0F4U

Contents

1. Introduction 1
2. Sections with compact support 1
3. Sections with finite support 5
4. Upper shriek for locally quasi-finite morphisms 13
5. Derived upper shriek for locally quasi-finite morphisms 15
6. Other chapters 15
References 17

1. Introduction

This chapter is the second in a series of chapter on the étale cohomology of schemes. To read the first chapter, please visit Étale Cohomology, Section 1.

The split with the previous chapter is roughly speaking that anything concerning “shriek functors” (cohomology with compact support and its right adjoint) and anything using this material goes into this chapter.

2. Sections with compact support

A reference for this section is [AGV71, Exposee XVII, Section 6]. Let $f : X \to Y$ be a morphism of schemes which is separated and locally of finite type. In this section we define a functor $f_! : \text{Ab} (X_{\text{étale}}) \to \text{Ab} (Y_{\text{étale}})$ by taking $f_! F \subset f_* F$ to be the subsheaf of sections which have proper support relative to $Y$ (suitably defined).

Warning: The functor $f_!$ is the zeroth cohomology sheaf of a functor $Rf_!$ on the derived category (insert future reference), but $Rf_!$ is not the derived functor of $f_!$.

Lemma 2.1. Let $f : X \to Y$ be a morphism of schemes which is locally of finite type. Let $\mathcal{F}$ be an abelian sheaf on $X_{\text{étale}}$. The rule

$$Y_{\text{étale}} \to \text{Ab}, \quad V \mapsto \{ s \in f_* \mathcal{F}(V) = \mathcal{F}(X_V) \mid \text{Supp}(s) \subset X_V \text{ is proper over } V \}$$

is an abelian subsheaf of $f_* \mathcal{F}$.

Warning: This sheaf isn’t the “correct one” if $f$ is not separated.

Proof. Recall that the support of a section is closed (Étale Cohomology, Lemma 31.4) hence the material in Cohomology of Schemes, Section 26 applies. By the lemma above and Cohomology of Schemes, Lemma 26.6 we find that our subset of $f_* \mathcal{F}(V)$ is a subgroup. By Cohomology of Schemes, Lemma 26.4 we see that our rule defines a sub presheaf. Finally, suppose that we have $s \in f_* \mathcal{F}(V)$ and an étale covering $\{ V_i \to V \}$ such that $s|_{V_i}$ has support proper over $V_i$. Observe that the
support of $s|_{V'}$ is the inverse image of the support of $s|_{V'}$ (use the characterization of the support in terms of stalks and Étale Cohomology, Lemma \[36.2\]). Whence the support of $s$ is proper of $V$ by Descent, Lemma \[22.5\]. This proves that our rule satisfies the sheaf condition.

\[\square\]

**Lemma 2.2.** Let $j : U \to X$ be a separated étale morphism. Let $F$ be an abelian sheaf on $U_{\text{étale}}$. The image of the injective map $j_* F \to j_* F$ of Étale Cohomology, Lemma \[69.5\], is the subsheaf of Lemma \[2.1\].

An alternative would be to move this lemma later and prove this using the description of the stalks of both sheaves.

**Proof.** The construction of $j_* F \to j_* F$ in the proof of Étale Cohomology, Lemma \[69.5\] is via the construction of a map $j_{pr} F \to j_* F$ of presheaves whose image is clearly contained in the subsheaf of Lemma \[2.1\]. Hence since $j_* F$ is the sheafification of $j_{pr} F$ we conclude the image of $j_* F \to j_* F$ is contained in this subsheaf. Conversely, let $s \in j_* F(V)$ have support $Z$ proper over $V$. Then $Z \to V$ is finite with closed image $Z' \subset V$, see More on Morphisms, Lemma \[39.1\]. The restriction of $s$ to $V \setminus Z'$ is zero and hence contained in the image of $j_* F \to j_* F$. On the other hand, if $v \in Z'$, then we can find an étale neighbourhood $(V', v') \to (V, v)$ such that we have a decomposition $U_{V'} = W \amalg U'_1 \amalg \ldots \amalg U'_n$ open and closed subschemes with $U'_i \to V'$ an isomorphism and with $T_{V'} \subset U'_1 \amalg \ldots \amalg U'_n$, see Étale Morphisms, Lemma \[18.2\]. Inverting the isomorphisms $U'_i \to V'$ we obtain $n$ morphisms $\varphi'_i : V' \to U$ and sections $s'_i \subset V'$ by pulling back $s$. Then the section $\sum (\varphi'_i, s'_i)$ of $j_{pr} F$ over $V'$, see formula for $j_{pr} F(V')$ in proof of Étale Cohomology, Lemma \[69.5\], maps to the restriction of $s$ to $V'$ by construction. We conclude that $s$ is étale locally in the image of $j_* F \to j_* F$ and the proof is complete. \[\square\]

**Definition 2.3.** Let $f : X \to Y$ be a morphism of schemes which is separated (!) and locally of finite type. Let $F$ be an abelian sheaf on $X_{\text{étale}}$. The subsheaf $f_* F \subset f_* F$ constructed in Lemma \[2.1\] is called the direct image with compact support.

By Lemma \[2.2\] this does not conflict with Étale Cohomology, Definition \[69.1\] as we have agreement when both definitions apply. Before extending the definition to more general morphisms we prove some basic lemmas about this notion.

**Lemma 2.4.** Let $f : X \to Y$ and $g : Y \to Z$ be composable morphisms of schemes which are separated and locally of finite type. Let $F$ be an abelian sheaf on $X_{\text{étale}}$. Then $g_* f_* F = (g \circ f)_* F$ as subsheaves of $(g \circ f)_* F$.

**Proof.** We strongly urge the reader to prove this for themselves. Let $W \in Z_{\text{étale}}$ and $s \in (g \circ f)_* F(W) = F(X_W)$. Denote $T \subset X_W$ the support of $s$; this is a closed subset. Observe that $s$ is a section of $(g \circ f)_* F$ if and only if $T$ is proper over $W$. We have $f_* F \subset f_* F$ and hence $g_* f_* F \subset g_* f_* F \subset g_* f_* F$. On the other hand, $s$ is a section of $g_* f_* F$ if and only if (a) $T$ is proper over $Y_W$ and (b) the support $T'$ of $s$ viewed as section of $f_* F$ is proper over $W$. If (a) holds, then the image of $T$ in $Y_W$ is closed and since $f_* F \subset f_* F$ we see that $T' \subset Y_W$ is the image of $T$ (details omitted; look at stalks).

The conclusion is that we have to show a closed subset $T \subset X_W$ is proper over $W$ if and only if $T$ is proper over $Y_W$ and the image of $T$ in $Y_W$ is proper over $W$. Let us endow $T$ with the reduced induced closed subscheme structure. If $T$ is proper
over $W$, then $T \rightarrow Y_W$ is proper by Morphisms, Lemma \[39.4\] and the image of $T$ in $Y_W$ is proper over $W$ by Cohomology of Schemes, Lemma \[26.5\] Conversely, if $T$ is proper over $Y_W$ and the image of $T$ in $Y_W$ is proper over $W$, then the morphism $T \rightarrow W$ is proper as a composition of proper morphisms (here we endow the closed image of $T$ in $Y_W$ with its reduced induced scheme structure to turn the question into one about morphisms of schemes), see Morphisms, Lemma \[39.4\] \[\square\]

0F51 Lemma 2.5. Let $f : X \rightarrow Y$ be a proper morphism of schemes. Then $f_! = f_*$. 

Proof. Immediate from the construction of $f_!$. \[\square\]

0F52 Lemma 2.6. Let $Y$ be a scheme. Let $j : X \rightarrow X$ be an open immersion of schemes over $Y$ with $X$ proper over $Y$. Denote $f : X \rightarrow Y$ and $j : X \rightarrow Y$ the structure morphisms. Then $f_! F = j_* f^! f$ for $F \in \text{Ab}(X_{\text{etale}})$. 

