depth}(A'_{\mathfrak{p}'}/\mathfrak{p}A'_{\mathfrak{p}'}) = \dim(A'_{\mathfrak{p}'}/\mathfrak{p}A'_{\mathfrak{p}'})$ . Thus we obtain

$$\begin{aligned} \text{depth}(M'_{\mathfrak{p}'}) + \dim((A'/\mathfrak{p}')_{\mathfrak{q}'}) &= \text{depth}(M_{\mathfrak{p}}) + \dim(A'_{\mathfrak{p}'}/\mathfrak{p}A'_{\mathfrak{p}'}) + \dim((A'/\mathfrak{p}')_{\mathfrak{q}'}) \\ &= \text{depth}(M_{\mathfrak{p}}) + \dim((A'/\mathfrak{p}A')_{\mathfrak{q}'}) \\ &= \text{depth}(M_{\mathfrak{p}}) + \dim((A/\mathfrak{p})_{\mathfrak{q}}) \end{aligned}$$

Second equality because  $A'$  is catenary and third equality by More on Algebra, Lemma 42.1 as  $(A/\mathfrak{p})_{\mathfrak{q}}$  and  $(A'/\mathfrak{p}A')_{\mathfrak{q}'}$  have the same  $I$ -adic completions.  $\square$

0BJT **Lemma 10.4.** *Let  $A$  be a universally catenary Noetherian local ring. Let  $I \subset A$  be an ideal. Let  $M$  be a finite  $A$ -module. Then*

$$s_{A,I}(M) \geq s_{A^{\wedge}, I^{\wedge}}(M^{\wedge})$$

*If the formal fibres of  $A$  are  $(S_n)$ , then  $\min(n+1, s_{A,I}(M)) \leq s_{A^{\wedge}, I^{\wedge}}(M^{\wedge})$ .*

**Proof.** Write  $X = \text{Spec}(A)$ ,  $X^{\wedge} = \text{Spec}(A^{\wedge})$ ,  $Z = V(I) \subset X$ , and  $Z^{\wedge} = V(I^{\wedge})$ . Let  $\mathfrak{p}' \subset \mathfrak{q}' \subset A^{\wedge}$  be primes with  $\mathfrak{p}' \notin Z^{\wedge}$  and  $\mathfrak{q}' \in Z^{\wedge}$ . Let  $\mathfrak{p} \subset \mathfrak{q}$  be the corresponding primes of  $A$ . Then  $\mathfrak{p} \notin Z$  and  $\mathfrak{q} \in Z$ . Picture

$$\begin{array}{ccccc} \mathfrak{p}' & \longrightarrow & \mathfrak{q}' & \longrightarrow & A^{\wedge} \\ \downarrow & & \downarrow & & \uparrow \\ \mathfrak{p} & \longrightarrow & \mathfrak{q} & \longrightarrow & A \end{array}$$

Let us write

$$\begin{aligned} a &= \dim(A/\mathfrak{p}) = \dim(A^\wedge/\mathfrak{p}A^\wedge), \\ b &= \dim(A/\mathfrak{q}) = \dim(A^\wedge/\mathfrak{q}A^\wedge), \\ a' &= \dim(A^\wedge/\mathfrak{p}'), \\ b' &= \dim(A^\wedge/\mathfrak{q}') \end{aligned}$$

Equalities by More on Algebra, Lemma 42.1. We also write

$$\begin{aligned} p &= \dim(A_{\mathfrak{p}'}^\wedge/\mathfrak{p}A_{\mathfrak{p}'}^\wedge) = \dim((A^\wedge/\mathfrak{p}A^\wedge)_{\mathfrak{p}'}) \\ q &= \dim(A_{\mathfrak{q}'}^\wedge/\mathfrak{p}A_{\mathfrak{q}'}^\wedge) = \dim((A^\wedge/\mathfrak{q}A^\wedge)_{\mathfrak{q}'}) \end{aligned}$$

Since  $A$  is universally catenary we see that  $A^\wedge/\mathfrak{p}A^\wedge = (A/\mathfrak{p})^\wedge$  is equidimensional of dimension  $a$  (More on Algebra, Proposition 94.5). Hence  $a = a' + p$ . Similarly  $b = b' + q$ . By Algebra, Lemma 157.1 applied to the flat local ring map  $A_{\mathfrak{p}} \rightarrow A_{\mathfrak{p}'}^\wedge$ , we have

$$\text{depth}(M_{\mathfrak{p}'}^\wedge) = \text{depth}(M_{\mathfrak{p}}) + \text{depth}(A_{\mathfrak{p}'}^\wedge/\mathfrak{p}A_{\mathfrak{p}'}^\wedge)$$

The quantity we are minimizing for  $s_{A,I}(M)$  is

$$s(\mathfrak{p}, \mathfrak{q}) = \text{depth}(M_{\mathfrak{p}}) + \dim((A/\mathfrak{p})_{\mathfrak{q}}) = \text{depth}(M_{\mathfrak{p}}) + a - b$$

(last equality as  $A$  is catenary). The quantity we are minimizing for  $s_{A^\wedge, I^\wedge}(M^\wedge)$  is

$$s(\mathfrak{p}', \mathfrak{q}') = \text{depth}(M_{\mathfrak{p}'}^\wedge) + \dim((A^\wedge/\mathfrak{p}')_{\mathfrak{q}'}) = \text{depth}(M_{\mathfrak{p}'}^\wedge) + a' - b'$$

(last equality as  $A^\wedge$  is catenary). Now we have enough notation in place to start the proof.

Let  $\mathfrak{p} \subset \mathfrak{q} \subset A$  be primes with  $\mathfrak{p} \notin Z$  and  $\mathfrak{q} \in Z$  such that  $s_{A,I}(M) = s(\mathfrak{p}, \mathfrak{q})$ . Then we can pick  $\mathfrak{q}'$  minimal over  $\mathfrak{q}A^\wedge$  and  $\mathfrak{p}' \subset \mathfrak{q}'$  minimal over  $\mathfrak{p}A^\wedge$  (using going down for  $A \rightarrow A^\wedge$ ). Then we have four primes as above with  $p = 0$  and  $q = 0$ . Moreover, we have  $\text{depth}(A_{\mathfrak{p}'}^\wedge/\mathfrak{p}A_{\mathfrak{p}'}^\wedge) = 0$  also because  $p = 0$ . This means that  $s(\mathfrak{p}', \mathfrak{q}') = s(\mathfrak{p}, \mathfrak{q})$ . Thus we get the first inequality.

Assume that the formal fibres of  $A$  are  $(S_n)$ . Then  $\text{depth}(A_{\mathfrak{p}'}^\wedge/\mathfrak{p}A_{\mathfrak{p}'}^\wedge) \geq \min(n, p)$ . Hence

$$s(\mathfrak{p}', \mathfrak{q}') \geq s(\mathfrak{p}, \mathfrak{q}) + q + \min(n, p) - p \geq s_{A,I}(M) + q + \min(n, p) - p$$

Thus the only way we can get in trouble is if  $p > n$ . If this happens then

$$\begin{aligned} s(\mathfrak{p}', \mathfrak{q}') &= \text{depth}(M_{\mathfrak{p}'}^\wedge) + \dim((A^\wedge/\mathfrak{p}')_{\mathfrak{q}'}) \\ &= \text{depth}(M_{\mathfrak{p}}) + \text{depth}(A_{\mathfrak{p}'}^\wedge/\mathfrak{p}A_{\mathfrak{p}'}^\wedge) + \dim((A^\wedge/\mathfrak{p}')_{\mathfrak{q}'}) \\ &\geq 0 + n + 1 \end{aligned}$$

because  $(A^\wedge/\mathfrak{p}')_{\mathfrak{q}'}$  has at least two primes. This proves the second inequality.  $\square$

The method of proof of the following lemma works more generally, but the stronger results one gets will be subsumed in Theorem 10.6 below.

0BJU **Lemma 10.5.** *Let  $A$  be a Gorenstein Noetherian local ring. Let  $I \subset A$  be an ideal and set  $Z = V(I) \subset \text{Spec}(A)$ . Let  $M$  be a finite  $A$ -module. Let  $s = s_{A,I}(M)$  as in (10.1.1). Then  $H_Z^i(M)$  is finite for  $i < s$ , but  $H_Z^s(M)$  is not finite.*

This is a special case of [Fal78, Satz 1].

**Proof.** Since a Gorenstein local ring has a dualizing complex, this is a special case of Proposition 10.1. It would be helpful to have a short proof of this special case, which will be used in the proof of a general finiteness theorem below.  $\square$



Observe that the hypotheses of the following theorem are satisfied by excellent Noetherian rings (by definition), by Noetherian rings which have a dualizing complex (Dualizing Complexes, Lemma 17.4 and Dualizing Complexes, Lemma 23.2), and by quotients of regular Noetherian rings.

0BJV **Theorem 10.6.** *Let  $A$  be a Noetherian ring and let  $I \subset A$  be an ideal. Set  $Z = V(I) \subset \text{Spec}(A)$ . Let  $M$  be a finite  $A$ -module. Set  $s = s_{A,I}(M)$  as in (10.1.1). Assume that*

- (1)  $A$  is universally catenary,
- (2) the formal fibres of the local rings of  $A$  are Cohen-Macaulay.

Then  $H_Z^i(M)$  is finite for  $0 \leq i < s$  and  $H_Z^s(M)$  is not finite.

**Proof.** By Lemma 6.2 we may assume that  $A$  is a local ring.

If  $A$  is a Noetherian complete local ring, then we can write  $A$  as the quotient of a regular complete local ring  $B$  by Cohen's structure theorem (Algebra, Theorem 154.8). Using Lemma 10.2 and Dualizing Complexes, Lemma 9.2 we reduce to the case of a regular local ring which is a consequence of Lemma 10.5 because a regular local ring is Gorenstein (Dualizing Complexes, Lemma 21.3).

