MORE ÉTALE COHOMOLOGY

0F4U

Contents

1. Introduction

0F4V This chapter is the second in a series of chapter on the étale cohomology of schemes. To read the first chapter, please visit Étale Cohomology, Section 1.

The split with the previous chapter is roughly speaking that anything concerning "shriek functors" (cohomology with compact support and its right adjoint) and anything using this material goes into this chapter.

2. Growing sections

0F71 In this section we discuss results of the following type.

0F6F **Lemma 2.1.** *Let X be a scheme. Let $\mathcal{F}$ be an abelian sheaf on $X_{\acute{e}tale}$. Let $\varphi : U' \to U$ be a morphism of $X_{\acute{e}tale}$. Let $Z' \subset U'$ be a closed subscheme such that $Z' \to U' \to U$ is a closed immersion with image $Z \subset U$. Then there is a canonical bijection*

$$\{s \in \mathcal{F}(U) \mid Supp(s) \subset Z\} = \{s' \in \mathcal{F}(U') \mid Supp(s') \subset Z'\}$$

which is given by restriction if $\varphi^{-1}(Z) = Z'$.

Proof. Consider the closed subscheme $Z'' = \varphi^{-1}(Z)$ of U'. Then $Z' \subset Z''$ is closed because Z' is closed in U'. On the other hand, $Z' \to Z''$ is an étale morphism (as a morphism between schemes étale over Z) and hence open. Thus $Z'' = Z' \amalg T$ for some closed subset T. The open covering $U' = (U' \setminus T) \cup (U' \setminus Z')$ shows that

$$\{s' \in \mathcal{F}(U') \mid \text{Supp}(s') \subset Z'\} = \{s' \in \mathcal{F}(U' \setminus T) \mid \text{Supp}(s') \subset Z'\}$$

This is a chapter of the Stacks Project, version 150ef3f2, compiled on Sep 15, 2019.

and the étale covering $\{U' \setminus T \to U, U \setminus Z \to U\}$ shows that

$$\{s \in \mathcal{F}(U) \mid \mathrm{Supp}(s) \subset Z\} = \{s' \in \mathcal{F}(U' \setminus T) \mid \mathrm{Supp}(s') \subset Z'\}$$

This finishes the proof. $\square$

0F6G **Lemma 2.2.** *Let X be a scheme. Let $Z \subset X$ be a locally closed subscheme. Let $\mathcal{F}$ be an abelian sheaf on $X_{\acute{e}tale}$. Given $U, U' \subset X$ open containing Z as a closed subscheme, there is a canonical bijection*

$$\{s \in \mathcal{F}(U) \mid Supp(s) \subset Z\} = \{s \in \mathcal{F}(U') \mid Supp(s) \subset Z\}$$

which is given by restriction if $U' \subset U$.

Proof. Since Z is a closed subscheme of $U \cap U'$, it suffices to prove the lemma when $U' \subset U$. Then it is a special case of Lemma 2.1. $\square$

Let us introduce a bit of nonstandard notation which will stand us in good stead later. Namely, in the situation of Lemma 2.2 above, let us denote

$$H_Z(\mathcal{F}) = \{s \in \mathcal{F}(U) \mid \mathrm{Supp}(s) \subset Z\}$$

where $U \subset X$ is any choice of open subscheme containing Z as a closed subscheme. The reader who is troubled by the lack of precision this entails may choose $U = X \setminus \partial Z$ where $\partial Z = \overline{Z} \setminus Z$ is the "boundary" of Z in X. However, in many of the arguments below the flexibility of choosing different opens will play a role. Here are some properties of this construction:

0F6H (1) If $Z \subset Z'$ are locally closed subschemes of X and Z is closed in Z', then there is a natural injective map

$$H_Z(\mathcal{F}) \to H_{Z'}(\mathcal{F}).$$

0F6I (2) If $f : Y \to X$ is a morphism of schemes and $Z \subset X$ is a locally closed subscheme, then there is a natural pullback map $f^* : H_Z(\mathcal{F}) \to H_{f^{-1}Z}(f^{-1}\mathcal{F})$.

It will be convenient to extend our notation to the following situation: suppose that we have $W \in X_{\acute{e}tale}$ and a locally closed subscheme $Z \subset W$. Then we will denote

$$H_Z(\mathcal{F}) = \{s \in \mathcal{F}(U) \mid \mathrm{Supp}(s) \subset Z\} = H_Z(\mathcal{F}|_{W_{\acute{e}tale}})$$

where $U \subset W$ is any choice of open subscheme containing Z as a closed subscheme, exactly as above[1].

3. Sections with compact support

0F4W A reference for this section is [AGV71, Exposee XVII, Section 6]. Let $f : X \to Y$ be a morphism of schemes which is separated and locally of finite type. In this section we define a functor $f_! : Ab(X_{\acute{e}tale}) \to Ab(Y_{\acute{e}tale})$ by taking $f_!\mathcal{F} \subset f_*\mathcal{F}$ to be the subsheaf of sections which have proper support relative to Y (suitably defined).

Warning: The functor $f_!$ is the zeroth cohomology sheaf of a functor $Rf_!$ on the derived category (insert future reference), but $Rf_!$ is not the derived functor of $f_!$.

[1]In fact, Lemma 2.1 shows, given Z over X which is isomorphic to a locally closed subscheme of some object W of $X_{\acute{e}tale}$, that the choice of W is irrelevant.

0F4X **Lemma 3.1.** *Let $f : X \to Y$ be a morphism of schemes which is locally of finite type. Let $\mathcal{F}$ be an abelian sheaf on $X_{étale}$. The rule*

$$Y_{étale} \longrightarrow Ab, \quad V \longmapsto \{s \in f_*\mathcal{F}(V) = \mathcal{F}(X_V) \mid Supp(s) \subset X_V \text{ is proper over } V\}$$

is an abelian subsheaf of $f_*\mathcal{F}$.

Warning: This sheaf isn't the "correct one" if f is not separated.

Proof. Recall that the support of a section is closed (Étale Cohomology, Lemma 31.4) hence the material in Cohomology of Schemes, Section 26 applies. By the lemma above and Cohomology of Schemes, Lemma 26.6 we find that our subset of $f_*\mathcal{F}(V)$ is a subgroup. By Cohomology of Schemes, Lemma 26.4 we see that our rule defines a sub presheaf. Finally, suppose that we have $s \in f_*\mathcal{F}(V)$ and an étale covering $\{V_i \to V\}$ such that $s|_{V_i}$ has support proper over V_i. Observe that the support of $s|_{V_i}$ is the inverse image of the support of $s|_V$ (use the characterization of the support in terms of stalks and Étale Cohomology, Lemma 36.2). Whence the support of s is proper over V by Descent, Lemma 22.5. This proves that our rule satisfies the sheaf condition. □

0F4Y **Lemma 3.2.** *Let $j : U \to X$ be a separated étale morphism. Let $\mathcal{F}$ be an abelian sheaf on $U_{étale}$. The image of the injective map $j_!\mathcal{F} \to j_*\mathcal{F}$ of Étale Cohomology, Lemma 69.6 is the subsheaf of Lemma 3.1.*

An alternative would be to move this lemma later and prove this using the descrition of the stalks of both sheaves.

Proof. The construction of $j_!\mathcal{F} \to j_*\mathcal{F}$ in the proof of Étale Cohomology, Lemma 69.6 is via the construction of a map $j_{p!}\mathcal{F} \to j_*\mathcal{F}$ of presheaves whose image is clearly contained in the subsheaf of Lemma 3.1. Hence since $j_!\mathcal{F}$ is the sheafification of $j_{p!}\mathcal{F}$ we conclude the image of $j_!\mathcal{F} \to j_*\mathcal{F}$ is contained in this subsheaf. Conversely, let $s \in j_*\mathcal{F}(V)$ have support Z proper over V. Then $Z \to V$ is finite with closed image $Z' \subset V$, see More on Morphisms, Lemma 39.1. The restriction of s to $V \setminus Z'$ is zero and the zero section is contained in the image of $j_!\mathcal{F} \to j_*\mathcal{F}$. On the other hand, if $v \in Z'$, then we can find an étale neighbourhood $(V', v') \to (V, v)$ such that we have a decomposition $U_{V'} = W \amalg U'_1 \amalg \ldots \amalg U'_n$ into open and closed subschemes with $U'_i \to V'$ an isomorphism and with $T_{V'} \subset U'_1 \amalg \ldots \amalg U'_n$, see Étale Morphisms, Lemma 18.2. Inverting the isomorphisms $U'_i \to V'$ we obtain n morphisms $\varphi'_i : V' \to U$ and sections s'_i over V' by pulling back s. Then the section $\sum(\varphi'_i, s'_i)$ of $j_{p!}\mathcal{F}$ over V', see formula for $j_{p!}\mathcal{F}(V')$ in proof of Étale Cohomology, Lemma 69.6, maps to the restriction of s to V' by construction. We conclude that s is étale locally in the image of $j_!\mathcal{F} \to j_*\mathcal{F}$ and the proof is complete. □

0F4Z **Definition 3.3.** Let $f : X \to Y$ be a morphism of schemes which is separated (!) and locally of finite type. Let $\mathcal{F}$ be an abelian sheaf on $X_{étale}$. The subsheaf $f_!\mathcal{F} \subset f_*\mathcal{F}$ constructed in Lemma 3.1 is called the *direct image with compact support*.

By Lemma 3.2 this does not conflict with Étale Cohomology, Definition 69.1 as we have agreement when both definitions apply. Here is a sanity check.

0F51 **Lemma 3.4.** *Let $f : X \to Y$ be a proper morphism of schemes. Then $f_! = f_*$.*

Proof. Immediate from the construction of $f_!$. □

A very useful observation is the following.

0F53 **Remark 3.5** (Covariance with respect to open embeddings)**.** Let $f : X \to Y$ be morphism of schemes which is separated and locally of finite type. Let $\mathcal{F}$ be an abelian sheaf on $X_{\acute{e}tale}$. Let $X' \subset X$ be an open subscheme. Denote $f' : X' \to Y$ the restriction of f. There is a canonical injective map

$$f'_!(\mathcal{F}|_{X'}) \longrightarrow f_!\mathcal{F}$$

Namely, let $V \in Y_{\acute{e}tale}$ and consider a section $s' \in f'_*(\mathcal{F}|_{X'})(V) = \mathcal{F}(X' \times_Y V)$ with support Z' proper over V. Then Z' is closed in $X \times_Y V$ as well, see Cohomology of Schemes, Lemma 26.5. Thus there is a unique section $s \in \mathcal{F}(X \times_Y V) = f_*\mathcal{F}(V)$ whose restriction to $X' \times_Y V$ is s' and whose restriction to $X \times_Y V \setminus Z'$ is zero, see Lemma 2.2. This construction is compatible with restriction maps and hence induces the desired map of sheaves $f'_!(\mathcal{F}|_{X'}) \to f_!\mathcal{F}$ which is clearly injective. By construction we obtain a commutative diagram

$$\begin{tikzcd} f'_!(\mathcal{F}|_{X'}) \arrow[r] \arrow[d] & f_!\mathcal{F} \arrow[d] \\ f'_*(\mathcal{F}|_{X'}) & f_*\mathcal{F} \arrow[l] \end{tikzcd}$$

functorial in $\mathcal{F}$. It is clear that for $X'' \subset X'$ open with $f'' = f|_{X''} : X'' \to Y$ the composition of the canonical maps $f''_!\mathcal{F}|_{X''} \to f'_!\mathcal{F}|_{X'} \to f_!\mathcal{F}$ just constructed is the canonical map $f''_!\mathcal{F}|_{X''} \to f_!\mathcal{F}$.

0F52 **Lemma 3.6.** *Let Y be a scheme. Let $j : X \to \overline{X}$ be an open immersion of schemes over Y with $\overline{X}$ proper over Y. Denote $f : X \to Y$ and $\overline{f} : \overline{X} \to Y$ the structure morphisms. For $\mathcal{F} \in Ab(X_{\acute{e}tale})$ there is a canonical isomorphism (see proof)*

$$f_!\mathcal{F} \longrightarrow \overline{f}_! j_!\mathcal{F}$$

As we have $\overline{f}_! = \overline{f}_*$ by Lemma 3.4 we obtain $\overline{f}_* \circ j_! = f_!$ as functors $Ab(X_{\acute{e}tale}) \to Ab(Y_{\acute{e}tale})$.

Proof. We have $(j_!\mathcal{F})|_X = \mathcal{F}$, see Étale Cohomology, Lemma 69.4. Thus the displayed arrow is the injective map $f_!(\mathcal{G}|_X) \to \overline{f}_!\mathcal{G}$ of Remark 3.5 for $\mathcal{G} = j_!\mathcal{F}$. The explicit nature of this map implies that it now suffices to show: if $V \in Y_{\acute{e}tale}$ and $s \in \overline{f}_!\mathcal{G}(V) = \overline{f}_*\mathcal{G}(V) = \mathcal{G}(\overline{X}_V)$ is a section, then the support of s is contained in the open $X_V \subset \overline{X}_V$. This is immediate from the fact that the stalks of $\mathcal{G}$ are zero at geometric points of $\overline{X} \setminus X$. □

We want to relate the stalks of $f_!\mathcal{F}$ to sections with compact support on fibres. In order to state this, we need a definition.

0F72 **Definition 3.7.** Let X be a separated scheme locally of finite type over a field k. Let $\mathcal{F}$ be an abelian sheaf on $X_{\acute{e}tale}$. We let $H^0_c(X, \mathcal{F}) \subset H^0(X, \mathcal{F})$ be the set of sections whose support is proper over k. Elements of $H^0_c(X, \mathcal{F})$ are called *sections with compact support*.

Warning: This definition isn't the "correct one" if X isn't separated over k.

0F73 **Lemma 3.8.** *Let X be a proper scheme over a field k. Then $H^0_c(X, \mathcal{F}) = H^0(X, \mathcal{F})$.*

Proof. Immediate from the construction of H^0_c. □

0F74 **Remark 3.9** (Open embeddings and compactly supported sections)**.** Let X be a separated scheme locally of finite type over a field k. Let $\mathcal{F}$ be an abelian sheaf on $X_{étale}$. Exactly as in Remark 3.5 there are injective maps

$$H^0_c(X', \mathcal{F}|_{X'}) \longrightarrow H^0_c(X, \mathcal{F})$$

which turn H^0_c into a "cosheaf" on the Zariski site of X.

0F75 **Lemma 3.10.** *Let k be a field. Let $j : X \to \overline{X}$ be an open immersion of schemes over k with $\overline{X}$ proper over k. For $\mathcal{F} \in Ab(X_{étale})$ there is a canonical isomorphism (see proof)*

$$H^0_c(X, \mathcal{F}) \longrightarrow H^0_c(\overline{X}, j_!\mathcal{F}) = H^0(\overline{X}, j_!\mathcal{F})$$

where we have the equality on the right by Lemma 3.8.

Proof. We have $(j_!\mathcal{F})|_X = \mathcal{F}$, see Étale Cohomology, Lemma 69.4. Thus the displayed arrow is the injective map $H^0_c(X, \mathcal{G}|_X) \to H^0_c(\overline{X}, \mathcal{G})$ of Remark 3.9 for $\mathcal{G} = j_!\mathcal{F}$. The explicit nature of this map implies that it now suffices to show: if $s \in H^0(\overline{X}, \mathcal{G})$ is a section, then the support of s is contained in the open X. This is immediate from the fact that the stalks of $\mathcal{G}$ are zero at geometric points of $\overline{X} \setminus X$. □

0F76 **Lemma 3.11.** *Let $f : X \to Y$ be a morphism of schemes which is separated and locally of finite type. Let $\mathcal{F}$ be an abelian sheaf on $X_{étale}$. Then there is a canonical isomorphism*

$$(f_!\mathcal{F})_{\overline{y}} \longrightarrow H^0_c(X_{\overline{y}}, \mathcal{F}|_{X_{\overline{y}}})$$

for any geometric point $\overline{y} : \mathrm{Spec}(k) \to Y$.