Proof. Follows immediately from Lemmas 2.4 and 2.5. \[\square\]

0F53 Remark 2.7 (Covariance with respect to open embeddings). Let $f : X \rightarrow Y$ be morphism of schemes which is separated and locally of finite type. Let $F$ be an abelian sheaf on $X_{\text{etale}}$. Let $X' \subset X$ be an open subscheme and denote $f' : X' \rightarrow Y$ the restriction of $f$. Denoting $j : X' \rightarrow X$ the inclusion morphism, we obtain a canonical map 

$$f'_!(F|_{X'}) = f'_!(f^{-1} F) = f j^! j^{-1} F \rightarrow f_! F$$

where we have used the identification $f'_! = f j_!$ of Lemma 2.4 and the adjunction map $j^! j^{-1} F \rightarrow F$ for the adjoint pair $j_!, j^{-1}$. Below we give a direct construction of this map.

Let $V \in Y_{\text{etale}}$ and consider a section $s' \in f'_!(F|_{X'}) = F(X' \times_Y V)$ with support $Z'$ proper over $V$. Then we have an open covering of $X' \times_Y V$ given by $X' \times_Y V$ and $X' \times_Y V \setminus Z'$ as $Z'$ is closed in $X \times_Y V$ as well, see for example Cohomology of Schemes, Lemma \[26.5\]. Thus there is a unique section $s \in F(X' \times_Y V)$ whose restriction to $X' \times_Y V$ is $s'$ and whose restriction to $X \times_Y V \setminus Z'$ is zero. This construction is compatible with restriction maps and induces an injective map of sheaves $f'_!(F|_{X'}) \rightarrow f_! F$. In fact, it is clear that we obtain a commutative diagram

$$
\begin{array}{ccc}
f'_!(F|_{X'}) & \longrightarrow & f_! F \\
\downarrow & & \downarrow \\
\left(f_!(F|_{X'}) \right) & \leftarrow & \left(f_! F \right)
\end{array}
$$

functorial in $F$.

0F54 Lemma 2.8. Let $f : X \rightarrow Y$ be morphism of schemes which is separated and locally of finite type. Let $I$ be a nonempty set and for $i \in I$ let $U_i \subset X$ be open such that $X = \bigcup_{i \in I} U_i$ and for all $i, j \in I$ there exists a $k$ with $U_i \cup U_j \subset U_k$. Denote $f_i : U_i \rightarrow Y$ the restriction of $f$. Then

$$f_! F = \text{colim}_{i \in I} f_i_!(F|_{U_i})$$

functorially in $F \in \text{Ab}(X_{\text{etale}})$ where the transition maps are the ones constructed in Remark 2.7.
Proof. It suffices to show that the canonical map from right to left is a bijection when evaluated on a quasi-compact object $V$ of $\mathcal{X}_{\text{etale}}$. Observe that the colimit on the right hand side is directed and has injective transition maps. Thus we can use Sites, Lemma 17.5 to evaluate the colimit. Hence, the statement comes down to the observation that a closed subset $Z \subset X_V$ proper over $V$ is quasi-compact and hence is contained in $U_{i,V}$ for some $i$. □

**Lemma 2.9.** Consider a cartesian square

$$
\begin{array}{ccc}
X' & \xrightarrow{g'} & X \\
\downarrow{f'} & & \downarrow{f} \\
Y' & \xrightarrow{g} & Y
\end{array}
$$

of schemes with $f$ separated and locally of finite type. For any abelian sheaf $\mathcal{F}$ on $X_{\text{etale}}$ we have $f'_!(g')^{-1} \mathcal{F} = g^{-1} f_! \mathcal{F}$.

**Proof.** In great generality there is a pullback map $g^{-1} f_* \mathcal{F} \to f'_!(g')^{-1} \mathcal{F}$, see Sites, Section 45. We claim that this map sends $g^{-1} f_* \mathcal{F}$ into the subsheaf $f'_!(g')^{-1} \mathcal{F}$ and induces the isomorphism in the lemma. This claim is local in the étale topology of $Y'$. Hence we may assume that $Y$ is affine.

First, assume $X$ is quasi-compact as well. By More on Flatness, Theorem 33.8 there exists a compactification $\overline{X} \subset X$ over $Y$. Then we have $f_1 = j_! \circ j_*$ by Lemma 2.6.

Both $f$ and $j_!$ commute with arbitrary base change by Étale Cohomology, Lemmas 87.5 and 69.4. This proves the result in this case.

General case where $Y$ is affine. Consider the set $X_i \subset X$, $i \in I$ of all quasi-compact open subschemes of $X$. Then $X = \bigcup X_i$ and for $i, j \in I$ we have $X_i \cup X_j = X_k$ for some $k \in I$. For each $i \in I$ consider the fibre product

$$
\begin{array}{ccc}
X'_i & \xrightarrow{g'_i} & X_i \\
\downarrow{f'_i} & & \downarrow{f_i} \\
Y'_i & \xrightarrow{g_i} & Y
\end{array}
$$

For notational convenience let us write $\mathcal{F}' = (g')^{-1} \mathcal{F}$, $\mathcal{F}_i = \mathcal{F}|_{X_i}$, and $\mathcal{F}'_i = \mathcal{F}'|_{X'_i} = (g'_i)^{-1} \mathcal{F}_i$. It follows from Lemma 2.8 that we have

$$
f_i \mathcal{F} = \text{colim}_{i \in I} f_{i,!*}(\mathcal{F}_i) \quad \text{and} \quad f'_i \mathcal{F}' = \text{colim}_{i \in I} f'_{i,!*}(\mathcal{F}'_i)
$$

Now the base change map above and the base change map for the last displayed cartesian square fit into commutative diagrams

$$
\begin{array}{ccc}
g^{-1} f_* \mathcal{F} & \xrightarrow{f'_i \mathcal{F}'} & g^{-1} f_{i,!*} \mathcal{F}_i \\
\downarrow{f'_i \mathcal{F}_i} & & \downarrow{f'_{i,!*} \mathcal{F}'_i} \\
g^{-1} f_{i,!*} \mathcal{F}_i & \xrightarrow{f'_{i,!*} \mathcal{F}'_i} & g^{-1} f'_{i,!*} \mathcal{F}'_i
\end{array}
$$

if $X'_i \subset X_i$. Using the claim for each $i$, the right hand commutative diagrams guarantee that the base change isomorphisms $g^{-1} f_{i,!*} \mathcal{F}_i \to f'_{i,!*} \mathcal{F}'_i$ are compatible for varying $i \in I$. Then we can prove the claim as follows: any local section of $g^{-1} f_* \mathcal{F}$ is locally a section of $g^{-1} f_{i,!*} \mathcal{F}_i$ for some $i$ and hence is mapped to a section of

\[1\] Appealing to this theorem can be avoided by using Chow’s lemma and h descent instead.
In this section we extend the construction of Section 2 to not necessarily separated locally of finite type. Then functor $f_!$ commutes with colimits as a left adjoint.)

**Lemma 2.10.** Let $f : X \to Y$ be a morphism of schemes which is separated and locally of finite type. Then functor $f_!$ commutes with direct sums.

**Proof.** Let $F = \bigoplus F_!$. To show that the map $\bigoplus f_! F_i \to f_! F$ is an isomorphism, it suffices to show that these sheaves have the same sections over a quasi-compact object $V$ of $\text{Y}_{\text{étale}}$. Replacing $Y$ by $V$ it suffices to show $H^0(Y, f_! F) \subset H^0(X, \mathcal{F})$ is equal to $\bigoplus H^0(Y, f_! F_i) \subset \bigoplus H^0(X, \mathcal{F}_i) \subset H^0(X, \bigoplus \mathcal{F}_i)$. In this case, by writing $X$ as the union of its quasi-compact opens and using Lemma 2.8 we reduce to the case where $X$ is quasi-compact as well. Then $H^0(X, \mathcal{F}) = \bigoplus H^0(X, \mathcal{F}_i)$ by Étale Cohomology, Theorem 51.3. Looking at supports of sections the reader easily concludes.