Let  $A$  be a Noetherian local ring. Let  $\mathfrak{m}$  be the maximal ideal. We may assume  $I \subset \mathfrak{m}$ , otherwise the lemma is trivial. Let  $A^\wedge$  be the completion of  $A$ , let  $Z^\wedge = V(IA^\wedge)$ , and let  $M^\wedge = M \otimes_A A^\wedge$  be the completion of  $M$  (Algebra, Lemma 96.1). Then  $H_Z^i(M) \otimes_A A^\wedge = H_{Z^\wedge}^i(M^\wedge)$  by Dualizing Complexes, Lemma 9.3 and flatness of  $A \rightarrow A^\wedge$  (Algebra, Lemma 96.2). Hence it suffices to show that  $H_{Z^\wedge}^i(M^\wedge)$  is finite for  $i < s$  and not finite for  $i = s$ , see Algebra, Lemma 82.2. Since we know the result is true for  $A^\wedge$  it suffices to show that  $s_{A,I}(M) = s_{A^\wedge,I^\wedge}(M^\wedge)$ . This follows from Lemma 10.4.  $\square$

0BJW **Remark 10.7.** The astute reader will have realized that we can get away with a slightly weaker condition on the formal fibres of the local rings of  $A$ . Namely, in the situation of Theorem 10.6 assume  $A$  is universally catenary but make no assumptions on the formal fibres. Suppose we have an  $n$  and we want to prove that  $H_Z^i(M)$  are finite for  $i \leq n$ . Then the exact same proof shows that it suffices that  $s_{A,I}(M) > n$  and that the formal fibres of local rings of  $A$  are  $(S_n)$ . On the other hand, if we want to show that  $H_Z^s(M)$  is not finite where  $s = s_{A,I}(M)$ , then our arguments prove this if the formal fibres are  $(S_{s-1})$ .

## 11. Finiteness of pushforwards, II

0BJX This section is the continuation of Section 7. In this section we reap the fruits of the labor done in Section 10.

0BJY **Lemma 11.1.** *Let  $X$  be a locally Noetherian scheme. Let  $j : U \rightarrow X$  be the inclusion of an open subscheme with complement  $Z$ . Let  $\mathcal{F}$  be a coherent  $\mathcal{O}_U$ -module. Let  $n \geq 0$  be an integer. Assume*

- (1)  $X$  is universally catenary,
- (2) for every  $z \in Z$  the formal fibres of  $\mathcal{O}_{X,z}$  are  $(S_n)$ .

In this situation the following are equivalent

- (a) for  $x \in \text{Supp}(\mathcal{F})$  and  $z \in Z \cap \overline{\{x\}}$  we have  $\text{depth}_{\mathcal{O}_{X,x}}(\mathcal{F}_x) + \dim(\mathcal{O}_{\overline{\{x\}},z}) > n$ ,
- (b)  $R^p j_* \mathcal{F}$  is coherent for  $0 \leq p < n$ .

This is a special case of [Fal81, Satz 2].

**Proof.** The statement is local on  $X$ , hence we may assume  $X$  is affine. Say  $X = \text{Spec}(A)$  and  $Z = V(I)$ . Let  $M$  be a finite  $A$ -module whose associated coherent  $\mathcal{O}_X$ -module restricts to  $\mathcal{F}$  over  $U$ , see Lemma 7.2. This lemma also tells us that  $R^p j_* \mathcal{F}$  is coherent if and only if  $H_Z^{p+1}(M)$  is a finite  $A$ -module. Observe that the minimum of the expressions  $\text{depth}_{\mathcal{O}_{X,x}}(\mathcal{F}_x) + \dim(\overline{\mathcal{O}_{\{x\},z}})$  is the number  $s_{A,I}(M)$  of (10.1.1). Having said this the lemma follows from Theorem 10.6 as elucidated by Remark 10.7.  $\square$

0BLT **Lemma 11.2.** *Let  $X$  be a locally Noetherian scheme. Let  $j : U \rightarrow X$  be the inclusion of an open subscheme with complement  $Z$ . Let  $n \geq 0$  be an integer. If  $R^p j_* \mathcal{O}_U$  is coherent for  $0 \leq p < n$ , then the same is true for  $R^p j_* \mathcal{F}$ ,  $0 \leq p < n$  for any finite locally free  $\mathcal{O}_U$ -module  $\mathcal{F}$ .*

**Proof.** The question is local on  $X$ , hence we may assume  $X$  is affine. Say  $X = \text{Spec}(A)$  and  $Z = V(I)$ . Via Lemma 7.2 our lemma follows from Lemma 6.4.  $\square$

0BM5 **Lemma 11.3.** *Let  $A$  be a ring and let  $J \subset I \subset A$  be finitely generated ideals. Let  $p \geq 0$  be an integer. Set  $U = \text{Spec}(A) \setminus V(I)$ . If  $H^p(U, \mathcal{O}_U)$  is annihilated by  $J^n$  for some  $n$ , then  $H^p(U, \mathcal{F})$  is annihilated by  $J^m$  for some  $m = m(\mathcal{F})$  for every finite locally free  $\mathcal{O}_U$ -module  $\mathcal{F}$ .* [BdJ14, Lemma 1.9]

**Proof.** Consider the annihilator  $\mathfrak{a}$  of  $H^p(U, \mathcal{F})$ . Let  $u \in U$ . There exists an open neighbourhood  $u \in U' \subset U$  and an isomorphism  $\varphi : \mathcal{O}_{U'}^{\oplus r} \rightarrow \mathcal{F}|_{U'}$ . Pick  $f \in A$  such that  $u \in D(f) \subset U'$ . There exist maps

$$a : \mathcal{O}_U^{\oplus r} \longrightarrow \mathcal{F} \quad \text{and} \quad b : \mathcal{F} \longrightarrow \mathcal{O}_U^{\oplus r}$$

whose restriction to  $D(f)$  are equal to  $f^N \varphi$  and  $f^N \varphi^{-1}$  for some  $N$ . Moreover we may assume that  $a \circ b$  and  $b \circ a$  are equal to multiplication by  $f^{2N}$ . This follows from Properties, Lemma 17.3 since  $U$  is quasi-compact ( $I$  is finitely generated), separated, and  $\mathcal{F}$  and  $\mathcal{O}_U^{\oplus r}$  are finitely presented. Thus we see that  $H^p(U, \mathcal{F})$  is annihilated by  $f^{2N} J^n$ , i.e.,  $f^{2N} J^n \subset \mathfrak{a}$ .

As  $U$  is quasi-compact we can find finitely many  $f_1, \dots, f_t$  and  $N_1, \dots, N_t$  such that  $U = \bigcup D(f_i)$  and  $f_i^{2N_i} J^n \subset \mathfrak{a}$ . Then  $V(I) = V(f_1, \dots, f_t)$  and since  $I$  is finitely generated we conclude  $I^M \subset (f_1, \dots, f_t)$  for some  $M$ . All in all we see that  $J^m \subset \mathfrak{a}$  for  $m \gg 0$ , for example  $m = M(2N_1 + \dots + 2N_t)n$  will do.  $\square$

## 12. Annihilators of local cohomology, II

0EHX We extend the discussion of annihilators of local cohomology in Section 9 to bounded below complexes with finite cohomology modules.

0EHY **Definition 12.1.** Let  $I$  be an ideal of a Noetherian ring  $A$ . Let  $K \in D_{\text{Coh}}^+(A)$ . We define the  $I$ -depth of  $K$ , denoted  $\text{depth}_I(K)$ , to be the maximal  $m \in \mathbf{Z} \cup \{\infty\}$  such that  $H_I^i(K) = 0$  for all  $i < m$ . If  $A$  is local with maximal ideal  $\mathfrak{m}$  then we call  $\text{depth}_{\mathfrak{m}}(K)$  simply the *depth* of  $K$ .

This definition does not conflict with Algebra, Definition 71.1 by Dualizing Complexes, Lemma 11.1.

0EHZ **Proposition 12.2.** *Let  $A$  be a Noetherian ring which has a dualizing complex. Let  $T \subset T' \subset \text{Spec}(A)$  be subsets stable under specialization. Let  $s \in \mathbf{Z}$ . Let  $K$  be an object of  $D_{\text{Coh}}^+(A)$ . The following are equivalent*

- (1) *there exists an ideal  $J \subset A$  with  $V(J) \subset T'$  such that  $J$  annihilates  $H_T^i(K)$  for  $i \leq s$ , and*
- (2) *for all  $\mathfrak{p} \notin T'$ ,  $\mathfrak{q} \in T$  with  $\mathfrak{p} \subset \mathfrak{q}$  we have*

$$\text{depth}_{A_{\mathfrak{p}}}(K_{\mathfrak{p}}) + \dim((A/\mathfrak{p})_{\mathfrak{q}}) > s$$

**Proof.** This lemma is the natural generalization of Proposition 9.1 whose proof the reader should read first. Let  $\omega_A^\bullet$  be a dualizing complex. Let  $\delta$  be its dimension function, see Dualizing Complexes, Section 17. An important role will be played by the finite  $A$ -modules

$$E^i = \text{Ext}_A^i(K, \omega_A^\bullet)$$

For  $\mathfrak{p} \subset A$  we will write  $H_{\mathfrak{p}}^i$  to denote the local cohomology of an object of  $D(A_{\mathfrak{p}})$  with respect to  $\mathfrak{p}A_{\mathfrak{p}}$ . Then we see that the  $\mathfrak{p}A_{\mathfrak{p}}$ -adic completion of

$$(E^i)_{\mathfrak{p}} = \text{Ext}_{A_{\mathfrak{p}}}^{\delta(\mathfrak{p})+i}(K_{\mathfrak{p}}, (\omega_A^\bullet)_{\mathfrak{p}}[-\delta(\mathfrak{p})])$$

is Matlis dual to

$$H_{\mathfrak{p}}^{-\delta(\mathfrak{p})-i}(K_{\mathfrak{p}})$$

by Dualizing Complexes, Lemma 18.4. In particular we deduce from this the following fact: an ideal  $J \subset A$  annihilates  $(E^i)_{\mathfrak{p}}$  if and only if  $J$  annihilates  $H_{\mathfrak{p}}^{-\delta(\mathfrak{p})-i}(K_{\mathfrak{p}})$ .