Proof. Recall that $(f_*\mathcal{F})_{\overline{y}} = \mathrm{colim}\, f_*\mathcal{F}(V)$ where the colimit is over the étale neighbourhoods $(V, \overline{v})$ of $\overline{y}$. If $s \in f_*\mathcal{F}(V) = \mathcal{F}(X_V)$, then we can pullback s to a section of $\mathcal{F}$ over $(X_V)_{\overline{v}} = X_{\overline{y}}$. Thus we obtain a canonical map

$$c_{\overline{y}} : (f_*\mathcal{F})_{\overline{y}} \longrightarrow H^0(X_{\overline{y}}, \mathcal{F}|_{X_{\overline{y}}})$$

We claim that this map induces a bijection between the subgroups $(f_!\mathcal{F})_{\overline{y}}$ and $H^0_c(X_{\overline{y}}, \mathcal{F}|_{X_{\overline{y}}})$. The claim implies the lemma, but is a little bit more precise in that it describes the identification of the lemma as given by pullbacks of sections of $\mathcal{F}$ to the geometric fibre of f.

Observe that any element $s \in (f_!\mathcal{F})_{\overline{y}} \subset (f_*\mathcal{F})_{\overline{y}}$ is mapped by $c_{\overline{y}}$ to an element of $H^0_c(X_{\overline{y}}, \mathcal{F}|_{X_{\overline{y}}}) \subset H^0(X_{\overline{y}}, \mathcal{F}|_{X_{\overline{y}}})$. This is true because taking the support of a section commutes with pullback and because properness is preserved by base change. This at least produces the map in the statement of the lemma. To prove that it is an isomorphism we may work Zariski locally on Y and hence we may and do assume Y is affine.

An observation that we will use below is that given an open subscheme $X' \subset X$ and if $f' = f|_{X'}$, then we obtain a commutative diagram

$$\begin{tikzcd}
(f'_!(\mathcal{F}|_{X'}))_{\overline{y}} \arrow[r] \arrow[d] & H^0_c(X'_{\overline{y}}, \mathcal{F}|_{X'_{\overline{y}}}) \arrow[d] \\
(f_!\mathcal{F})_{\overline{y}} \arrow[r] & H^0_c(X_{\overline{y}}, \mathcal{F}|_{X_{\overline{y}}})
\end{tikzcd}$$

where the horizontal arrows are the maps constructed above and the vertical arrows are given in Remarks 3.5 and 3.9. The reason is that given an étale neighbourhood $(V, \overline{v})$ of $\overline{y}$ and a section $s \in f_*\mathcal{F}(V) = \mathcal{F}(X_V)$ whose support Z happens to be contained in X'_V and is proper over V, so that s gives rise to an element of both $(f'_!(\mathcal{F}|_{X'}))_{\overline{y}}$ and $(f_!\mathcal{F})_{\overline{y}}$ which correspond via the vertical arrow of the diagram, then these elements are mapped via the horizontal arrows to the pullback $s|_{X_{\overline{y}}}$ of s to $X_{\overline{y}}$ whose support $Z_{\overline{y}}$ is contained in $X'_{\overline{y}}$ and hence this restriction gives rise to a compatible pair of elements of $H^0_c(X'_{\overline{y}}, \mathcal{F}|_{X'_{\overline{y}}})$ and $H^0_c(X_{\overline{y}}, \mathcal{F}|_{X_{\overline{y}}})$.

Suppose $s \in (f_!\mathcal{F})_{\overline{y}}$ maps to zero in $H^0_c(X_{\overline{y}}, \mathcal{F}|_{X_{\overline{y}}})$. Say s corresponds to $s \in f_*\mathcal{F}(V) = \mathcal{F}(X_V)$ with support Z proper over V. We may assume that V is affine and hence Z is quasi-compact. Then we may choose a quasi-compact open $X' \subset X$ containing the image of Z. Then Z is contained in X'_V and hence s is the image of an element $s' \in f'_!(\mathcal{F}|_{X'})(V)$ where $f' = f|_{X'}$ as in the previous paragraph. Then s' maps to zero in $H^0_c(X'_{\overline{y}}, \mathcal{F}|_{X'_{\overline{y}}})$. Hence in order to prove injectivity, we may replace X by X', i.e., we may assume X is quasi-compact. We will prove this case below.

Suppose that $t \in H^0_c(X_{\overline{y}}, \mathcal{F}|_{X_{\overline{y}}})$. Then the support of t is contained in a quasi-compact open subscheme $W \subset X_{\overline{y}}$. Hence we can find a quasi-compact open subscheme $X' \subset X$ such that $X'_{\overline{y}}$ contains W. Then it is clear that t is contained in the image of the injective map $H^0_c(X'_{\overline{y}}, \mathcal{F}|_{X'_{\overline{y}}}) \to H^0_c(X_{\overline{y}}, \mathcal{F}|_{X_{\overline{y}}})$. Hence in order to show surjectivity, we may replace X by X', i.e., we may assume X is quasi-compact. We will prove this case below.

In this last paragraph of the proof we prove the lemma in case X is quasi-compact and Y is affine. By More on Flatness, Theorem 33.8 there exists a compactification $j : X \to \overline{X}$ over Y. Set $\mathcal{G} = j_!\mathcal{F}$ so that $\mathcal{F} = \mathcal{G}|_X$ by Étale Cohomology, Lemma 69.4. By the disussion above we get a commutative diagram

$$\begin{tikzcd} (f_!\mathcal{F})_{\overline{y}} \arrow[r] \arrow[d] & H^0_c(X_{\overline{y}}, \mathcal{F}|_{X_{\overline{y}}}) \arrow[d] \\ (\overline{f}_!\mathcal{G})_{\overline{y}} \arrow[r] & H^0_c(\overline{X}_{\overline{y}}, \mathcal{G}|_{\overline{X}_{\overline{y}}}) \end{tikzcd}$$

By Lemmas 3.6 and 3.10 the vertical maps are isomorphisms. This reduces us to the case of the proper morphism $\overline{X} \to Y$. For a proper morphism our map is an isomorphism by Lemmas 3.4 and 3.8 and proper base change for pushforwards, see Étale Cohomology, Lemma 87.4. □

0F55 **Lemma 3.12.** *Consider a cartesian square*

$$\begin{tikzcd} X' \arrow[r, "g'"'] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

of schemes with f separated and locally of finite type. For any abelian sheaf $\mathcal{F}$ on $X_{étale}$ we have $f'_!(g')^{-1}\mathcal{F} = g^{-1}f_!\mathcal{F}$.

Proof. In great generality there is a pullback map $g^{-1}f_*\mathcal{F} \to f'_*(g')^{-1}\mathcal{F}$, see Sites, Section 45. We claim that this map sends $g^{-1}f_!\mathcal{F}$ into the subsheaf $f'_!(g')^{-1}\mathcal{F}$ and induces the isomorphism in the lemma.

Choose a geometric point $\overline{y}' : \mathrm{Spec}(k) \to Y'$ and denote $\overline{y} = g \circ \overline{y}'$ the image in Y. There is a commutative diagram

$$\begin{tikzcd}
(f_*\mathcal{F})_{\overline{y}} \arrow[r] \arrow[d] & H^0(X_{\overline{y}}, \mathcal{F}|_{X_{\overline{y}}}) \arrow[d] \\
(f'_*(g')^{-1}\mathcal{F})_{\overline{y}'} \arrow[r] & H^0(X'_{\overline{y}'}, (g')^{-1}\mathcal{F}|_{X'_{\overline{y}'}})
\end{tikzcd}$$

where the horizontal maps were used in the proof of Lemma 3.11 and the vertical maps are the pullback maps above. The diagram commutes because each of the four maps in question is given by pulling back local sections along a morphism of schemes and the underlying diagram of morphisms of schemes commutes. Since the diagram in the statement of the lemma is cartesian we have $X'_{\overline{y}'} = X_{\overline{y}}$. Hence by Lemma 3.11 and its proof we obtain a commutative diagram

$$\begin{tikzcd}
(f_*\mathcal{F})_{\overline{y}} \arrow[rrr] \arrow[ddd] & & & H^0(X_{\overline{y}}, \mathcal{F}|_{X_{\overline{y}}}) \arrow[ddd] \\
& (f_!\mathcal{F})_{\overline{y}} \arrow[ul] \arrow[r] \arrow[d, dotted] & H^0_c(X_{\overline{y}}, \mathcal{F}|_{X_{\overline{y}}}) \arrow[ur] \arrow[d] & \\
& (f'_!(g')^{-1}\mathcal{F})_{\overline{y}'} \arrow[r] \arrow[dl] & H^0_c(X'_{\overline{y}'}, (g')^{-1}\mathcal{F}|_{X'_{\overline{y}'}}) \arrow[dr] & \\
(f'_*(g')^{-1}\mathcal{F})_{\overline{y}'} \arrow[rrr] & & & H^0(X'_{\overline{y}'}, (g')^{-1}\mathcal{F}|_{X'_{\overline{y}'}})
\end{tikzcd}$$

where the horizontal arrows of the inner square are isomorphisms and the two right vertical arrows are equalities. Also, the se, sw, ne, nw arrows are injective. It follows that there is a unique bijective dotted arrow fitting into the diagram. We conclude that $g^{-1}f_!\mathcal{F} \subset g^{-1}f_*\mathcal{F} \to f'_*(g')^{-1}\mathcal{F}$ is mapped into the subsheaf $f'_!(g')^{-1}\mathcal{F} \subset f'_*(g')^{-1}\mathcal{F}$ because this is true on stalks, see Étale Cohomology, Theorem 29.10. The same theorem then implies that the induced map is an isomorphism and the proof is complete. $\square$

0F50 **Lemma 3.13.** *Let $f : X \to Y$ and $g : Y \to Z$ be composable morphisms of schemes which are separated and locally of finite type. Let $\mathcal{F}$ be an abelian sheaf on $X_{étale}$. Then $g_!f_!\mathcal{F} = (g \circ f)_!\mathcal{F}$ as subsheaves of $(g \circ f)_*\mathcal{F}$.*

Proof. We strongly urge the reader to prove this for themselves. Let $W \in Z_{étale}$ and $s \in (g \circ f)_*\mathcal{F}(W) = \mathcal{F}(X_W)$. Denote $T \subset X_W$ the support of s; this is a closed subset. Observe that s is a section of $(g \circ f)_!\mathcal{F}$ if and only if T is proper over W. We have $f_!\mathcal{F} \subset f_*\mathcal{F}$ and hence $g_!f_!\mathcal{F} \subset g_!f_*\mathcal{F} \subset g_*f_*\mathcal{F}$. On the other hand, s is a section of $g_!f_!\mathcal{F}$ if and only if (a) T is proper over Y_W and (b) the support T' of s viewed as section of $f_!\mathcal{F}$ is proper over W. If (a) holds, then the image of T in Y_W is closed and since $f_!\mathcal{F} \subset f_*\mathcal{F}$ we see that $T' \subset Y_W$ is the image of T (details omitted; look at stalks).

The conclusion is that we have to show a closed subset $T \subset X_W$ is proper over W if and only if T is proper over Y_W and the image of T in Y_W is proper over W. Let us endow T with the reduced induced closed subscheme structure. If T is proper over W, then $T \to Y_W$ is proper by Morphisms, Lemma 39.7 and the image of T in Y_W is proper over W by Cohomology of Schemes, Lemma 26.5. Conversely, if T is proper over Y_W and the image of T in Y_W is proper over W, then the morphism $T \to W$ is proper as a composition of proper morphisms (here we endow the closed image of T in Y_W with its reduced induced scheme structure to turn the question into one about morphisms of schemes), see Morphisms, Lemma 39.4. $\square$

0F77 **Remark 3.14.** The isomorphisms between functors constructed above satisfy the following two properties:

(1) Let $f : X \to Y$, $g : Y \to Z$, and $h : Z \to T$ be composable morphisms of schemes which are separated and locally of finite type. Then the diagram
$$\begin{tikzcd} (h \circ g \circ f)_! \arrow[r] \arrow[d] & (h \circ g)_! \circ f_! \arrow[d] \\ h_! \circ (g \circ f)_! \arrow[r] & h_! \circ g_! \circ f_! \end{tikzcd}$$
commutes where the arrows are those of Lemma 3.13.

(2) Suppose that we have a diagram of schemes
$$\begin{tikzcd} X' \arrow[r, "c"'] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "b"'] \arrow[d, "g'"'] & Y \arrow[d, "g"] \\ Z' \arrow[r, "a"] & Z \end{tikzcd}$$
with both squares cartesian and f and g separated and locally of finite type. Then the diagram
$$\begin{tikzcd} a^{-1} \circ (g \circ f)_! \arrow[rr] \arrow[d] & & (g' \circ f')_! \circ c^{-1} \arrow[d] \\ a^{-1} \circ g_! \circ f_! \arrow[r] & g'_! \circ b^{-1} \circ f_! \arrow[r] & g'_! \circ f'_! \circ c^{-1} \end{tikzcd}$$
commutes where the horizontal arrows are those of Lemma 3.12 the arrows are those of Lemma 3.13.

Part (1) holds true because we have a similar commutative diagram for pushforwards. Part (2) holds by the very general compatibility of base change maps for pushforwards (Sites, Remark 45.3) and the fact that the isomorphisms in Lemmas 3.12 and 3.13 are constructed using the corresponding maps fo pushforwards.

0F54 **Lemma 3.15.** *Let $f : X \to Y$ be morphism of schemes which is separated and locally of finite type. Let $X = \bigcup_{i \in I} X_i$ be an open covering such that for all $i, j \in I$ there exists a k with $X_i \cup X_j \subset X_k$. Denote $f_i : X_i \to Y$ the restriction of f. Then*
$$f_!\mathcal{F} = \mathrm{colim}_{i \in I}\, f_{i,!}(\mathcal{F}|_{X_i})$$
functorially in $\mathcal{F} \in Ab(X_{étale})$ where the transition maps are the ones constructed in Remark 3.5.

Proof. It suffices to show that the canonical map from right to left is a bijection when evaluated on a quasi-compact object V of $Y_{étale}$. Observe that the colimit on the right hand side is directed and has injective transition maps. Thus we can use Sites, Lemma 17.5 to evaluate the colimit. Hence, the statement comes down to the observation that a closed subset $Z \subset X_V$ proper over V is quasi-compact and hence is contained in $X_{i,V}$ for some i. □

0F56 **Lemma 3.16.** *Let $f : X \to Y$ be a morphism of schemes which is separated and locally of finite type. Then functor $f_!$ commutes with direct sums.*

Proof. Let $\mathcal{F} = \bigoplus \mathcal{F}_i$. To show that the map $\bigoplus f_!\mathcal{F}_i \to f_!\mathcal{F}$ is an isomorphism, it suffices to show that these sheaves have the same sections over a quasi-compact object V of $Y_{étale}$. Replacing Y by V it suffices to show $H^0(Y, f_!\mathcal{F}) \subset H^0(X, \mathcal{F})$ is equal to $\bigoplus H^0(Y, f_!\mathcal{F}_i) \subset \bigoplus H^0(X, \mathcal{F}_i) \subset H^0(X, \bigoplus \mathcal{F}_i)$. In this case, by writing X as the union of its quasi-compact opens and using Lemma 3.15 we reduce to the case where X is quasi-compact as well. Then $H^0(X, \mathcal{F}) = \bigoplus H^0(X, \mathcal{F}_i)$ by Étale Cohomology, Theorem 51.3. Looking at supports of sections the reader easily concludes. □

0F57 **Lemma 3.17.** *Let $f : X \to Y$ be a morphism of schemes which is separated and locally quasi-finite. Then*

1. *for $\mathcal{F}$ in $Ab(X_{étale})$ and a geometric point $\overline{y} : \mathrm{Spec}(k) \to Y$ we have*
$$(f_!\mathcal{F})_{\overline{y}} = \bigoplus\nolimits_{f(\overline{x}) = \overline{y}} \mathcal{F}_{\overline{x}}$$
functorially in $\mathcal{F}$, and
2. *the functor $f_!$ is exact.*

Proof. The functor $f_!$ is left exact by construction. Right exactness may be checked on stalks (Étale Cohomology, Theorem 29.10). Thus it suffices to prove part (1).