**Lemma 2.11.** Let $f : X \to Y$ be a morphism of schemes which is separated and locally quasi-finite. Then

1. for $\mathcal{F}$ in $\text{Ab}(X_{\text{étale}})$ and a geometric point $\overline{y} : \text{Spec}(k) \to Y$ we have
$$ (f_! \mathcal{F})_{\overline{y}} = \bigoplus_{f(x) = \overline{y}} \mathcal{F}_x $$
functorially in $\mathcal{F}$, and
2. the functor $f_!$ is exact.

**Proof.** The functor $f_!$ is left exact by construction. Right exactness may be checked on stalks (Étale Cohomology, Theorem 29.10). Thus it suffices to prove part (1).

Let $\overline{y} : \text{Spec}(k) \to Y$ be a geometric point. The scheme $X_{\overline{y}}$ has a discrete underlying topological space (Morphisms, Lemma 19.8) and all the residue fields at the points are equal to $k$ (as finite extensions of $k$). Hence $\{x : \text{Spec}(k) \to X : f(x) = \overline{y}\}$ is equal to the set of points of $X_{\overline{y}}$. The stalk at $\overline{y}$ is the pullback by $\overline{y}$. Hence by Lemma 2.9 we see that the stalk of $f_! \mathcal{F}$ at $\overline{y}$ can be identified with the subgroup of sections of $\mathcal{F}|_{X_{\overline{y}}}$ whose support is finite. On the other hand, the stalk of $\mathcal{F}|_{X_{\overline{y}}}$ at the point corresponding to $x$ is equal to $\mathcal{F}_x$. Thus the subgroup of sections of $\mathcal{F}|_{X_{\overline{y}}}$ whose support is finite is equal to $\bigoplus_{f(x) = \overline{y}} \mathcal{F}_x$ as desired.

### 3. Sections with finite support

In this section we extend the construction of Section 2 to not necessarily separated locally quasi-finite morphisms.

**Lemma 3.1.** Let $X$ be a scheme. Let $\mathcal{F}$ be an abelian sheaf on $X_{\text{étale}}$. Let $\varphi : U' \to U$ be a morphism of $X_{\text{étale}}$. Let $Z' \subset U'$ be a closed subscheme such that $Z' \to U' \to U$ is a closed immersion with image $Z \subset U$. Then there is a canonical bijection
$$ \{s \in \mathcal{F}(U) \mid \text{Supp}(s) \subset Z\} = \{s' \in \mathcal{F}(U') \mid \text{Supp}(s') \subset Z'\} $$
which is given by restriction if $\varphi^{-1}(Z) = Z'$.
Let $F$ be an abelian sheaf on $X_{\text{étale}}$. Given $U, U' \subset X$ open containing $Z$ as a closed subscheme, there is a canonical bijection

$$\{ s \in F(U) \mid \text{Supp}(s) \subset Z \} = \{ s \in F(U') \mid \text{Supp}(s) \subset Z \}$$

which is given by restriction if $U' \subset U$.

**Proof.** Since $Z$ is a closed subscheme of $U \cap U'$, it suffices to prove the lemma when $U' \subset U$. Then it is a special case of Lemma 3.1.

Let us introduce a bit of nonstandard notation which will stand us in good stead for the rest of this section. Namely, in the situation of Lemma 3.2 above, let us denote

$$H_Z(F) = \{ s \in F(U) \mid \text{Supp}(s) \subset Z \}$$

where $U \subset X$ is any choice of open subscheme containing $Z$ as a closed subscheme. The reader who is troubled by the lack of precision this entails may choose $U = X \setminus \partial Z$ where $\partial Z = Z \setminus Z$ is the “boundary” of $Z$ in $X$. However, in many of the arguments below the flexibility of choosing different opens will play a role. Here are some properties of this construction:

1. If $Z \subset Z'$ are locally closed subschemes of $X$ and $Z$ is closed in $Z'$, then there is a natural injective map

$$H_Z(F) \to H_{Z'}(F).$$

2. If $f : Y \to X$ is a morphism of schemes and $Z \subset X$ is a locally closed subscheme, then there is a natural pullback map $f^* : H_Z(F) \to H_{f^{-1}Z}(f^{-1}F)$.

It will be convenient to extend our notation to the following situation: suppose that we have $W \in X_{\text{étale}}$ and a locally closed subscheme $Z \subset W$. Then we will denote

$$H_Z(F) = \{ s \in F(U) \mid \text{Supp}(s) \subset Z \} = H_Z(F|_{W_{\text{étale}}})$$

where $U \subset W$ is any choice of open subscheme containing $Z$ as a closed subscheme, exactly as above.$^2$

Let $f : X \to Y$ be a locally quasi-finite morphism of schemes. Let $F$ be an abelian sheaf on $X_{\text{étale}}$. Given $V$ in $Y_{\text{étale}}$ denote $X_V = X \times_Y V$ the base change. We are going to consider the group of finite formal sums

$$s = \sum_{i=1,\ldots,n} (Z_i, s_i)$$

$^2$In fact, Lemma 3.1 shows, given $Z$ over $X$ which is isomorphic to a locally closed subscheme of some object $W$ of $X_{\text{étale}}$, that the choice of $W$ is irrelevant.
where $Z_i \subset X_V$ is a locally closed subscheme such that the morphism $Z_i \to V$ is finite and where $s_i \in H_Z(F)$. We are going to consider these formal sums modulo the following relations

0F6K (1) $(Z, s) + (Z, s') = (Z, s + s')$,  
0F6L (2) $(Z, s) = (Z', s)$ if $Z \subset Z'$.

Observe that the second relation makes sense: since $Z \to V$ is finite and $Z' \to V$ is separated, the inclusion $Z \to Z'$ is closed and we can use the map discussed in (1).

Let us denote $f_{pl}F(V)$ the quotient of the abelian group of formal sums (3.2.1) by these relations. The first relation tells us that $f_{pl}F(V)$ is a quotient of the direct sum of the abelian groups $H_Z(F)$ over all locally closed subschemes $Z \subset X_V$ finite over $V$. The second relation tells us that we are really taking the colimit

0F6M (3.2.2) $f_{pl}F(V) = \text{colim}_ZH_Z(F)$

This formula will be a convenient abstract way to think about our construction.

Next, we observe that there is a natural way to turn this construction into a presheaf $f_{pl}F$ of abelian groups on $Y_{\text{ étale}}$. Namely, given $V' \to V$ in $Y_{\text{ étale}}$ we obtain the base change morphism $X_{V'} \to X_V$. If $Z \subset X_V$ is a locally closed subscheme finite over $V$, then the scheme theoretic inverse image $Z' \subset X_{V'}$ is finite over $V'$. Moreover, if $U \subset X_V$ is an open such that $Z$ is closed in $U$, then the inverse image $U' \subset X_{V'}$ is an open such that $Z'$ is closed in $U'$. Hence the restriction mapping $F(U) \to F(U')$ of $F$ sends $H_Z(F)$ into $H_Z'(F)$; this is a special case of the functoriality discussed in (2) above. Clearly, these maps are compatible with inclusions $Z_1 \subset Z_2$ of such locally closed subschemes of $X_V$ and we obtain a map

$$f_{pl}F(V) = \text{colim}_ZH_Z(F) \longrightarrow \text{colim}_ZH_{Z'}(F) = f_{pl}F(V')$$

These maps indeed turn $f_{pl}F$ into a presheaf of abelian groups on $Y_{\text{ étale}}$. We omit the details.

A final observation is that the construction of $f_{pl}F$ is functorial in $F$ in $\text{Ab}(X_{\text{ étale}})$. We conclude that given a locally quasi-finite morphism $f : X \to Y$ we have constructed a functor

$$f_{pl} : \text{Ab}(X_{\text{ étale}}) \longrightarrow P\text{Ab}(Y_{\text{ étale}})$$

from the category of abelian sheaves on $X_{\text{ étale}}$ to the category of abelian presheaves on $Y_{\text{ étale}}$. Before we define $f_!$ as the sheafification of this functor, let us check that it agrees with the construction in Section 3 and with the construction in Étale Cohomology, Section 69 when both apply.