Set  $T_n = \{\mathfrak{p} \in T \mid \delta(\mathfrak{p}) \leq n\}$ . As  $\delta$  is a bounded function, we see that  $T_a = \emptyset$  for  $a \ll 0$  and  $T_b = T$  for  $b \gg 0$ .

Assume (2). Let us prove the existence of  $J$  as in (1). We will use a double induction to do this. For  $i \leq s$  consider the induction hypothesis  $IH_i$ :  $H_T^a(K)$  is annihilated by some  $J \subset A$  with  $V(J) \subset T'$  for  $a \leq i$ . The case  $IH_i$  is trivial for  $i$  small enough because  $K$  is bounded below.

Induction step. Assume  $IH_{i-1}$  holds for some  $i \leq s$ . Pick  $J'$  with  $V(J') \subset T'$  annihilating  $H_T^a(K)$  for  $a \leq i-1$  (the induction hypothesis guarantees we can do this). We will show by descending induction on  $n$  that there exists an ideal  $J$  with  $V(J) \subset T'$  such that the associated primes of  $JH_T^i(K)$  are in  $T_n$ . For  $n \ll 0$  this implies  $JH_T^i(K) = 0$  (Algebra, Lemma 62.7) and hence  $IH_i$  will hold. The base case  $n \gg 0$  is trivial because  $T = T_n$  in this case and all associated primes of  $H_T^i(K)$  are in  $T$ .

Thus we assume given  $J$  with the property for  $n$ . Let  $\mathfrak{q} \in T_n$ . Let  $T_{\mathfrak{q}} \subset \text{Spec}(A_{\mathfrak{q}})$  be the inverse image of  $T$ . We have  $H_T^j(K)_{\mathfrak{q}} = H_{T_{\mathfrak{q}}}^j(K_{\mathfrak{q}})$  by Lemma 4.7. Consider the spectral sequence

$$H_{\mathfrak{q}}^p(H_{T_{\mathfrak{q}}}^q(K_{\mathfrak{q}})) \Rightarrow H_{\mathfrak{q}}^{p+q}(K_{\mathfrak{q}})$$

of Lemma 4.8. Below we will find an ideal  $J'' \subset A$  with  $V(J'') \subset T'$  such that  $H_{\mathfrak{q}}^i(K_{\mathfrak{q}})$  is annihilated by  $J''$  for all  $\mathfrak{q} \in T_n \setminus T_{n-1}$ . Claim:  $J(J')^i J''$  will work for  $n-1$ . Namely, let  $\mathfrak{q} \in T_n \setminus T_{n-1}$ . The spectral sequence above defines a filtration

$$E_{\infty}^{0,i} = E_{i+2}^{0,i} \subset \dots \subset E_3^{0,i} \subset E_2^{0,i} = H_{\mathfrak{q}}^0(H_{T_{\mathfrak{q}}}^i(K_{\mathfrak{q}}))$$

The module  $E_{\infty}^{0,i}$  is annihilated by  $J''$ . The subquotients  $E_j^{0,i}/E_{j+1}^{0,i}$  for  $i+1 \geq j \geq 2$  are annihilated by  $J'$  because the target of  $d_j^{0,i}$  is a subquotient of

$$H_{\mathfrak{q}}^j(H_{T_{\mathfrak{q}}}^{i-j+1}(K_{\mathfrak{q}})) = H_{\mathfrak{q}}^j(H_T^{i-j+1}(K)_{\mathfrak{q}})$$

and  $H_T^{i-j+1}(K)_\mathfrak{q}$  is annihilated by  $J'$  by choice of  $J'$ . Finally, by our choice of  $J$  we have  $JH_T^i(K)_\mathfrak{q} \subset H_\mathfrak{q}^0(H_T^i(K)_\mathfrak{q})$  since the non-closed points of  $\text{Spec}(A_\mathfrak{q})$  have higher  $\delta$  values. Thus  $\mathfrak{q}$  cannot be an associated prime of  $J(J')^i J'' H_T^i(K)$  as desired.

By our initial remarks we see that  $J''$  should annihilate

$$(E^{-\delta(\mathfrak{q})-i})_\mathfrak{q} = (E^{-n-i})_\mathfrak{q}$$

for all  $\mathfrak{q} \in T_n \setminus T_{n-1}$ . But if  $J''$  works for one  $\mathfrak{q}$ , then it works for all  $\mathfrak{q}$  in an open neighbourhood of  $\mathfrak{q}$  as the modules  $E^{-n-i}$  are finite. Since every subset of  $\text{Spec}(A)$  is Noetherian with the induced topology (Topology, Lemma 9.2), we conclude that it suffices to prove the existence of  $J''$  for one  $\mathfrak{q}$ .

Since the ext modules are finite the existence of  $J''$  is equivalent to

$$\text{Supp}(E^{-n-i}) \cap \text{Spec}(A_\mathfrak{q}) \subset T'$$

This is equivalent to showing the localization of  $E^{-n-i}$  at every  $\mathfrak{p} \subset \mathfrak{q}$ ,  $\mathfrak{p} \notin T'$  is zero. Using local duality over  $A_\mathfrak{p}$  we find that we need to prove that

$$H_\mathfrak{p}^{i+n-\delta(\mathfrak{p})}(K_\mathfrak{p}) = H_\mathfrak{p}^{i-\dim((A/\mathfrak{p})_\mathfrak{q})}(K_\mathfrak{p})$$

is zero (this uses that  $\delta$  is a dimension function). This vanishes by the assumption in the lemma and  $i \leq s$  and our definition of depth in Definition 12.1.

To prove the converse implication we assume (2) does not hold and we work backwards through the arguments above. First, we pick a  $\mathfrak{q} \in T$ ,  $\mathfrak{p} \subset \mathfrak{q}$  with  $\mathfrak{p} \notin T'$  such that

$$i = \text{depth}_{A_\mathfrak{p}}(K_\mathfrak{p}) + \dim((A/\mathfrak{p})_\mathfrak{q}) \leq s$$

is minimal. Then  $H_\mathfrak{p}^{i-\dim((A/\mathfrak{p})_\mathfrak{q})}(K_\mathfrak{p})$  is nonzero by our definition of depth in Definition 12.1. Set  $n = \delta(\mathfrak{q})$ . Then there does not exist an ideal  $J \subset A$  with  $V(J) \subset T'$  such that  $J(E^{-n-i})_\mathfrak{q} = 0$ . Thus  $H_\mathfrak{q}^i(K_\mathfrak{q})$  is not annihilated by an ideal  $J \subset A$  with  $V(J) \subset T'$ . By minimality of  $i$  it follows from the spectral sequence displayed above that the module  $H_T^i(K)_\mathfrak{q}$  is not annihilated by an ideal  $J \subset A$  with  $V(J) \subset T'$ . Thus  $H_T^i(K)$  is not annihilated by an ideal  $J \subset A$  with  $V(J) \subset T'$ . This finishes the proof of the proposition.  $\square$

### 13. Finiteness of local cohomology, III

0EI0 We extend the discussion of finiteness of local cohomology in Sections 6 and 10 to bounded below complexes with finite cohomology modules.

0EI1 **Lemma 13.1.** *Let  $A$  be a Noetherian ring. Let  $T \subset \text{Spec}(A)$  be a subset stable under specialization. Let  $K$  be an object of  $D_{\text{Coh}}^+(A)$ . Let  $n \in \mathbf{Z}$ . The following are equivalent*

- (1)  $H_T^i(K)$  is finite for  $i \leq n$ ,
- (2) there exists an ideal  $J \subset A$  with  $V(J) \subset T$  such that  $J$  annihilates  $H_T^i(K)$  for  $i \leq n$ .

If  $T = V(I) = Z$  for an ideal  $I \subset A$ , then these are also equivalent to

- (3) there exists an  $e \geq 0$  such that  $I^e$  annihilates  $H_Z^i(K)$  for  $i \leq n$ .

**Proof.** This lemma is the natural generalization of Lemma 6.1 whose proof the reader should read first. Assume (1) is true. Recall that  $H_Z^j(K) = H_{V(J)}^j(K)$ , see Dualizing Complexes, Lemma 10.1. Thus  $H_T^i(K) = \text{colim } H_J^i(K)$  where the colimit

is over ideals  $J \subset A$  with  $V(J) \subset T$ , see Lemma 4.3. Since  $H_T^i(K)$  is finitely generated for  $i \leq n$  we can find a  $J \subset A$  as in (2) such that  $H_J^i(K) \rightarrow H_T^i(K)$  is surjective for  $i \leq n$ . Thus the finite list of generators are  $J$ -power torsion elements and we see that (2) holds with  $J$  replaced by some power.

Let  $a \in \mathbf{Z}$  be an integer such that  $H^i(K) = 0$  for  $i < a$ . We prove (2)  $\Rightarrow$  (1) by descending induction on  $a$ . If  $a > n$ , then we have  $H_T^i(K) = 0$  for  $i \leq n$  hence both (1) and (2) are true and there is nothing to prove.