Let $\overline{y} : \mathrm{Spec}(k) \to Y$ be a geometric point. The scheme $X_{\overline{y}}$ has a discrete underlying topological space (Morphisms, Lemma 19.8) and all the residue fields at the points are equal to k (as finite extensions of k). Hence $\{\overline{x} : \mathrm{Spec}(k) \to X : f(\overline{x}) = \overline{y}\}$ is equal to the set of points of $X_{\overline{y}}$. Thus the computation of the stalk follows from the more general Lemma 3.11. □

4. Sections with finite support

0F6E In this section we extend the construction of Section 3 to not necessarily separated locally quasi-finite morphisms.

Let $f : X \to Y$ be a locally quasi-finite morphism of schemes. Let $\mathcal{F}$ be an abelian sheaf on $X_{étale}$. Given V in $Y_{étale}$ denote $X_V = X \times_Y V$ the base change. We are going to consider the group of finite formal sums

0F6J (4.0.1)
$$s = \sum\nolimits_{i=1,\ldots,n} (Z_i, s_i)$$

where $Z_i \subset X_V$ is a locally closed subscheme such that the morphism $Z_i \to V$ is finite[2] and where $s_i \in H_{Z_i}(\mathcal{F})$. Here, as in Section 2, we set

$$H_{Z_i}(\mathcal{F}) = \{s \in \mathcal{F}(U_i) \mid \mathrm{Supp}(s) \subset Z_i\}$$

[2]Since f is locally quasi-finite, the morphism $Z_i \to V$ is finite if and only if it is proper.

where $U_i \subset X_V$ is an open subscheme containing Z as a closed subscheme. We are going to consider these formal sums modulo the following relations

0F6K (1) $(Z, s) + (Z, s') = (Z, s + s')$,
0F6L (2) $(Z, s) = (Z', s)$ if $Z \subset Z'$.

Observe that the second relation makes sense: since $Z \to V$ is finite and $Z' \to V$ is separated, the inclusion $Z \to Z'$ is closed and we can use the map discussed in (1).

Let us denote $f_{p!}\mathcal{F}(V)$ the quotient of the abelian group of formal sums (4.0.1) by these relations. The first relation tells us that $f_{p!}\mathcal{F}(V)$ is a quotient of the direct sum of the abelian groups $H_Z(\mathcal{F})$ over all locally closed subschemes $Z \subset X_V$ finite over V. The second relation tells us that we are really taking the colimit

0F6M (4.0.2)
$$f_{p!}\mathcal{F}(V) = \mathrm{colim}_Z\, H_Z(\mathcal{F})$$

This formula will be a convenient abstract way to think about our construction.

Next, we observe that there is a natural way to turn this construction into a presheaf $f_{p!}\mathcal{F}$ of abelian groups on $Y_{étale}$. Namely, given $V' \to V$ in $Y_{étale}$ we obtain the base change morphism $X_{V'} \to X_V$. If $Z \subset X_V$ is a locally closed subscheme finite over V, then the scheme theoretic inverse image $Z' \subset X_{V'}$ is finite over V'. Moreover, if $U \subset X_V$ is an open such that Z is closed in U, then the inverse image $U' \subset X_{V'}$ is an open such that Z' is closed in U'. Hence the restriction mapping $\mathcal{F}(U) \to \mathcal{F}(U')$ of $\mathcal{F}$ sends $H_Z(\mathcal{F})$ into $H_{Z'}(\mathcal{F})$; this is a special case of the functoriality discussed in (2) above. Clearly, these maps are compatible with inclusions $Z_1 \subset Z_2$ of such locally closed subschemes of X_V and we obtain a map

$$f_{p!}\mathcal{F}(V) = \mathrm{colim}_Z\, H_Z(\mathcal{F}) \longrightarrow \mathrm{colim}_{Z'}\, H_{Z'}(\mathcal{F}) = f_{p!}\mathcal{F}(V')$$

These maps indeed turn $f_{p!}\mathcal{F}$ into a presheaf of abelian groups on $Y_{étale}$. We omit the details.

A final observation is that the construction of $f_{p!}\mathcal{F}$ is functorial in $\mathcal{F}$ in $Ab(X_{étale})$. We conclude that given a locally quasi-finite morphism $f : X \to Y$ we have constructed a functor

$$f_{p!} : Ab(X_{étale}) \longrightarrow PAb(Y_{étale})$$

from the category of abelian sheaves on $X_{étale}$ to the category of abelian presheaves on $Y_{étale}$. Before we define $f_!$ as the sheafification of this functor, let us check that it agrees with the construction in Section 3 and with the construction in Étale Cohomology, Section 69 when both apply.

0F6N **Lemma 4.1.** *Let $f : X \to Y$ be a separated and locally quasi-finite morphism of schemes. Functorially in $\mathcal{F} \in Ab(X_{étale})$ there is a canonical isomorphism(!)*

$$f_{p!}\mathcal{F} \longrightarrow f_!\mathcal{F}$$

of abelian presheaves which identifies the sheaf $f_!\mathcal{F}$ of Definition 3.3 with the presheaf $f_{p!}\mathcal{F}$ constructed above.

Proof. Let V be an object of $Y_{étale}$. If $Z \subset X_V$ is locally closed and finite over V, then, since f is separated, we see that the morphism $Z \to X_V$ is a closed immersion. Moreover, if Z_i, $i = 1, \ldots, n$ are closed subschemes of X_V finite over V, then $Z_1 \cup \ldots \cup Z_n$ (scheme theoretic union) is a closed subscheme finite over V. Hence in this case the colimit (4.0.2) defining $f_{p!}\mathcal{F}(V)$ is directed and we find that $f_{!p}\mathcal{F}(V)$ is simply equal to the set of sections of $\mathcal{F}(X_V)$ whose support is finite over V. Since any closed subset of X_V which is proper over V is actually finite over V

(as f is locally quasi-finite) we conclude that this is equal to $f_!\mathcal{F}(V)$ by its very definition. $\square$

0F6P **Lemma 4.2.** *Let $f : X \to Y$ be a morphism of schemes which is locally quasi-finite. Let $\overline{y} : \mathrm{Spec}(k) \to Y$ be a geometric point. Functorially in $\mathcal{F}$ in $Ab(X_{étale})$ we have*

$$(f_{p!}\mathcal{F})_{\overline{y}} = \bigoplus\nolimits_{f(\overline{x})=\overline{y}} \mathcal{F}_{\overline{x}}$$

Proof. Recall that the stalk at $\overline{y}$ of a presheaf is defined by the usual colimit over étale neighbourhoods $(V, \overline{v})$ of $\overline{y}$, see Étale Cohomology, Definition 29.6. Accordingly suppose $s = \sum_{i=1,\ldots,n}(Z_i, s_i)$ as in (4.0.1) is an element of $f_{p!}\mathcal{F}(V)$ where $(V, \overline{v})$ is an étale neighbourhood of $\overline{y}$. Then since

$$X_{\overline{y}} = (X_V)_{\overline{v}} \supset Z_{i,\overline{v}}$$

and since s_i is a section of $\mathcal{F}$ on an open neighbourhood of Z_i in X_V we can send s to

$$\sum\nolimits_{i=1,\ldots,n} \sum\nolimits_{\overline{x} \in Z_{i,\overline{v}}} (\text{class of } s_i \text{ in } \mathcal{F}_{\overline{x}}) \quad \in \quad \bigoplus\nolimits_{f(\overline{x})=\overline{y}} \mathcal{F}_{\overline{x}}$$

We omit the verification that this is compatible with restriction maps and that the relations (1) $(Z, s) + (Z, s') - (Z, s + s')$ and (2) $(Z, s) - (Z', s)$ if $Z \subset Z'$ are sent to zero. Thus we obtain a map

$$(f_{p!}\mathcal{F})_{\overline{y}} \longrightarrow \bigoplus\nolimits_{f(\overline{x})=\overline{y}} \mathcal{F}_{\overline{x}}$$

Let us prove this arrow is surjective. For this it suffices to pick an $\overline{x}$ with $f(\overline{x}) = \overline{y}$ and prove that an element s in the summand $\mathcal{F}_{\overline{x}}$ is in the image. Let s correspond to the element $s \in \mathcal{F}(U)$ where $(U, \overline{u})$ is an étale neighbourhood of $\overline{x}$. Since f is locally quasi-finite, the morphism $U \to Y$ is locally quasi-finite too. By More on Morphisms, Lemma 36.3 we can find an étale neighbourhood $(V, \overline{v})$ of $\overline{y}$, an open subscheme

$$W \subset U \times_Y V,$$

and a geometric point $\overline{w}$ mapping to $\overline{u}$ and $\overline{v}$ such that $W \to V$ is finite and $\overline{w}$ is the only geometric point of W mapping to $\overline{v}$. (We omit the translation between the language of geometric points we are currently using and the language of points and residue field extensions used in the statement of the lemma.) Observe that $W \to X_V = X \times_Y V$ is étale. Choose an affine open neighbourhood $W' \subset X_V$ of the image $\overline{w}'$ of $\overline{w}$. Since $\overline{w}$ is the only point of W over $\overline{v}$ and since $W \to V$ is closed, after replacing V by an open neighbourhood of $\overline{v}$, we may assume $W \to X_V$ maps into W'. Then $W \to W'$ is finite and étale and there is a unique geometric point $\overline{w}$ of W lying over $\overline{w}'$. It follows that $W \to W'$ is an open immersion over an open neighbourhood of $\overline{w}'$ in W', see Étale Morphisms, Lemma 14.2. Shrinking V and W' we may assume $W \to W'$ is an isomorphism. Thus s may be viewed as a section s' of $\mathcal{F}$ over the open subscheme $W' \subset X_V$ which is finite over V. Hence by definition (W', s') defines an element of $j_{p!}\mathcal{F}(V)$ which maps to s as desired.

Let us prove the arrow is injective. To do this, let $s = \sum_{i=1,\ldots,n}(Z_i, s_i)$ as in (4.0.1) be an element of $f_{p!}\mathcal{F}(V)$ where $(V, \overline{v})$ is an étale neighbourhood of $\overline{y}$. Assume s maps to zero under the map constructed above. First, after replacing $(V, \overline{v})$ by an étale neighbourhood of itself, we may assume there exist decompositions

$Z_i = Z_{i,1} \amalg \ldots \amalg Z_{i,m_i}$ into open and closed subschemes such that each $Z_{i,j}$ has exactly one geometric point over $\overline{v}$. Say under the obvious direct sum decomposition

$$H_{Z_i}(\mathcal{F}) = \bigoplus H_{Z_{i,j}}(\mathcal{F})$$

the element s_i corresponds to $\sum s_{i,j}$. We may use relations (1) and (2) to replace s by $\sum_{i=1,\ldots,n} \sum_{j=1,\ldots,m_i} (Z_{i,j}, s_{i,j})$. In other words, we may assume Z_i has a unique geometric point lying over $\overline{v}$. Let $\overline{x}_1, \ldots, \overline{x}_m$ be the geometric points of X over $\overline{y}$ corresponding to the geometric points of our Z_i over $\overline{v}$; note that for one $j \in \{1, \ldots, m\}$ there may be multiple indices i for which $\overline{x}_j$ corresponds to a point of Z_i. By More on Morphisms, Lemma 36.3 applied to both $X_V \to V$ after replacing $(V, \overline{v})$ by an étale neighbourhood of itself we may assume there exist open subschemes

$$W_j \subset X \times_Y V, \quad j = 1, \ldots, m$$

and a geometric point $\overline{w}_j$ of W_j mapping to $\overline{x}_j$ and $\overline{v}$ such that $W_j \to V$ is finite and $\overline{w}_j$ is the only geometric point of W_j mapping to $\overline{v}$. After shrinking V we may assume $Z_i \subset W_j$ for some j and we have the map $H_{Z_i}(\mathcal{F}) \to H_{W_j}(\mathcal{F})$. Thus by the relation (2) we see that our element is equivalent to an element of the form

$$\sum\nolimits_{j=1,\ldots,m} (W_j, t_j)$$

for some $t_j \in H_{W_j}(\mathcal{F})$. Clearly, this element is mapped simply to the class of t_j in the summand $\mathcal{F}_{\overline{x}_j}$. Since s maps to zero, we find that t_j maps to zero in $\mathcal{F}_{\overline{x}_j}$. This implies that t_j restricts to zero on an open neighbourhood of $\overline{w}_j$ in W_j, see Étale Cohomology, Lemma 31.2. Shrinking V once more we obtain $t_j = 0$ for all j as desired. □

0F6Q **Lemma 4.3.** *Let $f = j : U \to X$ be an étale of schemes. Denote $j_{p!}$ the construction of Étale Cohomology, Equation (69.1.1) and denote $f_{p!}$ the construction above. Functorially in $\mathcal{F} \in Ab(X_{étale})$ there is a canonical map*

$$j_{p!}\mathcal{F} \longrightarrow f_{p!}\mathcal{F}$$

of abelian presheaves which identifies the sheaf $j_!\mathcal{F} = (j_{p!}\mathcal{F})^\#$ of Étale Cohomology, Definition 69.1 with $(f_{p!}\mathcal{F})^\#$.

Proof. Please read the proof of Étale Cohomology, Lemma 69.6 before reading the proof of this lemma. Let V be an object of $X_{étale}$. Recall that

$$j_{p!}\mathcal{F}(V) = \bigoplus\nolimits_{\varphi : V \to U} \mathcal{F}(V \xrightarrow{\varphi} U)$$

Given φ we obtain an open subscheme $Z_\varphi \subset U_V = U \times_X V$, namely, the image of the graph of φ. Via φ we obtain an isomorphism $V \to Z_\varphi$ over U and we can think of an element

$$s_\varphi \in \mathcal{F}(V \xrightarrow{\varphi} U) = \mathcal{F}(Z_\varphi) = H_{Z_\varphi}(\mathcal{F})$$

as a section of $\mathcal{F}$ over Z_φ. Since $Z_\varphi \subset U_V$ is open, we actually have $H_{Z_\varphi}(\mathcal{F}) = \mathcal{F}(Z_\varphi)$ and we can think of s_φ as an element of $H_{Z_\varphi}(\mathcal{F})$. Having said this, our map $j_{p!}\mathcal{F} \to f_{p!}\mathcal{F}$ is defined by the rule

$$\sum\nolimits_{i=1,\ldots,n} s_{\varphi_i} \longmapsto \sum\nolimits_{i=1,\ldots,n} (Z_{\varphi_i}, s_{\varphi_i})$$

with right hand side a sum as in (4.0.1). We omit the verification that this is compatible with restriction mappings and functorial in $\mathcal{F}$.