0F6N **Lemma 3.3.** Let $f : X \to Y$ be a separated and locally quasi-finite morphism of schemes. Functorially in $F \in \text{Ab}(X_{\text{ étale}})$ there is a canonical isomorphism(!)

$$f_{pl}F \longrightarrow f_!F$$

of abelian presheaves which identifies the sheaf $f_!F$ of Definition 2.3 with the presheaf $f_{pl}F$ constructed above.

**Proof.** Let $V$ be an object of $Y_{\text{ étale}}$. If $Z \subset X_V$ is locally closed and finite over $V$, then, since $f$ is separated, we see that the morphism $Z \to X_V$ is a closed immersion. Moreover, if $Z_i, i = 1, \ldots, n$ are closed subschemes of $X_V$ finite over $V$, then $Z_1 \cup \ldots \cup Z_n$ (scheme theoretic union) is a closed subscheme finite over $V$.

\[\text{Since } f \text{ is locally quasi-finite, the morphism } Z_i \to V \text{ is finite if and only if it is proper.}\]
Hence in this case the colimit (3.2.2) defining $f_{\mathfrak{p}} \mathcal{F}(V)$ is directed and we find that $f_{\mathfrak{p}} \mathcal{F}(V)$ is simply equal to the set of sections of $\mathcal{F}(X_V)$ whose support is finite over $V$. Since any closed subset of $X_V$ which is proper over $V$ is actually finite over $V$ (as $f$ is locally quasi-finite) we conclude that this is equal to $f_! \mathcal{F}(V)$ by its very definition.

\[ \square \]

\textbf{Lemma 3.4.} \ Let $f : X \to Y$ be a morphism of schemes which is locally quasi-finite. \ Let $\overline{y} : \text{Spec}(k) \to Y$ be a geometric point. \ Functorially in $\mathcal{F}$ in $\text{Ab}(X_{\text{étale}})$ we have

\[ (f_{\mathfrak{p}} \mathcal{F})_{\overline{y}} = \bigoplus_{f(\overline{y}) = \overline{y}} \mathcal{F}_{\overline{y}} \]

\textbf{Proof.} \ Recall that the stalk at $\overline{y}$ of a presheaf is defined by the usual colimit over \ étale neighbourhoods $(V, \overline{y})$ of $\overline{y}$, see Etale Cohomology, Definition 29.6. Accordingly suppose $s = \sum_{i=1,\ldots,n}(Z_i, s_i)$ as in (3.2.1) is an element of $f_{\mathfrak{p}} \mathcal{F}(V)$ where $(V, \overline{y})$ is an \ étale neighbourhood of $\overline{y}$. Then since

\[ X_{\overline{y}} = (X_V)_{\overline{y}} \supset Z_i, \overline{y} \]

and since $s_i$ is a section of $\mathcal{F}$ on an \ étale neighbourhood of $Z_i$ in $X_V$ we can send $s$ to

\[ \sum_{i=1,\ldots,n} \sum_{\overline{y} \in Z_i, \overline{y}} \text{(class of } s_i \text{ in } \mathcal{F}_{\overline{y}}) \in \bigoplus_{f(\overline{y}) = \overline{y}} \mathcal{F}_{\overline{y}} \]

We omit the verification that this is compatible with restriction maps and that the relations (1) $(Z, s) + (Z, s') - (Z, s + s')$ and (2) $(Z, s) - (Z', s)$ if $Z \subset Z'$ are sent to zero. Thus we obtain a map

\[ (f_{\mathfrak{p}} \mathcal{F})_{\overline{y}} \to \bigoplus_{f(\overline{y}) = \overline{y}} \mathcal{F}_{\overline{y}} \]

Let us prove this arrow is surjective. For this it suffices to pick an $\overline{y}$ with $f(\overline{y}) = \overline{y}$ and prove that an element $s$ in the summand $\mathcal{F}_{\overline{y}}$ is in the image. Let $s$ correspond to the element $s \in \mathcal{F}(U)$ where $(U, \overline{y})$ is an \ étale neighbourhood of $\overline{y}$. Since $f$ is locally quasi-finite, the morphism $U \to Y$ is locally quasi-finite too. By More on Morphisms, Lemma 36.3 we can find an \ étale neighbourhood $(V, \overline{y})$ of $\overline{y}$, an open subscheme

\[ W \subset U \times_Y V, \]

and a geometric point $\overline{w}$ mapping to $\overline{y}$ and $\overline{v}$ such that $W \to V$ is finite and $\overline{w}$ is the only geometric point of $W$ mapping to $\overline{v}$. (We omit the translation between the language of geometric points we are currently using and the language of points and residue field extensions used in the statement of the lemma.) Observe that $W \to X_V = X \times_Y V$ is \ étale. Choose an affine open neighbourhood $W' \subset X_V$ of the image $\overline{w}$ of $\overline{w}$. Since $\overline{w}$ is the only point of $W$ over $\overline{v}$ and since $W \to V$ is closed, after replacing $V$ by an open neighbourhood of $\overline{v}$, we may assume $W \to X_V$ maps into $W'$. Then $W \to W'$ is finite and \ étale and there is a unique geometric point $\overline{w}$ of $W$ lying over $\overline{w}'$. It follows that $W \to W'$ is an open immersion over an open neighbourhood of $\overline{w}'$ in $W'$, see Etale Morphisms, Lemma 14.2. Shrinking $V$ and $W'$ we may assume $W \to W'$ is an isomorphism. Thus $s$ may be viewed as a section $s'$ of $\mathcal{F}$ over the open subscheme $W' \subset X_V$ which is finite over $V$. Hence by definition $(W', s')$ defines an element of $f_{\mathfrak{p}} \mathcal{F}(V)$ which maps to $s$ as desired.

Let us prove the arrow is injective. To do this, let $s = \sum_{i=1,\ldots,n}(Z_i, s_i)$ as in (3.2.1) be an element of $f_{\mathfrak{p}} \mathcal{F}(V)$ where $(V, \overline{y})$ is an \ étale neighbourhood of $\overline{y}$. Assume $s$ maps to zero under the map constructed above. First, after replacing $(V, \overline{y})$
by an étale neighbourhood of itself, we may assume there exist decompositions
\( Z_i = Z_{i,1} \oplus \cdots \oplus Z_{i,m_i} \) into open and closed subschemes such that each
\( Z_{i,j} \) has exactly one geometric point over \( \overline{\eta} \). Say under the obvious direct sum decomposition
\[
H_{Z_i}(\mathcal{F}) = \bigoplus H_{Z_{i,j}}(\mathcal{F})
\]
the element \( s_i \) corresponds to \( \sum s_{i,j} \). We may use relations (1) and (2) to replace
\( s \) by \( \sum_{i=1}^n \sum_{j=1}^{m_i} (Z_{i,j}, s_{i,j}) \). In other words, we may assume \( Z_i \) has a
unique geometric point lying over \( \overline{\eta} \). Let \( \overline{\eta_1}, \ldots, \overline{\eta_m} \) be the geometric points of
\( X \) over \( \overline{\eta} \) corresponding to the geometric points of our \( Z_i \) over \( \overline{\eta} \); note that for
one \( j \in \{1, \ldots, m\} \) there may be multiple indices \( i \) for which \( \overline{\eta_j} \) corresponds to a
point of \( Z_i \). By More on Morphisms, Lemma \([36.3]\) applied to both \( X_\overline{\eta} \rightarrow V \) after
replacing \( (V, \overline{\eta}) \) by an étale neighbourhood of itself we may assume there exist open
subschemes
\[
W_j \subset X \times_V V, \quad j = 1, \ldots, m
\]
and a geometric point \( \overline{\eta_j} \) of \( W_j \) mapping to \( \overline{\eta_j} \) and \( \overline{\eta} \) such that \( W_j \rightarrow V \) is finite
and \( \overline{\eta_j} \) is the only geometric point of \( W_j \) mapping to \( \overline{\eta} \). After shrinking \( V \) we may
assume \( Z_i \subset W_j \) for some \( j \) and we have the map \( H_{Z_i}(\mathcal{F}) \rightarrow H_{W_j}(\mathcal{F}) \). Thus by
the relation (2) we see that our element is equivalent to an element of the form
\[
\sum_{j=1}^m (W_j, t_j)
\]
for some \( t_j \in H_{W_j}(\mathcal{F}) \). Clearly, this element is mapped simply to the class of \( t_j \)
in the summand \( \mathcal{F}_{\overline{\eta}_j} \). Since \( s \) maps to zero, we find that \( t_j \) maps to zero in \( \mathcal{F}_{\overline{\eta}_j} \).
This implies that \( t_j \) restricts to zero on an open neighbourhood of \( \overline{\eta}_j \) in \( W_j \), see
Étale Cohomology, Lemma \([31.2]\). Shrinking \( V \) once more we obtain \( t_j = 0 \) for all \( j \)
as desired. \( \square \)