Assume we have  $J$  as in (2). Observe that  $N = H_T^a(K) = H_T^0(H^a(K))$  is finite as a submodule of the finite  $A$ -module  $H^a(K)$ . If  $n = a$  we are done; so assume  $a < n$  from now on. By construction of  $R\Gamma_T$  we find that  $H_T^i(N) = 0$  for  $i > 0$  and  $H_T^0(N) = N$ , see Remark 4.6. Choose a distinguished triangle

$$N[-a] \rightarrow K \rightarrow K' \rightarrow N[-a+1]$$

Then we see that  $H_T^a(K') = 0$  and  $H_T^i(K) = H_T^i(K')$  for  $i > a$ . We conclude that we may replace  $K$  by  $K'$ . Thus we may assume that  $H_T^a(K) = 0$ . This means that the finite set of associated primes of  $H^a(K)$  are not in  $T$ . By prime avoidance (Algebra, Lemma 14.2) we can find  $f \in J$  not contained in any of the associated primes of  $H^a(K)$ . Choose a distinguished triangle

$$L \rightarrow K \xrightarrow{f} K \rightarrow L[1]$$

By construction we see that  $H^i(L) = 0$  for  $i \leq a$ . On the other hand we have a long exact cohomology sequence

$$0 \rightarrow H_T^{a+1}(L) \rightarrow H_T^{a+1}(K) \xrightarrow{f} H_T^{a+1}(K) \rightarrow H_T^{a+2}(L) \rightarrow H_T^{a+2}(K) \xrightarrow{f} \dots$$

which breaks into the identification  $H_T^{a+1}(L) = H_T^{a+1}(K)$  and short exact sequences

$$0 \rightarrow H_T^{i-1}(K) \rightarrow H_T^i(L) \rightarrow H_T^i(K) \rightarrow 0$$

for  $i \leq n$  since  $f \in J$ . We conclude that  $J^2$  annihilates  $H_T^i(L)$  for  $i \leq n$ . By induction hypothesis applied to  $L$  we see that  $H_T^i(L)$  is finite for  $i \leq n$ . Using the short exact sequence once more we see that  $H_T^i(K)$  is finite for  $i \leq n$  as desired.

We omit the proof of the equivalence of (2) and (3) in case  $T = V(I)$ . □

0EI2 **Proposition 13.2.** *Let  $A$  be a Noetherian ring which has a dualizing complex. Let  $T \subset \text{Spec}(A)$  be a subset stable under specialization. Let  $s \in \mathbf{Z}$ . Let  $K \in D_{\text{Coh}}^+(A)$ . The following are equivalent*

- (1)  $H_T^i(K)$  is a finite  $A$ -module for  $i \leq s$ , and
- (2) for all  $\mathfrak{p} \notin T$ ,  $\mathfrak{q} \in T$  with  $\mathfrak{p} \subset \mathfrak{q}$  we have

$$\text{depth}_{A_{\mathfrak{p}}}(K_{\mathfrak{p}}) + \dim((A/\mathfrak{p})_{\mathfrak{q}}) > s$$

**Proof.** Formal consequence of Proposition 12.2 and Lemma 13.1. □

### 14. Improving coherent modules

0DX2 Similar constructions can be found in [DG67] and more recently in [Kol15] and [Kol16].

0DX3 **Lemma 14.1.** *Let  $X$  be a Noetherian scheme. Let  $T \subset X$  be a subset stable under specialization. Let  $\mathcal{F}$  be a coherent  $\mathcal{O}_X$ -module. Then there is a unique map  $\mathcal{F} \rightarrow \mathcal{F}'$  of coherent  $\mathcal{O}_X$ -modules such that*

- (1)  $\mathcal{F} \rightarrow \mathcal{F}'$  is surjective,
- (2)  $\mathcal{F}_x \rightarrow \mathcal{F}'_x$  is an isomorphism for  $x \notin T$ ,
- (3)  $\text{depth}_{\mathcal{O}_{X,x}}(\mathcal{F}'_x) \geq 1$  for  $x \in T$ .

If  $f : Y \rightarrow X$  is a flat morphism with  $Y$  Noetherian, then  $f^*\mathcal{F} \rightarrow f^*\mathcal{F}'$  is the corresponding quotient for  $f^{-1}(T) \subset Y$  and  $f^*\mathcal{F}$ .

**Proof.** Condition (3) just means that  $\text{Ass}(\mathcal{F}') \cap T = \emptyset$ . Thus  $\mathcal{F} \rightarrow \mathcal{F}'$  is the quotient of  $\mathcal{F}$  by the subsheaf of sections whose support is contained in  $T$ . This proves uniqueness. The statement on pullbacks follows from Divisors, Lemma 3.1 and the uniqueness.

Existence of  $\mathcal{F} \rightarrow \mathcal{F}'$ . By the uniqueness it suffices to prove the existence and uniqueness locally on  $X$ ; small detail omitted. Thus we may assume  $X = \text{Spec}(A)$  is affine and  $\mathcal{F}$  is the coherent module associated to the finite  $A$ -module  $M$ . Set  $M' = M/H_T^0(M)$  with  $H_T^0(M)$  as in Section 4. Then  $M_{\mathfrak{p}} = M'_{\mathfrak{p}}$  for  $\mathfrak{p} \notin T$  which proves (1). On the other hand, we have  $H_T^0(M) = \text{colim} H_Z^0(M)$  where  $Z$  runs over the closed subsets of  $X$  contained in  $T$ . Thus by Dualizing Complexes, Lemmas 11.6 we have  $H_T^0(M') = 0$ , i.e., no associated prime of  $M'$  is in  $T$ . Therefore  $\text{depth}(M'_{\mathfrak{p}}) \geq 1$  for  $\mathfrak{p} \in T$ .  $\square$

0DX4 **Lemma 14.2.** *Let  $j : U \rightarrow X$  be an open immersion of Noetherian schemes. Let  $\mathcal{F}$  be a coherent  $\mathcal{O}_X$ -module. Assume  $\mathcal{F}' = j_*(\mathcal{F}|_U)$  is coherent. Then  $\mathcal{F} \rightarrow \mathcal{F}'$  is the unique map of coherent  $\mathcal{O}_X$ -modules such that*

- (1)  $\mathcal{F}|_U \rightarrow \mathcal{F}'|_U$  is an isomorphism,
- (2)  $\text{depth}_{\mathcal{O}_{X,x}}(\mathcal{F}'_x) \geq 2$  for  $x \in X$ ,  $x \notin U$ .

If  $f : Y \rightarrow X$  is a flat morphism with  $Y$  Noetherian, then  $f^*\mathcal{F} \rightarrow f^*\mathcal{F}'$  is the corresponding map for  $f^{-1}(U) \subset Y$ .

**Proof.** Let us show that  $\text{depth}_{\mathcal{O}_{X,x}}(\mathcal{F}'_x) \geq 2$  for  $x \in X$ ,  $x \notin U$ . Namely, let  $V$  be the punctured spectrum of  $\mathcal{O}_{X,x}$ . Then  $V$  contains the inverse image of  $U$  along  $\text{Spec}(\mathcal{O}_{X,x}) \rightarrow X$ . Since  $\mathcal{F}' = j_*(\mathcal{F}|_U) = j_*(\mathcal{F}'|_U)$ , the same is true after base change by the flat morphism  $\text{Spec}(\mathcal{O}_{X,x}) \rightarrow X$  (Cohomology of Schemes, Lemma 5.2). A fortiori, the canonical map  $\mathcal{F}'_x \rightarrow H^0(V, \mathcal{F}'|_V)$  is an isomorphism. This means that  $H_{\mathfrak{m}_x}^i(\mathcal{F}'_x)$  is zero for  $i = 0, 1$ , see Lemma 7.2. Thus the depth is at least 2 by Dualizing Complexes, Lemma 11.1.

The uniqueness of  $\mathcal{F} \rightarrow \mathcal{F}'$  follows from Divisors, Lemma 5.11. The compatibility with flat pullbacks follows from flat base change, see Cohomology of Schemes, Lemma 5.2.  $\square$

0DX5 **Lemma 14.3.** *Let  $X$  be a Noetherian scheme. Let  $Z \subset X$  be a closed subscheme. Let  $\mathcal{F}$  be a coherent  $\mathcal{O}_X$ -module. Assume  $X$  is universally catenary and the formal fibres of local rings have  $(S_1)$ . Then there exists a unique map  $\mathcal{F} \rightarrow \mathcal{F}''$  of coherent  $\mathcal{O}_X$ -modules such that*

- (1)  $\mathcal{F}_x \rightarrow \mathcal{F}''_x$  is an isomorphism for  $x \in X \setminus Z$ ,
- (2)  $\mathcal{F}_x \rightarrow \mathcal{F}''_x$  is surjective and  $\text{depth}_{\mathcal{O}_{X,x}}(\mathcal{F}''_x) = 1$  for  $x \in Z$  such that there exists an immediate specialization  $x' \rightsquigarrow x$  with  $x' \notin Z$  and  $x' \in \text{Ass}(\mathcal{F})$ ,
- (3)  $\text{depth}_{\mathcal{O}_{X,x}}(\mathcal{F}''_x) \geq 2$  for the remaining  $x \in Z$ .

If  $f : Y \rightarrow X$  is a Cohen-Macaulay morphism with  $Y$  Noetherian, then  $f^*\mathcal{F} \rightarrow f^*\mathcal{F}''$  satisfies the same properties with respect to  $f^{-1}(Z) \subset Y$ .

**Proof.** Let  $\mathcal{F} \rightarrow \mathcal{F}'$  be the map constructed in Lemma 14.1 for the subset  $Z$  of  $X$ . Recall that  $\mathcal{F}'$  is the quotient of  $\mathcal{F}$  by the subsheaf of sections supported on  $Z$ .

We first prove uniqueness. Let  $\mathcal{F} \rightarrow \mathcal{F}''$  be as in the lemma. We get a factorization  $\mathcal{F} \rightarrow \mathcal{F}' \rightarrow \mathcal{F}''$  since  $\text{Ass}(\mathcal{F}'') \cap Z = \emptyset$  by conditions (2) and (3). Let  $U \subset X$  be a maximal open subscheme such that  $\mathcal{F}'|_U \rightarrow \mathcal{F}''|_U$  is an isomorphism. We see that  $U$  contains all the points as in (2). Then by Divisors, Lemma 5.11 we conclude that  $\mathcal{F}'' = j_*(\mathcal{F}'|_U)$ . In this way we get uniqueness (small detail: if we have two of these  $\mathcal{F}''$  then we take the intersection of the opens  $U$  we get from either).