To finish the proof, we claim that given a geometric point $\overline{y} : \mathrm{Spec}(k) \to Y$ there is a commutative diagram

$$\begin{tikzcd} (j_{p!}\mathcal{F})_{\overline{y}} \arrow[r] \arrow[d] & \bigoplus_{j(\overline{x})=\overline{y}} \mathcal{F}_{\overline{x}} \arrow[d, equal] \\ (f_{p!}\mathcal{F})_{\overline{y}} \arrow[r] & \bigoplus_{f(\overline{x})=\overline{y}} \mathcal{F}_{\overline{x}} \end{tikzcd}$$

where the top horizontal arrow is constructed in the proof of Étale Cohomology, Proposition 69.3, the bottom horizontal arrow is constructed in the proof of Lemma 4.2, the right vertical arrow is the obvious equality, and the left veritical arrow is the map defined in the previous paragraph on stalks. The claim follows in a straightforward manner from the explicit description of all of the arrows involved here and in the references given. Since the horizontal arrows are isomorphisms we conclude so is the left vertical arrow. Hence we find that our map induces an isomorphism on sheafifications by Étale Cohomology, Theorem 29.10. □

0F6R **Definition 4.4.** Let $f : X \to Y$ be a locally quasi-finite morphism of schemes. We define the *direct image with compact support* to be the functor

$$f_! : Ab(X_{\acute{e}tale}) \longrightarrow Ab(Y_{\acute{e}tale})$$

defined by the formula $f_!\mathcal{F} = (f_{p!}\mathcal{F})^\#$, i.e., $f_!\mathcal{F}$ is the sheafification of the presheaf $f_{p!}\mathcal{F}$ constructed above.

By Lemma 4.1 this does not conflict with Definition 3.3 (when both definitions apply) and by Lemma 4.3 this does not conflict with Étale Cohomology, Definition 69.1 (when both definitions apply).

0F5F **Lemma 4.5.** *Let $f : X \to Y$ be a locally quasi-finite morphism of schemes. Then*

1. *for $\mathcal{F}$ in $Ab(X_{\acute{e}tale})$ and a geometric point $\overline{y} : \mathrm{Spec}(k) \to Y$ we have*
$$(f_!\mathcal{F})_{\overline{y}} = \bigoplus\nolimits_{f(\overline{x})=\overline{y}} \mathcal{F}_{\overline{x}}$$
functorially in $\mathcal{F}$, and
2. *the functor $f_! : Ab(X_{\acute{e}tale}) \to Ab(Y_{\acute{e}tale})$ is exact and commutes with direct sums.*

Proof. The formula for the stalks is immediate (and in fact equivalent) to Lemma 4.2. The exactness of the functor follows immediately from this and the fact that exactness may be checked on stalks, see Étale Cohomology, Theorem 29.10. □

0F6S **Remark 4.6** (Covariance with respect to open embeddings)**.** Let $f : X \to Y$ be locally quasi-finite morphism of schemes. Let $\mathcal{F}$ be an abelian sheaf on $X_{\acute{e}tale}$. Let $X' \subset X$ be an open subscheme and denote $f' : X' \to Y$ the restriction of f. We claim there is a canonical map

$$f'_!(\mathcal{F}|_{X'}) \longrightarrow f_!\mathcal{F}$$

Namely, this map will be the sheafification of a canonical map

$$f'_{p!}(\mathcal{F}|_{X'}) \to f_{p!}\mathcal{F}$$

constructed as follows. Let $V \in Y_{\acute{e}tale}$ and consider a section $s' = \sum_{i=1,\ldots,n}(Z'_i, s'_i)$ as in (4.0.1) defining an element of $f'_{p!}(\mathcal{F}|_{X'})(V)$. Then $Z'_i \subset X'_V$ may also be viewed as a locally closed subscheme of X_V and we have $H_{Z'_i}(\mathcal{F}|_{X'}) = H_{Z'_i}(\mathcal{F})$.

We will map s' to the exact same sum $s = \sum_{i=1,\ldots,n}(Z'_i, s'_i)$ but now viewed as an element of $f_{p!}\mathcal{F}(V)$. We omit the verification that this construction is compatible with restriction mappings and functorial in $\mathcal{F}$. This construction has the following properties:

(1) The maps $f'_{p!}\mathcal{F}' \to f_{p!}\mathcal{F}$ and $f'_!\mathcal{F}' \to f_!\mathcal{F}$ are compatible with the description of stalks given in Lemmas 4.2 and 4.5.
(2) If f is separated, then the map $f'_{p!}\mathcal{F}' \to f_{p!}\mathcal{F}$ is the same as the map constructed in Remark 3.5 via the isomorphism in Lemma 4.1.
(3) If $X'' \subset X'$ is another open, then the composition of $f''_{p!}(\mathcal{F}|_{X''}) \to f'_{p!}(\mathcal{F}|_{X'}) \to f_{p!}\mathcal{F}$ is the map $f''_{p!}(\mathcal{F}|_{X''}) \to f_{p!}\mathcal{F}$ for the inclusion $X'' \subset X$. Sheafifying we conclude the same holds true for $f''_!(\mathcal{F}|_{X''}) \to f'_!(\mathcal{F}|_{X'}) \to f_!\mathcal{F}$.
(4) The map $f'_!\mathcal{F}' \to f_!\mathcal{F}$ is injective because we can check this on stalks.

All of these statements are easily proven by representing elements as finite sums as above and considering what happens to these elements.

0F5H **Lemma 4.7.** *Let $f : X \to Y$ be a locally quasi-finite morphism of schemes. Let $X = \bigcup_{i \in I} X_i$ be an open covering. Then there exists an exact complex*

$$\ldots \to \bigoplus\nolimits_{i_0,i_1,i_2} f_{i_0i_1i_2,!}\mathcal{F}|_{X_{i_0i_1i_2}} \to \bigoplus\nolimits_{i_0,i_1} f_{i_0i_1,!}\mathcal{F}|_{X_{i_0i_1}} \to \bigoplus\nolimits_{i_0} f_{i_0,!}\mathcal{F}|_{X_{i_0}} \to f_!\mathcal{F} \to 0$$

functorial in $\mathcal{F} \in Ab(X_{\acute{e}tale})$, see proof for details.

Proof. Here as usual we set $X_{i_0\ldots i_p} = X_{i_0} \cap \ldots \cap X_{i_p}$ and we denote $f_{i_0\ldots i_p}$ the restriction of f to $X_{i_0\ldots i_p}$. The maps in the complex are the maps constructed in Remark 4.6 with sign rules as in the Čech complex. Exactness follows easily from the description of stalks in Lemma 4.5. Details omitted. □

0F5I **Remark 4.8** (Alternative construction)**.** Lemma 4.7 gives an alternative construction of the functor $f_!$ for locally quasi-finite morphisms f. Namely, given a locally quasi-finite morphism $f : X \to Y$ of schemes we can choose an open covering $X = \bigcup_{i \in I} X_i$ such that each $f_i : X_i \to Y$ is separated. For example choose an affine open covering of X. Then we can define $f_!\mathcal{F}$ as the cokernel of the penultimate map of the complex of the lemma, i.e.,

$$f_!\mathcal{F} = \text{Coker}\left(\bigoplus\nolimits_{i_0,i_1} f_{i_0i_1,!}\mathcal{F}|_{X_{i_0i_1}} \to \bigoplus\nolimits_{i_0} f_{i_0,!}\mathcal{F}|_{X_{i_0}}\right)$$

where we can use the construction of $f_{i_0,!}$ and $f_{i_0i_1,!}$ in Section 3 because the morphisms f_{i_0} and $f_{i_0i_1}$ are separated. One can then compute the stalks of $f_!$ (using the separated case, namely Lemma 3.17) and obtain the result of Lemma 4.5. Having done so all the other results of this section can be deduced from this as well.

0F78 **Remark 4.9.** Let $g : Y' \to Y$ be a morphism of schemes. For an abelian presheaf $\mathcal{G}'$ on $Y'_{\acute{e}tale}$ let us denote $g_*\mathcal{G}'$ the presheaf $V \mapsto \mathcal{G}'(Y' \times_Y V)$. If $\alpha : \mathcal{G} \to g_*\mathcal{G}'$ is a map of abelian presheaves on $Y_{\acute{e}tale}$, then there is a unique map $\alpha^\# : \mathcal{G}^\# \to g_*((\mathcal{G}')^\#)$ of abelian sheaves on $Y_{\acute{e}tale}$ such that the diagram

$$\begin{tikzcd} \mathcal{G} \arrow[r, "\alpha"'] \arrow[d] & g_*\mathcal{G}' \arrow[d] \\ \mathcal{G}^\# \arrow[r, "\alpha^\#"] & g_*((\mathcal{G}')^\#) \end{tikzcd}$$

is commutative where the vertical maps come from the canonical maps $\mathcal{G} \to \mathcal{G}^\#$ and $\mathcal{G}' \to (\mathcal{G}')^\#$. If $\alpha' : g^{-1}\mathcal{G}^\# \to (\mathcal{G}')^\#$ is the map adjoint to $\alpha^\#$, then for a geometric point $\overline{y}' : \mathrm{Spec}(k) \to Y'$ with image $\overline{y} = g \circ \overline{y}'$ in Y, the map

$$\alpha'_{\overline{y}'} : \mathcal{G}_{\overline{y}} = (\mathcal{G}^\#)_{\overline{y}} = (g^{-1}\mathcal{G}^\#)_{\overline{y}'} \longrightarrow (\mathcal{G}')^\#_{\overline{y}'} = \mathcal{G}'_{\overline{y}'}$$

is given by mapping the class in the stalk of a section s of $\mathcal{G}$ over an étale neighbourhood $(V, \overline{v})$ to the class of the section $\alpha(s)$ in $g_*\mathcal{G}'(V) = \mathcal{G}'(Y' \times_Y V)$ over the étale neighbourhood $(Y' \times_Y V, (\overline{y}', \overline{v}))$ in the stalk of $\mathcal{G}'$ at $\overline{y}'$.

0F5J **Lemma 4.10.** *Consider a cartesian square*

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

of schemes with f locally quasi-finite. For any abelian sheaf $\mathcal{F}$ on $X_{étale}$ we have $f'_!(g')^{-1}\mathcal{F} = g^{-1}f_!\mathcal{F}$.

Proof. With conventions as in Remark 4.9 we will explicitly construct a map

$$c : f_{p!}\mathcal{F} \longrightarrow g_*f'_{p!}(g')^{-1}\mathcal{F}$$

of abelian presheaves on $Y_{étale}$. By the discussion in Remark 4.9 this will determine a canonical map $g^{-1}f_!\mathcal{F} \to f'_!(g')^{-1}\mathcal{F}$. Finally, we will show this map induces isomorphisms on stalks and conclude by Étale Cohomology, Theorem 29.10.

Construction of the map c. Let $V \in Y_{étale}$ and consider a section $s = \sum_{i=1,\ldots,n}(Z_i, s_i)$ as in (4.0.1) defining an element of $f_{p!}\mathcal{F}(V)$. The value of $g_*f'_{p!}(g')^{-1}\mathcal{F}$ at V is $f'_{p!}(g')^{-1}\mathcal{F}(V')$ where $V' = V \times_Y Y'$. Denote $Z'_i \subset X'_{V'}$ the base change of Z_i to V'. By (2) there is a pullback map $H_{Z_i}(\mathcal{F}) \to H_{Z'_i}((g')^{-1}\mathcal{F})$. Denoting $s'_i \in H_{Z'_i}((g')^{-1}\mathcal{F})$ the image of s_i under pullback, we set $c(s) = \sum_{i=1,\ldots,n}(Z'_i, s'_i)$ as in (4.0.1) defining an element of $f'_{p!}(g')^{-1}\mathcal{F}(V')$. We omit the verification that this construction is compatible the relations (1) and (2) and compatible with restriction mappings. The construction is clearly functorial in $\mathcal{F}$.

Let $\overline{y}' : \mathrm{Spec}(k) \to Y'$ be a geometric point with image $\overline{y} = g \circ \overline{y}'$ in Y. Observe that $X'_{\overline{y}'} = X_{\overline{y}}$ by transitivity of fibre products. Hence g' produces a bijection $\{f'(\overline{x}') = \overline{y}'\} \to \{f(\overline{x}) = \overline{y}\}$ and if $\overline{x}'$ maps to $\overline{x}$, then $((g')^{-1}\mathcal{F})_{\overline{x}'} = \mathcal{F}_{\overline{x}}$ by Étale Cohomology, Lemma 36.2. Now we claim that the diagram

$$\begin{tikzcd} (g^{-1}f_!\mathcal{F})_{\overline{y}'} \arrow[r, equal] \arrow[d] & (f_!\mathcal{F})_{\overline{y}} \arrow[r] \arrow[dl] & \bigoplus_{f(\overline{x})=\overline{y}} \mathcal{F}_{\overline{x}} \arrow[d] \\ (f'_!(g')^{-1}\mathcal{F})_{\overline{y}'} \arrow[rr] & & \bigoplus_{f'(\overline{x}')=\overline{y}'} (g')^{-1}\mathcal{F}_{\overline{x}'} \end{tikzcd}$$

commutes where the horizontal arrows are given in the proof of Lemma 4.2 and where the right vertical arrow is an equality by what we just said above. The southwest arrow is described in Remark 4.9 as the pullback map, i.e., simply given by our construction c above. Then the simple description of the image of a sum $\sum(Z_i, z_i)$ in the stalk at $\overline{x}$ given in the proof of Lemma 4.2 immediately shows the diagram commutes. This finishes the proof of the lemma. $\square$

0F79 **Lemma 4.11.** *Let $f' : X \to Y'$ and $g : Y' \to Y$ be composable morphisms of schemes with f' and $f = g \circ f'$ locally quasi-finite and g separated and locally of finite type. Then there is a canonical isomorphism of functors $g_! \circ f'_! = f_!$. This isomorphism is compatible with*

(a) *covariance with respect to open embeddings as in Remarks 3.5 and 4.6,*
(b) *the base change isomorphisms of Lemmas 4.10 and 3.12, and*
(c) *equal to the isomorphism of Lemma 3.13 via the identifications of Lemma 4.1 in case f' is separated.*

Proof. Let $\mathcal{F}$ be an abelian sheaf on $X_{\acute{e}tale}$. With conventions as in Remark 4.9 we will explicitly construct a map

$$c : f_{p!}\mathcal{F} \longrightarrow g_* f'_{p!}\mathcal{F}$$

of abelian presheaves on $Y_{\acute{e}tale}$. By the discussion in Remark 4.9 this will determine a canonical map $c^\# : f_!\mathcal{F} \to g_* f'_!\mathcal{F}$. We will show that $c^\#$ has image contained in the subsheaf $g_! f'_!\mathcal{F}$, thereby obtaining a map $c' : f_!\mathcal{F} \to g_! f'_!\mathcal{F}$. Next, we will prove (a), (b), and (c) that. Finally, part (b) will allow us to show that c' is an isomorphism.

Construction of the map c. Let $V \in Y_{\acute{e}tale}$ and let $s = \sum(Z_i, s_i)$ be a sum as in (4.0.1) defining an element of $f_{p!}\mathcal{F}(V)$. Recall that $Z_i \subset X_V = X \times_Y V$ is a locally closed subscheme finite over V. Setting $V' = Y' \times_Y V$ we get $X_{V'} = X \times_{Y'} V' = X_V$. Hence $Z_i \subset X_{V'}$ is locally closed and Z_i is finite over V' because g is separated (Morphisms, Lemma 42.14). Hence we may set $c(s) = \sum(Z_i, s_i)$ but now viewed as an element of $f'_{p!}\mathcal{F}(V') = (g_* f'_{p!}\mathcal{F})(V)$. The construction is clearly compatible with relations (1) and (2) and compatible with restriction mappings and hence we obtain the map c.

Observe that in the discussion above our section $c(s) = \sum(Z_i, s_i)$ of $f'_!\mathcal{F}$ over V' restricts to zero on $V' \setminus \mathrm{Im}(\coprod Z_i \to V')$. Since $\mathrm{Im}(\coprod Z_i \to V')$ is proper over V (for example by Morphisms, Lemma 39.10) we conclude that $c(s)$ defines a section of $g_! f'_!\mathcal{F} \subset g_* f'_!\mathcal{F}$ over V. Since every local section of $f_!\mathcal{F}$ locally comes from a local section of $f_{p!}\mathcal{F}$ we conclude that the image of $c^\#$ is contained in $g_! f'_!\mathcal{F}$. Thus we obtain an induced map $c' : f_!\mathcal{F} \to g_! f'_!\mathcal{F}$ factoring $c^\#$ as predicted in the first paragraph of the proof.