0F6Q **Lemma 3.5.** Let \( f = j : U \rightarrow X \) be an étale of schemes. Denote \( j_{pl} \) the con-
struction of Étale Cohomology, Equation \([69.1.1]\) and denote \( f_{pl} \) the construction above.
Functorially in \( \mathcal{F} \in \text{Ab}(X_{étale}) \) there is a canonical map
\[
j_{pl}\mathcal{F} \rightarrow f_{pl}\mathcal{F}
\]
of abelian presheaves which identifies the sheaf \( j_{!}\mathcal{F} \) with \( (f_{pl}\mathcal{F})^\# \) of Étale Cohomology,
Definition \([69.4]\) with \( (f_{pl}\mathcal{F})^\# \).

**Proof.** Please read the proof of Étale Cohomology, Lemma \([69.5]\) before reading the
proof of this lemma. Let \( V \) be an object of \( X_{étale} \). Recall that
\[
j_{pl}\mathcal{F}(V) = \bigoplus_{\varphi : V \rightarrow U} \mathcal{F}(V \overset{\varphi}{\rightarrow} U)
\]
Given \( \varphi \) we obtain an open subscheme \( Z_\varphi \subset U_V = U \times_X V \), namely, the image of
the graph of \( \varphi \). Via \( \varphi \) we obtain an isomorphism \( V \rightarrow Z_\varphi \) over \( U \) and we can think of
an element
\[
s_\varphi \in \mathcal{F}(V \overset{\varphi}{\rightarrow} U) = \mathcal{F}(Z_\varphi) = H_{Z_\varphi}(\mathcal{F})
\]
as a section of \( \mathcal{F} \) over \( Z_\varphi \). Since \( Z_\varphi \subset U_V \) is open, we actually have \( H_{Z_\varphi}(\mathcal{F}) = \mathcal{F}(Z_\varphi) \) and we can think of \( s_\varphi \) as an element of \( H_{Z_\varphi}(\mathcal{F}) \). Having said this, our map
\( j_{pl}\mathcal{F} \rightarrow f_{pl}\mathcal{F} \) is defined by the rule
\[
\sum_{i=1}^n s_{\varphi_i} \mapsto \sum_{i=1}^n (Z_{\varphi_i}, s_{\varphi_i})
\]
with right hand side a sum as in \([3.2.1]\). We omit the verification that this is
compatible with restriction mappings and functorial in \( \mathcal{F} \).
To finish the proof, we claim that this map is compatible with the identifications of stalks given in Lemma 3.4 and Étale Cohomology, Proposition 69.3. We omit the details. □

**Definition 3.6.** Let $f : X \to Y$ be a locally quasi-finite morphism of schemes. We define the direct image with compact support to be the functor

$$f_! : \text{Ab}(X_{\text{étale}}) \to \text{Ab}(Y_{\text{étale}})$$

defined by the formula $f_! F = (f_ p F)^\#$, i.e., $f_! F$ is the sheafification of the presheaf $f_ p F$ constructed above.

By Lemma 3.3 this does not conflict with Definition 2.3 (when both definitions apply) and by Lemma 3.5 this does not conflict with Étale Cohomology, Definition 69.1 (when both definitions apply).

**Lemma 3.7.** Let $f : X \to Y$ be a locally quasi-finite morphism of schemes. Then

1. for $F$ in $\text{Ab}(X_{\text{étale}})$ and a geometric point $\overline{y} : \text{Spec}(k) \to Y$ we have

$$(f_! F)(\overline{y}) = \bigoplus_{f_ x(\overline{y}) = \overline{y}} F_x$$

functorially in $F$, and

2. the functor $f_! : \text{Ab}(X_{\text{étale}}) \to \text{Ab}(Y_{\text{étale}})$ is exact and commutes with direct sums.

**Proof.** The formula for the stalks is immediate (and in fact equivalent) to Lemma 3.4. The exactness of the functor follows immediately from this and the fact that exactness may be checked on stalks, see Étale Cohomology, Theorem 29.10 □

**Remark 3.8 (Covariance with respect to open embeddings).** Let $f : X \to Y$ be locally quasi-finite morphism of schemes. Let $\mathcal{F}$ be an abelian sheaf on $X_{\text{étale}}$. Let $X' \subset X$ be an open subscheme and denote $f' : X' \to Y$ the restriction of $f$. Denote $\mathcal{F}' = \mathcal{F}|_{X'_{\text{étale}}}$. We claim there is a canonical map

$$f'_! \mathcal{F}' \to f_! \mathcal{F}$$

In fact, we will construct a map $f'_ p \mathcal{F}' \to f_ p \mathcal{F}$ as follows. Let $V \in Y_{\text{étale}}$ and consider a section $s' = \sum_{i=1}^n (Z'_i, s'_i)$ as in (3.2.1) defining an element of $f'_ p \mathcal{F}'(V)$. Then $Z'_i \subset X'_V$ may also be viewed as a locally closed subscheme of $X_V$ and clearly we have $H_{Z'_i}(\mathcal{F}') = H_{Z'_i}(\mathcal{F})$. We will map $s'$ to the exact same sum $s = \sum_{i=1}^n (Z'_i, s'_i)$ but now viewed as an element of $f_ p \mathcal{F}(V)$. We omit the verification that this construction is compatible with restriction mappings and functorial in $\mathcal{F}$. This construction has the following properties:

1. The map $f'_! \mathcal{F}' \to f_! \mathcal{F}$ is compatible with the description of stalks given in Lemma 3.7.
2. If $X'' \subset X'$ is another open, then the composition of $f''_! \mathcal{F}'' \to f'_! \mathcal{F}' \to f_! \mathcal{F}$ is the map $f''_! \mathcal{F}'' \to f_! \mathcal{F}$ for the inclusion $X'' \subset X$. In fact, this already holds for the maps of presheaves defined above.
3. The map $f'_! \mathcal{F}' \to f_! \mathcal{F}$ is injective (because we can check on stalks).
4. The map $f'_! \mathcal{F}' \to f_! \mathcal{F}$ just constructed is the same as the map given in Remark 2.7 in case $f$ is separated.
Lemma 3.9. Let \( f : X \to Y \) be a locally quasi-finite morphism of schemes. Let \( X = \bigcup_{i \in I} X_i \) be an open covering. Then there exists an exact complex

\[
\cdots \to \bigoplus_{i_0, i_1, i_2} f_{i_0 i_1 i_2}^* F|_{X_{i_0 i_1 i_2}} \to \bigoplus_{i_0, i_1} f_{i_0 i_1}^* F|_{X_{i_0 i_1}} \to \bigoplus_{i_0} f_{i_0}^* F|_{X_{i_0}} \to f_* F \to 0
\]

functorial in \( F \in \text{Ab}(X_{\text{étale}}) \), see proof for details.

Proof. Here as usual we set \( X_{i_0 \ldots i_p} = X_{i_0} \cap \ldots \cap X_{i_p} \) and we denote \( f_{i_0 \ldots i_p} \) the restriction of \( f \) to \( X_{i_0 \ldots i_p} \). The maps in the complex are the maps constructed in Remark 3.8 with sign rules as in the Čech complex. Exactness follows easily from the description of stalks in Lemma 3.7. Details omitted.