Proof of existence. Recall that  $\text{Ass}(\mathcal{F}') = \{x_1, \dots, x_n\}$  is finite and  $x_i \notin Z$ . Let  $Y_i$  be the closure of  $\{x_i\}$ . Let  $Z_{i,j}$  be the irreducible components of  $Z \cap Y_i$ . Observe that  $\text{Supp}(\mathcal{F}') \cap Z = \bigcup Z_{i,j}$ . Let  $z_{i,j} \in Z_{i,j}$  be the generic point. Let

$$d_{i,j} = \dim(\mathcal{O}_{\overline{\{x_i\}, z_{i,j}}})$$

If  $d_{i,j} = 1$ , then  $z_{i,j}$  is one of the points as in (2). Thus we do not need to modify  $\mathcal{F}'$  at these points. Furthermore, still assuming  $d_{i,j} = 1$ , using Lemma 8.2 we can find an open neighbourhood  $z_{i,j} \in V_{i,j} \subset X$  such that  $\text{depth}_{\mathcal{O}_{X,z}}(\mathcal{F}'_z) \geq 2$  for  $z \in Z_{i,j} \cap V_{i,j}$ ,  $z \neq z_{i,j}$ . Set

$$Z' = X \setminus \left( X \setminus Z \cup \bigcup_{d_{i,j}=1} V_{i,j} \right)$$

Denote  $j' : X \setminus Z' \rightarrow X$ . By our choice of  $Z'$  the assumptions of Lemma 7.9 are satisfied. We conclude by setting  $\mathcal{F}'' = j'_*(\mathcal{F}'|_{X \setminus Z'})$  and applying Lemma 14.2.

The final statement follows from the formula for the change in depth along a flat local homomorphism, see Algebra, Lemma 157.1 and the assumption on the fibres of  $f$  inherent in  $f$  being Cohen-Macaulay. Details omitted.  $\square$

0E13 **Lemma 14.4.** *Let  $X$  be a Noetherian scheme which locally has a dualizing complex. Let  $T' \subset X$  be a subset stable under specialization. Let  $\mathcal{F}$  be a coherent  $\mathcal{O}_X$ -module. Assume that if  $x \rightsquigarrow x'$  is an immediate specialization of points in  $X$  with  $x' \in T'$  and  $x \notin T'$ , then  $\text{depth}(\mathcal{F}_x) \geq 1$ . Then there exists a unique map  $\mathcal{F} \rightarrow \mathcal{F}''$  of coherent  $\mathcal{O}_X$ -modules such that*

- (1)  $\mathcal{F}_x \rightarrow \mathcal{F}''_x$  is an isomorphism for  $x \notin T'$ ,
- (2)  $\text{depth}_{\mathcal{O}_{X,x}}(\mathcal{F}''_x) \geq 2$  for  $x \in T'$ .

*If  $f : Y \rightarrow X$  is a Cohen-Macaulay morphism with  $Y$  Noetherian, then  $f^*\mathcal{F} \rightarrow f^*\mathcal{F}''$  satisfies the same properties with respect to  $f^{-1}(T') \subset Y$ .*

**Proof.** Let  $\mathcal{F} \rightarrow \mathcal{F}'$  be the quotient of  $\mathcal{F}$  constructed in Lemma 14.1 using  $T'$ . Recall that  $\mathcal{F}'$  is the quotient of  $\mathcal{F}$  by the subsheaf of sections supported on  $T'$ .

Proof of uniqueness. Let  $\mathcal{F} \rightarrow \mathcal{F}''$  be as in the lemma. We get a factorization  $\mathcal{F} \rightarrow \mathcal{F}' \rightarrow \mathcal{F}''$  since  $\text{Ass}(\mathcal{F}'') \cap T' = \emptyset$  by condition (2). Let  $U \subset X$  be a maximal open subscheme such that  $\mathcal{F}'|_U \rightarrow \mathcal{F}''|_U$  is an isomorphism. We see that  $U$  contains all the points of  $T'$ . Then by Divisors, Lemma 5.11 we conclude that  $\mathcal{F}'' = j_*(\mathcal{F}'|_U)$ . In this way we get uniqueness (small detail: if we have two of these  $\mathcal{F}''$  then we take the intersection of the opens  $U$  we get from either).

Proof of existence. We will define

$$\mathcal{F}'' = \text{colim } j_*(\mathcal{F}'|_V)$$

where  $j : V \rightarrow X$  runs over the open subschemes such that  $X \setminus V \subset T'$ . Observe that the colimit is filtered as  $T'$  is stable under specialization. Each of the maps  $\mathcal{F}' \rightarrow j_*(\mathcal{F}'|_V)$  is injective as  $\text{Ass}(\mathcal{F}')$  is disjoint from  $T'$ . Thus  $\mathcal{F}' \rightarrow \mathcal{F}''$  is injective.

Suppose  $X = \text{Spec}(A)$  is affine and  $\mathcal{F}$  corresponds to the finite  $A$ -module  $M$ . Then  $\mathcal{F}'$  corresponds to  $M' = M/H_{T'}^0(M)$ , see proof of Lemma 14.1. Applying Lemmas 2.2 and 4.3 we see that  $\mathcal{F}''$  corresponds to an  $A$ -module  $M''$  which fits into the short exact sequence

$$0 \rightarrow M' \rightarrow M'' \rightarrow H_{T'}^1(M') \rightarrow 0$$

By Proposition 10.1 and our condition on immediate specializations in the statement of the lemma we see that  $M''$  is a finite  $A$ -module. In this way we see that  $\mathcal{F}''$  is coherent.

The final statement follows from the formula for the change in depth along a flat local homomorphism, see Algebra, Lemma 157.1 and the assumption on the fibres of  $f$  inherent in  $f$  being Cohen-Macaulay. Details omitted.  $\square$

0EI4 **Lemma 14.5.** *Let  $X$  be a Noetherian scheme which locally has a dualizing complex. Let  $T' \subset T \subset X$  be subsets stable under specialization such that if  $x \rightsquigarrow x'$  is an immediate specialization of points in  $X$  and  $x' \in T'$ , then  $x \in T$ . Let  $\mathcal{F}$  be a coherent  $\mathcal{O}_X$ -module. Then there exists a unique map  $\mathcal{F} \rightarrow \mathcal{F}''$  of coherent  $\mathcal{O}_X$ -modules such that*

- (1)  $\mathcal{F}_x \rightarrow \mathcal{F}''_x$  is an isomorphism for  $x \notin T$ ,
- (2)  $\mathcal{F}_x \rightarrow \mathcal{F}''_x$  is surjective and  $\text{depth}_{\mathcal{O}_{X,x}}(\mathcal{F}''_x) \geq 1$  for  $x \in T$ ,  $x \notin T'$ , and
- (3)  $\text{depth}_{\mathcal{O}_{X,x}}(\mathcal{F}''_x) \geq 2$  for  $x \in T'$ .

If  $f : Y \rightarrow X$  is a Cohen-Macaulay morphism with  $Y$  Noetherian, then  $f^*\mathcal{F} \rightarrow f^*\mathcal{F}''$  satisfies the same properties with respect to  $f^{-1}(T') \subset f^{-1}(T) \subset Y$ .

**Proof.** First, let  $\mathcal{F} \rightarrow \mathcal{F}'$  be the quotient of  $\mathcal{F}$  constructed in Lemma 14.1 using  $T$ . Second, let  $\mathcal{F}' \rightarrow \mathcal{F}''$  be the unique map of coherent modules construction in Lemma 14.4 using  $T'$ . Then  $\mathcal{F} \rightarrow \mathcal{F}''$  is as desired.  $\square$

### 15. Hartshorne-Lichtenbaum vanishing

0EB0 This and much else besides can be found in [Har68].

0EB1 **Lemma 15.1.** *Let  $A$  be a Noetherian ring of dimension  $d$ . Let  $I \subset I' \subset A$  be ideals. If  $I'$  is contained in the Jacobson radical of  $A$  and  $\text{cd}(A, I') < d$ , then  $\text{cd}(A, I) < d$ .*

**Proof.** By Lemma 3.7 we know  $\text{cd}(A, I) \leq d$ . We will use Lemma 2.6 to show

$$H_{V(I')}^d(A) \rightarrow H_{V(I)}^d(A)$$

is surjective which will finish the proof. Pick  $\mathfrak{p} \in V(I) \setminus V(I')$ . By our assumption on  $I'$  we see that  $\mathfrak{p}$  is not a maximal ideal of  $A$ . Hence  $\dim(A_{\mathfrak{p}}) < d$ . Then  $H_{\mathfrak{p}A_{\mathfrak{p}}}^d(A_{\mathfrak{p}}) = 0$  by Lemma 3.7.  $\square$

0EB2 **Lemma 15.2.** *Let  $A$  be a Noetherian ring of dimension  $d$ . Let  $I \subset A$  be an ideal. If  $H_{V(I)}^d(M) = 0$  for some finite  $A$ -module whose support contains all the irreducible components of dimension  $d$ , then  $\text{cd}(A, I) < d$ .*



**Proof.** By Lemma 3.7 we know  $\text{cd}(A, I) \leq d$ . Thus for any finite  $A$ -module  $N$  we have  $H_{V(I)}^i(N) = 0$  for  $i > d$ . Let us say property  $\mathcal{P}$  holds for the finite  $A$ -module  $N$  if  $H_{V(I)}^d(N) = 0$ . One of our assumptions is that  $\mathcal{P}(M)$  holds. Observe that  $\mathcal{P}(N_1 \oplus N_2) \Leftrightarrow (\mathcal{P}(N_1) \wedge \mathcal{P}(N_2))$ . Observe that if  $N \rightarrow N'$  is surjective, then  $\mathcal{P}(N) \Rightarrow \mathcal{P}(N')$  as we have the vanishing of  $H_{V(I)}^{d+1}$  (see above). Let  $\mathfrak{p}_1, \dots, \mathfrak{p}_n$  be the minimal primes of  $A$  with  $\dim(A/\mathfrak{p}_i) = d$ . Observe that  $\mathcal{P}(N)$  holds if the support of  $N$  is disjoint from  $\{\mathfrak{p}_1, \dots, \mathfrak{p}_n\}$  for dimension reasons, see Lemma 3.7. For each  $i$  set  $M_i = M/\mathfrak{p}_i M$ . This is a finite  $A$ -module annihilated by  $\mathfrak{p}_i$  whose support is equal to  $V(\mathfrak{p}_i)$  (here we use the assumption on the support of  $M$ ). Finally, if  $J \subset A$  is an ideal, then we have  $\mathcal{P}(JM_i)$  as  $JM_i$  is a quotient of a direct sum of copies of  $M$ . Thus it follows from Cohomology of Schemes, Lemma 12.8 that  $\mathcal{P}$  holds for every finite  $A$ -module.  $\square$