Proof of (a). Let $Y'_1 \subset Y'$ be an open subscheme and set $X_1 = (f')^{-1}(W')$. We obtain a diagram

$$\begin{tikzcd}
X_1 \arrow[r, "a"'] \arrow[d, "f'_1"'] \arrow[dd, bend right=60, "f_1"'] & X \arrow[d, "f'"] \arrow[dd, bend left=60, "f"] \\
Y'_1 \arrow[r, "b'"'] \arrow[d, "g_1"'] & Y' \arrow[d, "g"] \\
Y \arrow[r, equal] & Y
\end{tikzcd}$$

where the horizontal arrows are open immersions. Then our claim is that the diagram

$$\begin{tikzcd}
f_{1,!}\mathcal{F}|_{X_1} \arrow[r, "c'_1"'] \arrow[dd] & g_{1,!}f'_{1,!}\mathcal{F}|_{X_1} \arrow[d, equal] & \\
& g_{1,!}(f'_!\mathcal{F})|_{Y'_1} \arrow[d] & \\
f_!\mathcal{F} \arrow[r, "c'"] & g_!f'_!\mathcal{F} \arrow[r] & g_*f'_!\mathcal{F}
\end{tikzcd}$$

commutes where the left vertical arrow is Remark 4.6 and the right vertical arrow is Remark 3.5. The equality sign in the diagram comes about because f'_1 is the restriction of f' to Y'_1 and our construction of $f'_!$ is local on the base. Finally, to prove the commutativity we choose an object V of $Y_{\acute{e}tale}$ and a formal sum $s_1 = \sum(Z_{1,i}, s_{1,i})$ as in (4.0.1) defining an element of $f_{1,p!}\mathcal{F}|_{X_1}(V)$. Recall this means $Z_{1,i} \subset X_1 \times_Y V$ is locally closed finite over V and $s_{1,i} \in H_{Z_{1,i}}(\mathcal{F})$. Then we chase this section across the maps involved, but we only need to show we end up with the same element of $g_*f'_!\mathcal{F}(V) = f'_!\mathcal{F}(Y' \times_Y V)$. Going around both sides of the diagram the reader immediately sees we end up with the element $\sum(Z_{1,i}, s_{1,i})$ where now $Z_{1,i}$ is viewed as a locally closed subscheme of $X \times_{Y'} (Y' \times_Y V) = X \times_Y V$ finite over $Y' \times_Y V$.

Proof of (b). Let $b : Y_1 \to Y$ be a morphism of schemes. Let us form the commutative diagram

$$\begin{tikzcd}
X_1 \arrow[r, "a"] \arrow[d, "f'_1"'] \arrow[dd, bend right=60, "f_1"'] & X \arrow[d, "f'"] \arrow[dd, bend left=60, "f"] \\
Y'_1 \arrow[r, "b'"'] \arrow[d, "g_1"'] & Y' \arrow[d, "g"] \\
Y_1 \arrow[r, "b"] & Y
\end{tikzcd}$$

with cartesian squares. We claim that our construction is compatible with the base change maps of Lemmas 4.10 and 3.12, i.e., that the top rectangle of the diagram

$$\begin{tikzcd}
b^{-1}f_!\mathcal{F} \arrow[rr] \arrow[d, "b^{-1}c'"'] & & f_{1,!}a^{-1}\mathcal{F} \arrow[d, "c'_1"] \\
b^{-1}g_!f'_!\mathcal{F} \arrow[r] \arrow[d] & g_{1,!}(b')^{-1}f'_!\mathcal{F} \arrow[r] \arrow[d] & g_{1,!}f'_{1,!}a^{-1}\mathcal{F} \arrow[d] \\
b^{-1}g_*f'_!\mathcal{F} \arrow[r] & g_{1,*}(b')^{-1}f'_!\mathcal{F} \arrow[r] & g_{1,*}f'_{1,!}a^{-1}\mathcal{F}
\end{tikzcd}$$

commutes. The verification of this is completely routine and we urge the reader to skip it. Since the arrows going from the middle row down to the bottom row are injective, it suffices to show that the outer diagram commutes. To show this it suffices to take a local section of $b^{-1}f_!\mathcal{F}$ and show we end up with the same local section of $g_{1,*}f'_{1,!}a^{-1}\mathcal{F}$ going around either way. However, in fact it suffices to check this for local sections which are of the the pullback by b of a section $s = \sum(Z_i, s_i)$ of

$f_{p!}\mathcal{F}(V)$ as above (since such pullbacks generate the abelian sheaf $b^{-1}f_!\mathcal{F}$). Denote V_1, V_1', and $Z_{1,i}$ the base change of V, $V' = Y' \times_Y V$, Z_i by $Y_1 \to Y$. Recall that Z_i is a locally closed subscheme of $X_V = X_{V'}$ and hence $Z_{1,i}$ is a locally closed subscheme of $(X_1)_{V_1} = (X_1)_{V_1'}$. Then $b^{-1}c'$ sends the pullback of s to the pullback of the local section $c(s)\sum(Z_i, s_i)$ viewed as an element of $f'_{p!}\mathcal{F}(V') = (g_*f'_{p!}\mathcal{F})(V)$. The composition of the bottom two base change maps simply maps this to $\sum(Z_{i,1}, s_{1,i})$ viewed as an element of $f'_{1,p!}a^{-1}\mathcal{F}(V_1') = g_{1,*}f'_{1,p!}a^{-1}\mathcal{F}(V_1)$. On the other hand, the base change map at the top of the diagram sends the pullback of s to $\sum(Z_{1,i}, s_{1,i})$ viewed as an element of $f_{1,!}a^{-1}\mathcal{F}(V_1)$. Then finally c_1' by its very construction does indeed map this to $\sum(Z_{i,1}, s_{1,i})$ viewed as an element of $f'_{1,p!}a^{-1}\mathcal{F}(V_1') = g_{1,*}f'_{1,p!}a^{-1}\mathcal{F}(V_1)$ and the commutativity has been verified.

Proof of (c). This follows from comparing the definitions for both maps; we omit the details.

To finish the proof it suffices to show that the pullback of c' via any geometric point $\overline{y} : \mathrm{Spec}(k) \to Y$ is an isomorphism. Namely, pulling back by $\overline{y}$ is the same thing as taking stalks and $\overline{y}$ (Étale Cohomology, Remark 55.6) and hence we can invoke Étale Cohomology, Theorem 29.10. By the compatibility (b) just shown, we conclude that we may assume Y is the spectrum of k and we have to show that c' is an isomorphism. To do this it suffices to show that the induced map

$$\bigoplus\nolimits_{x \in X} \mathcal{F}_x = H^0(Y, f_!\mathcal{F}) \longrightarrow H^0(Y, g_!f'_!\mathcal{F}) = H^0_c(Y', f'_!\mathcal{F})$$

is an isomorphism. The equalities hold by Lemmas 4.5 and 3.11. Recall that X is a disjoint union of spectra of Artinian local rings with residue field k, see Varieties, Lemma 20.2. Since the left and right hand side commute with direct sums (details omitted) we may assume that $\mathcal{F}$ is a skyscraper sheaf x_*A supported at some $x \in X$. Then $f'_!\mathcal{F}$ is the skyscraper sheaf at the image y' of x in Y by Lemma 4.5. In this case it is obvious that our construction produces the identity map $A \to H^0_c(Y', y'_*A) = A$ as desired. ☐

0F6T **Lemma 4.12.** *Let $f : X \to Y$ and $g : Y \to Z$ be composable locally quasi-finite morphisms of schemes. Then there is a canonical isomorphism of functors*

$$(g \circ f)_! \longrightarrow g_! \circ f_!$$

These isomorphisms satisfy the following properties:

1. *If f and g are separated, then the isomorphism agrees with Lemma 3.13.*
2. *If g is separated, then the isomorphism agrees with Lemma 4.11.*
3. *For a geometric point $\overline{z} : \mathrm{Spec}(k) \to Z$ the diagram*

$$\begin{tikzcd}
((g \circ f)_!\mathcal{F})_{\overline{z}} \arrow[rr] \arrow[d] & & \bigoplus_{g(f(\overline{x}))=\overline{z}} \mathcal{F}_{\overline{x}} \arrow[d, equal] \\
(g_!f_!\mathcal{F})_{\overline{z}} \arrow[r] & \bigoplus_{g(\overline{y})=\overline{z}} (f_!\mathcal{F})_{\overline{y}} \arrow[r] & \bigoplus_{g(f(\overline{x}))=\overline{z}} \mathcal{F}_{\overline{x}}
\end{tikzcd}$$

is commutative where the horizontal arrows are given by Lemma 4.5.

(4) *Let $h : Z \to T$ be a third locally quasi-finite morphism of schemes. Then the diagram*

$$\begin{tikzcd} (h \circ g \circ f)_! \arrow[r] \arrow[d] & (h \circ g)_! \circ f_! \arrow[d] \\ h_! \circ (g \circ f)_! \arrow[r] & h_! \circ g_! \circ f_! \end{tikzcd}$$

commutes.

(5) *Suppose that we have a diagram of schemes*

$$\begin{tikzcd} X' \arrow[r, "c"'] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "b"'] \arrow[d, "g'"'] & Y \arrow[d, "g"] \\ Z' \arrow[r, "a"] & Z \end{tikzcd}$$

with both squares cartesian and f and g locally quasi-finite. Then the diagram

$$\begin{tikzcd} a^{-1} \circ (g \circ f)_! \arrow[rr] \arrow[d] & & (g' \circ f')_! \circ c^{-1} \arrow[d] \\ a^{-1} \circ g_! \circ f_! \arrow[r] & g'_! \circ b^{-1} \circ f_! \arrow[r] & g'_! \circ f'_! \circ c^{-1} \end{tikzcd}$$

commutes where the horizontal arrows are those of Lemma 4.10.

Proof. If f and g are separated, then this is a special case of Lemma 3.13. If g is separated, then this is a special case of Lemma 4.11 which moreover agrees with the case where f and g are separated.

Construction in the general case. Choose an open covering $Y = \bigcup Y_i$ such that the restriction $g_i : Y_i \to Z$ of g is separated. Set $X_i = f^{-1}(Y_i)$ and denote $f_i : X_i \to Y_i$ the restriction of f. Also denote $h = g \circ f$ and $h_i : X_i \to Z$ the restriction of h. Consider the following diagram

$$\begin{tikzcd} \bigoplus_{i_0,i_1} h_{i_0i_1,!}\mathcal{F}|_{X_{i_0i_1}} \arrow[r] \arrow[d] & \bigoplus_{i_0} h_{i_0,!}\mathcal{F}|_{X_{i_0}} \arrow[r] \arrow[d] & h_!\mathcal{F} \arrow[r] \arrow[dd, dotted] & 0 \\ \bigoplus_{i_0,i_1} g_{i_0i_1,!}f_{i_0i_1,!}\mathcal{F}|_{X_{i_0i_1}} \arrow[r] \arrow[d] & \bigoplus_{i_0} g_{i_0,!}f_{i_0,!}\mathcal{F}|_{X_{i_0}} \arrow[d] & & \\ \bigoplus_{i_0,i_1} g_{i_0i_1,!}(f_!\mathcal{F})|_{Y_{i_0i_1}} \arrow[r] & \bigoplus_{i_0} g_{i_0,!}(f_!\mathcal{F})|_{Y_{i_0}} \arrow[r] & g_!f_!\mathcal{F} \arrow[r] & 0 \end{tikzcd}$$

By Lemma 4.7 the top and bottom row in the diagram are exact. By Lemma 4.11 the top left square commutes. The vertical arrows in the lower left square come about because $(f_!\mathcal{F})|_{Y_{i_0i_1}} = f_{i_0i_1,!}\mathcal{F}|_{X_{i_0i_1}}$ and $(f_!\mathcal{F})|_{Y_{i_0}} = f_{i_0,!}\mathcal{F}|_{X_{i_0}}$ as the construction of $f_!$ is local on the base. Moreover, these equalities are (of course) compatible with the identifications $((f_!\mathcal{F})|_{Y_{i_0}})|_{Y_{i_0i_1}} = (f_!\mathcal{F})|_{Y_{i_0i_1}}$ and $(f_{i_0,!}\mathcal{F}|_{X_{i_0}})|_{Y_{i_0i_1}} = f_{i_0i_1,!}\mathcal{F}|_{X_{i_0i_1}}$ which are used (together with the covariance for open embeddings for $Y_{i_0i_1} \subset Y_{i_0}$) to define the horizontal maps of the lower left square. Thus this square commutes as well. In this way we conclude there is a unique dotted arrow as indicated in the diagram and moreover this arrow is an isomorphism.

Proof of properties (1) – (5). Fix the open covering $Y = \bigcup Y_i$. Observe that if $Y \to Z$ happens to be separated, then we get a dotted arrow fitting into the huge diagram above by using the map of Lemma 4.11 (by the very properties of that lemma). This proves (2) and hence also (1) by the compatibility of the maps of Lemma 4.11 and Lemma 3.13. Next, for any scheme Z' over Z, we obtain the compatibility in (5) for the map $(g' \circ f')_! \to g'_! \circ f'_!$ constructed using the open covering $Y' = \bigcup b^{-1}(Y_i)$. This is clear from the corresponding compatibility of the maps constructed in Lemma 4.11. In particular, we can consider a geometric point $\overline{z} : \text{Spec}(k) \to Z$. Since $X_{\overline{z}} \to Y_{\overline{z}} \to \text{Spec}(k)$ are separated maps, we find that the base change of $(g \circ f)_!\mathcal{F} \to g_! f_! \mathcal{F}$ by $\overline{z}$ is equal to the map of Lemma 3.13. The reader then immediately sees that we obtain property (3). Of course, property (3) guarantees that our transformation of functors $(g \circ f)_! \to g_! \circ f_!$ constructed using the open covering $Y = \bigcup Y_i$ doesn't depend on the choice of this open covering. Finally, property (4) follows by looking at what happens on stalks using the already proven property (3). $\square$

5. Upper shriek for locally quasi-finite morphisms

0F58 For a locally quasi-finite morphism $f : X \to Y$ of schemes, the functor $f_! : Ab(X_{étale}) \to Ab(Y_{étale})$ commutes with direct sums and is exact, see Lemma 4.5. This suggests that it has a right adjoint which we will denote $f^!$.

Warning: This functor is the non-derived version!

0F59 **Lemma 5.1.** *Let $f : X \to Y$ be a locally quasi-finite morphism of schemes. The functor $f_! : Ab(X_{étale}) \to Ab(Y_{étale})$ has a right adjoint $f^! : Ab(Y_{étale}) \to Ab(X_{étale})$. Moreover, we have $f^!(\overline{y}_*A) = \prod_{f(\overline{x})=\overline{y}} \overline{x}_*A$.*

Proof. Let $E \subset \text{Ob}(Ab(Y_{étale}))$ be the class consisting of products of skyscraper sheaves. We claim that

1. every $\mathcal{G}$ in $Ab(Y_{étale})$ is a subsheaf of an element of E, and
2. for every $\mathcal{G} \in E$ there exists an object $\mathcal{H}$ of $Ab(X_{étale})$ such that $\text{Hom}(f_!\mathcal{F}, \mathcal{G}) = \text{Hom}(\mathcal{F}, \mathcal{H})$ functorially in $\mathcal{F}$.

Once the claim has been verified, the dual of Homology, Lemma 27.6 produces the adjoint functor $f^!$.