Remark 3.10 (Alternative construction). Lemma 3.9 gives an alternative construction of the functor \( f_* \) for locally quasi-finite morphisms \( f \). Namely, given a locally quasi-finite morphism \( f : X \to Y \) of schemes we can choose an open covering \( X = \bigcup_{i \in I} X_i \) such that each \( f_i : X_i \to Y \) is separated. For example choose an affine open covering of \( X \). Then we can define \( f_* F \) as the cokernel of the penultimate map of the complex of the lemma, i.e.,

\[
f_* F = \text{Coker} \left( \bigoplus_{i_0, i_1} f_{i_0 i_1}^* F|_{X_{i_0 i_1}} \to \bigoplus_{i_0} f_{i_0}^* F|_{X_{i_0}} \right)
\]

where we can use the construction of \( f_{i_0 i_1} \) and \( f_{i_0 i_1}! \) in Section 2 because the morphisms \( f_{i_0} \) and \( f_{i_0 i_1} \) are separated. One can then compute the stalks of \( f_* \) (using the separated case, namely Lemma 2.11) and obtain the result of Lemma 3.7. Having done so all the other results of this section can be deduced from this as well.

Lemma 3.11. Consider a cartesian square

\[
\begin{array}{ccc}
X' & \xrightarrow{g} & X \\
\downarrow f' & & \downarrow f \\
Y' & \xrightarrow{g} & Y
\end{array}
\]

of schemes with \( f \) locally quasi-finite. For any abelian sheaf \( F \) on \( X_{\text{étale}} \) we have

\[
f_* (g')^{-1} F = g^{-1} f_* F.
\]

First proof. We will explicitly construct a map

\[
f_{p!} F \to g_* f_{p!} (g')^{-1} F
\]

of abelian presheaves on \( Y'_{\text{étale}} \). Since pushforward commutes with sheafification, we obtain a map \( f_* F \to g_* f_! (g')^{-1} F \). This map in turn is adjoint to a map \( g^{-1} f_* F \to f_! (g')^{-1} F \).

Construction of the map. Let \( V \in Y_{\text{étale}} \) and consider a section \( s = \sum_{i=1}^n (Z_i, s_i) \) as in (3.2.1) defining an element of \( f_{p!} F(V) \). The value of \( g_* f_{p!} (g')^{-1} F \) at \( V' = V \times_Y Y' \) is \( f_{p!} (g')^{-1} F(V') \). Denote \( Z'_i \subset X' \), the base change of \( Z_i \) to \( V' \). By (2) there is a pullback map \( H_{Z_i}(F) \to H_{Z'_i}((g')^{-1} F) \). Denoting \( s'_i \in H_{Z'_i}((g')^{-1} F) \) we send \( s \) to \( s' = \sum_{i=1}^n (Z'_i, s'_i) \) as in (3.2.1) defining an element of \( f_{p!} (g')^{-1} F(V') \). We omit the verification that this construction is compatible with restriction mappings and functorial in \( F \).

Let \( \overline{y} : \text{Spec}(k) \to Y' \) be a geometric point with image \( \overline{y} = g \circ \overline{x} \) in \( Y \). Observe that \( X'_{\overline{y}} = X_{\overline{y}} \) by transitivity of fibre products. Then \( g' \) produces a bijection
Lemma 3.12. Let \( f : X \to Y \) and \( g : Y \to Z \) be composable locally quasi-finite morphisms of schemes. Then there is a canonical isomorphism of functors \( g \circ f = (g \circ f)_! \). Given a third locally quasi-finite morphism \( h : Z \to T \) the diagram

\[
\begin{array}{ccc}
(h \circ g \circ f)_! & \longrightarrow & (h \circ g)_! \circ f \\
\downarrow & & \downarrow \\
(h_1 \circ (g \circ f))_! & \longrightarrow & h_1 \circ (g \circ f)_!
\end{array}
\]

commutes.

**Proof.** If \( f \) and \( g \) are separated, then this is a special case of Lemma 2.4. Assume only \( g \) is separated. Choose an open covering \( X = \bigcup X_i \) such that the restrictions \( f_i : X_i \to Y \) are separated. By Lemma 3.9 we get an exact sequence

\[
\bigoplus_{i_0, i_1} f_{i_0, i_1} !_! F|_{X_{i_0 i_1}} \to \bigoplus_{i_0} f_{i_0} ^! F|_{X_{i_0}} \to f_! F \to 0
\]

Applying the exact functor \( g_! \) we obtain the exact sequence

\[
\bigoplus_{i_0, i_1} g_! f_{i_0, i_1} !_! F|_{X_{i_0 i_1}} \to \bigoplus_{i_0} g_! f_{i_0} ^! F|_{X_{i_0}} \to g_! f_! F \to 0
\]

Using the case where both morphisms are separated we can rewrite this as

\[
\bigoplus_{i_0, i_1} h_{i_0, i_1} !_! F|_{X_{i_0 i_1}} \to \bigoplus_{i_0} h_{i_0} ^! F|_{X_{i_0}} \to g_! f_! F \to 0
\]
where $h = g \circ f$ and $h_{io_1}, h_{io_i}$ are the restrictions of $h$ to $X_{io_1}$ and $X_{io_i}$. Then by Lemma 3.9 again we see that the cokernel of the first arrow is $h_{io_1} F_{1,0}$ which gives us the desired isomorphism.

Proof in the general case. Choose an open covering $Y = \bigcup Y_i$ such that the restriction $g_i : Y_i \to Z$ of $g$ is separated. Set $X_i = f^{-1}(Y_i)$ and denote $f_i : X_i \to Y_i$ the restriction of $f$. Also denote $h = g \circ f$ and $h_i : X_i \to Z$ the restriction of $h$. By Lemma 3.8 we get an exact sequence

$$\bigoplus_{io_1} g_{io_1,i} F_{1,Y_{io_1}} \to \bigoplus_{io} g_{io,i} F_{1,Y_i} \to g_{1,0} F_{1,0} \to 0$$

Since formation of $f_1$ is local on the base (in fact étale local on the base by its very construction) we can write $f_1: F_{1,Y_{io_1}} = f_{io_1,i} F_{1,X_{io_1}}$ and $f_1 F_{1,Y_i} = f_{io,i} F_{1,X_i}$. Thus the exact sequence above can be rewritten as

$$\bigoplus_{io_1} g_{io_1,i} f_{io_1,i} F_{1,X_{io_1}} \to \bigoplus_{io} g_{io,i} f_{io,i} F_{1,X_i} \to g_{1,0} F_{1,0} \to 0$$

Using the case where the second morphism is separated proved above we can rewrite this as

$$\bigoplus_{io_1} h_{io_1,i} F_{1,X_{io_1}} \to \bigoplus_{io} h_{io,i} F_{1,X_i} \to g_{1,0} F_{1,0} \to 0$$

and we conclude as before.

We omit the proof of commutativity of the square of the lemma; hint: it follows because our isomorphisms are compatible with the description of stalks of the direct images with compact supports given in Lemma 3.7 \hfill \Box

4. Upper shriek for locally quasi-finite morphisms

**0F58** For a locally quasi-finite morphism $f : X \to Y$ of schemes, the functor $f_! : Ab(X_{etale}) \to Ab(Y_{etale})$ commutes with direct sums and is exact, see Lemma 3.7 \hfill \Box

This suggests that it has a right adjoint which we will denote $f^!$.

Warning: This functor is the non-derived version!

**Lemma 4.1.** Let $f : X \to Y$ be a locally quasi-finite morphism of schemes. The functor $f_! : Ab(X_{etale}) \to Ab(Y_{etale})$ has a right adjoint $f^! : Ab(Y_{etale}) \to Ab(X_{etale})$. Moreover, we have $f^! (\mathbb{G}, A) = \prod_{f(\mathbb{G}) = \mathbb{G}} \mathbb{G}^A$.