0EB3 **Lemma 15.3.** *Let  $A$  be a Noetherian local ring of dimension  $d$ . Let  $f \in A$  be an element which is not contained in any minimal prime of dimension  $d$ . Then  $f : H_{V(I)}^d(M) \rightarrow H_{V(I)}^d(M)$  is surjective for any finite  $A$ -module  $M$  and any ideal  $I \subset A$ .*

**Proof.** The support of  $M/fM$  has dimension  $< d$  by our assumption on  $f$ . Thus  $H_{V(I)}^d(M/fM) = 0$  by Lemma 3.7. Thus  $H_{V(I)}^d(fM) \rightarrow H_{V(I)}^d(M)$  is surjective. Since by Lemma 3.7 we know  $\text{cd}(A, I) \leq d$  we also see that the surjection  $M \rightarrow fM$ ,  $x \mapsto fx$  induces a surjection  $H_{V(I)}^d(M) \rightarrow H_{V(I)}^d(fM)$ .  $\square$

0EB4 **Lemma 15.4.** *Let  $A$  be a Noetherian local ring with normalized dualizing complex  $\omega_A^\bullet$ . Let  $I \subset A$  be an ideal. If  $H_{V(I)}^0(\omega_A^\bullet) = 0$ , then  $\text{cd}(A, I) < \dim(A)$ .*

**Proof.** Set  $d = \dim(A)$ . Let  $\mathfrak{p}_1, \dots, \mathfrak{p}_n \subset A$  be the minimal primes of dimension  $d$ . Recall that the finite  $A$ -module  $H^{-i}(\omega_A^\bullet)$  is nonzero only for  $i \in \{0, \dots, d\}$  and that the support of  $H^{-i}(\omega_A^\bullet)$  has dimension  $\leq i$ , see Lemma 8.4. Set  $\omega_A = H^{-d}(\omega_A^\bullet)$ . By prime avoidance (Algebra, Lemma 14.2) we can find  $f \in A$ ,  $f \notin \mathfrak{p}_i$  which annihilates  $H^{-i}(\omega_A^\bullet)$  for  $i < d$ . Consider the distinguished triangle

$$\omega_A[d] \rightarrow \omega_A^\bullet \rightarrow \tau_{\geq -d+1}\omega_A^\bullet \rightarrow \omega_A[d+1]$$

See Derived Categories, Remark 12.4. By Derived Categories, Lemma 12.5 we see that  $f^d$  induces the zero endomorphism of  $\tau_{\geq -d+1}\omega_A^\bullet$ . Using the axioms of a triangulated category, we find a map

$$\omega_A^\bullet \rightarrow \omega_A[d]$$

whose composition with  $\omega_A[d] \rightarrow \omega_A^\bullet$  is multiplication by  $f^d$  on  $\omega_A[d]$ . Thus we conclude that  $f^d$  annihilates  $H_{V(I)}^d(\omega_A)$ . By Lemma 15.3 we conclude  $H_{V(I)}^d(\omega_A) = 0$ . Then we conclude by Lemma 15.2 and the fact that  $(\omega_A)_{\mathfrak{p}_i}$  is nonzero (see for example Dualizing Complexes, Lemma 16.11).  $\square$

0EB5 **Lemma 15.5.** *Let  $(A, \mathfrak{m})$  be a complete Noetherian local domain. Let  $\mathfrak{p} \subset A$  be a prime ideal of dimension 1. For every  $n \geq 1$  there is an  $m \geq n$  such that  $\mathfrak{p}^{(m)} \subset \mathfrak{p}^n$ .*

**Proof.** Recall that the symbolic power  $\mathfrak{p}^{(m)}$  is defined as the kernel of  $A \rightarrow A_{\mathfrak{p}}/\mathfrak{p}^m A_{\mathfrak{p}}$ . Since localization is exact we conclude that in the short exact sequence

$$0 \rightarrow \mathfrak{a}_n \rightarrow A/\mathfrak{p}^n \rightarrow A/\mathfrak{p}^{(n)} \rightarrow 0$$

the support of  $\mathfrak{a}_n$  is contained in  $\{\mathfrak{m}\}$ . In particular, the inverse system  $(\mathfrak{a}_n)$  is Mittag-Leffler as each  $\mathfrak{a}_n$  is an Artinian  $A$ -module. We conclude that the lemma is equivalent to the requirement that  $\lim \mathfrak{a}_n = 0$ . Let  $f \in \lim \mathfrak{a}_n$ . Then  $f$  is an element of  $A = \lim A/\mathfrak{p}^n$  (here we use that  $A$  is complete) which maps to zero in the completion  $A_{\mathfrak{p}}^{\wedge}$  of  $A_{\mathfrak{p}}$ . Since  $A_{\mathfrak{p}} \rightarrow A_{\mathfrak{p}}^{\wedge}$  is faithfully flat, we see that  $f$  maps to zero in  $A_{\mathfrak{p}}$ . Since  $A$  is a domain we see that  $f$  is zero as desired.  $\square$

0EB6 **Proposition 15.6.** *Let  $A$  be a Noetherian local ring with completion  $A^{\wedge}$ . Let  $I \subset A$  be an ideal such that* [Har68, Theorem 3.1]

$$\dim V(IA^{\wedge} + \mathfrak{p}) \geq 1$$

for every minimal prime  $\mathfrak{p} \subset A^{\wedge}$  of dimension  $\dim(A)$ . Then  $cd(A, I) < \dim(A)$ .

**Proof.** Since  $A \rightarrow A^{\wedge}$  is faithfully flat we have  $H_{V(I)}^d(A) \otimes_A A^{\wedge} = H_{V(IA^{\wedge})}^d(A^{\wedge})$  by Dualizing Complexes, Lemma 9.3. Thus we may assume  $A$  is complete.

Assume  $A$  is complete. Let  $\mathfrak{p}_1, \dots, \mathfrak{p}_n \subset A$  be the minimal primes of dimension  $d$ . Consider the complete local ring  $A_i = A/\mathfrak{p}_i$ . We have  $H_{V(I)}^d(A_i) = H_{V(IA_i)}^d(A_i)$  by Dualizing Complexes, Lemma 9.2. By Lemma 15.2 it suffices to prove the lemma for  $(A_i, IA_i)$ . Thus we may assume  $A$  is a complete local domain.

Assume  $A$  is a complete local domain. We can choose a prime ideal  $\mathfrak{p} \supset I$  with  $\dim(A/\mathfrak{p}) = 1$ . By Lemma 15.1 it suffices to prove the lemma for  $\mathfrak{p}$ .

By Lemma 15.4 it suffices to show that  $H_{V(\mathfrak{p})}^0(\omega_A^{\bullet}) = 0$ . Recall that

$$H_{V(\mathfrak{p})}^0(\omega_A^{\bullet}) = \operatorname{colim} \operatorname{Ext}_A^0(A/\mathfrak{p}^n, \omega_A^{\bullet})$$

By Lemma 15.5 we see that the colimit is the same as

$$\operatorname{colim} \operatorname{Ext}_A^0(A/\mathfrak{p}^{(n)}, \omega_A^{\bullet})$$

Since  $\operatorname{depth}(A/\mathfrak{p}^{(n)}) = 1$  we see that these ext groups are zero by Lemma 8.4 as desired.  $\square$

0EB7 **Lemma 15.7.** *Let  $(A, \mathfrak{m})$  be a Noetherian local ring. Let  $I \subset A$  be an ideal. Assume  $A$  is excellent, normal, and  $\dim V(I) \geq 1$ . Then  $cd(A, I) < \dim(A)$ . In particular, if  $\dim(A) = 2$ , then  $\operatorname{Spec}(A) \setminus V(I)$  is affine.*

**Proof.** By More on Algebra, Lemma 51.6 the completion  $A^{\wedge}$  is normal and hence a domain. Thus the assumption of Proposition 15.6 holds and we conclude. The statement on affineness follows from Lemma 3.8.  $\square$

## 16. Frobenius action

0EBU Let  $p$  be a prime number. Let  $A$  be a ring with  $p = 0$  in  $A$ . The Frobenius endomorphism of  $A$  is the map

$$F: A \longrightarrow A, \quad a \longmapsto a^p$$

In this section we prove lemmas on modules which have Frobenius actions.

0EBV **Lemma 16.1.** *Let  $p$  be a prime number. Let  $(A, \mathfrak{m}, \kappa)$  be a Noetherian local ring with  $p = 0$  in  $A$ . Let  $M$  be a finite  $A$ -module such that  $M \otimes_{A, F} A \cong M$ . Then  $M$  is finite free.*

**Proof.** Choose a presentation  $A^{\oplus m} \rightarrow A^{\oplus n} \rightarrow M$  which induces an isomorphism  $\kappa^{\oplus n} \rightarrow M/\mathfrak{m}M$ . Let  $T = (a_{ij})$  be the matrix of the map  $A^{\oplus m} \rightarrow A^{\oplus n}$ . Observe that  $a_{ij} \in \mathfrak{m}$ . Applying base change by  $F$ , using right exactness of base change, we get a presentation  $A^{\oplus m} \rightarrow A^{\oplus n} \rightarrow M$  where the matrix is  $T = (a_{ij}^p)$ . Thus we have a presentation with  $a_{ij} \in \mathfrak{m}^p$ . Repeating this construction we find that for each  $e \geq 1$  there exists a presentation with  $a_{ij} \in \mathfrak{m}^e$ . This implies the fitting ideals (More on Algebra, Definition 8.3)  $\text{Fit}_k(M)$  for  $k < n$  are contained in  $\bigcap_{e \geq 1} \mathfrak{m}^e$ . Since this is zero by Krull's intersection theorem (Algebra, Lemma 50.4) we conclude that  $M$  is free of rank  $n$  by More on Algebra, Lemma 8.7.  $\square$

In this section, we say elements  $f_1, \dots, f_r$  of a ring  $A$  are *independent* if  $\sum a_i f_i = 0$  implies  $a_i \in (f_1, \dots, f_r)$ . In other words, with  $I = (f_1, \dots, f_r)$  we have  $I/I^2$  is free over  $A/I$  with basis  $f_1, \dots, f_r$ .