Part (1) is true because we can map $\mathcal{G}$ to the sheaf $\prod \overline{y}_*\mathcal{G}_{\overline{y}}$ where the product is over all geometric points of Y. This is an injection by Étale Cohomology, Theorem 29.10. (This is the first step in the Godement resolution when done in the setting of abelian sheaves on topological spaces.)

Part (2) and the final statement of the lemma can be seen as follows. Suppose that $\mathcal{G} = \prod \overline{y}_*A_{\overline{y}}$ for some abelian groups $A_{\overline{y}}$. Then

$$\text{Hom}(f_!\mathcal{F}, \mathcal{G}) = \prod \text{Hom}(f_!\mathcal{F}, \overline{y}_*A_{\overline{y}})$$

Thus it suffices to find abelian sheaves $\mathcal{H}_{\overline{y}}$ on $X_{étale}$ representing the functors $\mathcal{F} \mapsto \text{Hom}(f_!\mathcal{F}, \overline{y}_*A_{\overline{y}})$ and to take $\mathcal{H} = \prod \mathcal{H}_{\overline{y}}$. This reduces us to the case $\mathcal{H} = \overline{y}_*A$ for some fixed geometric point $\overline{y} : \text{Spec}(k) \to Y$ and some fixed abelian group A.

We claim that in this case $\mathcal{H} = \prod_{f(\overline{x})=\overline{y}} \overline{x}_* A$ works. This will finish the proof of the lemma. Namely, we have

$$\mathrm{Hom}(f_!\mathcal{F}, \overline{y}_*A) = \mathrm{Hom}_{Ab}((f_!\mathcal{F})_{\overline{y}}, A) = \mathrm{Hom}_{Ab}(\bigoplus\nolimits_{f(\overline{x})=\overline{y}} \mathcal{F}_{\overline{x}}, A)$$

by the description of stalks in Lemma 4.5 on the one hand and on the other hand we have

$$\mathrm{Hom}(\mathcal{F}, \mathcal{H}) = \prod\nolimits_{f(\overline{x})=\overline{y}} \mathrm{Hom}(\mathcal{F}, \overline{x}_*A) = \prod\nolimits_{f(\overline{x})=\overline{y}} \mathrm{Hom}_{Ab}(\mathcal{F}_{\overline{x}}, A)$$

We leave it to the reader to identify these as functors of $\mathcal{F}$. ☐

0F5A **Lemma 5.2.** *Let $j : U \to X$ be an étale morphism. Then $j^! = j^{-1}$.*

Proof. This is true because $j_!$ as defined in Section 4 agrees with $j_!$ as defined in Étale Cohomology, Section 69, see Lemma 4.3. Finally, in Étale Cohomology, Section 69 the functor $j_!$ is defined as the left adjoint of j^{-1} and hence we conclude by uniqueness of adjoint functors. ☐

0F5B **Lemma 5.3.** *Let $f : X \to Y$ and $g : Y \to Z$ be separated and locally quasi-finite morphisms. There is a canonical isomorphism $(g \circ f)^! \to f^! \circ g^!$. Given a third locally quasi-finite morphism $h : Z \to T$ the diagram*

$$\begin{tikzcd}
(h \circ g \circ f)^! \arrow[r] \arrow[d] & f^! \circ (h \circ g)^! \arrow[d] \\
(g \circ f)^! \circ h^! \arrow[r] & f^! \circ g^! \circ h^!
\end{tikzcd}$$

commutes.

Proof. By uniqueness of adjoint functors, this immediately translates into the corresponding (dual) statement for the functors $f_!$. See Lemma 4.12. ☐

0F5C **Lemma 5.4.** *Let $j : U \to X$ and $j' : V \to U$ be étale morphisms. The isomorphism $(j \circ j')^{-1} = (j')^{-1} \circ j^{-1}$ and the isomorphism $(j \circ j')^! = (j')^! \circ j^!$ of Lemma 5.3 agree via the isomorphism of Lemma 5.2.*

Proof. Omitted. ☐

0F6U **Lemma 5.5.** *Consider a cartesian square*

$$\begin{tikzcd}
X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\
Y' \arrow[r, "g"] & Y
\end{tikzcd}$$

of schemes with f locally quasi-finite. For any abelian sheaf $\mathcal{F}$ on $Y'_{étale}$ we have $(g')_*(f')^!\mathcal{F} = f^! g_*\mathcal{F}$.

Proof. By uniqueness of adjoint functors, this follows from the corresponding (dual) statement for the functors $f_!$. See Lemma 4.10. ☐

0F5L **Remark 5.6.** The material in this section can be generalized to sheaves of pointed sets. Namely, for a site $\mathcal{C}$ denote $Sh^*(\mathcal{C})$ the category of sheaves of pointed sets. The constructions in this and the preceding section apply, mutatis mutandis, to sheaves of pointed sets. Thus given a locally quasi-finite morphism $f : X \to Y$ of schemes we obtain an adjoint pair of functors

$$f_! : Sh^*(X_{étale}) \longrightarrow Sh^*(Y_{étale}) \quad \text{and} \quad f^! : Sh^*(Y_{étale}) \longrightarrow Sh^*(X_{étale})$$

such that for every geometric point $\overline{y}$ of Y there are isomorphisms

$$(f_!\mathcal{F})_{\overline{y}} = \coprod\nolimits_{f(\overline{x})=\overline{y}} \mathcal{F}_{\overline{x}}$$

(coproduct taken in the category of pointed sets) functorial in $\mathcal{F} \in Sh^*(X_{étale})$ and isomorphisms

$$f^!(\overline{y}_*S) = \prod\nolimits_{f(\overline{x})=\overline{y}} \overline{x}_*S$$

functorial in the pointed set S. If $F : Ab(X_{étale}) \to Sh^*(X_{étale})$ and $F : Ab(Y_{étale}) \to Sh^*(Y_{étale})$ denote the forgetful functors, compatibility between the constructions will guarantee the existence of canonical maps

$$f_!F(\mathcal{F}) \longrightarrow F(f_!\mathcal{F})$$

functorial in $\mathcal{F} \in Ab(X_{étale})$ and

$$F(f^!\mathcal{G}) \longrightarrow f^!F(\mathcal{G})$$

functorial in $\mathcal{G} \in Ab(Y_{étale})$ which produce the obvious maps on stalks, resp. skyscraper sheaves. In fact, the transformation $F \circ f^! \to f^! \circ F$ is an isomorphism (because $f^!$ commutes with products).

6. Derived upper shriek for locally quasi-finite morphisms

0F5M We can take the derived versions of the functors in Section 5 and obtain the following.

0F5N **Lemma 6.1.** *Let $f : X \to Y$ be a locally quasi-finite morphism of schemes. The functors $f_!$ and $f^!$ of Definition 4.4 and Lemma 5.1 induce adjoint functors $f_! : D(X_{étale}) \to D(Y_{étale})$ and $Rf^! : D(Y_{étale}) \to D(X_{étale})$ on derived categories.*

In the separated case the functor $f_!$ is defined in Section 3.

Proof. This follows immediately from Derived Categories, Lemma 29.5, the fact that $f_!$ is exact (Lemma 4.5) and hence $Lf_! = f_!$ and the fact that we have enough K-injective complexes of abelian sheaves on $Y_{étale}$ so that $Rf^!$ is defined. $\square$

7. Preliminaries to derived lower shriek via compactifications

0F7A In this section we prove some lemmas on the existence of certain natural isomorphisms of functors which follow immediately from proper base change.

0F7B **Lemma 7.1.** *Consider a commutative diagram of schemes*

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

with f and f' proper and g and g' separated and locally quasi-finite. For a torsion ring Λ and K in $D(X'_{étale}, \Lambda)$ there is a canonical isomorphism $g_!Rf'_*K \to Rf_*(g'_!K)$ in $D(Y_{étale}, \Lambda)$.

Proof. Represent K by a K-injective complex $\mathcal{J}^\bullet$ of sheaves of Λ-modules on $X'_{étale}$. Choose a quasi-isomorphism $g'_!\mathcal{J}^\bullet \to \mathcal{I}^\bullet$ to a K-injective complex $\mathcal{I}^\bullet$ of sheaves of Λ-modules on $X_{étale}$. Then we can consider the map

$$g_! f'_* \mathcal{J}^\bullet = g_! f'_! \mathcal{J}^\bullet = f_! g'_! \mathcal{J}^\bullet = f_* g'_! \mathcal{J}^\bullet \to f_* \mathcal{I}^\bullet$$

where the first and third equality come from Lemma 3.4 and the second equality comes from Lemma 3.13 which tells us that both $g_! \circ f'_!$ and $f_! \circ g'_!$ are equal to $(g \circ f')_! = (f \circ g')_!$ as subsheaves of $(g \circ f')_* = (f \circ g')_*$. To finish the proof it suffices to show that $f_* g'_! \mathcal{J}^\bullet \to f_* \mathcal{I}^\bullet$ is a quasi-isomorphism.

The question is local on Y. Hence we may assume that the dimension of fibres of f is bounded, see Étale Cohomology, Lemma 88.3. Then we see that Rf_* has finite cohomological dimension, see Étale Cohomology, Lemma 88.2. Hence by Derived Categories, Lemma 31.2, if we show that $R^q f_*(g'_! \mathcal{J}) = 0$ for $q > 0$ and any injective sheaf of Λ-modules $\mathcal{J}$ on $X'_{étale}$, then the result follows.

The stalk of $R^q f_*(g'_! \mathcal{J})$ at a geometric point $\overline{y}$ is equal to $H^q(X_{\overline{y}}, g'_! \mathcal{J}|_{X_{\overline{y}}})$ by Étale Cohomology, Lemma 87.13. Since formation of $g'_!$ commutes with base change this is equal to

$$H^q(X_{\overline{y}}, g'_{\overline{y},!}(\mathcal{J}|_{X'_{\overline{y}}}))$$

Since $Y' \to Y$ is locally quasi-finite, we see that $X'_{\overline{y}}$ is a disjoint union of the fibres $X'_{\overline{y}'}$ at geometric points $\overline{y}'$ of Y' lying over $\overline{y}$. Hence we get

$$H^q(X_{\overline{y}}, \bigoplus g'_{\overline{y}',!}(\mathcal{J}|_{X'_{\overline{y}'}}))$$

for example by Lemma 3.15 (but it is also obvious from the definition of $g'_{\overline{y},!}$ in Section 3). Since taking étale cohomology over $X_{\overline{y}}$ commutes with direct sums (Étale Cohomology, Theorem 51.3) we conclude it suffices to show that

$$H^q(X_{\overline{y}}, g'_{\overline{y}',!}(\mathcal{J}|_{X'_{\overline{y}'}}))$$

is zero. Observe that $g_{\overline{y}'} : X'_{\overline{y}'} \to X_{\overline{y}}$ is a morphism between proper scheme over $\overline{y}$ and hence is proper itself. As it is locally quasi-finite as well we conclude that $g_{\overline{y}'}$ is finite. Thus we see that $g'_{\overline{y}',!} = g'_{\overline{y}',*} = Rg'_{\overline{y}',*}$. By Leray we conlude that we have to show

$$H^q(X'_{\overline{y}'}, \mathcal{J}|_{X'_{\overline{y}'}}))$$

is zero. This of course follows from proper base change (in the form we quoted it before) as the higher direct images of $\mathcal{J}$ under f' are zero. $\square$

0F7C **Lemma 7.2.** *Consider a commutative diagram of schemes*

$$\begin{tikzcd}
X' \arrow[r, "k"'] \arrow[d, "f'"'] & X \arrow[d, "f"] \\
Y' \arrow[r, "l"'] \arrow[d, "g'"'] & Y \arrow[d, "g"] \\
Z' \arrow[r, "m"] & Z
\end{tikzcd}$$

with f, f', g and g' proper and k, l, and m separated and locally quasi-finite. Then the isomorphisms of Lemma 7.1 for the two squares compose to give the isomorphism for the outer rectangle (see proof for a precise statement).

Proof. The statement means that if we write $R(g\circ f)_* = Rg_*\circ Rf_*$ and $R(g'\circ f')_* = Rg'_*\circ Rf'_*$, then the isomorphism $m_!\circ Rg'_*\circ Rf'_* \to Rg_*\circ Rf_*\circ k_!$ of the outer rectangle is equal to the composition

$$m_! \circ Rg'_* \circ Rf'_* \to Rg_* \circ l_! \circ Rf'_* \to Rg_* \circ Rf_* \circ k_!$$

of the two maps of the squares in the diagram. To prove this choose a K-injective complex $\mathcal{J}^\bullet$ of Λ-modules on $X'_{\acute{e}tale}$ and a quasi-isomorphism $k_!\mathcal{J}^\bullet \to \mathcal{I}^\bullet$ to a K-injective complex $\mathcal{I}^\bullet$ of Λ-modules on $X_{\acute{e}tale}$. The proof of Lemma 7.1 shows that the canonical map

$$a : l_! f'_*\mathcal{J}^\bullet \to f_*\mathcal{I}^\bullet$$

is a quasi-isomorphism and this quasi-isomorphism produces the second arrow on applying Rg_*. By Cohomology on Sites, Lemma 21.9 the complex $f_*\mathcal{I}^\bullet$, resp. $f'_*\mathcal{J}^\bullet$ is a K-injective complex of Λ-modules on $Y_{\acute{e}tale}$, resp. $Y'_{\acute{e}tale}$. (Using this is cheating and could be avoided.) In particular, the same reasoning gives that the canonical map

$$b : m_! g'_* f'_*\mathcal{J}^\bullet \to g_* f_*\mathcal{I}^\bullet$$

is a quasi-isomorphism and this quasi-isomorphism represents the first arrow. Finally, the proof of Lemma 7.1 show that $g_*l_!f'_!\mathcal{J}^\bullet$ represents $Rg_*(l_!f'_*\mathcal{J}^\bullet)$ because $f'_*\mathcal{J}^\bullet$ is K-injective. Hence $Rg_*(a) = g_*(a)$ and the composition $g_*(a)\circ b$ is the arrow of Lemma 7.1 for the rectangle. □

0F7D **Lemma 7.3.** *Consider a commutative diagram of schemes*

$$\begin{tikzcd} X'' \arrow[r, "g'"'] \arrow[d, "f''"'] & X' \arrow[r, "g"'] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y'' \arrow[r, "h'"] & Y' \arrow[r, "h"] & Y \end{tikzcd}$$

with f, f', and f'' proper and g, g', h, and h' separated and locally quasi-finite. Then the isomorphisms of Lemma 7.1 for the two squares compose to give the isomorphism for the outer rectangle (see proof for a precise statement).