**Proof.** Let $E \subset \text{Ob}(Ab(Y_{etale}))$ be the class consisting of products of skyscraper sheaves. We claim that

(1) every $\mathbb{G}$ in $Ab(Y_{etale})$ is a subsheaf of an element of $E$, and

(2) for every $\mathbb{G} \in E$ there exists an object $H$ of $Ab(X_{etale})$ such that $\text{Hom}(f_!, \mathbb{G}) = \text{Hom}(f_!, H)$ functorially in $f_!$.

Once the claim has been verified, the dual of Homology, Lemma 26.6 produces the adjoint functor $f^!$.

Part (1) is true because we can map $\mathbb{G}$ to the sheaf $\prod \mathbb{G} \mathbb{G}^\mathbb{G}$ where the product is over all geometric points of $Y$. This is an injection by Étale Cohomology, Theorem

---

\footnote{We omit the argument showing that the map $h_{io_1,i} F_{1,X_{io_1}} \to h_{io,i} F_{1,Y_i}$ of Remark 3.8 agrees with the map $g h_{io_1,i} F_{1,X_{io_1}} \to g h_{io,i} F_{1,Y_i}$ coming from $g$ applied to the map $f h_{io_1,i} F_{1,X_{io_1}} \to f h_{io,i} F_{1,X_i}$ of Remark 3.8. To see this use that these maps are compatible with the description of stalks given in Lemma 3.7.}
29.10. (This is the first step in the Godement resolution when done in the setting of abelian sheaves on topological spaces.)

Part (2) and the final statement of the lemma can be seen as follows. Suppose that \( \mathcal{G} = \prod_{\tau} A_{\tau} \) for some abelian groups \( A_{\tau} \). Then

\[
\text{Hom}(f_! \mathcal{F}, \mathcal{G}) = \prod \text{Hom}(f_! \mathcal{F}, A_{\tau})
\]

Thus it suffices to find abelian sheaves \( \mathcal{H}_{\tau} \) on \( X_{\text{étale}} \) representing the functors \( F \mapsto \text{Hom}(f_! \mathcal{F}, A_{\tau}) \). Thus it suffices to find abelian sheaves \( H_{\tau} \) on \( X_{\text{étale}} \) representing the functors \( F \mapsto \text{Hom}(f_! \mathcal{F}, A_{\tau}) \).

Thus it suffices to find abelian sheaves \( H_{\tau} \) on \( X_{\text{étale}} \) representing the functors \( F \mapsto \text{Hom}(f_! \mathcal{F}, A_{\tau}) \). This reduces us to the case \( H = \prod_{f(\tau) = \overline{\eta}} A_{\tau} \) with some fixed geometric point \( \overline{\eta} : \text{Spec}(k) \to Y \) and some fixed abelian group \( A \).

We claim that in this case \( H = \prod f_!(x) = y_! x^* A \) works. This will finish the proof of the lemma. Namely, we have

\[
\text{Hom}(f_! \mathcal{F}, \overline{\eta}_* A) = \text{Hom}_{\text{Ab}}(f_! \mathcal{F}, A) = \text{Hom}_{\text{Ab}}(\bigoplus_{f(\tau) = \overline{\eta}} f_! \mathcal{F}, A)
\]

by the description of stalks in Lemma 3.7 on the one hand and on the other hand we have

\[
\text{Hom}(f_! \mathcal{F}, H) = \prod_{f(\tau) = \overline{\eta}} \text{Hom}(f_! \mathcal{F}, A) = \prod_{f(\tau) = \overline{\eta}} \text{Hom}_{\text{Ab}}(f_! \mathcal{F}, A)
\]

We leave it to the reader to identify these as functors of \( \mathcal{F} \).

\[\square\]

**Lemma 4.2.** Let \( j : U \to X \) be an étale morphism. Then \( j^! = j^{-1} \).

**Proof.** This is true because \( j^! \) as defined in Section 3 agrees with \( j^! \) as defined in Étale Cohomology, Section 69, see Lemma 3.5. Finally, in Étale Cohomology, Section 69 the functor \( j^! \) is defined as the left adjoint of \( j^{-1} \) and hence we conclude by uniqueness of adjoint functors. \[\square\]

**Lemma 4.3.** Let \( f : X \to Y \) and \( g : Y \to Z \) be separated and locally quasi-finite morphisms. There is a canonical isomorphism \( (g \circ f)^! \to f^! \circ g^! \). Given a third locally quasi-finite morphism \( h : Z \to T \) the diagram

\[
\begin{array}{ccc}
(h \circ g \circ f)^! & \to & f^! \circ (h \circ g)^! \\
\downarrow & & \downarrow \\
(g \circ f)^! \circ h^! & \to & f^! \circ g^! \circ h^!
\end{array}
\]

commutes.

**Proof.** By uniqueness of adjoint functors, this immediately translates into the corresponding (dual) statement for the functors \( f^! \). See Lemma 3.12. \[\square\]

**Lemma 4.4.** Let \( j : U \to X \) and \( j' : V \to U \) be étale morphisms. The isomorphism \( (j \circ j')^{-1} = (j')^{-1} \circ j^{-1} \) and the isomorphism \( (j \circ j')^! = (j')^! \circ j^! \) of Lemma 4.3 agree via the isomorphism of Lemma 4.2.

**Proof.** Omitted. \[\square\]

**Lemma 4.5.** Consider a cartesian square

\[
\begin{array}{ccc}
X' & \to & X \\
\downarrow^g & & \downarrow^f \\
Y' & \to & Y
\end{array}
\]

0F5A

0F5B

0F5C

0F6U
of schemes with \( f \) locally quasi-finite. For any abelian sheaf \( F \) on \( Y'_{\text{étale}} \) we have \((g')_*(f')! F = f'_! g_* F\).

**Proof.** By uniqueness of adjoint functors, this follows from the corresponding (dual) statement for the functors \( f'! \). See Lemma 3.11. \( \square \)

**Remark 4.6.** The material in this section can be generalized to sheaves of pointed sets. Namely, for a site \( \mathcal{C} \) denote \( \mathcal{S}h^* (\mathcal{C}) \) the category of sheaves of pointed sets. The constructions in this and the preceding section apply, mutatis mutandis, to sheaves of pointed sets. Thus given a locally quasi-finite morphism \( f : X \to Y \) of schemes we obtain an adjoint pair of functors 

\[
\begin{align*}
&f! : \mathcal{S}h^* (X'_{\text{étale}}) \to \mathcal{S}h^* (Y'_{\text{étale}}) \quad \text{and} \quad f^! : \mathcal{S}h^* (Y'_{\text{étale}}) \to \mathcal{S}h^* (X'_{\text{étale}})
\end{align*}
\]

such that for every geometric point \( \overline{y} \) of \( Y \) there are isomorphisms 

\[
(f_* F)_{\overline{y}} = \coprod_{f(\overline{x}) = \overline{y}} F_{\overline{x}}
\]

(coproduct taken in the category of pointed sets) functorial in \( F \in \mathcal{S}h^* (X'_{\text{étale}}) \) and isomorphisms 

\[
f^!(\overline{y}, S) = \prod_{f(\overline{x}) = \overline{y}} F_{\overline{x}, S}
\]

functorial in the pointed set \( S \). If \( F : Ab(X'_{\text{étale}}) \to \mathcal{S}h^* (X'_{\text{étale}}) \) and \( F : Ab(Y'_{\text{étale}}) \to \mathcal{S}h^* (Y'_{\text{étale}}) \) denote the forgetful functors, compatibility between the constructions will guarantee the existence of canonical maps 

\[
f_* F(F) \to F(f_* F)
\]

functorial in \( F \in Ab(X'_{\text{étale}}) \) and 

\[
F(f^! G) \to f^! F(G)
\]

functorial in \( G \in Ab(Y'_{\text{étale}}) \) which produce the obvious maps on stalks, resp. skyscraper sheaves. In fact, the transformation \( F \circ f^! \to f^! \circ F \) is an isomorphism (because \( f^! \) commutes with products).

5. Derived upper shriek for locally quasi-finite morphisms

We can take the derived versions of the functors in Section 4 and obtain the following.