0EBW **Lemma 16.2.** *Let  $A$  be a ring. If  $f_1, \dots, f_{r-1}, f_r g_r$  are independent, then  $f_1, \dots, f_r$  are independent.*

See [Lec64] and [Mat70, Lemma 1 page 299].

**Proof.** Say  $\sum a_i f_i = 0$ . Then  $\sum a_i g_r f_i = 0$ . Hence  $a_r \in (f_1, \dots, f_{r-1}, f_r g_r)$ . Write  $a_r = \sum_{i < r} b_i f_i + b f_r g_r$ . Then  $0 = \sum_{i < r} (a_i + b_i f_r) f_i + b f_r^2 g_r$ . Thus  $a_i + b_i f_r \in (f_1, \dots, f_{r-1}, f_r g_r)$  which implies  $a_i \in (f_1, \dots, f_r)$  as desired.  $\square$

0EBX **Lemma 16.3.** *Let  $A$  be a ring. If  $f_1, \dots, f_{r-1}, f_r g_r$  are independent and if the  $A$ -module  $A/(f_1, \dots, f_{r-1}, f_r g_r)$  has finite length, then*

See [Lec64] and [Mat70, Lemma 2 page 300].

$$\begin{aligned} \text{length}_A(A/(f_1, \dots, f_{r-1}, f_r g_r)) \\ = \text{length}_A(A/(f_1, \dots, f_{r-1}, f_r)) + \text{length}_A(A/(f_1, \dots, f_{r-1}, g_r)) \end{aligned}$$

**Proof.** We claim there is an exact sequence

$$0 \rightarrow A/(f_1, \dots, f_{r-1}, g_r) \xrightarrow{f_r} A/(f_1, \dots, f_{r-1}, f_r g_r) \rightarrow A/(f_1, \dots, f_{r-1}, f_r) \rightarrow 0$$

Namely, if  $a f_r \in (f_1, \dots, f_{r-1}, f_r g_r)$ , then  $\sum_{i < r} a_i f_i + (a + b g_r) f_r = 0$  for some  $b, a_i \in A$ . Hence  $\sum_{i < r} a_i g_r f_i + (a + b g_r) g_r f_r = 0$  which implies  $a + b g_r \in (f_1, \dots, f_{r-1}, f_r g_r)$  which means that  $a$  maps to zero in  $A/(f_1, \dots, f_{r-1}, g_r)$ . This proves the claim. To finish use additivity of lengths (Algebra, Lemma 51.3).  $\square$

0EBY **Lemma 16.4.** *Let  $(A, \mathfrak{m})$  be a local ring. If  $\mathfrak{m} = (x_1, \dots, x_r)$  and  $x_1^{e_1}, \dots, x_r^{e_r}$  are independent for some  $e_i > 0$ , then  $\text{length}_A(A/(x_1^{e_1}, \dots, x_r^{e_r})) = e_1 \dots e_r$ .*

See [Lec64] and [Mat70, Lemma 3 page 300].

**Proof.** Use Lemmas 16.2 and 16.3 and induction.  $\square$

0EBZ **Lemma 16.5.** *Let  $\varphi : A \rightarrow B$  be a flat ring map. If  $f_1, \dots, f_r \in A$  are independent, then  $\varphi(f_1), \dots, \varphi(f_r) \in B$  are independent.*

**Proof.** Let  $I = (f_1, \dots, f_r)$  and  $J = F(I)B$ . By flatness we have  $I/I^2 \otimes_A B = J/J^2$ . Hence freeness of  $I/I^2$  over  $A/I$  implies freeness of  $J/J^2$  over  $B/J$ .  $\square$

0EC0 **Lemma 16.6 (Kunz).** *Let  $p$  be a prime number. Let  $A$  be a Noetherian ring with  $p = 0$ . The following are equivalent*

[Kun69]

- (1)  $A$  is regular, and
- (2)  $F : A \rightarrow A, a \mapsto a^p$  is flat.

**Proof.** Observe that  $\text{Spec}(F) : \text{Spec}(A) \rightarrow \text{Spec}(A)$  is the identity map. Being regular is defined in terms of the local rings and being flat is something about local

rings, see Algebra, Lemma 38.19. Thus we may and do assume  $A$  is a Noetherian local ring with maximal ideal  $\mathfrak{m}$ .

Assume  $A$  is regular. Let  $x_1, \dots, x_d$  be a system of parameters for  $A$ . Applying  $F$  we find  $F(x_1), \dots, F(x_d) = x_1^p, \dots, x_d^p$ , which is a system of parameters for  $A$ . Hence  $F$  is flat, see Algebra, Lemmas 127.1 and 105.3.

Conversely, assume  $F$  is flat. Write  $\mathfrak{m} = (x_1, \dots, x_r)$  with  $r$  minimal. Then  $x_1, \dots, x_r$  are independent in the sense defined above. Since  $F$  is flat, we see that  $x_1^p, \dots, x_r^p$  are independent, see Lemma 16.5. Hence  $\text{length}_A(A/(x_1^p, \dots, x_r^p)) = p^r$  by Lemma 16.4. Let  $\chi(n) = \text{length}_A(A/\mathfrak{m}^n)$  and recall that this is a numerical polynomial of degree  $\dim(A)$ , see Algebra, Proposition 59.8. Choose  $n \gg 0$ . Observe that

$$\mathfrak{m}^{pn+pr} \subset F(\mathfrak{m}^n)A \subset \mathfrak{m}^{pn}$$

as can be seen by looking at monomials in  $x_1, \dots, x_r$ . We have

$$A/F(\mathfrak{m}^n)A = A/\mathfrak{m}^n \otimes_{A,F} A$$

By flatness of  $F$  this has length  $\chi(n)\text{length}_A(A/F(\mathfrak{m}^n)A)$  (Algebra, Lemma 51.13) which is equal to  $p^r\chi(n)$  by the above. We conclude

$$\chi(pn + pr) \geq p^r\chi(n) \geq \chi(pn)$$

Looking at the leading terms this implies  $r = \dim(A)$ , i.e.,  $A$  is regular. □

### 17. Structure of certain modules

0EC1 Some results on the structure of certain types of modules over regular local rings. These types of results and much more can be found in [HS93], [Lyu93], [Lyu97].

0EC2 **Lemma 17.1.** *Let  $k$  be a field of characteristic 0. Let  $d \geq 1$ . Let  $A = k[[x_1, \dots, x_d]]$  with maximal ideal  $\mathfrak{m}$ . Let  $M$  be an  $\mathfrak{m}$ -power torsion  $A$ -module endowed with additive operators  $D_1, \dots, D_d$  satisfying the leibniz rule* Special case of [Lyu93, Theorem 2.4]

$$D_i(fz) = \partial_i(f)z + fD_i(z)$$

for  $f \in A$  and  $z \in M$ . Here  $\partial_i$  is differentiation with respect to  $x_i$ . Then  $M$  is isomorphic to a direct sum of copies of the injective hull  $E$  of  $k$ .

**Proof.** Choose a set  $J$  and an  $A$ -module homomorphism  $\varphi : M \rightarrow \bigoplus_{j \in J} E$  which maps  $M[\mathfrak{m}]$  isomorphically onto  $(\bigoplus_{j \in J} E)[\mathfrak{m}] = \bigoplus_{j \in J} k$ . We claim that  $\varphi$  is an isomorphism, i.e., bijective.

Injective. Let  $z \in M$  be nonzero. Since  $M$  is  $\mathfrak{m}$ -power torsion we can choose an element  $f \in A$  such that  $fz \in M[\mathfrak{m}]$  and  $fz \neq 0$ . Then  $\varphi(fz) = f\varphi(z)$  is nonzero, hence  $\varphi(z)$  is nonzero.

Surjective. Let  $z \in M$ . Then  $x_1^n z = 0$  for some  $n \geq 0$ . We will prove that  $z \in x_1 M$  by induction on  $n$ . If  $n = 0$ , then  $z = 0$  and the result is true. If  $n > 0$ , then applying  $D_1$  we find  $0 = nx_1^{n-1}z + x_1^n D_1(z)$ . Hence  $x_1^{n-1}(nz + x_1 D_1(z)) = 0$ . By induction we get  $nz + x_1 D_1(z) \in x_1 M$ . Since  $n$  is invertible, we conclude  $z \in x_1 M$ . Thus we see that  $M$  is  $x_1$ -divisible. If  $\varphi$  is not surjective, then we can choose  $e \in \bigoplus_{j \in J} E$  not in  $M$ . Arguing as above we may assume  $\mathfrak{m}e \subset M$ , in particular  $x_1 e \in M$ . There exists an element  $z_1 \in M$  with  $x_1 z_1 = x_1 e$ . Hence  $x_1(z_1 - e) = 0$ .