Proof. The statement means that if we write $(h\circ h')_! = h_!\circ h'_!$ and $(g\circ g')_! = g_!\circ g'_!$ using the equalities of Lemma 3.13, then the isomorphism $h_!\circ h'_!\circ Rf''_* \to Rf_*\circ g_!\circ g'_!$ of the outer rectangle is equal to the composition

$$h_! \circ h'_! \circ Rf''_* \to h_! \circ Rf'_* \circ g'_! \to Rf_* \circ g_! \circ g'_!$$

of the two maps of the squares in the diagram. To prove this choose a K-injective complex $\mathcal{I}^\bullet$ of Λ-modules on $X''_{\acute{e}tale}$ and a quasi-isomorphism $g'_!\mathcal{I}^\bullet \to \mathcal{J}^\bullet$ to a K-injective complex $\mathcal{J}^\bullet$ of Λ-modules on $X'_{\acute{e}tale}$. Next, choose a quasi-isomorphism $g_!\mathcal{J}^\bullet \to \mathcal{K}^\bullet$ to a K-injective complex $\mathcal{K}^\bullet$ of Λ-modules on $X_{\acute{e}tale}$. The proof of Lemma 7.1 shows that the canonical maps

$$h'_! f''_*\mathcal{I}^\bullet \to f'_*\mathcal{J}^\bullet \quad \text{and} \quad h_! f'_*\mathcal{J}^\bullet \to f_*\mathcal{K}^\bullet$$

are quasi-isomorphisms and these quasi-isomorphisms define the first and second arrow above. Since $g_!$ is an exact functor (Lemma 3.17) we find that $g_!g'_!\mathcal{I}^\bullet \to \mathcal{K}^\bullet$ is a quasi-ismorphism and hence the canonical map

$$h_! h'_! f''_*\mathcal{I}^\bullet \to f_*\mathcal{K}^\bullet$$

is a quasi-isomorphism and represents the map for the outer rectangle in the derived category. Clearly this map is the composition of the other two and the proof is complete. □

0F7E **Remark 7.4.** Consider a commutative diagram

$$\begin{tikzcd} X'' \arrow[r, "k'"'] \arrow[d, "f''"'] & X' \arrow[r, "k"'] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y'' \arrow[r, "l'"] \arrow[d, "g''"'] & Y' \arrow[r, "l"] \arrow[d, "g'"'] & Y \arrow[d, "g"] \\ Z'' \arrow[r, "m'"] & Z' \arrow[r, "m"] & Z \end{tikzcd}$$

of schemes whose vertical arrows are proper and whose horizontal arrows are separated and locally quasi-finite. Let us label the squares of the diagram A, B, C, D as follows

$$\begin{matrix} A & B \\ C & D \end{matrix}$$

Then the maps of Lemma 7.1 for the squares are (where we use $Rf_* = f_*$, etc)

$$\begin{matrix} \gamma_A : l'_! \circ f''_* \to f'_* \circ k'_! & \gamma_B : l_! \circ f'_* \to f_* \circ k_! \\ \gamma_C : m'_! \circ g''_* \to g'_* \circ l'_! & \gamma_D : m_! \circ g'_* \to g_* \circ l_! \end{matrix}$$

For the 2×1 and 1×2 rectangles we have four further maps

$$\begin{matrix} \gamma_{A+B} : (l \circ l')_! \circ f''_* \to f_* \circ (k \circ k')_* \\ \gamma_{C+D} : (m \circ m')_! \circ g''_* \to g_* \circ (l \circ l')_! \\ \gamma_{A+C} : m'_! \circ (g'' \circ f'')_* \to (g' \circ f')_* \circ k'_! \\ \gamma_{B+D} : m_! \circ (g' \circ f')_* \to (g \circ f)_* \circ k_! \end{matrix}$$

By Lemma 7.3 we have

$$\gamma_{A+B} = \gamma_B \circ \gamma_A, \quad \gamma_{C+D} = \gamma_D \circ \gamma_C$$

and by Lemma 7.2 we have

$$\gamma_{A+C} = \gamma_A \circ \gamma_C, \quad \gamma_{B+D} = \gamma_B \circ \gamma_D$$

Here it would be more correct to write $\gamma_{A+B} = (\gamma_B \star \mathrm{id}_{k'_!}) \circ (\mathrm{id}_{l_!} \star \gamma_A)$ with notation as in Categories, Section 27 and similarly for the others. Having said all of this we find (a priori) two transformations

$$m_! \circ m'_! \circ g''_* \circ f''_* \longrightarrow g_* \circ f_* \circ k_! \circ k'_!$$

namely

$$\gamma_B \circ \gamma_D \circ \gamma_A \circ \gamma_C = \gamma_{B+D} \circ \gamma_{A+C}$$

and

$$\gamma_B \circ \gamma_A \circ \gamma_D \circ \gamma_C = \gamma_{A+B} \circ \gamma_{C+D}$$

The point of this remark is to point out that these transformations are equal. Namely, to see this it suffices to show that

$$\begin{tikzcd} m_! \circ g'_* \circ l'_! \circ f''_* \arrow[r, "\gamma_D"'] \arrow[d, "\gamma_A"'] & g_* \circ l_! \circ l'_! \circ f''_* \arrow[d, "\gamma_A"] \\ m_! \circ g'_* \circ f'_* \circ k'_! \arrow[r, "\gamma_D"] & g_* \circ l_! \circ f'_* \circ k'_! \end{tikzcd}$$

commutes. This is true because the squares A and D meet in only one point, more precisely by Categories, Lemma 27.2 or more simply the discussion preceding Categories, Definition 27.1.

0F7F **Lemma 7.5.** *Let $b : Y_1 \to Y$ be a morphism of schemes. Consider a commutative diagram of schemes*

$$\begin{tikzcd} X' \arrow[r, "g'"'] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd} \quad \text{and let} \quad \begin{tikzcd} X'_1 \arrow[r, "g'_1"'] \arrow[d, "f'_1"'] & X_1 \arrow[d, "f_1"] \\ Y'_1 \arrow[r, "g_1"] & Y_1 \end{tikzcd}$$

be the base change by b. Assume f and f' proper and g and g' separated and locally quasi-finite. For a torsion ring Λ and K in $D(X'_{\acute{e}tale}, \Lambda)$ there is commutative diagram

$$\begin{tikzcd} b^{-1}g_!Rf'_*K \arrow[r] \arrow[d] & g_{1,!}(b')^{-1}Rf'_*K \arrow[r] & g_{1,!}Rf'_{1,*}(a')^{-1}K \arrow[d] \\ b^{-1}Rf_*g'_!K \arrow[r] & Rf_{1,*}a^{-1}g'_!K \arrow[r] & Rf_{1,*}g'_{1,!}(a')^{-1}K \end{tikzcd}$$

in $D(Y_{1,\acute{e}tale}, \Lambda)$ where $a : X_1 \to X$, $a' : X'_1 \to X'$, $b' : Y'_1 \to Y'$ are the projections, the vertical maps are the arrows of Lemma 7.1 and the horizontal arrows are the base change map (from Étale Cohomology, Section 82) and the base change map of Lemma 3.12.

Proof. Represent K by a K-injective complex $\mathcal{J}^\bullet$ of sheaves of Λ-modules on $X'_{\acute{e}tale}$. Choose a quasi-isomorphism $g'_!\mathcal{J}^\bullet \to \mathcal{I}^\bullet$ to a K-injective complex $\mathcal{I}^\bullet$ of sheaves of Λ-modules on $X_{\acute{e}tale}$. The proof of Lemma 7.1 constructs $g_!Rf'_*K \to Rf_*g'_!K$ as

$$g_!f'_*\mathcal{J}^\bullet = g_!f'_!\mathcal{J}^\bullet = f_!g'_!\mathcal{J}^\bullet = f_*g'_!\mathcal{J}^\bullet \to f_*\mathcal{I}^\bullet$$

Choose a quasi-isomorphism $(a')^{-1}\mathcal{J}^\bullet \to \mathcal{J}_1^\bullet$ to a K-injective complex $\mathcal{J}_1^\bullet$ of sheaves of Λ-modules on $X'_{1,\acute{e}tale}$. Then we can pick a diagram of complexes

$$\begin{tikzcd} g'_{1,!}\mathcal{J}_1^\bullet \arrow[rr] & & \mathcal{I}_1^\bullet \\ g'_{1,!}(a')^{-1}\mathcal{J}^\bullet \arrow[u] \arrow[r, equal] & a^{-1}g'_!\mathcal{J}^\bullet \arrow[r] & a^{-1}\mathcal{I}^\bullet \arrow[u] \end{tikzcd}$$

commuting up to homotopy where all arrows are quasi-isomorphisms, the equality comes from Lemma 3.4, and $\mathcal{I}_1^\bullet$ is a K-injective complex of sheaves of Λ-modules on $X_{1,\acute{e}tale}$. The map $g_{1,!}Rf'_{1,*}(a')^{-1}K \to Rf_{1,*}g'_{1,!}(a')^{-1}K$ is given by

$$g_{1,!}f'_{1,*}\mathcal{J}_1^\bullet = g_{1,!}f'_{1,!}\mathcal{J}_1^\bullet = f_{1,!}g'_{1,!}\mathcal{J}_1^\bullet = f_{1,*}g'_{1,!}\mathcal{J}_1^\bullet \to f_{1,*}\mathcal{I}_1^\bullet$$

The identifications across the 3 equal signs in both arrows are compatible with pullback maps, i.e., the diagram

$$\begin{tikzcd} b^{-1}g_!f'_*\mathcal{J}^\bullet \arrow[r] \arrow[d, equal] & g_{1,!}(b')^{-1}f'_*\mathcal{J}^\bullet \arrow[r] & g_{1,!}f'_{1,*}(a')^{-1}\mathcal{J}^\bullet \arrow[d, equal] \\ b^{-1}f_*g'_!\mathcal{J}^\bullet \arrow[r] & f_{1,*}a^{-1}g'_!\mathcal{J}^\bullet \arrow[r] & f_{1,*}g'_{1,!}(a')^{-1}\mathcal{J}^\bullet \end{tikzcd}$$

of complexes of abelian sheaves commutes. To show this it is enough to show the diagram commutes with $g_!, g_{1,!}, g'_!, g'_{1,!}$ replaced by $g_*, g_{1,*}, g'_*, g'_{1,*}$ (because the shriek functors are defined as subfunctors of the $*$ functors and the base change maps are defined in a manner compatible with this, see proof of Lemma 3.12). For this new diagram the commutativity follows from the compatibility of pullback maps with horizontal and vertical stacking of diagrams, see Sites, Remarks 45.3 and 45.4 so that going around the diagram in either direction is the pullback map for the base change of $f \circ g' = g \circ f'$ by b. Since of course

$$\begin{tikzcd} g_{1,!}f'_{1,*}(a')^{-1}\mathcal{J}^\bullet \arrow[r] \arrow[d, equal] & g_{1,!}f'_{1,*}\mathcal{J}_1^\bullet \arrow[d, equal] \\ f_{1,*}g'_{1,!}(a')^{-1}\mathcal{J}^\bullet \arrow[r] & f_{1,*}g'_{1,!}\mathcal{J}_1^\bullet \end{tikzcd}$$

commutes, to finish the proof it suffices to show that

$$\begin{tikzcd} b^{-1}f_*g'_!\mathcal{J}^\bullet \arrow[r] \arrow[d] & f_{1,*}a^{-1}g'_!\mathcal{J}^\bullet \arrow[r] \arrow[d] & f_{1,*}g'_{1,!}(a')^{-1}\mathcal{J}^\bullet \arrow[r] & f_{1,*}g'_{1,!}\mathcal{J}_1^\bullet \arrow[d] \\ b^{-1}f_*\mathcal{I}^\bullet \arrow[r] & f_{1,*}a^{-1}\mathcal{I}^\bullet \arrow[rr] & & f_{1,*}\mathcal{I}_1^\bullet \end{tikzcd}$$

commutes in the derived category, which holds by our choice of maps earlier. □

0F7G **Lemma 7.6.** *Consider a commutative diagram of schemes*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "g"'] & Y \arrow[d, "h"] \\ & Z \end{tikzcd}$$

with f and g locally quasi-finite and h proper. For any torsion ring Λ and K in $D(X_{étale}, \Lambda)$ there is a canonical isomorphism $g_!K \to Rh_*(f_!K)$ in $D(Z_{étale}, \Lambda)$.

Proof. This is a special case of Lemma 7.1 if f and g are separated. We urge the reader to skip the proof in the general case as we'll mainly use the case where f and g are separated.

Represent K by a complex $\mathcal{K}^\bullet$ of sheaves of Λ-modules on $X_{étale}$. Choose a quasi-isomorphism $f_!\mathcal{K}^\bullet \to \mathcal{I}^\bullet$ into a K-injective complex $\mathcal{I}^\bullet$ of sheaves of Λ-modules on $Y_{étale}$. Consider the map

$$g_!\mathcal{K}^\bullet = h_!f_!\mathcal{K}^\bullet = h_*f_!\mathcal{K}^\bullet \longrightarrow h_*\mathcal{I}^\bullet$$

where the equalities are Lemmas 4.11 and 3.4. This map of complexes determines the map $g_!K \to Rh_*(f_!K)$ of the statement of the lemma. To check the map is an isomorphism we may work locally on Z. Hence we may assume that the dimension of fibres of h is bounded, see Étale Cohomology, Lemma 88.3. Then we see that Rh_* has finite cohomological dimension, see Étale Cohomology, Lemma 88.2. Hence by Derived Categories, Lemma 31.2, if we show that $R^qh_*(f_!\mathcal{F}) = 0$ for $q > 0$ and any sheaf $\mathcal{F}$ of Λ-modules on $X_{étale}$, then $h_*f_!\mathcal{K}^\bullet \to h_*\mathcal{I}^\bullet$ is a quasi-isomorphism.

Observe that $\mathcal{G} = f_!\mathcal{F}$ is a sheaf of Λ-modules on Y whose stalks are nonzero only at points $y \in Y$ such that $\kappa(y)/\kappa(h(y))$ is a finite extension. This follows from the description of stalks of $f_!\mathcal{F}$ in Lemma 4.5 and the fact that both f and g are locally quasi-finite. Hence by the proper base change theorem (Étale Cohomology, Lemma 87.13) it suffices to show that $H^q(Y_{\overline{z}}, \mathcal{H}) = 0$ where $\mathcal{H}$ is a sheaf on the proper scheme $Y_{\overline{z}}$ over $\kappa(\overline{z})$ whose support is contained in the set of closed points. Thus the required vanishing by Étale Cohomology, Lemma 91.3. □

8. Derived lower shriek via compactifications

0F7H Let $f : X \to Y$ be a finite type separated morphism of schemes with Y quasi-compact and quasi-separated. Choose a compactification $j : X \to \overline{X}$ over Y, see More on Flatness, Theorem 33.8. Given a torsion ring Λ we define

$$Rf_! = R\overline{f}_* \circ j_! : D(X_{étale}, \Lambda) \longrightarrow D(Y_{étale}, \Lambda)$$

Here is the obligatory lemma.

0F7I **Lemma 8.1.** *Let $f : X \to Y$ be a finite type separated morphism of quasi-compact and quasi-separated schemes. The functor $Rf_!$ is, up to canonical isomorphism, independent of the choice of the compactification.*

Proof. Consider the category of compactifications of X over Y, which is cofiltered according to More on Flatness, Theorem 33.8 and Lemmas 32.1 and 32.2. To every choice of a compactification

$$j : X \to \overline{X}, \quad \overline{f} : \overline{X} \to Y$$

the construction above associates the functor $R\overline{f}_* \circ j_! : D(X_{étale}, \Lambda) \to D(Y_{étale}, \Lambda)$. Let's be a little more explicit. Given a complex $\mathcal{K}^\bullet$ of sheaves of Λ-modules on $X_{étale}$, we choose a quasi-isomorphism $j_!\mathcal{K}^\bullet \to \mathcal{I}^\bullet$ into a K-injective complex of sheaves of Λ-modules on $\overline{X}_{étale}$. Then our functor sends $\mathcal{K}^\bullet$ to $\overline{f}_*\mathcal{I}^\bullet$.

Suppose given a morphism $g : \overline{X}_1 \to \overline{X}_2$ between compactifications $j_i : X \to \overline{X}_i$ over Y. Then we get an isomorphism

$$R\overline{f}_{2,*} \circ j_{2,!} = R\overline{f}_{2,*} \circ Rg_* \circ j_{1,!} = R\overline{f}_{1,*} \circ j_{1,!}$$

using Lemma 7.6 in the first equality.