**Lemma 5.1.** Let \( f : X \to Y \) be a locally quasi-finite morphism of schemes. The functors \( f_* \) and \( f^! \) of Definition 3.6 and Lemma 4.1 induce adjoint functors 

\[
\begin{align*}
&f_* : D(X'_{\text{étale}}) \to D(Y'_{\text{étale}}) \quad \text{and} \quad Rf^! : D(Y'_{\text{étale}}) \to D(X'_{\text{étale}})
\end{align*}
\]

on derived categories.

In the separated case the functor \( f_* \) is defined in Section 2.

**Proof.** This follows immediately from Derived Categories, Lemma 28.5, the fact that \( f_* \) is exact (Lemma 3.7) and hence \( Lf_* = f_* \) and the fact that we have enough K-injective complexes of abelian sheaves on \( Y'_{\text{étale}} \) so that \( Rf^! \) is defined. \( \square \)

6. Other chapters

Preliminaries

(1) Introduction
(2) Conventions
(3) Set Theory

(4) Categories
(5) Topology
(6) Sheaves on Spaces
(7) Sites and Sheaves
<table>
<thead>
<tr>
<th>(8) Stacks</th>
<th>(54) Étale Cohomology</th>
</tr>
</thead>
<tbody>
<tr>
<td>(9) Fields</td>
<td>(55) Crystalline Cohomology</td>
</tr>
<tr>
<td>(10) Commutative Algebra</td>
<td>(56) Pro-étale Cohomology</td>
</tr>
<tr>
<td>(11) Brauer Groups</td>
<td>(57) More Étale Cohomology</td>
</tr>
<tr>
<td>(12) Homological Algebra</td>
<td>(58) The Trace Formula</td>
</tr>
<tr>
<td>(13) Derived Categories</td>
<td>Algebraic Spaces</td>
</tr>
<tr>
<td>(14) Simplicial Methods</td>
<td>(59) Algebraic Spaces</td>
</tr>
<tr>
<td>(15) More on Algebra</td>
<td>(60) Properties of Algebraic Spaces</td>
</tr>
<tr>
<td>(16) Smoothing Ring Maps</td>
<td>(61) Morphisms of Algebraic Spaces</td>
</tr>
<tr>
<td>(17) Sheaves of Modules</td>
<td>(62) Decent Algebraic Spaces</td>
</tr>
<tr>
<td>(18) Modules on Sites</td>
<td>(63) Cohomology of Algebraic Spaces</td>
</tr>
<tr>
<td>(19) Injectives</td>
<td>(64) Limits of Algebraic Spaces</td>
</tr>
<tr>
<td>(20) Cohomology of Sheaves</td>
<td>(65) Divisors on Algebraic Spaces</td>
</tr>
<tr>
<td>(21) Cohomology on Sites</td>
<td>(66) Algebraic Spaces over Fields</td>
</tr>
<tr>
<td>(22) Differential Graded Algebra</td>
<td>(67) Topologies on Algebraic Spaces</td>
</tr>
<tr>
<td>(23) Divided Power Algebra</td>
<td>(68) Descent and Algebraic Spaces</td>
</tr>
<tr>
<td>(24) Hypercoverings</td>
<td>(69) Derived Categories of Spaces</td>
</tr>
<tr>
<td></td>
<td>(70) More on Morphisms of Spaces</td>
</tr>
<tr>
<td></td>
<td>(71) Flatness on Algebraic Spaces</td>
</tr>
<tr>
<td></td>
<td>(72) Groupoids in Algebraic Spaces</td>
</tr>
<tr>
<td></td>
<td>(73) More on Groupoids in Spaces</td>
</tr>
<tr>
<td></td>
<td>(74) Bootstrap</td>
</tr>
<tr>
<td></td>
<td>(75) Pushouts of Algebraic Spaces</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Schemes</th>
<th>Topics in Geometry</th>
</tr>
</thead>
<tbody>
<tr>
<td>(25) Schemes</td>
<td>(76) Chow Groups of Spaces</td>
</tr>
<tr>
<td>(26) Constructions of Schemes</td>
<td>(77) Quotients of Groupoids</td>
</tr>
<tr>
<td>(27) Properties of Schemes</td>
<td>(78) More on Cohomology of Spaces</td>
</tr>
<tr>
<td>(28) Morphisms of Schemes</td>
<td>(79) Simplicial Spaces</td>
</tr>
<tr>
<td>(29) Cohomology of Schemes</td>
<td>(80) Duality for Spaces</td>
</tr>
<tr>
<td>(30) Divisors</td>
<td>(81) Formal Algebraic Spaces</td>
</tr>
<tr>
<td>(31) Limits of Schemes</td>
<td>(82) Restricted Power Series</td>
</tr>
<tr>
<td>(32) Varieties</td>
<td>(83) Resolution of Surfaces Revisited</td>
</tr>
<tr>
<td>(33) Topologies on Schemes</td>
<td></td>
</tr>
<tr>
<td>(34) Descent</td>
<td>Deformation Theory</td>
</tr>
<tr>
<td>(35) Derived Categories of Schemes</td>
<td>(84) Formal Deformation Theory</td>
</tr>
<tr>
<td>(36) More on Morphisms</td>
<td>(85) Deformation Theory</td>
</tr>
<tr>
<td>(37) More on Flatness</td>
<td>(86) The Cotangent Complex</td>
</tr>
<tr>
<td>(38) Groupoid Schemes</td>
<td>(87) Deformation Problems</td>
</tr>
<tr>
<td>(39) More on Groupoid Schemes</td>
<td></td>
</tr>
<tr>
<td>(40) Étale Morphisms of Schemes</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Topics in Scheme Theory</th>
<th>Algebraic Stacks</th>
</tr>
</thead>
<tbody>
<tr>
<td>(41) Chow Homology</td>
<td>(88) Algebraic Stacks</td>
</tr>
<tr>
<td>(42) Intersection Theory</td>
<td>(89) Examples of Stacks</td>
</tr>
<tr>
<td>(43) Picard Schemes of Curves</td>
<td>(90) Sheaves on Algebraic Stacks</td>
</tr>
<tr>
<td>(44) Adequate Modules</td>
<td>(91) Criteria for Representability</td>
</tr>
<tr>
<td>(45) Dualizing Complexes</td>
<td>(92) Artin’s Axioms</td>
</tr>
<tr>
<td>(46) Duality for Schemes</td>
<td>(93) Quot and Hilbert Spaces</td>
</tr>
<tr>
<td>(47) Discriminants and Differents</td>
<td>(94) Properties of Algebraic Stacks</td>
</tr>
<tr>
<td>(48) Local Cohomology</td>
<td>(95) Morphisms of Algebraic Stacks</td>
</tr>
<tr>
<td>(49) Algebraic and Formal Geometry</td>
<td>(96) Limits of Algebraic Stacks</td>
</tr>
<tr>
<td>(50) Algebraic Curves</td>
<td></td>
</tr>
<tr>
<td>(51) Resolution of Surfaces</td>
<td></td>
</tr>
<tr>
<td>(52) Semistable Reduction</td>
<td></td>
</tr>
<tr>
<td>(53) Fundamental Groups of Schemes</td>
<td></td>
</tr>
<tr>
<td>Page</td>
<td>Title</td>
</tr>
<tr>
<td>------</td>
<td>--------------------------------</td>
</tr>
<tr>
<td>97</td>
<td>Cohomology of Algebraic Stacks</td>
</tr>
<tr>
<td>98</td>
<td>Derived Categories of Stacks</td>
</tr>
<tr>
<td>99</td>
<td>Introducing Algebraic Stacks</td>
</tr>
<tr>
<td>100</td>
<td>More on Morphisms of Stacks</td>
</tr>
<tr>
<td>101</td>
<td>The Geometry of Stacks</td>
</tr>
<tr>
<td></td>
<td>Topics in Moduli Theory</td>
</tr>
<tr>
<td>102</td>
<td>Moduli Stacks</td>
</tr>
<tr>
<td>103</td>
<td>Moduli of Curves</td>
</tr>
<tr>
<td></td>
<td>Miscellany</td>
</tr>
</tbody>
</table>

References