Replacing  $e$  by  $e - z_1$  we may assume  $e$  is annihilated by  $x_1$ . Thus it suffices to prove that

$$\varphi[x_1] : M[x_1] \longrightarrow \left( \bigoplus_{j \in J} E \right) [x_1] = \bigoplus_{j \in J} E[x_1]$$

is surjective. If  $d = 1$ , this is true by construction of  $\varphi$ . If  $d > 1$ , then we observe that  $E[x_1]$  is the injective hull of the residue field of  $k[[x_2, \dots, x_d]]$ , see Dualizing Complexes, Lemma 7.1. Observe that  $M[x_1]$  as a module over  $k[[x_2, \dots, x_d]]$  is  $\mathfrak{m}/(x_1)$ -power torsion and comes equipped with operators  $D_2, \dots, D_d$  satisfying the displayed Leibniz rule. Thus by induction on  $d$  we conclude that  $\varphi[x_1]$  is surjective as desired.  $\square$

0EC3 **Lemma 17.2.** *Let  $p$  be a prime number. Let  $(A, \mathfrak{m}, k)$  be a regular local ring with  $p = 0$ . Denote  $F : A \rightarrow A, a \mapsto a^p$  be the Frobenius endomorphism. Let  $M$  be a  $\mathfrak{m}$ -power torsion module such that  $M \otimes_{A,F} A \cong M$ . Then  $M$  is isomorphic to a direct sum of copies of the injective hull  $E$  of  $k$ .*

Follows from [HS93, Corollary 3.6] with a little bit of work. Also follows directly from [Lyu97, Theorem 1.4].

**Proof.** Choose a set  $J$  and an  $A$ -module homomorphism  $\varphi : M \rightarrow \bigoplus_{j \in J} E$  which maps  $M[\mathfrak{m}]$  isomorphically onto  $(\bigoplus_{j \in J} E)[\mathfrak{m}] = \bigoplus_{j \in J} k$ . We claim that  $\varphi$  is an isomorphism, i.e., bijective.

**Injective.** Let  $z \in M$  be nonzero. Since  $M$  is  $\mathfrak{m}$ -power torsion we can choose an element  $f \in A$  such that  $fz \in M[\mathfrak{m}]$  and  $fz \neq 0$ . Then  $\varphi(fz) = f\varphi(z)$  is nonzero, hence  $\varphi(z)$  is nonzero.

**Surjective.** Recall that  $F$  is flat, see Lemma 16.6. Let  $x_1, \dots, x_d$  be a minimal system of generators of  $\mathfrak{m}$ . Denote

$$M_n = M[x_1^{p^n}, \dots, x_d^{p^n}]$$

the submodule of  $M$  consisting of elements killed by  $x_1^{p^n}, \dots, x_d^{p^n}$ . So  $M_0 = M[\mathfrak{m}]$  is a vector space over  $k$ . Also  $M = \bigcup M_n$  by our assumption that  $M$  is  $\mathfrak{m}$ -power torsion. Since  $F^n$  is flat and  $F^n(x_i) = x_i^{p^n}$  we have

$$M_n \cong (M \otimes_{A, F^n} A)[x_1^{p^n}, \dots, x_d^{p^n}] = M[x_1, \dots, x_d] \otimes_{A, F^n} A = M_0 \otimes_k A/(x_1^{p^n}, \dots, x_d^{p^n})$$

Thus  $M_n$  is free over  $A/(x_1^{p^n}, \dots, x_d^{p^n})$ . A computation shows that every element of  $A/(x_1^{p^n}, \dots, x_d^{p^n})$  annihilated by  $x_1^{p^n-1}$  is divisible by  $x_1$ ; for example you can use that  $A/(x_1^{p^n}, \dots, x_d^{p^n}) \cong k[x_1, \dots, x_d]/(x_1^{p^n}, \dots, x_d^{p^n})$  by Algebra, Lemma 154.10. Thus the same is true for every element of  $M_n$ . Since every element of  $M$  is in  $M_n$  for all  $n \gg 0$  and since every element of  $M$  is killed by some power of  $x_1$ , we conclude that  $M$  is  $x_1$ -divisible.

Let  $x = x_1$ . Above we have seen that  $M$  is  $x$ -divisible. If  $\varphi$  is not surjective, then we can choose  $e \in \bigoplus_{j \in J} E$  not in  $M$ . Arguing as above we may assume  $\mathfrak{m}e \subset M$ , in particular  $xe \in M$ . There exists an element  $z_1 \in M$  with  $xz_1 = xe$ . Hence  $x(z_1 - e) = 0$ . Replacing  $e$  by  $e - z_1$  we may assume  $e$  is annihilated by  $x$ . Thus it suffices to prove that

$$\varphi[x] : M[x] \longrightarrow \left( \bigoplus_{j \in J} E \right) [x] = \bigoplus_{j \in J} E[x]$$

is surjective. If  $d = 1$ , this is true by construction of  $\varphi$ . If  $d > 1$ , then we observe that  $E[x]$  is the injective hull of the residue field of the regular ring  $A/xA$ , see

Dualizing Complexes, Lemma 7.1. Observe that  $M[x]$  as a module over  $A/xA$  is  $\mathfrak{m}/(x)$ -power torsion and we have

$$\begin{aligned} M[x] \otimes_{A/xA, F} A/xA &= M[x] \otimes_{A, F} A \otimes_A A/xA \\ &= (M \otimes_{A, F} A)[x^p] \otimes_A A/xA \\ &\cong M[x^p] \otimes_A A/xA \end{aligned}$$

Argue using flatness of  $F$  as before. We claim that  $M[x^p] \otimes_A A/xA \rightarrow M[x]$ ,  $z \otimes 1 \mapsto x^{p-1}z$  is an isomorphism. This can be seen by proving it for each of the modules  $M_n$ ,  $n > 0$  defined above where it follows by the same result for  $A/(x_1^{p^n}, \dots, x_d^{p^n})$  and  $x = x_1$ . Thus by induction on  $\dim(A)$  we conclude that  $\varphi[x]$  is surjective as desired.  $\square$

### 18. Additional structure on local cohomology

0EC4 Here is a sample result.

0EC5 **Lemma 18.1.** *Let  $A$  be a ring. Let  $I \subset A$  be a finitely generated ideal. Set  $Z = V(I)$ . For each derivation  $\theta : A \rightarrow A$  there exists a canonical additive operator  $D$  on the local cohomology modules  $H_Z^i(A)$  satisfying the Leibniz rule with respect to  $\theta$ .*

**Proof.** Let  $f_1, \dots, f_r$  be elements generating  $I$ . Recall that  $R\Gamma_Z(A)$  is computed by the complex

$$A \rightarrow \prod_{i_0} A_{f_{i_0}} \rightarrow \prod_{i_0 < i_1} A_{f_{i_0} f_{i_1}} \rightarrow \dots \rightarrow A_{f_1 \dots f_r}$$

See Dualizing Complexes, Lemma 9.1. Since  $\theta$  extends uniquely to an additive operator on any localization of  $A$  satisfying the Leibniz rule with respect to  $\theta$ , the lemma is clear.  $\square$

0EC6 **Lemma 18.2.** *Let  $p$  be a prime number. Let  $A$  be a ring with  $p = 0$ . Denote  $F : A \rightarrow A$ ,  $a \mapsto a^p$  the Frobenius endomorphism. Let  $I \subset A$  be a finitely generated ideal. Set  $Z = V(I)$ . There exists an isomorphism  $R\Gamma_Z(A) \otimes_{A, F}^L A \cong R\Gamma_Z(A)$ .*

**Proof.** Follows from Dualizing Complexes, Lemma 9.3 and the fact that  $Z = V(f_1^p, \dots, f_r^p)$  if  $I = (f_1, \dots, f_r)$ .  $\square$

0EC7 **Lemma 18.3.** *Let  $A$  be a ring. Let  $V \rightarrow \text{Spec}(A)$  be quasi-compact, quasi-separated, and étale. For each derivation  $\theta : A \rightarrow A$  there exists a canonical additive operator  $D$  on  $H^i(V, \mathcal{O}_V)$  satisfying the Leibniz rule with respect to  $\theta$ .*

**Proof.** If  $V$  is separated, then we can argue using an affine open covering  $V = \bigcup_{j=1, \dots, m} V_j$ . Namely, because  $V$  is separated we may write  $V_{j_0 \dots j_p} = \text{Spec}(B_{j_0 \dots j_p})$ . See Schemes, Lemma 21.8. Then we find that the  $A$ -module  $H^i(V, \mathcal{O}_V)$  is the  $i$ th cohomology group of the Čech complex

$$\prod B_{j_0} \rightarrow \prod B_{j_0 j_1} \rightarrow \prod B_{j_0 j_1 j_2} \rightarrow \dots$$

See Cohomology of Schemes, Lemma 2.6. Each  $B = B_{j_0 \dots j_p}$  is an étale  $A$ -algebra. Hence  $\Omega_B = \Omega_A \otimes_A B$  and we conclude  $\theta$  extends uniquely to a derivation  $\theta_B : B \rightarrow B$ . These maps define an endomorphism of the Čech complex and define the desired operators on the cohomology groups.

In the general case we use a hypercovering of  $V$  by affine opens, exactly as in the first part of the proof of Cohomology of Schemes, Lemma 7.3. We omit the details.  $\square$

0EC8 **Remark 18.4.** We can upgrade Lemmas 18.1 and 18.3 to include higher order differential operators. If we ever need this we will state and prove a precise lemma here.

0EC9 **Lemma 18.5.** *Let  $p$  be a prime number. Let  $A$  be a ring with  $p = 0$ . Denote  $F : A \rightarrow A, a \mapsto a^p$  the Frobenius endomorphism. If  $V \rightarrow \text{Spec}(A)$  is quasi-compact, quasi-separated, and étale, then there exists an isomorphism  $R\Gamma(V, \mathcal{O}_V) \otimes_{A,F}^{\mathbf{L}} A \cong R\Gamma(V, \mathcal{O}_V)$ .*

**Proof.** Observe that the relative Frobenius morphism

$$V \longrightarrow V \times_{\text{Spec}(A), \text{Spec}(F)} \text{Spec}(A)$$

of  $V$  over  $A$  is an isomorphism, see Étale Morphisms, Lemma 14.3. Thus the lemma follows from cohomology and base change, see Derived Categories of Schemes, Lemma 21.3. Observe that since  $V$  is étale over  $A$ , it is flat over  $A$ .  $\square$

## 19. Other chapters

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