To finish the proof, since the category of compactifications of X over Y is cofiltered, it suffices to show compositions of morphisms of compactifications of X over Y are turned into compositions of isomorphisms of functors[3]. To do this, suppose that $j_3 : X \to \overline{X}_3$ is a third compactification and that $h : \overline{X}_2 \to \overline{X}_3$ is a morphism of compactifications. Then we have to show that the composition

$$R\overline{f}_{3,*} \circ j_{3,!} = R\overline{f}_{3,*} \circ Rh_* \circ j_{2,!} = R\overline{f}_{2,*} \circ j_{2,!} = R\overline{f}_{2,*} \circ Rg_* \circ j_{1,!} = R\overline{f}_{1,*} \circ j_{1,!}$$

is equal to the isomorphism of functors constructed using simply j_3, $g \circ h$, and j_1. A calculation shows that it suffices to prove that the composition of the maps

$$j_{3,!} \to Rh_* \circ j_{2,!} \to Rh_* \circ Rg_* \circ j_{1,!}$$

of Lemma 7.6 agrees with the corresponding map $j_{3,!} \to R(h \circ g)_* \circ j_{1,!}$ via the identification $R(h \circ g)_* = Rh_* \circ Rg_*$. Since the map of Lemma 7.6 is a special case of the map of Lemma 7.1 (as j_1 and j_2 are separated) this follows immediately from Lemma 7.2. □

[3] Namely, if $\alpha, \beta : F \to G$ are morphisms of functors and $\gamma : G \to H$ is an isomorphism of functors such that $\gamma \circ \alpha = \gamma \circ \beta$, then we conclude $\alpha = \beta$.

0F7J **Lemma 8.2.** *Let $f : X \to Y$ and $g : Y \to Z$ be separated morphisms of finite type of quasi-compact and quasi-separated schemes. Then there is a canonical isomorphism $Rg_! \circ Rf_! \to R(g \circ f)_!$.*

Proof. Choose a compactification $i : Y \to \overline{Y}$ of Y over Z. Choose a compactification $X \to \overline{X}$ of X over $\overline{Y}$. This uses More on Flatness, Theorem 33.8 and Lemma 32.2 twice. Let U be the inverse image of Y in $\overline{X}$ so that we get the commutative diagram

$$\begin{tikzcd} X \arrow[r, "j"'] \arrow[d, "f"'] & U \arrow[r, "j'"'] \arrow[dl, "f'"] & \overline{X} \arrow[dl, "\overline{f}"] \\ Y \arrow[r, "i"'] \arrow[d, "g"'] & \overline{Y} \arrow[dl, "\overline{g}"] & \\ Z & & \end{tikzcd}$$

Then we have

$$\begin{aligned} R(g \circ f)_! &= R(\overline{g} \circ \overline{f})_* \circ (j' \circ j)_! \\ &= R\overline{g}_* \circ R\overline{f}_* \circ j'_! \circ j_! \\ &= R\overline{g}_* \circ i_! \circ Rf'_* \circ j_! \\ &= Rg_! \circ Rf_! \end{aligned}$$

The first equality is the definition of $R(g \circ f)_!$. The second equality uses the identifications $R(\overline{g} \circ \overline{f})_* = R\overline{g}_* \circ R\overline{f}_*$ and $(j' \circ j)_! = j'_! \circ j_!$ of Lemma 3.13. The identification $i_! \circ Rf'_* \to R\overline{f}_* \circ j_!$ used in the third equality is Lemma 7.1. The final fourth equality is the definition of $Rg_!$ and $Rf_!$. To finish the proof we show that this isomorphism is independent of choices made.

Suppose we have two diagrams

$$\begin{tikzcd} X \arrow[r, "j_1"'] \arrow[d] & U_1 \arrow[r, "j'_1"'] \arrow[dl, "f_1"] & \overline{X}_1 \arrow[dl, "\overline{f}_1"] \\ Y \arrow[r, "i_1"'] \arrow[d] & \overline{Y}_1 \arrow[dl, "\overline{g}_1"] & \\ Z & & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} X \arrow[r, "j_2"'] \arrow[d] & U_2 \arrow[r, "j'_2"'] \arrow[dl, "f_2"] & \overline{X}_2 \arrow[dl, "\overline{f}_2"] \\ Y \arrow[r, "i_2"'] \arrow[d] & \overline{Y}_2 \arrow[dl, "\overline{g}_2"] & \\ Z & & \end{tikzcd}$$

We can first choose a compactification $i : Y \to \overline{Y}$ of Y over Z which dominates both $\overline{Y}_1$ and $\overline{Y}_2$, see More on Flatness, Lemma 32.1. By More on Flatness, Lemma 32.3 and Categories, Lemmas 26.13 and 26.14 we can choose a compactification $X \to \overline{X}$ of X over $\overline{Y}$ with morphisms $\overline{X} \to \overline{X}_1$ and $\overline{X} \to \overline{X}_2$ and such that the composition $\overline{X} \to \overline{Y} \to \overline{Y}_1$ is equal to the composition $\overline{X} \to \overline{X}_1 \to \overline{Y}_1$ and such that the composition $\overline{X} \to \overline{Y} \to \overline{Y}_2$ is equal to the composition $\overline{X} \to \overline{X}_2 \to \overline{Y}_2$. Thus we see that it suffices to compare the maps determined by our diagrams when we have a commutative diagram as follows

$$
\begin{tikzcd}
X \arrow[r, "j_1"'] \arrow[d, equal] & U_1 \arrow[r, "j_1'"'] \arrow[d, "h'"] \arrow[ddl] & \overline{X}_1 \arrow[d, "h"] \arrow[ddl] \\
X \arrow[r, "j_2"] \arrow[d] & U_2 \arrow[r, "j_2'"] \arrow[ddl] & \overline{X}_2 \arrow[ddl] \\
Y \arrow[r, "i_1"] \arrow[d, equal] & \overline{Y}_1 \arrow[d, "k"] & \\
Y \arrow[r, "i_2"] \arrow[d] & \overline{Y}_2 \arrow[dl] & \\
Z & &
\end{tikzcd}
$$

Each of the squares

$$
\begin{tikzcd}
X \arrow[r, "j_1"'] \arrow[d, "\mathrm{id}"'] \arrow[dr, phantom, "A"] & U_1 \arrow[d, "h'"] \\
X \arrow[r, "j_2"] & U_2
\end{tikzcd}
\quad
\begin{tikzcd}
U_2 \arrow[r, "j_2'"'] \arrow[d, "f_2"'] \arrow[dr, phantom, "B"] & \overline{X}_2 \arrow[d, "\overline{f}_2"] \\
Y \arrow[r, "i_2"] & \overline{Y}_2
\end{tikzcd}
\quad
\begin{tikzcd}
U_1 \arrow[r, "j_1'"'] \arrow[d, "f_1"'] \arrow[dr, phantom, "C"] & \overline{X}_1 \arrow[d, "\overline{f}_1"] \\
Y \arrow[r, "i_1"] & \overline{Y}_1
\end{tikzcd}
\quad
\begin{tikzcd}
Y \arrow[r, "i_1"'] \arrow[d, "\mathrm{id}"'] \arrow[dr, phantom, "D"] & \overline{Y}_1 \arrow[d, "k"] \\
Y \arrow[r, "i_2"] & \overline{Y}_2
\end{tikzcd}
\quad
\begin{tikzcd}
X \arrow[r, "j_1' \circ j_1"'] \arrow[d, "\mathrm{id}"'] \arrow[dr, phantom, "E"] & \overline{X}_1 \arrow[d, "h"] \\
X \arrow[r, "j_2"] & \overline{X}_2
\end{tikzcd}
$$

gives rise to an isomorphism as follows

$$
\begin{aligned}
\gamma_A &: j_{2,!} \to Rh'_* \circ j_{1,!} \\
\gamma_B &: i_{2,!} \circ Rf_{2,*} \to R\overline{f}_{2,*} \circ j'_{2,!} \\
\gamma_C &: i_{1,!} \circ Rf_{1,*} \to R\overline{f}_{1,*} \circ j'_{1,!} \\
\gamma_D &: i_{2,!} \to Rk_* \circ i_{1,!} \\
\gamma_E &: j_{2,!} \to Rh_* \circ (j'_1 \circ j_1)_!
\end{aligned}
$$

by applying the map from Lemma 7.1 (which is the same as the map in Lemma 7.6 in case the left vertical arrow is the identity). Let us write

$$
\begin{aligned}
F_1 &= Rf_{1,*} \circ j_{1,!} \\
F_2 &= Rf_{2,*} \circ j_{2,!} \\
G_1 &= R\overline{g}_{1,*} \circ i_{1,!} \\
G_2 &= R\overline{g}_{2,*} \circ i_{2,!} \\
C_1 &= R(\overline{g}_1 \circ \overline{f}_1)_* \circ (j'_1 \circ j_1)_! \\
C_2 &= R(\overline{g}_2 \circ \overline{f}_2)_* \circ (j'_2 \circ j_2)_!
\end{aligned}
$$

The construction given in the first paragraph of the proof and in Lemma 8.1 uses

(1) γ_C for the map $G_1 \circ F_1 \to C_1$,
(2) γ_B for the map $G_2 \circ F_2 \to C_2$,
(3) γ_A for the map $F_2 \to F_1$,
(4) γ_D for the map $G_2 \to G_1$, and
(5) γ_E for the map $C_2 \to C_1$.

This implies that we have to show that the diagram

$$\begin{tikzcd} C_2 \arrow[r, "\gamma_E"'] & C_1 \\ G_2 \circ F_2 \arrow[u, "\gamma_B"] \arrow[r, "\gamma_D \circ \gamma_A"] & G_1 \circ F_1 \arrow[u, "\gamma_C"'] \end{tikzcd}$$

is commutative. We will use Lemmas 7.2 and 7.3 and with (abuse of) notation as in Remark 7.4 (in particular dropping $\star$ products with identity transformations from the notation). We can write $\gamma_E = \gamma_F \circ \gamma_A$ where

$$\begin{tikzcd} U_1 \arrow[r, "j_1'"'] \arrow[d, "h'"'] \arrow[dr, phantom, "F"] & \overline{X}_1 \arrow[d, "h"] \\ U_2 \arrow[r, "j_2'"] & \overline{X}_2 \end{tikzcd}$$

Thus we see that

$$\gamma_E \circ \gamma_B = \gamma_F \circ \gamma_A \circ \gamma_B = \gamma_F \circ \gamma_B \circ \gamma_A$$

the last equality because the two squares A and B only intersect in one point (similar to the last argument in Remark 7.4). Thus it suffices to prove that $\gamma_C \circ \gamma_D = \gamma_F \circ \gamma_B$. Since both of these are equal to the map for the square

$$\begin{tikzcd} U_1 \arrow[r] \arrow[d] & \overline{X}_1 \arrow[d] \\ Y \arrow[r] & \overline{Y}_2 \end{tikzcd}$$

we conclude. $\square$

0F7K **Lemma 8.3.** *Let $f : X \to Y$, $g : Y \to Z$, $h : Z \to T$ be separated morphisms of finite type of quasi-compact and quasi-separated schemes. Then the diagram*

$$\begin{tikzcd} Rh_! \circ Rg_! \circ Rf_! \arrow[r, "\gamma_C"'] \arrow[d, "\gamma_A"] & R(h \circ g)_! \circ Rf_! \arrow[d, "\gamma_{A+B}"'] \\ Rh_! \circ R(g \circ f)_! \arrow[r, "\gamma_{B+C}"] & R(h \circ g \circ f)_! \end{tikzcd}$$

of isomorphisms of Lemma 8.2 commutes (for the meaning of the γ's see proof).

Proof. To do this we choose a compactification $\overline{Z}$ of Z over T, then a compactification $\overline{Y}$ of Y over $\overline{Z}$, and then a compactification $\overline{X}$ of X over $\overline{Y}$. This uses More on Flatness, Theorem 33.8 and Lemma 32.2. Let $W \subset \overline{Y}$ be the inverse image of Z under $\overline{Y} \to \overline{Z}$ and let $U \subset V \subset \overline{X}$ be the inverse images of $Y \subset W$ under $\overline{X} \to \overline{Y}$.

This produces the following diagram

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "f"'] & U \arrow[r] \arrow[d] \arrow[dr, phantom, "A"] & V \arrow[r] \arrow[d] \arrow[dr, phantom, "B"] & \overline{X} \arrow[d] \\
Y \arrow[r] \arrow[d, "g"'] & Y \arrow[r] \arrow[d] & W \arrow[r] \arrow[d] \arrow[dr, phantom, "C"] & \overline{Y} \arrow[d] \\
Z \arrow[r] \arrow[d, "h"'] & Z \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d] & \overline{Z} \arrow[d] \\
T \arrow[r] & T \arrow[r] & T \arrow[r] & T
\end{tikzcd}$$

Without introducing tons of notation but arguing exactly as in the proof of Lemma 8.2 we see that the maps in the first displayed diagram use the maps of Lemma 7.1 for the rectangles $A+B$, $B+C$, A, and C as indicated in the diagram in the statement of the lemma. Since by Lemmas 7.2 and 7.3 we have $\gamma_{A+B} = \gamma_B \circ \gamma_A$ and $\gamma_{B+C} = \gamma_B \circ \gamma_C$ we conclude that the desired equality holds provided $\gamma_A \circ \gamma_C = \gamma_C \circ \gamma_A$. This is true because the two squares A and C only intersect in one point (similar to the last argument in Remark 7.4). $\square$

0F7L **Lemma 8.4.** *Consider a cartesian square*

$$\begin{tikzcd}
X' \arrow[r, "g'"'] \arrow[d, "f'"'] & X \arrow[d, "f"] \\
Y' \arrow[r, "g"] & Y
\end{tikzcd}$$

of quasi-compact and quasi-separated schemes with f separated and of finite type. Then there is a canonical isomorphism

$$g^{-1} \circ Rf_! \to Rf'_! \circ (g')^{-1}$$

Moreover, these isomorphisms are compatible with the isomorphisms of Lemma 8.2.

Proof. Choose a compactification $j : X \to \overline{X}$ over Y and denote $\overline{f} : \overline{X} \to Y$ the structure morphism. Let $j' : X' \to \overline{X}'$ and $\overline{f}' : \overline{X}' \to Y'$ denote the base changes of j and $\overline{f}$. Since $Rf_! = R\overline{f} \circ j_!$ and $Rf'_! = R\overline{f}' \circ j'_!$ the isomorphism can be constructed via

$$g^{-1} \circ R\overline{f} \circ j_! \to R\overline{f}' \circ (\overline{g}')^{-1} \circ j_! \to R\overline{f}' \circ j'_! \circ (g')^{-1}$$

where the first arrow is the isomorphism given to us by the proper base change theorem (Étale Cohomology, Lemma 88.4) and the second arrow is the isomorphism of Lemma 3.12.

To finish the proof we have to show two things: first we have to show that the isomorphism of functors so obtained does not depend on the choice of the compactification and second we have to show that if we vertically stack two base change diagrams as in the lemma, then these base change isomorphisms are compatible with the isomorphisms of Lemma 8.2. A straightforward argument which we omit shows that both follow if we can show that the isomorphisms

(1) $Rg_* \circ Rf_* = R(g \circ f)_*$ for $f : X \to Y$ and $g : Y \to Z$ proper,

(2) $g_! \circ f_! = (g \circ f)_!$ for $f : X \to Y$ and $g : Y \to Z$ separated and quasi-finite, and
(3) $g_! \circ Rf'_* = Rf_* \circ g'_!$ for $f : X \to Y$ and $f' : X' \to Y'$ proper and $g : Y' \to Y$ and $g' : X' \to X$ separated and quasi-finite with $f \circ g' = g \circ f'$

are compatible with base change. This holds for (1) by Cohomology on Sites, Remark 20.4, for (2) by Remark 3.14, and (3) by Lemma 7.5. $\square$

9. Other chapters

References

[AGV71] Michael Artin, Alexander Grothendieck, and Jean-Louis Verdier, *Theorie de topos et cohomologie etale des schemas I, II, III*, Lecture Notes in Mathematics, vol. 269, 270, 305, Springer, 1971.